UNIVERSITY OF NEW SOUTH WALES.
SCHOOL OF MATHEMATICS AND STATISTICS
MATH5535
TOPICS IN NUMBER THEORY

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6. PRIME NUMBER THEOREM:

1 **a.** Explain why $\psi(x)$ can be written in the form $\displaystyle\sum_{p\le x}\left[\frac{\log x}{\log p}\right]\log p$.

b. Deduce that $\psi(x) \le \pi(x)\log x$ and hence

$$\frac{\psi(x)}{x} \le \frac{\pi(x)}{\frac{x}{\log x}}.$$

c. Use Theorem 2.6 from section 2 to deduce that $\psi(x) \le 2x$.

2

a. Explain why $\psi(n)$ is the logarithm of the lowest common multiple of $2, 3, ..., n$.
b. Use this to find $\psi(10)$.

***3** In this question you may assume the following results from Sheet 2 Question 10.
Let n be an integer greater than 2. Write $N = \binom{2n+1}{2n}$. Let $p_{k+1}, p_{k+2}, ..., p_m$ be the primes p such that $n+2 \le p \le 2n+1$. We know that $N < 4^n$ and $\displaystyle\prod_{n+2\le p\le 2n+1} p < N$.

a. Prove that $\vartheta(2n+1) - \vartheta(n+1) \le n\log 4$.

b. By supposing that $\vartheta(k) \le k\log 4$ for all $k \le 2n$, $n > 1$ show that $\vartheta(2n+1) \le (2n+1)\log 4$.

c. Prove by induction that $\vartheta(n) \le n\log 4$ for all $n > 1$.

(Hence, since ϑ is increasing, we have $\vartheta(x) \le x\log 4$ for $x > 1$.)

***4** Assume that $\dfrac{\vartheta(x)}{x} \to 1$ as $x \to \infty$.

a. Prove that $\displaystyle\frac{\log x}{x}\int_2^x \frac{\vartheta(t)}{t(\log t)^2}\,dt \to 0$ as $x \to \infty$. (Hint: After replacing $\vartheta(x)$ as $O(x)$, split the integral over $[2, \sqrt{x}] \cup [\sqrt{x}, x]$.)

b. Use the formula $\displaystyle\pi(x) = \frac{\vartheta(x)}{logx} + \int_2^\infty \frac{\vartheta(t)}{t(\log t)^2}\,dt$ to prove that $\dfrac{\pi(x)\log x}{x} \to 1$ as $x \to \infty$.

5

a. If $f(s)$ has a pole of order k at $s = \alpha$, prove that $\dfrac{f'(s)}{f(s)}$ has a simple pole at $s = \alpha$ with residue $-k$.

b. Deduce that $F(s) = \dfrac{\zeta'(s)}{\zeta(s)} + \dfrac{1}{s-1}$ is analytic at $s = 1$.

6 Let $Z(s) = -\dfrac{\zeta'(s)}{\zeta(s)} = \displaystyle\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}$.

a. Use Abel's summation formula to prove that

$$Z(s) = s\int_1^\infty \frac{\psi(x)}{x^{s+1}}\,dx$$

where $\psi(x) = \displaystyle\sum_{n\le x}\Lambda(n)$.

(Note: This is the *Mellin transform* if $\psi(x)$.)

b. Use integration by parts to show that

$$Z(s) = s(s+1)\int_1^\infty \frac{\psi_1(x)}{x^{s+2}}\,dx$$

where $\psi_1(x) = \int_1^\infty \psi(x)\,dx$.

7 Let $s(x)$ denote the number of squares less or equal to x. Use the prime number theorem to prove that $\frac{s(x)}{\pi(x)}$ tends to 0 as $x \to \infty$. (This shows that the number of primes greatly exceeds the number of squares.)

8 Let $c > 1$

a. Prove that $\pi(cx) \sim c\pi(x)$ as $x \to \infty$.

b. Prove that $\pi(cx) - \pi(x) \sim (c-1)\dfrac{x}{\log x}$ as $x \to \infty$.

c. Deduce that for sufficiently large x, there are primes in the interval $(x, cx]$

9 Prove that every interval $[a, b]$, $0 < a < b$, contains a rational number $\dfrac{p}{q}$ with both p, q prime.

10 Prove that

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_p \frac{\log p}{p^s - 1},$$

for $s > 1$.
(Hint: Start with the Euler product for ζ.)

11 **a.** If $x \geq 2$, define the *logarithmic integral*,

$$Li(x) = \int_2^x \frac{dt}{\log t}.$$

Show that

$$Li(x) = \frac{x}{\log x} + \int_2^x \frac{dt}{(\log t)^2} - \frac{2}{\log 2}.$$

b. Show that

$$Li(x) = \frac{x}{\log x} + \frac{x}{(\log x)^2} + ... + (n-1)!\frac{x}{(\log x)^n} + r_{n+1}(x)$$

where $r_{n+1}(x) \sim n!\frac{x}{(\log x)^{n+1}}$ as $x \to \infty$.

BRIEF SOLUTIONS

1 **a.** $\psi(x) = \sum_{p^m \le x} \log p$. Now for each prime p, since we are summing $\log p$ for $p^m \le x \Rightarrow m \le \frac{\log x}{\log p}$, then each $\log p$ will appear in the sum $\left[\frac{\log x}{\log p}\right]$ times, the result follows. **b.** Using (a), we have $\psi(x) \le \sum_{p \le x} \frac{\log x}{\log p} \log p = \log x \sum_{p \le x} 1 = \pi(x) \log x$. **c.** This is immediate since $\pi(x) \le 1.7 \frac{x}{\log x}$.

2 Follows from the definition of ψ in terms of Λ. For (b) $\log 2520$.

3 **a.** $\vartheta(2n+1) - \vartheta(n+1) = \sum_{j=k+1}^{m} \log p_j = \log N \le n \log 4$. **b.** $\vartheta(n) \le n \log$ for $n = 2, 3, 4$ so suppose $\vartheta(k) \le k \log 4$ for all $k \le 2n$, $n > 1$. Then $\vartheta(2n+1) \le \vartheta(n+1) + n \log 4$ (by (a)), $\le (2n+1) \log 4$ (by assumption, since $n + 1 < 2n$. **c.** Also $\vartheta(2n+2) = \vartheta(2n+1)$ (since $2n+2$ is not prime). So the required result is true for all $k \le 2n+2$ and so the result follows by induction.

4 **a.** $\frac{\log x}{x} \int_2^x \frac{\vartheta(t)}{t(\log t)^2}\, dt = O\left(\frac{\log x}{x} \int_2^x \frac{dt}{(\log t)^2}\right)$. Now $\int_2^x \frac{dt}{(\log t)^2} = \int_2^{\sqrt{x}} \frac{dt}{(\log t)^2} + \int_{\sqrt{x}}^x \frac{dt}{(\log t)^2} \le \frac{\sqrt{x}}{(\log 2)^2} + \frac{x - \sqrt{x}}{(\log \sqrt{x})^2}$. Multiplying by $\frac{\log x}{x}$ and taking the limit as $x \to \infty$ gives the result. **b.** Multiply both sides of the given equation by $\frac{\log x}{x}$ and take the limit using (a).

6 **a.** Using $A(x) = \psi(x)$ and $a_n = \Lambda(n)$, Abel's summation formula gives $Z(s) = \frac{\psi(x)}{x^s} + s \int_1^\infty \frac{\psi(x)}{x^{s+1}}\, dx$ and the first term on the right goes to 0 as $x \to \infty$. **b.** Integration by parts gives $Z(s) = \frac{\psi_1(x)}{x^{s+1}}\Big|_1^\infty + s(s+1) \int_1^\infty \frac{\psi_1(x)}{x^{s+2}}\, dx$ and the result follows.

7 $s(x) < \sqrt{x}$, so $\frac{s(x)}{\pi(x)} = \frac{s(x)}{\frac{x}{\log x}} \frac{\frac{x}{\log x}}{\pi(x)} \le \frac{\log x}{\sqrt{x}} \frac{\frac{x}{\log x}}{\pi(x)} \to 0$ since $\frac{\frac{x}{\log x}}{\pi(x)} \to 1$ by the PNT.

8 see Jamieson p. 138

9 Look at the interval $[aq, bq]$ for sufficiently large prime q.

10 Using the product form for $\zeta(s)$ and taking logs we have $\log \zeta(s) = -\sum_p \log\left(\frac{p^s - 1}{p^s}\right)$. Now differentiate both sides w.r.t. s and the result follows.

11 Simple integration by parts.
