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Present for Stuart

Measure Theory

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Lemma 1. *Suppose that $q_n \to \lambda \in \mathbb{C}$ is a sequence of complex numbers. Then*

$$\lim_{n\to\infty} \left(1 + \frac{q_n}{n}\right)^n = e^\lambda$$

Proof. Fix n large enough such that $|q_n|/n < 1/2$.

Since q_n is a convergent sequence, it is bounded. Let M be large enough such that $|q_n| < M$ for all n.

Re-write $\left(1 + \frac{q_n}{n}\right)^n$ as $\exp(n \operatorname{Log}(1 + \frac{q_n}{n}))$.

The branch of the logarithm taken here is complex differentiable in the set $\mathbb{C} \setminus (-\infty, 0]$. Since $|q_n|/n < 1$, the above is valid.

So it is sufficient to show that,

$$\lim_{n\to\infty} n \operatorname{Log}\left(1 + \frac{q_n}{n}\right) = \lambda$$

The $z \mapsto \operatorname{Log}(1 + z)$ function is complex differentiable in the unit disc, and has a power series representation

$$\operatorname{Log}(1 + z) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{z^k}{k}$$

which converges uniformly on compact subsets of the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$.

Now, since $|q_n|/n < 1$, we have

$$n \operatorname{Log}(1 + \frac{q_n}{n}) = q_n + \sum_{k=2}^{\infty} (-1)^{k-1} \frac{q_n^k}{kn^{k-1}}$$

Now we consider the tail of the left hand side, let

$$L_n := \sum_{k=2}^{\infty} (-1)^{k-1} \frac{q_n^k}{kn^{k-1}}$$

By the triangle inequality,

$$|L_n| \le \sum_{k=2}^{\infty} \frac{M^k}{kn^{k-1}}$$

Thus,

$$|L_n| \leq M \sum_{k=1}^{\infty} \left(\frac{M}{n}\right)^k$$
$$= M \frac{M/n}{(1 - M/n)}$$

Hence, $L_n \to 0$ as $n \to \infty$. Thus, the limit

$$\lim_{n\to\infty} n \operatorname{Log}(1 + \frac{q_n}{n})$$

exists, and equals $\lim_{n\to\infty} q_n = \lambda$.

Hence, the limit

$$\lim_{n\to\infty} \left(1 + \frac{q_n}{n}\right)^n$$

exists, and equals e^λ. □

Now we let $\{p_n\}_{n=1}^{\infty}$ be a monotone decreasing sequence, such that $np_n \to \lambda$. We let $\mu_n = \operatorname{Bin}(n, p_n)$.

Theorem 1. *There is weak convergence, $\mu_n \to \nu$.*

Proof. By Lévy's continuity theorem, it is sufficient to show pointwise convergence of characteristic functions, $\hat{\mu}_n(u) \to \hat{\nu}(u)$ for all u. That is, we must show

$$\lim_{n\to\infty} (1 - p_n + p_n e^{iu})^n = \exp(\lambda(e^{iu} - 1)).$$

Rewrite $\hat{\mu}_n(u)$ as

$$\left(1 + \frac{np_n(e^{iu} - 1)}{n}\right)^n$$

Now by lemma **??**, we see

$$\lim_{n\to\infty} \hat{\mu}_n(u) = \exp(\lambda(e^{iu} - 1)).$$

Thus the result follows. □