University of New South Wales

School of Mathematics and Statistics

Assignment 1

Measure Theory

Author:
Edward McDonald

Student Number:
3375335

Question 2

Let X be a set, and $(Y, \mathcal{B})$ is a measurable space. $f : X \to Y$ is a function.

Lemma 1. *The collection of subsets of X, $\mathcal{A} = \{f^{-1}(B) \; : \; B \in \mathcal{B}\}$ is a σ-algebra and the smallest σ-algebra on X such that f is measurable.*

Proof. Clearly $X = f^{-1}(Y)$ and $\emptyset = f^{-1}(\emptyset)$ are in $\mathcal{A}$, and if $A = f^{-1}(B) \in \mathcal{A}$ then $A^c = f^{-1}(B^c) \in \mathcal{A}$.

Now if $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$, with $A_n = f^{-1}(B_n)$, then $\bigcup_n A_n = f^{-1}(\bigcup_n B_n) \in \mathcal{A}$.

Hence $\mathcal{A}$ is a σ-algebra.

If $\mathcal{A}'$ is any other σ-algebra on X such that f is $\mathcal{A}'/\mathcal{B}$ measurable, then by definition for any $B \in \mathcal{B}$, $f^{-1}(B) \in \mathcal{A}'$, so we must have $\mathcal{A} \subseteq \mathcal{A}'$.

□

Question 3

In this question, $(X, \mathcal{A}, \mu)$ is a measure space.

Suppose $\{A_n\} n \geq 0$ is a sequence of sets in $\mathcal{A}$ then the following holds:

Lemma 2.

$$\inf_n \chi_{A_n} = \chi_{\bigcap_n A_n}$$

and

$$\sup_n \chi_{A_n} = \chi_{\bigcup_n A_n}$$

Proof. Let $x \in X$. If $\inf_n \chi_{A_n}(x) = 1$ means that there is no k such that $x \notin A_n$. Hence $x \in \bigcap_n A_n$.

Similarly, if $x \in \bigcap_n A_n$, then $\inf_n \chi_{A_n}(x) = 1$ since for any n, $\chi_{A_n}(x) = 1$.

Now we write, using $\chi_{B^c} = 1 - \chi_B$,

$$\begin{aligned}
\inf_n \chi_{A_n^c} &= \chi_{\bigcup_n A_n^c} \\
1 - \inf_n \chi_{A_n^c} &= \chi_{\bigcap_n A_n} \\
\sup_n 1 - \chi_{A_n^c} &= \chi_{\bigcap_n A_n} \\
\sup_n \chi_{A_n} &= \chi_{\bigcap_n A_n}
\end{aligned}$$

□

Now we define

$$\liminf_n A_n := \bigcup_n \bigcap_{k\geq n} A_k.$$

Theorem 1. *The following are equivalent,*

$$x \in \liminf_n A_n$$

$$\liminf_n \chi_{A_n}(x) = 1$$

and $x \in A_n$ for all but finitely many n.

Proof. Using lemma 1, we write

$$\begin{aligned}\liminf_n \chi_{A_n}(x) &= \sup_n \inf_{k\geq n} \chi_{A_k} \\ &= \sup_n \chi_{\bigcap_{k\geq n}} A_k \\ &= \chi_{\liminf_n A_n}(x).\end{aligned}$$

Hence $x \in \liminf_n A_n$ if and only if $\liminf_n \chi_{A_n} = 1$.

If $\liminf_n \chi_{A_n}(x) = 1$. then 1 is the only limit point of the sequence $\chi_{A_n}(x)$, hence since χ_{A_n} takes only the values 0 and 1, it must take the value 0 only finitely many times. Hence $x \in A_n$ for all but finitely many n.

Conversely, if $x \in A_n$ for all but finitely many n, then the numerical sequence $\chi_{A_n}(x)$ takes the value 0 only finitely many times. Since it must have a limit point, we conclude $\liminf_n \chi_{A_n}(x) = 1$. □

Now we define

$$\limsup_n A_n = \bigcap_n \bigcup_{k\geq n} A_k$$

Theorem 2. *The following are equivalent:*

$$x \in \limsup_n A_n$$

$$\limsup_n \chi_{A_n}(x) = 1$$

and $x \notin A_n$ for infinitely many n.

Proof. The equivalence of the first two statements is identical to theorem 1.

For the third statement, if $\limsup_n \chi_{A_n}(x) = 1$ then the numerical sequence $\chi_{A_n}(x)$ has 1 as a limit point, so x must be in A_n infinitely often.

Conversely, if x is in A_n infinitely often then 1 is a limit point of the sequence $\chi_{A_n}(x)$. Hence it must be the largest limit point so $\limsup_n \chi_{A_n}(x) = 1$. □

Hence it is clear that $\liminf_n A_n \subseteq \limsup_n A_n$ since $\chi_{\liminf_n A_n}(x) = \liminf_n \chi_{A_n}(x) \leq \limsup_n \chi_{A_n}(x) = \chi_{\limsup_n A_n}$.