University of New South Wales

School of Mathematics and Statistics

Assignment 1

Measure Theory

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Question 1

Question 3

In this question, $(X, \mathcal{A}, \mu)$ is a measure space.

Suppose $\{A_n\}n \geq 0$ is a sequence of sets in $\mathcal{A}$ then the following holds:

Lemma 1.

$$\inf_n \chi_{A_n} = \chi_{\bigcap_n A_n}$$

and

$$\sup_n \chi_{A_n} = \chi_{\bigcup_n A_n}$$

Proof. Let $x \in X$. If $\inf_n \chi_{A_n}(x) = 1$ means that there is no k such that $x \notin A_n$. Hence $x \in \bigcap_n A_n$.

Similarly, if $x \in \bigcap_n A_n$, then $\inf_n \chi_{A_n}(x) = 1$ since for any n, $\chi_{A_n}(x) = 1$.

Now we write, using $\chi_{B^c} = 1 - \chi_B$,

$$\begin{aligned} \inf_n \chi_{A_n^c} &= \chi_{\bigcup_n A_n^c} \\ 1 - \inf_n \chi_{A_n^c} &= \chi_{\bigcap_n A_n} \\ \sup_n 1 - \chi_{A_n^c} &= \chi_{\bigcap_n A_n} \\ \sup_n \chi_{A_n} &= \chi_{\bigcap_n A_n} \end{aligned}$$

□

Now we define

$$\liminf_n A_n := \bigcup_n \bigcap_{k \geq n} A_k.$$

Theorem 1. *The following are equivalent,*

$$x \in \liminf_n A_n$$

$$\liminf_n \chi_{A_n}(x) = 1$$

and $x \in A_n$ for all but finitely many n.

Proof. Using lemma **??**, we write

$$\begin{aligned}
\liminf_n \chi_{A_n}(x) &= \sup_n \inf_{k\geq n} \chi_{A_k} \\
&= \sup_n \chi_{\bigcap_{k\geq n}} A_k \\
&= \chi_{\liminf_n A_n}(x).
\end{aligned}$$

Hence $x \in \liminf_n A_n$ if and only if $\liminf_n \chi_{A_n} = 1$.

If $\liminf_n \chi_{A_n}(x) = 1$. then 1 is the only limit point of the sequence $\chi_{A_n}(x)$, hence since χ_{A_n} takes only the values 0 and 1, it must take the value 0 only finitely many times. Hence $x \in A_n$ for all but finitely many n.

Conversely, if $x \in A_n$ for all but finitely many n, then the numerical sequence $\chi_{A_n}(x)$ takes the value 0 only finitely many times. Since it must have a limit point, we conclude $\liminf_n \chi_{A_n}(x) = 1$. □

Now we define

$$\limsup_n A_n = \bigcap_n \bigcup_{k\geq n} A_k$$

Theorem 2. *The following are equivalent:*

$$x \in \limsup_n A_n$$

$$\limsup_n \chi_{A_n}(x) = 1$$

and $x \notin A_n$ for infinitely many n.

Proof. The equivalence of the first two statements is identical to theorem **??**.

For the third statement, if $\limsup_n \chi_{A_n}(x) = 1$ then the numerical sequence $\chi_{A_n}(x)$ has 1 as a limit point, so x must be in A_n infinitely often.

Conversely, if x is in A_n infinitely often then 1 is a limit point of the sequence $\chi_{A_n}(x)$. Hence it must be the largest limit point so $\limsup_n \chi_{A_n}(x) = 1$.

□