University of New South Wales

School of Mathematics and Statistics

Assignment 2

Measure Theory

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Question 1

For this question, let μ and ν be probability measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$.

Lemma 1. *The function $x \mapsto \nu(B-x)$ is $\mathcal{B}(\mathbb{R}^d)$ measurable for any $B \in \mathcal{B}(\mathbb{R}^d)$.*

Proof. Let $B \in \mathcal{B}(\mathbb{R}^d)$ and $s(x) = \nu(B-x)$. Then,

$$\begin{aligned} s(x) &= \int_{\mathbb{R}^d} \chi_{B-x} \, d\nu \\ &= \int_{\mathbb{R}^d} \chi_B(y+x) \, d\nu(y). \end{aligned}$$

The function $(x,y) \mapsto \chi_B(y+x)$ is a composition of a continuous function, $(x,y) \mapsto y+x$ and a measurable function $x \mapsto \chi_B(x)$. Hence the function $(x,y) \mapsto \chi_B(y+x)$ is $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^d)$ measurable, and so by Tonelli's theorem s is $\mathcal{B}(\mathbb{R}^d)$ measurable. □

Lemma 2. *The convolution measure $\mu \star \nu(B) = \int_{\mathbb{R}^d} \nu(B-x) \, d\mu(x)$ is well defined and finite.*

Proof. Since $\mu \star \nu(B)$ is defined as an integral of a positive measurable function, the integral exists. Since $\nu(B-x) \leq 1$, we have $\mu \star \nu(B) \leq 1$. □

Theorem 1. *If there exists some bounded F such that $\mu \star \nu(F) = 1$, then there are bounded sets G and H such that $\mu(G) = 1$ and $\nu(H) = 1$. Similar results hold where "bounded" is replaced with "finite" or "countable".*

Proof. Suppose that there exists a bounded set $F \in \mathcal{B}(\mathbb{R}^d)$ such that $\mu \star \nu(F) = 1$, but for any bounded set B, $\mu(B), \nu(B) < 1$. Then,

$$\begin{aligned} \mu \star \nu(F) &= \int \nu(F-x) \, d\mu(x) \\ &< \int 1 \, d\mu \\ &= 1. \end{aligned}$$

since $F - x$ is bounded for any x. This is a contradiction. Hence we must have that ν attains the value 1 on some bounded set.

By symmetry, μ must also take the value 1 on some bounded set.

An identical argument holds if "bounded" is replaced by "finite" or "countable". □

Question 2

For this question, μ and ν are σ-finite positive measures on a measurable space $(\Omega, \mathcal{F})$.

Theorem 2. *The following are equivalent:*

1. *μ and ν have exactly the same null sets.*
2. *$\mu \ll \nu$ and $\nu \ll \mu$.*
3. *There is an $\mathcal{F}$-measurable function g with $0 < g < +\infty$ such that $\nu(A) = \int_A g \, d\mu$ for all $A \in \mathcal{F}$.*

Proof. First we prove (1) $\Rightarrow$ (2).

Assume that μ and ν have exactly the same null sets. Then if $\mu(A) = 0$, then $\nu(A) = 0$. That is $\nu \ll \mu$. Similarly, $\mu \ll \nu$.

Now we prove (2) $\Rightarrow$ (3).

Assume that $\nu \ll \mu$.

By the Radon-Nikodym Theorem, there is an $\mathcal{F}$-measurable function g such that $\nu(A) = \lambda(A) + \int_A g \, d\mu$, where λ is a measure mutually singular to μ. Suppose that $\mu(A) = 0$, then $\nu(A) = 0$ by assumption, hence $\lambda(A) = 0$. Thus $\lambda \ll \mu$ and $\lambda \perp \mu$, so $\lambda = 0$.

Suppose that $g = \infty$ only on a null set. Then we can modify g to $g = 0$ on this set since g is determined only up to μ-almost everywhere equivalence.

Suppose there is some set A with $\mu(A) > 0$ and $g(A) = \{\infty\}$.

Since ν is σ-finite, there exists B with $\nu(B) < \infty$ and $\nu(A \cap B) > 0$.

Hence $g(A \cap B) = \{\infty\}$, and we cannot have $\mu(A \cap B) = 0$ because then $\nu(A \cap B) = 0$.

But then $\nu(A \cap B) = \infty$, but this is a contradiction. Hence $g < \infty$ μ-almost everywhere.

Now suppose there is some set C with $\mu(C) > 0$ and $g(C) = \{0\}$. Hence $\nu(C) = 0$, but since $\mu \ll \nu$ this is a contradiction.

Now we prove that (3) $\Rightarrow$ (1).

Suppose that $\mu(A) = 0$. Then clearly since $\nu(A) = \int_A g \, d\mu(A)$, we have $\nu(A) = 0$.

Now suppose that $\nu(A) = 0$. Now,

$$\nu(A) \geq \frac{1}{n}\mu(A \cap g^{-1}([1/n, \infty))$$

Hence $\mu(A \cap g^{-1}([1/n, \infty)) = 0$. But since $g > 0$, we have

$$\mu(A) = \lim_{n\to\infty} \mu(A \cap g^{-1}([1/n, \infty)).$$

Hence $\mu(A) = 0$. Thus μ and ν have the same null sets so (1) is proved. $\square$

Theorem 3. *Suppose that μ is a σ-finite positive measure on $(\Omega, \mathcal{F})$. Then there is a finite positive measure ν on $(\Omega, \mathcal{F})$ such that $\nu \ll \mu$ and $\mu \ll \nu$.*

Proof. Suppose that $\{A_k\}_{k=1}^{\infty}$ is a disjoint sequence of sets with $A = \bigcup_k A_k$ and $0 < \mu(A_k) < \infty$. This can be chosen since μ is σ-finite.

Now define,

$$\nu(A) = \sum_{k=1}^{\infty} \frac{1}{2^k \mu(A_k)} \mu(A \cap A_k)$$

for $A \in \mathcal{F}$. This sum converges since each term is bounded by $1/2^k$, so the sum is bounded by a geometric series.

We wish to show that ν is a probability measure on $(\Omega, \mathcal{F})$ and than $\mu \ll \nu$ and $\nu \ll \mu$.

Note that $\nu(\Omega) = 1$ and $\nu(\emptyset) = 0$.

Suppose that $\{B_k\}_{k=1}^{\infty}$ is a disjoint sequence of sets in $\mathcal{F}$. Then

$$\nu(\bigcup_k B_k) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{2^j \mu(A_j)} \mu(B_k \cap A_j)$$

Then since this is a sum of positive numbers, we can change the order of summation by Tonelli's theorem,

$$\nu(\bigcup_k B_k) = \sum_{k=1}^{\infty} \nu(B_k).$$

Hence ν is countably additive, so is a measure on $(\Omega, \mathcal{F})$.

Now we wish to show that $\nu \ll \mu$. Suppose that $\mu(A) = 0$. Then clearly $\mu(A_k \cap A) = 0$ for all k, so $\nu(A) = 0$. Thus $\nu \ll \mu$.

Now to show that $\mu \ll \nu$, let $\nu(A) = 0$, then $\mu(A_k \cap A) = 0$ for all k. Hence,

$$\mu(A) = \sum_{k=1}^{\infty} \mu(A_k \cap A) = 0.$$

Thus $\mu \ll \nu$. $\square$

Question 3

For this question, X is a d-dimensional random vector with law μ and characteristic function $\hat{\mu}(u)$.

Lemma 3. *The characteristic function of cX is $\hat{\mu}(cu)$, for any $c \in \mathbb{R}$.*

Proof. This is a simple computation. By definition, the characteristic function of cX is

$$\mathbf{E}(e^{i\langle u, cX\rangle}$$

But since $\langle u, cX\rangle = \langle cu, X\rangle$, this is simply $\hat{\mu}(cu)$. $\square$

Theorem 4. *If X has moments up to order n, then $\hat{\mu}$ is differentiable at 0 up to nth order, and $\frac{\partial^\alpha}{\partial u^\alpha}\hat{\mu}(u)|_{u=0} = i^{|\alpha|}\mathbf{E}(X^\alpha)$ for multi-indices α, with $|\alpha| \leq n$.*

Proof. Suppose that X has moments up to order n. Then we can say that,

$$\frac{\partial}{\partial u_j}\mathbf{E}(e^{i\langle u, X\rangle}) = i\mathbf{E}(X_j e^{i\langle u, X\rangle})$$

by the Dominated convergence theorem, since $\mathbf{E}(|X_j|) < \infty$. Hence, by induction,

$$\frac{\partial^\alpha}{\partial u^\alpha}\mathbf{E}(e^{i\langle u, X\rangle}) = i^{|\alpha|}\mathbf{E}(X^\alpha e^{i\langle u, X\rangle}).$$

for all multi-indices α with $|\alpha| \leq n$ by the Dominated convergence theorem. This shows that the derivative exists.

So we simply evaluate this at zero to obtain the required result. $\square$

Now we let X be a random variable with Lebesgue density

$$f(x) = \frac{C}{(1+x^2)\log(e+x^2)}$$

for some normalising constant $C > 0$. Let $\hat{\mu}$ be the characteristic function of X.

Lemma 4. $\mathbf{E}(X)$ *is not defined. That is,* X *does not have moments up to order* 1.

Proof. Note that

$$\begin{gathered} 1+x^2 \leq 2x^2 \\ \log(e+x^2) \leq \log(2x^2) \end{gathered}$$

for $x > 2$, hence

$$\frac{Cx}{(1+x^2)\log(e+x^2)} \geq \frac{C}{2x\log(2x^2)}$$

for $x > 2$. Thus the integral,

$$\int_{[} 2,\infty) \frac{Cx}{(1+x^2)\log(e+x^2)}\, d\lambda(x)$$

where λ is Lebesgue measure, is bounded from below by

$$\int_{[} 2,\infty) \frac{C}{2x\log(2x^2)}\, d\lambda(x).$$

We use the change of variable $u = \sqrt{2}x$ to compute this as

$$\frac{C}{2}\int_{[} 2\sqrt{2},\infty) \frac{1}{u\log(u)}\, d\lambda(u).$$

However the integrand has antiderivative $\log(\log(u))$, which is unbounded. Hence the integral is infinite.

Thus, the integral

$$\int_{\mathbb{R}} \frac{Cx}{(1+x^2)\log(e+x^2)}\, d\lambda(x)$$

is not defined.

So $\mathbf{E}(X)$ is not defined. $\square$

Lemma 5. $\hat{\mu}$ *is differentiable at* 0.

Proof. By definition,

$$\hat{\mu}(u) = \int_{\mathbb{R}} \frac{Ce^{iux}}{(1+x^2)\log(e+x^2)}\, d\lambda(x).$$

where λ is Lebesgue measure.

Using integration by parts, we rewrite this as

$$\hat{\mu}(u) = \int_{\mathbb{R}} \frac{iuCe^{iux}}{[(1+x^2)\log(e+x^2)]^2}\left[2x\log(e+x^2) + (1+x^2)\frac{2x}{e+x^2}\right] d\lambda(x)$$

If we differentiate the integrand with respect to u, we get

$$F(x) := iC\frac{e^{iux} - xue^{iux}}{[(1+x^2)\log(e+x^2)]^2}\left[2x\log(e+x^2) + (1+x^2)\frac{2x}{e+x^2}\right].$$

We wish to show that this is in $L^1(\mathbb{R},\lambda)$. Write $F(x)$ as

$$F(x) = iC(F_1(x) + F_2(x) + F_3(x) + F_4(X)).$$

Where,

$$\begin{aligned}
F_1(x) &:= \frac{2e^{iux}x\log(e+x^2)}{[(1+x^2)\log(e+x^2)]^2}\\
F_2(x) &:= \frac{-2x^2ue^{iux}\log(e+x^2)}{[(1+x^2)\log(e+x^2)]^2}\\
F_3(x) &:= \frac{2x(1+x^2)e^{iux}}{(e+x^2)[(1+x^2)\log(e+x^2)]^2}\\
F_4(x) &:= \frac{-2x^2(1+x^2)ue^{iux}}{(e+x^2)[(1+x^2)\log(e+x^2)]^2}
\end{aligned}$$

So we can separately show that $F_1, F_2, F_3, F_4 \in L^1(\mathbb{R},\lambda)$.

First, see that

$$\begin{aligned}
|F_1(x)| &\leq 2\frac{x}{(1+x^2)^2\log(e+x^2)}\\
&\leq 2\frac{x}{(1+x^2)^2}
\end{aligned}$$

so $F_1 \in L^1(\mathbb{R},\lambda)$.

Now,

$$\begin{aligned}
|F_2(x)| &\leq 2u\frac{x^2}{(1+x^2)^2\log(e+x^2)}\\
&\leq 2u\frac{x^2}{(1+x^2)^2}\\
&\leq 2u\frac{1}{1+x^2}.
\end{aligned}$$

So $F_2 \in L^1(\mathbb{R},\lambda)$.

For F_3,

$$\begin{aligned} |F_3(x)| &\leq 2\frac{x}{(e+x^2)(1+x^2)\log(e+x^2)^2} \\ &\leq 2\frac{x}{(1+x^2)^2} \end{aligned}$$

So $F_3 \in L^1(\mathbb{R},\lambda)$.

Now F_4,

$$\begin{aligned} |F_4(x)| &\leq 2u\frac{x^2}{(e+x^2)(1+x^2)\log(e+x^2)^2} \\ &\leq 2u\frac{1}{e+x^2} \end{aligned}$$

Hence $F_4 \in L^1(\mathbb{R},\lambda)$.

Thus, $F \in L^1(\mathbb{R},\lambda)$.

Hence, by the dominated convergence theorem, $\hat{\mu}'(u)$ exists, so $\hat{\mu}$ is differentiable at 0. ☐

Question 4

For this question, μ is the binomial distribution $\mathrm{Bin}(n,p)$, and ν is the Poisson distribution with mean $\lambda > 0$.

Lemma 6. *The characteristic function of a Bernoulli random variable with probability p is*

$$1 - p + pe^{iu}$$

Proof. This is a simple computation, if X takes the value 1 with probability p and 0 with probability $1-p$, then

$$\mathbf{E}(e^{iuX}) = 1 - p + pe^{iu}.$$

☐

Lemma 7. $\hat{\mu}(u) = (1-p+pe^{iu})^n$.

Proof. A sum of n independent Bernoulli random variables with probability p is $\mathrm{Bin}(n,p)$ distributed. So,

$$\mu = \mathrm{Bern}(p)^{\star n}$$

Hence,

$$\hat{\mu}(u) = (1 - p + pe^{iu})^n.$$

□

Lemma 8. $\hat{\nu}(u) = \exp(\lambda(e^{iu} - 1))$.

Proof. We can compute,

$$\hat{\nu}(u) = \sum_{k=0}^{\infty} \frac{e^{iuk}e^{-\lambda}\lambda^k}{k!}$$

So we write this as,

$$\begin{aligned}\hat{\nu}(u) &= \sum_{k=0}^{\infty} e^{-\lambda}\frac{(e^{iu}\lambda)^k}{k!} \\ &= e^{-\lambda}\exp(\lambda e^{iu}) \\ &= \exp(\lambda(e^{iu} - 1)).\end{aligned}$$

□

Lemma 9. *Suppose that $q_n \to \lambda \in \mathbb{C}$ is a sequence of complex numbers. Then*

$$\lim_{n\to\infty}\left(1 + \frac{q_n}{n}\right)^n = e^{\lambda}$$

Proof. Fix n large enough such that $|q_n|/n < 1/2$.

Since q_n is a convergent sequence, it is bounded. Let M be large enough such that $|q_n| < M$ for all n.

Re-write $\left(1 + \frac{q_n}{n}\right)^n$ as $\exp(n\,\mathrm{Log}(1 + \frac{q_n}{n}))$.

The branch of the logarithm taken here is complex differentiable in the set $\mathbb{C} \setminus (-\infty, 0]$. Since $|q_n|/n < 1$, the above is valid.

So it is sufficient to show that,

$$\lim_{n\to\infty} n\,\mathrm{Log}\left(1 + \frac{q_n}{n}\right) = \lambda$$

The $z \mapsto \mathrm{Log}(1+z)$ function is complex differentiable in the unit disc, and has a power series representation

$$\mathrm{Log}(1+z) = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{z^k}{k}$$

which converges uniformly on compact subsets of the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$.

Now, since $|q_n|/n < 1$, we have

$$n\,\mathrm{Log}(1+\frac{q_n}{n}) = q_n + \sum_{k=2}^{\infty}(-1)^{k-1}\frac{q_n^k}{kn^{k-1}}$$

Now we consider the tail of the left hand side, let

$$L_n := \sum_{k=2}^{\infty}(-1)^{k-1}\frac{q_n^k}{kn^{k-1}}$$

By the triangle inequality,

$$|L_n| \leq \sum_{k=2}^{\infty}\frac{M^k}{kn^{k-1}}$$

Thus,

$$\begin{aligned}|L_n| &\leq M\sum_{k=1}^{\infty}\left(\frac{M}{n}\right)^k \\ &= M\frac{M/n}{(1-M/n)}\end{aligned}$$

Hence, $L_n \to 0$ as $n \to \infty$. Thus, the limit

$$\lim_{n\to\infty} n\,\mathrm{Log}(1+\frac{q_n}{n})$$

exists, and equals $\lim_{n\to\infty} q_n = \lambda$.

Hence, the limit

$$\lim_{n\to\infty}\left(1+\frac{q_n}{n}\right)^n$$

exists, and equals e^{λ}. $\square$

Now we let $\{p_n\}_{n=1}^{\infty}$ be a monotone decreasing sequence, such that $np_n \to \lambda$. We let $\mu_n = \mathrm{Bin}(n, p_n)$.

Theorem 5. *There is weak convergence, $\mu_n \to \nu$.*

Proof. By Lévy's continuity theorem, it is sufficient to show pointwise convergence of characteristic functions, $\hat{\mu}_n(u) \to \hat{\nu}(u)$ for all u. That is, we must show

$$\lim_{n\to\infty} (1 - p_n + p_n e^{iu})^n = \exp(\lambda(e^{iu} - 1)).$$

Rewrite $\hat{\mu}_n(u)$ as

$$\left(1 + \frac{np_n(e^{iu} - 1)}{n}\right)^n$$

Now by lemma 9, we see

$$\lim_{n\to\infty} \hat{\mu}_n(u) = \exp(\lambda(e^{iu} - 1)).$$

Thus the result follows. □

Theorem 6. *For $k \in \mathbb{N}$, we have the pointwise convergence*

$$\mu_n(\{k\}) \to \nu(\{k\}).$$

Proof. Let $F_n(x) = \mu_n(-\infty, x]$ be the cumulative distribution function of μ_n, and $G(x) = \nu(-\infty, x]$

Weak convergence implies that $F_n(x) \to G(x)$ at all points of continuity x. μ_n and ν are discrete, so the points of continuity are $\mathbb{R} \setminus \mathbb{N}$.

Now let $k \in \mathbb{N}$. Then

$$\begin{aligned}
\lim_{n\to\infty} \mu_n\{k\} &= \lim_{n\to\infty} F_n(k + 1/2) - F_n(k - 1/2) \\
&= G(k + 1/2) - G(k - 1/2) \\
&= \nu\{k\}.
\end{aligned}$$

□

Question 5

Now we let $(\Omega, \mathcal{F}, \mathbf{P}) = ([0,1], \mathcal{B}([0,1]), \lambda)$ where λ is Lebesgue measure.

For $\omega \in [0,1]$, we let $d_n(\omega)$ be the n binary digit of ω, where we take the binary expansion containing infinitely many ones. Let

$$B_n = \{\omega \in [0,1] \,:\, d_n(\omega) = 0\}.$$

Lemma 10. $\mathbf{P}(B_n) = 1/2$.

Proof. See that,

$$B_n = \bigcup_{k=0}^{2^{n-1}} k/2^{n-1} + [0, 2^{-n}]$$

So $B_n \in \mathcal{B}([0,1])$.

See that $B_n^c = \{\omega \in [0,1] \,:\, d_n(\omega) = 1\}$.

So that $B_n^c = B_n + 2^{-n}$. Since Lebesgue measure is translation invariant, we have $\mathbf{P}(B_n^c) = \mathbf{P}(B_n)$. Since $\mathbf{P}(\Omega) = \mathbf{P}(B_n) + \mathbf{P}(B_n^c) = 1$, we have $\mathbf{P}(B_n) = 1/2$. □

Lemma 11. *The events $\{B_n\}_{n=1}^{\infty}$ form an infinite sequence of independent events.*

Proof. Suppose we have some finite subset, $\{B_{n(k)}\}_{k=1}^{m}$. Then let

$$B = \bigcap_{k=1}^{m} B_{n(k)}$$

See that,

$$\bigcap_{k=2}^{m} B_{n(k)} = \left(B_{n(1)} \cap \bigcap_{k=2}^{m} B_{n(k)}\right) \cup \left(B_{n(1)}^c \cap \bigcap_{k=2}^{m} B_{n(k)}\right).$$

But $B_{n(1)}^c = B_{n(1)} + 2^{-n(1)}$.

Hence,

$$\mathbf{P}\left(\bigcap_{k=2}^{m} B_{n(k)}\right) = 2\mathbf{P}\left(\bigcap_{k=1}^{m} B_{n(k)}\right).$$

So by induction,

$$\mathbf{P}\left(\bigcap_{k=2}^{m} B_{n(k)}\right) = 2^{-m} = \prod_{k=1}^{m} \mathbf{P}(B_{n(k)}).$$

Hence the sequence $\{B_n\}_{n=1}^{\infty}$ is independent. □

Remark 1. *Precisely the same arguments would work if we replaced binary digits with decimal digits.*

Theorem 7. *Given any finite sequence of digits, the probability that a randomly sampled number in* $[0,1]$ *contained that sequence infinitely many times is* 1.

Proof. Suppose that the sequence has length L. Let E_n be the event that the sequence occurs in the nLth position. Then the E_n form an independent sequence of independent events, each with probability 10^{-L}. Then since

$$\sum_{n=1}^{\infty} \mathbf{P}(E_n) = \infty.$$

Hence, by the Borel-Cantelli lemma, the probability that infinitely many of the E_n occur is 1. ☐

Question 6

Lemma 12. *Let s be a continuous solution to the functional equation*

$$4s(2x) = 3s(x) + s(-x).$$

for x in a neighbourhood of 0. *Then s is constant.*

Proof. For any constant c, if s is a solution to the functional equation then so is $s + c$. So we may assume without loss of generality that $s(0) = 0$.

Suppose that $s(x) \neq 0$ for some $x \neq 0$. We may assume that $s(x) > 0$ since if s is a solution, then so is $-s$.

Suppose $s(x) > \varepsilon > 0$. Then

$$3s(x/2) + s(-x/2) > 4\varepsilon.$$

Hence since the average of $(s(x/2), s(x/2), s(x/2), s(-x/2))$ is larger than ε, at least one of these numbers must exceed ε Hence at least one of $s(x/2), s(-x/2)$ exceeds ε.

Thus we have a sequence of numbers approaching 0 which all exceeds ε. But this contradicts $s(0) = 0$ and continuity. ☐

Lemma 13. *Suppose that f is a solution of the functional equation*

$$f(2x) = f(x)^3 f(-x)$$

for x in a neighbourhood of 0, with $f(0) = 1$, and f is assumed to be twice continuously differentiable in a neighbourhood of the origin.

Then $f(x) = \exp(Ax^2 + Bx)$ for parameters A and B.

Proof. Since $f(0) = 1$, and f is continuous, then we may restrict x sufficiently small such that $f(x) > 0$. Hence $L(x) := \log(f(x))$ is well defined, and

$$L(2x) = 3L(x) + L(-x).$$

Differentiating twice, we have

$$4L''(2x) = 3L''(x) + L''(-x)$$

But by lemma 12, we have $L''(x)$ is constant.

Thus, L is a quadratic, so $f(x) = \exp(Ax^2 + Bx + C)$ for constants A,B and C. We see $C = 0$ since $f(0) = 1$. $\square$

Theorem 8. *Suppose that X and Y are independent identically distributed random variables, with finite variances.*

Also assume that $X + Y$ and $X - Y$ are independent.

Then X and Y are Gaussian.

Proof. Let $\varphi(u)$ be the characteristic function of X and Y. Then the characteristic function of $X + Y$ is $\varphi(u)^2$, and the characteristic function of $X - Y$ is $\varphi(u)\varphi(-u)$. The characteristic function of $2X$ is $\varphi(2u)$. Since $2X = X + Y + X - Y$, and $X + Y$ and $X - Y$ are independent, we have

$$\varphi(2u) = \varphi(u)^3 \varphi(-u)$$

and $\varphi(0) = 1$. Since X and Y have finite variances, φ is twice continuously differentiable in a neighbourhood of 0. Thus, by lemma 13 we have $\varphi(u) = \exp(Au^2 + Bu)$ for some parameters A and B.

This is the characteristic function of a Gaussian random variable. $\square$