UNIVERSITY OF NEW SOUTH WALES.
SCHOOL OF MATHEMATICS AND STATISTICS
MATH5645
TOPICS IN NUMBER THEORY

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2. DISTRIBUTION OF PRIMES:

1 Show that there are arbitrarily large sets of consecutive integers containing no primes.

2 Let $n \geq 1$ be a fixed, but arbitrary, positive integer and let $S = \{30n + k : 0 \leq k \leq 29\}$. Show that S contains at most 7 primes.

3 Suppose that $M > 1$ and $p_1, p_2, ..., p_s$ are all the primes less than or equal to M.

a. Explain why $\displaystyle\sum_{n=1}^{M} \frac{1}{n} < \frac{1}{\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) ... \left(1 - \frac{1}{p_s}\right)}$.

b. Deduce that $\displaystyle\prod_{j=1}^{\infty} \left(1 - \frac{1}{p_j}\right) = 0$.

***4** Another proof that $\displaystyle\sum \frac{1}{p_i}$ diverges.

Suppose the contrary, then there is an integer k such that $\displaystyle\sum_{m=k+1}^{\infty} \frac{1}{p_m} < \frac{1}{2}$.
Let $Q = p_1 \dots p_k$ and consider the numbers $1 + nQ$ for $n = 1, 2, \dots$.

a. Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1 + nQ}$ diverges.

b. Explain why every prime factor of $1 + nQ$ must be $\geq p_{k+1}$.

c. Explain why

$$S(r) = \sum_{n=1}^{r} \frac{1}{1 + nQ} < \sum_{t=1}^{\infty} \left(\sum_{m=k+1}^{\infty} \frac{1}{p_m} \right)^t.$$

d. Deduce that $S(r) < 1$.

e. Conclude that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1 + nQ}$ has bounded partial sums and arrive at a contradiction.

5 Let A denote the set of all integers of the form, $2^r 3^s 5^t$.
Evaluate $\displaystyle\sum_{n \in A} \frac{1}{n}$ and $\displaystyle\sum_{n \in A} \frac{1}{n^2}$.

6 A slightly generalised version of Bertrand's postulate states that, for $n \geq 6$, there are at least 2 primes between n and $2n$. Use this to prove that $p_{k+2} \leq p_k + p_{k+1}$.

7 **a.** Use Bertrand's Postulate to prove that for $m \geq 2$, if $m! = p_1^{\alpha_1} \dots p_r^{\alpha_r}$ then $\alpha_i = 1$ for at least one value of i.

(Hint: Write $m = 2k$ or $2k + 1$ and look at the interval $[k, 2k]$.)

b. Deduce that $m!$ is never a kth power for any $k \geq 2$.

8 Use Bertrand's postulate to prove that for every integer $k > 2$, there is a prime p such that $p < k < 2p$.
(Hint: It may be convenient to consider the cases k even and k odd separately.)

***9** Let $P(n)$ denote, for each positive integer $n \geq 2$, the proposition

$$\prod_{p \leq n} p < 4^{n-1}$$

where the product is over primes. For a fixed integer m suppose that $P(n)$ is true for all $n \leq m+1$.

a. Prove that $\displaystyle\prod_{m+1<p\leq 2m+1} p \leq \binom{2m+1}{m}$.

b. Prove that, for $m \geq 1$, $\binom{2m+1}{m} < 2^{2m}$.

c. Conclude that $\displaystyle\prod_{p\leq 2m+1} p < 4^{2m}$.

d. Deduce that $P(n)$ is true for all n.

10 Prove that there are infinitely many primes congruent to -1 mod 3.

***11** **a.** Suppose p is prime. Deduce that $x^4 \equiv -1 \bmod p$ is soluble iff $p \equiv 1 \bmod 8$.

b. Prove that there are infinitely many primes of the form $8k+1$.

(Hint: Use $N = (2p_1....p_n)^4 + 1$).

12 Prove that

a. $3.09\frac{n}{\log n} \leq 1.7\frac{2n}{\log 2n}$ for $n \geq 1001$.

b. $\frac{n\log 2}{\log n} - \frac{\log(n+1)}{\log n} > \frac{2}{3}\frac{n}{\log n}$ for $n > 220$. (Hint: Look at $f(x) = 3x\log 2 - 3\log(x+1) - 2x$.)

BRIEF SOLUTIONS

1 Let $S = \{n! + 2, n! + 3, \ldots, 2n!\}$. Each element of S is composite and S consists of $n! - 1$ consecutive integers.

2 Show there are only 8 possible values of k which could produce a prime and that (at least) one of these must be divisible by 7.

3 **a.** Expand out the RHS, so RHS $= (1 + \frac{1}{p_1} + ...)(1 + \frac{1}{p_2} + ...)... = \sum_{n \in \Lambda} \frac{1}{n} > \sum_{n \le M} \frac{1}{n}$, where Λ is the set of all integers each of whose prime factors is at most M. **b.** This follows from the divergence of the harmonic series.

4 **a.** The series diverges by the integral test. **b.** Since $1 + nQ$ is prime to Q, all the prime factors of all these denominators are from a finite set of primes, whose smallest element is p_{k+1}. **c.** Every term on the LHS also appears on the RHS. **d.** By our initial assumption, the inner sum is less than $\frac{1}{2}$. Thus for each r we have $S(r) < \sum_{t=1}^{\infty} \frac{1}{2^t} < 1$. **e.** Hence the partial sums of $\sum_{n=1}^{\infty} \frac{1}{1 + nQ}$ are bounded which is impossible in light of part (a).

5 Use Euler product. The answers are respectively $\frac{15}{4}$ and $\frac{25}{16}$.

6 For $p_k = 2, 3$ or 5 the statement is true, so suppose $p_k > 5$ then there are at least two primes between p_k and $2p_k$, so $p_{k+2} < 2p_k$. Also $2p_k = p_k + p_k < p_k + p_{k+1}$. The result follows.

7 **a.** The result it true for $m = 2$, so suppose $m > 2$. Write $m = 2k$ or $m = 2k + 1$ according as m is even or odd. By Bertrand's Postulate, there is a prime p between k and $2k$ and so $2p > m$. Hence $p||m!$. **b.** This follows directly from (a).

8 If k is even then given k there is a prime p with $\frac{k}{2} < p < k$. hence $k < 2p$ and $k > p$. Similarly, if k is odd, then there are a prime p, such that $\frac{k-1}{2} < p < k - 1 < k$. If $2p \le k - 1$ we have a contradiction, and so $2p > k - 1$. Now k cannot equal $2p$ since k is odd so $2p > k$ and the result follows.

9 **a.** $\binom{2m+1}{m} = \frac{(2m+1)!}{m!(m+1)!}$ so if $m + 1 < p \le 2m + 1$, p is a prime factor of the numerator but not the denominator. **b.** By the binomial theorem $\binom{2m+1}{m} < \frac{1}{2}\left[\binom{2m+1}{0} + \binom{2m+1}{1} + \cdots + \binom{2m+1}{m} + \binom{2m+1}{m+1} + \cdots + \binom{2m+1}{2m+1}\right] = \frac{1}{2}2^{2m+1} = 2^{2m}$. **c.** $\prod_{p \le 2m+1} p = \prod_{p \le m+1} p \prod_{m+1 < p \le 2m+1} p \le 4^m.2^{2m} = 4^{2m}$ (since we assumed $P(m+1)$ is true.) **d.** The result now follows by strong induction, since we have shown that $P(m + 1) \Rightarrow P(2m + 1)$, clearly $P(2)$ is true, and we note that $\prod_{p \le 2m-1} p = \prod_{p \le 2m} p$ so $P(2m - 1) \Rightarrow P(2m)$.

10 Suppose the contrary and let $S = \{q_1, q_2, \ldots, q_r\}$ be the set of all primes congruent to -1 mod 3 and let $P = 3q_1 \ldots q_r - 1$. Then $P \equiv -1$ mod 3. If P is prime we have a contradiction and if not then consider the prime factorisaton of P. If all the prime factors are 1 mod 3, then $P \equiv 1$ mod 3 which is false. Thus P has at least one prime factor which is -1 mod 3 but this cannot be in S. This contradiction completes the proof.

11 **a.** By Theorem 1.1 $x^4 \equiv -1$ mod p has solution iff $(-1)^{\frac{p-1}{(4,p-1)}} \equiv 1$ mod p. That is, iff $\frac{p-1}{(4,p-1)}$ is even. So if $p = 8k + r, r = 1, 3, 5, 7$ the condition holds iff $r = 1$. **b.** Suppose the contrary and let $S = \{p_1, p_2, ..., p_n\}$ be all the primes of the form $8k + 1$. Set $N = (2p_1....p_n)^4 + 1$ and let p be a prime factor of N. Since N is odd so is p. By (b), N is a solution of $x^4 + 1 \equiv 0$ mod p and so p is of the form $8k + 1$. But $p \notin S$ which gives the desired contradiction.

12 **a.** ('Discover' the proof by working backwards). For $n > 1200$ (in fact for $n > 1002$), we have $\log n > \frac{3.09 \log 2}{0.31}$ and so $3.09 \log 2 < 0.31 \log n$. This implies that $3.09(\log 2 + \log n) - 3.4 \log n < 0$. Multiplying by n and dividing by $\log n \log 2n$ we can write $\frac{3.09n \log 2n - 3.4n \log n}{\log n \log 2n} < 0$ which splits to give $\frac{3.09n}{\log n} - \frac{1.7 \times 2n}{\log 2n}$ and the result follows. **b.** Consider the function $f(x) = 3x \log 2 - 3 \log(x + 1) - 2x$. $f'(x) = 3 \log 2 - \frac{3}{1+x} - 2$ which is positive for $x \ge 37$. Hence for $x > 37$ the function is increasing. Also $f(220) > 0$ so for $x > 220$, $f(x) > 0$. Dividing by $3 \log x$ the result follows.