University of New South Wales

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Homework 4

Measure Theory

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Question 1

For this question, $(X, \mathcal{A}, \mu)$ is a measure space.

Lemma 1. *The set*

$$\{[a, \infty)\ :\ a \in \mathbb{R}\}$$

generates $\mathcal{B}(\mathbb{R})$.

Proof. Since $[a, b) = [c, \infty)^c \cap [a, \infty)$, by the result of homework week 2, this set generates $\mathcal{B}(\mathbb{R})$. □

Hence to prove that $f : X \to \mathbb{R}$ is Borel measurable, it is sufficient to show that $f^{-1}([a, \infty)) \in \mathcal{A}$ for all $a \in \mathbb{R}$.

Lemma 2. *If* $f, g : X \to \mathbb{R}$ *are Borel measurable functions, then* $f + g$ *is Borel measurable.*

Proof. Let $a \in \mathbb{R}$. Then for every $r \in \mathbb{Q}$, we have $f(x) + g(x) \geq a$ if and only if $f(x) \geq r$ and $g(x) \geq a - r$. Hence,

$$(f + g)^{-1}([a, \infty)) = \bigcup_{r \in \mathbb{Q}} \{f^{-1}([r, \infty]) \cap g^{-1}([a - r, \infty))\}$$

Since f and g are Borel measurable, the right hand side is in $\mathcal{A}$. Hence $f + g$ is Borel measurable. □

Lemma 3. *The following functions on* $\mathbb{R}$ *are Borel measurable:*

- $s_1(x) = x^2$
- $s_2(x) = \alpha x$, *for any* $\alpha \in \mathbb{R}$
- $s_3(x) = x^{-1}$, *is measurable on the subspace* $\mathbb{R} \setminus \{0\}$.
- $s_4(x) = |x|$

Proof. Let $a \in \mathbb{R}$. If $a < 0$, then $s_1^{-1}([a, \infty)) = \mathbb{R}$ and otherwise $s_1^{-1}([a, \infty)) = (-\infty, -\sqrt{a}) \cup (\sqrt{a}, \infty)$. Hence s_1 is Borel.

For s_2, the case $\alpha = 0$ is trivial. Suppose $\alpha > 0$. Then $s_2^{-1}([a, \infty)) = [a/\alpha, \infty)$. Similarly, if $\alpha < 0$, $s_2^{-1}([a, \infty)) = (-\infty, a/\alpha)$. Hence s_2 is Borel for any α.

Now if $a > 0$, $s_3^{-1}([a, \infty)) = (0, 1/a)$ and if $a = 0$ $s_3^{-1}([a, \infty)) = (0, \infty)$.

If $a < 0$, $s_3^{-1}([a, \infty)) = (-\infty, 1/a) \cup (0, \infty)$.

Hence s_3 is Borel since the set $(0, 1/a)$, $(0, \infty)$, $-\infty, 1/a)$ and $(0, \infty)$ are Borel on the subspace $\mathbb{R} \setminus \{0\}$.

Now $s_4^{-1}([a, \infty)) = (-\infty, -a] \cup [a, \infty)$ for any $a \in \mathbb{R}$, and so s_4 is measurable. □

Corollary 1. *Let $f : X \to \mathbb{R}$ be measurable. The following functions on X are measurable:*

- $f_1 = f^2$
- $f_2 = \alpha f$ *for any* $\alpha \in \mathbb{R}$.
- $f_3 = 1/f$, *measurable on the subspace of* X *given by* $X \setminus \{x \,:\, f(x) = 0\}$.
- $f_4 = |f|$.

Proof. We see that f_1, f_2 and f_4 are measurable since $f_1 = s_1 \circ f$, $f_2 = s_2 \circ f$ and $f_4 = s_4 \circ f$ using s_1, s_2 and s_4 from lemma 2.

For f_3, note that $X \setminus \{x \,:\, f(x) = 0\} = f^{-1}(\mathbb{R} \setminus \{0\})$, and by definition subsets of $X \setminus \{x \,:\, f(x) = 0\}$ are measurable if they are the intersection with some element of $\mathcal{A}$.

See that since $f_3 = s_3 \circ f$, it is measurable. □

Lemma 4. *If $f, g : X \to \mathbb{R}$ are measurable, then so is fg.*

Proof. We may write $fg = \frac{1}{2}((f + g)^2 - f^2 - g^2)$. Hence fg may be written as a composition of measurable functions and is hence measurable. □

Corollary 2. *If f and g are measurable functions on X, then f/g is measurable on $X \setminus \{x \,:\, g(x) = 0\}$.*

Proof. Since f is measurable on X, it is measurable on $X \setminus \{x \,:\, g(x) = 0\}$. Hence since $1/g$ is measurable on this set, their product is measurable. □

Lemma 5. *If $f, g : X \to \mathbb{R}$ are measurable, then so is* $\max\{f, g\}$ *and* $\min\{f, g\}$.

Proof. We may write $\max\{f, g\} = \frac{1}{2}(f + g + |f - g|)$ and $\min\{f, g\} = \frac{1}{2}(f + g - |f - g|)$. Hence these are compositions of measurable functions and so measurable. □

Lemma 6. *If $f, g : X \to \mathbb{R}$ are measurable, then the sets $\{x \,:\, f(x) = g(x)\}$, $\{x \,:\, f(x) > g(x)\}$ and $\{x \,:\, f(x) \geq g(x)\}$ are measurable.*

Proof. We may write these sets respectively as $(f - g)^{-1}(\{0\})$, $(f - g)^{-1}((0, \infty))$ and $(f - g)^{-1}([0, \infty))$. Hence since $f - g$ is measurable, the result follows. □

Lemma 7. *If $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions on X, then so is* $\inf_n f_n$.

Proof. Let $f = \inf_n f_n$ and $a \in \mathbb{R}$. Let $x \in f^{-1}([a,\infty))$. Then for all n, $f_n(x) \geq a$, so $x \in f_n^{-1}([a,\infty))$. Hence,

$$f^{-1}([a,\infty)) \subseteq \bigcap_{n=1}^{\infty} f_n^{-1}([a,\infty))$$

Similarly, if $f_n(x) \geq a$ for all n, then we must have $f(x) \geq a$. Hence,

$$f^{-1}((-\infty,a]) = \bigcap_{n=1}^{\infty} f_n^{-1}([a,\infty)).$$

Since each f_n is measurable, the right hand side is measurable. Hence f is measurable. □

Corollary 3. *If $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable real valued functions on X, then the following functions are measurable:*

- $\sup_n f_n$
- $\limsup_n f_n$
- $\liminf_n f_n$

Proof. We may write $\sup_n f_n = -\inf_n -f_n$, hence it is measurable.

$\limsup_n = \inf_n \sup_{k\geq n} f_k$ and $\liminf_n = \sup_n \inf_{k\geq n} f_n$. Hence these functions are measurable. □

Question 2