Quantised Calculus in One Variable

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Introduction

The purpose of this talk is to introduce the *quantised calculus*, which is a tool coming from non-commutative geometry which gives rigorous justification to computations involving "infinitesimals".

Infinitesimals

- Early calculus (e.g. Leibniz, Newton) made use of infinitesimal quantities:

Definition

A quantity x is called *infinitesimal* if for any integer $n > 0$,

$$|x| < \frac{1}{n}.$$

Definition

A quantity x is *infinite* if for any integer $n > 0$,

$$|x| > n.$$

Infinitesimals

This definition was good enough for 18th century calculus, and many definitions were formulated in terms of infinitesimals.
For example:

Definition

A function f is continuous if $df(x) := f(x+dx) - f(x)$ is infinitesimal for any infinitesimal quantity dx.

Definition

A function f is differentiable if the quantity

$$f'(x) := \frac{f(x+dx) - f(x)}{dx}$$

is not infinite when dx is infinitesimal.

Properties of infinitesimals

Lemma

If x is infinitesmal, then x^{-1} is infinite.

Lemma

For any f,

$$df = \frac{df}{dx}dx$$

Sizes of infinitesimals

Not all infinitesimals are equal. Intuitively we expect the following properties:

- If $x > 0$ is infinitesimal, than $x^2 < x$.
- If a function f is smoother than a function g, then $df < dg$.

Problem

There is a problem! These definitions make no sense.

Problems:

- If $x \neq 0$ is infinitesimal, then $\lfloor x \rfloor = 0$.
- If x is infinite, then $\lfloor x \rfloor > \lfloor 2\lfloor x \rfloor \rfloor$, so $1 > 2$.

So we cannot use the definitions given by 18th century mathematicians.

Infinitesimals were rightly banished from mathematics, and the definitions of continuity and differentiability were replaced with their modern definitions in terms of limits.

But what *is* an infinitesimal?

18th century mathematicians were still able to use infinitesimals even though they make no sense.

Question:

Why does calculus using infinitesmals "work"?

Question:

Is there a way to make sense of infinitesimals?

Infinitesimal Operators

An infinitesimal operator $T \in \mathcal{B}(\mathcal{H})$ should be an operator T such that for any $\varepsilon > 0$, we have

$$\|T\| < \varepsilon.$$

Again, this definition is useless as it implies that $T = 0$.

Compact Operators as Infinitesimals

We shall say that an operator $T \in \mathcal{B}(\mathcal{H})$ is *infinitesimal* if for any $\varepsilon > 0$ there exists a finite dimensional subspace E such that

$$\|T|_{E^\perp}\| < \varepsilon.$$

This is equivalent to saying that T is compact.

Infinitesimals in Non-commutative geometry

In a noncommutative geometry $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, the compact elements of $\mathcal{A}$ play a similar role to infinitesimals in classical real analysis.
We have the following dictionary:

Classical Analysis	Non-commutative Analysis
Function	Operator
One-form	Connes Differential
Range	Spectrum
Infinitesimal	Compact Operator

Quantised Differentials

In noncommutative geometry, we have a new object that is not present in classical analysis called a quantised derivative or quantised differential.

Definition

Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple. Let F be the unique operator such that $\mathcal{D} = F|\mathcal{D}|$, that is $F = \mathrm{sgn}(\mathcal{D})$. For $a \in \mathcal{A}$, define

$$đa := [F, a].$$

This is the *quantised differential* of a.

$đa$ is supposed to represent an infinitesimal variation of a. The motivation behind this definition is not clear, but as we work through examples it will become clear that this definition is the "correct" one.

Expected properties of infinitesimals

In classical 17-19th century analysis, infinitesimals were supposed to have a number of properties:

1. If $f : \mathbb{R} \to \mathbb{R}$ is a function, there is a function df representing infinitesimal variation in f. f is continuous if and only if df is infinitesimal.
2. If f is smoother than g, then df is smaller than dg.
3. If x is a positive infinitesimal, then x^2 is smaller than x.
4. If f is a differentiable function, then $df = f'dx$, provided that sufficiently small infinitesimals are ignored.

We shall see that the quantised differential satisfies all these properties, if they are interpreted correctly.

Sizes of Infinitesimals

Definition

Given $T \in \mathcal{B}(\mathcal{H})$, define the kth singular value of T to be

$$\mu_k(T) := \inf\{\|T - A\| \ : \ \text{rank}(A) \leq k\}.$$

For a compact operator, $\{\mu_k(T)\}_{k=0}^{\infty}$ is a vanishing sequence of positive numbers. We shall describe the *size* of an infinitesimal T as the *rate of decay* of $\{\mu_k(T)\}_{k=0}^{\infty}$.

Sizes of Infinitesimals

Theorem

Let T and S be compact operators in $\mathcal{B}(\mathcal{H})$. Then for every $k \geq 0$,

$$\mu_k(T^2) \leq \mu_k(T)^2.$$

Hence, if T is an infinitesimal, then T^2 is a smaller infinitesimal.

Sizes of Infinitesimals

We can quantify the sizes an infinitesimal T by placing conditions on the rate of decay of $\{\mu_k(T)\}_{k=0}^{\infty}$.

- The smallest infinitesimals have $\{\mu_k(T)\}_{k=0}^{\infty}$ of finite support. Then T is of finite rank.
- We say that $T \in \mathcal{L}^p$ if $\{\mu_k(T)\}_{k=0}^{\infty} \in \ell^p$.
- We say that $T \in \mathcal{L}^{p,\infty}$ if $\mu_k(T) = \mathcal{O}(k^{-1/p})$.
- We say that $T \in \mathcal{L}^{p,q}$ if $\{k^{1/p-1/q}\mu_k(T)\}_{k=0}^{\infty} \in \ell^q$.
- We say that $T \in \mathcal{M}^{1,\infty}$ if $\{\frac{1}{\log(k+1)}\sum_{n=0}^{k}\mu_k(T)\}_{k=0}^{\infty} \in \ell^{\infty}$.

Smoothness and rate of decay

We now enter the second phase of the talk: "Hard Analysis".
Recall that in classical analysis, we had the statement that if f is smoother than g, then df is smaller than dg.

The main question of this talk:

In what sense is it true that if f is smoother than g, then $\vec{d}f$ is smaller than $\vec{d}g$?

We shall restrict attention to functions on the circle $\mathbb{T}$ and the line $\mathbb{R}$.

Differentials on $\mathbb{T}$

The Dirac operator on the circle is $\mathcal{D} = \frac{1}{i}\frac{d}{d\theta}$, i.e., $\mathcal{D}(z^n) = nz^n$. So we may describe $F = \operatorname{sgn}\mathcal{D}$ as the *Hilbert transform*,

$$F\left(\sum_{n\in\mathbb{Z}} a_n z^n\right) = \sum_{n\in\mathbb{Z}} \operatorname{sgn}(n) a_n z^n.$$

Differentials on $\mathbb{T}$

Hence for $f \in L^2(\mathbb{T})$,

$$Ff = \varphi * f$$

where

$$\varphi = \sum_{n\in\mathbb{Z}} \mathrm{sgn}(n) z^n = \frac{1}{1-z} - \frac{z^{-1}}{1-z^{-1}} = \frac{2}{1-z}.$$

Thus,

$$\begin{aligned}(\bar{d} f)g &= ([F, f]g)(t) \\ &= 2\lim_{\varepsilon\to 0} \int_{|\tau - t| > \varepsilon} \frac{f(t) - f(\tau)}{t - \tau} g(\tau)\, d\mathbf{m}(\tau).\end{aligned}$$

Finite rank differentials

Let $f : \mathbb{T} \to \mathbb{C}$. The strictest condition we can put on the smoothness of f is that f is a rational function. The strictest condition we can put on the size of $đf$ is that $đf$ is finite rank. These two conditions are equivalent.

Theorem (Kronecker)

If $f : \mathbb{T} \to \mathbb{C}$, then $đf$ is finite rank if and only if f is a rational function.

Bounded differentials

Let $f : \mathbb{T} \to \mathbb{C}$. The weakest condition that we can place on $\eth f$ is that $\eth f$ is bounded.

Definition

Let $f : \mathbb{T} \to \mathbb{C}$ be measureable. We say that f is of *bounded mean oscillation* if for an arc $I \subseteq \mathbb{T}$, define

$$f_I = \frac{1}{\mathbf{m}(I)} \int_I f \, d\mathbf{m}$$

and

$$\sup_I \frac{1}{\mathbf{m}(I)} \int_I |f - f_I| \, d\mathbf{m} < \infty$$

where the supremum runs over all arcs I. The set of functions with bounded mean oscillation is denoted $\mathrm{BMO}(\mathbb{T})$.

Bounded differentials

Theorem (Nehari)

Let $f : \mathbb{T} \to \mathbb{C}$. *Then* $đf$ *is bounded if and only if* $f \in \mathrm{BMO}(\mathbb{T})$.

Compact differentials

We define the space $\mathrm{VMO}(\mathbb{T})$:

Definition

We say that $f \in \mathrm{VMO}(\mathbb{T})$ if $f \in \mathrm{BMO}(\mathbb{T})$ and

$$\lim_{\mathbf{m}(I)\to 0} \frac{1}{\mathbf{m}(I)} \int_I |f - f_I| \, d\mathbf{m} = 0.$$

Theorem

If $f : \mathbb{T} \to \mathbb{C}$, then $\bar{d}f$ is compact if and only if $f \in \mathrm{VMO}(\mathbb{T})$.

Can we do better?

We seek a more precise characterisation of the relationship between the smoothness of f and the size of $\bar{\partial} f$. To this end, we define the *Besov classes* B^s_{pq}.

Definition of B^s_{pq}

We define a sequence of polynomials $\{W_n\}_{n\in\mathbb{Z}}$ on $\mathbb{T}$ as follows.

1. $W_0 = z^{-1} + 1 + z$.
2. For $n > 0$ and $k > 0$,

$$\widehat{W}_n(k) = \begin{cases} 1, & \text{if } k = 2^n \\ \text{a linear function on } [2^{n-1}, 2^n] \text{ and } [2^n, 2^{n+1}] \\ 0, & \text{otherwise.} \end{cases}$$

and $\widehat{W}_n(-k) = \widehat{W}_n(k)$, and $\widehat{W}_n(0) = 0$.

Definition of B^s_{pq}

Definition

For an integrable function φ on $\mathbb{T}$, we say that $f \in B^s_{pq}(\mathbb{T})$ if

$$\sum_{n\in\mathbb{Z}} 2^{|n|sp} \|W_n * \varphi\|_p^q < \infty.$$

Quantised differentials of Schatten-Von Neumann class

Theorem (Peller)

Let f be a measurable function on $\mathbb{T}$ and $p > 0$. Then $đf \in \mathcal{L}^p$ if and only if $f \in B^{1/p}_{pp}(\mathbb{T})$.

The End

In summary:

1. The quantised differential has meaning in analysis on $\mathbb{T}$ and $\mathbb{R}$.
2. The quantised differential has many properties that we would anticipate for "infinitesimal differences".

Thank you for listening!