University of New South Wales

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Homework 3

Measure Theory

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Question 1

In this question we work over $\mathbb{R}^d$, with $d > 1$. λ_d is the Lebesgue measure on $\mathbb{R}^d$.

Theorem 1. *Let $\ell \subset \mathbb{R}^d$ be a line. Then $\lambda_d(\ell) = 0$.*

Proof. Since ℓ is closed, ℓ is Borel and hence Lebesgue measurable.

Since λ_d is translation invariant, we may assume that $0 \in \ell$. Let $v \in \ell$ with $v \neq 0$. Consider half open line segment,

$$S = \{tv \ : \ t \in [0,1)\ \}.$$

S is Borel, hence measurable. Suppose $v = (v_1, v_2, \ldots, v_d)$, and define the box,

$$B_n = \prod_{k=1}^{d} [0, \frac{v_k}{2^n}).$$

If $t \in [0,1)$ then for any $n > 0$ there exists an integer k such that $\frac{k}{2^n} \leq v \leq \frac{k+1}{2^n}$.

Hence,

$$S \subset \bigcup_{k=0}^{2^n-1} (B_n + \frac{k}{2^n} v).$$

So by translation invariance,

$$\lambda_d(S) \leq 2^n \lambda_d(B_n).$$

But by definition, $\lambda_d(B_n) = \frac{1}{2^{nd}} \lambda_d(B_0)$. Hence,

$$\lambda_d(S) \leq 2^{n(1-d)} \lambda_d(B_0).$$

Since $d > 1$, n is arbitrary and $\lambda_d(B_0)$ is finite, we conclude that $\lambda_d(S) = 0$.

Now since

$$\ell = \bigcup_{n \in \mathbb{Z}} (nv + S)$$

by translation invariance we conclude that $\lambda_d(\ell) = 0$. □

Question 2

Let A,B and C be sets, and define $A \triangle B := (A \setminus B) \cup (B \setminus A)$.

(a)

Lemma 1. $A^c \triangle B^c = A \triangle B$.

Proof. This is a simple computation, since by definition,

$$\begin{aligned} A^c \triangle B^c &= (A^c \setminus B^c) \cup (B^c \setminus A^c) \\ &= (A^c \cap B) \cup (B^c \cap A) \\ &= (B \cap A^c) \cup (A \cap B^c) \\ &= (B \setminus A) \cup (A \setminus B) \\ &= (A \triangle B). \end{aligned}$$

□

Lemma 2. $A \triangle C \subseteq (A \triangle B) \cup (B \triangle C)$

Proof. See that

$$\begin{aligned} A \triangle C &= (A \triangle C) \cap B \cup (A \triangle C) \cap B^c \\ &= (A \cap B \cap C^c) \cup (A^c \cap B \cap C) \cup (A \cap B^c \cap C^c) \cup (A^c \cap B^c \cap C) \\ &\subseteq (A^c \cap B) \cup (B^c \cap A) \cup (B \cap C^c) \cup (B^c \cap C) \\ &= (A \triangle B) \cup (B \triangle C). \end{aligned}$$

□

(b)

Now we define

$$\mathcal{G} := \{\ B \in \mathcal{F}\ :\ \forall \varepsilon > 0\ \exists B_\varepsilon \in \mathcal{A} \text{ such that } \mu(B \triangle B_\varepsilon) < \varepsilon\ \}.$$

Lemma 3. *If $A \in \mathcal{G}$, then $A^c \in \mathcal{G}$.*

Proof. Let $A \in \mathcal{G}$, and let $\varepsilon > 0$. Then choose $B_\varepsilon \in \mathcal{A}$ such that $\mu(A \triangle B_\varepsilon) < \varepsilon$. Hence $\mu(A^c \triangle B_\varepsilon^c) < \varepsilon$, and since $B_\varepsilon^c \in \mathcal{A}$, we conclude $A^c \in \mathcal{G}$. □

(c)

Lemma 4. *Let $A_n \in \mathcal{G}, n \geq 1$, with $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots$. If $A = \bigcup_{n\geq 1} A_n$, then for any $\varepsilon > 0$ there exists $N > 0$ such that $\mu(A \triangle A_N) < \varepsilon$.*

Proof. We compute,

$$\begin{aligned}\mu(A) &= \mu(\bigcup_{n=1}^{\infty} A_n)\\ &= \lim_{n\to\infty} \mu(A_n).\end{aligned}$$

And,

$$\mu(A \triangle A_n) = \mu(A \setminus A_n) = \mu(A) - \mu(A_n).$$

Hence,

$$\lim_{n\to\infty} \mu(A \triangle A_n) = 0.$$

□

(d)

Corollary 1. $A \in \mathcal{G}$

Proof. Let $\varepsilon > 0$. Choose $N > 0$ such that $\mu(A \triangle A_n) < \varepsilon/2$ and select $B \in \mathcal{A}$ such that $\mu(B \triangle A_n) < \varepsilon/2$. Hence,

$$\mu(A \triangle B) \leq \mu(A \triangle A_n \cup B \triangle A_n) \leq \mu(A \triangle A_n) + \mu(B \triangle A_n) < \varepsilon.$$

Hence $A \in \mathcal{G}$. □

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Theorem 2. $\mathcal{G} = \mathcal{F}$.

Proof. Suppose $A, B \in \mathcal{G}$. Then choose $A_\varepsilon, B_\varepsilon \in \mathcal{A}$ such that $\mu(A \triangle A_\varepsilon) < \varepsilon/2$ and $\mu(B \triangle B_\varepsilon) < \varepsilon/2$.

Then

$$\begin{aligned}
\mu((A \cap B) \triangle (A_\varepsilon \cap B_\varepsilon)) &= \mu((A \cap B) \setminus (A_\varepsilon \cap B_\varepsilon) \cup ((A_\varepsilon \cap B_\varepsilon) \setminus (A \cap B)) \\
&= \mu((A \cap B) \setminus A_\varepsilon \cup (A \cap B) \setminus B_\varepsilon \cup (A_\varepsilon \cap B_\varepsilon) \setminus A \cup (A_\varepsilon \cap B_\varepsilon) \setminus B) \\
&\leq \mu(A \triangle A_\varepsilon \cap B \triangle B_\varepsilon) \\
&\leq \varepsilon.
\end{aligned}$$

Hence $\mathcal{G}$ is closed under intersection. Thus $\mathcal{G}$ is closed under relative complement since it is closed under complement.

Hence $\mathcal{G}$ is a d-class containing $\mathcal{A}$ since we have shown that it is closed under complement and increasing countable union, hence $d(\mathcal{A}) \subseteq \mathcal{G}$. But since $\mathcal{A}$ is an algebra, it is a π-class. Hence $\mathcal{F} \subseteq \mathcal{G}$ since $\sigma(\mathcal{A}) = d(\mathcal{A})$ by the monotone class theorem.

Since by definition $\mathcal{G} \subseteq \mathcal{F}$, we conclude $\mathcal{F} = \mathcal{G}$. □

Question 3

In this question we consider the measure space $(X, \mathcal{A}, \mu)$ and the completed measure $\overline{\mu}$ with associated algebra $\mathcal{A}_\mu$.

Lemma 5. *$\mathcal{A}_\mu$ is a σ-algebra on X.*

Proof. Since $\mathcal{A} \subseteq \mathcal{A}_\mu$, we have $X \in \mathcal{A}_\mu$.

Suppose $A \in \mathcal{A}_\mu$. Then by definition there are $E, F \in \mathcal{A}$ with $E \subseteq A \subseteq F$ and $\mu(F \setminus E) = 0$. Hence we have $F^c \subseteq A^c \subseteq E^c$, and $\mu(E^c \setminus A^c) = \mu(F \setminus E) = 0$. Since $F^c, E^c \in \mathcal{A}$, we conclude that $A^c \in A_\mu$.

Now let $\{A_n\}_{n=1}^\infty$ be a countable subcollection of $\mathcal{A}_\mu$. Choose $E_n, F_n \in \mathcal{A}$ for each $n \geq 1$ such that $E_n \subseteq A_n \subseteq F_n$ and $\mu(F_n \setminus E_n) = 0$. Hence,

$$\bigcup_{n=1}^\infty E_n \subseteq \bigcup_{n=1}^\infty A_n \subseteq \bigcup_{n=1}^\infty F_n$$

where the left and right hand sides are in $\mathcal{A}$, and

$$\mu(\bigcup_{n=1}^\infty F_n \setminus \bigcup_{n=1}^\infty E_n) \leq \sum_{n=1}^\infty \mu(F_n \setminus E_n) = 0.$$

Hence $\mathcal{A}_\mu$ is a σ-algebra. □

Lemma 6. *$\overline{\mu}$ is a measure on $(X, \mathcal{A}_\mu)$.*

Proof. We need to prove that μ is countably additive on $\mathcal{A}_\mu$. Suppose

that $\{A_n\}_{n=1}^{\infty}$ is a sequence of $\overline{\mu}$-measurable sets that is pairwise disjoint. Then choose $E_n, F_n \in \mathcal{A}$ such that $E_n \subseteq A_n \subseteq F_n$ and $\mu(F_n \setminus E_n) = 0$.

Hence $\overline{\mu}(A_n) = \mu(E_n)$. Then we have

$$\bigcup E_n \subseteq_{n=1}^{\infty} \bigcup A_n \subseteq_{n=1}^{\infty} \bigcup_{n=1}^{\infty} F_n$$

Then since $\mu(\bigcup_{n=1}^{\infty} F_n \setminus \bigcup_{n=1}^{\infty} E_n) = 0$, we have

$$\overline{\mu}(\bigcup_{n=1}^{\infty} A_n) = \mu(\bigcup_{n=1}^{\infty} E_n).$$

But since the A_n are pairwise disjoint, so are the E_n, and so

$$\begin{aligned}\overline{\mu}(\bigcup_{n=1}^{\infty} A_n) &= \sum_{n=1}^{\infty} \mu(E_n) \\ &= \sum_{n=1}^{\infty} \overline{\mu}(A_n).\end{aligned}$$

Hence $\overline{\mu}$ is countably additive, hence a measure.

□

Lemma 7. *The restriction of $\overline{\mu}$ to $\mathcal{A}$ is μ.*

Proof. Let $E \in \mathcal{A}$. Then $E \in \mathcal{A}_\mu$ since $E \subseteq E \subseteq E$, and $\mu(E \setminus E) = 0$.

Then $\overline{\mu} = \mu(E)$. Hence $\overline{\mu}$ restricted to $\mathcal{A}$ is μ. □

Lemma 8. *The measure space $(X, \mathcal{A}_\mu, \overline{\mu})$ is complete.*

Proof. We need to show that every subset of a $\overline{\mu}$-null set is measurable.

Suppose $E \in 2^X$ with $E \subseteq F \in \mathcal{A}_\mu$ and $\overline{\mu}(F) = 0$. Then since $\emptyset \in \mathcal{A}$, and there is some set $B \in \mathcal{A}$ with $F \subseteq B$ and $\mu(B \setminus \emptyset) = \mu(B) = 0$, then $F \in \mathcal{A}_\mu$ and $\overline{\mu}(F) = \mu(B) = 0$.

Hence $(X, \mathcal{A}_\mu, \overline{\mu})$ is complete. □