# UNSW AUSTRALIA.
# SCHOOL OF MATHEMATICS AND STATISTICS
# MATH5645
# TOPICS IN NUMBER THEORY

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## 3. ARITHMETIC FUNCTIONS AND DIRICHLET MULTIPLICATION:

**1** **a.** Show that $\prod_{d|n} d = n^{\frac{\tau(n)}{2}}$.

**b.** Use the AM/GM inequality to show that $\dfrac{\sigma(n)}{\tau(n)} \geq \prod_{d|n} d^{\frac{1}{\tau(n)}}$.

**c.** Deduce that $\dfrac{\sigma(n)}{\tau(n)} \geq n^{\frac{1}{2}}$.

**2** Show that for $n > 1$, $\tau(n) \leq 2\sqrt{n}$.

**3** **a.** Prove that $S = \sum_{d|n} \dfrac{1}{d} = \dfrac{\sigma(n)}{n}$ and find the value of $S$ in the case when $n$ is a perfect number.

**b.** Hence show that if $m|n$, $\dfrac{\sigma(n)}{n} \geq \dfrac{\sigma(m)}{m}$.

**c.** Deduce that no proper factor of a perfect number can be perfect.

**4** **a.** Let $n$ be of the form $12k+2$. Suppose that $6k+1$ and $4k+1$ are both prime. Prove that $\tau(n) = \tau(n+1)$. (It is an unsolved problem to show there are infinitely many $n$ such that $\tau(n) = \tau(n+1)$.)

**b.** Prove that $\tau(n)$ is odd, if and only if $n$ is a square.

**5** Let $\sigma^k(n)$ denote the sum of the $k$th powers of the divisors of $n$. Find the formula for $\sigma^k(n)$ in terms of the prime decomposition of $n$.

**6** Use the Möbius Inversion Formula to show that for $n > 1$,

**a.** $\sum_{d|n} \mu(d)\tau(\frac{n}{d}) = 1$ **b.** $\sum_{d|n} \mu(d)\sigma(\frac{n}{d}) = n$ **c.** $\sum_{d|n} \mu(d)\log(d) = -\Lambda(n)$.

**7** For any positive integer $n$, show that

**a.** $\sum_{d|n} \sigma(d) = \sum_{d|n} \frac{n}{d}\tau(d)$ **b.** $\sum_{d|n} \frac{n}{d}\sigma(d) = \sum_{d|n} d\tau(d)$

**8** For any positive integer $n$, show that $\sum_{d|n} (\tau(d))^3 = \left(\sum_{d|n} (\tau(d)\right)^2$.

**9** Suppose $p$ is a fixed prime and $n$ a positive integer. It is known from Algebra, that there is an irreducible polynomial in $\mathbb{Z}_p[x]$ of degree $n$. Also, the product of all monic irreducible polynomials in $\mathbb{Z}_p[x]$ of degree $d$, for each divisor $d$ of $n$, is given by $x^{p^n} - x$. Let $N_d$ be the number of such irreducible polynomials of degree $d$.

**a.** Explain by $p^n = \sum_{d|n} dN_d$.

**b.** Hence find a formula for $N_n$.

**10**

**a.** If $f, g, h$ are arithmetic functions, show that $f * g = g * f$ and $(f * g) * h = f * (g * h)$, where * denotes the Dirichlet product.

**b.** Suppose further that $f, g$ are multiplicative functions, and define

$$F(n) = \sum_{d|n} f(d)g(\frac{n}{d}).$$

Prove that $F$ is multiplicative.

**c.** Let $I(n)$ be the arithmetic function which is identically zero for all $n$ except $n = 1$ and define $I(1) = 1$. Suppose we have an arithmetic function $f$ such that $f(1) \neq 0$, show that there is a unique arithmetic function $f^{-1}$ such that $f * f^{-1}$ and $f^{-1} * f = I$. Show that a recursive formula for $f^{-1}$is given by $f^{-1}(n) = -\frac{1}{f(1)} \sum_{d<n,d|n} f(\frac{n}{d})f^{-1}(d)$..

**d.** Show that if $f$ is multiplicative, then $f^{-1}(p^2) = (f(p))^2 - f(p^2)$, when $p$ is prime.

**e.** Deduce that the set of all arithmetic functions $f$, with $f(1) \neq 0$, is an abelian group under the Dirichlet product.

**11** **a.** Suppose that $f(n)$ is multiplicative. Show that $\sum_{d|n} \mu(d)f(d) = \prod_{p|n}(1 - f(p))$.

**b.** Deduce that $\sum_{d|n} \mu(d)\frac{n}{d} = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$.

**c.** Also deduce that $\sum_{d|n} \mu(d)\phi(d) = \prod_{p|n} (2 - p)$.

**12** Suppose $p > 2$ is a prime. Define $\psi(d)$ to be the number of elements in $\mathbb{Z}_p^*$ with order $d$.

**a.** If $d|p-1$ explain why $\sum_{c|d} \psi(c) = d$.

**b.** Deduce that $\psi(d) = \phi(d)$.

**c.** Hence show that there are $\phi(p-1)$ primitive roots modulo $p$.

**13** If $F(n) = \sum_{d|n} f(d)$ is multiplicative for some arithmetic function $f$, show that $f$ is also multiplicative. (Hint: Use the Möbius inversion formula.)

***14** Give a direct proof of the result: For $n \geq 1$, $\sum_{d|n} \phi(d) = n$.

(Hint: Consider the sets $N_d = \{k : (k, n) = \frac{n}{d}\}$. Show that $|N_d| = \phi(d)$.)

***15**

**a.** Let $\phi_2(n)$ denote the sum of the squares of the numbers $\leq n$ and relatively prime to $n$. Given

$$\sum_{d|n} \frac{\phi_2(d)}{d^2} = \frac{1^2 + 2^2 + \cdots + n^2}{n^2}$$

show that, for $n > 1$, $\phi_2(n) = \frac{1}{3}n^2\phi(n) + \frac{n}{6}\prod_{p|n}(1-p) = \frac{1}{3}n^2\phi(n) + \frac{n}{6}\phi^{-1}(n)$.

**b.** ** Generalise to $\phi_3(n)$.

**16** **a.** Show that $u * u = \tau$.

**b.** Start with the product rule for Dirichlet derivatives, to deduce that $\tau(n)\log n = 2\sum_{d|n} \log d$.

***17** Suppose $f$ and $g$ are arithmetic functions. Prove the following so-called *divisor to sum identities:*

**a.**

$$\sum_{n\leq x} \sum_{d|n} f(d) = \sum_{n\leq x} \sum_{d\leq \frac{x}{n}} f(d). \qquad (DSI\ 1)$$

**b.**

$$\sum_{n\leq x} f(n) \sum_{d|n} g(d) = \sum_{d\leq x} g(d) \sum_{j\leq \frac{x}{d}} f(dj). \qquad (DSI\ 2)$$

**18** Prove that for $x \geq 1$, $\sum_{n \leq x} \mu(n) \lfloor \frac{x}{n} \rfloor = 1$.

(Hint: Write $\lfloor \frac{x}{n} \rfloor$ as $\sum_{j \leq \frac{x}{n}} 1$ and use the DSI 2.)

**19** **a.** Prove that for $x \geq 2$ and $\alpha \geq 0$,

$$\sum_{n \leq x} n^{\alpha} = \frac{x^{\alpha+1}}{\alpha+1} + O(x^{\alpha}).$$

**b.** Let $\sigma_{\alpha}(n) = \sum_{d|n} d^{\alpha}$. Prove that for $x \geq 2, \alpha \geq 2$, we have

$$\sum_{n \leq x} \sigma_{\alpha}(n) = \frac{\zeta(\alpha+1)}{\alpha+1} x^{\alpha+1} + O(x^{\alpha}).$$

**c.** Prove that for $x \geq 2$,

$$\sum_{n \leq x} \log^2 n = x \log^2 x - 2x \log x + 2x + O(\log^2 x).$$

**20** Suppose $f$ is a strictly increasing function from $\mathbb{Z}^+ \to \mathbb{R}$ with $0 < f(1) \leq 1$. Let $n$ be a positive integer and set $m = \lfloor f(n) \rfloor$.
**a.** Prove that

$$\sum_{j=1}^{n} \lfloor f(j) \rfloor + \sum_{j=1}^{m} \lfloor f^{-1}(j) \rfloor = mn + d,$$

where $d$ is the number of lattice points on the graph of $f$. **b.** Find

$$\sum_{j=1}^{n} \lfloor j^{1/3} \rfloor.$$

***21** **a.** Let $S_2(n)$ denote the number of ways to write $n$ as a sum of two integer squares. (Here we count order and negatives as different. For example, $(-1)^2 + 2^2, 2^2 + 1^2$ etc are counted as distinct representations. Thus $S(5) = 8$.

Prove that $\sum_{n=0}^{N} S_2(n) = N\pi + O(\sqrt{N})$.

Note that the error term has been improved to $O(N^{\frac{27}{82}})$. The true order is not known, but it is known that the exponent cannot be replaced by $\frac{1}{4}$.

(Hint: Look at the lattice points in the circle of radius $\sqrt{N}$. Under and over approximate using unit squares.)

**b.** Let $S_3(n)$ denote the number of representations of $n$ as a sum of three squares.

Prove that $\sum_{n=0}^{N} S_3(n) = \frac{4}{3}\pi N^{\frac{3}{2}} + O(N)$.

**22** (A slightly different derivation of the order of $\sum_{n \leq x} \phi(n)$.)

**a.** Prove that for $x \geq 2$, $\sum_{n \leq x} \phi(n) = \frac{1}{2} \sum_{n \leq x} \mu(n) \lfloor \frac{x}{n} \rfloor^2 + \frac{1}{2}$.

(Hint: Write $\phi(n) = \sum_{d|n} \mu(d) \frac{n}{d}$ and apply the DSI 2.)

**b.** Prove that $\sum_{n \leq x} \mu(n) \lfloor \frac{x}{n} \rfloor^2 = \frac{x^2}{\zeta(2)} + O(x \log x)$.

**c.** Deduce that $\sum_{n \leq x} \phi(n) = \frac{1}{2} \frac{x^2}{\zeta(2)} + O(x \log x)$.

**23** **a.** Prove that for $x \geq 2$, $\displaystyle\sum_{n \leq x} \frac{\phi(n)}{n} = \sum_{n \leq x} \frac{\mu(n)}{n} \lfloor \frac{x}{n} \rfloor$.

**b.** Prove that $\displaystyle\sum_{n \leq x} \frac{\mu(n)}{n} \lfloor \frac{x}{n} \rfloor = \frac{x}{\zeta(2)} + O(\log x)$.

**c.** Deduce that $\displaystyle\sum_{n \leq x} \frac{\phi(n)}{n} = \frac{x}{\zeta(2)} + O(\log x)$.

# BRIEF SOLUTIONS

**1** **a.** Let $P = \prod_{d|n} d = \prod_{d|n} \frac{n}{d}$. Thus $P^2 = \prod_{d|n} n = n^{\tau(n)}$. **b.** Follows immediately from AM/GM inequality. **c.** Follows easily.

**2** If $n = k\ell$ then $k \le \sqrt{n}$ or $\ell \le \sqrt{n}$. Suppose the divisors of $n$ are $a_1, a_2, \cdots, a_{\tau(n)}$, in increasing order, then at least half of these are less than or equal to $\sqrt{n}$ (factors and co-factors), hence $\frac{\tau(n)}{2} \le a_{\lceil \frac{\tau(n)}{2} \rceil} \le \sqrt{n}$.

**3** **a.** $S = \sum_{d|n} \frac{1}{d} = \sum_{d|n} \frac{d}{n} = \frac{1}{n}\sum_{d|n} d = \frac{\sigma(n)}{n} = 2$ if $n$ is perfect. **b.** Easy given (a).

**4** Both very easy. **5** $\prod_{i=1}^{r} \frac{(p_i^k)^{\alpha_i+1} - 1}{p_i^k - 1}$. **6** All easy!

**7** Since multiplicative, show true for prime powers. A lot of algebra involved - hand written solutions in folder.

**8** Same method as for previous question.

**9** **a.** Consider the degree of the product of all the monic irreducible polys. **b.** By Möbius inversion, $N_n = \frac{1}{n}\sum_{d|n} \mu(\frac{n}{d}p^d$.

**10** **a.** $(f * g)(n) = \sum_{d|n} f(d)g(\frac{n}{d}) = \sum_{k|n} g(k)f(\frac{n}{k}) = (g * f)(n)$, where we put $kd = n$ and $((f * g) * h)(n) = \sum_{n=q_1q_2} (f{*}g)(q_1)h(q_2) = \sum_{n=q_1q_2}\sum_{q_1=r_1r_2} f(r_1)g(r_2)h(q_2) = \sum_{n=r_1r_2q_2} f(r_1)g(r_2)h(q_2) = \sum_{n=r_1r_2q_1} f(q_1)g(r_1)h(r_2) = \sum_{n=q_1q_2} f(q_1) \sum_{q_2=r_1r_2} g(r_1)h(r_2) = (f * (g * h))(n)$. **b.** $F(mn) = \sum_{d|mn} f(d)g(\frac{mn}{d}) = \sum_{\substack{d_1|m \\ d_2|n}} f(d_1d_2)g(\frac{mn}{d_1d_2}) = \sum_{d_1|m} f(d_1)g(\frac{m}{d_1})\sum_{d_2|n} f(d_2)g(\frac{n}{d_2}) = F(m)F(n)$. **c.** Since $f(1) \ne 0$, define $f^{-1}(1) = \frac{1}{f(1)}$. Suppose that $f^{-1}(k)$ has been defined for all $k < n$. Then we require $\sum_{d|n} f(\frac{n}{d})f^{-1}(d) = I(n) = 0$ so write $\sum_{\substack{d<n \\ d|n}} f(\frac{n}{d})f^{-1}(d) + f(1)f^{-1}(n) = 0$ whence $f^{-1}(n) = -\frac{1}{f(1)}\sum_{\substack{d<n \\ d|n}} f(\frac{n}{d})f^{-1}(d)$ which is defined by assumption. The result follows by induction.**d.** Easy **e.** Follows by (a) and (c).

**11** **a.** Since $F$ is multiplicative, we can evaluate $F$ at each of the prime powers. See Andrewes, p. 244. **b.** easy **c.** easy

**12** **a.** If $d|p-1$ then $x^d \equiv 1 \bmod p$ has exactly $d$ solutions. Also, if $c|d$, then $x^c \equiv 1 \bmod p \Rightarrow x^c \equiv 1 \bmod p$. The result follows. **b.** By Möbius inversion $\psi(d) = \sum_{c|d} \mu(c)\frac{d}{c} = \phi(d)$. **c.** Put $d = p-1$ and the result follows.

**13** Use the Möbius inversion formula to obtain $f(n) = \sum_{d|n} F(d)\mu(\frac{n}{d})$. Now $F$ and $\mu$ are multiplicative and hence so is $F * \mu$.

**14** Let $N = \{0, 1, 2, \cdots, n-1\}$. If $d|n$ then let $N_d = \{k : (k, n) = \frac{n}{d}\}$. Thus $N_d$ consists of all numbers of the form $e\left(\frac{n}{d}\right)$, with $(d, e) = 1$. Then $|N_d| = \phi(d)$ and $N_d \ne N_{d'}$ if $d \ne d'$. Thus as $d$ varies across the divisors of $n$ the (disjoint) sets $N_d$ have $N$ as their union, so $\sum_{d|n} \phi(d) = n$.

**15** **a.** Using the well-known formula for the sums of squares and MIF, $\phi_2(n) = n^2\sum_{d|n}\left(\frac{2d^2+3d+1}{6d}\right)\mu(\frac{n}{d}) = n^2\left(\frac{1}{3}\sum_{d|n} d\mu(\frac{n}{d}) + \frac{1}{2}\sum_{d|n}\mu(\frac{n}{d}) + \frac{1}{6}\sum_{d|n}\frac{\mu(\frac{n}{d})}{d}\right) = \frac{1}{3}n^2\phi(n) + \frac{n^2}{6}\sum_{d'|n}\mu(d)\frac{d'}{n} = \frac{1}{3}n^2\phi(n) + \frac{n}{6}\phi^{-1}(n)$. **b.** Similar argument results in $\phi_3(n) = \frac{n^3}{4}\phi(n) + \frac{n^2}{4}\phi^{-1}(n)$.

**16** **a.** $u * u(n) = \sum_{d|n} 1 = \tau(n)$, **b.** Take Dirichlet derivative and re-arrange.

**17** **a.** $\sum_{n\le x}\sum_{d|n} f(d) = f(1)+(f(1)+f(2))+(f(1)+f(3))+\cdots = f(1)x+f(2)\lfloor\frac{x}{2}\rfloor+f(3)\lfloor\frac{x}{3}\rfloor+\cdots$. In the sum $\sum_{n\le x}\sum_{d\le\frac{x}{n}} f(d)$, $f(1)$ occurs $x$ times, $f(2)$ occurs $\lfloor\frac{x}{2}\rfloor$ times, $f(3)$ occurs $\lfloor\frac{x}{3}\rfloor$ times and so on. Hence the sums are equal. **b.** Expanding out the LHS we have $f(1)g(1)+f(2)(g(1)+g(2))+f(3)(g(1)+g(3))+....$ Now collect terms involving $g(1), g(2)$, etc giving $g(1)(f(1)+f(2)+...)+g(2)(f(2)+f(4)+...)+g(3)(f(3)+...)$ which equals the RHS.

**18** $\sum_{n\le x}\mu(n)\lfloor\frac{x}{n}\rfloor = \sum_{n\le x}\mu(n)\sum_{j\le\frac{x}{n}}1 = \sum_{n\le x}\sum_{d|n}\mu(d)$ by DSI. But this sum is 1 since the inner sum is zero for $n>1$.

**19** **a.** $\sum_{n\le x} n^\alpha = \int_1^x t^\alpha\,dt + k(\alpha) + O(x^\alpha)$ using Theorem 3.11. The result follows. **b.** $\sum_{n\le x}\sigma_\alpha(n) = \sum_{n\le x}\sum_{d|n} d^\alpha = \sum_{n\le x}\sum_{d\le\frac{x}{n}} d^\alpha$ using DSI. This last sum is equal to $\sum_{n\le x}\frac{\left(\frac{x}{n}\right)^{\alpha+1}}{\alpha+1} + O\left(\frac{x^\alpha}{n^\alpha}\right)$ (by (a)), and by Lemma 1a, this is equal to $\frac{x^{\alpha+1}}{\alpha+1}\sum_{n\le x}\frac{1}{n^{\alpha+1}} + O\left(x^\alpha\sum_{n\le x}\frac{1}{n^\alpha}\right) = \frac{x^{\alpha+1}}{\alpha+1}\left(\frac{x^{-\alpha}}{-\alpha}+\zeta(\alpha+1)+O(x^{-\alpha})\right) + O\left(x^\alpha\left(\frac{x^{1-\alpha}}{1-\alpha}+\zeta(\alpha)+O(x^\alpha)\right)\right) = \frac{\zeta(\alpha+1)}{\alpha+1}x^{\alpha+1} + O(x^\alpha)$. **c.** Straighforward application of Theorem 3:11. Use integration by parts.

**20** **a.** Draw a rectangle bounded by the lines $x=1/2, x=n$; $y=f(1/2), y=f(n)$ and count lattice points. The sum $\sum_{j=1}^n\lfloor f(j)\rfloor$ counts the number of lattice points on and below the graph of $f$ in the rectangle and the second sum counts those on and above. Hence their sum is $mn+d$, since there are $d$ lattice points on $f$ and they are counted twice. **b.** Let $m=\lfloor n^{1/3}\rfloor(=d)$ then the sum is $m(n+1)-\frac{1}{4}m^2(m+1)^2$.

**21** **a.** The desired sum gives the number of lattice points inside and on a circle of radius $\sqrt{N}$. Associate with each lattice point $L$ in the circle a unit square which has $L$ as its north west corner. We obtain an estimate of the total area of the squares by noting that each point outside the circle lies no more than $\sqrt{2}$ units away from the boundary of the circle and each point inside the circle lies no more than $\sqrt{2}$ units from the boundary of the circle so $\pi(\sqrt{N}-\sqrt{2})^2 < \sum_{n=0}^N S_2(n) < \pi(\sqrt{N}+\sqrt{2})^2$. Thus $-2\sqrt{2N}\pi < \sum_{n=0}^N S_2(n) - (N+2)\pi < 2\sqrt{2N}\pi$ and so $\sum_{n=0}^N S_2(n) - N\pi - 4\pi = O(\sqrt{N})$ and the result follows. The term $O(N^{\frac{1}{2}})$ has been replaced with $O(N^{\frac{23}{73}})$ (Huxley 1997). It is known that the power cannot be replaced with $\frac{1}{4}$. **b.** Is done similarly, but we look at a sphere and replace the unit squares with unit cubes.

**22** **a.** From $\phi(n)=\sum_{d|n}\mu(d)\frac{n}{d}$ we have $\sum_{n\le x}\phi(n) = \sum_{n\le x}\sum_{d|n}\mu(d)\frac{n}{d} = \sum_{d\le x}\mu(d)\sum_{q\le\frac{x}{d}} q$ (using DSI 2), $= \sum_{d\le x}\mu(d)\frac{1}{2}\left(\lfloor\frac{x}{d}\rfloor^2+\lfloor\frac{x}{d}\rfloor\right) = \frac{1}{2}\sum_{d\le x}\mu(d)\lfloor\frac{x}{d}\rfloor^2 + \frac{1}{2}\sum_{d\le x}\mu(d)\lfloor\frac{x}{d}\rfloor = \frac{1}{2}\sum_{d\le x}\mu(d)\lfloor\frac{x}{d}\rfloor + \frac{1}{2}$ (from Question 16). **b.** $\sum_{n\le x}\mu(n)\lfloor\frac{x}{n}\rfloor^2 = \sum_{n\le x}\mu(n)\left(\frac{x}{n}+O(1)\right)^2 = \sum_{n\le x}\mu(n)\frac{x^2}{n^2} + O\left(\sum_{n\le x}\frac{x}{n}\right) = x^2\sum_{n\le x}\frac{\mu(n)}{n^2} + O(x\sum_{n\le x}\frac{1}{n}) = x^2\frac{1}{\zeta(2)} + O(x\log x)$. **c.** This now follows immediately.

**23** **a.** From $\phi(n)=\sum_{d|n}\mu(d)\frac{n}{d}$ we have $\sum_{n\le x}\frac{\phi(n)}{n} = \sum_{n\le x}\sum_{d|n}\frac{\mu(d)}{d} = \sum_{d\le x}\frac{\mu(d)}{d}\sum_{q\le\frac{x}{d}}1$ (using DSI 2), $= \sum_{n\le x}\frac{\mu(n)}{n}\lfloor\frac{x}{n}\rfloor$ **b.** $\sum_{n\le x}\frac{\mu(n)}{n}\lfloor\frac{x}{n}\rfloor = \sum_{n\le x}\frac{\mu(n)}{n}\left(\frac{x}{n}+O(1)\right) = x\sum_{n\le x}\frac{\mu(n)}{n^2} + O\left(\sum_{n\le x}\frac{\mu(n)}{n}\right) = \frac{x}{\zeta(2)} + O(\log x)$. **c.** This now follows immediately.

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