UNIVERSITY OF NEW SOUTH WALES.
SCHOOL OF MATHEMATICS AND STATISTICS
MATH5535
TOPICS IN NUMBER THEORY

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5. THE RIEMANN ZETA FUNCTION:

1 **a.** Show that for $\Re(s) > 1$, $\displaystyle\sum_{n=1}^{\infty} \frac{(\tau(n))^2}{n^s} = \frac{\zeta^4(s)}{\zeta(2s)}$.

b. Recall the definition of $\omega(n)$ from question 2 of Sheet 3. Show that for $\Re(s) > 1$, $\displaystyle\sum_{n=1}^{\infty} \frac{2^{\omega(n)}}{n^s} = \frac{\zeta^2(s)}{\zeta(2s)}$.

2 Prove that the integral $\displaystyle\int_0^{\infty} x^{s-1} e^{-x}\, dx$, which defines the Gamma function, converges for $Re(s) > 0$.

3 Prove that **a.** $\Gamma(s+1) = s\Gamma(s)$ **b.** $\Gamma(n+1) = n!$ **c.** $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

4 Show that $\displaystyle\frac{1}{e^t - 1} - \frac{1}{t}$ can be written as $\displaystyle -\frac{1}{2} + \sum_{n=1}^{\infty} a_n t^n$ with $a_{2n} = 0$.

5 Prove that $\Gamma(s)$ has simple poles only for $s = 0, -1, -2, \ldots$ and show that the residue at the pole $s = -k$ is $\frac{(-1)^k}{k!}$, where k is a non-negative integer.

6 Evaluate **a.** $\zeta(0)$, **b.** $\zeta(-1)$, **c.** $\zeta(-2)$, **d.** $\zeta(-3)$
and show that $\zeta(-2n) = 0$ for $n = 1, 2, 3, \cdots$.

7 Given the well-known infinite product $\displaystyle\sin x = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right)$, use logarithmic differentiation to show that

$$\cot x = \frac{1}{x} - 2\sum_{n=1}^{\infty} \frac{x}{n^2\pi^2 - x^2}.$$

***8** Prove that $\displaystyle\int_0^{\infty} \frac{u^s}{u^2+1}\, du = \frac{\pi}{2} \sec\left(\frac{s\pi}{2}\right)$ for $-1 < s < 1$.

***9** Suppose $\Re(s) > 1$.

a. Use the Euler product to show that $\displaystyle\log\zeta(s) = -\lim_{k\to\infty} \sum_{n=2}^{k} [\pi(n) - \pi(n-1)] \log(1 - \frac{1}{n^s})$.

b. Deduce that $\displaystyle\log\zeta(s) = s\int_2^{\infty} \frac{\pi(x)}{x(x^s - 1)}\, dx$.

10 You may assume the (well-known) result $\displaystyle\int_0^{\infty} \frac{x^{p-1}}{1+x}\, dx = \frac{\pi}{\sin(p\pi)}$, for $0 < p < 1$, from Complex Analysis.

a. Show that $\displaystyle\Gamma(m) = 2\int_0^{\infty} x^{2m-1} e^{-x^2}\, dx$.

b. Use the formula in (a) to show that for $0 < m < 1$,

$$\Gamma(m)\Gamma(1-m) = 2\int_0^{\frac{\pi}{2}} \tan^{1-2m}\theta\; d\theta.$$

(Hint: Express the product as the limit of a double integral over a quarter circle in the first quadrant.)

c. By putting $x = \tan^2\theta$ and $p = 1 - m$, deduce that

$$\Gamma(m)\Gamma(1-m) = \frac{\pi}{\sin(m\pi)}.$$

11 You may assume that $2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2}) = \sqrt{\pi}\Gamma(2z)$ (this is the Legendre duplication formula).
a. Put $z = \frac{1-s}{2}$ to obtain

$$\Gamma(1-s)\sin\frac{\pi s}{2} = \frac{2^{-s}\pi^{\frac{1}{2}}\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}.$$

b. Use the Functional Equation for $\zeta(s)$ to show that

$$\pi^{\frac{-s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{(1-s)}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

c. Now let $\Phi(s)$ denote the left hand side of the formula in (b) above.

Show that $\Phi(s) = \Phi(1-s)$ and that Φ has simple poles at $s = 0$ and $s = 1$.

d. Define $\xi(s) = \frac{1}{2}s(s-1)\Phi(s)$. Explain why $\xi(s)$ is an entire function, with functional equation $\xi(s) = \xi(1-s)$.

***12** Let ξ be the function defined in the previous question.

a. Explain why $\xi(s)$ satisfies the equation $\overline{\xi(s)} = \xi(\bar{s})$.

b. Prove that $\xi(s)$ is real on the lines $t = 0$ and $\sigma = \frac{1}{2}$ and that $\xi(0) = \xi(1) = \frac{1}{2}$.

c. Prove that the zeros of $\xi(s)$ (if any exist) are all situated in the strip $0 < \sigma < 1$ and lie symmetrically about the lines $t = 0$ and $\sigma = \frac{1}{2}$.

13 Let $f(s) = \dfrac{1}{\zeta(s)}$, for $s \neq 1$ and define $f(1) = 0$.
a. Show that the power series for $f(s)$ about $s = 1$ is

$$f(s) = (s-1) - \gamma(s-1)^2 + ...$$

where γ is Euler's constant.
b. Write down the values of $f'(1)$ and $f''(1)$.

14 Show the function $f(t) = \dfrac{t}{e^t - 1} + \dfrac{t}{2}$ is even, thereby confirming that $B_{2n+1} = 0$. Use the formula relating Bernoulli numbers and the zeta function to show that the non-zero Bernoulli numbers alternate in sign.

15 **a.** Starting from the functional equation for $\zeta(s)$, show that

$$\zeta(s) = \frac{(2\pi)^s}{2}\frac{\zeta(1-s)}{2\Gamma(s)\cos(\pi s/2)}.$$

b. Deduce that for $n \geq 1$,

$$\zeta(1-2n) = -\frac{B_{2n}}{2n}.$$

c. Find the value of $\zeta(-3), \zeta(-5), \zeta(-7)$.

16 Use the Bernoulli numbers to find a formula for $\displaystyle\sum_{x=1}^{k-1} x^4$ and $\displaystyle\sum_{x=1}^{k-1} x^5$.

17 Prove that $\displaystyle\underbrace{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}}_{(m,n)=1} \frac{1}{m^2n^2} = \frac{\zeta^2(2)}{\zeta(4)} = \frac{5}{2}.$

BRIEF SOLUTIONS

1 **a.** Easy. You need the power series for $1/(1-x)^3$ and the sum of two consecutive triangular numbers is a square. **b.** Easy

3 Easy!

4 Let $f(t) = \frac{1}{e^t - 1} - \frac{1}{t}$. Show that $f(t) \to -\frac{1}{2}$ as $t \to 0$ and $g(t) = f(t) + \frac{1}{2}$ is odd.

5 For $s < 0$, $\Gamma(s) = \frac{1}{\Gamma(1-s)} \frac{\pi}{\sin(\pi s)}$. Now $\Gamma(1-s) \neq 0$ for $s < 0$, hence we have singularities at $s = 0, -1, -2, \cdots$. and $\lim_{s \to -k} (s+k) \dfrac{1}{\Gamma(1-s)} \dfrac{\pi}{\sin(\pi s)} = \dfrac{1}{\Gamma(1+k)} \dfrac{1}{\cos(\pi k)} = \dfrac{(-1)^k}{k!}$.

6 **a.** $\frac{-1}{2}$ **b.** $\frac{-1}{12}$ **c.** 0 **d.** $\frac{1}{12}$. **7** Easy!

8 Use the standard keyhole contour (C shape) with branch cut along the positive real axis. See hand written solutions.

9 See Rose p. 227.

10 **a.** Put $t = x^2$ in $\Gamma(m) = \displaystyle\int_0^\infty t^{m-1} e^{-t}\, dt$. **b.** From (a), $\Gamma(m)\Gamma(1-m) = 4 \int\int_\Omega x^{2m-1} y^{1-2m} e^{x^2+y^2}\, dxdy$ where Ω is a first quadrant. Pass to polar co-ordinates and the result follows. **c.** Put $x = \tan^2\theta$ and $p = 1 - m$ and use the integral given in the question.

11 **a.** Straightforward. You will need to use the result of the previous question as well. **b.** etc are easy.

12 **a.** Since ξ is real on the real line, $\xi(s) - \overline{\xi(\overline{s})}$ is an analytic function vanishing on the real line and so must be identically zero in the complex plane by analytic continuation. **b.** $\overline{\xi(\sigma + it)} = \xi(\sigma - it) = \xi(1 - \sigma + it)$. The latter will equal $\xi(\sigma + it)$ precisely when $t = 0$ or $1 - \sigma = \sigma$, i.e. $\sigma = \frac{1}{2}$. **c.** Neither of the functions $(s-1)\zeta(s)$ and $\Gamma(\frac{s}{2})$ vanish in the half plane $\Re(s) \geq 1$, so $\xi(s)$ is non-zero in this half plane. The functional equation for $\xi(s)$ implies that $\xi(s) \neq 0$ for $\Re(s) \leq 0$. Hence all the zeros lie in the critical strip. If $\xi(\beta + i\gamma) = 0$ then $\xi(\beta - i\gamma) = 0$ which implies that $\xi(1 - \beta \pm i\gamma) = 0$. The zeros (if they exist) are symmetrically located about the line $\beta = \frac{1}{2}$.

13 **a.** Since $\zeta(s) = g(s)/(s-1)$, where $g(s)$ is differentiable and non-zero at $s = 1$. Hence $f(s) = (s-1)/g(s)$ is well-defined and differentiable at 1, so has a power series about $s = 1$. Write $f(s) = a_0 + a_1(s-1) + a_2(s-1)^2 + O((s-1)^3)$. Using $\zeta(s)f(s) = 1$, expanding and comparing co-efficients, we have $a_0 = 1, a_1 = 1$ and $a_2 = -\gamma$. **b.** $f'(1) = 1, f''(1) = -2\gamma$.

14 Both parts very easy.

15 **a.** Get rid of $\Gamma(1-s)$ and use $\sin 2\theta$. **b.** Get rid of $\zeta(2n)$ in terms of B_{2n}.

16 $\frac{k}{30} k(k-1)(2k-1)(3k^2 - 3k - 1)$, $\frac{1}{12} k^2 (k-1)^2 (2k^2 - 2k - 1)$.

17 For each pair of positive integers p, q, take $r = (p, q)$ then $m = \frac{p}{r}, n = \frac{q}{r}$. Then $\displaystyle\sum_{p=1}^\infty \frac{1}{p^2} \sum_{q=1}^\infty \frac{1}{q^2} =$

$$\left(\underbrace{\sum_{m=1}^\infty \sum_{n=1}^\infty}_{(m,n)=1} \frac{1}{m^2 n^2} \right) \left(\sum_{r=1}^\infty \frac{1}{r^4} \right)$$. The result follows.
