School of Mathematics and Statistics University of New South Wales

MATH5735: Modules and Representation Theory (2014,S1) Problem Set 3 [1]

This problem set covers lectures 11-14. Throughout, k will denote some field and R a ring.

1. Consider the sequence of $\mathbb{Z}$-modules
$$\dots \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \dots$$
where the maps are multiplication by 2. Show that the sequence is exact.

2. Let $R = M \oplus M'$ as R-modules. Hence we may write $1_R = e + e'$ where $e \in M, e' \in M'$. Show that e is an idempotent of R.

3. A set $\{e_1, \dots, e_n\}$ of idempotents for R is said to be *complete* if $1 = e_1 + \dots + e_n$ and *orthogonal* if $e_i e_j = 0$ for $i \neq j$. Show that if $\{e_1, \dots, e_n\}$ is a complete, orthogonal set of idempotents and B is an (R, S)-bimodule then we have a direct sum decomposition of right S-modules
$$B = e_1 B \oplus \dots \oplus e_n B.$$

4. Let k be a field and A be a finite dimensional k-algebra. Show that any finitely generated A-module M is both noetherian and artinian.

5. Consider the subring $\mathbb{Z}[\frac{1}{2}] = \{\frac{n}{2^m} | n, m \in \mathbb{Z}\}$ of $\mathbb{Q}$. Note that since $\mathbb{Z}$ is a subring of $\mathbb{Z}[\frac{1}{2}]$, we may consider it as a $\mathbb{Z}$-module. Consider the quotient $\mathbb{Z}$-module $M = \mathbb{Z}[\frac{1}{2}]/\mathbb{Z}$. Is M noetherian? Is M artinian? Justify your answer fully in either case. Hint: $\mathbb{Z}[\frac{1}{2}]$ is the union of submodules $\frac{1}{2^m}\mathbb{Z}$.

6. Which of the following rings are noetherian and/or artinian? Give full reasons for your answer. i) $\mathbb{Q}(\sqrt{7})$, ii) $M_3(\mathbb{R})$, iii) $M_3(\mathbb{Z}[i])$, iv) $k[x, y, z]/\langle y^2 - xyz^5 \rangle$, v) $k\langle x, y, z\rangle/\langle y^2 - xyz^5\rangle$, vi) $k\langle x, y\rangle/\langle yx - qxy\rangle$ for some $q \in k^\times$, vii) $\mathbb{Z}G$ for a finite group G.

7. Show that a module has finite length if and only if it is both noetherian and artinian.

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8. Consider the $k[x]$-module $M = k[x]/\langle x^3 - x^2 \rangle$. Show that M has finite length by constructing a composition series for it. Write down the composition factors.

9. Let $R = k[x, y]$ and consider the R-modules $M = (R/\langle x, y\rangle)^3, N = R/\langle x^2, xy, y^2\rangle$. Show that they have the same composition factors but are not isomorphic as R-modules.

10. Let $R = \mathbb{Z}[x]$ and $M = R/\langle 6, x^2 \rangle$. Find the composition factors of M. Is M a direct sum of simple modules?

11. This question concerns the first Weyl algebra A_1 which is the subalgebra of $E = \mathrm{End}_{\mathbb{C}}\, \mathbb{C}[x]$ defined as follows. Let $\partial \in E$ be differentiation and $\lambda_x \in E$ be multiplication by x. Then by the universal property of free algebras, there is a unique $\mathbb{C}$-algebra homomorphism $\phi : \mathbb{C}\langle x, y\rangle \longrightarrow E$ which sends x to λ_x and y to ∂. Then A_1 is the image of ϕ and is often written as $\mathbb{C}\langle x, \partial\rangle$. (Note that in this notation, the algebra is not the free algebra on x, ∂ and that x really means λ_x.)

   (a) Show that $\partial x = x\partial + 1$ so A_1 is non-commutative.

   (b) Prove that $A_1 \simeq \mathbb{C}\langle x, y\rangle/\langle yx - xy - 1\rangle$. Hint: It may be useful to show that a $\mathbb{C}$-basis for A_1 is $\{x^i \partial^j | i, j \in \mathbb{N}\}$.

   (c) Prove that A_1 is noetherian.

12. Consider the matrices

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, e_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, f = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

   in $M_2(k)$ where k is a field. They span the subalgebra A of upper triangular matrices (see problem set 1, question 2).

   (a) Show that $\{e_1, e_1\}$ form a complete set of orthogonal idempotents so any right A-module M decomposes as a vector space in the form $M = Me_1 \oplus Me_2$. Show also that $(Me_1)f \subseteq Me_2$.

   (b) Show that up to isomorphism, there are only two simple modules.

   (c) Show that A_A is the direct sum of two non-isomorphic indecomposable submodules and find their composition factors.

   (d) Show that up to isomorphism, there are only 3 indecomposable A-modules. Hint: If $m \in Me_1$ is non-zero, show that $km + kmf$ is a direct summand of M.