School of Mathematics and Statistics UNSW Australia

MATH5735: Modules and Representation Theory (2014,S1)
Problem Set 1 [1]

Throughout, k will denote some field, R a ring and n some positive integer.

1. Let R be a commutative ring. Prove that $M_n(R)^{\mathrm{op}} \simeq M_n(R)$.

2. Let R be a ring and consider the subset T, respectively U, of $M_n(R)$ consisting of upper triangular, respectively, strictly upper triangular, matrices. Are either of T or U subrings of $M_n(R)$? If so, compute their centres. Are either of T or U ideals of $M_n(R)$?

3. Let G be a group and consider the subset J of kG consisting of elements of the form $\sum_{g\in G} \alpha_g g$ where $\sum \alpha_g = 0$. Show that J is an ideal of kG. Determine the algebra kG/J.

4. Let A be an R-algebra where R is a commutative ring. If I is an ideal of A, show that A/I is naturally an R-algebra too with unit map the composite $R \longrightarrow A \longrightarrow A/I$.

5. Suppose k is algebraically closed. Prove that every two-dimensional k-algebra A is isomorphic to $k \times k$ or $k[x]/(x^2)$. Hint: Let $\{1, x\}$ be a basis for A and show x satisfies a quadratic equation over k. Show this result no longer holds if k is not algebraically closed.

6. The *quaternion algebra* is

$$\mathbb{H} = \mathbb{R}\langle i, j\rangle/(i^2 + 1, j^2 + 1, ij + ji).$$

   (a) Show that there is an injective $\mathbb{R}$-algebra homomorphism $\phi : \mathbb{H} \longrightarrow M_2(\mathbb{C})$ which sends

$$i \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad j \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

   (b) Hence or otherwise show that $\mathbb{H}$ is a division ring. Hint: First show that for any $h \in \mathbb{H} - 0$ we have $\phi(h)$ is invertible in $M_2(\mathbb{C})$ and then show that the inverse lies in $\phi(\mathbb{H})$.

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(c) Show that the centre of $\mathbb{H}$ is $\mathbb{R}$.

7. Prove that the 3-dimensional algebras $k[x]/(x^3)$ and $k[x,y]/(x^2, xy, y^2)$ are not isomorphic.

8. Let M be an R-module. For any $m \in M$, show that $m0 = 0$ and $m(-1) = -m$.

9. Consider the subring $R = k[x^2]$ of $S = k[x]$. Note that there is a ring homomorphism $\iota : R \longrightarrow S$ given by inclusion so we can change scalars from S to R. Hence we may consider S not just as an S-module but also as an R-module. Show that R is an R-submodule of S and that $S/R \simeq R$ as R-modules.

10. Let H be a subgroup of G and $C = gH$ be a left coset of H in G. As in the previous question note that kG can also be considered as a kH-module. Show that $kC = \oplus_{g \in C} kg$ is a right kH-submodule of kG.

11. Let $R = kG$ where $G = \langle \sigma \rangle$ is the cyclic group of order 6. Consider the R-module homomorphism $\phi : R \longrightarrow R$ given by left multiplication by $1 + \sigma^2 + \sigma^4$.

    (a) Compute the kernel of ϕ.

    (b) Compute the image of ϕ.

    (c) What does the first isomorphism theorem applied to ϕ say in this case?

12. Using the universal property for quotient modules or otherwise, find all $k[x]$-module homomorphisms from $k[x]/(x^2)$ to $k[x]/(x^3)$.