UNSW School of Mathematics and Statistics

MATH5825 Measure, Integration and Probability

Semester 2/2014

Assignment 1

1. Riemann and Lebesgue Integrals in $\mathbb{R}^d$. You will show that

   - if a function is Riemann integrable, then it is also Lebesgue integrable, and the integrals are the same
   - a Riemann integrable function is continuous a.e. (with respect to Lebesgue measure).

   Let $S = (a_1, b_1] \times \ldots \times (a_d, b_d]$ be a block in $\mathbb{R}^d$, and let $f : S \to \mathbb{R}$ be bounded and Riemann integrable.

   Here is a quick reminder of when a function is Riemann integrable. A finite collection of subsets of S, $\mathcal{P}_n = \{C_1, \ldots, C_{N_n}\}$, is called a partitioning of S if $S = \bigcup_{k=1}^{N_n} C_n$ and the union is disjoint. $\mathcal{P}_{n+1}$ is a refinement of $\mathcal{P}_n$ if each set in $\mathcal{P}_n$ is equal to a disjoint union of sets in $\mathcal{P}_{n+1}$. The diameter of a set $C \subset S$ is $D(C) = \sup\{y - x : x, y \in C\}$. The lower and upper Riemann sums are

   $$L(f; \mathcal{P}_n) = \sum_{k=1}^{N_n} \alpha_k \lambda(C_k), \quad \alpha_k := \inf\{f(x) : x \in C_k\}$$
   $$U(f; \mathcal{P}_n) = \sum_{k=1}^{N_n} \beta_k \lambda(C_k), \quad \beta_k := \sup\{f(x) : x \in C_k\}$$

   (where $C_k \in \mathcal{P}_n$). If for any sequence $\{\mathcal{P}_n\}_{n \in \mathbb{N}}$ which satisfies that

   - $\mathcal{P}_{n+1}$ is a refinement of $\mathcal{P}_n$ and
   - $\lim_{n\to\infty} \max\{D(C) : C \in \mathcal{P}_n\} = 0$

   one has that $\lim_n [U(f; \mathcal{P}_n) - L(f; \mathcal{P}_n)] = 0$, then f is called Riemann integrable.

   (a) (5 marks) Show that f is also Lebesgue integrable and that $\int f d\lambda = \int_S f(x)dx$ (where the right-hand side denotes the Riemann integral and λ denotes Lebesgue measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$).
   Hints: Write down two sequences of simple functions, ℓ_n and $u_n : S \to \mathbb{R}$, $n \in \mathbb{N}$, such that

   $$\int \ell_n \, d\lambda = L(\mathcal{P}_n), \quad \int u_n \, d\lambda = U(\mathcal{P}_n).$$

   Then use the Dominated Convergence Theorem to show that $\lim_n \ell_n = \lim_n u_n = f$ holds λ-a.e. Use Dominated Convergence again to show the equality of the integrals.

   (b) (5 marks) Show that f is continuous λ-a.e. on S.
   Hints: From part (a) you know that $\lim_n \ell_n = f = \lim_n u_n$, λ-a.e. Exclude all points for which this limit does not hold and which lie on the boundary of any block. Then show that for any remaining x and $\varepsilon > 0$, there is an open neighbourhood of x in which f doesn't vary more than ϵ.

2. Let X be a non-empty set, and let $(Y, \mathcal{B})$ be a measurable space.

   (a) (4 marks) Explain why $\mathcal{A} := \{f^{-1}(B) : B \in \mathcal{B}\}$ is a σ-algebra, and why it is the smallest σ-algebra for which f is measurable from $(X, \mathcal{A})$ to $(Y, \mathcal{B})$.

   (b) (4 marks) Let $(Z, \mathcal{C})$ be another measure space, and let h be a measurable mapping from $(X, \mathcal{A})$ to $(Z, \mathcal{C})$ which takes countably many values $a_1, a_2, \ldots \in Z$. Further assume that $\{a_n\} \in \mathcal{C}$ for every n. Show that there exists a measurable mapping g from $(Y, \mathcal{B})$ to $(Z, \mathcal{C})$ such that $h = g \circ f$ (composition of mappings).

3. Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $\{A_n\}_{n \in \mathbb{N}}$ be a sequence in $\mathcal{A}$. The limit inferior of this sequence is

$$\liminf_n A_n := \bigcup_n \bigcap_{k \geq n} A_n.$$

   (a) (6 marks) Show that the following statements are equivalent:

      (i) $x \in \liminf_n A_n$

      (ii) $\liminf_n \chi_{A_n}(x) = 1$

      (iii) $x \in A_n$ for all but finitely many n.

   (b) (3 marks) Give similar three equivalent statements for the limit superior of A_n,

$$\limsup_n A_n := \bigcap_n \bigcup_{k \geq n} A_n$$

   and explain why $\liminf_n A_n \subset \limsup_n A_n$.

4. Let $(X, \mathcal{A}, \mu)$ be a finite measure space (that is, $\mu(X) < \infty$). Let C be a subset of X, not necessarily such that $C \in \mathcal{A}$. The aim is to construct a measure space $(C, \mathcal{A}_C, \mu_C)$ restricted to C, called the "trace" of $(X, \mathcal{A}, \mu)$ on C.

   (a) (2 marks) Show that $\mathcal{A}_C := \{A \cap C : A \in \mathcal{A}\}$ is a σ-algebra on C.

   (b) (3 marks) Define the outer measure

$$\mu^*(B) := \inf\{\mu(A) : A \supset B\}$$

   and the inner measure

$$\mu_*(B) := \sup\{\mu(A) : A \subset B\}$$

   for all subsets $B \in 2^X$. Show that for each $B \in 2^X$ there are sets $A_0, A_1 \in \mathcal{A}$ which satisfy $A_0 \subset B \subset A_1$, $\mu_*(B) = \mu(A_0)$ and $\mu^*(B) = \mu(A_1)$.

   (c) (4 marks) Now let $C_1 \in \mathcal{A}$ be such that $C_1 \supset C$ and $\mu^*(C) = \mu(C_1)$. Show that for $A_1, A_2 \in \mathcal{A}$ which satisfy $A_1 \cap C = A_2 \cap C$, we have $\mu(A_1 \cap C_1) = \mu(A_2 \cap C_1)$. (Hints: Write down what it means that $A_1 \cap C_1$ and $A_2 \cap C_1$ are $\mathcal{M}(\mu^*)$ measurable. Then show $\mu^*(C_1 \setminus C) = 0$ using the completeness property of μ^*.)
   Hence define $\mu_C : \mathcal{A}_C \to [0, \infty]$ by

$$\mu_C(A \cap C) := \mu(A \cap C_1).$$

   (d) (2 marks) Show that $\mu_C(B) = \mu^*(B)$ for every $B \in \mathcal{A}_C$. (This means that μ_C does not depend on the choice of C_1.)

   (e) (2 marks) Show that μ_C is a measure on $\mathcal{A}_C$.