UNIVERSITY OF NEW SOUTH WALES.
SCHOOL OF MATHEMATICS AND STATISTICS
MATH5645
TOPICS IN ANALYTIC NUMBER THEORY

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4. THE RIEMANN ZETA FUNCTION:

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a. Show that for $\Re(s) > 1$, $\displaystyle\sum_{n=1}^{\infty} \frac{(\tau(n))^2}{n^s} = \frac{\zeta^4(s)}{\zeta(2s)}$.

b. Recall the definition of $\omega(n)$ from question 2 of Sheet 2. Show that for $\Re(s) > 1$, $\displaystyle\sum_{n=1}^{\infty} \frac{2^{\omega(n)}}{n^s} = \frac{\zeta^2(s)}{\zeta(2s)}$.

2

a. Use the comparison test for series to prove that for $s > 1$, (s real),

$$\frac{1}{s-1} \le \zeta(s) \le 1 + \frac{1}{s-1}.$$

b. Deduce that $\displaystyle\lim_{s\to 1^+} (s-1)\zeta(s) = 1$.

c. Further show that $\dfrac{\log \zeta(s)}{\log(\frac{1}{s-1})} \to 1$ as $s \to 1^+$.

3 Prove that the integral $\displaystyle\int_0^{\infty} x^{s-1} e^{-x}\, dx$, which defines the Gamma function, converges for $Re(s) > 0$.

4 Prove that **a.** $\Gamma(s+1) = s\Gamma(s)$ **b.** $\Gamma(n+1) = n!$ **c.** $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

5 Show that $\dfrac{1}{e^t - 1} - \dfrac{1}{t}$ can be written as $-\dfrac{1}{2} + \displaystyle\sum_{n=1}^{\infty} a_n t^n$ with $a_{2n} = 0$.

6 Prove that $\Gamma(s)$ has simple poles only for $s = 0, -1, -2, \ldots$ and show that the residue at the pole $s = -k$ is $\frac{(-1)^k}{k!}$, where k is a non-negative integer.

7 Evaluate
a. $\zeta(0)$, **b.** $\zeta(-1)$, **c.** $\zeta(-2)$, **d.** $\zeta(-3)$

and show that $\zeta(-2n) = 0$ for $n = 1, 2, 3, \cdots$.

8 Given the well-known infinite product $\sin x = x \displaystyle\prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2\pi^2}\right)$, use logarithmic differentiation to show that

$$\cot x = \frac{1}{x} - 2\sum_{n=1}^{\infty} \frac{x}{n^2\pi^2 - x^2}.$$

***9** Prove that $\displaystyle\int_0^{\infty} \frac{u^s}{u^2+1}\, du = \frac{\pi}{2}\sec\left(\frac{s\pi}{2}\right)$ for $-1 < s < 1$.

10 **a.** Prove that if p is prime and $s > 2$,

$$1 + \frac{\phi(p)}{p^s} + \frac{\phi(p^2)}{p^{2s}} + \cdots = \frac{1-p^{-s}}{1-p^{1-s}}.$$

b. Deduce that for $s > 2$, $\displaystyle\sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)}$.

***11**

Suppose $\Re(s) > 1$.

a. Use the Euler product to show that $\log\zeta(s) = -\displaystyle\lim_{k\to\infty}\sum_{n=2}^{k}[\pi(n) - \pi(n-1)]\log(1 - \frac{1}{n^s})$.

b. Deduce that $\log\zeta(s) = s\displaystyle\int_2^{\infty} \frac{\pi(x)}{x(x^s - 1)}\,dx$.

12 You may assume the (well-known) result $\displaystyle\int_0^{\infty} \frac{x^{p-1}}{1+x}\,dx = \frac{\pi}{\sin(p\pi)}$, for $0 < p < 1$, from Complex Analysis.

a. Show that $\Gamma(m) = 2\displaystyle\int_0^{\infty} x^{2m-1}e^{-x^2}\,dx$.

b. Use the formula in (a) to show that for $0 < m < 1$,

$$\Gamma(m)\Gamma(1-m) = 2\int_0^{\frac{\pi}{2}} \tan^{1-2m}\theta\;d\theta.$$

(Hint: Express the product as the limit of a double integral over a quarter circle in the first quadrant.)

c. By putting $x = \tan^2\theta$ and $p = 1 - m$, deduce that

$$\Gamma(m)\Gamma(1-m) = \frac{\pi}{\sin(m\pi)}.$$

13 You may assume that $2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2}) = \sqrt{\pi}\Gamma(2z)$ (this is the Legendre duplication formula).

a. Put $z = \frac{1-s}{2}$ to obtain

$$\Gamma(1-s)\sin\frac{\pi s}{2} = \frac{2^{-s}\pi^{\frac{1}{2}}\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}.$$

b. Use the Functional Equation for $\zeta(s)$ to show that

$$\pi^{\frac{-s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{(1-s)}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

c. Now let $\Phi(s)$ denote the left hand side of the formula in (b) above.

Show that $\Phi(s) = \Phi(1-s)$ and that Φ has simple poles at $s = 0$ and $s = 1$.

d. Define $\xi(s) = \frac{1}{2}s(s-1)\Phi(s)$. Explain why $\xi(s)$ is an entire function, with functional equation $\xi(s) = \xi(1-s)$.

***14** Let ξ be the function defined in the previous question.

a. Explain why $\xi(s)$ satisfies the equation $\overline{\xi(s)} = \xi(\overline{s})$.

b. Prove that $\xi(s)$ is real on the lines $t = 0$ and $\sigma = \frac{1}{2}$ and that $\xi(0) = \xi(1) = \frac{1}{2}$.

c. Prove that the zeros of $\xi(s)$ (if any exist) are all situated in the strip $0 < \sigma < 1$ and lie symmetrically about the lines $t = 0$ and $\sigma = \frac{1}{2}$.

15 Let $f(s) = \dfrac{1}{\zeta(s)}$, for $s \neq 1$ and define $f(1) = 0$.

a. Explain why f is holomorphic and show that the power series for $f(s)$ about $s = 1$ is

$$f(s) = (s-1) - \gamma(s-1)^2 + \ldots$$

where γ is Euler's constant.

b. Write down the values of $f'(1)$ and $f''(1)$.

***16** Let $M(x) = \sum_{n \le x} \mu(n)$.

a. Use the result $\dfrac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \dfrac{\mu(n)}{n^s}$ for $\sigma > 1$, to deduce that

$$\frac{1}{\zeta(s)} = s \int_1^{\infty} M(x) x^{-s-1}\, dx.$$

b. Assume that $M(x) = O(x^{\frac{1}{2}+\epsilon})$, for any $\epsilon > 0$ is true and deduce that the integral on the left converges for $\sigma > \frac{1}{2} + \epsilon$.

c. Conclude that $M(x) = O(x^{\frac{1}{2}+\epsilon})$, for any $\epsilon > 0$ implies the Riemann Hypothesis. (Note that the converse is also true.)

17 Show the function $f(t) = \dfrac{t}{e^t - 1} + \dfrac{t}{2}$ is even, thereby confirming that $B_{2n+1} = 0$. Use the formula relating Bernoulli numbers and the zeta function to show that the non-zero Bernoulli numbers alternate in sign.

18 **a.** Starting from the functional equation for $\zeta(s)$, show that

$$\zeta(s) = \frac{(2\pi)^s}{2} \frac{\zeta(1-s)}{2\Gamma(s)\cos(\pi s/2)}.$$

b. Deduce that for $n \ge 1$,

$$\zeta(1-2n) = -\frac{B_{2n}}{2n}.$$

c. Find the value of $\zeta(-3), \zeta(-5), \zeta(-7)$.

19 Use the Bernoulli numbers to find a formula for $\sum_{x=1}^{k-1} x^4$ and $\sum_{x=1}^{k-1} x^5$.

20 Prove that $\underbrace{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}}_{(m,n)=1} \dfrac{1}{m^2 n^2} = \dfrac{\zeta^2(2)}{\zeta(4)} = \dfrac{5}{2}$.