UNSW AUSTRALIA

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Assignment 1

Measure Theory

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Question 1

Let $d \geq 1$ be an integer, and $S = \prod_{k=1}^{d}(a_k, b_k]$.

Theorem 1. *If $f : S \to \mathbb{R}$ is Riemann integrable, then f is Lebesgue integrable and $\int_S f \, d\lambda = \int_S f(x) \, dx$.*

Proof. Choose a sequence of partitions $\mathcal{P}_n = \{C_{nk}\}$ of S such that the maximum diameter of C_{nk} vanishes as n goes to infinity, and $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$. Define the simple functions,

$$\ell_n = \sum_k \inf_{x \in C_{nk}} f(x) \chi_{C_{nk}}$$
$$u_n = \sum_k \sup_{x \in C_{nk}} f(x) \chi_{C_{nk}}.$$

By assumption these suprema and infima exist since f is Riemann integrable. Then since the partition consists of disjoint sets, we may write

$$\int \ell_n \, d\lambda = \sum_k \inf_{x \in C_{nk}} f(x) \lambda(C_{nk}) = L(f; \mathcal{P}_n)$$
$$\int u_n \, d\lambda = \sum_k \sup_{x \in C_{nk}} f(x) \lambda(C_{nk}) = U(f; \mathcal{P}_n).$$

By the assumption that f is Riemann integrable, these integrals are finite.

Since $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$, we must have that $\ell_{n+1} \geq \ell_n$ and $u_{n+1} \leq u_n$. Hence the sequences $\{\ell_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ are monotonically increasing and decreasing respectively.

Since for each $x \in S$, we have $\ell_n(x) \leq u_k(x) < \infty$ for all k, the supremum $\sup_n \ell_n$ is a well defined function, and is measurable, and we have the bound $\sup_n \ell_n \leq u_k$ for all k. Similarly, $\inf_n u_n$ is well defined and measurable, and we have $\sup_n \ell_n \leq \inf_n u_n$.

For all n, the we have the bound $|\ell_n| \leq \max\{|\ell_1|, |u_1|\} \in L^1$ and $|u_n| \leq \max\{|u_1|, |\ell_1|\} \in L^1$, so by the dominated convergence theorem,

$$\lim_{n \to \infty} \int_S \ell_n \, d\lambda = \int_S \sup_n \ell_n \, d\lambda \leq \int_S \inf_n u_n \, d\lambda = \lim_{n \to \infty} \int_S u_n \, d\lambda.$$

However the left and right hand sides must be equal since they are the limit of the lower and upper Riemann sums respectively. Hence,

$$\int_S |\inf_n u_n - \sup_n \ell_n| \, d\lambda = 0.$$

Thus, $\inf_n u_n = \sup_n \ell_n$ λ-almost everywhere.

Note that for any x, $\inf_n u_n(x) \geq f(x) \geq \sup_n \ell_n(x)$.

Thus $f(x) = \inf_n u_n(x) = \sup_n \ell_n(x)$ for λ-almost all x.

Hence, f agrees almost everywhere with a Lebesgue integrable function, so we may regard it as Lebesgue integrable, and

$$\int_S f \, d\lambda = \lim_{n\to\infty} \int_S \ell_n \, d\lambda = \int_S f(x) \, dx.$$

□

Theorem 2. *f is continuous λ-almost everywhere on S.*

Proof. Choose a specific sequence of partitions of S, $\mathcal{P}_n = \{C_{nk}\}_k$ such that each C_{nk} is a box. Let E denote the set of points $x \in S$ such that $\lim_n u_n(x) = f(x) = \lim_n \ell_n(x)$ and also such that x does not lie on the boundary of C_{nk} for any n or k.

We know that $\lambda(S \setminus E) = 0$, and E is Borel measurable.

Let $\varepsilon > 0$, and $x \in E$. Choose N large enough such that for $n > N$, $u_n(x) - \ell_n(x) < \varepsilon$. Since x lies in the interior of some C_{Nk}, choose δ small enough such that $\|y - x\| < \delta$ implies that $y \in C_{Nk}$.

Since u_N and ℓ_N are constant on C_{Nk}, we have $\ell_N(x) \leq f(y) \leq u_N(x)$. Hence $|f(x) - f(y)| < \varepsilon$. □

Question 2

Let X be a set, and $(Y, \mathcal{B})$ is a measurable space. $f : X \to Y$ is a function.

(a)

Lemma 1. *The collection of subsets of X, $\mathcal{A} = \{f^{-1}(B) \ : \ B \in \mathcal{B}\}$ is a σ-algebra and the smallest σ-algebra on X such that f is measurable.*

Proof. Clearly $X = f^{-1}(Y)$ and $\emptyset = f^{-1}(\emptyset)$ are in $\mathcal{A}$, and if $A = f^{-1}(B) \in \mathcal{A}$ then $A^c = f^{-1}(B^c) \in \mathcal{A}$.

Now if $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$, with $A_n = f^{-1}(B_n)$, then $\bigcup_n A_n = f^{-1}(\bigcup_n B_n) \in \mathcal{A}$.

Hence $\mathcal{A}$ is a σ-algebra.

If $\mathcal{A}'$ is any other σ-algebra on X such that f is $\mathcal{A}'/\mathcal{B}$ measurable, then by definition for any $B \in \mathcal{B}$, $f^{-1}(B) \in \mathcal{A}'$, so we must have $\mathcal{A} \subseteq \mathcal{A}'$.

□

(ii)

Now let $(Z, \mathcal{C})$ be a measurable space, and $h : (X, \mathcal{A}) \to (Z, \mathcal{C})$ is measurable and takes only countably many values $\{c_n\}_{n=1}^{\infty} \subseteq \mathcal{C}$.

Lemma 2. *There exists a measurable function $g : (Y, \mathcal{B}) \to (Z, \mathcal{C})$ such that $h = g \circ f$.*

Proof. Since for each n, $\{c_n\} \in \mathcal{C}$, by the measurability of h we have $h^{-1}(\{c_n\}) \in \mathcal{A}$.

Hence there exist $B_n \in \mathcal{B}$ such that $h^{-1}(\{c_n\}) = f^{-1}(B_n)$.

Define $g : Y \to Z$ by $g(x) = c_n$ if $x \in B_n$.

This is unambiguous, since if $B_n \cap B_m \neq \emptyset$, then $c_n = c_m$,

and $h^{-1}(\{c_n\}_{n=1}^{\infty}) = X = f^{-1}(\bigcup_n B_n)$, so $g(x)$ is uniquely defined for any $x \in Y$.

By construction g is measurable, since if $C \in \mathcal{C}$ contains the points $\{c_{n_k}\}_{k=1}^{\infty}$, then $g^{-1}(C) = \bigcup_{n_k} B_{n_k} \in \mathcal{B}$.

If $x \in X$, then $f(x) \in B_n$ for some n since $X = f^{-1}(\bigcup_n B_n)$. Then $g(f(x)) = c_n$, but since $f(x) \in B_n$, $x \in f^{-1}(B_n) = h^{-1}(\{c_n\})$.

So $h(x) = c_n = g(f(x))$.

Hence we have $h = g \circ f$, as required. □

Question 3

In this question, $(X, \mathcal{A}, \mu)$ is a measure space.

Suppose $\{A_n\}n \geq 0$ is a sequence of sets in $\mathcal{A}$ then the following holds:

Lemma 3.

$$\inf_n \chi_{A_n} = \chi_{\bigcap_n A_n}$$

and

$$\sup_n \chi_{A_n} = \chi_{\bigcup_n A_n}$$

Proof. Let $x \in X$. If $\inf_n \chi_{A_n}(x) = 1$ means that there is no k such that $x \notin A_n$. Hence $x \in \bigcap_n A_n$.

Similarly, if $x \in \bigcap_n A_n$, then $\inf_n \chi_{A_n}(x) = 1$ since for any n, $\chi_{A_n}(x) = 1$.

Now we write, using $\chi_{B^c} = 1 - \chi_B$,

$$\begin{aligned} \inf_n \chi_{A_n^c} &= \chi_{\bigcup_n A_n^c} \\ 1 - \inf_n \chi_{A_n^c} &= \chi_{\bigcap_n A_n} \\ \sup_n 1 - \chi_{A_n^c} &= \chi_{\bigcap_n A_n} \\ \sup_n \chi_{A_n} &= \chi_{\bigcap_n A_n} \end{aligned}$$

□

Now we define

$$\liminf_n A_n := \bigcup_n \bigcap_{k \geq n} A_k.$$

Theorem 3. *The following are equivalent,*

$$x \in \liminf_n A_n$$

$$\liminf_n \chi_{A_n}(x) = 1$$

and $x \in A_n$ for all but finitely many n.

Proof. Using lemma 3, we write

$$\begin{aligned}\liminf_n \chi_{A_n}(x) &= \sup_n \inf_{k\geq n} \chi_{A_k}\\ &= \sup_n \chi_{\bigcap_{k\geq n} A_k}\\ &= \chi_{\liminf_n A_n}(x).\end{aligned}$$

Hence $x \in \liminf_n A_n$ if and only if $\liminf_n \chi_{A_n} = 1$.

If $\liminf_n \chi_{A_n}(x) = 1$. then 1 is the only limit point of the sequence $\chi_{A_n}(x)$, hence since χ_{A_n} takes only the values 0 and 1, it must take the value 0 only finitely many times. Hence $x \in A_n$ for all but finitely many n.

Conversely, if $x \in A_n$ for all but finitely many n, then the numerical sequence $\chi_{A_n}(x)$ takes the value 0 only finitely many times. Since it must have a limit point, we conclude $\liminf_n \chi_{A_n}(x) = 1$. □

Now we define

$$\limsup_n A_n = \bigcap_n \bigcup_{k\geq n} A_k$$

Theorem 4. *The following are equivalent:*

$$x \in \limsup_n A_n$$

$$\limsup_n \chi_{A_n}(x) = 1$$

and $x \notin A_n$ for infinitely many n.

Proof. The equivalence of the first two statements is identical to theorem 3.

For the third statement, if $\limsup_n \chi_{A_n}(x) = 1$ then the numerical sequence $\chi_{A_n}(x)$ has 1 as a limit point, so x must be in A_n infinitely often.

Conversely, if x is in A_n infinitely often then 1 is a limit point of the sequence $\chi_{A_n}(x)$. Hence it must be the largest limit point so $\limsup_n \chi_{A_n}(x) = 1$. □

Hence it is clear that $\liminf_n A_n \subseteq \limsup_n A_n$ since $\chi_{\liminf_n A_n}(x) = \liminf_n \chi_{A_n}(x) \leq \limsup_n \chi_{A_n}(x) = \chi_{\limsup_n A_n}$.

Question 4

For this question, $(X, \mathcal{A}, \mu)$ is a measure space with $\mu(X) < \infty$ and $C \subseteq X$.

(a)

Lemma 4. $\mathcal{A}_C = \{A \cap C \,:\, A \in \mathcal{A}\}$ *is a σ-algebra on C.*

Proof. We have $C = C \cap X \in \mathcal{A}_C$, and if $A \cap C \in \mathcal{A}$, then $A^c \cap C = C \setminus A \in \mathcal{A}_C$.

Now if $\{C_n\}_{n=1}^{\infty}$ is a countable subset of $\mathcal{A}_C$, with $C_n = C \cap A_n$ for $A_n \in \mathcal{C}$, then $\bigcup_n C_n = \bigcup_n C \cap A_n = C \cap \bigcup_n A_n \in \mathcal{A}_C$.

Hence $\mathcal{A}_C$ is a σ-algebra. □

(b)

Now define the outer and inner measures $\mu_*, \mu^* : 2^X \to [0, \infty]$ by the usual formualae,

$$\mu_*(B) = \sup\{\mu(A) \,:\, A \subseteq B, A \in \mathcal{A}\}$$
$$\mu^*(B) = \inf\{\mu(A) \,:\, A \supseteq B, A \in \mathcal{A}\}.$$

Lemma 5. *For each $B \in 2^X$ there exist $A_0 \subseteq B \subseteq A_1$ with $A_0, A_1 \in \mathcal{A}$ and $\mu(A_0) = \mu_*(B)$ and $\mu(A_1) = \mu^*(B)$.*

Proof. We may choose $C_n \in \mathcal{A}$ such that $\mu^*(B) \leq \mu(C_n) \leq \mu^*(B) + \frac{1}{n}$

Then $B \subseteq \bigcap_n C_n \in \mathcal{A}$, and for every n,

$$\mu(\bigcap_n C_n) \leq \mu^*(B) + \frac{1}{n}$$

Hence, if $A_1 = \bigcap_n C_n$, we have $\mu(A_1) = \mu^*(B)$.

Similarly, choose $C_n' \in \mathcal{A}$ $C_n' \subseteq B$ and $\mu_*(B) \geq \mu(C_n') \geq \mu * (B) - \frac{1}{n}$, and define $A_0 = \bigcup_n C_n'$. □

(c)

Let $C_1 \in \mathcal{A}$ with $C \subseteq C_1$ be such that $\mu(C_1) = \mu^*(C)$.

Lemma 6. $\mu^*(C_1 \setminus C) = 0$.

Proof. There exists some $B \in \mathcal{A}$ with $B \subseteq C_1 \setminus C$ such that $\mu(B) = \mu^*(C_1 \setminus C)$. If $\mu(B) > 0$, then $\mu(C_1) = \mu(B) + \mu(C_1 \setminus B) > \mu(C_1 \setminus B)$.

But $C \subseteq C_1 \setminus B$, so $\mu^*(C) \leq \mu(C_1 \setminus B)$. Hence $\mu(C_1) > \mu^*(C)$, but this is a contradiction. □

Lemma 7. *If $A \in \mathcal{A}$, then $\mu(A \cap C_1) = \mu^*(A \cap C)$.*

Proof. Since $C \subseteq C_1$, we have $\mu^*(A \cap C) \leq \mu(A \cap C_1)$.

Now by the subadditivity of μ^*, $\mu(C_1 \cap A) \leq \mu^*(C \cap A) + \mu^*((C_1 \setminus C) \cap A)$. But $(C_1 \setminus C) \cap A \subseteq C_1 \setminus C$, so $\mu^*((C_1 \setminus C) \cap A) = 0$.

Hence $\mu(A \cap C_1) = \mu^*(A \cap C)$. □

Theorem 5. *Suppose $A_1, A_2 \in \mathcal{A}$ with $A_1 \cap C = A_2 \cap C$. Then $\mu(A_1 \cap C_1) = \mu(A_2 \cap C_1)$.*

Proof. Since $A_1 \cap C = A_2 \cap C$, we have $\mu(A_1 \cap C_1) = \mu^*(A_1 \cap C) = \mu^*(A_2 \cap C) = \mu(A_2 \cap C_1)$.

□

(c)

Now define $\mu_C : \mathcal{A}_C \to [0, \infty]$ by $\mu_C(A \cap C) = \mu(A \cap C_1)$,

Theorem 6. $\mu_C(B) = \mu^*(B)$ *for every* $B \in \mathcal{A}_C$.

Proof. Let $B \in \mathcal{A}_C$, then $B = C \cap A$ for some $A \in \mathcal{A}$.

Then $\mu_C(B) = \mu(A \cap C_1)$. This does not depend on the choice of A by lemma 5.

Then $\mu_C(B) = \mu^*(A \cap C) = \mu^*(B)$.

□

(d)

Lemma 8. *Suppose $(X, \mathcal{A}, \mu)$ is now any measure space, and $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ is a sequence with $\mu(A_n \cap A_m) = 0$ for $n \neq m$. Then*

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

Proof. We disjointify the sequence. Recursively define the sequence $\{B_n\}_{n=1}^{\infty}$ by

$$\begin{aligned} B_1 &= A_1 \\ B_n &= A_n \setminus \left(\bigcup_{k=1}^{n-1} A_k\right), n > 1. \end{aligned}$$

Then we have

$$\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n.$$

And if $n > m$, $B_n \cap B_m = \emptyset$.

Hence,

$$\mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n).$$

Note that,

$$\begin{aligned} \mu(A_n) &= \mu(B_n) + \mu(A_n \cap \bigcup_{k=1}^{n-1}) \\ &= \mu(B_n) + \mu(\bigcup_{k=1}^{n-1} A_n \cap A_k). \end{aligned}$$

But

$$\mu(\bigcup_{k=1}^{n-1} A_n \cap A_k) \leq \sum_{k=1}^{n-1} \mu(A_n \cap A_k) = 0.$$

Hence $\mu(A_n) = \mu(B_n)$ so the result follows.

□

Corollary 1. *μ_C is a measure on A_C.*

Proof. We have proved that μ_C is the restriction of μ^* to $\mathcal{A}_C$. It is therefore sufficient to prove that μ^* is countably additive on $\mathcal{A}_C$.

Let $\{B_n\}_{n=1}^{\infty}$ be a sequence of pairwise disjoint sets in $\mathcal{A}_C$. Let $B = \bigcup_n B_n$. Let $A_n \in \mathcal{A}$ be such that $B_n = C \cap A_n$.

Then,

$$\begin{aligned} \mu_C(B) &= \mu(C_1 \cap \bigcup_{n=1}^{\infty} A_n) \\ &= \mu(\bigcup_{n=1}^{\infty} C_1 \cap A_n). \end{aligned}$$

The sets $\{C_1 \cap A_n\}$ are not necessarily pairwise disjoint, however we may say that

$$\mu(C_1 \cap A_n \cap C_1 \cap A_m) = 0$$

unless $n = m$. So by lemma 8,

$$\mu_C(B) = \sum_{n=1}^{\infty} \mu_C(B_n).$$

Hence μ_C is a measure.

□