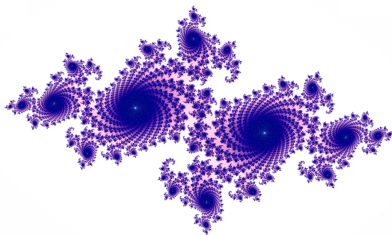


# Quantised Calculus in One Variable

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# Motivation

Problem:

What is an infinitesimal?

Problem:

What is the derivative of a nondifferentiable function?

- Early calculus (e.g. Leibniz, Newton) made use of infinitesimal quantities:

## Definition

A quantity  $x$  is called *infinitesimal* if for any  $\varepsilon > 0$ ,

$$|x| < \varepsilon.$$

## Definition

A quantity  $x$  *infinite* if for any  $N > 0$ ,

$$|x| > N.$$

# Infinitesimals

This definition was good enough for 18th century calculus, and many definitions were formulated in terms of infinitesimals.

For example:

## Definition

A function  $f$  is continuous if  $df(x) := f(x + dx) - f(x)$  is infinitesimal for any infinitesimal quantity  $dx$ .

## Definition

A function  $f$  is differentiable if the quantity

$$f'(x) := \frac{f(x + dx) - f(x)}{dx}$$

is not infinite when  $dx$  is infinitesimal.

# Properties of infinitesimals

## Lemma

*If  $x$  is infinitesimal, then  $x^{-1}$  is infinite.*

## Lemma

*For any  $f$ ,*

$$df = \frac{df}{dx} dx$$

# Sizes of infinitesimals

Not all infinitesimals are equal. Intuitively, some should be bigger than others. Intuitively we expect the following properties:

- If  $x > 0$  is infinitesimal, then  $x^2 < x$ .
- If a function  $f$  is smoother than a function  $g$ , then  $df < dg$ .

# Problem

There is a problem! These definitions make no sense.

## Problems:

- If  $x \neq 0$  is infinitesimal, then  $|x| < |x|/2$ , so  $1 < 1/2$ .
- If  $x$  is infinite, then  $|x| > 2|x|$ , so  $1 > 2$ .

So we cannot use the definitions given by 18th century mathematicians.

Infinitesimals were rightly banished from mathematics, and the definitions of continuity and differentiability were replaced with their modern definitions in terms of limits.



# But what *is* an infinitesimal?

18th century mathematicians were still able to use infinitesimals even though they make no sense.

Question:

Why does calculus using infinitesimals “work”?

Question:

Is there a way to make sense of infinitesimals?

An answer is provided by physics.

## Classical Mechanics

- State space is a manifold  $\mathcal{M}$
- States are points  $p \in \mathcal{M}$
- Quantities are functions  $f : \mathcal{M} \rightarrow \mathbb{C}$ .
- Quantities evolve according to

$$\frac{df}{dt} = \{f, H\}$$

## Quantum Mechanics

- State space is a Hilbert space  $\mathcal{H}$
- States are vectors  $\psi \in \mathcal{H}$
- Quantities are operators  $O : \mathcal{H} \rightarrow \mathcal{H}$
- Quantities evolve according to

$$\frac{dO}{dt} = \frac{i}{\hbar}[O, H]$$

- Quantisation is the process of converting a classical mechanical system to a quantum mechanical system
- In general, doing this in a satisfying way is an unresolved problem.

Quantisation should look like this:

- Manifolds should be replaced with Hilbert spaces (e.g, replace  $\mathcal{M}$  with  $L^2(\mathcal{M})$ )
- Functions  $f : \mathcal{M} \rightarrow \mathbb{C}$  should be replaced with operators (e.g. replace  $f$  with  $M_f$ )

- The setting for calculus (in one variable) is a manifold,  $\mathbb{R}$ .
- The objects of study in calculus are functions  $f : \mathbb{R} \rightarrow \mathbb{C}$ .

This is like “classical” physics. Can we find a “quantum” counterpart?

- Instead of  $\mathbb{R}$ , we should talk about  $L^2(\mathbb{R})$
- Instead of functions on  $\mathbb{R}$ , we should talk about operators on  $L^2(\mathbb{R})$ .

This includes ordinary calculus, since any  $f : \mathbb{R} \rightarrow \mathbb{C}$  can be encoded as a pointwise multiplication operator  $M_f$ .

Our class of objects of study is therefore massively expanded.

## Classical Calculus

- $\mathbb{R}$
- Complex valued function
- Real valued function
- Range of a function

## Quantised Calculus

- $L^2(\mathbb{R})$
- Operator  $F : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$
- Self adjoint operator
- Spectrum of an operator

### Question:

We now have many more objects to consider in calculus. Do we have something like infinitesimals?



## Definition

An operator  $T$  on a Hilbert space  $\mathcal{H}$  is called *compact* if for any  $\varepsilon > 0$  there is a finite dimensional subspace  $E$  such that

$$\|T|_{E^\perp}\| < \varepsilon$$

This is very much like the 18th century definition of infinitesimal! A compact operator is “infinitesimal modulo finite dimensional subspaces”.

So we have something in quantised calculus that looks like an 18th century infinitesimal. How much of the classical use of infinitesimals can we recover?

Question:

How can we measure the size of a compact operator?

The answer should be analogous to the “size” of an infinitesimal.

If  $T$  is a compact operator, we define

## Definition

The  $n$ th singular value of  $T$  is

$$\mu_n(T) := \inf\{\|T - R\| : \text{rank}(R) \leq n\}.$$

The sequence  $\{\mu_n(T)\}_{n=1}^{\infty}$  is a decreasing sequence of real numbers approaching 0.

Idea:

We can quantify the size of  $T$  by the rate of decay of  $\{\mu_n(T)\}_{n=1}^{\infty}$ .

Does this correspond to the 18th century notions of the size of an infinitesimal?

## Theorem

*If  $T$  and  $S$  are compact operators, then*

$$\mu_n(TS) \leq \mu_n(T)\mu_n(S)$$

So if  $T$  is infinitesimal (i.e. compact), then “ $T^2 < T$ ” (in this sense).

## Definition

If  $f : \mathbb{R} \rightarrow \mathbb{C}$ , then the *operator*  $df$  is defined as

$$df(g)(x) = \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R} \setminus (x-\varepsilon, x+\varepsilon)} \frac{f(x) - f(t)}{x - t} g(t) dt.$$

for  $g \in L^2(\mathbb{R})$ .

This is the topic of my thesis.

Question:

What is the relationship between the smoothness of  $f$  and the size of  $df$ ?



# Smoothness and differentials

There are some initial results:

- $df$  is of finite rank if and only if  $f$  is rational.
- If  $f$  is continuous, then  $df$  is compact.

This is exactly what we wanted. Can we do better?

# More advanced results

- $df$  is compact if and only if  $f$  has vanishing mean oscillation.
- $df$  is bounded if and only if  $f$  has bounded mean oscillation.

## Definition

The set  $\mathcal{S}_p$  is the set of compact operators such that

$$\{\mu_n(T)\}_{n=1}^{\infty} \in \ell^p.$$

The following is due to V.V. Peller (1984)

## Theorem

$df \in S_p$  if and only if  $f \in B_{pp}^{1/p}$ .

$B_{pp}^{1/p}$  is a Besov space.

Can we do better?

The Schatten-Lorentz class  $\ell^{p,q}$ , for  $p \in (0, \infty)$  and  $q \in (0, \infty]$  is defined as

## Definition

A sequence  $\{x_n\}_{n=0}^{\infty}$  is  $\ell^{p,q}$  if

$$\sum_{n=0}^{\infty} x_n^q (1+n)^{q/p-1} < \infty$$

for  $q < \infty$  and

$$\sup_{n \geq 0} (1+n)^{1/p} x_n < \infty.$$

## Definition

The Schatten-Lorentz class  $S_{p,q}$  is the set of compact operators  $T$  such that  $\{\mu_n(T)\}_{n=1}^{\infty} \in \ell^{p,q}$ .

For what functions  $f$  is  $df \in S_{p,q}$ .