UNSW AUSTRALIA.
SCHOOL OF MATHEMATICS AND STATISTICS.
MATH5645: TOPICS IN NUMBER THEORY.

§5 THE RIEMANN ZETA FUNCTION:

ζ and the Arithmetic Functions:

We have previously seen that, for $s > 1$,

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} = \sum_{k=1}^{\infty} \frac{1}{k^s} \qquad \text{and} \quad \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}.$$

These can be extended to s complex as long as $\Re(s) > 1$.
We will write once and for all $s = \sigma + it$. The first series converges uniformly for $\sigma > 1$ and so defines a analytic function $\zeta(s)$ which we have previously called the *Riemann Zeta Function*. We have also seen a relationship between the Riemann Zeta function and $\Lambda(n)$, viz:

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}.$$

The Riemann Zeta function is intimately related to many of the arithmetic functions we have previously encountered.
Here are some examples of this:

$$(1) \qquad \frac{\zeta(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^s}$$

$$(2) \qquad \frac{\zeta^3(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{\tau(n^2)}{n^s}$$

$$(3) \qquad \frac{\zeta^4(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{(\tau(n))^2}{n^s}.$$

Here are the proofs of (1) and (2). They rely on the general fact that if $f(n)$ is multiplicative then

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_{p \text{ prime}} \left(1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \cdots\right).$$

Formula 3 and the corresponding sum for $\displaystyle\sum_{n=1}^{\infty} \frac{\zeta^2(s)}{\zeta(2s)}$ are in the tutorial problems.

For (1),

The zeta function is also related to the arithmetic functions ϕ and σ. For example, for $s > 2$,

$$
\begin{aligned}
1 + \frac{\sigma(p)}{p^s} + \frac{\sigma(p^2)}{p^{2s}} + \ldots &= 1 + \frac{1}{p^s}(1+p) + \frac{1}{p^{2s}}(1+p+p^2) + \ldots \\
&= \frac{p-1}{p-1} + \frac{1}{p^s}\frac{p^2-1}{p-1} + \frac{1}{p^{2s}}\frac{p^3-1}{p-1} + \ldots \\
= \frac{1}{p-1}\left[p + \frac{1}{p^{s-2}} + \frac{1}{p^{2s-3}} + \ldots\right] &- \frac{1}{p-1}\left[1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \ldots\right] \\
&= \frac{1}{p-1}\left(\frac{p}{1-p^{1-s}} - \frac{1}{1-p^{-s}}\right) = \frac{1}{(1-p^{1-s})(1-p^{-s})}.
\end{aligned}
$$

Thus for $s > 2$,

$$
\sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s} = \prod_p \left(\frac{1}{1-p^{1-s}}\right) \prod_p \left(\frac{1}{1-p^{-s}}\right) = \zeta(s-1)\zeta(s).
$$

Similarly, $\sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)}$, whence we can write $\sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s} = \zeta^2(s) \sum_{n=1}^{\infty} \frac{\phi(n)}{n^s}$.

A more interesting result, is

$$
\log \zeta(s) = s \int_2^{\infty} \frac{\pi(x)}{x(x^s-1)}\, dx.
$$

(These last few results are tutorial exercises).

There are proofs of the Prime Number Theorem which begin from this equation and 'solve' for $\pi(x)$ using a version of the Mellin Transform. (eg. Grosswald 1984).

Extending the Domain:

Notice that for $\sigma > 1$, we can re-arrange the terms of the zeta function as follows:

This gives another proof that $s = 1$ is a simple pole with residue 1. (Compare with Theorem 4.4 of Chapter 4.)

The Gamma Function:

We seek to extend the ζ function further, so that is has an analytic continuation on the whole of the complex plane. To this end we will need to introduce the *Gamma Function* which is defined for $\Re(s) > 0$ by

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x}\, dx,$$

where the integration is carried out along the positive real axis in the x-plane. (It requires checking (tutorial exercise) that the improper integral does in fact converge for $\Re(s) > 0$.)

The notation Γ goes back to Legendre.
This function has the following properties:

(a) $\Gamma(s+1) = s\Gamma(s), \qquad \Gamma(1) = 1.$

(b) $\Gamma(n+1) = n!$, for n a non-negative integer.

(c) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

(d) $\Gamma(s)\Gamma(1-s) = \dfrac{\pi}{\sin \pi s}.$

(a) and (b) are easy to show, and (c) follows from (d). Result (d) is more difficult to show and is left as a tutorial problem. It is in fact this functional equation (d) which enables us to extend Γ analytically to a meromorphic function on $\mathbb{C}$, with simple poles at $s = -n, n = 0, 1, 2, \cdots$ and corresponding residues $\dfrac{(-1)^n}{n!}$.

(This is also left as a tutorial exercise).

The Re-Duplication Formula: The following formula was discovered by Legendre and generalised by Gauss, and will be of use later.

Lemma: For $s, t > 0$

$$\Gamma(s)\Gamma(t) = \Gamma(s+t) \times 2\int_0^{\frac{\pi}{2}} \cos^{2s-1}\theta \sin^{2t-1}\theta \, d\theta.$$

(Note this is often written as

$$\frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)} = B(s,t) = 2\int_0^{\frac{\pi}{2}} \cos^{2s-1}\theta \sin^{2t-1}\theta \, d\theta$$

where $B(s,t)$ is known as the beta function, $B(s,t)$.)

Proof:

Theorem 5.1: (Legendre)
For $\Re(s) > 0$,

$$2\sqrt{\pi}2^{-2s}\Gamma(2s) = \Gamma(s)\Gamma(s+\frac{1}{2}).$$

Proof: As usual, we prove that this for s real, and $s > 0$ and invoke analytic continuation.

Write the previous lemma as

$$\frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)} = 2\int_0^{\frac{\pi}{2}} \cos^{2s-1}\theta \sin^{2t-1}\theta \, d\theta = B(s,t)$$

then $\frac{1}{2}B(\frac{1}{2}, s+\frac{1}{2}) = \int_0^{\frac{\pi}{2}} \sin^{2s}\theta \, d\theta$.

Now let $J = \int_0^{\frac{\pi}{2}} \sin^{2s} 2\theta \, d\theta = \frac{1}{2}\int_0^{\pi} \sin^{2s} u \, du = \int_0^{\frac{\pi}{2}} \sin^{2s} u \, du$, by putting $u = 2\theta$ and using symmetry. Hence $J = \frac{1}{2}B(\frac{1}{2}, s+\frac{1}{2})$. But

$$J = \int_0^{\frac{\pi}{2}} (2\sin\theta\cos\theta)^{2s} \, d\theta = 2^{2s}\int_0^{\frac{\pi}{2}} \sin^{2s}\theta\cos^{2s}\theta \, d\theta = 2^{2s-1}B(s+\frac{1}{2}, s+\frac{1}{2}).$$

Hence

$$\frac{1}{2}B(\frac{1}{2}, s+\frac{1}{2}) = 2^{2s-1}B(s+\frac{1}{2}, s+\frac{1}{2}).$$

Re-writing these in terms of Γ and using properties (a) and (c) above, the result follows.

The following theorem shows the close relationship between the zeta and gamma functions:

Theorem 5.2: For $\Re(s) > 1$,

$$\zeta(s)\Gamma(s) = \int_0^{\infty} (e^t - 1)^{-1} t^{s-1} \, dt. \qquad (*)$$

Proof:

(N.B. The interchange of limit and integral can be justified by invoking the dominated convergence theorem from integration theory. This says that if $\int g < \infty$ and $|f_n| \leq g$ for all n, then $\int f_n \to \int f$ under the assumption that $f_n(x) \to f(x)$ for (almost) all x.

Here we take $f_n = \displaystyle\sum_{k=1}^{n+1} e^{-kt}t^{x-1}$and $|f_n| = t^{x-1}e^{-t}\dfrac{(1-e^{-(n+1)t})}{1-e^{-t}} < t^{x-1}\dfrac{e^{-t}}{1-e^{-t}}$ for all n.)

This integral arises in statistical mechanics and is known as the *Bose-Einstein Integral.*

For example, $\displaystyle\int_0^\infty (e^t-1)^{-1}t^3\,dt = \frac{\pi^4}{15}$.

The Functional Equation:

Our initial aim is to extend the definition of ζ to include the region $-1 < \Re(s) < 0$.

We now proceed by observing that $\dfrac{1}{e^t-1}$ has a simple pole at $t=0$ with residue 1 and using L'Hopital's rule, we find that $\dfrac{1}{e^t-1} - \dfrac{1}{t} \to -\dfrac{1}{2}$ as $t \to 0$. Hence we can write

$$\frac{1}{e^t-1} = \frac{1}{t} - \frac{1}{2} + \sum_{n=1}^{\infty} a_n t^n \qquad \text{for all } t \neq 0.$$

It is left as an exercise to show $a_{2n} = 0$.

Thus, for $\Re(s) > 0$, $\displaystyle\int_0^1 \left(\frac{1}{e^t-1} - \frac{1}{t}\right) t^{s-1}\,dt$ is analytic and for $\Re(s) > 1$ we can write $\int_0^1 t^{s-2}\,dt = \frac{1}{s-1}$.

Write (*) above as

$$\zeta(s)\Gamma(s) = \int_0^1 \frac{1}{e^t-1}t^{s-1}\,dt + \int_1^\infty \frac{1}{e^t-1}t^{s-1}\,dt$$

which implies

$$\zeta(s) = \frac{1}{\Gamma(s)}\left[\int_0^1 \left(\frac{1}{e^t-1} - \frac{1}{t}\right)t^{s-1}\,dt + \frac{1}{s-1} + \int_1^\infty \frac{1}{e^t-1}t^{s-1}\,dt\right]$$

and so we can (once again) extend the definition of ζ from $\Re(s) > 1$ to $\Re(s) > 0$.

Now suppose that $0 < \Re(s) < 1$, then $\dfrac{1}{s-1} = -\displaystyle\int_1^\infty t^{s-2}\,dt$ so taking the $\frac{1}{s-1}$ term into the second integral and combining we can write (*) as

$$\zeta(s)\Gamma(s) = \int_0^\infty \left[\frac{1}{e^t-1} - \frac{1}{t}\right] t^{s-1}\,dt$$

and hence as

$$\zeta(s)\Gamma(s) = \int_0^1 \left[(e^t-1)^{-1} - t^{-1} + \frac{1}{2}\right] t^{s-1}\,dt - \frac{1}{2s} + \int_1^\infty \left[(e^t-1)^{-1} - \frac{1}{t}\right] t^{s-1}\,dt$$

Both of the integrals are convergent for $-1 < \Re(s) < 0$ whilst the function $\dfrac{1}{s\Gamma(s)} = \dfrac{1}{\Gamma(s+1)}$ is analytic at 0. Consequently, we can extend $\zeta(s)$ to a meromorphic function defined on $Re(s) > -1$.
Now observe that for $-1 < \Re(s) < 0$, $\displaystyle\int_1^\infty t^{s-1}\,dt = -\frac{1}{s}$. Hence we can again take the $-\frac{1}{2s}$ term into the second integral and re-combine to obtain

$$\zeta(s)\Gamma(s) = \int_0^\infty \left[\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\right] t^{s-1}\,dt$$

Now recall that $\pi\cot(\pi z) = \dfrac{1}{z} + \displaystyle\sum_{n=1}^\infty \frac{2z}{z^2-n^2}$ which is a standard result obtained using residues (tutorial exercise), whence $\cot z = \dfrac{1}{z} + \displaystyle\sum_{n=1}^\infty \frac{2z}{z^2-n^2\pi^2}$. Then

$$\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2} = \frac{1}{2}\left(\frac{e^t+1}{e^t-1}\right) - \frac{1}{t} = \frac{i}{2}\cot(\frac{it}{2}) - \frac{1}{t}$$

$$= \frac{i}{2}(\frac{2}{it}) - \frac{1}{t} + \sum_{n=1}^\infty \frac{2i(\frac{it}{2})}{2\left((\frac{it}{2})^2 - n^2\pi^2\right)}$$

$$= \sum_{n=1}^\infty \frac{2t}{t^2+4n^2\pi^2}.$$

Hence for $-1 < \Re(s) < 0$ we can write

$$\zeta(s)\Gamma(s) = 2\int_0^\infty \left(\sum_{n=1}^\infty \frac{1}{t^2+4n^2\pi^2}\right) t^s\,dt = 2\sum_{n=1}^\infty \int_0^\infty \frac{t^s}{t^2+4n^2\pi^2}\,dt.$$

Now let $t = 2\pi nu$, then

$$\zeta(s)\Gamma(s) = 2\sum_{n=1}^\infty \int_0^\infty \frac{(2n\pi)^s u^s}{4n^2\pi^2(u^2+1)} 2n\pi\,du = 2\sum_{n=1}^\infty (2n\pi)^{s-1} \int_0^\infty \frac{u^s}{u^2+1}\,du$$

$$= 2(2\pi)^{s-1} \sum_{n=1}^\infty \frac{1}{n^{1-s}} \int_0^\infty \frac{u^s}{u^2+1}\,du = 2(2\pi)^{s-1}\zeta(1-s)\int_0^\infty \frac{u^s}{u^2+1}\,du.$$

If we again look at the case $s = x$, where $0 < x < 1$ is real, then it can be shown using complex anaylsis (exercise), that the last integral is equal to $\frac{\pi}{2}\sec(\frac{\pi s}{2})$ and so

$$\zeta(x)\Gamma(x) = 2(2\pi)^{x-1}\zeta(1-x)\frac{\pi}{2\cos(\frac{\pi x}{2})} = 2(2\pi)^{x-1}\zeta(1-x)\frac{\pi\sin(\frac{1}{2}\pi x)}{\sin(\pi x)}$$

(using double angle formula)

$$= 2(2\pi)^{x-1}\zeta(1-x)\Gamma(x)\Gamma(1-x)\sin(\frac{1}{2}\pi x)$$

by the functional equation for the Gamma function above.
'Thus' we have the so-called *Functional Equation* of Riemann, which states:

$$\zeta(s) = 2(2\pi)^{s-1}\Gamma(1-s)\zeta(1-s)\sin\left(\frac{\pi s}{2}\right).$$

We have established the functional equation for $-1 < \Re(s) < 0$. Now the right-hand side is defined and analytic for

(i) $1 - s \neq 0, -1, -2, \cdots$ (these are the singularities of Γ).

(ii) $\Re(1-s) > -1$ (since this is the domain of ζ at this stage) and for $1 - s \neq 1$ (which is the singularity of ζ.)
So it is analytic provided $\Re(s) < 2$ and $s \neq 0, 1$. Thus both sides are analytic for $-1 < \Re(s) < 1$, (provided $s \neq 0$). Thus this gives us an analytic continuation of ζ to the whole of the complex plane, with only a simple pole at $s = 1$ as we have seen before.

Ex: $\zeta(-1) = -\frac{1}{12}, \zeta(-2) = 0$.

To find $\zeta(0)$, we use the functional equation to write

$$\zeta(0) = \lim_{s \to 0^+} 2(2\pi)^{s-1}\Gamma(1-s)\zeta(1-s)\sin(\frac{\pi s}{2}) = \lim_{s \to 0^+} \frac{1}{\pi}\zeta(1-s)\sin(\frac{\pi s}{2})$$

$$= \lim_{s \to 0^+} \frac{1}{\pi}\left(\frac{-1}{s} + G(1-s)\right)\left(\frac{\pi}{2}s + O(s^3)\right) = -\frac{1}{2},$$

since $\zeta(s) = \frac{1}{s-1} + G(s)$ with $G(s)$ bounded as $s \to 1$.

The Zeros of Zeta:

We will firstly investigate $\zeta(s)$ for $\Re(s) > 1$.

Lemma: If $a_n \neq -1$ for all n and $\displaystyle\sum_{n=1}^{\infty} |a_n|$ converges, then $\displaystyle\prod_{n=1}^{\infty}(1 + a_n)$ converges and is not equal to zero.

Proof: Let $P_n = \displaystyle\prod_{k=1}^{n}(1 + a_k)$.

For $n \geq 2, P_n = (1 + a_n)P_{n-1}$ so

$$P_n - P_{n-1} = a_n P_{n-1} \qquad (*).$$

Suppose $\displaystyle\sum_{n=1}^{\infty} |a_n| = S$. Now

$$|1 + a_n| \leq 1 + |a_n| \leq e^{|a_n|}$$

so

$$|P_n| \leq e^S.$$

Hence $|a_n P_{n-1}| \leq e^S |a_n|$ so $\displaystyle\sum_{n=2}^{\infty} |a_n P_{n-1}| < \infty$ by the comparison test.

By (*), this implies that $\displaystyle\sum_{n=2}^{\infty}(P_n - P_{n-1}) < \infty$ so

$$\sum_{r=2}^{n}(P_r - P_{r-1}) = P_n - P_1$$

tends to a limit as $n \to \infty$. Hence $P_n \to$ a limit P say, as $n \to \infty$.

To show that $P \neq 0$, we show that the product $\displaystyle\prod_{n=1}^{\infty}(1 + a_n)^{-1}$ also converges to some limit Q and that $PQ = 1$ so that neither P nor Q can be zero.

Write $\dfrac{1}{1 + a_n} = 1 - b_n$ then $b_n = \dfrac{a_n}{1 + a_n}$. Now $1 + a_n \to 1$ as $n \to \infty$ so for sufficiently large n, $|1 + a_n| > \dfrac{1}{2}$ and hence $|b_n| < 2|a_n|$ - thus $\displaystyle\sum^{\infty} |b_n| < \infty$ and so $\displaystyle\prod_{n=1}^{\infty}(1 - b_n)$ converges to Q say. Finally, it is obvious that $PQ = 1$ and the result follows.

Specialising to $a_n = -\dfrac{1}{p^s}$, if $n = p$ is a prime and zero otherwise, tells us that the Euler product $\displaystyle\prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$ does not equal zero for any s in the half plane $\Re(s) > 1$, which in turn implies that $\zeta(s) \neq 0$ for $\Re(s) > 1$.

The Riemann Hypothesis:

We can see from the functional equation that $\zeta(-2n) = 0$ for $n = 1, 2, \cdots$.
These are the so-called *trivial zeros* for $\zeta(s)$. We know that from the above discussion that $\zeta(s) \neq 0$ if $\sigma > 1$. Also the functional equation shows that $\zeta(s) \neq 0$ if $\sigma \leq 0$, except for the trivial zeros.

Moreover we showed earlier that if s is real and $s > 0$, $\zeta(s) = \dfrac{s}{s-1} - s\displaystyle\int_1^\infty \frac{\{x\}}{x^{s+1}}\,dx$ so
$\left|\zeta(s) - \dfrac{s}{s-1}\right| < s\displaystyle\int_1^\infty \frac{dx}{x^{s+1}} = 1$. Thus

$$-1 + \frac{s}{s-1} < \zeta(s) < 1 + \frac{s}{s-1} = \frac{2s-1}{s-1}.$$

Now for $\frac{1}{2} < s < 1$, both the left and right-hand sides are negative, so $\zeta(s) \neq 0$ for $\frac{1}{2} < s < 1$ and using the functional equation (and $\zeta(\frac{1}{2}) \neq 0$), we have $\zeta(s) < 0$ if s is real and $0 < s < 1$. We shall soon show that ζ is not zero on the line $1 + it$ for any real t. Hence $\zeta(s)$ is no-where zero outside the critical strip except for the trivial zeros, nor is it zero on the real line in the critical strip.

The functional equation can be written more compactly by using the Reduplication formula (Theorem 5.1), which says

$$2\sqrt{\pi}2^{-2s}\Gamma(2s) = \Gamma(s)\Gamma(s + \frac{1}{2}).$$

We replace s by $\frac{1-s}{2}$ giving

$$2^s\sqrt{\pi}\Gamma(1-s) = \Gamma\left(\frac{1-s}{2}\right)\Gamma\left(1 - \frac{s}{2}\right).$$

Now since $\Gamma(\dfrac{s}{2})\Gamma(1 - \dfrac{s}{2}) = \dfrac{\pi}{\sin(\frac{\pi s}{2})}$, this gives us

$$\Gamma(1-s)\sin(\frac{\pi s}{2}) = \frac{2^{-s}\sqrt{\pi}\Gamma\left(\frac{1-s}{2}\right)}{\Gamma(\frac{s}{2})}.$$

Substituting this into the functional equation we obtain

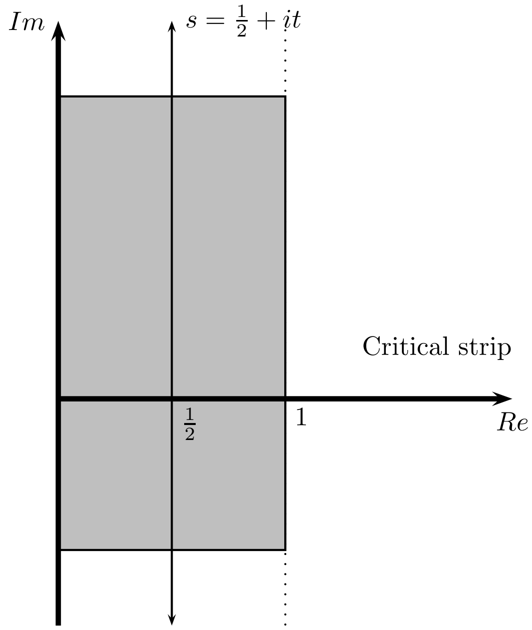
$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

This can now be written as

$$\Phi(s) = \Phi(1-s)$$

where $\Phi(s) = \pi^{-\frac{s}{2}}\Gamma(\dfrac{s}{2})\zeta(s)$, which has simple poles at $s = 0$ and $s = 1$. To remove these, Riemann wrote $\xi(s) = \frac{1}{2}s(s-1)\Phi(s)$ giving the entire function $\xi(s)$, which satisfies the functional equation $\xi(s) = \xi(1-s) = \overline{\xi(\overline{s})}$ and so is real on the line $s = \frac{1}{2} + it$. (See Tutorial exercise).
It is clear that in the critical strip $\zeta(s) = 0 \Leftrightarrow \xi(s) = 0$ and it is also clear (tutorial exercise) that if ξ has any zeros in the critical strip they lie symmetrically about the line $\sigma = \frac{1}{2}$.
We shall presently see that $\xi(s)$ has infinitely many zeros in the critical strip.

At $s = \frac{1}{2}$, the functional equation collapses trivially. Riemann conjectured that in the critical strip, the only zeros are on the line $\Re(s) = \frac{1}{2}$. This conjecture has never been proven and is known as the *Riemann Hypothesis.* Both it and generalisations of it have important ramefications in many branches of Mathematics. Hardy proved that $\zeta(s)$ has infinitely many zeros on the above line and indeed millions of such zeros have been calculated.



Generalisations of the Zeta Function:

There are a number of different ways to generalise the zeta function.

(i) As we have seen, if χ is a Dirichlet character, then $L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$ defines an analytic function for $\Re(s) > 1$. This function can also be analytically extended and the case when χ is a non-principal character leads to another version of the Riemann Hypothesis, (the so-called Generalised Hypothesis).

(ii) The Hurwitz Zeta function, $\zeta(s,a)$ is defined by

$$\zeta(s,a) = \sum_{n=1}^{\infty} \frac{1}{(n+a)^s}$$

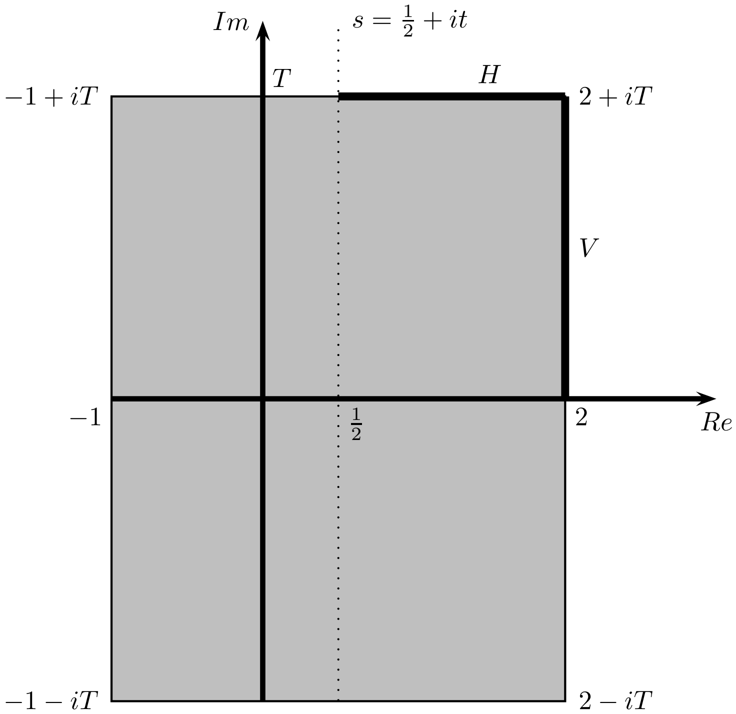
for $\sigma > 1$. As you would expect, results analogous to those for $\zeta(s)$ (including the functional equation) can be obtained for this function.

(iii) There are other generalisations involving the norms of ideals in number fields.

Number of Zeros of $\zeta(s)$ in the Critical Strip:

Since ξ is real on the line $s = \frac{1}{2} + it$, we can use the Intermediate Value Theorem to locate zeros. For example, $\xi(\frac{1}{2} + 14i) = 2 \times 10^{-4}$ and $\xi(\frac{1}{2} + 15i) = -7 \times 10^{-4}$, so there is a zero in this interval. In fact the 'first' zero is at $\frac{1}{2} + i14.134725....$ The next is somewhere between $\frac{1}{2} + 21i$ and $\frac{1}{2} + 22i$.

We will thus look at the number of zeros of $\xi(s)$ rather than those of $\zeta(s)$. We let $N(s)$ denote the number of zeros of $\xi(s)$ in the **upper half plane** intersected with the region R, where R denotes the rectangle with vertices $2 \pm iT$ and $-1 \pm iT$.



$L = V \cup H$

Recall that

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s).$$

Now we know from Complex Analysis that the number N of zeros of a function f, which is analytic in and on a closed contour γ, is given by

$$N = \frac{1}{2\pi i}\oint_{\gamma}\frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi}\left[argf(z)\right]_{\gamma} = \frac{1}{2\pi}\left[Im(\log f(z))\right]_{\gamma},$$

where $[arg(f)]_{\gamma}$ denotes the change in the argument of f as z moves around the contour γ. Let L be the contour $V \cup H$ as shown in the above diagram.

We state without proof, the following results.

Lemma: (a) For any $\delta > 0$,

$$\log\Gamma(z+\alpha) = (z+\alpha-\frac{1}{2})\log z - z + \frac{1}{2}\log 2\pi + O(\frac{1}{|z|}),$$

uniformly for $-\pi + \delta \leq arg(z) \leq \pi - \delta$, and for any bounded α.

This can be obtained using a generalisation of Stirling's formula, $\Gamma(x+1) \sim e^{-x}x^{x-\frac{1}{2}}\sqrt{2\pi}$.

(b) $[arg(\zeta(s)]_L = O(\log T)$, where L is defined above.

Theorem 5.3: For $T \geq 2$, (and T not equal to any of the zeros of $\xi(s)$), we have

$$N(T) = \frac{T}{2\pi}\log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + O(\log T),$$

where the constant implied in the error term is absolute.

Proof: From the above comments, we have

$$2N(T) = \frac{1}{2\pi}\left[arg(\xi(s))\right]_R = \frac{1}{2\pi}\left[arg(\frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s))\right]_R$$

$$= \frac{1}{2\pi}\left[arg(\frac{1}{2}s(s-1))\right]_R + \frac{1}{2\pi}\left[arg(\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s))\right]_R.$$

Clearly $\left[arg(\frac{1}{2}s(s-1))\right]_R = 4\pi$.
Now since $\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$ takes equal values at the points s and $1-s$ and conjugate values at the points $\sigma \pm it$, it follows that

$$\left[arg(\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s))\right]_R = 4\left[arg(\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s))\right]_L,$$

where $L = V \cup H$ as shown in the diagram.

So far then, we have

$$N(T) = 1 + \frac{1}{\pi}\left[arg(\pi^{-\frac{s}{2}})\right]_L + \frac{1}{\pi}\left[arg(\Gamma\left(\frac{s}{2}\right))\right]_L + \frac{1}{\pi}\left[arg(\zeta(s))\right]_L.$$

Clearly, $\left[arg(\pi^{-\frac{s}{2}})\right]_L = \left[Im(\log(\pi^{-\frac{s}{2}}))\right]_L = \left[-\frac{t}{2}\log\pi\right]_0^T = -\frac{1}{2}T\log\pi.$

Also,

$$\left[arg\Gamma(\frac{s}{2})\right]_L = \left[Im(\log\Gamma(\frac{s}{2}))\right]_{s=2}^{s=\frac{1}{2}+iT} = Im(\log\Gamma(\frac{1}{4}+i\frac{T}{2})) - \log\Gamma(1))$$

$$= Im\left\{(-\frac{1}{4}+i\frac{T}{2})\log(\frac{iT}{2}) - i\frac{T}{2} + \frac{1}{2}\log 2\pi\right\} + O\left(\frac{1}{T}\right),$$

using the Lemma (a), with $z = i\frac{T}{2}, \alpha = \frac{1}{4}$.

Simplifying this, we have $\frac{1}{2}T\log\frac{T}{2} - \frac{T}{2} - \frac{\pi}{8} + O(\frac{1}{T})$.

Thus $N(T) = \frac{T}{2\pi}\log(\frac{T}{2\pi}) - \frac{T}{2\pi} + \frac{7}{8} + \frac{1}{\pi}\left[arg\zeta(s)\right]_L + O(\frac{1}{T})$. Using the above Lemma, part (b) gives the desired result.

This shows that there are infinitely many zeros in the critical strip. Furthermore, Selberg was able to show that there is a constant $c > 0$ such that for all sufficiently large T, the number of zeros on the line $\Re(s) = \frac{1}{2}$, with $0 < t < T$ is at least $c\frac{T}{2\pi}\log\frac{T}{2\pi}$. This means that a positive proportion of the zeros are on the critical line. Also, all the *known* zeros are simple and in 1970 it was shown by Montgomery that if the RH is true, then the proportion of simple zeros on the critical line is at least $\frac{2}{3}$.

Merten's Conjecture:

Let $M(x) = \sum_{n \le x}\mu(n)$, where μ is the Möbius function. By definition $|\mu(n)| \le 1$ for all n so $M(x) = O(x)$.
In 1897 Mertens conjectured that $|M(x)| \le \sqrt{x}$, but this was shown to be false in 1985 by Odlyzko and te Riele. A weaker conjecture is that for all $\epsilon > 0$ and $x > 0$,

$$M(x) = o(x^{\frac{1}{2}+\epsilon})$$

in the sense that $\frac{M(x)}{x^{\frac{1}{2}+\epsilon}} \to 0$ as $x \to \infty$. This is, in fact, equivalent to the Riemann Hypothesis.
It is probably false that $M(x) = O(\sqrt{x})$, (Stieltjes claimed this in 1885) but no proof has been found.

Furthermore, it can be shown that the Prime Number Theorem is equivalent to the statement $M(x) = o(x)$ and so the Riemann Hypothesis implies the Prime Number Theorem, (but not, of course, conversely.)

The Line $1 + it$.

In this section we show that $\zeta(1 + it) \neq 0$ for any real t.

Lemma: For $\sigma > 1$, we have

$$\zeta^3(\sigma)|\zeta(\sigma + it)|^4|\zeta(\sigma + 2it)| \geq 1.$$

Proof:
For $\Re(s) > 1$, we have the representation

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}.$$

So for $\sigma > 1$, we can take logarithms and write

$$\log\{\zeta(\sigma + it)\} = -\sum_p \log(1 - p^{-\sigma - it})$$

$$= \sum_p \sum_{k=1}^{\infty} \frac{1}{k} p^{-k(\sigma + it)}.$$

Now equating the real parts

$$\log|\zeta(\sigma + it)| = \sum_p \sum_{k=1}^{\infty} \frac{1}{k} p^{-k\sigma} \cos(kt \log p).$$

Now it is easy to show that $3 + 4\cos\phi + \cos 2\phi = 2(1 + \cos\phi)^2 \geq 0$ and so using $\phi = kt\log p$, multiplying by $\frac{1}{k}p^{-k\sigma}$ and summing, we can write

$$\sum_p \sum_{k=1}^{\infty} \frac{1}{k} p^{-k\sigma}[3 + 4\cos(kt\log p) + \cos(2kt\log p)] \geq 0.$$

Writing this in terms of logs of zeta, we have

$$3\log\zeta(\sigma) + 4\log|\zeta(\sigma + it)| + \log|\zeta(\sigma + 2it)| \geq 0$$

and exponentiating, for $\sigma > 1$ and all real t, it follows that

$$\zeta^3(\sigma)|\zeta(\sigma + it)|^4|\zeta(\sigma + 2it)| \geq 1.$$

Theorem 5.4: $\zeta(1 + it) \neq 0$ for all real t.

Proof:

Rewrite the formula from the preceeding lemma as

$$[(\sigma - 1)\zeta(\sigma)]^3 \left|\frac{\zeta(\sigma + it)}{\sigma - 1}\right|^4 |\zeta(\sigma + 2it)| \geq \frac{1}{\sigma - 1}, \qquad (\#)$$

which is valid for $\sigma > 1$.

Now let $\sigma \to 1^+$. For $t = 0$, $\zeta(\sigma) \to \infty$ as $\sigma \to 1^+$, so we may suppose that $t \neq 0$.

Since ζ has a simple pole with residue 1 at $s = 1$, the first factor tends to 1. The third factor tends to $|\zeta(1 + 2it)|$ which is finite (the only pole is at $s = 1$). Now IF $\zeta(1 + it) = 0$, then we can write

$$\frac{\zeta(\sigma + it)}{\sigma - 1} = \frac{\zeta(\sigma + it) - \zeta(1 + it)}{\sigma - 1} \to \zeta'(1 + it) \quad \text{as } \sigma \to 1^+.$$

Thus the LHS of (#) is bounded (for $t \neq 0$) as $\sigma \to 1^+$, while the RHS tends to infinity. This is a contradiction.

Bounds on ζ and its derivative:

We now proceed to find some bounds on $\zeta(s)$ and its derivative which we will need in our proof of the Prime Number Theorem.

We begin with the following summation formula due to Euler.

Lemma: (*Euler-Summation Formula*).

If f has a continuous derivative on the interval $[0, N]$, where N is a positive integer, then

$$\sum_{n=1}^{N} f(n) = \int_0^N f(t)\, dt + \int_0^N (t - \lfloor t \rfloor) f'(t)\, dt.$$

Proof: Clearly,

$$\int_{n-1}^{n} \lfloor t \rfloor f'(t)\, dt = \int_{n-1}^{n} (n - 1) f'(t)\, dt$$
$$= (n - 1)[f(n) - f(n - 1)] = [nf(n) - (n - 1)f(n - 1)] - f(n).$$

Summing from $n = 1$ to N,

$$\int_0^N \lfloor t \rfloor f'(t)\, dt = \sum_{n=1}^{N} \int_{n-1}^{n} \lfloor t \rfloor f'(t)\, dt$$
$$= \sum_{n=1}^{N} [nf(n) - (n - 1)f(n - 1)] - \sum_{n=1}^{N} f(n)$$
$$= Nf(N) - \sum_{n=1}^{N} f(n).$$

Thus

$$\sum_{n=1}^{N} f(n) = Nf(N) - \int_0^N \lfloor t \rfloor f'(t)\, dt.$$

Now, using integration by parts,

$$\int_0^N t f'(t)\, dt = Nf(N) - \int_0^N f(t)\, dt$$

and substituting back we have the desired result.

Now assuming the necessary integrals and series converge, we can change variables and take limits to obtain:

$$\sum_{n=M+1}^{\infty} f(n) = \int_M^\infty f(t)\, dt + \int_M^\infty (t - \lfloor t \rfloor) f'(t)\, dt.$$

Hence for $\sigma > 1$,

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} + \sum_{n=N+1}^{\infty} \frac{1}{n^s}$$

$$= \sum_{n=1}^{N} \frac{1}{n^s} + \int_N^\infty \frac{1}{x^s}\,dx - s\int_N^\infty \frac{x-\lfloor x \rfloor}{x^{s+1}}\,dx$$

$$\text{so} \qquad \zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - s\int_N^\infty \frac{x-\lfloor x \rfloor}{x^{s+1}}\,dx. \qquad \dagger$$

We can now state some bounds on ζ and its derivative.

Theorem 5.5:

A: When $\sigma \geq 1$ and $t \geq 2$ we have

$$|\zeta(\sigma + it)| \leq \log t + 4 \qquad (\leq M \log t).$$

B: When $\sigma \geq 1$ and $t \geq 2$ we have

$$|\zeta'(\sigma + it)| \leq \frac{1}{2}(\log t + 3)^2 \qquad (\leq M(\log t)^2).$$

Proof: We prove only part A. Part B is similar but more intricate.

From the above formula (†),

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - s\int_N^\infty \frac{x-\lfloor x \rfloor}{x^{s+1}}\,dx$$

$$= \sum_{n=1}^{N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} + r_N(s).$$

Let $N = \lfloor t \rfloor$, then $N \leq t < N+1$, hence $N \geq 2$. Recall also that $\sigma \geq 1, t \geq 2$.

Then

$$|r_N(s)| = \left| -s\int_N^\infty \frac{x-\lfloor x \rfloor}{x^{s+1}}\,dx \right| \leq |s| \int_N^\infty \frac{1}{x^{\sigma+1}}\,dx \leq \frac{|s|}{\sigma N^\sigma}$$

$$\leq \left(\frac{\sigma + t}{\sigma}\right)\frac{1}{N^\sigma} \leq (1+t)\frac{1}{N} \leq \frac{N+2}{N} \leq 2.$$

Also,

$$\left| \sum_{n=1}^{N} \frac{1}{n^s} \right| \leq \sum_{n=1}^{N} \frac{1}{n} \leq \log N + 1 \leq \log t + 1.$$

And finally,

$$\left| \frac{N^{1-s}}{s-1} \right| \leq \frac{1}{t} \leq \frac{1}{2}$$

since $|s-1| \geq t$.

Thus, $|\zeta(s)| = |\zeta(\sigma + it)| \leq (\log t + 1) + 2 + \dfrac{1}{2} \leq \log t + 4.$

Theorem 5.6: There is a constant M such that

$$\frac{1}{|\zeta(s)|} < M(\log t)^7 \quad \text{and} \quad \left| \frac{\zeta'(s)}{\zeta(s)} \right| < M(\log t)^9,$$

whenever $\sigma \geq 1$ and $t \geq e$.

The proof of this is rather long, see Apostol pp. 287-9, or Jamieson p. 108. The logarithmic derivative of $\zeta(s)$ plays an important role in the proof of the PNT.

Laurent Series for $\zeta(s)$.

Since $\zeta(s)$ has a simple pole at $s = 1$ with residue 1, we have

$$\zeta(s) = \frac{1}{s-1} + a_0 + a_1(s-1) + ...$$

What is the value of a_0?

Lemma
The expression $\displaystyle\sum_{r=1}^{n} \frac{1}{r} - \log n$ tends to the limit γ as $n \to \infty$.
The number γ may be written as

$$\gamma = 1 - \int_1^\infty \frac{t-[t]}{t^2}\,dt.$$

Proof: The existence of γ as the limit of $\displaystyle\sum_{r=1}^{n} \frac{1}{r} - \log n$ is well-known. Using the euler-summation formula with $f(t) = \frac{1}{t}$,

$$\sum_{n=2}^{N} \frac{1}{n} = \int_1^N \frac{1}{t}\,dt - \int_1^N \frac{(t-[t])}{t^2}\,dt$$

so

$$\sum_{n=1}^{N} \frac{1}{n} - \log N = 1 - \int_1^N \frac{(t-[t])}{t^2}\,dt.$$

As $N \to \infty$, the LHS approaches γ and the result follows.

Theorem 5.7: $\zeta(s) - \frac{1}{s-1} \to \gamma$ as $s \to 1$.

Hence $\zeta(s)$ has Laurent expansion

$$\zeta(s) = \frac{1}{s-1} + \gamma + \sum_{n=1}^{\infty} c_n(s-1)^n.$$

The constants c_n are called the Stieltjes Constants. It can be shown using CIF that

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-inx}\zeta(e^{ix}+1)\,dx.$$

Proof: This follows from equation † above Theorem 5.5.

Corollary: On some punctured disc with centre $s = 1$, we have

$$\frac{1}{\zeta(s)} = (s-1) - \gamma(s-1)^2 + ...$$

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s-1} + \gamma + a_1(s-1) + ...$$

Proof:

Bernoulli Numbers:

In this final section we will develop a nice connection between the values of the zeta function at even integer arguments and the Bernoulli numbers.

There are a number of ways to motivate the definition of the Bernoulli numbers. For our purposes we define them by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n t^n}{n!} = B_0 + B_1 t + \frac{B_2 t^2}{2!} + \cdots$$

Observe that multiplying by $e^t - 1$ gives

$$t = \sum_{m=1}^{\infty} \frac{t^m}{m!} \sum_{n=0}^{\infty} \frac{B_n t^n}{n!}$$

and equating co-efficients, we have $B_0 = 1$ and (if $k+1 = m+n$), the co-efficient of t^{k+1} on both sides gives:

$$0 = \frac{B_0}{(k+1)!} + \frac{B_1}{k!1!} + \frac{B_2}{(k-1)!2!} + \cdots + \frac{B_k}{(k)!1!}$$

$$= \frac{1}{(k+1)!}\left[\frac{(k+1)!B_0}{(k+1)!} + \frac{(k+1)!B_1}{k!1!} + \cdots + \frac{B_k(k+1)!}{k!1!}\right]$$

$$= \frac{1}{(k+1)!} \sum_{j=0}^{k} B_j \binom{k+1}{j}.$$

So, with $B_0 = 1$, and, for $k \geq 1$, $\sum_{j=0}^{k} B_j \binom{k+1}{j} = 0$, we can compute $B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, B_5 = 0, B_6 = \frac{1}{42}, \cdots$.

From these we surmise that:

Corollary: If $n > 0$ then $B_{2n+1} = 0$.

Proof: $B_1 = -\frac{1}{2}$ so

$$\frac{t}{e^t - 1} + \frac{t}{2} = 1 + \sum_{n=2}^{\infty} \frac{B_n t^n}{n!}.$$

Now $f(t) = \frac{t}{e^t-1} + \frac{t}{2}$ is an even function (!) and the result follows.

Our main reason for introducing the Bernoulli numbers here is to show their connection with the Riemann zeta function.

Theorem 5.8: (Euler). If t is a positive integer,

$$2(2t)!\zeta(2t) = (-1)^{t+1}(2\pi)^{2t} B_{2t}.$$

Proof: As before, we use the standard result, for $x \neq k\pi$, $k \in \mathbb{Z}$,

$$\cot x = \frac{1}{x} - 2\sum_{n=1}^{\infty} \frac{x}{n^2\pi^2 - x^2}.$$

Multiplying by x and using geometric series, we have

$$x \cot x = 1 - 2\sum_{n=1}^{\infty} \frac{x^2}{n^2\pi^2 - x^2}$$

$$= 1 - 2\sum_{n=1}^{\infty}\sum_{t=1}^{\infty}\left(\frac{x}{n\pi}\right)^{2t} = 1 - 2\sum_{t=1}^{\infty}\zeta(2t)\left(\frac{x}{\pi}\right)^{2t}.$$

Now $\cot x = i\dfrac{(e^{ix}+e^{-ix})}{(e^{ix}-e^{-ix})}$ so

$$x\cot x = ix\left(\frac{e^{2ix}+1}{e^{2xi}-1}\right) = \frac{1}{2}2ix + \frac{2ix}{e^{2ix}-1} = 1 + \sum_{s=2}^{\infty}\frac{B_s(2ix)^s}{s!} = 1 + \sum_{t=1}^{\infty}\frac{B_{2t}(2ix)^{2t}}{(2t)!},$$

since $B_s = 0$ for s odd $(s \neq 1)$.
Equating the coefficients of $x^{2t}, (t > 0)$ in these expressions for $x\cot x$, yields,
$-2\zeta(2t)\dfrac{1}{\pi^{2t}} = \dfrac{B_{2t}(2i)^{2t}}{(2t)!}$ and the result follows.

Hence $\zeta(2n)$ may be determined for positive integers n,
viz: $\zeta(2) = \dfrac{\pi^2}{6}$, $\quad\zeta(4) = \dfrac{\pi^4}{90}$, $\quad\zeta(6) = \dfrac{\pi^6}{945}$, $\quad$....

It can also be shown (tutorial problem) that for $n \geq 1$, $\zeta(1-2n) = -\dfrac{B_{2n}}{2n}$.

Little is known about $\zeta(2n+1)$. Apery (1979) proved that $\zeta(3)$ is irrational and more recently Rivoal (2001) proved that $\zeta(2n+1)$ is irrational for infinitely many n and that at least one of $\zeta(5), \zeta(7), \zeta(9), \cdots, \zeta(21)$, is irrational.

Asymptotic Formula for B_{2n}.

Theorem 5.8 can be written as

$$|B_{2n}| = \frac{2(2n)!}{(2\pi)^{2n}}\zeta(2n).$$

Now recall Stirling's approximation formula for $n!$, which says $n! \sim \sqrt{2\pi n}(\frac{n}{e})^n$.
Thus for large n,

$$|B_{2n}| \approx \frac{2}{(2\pi)^{2n}}\zeta(2n)\sqrt{4n\pi}\left(\frac{2n}{e}\right)^{2n}.$$

So

$$\frac{|B_{2n}|}{4\sqrt{n\pi}\left(\frac{n}{e\pi}\right)^{2n}} \sim \zeta(2n).$$

Now since $\zeta(2n) = \sum_{k=1}^{\infty}\dfrac{1}{k^{2n}} \to 1$ as $n \to \infty$, it follows that

$$|B_{2n}| \sim 4\sqrt{n\pi}\left(\frac{n}{e\pi}\right)^{2n}.$$

It was shown (by D.J. Leeming in 1989) that $|B_{2n}| < 5\sqrt{n\pi}\left(\dfrac{n}{e\pi}\right)^{2n}$ for $n \geq 2$.

Notes:

1. A prime p is said to be *regular* if p does not divide the numerators of $B_2, B_4, \cdots, B_{p-3}$. In his attempt to prove FLT, E. Kummer showed that the result was true for all regular primes.

2. It can be shown that $\displaystyle\sum_{x=1}^{k-1}x^n = \frac{1}{n+1}\sum_{j=0}^{n}B_j\binom{n+1}{j}k^{n+1-j}$.

So for example, we obtain $\displaystyle\sum_{x=1}^{k-1}x^2 = \frac{1}{3}(B_0k^3 + 3B_1k^2 + 3B_2k) = \frac{1}{6}k(k-1)(2k-1)$,

$\sum_{x=1}^{k-1} x^3 = \frac{1}{4}k^2(k-1)^2$ and so on.