MATH5735: Modules and Representation Theory (2014,S1) Problem Set 4 [1]

Throughout, k will denote some field, R a ring and n some positive integer. This problem sets concerns lectures 15-20.

1. This question gets you to prove proposition 2 in lecture 16. Let $R = R_1 \times R_2 \times \ldots \times R_n$ and $\pi_j : R \longrightarrow R_j$ be the canonical projection onto the i-th component. Note that π_j is a ring homomorphism, so by change of scalars, any R_j-module is also naturally an R-module. Let $\iota_j : R_j \longrightarrow R$ be the canonical injection into the i-th component.

   (a) Let $e_j = \iota_j(1_{R_j}) \in R$. Show that $\{e_1, \ldots, e_n\}$ is a complete set of orthogonal idempotents in the centre of R.

   (b) Show that $R_j = Re_j$. More precisely
   $$Re_j = 0 \times \ldots \times 0 \times R_j \times 0 \times \ldots \times 0.$$

   (c) Let M be an R-module. Show that M can be gotten by change of scalars from an R_j-module if and only if right multiplication by e_j acts as the identity on M, or equivalently, $M(1 - e_j) = 0$.

   (d) Show that Me_j is an R-submodule of M which comes from an R_j-module structure. Moreover, show Me_j contains all other such R_j-submodules of M.

   (e) Show that $M = Me_1 \oplus \ldots \oplus Me_n$ and that this is the unique way of writing M as an internal direct sum $M = M_1 \oplus \ldots \oplus M_n$ with M_j an R_j-module.

2. This question gets you to prove proposition 3 of lecture16. Let R be a commutative ring and G be a group. Prove that the following defines a one-to-one correspondence between the class of R-linear representations of G and the class of RG-modules. Given a representation $\rho : G \longrightarrow \mathrm{Aut}_R V$, the corresponding RG-module is the abelian group V with scalar multiplication given by
   $$(\sum_{g \in G} r_g g)v = \sum_g r_g \rho(g)v.$$
   for any $v \in V, r_g \in R$.

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3. Let $G = \langle \sigma \rangle$ be a cyclic group of order n. We identify the vector space $\mathbb{C}G = \mathbb{C}^n$ using the basis $\{1, \sigma, \sigma^2, \ldots, \sigma^{n-1}\}$. Then the $\mathbb{C}G$-module $\mathbb{C}G$ corresponds to a group representation of the form $\rho : G \longrightarrow GL_n(\mathbb{C})$. Determine ρ explicitly.

4. Let V be the $\mathbb{R}$-space of $\mathbb{R}$-valued functions on $\mathbb{R}$. Consider the vector space automorphism $\sigma : V \longrightarrow V : f(x) \mapsto f(-x)$.
   (a) Show that σ generates a group G of order 2.
   (b) The inclusion $G \hookrightarrow \mathrm{Aut}_{\mathbb{R}} V$ is a group representation. Identify the fixed module V^G with a well-known set.
   (c) Write out explicitly the Reynolds operator in this case.
   (d) Hence or otherwise, show that every function in V can be written uniquely as the sum of an even and an odd function.

5. Let R be a right semisimple ring. Show that the decomposition into simple modules $R_R = \oplus M_i$ must be finite. Hint: Show that the image of 1_R in each component must be non-zero.

6. Which of the following rings are semisimple? Justify your answers. i) $M_3(\mathbb{R}) \times \mathbb{C}$, ii) the ring of upper triangular matrices in $M_n(\mathbb{Q})$ in problem set 1 , iii) the ring of diagonal matrices in $M_n(\mathbb{Q})$, iv) $\mathbb{F}_7 G$ where G is the dihedral group of order 10 , v) $\mathbb{Z}G$ where G is the symmetric group on 4 symbols.

7. Show for any finite abelian group G, that $\mathbb{C}G \simeq \prod_{i=1}^{|G|} \mathbb{C}$ as $\mathbb{C}$-algebras.

8. For G the cyclic group of order 3, find an explicit isomorphism $\mathbb{C}G \simeq \mathbb{C} \times \mathbb{C} \times \mathbb{C}$.

9. For the cyclic group G of order 4, find the Wedderburn components of $\mathbb{R}G$.

10. Let A be a semisimple ring and S be a simple A-module. Show that the isotypic component of an A-module M corresponding to the simple S, is given by the sum of all submodules N of M which are isomorphic to S.

11. Let G be a finite group and $A = \mathbb{C}G$. Given a left A-module M, show that M^G is the isotypic component of M corresponding to the trivial representation.

12. Let $G = S_4$. Find the abelianisation G_{ab} of G.

13. Let G be one of the two non-abelian groups of order 8 (either the dihedral group or the quaternion group). Show that $\mathbb{C}G \simeq \mathbb{C} \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C})$.

14. As mentioned in lecture 19, show that for a finite group G, isomorphism classes of n-dimensional kG-modules correspond to equivalence classes of representations $\rho : G \longrightarrow GL_n(k)$ where $\rho \sim \rho'$ if there is some $T \in GL_n(k)$ such that $\rho'(g) = T\rho(g)T^{-1}$ for all $g \in G$.

15. Consider the dihedral group D_n or order $2n$ where n is even. Find the Wedderburn components of $\mathbb{C}D_n$ and the irreducible representations of D_n.