School of Mathematics and Statistics University of New South Wales

MATH5735: Modules and Representation Theory (2014,S1) Problem Set 2 [1]

Throughout, k will denote some field, R a ring and n some positive integer. This problem set covers lectures 6-10.

1. Use universal properties to show that the abelian group $\mathrm{Hom}_R(R^n, R^m)$ is naturally isomorphic to the additive group of $m \times n$-matrices with entries in R.

2. Let G be the cyclic group of order 2 and $R = \mathbb{C}G$. Show that $\mathbb{C}G$ is decomposable as a right R-module.

3. Let $R = \mathbb{R}[x]$. Is $M = \mathbb{R}[x]/\langle x^2+x-2\rangle$ decomposable as an R-module? What about $M = \mathbb{R}[x]/\langle x^2 + x + 1\rangle$?

4. Show that $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$ is a cyclic $\mathbb{Z}$-module.

5. Let $R = \mathrm{End}_{\mathbb{C}}\, \mathbb{C}[x]$ and $\partial \in R$ be the differentiation operator. Show that the right R-module generated by ∂ is all of R but the left R-module generated by ∂ is not.

6. Recall that since $\mathbb{Z}[\sqrt{2}]$ is a subring of $\mathbb{Z}[\sqrt[4]{2}]$ we may consider $\mathbb{Z}[\sqrt[4]{2}]$ as a $\mathbb{Z}[\sqrt{2}]$-module. Show that it is free.

7. Let $R = k[x]$ and M be the R-module $k[x]/\langle x\rangle$. Let $\bar{1}$ denote the element $1 + \langle x\rangle \in M$. Recall that the matrix $(\bar{1} \quad -\bar{1})$ defines an R-linear homomorphism $\phi : R^2 \longrightarrow M$. Is $\ker \phi$ cyclic? If so, find a generator for it, otherwise find a minimal generating set.

8. Let $G = \langle \sigma, \tau\rangle$ be the dihedral group of order $2n$ where σ is an order n rotation and τ is a reflection. Let $H = \langle \sigma\rangle$ and $R = kH$. Recall that $M = kG$ is a free kH-module. Write down an explicit isomorphism $\phi : R^2 \longrightarrow M$. Recall that left multiplication by τ on M is (right) kG-linear and hence also kH-linear. This induces an R-linear map $\psi : R^2 \longrightarrow R^2$ via ϕ. Write down the 2×2-matrix representing this homomorphism.

[1] by Daniel Chan

9. Consider the submodule N of $\mathbb{Z}^3$ generated by $(4,4,6)$ and $(8,4,4)$. Use EROs and ECOs to write $\mathbb{Z}^3/N$ as a direct sum of cyclic groups. (Your answer should be of the form $\mathbb{Z}^3/N \simeq \mathbb{Z}^r \oplus \mathbb{Z}/a_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/a_s\mathbb{Z}$ for integers $r, s, a_1, \ldots, a_s$ which you need to determine).

10. Let $R = k[x]$ and M be the R-submodule of R^2 generated by $(x^2 - 1, x-1)$ and $(2-2x, x^3-1)$. Write R^2/M as a direct sum of cyclic modules.

11. How close can you get to a Jordan canonical form theorem over $\mathbb{R}$? What's the new theorem?

12. Prove the rational canonical form theorem (below) in linear algebra which works over an arbitrary field k, even if it is not algebraically closed.

Theorem 0.1 *Let A be an $n \times n$-matrix over k. Then A is similar to a direct sum of blocks, each of which has the form*

$$B = \begin{pmatrix} 0 & 0 & \cdots & 0 & \beta_1 \\ 1 & 0 & & \vdots & \beta_2 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & & 0 & \vdots \\ 0 & \cdots & 0 & 1 & \beta_n \end{pmatrix}$$

for some $\beta_1, \ldots, \beta_n \in k$.