UNSW School of Mathematics and Statistics

MATH5825 Measure, Integration and Probability

Semester 2/2014

Homework Week 4

1. Prove the statements from Proposition 2.6.

2. Let f be a non-negative, bounded and measurable function defined on a measure space $(X, \mathcal{A}, \mu)$, and define the function F on the interval $[0, \infty)$ by
$$F(t) = \mu(\{x : f(x) > t\}).$$
   (a) Show that F is a decreasing function and that $F(t) = 0$ for t outside some interval $[0, N]$. Under what condition on f is the limit $\lim_{t\downarrow 0} F(t)$ finite? (Hint: look up "support of a function") Suppose now that $\lim_{t\downarrow 0} F(t)$ is finite. Show that the indefinite Riemann-Integral
$$\int_0^\infty F(t)dt$$
   is well defined; that is, show that the limit $\lim_n \int_0^n F(t)dt$ exists. (You may assume that bounded monotone functions defined on bounded intervals are Riemann integrable.)

   (b) Consider the partition $\mathcal{P}_n = \{0, 1/2^n, 2/2^n, \dots, N2^n/2^n\}$ and let L_n be the corresponding lower Riemann sum of F. Show that L_n is the Lebesgue integral of a measurable simple function s_n which is defined on X. (Drawing a picture might help.) Write down the canonical representation of this simple function.

   (c) Show that $\lim_n s_n(x) = f(x)$ for every $x \in X$ where $s_n(x) \leq s_{n+1}(x)$ for all $n \in \mathbb{N}$. Apply Beppo Levi's theorem to show that
$$\int f\, d\mu = \int_0^\infty F(t)\, dt$$

3. Let f be a non-negative measurable function on a measure space $(X, \mathcal{A}, \mu)$ which is integrable, that is
$$\int f\, d\mu < \infty$$
   Show that
$$\lim_{n\to\infty} n\, \mu(\{x : f(x) > n\}) = 0.$$
   (Hint: Define the set $A_n := f^{-1}([0, n])$ and consider the approximation $\chi_{A_n} f + n\chi_{A_n^\complement}$ of f.)