UNSW AUSTRALIA.
SCHOOL OF MATHEMATICS AND STATISTICS.
MATH5645: TOPICS IN NUMBER THEORY.

§6 THE PRIME NUMBER THEOREM:

This section will be devoted to studying the proof of the Prime Number Theorem (PNT), using analytic methods. In 1949 Selberg and Erdös independently found *elementary* (but certainly not *easy*) proofs of the PNT starting with the so-called *Selberg* identity

$$\psi(x)\log x + \sum_{n\le x}\Lambda(n)\psi(\frac{x}{n}) = 2x\log x + O(x)$$

where $\Lambda(x)$ is the von Mangoldt function and $\psi(x) = \sum_{n\le x}\Lambda(x)$.

The other known proofs are analytic in nature.

The original proof, given by de la Vallée Poussin (and independently by Hadamard), was modified by Ingham. It is rather long and is given as an Appendix to this chapter. It uses the Riemann-Lebesque Lemma.

A second proof, similar to but simpler than the above, invokes a result known as the Wiener-Ikehara theorem, in place of the Riemann-Lebesque Lemma. The proof of the Wiener-Ikehara theorem, which is technical, will not be given.

The most recent analytic proof is due to Newman (1980). It will appear as an Appendix, but we may go through it briefly, skipping over the more technical parts.

All analytic versions of the proof require some kind of *Tauberian* theorem from Measure Theory in the final step.

As is often the case in Mathematics, we try to write the problem in terms of an equivalent problem and solve the latter. This *equivalent probem* will involve the function $\psi(x)$ so we define:

The Chebyshev Functions:

In this section we introduce the Chebyshev functions ψ and ϑ.

Let $\psi(x) = \sum_{n\le x}\Lambda(x)$ for $x > 0$, where $\Lambda(n)$ is the Von Mangoldt function which takes the value zero unless $n = p^m$ for some prime p, whereat it takes the value $\log p$.
Noting that $\Lambda(n) = 0$ unless n is a prime power, we have

$$\psi(x) = \sum_{n\le x}\Lambda(n) = \sum_{m=1}^{\infty}\sum_{\substack{p \\ p^m\le x}}\Lambda(p^m) = \sum_{m=1}^{\infty}\sum_{p\le x^{\frac{1}{m}}}\log p.$$

Now the sum on m is, in fact, finite, since it is empty if $x^{\frac{1}{m}} < 2$, i.e. if $m > \log_2 x$. Thus we can write

$$\psi(x) = \sum_{m\le\log_2 x}\sum_{p\le x^{\frac{1}{m}}}\log p.$$

This form motivates:

Definition: If $x > 0$, define $\vartheta(x)$ by

$$\vartheta(x) = \sum_{p\le x}\log p,$$

where the sum is taken over all **primes** p less or equal to x.
Thus we can write

$$\psi(x) = \sum_{m \le \log_2 x} \vartheta(x^{\frac{1}{m}}). \qquad (1)$$

Lemma 6.1: For $x > 0$, we have

$$0 \le \frac{\psi(x)}{x} - \frac{\vartheta(x)}{x} \le \frac{(\log x)^2}{2\sqrt{x}\log 2}.$$

Proof:

Note that this implies the equivalence:

$$\lim_{x\to\infty} \frac{\vartheta(x)}{x} = 1 \Longleftrightarrow \lim_{x\to\infty} \frac{\psi(x)}{x} = 1.$$

Statements Equivalent to the PNT:

We now develop a sequence of statements which are equivalent to the PNT.

To proceed, we need the following technical result, which is also referred to as *Abel's Identity.*

Theorem 6.1: Suppose $a(n)$ is any arithmetic function and let $A(x) = \sum_{n \le x} a(n)$, where we take $A(x) = 0$ if $x < 1$.
Suppose further that $f \in C^1[y, x]$ where $0 \le y < x$. Then we have

$$\sum_{y<n\le x} a(n)f(n) = A(x)f(x) - A(y)f(y) - \int_y^x A(t)f'(t)\,dt.$$

Proof: I will prove a slightly easier version with $y = 0$. Let $k = \lfloor x \rfloor$. Then

$$\sum_{n\le x} a(n)f(n) = \sum_{n=1}^{k}(A(n) - A(n-1))f(n)$$

$$= \sum_{n=1}^{k-1} A(n)(f(n) - f(n+1)) + A(k)f(k)$$

$$= -\sum_{n=0}^{k-1} A(n)\int_n^{n+1} f'(t)\,dt + A(k)f(k) \qquad \text{since } A(0) = 0,$$

$$= -\int_0^k A(t)f'(t)\,dt + A(k)f(k).$$

Hence

$$\sum_{n\le x} a(n)f(n) = A(x)f(x) - \int_0^x A(t)f'(t)\,dt.$$

This enables us to express the Chebychev function $\vartheta(x)$ in terms of an integral and relate it to the function $\pi(x)$.

Theorem 6.2: For $x \ge 2$, we have

$$\text{(a)} \qquad \vartheta(x) = \pi(x)\log x - \int_2^x \frac{\pi(t)}{t}\,dt$$

$$\text{(b)} \qquad \pi(x) = \frac{\vartheta(x)}{\log x} + \int_2^x \frac{\vartheta(t)}{t(\log t)^2}\,dt.$$

Proof:
(a) Define $a(n) = \begin{cases} 1 & \text{if } n \text{ is prime} \\ 0 & \text{otherwise} \end{cases}$ then

$$\pi(x) = \sum_{p\le x} 1 = \sum_{1<n\le x} a(n)$$

and

$$\vartheta(x) = \sum_{p\le x} \log p = \sum_{1<n\le x} a(n)\log n.$$

Applying Abel's identity, with $A(x) = \pi(x)$, $f(x) = \log x$ and $y = 1$, we have part (a) of the theorem. (Note that $\pi(x) = 0$ for $x < 2$.)

For (b), let $b(n) = a(n)\log n$ then

$$\pi(x) = \sum_{\frac{3}{2}<n\le x} \frac{b(n)}{\log n}, \qquad \vartheta(x) = \sum_{n\le x} b(n)$$

so taking $f(x) = \frac{1}{\log x}$, $y = \frac{3}{2}$ and $A(x) = \vartheta(x) = \sum_{n\le x} b(n)$ a second application of Abel's identity yields

$$\pi(x) = \frac{\vartheta(x)}{\log x} - \frac{\vartheta(\frac{3}{2})}{\log \frac{3}{2}} + \int_{\frac{3}{2}}^x \frac{\vartheta(t)}{t(\log t)^2}\,dt$$

$$= \frac{\vartheta(x)}{\log x} + \int_2^x \frac{\vartheta(t)}{t(\log t)^2}\,dt$$

since $\vartheta(t) = 0$ for $t < 2$.

We can now prove some equivalent forms of the prime number theorem, one of which will be used to prove the PNT.

Theorem 6.3: As $x \to \infty$

$$\frac{\pi(x)\log x}{x} \to 1 \iff \frac{\vartheta(x)}{x} \to 1 \iff \frac{\psi(x)}{x} \to 1.$$

$$\text{((a)} \qquad \iff \qquad \text{(b)} \qquad \iff \qquad \text{(c))}$$

Proof: We have already shown that (b) $\iff$ (c), so we suppose (a) holds.
Recall from Theorem 6.2 (a) that

$$\frac{\vartheta(x)}{x} = \frac{\pi(x)\log x}{x} - \frac{1}{x}\int_2^x \frac{\pi(t)}{t}\,dt$$

and so we need to show that $\dfrac{1}{x}\displaystyle\int_2^x \frac{\pi(t)}{t}\,dt \to 0$ as $x \to \infty$.

Now (a) implies that $\frac{\pi(t)}{t} = O(\frac{1}{\log t})$ for $t \geq 2$, so

$$\frac{1}{x}\int_2^x \frac{\pi(t)}{t}\,dt = O\left(\frac{1}{x}\int_2^x \frac{dt}{\log t}\right)$$

and

$$\frac{1}{x}\int_2^x \frac{dt}{\log t} = \frac{1}{x}\int_2^{\sqrt{x}} \frac{dt}{\log t} + \frac{1}{x}\int_{\sqrt{x}}^x \frac{dt}{\log t} \leq \frac{1}{x}\left(\frac{\sqrt{x}}{\log 2} + \frac{x-\sqrt{x}}{\log\sqrt{x}}\right)$$

and so $\dfrac{1}{x}\displaystyle\int_2^x \frac{\pi(t)}{t}\,dt \to 0$ as $x \to \infty$.

Now suppose (b), then by a similar argument we need to show that

$$\frac{\log x}{x}\int_2^x \frac{\vartheta(t)}{t(\log t)^2}\,dt \to 0$$

as $x \to \infty$. This is left as exercise 3 on the problem sheet.

We now have a number of equivalent forms for the PNT (there are many others as well) and it is (c) that we will use in our proof.

Proof that $\psi(x) \sim x$.

Recall from the Corollary to Theorem 2.8 that $-\dfrac{\zeta'(s)}{\zeta(s)} = \displaystyle\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$, where $\Lambda(n)$ is the Von Mangold function.

Theorem 6.4: For $s > 1$ and real,

$$-\frac{\zeta'(s)}{\zeta(s)} = s\int_1^{\infty} \frac{\psi(x)}{x^{s+1}}\,dx.$$

Proof:

We now extend the above formula, by analytic continuation to the complex plane for $\Re(s) > 1$.

Put $x = e^u$, then we have
$$-\frac{\zeta'(s)}{\zeta(s)} = s\int_0^\infty \psi(e^u)e^{-us}\,du$$
for $\sigma > 1, s = \sigma + it$.

By Theorem 4.4, we know that $\zeta(1+it) \neq 0$ for all $t \neq 0$ and by Theorem 4.5 both ζ and ζ' are bounded for $\sigma \geq 1$. Thus $-\dfrac{\zeta'(s)}{\zeta(s)}$ represents an analytic function for $\sigma \geq 1$ except at $s = 1$ where we have a simple pole with residue 1.

Theorem 6.5: (Wiener-Ikehara Theorem)

Let $A(x)$ be a non-negative non-decreasing function of x, $(0 \leq x < \infty)$ and suppose $A(0) \leq 1$. Suppose further that $\displaystyle\int_0^\infty A(x)e^{-sx}\,dx$, $s = \sigma + it$, converges for $\sigma > 1$ to the function $f(s)$ which is analytic for $\sigma \geq 1$ except for a simple pole at $s = 1$ with residue 1, then
$$\lim_{x\to\infty} e^{-x}A(x) = 1.$$

Proof: Very long and technical. It will not be given here.

Theorem 6.6: $\dfrac{\psi(x)}{x} \to 1$ as $x \to \infty$.

Proof:

Applying Theorem 6.5 to $-\dfrac{\zeta'(s)}{\zeta(s)} = s\displaystyle\int_0^\infty \psi(e^u)e^{-us}\,du$ with $A(u) = \psi(e^u)$ and $f(s) = -\frac{\zeta'(s)}{\zeta(s)}$, which satisfy the required conditions, we conclude that $\dfrac{\psi(e^u)}{e^u} \to 1$ as $u \to \infty$ and hence the result follows.

The PNT now follows from Theorem 6.3.

The Error Term:

We have thus proven that $\psi(x) = x + o(x)$. If we write $\psi(x) = x + r(x)$, what can be said about $r(x)$? De la Vallée Poussin showed that $r(x) = O(xe^{-\alpha\sqrt{\log x}})$, for a certain positive constant α, by finding a zero free region in the critical strip. The best known (to me) estimate is $r(x) = xexp\left(-\alpha\dfrac{\log x^{\frac{4}{7}}}{\log\log x^{\frac{3}{7}}}\right)$ which goes back to Vinogradov and Korobov (1958). On the other hand, if the Riemann Hypothesis is true, then it can be shown that $\psi_1(x) = \displaystyle\int_1^x \psi(t)\,dt = \frac{1}{2}x^2 + O(x^{\frac{3}{2}})$. Despite this, the exact order of magnitude of $\psi(x)$ is unknown, even assuming the Riemann Hypothesis.

Merten's Theorems:

We proved back in the first chapter that $\sum_{p} \frac{1}{p}$ diverges. We now state and prove a number of theorems, due to Mertens, which culminate in finding the true order of the sum $\sum_{p \leq x} \frac{1}{p}$.

We will assume the following two results which were tutorial problems. The first is from the problems from Chapter 2.

Result (1): Define $u(n) = 1$ for all n, then for $x > 1$, $\sum_{n \leq x} (f * u)(n) = \sum_{j \leq x} f(j) \left[\frac{x}{j} \right]$

Result (2): From the problems from this section, $\psi(x) \leq 2x$.

Theorem 6.7:
For $x > 1$,

$$\sum_{n \leq x} \frac{\Lambda(x)}{n} = \log x + O(1).$$

Proof:

Theorem 6.8:

For $x > 1$, and p always prime,

$$\sum_{p \leq x} \frac{\log p}{p} = \log x + O(1).$$

Proof:

We now finally have:

Theorem 6.9:
For $x > 2$, and p always prime,

$$\sum_{p \leq x} \frac{1}{p} = \log \log x + C + O\left(\frac{1}{\log x}\right),$$

where C is a constant.

Proof:
Define

$$a(n) = \begin{cases} \frac{\log n}{n} & \text{if } n \text{ is prime} \\ 0 & \text{otherwise} \end{cases}$$

then $\sum_{p \leq x} \frac{\log p}{p} = \sum_{n \leq x} a(n) = A(x)$.
From Theorem 6.9 above, $A(x) = \log x + O(1) = \log x + R(x)$, where $R(x) = O(1)$.

We now write

$$\sum_{p \leq x} \frac{1}{p} = \sum_{2 \leq n \leq x} \frac{a(n)}{\log n}$$

and apply Abel's identity (Theorem 6.1) with $f(n) = \frac{1}{\log n}$ and $y = 1$, noting that $a(1) = 0$. Thus,

$$\sum_{p \leq x} \frac{1}{p} = \frac{A(x)}{\log x} + \int_2^x \frac{A(t)}{t(\log t)^2}\, dt = 1 + \frac{R(x)}{\log x} + \int_2^x \frac{\log t + R(t)}{t(\log t)^2}\, dt$$

$$= 1 + O\left(\frac{1}{\log x}\right) + \int_2^x \frac{1}{t \log t}\, dt + \int_2^x \frac{R(t)}{t(\log t)^2}\, dt = 1 + O\left(\frac{1}{\log x}\right) + \log \log x - \log \log 2 + \int_2^x \frac{R(t)}{t(\log t)^2}\, dt.$$

Now since $R(t) = O(1)$ and $\int_2^x \frac{1}{t(\log t)^2}\, dt$ converges (by the integral test), $\int_2^x \frac{R(t)}{t(\log t)^2}\, dt$ has a finite limit as $x \to \infty$. Hence

$$\sum_{p \leq x} \frac{1}{p} = \log \log x + C + O\left(\frac{1}{\log x}\right)$$

where $C = 1 - \log \log 2 + \int_2^\infty \frac{R(t)}{t(\log t)^2}\, dt$.

Notes: The proof does not give much idea about the value of C, but it can be shown that $C \approx 0.26150$.

As a numerical example, MAPLE reports that

$$\sum_{p \leq 1000} \frac{1}{p} \approx 2.1980801, \qquad \log\log 1000 + C \approx 2.194145.$$

The Logarithmic Integral:

The approximation $\pi(x) \sim \frac{x}{\log x}$ is, of course, asymptotic and so does not really give a particularly good approximation to the actual value of $\pi(x)$. Gauss suggested that, to get a better estimate, one should 'average out', i.e. look at

$$Li(x) = \int_2^x \frac{1}{\log t}\, dt.$$

This (non-elementary) integral is called the *Logarithmic integral.*

The following table shows that Gauss had the right idea (as usual!). (In the table the values are rounded to the nearest integer).

n	$\pi(n)$	$\frac{n}{logn}$	Li(n)
1,000	168	145	177
10,000	1,229	1,068	1,246
50,000	5,133	4,621	5,166
100,000	9,592	8,686	9,630
500,000	41,538	38,103	41,607
1,000,000	78,498	72,382	78,628
10,000,000	664,579	620,421	664,918

In this section we will show that $Li(x) \sim \frac{x}{\log x}$ and so $Li(x) \sim \pi(x)$.

Integration by parts gives:

$$Li(x) = \frac{x}{\log x} - \frac{2}{\log 2} + \int_2^x \frac{1}{(\log t)^2}\, dt.$$

This suggests two things. Firstly, replace 2 by e to make the constants easier and secondly, look at the family of integrals:

$$I_n(x) = \int_e^x \frac{1}{(\log t)^n}\, dt.$$

A simple integration by parts gives

$$I_n(x) = \frac{x}{(\log x)^n} - e + nI_{n+1}(x).$$

Using this idea, it can be shown (tutorial problem) that

$$Li(x) = \frac{x}{\log x} + \frac{x}{(\log x)^2} + ... + (n-1)!\frac{x}{(\log x)^n} + r_{n+1}(x)$$

where $r_{n+1}(x) \sim n!\frac{x}{(\log x)^{n+1}}$ as $x \to \infty$.

Lemma: As $x \to \infty$,

$$I_n(x) \sim \frac{x}{(\log x)^n}.$$

Proof:
By the above recurrence, our desired result is equivalent to

$$I_{n+1}(x)\frac{(\log x)^n}{x} \to 0$$

as $x \to \infty$.

Divide the interval $[e, x]$ into $[e, \sqrt{x}] \cup [\sqrt{x}, x] = A \cup B$. Then for $t \in A$, $\log t \geq 1$ and for $t \in B$, $\log t \geq \frac{1}{2} \log x$.

Hence,

$$I_{n+1}(x) = \int_e^{\sqrt{x}} \frac{1}{(\log t)^{n+1}}\, dt + \int_{\sqrt{x}}^{x} \frac{1}{(\log t)^{n+1}}\, dt$$

$$\leq \int_e^{\sqrt{x}} 1\ dt + \int_{\sqrt{x}}^{x} \left(\frac{2}{\log x} \right)^{n+1} dt = \sqrt{x} - e + \left(\frac{2}{\log x} \right)^{n+1} (x - \sqrt{x})$$

$$\leq \sqrt{x} + x \left(\frac{2}{\log x} \right)^{n+1}.$$

Hence

$$I_{n+1}(x) \frac{(\log x)^n}{x} \leq \frac{(\log x)^n}{\sqrt{x}} + \frac{2^{n+1}}{\log x} \to 0$$

as $x \to \infty$.

Theorem 6.10:

$$Li(x) \sim \frac{x}{\log x}$$

as $\to \infty$. More precisely,

$$Li(x) = \frac{x}{\log x} + r(x)$$

where $r(x) \sim \dfrac{x}{(\log(x))^2}$ as $x \to \infty$.

Proof: It suffices to prove the equivalent statement for $I_1(x)$ instead of $Li(x)$.

Now $I_1(x) = \dfrac{x}{\log x} + I_2(x) - e$. By the Lemma, $I_2(x) - e \sim \frac{x}{(\log x)^2}$ as $x \to \infty$ and the result follows.

There are many deep connections between $Li(x)$ and $\pi(x)$. For example, it was proven a long time ago that

$$|\pi(x) - Li(x)| \le Kxe^{-c(\log x)^{\frac{1}{2}}}$$

for some constants K and c. This result has not been bettered, but it is known that

$$|\pi(x) - Li(x)| \le \sqrt{x} \log x$$

(for $x \geq 3$) is equivalent to the Riemann Hypothesis!! Hence the nature of the zeros of the zeta function in the critical strip impose a limitation on the accuracy with which $Li(x)$ can represent $\pi(x)$.

Riemann (and others) believed that $\pi(x) > Li(x)$ for all x (as the table above suggests). This was proven to be false. Indeed the two curves cross infinitely often (!) (Littlewood). The smallest x at which the curves cross is known as *Skewes number* but is not explicity known. Its size (given initially by Skewes in 1933) was shown to be less than $e^{e^{e^{79}}} \approx 10^{10^{10^{34}}}$. The Skewes number has since been reduced to 1.165×10^{1165} by Lehman in 1966, $e^{e^{27/4}} \approx 8.185 \times 10^{370}$ by te Riele (1987), and less than 1.39822×10^{316} (Bay and Hudson 2000.)

More recent work (2005) by Demichel establishes that the first crossover occurs around $1.397162914 \times 10^{316}$.

Hence we have come from the astronomical to the merely ‘ginormous’.

Appendix 1: Newman's Proof.

This proof was given by D.J. Newman in 1980. If uses the equivalence between the PNT and $\dfrac{\vartheta(x)}{x} \to 1$. The Wiener-Ikehara Theorem is replaced by the so-called *Analytic Theorem* which has the advantage that it can be proven using only a simple contour integral.

Define $\Phi(s) = \displaystyle\sum_p \frac{\log p}{p^s}$.

Theorem 1: $\Phi(s) - \dfrac{1}{s-1}$ is holomorphic for $Re(s) \geq 1$.

Proof: For s real and $s > 1$, $-\dfrac{\zeta'(s)}{\zeta(s)} = \displaystyle\sum_p \frac{\log p}{p^s - 1}$ (exercise).

The right-hand side of this is also equal to $\Phi(s) + \displaystyle\sum_p \frac{\log p}{p^s(p^s-1)}$. The latter sum converges for $s > \frac{1}{2}$ and so by analytic continuation it converges for complex s, provided $Re(s) > \frac{1}{2}$. Now since $-\frac{\zeta'(s)}{\zeta(s)}$ has a simple pole at $s = 1$ with residue 1, the result follows.

Theorem 2: (Analytic Theorem).

Let $f(t)$, $(t \geq 0)$, be a bounded integrable function and suppose that

$$g(z) = \int_0^\infty f(t)e^{-zt}\,dt,$$

for $Re(z) > 0$, extends holomorphically for $Re(z) \geq 0$, Then $\displaystyle\int_0^\infty f(t)\,dt$ exists and is equal to $g(0)$.

(Notice that g is simply the Laplace transform of f.)

Theorem 3:

$$\int_1^\infty \frac{\vartheta(x) - x}{x^2}\,dx \ \text{ converges.}$$

Proof: Let $b(n) = 1$ if n is prime and zero otherwise, then $\vartheta(x) = \displaystyle\sum_{n \leq x} b(n)\log n$ and $\Phi(s) = \displaystyle\sum_p \frac{\log p}{p^s} = \sum_n \frac{b(n)\log n}{n^s}$.

Now for s real and $s > 1$, $\dfrac{\vartheta(x)}{x^2} \leq \dfrac{x \log x}{x^s} \to 0$ as $x \to \infty$. Again using Abel's lemma with $a(n) = b(n)\log n$, so $A(x) = \vartheta(x)$, and $f(n) = \dfrac{1}{n^s}$ and putting $x = e^t$, we have, (for $Re(s) > 1$),

$$\Phi(s) = \sum_p \frac{\log p}{p^s} = \lim_{x\to\infty} \frac{\vartheta(x)}{x^s} + s\int_1^\infty \frac{\vartheta(x)}{x^{s+1}}\,dx = s\int_0^\infty e^{-st}\vartheta(e^t)\,dt.$$

Now putting $f(t) = \vartheta(e^t)e^{-t} - 1$ and $g(z) = \dfrac{\Phi(z+1)}{z+1} - \dfrac{1}{z}$, then the conditions of Theorem 2 are satisfied and so $\displaystyle\int_0^\infty f(t)\,dt$ exists. Replacing e^t with x we arrive at the required integral.

Theorem 4: $\vartheta(x) \sim x$.

Proof: Suppose $\lambda > 1$ and there exist arbitrarily large x such that $\vartheta(x) \geq \lambda x$,. For $t \geq x$ we have $\vartheta(t) \geq \vartheta(x) \geq \lambda x$ so,

$$\int_x^{\lambda x} \frac{\vartheta(t) - t}{t^2}\,dt \geq \int_x^{\lambda x} \frac{\lambda x - t}{t^2}\,dt = \int_1^\lambda \frac{\lambda - t}{t^2}\,dt = \lambda + 1 - \log\lambda > 0.$$

(where we put $u = \frac{t}{x}$). Hence the tail of the integral is not going to zero, but this implies that the improper integral diverges contradicting Theorem 3.

Similarly, the inequality $\vartheta(x) \leq \lambda x$ with $\lambda < 1$ would imply that

$$\int_{\lambda x}^{x} \frac{\vartheta(t) - t}{t^2}\, dt < 0$$

again contradicting Theorem 3. The conclusion is that $\vartheta(x) \sim x$ so the PNT follows.

Appendix 2: The Ingham version of the first proof.

We introduce the function ψ_1 and get yet another equivalence to the PNT.

The function $\psi(x)$ is a step function, so as usual we will smooth it out by integrating and define

$$\psi_1(x) = \int_1^x \psi(t)\,dt.$$

We need to show that

$$\psi_1(x) \sim \frac{1}{2}x^2 \Longrightarrow \psi(x) \sim x$$

so that we can then concentrate on showing the left hand side of this implication.

To obtain this result, we need to following lemma:

Lemma 1. Let $A(x) = \sum_{n \leq x} a(n)$, and $A_1(x) = \int_1^x A(t)\,dt$.
Assume that $a(n) \geq 0$ for all n, then if $A_1(x) \sim Lx^c$ as $x \to \infty$ for some $c > 0$ and $L > 0$, then $A(x) \sim cLx^{c-1}$ as $x \to \infty$.

(In other words, formal differentiation of the first asymptotic formula gives a correct asymptotic result.)

Proof: $A(x)$ is increasing since $a(n) \geq 0$. Now choose $\beta > 1$ and consider

$$A_1(\beta x) - A_1(x) = \int_x^{\beta x} A(u)\,du \geq A(x)(\beta x - x) = x(\beta - 1)A(x).$$

Thus

$$xA(x) \leq \frac{1}{\beta - 1}\left\{A_1(\beta x) - A_1(x)\right\}$$

so

$$\frac{A(x)}{x^{c-1}} \leq \frac{1}{\beta - 1}\left\{\frac{A_1(\beta x)}{(\beta x)^c}\beta^c - \frac{A_1(x)}{x^c}\right\}.$$

Now fix β and let $x \to \infty$, so using the asymptotic approximation to $A_1(x)$ we have

$$\limsup_{x\to\infty} \frac{A(x)}{x^{c-1}} \leq \frac{1}{\beta - 1}(L\beta^c - L) = L\frac{\beta^c - 1}{\beta - 1}.$$

Now let $\beta \to 1^+$ then $\dfrac{\beta^c - 1}{\beta - 1} \to c$ and so

$$\limsup_{x\to\infty} \frac{A(x)}{x^{c-1}} \leq cL.$$

Now consider any α strictly between 0 and 1, then a similar argument applied to $A_1(x) - A_1(\alpha x)$ gives

$$\liminf_{x\to\infty} \frac{A(x)}{x^{c-1}} \geq L\frac{(1 - \alpha^c)}{1 - \alpha}$$

and as $\alpha \to 1^-$, the righthand side tends to cL and hence $\dfrac{A(x)}{x^{c-1}} \to cL$ as $x \to \infty$.
Thus we have

Theorem 1:
$\psi_1(x) \sim \dfrac{x^2}{2} \Longrightarrow \psi(x) \sim x$ as $x \to \infty$.

Proof: This follows immediately from the Lemma.

Our next step is to represent $\psi_1(x)$ in terms of a complex contour integral involving the Riemann Zeta function. For this task we will need the following Lemmata. (Note that the first Lemma can be generalised, but we only require the simplest cases which are given here.) We use the (standard) notation

$$\int_{c-\infty i}^{c+\infty i} f(t)\,dt = \lim_{L\to\infty}\int_{c-Li}^{c+Li} f(t)\,dt$$

assuming the limit exists.

Lemma 2 Suppose $c > 0$ and $u > 0$ then

$$\text{a)} \qquad \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} \frac{u^{-z}}{z(z+1)}\,dz = \begin{cases} 1-u & \text{if } 0 < u \le 1 \\ 0 & \text{if } u > 1 \end{cases}$$

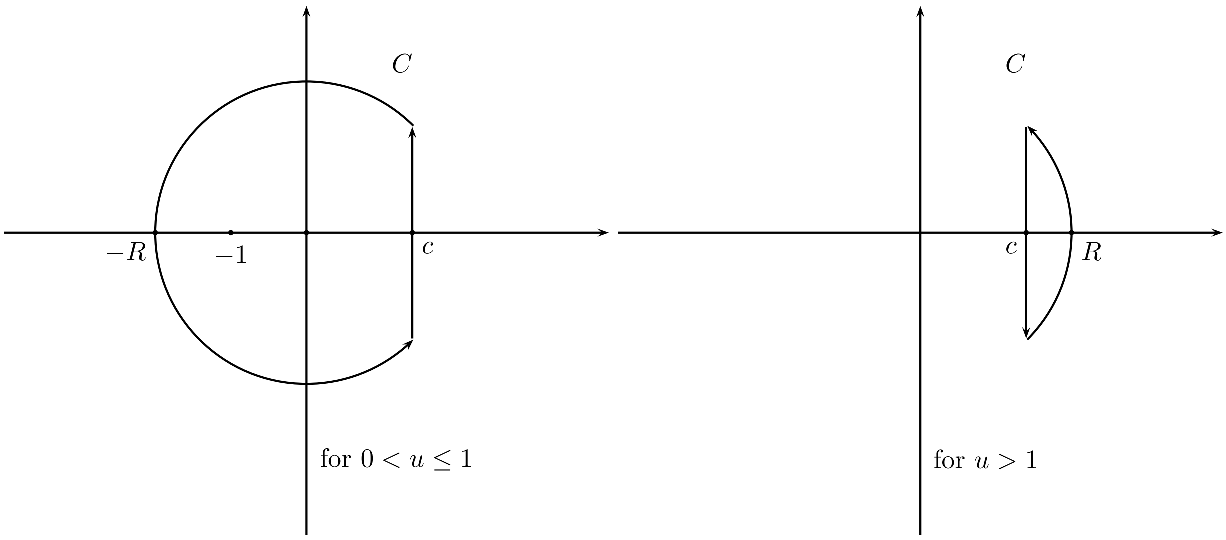
$$\text{b)} \qquad \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} \frac{u^{-z}}{z(z+1)(z+2)}\,dz = \begin{cases} \frac{1}{2}(1-u)^2 & \text{if } 0 < u \le 1 \\ 0 & \text{if } u > 1 \end{cases}$$

Proof: We will only prove (a) here, and note that (b) is done similarly.

Consider the integral

$$I = \frac{1}{2\pi i}\int_C \frac{u^{-z}}{z(z+1)}\,dz$$

where C is the appropriate contour shown below according as $0 < u \le 1$ or $u \ge 1$.



The segments shown are of radius $R > 2$ and we let C' refer to the curved section of either contour. Also let $f(z) = \frac{u^{-z}}{z(z+1)}$.

Now in either diagram, for $z \in C'$, we can estimate:

$$\left|\frac{u^{-z}}{z(z+1)}\right| \le \frac{u^{-x}}{|z||z+1|} \le \frac{u^{-c}}{R|z+1|}$$

since u^{-x} is increasing for $0 < u \le 1$ and decreasing for $u > 1$. Now again for $z \in C'$, $|z+1| \ge |z| - 1 = R - 1 \ge \frac{R}{2}$ and so

$$\left|\frac{1}{2\pi i}\int_{C'} \frac{u^{-z}}{z(z+1)}\,dz\right| \le \frac{1}{2\pi}.2\pi R.\frac{2u^{-c}}{R^2} \to 0$$

as $R \to \infty$.

Now for $u > 1$, $f(z)$ is analytic in C so the integral is zero.

For $0 < u \le 1$, there are simple poles in C at $z = 0, z = -1$. The corresponding residues are clearly 1 and $-u$ and so, in this case

$$\frac{1}{2\pi i} \int_C \frac{u^{-z}}{z(z+1)}\, dz = (1-u)$$

by the residue theorem. Letting $R \to \infty$ we have the desired result.

Lemma 3.

a) For any arithmetic function $a(n)$, let $A(x) = \sum_{n \le x} a(n)$, where $A(x) = 0$ for $x < 1$, then

$$\sum_{n \le x} (x-n)a(n) = \int_1^x A(t)\, dt.$$

b) $\psi_1(x) = \sum_{n \le x} (x-n)\Lambda(n)$

Proof: (a) Using Abel's identity (Theorem 6.1), with $f(t) = t$ we have

$$\sum_{n \le x} a(n)f(n) = \sum_{n \le x} na(n) = xA(x) - \int_1^x A(t)f'(t)\, dt$$

and so

$$-\sum_{n \le x} na(n) + x \sum_{n \le x} a(n) = \int_1^x A(t)\, dt$$

and the result follows.

(b) Put $a(n) = \Lambda(n)$ and $A(x) = \psi(x)$ in (a).

Recall from the Corollary to Theorem 2.8 that

$$\text{(a)} \qquad \zeta'(s) = -\sum_{n=1}^{\infty} \frac{\log n}{n^s}$$

$$\text{(b)} \qquad -\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}.$$

We can now state and prove the following key result:

Theorem 2: If $c > 1$ and $x \ge 1$ then

$$\frac{\psi_1(x)}{x^2} = \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} \frac{x^{s-1}}{s(s+1)} \left(-\frac{\zeta'(s)}{\zeta(s)} \right) ds.$$

Proof: From Lemma 3(b), we know that $\dfrac{\psi_1(x)}{x} = \sum_{n \le x} (1 - \dfrac{n}{x})\Lambda(n)$ and from Lemma 2(a) above, with $u = \frac{n}{x}$, we obtain

$$1 - \frac{n}{x} = \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} \frac{\left(\frac{x}{n}\right)^s}{s(s+1)}\, ds.$$

Multiplying by $\Lambda(n)$ and summing over all $n \le x$, we have

$$\frac{\psi_1(x)}{x} = \sum_{n \le x} \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} \frac{\Lambda(n)\left(\frac{x}{n}\right)^s}{s(s+1)}\, ds$$

$$= \sum_{n=1}^{\infty} \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} \frac{\Lambda(n) \left(\frac{x}{n}\right)^s}{s(s+1)}\, ds$$

since if $n > x$ then $u = \frac{n}{x} > 1$ and so by Lemma 3 the integral vanishes.

We now wish to interchange the summation and the integral which is allowable, by the dominated convergence theorem, provided $\sum_{n=1}^{\infty} \int_{c-\infty i}^{c+\infty i} \left| \frac{\Lambda(n) \left(\frac{x}{n}\right)^s}{s(s+1)} \right| ds$ converges. It is left as an exercise to show that the partial sums of the above series are bounded by $A \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^c}$, where A is a constant. (See Apostol p.283 for details). Hence

$$\frac{\psi_1(x)}{x} = \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} \sum_{n=1}^{\infty} \frac{\Lambda(n) \left(\frac{x}{n}\right)^s}{s(s+1)}\, ds = \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} \frac{x^s}{s(s+1)} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}\, ds$$

$$= \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} \frac{x^s}{s(s+1)} \left(-\frac{\zeta'(s)}{\zeta(s)} \right) ds.$$

Dividing by x gives the desired result.

We now use the above integral representation to show that

$$\frac{\psi_1(x)}{x^2} - \frac{1}{2}\left(1 - \frac{1}{x}\right)^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(1+it) e^{it \log x}\, dt,$$

where $\int_{-\infty}^{\infty} |h(1+it)|\, dt$ converges.

Once again we need a succession of technical lemmata to achieve this.

Lemma 4: If $f(s)$ has a pole of order k at $s = \alpha$ then $\frac{f'(s)}{f(s)}$ has a simple pole at $s = \alpha$ with residue $-k$.

Proof: Tutorial Exercise

Lemma 6: The function $F(s) = \frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1}$ is analytic at $s = 1$.

Proof: Exercise. (Follows from Lemma 4).

Theorem 3: For $x \geq 1$ we have

$$\frac{\psi_1(x)}{x^2} - \frac{1}{2}(1 - \frac{1}{x})^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(1+it) e^{it \log x}\, dt,$$

where $\int_{-\infty}^{\infty} |h(1+it)|\, dt$ converges.

Proof:
From Lemma 3(b), with $u = \frac{1}{x}$, we have

$$\frac{1}{2}\left(1 - \frac{1}{x}\right)^2 = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \frac{x^s}{s(s+1)(s+2)}\, ds$$

where $c > 0$.

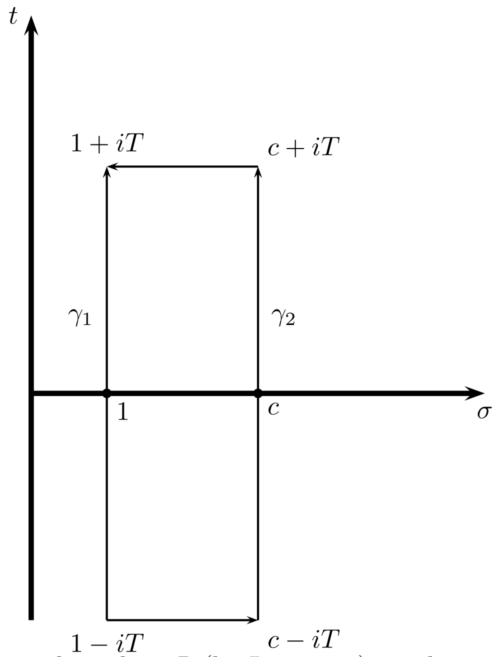
Replacing s by $s-1$ and subtracting the result from the formula in Theorem 2 gives

$$\frac{\psi_1(x)}{x^2} - \frac{1}{2}\left(1 - \frac{1}{x}\right)^2 = \frac{1}{2i\pi} \int_{c'-i\infty}^{c'+i\infty} \frac{x^{s-1}}{s(s+1)} \left[\frac{-1}{s-1} - \frac{\zeta'(s)}{\zeta(s)} \right] ds, \text{ where } c' > 1,$$

$$= \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} x^{s-1} h(s)\, ds \qquad \text{(relabelling } \; c' = c)$$

where $h(s) = \dfrac{-1}{s(s+1)} \left(\dfrac{1}{s-1} + \dfrac{\zeta'(s)}{\zeta(s)} \right)$.

We now use Cauchy's theorem to move the path of integration to the line $\sigma = 1$.
Consider the rectangular contour R shown with $T > e$ (recalling that $c > 1$).



Now $x^{s-1}h(s)$ is analytic inside and on R (by Lemma 6), so the integral around R is zero.
Now consider the integral along the top horizontal segment $-\gamma_3$, where $t = T$. For $s \in -\gamma_3$, clearly, $|s| \geq T, |s+1| \geq |s|$ and $|s-1| \geq T$, so

$$\left| \frac{1}{s(s+1)} \right| \leq \frac{1}{T^2} \quad \text{and} \quad \left| \frac{1}{s(s+1)(s-1)} \right| \leq \frac{1}{T^3} \leq \frac{1}{T^2}.$$

Recall now, from Theorem 4.6, that $\left| \dfrac{\zeta'(s)}{\zeta(s)} \right| \leq M(\log t)^9$, provided $\sigma \geq 1$ and $t \geq e$. Hence for $T \geq e$,

$$|h(s)| \leq \frac{2M(\log T)^9}{T^2} \qquad \text{for sufficiently large} T$$

so, on the top horizontal segment $-\gamma_3$, $s = t + iT$, $1 \leq t \leq c$,

$$\left| \int_{-\gamma_3} x^{s-1} h(s)\, ds \right| \leq \int_1^c x^{t-1} \left| h(t+iT) \right| \, dt \leq \frac{2Mx^{c-1}(\log T)^9}{T^2}(c-1) \to 0$$

as $T \to \infty$.
The same argument holds for the lower horizontal segment.

Hence, as $T \to \infty$,

$$\int_{\gamma_2} x^{s-1} h(s)\, ds = \int_{-\gamma_1} x^{s-1} h(s)\, ds$$

in other words

$$\int_{c-i\infty}^{c+i\infty} x^{s-1} h(s)\, ds = \int_{1-i\infty}^{1+i\infty} x^{s-1} h(s)\, ds.$$

Now on the line $\sigma = 1$, put $s = 1 + it$, then $x^{s-1} = x^{it} = e^{it\log x}$, so

$$\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} x^{s-1}h(s)\,ds = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(1+it)e^{it\log x}\,dt.$$

We can split this integral into the three parts

$$\int_e^{\infty} + \int_{-\infty}^{-e} + \int_{-e}^{e}.$$

Now for $e \le t < \infty$, $|h(1+it)| \le \frac{M(\log t)^9}{t^2}$ so $\displaystyle\int_e^{\infty} |h(1+it)|\,dt$ converges.

Similarly, $\displaystyle\int_{-\infty}^{-e} |h(1+it)|\,dt$ converges, and the third integral is finite. Thus $\displaystyle\int_{-\infty}^{\infty} |h(1+it)|\,dt$ converges.

The final piece in the puzzle is to show that the integral on the right in Theorem 3 goes to zero as $x \to \infty$ and this will complete the proof. To do this we need the follow result from Fourier Analysis.

Lemma 6: (Riemann-Lebesque Lemma).

If $\displaystyle\int_{-\infty}^{\infty} |f(t)|\,dt$ converges then $\displaystyle\int_{-\infty}^{\infty} f(t)e^{itx}\,dt \to 0$ as $x \to \infty$.

Proof: See Analysis 2 course.

Note that we can replace the x in the above by $\log x$.

Applying the Riemann Lebesque lemma to the formula in Theorem 3, we have

$$\int_{-\infty}^{\infty} h(1+it)e^{it\log x}\,dt \to 0 \quad \text{as} \quad x \to \infty$$

and so $\frac{\psi_1(x)}{x^2} \to \frac{1}{2}$ giving $\psi_1(x) \sim \frac{1}{2}x^2$ which implies the PNT.