Conformal Trace theorem for Julia sets of quadratic polynomials

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June 6, 2017

What this talk is about

This talk is based on a forthcoming collaboration between A Connes, F Sukochev, D Zanin and myself.

I will cover:

1. Motivation: computing line integrals
2. What is a Julia set?
3. What is the Hausdorff measure?
4. What is the conformal trace formula?

The actual topic of the paper concerns a certain formula for computing the Hausdorff measure on certain Julia sets, but this talk will only introduce the context.

Motivation: Integration with respect to curve length

Suppose that $\gamma : [0,1] \to \mathbb{C}$ is a parametrised curve in the complex plane, and let f be a continuous function on $\mathbb{C}$. If γ has finite length, that is,

$$\sup_{0=t_0<t_1\cdots<t_n=1} \sum_{j=1}^{n} |\gamma(t_{j+1}) - \gamma(t_j)| < \infty$$

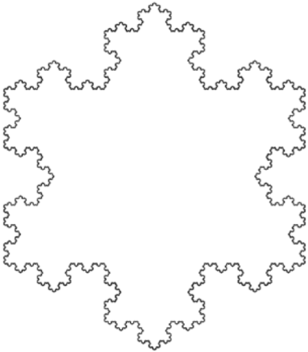
(where the supremum is taken over all partitions $0 = t_0 < t_1 < \cdots < t_n = 1$ of all sizes) then the line integral,

$$\int_\gamma f(t)|d\gamma(t)| = \int_\gamma f\, ds$$

is well defined (as a limit of Riemann sums). If γ does not have finite length, then this is not necessarily the case.

Motivation: Integration with respect to p-dimensional "length"

There are continuous curves in the plane with infinite length. E.g., a Koch snowflake,



Motivation: Integration with respect to p-dimensional "length"

- The Koch curve does not have finite length, but it does have a "Hausdorff measure": a kind of p-dimensional "length" with dimension $p = \frac{\log(4)}{\log(3)}$. How can we compute "integrals" with respect to this fractional-dimensional measure?
- For certain very special curves in the plane, there is a formula for the Hausdorff measure in terms of singular traces.
- We focus in particular on Julia sets of quadratic polynomials (a particular source of curves in the plane with infinite length).

Complex polynomial dynamics

Let f be a polynomial with complex coefficients, and let $z_0 \in \mathbb{C}$. We are interested in studying the behaviour of the recursively defined sequence:

$$z_{n+1} := f(z_n) \quad n \geq 0.$$

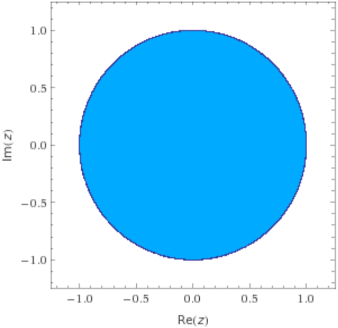
We are especially interested in studying the asymptotic behaviour of $\{z_n\}_{n\geq 0}$ for different $z_0 \in \mathbb{C}$. In particular, given any complex number, it can be shown that exactly one of the following happens:

1. Either $|z_n| \to \infty$.
2. $\{z_n\}_{n\geq 0}$ remains bounded.

Complex polynomial dynamics

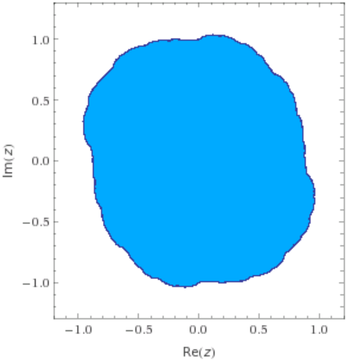
The simplest nontrivial example is $f(z) = z^2$. Then $z_k = f^k(z_0) = {z_0}^{2^k}$, and the behaviour of $f^k(z_0)$ neatly splits into three separate cases:

1. If $|z_0| < 1$, then $f^k(z_0) \to 0$ as $k \to \infty$.
2. If $|z_0| = 1$, then $|f^k(z_0)| = 1$ for all $k \geq 0$.
3. If $|z_0| > 1$, then $|f^k(z_0)| \to \infty$ as $k \to \infty$.



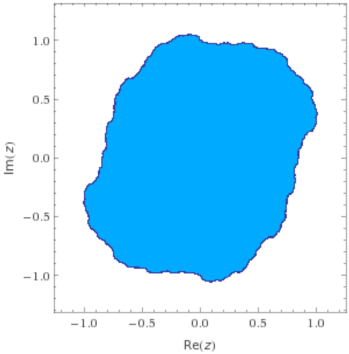
Complex polynomial dynamics

What if we perturb the polynomial $f(z) = z^2$ slightly... Consider $f(z) = z^2 + 0.1 + 0.1i$. A numerical approximation looks like this:



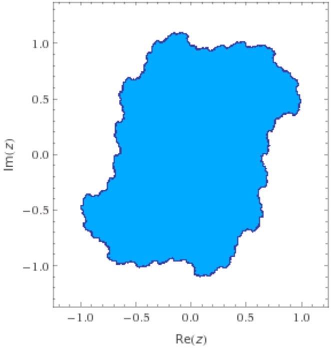
Complex polynomial dynamics

Try $f(z) = z^2 + 0.1 - 0.2i$,



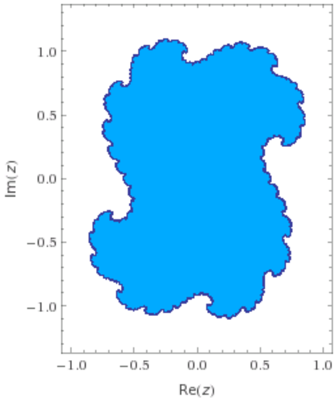
Complex polynomial dynamics

Try $f(z) = z^2 + 0.2 - 0.3i$,



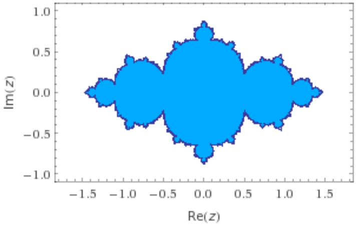
Complex polynomial dynamics

Try $f(z) = z^2 + 0.3 - 0.1i$,



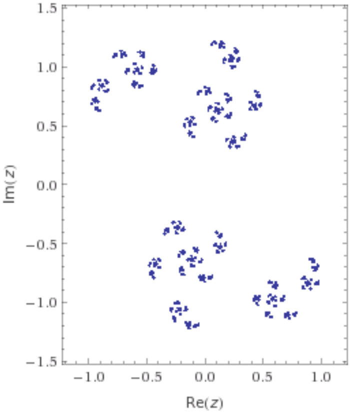
Complex polynomial dynamics

Try $f(z) = z^2 - 0.7 + 0.001i$,



Complex polynomial dynamics

Let try a slightly bigger parameter. Consider $f(z) = z^2 + 0.5 + 0.5i$,



The Mandelbrot set

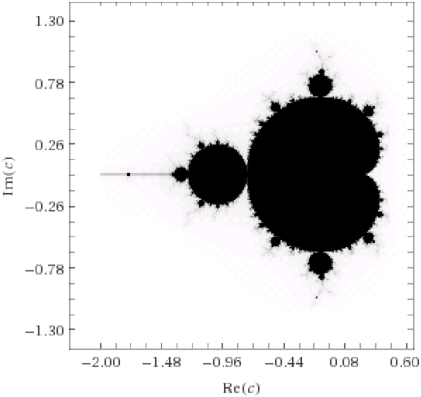
What is going on here? Consider the general polynomial:

$$f_c(z) := z^2 + c.$$

Note: any quadratic polynomial can be transformed into some f_c by an affine change of coordinates.

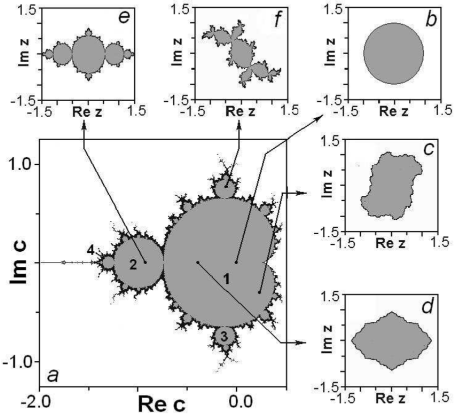
The Mandelbrot set

Consider the case $z_0 = 0$ (for simplicity). For which c is $\{f_c^k(0)\}_{k\geq 0}$ bounded? Define the Mandelbrot set $M := \{c \in \mathbb{C} \ : \ \{f_c^k(0)\}_{k\geq 0} \text{ is bounded}\}$. M can be approximated by a computer:



The Mandelbrot set

A more informative image is this one:



The Julia set

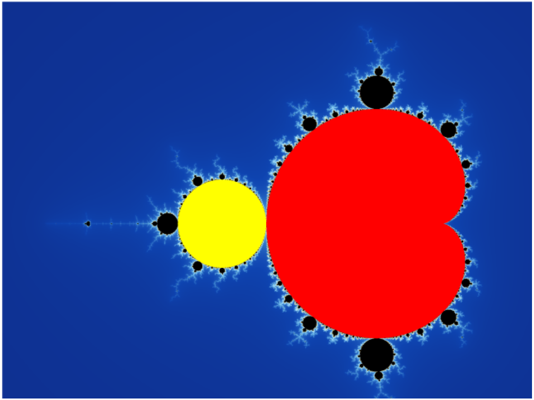
Let $c \in \mathbb{C}$, and consider $f_c(z) = z^2 + c$. The *Julia set* of f_c is the boundary of the set of points $z \in \mathbb{C}$ such that $\{f_c^n(z)\}_{n \geq 0}$ is bounded.

Theorem

The Julia set $J(f_c)$ is connected if and only if $c \in M$ (the Mandelbrot set).

The main cardioid

Let M_0 be the set $\{\frac{z}{2}(1 - \frac{z}{2}) \ : \ |z| < 1\}$. M_0 is an open subset of the Mandelbrot set M called the *main cardioid*, shown below in red:



The main cardioid

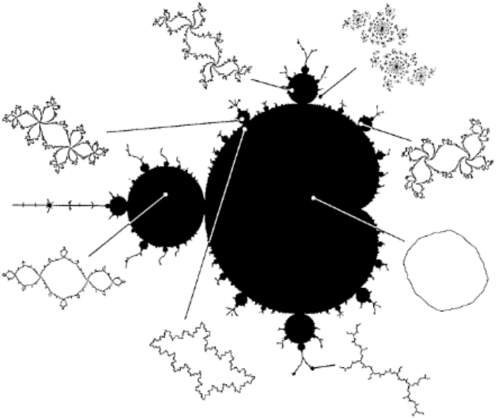
The significance of the main cardioid is the following theorem:

Theorem

The Julia set $J(f_c)$ of f_c is a Jordan curve (i.e. homeomorphic to a circle) if and only if c is in the main cardioid M_0.

Mandelbrot and Julia sets

Let's look at some more pictures:

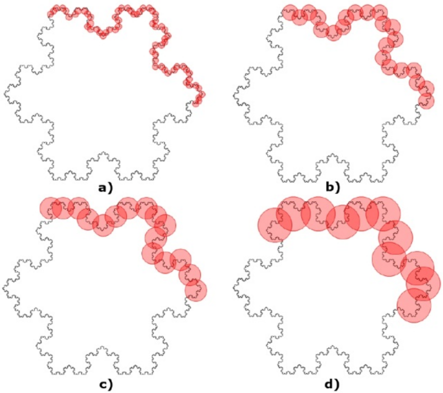


Hausdorff dimension

Let (X, d) be a metric space. Let $\mathcal{C}$ be the set of $q \in [0, \infty)$ such that there exists a covering of X by balls $\{B(a_j, r_j)\}_j$ such that $\sum_j r_j^q < \infty$.
The Hausdorff dimension of X is defined to be the infimum of $\mathcal{C}$.

Hausdorff dimension

The Hausdorff dimension is a kind of “scaling dimension”:



Hausdorff dimension of Julia sets

Fact: If c is in the main cardioid M_0 of the Mandelbrot set M, then the Julia set $J(f_c)$ is a Jordan curve with Hausdorff dimension $p \in [1, 2)$. In fact $p = 1$ if and only if $c = 0$.

Hausdorff measure

Associated to the Hausdorff dimension p there is a p-dimensional Hausdorff measure m_p. In many cases, the Hausdorff measure of a ball of radius r scales like r^p,

$$cr^p \leq m_p(B(z,r)) \leq Cr^p$$

for some fixed c, C and all $r > 0$.
Fact: If $c \in M_0$, and p is the Hausdorff dimension of $J(f_c)$, then the Hausdorff measure m_p on $J(f_c)$ is uniquely specified by the above identity.

The main task of the paper

Let $g : J(f_c) \to \mathbb{C}$ be a continuous function. In his 1994 book *Noncommutative geometry*, Alain Connes claimed that there is a formula for $\int_J g \, dm_p$ given in terms of his "quantised calculus". No proof was ever published. Finally, today we have completed a proof.