University of New South Wales

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Homework 4

Measure Theory

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Question 1

For this question, $(X, \mathcal{A}, \mu)$ is a measure space.

Lemma 1. *The set*
$$\{[a, \infty) \ : \ a \in \mathbb{R}\}$$
generates $\mathcal{B}(\mathbb{R})$*.*

Proof. Since $[a, b) = [c, \infty)^c \cap [a, \infty)$, by the result of homework week 2, this set generates $\mathcal{B}(\mathbb{R})$. □

Hence to prove that $f : X \to \mathbb{R}$ is Borel measurable, it is sufficient to show that $f^{-1}([a, \infty)) \in \mathcal{A}$ for all $a \in \mathbb{R}$.

Lemma 2. *If $f, g : X \to \mathbb{R}$ are Borel measurable functions, then $f + g$ is Borel measurable.*

Proof. Let $a \in \mathbb{R}$. Then for every $r \in \mathbb{Q}$, we have $f(x) + g(x) \geq a$ if and only if $f(x) \geq r$ and $g(x) \geq a - r$. Hence,
$$(f + g)^{-1}([a, \infty)) = \bigcup_{r \in \mathbb{Q}} \{f^{-1}([r, \infty]) \cap g^{-1}([a - r, \infty))\}$$
Since f and g are Borel measurable, the right hand side is in $\mathcal{A}$. Hence $f + g$ is Borel measurable. □

Lemma 3. *The following functions on $\mathbb{R}$ are Borel measurable:*

- $s_1(x) = x^2$
- $s_2(x) = \alpha x$*, for any* $\alpha \in \mathbb{R}$
- $s_3(x) = x^{-1}$*, is measurable on the subspace* $\mathbb{R} \setminus \{0\}$*.*
- $s_4(x) = |x|$

Proof. Let $a \in \mathbb{R}$. If $a < 0$, then $s_1^{-1}([a, \infty)) = \mathbb{R}$ and otherwise $s_1^{-1}([a, \infty)) = (-\infty, -\sqrt{a}) \cup (\sqrt{a}, \infty)$. Hence s_1 is Borel.

For s_2, the case $\alpha = 0$ is trivial. Suppose $\alpha > 0$. Then $s_2^{-1}([a, \infty)) = [a/\alpha, \infty)$. Similarly, if $\alpha < 0$, $s_2^{-1}([a, \infty)) = (-\infty, a/\alpha)$. Hence s_2 is Borel for any α.

Now if $a > 0$, $s_3^{-1}([a, \infty)) = (0, 1/a)$ and if $a = 0$ $s_3^{-1}([a, \infty)) = (0, \infty)$.

If $a < 0$, $s_3^{-1}([a, \infty)) = (-\infty, 1/a) \cup (0, \infty)$.

Hence s_3 is Borel since the set $(0, 1/a)$, $(0, \infty)$, $-\infty, 1/a)$ and $(0, \infty)$ are Borel on the subspace $\mathbb{R} \setminus \{0\}$.

Now $s_4^{-1}([a, \infty)) = (-\infty, -a] \cup [a, \infty)$ for any $a \in \mathbb{R}$, and so s_4 is measurable. □

Corollary 1. *Let $f : X \to \mathbb{R}$ be measurable. The following functions on X are measurable:*

- $f_1 = f^2$
- $f_2 = \alpha f$ *for any* $\alpha \in \mathbb{R}$.
- $f_3 = 1/f$, *measurable on the subspace of* X *given by* $X \setminus \{x : f(x) = 0\}$.
- $f_4 = |f|$.

Proof. We see that f_1, f_2 and f_4 are measurable since $f_1 = s_1 \circ f$, $f_2 = s_2 \circ f$ and $f_4 = s_4 \circ f$ using s_1, s_2 and s_4 from lemma 2.

For f_3, note that $X \setminus \{x : f(x) = 0\} = f^{-1}(\mathbb{R} \setminus \{0\})$, and by definition subsets of $X \setminus \{x : f(x) = 0\}$ are measurable if they are the intersection with some element of $\mathcal{A}$.

See that since $f_3 = s_3 \circ f$, it is measurable. □

Lemma 4. *If $f, g : X \to \mathbb{R}$ are measurable, then so is fg.*

Proof. We may write $fg = \frac{1}{2}((f+g)^2 - f^2 - g^2)$. Hence fg may be written as a composition of measurable functions and is hence measurable. □

Corollary 2. *If f and g are measurable functions on X, then f/g is measurable on $X \setminus \{x : g(x) = 0\}$.*

Proof. Since f is measurable on X, it is measurable on $X \setminus \{x : g(x) = 0\}$. Hence since $1/g$ is measurable on this set, their product is measurable. □

Lemma 5. *If $f, g : X \to \mathbb{R}$ are measurable, then so is* $\max\{f, g\}$ *and* $\min\{f, g\}$.

Proof. We may write $\max\{f, g\} = \frac{1}{2}(f + g + |f - g|)$ and $\min\{f, g\} = \frac{1}{2}(f + g - |f - g|)$. Hence these are compositions of measurable functions and so measurable. □

Lemma 6. *If $f, g : X \to \mathbb{R}$ are measurable, then the sets $\{x : f(x) = g(x)\}$, $\{x : f(x) > g(x)\}$ and $\{x : f(x) \geq g(x)\}$ are measurable.*

Proof. We may write these sets respectively as $(f-g)^{-1}(\{0\})$, $(f-g)^{-1}((0, \infty))$ and $(f-g)^{-1}([0, \infty))$. Hence since $f - g$ is measurable, the result follows. □

Lemma 7. *If $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions on X, then so is* $\inf_n f_n$.

Proof. Let $f = \inf_n f_n$ and $a \in \mathbb{R}$. Let $x \in f^{-1}([a,\infty))$. Then for all n, $f_n(x) \geq a$, so $x \in f_n^{-1}([a,\infty))$. Hence,

$$f^{-1}([a,\infty)) \subseteq \bigcap_{n=1}^{\infty} f_n^{-1}([a,\infty))$$

Similarly, if $f_n(x) \geq a$ for all n, then we must have $f(x) \geq a$. Hence,

$$f^{-1}((-\infty,a]) = \bigcap_{n=1}^{\infty} f_n^{-1}([a,\infty)).$$

Since each f_n is measurable, the right hand side is measurable. Hence f is measurable. □

Corollary 3. *If $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable real valued functions on X, then the following functions are measurable:*

- $\sup_n f_n$
- $\limsup_n f_n$
- $\liminf_n f_n$

Proof. We may write $\sup_n f_n = -\inf_n -f_n$, hence it is measurable.

$\limsup_n = \inf_n \sup_{k\geq n} f_k$ and $\liminf_n = \sup_n \inf_{k\geq n} f_n$. Hence these functions are measurable. □

Question 2

For this question, we consider the measure space $(X, \mathcal{A}, \mu)$ and a bounded measurable measurable function $f : X \to [0,\infty)$.

Define $F : [0,\infty) \to [0,\infty]$ by $F(t) = \mu(\{x \,:\, f(x) > t\})$.

Lemma 8. *F is a decreasing function and F vanishes outside some bounded interval.*

Proof. It is clear that F is a decreasing function, since if $s > t$,

$$\{x : f(x) > s\} \subseteq \{x : f(x) > t\}.$$

Hence $\mu\{x : f(x) > s\} \leq \mu\{x : f(x) > t\}$, so $F(s) \leq F(t)$.

Since f is bounded, there is some M such that $f(x) < M$ for all x. Hence for $x > M$, $F(x) = \mu(\emptyset) = 0$. Thus the support of F is contained in $[0, M]$. □

Lemma 9. *The limit* $\lim_{t\to 0} F(t)$ *is finite if and only if the support of* f *has finite measure.*

Proof. We may write $\lim_{t\to 0} F(t) = \lim_{n\to\infty} \mu\{x : F(x) > 1/n\} = \mu(\bigcap_{n=1}^{\infty}\{x : F(x) > 1/n\}) = \mu(\{x : F(x) > 0\})$.

Hence this limit is finite if and only if F is nonzero on a set of finite measure. □

Theorem 1. *The indefinite Riemann integral,*

$$\int_0^\infty F(t)\, dt$$

exists if f *has support of finite measure.*

Proof. Since F vanishes outside some interval $[0, M]$, we have that

$$\lim_{N\to\infty} \int_0^N F(t)\, dt = \int_0^M F(t)\, dt.$$

Hence we need only consider the integrability of F on $[0, M]$.

Since f has support of finite measure, and F is a decreasing function, we have that F is bounded.

Hence F is a bounded monotone function, so the Riemann integral exists. □

Suppose f has support of finite measure and let F have support contained in $[0, N]$. Consider the partition of $[0, N]$ given by $\mathcal{P}_n = \{0, 1/2^n, 2/2^n, \ldots, N2^n/2^n\}$. Let L_n be the corresponding lower Riemann sum of F.

Lemma 10. L_n *is given by the integral of a simple function bounded above by* F*,* s_n *given by*

$$s_n = \sum_{k=0}^{2^n N-1} \chi_{f^{-1}([k/2^n,(k+1)/2^n))}(k/2^n).$$

Proof. Since F is a decreasing function, the infimum of F on the interval $[k/2^n, (k+1)/2^n)$ is $F((k+1)/2^n)$. Hence,

$$L_n = \sum_{k=0}^{2^n N-1} F((k+1)/2^n)\frac{1}{2^n}$$

Now we compute the integral of s_n,

$$\begin{aligned}
\int s_n \, d\mu &= \sum_{k=0}^{2^n N-1} \mu(f^{-1}([k/2^n,(k+1)/2^n))\frac{k}{2^n} \\
&= \sum_{k=0}^{2^n N-1} [F(k/2^n) - F((k+1)/2^n)]\frac{k}{2^n} \\
&= \sum_{k=0}^{2^n N-1} \sum_{j=1}^{k} \frac{1}{2^n}[F(k/2^n) - F((k+1)/2^n)] \\
&= \sum_{j=0}^{2^n N-1} \sum_{k=j}^{2^n N-1} \frac{1}{2^n}[F(k/2^n) - F((k+1)/2^n)] \\
&= \sum_{j=0}^{2^n N-1} \frac{1}{2^n}[F(j/2^n) - F(N+1)] \\
&= \sum_{j=1}^{2^n N-1} \frac{1}{2^n} F(j/2^n) \\
&= L_n.
\end{aligned}$$

□

Lemma 11. *The simple functions s_n in the above lemma form a monotonic increasing sequence and converge pointwisely to f.*

Proof. Fix $x \in X$ and $n \geq 1$. Then $x \in f^{-1}([k/2^n, (k+1)/2^n))$ for some $0 \leq k \leq 2^n N$, since f is bounded above by n.

Then $s_n(x) = k/2^n$.

And $x \in f^{-1}([2k/2^{n+1}, (2k+1)/2^{n+1})) \cup f^{-1}([(2k+1)/2^{n+1}, (2k+2)/2^{n+2}))$, so $s_{n+1}(x) = k/2^n$ or $s_n(x) = k/2^n + 1/2^n$.

Hence $s_{n+1}(x) \geq s_n(x)$, and so the simple functions s_n form a non-decreasing sequence.

Now $|f(x) - s_n(x)| \leq 1/2^n$, so s_n converges pointwisely to f. □

Theorem 2.

$$\int f\ d\mu = \int_0^\infty F(t)\ dt.$$

Proof. Since the simple functions s_n are a non-decreasing sequence that converges pointwisely to f, we have by the Beppo-Levi theorem that

$$\int f\ d\mu = \lim_{n\to\infty} \int s_n\ d\mu = \lim_{n\to\infty} L_n = \int_0^\infty F(t)\ dt.$$

□

Question 3

Again in this question $(X, \mathcal{A}, \mu)$ is a measure space and $f : X \to [0,\infty]$ is a measurable function, with

$$\int f\ d\mu < \infty.$$

Theorem 3.

$$\lim_{n\to\infty} n\mu(\{x\ :\ f(x) > n\}) = 0.$$

Proof. We assume that f takes only finite values. This does not change the result since f is integrable, so it can only take the value ∞ on a set of measure zero. Since the value of f on a set of measure zero does not change the integral of f or the value of $\mu(\{x\ :\ f(x) > n\})$ for any n, we are free to assume that $f < \infty$.

Let $A_n = f^{-1}([0,n])$, and

$$\begin{aligned} f_n &= f\chi_{A_n} + n\chi_{A_n^c} \\ g_n &= f\chi_{A_n} \end{aligned}$$

Then by definition $f_n \leq f$.

Since by assumption $f < \infty$ everywhere, we have

$$\bigcup_{n=1}^\infty A_n = X.$$

Hence the sequences f_n and g_n converges pointwisely to f. So by the Beppo-Levi theorem,

$$\begin{aligned}\int f\ d\mu = \lim_{n\to\infty}\int f_n\ d\mu &= \lim_{n\to\infty}\int g_n\ d\mu + \lim_{n\to\infty}\int n\chi_{A_n^c}\ d\mu \\ &= \int f\ d\mu + \lim_{n\to\infty} n\mu(\{x\ :\ f(x) > n\}).\end{aligned}$$

Since by assumption $\int f\ d\mu < \infty$, we conclude the required result. $\square$