MATH 5645

TOPICS IN NUMBER THEORY. 2014

Assignment 3. Due Week 7.

Marks will be deducted for poor and illogical setting out or for solutions that are unnecessarily complicated or obscure.

1. Write $\omega(n)$ for the number of (distinct) prime divisors of n, $\Omega(n)$ for the number of prime factors of n, counted with repetition. Thus if, $n = \prod_{j=1}^{m} p_j^{k_j}$, then $\omega(n) = m$, and $\Omega(n) = \sum_{j=1}^{m} k_j$.

a. Prove that $2^{\omega(n)} \leq \tau(n) \leq 2^{\Omega(n)} \leq n$ for $n \geq 2$.

b. When does $\tau(n) = 2^{\omega(n)}$?

2. Define the **Jordan totient function** by

$$J_k(n) = n^k \prod_{p|n} (1 - p^{-k}),$$

where, as usual, the product is taken over primes. This is a generalisation of Euler's totient function.

a. Prove that J is multiplicative.

b. Show that $J_k(n) = \sum_{d|n} \mu(d) \left(\frac{n}{d}\right)^k$.

c. Find a simple expression (as a product over primes) for $J^{-1}(n)$.

3. Write $M_2(x) = \sum_{n \leq x} (\mu(n))^2$.
Hence $M_2(x)$ counts the number of square free integers $\leq x$.

a. Explain why $(\mu(n))^2 = \sum_{m^2|n} \mu(m)$.

b. Prove that $M_2(x) = x \sum_{m \leq x^{\frac{1}{2}}} \frac{\mu(m)}{m^2} - \sum_{m \leq x^{\frac{1}{2}}} \mu(m) \left\{ \frac{x}{m^2} \right\}$.

c. Deduce that

$$M_2(x) = \frac{6}{\pi^2} x + O(x^{\frac{1}{2}}).$$

d. Interpret this result in terms of the proportion of square-free numbers in the interval $[1, x]$.

e. Use MAPLE (or otherwise) to count the number of square-free numbers between 1 and 1000. (Comment!)