MATH 5645

TOPICS IN NUMBER THEORY. 2014

Assignment 2. Due Week 5.

Marks will be deducted for poor and illogical setting out or for solutions that are unnecessarily complicated or obscure.

1.Here is another proof that $\sum \dfrac{1}{p_i}$ diverges.

For a contradiction, suppose that there is an integer k such that $\displaystyle\sum_{m=k+1}^{\infty} \frac{1}{p_m} < \frac{1}{2}$.

a. Let p_k denote, as usual, the kth prime and let $\alpha_k(x)$ be the number of positive integers not exceeding x all of whose prime factors are less or equal to p_k. Show that there can be no more than 2^k such squarefree integers and then prove that $\alpha_k(x) \leq 2^k\sqrt{x}$.

b. By noting that the number of positive integers less than a given x that are divisible by the prime p is no more than $\frac{x}{p}$, show that $x - \alpha_k(x) < \frac{x}{2}$. Deduce that $x < 2^{2k+2}$ and arrive at a contradiction.

2a. Show that if p is a prime and $p|x^2 + 2$ then $p \equiv 1$ or $3 \bmod 8$.

b. By considering $N = (p_1 \dots p_r)^2 + 2$, where $p_i \equiv 3 \bmod 8$, prove that there are infinitely many primes congruent to 3 mod 8.

3. Write $\pi^*(x)$ for the number of integers not greater than x, that are of the form p^k for some prime p and some integer k. (Hence $\pi^*(x)$ counts primes and prime powers.)

a. Explain why

$$\pi^*(x) = \pi(x) + \pi(x^{\frac{1}{2}}) + \pi(x^{\frac{1}{3}}) + \cdots + \pi(x^{\frac{1}{m}}),$$

where m is the largest integer such that $2^m \leq x$.

b. Suppose that C is a constant such that $\pi(x) \leq \dfrac{Cx}{\log x}$ for all $x \geq 2$.
Explain why $\pi^*(x) - \pi(x) \leq \pi(x^{\frac{1}{2}}) + m\pi(x^{\frac{1}{3}})$, with m as in (a), and hence prove that

$$\pi^*(x) - \pi(x) \leq 12C\frac{x^{\frac{1}{2}}}{\log x}$$

for all $x \geq 2$. (Hint: The inequality $x^{\frac{1}{3}} \log x \leq 6e^{-1}x^{\frac{1}{2}}$ may be useful.)

c. What does this tell us about prime powers? (Hint: Get MAPLE to plot $\frac{x^{\frac{1}{2}}}{\log x}$.)