Question 1

Let R be a ring, and M and N are R modules. M' is an R submodule of M. $\pi : M \to M/M'$ is the canonical quotient map.

Part (a)

Theorem 1 *The map $\Psi : \mathrm{Hom}_R(M/M', N) \to \mathrm{Hom}_R(M, N)$ given by $\Psi(\varphi) = \varphi \circ \pi$ is a homomorphism of abelian groups.*

Proof The group operation on $\mathrm{Hom}_R(M/M', N)$ is pointwise multiplication: $(\varphi + \psi)(m) = \varphi(m) + \psi(m)$, for $m \in M/M'$.

So we simply compute, for $\varphi, \psi \in \mathrm{Hom}_R(M/M', N)$ and $m \in M$

$$\begin{aligned}\Psi(\varphi + \psi)(m) &= ((\varphi + \psi) \circ \pi)(m) \\ &= \varphi(\pi(m)) + \varphi(\pi(m)) \\ &= (\varphi \circ \pi)(m) + (\psi \circ \pi)(m) \\ &= (\varphi \circ \pi + \psi \circ \pi)(m) \\ &= (\Psi(\varphi) + \Psi(\psi))(m).\end{aligned}$$

Hence, $\Psi(\varphi + \psi) = \Psi(\varphi) + \Psi(\psi)$. So Ψ is a homomorphism of abelian groups. $\square$

Part (d)

Theorem 2 *The kernel of Ψ is trivial. That is, the only $\varphi \in \mathrm{Hom}_R(M/M', N)$ with $\Psi(\varphi) = 0$ is $\varphi = 0$. That is, Ψ is injective.*

Proof Suppose that $\varphi \in \ker \Psi$. Then for all $m \in M$, we have

$$\Psi(\varphi)(m) = 0_N.$$

Hence,

$$\varphi(m + M') = 0_N.$$

However, this means that for any $m + M' \in M/M'$, $\varphi(m + M') = 0$. So φ is identically zero. $\square$

Part (c)

Theorem 3 *The image of Ψ is precisely the set*

$$\mathrm{im}\, \Psi = \{\varphi \in \mathrm{Hom}_R(M, N) \ : \ M' \subset \ker \varphi \ \}$$

Proof Let $S = \{\varphi \in \mathrm{Hom}_R(M, N) \ : \ \ker \varphi \subset M' \ \}$.

Let $\Psi(\varphi) \in \mathrm{im}\, \Psi$. Then if $m \in M'$,

$$\begin{aligned}\Psi(\varphi)(m) &= \varphi \circ \pi(m) \\ &= \varphi(m + M') \\ &= \varphi(M') \\ &= \varphi(0_{M/M'}) \\ &= 0_N.\end{aligned}$$

since M' is the identity of the additive group of M/M'. Hence $M' \subset \ker \Psi\varphi$ and so $\operatorname{im} \Psi \subset S$.

Now suppose that $\varphi \in S$. Consider the function $\psi : M/M' \to N$ given by $\psi(m + M') = \varphi(m)$. This is well defined, since if we choose $m' \in M'$, then $\psi(m + m' + M') = \psi(m) + \psi(m') = \psi(m)$, since $M' \subset \ker \psi$, so $\psi(m') = 0$.

Then $\Psi(\psi) = \psi \circ \pi : M \to N$, and if $m' \in M'$, then $\Psi(\psi)(m) = \psi(m + M') = \psi(M') = 0$.

Hence, $\Psi(\psi) \in S$, so $\operatorname{im} \Psi \subseteq S$.

Therefore, $\operatorname{im} \Psi = S$. $\square$

Part (d)

Theorem 4 *The finitely generated abelian groups A such that $\operatorname{Hom}_{\mathbb{Z}}(A, \mathbb{Z}/24\mathbb{Z})$ is a group of order 12 are exactly those that can be expressed in the form*

$$A \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \prod_{i=1}^{n} \mathbb{Z}/b_i\mathbb{Z}$$

or

$$A \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \prod_{i=1}^{n} \mathbb{Z}/b_i\mathbb{Z}$$

where the collection $\{b_1, \ldots, b_n\}$ is such that $\gcd(b_i, 24) = 1$ for $i = 1, \ldots, n$.

Proof If A is a finitely generated abelian group, then it can be expressed in the form

$$A \equiv \prod_{i=0}^{n} \mathbb{Z}/a_i\mathbb{Z}$$

where each $a_i \in \mathbb{Z}$ is either 0 or a power of a prime. By the universal property of direct sums, we have

$$\operatorname{Hom}_{\mathbb{Z}}(A, \mathbb{Z}/24\mathbb{Z}) \cong \prod_{i=1}^{n} \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/a_i\mathbb{Z}, \mathbb{Z}/24\mathbb{Z}).$$

Hence the order of $\operatorname{Hom}_{\mathbb{Z}}(A, \mathbb{Z}/24\mathbb{Z})$ is exactly

$$\prod_{i=1}^{n} |\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/a_i\mathbb{Z}, \mathbb{Z}/24\mathbb{Z})|.$$

By the universal property of quotient modules, we have

$$|\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/a_i\mathbb{Z}, \mathbb{Z}/24\mathbb{Z})| = |\{\varphi \in \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/24\mathbb{Z}) \ : \ a_i\mathbb{Z} \subseteq \ker \varphi \}|.$$

We can compute the right hand side by noting that the set $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/24\mathbb{Z})$ is exactly a set of multiplication operators, for $x \in \mathbb{Z}$, define $\lambda_x \in \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/24\mathbb{Z})$ by $\lambda_x(n) = nx + 24\mathbb{Z}$.

See that $a_i\mathbb{Z} \subseteq \ker \lambda_x$ precisely when $xa_i \in 24\mathbb{Z}$.

So the possible values of x such that $a_i\mathbb{Z} \subseteq \ker \lambda_x$ correspond to solutions of the linear congruence,

$$a_i x \equiv 0 \mod 24.$$

The number of solutions is given by $\gcd(a_i, 24)$. Hence,

$$|\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/a_i\mathbb{Z}, \mathbb{Z}/24\mathbb{Z})| = \gcd(a_i, 24).$$

So we have

$$|\operatorname{Hom}_{\mathbb{Z}}(A, \mathbb{Z}/24\mathbb{Z})| = \prod_{i=1}^{n} \gcd(a_i, 24).$$

So we wish to find choices for the set $\{a_1, a_2, \ldots, a_n\}$ such that

$$\prod_{i=1}^{n} \gcd(a_i, 24) = 12.$$

We cannot have $a_i = 0$ for any i, since $\gcd(0, 24) = 24$.

The only possible cases for $\gcd(a_i, 24) \neq 1$ are when $a_i|24$, since by assumption a_i is a power of a prime. Hence we can have $a_i = 2, 4, 8, 3$.

Since $3|12$, we must have $a_i = 3$ for some i. Reorder the set $\{a_1, \ldots, a_n\}$ if necessary so that $a_1 = 3$.

Now we require

$$\prod_{i=2}^{n} \gcd(a_i, 24) = 4.$$

So we cannot have $a_i = 8$. The only possible cases are $a_i = 4$ for some $i \geq 2$ and $\gcd(a_i, 24) = 1$ otherwise, or we have $a_i = 2$ and $a_j = 2$ for some distinct i and j.

Reorder indices if necessary so that the two possible cases are $a_2 = 4$ or $a_2 = 2$ and $a_3 = 2$. Then we have two possible decompositions for A,

$$A \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \prod_{i=3}^{n} \mathbb{Z}/a_i\mathbb{Z}$$

$$A \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \prod_{i=4}^{n} \mathbb{Z}/a_i\mathbb{Z}$$

with $\gcd(a_i, 24) = 1$ for $i \geq 3$ in the first case and $\gcd(a_i, 24) = 1$ for $i \geq 4$ in the second case. Thus the required result follows. □

Question 2

For this question, we consider the quaternion group,

$$Q = \langle i, j \mid i^4 = 1, j^2 = i^2, ji = i^3 j \rangle$$

and the associated real group algebra $\mathbb{R}Q$.

Lemma 5 $\mathbb{R}Q \cong \frac{\mathbb{R}\langle i,j\rangle}{\langle i^4-1, j^2-i^2, ji-i^3j\rangle}$

Proof Let $I = \langle i^4 - 1, j^2 - i^2, ji - i^3 j\rangle \lhd \mathbb{R}\langle i, j\rangle$.

By the universal property of free algebras, there is a unique $\mathbb{R}$-algebra homomorphism

$$\psi : \mathbb{R}\langle i, j\rangle \to \mathbb{R}Q$$

satisfying $\psi(i) = i$ and $\psi(j) = j$.

Since ψ is an algebra homomorphism, we have $\psi(1) = 1$, and since $\mathbb{R}Q$ is generated by i,j and 1 ψ is a surjection.

See that

$$\psi(i^4 - 1) = i^4 - 1 = 0$$
$$\psi(j^2 - i^2) = j^2 - i^2 = 0$$
$$\psi(ji - i^3 j) = ji - i^3 j = 0$$

Hence, $I \subseteq \ker \psi$. By the universal property of quotient modules, there is an isomorphism of abelian groups

$$\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}\langle i, j\rangle / I, \mathbb{R}Q) \cong \{\varphi \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}\langle i, j\rangle, \mathbb{R}Q) \ : \ I \subseteq \ker \varphi\}$$

The map ψ is in the left hand side of the above isomorphism. Hence, there is a corresponding $\varphi \in \mathrm{Hom}(\mathbb{R}\langle i, j\rangle, \mathbb{R}Q)$ so that $\psi = \varphi \circ \pi$, where π is the canonical projection $\mathbb{R}\langle i, j\rangle \to \mathbb{R}\langle i, j\rangle / I$.

Since ψ is a surjection, and $\psi = \varphi \circ \pi$, φ must also be a surjection.

Now we estimate the dimension of $\mathbb{R}\langle i, j\rangle / I$. Each element of this algebra is an $\mathbb{R}$-linear combination of terms of the form

$$i^{n_1} j^{n_2} i^{n_3} \cdots j^{n_k}$$

for some choice of non-negative integers $\{n_1, \ldots, n_k\}$.

Since we work modulo the ideal I, we can use the relationship $ji = i^3 j$ to reduce this to

$$i_{n_1} j^{n_2}$$

for some non- negative integers n_1 and n_2. Since $i^4 = 1$ and $j^2 = i^2$, we can reduce this to $n_1 \in \{0, 1, 2, 3\}$ and $n_2 \in 0, 1$.

Hence, there are at most 8 possible linearly independent terms of the form $i^{n_1} j^{n_2}$. Since these terms span $\mathbb{R}\langle i, j\rangle / I$, the dimension of $\mathbb{R}\langle i, j\rangle / I$ must be less than or equal to 8.

The algebra $A = \mathbb{R}Q$ is 8 dimensional since Q is a group of order 8.

Hence the map φ is a surjective linear map from a vector space of at most dimension 8 to a vector space of dimension 8.

Hence, φ is injective and is thus an isomorphism of $\mathbb{R}$-algebras. $\square$

Part (a)

Let $G = (\mathbb{Z}/2\mathbb{Z})^2$. Now we consider the group algebra $\mathbb{R}G$.

Theorem 6 $\mathbb{R}G \cong \frac{\mathbb{R}\langle i,j\rangle}{\langle i^2-1, j^2-1, ij-ji\rangle}$

Proof The group G has two generators, label them i and j so that $G = \{1, i, j, ij\}$ and $i^2 = 1$, $j^2 = 1$ where 1 is the group identity.

Define the ideal $J = \langle i^2 - 1, j^2 - 1, ij - ji\rangle \lhd \mathbb{R}\langle i, j\rangle$.

By the universal property of free algebras, there is a unique $\mathbb{R}$-algebra homomorphism

$$\psi : \mathbb{R}\langle i, j\rangle \to \mathbb{R}G$$

such that $\psi(i) = i$ and $\psi(j) = j$.

Since 1, i and j generate the algebra $\mathbb{R}G$, ψ is surjective.

Since we have the relationships,

$$\psi(i^2-1) = i^2 - 1 = 0$$
$$\psi(j^2-1) = j^2 - 1 = 0$$
$$\psi(ij-ji) = ij - ji = 0$$

we have $J \subseteq \ker \psi$.

Now by the universal property of quotient modules, there is an isomorphism of abelian groups

$$\{\varphi \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}\langle i,j\rangle, \mathbb{R}G) \ : \ I \subseteq \ker \varphi \ \} \cong \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}\langle i,j\rangle/J, \mathbb{R}G).$$

Note that ψ is in the left hand side of this isomorphism. Hence, there is a corresponding $\varphi \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}\langle i,j\rangle, \mathbb{R}G)$ so that $\psi = \varphi \circ \pi$.

Since ψ is surjective, φ is surjective.

Now we estimate the dimension of $\mathbb{R}\langle i,j\rangle/J$. Each element in this algebra is an $\mathbb{R}$-linear combination of terms of the form

$$i^{n_1} j^{n_2} i^{n_3} \cdots j^{n_k}$$

for some choice of non-negative integers $\{n_1, \ldots, n_k\}$. Since we have the relationship $ij = ji$, we can reduce this to

$$i^{n_1} j^{n^2}$$

for some non-negative integers n_1 and n_2. Since $i^2 = j^2 = 1$, we need only consider $n_1, n_2 \in \{0, 1\}$. Hence there are at most four distinct terms of the form $i^{n_1} j^{n_k}$.

Since terms of these form span $\mathbb{R}\langle i,j\rangle/J$, we have at most four linearly independnt terms in $\mathbb{R}\langle i,j\rangle/J$.

So the dimension of $\mathbb{R}\langle i,j\rangle/J$ is at most 4.

However, since the order of G is 4, the dimension of $\mathbb{R}G$ is 4.

Hence φ is a surjective linear map between a vector space of dimension 4 and a vector space of dimension 4. So φ must be bijective.

Hence φ is an isomorphism of $\mathbb{R}$-algebras. $\square$

Theorem 7 $\mathbb{R}/\langle i^2 - 1\rangle \cong \mathbb{R}G$.

Proof The ideal $\langle i^2 - 1\rangle$ of A corresponds to the ideal $\langle i^2 - 1\rangle + I/I$ of $\mathbb{R}\langle i,j\rangle/I$. Hence it is sufficient to prove that

$$\frac{\mathbb{R}\langle i,j\rangle/I}{(\langle i^2-1\rangle + I)/I} \cong \frac{\mathbb{R}\langle i,j\rangle}{J}.$$

By the second isomorphism theorem, the left hand side is isomorphic to

$$\frac{\mathbb{R}\langle i,j\rangle}{\langle i^2-1\rangle + I}.$$

Hence it is sufficient to prove that $J = \langle i^2 - 1\rangle + I$. That is, we must prove that

$$\langle i^2-1, j^2-1, ij-ji\rangle = \langle i^2-1\rangle + \langle i^4-1, ji-i^3j, i^2-j^2\rangle.$$

Clearly $i^2 - 1 \in \langle i^2 - 1\rangle + \langle i^4 - 1, ji - i^3 j, i^2 - j^2\rangle$.

Since we have

$$j^2 - 1 = (j^2 - i^2) + (i^2 - 1)$$
$$ij - ji = -(ji - i^3 j) + (1 - i^2)ij$$

The inclusion $\langle i^2-1, j^2-1, ij-ji\rangle \subseteq \langle i^2-1\rangle + \langle i^4-1, ji-i^3j, i^2-j^2\rangle$ follows.

Now we prove the reverse inclusion. Clearly we have $i^2-1 \in \langle i^2-1, j^2-1, ij-ji\rangle$, and we have the following identities:

$$\begin{aligned} i^4-1 &= (i^2-1)(i^2+1) \\ ji-i^3j &= (ji-ij)+(1-i^2)ij \\ i^2-j^2 &= (i^2-1)-(j^2-1) \end{aligned}$$

Hence $\langle i^4-1, ji-i^3j, i^2-j^2\rangle \subseteq \langle i^2-1, j^2-1, ij-ji\rangle$, and also since $\langle i^2-1\rangle \subseteq \langle i^2-1, j^2-1, ij-ji\rangle$. we have $\langle i^2-1\rangle + \langle i^4-1, ji-i^3j, i^2-j^2\rangle \subseteq \langle i^2-1, j^2-1, ij-ji\rangle$.

So we have proven the required equality. $\square$

Part (b)

Now we define the quaternion algebra,

$$\mathbb{H} = \frac{\mathbb{R}\langle i, j\rangle}{\langle i^2+1, j^2+1, ij+ji\rangle}$$

Theorem 8 $A/\langle i^2+1\rangle \cong \mathbb{H}$

Proof Again we use the isomorphism $A \cong \mathbb{R}\langle i,j\rangle/I$. The ideal $\langle i^2+1\rangle \lhd A$ corresponds to $\langle i^2+1\rangle + I/I$. So it is sufficient to prove that

$$\frac{\mathbb{R}\langle i,j\rangle/I}{(\langle i^2+1\rangle + I)/I} \cong \frac{\mathbb{R}\langle i,j\rangle}{\langle i^2+1, j^2+1, ij+ji\rangle}.$$

By the second isomorphism theorem, this is equivalent to proving that

$$\frac{\mathbb{R}\langle i,j\rangle}{\langle i^2+1\rangle + I} \cong \frac{\mathbb{R}\langle i,j\rangle}{\langle i^2+1, j^2+1, ij+ji\rangle}.$$

Hence is it required to prove that

$$\langle i^2+1\rangle + I = \langle i^2+1, j^2+1, ij+ji\rangle.$$

That is, we must prove

$$\langle i^2+1\rangle + \langle i^4-1, ji-i^3j, j^2-i^2\rangle = \langle i^2+1, j^2+1, ji+ij\rangle.$$

Firstly, it is clear that $i^2+1 \in \langle i^2+1\rangle + \langle i^4-1, ji-i^3j, j^2-i^2\rangle$.

We also have the following relations,

$$\begin{aligned} j^2+1 &= (j^2-i^2)+(i^2+1) \\ ji+ij &= ji-i^3j+(i^2+1)ij \end{aligned}$$

Hence, $\langle i^2+1, j^2+1, ji+ij\rangle \subseteq \langle i^2+1\rangle + \langle i^4-1, ji-i^3j, j^2-i^2\rangle$.

Now we wish to prove the opposite inclusion. Clearly $i^2+1 \in \langle i^2+1, j^2+1, ji+ij\rangle$.

We have the following identities:

$$\begin{aligned} i^4-1 &= (i^2-1)(i^2+1) \\ ji-i^3j &= ji+ij-(i^2+1)ij \\ j^2-i^2 &= (j^2+1)-(i^2+1). \end{aligned}$$

Hence we have $\langle i^2+1\rangle + \langle i^4-1, ji-i^3j, j^2-i^2\rangle = \langle i^2+1, j^2+1, ji+ij\rangle$, as required.

So the result follows. $\square$

Part (c)

Lemma 9 *Suppose that R is a ring and $I \trianglelefteq R$ is a two sided ideal. R/I is both a left and right R module, and also both a left R/I module and a right R/I module.*

The R submodules of R/I are exactly the R/I submodules of R/I.

Proof Consider first R/I as a left R module. Suppose that M is an R submodule of R/I. Let $r \in R$, then $(r+I)M = rM \in M$. Hence M is an R/I submodule of R/I. Similarly, if R/I is considered as a right R module then any R-submodule of R/I is an R/I submodule of R/I considered as a right R/I submodule.

Conversely, suppose that N is an R/I submodule of R/I, considered as a left R/I-module. Then let $r \in R$. Then we have $rM = (r+I)M$ since $IM \subseteq M$. Hence $rM \in M$, and so M is an R-submodule of R/I. Similarly, if we considered R/I as a right R/I-module, then R/I submodules of R/I are R submodules of R/I considered as a right R module. $\square$

Theorem 10 *$A/\langle i^2+1\rangle$ is indecomposable as an A module.*

Proof We wish to prove that $A/\langle i^2+1\rangle$ has no nontrivial A-submodules. It is sufficient to prove that $A/\langle i^2+1\rangle$ has no nontrivial $A/\langle i^2+1\rangle$ submodules.

However, since $A/\langle i^2+1\rangle \cong \mathbb{H}$, we need only prove that $\mathbb{H}$ has no nontrivial ideals.

Indeed, since $\mathbb{H}$ is a division ring, if $I \trianglelefteq \mathbb{H}$ is a nonzero ideal with $a \in I$, then $1 = aa^{-1} \in I$, and so $I = \mathbb{H}$.

Hence $\mathbb{H}$ has no nontrivial ideals and so $A/\langle i^2+1\rangle$ is indecomposible.

Part (d)

Theorem 11 *$A/\langle i^2-1\rangle$ is decomposible as an A-module.*

Proof We may write $A/\langle i^2-1\rangle$ as $\mathbb{R}G$. It is required to find two A-submodules of $A/\langle i^2-1\rangle$ which span $A/\langle i^2-1\rangle$ and have trivial intersection.

It is sufficient to find two ideals S and T of $R\mathbb{G}$ such that $\mathbb{R}G = S+T$, and $S \cap T = \{0\}$.

Consider

$$S = \langle i+j\rangle$$
$$T = \langle i-j\rangle$$

It is easy to see that $S+T = \mathbb{R}G$, since

$$\frac{1}{2}[i(i+j) + i(i-j)] = 1.$$

Now we need to show that $S \cap T = \{0\}$. Suppose that there are $p, q \in \mathbb{R}G$ such that

$$p(i-j) = q(i+j)$$

Since $\mathbb{R}G$ is spanned by the elements of G, write $p = p_0 + p_1 i + p_2 j + p_3 ij$ and $q = q_0 + q_1 i + q_2 j + q_3 ij$. Then we have

$$(p_0 + p_1 i + p_2 j + p_3 ij)(i-j) = (q_0 + q_1 i + q_2 j + q_3 ij)(i+j)$$

Expanding this out,

$$(p_1 - p_2) + (p_0 - p_3)i + (-p_0 + p_3)j + (p_2 - p_1)ij = (q_1 + q_2) + (q_0 + q_3)i + (q_0 + q_3)j + (q_1 + q_2)ij$$

Hence by comparing coefficients, we have

$$\begin{aligned} p_1 - p_2 &= q_1 + q_2 \\ p_0 - p_3 &= q_0 + q_3 \\ -p_0 + p_3 &= q_0 + q_3 \\ p_2 - p_1 &= q_1 + q_2. \end{aligned}$$

Hence $p_1 = p_2$ and $p_0 = p_3$, so we can write $p = a + bi + bj + aij$ for some $a, b \in \mathbb{R}$.

Now,

$$\begin{aligned} p(i - j) &= (a + bi + bj + aij)(i - j) \\ &= 0. \end{aligned}$$

So therefore, $S \cap T = \{0\}$. $\square$

Question 3

For this question k is a field, and R denotes the subring of $k[x]$ given by

$$R = \{p(x) \in k[x] \ : \ p'(0) = 0 \ \}.$$

Part (a)

Theorem 12 *The R-module $k[x]/R$ is 1-dimensional as a vector space over k, and cyclic as an R-module.*

Proof Consider the k-linear map $\varphi : k[x] \to k$ given by $p(x) \mapsto p'(0)$. The image of φ is k, since for any $\lambda \in$k, $\varphi(\lambda x) = \lambda$, and the kernel of φ is precisely R. Hence, we have an isomorphism of k-vector spaces,

$$k[x]/R \cong k$$

by the first isomorphism theorem. Thus, $k[x]/R$ is a 1 dimensional vector space over k.

Hence, $k[x]/R = km$, for some $m \in k[x]/R$. For $p(x) \in R$, we have $p(x)m = \lambda m$ for some $\lambda \in k$. Hence m is a generator for $k[x]/R$ considered as an R module, and so $k[x]/R$ is cyclic. $\square$

Part (b)

Theorem 13 *The element $x + R \in k[x]/R$ is a generator for $k[x]/R$ as an R-module. Hence, $k[x]/R \cong R/I$, where I is the ideal of R given by*

$$I = \{p(x) \in k[x] \ \ p(0) = p'(0) = 0 \ \}.$$

Proof Let $p(x) + R \in k[x]/R$. Write $p(x)$ as

$$p(x) = p_0 + p_1 x + p_2 x^2 + \cdots + p_n x^n$$

Then since

$$p_0 + p_2x^2 + p_3x^3 + \cdots + p_nx^n \in R,$$

we have $p(x) + R = p_1x + R$. Hence any element of $k[x]/R$ can be written as $\lambda x + R$ for some $\lambda \in k$.

That is,

$$k[x]/R = k(x + R).$$

Hence $x + R$ is a generator for $k[x]/R$. So we can define a surjective R-module homomorphism,

$$\varphi : R \to k[x]/R$$

given by $\varphi(p(x)) = p(x)(x + R)$.

The kernel of this map is

$$\ker\varphi = \{p(x) \in R \ : \ xp(x) \in R\}$$

We can determine when $xp(x)$ is in R by differentiating, so $xp(x) \in R$ when $xp'(x) + p(x) = 0$ when $x = 0$. Hence,

$$\ker\varphi = \{p(x) \in k[x] \ : \ p'(0) = 0, p(0) = 0\}.$$

So by the first isomorphism theorem we have an isomorphism of R-modules,

$$R/I \cong k[x]/R$$

where $I = \{p(x) \in k[x] \ : \ p(0) = p'(0) = 0 \}$. □

Part (c)

Theorem 14 *The ideal $I = \{p(x) \in R \ : \ p(0) = 0\}$ is generated by x^2 and x^3, $I = \langle x^2, x^3\rangle$.*

Proof Suppose that $p(x) \in I$. Then $p(x)$ can be written in the form

$$p(x) = p_2x^2 + p_3x^3 + \cdots + p_nx^n$$

since by assumption, $p(0) = p'(0) = 0$.

Each term in $p(x)$ can therefore be written as αx^r, for $r \geq 2$ and $\alpha \in k$. If r is even, then write $\alpha x^r = x^2(\alpha x^{r-2})$. Then since r is even, $\alpha x^{r-2} \in R$. Hence, $\alpha x^r \in \langle x^2, x^3\rangle$.

If r is odd, write $\alpha x^r = x^3\alpha x^{r-3}$. Since r is odd, $r - 3$ is even and so $\alpha x^{r-2} \in R$. Hence $\alpha x^r \in \langle x^2, x^3\rangle$.

Thus, every term in $p(x)$ is in $\langle x^2, x^3\rangle$ and so $p(x) \in \langle x^2, x^3\rangle$.

Since every polynomial in $\langle x^2, x^3\rangle$ has degree at least 2, we see that $\langle x^2, x^3\rangle \subseteq I$.

Hence, $I = \langle x^2, x^3\rangle$.

Corollary 15 *$I = \{p(x) \in R \ : \ p(0) = 0 \} \lhd R$ can be generated by no fewer than 2 elements.*

Proof Suppose that $I = \langle q(x)\rangle$, that is suppose that I is generated by a single polynomial q. Since $x^2 \in I$, we must have $q(x)|x^2$.

We must have $q(x) = x^2r(x)$ for some $r \in k[x]$. Thus in $k[x]$, $q(x)|x^2$ and $x^2|q(x)$, so $q(x) = \alpha x^2$ for some $\alpha \in k$.

However, $x^3 \in I$. However no polynomial in x^2R has degree 3, since for any $s(x) = s_0+s_2x^2+\cdots+s_nx^n \in R$, we have $x^2s(x) = s_0x^2 + s_2x^4 + \cdots + s_nx^{n+2}$.

Therefore, $x^3 \notin \langle q(x)\rangle$.

Hence I can not have a single generator. □

Part (c)

The ideal $I \trianglelefteq R$ is not principal, so R cannot be a principal ideal domain.