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Chapter 1

Introduction to Measures

1.1 Lebesgue's Problem of measure

(This section is taken from Hendrik Grundling's 2011 lecture.) Assigning a measure (length, area, volume, ...) to a subset S of \mathbb{R}^d is necessary in mathematics, e.g. for any type of integral. A reasonable measure $m(S)$ of S should satisfy the following requirements:

1. If $S \subset \mathbb{R}^d$ is congruent (after shifts, rotations & reflections) to $T \subset \mathbb{R}^d$, then $m(S) = m(T)$
2. If $S = \bigcup_{i=1}^{\infty} S_i$ where $S_i \cap S_j = \emptyset$ for $i \neq j$, then $m(S) = \sum_{i=1}^{\infty} m(S_i)$ (countable additivity)
3. $m(I) = 1$ for $I = [0, 1]^d \subset \mathbb{R}^d$ (normedness)
4. $m(S) \geq 0$

However this humble attempt is still too much to ask as the following shall show. Below, we construct a set S which cannot satisfy all four

requirements.

For $x, y \in [0, 1]$ define $x \sim y$ if and only if $x - y \in \mathbb{Q}$. This is an equivalence relation, so $[0, 1]$ is partitioned into (disjoint) equivalence classes. Define the “Vitali set” $S \subset [0, 1]$ by choosing one point from each equivalence class (this assumes the Axiom of Choice).

Lemma 1.1. *If $r, q \in \mathbb{Q} \cap [0, 1]$, $r \neq q$ then $(S + r) \cap (S + q) = \emptyset$.*

Proof. If $x \in (S + r) \cap (S + q)$ then $x = p + r = t + q$ for $p, t \in S$. Then $p - t = q - r \neq 0$ and $q - r \in \mathbb{Q}$. So $p \sim t$, $p \neq t$. Since $p, t \in S$, this violates the definition of S . \square

Lemma 1.2. $[0, 1] \subseteq \bigcup \{S + r \mid r \in \mathbb{Q} \cap [-1, 1]\} =: T \subseteq [-1, 2]$.

Proof. Let $x \in [0, 1]$. Then $x \sim p$ for some $p \in S$. Thus $x - p =: r \in \mathbb{Q}$ and as $x, p \in [0, 1]$ it follows that $r \in \mathbb{Q} \cap [-1, 1]$. So $x \in S + r$, $r \in \mathbb{Q} \cap [-1, 1]$. \square

Proposition 1.3. *In \mathbb{R} , Lebesgue's problem of measure has no solution.*

Proof. Let S be as above. We show a contradiction to the requirement $m(S + r) = m(S)$ for all r .

$$\begin{aligned} 1 = m([0, 1]) &= m(T) - m(T \setminus [0, 1]) \leq m(T) \\ &= \sum_{r \in \mathbb{Q} \cap [-1, 1]} m(S + r) = \sum_{r \in \mathbb{Q} \cap [-1, 1]} m(S) \end{aligned}$$

$m(S)$ can't be 0, hence the right-hand side must be $+\infty$. But this contradicts

$$\begin{aligned} \infty = m(T) &= m([-1, 2]) - m([-1, 2] \setminus T) \leq m([-1, 2]) \\ &= m([-1, 0]) + m([0, 1]) + m([1, 2]) - m(\{0, 1\}) \leq 3m([0, 1]) = 3 \end{aligned}$$

□

In higher dimensions, things are even worse:

Theorem 1.4 (Banach-Tarski). *Let $S, T \subset \mathbb{R}^d$ be bounded with non-empty interiors, $d \geq 3$. Then there is a $k \in \mathbb{N}$ and partitions $\{E_1, \dots, E_k\}$, $\{F_1, \dots, F_k\}$ of S , T respectively such that E_i is congruent to $F_i \forall i$. (That is, S and T are equidecomposable.)*

Thus we can take e.g. a unit sphere S , cut it up into finitely many pieces, and reassemble it to a set T of two unit spheres. If Lebesgue's problem of measure had a solution in \mathbb{R}^d , $d \geq 3$, then by congruence $m(S) = m(T)$ which contradicts $m(T) = 2m(S)$ and $m(S) > 0$.

A way out of this situation is to restrict attention to a smaller set of subsets, on which Lebesgue's problem of measure does have a solution.

1.2 σ -algebras

Definition 1.5. Let X be any non-empty set. A system \mathcal{A} of subsets of X is called an *algebra* on X if it has the following properties:

1. $X \in \mathcal{A}$,
2. If $\{A, B\} \subset \mathcal{A}$ then $A \cup B \in \mathcal{A}$
3. If $A \in \mathcal{A}$ then $A^c \in \mathcal{A}$ (where $A^c = X \setminus A$).

If additionally the following condition holds, then \mathcal{A} is called a *σ -algebra*:

4. If $\{A_1, A_2, \dots\} \subset \mathcal{A}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$.

A *measurable space* is any pair (X, \mathcal{A}) where \mathcal{A} is a σ -algebra on X .

Algebras are closed under finite intersections: If $\{A, B\} \subset \mathcal{A}$, then

$$(A \cap B)^c = A^c \cup B^c \in \mathcal{A}.$$

Similarly, σ -algebras are closed under countably infinite intersections.

Definition 1.6. Let X be a non-empty set. A system $\mathcal{C} \subset 2^X$ of subsets of X is called a *ring* if $\emptyset \in \mathcal{C}$ and $A \cup B, A \setminus B \in \mathcal{C}$ for any $A, B \in \mathcal{C}$.

Every σ -algebra is an algebra; every finite algebra is a σ -algebra.

Example 1.7. Let X be any non-empty set, and consider the following subsets of 2^X :

1. $\mathcal{A} = \{\emptyset, X\}$. This is the “coarsest”, “smallest” or “trivial” σ -algebra on X (and thus also an algebra and a ring).
2. $\mathcal{A} = \{\emptyset, A, A^c, X\}$, where $A \subset X$. This is a σ -algebra, an algebra and a ring.
3. The power set 2^X , i.e. the family of all subsets of X , where X is any set (the “finest” or “largest” σ -algebra on X).

4. Let $X = \mathbb{R}$. Then

$$\mathcal{C} := \left\{ \bigcup_{k=1}^n (a_k, b_k] : n \in \mathbb{N}, a_k, b_k \in \mathbb{R}, a_k \leq b_k, b_k \leq a_{k+1} \right\} \subset 2^{\mathbb{R}}. \quad (1.8)$$

is a ring, but not an algebra. $\mathcal{C} \cup \mathbb{R}$ is an algebra, but not a σ -algebra. We shall later see that the σ -algebra generated by \mathcal{C} is the Borel- σ -algebra on \mathbb{R} . \square

If two algebras (or σ -algebras) $\mathcal{A}_1, \mathcal{A}_2$ are defined on the same set X , then their intersection (not necessarily their union) is again an algebra (or σ -algebra) (try a simple example). This even holds for *arbitrary* (uncountably infinite) intersections.

Lemma 1.9. *Let X be any set, and let \mathcal{E} be a system of subsets of X . Then there exist*

1. *a smallest algebra $\alpha(\mathcal{E}) \supset \mathcal{E}$*
2. *a smallest σ -algebra $\sigma(\mathcal{E}) \supset \mathcal{E}$*

Proof. Let \mathcal{S}_α be the set of algebras which contain \mathcal{E} , and let \mathcal{S}_σ be the set of σ -algebras which contain \mathcal{E} . Then

$$2^X \in \mathcal{S}_\sigma \subset \mathcal{S}_\alpha,$$

where 2^X is the power set of X (the set of all possible subsets). In particular neither \mathcal{S}_α nor \mathcal{S}_σ are empty. Then

$$\alpha(\mathcal{E}) = \bigcap_{\mathcal{A} \in \mathcal{S}_\alpha} \mathcal{A} \qquad \sigma(\mathcal{E}) = \bigcap_{\mathcal{A} \in \mathcal{S}_\sigma} \mathcal{A}$$

are as required. □

The algebra and σ -algebra $\alpha(\mathcal{E})$ and $\sigma(\mathcal{E})$ are said to be **generated** by \mathcal{E} .

1.3 Measures

Definition 1.10. A countably additive function μ from a σ -algebra \mathcal{A} of subsets of X into $[0, \infty]$ is called a *measure*. Then (X, \mathcal{A}, μ) is called a *measure space*.

Example 1.11. 1. For a set X with σ -algebra \mathcal{A} , define

$$\mu : \mathcal{A} \rightarrow [0, \infty]$$

$$A \mapsto \mu(A) = \text{number of elements in } A$$

Such μ is called the counting measure.

2. For a σ -algebra \mathcal{A} of X , fix $x \in X$ and define

$$\delta_x : \mathcal{A} \rightarrow [0, \infty]$$

$$A \mapsto \delta_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Then δ_x is called the Dirac measure concentrated at x .

3. For a σ -algebra \mathcal{A} on X let $\mu(\emptyset) = 0$ and $\mu(A) = \infty$ if $A \neq \emptyset$, $A \in \mathcal{A}$. Then μ is called the trivial measure.

The aim of this section is to construct a measure on a suitable σ -algebra of \mathbb{R} which solves Lebesgue's problem of measure. Other measurable spaces which are important in probability theory are:

- $(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty))$: This space is used for probabilistic models of experiments with infinitely many steps, that is, stochastic processes in discrete time steps.
- $(D(\mathbb{R}^d), \mathcal{D})$: Here $D(\mathbb{R}^d)$ denotes the set of càdlàg (right-continuous with left limits) paths (mappings from $[0, \infty)$ to \mathbb{R}^d) with countably many jumps. This set can be equipped with a metric and hence a topology. The corresponding Borel- σ -algebra is \mathcal{D} . A probability measure ($\mu(D(\mathbb{R}^d)) = 1$) on this measurable space then governs the behaviour of a stochastic process in continuous time.

Theorem 1.12 (Basic properties of measures). *Let (X, \mathcal{A}, μ) be a measure space. Then*

1. $\mu(\emptyset) = 0$

2. $A \subset B \Rightarrow \mu(A) \leq \mu(B)$

3. *If $A_1 \subset A_2 \subset A_3 \subset \dots$, then*

$$\lim_{n \rightarrow \infty} \mu(A_n) = \mu \left(\bigcup_{n=1}^{\infty} A_n \right)$$

4. *If $A_1 \supset A_2 \supset A_3 \supset \dots$ where $\mu(A_1) < \infty$, then*

$$\lim_{n \rightarrow \infty} \mu(A_n) = \mu \left(\bigcap_{n=1}^{\infty} A_n \right)$$

Outer Measure

Definition 1.13. Let X be a non-empty set. Let $\mathcal{A} \subset 2^X$, $\emptyset \in \mathcal{A}$, and for any $E \in 2^X$, there exists a covering $\{A_n\}_{n=1}^{\infty} \subset \mathcal{A}$ so that $E \subset \bigcup_{n=1}^{\infty} A_n$. Let $\mu : \mathcal{A} \rightarrow [0, \infty]$ be any map such that $\mu(\emptyset) = 0$. Then

$$\mu^* : 2^X \rightarrow [0, \infty]$$
$$E \mapsto \inf \left\{ \sum_{n=1}^{\infty} \mu(A_n) \mid A_n \in \mathcal{A}, E \subset \bigcup_{n=1}^{\infty} A_n \right\}$$

is called the *outer measure defined by μ and \mathcal{A}* . (A μ as above is called a pre-measure.)

If $A \in 2^X$ satisfies

$$\mu^*(B) = \mu^*(B \cap A) + \mu^*(B \setminus A) \quad \forall B \in 2^X,$$

then A is called *μ^* -measurable*.

Lemma 1.14 (Subadditivity of μ^*). For any sets E and $E_n \subset X$, if $E \subset \bigcup_n E_n$, then $\mu^*(E) \leq \sum_n \mu^*(E_n)$.

Proof. WLOG assume the right-hand side is not $+\infty$. Let $\varepsilon > 0$, and suppose $A_{nm} \in \mathcal{A}$ are such that $\bigcup_m A_{nm} \supset E_n$, and

$$\sum_m \mu(A_{nm}) < \mu^*(E_n) + \frac{\varepsilon}{2^n}.$$

Then $E \subset \bigcup_n \bigcup_m A_{nm}$, and

$$\mu^*(E) \leq \sum_n \sum_m \mu(A_{nm}) < \sum_n \left(\mu^*(E_n) + \frac{\varepsilon}{2^n} \right) = \sum_n \mu^*(E_n) + \varepsilon$$

Since the above holds for arbitrarily small ε , the statement follows. \square

Theorem 1.15 (Carathéodory). *The μ^* -measurable sets $\mathcal{M}(\mu^*)$ form a σ -algebra on X , and μ^* is a measure on $\mathcal{M}(\mu^*)$.*

Proof. Check that $F \in \mathcal{M}(\mu^*)$ iff $F^c \in \mathcal{M}(\mu^*)$. Suppose that $A, B \in \mathcal{M}(\mu^*)$. For any $E \in 2^X$, we have

$$\begin{aligned}\mu^*(E) &= \mu^*(E \cap A) + \mu^*(E \setminus A) \\ &= \mu^*(E \cap A \cap B) + \mu^*(E \cap A \setminus B) + \mu^*(E \setminus A) \\ &= \mu^*(E \cap (A \cap B)) + \mu^*(E \setminus (A \cap B)),\end{aligned}$$

using that $A \in \mathcal{M}(\mu^*)$ in equality 1 & 3 and that $B \in \mathcal{M}(\mu^*)$ in equality 2. Thus $\mathcal{M}(\mu^*)$ is an algebra (right?). Now let $E_n \in \mathcal{M}(\mu^*)$ for $n = 1, 2, \dots$, $F := \bigcup_{j=1}^{\infty} E_j$, and $F_n := \bigcup_{j=1}^n E_j \in \mathcal{M}(\mu^*)$. Since $E_n \setminus \bigcup_{j < n} E_j \in \mathcal{M}(\mu^*)$ for all n , we may assume E_n disjoint in proving F measurable. For

any $E \subset X$ we have

$$\begin{aligned}\mu^*(E) &= \mu^*(E \setminus F_n) + \mu^*(E \cap F_n) \\ &= \mu^*(E \setminus F_n) + \mu^*(E \cap F_n \cap E_n) + \mu^*(E \cap F_n \setminus E_n) \\ &= \mu^*(E \setminus F_n) + \mu^*(E \cap E_n) + \mu^*(E \cap F_{n-1})\end{aligned}$$

by μ^* -measurability of F_n and E_n . We have shown

$$\mu^*(E \cap F_n) = \mu^*(E \cap E_n) + \mu^*(E \cap F_{n-1})$$

which inductively means $\mu^*(E \cap F_n) = \sum_{j=1}^n \mu^*(E \cap E_j)$. Thus

$$\mu^*(E) = \mu^*(E \setminus F_n) + \sum_{j=1}^n \mu^*(E \cap E_j) \geq \mu^*(E \setminus F) + \sum_{j=1}^n \mu^*(E \cap E_j).$$

As this holds for every n , we have

$$\mu^*(E) \geq \mu^*(E \setminus F) + \sum_{j=1}^{\infty} \mu^*(E \cap E_j) \geq \mu^*(E \setminus F) + \mu^*(E \cap F) \quad (1.16)$$

using Lemma 1.14. Again using Lemma 1.14, we have $\mu^*(E) = \mu^*(E \setminus F) + \mu^*(E \cap F)$, and thus $F \in \mathcal{M}(\mu^*)$. Hence $\mathcal{M}(\mu^*)$ is a σ -algebra. Setting $E = F$ in (1.16) shows $\mu^*(F) = \sum_{j=1}^n \mu^*(F \cap E_j)$ and hence that μ^* is a measure on F . \square

1.4 Lebesgue Measure

Consider the ring of subsets

$$\mathcal{C} = \{(a, b] : a, b \in \mathbb{R}, a \leq b\} \quad (1.17)$$

and the pre-measure $\gamma : \mathcal{C} \rightarrow [0, \infty)$ given by

$$\gamma((a, b]) = b - a. \quad (1.18)$$

Then \mathcal{C} and γ define an outer measure λ^* on $X = \mathbb{R}$. This outer measure is called the **Lebesgue outer measure**. The **Lebesgue measure** can then be defined as the measure λ on the measurable space $(\mathbb{R}, \mathcal{M}(\lambda^*))$, which is constructed as in Theorem 1.15. The elements of $\mathcal{M}(\lambda^*)$ are called the Lebesgue-measurable sets.

Below we define the Borel σ -algebra $\mathcal{B}(\mathbb{R})$, the “standard” σ -algebra on \mathbb{R} . Its elements are the Borel-measurable sets, or Borel sets, or simply “measurable” sets. It is possible to develop an intuition for the Borel- σ -algebra, since it is generated by the open intervals: We can construct Borel

sets through countable unions and intersections of open and closed intervals. As it turns out (in this section and the next), $\mathcal{M}(\lambda^*)$ is almost equal to $\mathcal{B}(\mathbb{R})$.

Definition 1.19 (Topology). *Let X be a non-empty set, and let $\mathcal{T} \subset 2^X$ be a system of subsets of X with the following properties:*

1. $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$.
2. If $A, B \in \mathcal{T}$, then $A \cap B \in \mathcal{T}$.
3. If $A_i \in \mathcal{T}$ for all $i \in I$ (I is not necessarily countable), then $\bigcup_{i \in I} A_i \in \mathcal{T}$.

Then \mathcal{T} is called a **topology** on X . The pair (X, \mathcal{T}) is called a topological space, and the elements of \mathcal{T} are called the **open sets** in X . A set $A \subset X$ is called **closed** if A^c is open. The σ -algebra $\sigma(\mathcal{T})$ is called the **Borel σ -field** on X . The elements of $\sigma(\mathcal{T})$ are called **Borel sets**.

Example 1.20. Let $X = \mathbb{R}$, and let \mathcal{T} be the system of countably infinite unions of intervals $\{(a, b) : a, b \in \mathbb{R} \cup \{-\infty, +\infty\}\}$. Then $(\mathbb{R}, \mathcal{T})$ is a topological space, and $\sigma(\mathcal{T})$ is the Borel σ -field $\mathcal{B}(\mathbb{R})$ on \mathbb{R} . The Borel σ -algebra is generated by any of the following systems:

$$\begin{aligned}\mathcal{E}_1 &= \{(a, b) : a, b \in \mathbb{R}\}, & \mathcal{E}_2 &= \{[a, b] : a, b \in \mathbb{R}\}, \\ \mathcal{E}_3 &= \{(a, b] : a, b \in \mathbb{R}\}, & \mathcal{E}_4 &= \{[a, b) : a, b \in \mathbb{R}\}\end{aligned}$$

Theorem 1.21. *Borel sets are Lebesgue-measurable: $\mathcal{B}(\mathbb{R}) \subset \mathcal{M}(\lambda^*)$.*

Proof. Since we know that $\mathcal{M}(\mu^*)$ is a σ -algebra, it suffices to show that it contains a generator of $\mathcal{B}(\mathbb{R})$, such as the ring \mathcal{C} of half-open intervals $(a, b]$ (1.17). That is, we show

$$\mu^*(B) \geq \mu^*(B \cap I) + \mu^*(B \setminus I)$$

for any interval $I = (a, b] \in \mathcal{C}$ and any subset $B \in 2^X$.

Let $\{I_k\}_{k \in \mathbb{N}} \subset \mathcal{C}$ be a covering of B such that $\sum_k \gamma(I_k) \leq \mu^*(B) + \varepsilon$, where γ is the pre-measure (1.18) and ε is any positive small number. For each k , let $\{E_{kl}\}_{l \in \mathbb{N}} \subset \mathcal{C}$ and $\{F_{kl}\}_{l \in \mathbb{N}} \subset \mathcal{C}$ be a covering of $B \cap I_k$ and $B \setminus I_k$, respectively, such that

$$\sum_{\ell} \gamma(E_{kl}) \leq \mu^*(B \cap I_k) + 2^{-k} \varepsilon \quad \sum_{\ell} \gamma(F_{kl}) \leq \mu^*(B \setminus I_k) + 2^{-k} \varepsilon.$$

Then

$$\begin{aligned}
 \mu^*(B \cap I) + \mu^*(B \setminus I) &\leq \sum_k \sum_\ell [\gamma(E_{k\ell}) + \gamma(F_{k\ell})] \\
 &\leq \sum_k \left[\gamma(I_k \cap I) + 2^{-k}\varepsilon + \gamma(I_k \setminus I) + 2^{-k}\varepsilon \right] \\
 &= \sum_k \left[\gamma(I_k) + 2^{-k+1}\varepsilon \right] \leq \mu^*(B) + 2\varepsilon.
 \end{aligned}$$

and as the above inequality holds for arbitrary $\varepsilon > 0$, the result follows. \square

In the above proof, the decisive step was the equality $\gamma(I_k) = \gamma(I_k \cap I) + \gamma(I_k \setminus I)$, which relies on the assumption that \mathcal{C} is a ring; the remaining steps work for any system \mathcal{C} and any pre-measure γ .

The Lebesgue measure $\lambda(A)$ is defined for all Lebesgue sets $A \in \mathcal{M}(\mu^*)$; by restricting the domain of λ from $\mathcal{M}(\mu^*)$ to $\mathcal{B}(\mathbb{R})$, we define the Lebesgue measure for the Borel sets. This is a different measure (since the domains are different), but we still denote it by λ .

Uniqueness of Lebesgue Measure

To uniquely determine a measure μ on a measurable space (X, \mathcal{A}) , it is not necessary to know all values $\{\mu(A) : A \in \mathcal{A}\}$. As it turns out, it suffices to determine $\mu(A)$ only for A from a much smaller subset of \mathcal{A} . This result is proved using the *monotone class theorem* due to Dynkin. It is a powerful tool for the characterisation of measures, and we will use it to prove the uniqueness of Lebesgue measure λ .

Definition 1.22. Let X be a non-empty set. A *d-class* on X is a system of subsets of X , $\mathcal{D} \subset 2^X$, which satisfies the following:

1. $X \in \mathcal{D}$
2. If $A, B \in \mathcal{D}$ such that $B \subset A$, then $A \setminus B \in \mathcal{D}$
3. For any increasing sequence $A_1 \subset A_2 \subset \dots$ where $A_j \in \mathcal{D}$, one has $\bigcup_{j=1}^{\infty} A_j \in \mathcal{D}$

A *π -class* on X is a system of subsets of X , $\pi \subset 2^X$, which is closed with respect to finite intersections. That is, it satisfies $A, B \in \pi \Rightarrow A \cap B \in \pi$.

Any σ -algebra is a d-class. Similarly to algebras and σ -algebras, the intersection of arbitrarily many d-classes on X is again a d-class. Hence we can say that a d-class $\mathcal{D} = d(\mathcal{C})$ is generated by $\mathcal{C} \subset 2^X$ if \mathcal{D} is the smallest d-class containing \mathcal{C} .

Theorem 1.23 (Monotone Class Theorem, Dynkin). *Let X be a non-empty set, and let \mathcal{C} be a π -class on X . Then $\sigma(\mathcal{C}) = d(\mathcal{C})$.*

Proof. Since $\sigma(\mathcal{C})$ is a d-class, we have $\sigma(\mathcal{C}) \supset d(\mathcal{C})$. It remains to show $\sigma(\mathcal{C}) \subset d(\mathcal{C})$, which follows if we can show that $d(\mathcal{C})$ is a σ -algebra. We check that $d(\mathcal{C})$ indeed satisfies the first two properties required of a σ -algebra. For the last property, let E_1, E_2, \dots be a sequence (not necessarily increasing) of sets in $d(\mathcal{C})$. Since

$$\bigcup_{j=1}^{\infty} E_j = \bigcup_{k=1}^{\infty} F_k$$

where $F_k = \bigcup_{j=1}^k E_j$, we can write the countable union of E_j as an *increasing* countable union of F_j . Since $d(\mathcal{C})$ is a d-class, $\bigcup_{k=1}^{\infty} F_k$ will lie in $d(\mathcal{C})$; but we need to check that $F_k \in d(\mathcal{C})$. Since $F_k^c = \bigcap_{j=1}^k E_j^c$, it hence suffices to show that $d(\mathcal{C})$ is closed under *finite* intersections. First, define

$$\mathcal{D}_1 := \{E \in d(\mathcal{C}) : E \cap C \in d(\mathcal{C}) \forall C \in \mathcal{C}\}$$

By definition of \mathcal{C} , we have $\mathcal{C} \subset \mathcal{D}_1$. The identities

$$(A \setminus B) \cap C = (A \cap C) \setminus (B \cap C)$$
$$\left(\bigcup_{n=1}^{\infty} A_n \right) \cap C = \bigcup_{n=1}^{\infty} (A_n \cap C)$$

show that \mathcal{D}_1 is closed with respect to proper set differences and countable unions of increasing sets, that is, \mathcal{D}_1 is a d-class containing \mathcal{C} and $\mathcal{D}_1 \supset d(\mathcal{C})$. By definition, $\mathcal{D}_1 \subset d(\mathcal{C})$, and so $\mathcal{D}_1 = d(\mathcal{C})$. Next, define

$$\mathcal{D}_2 := \{E \in d(\mathcal{C}) : E \cap F \in d(\mathcal{C}) \forall F \in d(\mathcal{C})\}$$

We see that $\mathcal{C} \subset \mathcal{D}_2$ and $X \in \mathcal{D}_2$. By the same arguments as above, $\mathcal{D}_2 = d(\mathcal{C})$. This shows that $d(\mathcal{C})$ is closed with respect to finite intersections. \square

Definition 1.24. Let (X, \mathcal{A}, μ) be a measure space. If there is a sequence $\{E_n\}_{n \in \mathbb{N}} \subset \mathcal{A}$ such that $X = \bigcup_n E_n$ and $\mu(E_n) < \infty$ for all n , then the measure μ is said to be *σ -finite*.

For instance, Lebesgue measure is σ -finite on the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$; See this by setting $E_n = (-n, n]$. Moreover, any finite measure space with $\mu(X) < \infty$ is σ -finite.

Theorem 1.25. Let (X, \mathcal{A}) be a measurable space, and let μ and ν be two σ -finite measures on it. Let $\mathcal{C} \subset 2^X$ be a π -class such that $\mathcal{A} = \sigma(\mathcal{C})$. If $\mu(C) = \nu(C)$ for all $C \in \mathcal{C}$, then $\mu = \nu$.

Proof. First, let's assume that μ and ν are finite measures, that is, $\mu(X) < \infty$ and $\nu(X) < \infty$. Define $\mathcal{D} = \{A \in \mathcal{A} : \mu(A) = \nu(A)\}$. We see $\mathcal{C} \subset \mathcal{D}$. Then \mathcal{D} is a d-class because:

1. $\mu(X) = \nu(X) \Rightarrow X \in \mathcal{D}$
2. Let $A, B \in \mathcal{D}$ such that $A \subset B$. Then $\mu(A \setminus B) = \mu(A) - \mu(B) = \nu(A) - \nu(B) = \nu(A \setminus B)$, and thus $A \setminus B \in \mathcal{D}$. (This step assumes that the measures are finite, since $\infty - \infty$ is to be avoided.)
3. Let $E_n \in \mathcal{D}$ be an increasing sequence in \mathcal{D} . By Theorem [1.12](#),

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n \rightarrow \infty} \mu(E_n) = \lim_{n \rightarrow \infty} \nu(E_n) = \nu\left(\bigcup_{n=1}^{\infty} E_n\right)$$

and hence $\bigcup_{n=1}^{\infty} E_n \in \mathcal{D}$.

By Theorem 1.23, $\mathcal{A} = \mathcal{D}$, thus $\mu = \nu$.

Now assume that μ and ν are σ -finite with respect to the sequences $\{E_n\}_{n \in \mathbb{N}}$ and $\{F_n\}_{n \in \mathbb{N}}$; then both are σ -finite with respect to the the sequence $\{C_n\}_{n \in \mathbb{N}}$ where $C_n = E_n \cap F_n$. For each n , define measures μ_n and ν_n on (X, \mathcal{A}) by $\mu_n(A) = \mu(A \cap C_n)$ and $\nu_n(A) = \nu(A \cap C_n)$. From the first part of the proof, we have $\mu_n = \nu_n$, and thus using Th 1.12

$$\mu(A) = \lim_n \mu_n(A) = \lim_n \nu_n(A) = \nu(A)$$

for every $A \in \mathcal{A}$.

□

Theorem 1.26. *The Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is the unique measure which satisfies $\lambda((a, b]) = b - a$ for all $a, b \in \mathbb{R}$, $a \leq b$.*

Proof. First, we show $\lambda((a, b]) = b - a$. Indeed, consider an interval $I = (a, b]$. It is covered by the sequence $I \cup \emptyset \cup \emptyset \cup \dots$ of elements in the ring \mathcal{C} , which shows $\lambda^*(I) \leq \gamma(I) = b - a$. It is intuitively clear (but technically requires a proof) that this covering is optimal. That is: *any* countable covering of I with sets $\{C_n\}_{n \in \mathbb{N}} \subset \mathcal{C}$ is such that $\sum_n \gamma(C_n) \geq b - a$. But this means $\lambda^*(I) \geq b - a$, and so $\lambda^*(I) = b - a$. I is a Borel set, and thus also a Lebesgue set, hence $\lambda(I) = \lambda^*(I) = b - a$. We know that λ is σ -finite; any other measure which maps $(a, b]$ to $b - a$ is also σ -finite. Th 1.25 completes the proof. \square

Since the pre-measure γ and thus the outer Lebesgue-measure λ^* are translation invariant, it follows that Lebesgue measure is translation invariant. In fact, up to a multiplicative constant, Lebesgue measure is the only translation invariant measure on \mathbb{R} :

Theorem 1.27. *Let ν be a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ which is finite on bounded sets and translation invariant (that is, $\nu(A+x) = \nu(A) \forall A \in \mathcal{B}(\mathbb{R}), x \in \mathbb{R}$). Then ν is a multiple of the Lebesgue measure.*

Proof. The interval $I = (0, 1]$ is the pairwise disjoint union of n intervals $((k-1)/n, k/n]$, $k = 1, \dots, n$. These are all translates of each other and hence all have the same measure as any fixed one, say B . Set $c := \nu(I)$, then

$$n\nu(B) = c = c \times 1 = c\lambda(I) = c n \lambda(B),$$

and $\nu(B) = c \lambda(B)$. Any $C \in \mathcal{C}$ can be represented as a countable union of intervals of length $1/n$ ($n \in \mathbb{N}$). The family of such intervals hence generates $\mathcal{B}(\mathbb{R})$. Hence by Th 1.25, $\nu = c\lambda$. \square

Finally, here are a few properties of Lebesgue measure:

Theorem 1.28. *Lebesgue measure on \mathbb{R} has the following properties:*

1. *If $A \in 2^X$ is countable, then it is a Borel set, and $\lambda(A) = 0$*
2. *If $A \in \mathcal{M}(\lambda^*)$ is bounded, then $\lambda(A) < \infty$*
3. *For every $A \in \mathcal{M}(\lambda^*)$ we have $\lambda(A) = \inf\{\lambda(U) : U \text{ open}, U \supset A\}$*
4. *For every $A \in \mathcal{M}(\lambda^*)$ we have $\lambda(A) = \sup\{\lambda(K) : K \text{ compact}, K \subset A\}$. (Recall that a set $K \in 2^X$ is compact iff it is both closed and bounded.)*

Proof. Properties 1. and 2. are left as an exercise. For 3. and 4., note that monotonicity of measures ($A \subset B \Rightarrow \lambda(A) \leq \lambda(B)$) implies

$$\lambda(A) \leq \inf\{\lambda(U) : U \text{ open}, U \supset A\},$$

$$\lambda(A) \geq \sup\{\lambda(K) : K \text{ compact}, K \subset A\}$$

and it remains to show the reverse inequalities. For part 3, we may assume $\lambda(A) < \infty$. Let $\varepsilon > 0$, and find a covering $\bigcup C_k$ of A where $C_k \in \mathcal{C}$ are half open intervals, such that

$$\sum_k \gamma(C_k) < \lambda(A) + \varepsilon.$$

Replace $C_k = (a_k, b_k]$ by $R_k = (a_k, b_k + 2^{-k}\varepsilon]$ to get a covering such that

$$\sum_k \gamma(R_k) < \lambda(A) + 2\varepsilon.$$

The open set $U = \bigcup_k (a_k, b_k + 2^{-k}\varepsilon)$ contains A , and $\lambda(U) \leq \lambda(A) + 2\varepsilon$. As ε was arbitrary, 3. is proved.

For part 4., assume first that A is bounded. Let C be a closed and bounded set that includes A , and let ε be an arbitrary positive number. Use part 3. to choose an open set U that includes $C \setminus A$ and satisfies

$$\lambda(U) < \lambda(C \setminus A) + \varepsilon.$$

Let $K = C \setminus U$. (Draw a picture!) Then K is a closed and bounded (and hence compact) subset of A ; it satisfies $C \subset K \cup U$, and so satisfies

$$\lambda(C) \leq \lambda(K) + \lambda(U)$$

The two inequalities (together with $\lambda(C \setminus A) = \lambda(C) - \lambda(A)$) now imply that $\lambda(A) - \varepsilon < \lambda(K)$. Since ε was arbitrary, part 4 is proved in the case where A is bounded. Finally, consider the case where A is not bounded. Suppose that b is a real number that satisfies $b < \lambda(A)$; we shall produce a compact subset K of A such that $b < \lambda(K)$. Let $\{A_j\}$ be an increasing sequence of bounded measurable subsets of A such that $A = \bigcup_j A_j$ (for example, A_j could be defined to be $A \cap [-j, +j]$). By Th 1.12, $\lambda(A) = \lim_j \lambda(A_j)$, and so we can choose j_0 so that $\lambda(A_{j_0}) > b$. Now apply to A_{j_0} what was shown above for bounded A , obtaining a compact subset of A_{j_0} such that $\lambda(K) > b$. Since $K \subset A$ and since b was an arbitrary number less than $\lambda(A)$, the proof is complete. \square

1.5 Completion of Measures

In this section, we establish a result which shows that Lebesgue sets can be very well approximated by Borel sets: For every Lebesgue set, there is a Borel subset and a Borel superset of the same measure. We show that the Lebesgue σ -algebra results from the Borel σ -algebra by completion. Completed probability spaces are of interest in probability theory.

Definition 1.29. Let (X, \mathcal{A}, μ) be a measure space.

1. A subset $B \in 2^X$ is called μ -null if there is $A \in \mathcal{A}$ such that $B \subset A$ and $\mu(A) = 0$.
2. If \mathcal{A} contains all μ -null sets, then μ and (X, \mathcal{A}, μ) are called complete.
3. A property holds μ -almost everywhere if the set of $x \in X$ for which it doesn't hold is a μ -null set.

A measure space (X, \mathcal{A}, μ) which is not complete can be completed, in a unique well-defined way. First, define the **completion of \mathcal{A} under μ** as the system \mathcal{A}_μ of all subsets $A \subset X$ for which there exist $E, F \in \mathcal{A}$ such that $E \subset A \subset F$ and $\mu(F \setminus E) = 0$. Members of \mathcal{A}_μ are sometimes called μ -measurable.

To define a candidate for a measure $\bar{\mu}$ on \mathcal{A}_μ , let $A \in \mathcal{A}_\mu$, where $E, F \in \mathcal{A}$ are such that $E \subset A \subset F$, and set $\bar{\mu}(A) := \mu(E)$. This does not depend on the choice of E or F , hence $\bar{\mu}$ is well defined.

Proposition 1.30. *Let (X, \mathcal{A}, μ) be a measure space. Then \mathcal{A}_μ is a σ -algebra on X , and $\bar{\mu}$ is a measure on (X, \mathcal{A}_μ) . The restriction of $\bar{\mu}$ from \mathcal{A}_μ to \mathcal{A} is μ .*

Proof. This is left as an exercise. □

Proposition 1.31. *Suppose $(X, \mathcal{M}(\mu^*), \mu)$ is the measure space obtained from an outer measure by the Carathéodory construction (Th 1.15). That is, μ is the restriction of μ^* from 2^X to $\mathcal{M}(\mu^*)$. Then the measure μ is complete.*

Proof. If $B \in 2^X$ is μ -null, then $B \subset A$ for some $A \in \mathcal{M}(\mu^*)$ with $\mu^*(A) = 0$. Outer measures are monotone, thus $\mu^*(B) = 0$. For arbitrary $C \in 2^X$,

$$\mu^*(C) \leq \mu^*(C \cap B) + \mu^*(C \setminus B) \leq \mu^*(C \setminus B) \leq \mu^*(C);$$

the first inequality follows from subadditivity of outer measures, the second from $\mu^*(C \cap B) \leq \mu^*(B) = 0$ and the third from $C \setminus B \subset C$. It follows that $B \in \mathcal{M}(\mu^*)$. \square

Lemma 1.32. *Let A be a Lebesgue set in \mathbb{R} . Then there exist Borel subsets E and F such that $E \subset A \subset F$ and $\lambda(F \setminus E) = 0$.*

Proof. First suppose that A is a Lebesgue set such that $\lambda(A) < \infty$. For each positive integer n use Th 1.28 to choose a compact set K_n that satisfies $K_n \subset A$ and $\lambda(A) - 1/n < \lambda(K_n)$ and an open set U_n that satisfies $A \subset U_n$ and $\lambda(U_n) \leq \lambda(A) + 1/n$. Let $E = \bigcup_n K_n$ and $F = \bigcap_n U_n$. Then E and F belong to $\mathcal{B}(\mathbb{R})$ and satisfy $E \subset A \subset F$. The relation

$$\lambda(F \setminus E) \leq \lambda(U_n \setminus K_n) = \lambda(U_n \setminus A) + \lambda(A \setminus K_n) \leq 2/n$$

holds for each n , and so $\lambda(F \setminus E) = 0$. Thus the lemma is proved in the case where $\lambda(A) < \infty$. If A is an arbitrary Lebesgue set, then A is the union of a sequence $\{A_n\}$ of Lebesgue measurable sets each of which satisfies $\lambda(A_n) < \infty$. For each n we can choose Borel sets E_n and F_n such that $E_n \subset A_n \subset F_n$ and $\lambda(F_n \setminus E_n) = 0$. The sets E and F defined by $E = \bigcup_n E_n$ and $F = \bigcup_n F_n$ then satisfy $E \subset A \subset F$ and $\lambda(F \setminus E) = 0$ (note that $F \setminus E \subset \bigcup_n (F_n \setminus E_n)$). \square

Theorem 1.33. *The measure space $(\mathbb{R}, \mathcal{M}(\lambda^*), \lambda)$ is the completion of the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$.*

Proof. Let λ be Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, let $\bar{\lambda}$ be the completion of λ , and let λ_m be Lebesgue measure on $(\mathbb{R}, \mathcal{M}(\lambda^*))$. Lemma 1.32 implies that $\mathcal{M}(\lambda^*)$ is included in the completion of $\mathcal{B}(\mathbb{R})$ under λ and that λ_m is the restriction of λ to $\mathcal{M}(\lambda^*)$. Thus we need only check that each set A that belongs to the completion of $\mathcal{B}(\mathbb{R})$ under λ is Lebesgue measurable. For such a set A there exist Borel sets E and F such that $E \subset A \subset F$ and $\lambda(F \setminus E) = 0$. Since $A \setminus E \subset F \setminus E$ and $\lambda_m(F \setminus E) = \lambda(F \setminus E) = 0$, the completeness of Lebesgue measure on $\mathcal{M}(\lambda^*)$ (Prop 1.31) implies that $A \setminus E \in \mathcal{M}(\lambda^*)$. Thus A , since it is the union of $A \setminus E$ and E , must belong to $\mathcal{M}(\lambda^*)$. \square

This completes our discussion of Lebesgue measure. We have only constructed and studied it on \mathbb{R}^d for $d = 1$, to keep notation simple. The entire theory above remains correct if the ring \mathcal{C} of half-open intervals $(a, b]$ in \mathbb{R} is replaced by the ring of “blocks” $(a_1, b_1] \times (a_2, b_2] \times \dots \times (a_d, b_d]$ in \mathbb{R}^d , and the function $\gamma((a, b]) = b - a$ is replaced by $\gamma((a_1, b_1] \times (a_2, b_2] \times \dots \times (a_d, b_d]) = \prod_{k=1}^d (b_k - a_k)$. This results in Lebesgue measure on \mathbb{R}^d , Lebesgue sets $\mathcal{M}(\lambda^*)$ in \mathbb{R}^d and Borel sets $\mathcal{B}(\mathbb{R}^d)$ in \mathbb{R}^d . The same results regarding completeness remain true.