University of New South Wales

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Homework 3

Measure Theory

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Question 1

In this question we work over $\mathbb{R}^d$, with $d > 1$. λ_d is the Lebesgue measure on $\mathbb{R}^d$.

Theorem 1. *Let $\ell \subset \mathbb{R}^d$ be a line. Then $\lambda_d(\ell) = 0$.*

Proof. Since ℓ is closed, ℓ is Borel and hence Lebesgue measurable.

Since λ_d is translation invariant, we may assume that $0 \in \ell$. Let $v \in \ell$ with $v \neq 0$. Consider half open line segment,

$$S = \{tv \ : \ t \in [0,1)\ \}.$$

S is Borel, hence measurable. Suppose $v = (v_1, v_2, \ldots, v_d)$, and define the box,

$$B_n = \prod_{k=1}^{d} [0, \frac{v_k}{2^n}).$$

If $t \in [0,1)$ then for any $n > 0$ there exists an integer k such that $\frac{k}{2^n} \leq v \leq \frac{k+1}{2^n}$.

Hence,

$$S \subset \bigcup_{k=0}^{2^n-1} (B_n + \frac{k}{2^n}v).$$

So by translation invariance,

$$\lambda_d(S) \leq 2^n \lambda_d(B_n).$$

But by definition, $\lambda_d(B_n) = \frac{1}{2^{nd}}\lambda_d(B_0)$. Hence,

$$\lambda_d(S) \leq 2^{n(1-d)}\lambda_d(B_0).$$

Since $d > 1$, n is arbitrary and $\lambda_d(B_0)$ is finite, we conclude that $\lambda_d(S) = 0$.

Now since

$$\ell = \bigcup_{n\in\mathbb{Z}} (nv + S)$$

by translation invariance we conclude that $\lambda_d(\ell) = 0$. $\square$

Question 2

(a)

Lemma 1.

(b)

Now we define

$$\mathcal{G} := \{\ B \in \mathcal{F}\ :\ \forall \varepsilon > 0\ \exists B_\varepsilon \in \mathcal{A} \text{ such that } \mu(B \triangle B_\varepsilon) < \varepsilon\ \}.$$

Lemma 2. *If $A \in \mathcal{G}$, then $A^c \in \mathcal{G}$.*

Proof. Let $A \in \mathcal{G}$, and let $\varepsilon > 0$. Then choose $B_\varepsilon \in \mathcal{A}$ such that $\mu(A \triangle B_\varepsilon) < \varepsilon$. Hence $\mu(A^c \triangle B_\varepsilon^c) < \varepsilon$, and since $B_\varepsilon^c \in \mathcal{A}$, we conclude $A \in \mathcal{G}$. □

(c)

Lemma 3. *Let $A_n \in \mathcal{G}$,$n \geq 1$, with $A_1 \subseteq A_2 \subseteq \cdots A_n \cdots$. If $A = \bigcup_{n \geq 1} A_n$, then for any $\varepsilon > 0$ there exists $N > 0$ such that $\mu(A \triangle A_N) < \varepsilon$.*

Proof. We compute,

$$\begin{aligned} \mu(A) &= \mu(\bigcup_{n=1}^{\infty} A_n) \\ &= \lim_{n \to \infty} \mu(A_n). \end{aligned}$$

And,

$$\mu(A \triangle A_n) = \mu(A \setminus A_n) = \mu(A) - \mu(A_n).$$

Hence,

$$\lim_{n \to \infty} \mu(A \triangle A_n) = 0.$$

□

(d)

Corollary 1. $A \in \mathcal{G}$

Proof. Let $\varepsilon > 0$. Choose $N > 0$ such that $\mu(A \triangle A_n) < \varepsilon/2$ and select $B \in \mathcal{A}$ such that $\mu(B \triangle A_n) < \varepsilon/2$. Hence,

$$\mu(A \triangle B) \leq \mu(A \triangle A_n \cup B \triangle A_n) \leq \mu(A \triangle A_n) + \mu(B \triangle A_n) < \varepsilon.$$

Hence $A \in \mathcal{G}$. □

e

Theorem 2. *$\mathcal{G} = \mathcal{F}$.*

Proof. $\mathcal{G}$ is a d-class containing $\mathcal{A}$, hence $d(\mathcal{A}) \subseteq \mathcal{G}$. But since $\mathcal{A}$ is an algebra, it is a π-class. Hence $\mathcal{F} \subseteq \mathcal{G}$ since $\sigma(\mathcal{A}) = d(\mathcal{A})$ by the monotone class theorem. □

Question 3

In this question we consider the measure space $(X, \mathcal{A}, \mu)$ and the completed measure $\overline{\mu}$ with associated algebra $\mathcal{A}_\mu$.

Lemma 4. *$\mathcal{A}_\mu$ is a σ-algebra on X.*

Proof. Since $\mathcal{A} \subseteq \mathcal{A}_\mu$, we have $X \in \mathcal{A}_\mu$.

Suppose $A \in \mathcal{A}_\mu$. Then by definition there are $E, F \in \mathcal{A}$ with $E \subseteq A \subseteq F$ and $\mu(F \setminus E) = 0$. Hence we have $F^c \subseteq A^c \subseteq E^c$, and $\mu(E^c \setminus A^c) = \mu(F \setminus E) = 0$. Since $F^c, E^c \in \mathcal{A}$, we conclude that $A^c \in A_\mu$.

Now let $\{A_n\}_{n=1}^\infty$ be a countable subcollection of $\mathcal{A}_\mu$. Choose $E_n, F_n \in \mathcal{A}$ for each $n \geq 1$ such that $E_n \subseteq A_n \subseteq F_n$ and $\mu(F_n \setminus E_n) = 0$. Hence,

$$\bigcup_{n=1}^\infty E_n \subseteq \bigcup_{n=1}^\infty A_n \subseteq \bigcup_{n=1}^\infty F_n$$

where the left and right hand sides are in $\mathcal{A}$, and

$$\mu(\bigcup_{n=1}^\infty F_n \setminus \bigcup_{n=1}^\infty E_n) \leq \sum_{n=1}^\infty \mu(F_n \setminus E_n) = 0.$$

Hence $\mathcal{A}_\mu$ is a σ-algebra. □

Lemma 5. *$\overline{\mu}$ is a measure on $(X, \mathcal{A}_\mu)$.*

Proof. We need to prove that μ is countably additive on $\mathcal{A}_\mu$. Suppose

□