UNSW AUSTRALIA.
SCHOOL OF MATHEMATICS AND STATISTICS.
MATH5645: TOPICS IN NUMBER THEORY.

§4 CHARACTERS, L-SERIES and DIRICHLET'S THEOREM:

In Chapter 2 we used the properties of integers and primes to prove that certain arithmetic progressions contained infinitely many primes. These proofs relied on subtle observations and did not appear to generalise. We now seek to find a more general way of approaching such problems using analytic tools.

In order to accomplish this we need some way to *lift* the integers to the complex numbers. This is done via the notion of **characters** which are simply homomorphisms from a group G to $\mathbb{C}$. Although there is a general theory of characters for a general finite group, in what follows we will suppose that G is a finite **abelian** group.

Definition: A **character** is a function $\chi : G \to \mathbb{C}$, such that $\chi \neq 0$, and

$$\chi(xy) = \chi(x)\chi(y) \text{ for all } x, y \in G.$$

Ex: Suppose $G = C_n = \{1, x, \cdots, x^{n-1}\}$ with $x^n = 1$, and $\chi(x^a) = e^{\frac{2\pi i}{n}a}$.

Then $\chi(x^a)\chi(x^b) = e^{\frac{2\pi i(a+b)}{n}} = \chi(x^{a+b}) = \chi(x^a x^b)$. So χ is a character.

Ex: Given any finite abelian group G, define $\chi : G \to \mathbb{C}$ by $\chi(x) = 1$ for all $x \in G$.

χ is clearly a character and is called the **principal character** for G.

We usually write it as χ_1.

Theorem 4.1: If $\chi : G \to \mathbb{C}$ is a character then

(i) $\chi(1) = 1$

(ii) $\chi(x)$ is a root of unity and $(\chi(x))^{|G|} = 1$

(iii) $\chi(x^{-1}) = (\chi(x))^{-1} = \overline{\chi(x)}$.

Proof:

Theorem 4.2: If χ is a non-principal character then

$$\sum_{x \in G} \chi(x) = 0.$$

Proof:

Observe that for fixed G, the set of **all** characters on G is itself a finite abelian group under the operation

$$(\chi_1.\chi_2)(x) = \chi_1(x)\chi_2(x) \quad \text{for} \quad x \in G.$$

The principal character χ_1 is the identity of the group and for each χ, we have $\chi^{-1} = \overline{\chi}$.

Note also that if G is cyclic of order n then there are exactly n characters for G, since once we have specified the value of χ for a generator α of G, then, for some k, we have $\chi(x) = \chi(\alpha^k) = (\chi(\alpha))^k$.

More generally, given any finite abelian group G of order n, there are precisely n characters on G. (For a proof of this see the Tutorial problems.)

We will write $\hat{G}$ for the group of characters on G.

Recall that $\mathbb{U}_n$ denotes the set of units in $\mathbb{Z}_n$, i.e.

$$\mathbb{U}_n = \{x \in \mathbb{Z}_n : (x, n) = 1\}.$$

You will recall that these are simply the invertible elements in $\mathbb{Z}_n$, that they form a group under multiplication and that $|\mathbb{U}_n| = \phi(n)$, where ϕ is Euler's totient function. You should also recall that $\mathbb{U}_n$ is cyclic precisely when $n = 1, 2, 4, p^\alpha, 2p^\alpha$, where p is an odd prime and $\alpha \geq 1$.

A character can be completely specified by drawing up a character table.

Examples:

1. $G = \mathbb{U}_4 = \{1, 3\}$. $\hat{G} = \{\chi_1, \chi_2\}$ with character table

	1	3
χ_1		
χ_2		

2. $G = \mathbb{U}_5 = \{1, 3, 4, 2\}$. $\hat{G} = \{\chi_1, \chi_2, \chi_3, \chi_4\}$.

Note that 3 is a generator (hence the order in which I have listed the elements).

Since the group is cyclic, if χ is not principal, then $\chi(3)$ is any of the 4 fourth roots of unity. Once $\chi(3)$ is decided, then the remainder of the row is determined.
The character table is

	1	3	4	2
χ_1				
χ_2				
χ_3				
χ_4				

3. $G = \mathbb{U}_8 = \{1, 3, 5, 7\} \cong C_2 \times C_2$. Each (non-identity) element is of order 2 and $3 \times 5 \equiv 7$. Hence the character table is:

	1	3	5	7
χ_1				
χ_2				
χ_3				
χ_4				

(The character tables for $\mathbb{U}_9$ and $\mathbb{U}_{16}$ are in the Tutorial problems.)

4. $G = \mathbb{U}_7 =< 3 >= \{1, 3, 2, 6, 4, 5\}$. 3 is a generator so $\chi(3)$ can be any 6th root of unity.
Let $\omega = e^{\frac{i\pi}{3}}$. The character table is:

	1	3	2	6	4	5
χ_1	1	1	1	1	1	1
χ_2	1	-1	1	-1	1	-1
χ_3	1	ω	ω^2	-1	$-\omega$	$-\omega^2$
χ_4	1	$-\omega$	ω^2	1	$-\omega$	ω^2
χ_5	1	ω^2	$-\omega$	1	ω^2	$-\omega$
χ_6	1	$-\omega^2$	$-\omega$	-1	ω^2	ω

Direct Products:

It is very easy to write down the character table for cyclic groups.

If G is a cyclic group of order m then $\hat{G}$ has exactly m characters given by

$$\chi_\rho(x^i) = \rho^i$$

where ρ is any m-th root of unity.

For non-cyclic abelian groups, we recall the Fundamental Theorem of Abelian groups which states that every finite abelian group is a direct product (or written additively, a direct sum) of cyclic groups.

Suppose $G = A_1 \otimes A_2 \otimes \cdots \otimes A_r$, where each A_i is a cyclic group of order n_i, and $A_i = \langle a_i \rangle$. For each i, select a complex n_i-th root of unity in $\mathbb{C}$, ρ_i.
Define $\chi_{\rho_i} : G \to \mathbb{C}$ by

$$\chi_{\rho_i}(a_1^{\alpha_1}, a_2^{\alpha_2}, \ldots, a_i^{\alpha_i}, \ldots, a_r^{\alpha_r}) = \rho_i^{\alpha_i}.$$

For each choice of ρ_i, for $i = 1, 2, \ldots, r$ we obtain a character

$$\chi = \chi_{\rho_1}\chi_{\rho_2}\cdots\chi_{\rho_r}$$

There are n_i choices for ρ_i and so this gives $n_1 n_2 \cdots n_r = |G|$ such characters.
The fact that there are no others requires a little proof. (Tutorial problem.)

The smallest 'interesting' example is:

5. $G = \mathbb{U}_{35} \cong \mathbb{U}_5 \otimes \mathbb{U}_7$.
Now $\mathbb{U}_5 = \langle 3 \rangle = \{1, 3, 4, 2\}$ and $\mathbf{U_7} = \langle 3 \rangle = \{1, 3, 2, 6, 4, 5\}$. Let ρ be a 4th root of unity (so $\rho \in \{1, -1, i, -i\}$), and σ be a 6th root of unity. We can write the valuations of the group as follows:

$\mathbb{U}_{35}$	$\cong \mathbb{U}_5 \otimes \mathbb{U}_7$	χ_ρ	ψ_σ	$\chi_\rho\psi_\sigma$
1	$(1,1)$	1	1	1
2	$(2,2)$	ρ^3	σ^2	$\rho^3\sigma^2$
3	$(3,3)$	ρ	σ	$\rho\sigma$
4	$(4,4)$	ρ^2	σ^4	$\rho^2\sigma^4$
6	$(1,6)$	1	σ^3	σ^3
8	$(3,1)$	ρ	1	ρ
9	$(4,2)$	ρ^2	σ^2	$\rho^2\sigma^2$
11	$(1,4)$	1	σ^4	σ^4
12	$(2,5)$	ρ^3	σ^5	$\rho^3\sigma^5$
13	$(3,6)$	ρ	σ^3	$\rho\sigma^3$
16	$(1,2)$	1	σ^2	σ^2
17	$(2,3)$	ρ^3	σ	$\rho^3\sigma$
18	$(3,4)$	ρ	σ^4	$\rho\sigma^4$
19	$(4,5)$	ρ^2	σ^5	$\rho^2\sigma^5$
22	$(2,1)$	ρ^3	1	ρ^3
23	$(3,1)$	ρ	σ^2	$\rho\sigma^2$
24	$(4,3)$	ρ^2	σ	$\rho^2\sigma$
26	$(1,5)$	1	σ^5	σ^5
27	$(2,6)$	ρ^3	σ^3	$\rho^3\sigma^3$
29	$(4,1)$	ρ^2	1	ρ^2
31	$(1,3)$	1	σ	σ
32	$(2,4)$	ρ^3	σ^4	$\rho^3\sigma^4$
33	$(3,5)$	ρ	σ^5	$\rho\sigma^5$
34	$(4,6)$	ρ^2	σ^3	$\rho^2\sigma^3$

There are 4 choices of ρ and 6 choices for σ giving 24= $\phi(35)$ characters.

Orthogonality Relations.

A quick glance at some of the smaller examples, suggests that the rows and columns satisfy some nice orthogonality relations.

Theorem 4.3:

(a) (Rows) If ϕ, χ are characters of G then

$$\sum_{x \in G} \phi(x)\overline{\chi(x)} = \begin{cases} |\mathrm{G}| & \text{if } \phi = \chi \\ 0 & \text{if } \phi \neq \chi \end{cases}$$

(b) (Columns) If $x, y \in G$ then

$$\sum_{\chi \in \hat{G}} \chi(x)\overline{\chi(y)} = \begin{cases} |\mathrm{G}| & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

Proof:

(a)

(b) Let $A = \sum_{\chi \in \hat{G}} \chi(x)\overline{\chi(y)} = \sum_{\chi \in \hat{G}} \chi(x)\chi(y^{-1}) = \sum_{\chi \in \hat{G}} \chi(xy^{-1})$.
Now if $x \neq y$ then $xy^{-1} \neq 1$ so there is a $\psi \in \hat{G}$ such that $\psi(xy^{-1}) \neq 1$. Hence

$$A\psi(xy^{-1}) = \sum_{\chi \in \hat{G}} \chi(xy^{-1})\psi(xy^{-1})$$

$$= \sum_{\chi \in \hat{G}} (\chi\psi)(xy^{-1}).$$

Now as χ runs through $\hat{G}$ so does $\chi\psi$, thus

$$A\psi(xy^{-1}) = \sum_{\chi \in \hat{G}} \chi(xy^{-1}) = A.$$

Now $\psi(xy^{-1}) \neq 1$ so $A = 0$.

Finally, if $x = y$ then $xy^{-1} = 1$ so $\chi(xy^{-1}) = 1$ giving $\sum_{\chi \in \hat{G}} \chi(xy^{-1}) = |\hat{G}| = |G|$.

Dirichlet Characters: We now try to lift the integers into $\mathbb{C}$ by using the fact that $\mathbb{Z}$ can be reduced to $\mathbb{Z}_n$ (by reading each integer modulo n) and then ignoring those x for which $(x, n) \neq 1$.

Definition: Suppose $\chi : \mathbb{U}_n \to \mathbb{C}$ is a character for the group $\mathbb{U}_n$.
We define a **Dirichlet character** χ^o modulo n, $\chi^o : \mathbb{Z} \to \mathbb{C}$, by

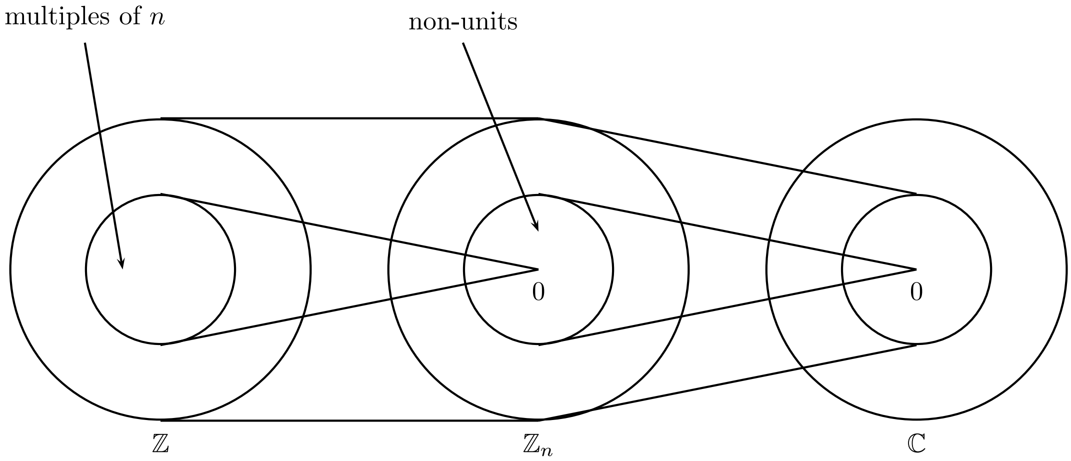
$$\chi^o(x) = \begin{cases} \chi(\hat{x}) & \text{if } (x, n) = 1 \\ 0 & \text{if } (x, n) \neq 1 \end{cases}$$

where $\hat{x} \in \mathbb{Z}_n$ and $\hat{x} \equiv x \bmod n$.

The **principal Dirichlet character**, χ_1^o is defined by

$$\chi_1^o(x) = \begin{cases} 1 & \text{if } (x, n) = 1 \\ 0 & \text{if } (x, n) \neq 1 \end{cases}$$

In simple terms χ^o maps $\mathbb{Z}$ into $\mathbb{Z}_n$, (in the usual way), kills all the non-units and maps the units into $\mathbb{C}$ via the character χ.



Note also that a Dirichlet character modulo n satisfies

$$(i) \quad \chi^o(xy) = \chi^o(x)\chi^o(y) \text{ for all } x, y \in \mathbb{Z}$$

$$(ii) \quad \chi^o(x + n) = \chi^o(x) \text{ for all integers } x.$$

Hence χ^o is completely multiplicative and periodic of period n.

Moreover, there are $\phi(n)$ distinct characters for $\mathbb{U}_n$, so it is clear that there will be $\phi(n)$ distinct Dirichlet characters modulo n.

At the risk of confusion, we will drop the ^{o} and refer to a Dirichlet character modulo n as simply a **character** χ **modulo** n**.**

Note that Theorem 4.3(b) can now be written as

$$\sum_{r=1}^{\phi(n)} \chi_r(x)\overline{\chi_r(y)} = \begin{cases} \phi(n) & \text{if } x \equiv y \bmod n \\ 0 & \text{otherwise.} \end{cases}$$

L- functions and the Generalised Euler Product:

Dirichlet generalised the zeta function and found an analogue to the Euler product,

$$\prod_{p\ prime} \left(1 - \frac{1}{p^s}\right)^{-1} = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

From now on, we will take s to be a complex variable. Hence the above series converges for $\sigma = \Re(s) > 1$.

Using this, Dirichlet was able to generalise the proof of the divergence of $\displaystyle\sum_{p \text{ prime}} \frac{1}{p}$, from the set of all primes, to sets of primes of a certain shape.

Given a Dirichlet character χ modulo n, we define the function $L(s, \chi)$, called an L-function for χ, by

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

Now since $\displaystyle\left|\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}\right| \leq \sum_{n=1}^{\infty} \frac{1}{n^\sigma}$, the above series converges uniformly on any compact subset of the region $\Re(s) > 1$.

Now consider the product, for $\Re(s) > 1$,

$$\left(1 + \frac{\chi(2)}{2^s} + \frac{\chi(2^2)}{2^{2s}} + \cdots\right)\left(1 + \frac{\chi(3)}{3^s} + \frac{\chi(3^2)}{3^{2s}} + \cdots\right)\left(1 + \frac{\chi(5)}{5^s} + \cdots\right)\left(1 + \frac{\chi(7)}{7^s} + \cdots\right)\cdots$$

$$= \prod_{p \text{ prime}} \frac{1}{1 - \frac{\chi(p)}{p^s}}.$$

Expanding out the brackets and using the fact that χ is complelely multiplicative, we obtain

$$1 + \frac{\chi(2)}{2^s} + \frac{\chi(3)}{3^s} + \frac{\chi(4)}{4^s} + \frac{\chi(5)}{5^s} + \cdots$$

$$= L(s, \chi).$$

Thus we have, for $\Re(s) > 1$,

$$L(s, \chi) = \prod_{p \text{ prime}} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

A Warm up to Dirichlet's Theorem:

Rather than launch into a proof of Dirichlet's theorem, let us take some time to see the key ideas displayed in some examples. In particular, we return to the problem of showing that there are infinitely many primes congruent to 1 mod 4 and infinitely many primes congruent to -1 mod 4.

Dirichlet Density:

Definition: A set of positive primes $\mathcal{P}$ is said to have Dirichlet density $d(\mathcal{P})$ if

$$\lim_{s\to 1^+} \frac{\sum_{p\in\mathcal{P}} \frac{1}{p^s}}{\log\left(\frac{1}{s-1}\right)}$$

exists.

Notes:
1. This rather 'strange' definition is motivated by the fact that

$$\lim_{s\to 1^+} \frac{\log\zeta(s)}{\log\left(\frac{1}{s-1}\right)} = 1.$$

(see the assignment for the next chapter), so the set of all primes has Dirichlet density 1, as we would expect.

2. If $\mathcal{P}$ is any finite set of primes then $d(\mathcal{P}) = 0$ and if $\mathcal{P}$ contains all but a finite number of primes then $d(\mathcal{P}) = 1$.

3. The more natural definition of density would be to take the number of primes in $\mathcal{P}$ less than some number N, divide by $\pi(N)$ and take a limit as $N \to \infty$. This *natural* density, when it exists can be shown to be the same as the Dirichlet density, but there are sets which do not have natural density for which the Dirichlet density does exist. (For example, the set of all primes whose first digit is 1.)

4. Dirichlet proved that if $(a, b) = 1$ and $\mathcal{P} = \{$ primes $\equiv b \bmod a\}$, then $d(\mathcal{P}) = \frac{1}{\phi(a)}$. We will show this later on.

Hence in the above example, if $\mathcal{P} = \{$ primes $\equiv 1 \bmod 4\}$, then $d(\mathcal{P}) = \frac{1}{2}$.

To verify that this is correct, we note that (with χ_1 and χ_2 as in the above example),

$$L(s,\chi_1) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \cdots = \zeta(s) - \left(\frac{1}{2^s} + \frac{1}{4^s} + \cdots\right)$$

$$= \zeta(s) - \frac{1}{2^s}\left(1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots\right) = \left(1 - \frac{1}{2^s}\right)\zeta(s).$$

Hence $\log L(s,\chi_1) = \log\left(1 - \frac{1}{2^s}\right) + \log\zeta(s)$.

We saw above that

$$\log L(s,\chi_2) + \log L(s,\chi_1) = 2\sum_{p\equiv 1 \bmod 4} \frac{1}{p^s} + R_3(s)$$

so

$$d(\mathcal{P}) = \frac{1}{2}\lim_{s\to 1^+} \frac{\log L(s,\chi_2) + \log L(s,\chi_1) - R_3(s)}{\log\left(\frac{1}{s-1}\right)}$$

$$= \frac{1}{2}\lim_{s\to 1^+} \frac{\log(1-\frac{1}{2^s}) + \log\zeta(s) + \log L(s,\chi_2) - R_3(s)}{\log\left(\frac{1}{s-1}\right)} = \frac{1}{2}.$$

Another Example:

We now look at the primes in the congruence classes modulo 8. The character table for $\mathbb{U}_8$ is

	1	3	5	7
χ_1	1	1	1	1
χ_2	1	-1	1	-1
χ_3	1	1	-1	-1
χ_4	1	-1	-1	1

We define a Dirichlet character mod 8 by

$$\chi_i^o(n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ \chi_i(\hat{n}) & \text{if } n \text{ is odd} \end{cases}$$

and as usual we will immediately drop the ^{0}. Since the characters are real, we will take s real and $s > 1$.

As before, define $L(s, \chi_i) = \displaystyle\sum_{n=1}^{\infty} \frac{\chi_i(n)}{n^s}$ and as in the previous example

$$\log L(s, \chi_i) = \sum_{p \text{ odd prime}} \frac{\chi_i(p)}{p^s} + R_i(s)$$

for $i = 1, 2, 3, 4$, where $R_i(s)$ is bounded as $s \to 1^+$.

Now observe that

$$(\chi_1 + \chi_2 + \chi_3 + \chi_4)(n) = \begin{cases} 0 & \text{if } n \not\equiv 1 \bmod 8 \\ 4 & \text{if } n \equiv 1 \bmod 8 \end{cases}$$

$$(\chi_1 - \chi_2 + \chi_3 - \chi_4)(n) = \begin{cases} 0 & \text{if } n \not\equiv 3 \bmod 8 \\ 4 & \text{if } n \equiv 3 \bmod 8 \end{cases}$$

$$(\chi_1 + \chi_2 - \chi_3 - \chi_4)(n) = \begin{cases} 0 & \text{if } n \not\equiv 5 \bmod 8 \\ 4 & \text{if } n \equiv 5 \bmod 8 \end{cases}$$

$$(\chi_1 - \chi_2 - \chi_3 + \chi_4)(n) = \begin{cases} 0 & \text{if } n \not\equiv 7 \bmod 8 \\ 4 & \text{if } n \equiv 7 \bmod 8 \end{cases}$$

Thus,

$$\log L(s, \chi_1) + \log L(s, \chi_2) + \log L(s, \chi_3) + \log L(s, \chi_4) = 4 \sum_{p \equiv 1 \bmod 8} \frac{1}{p^s} + A(s)$$

where $A(s)$ is bounded as $s \to 1^+$,

$$\log L(s, \chi_1) - \log L(s, \chi_2) + \log L(s, \chi_3) - \log L(s, \chi_4) = 4 \sum_{p \equiv 3 \bmod 8} \frac{1}{p^s} + B(s)$$

where $B(s)$ is bounded as $s \to 1^+$, and so on for the other sums.

Now once again $L(s, \chi_1) = \displaystyle\sum_{n=0}^{\infty} \frac{1}{(2n+1)^s}$ diverges to ∞ as $s \to 1^+$. We need to show that all the other L-functions are bounded and away from 0 as $s \to 1^+$. This is left as an exercise. It then follows that there are infinitely many primes congruent to 1 mod 8, to 3 mod 8 etc.

Outline of the General Proof:

The general result, known as Dirichlet's theorem, states that there are infinitely many primes congruent to b mod a, whenever $(a, b) = 1$.

The key steps in proving this result are to show that:

- $L(s, \chi)$ is bounded away from 0 as $s \to 1^+$, for each non-principal character χ.

  This is done by showing that the series for $L(s, \chi)$ converges for $Re(s) > 0$, and (more difficult) also showing that $L(1, \chi) \neq 0$.

- $L(s, \chi_1)$ diverges as $s \to 1^+$.

  This is done by extending the zeta function to an analytic function, valid for $Re(s) > 0$, with a simple pole at $s = 1$, and relating $L(s, \chi_1)$ to this new function.

- For each b, $\displaystyle\sum_{p \equiv b \text{ mod a}} \frac{1}{p^s}$, where p is prime, is some linear combination of the logs of these L-functions.

  This is done using the orthogonality properties of characters.

Then, as $s \to 1^+$, it follows that this sum diverges, so there are infinitely many primes in the arithmetic progression $\{b + ka : k \in \mathbb{Z}\}$.

One may compare this idea with Euler's proof that $\displaystyle\sum \frac{1}{p}$ diverges, thus showing the infinitude of ordinary primes.

In the previous examples, the characters involved were real-valued. In general, this is, of course, not the case and so we will often need to think of s as a complex variable.

Some Notes on Complex Function Theory:

Since we will now regard s as a complex variable, we need a few results and ideas from complex function theory.

Weierstrass M-test and Uniform Convergence:

Recall from analysis the Weierstrass M-test.

Let $\{a_j(z)\}$ be a sequence of functions of a complex variable z.

Suppose $|a_j(z)| \leq M_j$ in some region G, where the M_j's are constants independent of z, and suppose

$$\sum_{j=1}^{\infty} M_j < \infty,$$

then the series $\displaystyle\sum_{j=1}^{\infty} a_j(z)$ converges uniformly in G.

Furthermore, if each $a_j(z)$ is analytic in G and the series $\displaystyle\sum_{j=1}^{\infty} a_j(z)$ converges uniformly in G, then this series represents an analytic function.

We can define the complex zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$.

Note that if we write $s = \sigma + it$, then $\left|\frac{1}{n^s}\right| = \frac{1}{n^\sigma}$.

If we fix $\sigma, (\sigma > 1)$, then, for each fixed σ,

$$\sum_{n=1}^{\infty} \frac{1}{n^\sigma} < \infty.$$

It then follows, by the Weierstrass M test, that on any compact subset of the region $\sigma > 1$, the series converges uniformly and $\zeta(s)$ defines an analytic function in the complex plane which can be differentiated term by term as often as we please.

Analytic Continuation:

1. If two entire functions, f and g, agree on some open interval on the real axis, (no matter how small), then f and g agree everywhere in the complex plane.

This can be generalised as follows: Suppose two functions f and g are analytic in some domain D, which contains a portion of the real axis. If $f = g$ on the real axis then $f = g$ everywhere in D. We can use this to prove identities by specialising them to the real axis.

For example, since $\sin 2x = 2 \sin x \cos x$ for $x \in \mathbb{R}$, it follows that $\sin 2z = 2 \sin z \cos z$ for all $z \in \mathbb{C}$.

2. Consider the function $f(z) = 1 + z + z^2 + \cdots$.

This series converges (uniformly) in the open disc $|z| < 1$, but not outside this disc. However, for $|z| < 1$ we can sum this series to obtain $\frac{1}{1-z}$.

If we write $g(z) = \frac{1}{1-z}$, then g is a meromorphic function with a simple pole at $z = 1$ (and residue -1). Moreover, for $|z| < 1$, we have $f(z) = g(z)$.

We say that g is the **analytic continuation** of f with a simple pole at $z = 1$.

Given a function f which is analytic in some domain D, then if there is a meromorphic function g, defined on $E \supset D$, which agrees with f in D, then we say that g is the **analytic continuation** of f and such a g, if it exists, is unique.

In this section we will see how to analytically continue the zeta function from $\sigma > 1$ to $\sigma > 0$. In the next chapter, we will analytically continue ζ to the whole of the complex plane.

We now embark on the proof of Dirichlet's theorem, which requires a number of technical results.

The Proof of Dirichlet's Theorem:

Fix a, a positive integer greater than 1, once and for all. Let $\chi_i : \mathbb{Z}_a \to \mathbb{C}$ be a character and, as above, define the Dirichlet character $\chi_i^o : \mathbb{Z} \to \mathbb{C}$, for $i = 1, 2, \cdots, \phi(a)$ by

$$\chi_i^o(n) = \begin{cases} 0 & \text{if } (n,a) \neq 1 \\ \chi_i(\hat{n}) & \text{if } (n,a) = 1,\ \hat{n} \equiv n \bmod a \end{cases}$$

and as usual we will immediately drop the ^{o}.

As before, define the Dirichlet L-function by $L(s,\chi) = \displaystyle\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$, where we will take s to be a complex variable whose real part is greater than 1. By the Weierstrass M-test, $L(s,\chi)$, converges absolutely and uniformly to an analytic function in this region.

Using the multiplicative property of χ we have, (as before):

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

Abel's Summation Formulae:

a. Suppose $\{a_n\}$ and $\{b_n\}$ are sequences of complex numbers and let $A_n = a_1 + a_2 + \cdots + a_n$, then for $n \geq 0$, (setting $A_0 = 0$), and for $1 \leq M < N$, we have

$$\sum_{n=M}^{N} a_n b_n = \sum_{n=M}^{N-1} A_n(b_n - b_{n+1}) + A_N b_N - A_{M-1} b_M.$$

b. Suppose further that $\displaystyle\sum_{n=1}^{\infty} a_n b_n$ converges and that $A_n b_n \to 0$ as $n \to \infty$, then

$$\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} A_n(b_n - b_{n+1}).$$

Proof:
a. The steps are similar to the proof of (b) which follows.

b. Let $S_k = \displaystyle\sum_{n=1}^{k} a_n b_n$ and set $A_0 = 0$ then

$$\begin{aligned} S_k &= \sum_{n=1}^{k} (A_n - A_{n-1}) b_n = \sum_{n=1}^{k} A_n b_n - \sum_{n=1}^{k} A_{n-1} b_n \\ &= \sum_{n=1}^{k} A_n b_n - \sum_{n=1}^{k-1} A_n b_{n+1} \qquad \text{(since } A_0 = 0\text{)} \\ &= A_k b_k + \sum_{n=1}^{k-1} A_n(b_n - b_{n+1}) \to \sum_{n=1}^{\infty} A_n(b_n - b_{n+1}) \end{aligned}$$

as $k \to \infty$.
(Note that (a) is simply a form of 'summation by parts'.)

We now show that $\zeta(s) - \frac{1}{s-1}$ can be continued to an analytic function on the region $S = \{s \in \mathbb{C} : Re(s) > 0\}$.

This will follow from:

Theorem 4.4:

For $\Re(s) > 1$,

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{\{x\}}{x^{s+1}}\, dx$$

where $\{x\}$ denotes the fractional part of x.

Proof:

Now for $\sigma > 1$, $\zeta(s)$ agrees with $\dfrac{s}{s-1} - s\displaystyle\int_1^\infty \frac{\{x\}}{x^{s+1}}\, dx$ and so we can re-define $\zeta(s)$ to be $\dfrac{s}{s-1} - s\displaystyle\int_1^\infty \frac{\{x\}}{x^{s+1}}\, dx$ which is valid for $\sigma > 0, s \neq 1$.

In other words $\zeta(s)$ has an analytic continuation into the region $\sigma > 0$ and this new zeta function has a simple pole at $s = 1$.

We now try to do the same thing for $L(s, \chi)$, where $\chi \neq \chi_1$.

Lemma:
Let χ be a non-trivial character modulo a, then for all $N > 0$ we have

$$\left|\sum_{n=0}^{N} \chi(n)\right| \leq \phi(a).$$

Proof: For $\chi \neq \chi_1$, write $N = aq + r, 0 \leq r < a$, and note that $\chi(n + a) = \chi(n)$ for all positive integers n. Hence

$$\sum_{n=1}^{N} \chi(n) = q \sum_{n=0}^{a-1} \chi(n) + \sum_{n=1}^{r} \chi(n) = \sum_{n=1}^{r} \chi(n)$$

since the first sum is 0. Thus,

$$\left|\sum_{n=1}^{N} \chi(n)\right| = \left|\sum_{n=1}^{r} \chi(n)\right| \leq \sum_{n=1}^{a-1} |\chi(n)| = \phi(a).$$

Theorem 4.5:

If χ is not principal then the series for $L(s, \chi)$ is convergent for $Re(s) > 0$ and the sum is analytic in that region.

Thus, for $\chi \neq \chi_1$, $L(s, \chi)$ is bounded as $s \to 1^+$.

Proof:

Theorem 4.6:

For the principal character χ_1, $L(s, \chi_1)$ extends to a meromorphic function for $Re(s) > 0$, with a simple pole at $s = 1$, and

$$L(s, \chi_1) = \zeta(s) \prod_{p|a} \left(1 - \frac{1}{p^s}\right).$$

Hence, $L(s, \chi_1)$ **diverges** as $s \to 1^+$.

Proof:

The most difficult part of the proof of Dirichlet's theorem, is to show that for $\chi \neq \chi_1$, $L(1, \chi) \neq 0$ so its logarithm is finite. That is, $L(s, \chi)$ is bounded away from 0 as $s \to 1^+$.
We will assume this result for the time being and finish the proof of the theorem.

Theorem 4.7: If χ is not principal then $L(1, \chi) \neq 0$.

Theorem 4.8: (Dirichlet's Theorem.)
If a, b are relatively prime integers with $a > 0$ then there exist infinitely many primes of the form $ka + b$ with k a positive integer.

Proof: We need to show that for each non-principal Dirichlet character χ mod a, we have

$$\log L(s, \chi) = \sum_{p \text{ prime}} \frac{\chi(p)}{p^s} + R(s, \chi)$$

where $R(s, \chi)$ is bounded as $s \to 1^+$.

Even if we restrict s to be real, the values of $L(s,\chi)$ are in general complex numbers so it is necessary to worry about the fact that $\log z$ is multivalued in the complex plane. One way around this is to define $\log L(s,\chi)$ by an infinite series.

Let χ be a Dirichlet character and s be real. We define $G(s,\chi)$ by

$$G(s,\chi) = \sum_p \sum_{k=1}^{\infty} \frac{\chi(p^k)}{kp^{ks}}.$$

Note that this is simply the Taylor series of $\log \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}$.

Since $\left|\frac{\chi(p^k)}{kp^{ks}}\right| \leq \frac{1}{p^{ks}}$, so $|G(s,\chi)| < \sum_p \frac{1}{p^s - 1} < 2\zeta(s)$ and since $\zeta(s)$ converges for $s > 1$, the same is true for $G(s,\chi)$. Thus $G(s,\chi)$ is continuous for $s > 1$.

Now if z is complex with $|z| < 1$ then $exp\left(\sum_{k=1}^{\infty} \frac{z^k}{k}\right) = \frac{1}{1-z}$.

Substituting $z = \chi(p)p^{-s}$ we have

$$\exp\left(\sum_{k=1}^{\infty} \frac{\chi(p^k)}{kp^{ks}}\right) = \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

Hence

$$exp\left(\sum_p \sum_{k=1}^{\infty} \frac{\chi(p^k)}{kp^{ks}}\right) = \prod_p exp\left(\sum_{k=1}^{\infty} \frac{\chi(p^k)}{kp^{ks}}\right) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} = L(s,\chi)$$

for $s > 1$.
Thus $G(s,\chi)$ provides an unambiguous definition for $\log L(s,\chi)$.

Moreover, for $\chi \neq \chi_1$, $L(s,\chi)$ is bounded as $s \to 1^+$ so $G(s,\chi)$ is bounded as $s \to 1^+$ and, $L(1,\chi) \neq 0$ tells us that $G(s,\chi)$ is defined at $s = 1$.

Now for any character χ, each of the terms in the inner sum of $G(s,\chi)$ satisfies,

$$\sum_{k=1}^{\infty} \frac{\chi(p^k)}{kp^{ks}} = \frac{\chi(p)}{p^s} + \frac{1}{2}\frac{\chi(p^2)}{p^{2s}} + \cdots$$

$$= \frac{\chi(p)}{p^s} + r(s,p,\chi).$$

where,

$$|r(s,p,\chi)| = \left|\frac{1}{2}\frac{\chi(p^2)}{p^{2s}} + \frac{1}{3}\frac{\chi(p^3)}{p^{3s}} \cdots\right|$$

$$\leq \frac{1}{2p^{2s}} + \frac{1}{3p^{3s}} + \cdots$$

$$\leq \frac{1}{2p^{2s}}\left(1 + \frac{1}{p^s} + \cdots\right)$$

$$= \frac{1}{2p^s(p^s - 1)}.$$

Hence,

$$|R(s,\chi)| = \left|\sum_p r(s,p,\chi)\right| \leq \sum_p \frac{1}{2p^s(p^s-1)} \leq \frac{1}{2}\sum_p \frac{1}{p(p-1)} < \frac{1}{2}\sum_{n=2}^{\infty} \frac{1}{n(n-1)} = \frac{1}{2}.$$

Thus $R(s,\chi)$ is bounded as $s \to 1^+$.

We can thus write $G(s,\chi) = \sum_p \dfrac{\chi(p)}{p^s} + R(s,\chi)$, with the last term bounded as $s \to 1^+$.

The next step in the proof is to pick out the primes congruent to b mod a.

Multiplying the above formula for $G(s,\chi)$ by $\overline{\chi(b)}$ and summing over χ, we obtain

$$\sum_\chi \sum_{p \text{ prime}} \frac{\chi(p)\overline{\chi(b)}}{p^s} = \sum_\chi \overline{\chi(b)} G(s,\chi) - \sum_\chi \overline{\chi(b)} R(s,\chi).$$

Now reversing the order of summation on the left hand side and using the orthogonality relation in Theorem 4.3b, we have

$$\phi(a) \sum_{\substack{p \text{ prime} \\ p \equiv b \bmod a}} \frac{1}{p^s} = \sum_\chi \overline{\chi(b)} G(s,\chi) - \sum_\chi \overline{\chi(b)} R(s,\chi).$$

We have shown that the (finite) sum $\sum_\chi \overline{\chi(b)} R(s,\chi)$ is bounded as $s \to 1^+$.

Now $\sum_\chi \overline{\chi(b)} G(s,\chi) = \sum_{\chi \neq \chi_1} \overline{\chi(b)} G(s,\chi) + G(s,\chi_1)$ with the first sum defined and bounded as $s \to 1^+$. Also by Theorem 4.6, $\log L(s,\chi_1)$ is unbounded as $s \to 1^+$, thus $G(s,\chi_1)$ is unbounded as $s \to 1^+$ and it follows that the series

$$\sum_{\substack{p \text{ prime} \\ p \equiv b \bmod a}} \frac{1}{p^s}$$

diverges as $s \to 1^+$, giving the desired result.

Dirichlet's Density Formula:

Recall that we defined the Dirichlet density for a set of primes $\mathcal{P}$ as

$$\lim_{s \to 1^+} \frac{\sum_{p \in \mathcal{P}} \frac{1}{p^s}}{\log\left(\frac{1}{s-1}\right)}$$

whenever it exists. Take s real.

From the above, if we take $\mathcal{P} = \{$ primes $\equiv b \bmod a\}$, and recall that $L(s,\chi_1) = \zeta(s) \prod_{p|a} \left(1 - \dfrac{1}{p^s}\right)$ so that $\log L(s,\chi_1) = \log \zeta(s) + \log \prod_{p|a} \left(1 - \dfrac{1}{p^s}\right)$ then

$$d(\mathcal{P}) = \lim_{s \to 1^+} \frac{\sum_{p \equiv b \bmod a} \frac{1}{p^s}}{\log(\frac{1}{s-1})}$$

$$= \frac{1}{\phi(a)} \lim_{s \to 1^+} \frac{\log \zeta(s) + \log \prod_{p|a} \left(1 - \frac{1}{p^s}\right) + \sum_{\chi \neq \chi_1} \overline{\chi(b)} G(s,\chi) - \sum_\chi \overline{\chi(b)} R(s,\chi)}{\log(\frac{1}{s-1})}$$

$$= \frac{1}{\phi(a)},$$

since the last three terms in the numerator above are bounded as $s \to 1^+$.

Proof of Theorem 4.7:

Dirichlet used some deep results from the theory of quadratic forms to prove this result. These results were connected with the class numbers for quadratic extensions of $\mathbb{Q}$. Given a quadratic form $ax^2 + bxy + cy^2$, where a, b, c are integers, we define the discriminant by $d = b^2 - 4ac$. For fixed d, we take all the possible quadratic forms and look at the set of integers generated by these forms as x, y take all integer values. This partitions the forms into equivalence classes, and the number of such equivalence classes is called the *class number* corresponding to d. It is not at all obvious *a priori* that the class number is even finite, but this was shown to be true by Gauss. For example, the quadratics $x^2 + y^2$ and $x^2 + 2xy + 2y^2$ both have discriminant -4, and they represent the same integers, while $x^2 + 5y^2$ and $2x^2 + 2xy + 3y^2$ both have discriminant -20, but are inequivalent since the latter represents and 3 and 7, which are not representable by the first. Hence the class number of -20 is at least 2. Dirichlet showed that if $p > 3$ is a prime then the class number is a multiple of $L(1, \chi)$ and since the class number is at least one, then $L(1, \chi)$ is not zero.

Case 1: χ is real:

All known proofs of this are difficult. The proof given here is the easiest one to follow and is taken from Jamieson's book *The Prime Number Theorem.* Another proof is given in the Appendix at the end of this chapter.

We begin with some Lemmas.

Lemma 1: Let χ be a real character. Then for all n,

$$(\chi * u)(n) \geq 0 \qquad \text{and } (\chi * u)(n^2) \geq 1.$$

Proof: Since χ and u are multiplicative, so is $\chi * u$. Hence it suffices to show that if p is prime, then $(\chi * u)(p^n) \geq 0$ for all n and that $(\chi * u)(p^n) \geq 1$ if n is even. Now

$$(\chi * u)(p^n) = \sum_{d|p^n} \chi(d) u\left(\frac{p^n}{d}\right) = 1 + \sum_{r=1}^{n} (\chi(p))^r .$$

Hence $(\chi * u)(p^n) = n + 1$ if $\chi(p) = 1$ and 1 if $\chi(p) = 0$, while if $\chi(p) = -1$, the Dirichlet product takes the value 0 when n is odd and 1 when n is even.

Lemma 2: Let $g(x) = \dfrac{1}{x} - \dfrac{1}{e^x - 1}$.

Then $g(x)$ is decreasing and $0 < g(x) < \frac{1}{2}$ for $x > 0$.

Proof: A MAPLE plot is convincing, but a formal proof is in the Tutorial problems.

Given a non-principal character χ, defined on $\mathbb{U}_k$, we will write, for $x \geq 1$,

$$S_\chi(x) = \sum_{1 \leq r \leq x} \chi(r) \text{ and } M_\chi = \sup_{x \geq 1} |S_\chi(x)|.$$

That such an M_χ exists follows from the bound $M_\chi \leq \frac{\phi(k)}{2}$.
This is done in the Tutorial problems.

Main result:

If χ is a real non-principal character then

$$L(1, \chi) \geq \frac{\pi}{8M_\chi + 16}.$$

So in particular, $L(1, \chi) > 0$.

Proof:

For any $\alpha > 0$, let $F(\alpha) = \sum_{n=1}^{\infty} (\chi * u)(n) e^{-\alpha n}$.

By Lemma 1,

$$F(\alpha) \geq \sum_{n=1}^{\infty} e^{-\alpha n^2} = \sum_{n=0}^{\infty} e^{-\alpha n^2} - 1$$
$$\geq \int_0^{\infty} e^{-\alpha x^2}\, dx - 1 = \frac{1}{2}\left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}} - 1.$$

Also, reversing summation, and putting $n = mj$,

$$F(\alpha) = \sum_{n=1}^{\infty} e^{-\alpha n} \sum_{j|n} \chi(j) = \sum_{j=1}^{\infty} \chi(j) \sum_{m=1}^{\infty} e^{-\alpha m j} = \sum_{j=1}^{\infty} \chi(j) \frac{1}{e^{\alpha j} - 1}. \quad \text{(G. Series.)}$$

Now in the notation of Lemma 2, $g(\alpha x) = \dfrac{1}{\alpha x} - \dfrac{1}{e^{\alpha x} - 1}$ so $\dfrac{1}{e^{\alpha j} - 1} = \dfrac{1}{\alpha j} - g(\alpha j) := \dfrac{1}{\alpha j} - h(j)$. Hence

$$F(\alpha) = \sum_{j=1}^{\infty} \frac{\chi(j)}{\alpha j} - \sum_{j=1}^{\infty} \chi(j) h(j)$$
$$= \frac{1}{\alpha} L(1, \chi) - \sum_{j=1}^{\infty} \chi(j) h(j).$$

Writing M for M_χ and noting that $h(j)$ is decreasing (Lemma 2), we can use the Abel Summation formula to write:

$$\left| \sum_{j=1}^{\infty} \chi(j) h(j) \right| \leq \sum_{j=1}^{\infty} M\left(h(j) - h(j+1)\right) = M h(1) = M g(\alpha) \leq \frac{1}{2} M$$

from Lemma 2.

Thus,

$$\frac{1}{\alpha} L(1, \chi) = \left| \frac{1}{\alpha} L(1, \chi) \right| = \left| F(\alpha) + \sum_{j=1}^{\infty} \chi(j) h(j) \right| \geq |F(\alpha)| - \left| \sum_{j=1}^{\infty} \chi(j) h(j) \right|$$
$$\geq \frac{1}{2}\left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}} - 1 - \frac{1}{2} M.$$

Thus we can write $L(1, \chi) \geq a\alpha^{\frac{1}{2}} - b\alpha$, where $a = \dfrac{1}{2}\pi^{\frac{1}{2}}$ and $b = 1 + \frac{1}{2}M$.

The right-hand side, (essentially a quadratic in $\alpha^{\frac{1}{2}}$), has a maximum if we choose $\alpha^{\frac{1}{2}} = \dfrac{a}{2b}$, which gives

$$L(1, \chi) \geq \frac{a^2}{4b} = \frac{\pi}{8M + 16}.$$

Case 2: χ is complex: (i.e. $\overline{\chi} \neq \chi$).

Let $F(s) = \prod_{\chi} L(s, \chi)$, where the product is over all Dirichlet characters modulo m. Assume s is real and $s > 1$ and recall the function

$$G(s, \chi) = \sum_{p} \sum_{k=1}^{\infty} \frac{1}{k} \chi(p^k) p^{-ks}.$$

Recall also that $\sum_{\chi} \chi(p^k) = 0$ unless $p^k = 1$ in which case the sum is $\phi(m)$.

Hence summing over χ and changing the order of summation, we have

$$\sum_{\chi} G(s, \chi) = \phi(m) \sum_{\substack{p^k \equiv 1 \bmod m \\ p \text{ prime}, k \geq 1}} \frac{1}{k} p^{-ks}.$$

Now the righthand side is non-negative and so taking the exponential of both sides and recalling that $exp(G(s, \chi) = L(s, \chi)$, we see that $F(s) = \prod_{\chi} L(s, \chi) \geq 1$.

Now since s is real, $\overline{L(s, \chi)} = L(s, \overline{\chi})$ and so if $L(1, \chi) = 0$ then $L(1, \overline{\chi}) = 0$.

Assume then that $L(1, \chi) = 0$, where χ is a complex character, then the product $L(s, \chi)L(s, \overline{\chi})$ has a zero of order (at least) 2 at $s = 1$. Also $L(s, \chi_1)$ has a pole of order 1 at $s = 1$.

Thus the product $F(s)$ is analytic at $s = 1$ and has a zero there, since one zero cancels with the simple pole, leaving a zero. That is, $F(1) = 0$ which contradicts the above lower bound on F.

Thus $L(1, \chi) \neq 0$ in this case.

Evaluation of $L(1, \chi)$.

If χ is a non-principal character modulo k, then for any positive integer N, we have

Examples: If χ_3 is the (unique) non-principal character modulo 3, then

$$L(1,\chi_3) = \int_0^1 \frac{1-t}{1-t^3}\,dt = \frac{\pi}{3\sqrt{3}}.$$

Similarly, if χ is the real non-principal Dirichlet character modulo 5, then

$$L(1,\chi_5) = \int_0^1 \frac{1-x-x^2+x^3}{1-x^5}\,dx = \int_0^1 \frac{1-x^2}{1+x+x^2+x^3+x^4}\,dx.$$

Putting $y = x + \frac{1}{x}$ we obtain $L(1,\chi_5) = \displaystyle\int_2^\infty \frac{dy}{y^2+y-1} = \frac{2}{\sqrt{5}}\log\left(\frac{1+\sqrt{5}}{2}\right)$.

It is an exercise to show that if χ_4 and χ_6 are (unique) non-principal characters modulo 4 and 6 respectively, then $L(1,\chi_4) = \frac{\pi}{4}$ and $L(1,\chi_6) = \frac{\pi}{2\sqrt{3}}$.

More of these are given in the Tutorial problems. There are very surprising connections here between the evaluations of these L functions and the fundamental units in the rings $\mathbb{Z}(\sqrt{k})$. Alas, we do not have time to explore any further in that direction.

Appendix:

Here is another proof proof given by Shapiro (1950) that $L(1,\chi) \neq 0$ when χ is a real non-principal character. Since it appears in many books, I have included it here.

It is easy to show that

$$\sum_{n\leq x} \frac{1}{\sqrt{n}} \leq 2\sqrt{x} + C \qquad (1)$$

where C is a constant. (Draw diagram and over-approximate the area.)

We need now to get a bound on the tail of $L(s,\chi)$.

Set $S(n) = \sum_{k=1}^{n} \chi(k)$, then by the Lemma just prior to Theorem 4.5, we have $|S(n)| \leq \phi(a)$. Using summation by parts, for integers $M \leq N$, we can write

$$\left|\sum_{n=M}^{N} \frac{\chi(n)}{n^s}\right| = \left|\sum_{n=M}^{N} (S(n) - S(n-1))\frac{1}{n^s}\right|$$

$$\left|-\frac{S(M-1)}{M^s} + \sum_{n=M}^{N-1} S(n)\left(\frac{1}{n^s} - \frac{1}{(n+1)^s}\right) + S(N)\frac{1}{N^s}\right|$$

$$\leq \phi(a)\left(\frac{1}{M^s} + \sum_{n=M}^{N-1}\left(\frac{1}{n^s} - \frac{1}{(n+1)}\right) + \frac{1}{N^s}\right) \leq \frac{2\phi(a)}{M^s}.$$

Thus,

$$\left|L(s,\chi) - \sum_{n\leq x} \frac{\chi(n)}{n^s}\right| = \left|\sum_{n>x} \frac{\chi(n)}{n^s}\right| \leq \frac{2\phi(a)}{x^s} = \frac{2c}{x^s}. \qquad (2)$$

Now fix χ, a real, non-trivial character, and let

$$F(n) = \sum_{d|n} \chi(d).$$

Note that F is multiplicative and since χ is real, it is easy to show that for any prime p and positive integer α,

$$F(p^\alpha) = \begin{cases} \alpha+1 & \text{if } \chi(p) = 1 \\ 0 & \text{if } \chi(p) = -1 \text{ and } \alpha \text{ is odd} \\ 1 & \text{if } \chi(p) = 0 \\ 1 & \text{if } \chi(p) = -1 \text{ and } \alpha \text{ is even} \end{cases}.$$

Hence $F(n) \geq 0$ for all n and $F(n) \geq 1$ if n is a square.

So

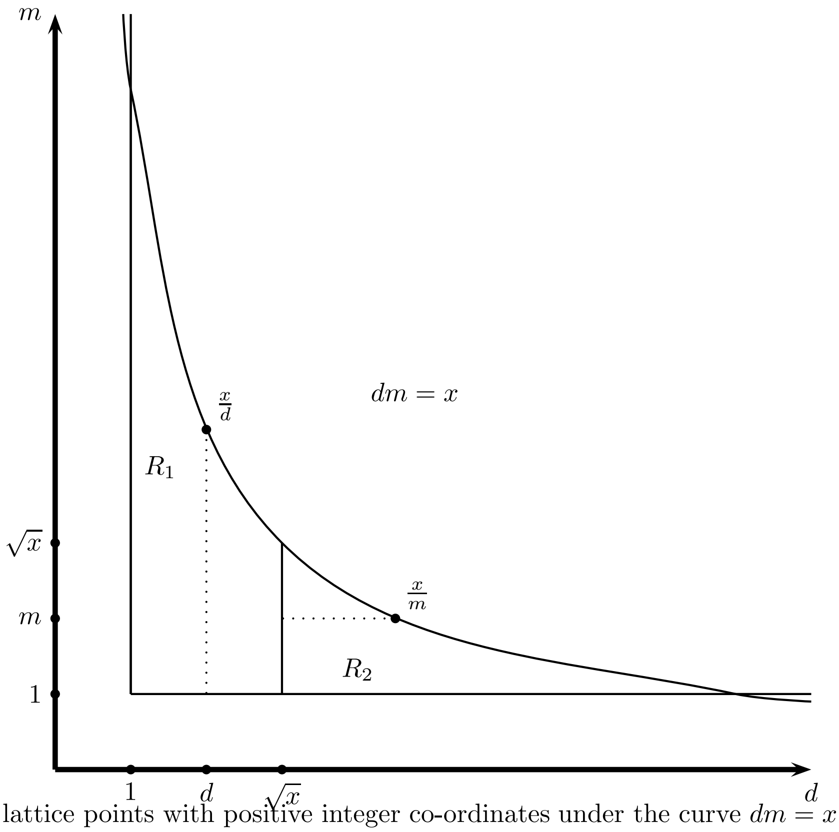
$$\sum_{n\leq x} \frac{F(n)}{\sqrt{n}} \geq \sum_{m^2\leq x} \frac{1}{m^2} \geq \sum_{m\leq\sqrt{x}} \frac{1}{m}.$$

Thus $S(x) = \sum_{n\leq x} \frac{F(n)}{\sqrt{n}}$ **diverges** as $x \to \infty$ (3).

Now

$$S(x) = \sum_{n\leq x} \frac{1}{\sqrt{n}} \sum_{d|n} \chi(d) = \sum_{d\leq x} \sum_{m\leq \frac{x}{d}} \frac{\chi(d)}{\sqrt{dm}} \quad \text{using DSI 2}$$

$$= \sum_{\substack{m,d \\ md \le x}} \frac{\chi(d)}{\sqrt{dm}}$$



The sum is over lattice points with positive integer co-ordinates under the curve $dm = x$. We split this into two regions R_1, R_2 where

$$R_1 = \{(d,m) : d \le \sqrt{x}, m \le \frac{x}{d}\}, R_2 = \{(d,m) : m \le \sqrt{x}, \sqrt{x} < d \le \frac{x}{m}\}.$$

Hence

$$S(x) = \sum_{d \le \sqrt{x}} \sum_{m \le \frac{x}{d}} \frac{\chi(d)}{\sqrt{dm}} + \sum_{m \le \sqrt{x}} \sum_{\sqrt{x} < d \le \frac{x}{m}} \frac{\chi(d)}{\sqrt{dm}} = S_1 + S_2.$$

Using (1), we have

$$S_1 \le \sum_{d \le \sqrt{x}} \frac{\chi(d)}{\sqrt{d}} \left(\frac{2\sqrt{x}}{\sqrt{d}} + C \right) = 2\sqrt{x} \sum_{d \le \sqrt{x}} \frac{\chi(d)}{d} + C \sum_{d \le \sqrt{x}} \frac{\chi(d)}{\sqrt{d}},$$

so for large x, the right-hand side approaches $2\sqrt{x}L(1,\chi) + CL(\frac{1}{2},\chi)$. Now if $L(1,\chi)$ were equal to 0, then S_1 would be bounded as $x \to \infty$.

Finally,

$$S_2 = \sum_{m \le \sqrt{x}} \frac{1}{\sqrt{m}} \sum_{\sqrt{x} < d \le \frac{x}{m}} \frac{\chi(d)}{\sqrt{d}}$$

$$= \sum_{m \le \sqrt{x}} \frac{1}{\sqrt{m}} \left(\sum_{d=1}^{\infty} \frac{\chi(d)}{\sqrt{d}} - \sum_{d \le \sqrt{x}} \frac{\chi(d)}{\sqrt{d}} + \sum_{d \le \frac{x}{m}} \frac{\chi(d)}{\sqrt{d}} - \sum_{d=1}^{\infty} \frac{\chi(d)}{\sqrt{d}} \right)$$

$$= \sum_{m \le \sqrt{x}} \frac{1}{\sqrt{m}} \left(L(\frac{1}{2},\chi) - \sum_{d \le \sqrt{x}} \frac{\chi(d)}{\sqrt{d}} - L(\frac{1}{2},\chi) + \sum_{d \le \frac{x}{m}} \frac{\chi(d)}{\sqrt{d}} \right)$$

$$\leq \sum_{m \leq \sqrt{x}} \frac{1}{\sqrt{m}} \left(\frac{2c_1}{x^{\frac{1}{4}}} - \frac{2c_2}{x^{\frac{1}{2}}} \right) \quad \text{by (2)}$$

$$\leq (2x^{\frac{1}{4}} + C) \left(\frac{2c_1}{x^{\frac{1}{4}}} - \frac{2c_2}{x^{\frac{1}{2}}} \right) = O\left(\frac{1}{x^{\frac{1}{2}}} + O(1) \right)$$

and so is bounded. Thus S is bounded, which contradicts (3).