

# Quantised Calculus and Hankel Operators

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# Introduction

This talk will be divided into two parts:

## Soft Analysis:

The motivation for quantised calculus from non-commutative geometry.

## Hard Analysis:

Operator ideal membership of quantised derivatives.

## What is Non-commutative geometry?

The study of noncommutative algebras which are similar to algebras of functions on geometric spaces, using the methods and language of geometry.

# Why Non-commutative Geometry?

Non-commutative geometry gives us new insights on non-commutative algebra, but historically it was motivated by quantum mechanics.

# A very brief description of quantum mechanics

## Definition

A *quantum mechanical system* is a pair  $(\mathcal{A}, \mathcal{H})$  where  $\mathcal{H}$  is a complex separable Hilbert space and  $\mathcal{A}$  is a  $*$ -algebra of (potentially unbounded, densely defined) operators on  $\mathcal{H}$ . The self adjoint elements of  $\mathcal{A}$  are called *observables*. The non-zero elements of  $\mathcal{H}$  are called *states*.

- The states correspond to potential configurations of a physical system. States which are non-zero scalar multiples of each other are considered physically indistinguishable.
- The observables correspond to physical quantities that can be measured. The range of potential measurements for an observable  $A \in \mathcal{A}$  is  $\sigma(A)$ .

# Reminder: The spectral theorem

## Theorem

*Let  $(\mathcal{A}, \mathcal{H})$  be a quantum mechanical system. If  $A \in \mathcal{A}$  is an observable, then there exists a projection valued measure  $E_A$  on  $\sigma(A)$  such that*

$$A = \int_{\sigma(A)} \lambda \, dE_A(\lambda).$$

# Measurement in quantum mechanics

Let  $(\mathcal{A}, \mathcal{H})$  be a quantum mechanical system, currently in a state  $\psi \in \mathcal{H}$ . Let  $A \in \mathcal{A}$  be an observable. The probability that the measured value of the physical quantity corresponding to  $A$  lies in a borel set  $\Delta \subseteq \sigma(A)$  is

$$P(\Delta; \psi) := \frac{(\psi, E_A(\Delta)\psi)}{\|\psi\|^2} = \frac{\|E_A(\Delta)\psi\|^2}{\|\psi\|^2}.$$

If the measured value lies in  $\Delta$ , then the state of the system changes to

$$E_A(\Delta)\psi.$$

# Measurement in quantum mechanics

There are two surprising features of measurement in quantum mechanics:

- 1 The act of observation changes the state of the system.
- 2 The order of observation is important, since if  $A$  and  $B$  are observables, and  $\Delta \times \Sigma \subseteq \sigma(A) \times \sigma(B)$ , then in general it is not true that

$$E_A(\Delta)E_B(\Sigma)\psi = E_B(\Sigma)E_A(\Delta)\psi.$$



# The beginnings of noncommutative geometry

A noncommutative space is a “space” with “coordinates” such that the order of measurement of coordinates changes the measured values. Inspired by quantum mechanics, we define a noncommutative space as a pair  $(\mathcal{A}, \mathcal{H})$  exactly like a quantum mechanical system.

# Relation to Ordinary Geometry

The usual setting for geometry is a Riemannian manifold  $M$ . Let  $\mathcal{A} = C^\infty(M)$  and  $\mathcal{H} = L^2(M)$ .

However, we need more information to specify the geometry of the manifold. One way to do this is by giving a *Dirac operator*.

Associated to a Riemannian manifold  $(M, g)$  is a *clifford bundle*.

$$\text{Cliff}(M, g) = \frac{T(TM)}{\langle xy + yx = -2g(x, y) \rangle}.$$

A vector bundle  $V$  on  $M$  is called a *clifford module* if there is a left multiplication action  $m : \text{Cliff}(M, g) \otimes V \rightarrow V$ .

## Definition

Suppose that  $(M, g)$  is a Riemannian manifold with a Clifford module  $V$ , and a  $V$ -valued connection  $\nabla : V \rightarrow T^*M \otimes V$ . Then the *Dirac Operator* on  $V$  is given by the composition of maps

$$V \xrightarrow{\nabla} T^*M \otimes V \xrightarrow{\sharp \otimes I} TM \otimes V \xrightarrow{m} V$$

where  $\sharp$  is the canonical isomorphism from  $T^*M$  to  $TM$  given by  $g$ .

## Theorem

*Let  $(M, g)$  be a Riemannian manifold with Clifford module  $V$  and Dirac operator  $\mathcal{D}$ . Then we have for every  $a \in C^\infty(M)$ ,*

$$da = [\mathcal{D}, a]$$

*as an equality of operators on the sections of  $V$ .*

To specify the geometry of non-commutative space  $(\mathcal{A}, \mathcal{H})$ , we need additional data to tell us how observables relate to each other. An elegant way of doing this is by giving a Dirac operator.

## Definition

A *spectral triple* is a triple  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  where  $\mathcal{H}$  is a complex separable Hilbert space and  $\mathcal{A}$  is a  $*$ -algebra of bounded operators on  $\mathcal{H}$ .  $\mathcal{D}$  is a densely defined unbounded self adjoint operator on  $\mathcal{H}$  such that:

- 1  $[\mathcal{D}, a]$  is bounded for all  $a \in \mathcal{A}$ .
- 2  $(\lambda - \mathcal{D})^{-1}$  is compact for all  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ .

The prototypical example is a Riemannian manifold  $M$  with a Clifford module, where  $\mathcal{A} = C^\infty(M)$ ,  $\mathcal{H} = L^2(V)$  and  $\mathcal{D}$  is the Dirac operator of the module.

# Properties of Spectral Triples

## Definition

Let  $k \geq 0$  be an integer. We say that a spectral triple  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  is  $QC^k$  if for any  $a \in \mathcal{A}$ ,  $a$  and  $[\mathcal{D}, a]$  are in the domain of  $\delta^k(a)$ , where  $\delta(a) := [|\mathcal{D}|, a]$ .

## Definition

Let  $p > 0$ . A spectral triple  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  is called  $(p, \infty)$  summable if  $(\lambda - \mathcal{D})^{-1} \in \mathcal{L}^{p, \infty}$  for  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ .  $p$  is analogous to the dimension of the space.

# Differentials in abstract algebra

## Definition

Let  $A$  be a  $\mathbb{C}$ -algebra, and  $M$  is an  $A$ -bimodule. A  $\mathbb{C}$ -linear map  $\theta : A \rightarrow M$  is called a derivation if  $\theta(ab) = a\theta(b) + \theta(a)b$  for all  $a, b \in A$ .

## Example

Let  $N$  be a manifold. We can take  $A = C^\infty(N)$  and  $M = \Omega^1(N)$ , and  $\theta = d$ .



For a  $\mathbb{C}$ -algebra  $\mathcal{A}$ , there is a “largest”  $A$ -bimodule that is the image of a derivation on  $\mathcal{A}$ .

## Definition

Let  $A$  be a  $\mathbb{C}$ -algebra. There is a multiplication map  $\gamma : A \otimes A \rightarrow A$ . Define

$$\Omega^1(A) := \ker \gamma.$$

And  $d : A \rightarrow \Omega^1(A)$  is given by  $a \mapsto 1 \otimes a - a \otimes 1$ . This is the algebra of *Kähler differentials*.

The utility of Kähler differentials comes from the universal property:

## Theorem

*Let  $A$  be a  $\mathbb{C}$ -algebra, and  $M$  is an  $A$ -bimodule with a derivation  $\theta : A \rightarrow M$ . Then there exists a unique  $A$ -module homomorphism  $\tilde{\theta} : \Omega^1(A) \rightarrow M$  such that  $\theta = \tilde{\theta} \circ d$ .*

Given a spectral triple,  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ , and  $a \in \mathcal{A}$ , we define

$$\pi : \Omega^1(\mathcal{A}) \rightarrow \mathcal{B}(\mathcal{H})$$

by

$$da \mapsto [\mathcal{D}, a].$$

Define the two-sided ideal of  $\mathcal{B}(\mathcal{H})$ ,  $J$  as

$$J = \{t \in \Omega^1(\mathcal{A}) : \pi(t) = 0\}.$$

and  $J_0 = J + dJ$ . Then,

$$\Omega_{\mathcal{D}}^1(\mathcal{A}) := \frac{\pi(\Omega^1(\mathcal{A}))}{\pi(dJ_0)}.$$

17-19th century mathematicians made frequent use of *infinitesimals*. An infinitesimal was a quantity  $x$  such that for all  $\varepsilon > 0$ , we have

$$|x| < \varepsilon.$$

Of course, this implies that  $x = 0$ , so the use of non-zero infinitesimals was abolished.

# Infinitesimal Operators

An infinitesimal operator  $T \in \mathcal{B}(\mathcal{H})$  should be an operator  $T$  such that for any  $\varepsilon > 0$ , we have

$$\|T\| < \varepsilon.$$

Again, this definition is useless as it implies that  $T = 0$ .

# Compact Operators as Infinitesimals

We shall say that an operator  $T \in \mathcal{B}(\mathcal{H})$  is *infinitesimal* if for any  $\varepsilon > 0$  there exists a finite dimensional subspace  $E$  such that

$$\|T|_{E^\perp}\| < \varepsilon.$$

This is equivalent to saying that  $T$  is compact.

# Infinitesimals in Non-commutative geometry

In a noncommutative geometry  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ , the compact elements of  $\mathcal{A}$  play a similar role to infinitesimals in classical real analysis.

We have the following dictionary:

Classical Analysis	Non-commutative Analysis
Function	Operator
One-form	Connes Differential
Range	Spectrum
Infinitesimal	Compact Operator

# Quantised Differentials

In noncommutative geometry, we have a new object that is not present in classical analysis called a quantised derivative or quantised differential.

## Definition

Let  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  be a spectral triple. Let  $F$  be the unique operator such that  $\mathcal{D} = F|\mathcal{D}|$ , that is  $F = \text{sgn}(\mathcal{D})$ . For  $a \in \mathcal{A}$ , define

$$\overline{d}a := [F, a].$$

This is the *quantised differential* of  $a$ .

$\overline{d}a$  is supposed to represent an infinitesimal variation of  $a$ . The motivation behind this definition is not clear, but as we work through examples it will become clear that this definition is the “correct” one.



# Expected properties of infinitesimals

In classical 17-19th century analysis, infinitesimals were supposed to have a number of properties:

- 1 If  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a function, there is a function  $df$  representing infinitesimal variation in  $f$ .  $f$  is continuous if and only if  $df$  is infinitesimal.
- 2 If  $f$  is smoother than  $g$ , then  $df$  is smaller than  $dg$ .
- 3 If  $x$  is a positive infinitesimal, then  $x^2$  is smaller than  $x$ .
- 4 If  $f$  is a differentiable function, then  $df = f'dx$ , provided that sufficiently small infinitesimals are ignored.

We shall see that the quantised differential satisfies all these properties, if they are interpreted correctly.

## Definition

Given  $T \in \mathcal{B}(\mathcal{H})$ , define the  $k$ th singular value of  $T$  to be

$$\mu_k(T) := \inf \{ \|T - A\| : \text{rank}(A) \leq k \}.$$

For a compact operator,  $\{\mu_k(T)\}_{k=0}^{\infty}$  is a vanishing sequence of positive numbers. We shall describe the *size* of an infinitesimal  $T$  as the *rate of decay* of  $\{\mu_k(T)\}_{k=0}^{\infty}$ .

## Theorem

*Let  $T$  and  $S$  be compact operators in  $\mathcal{B}(\mathcal{H})$ . Then for every  $k \geq 0$ ,*

$$\mu_k(TS) \leq \mu_k(T)\mu_k(S).$$

Hence, if  $T$  is an infinitesimal, then  $T^2$  is a smaller infinitesimal.

# Sizes of Infinitesimals

We can quantify the sizes an infinitesimal  $T$  by placing conditions on the rate of decay of  $\{\mu_k(T)\}_{k=0}^\infty$ .

- The smallest infinitesimals have  $\{\mu_k(T)\}_{k=0}^\infty$  of finite support. Then  $T$  is of finite rank.
- We say that  $T \in \mathcal{L}^p$  if  $\{\mu_k(T)\}_{k=0}^\infty \in \ell^p$ .
- We say that  $T \in \mathcal{L}^{p,\infty}$  if  $\mu_k(T) = \mathcal{O}(k^{-1/p})$ .
- We say that  $T \in \mathcal{L}^{p,q}$  if  $\{k^{1/p-1/q}\mu_k(T)\}_{k=0}^\infty \in \ell^q$ .
- We say that  $T \in \mathcal{M}^{1,\infty}$  if  $\{\frac{1}{\log(k+1)} \sum_{n=0}^k \mu_n(T)\}_{k=0}^\infty \in \ell^\infty$ .

# An initial result about quantised differentials

## Lemma

*Suppose that  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  is a  $QC^1$ ,  $(p, \infty)$ -summable spectral triple, where we assume that  $\mathcal{D}$  is invertible. Then for all  $a \in \mathcal{A}$ ,  $\overleftarrow{d}a \in \mathcal{L}^{p, \infty}$ .*

## Proof.

It can be proved that  $|\mathcal{D}|^{-1} \in \mathcal{L}^{p, \infty}$ . Now we write,

$$\begin{aligned} [\mathcal{D}, a] &= [F|\mathcal{D}|, a] \\ &= F[|\mathcal{D}|, a] + [F, a]|\mathcal{D}|. \end{aligned}$$

Hence,

$$\begin{aligned} \overleftarrow{d}a &= [\mathcal{D}, a]|\mathcal{D}|^{-1} - F[|\mathcal{D}|, a]|\mathcal{D}|^{-1} \\ &= (da - F\delta(a))|\mathcal{D}|^{-1}. \end{aligned}$$

Therefore  $\overleftarrow{d}a \in \mathcal{L}^{p, \infty}$ . □

# Smoothness and rate of decay

We now enter the second phase of the talk “Hard Analysis”.

Recall that in classical analysis, we had the statement that if  $f$  is smoother than  $g$ , then  $df$  is smaller than  $dg$ .

The main question of this talk:

In what sense is it true that if  $f$  is smoother than  $g$ , then  $\vec{\partial}f$  is smaller than  $\vec{\partial}g$ ?

We shall restrict attention to functions on the circle  $\mathbb{T}$  and the line  $\mathbb{R}$ .

# Revision of Classical Fourier Analysis and Notation

We define  $\mathbb{T} = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$ . Let  $z : \mathbb{T} \rightarrow \mathbb{T}$  be the identity function. Denote the normalised Haar (or arc length) measure on  $\mathbb{T}$  by  $\mathbf{m}$ . For  $f \in L^1(\mathbb{T}, \mathbf{m})$ , define for  $n \in \mathbb{Z}$ ,

$$\widehat{f}(n) := \int_{\mathbb{T}} z^{-n} f \, d\mathbf{m}.$$

Recall that any  $f \in L^2(\mathbb{T}, \mathbf{m})$  can be written as

$$f = \sum_{n \in \mathbb{Z}} \widehat{f}(n) z^n.$$

The sum converges in the  $L^2$  sense. This effects an isometric isomorphism between  $L^2(\mathbb{T})$  and  $\ell^2(\mathbb{Z})$ .

# Revision of Classical Fourier Analysis and Notation

The closed linear span of  $\{z^n\}_{n=0}^\infty$  in  $L^p$  is denoted  $H^2(\mathbb{T})$ , and the orthogonal complement is denoted  $H_-^2(\mathbb{T})$ .

We define the space of polynomials  $\mathcal{P}(\mathbb{T})$  to be the finite linear span of  $\{z^n\}_{n \in \mathbb{Z}}$ .  $\mathcal{P}_A(\mathbb{T}) = \text{span}\{z^n\}_{n \geq 0}$ .



The Dirac operator on the circle is  $\mathcal{D} = \frac{1}{i} \frac{d}{d\theta}$ , i.e.,  $\mathcal{D}(z^n) = nz^n$ . So we may describe  $F = \text{sgn } \mathcal{D}$  as the *Hilbert transform*,

$$F \left( \sum_{n \in \mathbb{Z}} a_n z^n \right) = \sum_{n \in \mathbb{Z}} \text{sgn}(n) a_n z^n.$$

# Differentials on $\mathbb{T}$

Hence for  $f \in L^2(\mathbb{T})$ ,

$$Ff = \varphi * f$$

where

$$\varphi = \sum_{n \in \mathbb{Z}} \operatorname{sgn}(n) z^n = \frac{1}{1-z} - \frac{z^{-1}}{1-z^{-1}} = \frac{2}{1-z}.$$

Thus,

$$\begin{aligned} (\bar{\partial} f)g &= ([F, f]g)(t) \\ &= 2 \lim_{\varepsilon \rightarrow 0} \int_{|\tau-t| > \varepsilon} \frac{f(t) - f(\tau)}{t - \tau} g(\tau) \, d\mathbf{m}(\tau). \end{aligned}$$

Let  $f : \mathbb{T} \rightarrow \mathbb{C}$ . The strictest condition we can put on the smoothness of  $f$  is that  $f$  is a rational function. The strictest condition we can put on the size of  $\vec{\partial}f$  is that  $\vec{\partial}f$  is finite rank. These two conditions are equivalent.

## Theorem (Kronecker)

*If  $f : \mathbb{T} \rightarrow \mathbb{C}$ , then  $\vec{\partial}f$  is finite rank if and only if  $f$  is a rational function.*

# Bounded differentials

Let  $f : \mathbb{T} \rightarrow \mathbb{C}$ . The weakest condition that we can place on  $\overline{\partial}f$  is that  $\overline{\partial}f$  is bounded.

## Definition

Let  $f : \mathbb{T} \rightarrow \mathbb{C}$  be measurable. We say that  $f$  is of *bounded mean oscillation* if for an arc  $I \subseteq \mathbb{T}$ , define

$$f_I = \frac{1}{\mathbf{m}(I)} \int_I f \, d\mathbf{m}$$

and

$$\sup_I \frac{1}{\mathbf{m}(I)} \int_I |f - f_I| \, d\mathbf{m} < \infty$$

where the supremum runs over all arcs  $I$ . The set of functions with bounded mean oscillation is denoted  $\text{BMO}(\mathbb{T})$ .

## Theorem (Nehari)

*Let  $f : \mathbb{T} \rightarrow \mathbb{C}$ . Then  $\bar{\partial}f$  is bounded if and only if  $f \in \text{BMO}(\mathbb{T})$ .*

# Compact differentials

We define the space  $VMO(\mathbb{T})$ ,

## Definition

We say that  $f \in VMO(\mathbb{T})$  if  $f \in BMO(\mathbb{T})$  and

$$\lim_{m(I) \rightarrow 0} \frac{1}{m(I)} \int_I |f - f_I| \, d\mathbf{m} = 0.$$

## Theorem

*If  $f : \mathbb{T} \rightarrow \mathbb{C}$ , then  $\vec{d}f$  is compact if and only if  $f \in VMO(\mathbb{T})$ .*

# Can we do better?

We seek a more precise characterisation of the relationship between the smoothness of  $f$  and the size of  $\vec{\sigma}f$ . To this end, we define the *Besov classes*  $B_{pq}^s$ .

# Definition of $B_{pq}^s$

We define a sequence of polynomials  $\{W_n\}_{n \in \mathbb{Z}}$  on  $\mathbb{T}$  as follows.

- ①  $W_0 = z^{-1} + 1 + z$ .
- ② For  $n > 0$  and  $k > 0$ ,

$$\widehat{W}_n(k) = \begin{cases} 1, & \text{if } k = 2^n \\ \text{a linear function on } [2^{n-1}, 2^n] \text{ and } [2^n, 2^{n+1}] \\ 0, & \text{otherwise.} \end{cases}$$

and  $\widehat{W}_n(-k) = \widehat{W}_n(k)$ , and  $\widehat{W}_n(0) = 0$ .



# Definition of $B_{pq}^s$

## Definition

For an integrable function  $\varphi$  on  $\mathbb{T}$ , we say that  $f \in B_{pq}^s(\mathbb{T})$  if

$$\sum_{n \in \mathbb{Z}} 2^{|n|sp} \|W_n * \varphi\|_p^q < \infty.$$

## Theorem (Peller)

*Let  $f$  be a measurable function on  $\mathbb{T}$  and  $p > 0$ . Then  $\overline{\partial}f \in \mathcal{L}^p$  if and only if  $f \in B_{pp}^{1/p}(\mathbb{T})$ .*

# Can we do better?

Similar necessary and sufficient conditions can be found on  $f$  such that  $\overline{\partial}f \in \mathcal{L}^{p,q}$ .

Let us see how to prove these results.

# Hankel Operators

A *Hankel matrix* is an infinite matrix  $\{a_{j+k}\}_{j,k \geq 0}$ ,

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_1 & a_2 & a_3 & a_4 & \cdots \\ a_2 & a_3 & a_4 & a_5 & \cdots \\ a_3 & a_4 & a_5 & a_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

There is a direct link between Hankel operators and quantised differentials on the circle.

## Theorem

Let  $\varphi \in L^1(\mathbb{T})$ , and  $M_\varphi$  is the densely defined pointwise multiplication operator on  $L^2(\mathbb{T})$ . Let  $\mathbf{P}_-$  be the projection,

$$\mathbf{P}_- \left( \sum_{n \in \mathbb{Z}} a_n z^n \right) = \sum_{n < 0} a_n z^n.$$

Then  $H_\varphi := \mathbf{P}_- M_\varphi|_{H^2(\mathbb{T})}$  is a Hankel matrix when represented in the bases  $\{z^n\}_{n=0}^\infty$  and  $\{z^n\}_{n < 0}$  with  $(j, k)$ th entry  $\widehat{\varphi}(-j - k)$  for  $j \geq 0$  and  $k \geq 1$ .

# Hankel operators and quantised differentials

The link between Hankel operators and quantised differentials is provided by the following result:

## Theorem

Let  $\varphi \in L^1(\mathbb{T})$ . Let  $\varphi_- = \mathbf{P}_-\varphi$  and  $\varphi_+ = (1 - \mathbf{P}_-)\varphi$ . Then,

$$\vec{\partial}\varphi = 2((H_{\varphi_+})^* - H_{\varphi_-})$$

# Symbol of a Hankel Operator

## Definition

Let  $\varphi \in L^1(\mathbb{T})$ . Let  $\Gamma_\varphi$  be the infinite Hankel matrix with  $(j, k)$ th entry  $\widehat{\varphi}(j+k)$ .

From now on, it suffices to study the properties of  $\Gamma_\varphi$ .

## Theorem (Nehari)

*The Hankel operator  $\Gamma_\varphi$  defines a bounded linear operator on  $\ell^2(\mathbb{N})$  if and only if  $\varphi \in \text{BMO}(\mathbb{T})$ .*



# A result of Fefferman

To begin proving the Nehari theorem, we need the following result of C. Fefferman.

## Theorem

*The set  $\text{BMO}(\mathbb{T})$  is exactly*

$$\text{BMO}(\mathbb{T}) = \{f + \mathbf{P}_-g : f, g \in L^\infty(\mathbb{T})\}.$$

# The Boundedness of Hankel operators

## Theorem (Nehari)

Let  $T = \{\alpha_{j+k}\}_{j,k \geq 0}$  be an infinite Hankel matrix defined by the sequence  $\alpha$ .  $T$  defines a bounded linear operator on  $\ell^2(\mathbb{N})$  if and only if there exists a function  $\psi \in L^\infty(\mathbb{T})$  such that  $\alpha_n = \widehat{\psi}(n)$  for all  $n \geq 0$ , and

$$\|T\| \leq \|\psi\|_\infty.$$

## Proof.

Suppose that there exists such a  $\psi$ . Let  $a$  and  $b$  be finitely supported sequences, and compute the inner product:

$$(Ta, b) = \sum_{j,k \geq 0} \alpha_{j+k} a_j \overline{b_k}.$$

## Proof (Cont.)

Let

$$f = \sum_{n=0}^{\infty} a_n z^n, \quad g = \sum_{n=0}^{\infty} \overline{b_n} z^n$$

and let  $q = fg$ . Hence,

$$\begin{aligned} (Ta, b) &= \sum_{j,k \geq 0} a_j \overline{b_k} \hat{\psi}(j+k) \\ &= \sum_{n \geq 0} \hat{\psi}(n) \sum_{k=0}^n a_k \overline{b_{n-k}} \\ &= \sum_{n \geq 0} \hat{\psi}(n) \hat{q}(n) \\ &= \int_{\mathbb{T}} \psi(\zeta) q(\overline{\zeta}) \, d\mathbf{m}(\zeta). \end{aligned}$$

## Proof (Cont.)

Hence,

$$|(Ta, b)| \leq \|\psi\|_{\infty} \|q\|_1 \leq \|\psi\|_{\infty} \|f\|_2 \|g\|_2 = \|\psi\|_{\infty} \|a\|_2 \|b\|_2.$$

Hence  $T$  is bounded on  $\ell^2(\mathbb{N})$ .

## Proof (Cont.)

Now we prove the converse. Suppose that  $T$  is bounded on  $\ell^2(\mathbb{N})$ . Define  $\mathcal{L}$  on  $\mathcal{P}_A(\mathbb{T})$

$$\mathcal{L}q = \sum_{n \geq 0} \alpha_n \widehat{q}(n).$$

Assume  $\alpha \in \ell^1(\mathbb{N})$ . In this case,  $\mathcal{L}$  is continuous on  $H^1(\mathbb{T})$ . Let  $q \in H^1(\mathbb{T})$  with  $\|q\|_1 \leq 1$ . Then  $q = fg$  for  $f, g \in H^2(\mathbb{T})$  with  $\|f\|_2, \|g\|_2 \leq 1$ . Hence,

$$\begin{aligned} \mathcal{L}q &= \sum_{m \geq 0} \alpha_m \widehat{q}(m) \\ &= \sum_{m \geq 0} \alpha_m \sum_{j=0}^m \widehat{f}(j) \widehat{g}(m-j). \end{aligned}$$

## Proof (Cont.)

Hence, if we define  $a_j = \widehat{f}(j)$  and  $b_j = \overline{\widehat{b}(j)}$ ,

$$\begin{aligned}\mathcal{L}q &= \sum_{m \geq 0} \alpha_m \sum_{j=0}^m \widehat{f}(j) \widehat{g}(m-j) \\ &= (Ta, b).\end{aligned}$$

Thus,

$$|\mathcal{L}q| \leq \|T\| \|f\|_2 \|g\|_2 \leq \|T\|.$$

## Proof (Cont.)

Now let  $\alpha$  be an arbitrary sequence such that  $\|T\|$  is bounded. Let  $0 < r < 1$ . Define the sequence  $\alpha^{(r)}$  by

$$\alpha_j^{(r)} = r^j \alpha_j$$

for  $j \geq 0$ .

Let  $T^{(r)}$  be the Hankel matrix  $\{\alpha_{j+k}^{(r)}\}_{j,k \geq 0}$ .

Let  $D_r$  be the operator of multiplication by  $\{r^j\}_{j \geq 0}$  on  $\ell^2(\mathbb{N})$ . Then we can compute that  $T^{(r)} = D_r T D_r$ . Hence  $T^{(r)}$  is bounded, and  $\|T^{(r)}\| \leq \|T\|$ .

## Proof (Cont.)

Let

$$\mathcal{L}_r q = \sum_{n \geq 0} \alpha_n^{(r)} \widehat{q}(n).$$

Since  $\alpha^{(r)} \in \ell^2(\mathbb{N})$ , we have already proved that

$$\|\mathcal{L}\|_{H^1 \rightarrow \mathbb{C}} \leq \|T^{(r)}\| \leq \|T\|.$$

Now the functionals  $\{\mathcal{L}^{(r)}\}_{r \in (0,1)}$  are uniformly bounded and converge strongly to  $\mathcal{L}$ .

Hence,  $\mathcal{L}$  is continuous.



## Proof (Cont.)

We have shown that  $\mathcal{L}$  is bounded on  $H^1(\mathbb{T})$  with norm bounded by  $\|T\|$ . Hence by the Hahn Banach theorem there is  $\psi \in L^\infty$  such that  $\mathcal{L}q = (\psi, q)$ , and  $\|\psi\|_\infty \leq \|T\|$ . This completes the proof. □

# Reformulation in terms of $H_\varphi$

Recall that we defined

$$H_\varphi = \mathbf{P}_- M_\varphi|_{H^2(\mathbb{T})}.$$

For  $\varphi \in L^1(\mathbb{T})$ . This has matrix representation  $\{\widehat{\varphi}(-j-k)\}_{j \geq 1, k \geq 0}$ . Hence,  $H_\varphi$  is bounded if and only if there exists  $\psi \in L^\infty(\mathbb{T})$  such that

$$\widehat{\varphi}(-j) = \widehat{\psi}(j).$$

And hence  $\mathbf{P}_- \varphi \in \text{BMO}(\mathbb{T})$ . Conversely, if  $\mathbf{P}_- \varphi \in \text{BMO}(\mathbb{T})$ , then such a  $\psi$  exists.

# Back to quantised derivatives

Recall that

$$\vec{\partial}\varphi = 2((H_{\varphi_-})^* - H_{\varphi_+}).$$

Since the two Hankel operators act on orthogonal complements, we see that  $\vec{\partial}\varphi$  is bounded if and only if  $H_{\varphi_+}$  and  $H_{\varphi_-}$  are bounded. Hence

## Theorem

*$\vec{\partial}\varphi$  is bounded if and only if  $\varphi \in \text{BMO}(\mathbb{T})$ .*

# Finite rank Hankel operators

Now let us prove Kronecker's theorem.

## Theorem

*Let  $\varphi \in \text{BMO}(\mathbb{T})$ .  $\Gamma_\varphi$  is of finite rank if and only if  $\varphi$  is a rational function.*

Recall that  $\Gamma_\varphi$  is the Hankel matrix with  $(j, k)$ th entry  $\widehat{\varphi}(j + k)$ .

## Proof.

First assume that  $\Gamma_\varphi$  is a finite rank operator on  $\ell^2(\mathbb{N})$ . Let  $\text{rank}(\Gamma_\varphi) = n$ . Then the first  $n + 1$  columns of  $\Gamma_\varphi$  are linearly dependent.

## Proof (Cont.)

Let  $F$  be the operator on  $H^2(\mathbb{T})$  that is “forward shift”, that is

$$F \left( \sum_{n=0}^{\infty} a_n z^n \right) = \sum_{n=0}^{\infty} a_{n+1} z^n.$$

and  $B$  is “backward shift”, so  $Bf(\zeta) = \zeta f(\zeta)$ .

The first  $n+1$  columns of  $\Gamma_\varphi$  being linearly dependent means that there are non-trivial scalars  $\{c_j\}_{j=0}^n$  such that

$$c_0\varphi + c_1F\varphi + \cdots + c_nF^n\varphi = 0.$$

## Proof (Cont.)

For a function  $f \in L^2(\mathbb{T})$ , it is easy to see that

$$B^n F^k f = B^{n-k} f - B^{n-k} \sum_{j=0}^{k-1} \widehat{f}(j) z^j.$$

Hence, since

$$\sum_{k=0}^n c_k F^k \varphi = 0$$

we conclude that

$$0 = \sum_{k=0}^n c_k B^n F^k \varphi = \sum_{k=0}^n c_k B^{n-k} \varphi - p,$$

where  $p$  is a polynomial. Let  $q = \sum_{k=0}^n c_{n-k} z^k$ . Hence  $q\varphi = p$ , so  $\varphi$  is a rational function.

## Proof (Cont.)

Now we prove the converse. Suppose that  $\varphi = p/q$  for polynomials  $p$  and  $q$  such that  $\deg(p) \leq n-1$  and  $\deg(q) \leq n$ . Again, let

$$q = \sum_{j=0}^n c_{n-j} z^j.$$

Hence since  $\varphi q = p$ , we have

$$\sum_{j=0}^n c_j B^{n-j} \varphi = 0.$$

## Proof (Cont.)

Hence,

$$\begin{aligned} F^n \sum_{j=0}^n c_j B^{n-j} \varphi &= \sum_{j=0}^n c_j F^j \alpha \\ &= 0. \end{aligned}$$

Which means that the first  $n + 1$  rows of  $\Gamma_\varphi$  are linearly dependent.

Let  $m \leq n$  be the largest number such that  $c_m \neq 0$ . Then  $F^m \varphi$  is a linear combination of the  $F^j \varphi$  with  $j \leq m - 1$ :

$$F^m \varphi = \sum_{j=0}^{m-1} d_j F^j \varphi,$$

for some nontrivial choice of scalars  $\{d_j\}_{j=0}^{m-1}$ .



## Proof.

Let us proceed by induction to show that any row of  $\Gamma_\varphi$  is a linear combination of the first  $m$  rows. Let  $k > m$ . We have

$$\begin{aligned} F^k \varphi &= F^{k-m} F^m \varphi \\ &= \sum_{j=0}^{m-1} d_j F^{k-m+j} \varphi \end{aligned}$$

Since for each  $0 \leq j \leq m-1$ , we have  $k-m+j < k$ , by the inductive hypothesis the right hand side is a linear combination of the first  $m$  rows. Thus  $\text{rank}(\Gamma_\varphi) \leq m$ . □

It is now straightforward to show that:

## Theorem

*$\bar{\partial}\varphi$  is finite rank if and only if  $\varphi$  is a rational function.*

# Compactness of quantised derivatives

Recall that in classical analysis, we could claim that  $f$  is continuous if and only if  $df$  is infinitesimal.

Since,

$$\|\vec{\partial}\varphi\| = \|[\operatorname{sgn}(\mathcal{D}), M_\varphi]\| \leq 2\|\varphi\|_\infty,$$

and  $\varphi$  a rational function implies that  $\vec{\partial}\varphi$  is of finite rank, we obtain

## Theorem

*Let  $\varphi \in C(\mathbb{T})$ . Then  $\vec{\partial}\varphi$  is compact (i.e., infinitesimal).*

It can be proved that

## Theorem

*Let  $\varphi \in L^2(\mathbb{T})$ . Then  $H_\varphi$  is compact if and only if  $\mathbf{P}_-\varphi \in \text{VMO}(\mathbb{T})$ .*

Consequently  $\bar{\partial}\varphi$  is compact if and only if  $\varphi \in \text{VMO}(\mathbb{T})$ .

## Theorem

*Let  $\varphi$  be a function holomorphic in the unit disc. Then  $\Gamma_\varphi \in \mathcal{L}^1$  if and only if  $\varphi \in B_{11}^1(\mathbb{T})$ .*

Recall that  $\varphi \in B_{11}^1(\mathbb{T})$  means that

$$\sum_{n \geq 0} 2^{|n|} \|W_n * \varphi\|_1 < \infty.$$

# A lemma about $\mathcal{L}^1$ norms

## Lemma

*Let  $f \in \mathcal{P}_A(\mathbb{T})$ , an analytic polynomial of degree at most  $m$ . Then*

$$\|\Gamma_f\|_1 \leq (m+1)\|f\|_1.$$