A Gentle Introduction to Non-commutative Harmonic Analysis

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Introduction

Question for this lecture:

How do Fourier Series work?

Another Question:

What is the non-commutative torus?

Fourier Analysis In Kindergarten

- In second year we learn the following situation:

Definition

If $f : \mathbb{R} \to \mathbb{C}$ is a function that is periodic with period P, then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{2\pi nx}{P}\right) + b_n \sin\left(\frac{2\pi nx}{P}\right) \right)$$

Definition

$$a_n = \frac{2}{P} \int_0^P f(x) \cos\left(\frac{2\pi nx}{P}\right) \, dx$$

$$b_n = \frac{2}{P} \int_0^P f(x) \sin\left(\frac{2\pi nx}{P}\right) \, dx$$

Fourier Analysis In Kindergarten

We normally describe the coefficients $\{a_n, b_n\}_{n=0}^{\infty}$ as the "Fourier Transform" of f.

There are many questions left unanswered...

- Does this really work for *any* function f?
- In what sense is the Fourier series meant to converge?

A better notation

The notation that was used in second year,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{2\pi nx}{P}\right) + b_n \sin\left(\frac{2\pi nx}{P}\right)\right),$$

is rather silly. This;

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{2\pi inx}{P}\right)$$

is much better.

An even better notation still

Let

$$\mathbb{T} = \{\zeta \in \mathbb{C} \; : \; |\zeta| = 1\}.$$

We can identify periodic functions on $\mathbb{R}$ with functions on $\mathbb{T}$. Let

$$z : \mathbb{T} \to \mathbb{T}$$

be the identity function. If we are identifying P-periodic functions on $\mathbb{R}$ with functions on $\mathbb{T}$, then we identify

$$\exp\left(\frac{2\pi inx}{P}\right) = z^n.$$

Integrating functions on $\mathbb{T}$

$\mathbb{T}$ has a measure: normalised arc length. We call this measure $\mathbf{m}$, so that if $f : \mathbb{T} \to \mathbb{C}$,

$$\int_{\mathbb{T}} f \; d\mathbf{m} = \int_0^1 f\left(\exp(2\pi i t)\right) \; dt.$$

An even better notation still

So with our new notation, Fourier series look like this:

Definition

If $f : \mathbb{T} \to \mathbb{C}$, then the fourier series of f is

$$\sum_{n \in \mathbb{Z}} c_n z^n.$$

where

$$c_n = \int_{\mathbb{T}} z^{-n} f \, d\mathbf{m}.$$

Fourier Series for Grown-ups

For $n \in \mathbb{Z}$, and $f \in L^1(\mathbb{T}, \mathbf{m})$, let

$$\widehat{f}(n) := \int_{\mathbb{T}} z^{-n} f \, d\mathbf{m}.$$

So we have a "correspondence",

$$f \sim \sum_{n \in \mathbb{Z}} \widehat{f}(n) z^n.$$

What does "$\sim$" mean?

What does the Fourier Series mean?

At the moment when we write

$$f \sim \sum_{n\in\mathbb{Z}} \widehat{f}(n)z^n$$

the right hand side is just a formal power series. What do we have to do to make the "$\sim$" into an "$=$"?

How shall the sum on the right hand side be interpreted? What kind of object is this?

Divergent Series

Normally in mathematics, when we write

$$\sum_{n=0}^{\infty} a_n,$$

we mean

$$\lim_{N\to\infty} \sum_{n=0}^{N} a_n.$$

This means that series like

$$\sum_{n=0}^{\infty} (-1)^n$$

$$\sum_{n=0}^{\infty} n$$

do not make sense. This is "classical summation".

Divergent Series

It is often desirable to assign a "sensible" value to a series that is divergent. This is called *regularisation*.
What this means is redefining what we mean by

$$\sum_{n=0}^{\infty} a_n.$$

Abel Summation

Abel Summation is a method of regularisation that is motivated by *Abel's theorem*.

Theorem

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *be a power series that converges in* $(-1, 1)$. *Suppose that*

$$\sum_{n=0}^{\infty} a_n$$

exists. Then

$$\sum_{n=0}^{\infty} a_n = \lim_{x \to 1^-} \sum_{n=0}^{\infty} a_n x^n.$$

Abel summation

Definition

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence, the *Abel sum* of $\{a_n\}_{n=0}^{\infty}$ is

$$(A)\sum_{n=0}^{\infty} a_n := \lim_{r\to 1^-} \sum_{n=0}^{\infty} a_n r^n.$$

Abel summation

For example,

$$
\begin{aligned}
(A)\sum_{n=0}^{\infty}(-1)^n &= \lim_{r\to 1^-}\sum_{n=0}^{\infty}(-r)^n \\
&= \lim_{r\to 1^-}\frac{1}{1+r} \\
&= \frac{1}{2}.
\end{aligned}
$$

Abel's theorem guarantees that if a series is summable in the classical sense, then it is Abel summable and the Abel and classical sums agree.

Aside

Using a completely different method of regularisation, we can give rigorous justification to the (infamous) expression,

$$1 + 2 + 3 + 4 + \cdots = -\frac{1}{12}.$$

Back to Fourier Series

Let $f \in L^1(\mathbb{T})$, define

$$A_r f := \sum_{n\in\mathbb{Z}} \widehat{f}(n) r^n z^n.$$

This exists for any r.

The question of whether a Fourier series is Abel summable is equivalent to considering the limit $\lim_{r\to 1^-} A_r f$.

Back to Fourier Series

The following theorem is remarkably easy to prove:

Theorem

Suppose that $f \in L^p(\mathbb{T}, \mathbf{m})$, with $p \in [1, \infty)$. Then

$$f = \lim_{r \to 1^-} A_r f =_{(A)} \sum_{n \in \mathbb{Z}} \widehat{f}(n) z^n$$

where the limit is in the L^p sense.

The final word on Fourier Series?

So at last we have rigorous meaning for the expression,

$$f = \sum_{n \in \mathbb{Z}} \widehat{f}(n) z^n.$$

We simply need to interpret the right hand side as an Abel sum, converging in the L^p sense, if $f \in L^p(\mathbb{T}, \mathbf{m})$ and $p \in [1, \infty)$.

Can we do better?

Abel summation is nice, but what really interests us is classical summation. Carleson and Hunt proved the following,

Theorem

Suppose that $f \in L^p(\mathbb{T}, \mathbf{m})$, for $p \in (1, \infty)$, then

$$f = \sum_{n \in \mathbb{Z}} \widehat{f}(n) z^n$$

where the sum is a classical sum, converging in the L^p sense.

What about the $p = 1$ case?

The theorem spectacularly fails to be true when $p = 1$. The following is a result of Kolmogorov,

Theorem

There is a function $k \in L^1(\mathbb{T}, \mathbf{m})$ such that

$$\zeta \mapsto \sum_{n \in \mathbb{Z}} \widehat{k}(n) \zeta^n$$

diverges for every $\zeta \in \mathbb{T}$.

What about higher dimensions?

Let $f \in L^1(\mathbb{T}^2, \mathbf{m} \times \mathbf{m})$. Let $z, w : \mathbb{T}^2 \to \mathbb{T}$ be the first and second coordinate functions.
Then we can write,

$$f = \lim_{r,s \to 1^-} \sum_{(n,m) \in \mathbb{Z}^2} \widehat{f}(n,m) r^n s^m z^n w^m.$$

where

$$\widehat{f}(n,m) = \int_{\mathbb{T}^2} z^{-n} w^{-m} f \; d(\mathbf{m} \times \mathbf{m}).$$

What about higher dimensions?

In other words, most functions on $\mathbb{T}^2$ can be written as a doubly-indexed power series,

$$f = \sum_{(n,m)\in\mathbb{Z}^2} c_{n,m} z^n w^m$$

for some coefficients $c_{n,m}$, and the series converges in "some sense".
In particular, the subspace

$$\left\{ \sum_{(n,m)\in\mathbb{Z}^2} c_{n,m} z^n w^m \;:\; \sup_{n,m}(1+|n|+|m|)^k |c_{n,m}| < \infty \text{ for all } k \geq 0 \right\}.$$

is exactly $C^\infty(\mathbb{T}^2)$.

Non-commutative Geometry

Question: What is non-commutative geometry?

Answer:

The study of non-commutative algebras which are somehow similar to algebras of functions on geometric spaces, using the methods and language of geometry.

Introducing the non-commutative torus

Let θ be an irrational number.

Definition

The non-commutatative torus $\mathcal{A}_\theta$ is a $\mathbb{C}$-$*$-algebra generated by two elements U and V satisfying the commutation relation

$$UV = e^{2\pi i\theta} VU.$$

and $U^{-1} = U^*, V^{-1} = V^*$ and every element of $\mathcal{A}_\theta$ can be written as

$$\sum_{(n,m)\in\mathbb{Z}^2} c_{n,m} U^n V^m.$$

where $c_{n,m}$ is a sequence satisfying $\sup_{n,m}(1+|n|+|m|)^k|c_{n,m}| < \infty$.

The Upshot

The algebra $\mathcal{A}(\mathbb{T}^2_\theta)$ is a lot like an algebra of functions on $\mathbb{T}^2$, and many techniques of harmonic analysis can be used on $\mathcal{A}(\mathbb{T}^2_\theta)$. So this justifies the term "non-commutative harmonic analysis".

The end!
(p.s. Come to my seminar next month to learn a lot more about non-commutative geometry!)