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MATH5735: Modules and Representation Theory (2014,S1) Problem Set 0 [1]

The main purpose of this problem set is to review some basic group theory and ring theory from MATH3711. Hopefully most of this is old stuff.

Generators and relations

We first recall some simple examples of groups and their description via generators and relations. In this course, D_n will denote the dihedral group of order $2n$ which is just the symmetry group of a regular n-gon. Recall that this is generated by a rotation σ through angle $\frac{2\pi}{n}$ and a reflection τ. We also have the following relations in D_n,

$$\sigma^n = 1, \ \tau^2 = 1, \ \tau\sigma = \sigma^{-1}\tau.$$

These are actually *defining* relations in the sense that all other relations between σ and τ can be gotten from these by using the group axioms. (We'll make this more precise later). Hence we use the generators and relations notation to express this fact:

$$D_n = \langle \sigma, \tau | \sigma^n = 1, \tau^2 = 1, \tau\sigma = \sigma^{-1}\tau \rangle.$$

More precisely, we use the notation $\langle s, t | s^n = 1, t^2 = 1, ts = s^{-1}t \rangle$ to denote the group G generated by s, t satisfying the relations $s^n = 1, t^2 = 1, ts = s^{-1}t$ and also the following universal property:

Given any group $\bar{G}$ with elements $\bar{s}, \bar{t} \in \bar{G}$ satisfying relations $\bar{s}^n = 1$, $\bar{t}^2 = 1, \bar{t}\bar{s} = \bar{s}^{-1}\bar{t}$, there exists a unique group homomorphism $\phi : G \longrightarrow \bar{G}$ with $\phi(s) = \bar{s}, \phi(t) = \bar{t}$.

Remark: a) Firstly, the notation naturally generalises to any set of generators and choice of relations.

b) A group satisfying the universal property always exists for any choice of generators and relations, and it is unique. This is a rather tedious theorem to prove.

1. Use the universal property definition to show that D_n is indeed the group $\langle \sigma, \tau | \sigma^n = 1, \tau^2 = 1, \tau\sigma = \sigma^{-1}\tau \rangle$. Hint: Show that the latter has at most $2n$ elements.

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2. Show that we can describe the cyclic group of order n via generators and relations as $\langle g|g^n = 1\rangle$.

3. The quaternion group Q is often defined by generators and relations by
$$Q = \langle i, j|i^4 = 1, \ j^2 = i^2, \ ji = i^3 j\rangle.$$
Show that Q is isomorphic to the subgroup of $GL_2(\mathbb{C})$ generated by the invertible matrices
$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$
Conclude that Q has order 8. **Remark:** Up to isomorphism there are only 2 non-abelian groups of order 8, the other one being D_4.

PIDs, Chinese remainder theorem, constructing fields

Recall that the following rings are Euclidean domains and hence principal ideal domains (PIDs):
$$\mathbb{Z}, \ \mathbb{Z}[i], \ \mathbb{Z}[e^{\pi i/3}], \ k[x] \text{ for } k \text{ a field.}$$
Also recall the following Chinese remainder theorem

Theorem 0.1 *Let R be a commutative ring and $I_1, \ldots, I_r \lhd R$ be ideals which are co-maximal in the sense that $I_a + I_b = R$ whenever $a \neq b$. Then the natural ring homomorphism*
$$\phi : R/(I_1 \cap \ldots \cap I_r) \longrightarrow R/I_1 \times \ldots \times R/I_r$$
is an isomorphism.

1. Show that $\mathbb{Z}/60\mathbb{Z} \simeq \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$.

2. Show that $\mathbb{R}[x]/(x^2 + 5)$ is a field but $\mathbb{C}[x]/(x^2 + 5)$ is a product of fields.

3. Let p be a prime and $\mathbb{F}_p$ denote the finite field with p elements. Show that $\mathbb{F}_7[x]/(x^2 - 3)$ is a field but $\mathbb{F}_7[x]/(x^2 - 2)$ is a product of fields. Identify all the fields.