Quantised Calculus and Hankel Operators

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Introduction

This talk will be divided into two parts:

Soft Analysis:

The motivation for quantised calculus from non-commutative geometry.

Hard Analysis:

Operator ideal membership of quantised derivatives.

Non-commutative Geometry

What is Non-commutative geometry?

The study of noncommutative algebras which are similar to algebras of functions on geometric spaces, using the methods and language of geometry.

Why Non-commutative Geometry?

Non-commutative geometry gives us new insights on non-commutative algebra, but historically it was motivated by quantum mechanics.

A very brief description of quantum mechanics

Definition

A *quantum mechanical system* is a pair $(\mathcal{A}, \mathcal{H})$ where $\mathcal{H}$ is a complex separable Hilbert space and $\mathcal{A}$ is a $*$-algebra of (potentially unbounded, densely defined) operators on $\mathcal{H}$. The self adjoint elements of $\mathcal{A}$ are called *observables*. The non-zero elements of $\mathcal{H}$ are called *states*.

- The states correspond to potential configurations of a physical system. States which are non-zero scalar multiples of each other are considered physically indistinguishable.
- The observables correspond to physical quantities that can be measured. The range of potential measurements for an observable $A \in \mathcal{A}$ is $\sigma(A)$.

Reminder: The spectral theorem

Theorem

Let $(\mathcal{A}, \mathcal{H})$ be a quantum mechanical system. If $A \in \mathcal{A}$ is an observable, then there exists a projection valued measure E_A on $\sigma(A)$ such that

$$A = \int_{\sigma(A)} \lambda \, dE_A(\lambda).$$

Measurement in quantum mechanics

Let $(\mathcal{A}, \mathcal{H})$ be a quantum mechanical system, currently in a state $\psi \in \mathcal{H}$. Let $A \in \mathcal{A}$ be an observable. The probability that the measured value of the physical quantity corresponding to A lies in a borel set $\Delta \subseteq \sigma(A)$ is

$$P(\Delta; \psi) := \frac{(\psi, E_A(\Delta)\psi)}{\|\psi\|^2} = \frac{\|E_A(\Delta)\psi\|^2}{\|\psi\|^2}.$$

If the measured value lies in Δ, then the state of the system changes to

$$E_A(\Delta)\psi.$$

Measurement in quantum mechanics

There are two surprising features of measurement in quantum mechanics:

1. The act of observation changes the state of the system.
2. The order of observation is important, since if A and B are observables, and $\Delta \times \Sigma \subseteq \sigma(A) \times \sigma(B)$, then in general it is not true that
$$E_A(\Delta)E_B(\Sigma)\psi = E_B(\Sigma)E_A(\Delta)\psi.$$

The beginnings of noncommutative geometry

A noncommutative space is a "space" with "coordinates" such that the order of measurement of coordinates changes the measured values.

Inspired by quantum mechanics, we define a noncommutative space as a pair $(\mathcal{A}, \mathcal{H})$ exactly like a quantum mechanical system.

Relation to Ordinary Geometry

The usual setting for geometry is a Riemannian manifold M. Let $\mathcal{A} = C^\infty(M)$ and $\mathcal{H} = L^2(M)$.
However, we need more information to specify the geometry of the manifold. One way to do this is by giving a *Dirac operator*.

Associated to a Riemannian manifold (M, g) is a *clifford bundle*.

$$\mathrm{Cliff}(M, g) = \frac{T(TM)}{\langle xy + yx = -2g(x, y)\rangle}.$$

A vector bundle V on M is called a *clifford module* if there is a left multiplication action $m : \mathrm{Cliff}(M, g) \otimes V \to V$.

Dirac Operators

Definition

Suppose that (M, g) is a Riemannian manifold with a Clifford module V, and a V-valued connection $\nabla : V \to T^*M \otimes V$. Then the *Dirac Operator* on V is given by the composition of maps

$$V \xrightarrow{\nabla} T^*M \otimes V \xrightarrow{\sharp \otimes I} TM \otimes V \xrightarrow{m} V$$

where $\sharp$ is the canonical isomorphism from T^*M to TM given by g.

Theorem

Let (M, g) be a Riemannian manifold with Clifford module V and Dirac operator $\mathcal{D}$. Then we have for every $a \in C^\infty(M)$,

$$da = [\mathcal{D}, a]$$

as an equality of operators on the sections of V.

Spectral Triples

To specify the geometry of non-commutative space $(\mathcal{A}, \mathcal{H})$, we need additional data to tell us how observables relate to each other. An elegant way of doing this is by giving a Dirac operator.

Definition

A *spectral triple* is a triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ where $\mathcal{H}$ is a complex separable Hilbert space and $\mathcal{A}$ is a $*$-algebra of bounded operators on $\mathcal{H}$. $\mathcal{D}$ is a densely defined unbounded self adjoint operator on $\mathcal{H}$ such that:

1. $[\mathcal{D}, a]$ is bounded for all $a \in \mathcal{A}$.
2. $(\lambda - \mathcal{D})^{-1}$ is compact for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

The prototypical example is a Riemannian manifold M with a Clifford module, where $\mathcal{A} = C^\infty(M)$, $\mathcal{H} = L^2(V)$ and $\mathcal{D}$ is the Dirac operator of the module.

Properties of Spectral Triples

Definition

Let $k \geq 0$ be an integer. We say that a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is QC^k if for any $a \in \mathcal{A}$, a and $[\mathcal{D}, a]$ are in the domain of $\delta^k(a)$, where $\delta(a) := [|\mathcal{D}|, a]$.

Definition

Let $p > 0$. A spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is called (p, ∞) summable if $(\lambda - \mathcal{D})^{-1} \in \mathcal{L}^{p,\infty}$ for $\lambda \in \mathbb{C} \setminus \mathbb{R}$. p is analogous to the dimension of the space.

Differentials in abstract algebra

Definition

Let A be a $\mathbb{C}$-algebra, and M is an A-bimodule. A $\mathbb{C}$-linear map $\theta : A \to M$ is called a derivation if $\theta(ab) = a\theta(b) + \theta(a)b$ for all $a, b \in A$.

Example

Let N be a manifold. We can take $A = C^\infty(N)$ and $M = \Omega^1(N)$, and $\theta = d$.

Kähler Differentials

For a $\mathbb{C}$-algebra $\mathcal{A}$, there is a "largest" A-bimodule that is the image of a derivation on $\mathcal{A}$.

Definition

Let A be a $\mathbb{C}$-algebra. There is a multiplication map $\gamma : A \otimes A \to A$. Define

$$\Omega^1(A) := \ker \gamma.$$

And $d : A \to \Omega^1(A)$ is given by $a \mapsto 1 \otimes a - a \otimes 1$. This is the algebra of *Kähler differentials*.

Kähler differentials

The utility of Kähler differentials comes from the universal property:

Theorem

Let A be a $\mathbb{C}$-algebra, and M is an A-bimodule with a derivation $\theta : A \to M$. Then there exists a unique A-module homomorphism $\tilde{\theta} : \Omega^1(A) \to M$ such that $\theta = \tilde{\theta} \circ d$.

Given a spectral triple, $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, and $a \in \mathcal{A}$, we define

$$\pi : \Omega^1(\mathcal{A}) \to \mathcal{B}(\mathcal{H})$$

by

$$da \mapsto [\mathcal{D}, a].$$

Define the two-sided ideal of $\mathcal{B}(\mathcal{H})$, J as

$$J = \{t \in \Omega^1(\mathcal{A}) \ : \ \pi(t) = 0\}.$$

and $J_0 = J + dJ$. Then,

$$\Omega^1_{\mathcal{D}}(\mathcal{A}) := \frac{\pi(\Omega^1(\mathcal{A}))}{\pi(dJ_0)}.$$

Infinitesimals

17-19th century mathematicians made frequent use of *infinitesimals*. An infintesimal was a quantity x such that for all $\varepsilon > 0$, we have

$$|x| < \varepsilon.$$

Of course, this implies that $x = 0$, so the use of non-zero infinitesimals was abolished.

Infinitesimal Operators

An infinitesimal operator $T \in \mathcal{B}(\mathcal{H})$ should be an operator T such that for any $\varepsilon > 0$, we have

$$\|T\| < \varepsilon.$$

Again, this definition is useless as it implies that $T = 0$.

Compact Operators as Infinitesimals

We shall say that an operator $T \in \mathcal{B}(\mathcal{H})$ is *infinitesimal* if for any $\varepsilon > 0$ there exists a finite dimensional subspace E such that

$$\|T|_{E^\perp}\| < \varepsilon.$$

This is equivalent to saying that T is compact.

Infinitesimals in Non-commutative geometry

In a noncommutative geometry $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, the compact elements of $\mathcal{A}$ play a similar role to infinitesimals in classical real analysis.
We have the following dictionary:

Classical Analysis	Non-commutative Analysis
Function	Operator
One-form	Connes Differential
Range	Spectrum
Infinitesimal	Compact Operator

Quantised Differentials

In noncommutative geometry, we have a new object that is not present in classical analysis called a quantised derivative or quantised differential.

Definition

Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple. Let F be the unique operator such that $\mathcal{D} = F|\mathcal{D}|$, that is $F = \mathrm{sgn}(\mathcal{D})$. For $a \in \mathcal{A}$, define

$$đa := [F, a].$$

This is the *quantised differential* of a.

$đa$ is supposed to represent an infinitesimal variation of a. The motivation behind this definition is not clear, but as we work through examples it will become clear that this definition is the "correct" one.

Expected properties of infinitesimals

In classical 17-19th century analysis, infinitesimals were supposed to have a number of properties:

1. If $f : \mathbb{R} \to \mathbb{R}$ is a function, there is a function df representing infinitesimal variation in f. f is continuous if and only if df is infinitesimal.
2. If f is smoother than g, then df is smaller than dg.
3. If x is a positive infinitesimal, then x^2 is smaller than x.
4. If f is a differentiable function, then $df = f'dx$, provided that sufficiently small infinitesimals are ignored.

We shall see that the quantised differential satisfies all these properties, if they are interpreted correctly.

Sizes of Infinitesimals

Definition

Given $T \in \mathcal{B}(\mathcal{H})$, define the kth singular value of T to be

$$\mu_k(T) := \inf\{\|T - A\| \ : \ \mathrm{rank}(A) \leq k\}.$$

For a compact operator, $\{\mu_k(T)\}_{k=0}^{\infty}$ is a vanishing sequence of positive numbers. We shall describe the *size* of an infinitesimal T as the *rate of decay* of $\{\mu_k(T)\}_{k=0}^{\infty}$.

Sizes of Infinitesimals

Theorem

Let T and S be compact operators in $\mathcal{B}(\mathcal{H})$. Then for every $k \geq 0$,

$$\mu_k(TS) \leq \mu_k(T)\mu_k(S).$$

Hence, if T is an infinitesimal, then T^2 is a smaller infinitesimal.

Sizes of Infinitesimals

We can quantify the sizes an infinitesimal T by placing conditions on the rate of decay of $\{\mu_k(T)\}_{k=0}^{\infty}$.

- The smallest infinitesimals have $\{\mu_k(T)\}_{k=0}^{\infty}$ of finite support. Then T is of finite rank.
- We say that $T \in \mathcal{L}^p$ if $\{\mu_k(T)\}_{k=0}^{\infty} \in \ell^p$.
- We say that $T \in \mathcal{L}^{p,\infty}$ if $\mu_k(T) = \mathcal{O}(k^{-1/p})$.
- We say that $T \in \mathcal{L}^{p,q}$ if $\{k^{1/p-1/q}\mu_k(T)\}_{k=0}^{\infty} \in \ell^q$.
- We say that $T \in \mathcal{M}^{1,\infty}$ if $\{\frac{1}{\log(k+1)}\sum_{n=0}^{k}\mu_k(T)\}_{k=0}^{\infty} \in \ell^\infty$.

An initial result about quantised differentials

Lemma

Suppose that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a QC^1, (p, ∞)-summable spectral triple, where we assume that $\mathcal{D}$ is invertible. Then for all $a \in \mathcal{A}$, $\text{đ}a \in \mathcal{L}^{p,\infty}$.

Proof.

It can be proved that $|\mathcal{D}|^{-1} \in \mathcal{L}^{p,\infty}$. Now we write,

$$\begin{aligned}[\mathcal{D}, a] &= [F|\mathcal{D}|, a] \\ &= F[|\mathcal{D}|, a] + [F, a]|\mathcal{D}|.\end{aligned}$$

Hence,

$$\begin{aligned}\text{đ}a &= [\mathcal{D}, a]|\mathcal{D}|^{-1} - F[|\mathcal{D}|, a]|\mathcal{D}|^{-1} \\ &= (da - F\delta(a))|\mathcal{D}|^{-1}.\end{aligned}$$

Therefore $\text{đ}a \in \mathcal{L}^{p,\infty}$. $\square$

Smoothness and rate of decay

We now enter the second phase of the talk "Hard Analysis".
Recall that in classical analysis, we had the statement that if f is smoother than g, then df is smaller than dg.

The main question of this talk:

In what sense is it true that if f is smoother than g, then $\text{đ}f$ is smaller than $\text{đ}g$?

We shall restrict attention to functions on the circle $\mathbb{T}$ and the line $\mathbb{R}$.

We define $\mathbb{T} = \{\zeta \in \mathbb{C} \ : \ |\zeta| = 1\}$. Let $z : \mathbb{T} \to \mathbb{T}$ be the identity function. Denote the normalised Haar (or arc length) measure on $\mathbb{T}$ by $\mathbf{m}$.
For $f \in L^1(\mathbb{T}, \mathbf{m})$, define for $n \in \mathbb{Z}$,

$$\widehat{f}(n) := \int_{\mathbb{T}} z^{-n} f \, d\mathbf{m}.$$

Recall that any $f \in L^2(\mathbb{T}, \mathbf{m})$ can be written as

$$f = \sum_{n \in \mathbb{Z}} \widehat{f}(n) z^n.$$

The sum converges in the L^2 sense. This effects an isometric isomorphism between $L^2(\mathbb{T})$ and $\ell^2(\mathbb{Z})$.

The closed linear span of $\{z^n\}_{n=0}^{\infty}$ in L^p is denoted $H^2(\mathbb{T})$, and the orthogonal complement is denoted $H^2_-(\mathbb{T})$.
We define the space of polynomials $\mathcal{P}(\mathbb{T})$ to be the finite linear span of $\{z^n\}_{n\in\mathbb{Z}}$. $P_A(\mathbb{T}) = \text{span}\{z^n\}_{n\geq 0}$.

Differentials on $\mathbb{T}$

The Dirac operator on the circle is $\mathcal{D} = \frac{1}{i}\frac{d}{d\theta}$, i.e., $\mathcal{D}(z^n) = nz^n$. So we may describe $F = \operatorname{sgn}\mathcal{D}$ as the *Hilbert transform*,

$$F\left(\sum_{n\in\mathbb{Z}} a_n z^n\right) = \sum_{n\in\mathbb{Z}} \operatorname{sgn}(n) a_n z^n.$$

Hence for $f \in L^2(\mathbb{T})$,

$$Ff = \varphi * f$$

where

$$\varphi = \sum_{n\in\mathbb{Z}} \mathrm{sgn}(n) z^n = \frac{1}{1-z} - \frac{z^{-1}}{1-z^{-1}} = \frac{2}{1-z}.$$

Thus,

$$\begin{aligned}(\,đf)g &= ([F,f]g)(t)\\ &= 2\lim_{\varepsilon\to 0}\int_{|\tau-t|>\varepsilon} \frac{f(t)-f(\tau)}{t-\tau} g(\tau)\,d\mathbf{m}(\tau).\end{aligned}$$

Finite rank differentials

Let $f : \mathbb{T} \to \mathbb{C}$. The strictest condition we can put on the smoothness of f is that f is a rational function. The strictest condition we can put on the size of $\overline{d}f$ is that $\overline{d}f$ is finite rank. These two conditions are equivalent.

Theorem (Kronecker)

If $f : \mathbb{T} \to \mathbb{C}$, then $\overline{d}f$ is finite rank if and only if f is a rational function.

Bounded differentials

Let $f : \mathbb{T} \to \mathbb{C}$. The weakest condition that we can place on $\overline{\partial} f$ is that $\overline{\partial} f$ is bounded.

Definition

Let $f : \mathbb{T} \to \mathbb{C}$ be measureable. We say that f is of *bounded mean oscillation* if for an arc $I \subseteq \mathbb{T}$, define

$$f_I = \frac{1}{\mathbf{m}(I)} \int_I f \, d\mathbf{m}$$

and

$$\sup_I \frac{1}{\mathbf{m}(I)} \int_I |f - f_I| \, d\mathbf{m} < \infty$$

where the supremum runs over all arcs I. The set of functions with bounded mean oscillation is denoted $\mathrm{BMO}(\mathbb{T})$.

Bounded differentials

Theorem (Nehari)

Let $f : \mathbb{T} \to \mathbb{C}$. Then $đf$ is bounded if and only if $f \in \mathrm{BMO}(\mathbb{T})$.

Compact differentials

We define the space $\mathrm{VMO}(\mathbb{T})$,

Definition

We say that $f \in \mathrm{VMO}(\mathbb{T})$ if $f \in \mathrm{BMO}(\mathbb{T})$ and

$$\lim_{\mathbf{m}(I)\to 0} \frac{1}{\mathbf{m}(I)} \int_I |f - f_I| \, d\mathbf{m} = 0.$$

Theorem

If $f : \mathbb{T} \to \mathbb{C}$, then $\overline{d}f$ is compact if and only if $f \in \mathrm{VMO}(\mathbb{T})$.

Can we do better?

We seek a more precise characterisation of the relationship between the smoothness of f and the size of $\overline{d}f$. To this end, we define the *Besov classes* B^s_{pq}.

We define a sequence of polynomials $\{W_n\}_{n\in\mathbb{Z}}$ on $\mathbb{T}$ as follows.

1. $W_0 = z^{-1} + 1 + z$.
2. For $n > 0$ and $k > 0$,

$$\widehat{W}_n(k) = \begin{cases} 1, & \text{if } k = 2^n \\ \text{a linear function on } [2^{n-1}, 2^n] \text{ and } [2^n, 2^{n+1}] \\ 0, & \text{otherwise.} \end{cases}$$

and $\widehat{W}_n(-k) = \widehat{W}_n(k)$, and $\widehat{W}_n(0) = 0$.

Definition of B^s_{pq}

Definition

For an integrable function φ on $\mathbb{T}$, we say that $f \in B^s_{pq}(\mathbb{T})$ if

$$\sum_{n\in\mathbb{Z}} 2^{|n|sp}\|W_n * \varphi\|_p^q < \infty.$$

Quantised differentials of Schatten-Von Neumann class

Theorem (Peller)

Let f be a measurable function on $\mathbb{T}$ and $p > 0$. Then $đf \in \mathcal{L}^p$ if and only if $f \in B^{1/p}_{pp}(\mathbb{T})$.

Can we do better?

Similar necessary and sufficient conditions can be found on f such that $\bar{d}f \in \mathcal{L}^{p,q}$.
Let us see how to prove these results.

Hankel Operators

A *Hankel matrix* is an infinite matrix $\{a_{j+k}\}_{j,k\geq 0}$,

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_1 & a_2 & a_3 & a_4 & \cdots \\ a_2 & a_3 & a_4 & a_5 & \cdots \\ a_3 & a_4 & a_5 & a_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

There is a direct link between Hankel operators and quantised differentials on the circle.

Hankel operators and quantised differentials

Theorem

Let $\varphi \in L^1(\mathbb{T})$, and M_φ is the densely defined pointwise multiplication operator on $L^2(\mathbb{T})$. Let $\mathbf{P}_-$ be the projection,

$$\mathbf{P}_-\left(\sum_{n\in\mathbb{Z}} a_n z^n\right) = \sum_{n<0} a_n z^n.$$

Then $H_\varphi := \mathbf{P}_- M_\varphi|_{H^2(\mathbb{T})}$ is a Hankel matrix when represented in the bases $\{z^n\}_{n=0}^\infty$ and $\{z^n\}_{n<0}$ with (j,k)th entry $\widehat{\varphi}(-j-k)$ for $j \geq 0$ and $k \geq 1$.

Hankel operators and quantised differentials

The link between Hankel operators and quantised differentials is provided by the following result:

Theorem

Let $\varphi \in L^1(\mathbb{T})$. Let $\varphi_- = \mathbf{P}_-\varphi$ and $\varphi_+ = (1-\mathbf{P}_-)\varphi$. Then,

$$\overline{đ}\varphi = 2((H_{\varphi_+})^* - H_{\varphi_-})$$

Symbol of a Hankel Operator

Definition

Let $\varphi \in L^1(\mathbb{T})$. Let Γ_φ be the infinite Hankel matrix with (j,k)th entry $\widehat{\varphi}(j+k)$.

From now on, it suffices to study the properties of Γ_φ.

Bounded Hankel Operators

Theorem (Nehari)

The Hankel operator Γ_φ *defines a bounded linear operator on* $\ell^2(\mathbb{N})$ *if and only if* $\varphi \in \mathrm{BMO}(\mathbb{T})$.

A result of Fefferman

To begin proving the Nehari theorem, we need the following result of C. Fefferman.

Theorem

The set $\mathrm{BMO}(\mathbb{T})$ *is exactly*

$$\mathrm{BMO}(\mathbb{T}) = \{f + \mathbf{P}_{-}g \ : \ f, g \in L^{\infty}(\mathbb{T})\}.$$

The Boundedness of Hankel operators

Theorem (Nehari)

Let $T = \{\alpha_{j+k}\}_{j,k\geq 0}$ be an infinite Hankel matrix defined by the sequence α. T defines a bounded linear operator on $\ell^2(\mathbb{N})$ if and only if there exists a function $\psi \in L^\infty(\mathbb{T})$ such that $\alpha_n = \widehat{\psi}(n)$ for all $n \geq 0$, and

$$\|T\| \leq \|\psi\|_\infty.$$

Proof.

Suppose that there exists such a ψ. Let a and b be finitely supported sequences, and compute the inner product:

$$(Ta, b) = \sum_{j,k\geq 0} \alpha_{j+k} a_j \overline{b_k}.$$

Proof (Cont.)

Let

$$f = \sum_{n=0}^{\infty} a_n z^n,\ g = \sum_{n=0}^{\infty} \overline{b_n} z^n$$

and let $q = fg$. Hence,

$$\begin{aligned}
(Ta, b) &= \sum_{j,k \geq 0} a_j \overline{b_k} \widehat{\psi}(j+k) \\
&= \sum_{n \geq 0} \widehat{\psi}(n) \sum_{k=0}^{n} a_k \overline{b_{n-k}} \\
&= \sum_{n \geq 0} \widehat{\psi}(n) \widehat{q}(n) \\
&= \int_{\mathbb{T}} \psi(\zeta) q(\overline{\zeta})\, d\mathbf{m}(\zeta).
\end{aligned}$$

Proof (Cont.)

Hence,

$$|(Ta, b)| \leq \|\psi\|_\infty \|q\|_1 \leq \|\psi\|_\infty \|f\|_2 \|g\|_2 = \|\psi\|_\infty \|a\|_2 \|b\|_2.$$

Hence T is bounded on $\ell^2(\mathbb{N})$.

Proof (Cont.)

Now we prove the converse. Suppose that T is bounded on $\ell^2(\mathbb{N})$. Define $\mathcal{L}$ on $\mathcal{P}_A(\mathbb{T})$

$$\mathcal{L}q = \sum_{n\geq 0} \alpha_n \widehat{q}(n).$$

Assume $\alpha \in \ell^1(\mathbb{N})$. In this case, $\mathcal{L}$ is continuous on $H^1(\mathbb{T})$. Let $q \in H^1(\mathbb{T})$ with $\|q\|_1 \leq 1$. Then $q = fg$ for $f, g \in H^2(\mathbb{T})$ with $\|f\|_2, \|g\|_2 \leq 1$. Hence,

$$\begin{aligned}
\mathcal{L}q &= \sum_{m\geq 0} \alpha_m \widehat{q}(m) \\
&= \sum_{m\geq 0} \alpha_m \sum_{j=0}^{m} \widehat{f}(j)\widehat{g}(m-j).
\end{aligned}$$

Proof (Cont.)

Hence, if we define $a_j = \widehat{f}(j)$ and $b_j = \overline{\widehat{b}(j)}$,

$$\begin{aligned}\mathcal{L}q &= \sum_{m\geq 0} \alpha_m \sum_{j=0}^{m} \widehat{f}(j)\widehat{g}(m-j) \\ &= (Ta, b).\end{aligned}$$

Thus,

$$|\mathcal{L}q| \leq \|T\| \|f\|_2 \|g\|_2 \leq \|T\|.$$

Proof (Cont.)

Now let α be an arbitrary sequence such that $\|T\|$ is bounded.
Let $0 < r < 1$. Define the sequence $\alpha^{(r)}$ by

$$\alpha_j^{(r)} = r^j \alpha_j$$

for $j \geq 0$.
Let $T^{(r)}$ be the Hankel matrix $\{\alpha_{j+k}^{(r)}\}_{j,k\geq 0}$.
Let D_r be the operator of multiplication by $\{r^j\}_{j\geq 0}$ on $\ell^2(\mathbb{N})$. Then we can compute that $T^{(r)} = D_r T D_r$. Hence $T^{(r)}$ is bounded, and $\|T^{(r)}\| \leq \|T\|$.

Proof (Cont.)

Let

$$\mathcal{L}_r q = \sum_{n\geq 0} \alpha_n^{(r)} \widehat{q}(n).$$

Since $\alpha^{(r)} \in \ell^2(\mathbb{N})$, we have already proved that $\|\mathcal{L}\|_{H^1\to\mathbb{C}} \leq \|T^{(r)}\| \leq \|T\|$.
Now the functionals $\{\mathcal{L}^{(r)}\}_{r\in(0,1)}$ are uniformly bounded and converge strongly to $\mathcal{L}$.
Hence, $\mathcal{L}$ is continuous.

Proof (Cont.)

We have shown that $\mathcal{L}$ is bounded on $H^1(\mathbb{T})$ with norm bounded by $\|T\|$.
Hence by the Hahn Banach theorem there is $\psi \in L^\infty$ such that
$\mathcal{L}q = (\psi, q)$, and $\|\psi\|_\infty \leq \|T\|$.
This completes the proof. $\square$

Reformulation in terms of H_φ

Recall that we defined

$$H_\varphi = \mathbf{P}_- M_\varphi|_{H^2(\mathbb{T})}.$$

For $\varphi \in L^1(\mathbb{T})$. This has matrix representation $\{\widehat{\varphi}(-j-k)\}_{j\geq 1, k\geq 0}$.
Hence, H_φ is bounded if and only if there exists $\psi \in L^\infty(\mathbb{T})$ such that

$$\widehat{\varphi}(-j) = \widehat{\psi}(j).$$

And hence $\mathbf{P}_-\varphi \in \mathrm{BMO}(\mathbb{T})$. Conversely, if $\mathbf{P}_-\varphi \in \mathrm{BMO}(\mathbb{T})$, then such a ψ exists.

Back to quantised derivatives

Recall that

$$\bar{d}\varphi = 2((H_{\varphi_-})^* - H_{\varphi_+}).$$

Since the two Hankel operators act on orthogonal complements, we see that $\bar{d}\varphi$ is bounded if and only if H_{φ_+} and H_{φ_-} are bounded. Hence

Theorem

$\bar{d}\varphi$ is bounded if and only is $\varphi \in \mathrm{BMO}(\mathbb{T})$.

Finite rank Hankel operators

Now let us prove Kronecker's theorem.

Theorem

Let $\varphi \in \mathrm{BMO}(\mathbb{T})$. Γ_φ is of finite rank if and only if φ is a rational function.

Recall that Γ_φ is the Hankel matrix with (j,k)th entry $\widehat{\varphi}(j+k)$.

Proof.

First assume that Γ_φ is a finite rank operator on $\ell^2(\mathbb{N})$. Let $\mathrm{rank}(\Gamma_\varphi) = n$. Then the first $n+1$ columns of Γ_φ are linearly dependent.

Proof (Cont.)

Let F be the operator on $H^2(\mathbb{T})$ that is "forward shift", that is

$$F\left(\sum_{n=0}^{\infty} a_n z^n\right) = \sum_{n=0}^{\infty} a_{n+1} z^n.$$

and B is "backward shift", so $Bf(\zeta) = \zeta f(\zeta)$.
The first $n+1$ columns of Γ_φ being linearly dependent means that there are non-trivial scalars $\{c_j\}_{j=0}^n$ such that

$$c_0\varphi + c_1 F\varphi + \cdots + c_n F\varphi = 0.$$

Proof (Cont.)

For a function $f \in L^2(\mathbb{T})$, it is easy to see that

$$B^n F^k f = B^{n-k} f - B^{n-k} \sum_{j=0}^{k-1} \widehat{f}(j) z^j.$$

Hence, since

$$\sum_{k=0}^{n} c_k F^k \varphi = 0$$

we conclude that

$$0 = \sum_{k=0}^{n} c_k B^n F^k \varphi = \sum_{k=0}^{n} c_k B^{n-k} \varphi - p,$$

where p is a polynomial. Let $q = \sum_{k=0}^{n} c_{n-k} z^k$. Hence $q\varphi = p$, so φ is a rational function.

Proof (Cont.)

Now we prove the converse. Suppose that $\varphi = p/q$ for polynomials p and q such that $\deg(p) \leq n-1$ and $\deg(q) \leq n$. Again, let

$$q = \sum_{j=0}^{n} c_{n-j} z^j.$$

Hence since $\varphi q = p$, we have

$$\sum_{j=0}^{n} c_j B^{n-j}\varphi = 0.$$

Proof (Cont.)

Hence,

$$F^n \sum_{j=0}^{n} c_j B^{n-j}\varphi = \sum_{j=0}^{n} c_j F^j \alpha$$
$$= 0.$$

Which means that the first $n+1$ rows of Γ_φ are linearly dependent. Let $m \leq n$ be the largest number such that $c_m \neq 0$. Then $F^m\varphi$ is a linear combination of the $F^j\varphi$ with $j \leq m-1$:

$$F^m\varphi = \sum_{j=0}^{m-1} d_j F^j\varphi,$$

for some nontrivial choice of scalars $\{d_j\}_{j=0}^{m-1}$.

Proof.

Let us proceed by induction to show that any row of Γ_φ is a linear combination of the first m rows. Let $k > m$. We have

$$\begin{aligned} F^k\varphi &= F^{k-m}F^m\varphi \\ &= \sum_{j=0}^{m-1} d_j F^{k-m+j}\varphi \end{aligned}$$

Since for each $0 \leq j \leq m-1$, we have $k - m + j < k$, by the inductive hypothesis the right hand side is a linear combination of the first m rows. Thus $\mathrm{rank}(\Gamma_\varphi) \leq m$. $\square$

Finite rank quantised derivatives

It is now straightforward to show that:

Theorem

$\bar{d}\varphi$ is finite rank if and only if φ is a rational function.

Compactness of quantised derivatives

Recall that in classical analysis, we could claim that f is continuous if and only if df is infinitesimal.
Since,

$$\|đ\varphi\| = \|[\mathrm{sgn}(\mathcal{D}), M_\varphi]\| \leq 2\|\varphi\|_\infty,$$

and φ a rational function implies that $đ\varphi$ is of finite rank, we obtain

Theorem

Let $\varphi \in C(\mathbb{T})$. Then $đ\varphi$ is compact (i.e., infinitesimal).

It can be proved that

Theorem

Let $\varphi \in L^2(\mathbb{T})$. Then H_φ is compact if and only if $\mathbf{P}_-\varphi \in \mathrm{VMO}(\mathbb{T})$.

Consequently $đ\varphi$ is compact if and only if $\varphi \in \mathrm{VMO}(\mathbb{T})$.

Trace class Hankel operators

Theorem

Let φ be a function holomorphic in the unit disc. Then $\Gamma_\varphi \in \mathcal{L}^1$ if and only if $\varphi \in B^1_{11}(\mathbb{T})$.

Recall that $\varphi \in B^1_{11}(\mathbb{T})$ means that

$$\sum_{n\geq 0} 2^{|n|}\|W_n * \varphi\|_1 < \infty.$$

A lemma about $\mathcal{L}^1$ norms

Lemma

Let $f \in \mathcal{P}_A(\mathbb{T})$, an analytic polynomial of degree at most m. Then

$$\|\Gamma_f\|_1 \leq (m+1)\|f\|_1.$$

Proof.

Let $\zeta \in \mathbb{T}$. Define the two vectors $x_\zeta, y_\zeta \in \ell^2(\mathbb{N})$,

$$x_\zeta = (1, \zeta, \zeta^2, \ldots, \zeta^m, 0, 0, \ldots)$$
$$y_\zeta = f(\zeta)(1, \zeta^{-1}, \zeta^{-2}, \ldots, \zeta^{-m}, 0, 0, \ldots).$$

And define the rank one operator $A_\zeta = y_\zeta(x, x_\zeta)$.

Proof (Cont.)

We can see that the (j,k)th entry of the matrix for A_ζ is

$$f(\zeta)\zeta^{-j-k}.$$

Hence we have an entrywise equality,

$$\Gamma_f = \int_{\mathbb{T}} A_\zeta \, d\mathbf{m}(\zeta).$$

But since these matrices only have finitely many non-zero entries, this can be considered as a Bochner integral. Therefore,

$$\|\Gamma_f\| \leq \int_{\mathbb{T}} \|A_\zeta\|_1 \, d\mathbf{m}(\zeta).$$

Proof (Cont.)

Since A_ζ is rank one, we can compute its $\mathcal{L}^1$ norm,

$$\|A_\zeta\|_1 = \|x_\zeta\|_2\|y_\zeta\|_2 = (m+1)|f(\zeta)|.$$

Therefore,

$$\|\Gamma_f\|_1 \leq \int_{\mathbb{T}} (m+1)|f(\zeta)|\, d\mathbf{m}(\zeta) = (m+1)\|f\|_1.$$

$\square$

Trace class Hankel operators

Finally we can prove the following,

Theorem

Let $\varphi \in B^1_{11}(\mathbb{T})$ and analytic in the unit disc. Then $\Gamma_\varphi \in \mathcal{L}^1$.

Proof.

By the definition of the sequence $\{W_n\}_{n\geq 0}$, we have

$$\varphi = \sum_{n\geq 0} W_n * \varphi.$$

This series converges uniformly since $\varphi \in \mathcal{B}^1_{11}(\mathbb{T}) \subseteq C^1(\mathbb{T})$. Hence,

$$\Gamma_\varphi = \sum_{n\geq 0} \Gamma_{W_n*\varphi}$$

is a series converging in the operator norm topology.

Proof (Cont.)

Hence,

$$\|\Gamma_\varphi\|_1 \leq \sum_{n\geq 0} \|\Gamma_{W_n * \varphi}\|_1.$$

But since $W_n * \varphi$ is a polynomial of degree at most 2^{n+1}, we get

$$\|\Gamma_\varphi\|_1 \leq \sum_{n\geq 0} 2^{n+1} \|W_n * \varphi\|_1.$$

□

Trace class quantised differentials

As a consequence, we obtain,

Theorem

Let $\varphi \in B^1_{11}(\mathbb{T})$, then $\bar{d}\varphi \in \mathcal{L}^1$.

The converse is also true. In fact, it is true that

$$\frac{1}{6}\sum_{n\geq 0} 2^n \|W_n * \varphi\|_1 \leq \|\Gamma_\varphi\|_1 \leq 2\sum_{n\geq 0} 2^n \|W_n * \varphi\|_1.$$