Quantised Calculus in One Variable

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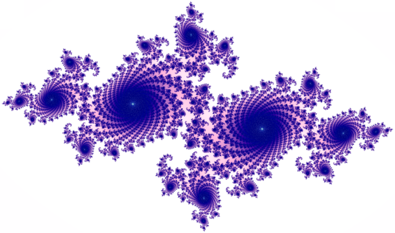
September 17, 2014



Motivation

Problem:

What is an infinitesimal?

Problem:

What is the derivative of a nondifferentiable function?

Infinitesimals

- Early calculus (e.g. Leibniz, Newton) made use of infinitesimal quantities:

Definition

A quantity x is called *infinitesimal* if for any $\varepsilon > 0$,

$$|x| < \varepsilon.$$

Definition

A quantity x *infinite* if for any $N > 0$,

$$|x| > N.$$

Infinitesimals

This definition was good enough for 18th century calculus, and many definitions were formulated in terms of infinitesimals.
For example:

Definition

A function f is continuous if $df(x) := f(x+dx) - f(x)$ is infinitesimal for any infinitesimal quantity dx.

Definition

A function f is differentiable if the quantity

$$f'(x) := \frac{f(x+dx) - f(x)}{dx}$$

is not infinite when dx is infinitesimal.

Properties of infinitesimals

Lemma

If x is infinitesmal, then x^{-1} is infinite.

Lemma

For any f,

$$df = \frac{df}{dx}dx$$

Sizes of infinitesimals

Not all infinitesimals are equal. Intuitively, some should be bigger than others. Intuitively we expect the following properties:

- If $x > 0$ is infinitesimal, than $x^2 < x$.
- If a function f is smoother than a function g, then $df < dg$.

Problem

There is a problem! These definitions make no sense.

Problems:

- If $x \neq 0$ is infinitesimal, then $|x| < |x|/2$, so $1 < 1/2$.
- If x is infinite, then $|x| > 2|x|$, so $1 > 2$.

So we cannot use the definitions given by 18th century mathematicians.

Infinitesimals were rightly banished from mathematics, and the definitions of continuity and differentiability were replaced with their modern definitions in terms of limits.

But what *is* an infinitesimal?

18th century mathematicians were still able to use infinitesimals even though they make no sense.

Question:

Why does calculus using infinitesmals "work"?

Question:

Is there a way to make sense of infinitesimals?

An answer is provided by physics.

Physics

Classical Mechanics

- State space is a Manifold $\mathcal{M}$
- States are points $p \in \mathcal{M}$
- Quantities are functions $f : \mathcal{M} \to \mathbb{C}$.
- Quantities evolve according to

$$\frac{df}{dt} = \{f, H\}$$

Quantum Mechanics

- State space is a Hilbert space $\mathcal{H}$
- States are vectors $\psi \in \mathcal{H}$
- Quantities are operators $O : \mathcal{H} \to \mathcal{H}$
- Quantities evolve according to

$$\frac{dO}{dt} = \frac{i}{\hbar}[O, H]$$

Quantisation

- Quantisation is the process of converting a classical mechanical system to a quantum mechanical system
- In general, doing this in a satisfying way is an unresolved problem.

Quantisation

Quantisation should look like this:

- Manifolds should be replaced with Hilbert spaces (e.g, replace $\mathcal{M}$ with $L^2(\mathcal{M})$)
- Functions $f : \mathcal{M} \to \mathbb{C}$ should be replaced with operators (e.g. replace f with M_f)

Quantising calculus

- The setting for calculus (in one variable) is a manifold, $\mathbb{R}$.
- The objects of study in calculus are functions $f : \mathbb{R} \to \mathbb{C}$.

This is like "classical" physics. Can we find a "quantum" counterpart?

Quantising calculus

- Instead of $\mathbb{R}$, we should talk about $L^2(\mathbb{R})$
- Instead of functions on $\mathbb{R}$, we should talk about operators on $L^2(\mathbb{R})$.

This includes ordinary calculus, since any $f : \mathbb{R} \to \mathbb{C}$ can be encoded as a pointwise multiplication operator M_f.

Quantising calculus

Our class of objects of study is therefore massively expanded.

Classical Calculus

- $\mathbb{R}$
- Complex valued function
- Real valued function
- Range of a function

Quantised Calculus

- $L^2(\mathbb{R})$
- Operator $F : L^2(\mathbb{R}) \to L^2(\mathbb{R})$
- Self adjoint operator
- Spectrum of an operator

Question:

We now have many more objects to consider in calculus. Do we have something like infinitesimals?

Definition

An operator T on a Hilbert space $\mathcal{H}$ is called *compact* if for any $\varepsilon > 0$ there is a finite dimensional subspace E such that

$$\|T|_{E^\perp}\| < \varepsilon$$

This is very much like the 18th century definition of infinitesimal! A compact operator is "infinitesimal modulo finite dimensional subspaces".

So we have something in quantised calculus that looks like an 18th century infinitesimal. How much of the classical use of infinitesimals can we recover?

Compact operators

Question:

How can we measure the size of a compact operator?

The answer should be analogous to the “size” of an infinitesimal.

Compact operators

If T is a compact operator, we define

Definition

The nth singular value of T is

$$\mu_n(T) := \inf\{\|T - R\| \ : \ \mathrm{rank}(R) \leq n\}.$$

The sequence $\{\mu_n(T)\}_{n=1}^{\infty}$ is a decreasing sequence of real numbers approacing 0.

Idea:

We can quantify the size of T by the rate of decay of $\{\mu_n(T)\}_{n=1}^{\infty}$.

Does this correspond to the 18th century notions of the size of an infinitesimal?

Submultiplicativity

Theorem

If T and S are compact operators, then

$$\mu_n(TS) \leq \mu_n(T)\mu_n(S)$$

So if T is infinitesimal (i.e. compact), then "$T^2 < T$" (in this sense).

Quantised differential

Definition

If $f : \mathbb{R} \to \mathbb{C}$, then the *operator df* is defined as

$$df(g)(x) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}\setminus(x-\varepsilon, x+\varepsilon)} \frac{f(x) - f(t)}{x - t} g(t)\, dt.$$

for $g \in L^2(\mathbb{R})$.

Smoothness and differentials

This is the topic of my thesis.

Question:

What is the relationship between the smoothness of f and the size of df?

Smoothness and differentials

There are some initial results:

- df is of finite rank if and only if f is rational.
- If f is continuous, then df is compact.

This is exactly what we wanted. Can we do better?

More advanced results

- df is compact if and only if f has vanishing mean oscillation.
- df is bounded if and only if f has bounded mean oscillation.

Schatten-Von Neumann Classes

Definition

The set $\mathcal{S}_p$ is the set of compact operators such that

$$\{\mu_n(T)\}_{n=1}^{\infty} \in \ell^p.$$

The following is due to V.V. Peller (1984)

Theorem

$df \in S_p$ if and only if $f \in B^{1/p}_{pp}$.

$B^{1/p}_{pp}$ is a Besov space.

Can we do better?

Schatten-Lorentz classes

The Schatten-Lorentz class $\ell^{p,q}$, for $p \in (0, \infty)$ and $q \in (0, \infty]$ is defined as

Definition

A sequence $\{x_n\}_{n=0}^{\infty}$ is $\ell^{p,q}$ if

$$\sum_{n=0}^{\infty} x_n^q (1+n)^{q/p-1} < \infty$$

for $q < \infty$ and

$$\sup_{n \geq 0} (1+n)^{1/p} x_n < \infty.$$

Definition

The Schatten-Lorentz class $S_{p,q}$ is the set of compact operators T such that $\{\mu_n(T)\}_{n=1}^{\infty} \in \ell^{p,q}$.

For what functions f is in $df \in S_{p,q}$.