University of New South Wales

School of Mathematics and Statistics

Assignment 1

Lie Groups

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Question 1

Let R be a commutative ring, and A is an R algebra. Suppose X and Y are R-linear derivations on A, that is $X, Y : A \to A$ and for all $f, g \in A$ and $r \in R$,

$$\begin{aligned} X(fg) &= X(f)g + fX(g) \\ X(f+g) &= X(f) + X(g) \\ X(rf) &= rX(f) \end{aligned}$$

and similarly for Y.

Theorem 1. *The commutator $[X,Y] = XY - YX$ is an R-linear derivation on A.*

Proof. Clearly $[X,Y]$ is R-linear, since it is a sum of compositions of R-linear maps.

Now suppose $f, g \in A$. Then we simply must expand out,

$$\begin{aligned} [X,Y](fg) &= XY(fg) - YX(fg) \\ &= X(Y(f)g + fY(g)) - Y(X(f)g + fX(g)) \\ &= X(Y(f)g) + X(fY(g)) - Y(X(f)g) - Y(fX(g)) \\ &= (XY)(f)g + Y(f)X(g) + X(f)Y(g) + f(XY)(g) - (YX)(f)g - X(f)Y(g) - \\ &\quad Y(f)X(g) - f(YX)(g) \\ &= [X,Y](f)g + f[X,Y](g) \end{aligned}$$

□

Note that we did not need to assume the commutativity of A.

Question 2

For this question, M and M' are smooth manifolds and $f : M \to M'$ is smooth. We say that vector fields $X \in \mathfrak{X}(M)$ and $X' \in \mathfrak{X}(M')$ are f-related if $df_p(X) = X'_{f(p)}$ for all $p \in M$.

Theorem 2. *If $X, Y \in \mathfrak{X}(M)$ are f-related to $X', Y' \in \mathfrak{X}(M')$ respectively, then $[X,Y]$ is f-related to $[X', Y']$.*

Proof. We must compute $df_p[X,Y]$. To do this, let $h \in C^\infty(M')$ and $p \in M$. Then by definition,

$$\begin{aligned}
df_p[X,Y](h) &= [X,Y](h\circ f)(p)\\
&= (XY)(h\circ f)(p) - (YX)(h\circ f)(p)\\
&= X(df(Y)(h))(p) - Y(df(X)(h))(p)\\
&= X(Y'(h)\circ f)(p) - Y(X'(h)\circ f)(p)\\
&= df_p(X)(Y'(h)) - df_p(Y)(X'(h))\\
&= X'Y'(h\circ f) - Y'X'(h\circ f)\\
&= [X',Y'](h(f(p)))
\end{aligned}$$

Hence, $df_p[X,Y] = [X',Y']_{f(p)}$. □

Question 3

For this question, G and H are Lie groups with identity elements e_G and e_H respectively. We denote the Lie algebras $T_{e_G}G$ and $T_{e_H}H$ as $\mathfrak{g}$ and $\mathfrak{h}$ respectively.

Theorem 3. *If $f : G \to H$ is a Lie group homomorphism, then $df_{e_G} : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.*

Proof. We already know that df_{e_G} is a linear map between vector spaces $\mathfrak{g}$ and $\mathfrak{h}$. It remains to prove that

$$df_{e_G}[X,Y] = [df_{e_G}X, df_{e_H}Y].$$

For $X, Y \in \mathfrak{g}$. However this is precisely the result of question 2, so df_{e_G} is a Lie algebra homomorphism. □

Question 4

For this question, $\mathbb{H}$ is the set of real 3×3 upper triangular matrices equal to 1 on the main diagonal.

(a)

Lemma 1. $\mathbb{H}$ *is a Lie subgroup of* $GL(n,\mathbb{R})$.

Proof. Clearly $\mathbb{H} \subset GL(n,\mathbb{R})$ since each element of $\mathbb{H}$ has determinant 1. We also see the identity matrix is in $\mathbb{H}$.

Thus it is sufficient to prove that $\mathbb{H}$ is closed under matrix multiplication. Let $a, b, c, d, e, f \in \mathbb{R}$, then

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & d & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+d & e+b+af \\ 0 & 1 & f+c \\ 0 & 0 & 1 \end{pmatrix}$$

Hence $\mathbb{H}$ is closed under matrix multiplication so is a Lie subgroup of $GL(n,\mathbb{R})$ □

(b)

Now that we know $\mathbb{H}$ is a Lie group, denote its Lie algebra by $\mathfrak{h}$.

Lemma 2. *The tangent space at the identity* $T_I\mathbb{H}$ *can be described by the set of matrices*

$$\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$$

for $a, b, c \in \mathbb{R}$.

Proof. □

Theorem 4. *Thus the Lie Bracket on* $\mathfrak{h}$ *is given by*

(c)

Lemma 3. $[\mathfrak{h}, \mathfrak{h}] = \mathbb{R}e_2$

Proof. □

Lemma 4. $[\mathfrak{h}, [\mathfrak{h}, \mathfrak{h}]] = 0$.

Proof. □