UNIVERSITY OF NEW SOUTH WALES.
SCHOOL OF MATHEMATICS AND STATISTICS
MATH5645
TOPICS IN ANALYTIC NUMBER THEORY

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5. PRIME NUMBER THEOREM:

1 **a.** Explain why $\psi(x)$ can be written in the form $\displaystyle\sum_{p\le x}\left[\frac{\log x}{\log p}\right]\log p$.

b. Deduce that $\psi(x) \le \pi(x)\log x$ and hence

$$\frac{\psi(x)}{x} \le \frac{\pi(x)}{\frac{x}{\log x}}.$$

c. Use Theorem 1.6 from section 1 to deduce that $\psi(x) \le 2x$.

***2** Suppose $x > 2$ and let m be the largest integer such that $2^m \le x$.

a. By considering $\psi(x) - \vartheta(x)$, show that $\psi(x) = \vartheta(x) + \vartheta(x^{\frac{1}{2}}) + \vartheta(x^{\frac{1}{3}}) + ... + \vartheta(x^{\frac{1}{m}})$.

b. Deduce that $\psi(x) \ge \vartheta(x)$ and so $\vartheta(x) \le 2x$.

c. Show that $\dfrac{\log x}{x^\alpha}$ has a maximum of $\frac{1}{\alpha e}$.

d. Deduce that $\psi(x) - \vartheta(x) \le 9x^{\frac{1}{2}}$.

e. Conclude that, as $x \to \infty$, $\dfrac{\psi(x)}{x} \to 1 \iff \dfrac{\vartheta(x)}{x} \to 1$.

3

a. Explain why $\psi(n)$ is the logarithm of the lowest common multiple of $2, 3, ..., n$.
b. Use this to find $\psi(10)$.

***4** In this question you may assume the following results from Sheet 1 Question 10.
Let n be an integer greater than 2. Write $N = \dbinom{2n+1}{2n}$. Let $p_{k+1}, p_{k+2}, ..., p_m$ be the primes p such that $n+2 \le p \le 2n+1$. We know that $N < 4^n$ and $\displaystyle\prod_{n+2\le p\le 2n+1} p < N$.

a. Prove that $\vartheta(2n+1) - \vartheta(n+1) \le n\log 4$.

b. By supposing that $\vartheta(k) \le k\log 4$ for all $k \le 2n$, $n > 1$ show that $\vartheta(2n+1) \le (2n+1)\log 4$.

c. Prove by induction that $\vartheta(n) \le n\log 4$ for all $n > 1$.

(Hence, since ϑ is increasing, we have $\vartheta(x) \le x\log 4$ for $x > 1$.)

5 **a.** Assuming that $\displaystyle\lim_{n\to\infty}\frac{\pi(x)\log x}{x} = 1$ show that $\displaystyle\lim_{n\to\infty}\frac{\log \pi(x)}{\log x} = 1$.

b. Deduce that $\displaystyle\lim_{n\to\infty}\frac{\pi(x)\log \pi(x)}{x} = 1$.

c. If p_n denotes the nth prime, show that the PNT implies $\displaystyle\lim_{n\to\infty}\frac{n\log n}{p_n} = 1$.

(This says that the nth prime is 'roughly' $n\log n$ for large n.)

***6** Assume that $\dfrac{\vartheta(x)}{x} \to 1$ as $x \to \infty$.

a. Prove that $\displaystyle\frac{\log x}{x}\int_2^x \frac{\vartheta(t)}{t(\log t)^2}\,dt \to 0$ as $x \to \infty$. (Hint: After replacing $\vartheta(x)$ as $O(x)$, split the integral over $[2, \sqrt{x}] \cup [\sqrt{x}, x]$.)

b. Use the formula $\displaystyle\pi(x) = \frac{\vartheta(x)}{logx} + \int_2^\infty \frac{\vartheta(t)}{t(\log t)^2}\,dt$ to prove that $\dfrac{\pi(x)\log x}{x} \to 1$ as $x \to \infty$.

7

a. If $f(s)$ has a pole of order k at $s = \alpha$, prove that $\dfrac{f'(s)}{f(s)}$ has a simple pole at $s = \alpha$ with residue $-k$.

b. Deduce that $F(s) = \dfrac{\zeta'(s)}{\zeta(s)} + \dfrac{1}{s-1}$ is analytic at $s = 1$.

8 Let $Z(s) = -\dfrac{\zeta'(s)}{\zeta(s)} = \displaystyle\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$.

a. Use Abel's summation formula to prove that

$$Z(s) = s \int_1^\infty \frac{\psi(x)}{x^{s+1}}\, dx$$

where $\psi(x) = \displaystyle\sum_{n \le x} \Lambda(n)$.

(Note: This is the *Mellin transform* if $\psi(x)$.)

b. Use integration by parts to show that

$$Z(s) = s(s+1) \int_1^\infty \frac{\psi_1(x)}{x^{s+2}}\, dx$$

where $\psi_1(x) = \displaystyle\int_1^\infty \psi(x)\, dx$.

9 Prove Bertrand's postulate assuming the PNT.

10 Let $s(x)$ denote the number of squares less or equal to x. Use the prime number theorem to prove that $\frac{s(x)}{\pi(x)}$ tends to 0 as $x \to \infty$. (This shows that the number of primes greatly exceeds the number of squares.)

11 Let $c > 1$

a. Prove that $\pi(cx) \sim c\pi(x)$ as $x \to \infty$.

b. Prove that $\pi(cx) - \pi(x) \sim (c-1)\dfrac{x}{\log x}$ as $x \to \infty$.

c. Deduce that for sufficiently large x, there are primes in the interval $(x, cx]$

12 Prove that every interval $[a, b]$, $0 < a < b$, contains a rational number $\dfrac{p}{q}$ with both p, q prime.

13 Prove that

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_p \frac{\log p}{p^s - 1},$$

for $s > 1$.

(Hint: Start with the Euler product for ζ.)

14 **a.** If $x \ge 2$, define the *logarithmic integral*,

$$Li(x) = \int_2^x \frac{dt}{\log t}.$$

Show that

$$Li(x) = \frac{x}{\log x} + \int_2^x \frac{dt}{(\log t)^2} - \frac{2}{\log 2}.$$

b. Show that

$$Li(x) = \frac{x}{\log x} + \frac{x}{(\log x)^2} + \ldots + (n-1)!\frac{x}{(\log x)^n} + r_{n+1}(x)$$

where $r_{n+1}(x) \sim n!\frac{x}{(\log x)^{n+1}}$ as $x \to \infty$.

15 It is an unsolved problem to prove that there is always a prime between n^2 and $(n+1)^2$. Use the PNT to show heuristically that there should be about $\pi(n)$ primes between n^2 and $(n+1)^2$.
For example, there are 7 primes between 400 and 441, while $\pi(20) = 8$.
(You may find that $\log(x+1) \sim \log(x) + 1/x$ is useful here.)