UNSW School of Mathematics and Statistics

MATH5825 Measure, Integration and Probability

Semester 2/2014

Homework Week 2

1. Let $X = \{1, 2, 3, 4, 5\}$.
   (a) Let
   $$\mathcal{C} = \{\emptyset, X, \{1\}, \{2, 3\}, \{1, 2, 3\}, \{4, 5\}\}$$
   Write down $\sigma(\mathcal{C})$.
   (b) Define γ on $\mathcal{C}$ by:
   $$\gamma(\emptyset) = 0, \qquad \gamma(\{1\}) = 1, \qquad \gamma(\{2, 3\}) = 1$$
   $$\gamma(\{1, 2, 3\}) = 2, \qquad \gamma(\{4, 5\}) = 1, \qquad \gamma(X) = 3$$
   Let μ^* be the outer measure, defined on 2^X. Is μ^*, if restricted to $\sigma(\mathcal{C})$, a measure?

2. Let μ^* be an outer measure as in Definition 1.12. Show the properties
   (a) $\mu^*(\emptyset) = 0$
   (b) If $A, B \in 2^X$ such that $A \subset B$, then $\mu^*(A) \leq \mu^*(B)$.

3. Let $X = \mathbb{R}$, and let $\mathcal{T}$ be the system of countably infinite unions of intervals $\{(a, b) : a, b \in \mathbb{R} \cup \{-\infty, +\infty\}\}$. Show that $(\mathbb{R}, \mathcal{T})$ is a topological space. The σ-algebra $\sigma(\mathcal{T})$ is called the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ on $\mathbb{R}$. Show that the Borel σ-algebra is generated by any of the following systems:
   $$\mathcal{E}_1 = \{(a, b) : a, b \in \mathbb{R}\}, \quad \mathcal{E}_2 = \{[a, b] : a, b \in \mathbb{R}\},$$
   $$\mathcal{E}_3 = \{(a, b] : a, b \in \mathbb{R}\}, \quad \mathcal{E}_4 = \{[a, b) : a, b \in \mathbb{R}\}$$
   (Hint: Show that a set U is open iff $\forall x \in U \, \exists \varepsilon > 0 : (x - \varepsilon, x + \varepsilon) \subset U$ using that $\mathbb{Q}$ is dense in $\mathbb{R}$.) Is $\mathcal{B}(\mathbb{R})$ generated also generated by
   $$\mathcal{E}_5 = \{(a, b) : a, b \in \mathbb{Q}\}?$$

4. In older textbooks, the Borel sets are often introduced as the smallest family $\mathcal{M}$ of sets which
   - contains all open sets (that is, elements of $\mathcal{T}$)
   - is closed under countable intersections and countable unions.

   Show that
   (a) $\mathcal{M}$ is well-defined and $\mathcal{M} \subset \sigma(\mathcal{T})$; (Hint: proceed as with σ-algebras)
   (b) If $U \in \mathcal{T}$, then $U^{\complement} \in \mathcal{M}$, that is, $\mathcal{M}$ contains all closed sets;
   (hint: any closed set can be written as the countable intersection of suitable open sets)
   (c) $\{B \in \mathcal{M} : B^{\complement} \in \mathcal{M}\}$ is a σ-algebra;
   (d) $\sigma(\mathcal{T}) \subset \{B \in \mathcal{M} : B^{\complement} \in \mathcal{M}\} \subset \mathcal{M}$

   Hence $\sigma(\mathcal{T}) = \mathcal{M}$.