Quantised Calculus in One Dimension

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Introduction

The purpose of this talk is to introduce the *quantised calculus*, which is a tool coming from non-commutative geometry which gives rigorous justification to computations involving "infinitesimals".

Many thanks go to my supervisor, who has put up with me for a very long time.

Infinitesimals

- Early calculus (e.g. Leibniz, Newton) made use of infinitesimal quantities:

Definition

A quantity x is called *infinitesimal* if for any integer $n > 0$,

$$|x| < \frac{1}{n}.$$

Definition

A quantity x is *infinite* if for any integer $n > 0$,

$$|x| > n.$$

Infinitesimals

This definition was good enough for 18th century calculus, and many definitions were formulated in terms of infinitesimals.
For example:

Definition

A function f is continuous if $df(x) := f(x+dx) - f(x)$ is infinitesimal for any infinitesimal quantity dx.

Definition

A function f is differentiable if the quantity

$$f'(x) := \frac{f(x+dx) - f(x)}{dx}$$

is not infinite when dx is infinitesimal.

Properties of infinitesimals

Lemma

If x is infinitesmal, then x^{-1} is infinite.

Lemma

For any f,

$$df = \frac{df}{dx}dx.$$

Sizes of infinitesimals

Not all infinitesimals are equal. Intuitively we expect the following properties:

- If $x > 0$ is infinitesimal, then $x^2 < x$.
- If a function f is smoother than a function g, then $df < dg$... in some sense.

Problem

There is a problem! These definitions make no sense.

Problems:

- If x is infinitesimal, then $x = 0$.
- If x is infinite, then $|x| > 2|x|$, which is absurd.

So we cannot use the definitions given by 18th century mathematicians.

Infinitesimals were rightly banished from mathematics, and the definitions of continuity and differentiability were replaced with their modern definitions in terms of limits.

But what *is* an infinitesimal?

18th century mathematicians were still able to use infinitesimals even though they make no sense.

Question:

Why does calculus using infinitesmals "work"?

Question:

Is there a way to make sense of infinitesimals?

Introducing Non-commutative Geometry

What is noncommutative geometry?

The study of noncommutative algebras which resemble algebras of functions on geometric spaces, using the methods and the language of geometry.

Non-commutative geometry

Non-commutative geometry in analysis is usually the study of algebras of operators on Hilbert space. Algebras of operators are considered the generalisation of algebras of functions.

Example

The algebra $L^\infty(\mathbb{R})$ is an algebra of bounded operators on $L^2(\mathbb{R})$. The full algebra $\mathcal{B}(L^2(\mathbb{R}))$ is the "noncommutative extension" of the study of functions on $\mathbb{R}$.

Question:

We would like to think of all objects of interest as being operators on a Hilbert space.

Does this give us a way to define infinitesimals?

Infinitesimal Operators

Let $\mathcal{H}$ be a (complex, separable) Hilbert space. An infinitesimal operator $T \in \mathcal{B}(\mathcal{H})$ should be an operator T such that for any $\varepsilon > 0$, we have

$$\|T\| < \varepsilon.$$

Again, this definition is useless as it implies that $T = 0$.

Compact Operators as Infinitesimals

We shall say that an operator $T \in \mathcal{B}(\mathcal{H})$ is *infinitesimal* if for any $\varepsilon > 0$ there exists a finite dimensional subspace E such that

$$\|T|_{E^\perp}\| < \varepsilon.$$

This is equivalent to saying that T is compact.

Sizes of Infinitesimals

Definition

Given $T \in \mathcal{B}(\mathcal{H})$, define the kth singular value of T to be

$$\mu_k(T) := \inf\{\|T - A\| \ : \ \mathrm{rank}(A) \leq k\}.$$

For a compact operator, $\{\mu_k(T)\}_{k=0}^{\infty}$ is a vanishing sequence of positive numbers. We shall describe the *size* of an infinitesimal T as the *rate of decay* of $\{\mu_k(T)\}_{k=0}^{\infty}$.

Sizes of Infinitesimals

We can quantify the sizes an infinitesimal T by placing conditions on the rate of decay of $\{\mu_k(T)\}_{k=0}^{\infty}$.

- The smallest infinitesimals have $\{\mu_k(T)\}_{k=0}^{\infty}$ of finite support. Then T is of finite rank.
- We say that $T \in \mathcal{L}^p$ if $\{\mu_k(T)\}_{k=0}^{\infty} \in \ell^p$.
- We say that $T \in \mathcal{L}^{p,\infty}$ if $\mu_k(T) = \mathcal{O}(k^{-1/p})$.
- We say that $T \in \mathcal{L}^{p,q}$ if $\{k^{1/p-1/q}\mu_k(T)\}_{k=0}^{\infty} \in \ell^q$.
- We say that $T \in \mathcal{M}_{1,\infty}$ if $\{\frac{1}{\log(k+1)}\sum_{n=0}^{k}\mu_k(T)\}_{k=0}^{\infty} \in \ell^\infty$.

These last four conditions in fact correspond to ideals of $\mathcal{B}(\mathcal{H})$.

Sizes of Infinitesimals

Theorem

Let T be a compact operator in $\mathcal{B}(\mathcal{H})$. Then for every $k \geq 0$,

$$\mu_k(T^2) \leq \|T\|\mu_k(T).$$

Hence, if T is an infinitesimal, then T^2 is a smaller infinitesimal.

Quantised Differentials

In noncommutative geometry, we have a new object that is not present in classical analysis called a quantised derivative or quantised differential.

Definition

Let $f \in L^\infty(\mathbb{R})$. M_f is the operator on $L^2(\mathbb{R})$ of pointwise multiplication: $M_f g(x) = f(x)g(x)$ for almost all $x \in \mathbb{R}$. F denotes the Hilbert transform:

$$Fg(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \operatorname{sgn}(\xi) e^{ix\xi} \widehat{g}(\xi)\, d\xi.$$

(Defined at least initially for g a smooth function of compact support, then extended by continuity to $g \in L^2(\mathbb{R})$). Then we define:

$$đf := [F, M_f].$$

What is a quantised differential?

$\text{đ}f$ is supposed to represent the "infinitesimal variation", like df in classical analysis.

Warning:

$\text{đ}f$ is not a one-form, or a derivative. It is an infinitesimal deviation. It should be thought of as $f(x+h) - f(x)$ for infinitesimal epsilon.

It is very hard to motivate the definition of $\text{đ}f$. Instead, we will show that is satisfies a number of "headline properties" of a classical differential.

A dictionary

Classical Analysis	Non-commutative Analysis
Function	Operator
Range	Spectrum
Infinitesimal	Compact Operator
Differential	Quantised differential

Headline properties of infinitesimals

In classical 17-19th century analysis, infinitesimals were supposed to have a number of properties:

1. If $f : \mathbb{R} \to \mathbb{R}$ is a function, there is a function df representing infinitesimal variation in f. f is continuous if and only if df is infinitesimal.
2. If f is smoother than g, then df is smaller than dg.
3. If x is a positive infinitesimal, then x^2 is smaller than x.
4. If f is a differentiable function, then $df = f'dx$, provided that sufficiently small infinitesimals are ignored.

We shall see that the quantised differential satisfies all of these conditions, if they are interpreted correctly.

Smoothness and rate of decay

In classical analysis, we had the statement that if f is smoother than g, then df is smaller than dg.

The main question of this talk:

In what sense is it true that if f is smoother than g, then $đf$ is smaller than $đg$?

We shall restrict attention to functions on the circle $\mathbb{T}$ and the line $\mathbb{R}$.

We define $\mathbb{T} = \{\zeta \in \mathbb{C} \ : \ |\zeta| = 1\}$. Let $z : \mathbb{T} \to \mathbb{T}$ be the identity function. Denote the normalised Haar (or arc length) measure on $\mathbb{T}$ by $\mathbf{m}$.
For $f \in L^1(\mathbb{T}, \mathbf{m})$, define for $n \in \mathbb{Z}$,

$$\widehat{f}(n) := \int_{\mathbb{T}} z^{-n} f \, d\mathbf{m}.$$

Recall that any $f \in L^2(\mathbb{T}, \mathbf{m})$ can be written as

$$f = \sum_{n \in \mathbb{Z}} \widehat{f}(n) z^n.$$

The sum converges in the L^2 sense. This effects an isometric isomorphism between $L^2(\mathbb{T})$ and $\ell^2(\mathbb{Z})$.

Revision of Classical Fourier Analysis and Notation

The closed linear span of $\{z^n\}_{n=0}^{\infty}$ in L^2 is denoted $H^2(\mathbb{T})$, and the orthogonal complement is denoted $H^2_-(\mathbb{T})$.
We define the space of polynomials $\mathcal{P}(\mathbb{T})$ to be the finite linear span of $\{z^n\}_{n\in\mathbb{Z}}$. $P_A(\mathbb{T}) = \mathrm{span}\{z^n\}_{n\geq 0}$.

Differentials on $\mathbb{T}$

Defining quantised differentials for functions on $\mathbb{T}$ is exactly like for functions on $\mathbb{R}$.

Definition

The Hilbert transform, for $g \in L^2(\mathbb{T})$, is defined to be

$$Fg := \sum_{n\in\mathbb{Z}} \operatorname{sgn}(n)\widehat{g}(n)z^n.$$

We define the quantised differential:

$$đf := [F, M_f].$$

Differentials on $\mathbb{T}$

Hence for $f \in L^2(\mathbb{T})$,

$$Ff = \varphi * f$$

where

$$\varphi = \sum_{n\in\mathbb{Z}} \mathrm{sgn}(n) z^n = \frac{1}{1-z} - \frac{z^{-1}}{1-z^{-1}} = \frac{2}{1-z}.$$

Thus,

$$\begin{aligned}(\bar{d}f)g &= ([F,f]g)(t)\\ &= 2\lim_{\varepsilon\to 0}\int_{|\tau-t|>\varepsilon} \frac{f(t)-f(\tau)}{t-\tau} g(\tau)\, d\mathbf{m}(\tau).\end{aligned}$$

Finite rank differentials

Let $f : \mathbb{T} \to \mathbb{C}$. The strictest condition we can put on the smoothness of f is that f is a rational function. The strictest condition we can put on the size of $đf$ is that $đf$ is finite rank. These two conditions are equivalent.

Theorem (Kronecker)

If $f : \mathbb{T} \to \mathbb{C}$, then $đf$ is finite rank if and only if f is a rational function.

Bounded differentials

Let $f : \mathbb{T} \to \mathbb{C}$. The weakest condition that we can place on $\bar{\partial} f$ is that $\bar{\partial} f$ is bounded.

Definition

Let $f : \mathbb{T} \to \mathbb{C}$ be measureable. We say that f is of *bounded mean oscillation* if for an arc $I \subseteq \mathbb{T}$, define

$$f_I = \frac{1}{\mathbf{m}(I)} \int_I f \, d\mathbf{m}$$

and

$$\sup_I \frac{1}{\mathbf{m}(I)} \int_I |f - f_I| \, d\mathbf{m} < \infty$$

where the supremum runs over all arcs I. The set of functions with bounded mean oscillation is denoted $\mathrm{BMO}(\mathbb{T})$.

Bounded differentials

Theorem (Nehari)

Let $f : \mathbb{T} \to \mathbb{C}$. Then $đf$ is bounded if and only if $f \in \mathrm{BMO}(\mathbb{T})$.

Compact differentials

We define the space $\mathrm{VMO}(\mathbb{T})$:

Definition

We say that $f \in \mathrm{VMO}(\mathbb{T})$ if $f \in \mathrm{BMO}(\mathbb{T})$ and

$$\lim_{\mathbf{m}(I)\to 0} \frac{1}{\mathbf{m}(I)} \int_I |f - f_I| \, d\mathbf{m} = 0.$$

Theorem

If $f : \mathbb{T} \to \mathbb{C}$, then $\eth f$ is compact if and only if $f \in \mathrm{VMO}(\mathbb{T})$.

Can we do better?

We seek a more precise characterisation of the relationship between the smoothness of f and the size of $\bar{\partial} f$. To this end, we define the *Besov classes* B^s_{pq}.

Definition of B^s_{pq}

We define a sequence of polynomials $\{W_n\}_{n\in\mathbb{Z}}$ on $\mathbb{T}$ as follows.

1. $W_0 = z^{-1} + 1 + z$.
2. For $n > 0$ and $k > 0$,

$$\widehat{W_n}(k) = \begin{cases} 1, \text{ if } k = 2^n \\ \text{a linear function on } [2^{n-1}, 2^n] \text{ and } [2^n, 2^{n+1}] \\ 0, \text{ otherwise.} \end{cases}$$

and $\widehat{W_n}(-k) = \widehat{W_n}(k)$, and $\widehat{W_n}(0) = 0$.

Definition of B^s_{pq}

Definition

For an integrable function φ on $\mathbb{T}$, we say that $f \in B^s_{pq}(\mathbb{T})$ if

$$\sum_{n\in\mathbb{Z}} 2^{|n|sp}\|W_n * \varphi\|_p^q < \infty.$$

Quantised differentials of Schatten-Von Neumann class

Theorem (Peller)

Let f be a measurable function on $\mathbb{T}$ and $p > 0$. Then $ đf \in \mathcal{L}^p$ if and only if $f \in B^{1/p}_{pp}(\mathbb{T})$.

Can we do better?

What about finding conditions on φ such that $\eth\varphi \in \mathcal{L}^{p,q}$?
This problem can be solved with an interpolation functor:

$$F(\mathcal{L}^1, \mathcal{B}(L^2(\mathbb{T}))) = \mathcal{L}^{p,q}.$$

Hence, if $\varphi \in F(B^1_{11}, \mathrm{BMO}(\mathbb{T}))$, then

$$\eth\varphi \in \mathcal{L}^{p,q}.$$

There indeed exists such a functor: $F = K(-,-)_{(1-p)^{-1},q}$. This is highly technical and explained in detail in the thesis.

Summary

In summary we have:

f	Γf
Rational	Finite rank
$B_{pp}^{1/p}$	$\mathcal{L}^p$
$K(B_{11}^1, VMO)_{(1-p)^{-1},q}$	$\mathcal{L}^{p,q}$
VMO	Compact
BMO	Bounded

You should now be convinced that the statement "If f is smoother than g, then Γf is smaller than Γg" has at least some rigorous justification for functions on $\mathbb{T}$.

Quantised differentials on $\mathbb{R}$

So far we have talked about quantised differentials exclusively on $\mathbb{T}$. However, many of the same results hold for $\mathbb{R}$, since we have the *Cayley transform*.

Let $\zeta \in \mathbb{T} \setminus \{1\}$. Let

$$\omega(\zeta) = i\frac{1+\zeta}{1-\zeta}.$$

Let $g \in L^0(\mathbb{R})$. Define for $\zeta \in \mathbb{T}$,

$$\mathcal{C}(g)(\zeta) = \frac{\sqrt{\pi}}{2i}\frac{(g\circ\omega)(\zeta)}{1-\zeta}$$

Quantised differentials on $\mathbb{R}$

Theorem

$\mathcal{C} : L^2(\mathbb{R}) \to L^2(\mathbb{T})$, and $\mathcal{C}$ is unitary.
Moreover, for $f \in L^2(\mathbb{R})$,

$$\mathcal{C} đf \mathcal{C}^{-1} = đ(f \circ \omega).$$

Hence results on $\mathbb{T}$ can be transferred to results on $\mathbb{R}$.

Proving these statements

To prove all the preceding results, we need to introduce the theory of Hankel operators.

Definition

Let $\varphi \in L^1(\mathbb{T})$, and M_φ is the densely defined pointwise multiplication operator on $L^2(\mathbb{T})$. Let $\mathbf{P}_-$ be the projection,

$$\mathbf{P}_- \left(\sum_{n \in \mathbb{Z}} a_n z^n \right) = \sum_{n<0} a_n z^n.$$

Then $H_\varphi := \mathbf{P}_- M_\varphi|_{H^2(\mathbb{T})}$

Hankel operators and quantised differentials

The link between Hankel operators and quantised differentials is provided by the following result:

Theorem

Let $\varphi \in L^1(\mathbb{T})$. Let $\varphi_- = \mathbf{P}_-\varphi$ and $\varphi_+ = (1-\mathbf{P}_-)\varphi$. Then,

$$đ\varphi = 2((H_{\varphi_+})^* - H_{\varphi_-})$$

Since the two Hankel operators act on orthogonal complements, we see that $đ\varphi$ is bounded if and only if H_{φ_+} and H_{φ_-} are bounded.

Reformulation in terms of H_φ

Recall:

$$H_\varphi = \mathbf{P}_- M_\varphi|_{H^2(\mathbb{T})}.$$

For $\varphi \in L^1(\mathbb{T})$. This has matrix representation $\{\widehat{\varphi}(-j-k)\}_{j\geq 1, k\geq 0}$.

Hankel Operators

A *Hankel matrix* is an infinite matrix $\{a_{j+k}\}_{j,k\geq 0}$,

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_1 & a_2 & a_3 & a_4 & \cdots \\ a_2 & a_3 & a_4 & a_5 & \cdots \\ a_3 & a_4 & a_5 & a_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

All of our results about quantised differentials come from the (very old) theory of Hankel matrices.

Quantised differentials and classical derivatives

Recall that 17th century differentials were supposed to have the property that

$$df = f' dx$$

provided that sufficiently small infinitesimals are ignored.
We can now interpret this in a rigorous sense. First, we replace infinitesimals with quantised differentials. To "ignore sufficiently small infinitesimals" means that we work modulo some ideal of compact operators.

Quantised differentials and classical derivatives

Let $\mathcal{L}_0^{p,\infty}$ be the closure in $\mathcal{L}^{p,\infty}$ of the ideal of finite rank operators.

Theorem (Connes)

Let $f \in C(\mathbb{T})$ be such that $\bar{d}f \in \mathcal{L}^{p,\infty}$ and $\varphi \in C^\infty(\sigma(f))$, and $p \in [1,\infty)$. Then

$$\bar{d}\varphi(f) \equiv \varphi'(f)\bar{d}f \mod \mathcal{L}_0^{p,\infty}.$$

Further results

What more can be done?

Better ideal membership results

Can we find conditions on φ (a function either on $\mathbb{T}$ or $\mathbb{R}$) such that $đ\varphi \in \mathcal{M}_{1,\infty}$?

More general forms of the chain rule

In how much generality can we prove that $đ\varphi(a) \equiv \varphi'(a)đa$?

Higher dimensions

Can we prove analogues of these results for functions on $\mathbb{R}^d$ or $\mathbb{T}^d$?

Better ideal membership results

It can be shown that for an operator T, that $T \in \mathcal{M}_{1,\infty}$ if and only if

$$\sup_{s\in(0,1)} s\|T\|_{s+1} < \infty$$

where

$$\|T\|_p = \left(\sum_{n\geq 0} \mu_n(T)^p\right)^{1/p}.$$

Since we know conditions on $\bar{d}f$ such that $\|\bar{d}f\|_p$ is finite, we (with some effort) can find conditions such that $\bar{d}f \in \mathcal{M}_{1,\infty}$.

More general forms of the chain rule

Further generalisations of the chain rule can be found.
In particular it can be proved in the setting of certain *spectral triples*.

Higher dimensions

It would appear that there an immediate generalisation of these results in higher dimensions fails to work.
In particular, even for polynomial functions f, $\bar{d}f$ is not even $\mathcal{L}^2$.

The End

Thank you for listening!