UNSW AUSTRALIA.
SCHOOL OF MATHEMATICS AND STATISTICS.
MATH5645: TOPICS IN NUMBER THEORY.

§3 ARITHMETIC FUNCTIONS, DIRICHLET MULTIPLICATION:

A function $f : \mathbb{N} \to \mathbb{C}$ is called an *arithmetic* function.

If an arithmetic function f has the property that $f(ab) = f(a)f(b)$, whenever $(a, b) = 1$, then f is said to be *multiplicative.* If $f(ab) = f(a)f(b)$ for **any** positive integers a, b then f is said to be *completely multiplicative.* It is easy to see that $f(1) = 1$ for any multiplicative function f, provided $f \neq 0$.

A multiplicative function is completely determined by its values at prime powers.

In this section we are going to look at a number of important examples of multiplicative functions and define a special multiplication on the set of such functions. The function values of these functions tend to be rather spasmodic but we can *smooth* these out by averaging and then look at developing estimates of their average rate of growth.

Divisor Functions:

We define the well known *number of divisors* and *sum of divisor* functions τ and σ.

$$\tau(n) = \sum_{d|n} 1, \qquad \sigma(n) = \sum_{d|n} d.$$

For example, $\tau(12) = 6, \sigma(12) = 28$.

Note that we may sometimes refer to $\sigma_0(n)$, which is the sum of the *aliquot* factors of n, i.e. $\sigma_0(n) = \sigma(n)-n$.

Also, $\sigma^k(n) = \sum_{d|n} d^k$ is the sum of the kth powers of the divisors.

If $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$ then clearly

$$\tau(n) = \prod_{i=1}^{r} (\alpha_i + 1),$$

since any divisor of n has the form $p_1^{\beta_1} p_2^{\beta_2} \dots p_r^{\beta_r}$, where $0 \le \beta_i \le \alpha_i$, and so there are $\alpha_i + 1$ choices for each exponent.

Similarly

$$\sigma(n) = \prod_{i=1}^{r} (1 + p_i + \cdots + p_i^{\alpha_i}) = \prod_{i=1}^{r} \left(\frac{p_i^{\alpha_i+1} - 1}{p_i - 1} \right).$$

This becomes clear when we think of expanding out

$$(1 + p_1 + p_1^2 + \cdots + p_1^{\alpha_1})(1 + p_2 + p_2^2 + \cdots + p_2^{\alpha_2}) \cdots (1 + p_r + p_r^2 + \cdots + p_r^{\alpha_r})$$

and using the Fundamental Theorem of Arithmetic.

Ex: $n = 1341648 = 2^4 \times 3^2 \times 7 \times 11^3$.

So $\tau(n) = 5 \times 3 \times 2 \times 4 = 120$
and $\sigma(n) = (1 + 2 + 4 + 8 + 16)(1 + 3 + 9)(1 + 7)(1 + 11 + 121 + 1331) = 4719936$.

It is easy to show that both σ and τ are multiplicative (but not completely multiplicative, since $3 = \tau(4) \neq$

$\tau(2)\tau(2) = 4$ and $7 = \sigma(4) \neq \sigma(2)\sigma(2) = 9$.)

The Möbius Function: $\mu(n)$.

This is a very important function in both number theory in general and in analytic number theory in particular.
To motivate the definition, recall that for $s > 1$,

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}.$$

Hence we define the Möbius function $\mu(n)$ by:

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } p^2|n \\ (-1)^r & \text{if } n = p_1 p_2 \ldots p_r \end{cases}$$

Thus $\mu(n)$ is zero if n has a square factor.

We can thus write $\dfrac{1}{\zeta(s)} = \displaystyle\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$.
Ex: $\mu(12) = 0, \mu(30) = -1$.

The Möbius function is, in some sense, a discrete version of the Dirac δ function.

Theorem 3.1:

$$\sum_{d|n} \mu(d) = \lfloor \frac{1}{n} \rfloor = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$$

Proof:

Euler's Totient Function: $\phi(n)$.

We previously defined the Euler *totient* function ϕ by $\phi(1) = 1$ and for $n > 1$,

$$\phi(n) = |\{x : (n, x) = 1\}| = \sum_{k=1}^{n}{}' 1,$$

where the $'$ denotes that the sum is taken over all positive integers k, that are less than and relatively prime to n. We also stated the result:

$$\phi(p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}) = \prod_{i=1}^{r} (p_i^{\alpha_i} - p_i^{\alpha_i - 1}) = n \prod_{p|n} \left(1 - \frac{1}{p}\right),$$

so ϕ is multiplicative.

Multiplicative Functions:

Theorem 3.2: If f is multiplicative then so is $F(n) = \sum_{d|n} f(d)$.

Proof: Let $(m, n) = 1$. A divisor d of mn can be uniquely expressed as $d = d_1 d_2$ where $d_1|m$ and $d_2|n$, with $(d_1, d_2) = 1$. Hence

(Here we have written a sum of products as a product of sums.)

We can use this important result to prove:

Theorem 3.3: For $n \geq 1$, $\sum_{d|n} \phi(d) = n$.

Proof:

Dirichlet Multiplication.

Sums of the form $\sum_{d|n} f(d)g(\frac{n}{d})$ or $\sum_{n=d_1 d_2} f(d_1)g(d_2)$ occur frequently.
Hence we define the *Dirichlet Product*, $f * g$ of two arithmetic functions f, g by:

$$(f * g)(n) = \sum_{d|n} f(d)g(\frac{n}{d}).$$

Theorem 3.4:
(i) $f * g = g * f$
(ii) $(f * g) * h = f * (g * h)$

Proof: Tutorial Exercise.

We define two simple arithmetic functions, the *identity* function and the *unit* function by:

$I(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$, and $u(n) = 1$ for all n.

These functions are trivially completely multiplicative. It is easy to show that $f * I = I * f = f$ and we can write Theorem 3.1 as: $\mu * u = I$, since if $n = 1$ the sum takes the value 1 and is zero otherwise.

Using these ideas we can state and easily prove the important relationship between the Dirichlet Product and the Möbius function.

Theorem 3.5: (Möbius Inversion Formula)

Let $F(n) = \sum_{d|n} f(d)$ then $f(n) = \sum_{d|n} \mu(d)F(\frac{n}{d})$.

Proof: The statement $F(n) = \sum_{d|n} f(d)$ can be translated into $F = f * u$ so $F * \mu = (f * u) * \mu = f * (u * \mu) = f * I = f$.

Translating back we have $f(n) = (F * \mu)(n) = (\mu * F)(n) = \sum_{d|n} \mu(d) F(\frac{n}{d})$.

As an immediate consequence we have:

Theorem 3.6: For $n \geq 1$,

$$\phi(n) = \sum_{d|n} \mu(d) \frac{n}{d}.$$

Proof: Apply Theorem 3.5 to the result $\sum_{d|n} \phi(d) = n$.

Inverses:

We saw above that $\mu * u = I$. We can interpret this as saying that μ and u are **inverses** with respect to Dirichlet multiplication.
Is is true that any (arithmetic) function f has an inverse, f^{-1}, such that $(f * f^{-1})(n) = I(n)$? It can be shown (tutorial exercise), by induction on n, that such a function exists iff $f(1) \neq 0$, and that

$$f^{-1}(n) = -\frac{1}{f(1)} \sum_{d<n, d|n} f(\frac{n}{d}) f^{-1}(d).$$

Corollary: If f is completely multiplicative, then $f^{-1}(n) = \mu(n) f(n)$ for $n \geq 1$.

Proof: Let $g = \mu f$, then $(g * f)(n) = \sum_{d|n} \mu(d) f(d) f(\frac{n}{d}) = f(n) \sum_{d|n} \mu(d) = I(n)$. So $g = f^{-1}$.

Note that $(f * g)^{-1} = f^{-1} * g^{-1}$, whenever the inverses exist.

Introduce the completely multiplicative function N defined by $N(n) = n$ for all $n \geq 1$.

Then the result $\phi(n) = \sum_{d|n} \mu(d) \frac{n}{d}$ can be written as $\phi = \mu * N$.

Thus $\phi^{-1} =$

The Von Mangoldt Function: $\Lambda(n)$.

The Von Mangoldt function Λ plays an important role in the theory of the distribution of primes. For each integer $n \geq 1$, we define

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^\alpha \text{ for some prime } p \text{ and some } \alpha \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

For example $\Lambda(12) = 0, \qquad \Lambda(125) = \log 5$.

Observe that since $\Lambda(1) \neq 1$, Λ is not multiplicative.

Theorem 3.7: For $n \geq 1$, $\displaystyle\sum_{d|n} \Lambda(d) = \log n$.

Proof:

Note also that we can apply the Möbius Inversion Formula to obtain

$$\Lambda(n) = \sum_{d|n} \mu(d) \log\left(\frac{n}{d}\right)$$

$$= \sum_{d|n} \mu(d) \log n - \sum_{d|n} \mu(d) \log d$$

$$= \log n \sum_{d|n} \mu(d) - \sum_{d|n} \mu(d) \log d = -\sum_{d|n} \mu(d) \log d, \quad \text{for } n > 1.$$

Derivatives of Arithmetic Functions:

Definition:
If f is an arithmetic function, we define its derivative f' by the formula

$$f'(n) = f(n) \log n.$$

Ex: $I'(n) = I(n)\log n = 0$ for all n, $\quad u'(n) = \log n$.

Thus the identity $\sum_{d|n} \Lambda(d) = \log n$ (Theorem 3.7) can be written as $\Lambda * u = u'$.

To see that this definition has some point to it, we note:

Theorem 3.8: If f and g are arithmetic functions then

(a) $(f+g)' = f' + g'$.

(b) $(f * g)' = f' * g + f * g'$.

(c) $(f^{-1})' = -f' * (f * f)^{-1}$ provided $f(1) \neq 0$.

Proof: (a) and (c) are left as exercises, (note that for (c) you should start from $f * f^{-1} = I$.)

(b)

$$(f * g)'(n) = \sum_{d|n} f(d)g(\frac{n}{d})\log n = \sum_{d|n} f(d)g(\frac{n}{d})\log(d.\frac{n}{d})$$

$$= \sum_{d|n}(f(d)\log d)g(\frac{n}{d}) + \sum_{d|n} f(d)(g(\frac{n}{d})\log(\frac{n}{d})) = (f' * g)(n) + (f * g')(n).$$

Using this we can prove the celebrated *Selberg Symmetry formula* which was the starting point for the so-called *elementary* proof of the PNT given 'independently' by Selberg and Erdös.

Theorem 3.9: If $n \geq 1$, we have

$$\Lambda(n)\log n + \sum_{d|n} \Lambda(d)\Lambda\left(\frac{n}{d}\right) = \sum_{d|n} \mu(d)\left(\log\frac{n}{d})\right)^2.$$

Proof: We have $\Lambda * u = u'$ so applying the 'product rule', we can write

$$\Lambda' * u + \Lambda * u' = u''$$

Using the formula for u' above we write this as

$$\Lambda' * u + \Lambda * (\Lambda * u) = u''.$$

Recall that $\mu = u^{-1}$, so Dirichlet multiplying both sides by u^{-1} we obtain

$$\Lambda' + \Lambda * \Lambda = u'' * \mu$$

and translating back, we have the desired result.

Dirichlet Series:

In the next few chapters we will be looking in detail at functions defined by series of the form $\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$, where $f(n)$ is an arithmetic function. Such a series is called a *Dirichlet Series*, provided it converges. The following theorem relates Dirichlet series and Dirichlet Multiplication.

Theorem 3.10: Given two Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}, \qquad G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$$

which converge absolutely for $s > \sigma_0$, then for $s > \sigma_0$ we have

$$F(s)G(s) = \sum_{n=1}^{\infty} \frac{h(n)}{n^s}$$

where $h(n) = \sum_{d|n} f(d)g(\frac{n}{d})$, i.e. $h = f * g$.

Proof: For $s > \sigma_0$,

$$F(s)G(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \sum_{m=1}^{\infty} \frac{g(m)}{m^s} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{f(n)g(m)}{(nm)^s}.$$

Now collect the terms for which $mn = k$ for $k = 1, 2, \cdots$ and so

$$F(s)G(s) = \sum_{k=1}^{\infty} \left(\sum_{mn=k} f(n)g(m) \right) k^{-s} = \sum_{k=1}^{\infty} h(k)k^{-s}$$

where $h(k) = \sum_{mn=k} f(n)g(m) = (f * g)(k)$.

We can now prove the following Corollary which will be of importance in later work.

Let $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ for $s > 1$. Then since this series converges uniformly, (by Weierstrass M- test), we can differentiate to obtain $\zeta'(s) = -\sum_{n=1}^{\infty} \frac{\log n}{n^s}$ and hence:

Corollary:

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}.$$

Proof:

Averages of Arithmetic Functions:

As mentioned in the introduction to this chapter, the values of the arithmetic functions behave very erratically. For example the function $\tau(n)$ takes the value 2 infinitely often. We can 'smooth out' these functions by looking at their average rate of growth. If f is an arithmetic function, then we will study the mean,

$$\tilde{f}(n) = \frac{1}{n}\sum_{k=1}^{n} f(k),$$

which we do by looking at $\displaystyle\sum_{k\leq x} f(k)$ for any positive real number x. It is understood that k varies from 1 to $\lfloor x \rfloor$.

To describe the growth rate of these functions, we introduce the (standard) notation $O(f(x))$, also known as Landau's notation.

Definition: Suppose $g(x) > 0$ for all $x \geq a$. If there is a constant M such that $\frac{|f(x)|}{g(x)} < M$ for all $x \geq a$, then we write $f(x) = O(g(x))$, which we read as 'f is of order g', or 'f is big-O g'.

Furthermore, if $\displaystyle\lim_{x\to\infty}\frac{f(x)}{g(x)} = 1$, then we say that f is asymptotic to g and write $f \sim g$.

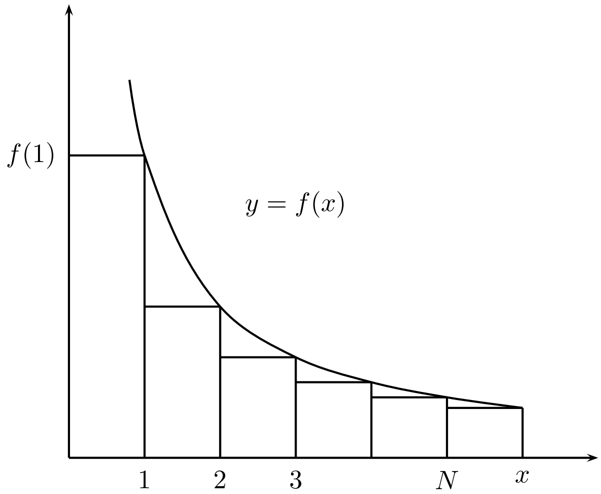
Examples:

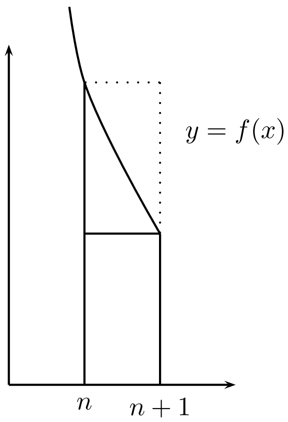
In dealing with sums of the form $\displaystyle\sum_{k\leq x} f(k)$, we will frequently use the following result.

Theorem 3.11 If $f(x)$ is a positive decreasing function, then there is a constant k such that

$$\sum_{n \le x} f(n) = \int_1^x f(t)\,dt + k + O(f(x)).$$

Proof:





On $[n, n+1]$, since f is decreasing, $f(n+1) \le \displaystyle\int_n^{n+1} f(t)\,dt \le f(n)$,
so we define $a_n = f(n) - \int_n^{n+1} f(t)\,dt \le f(n) - f(n+1)$, noting that $a_n \ge 0$.

The number a_n is the area between the curve and rectangle from n to $n+1$.
Now, for positive integers, $0 < M < N$, we have

$$\sum_{n=M}^{N} a_n \le (f(M) - f(M+1)) + (f(M+1) - f(M+2)) + ... + (f(N) - f(N+1)) = f(M) - f(N+1) \le f(M),$$

since f is positive. Thus

$$0 \le \sum_{n=M}^{N} a_n \le f(M) \qquad (*).$$

In the case when $M = 1$, we have $\displaystyle\sum_{n=1}^{N} a_n \le f(1)$ and so the infinite series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges since its partial sums are bounded and increasing. Let $k = \displaystyle\sum_{n=1}^{\infty} a_n$, then k is a constant determined solely by f and not N.

Now $k = \displaystyle\sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} a_n$, and by (*), $\displaystyle\sum_{n=N+1}^{\infty} a_n \le f(N+1)$. Hence $\displaystyle\sum_{n=N+1}^{\infty} a_n = O(f(N+1))$.
Now, referring back to the definition of a_n,

$$k = \sum_{n=1}^{N} a_n + O(f(N+1)) = \sum_{n=1}^{N} f(n) - \int_1^{N+1} f(t)\,dt + O(f(N+1)).$$

Re-arranging,

$$\sum_{n=1}^{N} f(n) = \int_1^{N+1} f(t)\,dt + k + O(f(N+1)).$$

Setting $N = \lfloor x \rfloor$,

$$\sum_{n \le x} f(n) = \int_1^{\lfloor x \rfloor + 1} f(t)\,dt + k + O(f(\lfloor x \rfloor + 1)).$$

Now f is decreasing, so $\displaystyle\int_x^{\lfloor x \rfloor + 1} f(t)\,dt \le f(x)$ and $0 \le f(\lfloor x \rfloor + 1) \le f(x)$ and so

$$\sum_{n \le x} f(n) = \int_1^x f(t)\,dt + \int_x^{\lfloor x \rfloor + 1} f(t)\,dt + k + O(f(x)) = \int_1^x f(t)\,dt + k + O(f(x)).$$

Dirichlet's Divisor Problem:

Lemma: $\displaystyle\sum_{n \le x} \frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right)$, where γ is Euler's constant.

Proof: This follows immediately from Theorem 3.11, noting that as $N \to \infty$, $\displaystyle\sum_{n=1}^{N} \frac{1}{n} - \log N \to \gamma$ and so the K above is Euler's constant.

Hence we have the rough approximation $\displaystyle\sum_{n \le x} \frac{1}{n} = O(\log x)$.

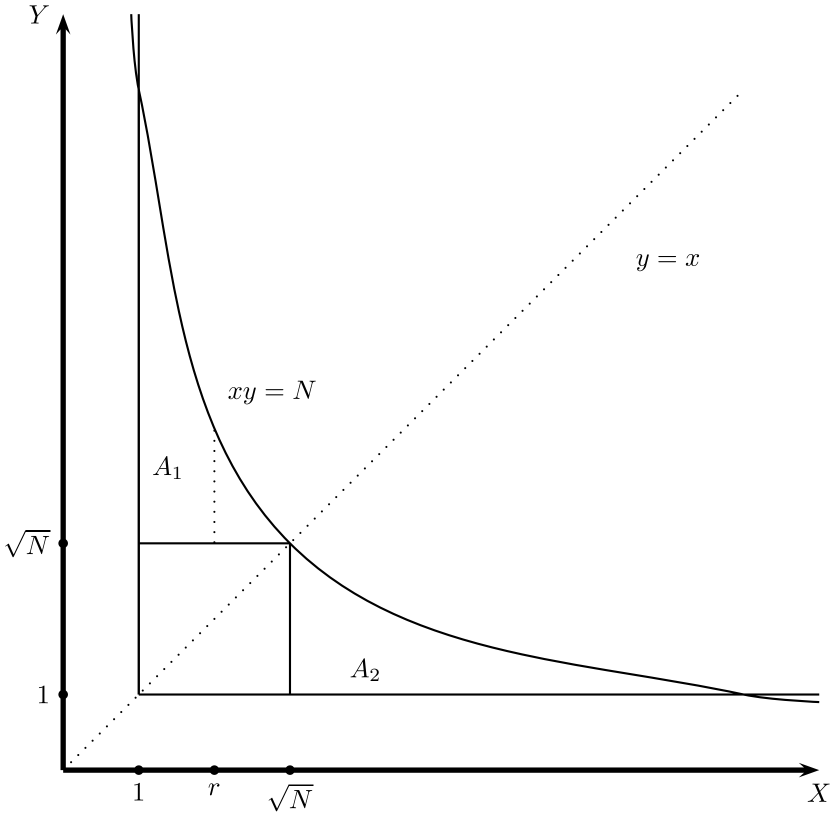
The following result goes back to Dirichlet (1838),

Theorem 3.12: For all $x \ge 1$, we have

$$\sum_{n \le x} \tau(n) = x \log x + (2\gamma - 1)x + O(\sqrt{x}).$$

(Note that this implies that $\tilde{\tau}(n) \sim \log n$.)

Proof:
The function $\tau(n)$ counts the number of lattice points of the form (x, y), lying on the hyperbola $xy = n$ in the first quadrant. Hence the sum $\displaystyle\sum_{n \le N} \tau(n)$ is simply the number of lattice points with positive co-ordinates lying under (and on if N is an integer) the hyperbola $xy = N$.



In the diagram, the number of lattice points lying in the square is $\lfloor\sqrt{N}\rfloor^2$. By symmetry, the number of lattice points in regions A_1 and A_2 are equal, so concentrate on A_1.
On any line $x = r$, $(1 \le r \le \sqrt{N})$, the number of lattice points in A_1 is clearly $\left\lfloor \dfrac{N}{r} \right\rfloor - \lfloor\sqrt{N}\rfloor$. Hence the total number of lattice points is

$$\sum_{n=1}^{N} \tau(n) = 2 \sum_{r=1}^{\lfloor\sqrt{N}\rfloor} \left(\left\lfloor \frac{N}{r} \right\rfloor - \lfloor\sqrt{N}\rfloor \right) + \lfloor\sqrt{N}\rfloor^2.$$

Using the estimate, $\lfloor y \rfloor = y + O(1)$, we can write this as

$$2 \sum_{n=1}^{\lfloor\sqrt{N}\rfloor} \left(\frac{N}{n} + O(1) \right) - 2\lfloor\sqrt{N}\rfloor^2 + \lfloor\sqrt{N}\rfloor^2$$

$$= 2N \sum_{n=1}^{\lfloor\sqrt{N}\rfloor} \frac{1}{n} - \lfloor\sqrt{N}\rfloor^2 + O(\sqrt{N})$$

$$= 2N \sum_{n=1}^{\lfloor\sqrt{N}\rfloor} \frac{1}{n} - N + O(\sqrt{N}).$$

Thus

$$\sum_{n \le x} \tau(n) = 2x \sum_{n=1}^{\lfloor\sqrt{x}\rfloor} \frac{1}{n} - x + O(\sqrt{x})$$

$$= 2x \left(\log x^{\frac{1}{2}} + \gamma + O(\frac{1}{\sqrt{x}}) \right) - x + O(\sqrt{x})$$

$$= x \log x + (2\gamma - 1)x + O(\sqrt{x}).$$

Notes: The error term $O(\sqrt{x})$ has been improved. The determination of the best possible error bound is known as the *Dirichlet Divisor Problem.* The most recent results (of which I am aware) are that $x^{\frac{1}{2}}$ can be replaced by $x^{\frac{7}{22}}$ (Iwaniec and Mossochi 1988, J. No. Th. 29(1), pp. 60-93) and in 1990 Van de Lune and Wattel conjectured $x^{\frac{1}{4}}\log x$. In 2003 Huxley reduced the power of x to $\frac{131}{416}$. Hardy and Landau showed (1915) that $\frac{1}{4}$ was a lower bound for this fraction and $x^{\frac{1}{4}}$ is widely believed to be the true order.

The order of σ:

Lemma 1:
a. $\displaystyle\sum_{n\le x}\frac{1}{n^s}=\frac{x^{1-s}}{1-s}+\zeta(s)+O(x^{-s})$ if $s>1$.
b. $\displaystyle\sum_{n>x}\frac{1}{n^s}=O(x^{1-s})$ for $s>1$.
Proof:

Dirichlet Sum Identities:

The following identities are extremely important in approximating sums over divisors.

Lemma 2: (Dirichlet Sum Identity, DSI)

a. If g is an arithmetic function, then

$$\sum_{n\le x}\sum_{d|n}g(d)=\sum_{n\le x}\sum_{d\le\frac{x}{n}}g(d),\qquad (DSI\quad 1).$$

b. If f,g are arithmetic functions, then

$$\sum_{n\le x}f(n)\sum_{d|n}g(d)=\sum_{d\le x}g(d)\sum_{j\le\frac{x}{d}}f(dj),\qquad (DSI\quad 2).$$

Proof:

We part (b) first and use it to do (a).

$$\sum_{n\le x} f(n) \sum_{d|n} g(d)$$

$$= f(1)g(1) + f(2)[g(1)+g(2)] + f(3)[g(1)+g(3)] + f(4)[g(1)+g(2)+g(4)] + ...$$

$$g(1)[f(1)+f(2)+...] + g(2)[f(2)+f(4)+...] + g(3)[f(3)+f(6)+...] + ...$$

$$= \sum_{d\le x} g(d) \sum_{dj\le x} f(dj) \qquad \text{(take all the multiples of } d \text{ less than } x\text{).}$$

This gives the result.

For (a), take the result in (b) and let $f = 1$, then

$$\sum_{n\le x}\sum_{d|n} g(d) = \sum_{d\le x} g(d) \sum_{j\le \frac{x}{d}} 1.$$

Now the RHS of (a) says

$$\sum_{n\le x}\sum_{d\le \frac{x}{n}} g(d) = \sum_{\substack{n,d \\ nd\le x}} g(d) = \sum_{d\le x} g(d) \sum_{n\le \frac{x}{d}} 1 = \sum_{d\le x} g(d) \sum_{j\le \frac{x}{d}} 1.$$

The result follows.

Theorem 3.13:

$$\sum_{n\le x} \sigma(n) = \frac{\pi^2 x^2}{12} + O(x\log x).$$

Proof:

Thus the average value of $\sigma(x)$ is $\dfrac{\pi^2}{12}x + O(\log x)$, i.e. roughly linear(!).

The order of $\phi(n)$:

Recall that we defined $\mu(n)$ so that

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}$$

for $s > 1$.

In particular we have

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} = \frac{6}{\pi^2} \text{ and } \sum_{n \le x} \frac{\mu(n)}{n^2} = \frac{6}{\pi^2} - \sum_{n > x} \frac{\mu(n)}{n^2}$$

$$= \frac{6}{\pi^2} + O(\sum_{n>x} \frac{1}{n^2}) = \frac{6}{\pi^2} + O(\frac{1}{x})$$

by part (b) of Lemma 1 above.

Theorem 3.14: $\displaystyle\sum_{n \le x} \phi(n) = \frac{3x^2}{\pi^2} + O(x \log x)$.

(Hence the average order of $\phi(x)$ is $\dfrac{3}{\pi^2}x + O(\log x)$.)

Proof: Using the relation $\phi(n) = \displaystyle\sum_{d|n} \mu(d)\frac{n}{d}$, we have

$$\sum_{n \le x} \phi(n) = \sum_{n \le x} \sum_{d|n} \mu(d) \frac{n}{d}$$

$$= \sum_{n \le x} n \sum_{d|n} \frac{\mu(d)}{d} = \sum_{d \le x} \frac{\mu(d)}{d} \sum_{q \le \frac{x}{d}} dq \qquad (DSI \quad 2)$$

$$= \sum_{d \le x} \mu(d) \sum_{q \le \frac{x}{d}} q$$

$$= \sum_{d \le x} \mu(d) \left\{ \frac{1}{2} \left(\frac{x}{d}\right)^2 + O\left(\frac{x}{d}\right) \right\}$$

$$= \frac{1}{2}x^2 \sum_{d \le x} \frac{\mu(d)}{d^2} + O\left(x \sum_{d \le x} \frac{1}{d} \right)$$

$$= \frac{1}{2}x^2 \left\{ \frac{6}{\pi^2} + O(\frac{1}{x}) \right\} + O(x \log x) = \frac{3x^2}{\pi^2} + O(x \log x).$$

Corollary: The probability that two positive integers chosen at random are coprime is $\frac{6}{\pi^2}$.

Proof: The number of pairs of positive integers $\{r, s\}$ satisfying $1 \leq r \leq s \leq n$ is $\binom{n+1}{2} = \frac{1}{2}n(n+1)$. (Choose 2 distinct numbers from the list $\{1, 2, ..., n+1\}$ and subtract 1 from the larger. Alternatively, the answer is $1 + 2 + ... + (n-1) + n = \frac{1}{2}n(n+1)$.)

Also $\sum_{i \leq n} \phi(i)$ equals the number of such pairs which are coprime (by the definition of ϕ). Hence we can define the probability of two integers being coprime to be

$$\lim_{n\to\infty} \frac{\sum_{i\leq n} \phi(i)}{\frac{1}{2}n(n+1)} = \frac{6}{\pi^2} \approx 0.608$$

by the previous theorem.

(Note: The answer above is simply $\frac{1}{\zeta(2)}$. It has been shown that the probability (as defined above) that N positive integers chosen at random are co-prime is $\frac{1}{\zeta(N)}$.)

Orders of μ and Λ:

The average orders of the Möbius and Von Mangoldt functions are more difficult to determine and are in fact quite deep results. We will show in a later section that

$$\text{(i)} \lim_{x\to\infty} \frac{1}{x} \sum_{n\leq x} \mu(n) = 0$$

and

$$\text{(ii)} \lim_{x\to\infty} \frac{1}{x} \sum_{n\leq x} \Lambda(n) = 1.$$

Each of these results is **equivalent** to the prime number theorem.

Furthermore, the sum $\sum_{n=1}^{\infty} \frac{\mu(n)}{n}$ converges and has a sum of zero, but this also is equivalent to the prime number theorem. Here we will show that this series has bounded partial sums.

Theorem 3.15: For all $x \geq 1$, we have

$$\left| \sum_{n\leq x} \frac{\mu(n)}{n} \right| \leq 1,$$

with equality only if $x < 2$.

Proof: