University of New South Wales

School of Mathematics and Statistics

Homework 2

Measure Theory

Author:
Edward McDonald

Student Number:
3375335

Question 1

(a)

We are given,
$$\mathcal{C} = \{\emptyset, X, \{1\}, \{2,3\}, \{1,2,3\}, \{4,5\}\}.$$
See that $\mathcal{C}$ is closed under intersection, so we only need to consider unions of elements of $\mathcal{C}$ Hence,
$$\sigma(\mathcal{C}) = \{\emptyset, X, \{1\}, \{2,3\}, \{4,5\}, \{1,2,3\}, \{1,4,5\}, \{2,3,4,5\}\}.$$

(b)

Then we define $\gamma : \mathcal{C} \to [0,\infty]$ as follows,
$$\begin{aligned} \gamma(\emptyset) &= 0 \\ \gamma(\{1\}) &= 1 \\ \gamma(\{2,3\}) &= 1 \\ \gamma(\{1,2,3\}) &= 2 \\ \gamma(\{4,5\}) &= 1 \\ \gamma(X) &= 3. \end{aligned}$$
Let μ^* be the outer measure on 2^X defined by γ.

Lemma 1. *μ^* restricted to $\sigma(\mathcal{C})$ is a measure.*

Proof. Since μ^* is a measure on $\mathcal{M}(\mu^*)$, it is sufficient to show that $\mathcal{C} \subseteq \mathcal{M}(\mu^*)$. Then since $\mathcal{M}(\mu^*)$ is a σ-algebra, $\sigma(\mathcal{C}) \subseteq \mathcal{M}(\mu^*)$.

Clearly $\emptyset$ and X are in $\mathcal{M}(\mu^*)$, so we need only show that $\{1\}, \{2,3\}, \{1,2,3\}$ and $\{4,5\}$ are in $\mathcal{M}(\mu^*)$. We do not need to show that $\{1,2,3\} \in \mathcal{M}(\mu^*)$ since $\{1,2,3\} = \{1\} \cup \{2,3\}$.

Let $A \subset X$. Then
$$\mu^*(A) = \mu^*(A \cap \{1\}) + \mu^*(A \setminus \{1\})$$
since if $1 \in A$, $\mu^*(A \cap \{1\}) = 1$ and $\mu^*(A \setminus \{1\})$,

□

Question 2

For this question, μ^* is an outer measure on the set X defined by the function $\gamma : \mathcal{C} \subseteq 2^X \to [0, \infty]$.

Lemma 2. $\mu^*(\emptyset) = 0$.

Proof. By definition,

$$\mu * (\emptyset) = \inf\{\sum_{n=0}^{\infty} \gamma(A_n) \ : \ \emptyset \subseteq \bigcup_{n=0}^{\infty} A_n, A_n \in \mathcal{C} \ \}.$$

However this is simply

$$\mu^*(\emptyset) = \inf\{\gamma(A) \ A \in \mathcal{C}\}$$

since $\emptyset$ is a subset of any set. However $\gamma(\emptyset) = 0$, so $\mu^*(\emptyset) = 0$. $\square$

Lemma 3. *If $A \subseteq B \subseteq X$, then $\mu^*(A) \leq \mu^*(B)$.*

Proof. Since $A \subseteq B$, we have the set inclusion

$$\{\{E_i\}_{i=0}^{\infty} \subseteq \mathcal{C} \ : \ A \subseteq \bigcup_{i=0}^{\infty} E_i\} \subseteq \{\{E_i\}_{i=0}^{\infty} \subseteq \mathcal{C} \ : \ B \subseteq \bigcup_{i=0}^{\infty} E_i\}$$

Hence,

$$\inf\{\sum_{i=0}^{\infty} \gamma(E_i) \subseteq \mathcal{C} \ : \ A \subseteq \bigcup_{i=0}^{\infty} E_i\} \leq \inf\{\sum_{i=0}^{\infty} \gamma(E_i) \subseteq \mathcal{C} \ : \ B \subseteq \bigcup_{i=0}^{\infty} E_i\}$$

So by definition,

$$\mu^*(A) \leq \mu^*(B).$$

$\square$

Question 3

The euclidean topology τ on $\mathbb{R}$ is defined as the toplogy generated by open intervals (a, b),$a < b \in \mathbb{R}$

Lemma 4. *Every open set in τ is a countable union of intervals (a, b),$a, b \in \mathbb{R}$.*

Proof. Let $U \in \tau$, and $x \in U$. Then there is some $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subset U$. By the density of $\mathbb{Q}$ in $\mathbb{R}$, there is some $p, q \in \mathbb{Q}$ such that $q \in (x-\varepsilon, x)$ and $q \in (x, x+\varepsilon)$. Then $x \in (p, q) \subset U$.

Hence U can be expressed as a union of intervals of the form (p, q) for $p, q \in \mathbb{Q}$.

However the set $\{(p, q) \ : \ p, q \in \mathbb{Q}\}$ is countable, so any open set in τ is a countable union of intervals. □

Theorem 1. *The set $\mathcal{T}$ consisting of all countable unions of open intervals (a, b), $a, b \in \mathbb{R}$ is a topology.*

Proof. Since any open set in τ is a countable union of intervals, $\tau \subseteq \mathcal{T}$, and since any countable union of intervals is in τ, $\mathcal{T} \subseteq \tau$. Hence $\mathcal{T} = \tau$. □

We now consider the Borel σ-algebra $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{T})$.

Theorem 2. *$\mathcal{B}(\mathbb{R})$ can be generated by the system*