UNSW School of Mathematics and Statistics

MATH5825 Measure, Integration and Probability

Semester 2/2014

Homework Week 3

1. In much the same way as done in class, Lebesgue measure can be defined in $\mathbb{R}^d$, for $d > 1$; see the remark at the end of Lecture notes, week 2. Show that every straight line has measure zero.

2. Let $(X, \mathcal{F}, \mu)$ be a finite measure space. Assume that $\mathcal{F} = \sigma(\mathcal{A})$ for a certain algebra $\mathcal{A}$ of subsets of X.

   (a) Recall the symmetric set difference $A\Delta B = (A \setminus B) \cup (B \setminus A)$. Verify that $A^{\complement}\Delta B^{\complement} = A\Delta B$ and $A\Delta C \subset (A\Delta B) \cup (B\Delta C)$.

   (b) Define
   $$\mathcal{G} := \{B \in \mathcal{F} : \forall\varepsilon > 0\, \exists B_\varepsilon \in \mathcal{A} \text{ such that } \mu(B_\varepsilon \Delta B) < \varepsilon\}$$
   Show that $B \in \mathcal{G} \Leftrightarrow B^{\complement} \in \mathcal{G}$

   (c) Let $A_1 \subset A_2 \subset \ldots$ be an increasing sequence of sets in $\mathcal{G}$. Show that $\forall\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $\mu(A\Delta A_N) < \varepsilon$.

   (d) Show that $A = \bigcup_n A_n \in \mathcal{G}$.

   (e) Deduce that $\mathcal{G} = \mathcal{F}$ using the monotone class theorem.

   Hence any set B from a σ-algebra $\mathcal{F}$ can be "approximated" by a member B_ε of an algebra $\mathcal{A}$ which generates $\mathcal{F}$.

3. Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $\mathcal{A}_\mu$ and $\bar{\mu}$ be defined as in lecture. Prove that

   (a) $\mathcal{A}_\mu$ is a σ-algebra on X

   (b) $\bar{\mu}$ is a measure on $(X, \mathcal{A}_\mu)$

   (c) The restriction of $\bar{\mu}$ from $\mathcal{A}_\mu$ to $\mathcal{A}$ is μ.

   (d) The measure space $(X, \mathcal{A}_\mu, \bar{\mu})$ is complete.

   (Compare Proposition 1.30.)