UNSW School of Mathematics and Statistics

MATH5825 Measure, Integration and Probability

Semester 2/2014

Assignment 2

(1) **[7 marks]** Let μ and ν be probability measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$.

(a) Show that the convolution $\mu \star \nu(B) = \int \nu(B-x)\mu(dx)$ of two finite measures μ and ν on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)$ is well defined, that is:

i. the mapping $x \mapsto \nu(B-x)$ is measurable

ii. the integral exists.

(Hint: Tonelli.)

(b) Show that if there is a bounded (resp. countable, resp. finite) set $F \in \mathcal{B}(\mathbb{R}^d)$ such that $\mu \star \nu(F) = 1$, then there are bounded (resp. countable, resp. finite) sets $G, H \in \mathcal{B}(\mathbb{R}^d)$ such that $\mu(G) = 1$ and $\nu(H) = 1$.

(2) **[7 marks]** Let μ and ν be σ-finite positive measures on $(\Omega, \mathcal{F})$.

(a) Show that the following conditions are equivalent:

(i) $\mu \ll \nu$ and $\nu \ll \mu$

(ii) μ and ν have exactly the same set of measure zero, and

(iii) there is an $\mathcal{F}$-measurable function g that satisfies $0 < g(\omega) < +\infty$ at each $\omega \in \Omega$ and is such that $\nu(A) = \int_A g \, d\mu$ holds for each $A \in \mathcal{F}$.

(b) Show that if μ is a σ-finite measure on $(\Omega, \mathcal{F})$ then there is a finite measure ν on $(\Omega, \mathcal{F})$ such that $\nu \ll \mu$ and $\mu \ll \nu$.

(3) **[6 marks]** Let X be a d-dimensional random vector with law μ.

(a) For any $c \in \mathbb{R}$, the characteristic function of cX is $\hat{\mu}(cu)$.

(b) X is said to have moments up to order n if the following holds: For all $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ such that $|\alpha| := \sum_{k=1}^d \alpha_k \leqslant n$

$$\mathbf{E}(|X|^\alpha) := \mathbf{E}\left(\prod_{k=1}^d |X_k|^{\alpha_k}\right) < \infty$$

Show that if X has moments up to order n, then

$$\frac{\partial^\alpha}{\partial u^\alpha}\hat{\mu}(u) := \frac{\partial^{\alpha_1}}{\partial u_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_d}}{\partial u_d^{\alpha_d}}\hat{\mu}(u)$$

evaluated at $u = 0$ equals $i^{|\alpha|}\mathbf{E}(X^\alpha)$ where $X^\alpha := \prod_{k=1}^d X_k^{\alpha_k}$.

(c) Let $d = 1$ and let μ have the Lebesgue density

$$f(x) = \frac{C}{(1+x^2)\log(e+x^2)}, \quad x \in \mathbb{R}.$$

Show that $\mathbf{E}(X)$ is not defined but $\hat{\mu}(u)$ is differentiable at 0.
(That is, the converse to (b) is not necessarily true.)

(4) **[7 marks]** Let μ be the binomial distribution with n trials and probability of success p, that is, $\mu = \mathrm{Bin}(n, p)$, and let ν be the Poisson distribution with mean $\lambda > 0$.

(a) Verify that $\hat{\mu}(u) = (1 - p + pe^{iu})^n$. (Hint: μ is the convolution of n much easier measures.)

(b) Verify that $\hat{\nu}(u) = \exp\big(\lambda(e^{iu} - 1)\big)$.

(c) Let p_n be a sequence in $[0,1]$ such that $p_n \downarrow 0$ and $np_n \to \lambda$. Let $\mu_n = \mathrm{Bin}(n, p_n)$. Show that the weak convergence $\mu_n \to \nu$ holds.

(d) Is it true that $\mu_n(\{k\}) \to \nu(\{k\})$ for every $k \in \mathbb{N} \cup \{0\}$? Why or why not?

(5) **[7 marks]** Consider the probability space $(\Omega, \mathcal{F}, \mathbf{P}) = ([0,1], \mathcal{B}([0,1]), \lambda)$, where $\mathcal{B}([0,1])$ is the Borel-σ-algebra generated by open intervals $(a,b) \subset [0,1]$ and where λ is Lebesgue measure. Every $\omega \in [0,1]$ has a dyadic expansion

$$\omega = \sum_{n=1}^{\infty} \frac{d_n(\omega)}{2^n}.$$

If ω has two different dyadic expansions then it can be shown that one of the two has only a finite number of ones; in that case we choose the other expansion which has infinitely many ones. Let

$$B_n = \{\omega \in [0,1] : d_n(\omega) = 0\}, \quad n \geqslant 1.$$

(a) Show that $\mathbf{P}(B_n) = 1/2$ for every $n \geqslant 1$.

(b) Show that the events B_n form an infinite sequence of independent events.

(c) What is the probability that a randomly sampled number ω has the sequence 5825 occur infinitely often in its decimal expansion? Prove your answer.

(6) **[6 marks]** Let X and Y be independent and identically distributed random variables with finite variances. Show that if $X + Y$ and $X - Y$ are independent, then X and Y are Gaussian.