Quantised Calculus and Hankel Operators

McDonald, E.
Supervisor: Sukochev, F.

UNSW Australia

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Introduction

The purpose of this talk is to introduce the *quantised calculus*, which is a tool coming from non-commutative geometry which gives rigorous justification to computations involving “infinitesimals”.

Infinitesimals

- Early calculus (e.g. Leibniz, Newton) made use of infinitesimal quantities:

Definition

A quantity x is called *infinitesimal* if for any integer $n > 0$,

$$|x| < \frac{1}{n}.$$

Definition

A quantity x is *infinite* if for any integer $n > 0$,

$$|x| > n.$$

Infinitesimals

This definition was good enough for 18th century calculus, and many definitions were formulated in terms of infinitesimals.
For example:

Definition

A function f is continuous if $df(x) := f(x+dx) - f(x)$ is infinitesimal for any infinitesimal quantity dx.

Definition

A function f is differentiable if the quantity

$$f'(x) := \frac{f(x+dx) - f(x)}{dx}$$

is not infinite when dx is infinitesimal.

Properties of infinitesimals

Lemma

If x is infinitesmal, then x^{-1} is infinite.

Lemma

For any f,

$$df = \frac{df}{dx}dx.$$

Sizes of infinitesimals

Not all infinitesimals are equal. Intuitively we expect the following properties:

- If $x > 0$ is infinitesimal, than $x^2 < x$.
- If a function f is smoother than a function g, then $df < dg$.

Problem

There is a problem! These definitions make no sense.

Problems:

- If x is infinitesimal, then $x = 0$.
- If x is infinite, then $|x| > 2|x|$, which is absurd.

So we cannot use the definitions given by 18th century mathematicians.

Infinitesimals were rightly banished from mathematics, and the definitions of continuity and differentiability were replaced with their modern definitions in terms of limits.

But what *is* an infinitesimal?

18th century mathematicians were still able to use infinitesimals even though they make no sense.

Question:

Why does calculus using infinitesmals “work”?

Question:

Is there a way to make sense of infinitesimals?

Expected properties of infinitesimals

In classical 17-19th century analysis, infinitesimals were supposed to have a number of properties:

1. If $f : \mathbb{R} \to \mathbb{R}$ is a function, there is a function df representing infinitesimal variation in f. f is continuous if and only if df is infinitesimal.
2. If f is smoother than g, then df is smaller than dg.
3. If x is a positive infinitesimal, then x^2 is smaller than x.
4. If f is a differentiable function, then $df = f'dx$, provided that sufficiently small infinitesimals are ignored.

Introducing Non-commutative Geometry

What is noncommutative geometry?

The study of noncommutative algebras which resemble algebras of functions on geometric spaces, using the methods and the language of geometry.

Why Non-commutative Geometry?

Non-commutative geometry gives us new insights on non-commutative algebra, but historically it was motivated by quantum mechanics.

A very brief description of quantum mechanics

Definition

A *quantum mechanical system* is a pair $(\mathcal{A}, \mathcal{H})$ where $\mathcal{H}$ is a complex separable Hilbert space and $\mathcal{A}$ is a $*$-algebra of (potentially unbounded, densely defined) operators on $\mathcal{H}$. The self adjoint elements of $\mathcal{A}$ are called *observables*. The non-zero elements of $\mathcal{H}$ are called *states*.

- The states correspond to potential configurations of a physical system. States which are non-zero scalar multiples of each other are considered physically indistinguishable.
- The observables correspond to physical quantities that can be measured. The range of potential measurements for an observable $A \in \mathcal{A}$ is $\sigma(A)$.

Reminder: The spectral theorem

Theorem

Let $(\mathcal{A}, \mathcal{H})$ be a quantum mechanical system. If $A \in \mathcal{A}$ is an observable, then there exists a projection valued measure E_A on $\sigma(A)$ such that

$$A = \int_{\sigma(A)} \lambda \, dE_A(\lambda).$$

Measurement in quantum mechanics

Let $(\mathcal{A}, \mathcal{H})$ be a quantum mechanical system, currently in a state $\psi \in \mathcal{H}$. Let $A \in \mathcal{A}$ be an observable. The probability that the measured value of the physical quantity corresponding to A lies in a borel set $\Delta \subseteq \sigma(A)$ is

$$P(\Delta; \psi) := \frac{(\psi, E_A(\Delta)\psi)}{\|\psi\|^2} = \frac{\|E_A(\Delta)\psi\|^2}{\|\psi\|^2}.$$

If the measured value lies in Δ, then the state of the system changes to

$$E_A(\Delta)\psi.$$

Measurement in quantum mechanics

There are two surprising features of measurement in quantum mechanics:

1. The act of observation changes the state of the system.
2. The order of observation is important, since if A and B are observables, and $\Delta \times \Sigma \subseteq \sigma(A) \times \sigma(B)$, then in general it is not true that
$$E_A(\Delta)E_B(\Sigma)\psi = E_B(\Sigma)E_A(\Delta)\psi.$$

The beginnings of noncommutative geometry

A noncommutative space is a “space” with “coordinates” such that the order of measurement of coordinates changes the measured values.
Inspired by quantum mechanics, we define a noncommutative space as a pair $(\mathcal{A}, \mathcal{H})$ exactly like a quantum mechanical system.

Relation to Ordinary Geometry

The usual setting for geometry is a Riemannian manifold M. Let $\mathcal{A} = C^\infty(M)$ and $\mathcal{H} = L^2(M)$.
However, we need more information to specify the geometry of the manifold. One way to do this is by giving a *Dirac operator*.

Dirac Operators

Associated to a Riemannian manifold (M, g) is a *clifford bundle*.

$$\mathrm{Cliff}(M, g) = \frac{T(TM)}{\langle xy + yx = -2g(x, y)\rangle}.$$

A vector bundle V on M is called a *clifford module* if there is a left multiplication action $m : \mathrm{Cliff}(M, g) \otimes V \to V$.

Definition

Suppose that (M, g) is a Riemannian manifold with a Clifford module V, and a V-valued connection $\nabla : V \to T^*M \otimes V$. Then the *Dirac Operator* on V is given by the composition of maps

$$V \xrightarrow{\nabla} T^*M \otimes V \xrightarrow{\sharp \otimes I} TM \otimes V \xrightarrow{m} V$$

where $\sharp$ is the canonical isomorphism from T^*M to TM given by g.

Theorem

Let (M,g) be a Riemannian manifold with Clifford module V and Dirac operator $\mathcal{D}$. Then for every $a \in C^\infty(M)$,

$$da = [\mathcal{D}, a]$$

as an equality of operators on the sections of V.

Spectral Triples

To specify the geometry of non-commutative space $(\mathcal{A}, \mathcal{H})$, we need additional data to tell us how observables relate to each other. An elegant way of doing this is by giving a Dirac operator.

Definition

A *spectral triple* is a triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ where $\mathcal{H}$ is a complex separable Hilbert space and $\mathcal{A}$ is a $*$-algebra of bounded operators on $\mathcal{H}$. $\mathcal{D}$ is a densely defined unbounded self adjoint operator on $\mathcal{H}$ such that:

1. $[\mathcal{D}, a]$ is bounded for all $a \in \mathcal{A}$.
2. $a(\lambda - \mathcal{D})^{-1}$ is compact for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and $a \in \mathcal{A}$.

If in addition there is a $\mathbb{Z}_2$-grading on $\mathcal{H}$, with grading operator Γ, $\Gamma^2 = \mathbb{1}_{\mathcal{H}}$, we assert that $\Gamma\mathcal{D} + \mathcal{D}\Gamma = 0$, and call $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ *even*. If no such grading exists, we call $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ *odd*.

The prototypical example is a Riemannian manifold M with a Clifford module, where $\mathcal{A} = C^\infty(M)$, $\mathcal{H} = L^2(V)$ and $\mathcal{D}$ is the Dirac operator of the module.

Properties of Spectral Triples

Definition

Let $k \geq 0$ be an integer. We say that a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is QC^k if for any $a \in \mathcal{A}$, a and $[\mathcal{D}, a]$ are in the domain of $\delta^k(a)$, where $\delta(a) := [|\mathcal{D}|, a]$.

Definition

Let $p > 0$. A spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is called (p, ∞) summable if $(\lambda - \mathcal{D})^{-1} \in \mathcal{L}^{p,\infty}$ for $\lambda \in \mathbb{C} \setminus \mathbb{R}$. p is analogous to the dimension of the space.

Derivations in abstract algebra

Definition

Let A be a $\mathbb{C}$-algebra, and M is an A-bimodule. A $\mathbb{C}$-linear map $\theta : A \to M$ is called a derivation if $\theta(ab) = a\theta(b) + \theta(a)b$ for all $a, b \in A$.

Example

Let N be a manifold. We can take $A = C^\infty(N)$ and $M = \Omega^1(N)$, and $\theta = d$.

For a $\mathbb{C}$-algebra $\mathcal{A}$, there is a "largest" A-bimodule that is the image of a derivation on $\mathcal{A}$.

Definition

Let A be a $\mathbb{C}$-algebra. There is a multiplication map $\gamma : A \otimes A \to A$. Define

$$\Omega^1(A) := \ker \gamma.$$

And $d : A \to \Omega^1(A)$ is given by $a \mapsto 1 \otimes a - a \otimes 1$. This is the algebra of *Kähler differentials*.

Kähler differentials

The utility of Kähler differentials comes from the universal property:

Theorem

Let A be a $\mathbb{C}$-algebra, and M is an A-bimodule with a derivation $\theta : A \to M$. Then there exists a unique A-module homomorphism $\tilde{\theta} : \Omega^1(A) \to M$ such that $\theta = \tilde{\theta} \circ d$.

Connes Differentials

Given a spectral triple, $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, and $a \in \mathcal{A}$, we define

$$\pi : \Omega^1(\mathcal{A}) \to \mathcal{B}(\mathcal{H})$$

by

$$da \mapsto [\mathcal{D}, a].$$

Define the two-sided ideal of $\mathcal{B}(\mathcal{H})$, J as

$$J = \{t \in \Omega^1(\mathcal{A}) \ : \ \pi(t) = 0\}.$$

and $J_0 = J + dJ$. Then,

$$\Omega^1_{\mathcal{D}}(\mathcal{A}) := \frac{\pi(\Omega^1(\mathcal{A}))}{\pi(dJ_0)}.$$

Upshot:

The noncommutative perspective teaches us that all objects of interest can be thought of as operators on an appropriate Hilbert space.
Does this give us a way to define infinitesimals?

Infinitesimal Operators

An infinitesimal operator $T \in \mathcal{B}(\mathcal{H})$ should be an operator T such that for any $\varepsilon > 0$, we have

$$\|T\| < \varepsilon.$$

Again, this definition is useless as it implies that $T = 0$.

Compact Operators as Infinitesimals

We shall say that an operator $T \in \mathcal{B}(\mathcal{H})$ is *infinitesimal* if for any $\varepsilon > 0$ there exists a finite dimensional subspace E such that

$$\|T|_{E^\perp}\| < \varepsilon.$$

This is equivalent to saying that T is compact.

Infinitesimals in Non-commutative geometry

In noncommutative geometry, all objects of interest arise as operators on Hilbert space.

We have the following dictionary:

Classical Analysis	Non-commutative Analysis
Function	Operator
One-form	Connes Differential
Range	Spectrum
Infinitesimal	Compact Operator

Quantised Differentials

In noncommutative geometry, we have a new object that is not present in classical analysis called a quantised derivative or quantised differential.

Definition

Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple. By the Borel functional calculus, we can define $F = \mathrm{sgn}(\mathcal{D})$.
We define
$$đa = [F, a].$$

$đa$ is supposed to represent an infinitesimal variation of a. The motivation behind this definition is not clear, but as we work through examples it will become clear that this definition is the "correct" one.

Sizes of Infinitesimals

Definition

Given $T \in \mathcal{B}(\mathcal{H})$, define the kth singular value of T to be

$$\mu_k(T) := \inf\{\|T - A\| \ : \ \mathrm{rank}(A) \leq k\}.$$

For a compact operator, $\{\mu_k(T)\}_{k=0}^{\infty}$ is a vanishing sequence of positive numbers. We shall describe the *size* of an infinitesimal T as the *rate of decay* of $\{\mu_k(T)\}_{k=0}^{\infty}$.

Sizes of Infinitesimals

Theorem

Let T be a compact operator in $\mathcal{B}(\mathcal{H})$. Then for every $k \geq 0$,

$$\mu_k(T^2) \leq \|T\|\mu_k(T).$$

Hence, if T is an infinitesimal, then T^2 is a smaller infinitesimal.

Sizes of Infinitesimals

We can quantify the sizes an infinitesimal T by placing conditions on the rate of decay of $\{\mu_k(T)\}_{k=0}^\infty$.

- The smallest infinitesimals have $\{\mu_k(T)\}_{k=0}^\infty$ of finite support. Then T is of finite rank.
- We say that $T \in \mathcal{L}^p$ if $\{\mu_k(T)\}_{k=0}^\infty \in \ell^p$.
- We say that $T \in \mathcal{L}^{p,\infty}$ if $\mu_k(T) = \mathcal{O}(k^{-1/p})$.
- We say that $T \in \mathcal{L}^{p,q}$ if $\{k^{1/p-1/q}\mu_k(T)\}_{k=0}^\infty \in \ell^q$.
- We say that $T \in \mathcal{M}^{1,\infty}$ if $\{\frac{1}{\log(k+1)}\sum_{n=0}^k \mu_k(T)\}_{k=0}^\infty \in \ell^\infty$.

Smoothness and rate of decay

Recall that in classical analysis, we had the statement that if f is smoother than g, then df is smaller than dg.

The main question of this talk:

In what sense is it true that if f is smoother than g, then $\bar{d}f$ is smaller than $\bar{d}g$?

We shall restrict attention to functions on the circle $\mathbb{T}$ and the line $\mathbb{R}$.

We define $\mathbb{T} = \{\zeta \in \mathbb{C} \ : \ |\zeta| = 1\}$. Let $z : \mathbb{T} \to \mathbb{T}$ be the identity function. Denote the normalised Haar (or arc length) measure on $\mathbb{T}$ by $\mathbf{m}$.
For $f \in L^1(\mathbb{T}, \mathbf{m})$, define for $n \in \mathbb{Z}$,

$$\widehat{f}(n) := \int_{\mathbb{T}} z^{-n} f \, d\mathbf{m}.$$

Recall that any $f \in L^2(\mathbb{T}, \mathbf{m})$ can be written as

$$f = \sum_{n\in\mathbb{Z}} \widehat{f}(n) z^n.$$

The sum converges in the L^2 sense. This effects an isometric isomorphism between $L^2(\mathbb{T})$ and $\ell^2(\mathbb{Z})$.

The closed linear span of $\{z^n\}_{n=0}^{\infty}$ in L^p is denoted $H^2(\mathbb{T})$, and the orthogonal complement is denoted $H^2_-(\mathbb{T})$.
We define the space of polynomials $\mathcal{P}(\mathbb{T})$ to be the finite linear span of $\{z^n\}_{n\in\mathbb{Z}}$. $P_A(\mathbb{T}) = \text{span}\{z^n\}_{n\geq 0}$.

Differentials on $\mathbb{T}$

The Dirac operator on the circle is $\mathcal{D} = \frac{1}{i}\frac{d}{d\theta}$, i.e., $\mathcal{D}(z^n) = nz^n$. So we may describe $F = \operatorname{sgn}\mathcal{D}$ as the *Hilbert transform*,

$$F\left(\sum_{n\in\mathbb{Z}} a_n z^n\right) = \sum_{n\in\mathbb{Z}} \operatorname{sgn}(n) a_n z^n.$$

Differentials on $\mathbb{T}$

Hence for $f \in L^2(\mathbb{T})$,

$$Ff = \varphi * f$$

where

$$\varphi = \sum_{n \in \mathbb{Z}} \mathrm{sgn}(n) z^n = \frac{1}{1-z} - \frac{z^{-1}}{1-z^{-1}} = \frac{2}{1-z}.$$

Thus,

$$\begin{aligned}(\text{đ} f)g &= ([F,f]g)(t) \\ &= 2 \lim_{\varepsilon \to 0} \int_{|\tau - t| > \varepsilon} \frac{f(t) - f(\tau)}{t - \tau} g(\tau)\, d\mathbf{m}(\tau).\end{aligned}$$

Finite rank differentials

Let $f : \mathbb{T} \to \mathbb{C}$. The strictest condition we can put on the smoothness of f is that f is a rational function. The strictest condition we can put on the size of $đf$ is that $đf$ is finite rank. These two conditions are equivalent.

Theorem (Kronecker)

If $f : \mathbb{T} \to \mathbb{C}$, then $đf$ is finite rank if and only if f is a rational function.

Bounded differentials

Let $f : \mathbb{T} \to \mathbb{C}$. The weakest condition that we can place on $\bar{\partial} f$ is that $\bar{\partial} f$ is bounded.

Definition

Let $f : \mathbb{T} \to \mathbb{C}$ be measureable. We say that f is of *bounded mean oscillation* if for an arc $I \subseteq \mathbb{T}$, define

$$f_I = \frac{1}{\mathbf{m}(I)} \int_I f \, d\mathbf{m}$$

and

$$\sup_I \frac{1}{\mathbf{m}(I)} \int_I |f - f_I| \, d\mathbf{m} < \infty$$

where the supremum runs over all arcs I. The set of functions with bounded mean oscillation is denoted $\mathrm{BMO}(\mathbb{T})$.

Bounded differentials

Theorem (Nehari)

Let $f : \mathbb{T} \to \mathbb{C}$. Then $\bar{d}f$ is bounded if and only if $f \in \mathrm{BMO}(\mathbb{T})$.

Compact differentials

We define the space $\mathrm{VMO}(\mathbb{T})$:

Definition

We say that $f \in \mathrm{VMO}(\mathbb{T})$ if $f \in \mathrm{BMO}(\mathbb{T})$ and

$$\lim_{\mathbf{m}(I)\to 0} \frac{1}{\mathbf{m}(I)} \int_I |f - f_I| \, d\mathbf{m} = 0.$$

Theorem

If $f : \mathbb{T} \to \mathbb{C}$, then $\bar{d}f$ is compact if and only if $f \in \mathrm{VMO}(\mathbb{T})$.

Can we do better?

We seek a more precise characterisation of the relationship between the smoothness of f and the size of $đf$. To this end, we define the *Besov classes* B^s_{pq}.

Definition of B^s_{pq}

We define a sequence of polynomials $\{W_n\}_{n\in\mathbb{Z}}$ on $\mathbb{T}$ as follows.

1. $W_0 = z^{-1} + 1 + z$.
2. For $n > 0$ and $k > 0$,

$$\widehat{W}_n(k) = \begin{cases} 1, & \text{if } k = 2^n \\ \text{a linear function on } [2^{n-1}, 2^n] \text{ and } [2^n, 2^{n+1}] \\ 0, & \text{otherwise.} \end{cases}$$

and $\widehat{W}_n(-k) = \widehat{W}_n(k)$, and $\widehat{W}_n(0) = 0$.

Definition of B^s_{pq}

Definition

For an integrable function φ on $\mathbb{T}$, we say that $f \in B^s_{pq}(\mathbb{T})$ if

$$\sum_{n\in\mathbb{Z}} 2^{|n|sp}\|W_n * \varphi\|_p^q < \infty.$$

Quantised differentials of Schatten-Von Neumann class

Theorem (Peller)

Let f be a measurable function on $\mathbb{T}$ and $p > 0$. Then $\text{đ}f \in \mathcal{L}^p$ if and only if $f \in B^{1/p}_{pp}(\mathbb{T})$.

Can we do better?

We claim that there exists an interpolation functor F such that

$$F(\mathcal{L}^1, K) = \mathcal{L}^{p,q}.$$

Hence, if $\varphi \in F(B^1_{11}, \mathrm{VMO}(\mathbb{T}))$, then

$$đ\varphi \in \mathcal{L}^{p,q}.$$

Summary

Let us see how to prove these results. In summary we have:

f	$\bar{d}f$
Rational	Finite rank
$B^{1/p}_{pp}$	$\mathcal{L}^p$
$K(B^1_{11}, VMO)_{(1-p)^{-1},q}$	$\mathcal{L}^{p,q}$
VMO	Compact
BMO	Bounded

You should now be convinced that the statement "If f is smoother than g, then $\bar{d}f$ is smaller than $\bar{d}g$" has at least some rigorous justification for functions on $\mathbb{T}$.

Quantised differentials and classical derivatives

Recall that 17th century differentials were supposed to have the property that

$$df = f' dx$$

provided that sufficiently small infinitesimals are ignored.
We can now interpret this in a rigorous sense. First, we replace infinitesimals with quantised differentials. To "ignore sufficiently small infinitesimals" means that we work modulo some ideal of compact operators.

Quantised differentials and classical derivatives

Let $\mathcal{L}_0^{p,\infty}$ be the closure in $\mathcal{L}^{p,\infty}$ of the ideal of finite rank operators.

Theorem (Connes)

Let $f \in C(\mathbb{T})$ be such that $đf \in \mathcal{L}^{p,\infty}$ and $\varphi \in C^\infty(\sigma(f))$, and $p \in [1,\infty)$. Then

$$đ\varphi(f) \equiv \varphi'(f)đf \mod \mathcal{L}_0^{p,\infty}.$$

So far we have talked about quantised differentials exclusively on $\mathbb{T}$. However, many of the same results hold for $\mathbb{R}$, since we have the *Cayley transform*.

Let $\zeta \in \mathbb{T} \setminus \{1\}$. Let

$$\omega(\zeta) = i\frac{1+\zeta}{1-\zeta}.$$

Let $g \in L^0(\mathbb{R})$. Define for $\zeta \in \mathbb{T}$,

$$\mathcal{C}(g)(\zeta) = \frac{\sqrt{\pi}}{2i}\frac{(g \circ \omega)(\zeta)}{1-\zeta}$$

This allows results on $\mathbb{T}$ to be transferred to results on $\mathbb{R}$.

Proving these statements

To prove all the preceding results, we need to introduce the theory of Hankel operators.

Hankel Operators

A *Hankel matrix* is an infinite matrix $\{a_{j+k}\}_{j,k\geq 0}$,

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_1 & a_2 & a_3 & a_4 & \cdots \\ a_2 & a_3 & a_4 & a_5 & \cdots \\ a_3 & a_4 & a_5 & a_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

There is a direct link between Hankel operators and quantised differentials on the circle.

Hankel operators and quantised differentials

Theorem

Let $\varphi \in L^1(\mathbb{T})$, and M_φ is the densely defined pointwise multiplication operator on $L^2(\mathbb{T})$. Let $\mathbf{P}_-$ be the projection,

$$\mathbf{P}_-\left(\sum_{n\in\mathbb{Z}} a_n z^n\right) = \sum_{n<0} a_n z^n.$$

Then $H_\varphi := \mathbf{P}_- M_\varphi|_{H^2(\mathbb{T})}$ is a Hankel matrix when represented in the bases $\{z^n\}_{n=0}^\infty$ and $\{z^n\}_{n<0}$ with (j,k)th entry $\widehat{\varphi}(-j-k)$ for $j \geq 0$ and $k \geq 1$.

The link between Hankel operators and quantised differentials is provided by the following result:

Theorem

Let $\varphi \in L^1(\mathbb{T})$*. Let* $\varphi_- = \mathbf{P}_-\varphi$ *and* $\varphi_+ = (1-\mathbf{P}_-)\varphi$*. Then,*

$$\bar{d}\varphi = 2((H_{\varphi_+})^* - H_{\varphi_-})$$

Symbol of a Hankel Operator

Definition

Let $\varphi \in L^1(\mathbb{T})$. Let Γ_φ be the infinite Hankel matrix with (j,k)th entry $\widehat{\varphi}(j+k)$.

From now on, it suffices to study the properties of Γ_φ.

Bounded Hankel Operators

Theorem (Nehari)

The Hankel operator Γ_φ defines a bounded linear operator on $\ell^2(\mathbb{N})$ if and only if $\varphi \in \mathrm{BMO}(\mathbb{T})$.

A result of Fefferman

To begin proving the Nehari theorem, we need the following result of C. Fefferman.

Theorem

The set $\mathrm{BMO}(\mathbb{T})$ *is exactly*

$$\mathrm{BMO}(\mathbb{T}) = \{f + \mathbf{P}_{-} g \ :\ f, g \in L^{\infty}(\mathbb{T})\}.$$

The Boundedness of Hankel operators

Theorem (Nehari)

Let $T = \{\alpha_{j+k}\}_{j,k\geq 0}$ *be an infinite Hankel matrix defined by the sequence* α. T *defines a bounded linear operator on* $\ell^2(\mathbb{N})$ *if and only if there exists a function* $\psi \in L^\infty(\mathbb{T})$ *such that* $\alpha_n = \widehat{\psi}(n)$ *for all* $n \geq 0$, *and*

$$\|T\| \leq \|\psi\|_\infty.$$

Proof.

Suppose that there exists such a ψ. Let a and b be finitely supported sequences, and compute the inner product:

$$(Ta, b) = \sum_{j,k\geq 0} \alpha_{j+k} a_j \overline{b_k}.$$

Proof (Cont.)

Let

$$f = \sum_{n=0}^{\infty} a_n z^n, \; g = \sum_{n=0}^{\infty} \overline{b_n} z^n$$

and let $q = fg$. Hence,

$$\begin{aligned}
(Ta, b) &= \sum_{j,k\geq 0} a_j \overline{b_k} \widehat{\psi}(j+k) \\
&= \sum_{n\geq 0} \widehat{\psi}(n) \sum_{k=0}^{n} a_k \overline{b_{n-k}} \\
&= \sum_{n\geq 0} \widehat{\psi}(n) \widehat{q}(n) \\
&= \int_{\mathbb{T}} \psi(\zeta) q(\overline{\zeta}) \, d\mathbf{m}(\zeta).
\end{aligned}$$

Proof (Cont.)

Hence,

$$|(Ta, b)| \leq \|\psi\|_\infty \|q\|_1 \leq \|\psi\|_\infty \|f\|_2 \|g\|_2 = \|\psi\|_\infty \|a\|_2 \|b\|_2.$$

Hence T is bounded on $\ell^2(\mathbb{N})$.

Proof (Cont.)

Now we prove the converse. Suppose that T is bounded on $\ell^2(\mathbb{N})$. Define $\mathcal{L}$ on $\mathcal{P}_A(\mathbb{T})$

$$\mathcal{L}q = \sum_{n\geq 0} \alpha_n \widehat{q}(n).$$

Assume $\alpha \in \ell^1(\mathbb{N})$. In this case, $\mathcal{L}$ is continuous on $H^1(\mathbb{T})$. Let $q \in H^1(\mathbb{T})$ with $\|q\|_1 \leq 1$. Then $q = fg$ for $f, g \in H^2(\mathbb{T})$ with $\|f\|_2, \|g\|_2 \leq 1$. Hence,

$$\begin{aligned}\mathcal{L}q &= \sum_{m\geq 0} \alpha_m \widehat{q}(m) \\ &= \sum_{m\geq 0} \alpha_m \sum_{j=0}^{m} \widehat{f}(j)\widehat{g}(m-j).\end{aligned}$$

Proof (Cont.)

Hence, if we define $a_j = \widehat{f}(j)$ and $b_j = \overline{\widehat{b}(j)}$,

$$\begin{aligned}\mathcal{L}q &= \sum_{m\geq 0} \alpha_m \sum_{j=0}^{m} \widehat{f}(j)\widehat{g}(m-j) \\ &= (Ta, b).\end{aligned}$$

Thus,

$$|\mathcal{L}q| \leq \|T\| \|f\|_2 \|g\|_2 \leq \|T\|.$$

Proof (Cont.)

Now let α be an arbitrary sequence such that $\|T\|$ is bounded.
Let $0 < r < 1$. Define the sequence $\alpha^{(r)}$ by

$$\alpha_j^{(r)} = r^j \alpha_j$$

for $j \geq 0$.
Let $T^{(r)}$ be the Hankel matrix $\{\alpha_{j+k}^{(r)}\}_{j,k\geq 0}$.
Let D_r be the operator of multiplication by $\{r^j\}_{j\geq 0}$ on $\ell^2(\mathbb{N})$. Then we can compute that $T^{(r)} = D_r T D_r$. Hence $T^{(r)}$ is bounded, and $\|T^{(r)}\| \leq \|T\|$.

Proof (Cont.)

Let

$$\mathcal{L}_r q = \sum_{n\geq 0} \alpha_n^{(r)} \widehat{q}(n).$$

Since $\alpha^{(r)} \in \ell^2(\mathbb{N})$, we have already proved that $\|\mathcal{L}\|_{H^1\to\mathbb{C}} \leq \|T^{(r)}\| \leq \|T\|$.
Now the functionals $\{\mathcal{L}^{(r)}\}_{r\in(0,1)}$ are uniformly bounded and converge strongly to $\mathcal{L}$.
Hence, $\mathcal{L}$ is continuous.

Proof (Cont.)

We have shown that $\mathcal{L}$ is bounded on $H^1(\mathbb{T})$ with norm bounded by $\|T\|$.
Hence by the Hahn Banach theorem there is $\psi \in L^\infty$ such that
$\mathcal{L}q = (\psi, q)$, and $\|\psi\|_\infty \leq \|T\|$.
This completes the proof. $\square$

Reformulation in terms of H_φ

Recall that we defined

$$H_\varphi = \mathbf{P}_- M_\varphi|_{H^2(\mathbb{T})}.$$

For $\varphi \in L^1(\mathbb{T})$. This has matrix representation $\{\widehat{\varphi}(-j-k)\}_{j\geq 1, k\geq 0}$. Hence, H_φ is bounded if and only if there exists $\psi \in L^\infty(\mathbb{T})$ such that

$$\widehat{\varphi}(-j) = \widehat{\psi}(j).$$

And hence $\mathbf{P}_-\varphi \in \mathrm{BMO}(\mathbb{T})$. Conversely, if $\mathbf{P}_-\varphi \in \mathrm{BMO}(\mathbb{T})$, then such a ψ exists.

Back to quantised derivatives

Recall that

$$\bar{d}\varphi = 2((H_{\varphi_-})^* - H_{\varphi_+}).$$

Since the two Hankel operators act on orthogonal complements, we see that $\bar{d}\varphi$ is bounded if and only if H_{φ_+} and H_{φ_-} are bounded. Hence

Theorem

$\bar{d}\varphi$ is bounded if and only is $\varphi \in \mathrm{BMO}(\mathbb{T})$.

A classical definition of continuity

It follows from Nehari's theorem that,

$$\|\bar{d}\varphi\| \leq 2\|\varphi\|_\infty. \tag{1}$$

It is easy to prove that if φ is a polynomial, then $\bar{d}\varphi$ is finite rank. Hence, if $\varphi \in C(\mathbb{T})$, then $\bar{d}\varphi$ is compact.

The End

In summary:

1. Quantum mechanics causes us to look for generalisations of differential geometry in non-commutative algebra.
2. If we define non-commutative spaces as spectral triples, a new object appears: the quantised differential.
3. The quantised differential has meaning in analysis on $\mathbb{T}$ and $\mathbb{R}$.
4. The quantised differential has many properties that we would anticipate for "infinitesimal differences".

Thank you for listening!