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Honours Thesis

Quantised Calculus in One Dimension

Author:
Edward McDonald

Supervisor:
Prof. Fedor Sukochev

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Declaration of Authorship

I, Edward McDonald, declare that this thesis titled, 'Quantised Calculus in One Dimension' and the work presented in it are my own. I confirm that:

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"Non-Euclidean calculus and quantum physics are enough to stretch any brain; and when one mixes them with folklore, and tries to trace a strange background of multi-dimensional reality behind the ghoulish hints of the Gothic tales and the wild whispers of the chimney-corner, one can hardly expect to be wholly free from mental tension."

H.P. Lovecraft, *The Dreams in the Witch House* (1933)

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Abstract

Faculty of Science
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Quantised Calculus in One Dimension

by Edward McDonald

The quantised calculus is a tool originating from noncommutative geometry that provides a rigorous "calculus of infinitesimals". We give an in depth study of the quantised calculus in two situations: on the real line and on the circle.

The appendices of this thesis cover background material necessary for the text, for the convenience of the reader. A reader familiar with these topics should not need to consult them, and they are consequently not considered part of the thesis itself.

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Contents

Chapter 1

Introduction

1.1 Classical Infinitesimals

According to the mathematicians of the 17th century, an "infinitesimal" is a quantity x that is smaller than any positive magnitude. In other words, for all $\varepsilon > 0$,

$$|x| < \varepsilon. \tag{1.1}$$

Mathematicians manipulated infinitesimals as though they were real numbers: addition, multiplication and division of infinitesimals were permitted.

Numerous definitions relied on the use of infinitesimals. For example, a function $f : \mathbb{R} \to \mathbb{R}$ was said to be continuous at x if $f(x+h) - f(x)$ is infinitesimal for all infinitesimals h.

Another example is that f was said to be differentiable when the quantity

$$\frac{df}{dx} := \frac{f(x+h) - f(x)}{h} \tag{1.2}$$

exists for all infinitesimals h, and is constant in h if one ignores sufficiently small infinitesimals.

Mathematicians distinguished between infinitesimals of different "sizes", and if x is infinitesimal, then x^2 was said to be smaller than x, and algebraic manipulations could be performed with "sufficiently small infinitesimals ignored".

However, as we are now aware, the condition in equation 1.1 implies that $x = 0$, so all manipulations involving infinitesimals are either trivial or impossible. Hence, the use of infinitesimals was banned from mathematics and their role in analysis was replaced with the concept of a *limit*.

A comprehensive history of the use of infinitesimals prior to the 19th century is given in [1].

Despite the difficulties with giving a non-contradictory definition of infinitesimals, there are numerous reasons that the concept is appealing. Using infinitesimals makes some definitions in analysis seem simpler, for example the definition of continuity given above. Infinitesimals can also be more intuitive than limits. For these reasons, some mathematicians have attempted to revive the concept of infinitesimals by defining them in a rigorous manner. Most famously, one can use model theory to define and analyse a field of *hyperreal numbers* which strictly contains the real numbers and includes a plentiful supply of infinitesimals. This is called non-standard analysis, see [2] for details. Other approaches to rigorously defining infinitesimals include smooth infinitesimal analysis, which is simply the observation that infinitesimals in the classical sense are not self-contradictory objects if certain axioms of logic are ignored, see [3] for an introduction to this approach. A third approach is the Levi-Civita field [4].

The quantised calculus is a relatively new approach to giving a rigorous foundation to mathematics with infinitesimals. Quantised calculus is an invention of A. Connes and D. Sullivan and was first introduced in the 1994 paper "Quantized calculus on S^1 and quasi-fuchsian groups.", [5]. The ideas were then presented in chapter 4 of Connes' book "Noncommutative Geometry", [6]. Connes has further described quantised calculus in the 1995 article "Noncommutative Geometry and Reality", [7]. There is a brief discussion of quantised calculus in the 2001 book "Elements of Noncommutative geometry", [8]. However, there has otherwise been very little work on this topic.

1.2 Compact Operators as Infinitesimals

Quantised calculus comes from noncommutative geometry. In this setting, all objects of interest such as functions, vector fields, differential forms, etc., are thought of as operators on a Hilbert space.

In what follows, let $\mathcal{H}$ be a complex separable Hilbert space.

Suppose we wish to find a good definition of an "infinitesimal operator" on $\mathcal{H}$. A preliminary definition would be to say that an operator T is infinitesimal if for every $\varepsilon > 0$,

$$\|T\| < \varepsilon. \tag{1.3}$$

This definition is useless, as it implies that $T = 0$. However, we can get something close:

Definition 1.1. Let $T \in \mathcal{B}(\mathcal{H})$. We say that T is *infinitesimal* if for every $\varepsilon > 0$, there is a finite dimensional subspace E such that

$$\|T|_{E^\perp}\| < \varepsilon. \tag{1.4}$$

(here $T|_{E^\perp}$ denotes the restriction of T to the orthogonal complement of E.)

Remark 1.2. Recall that the ideal of compact operators $\mathcal{K}(\mathcal{H}) \subseteq \mathcal{B}(\mathcal{H})$ is the closure of the set of finite rank operators. It is easy to see that T is infinitesimal (in the sense of definition 1.1) if and only if T is compact.

We require a way of measuring the "size" of an infinitesimal. According to definition 1.1, an infinitesimal is "zero modulo finite dimensional subspaces", and it is sensible to consider the "size" of the infinitesimal as being measured by the speed at which the dimension of the subspaces E must increase as ε moves towards zero. To this end, we define the singular values.

Definition 1.3. Let $T \in \mathcal{B}(\mathcal{H})$, and $n \geq 0$. Define

$$\mu_n(T) := \inf\{\|T - F\| \ : \ \mathrm{rank}(F) \leq n\}. \tag{1.5}$$

Then $\mu_n(T)$ is called the nth singular value of T, and the sequence $\{\mu_n(T)\}_{n=0}^{\infty}$ is called the sequence of singular values.

Remark 1.4. For any operator T, the sequence $\{\mu_n(T)\}_{n=0}^{\infty}$ is strictly non-increasing. If T is compact, the sequence of singular values of T is decreasing and approaches 0.

We shall regard the *size* of T as being given by the *rate of decay* of $\{\mu_n(T)\}_{n=0}^{\infty}$.

The philosophy of measuring the size of a compact operator by the rate of decay of its singular values is given formal justification by the "Calkin Correspondence", explained in Appendix E.

1.3 Expected Properties of infinitesimals

According to 17th century mathematicians, infinitesimals were supposed to have a number of remarkable properties:

1. There exist non-zero infinitesimals.
2. Infinitesimals can be added, subtracted, multiplied and divided just like real numbers.
3. A real number multiplied by an infinitesimal produces an infinitesimal.
4. Infinitesimals can be split into "sizes", and we can work modulo a particular size of infinitesimal.
5. If x is an infinitesimal, then x^2 is a smaller infinitesimal.
6. Given a function $f : \mathbb{R} \to \mathbb{R}$, there is a function df representing the infinitesimal variation in f, that is $df(x) = f(x + \delta) - f(x)$ for some fixed infinitesimal δ.

7. If a function f is smoother than a function g, then df is smaller than dg.
8. If f is a smooth function of x, we can write

$$df = f'(x)dx \tag{1.6}$$

provided that sufficiently small infinitesimals are ignored.

If we interpret compact operators as infinitesimals, we see that analogues of these statements remain true:

1. There indeed exist non-zero compact operators.
2. Compact operators can be added and subtracted like numbers, they can also be multiplied (although multiplication is not commutative). We cannot divide by a compact operator on an infinite dimensional Hilbert space. However it is true that if $XT = YT$ for operators $X, Y \in \mathcal{B}(\mathcal{H})$ for all $T \in \mathcal{K}(\mathcal{H})$, then $X = Y$.
3. $\mathcal{K}(\mathcal{H})$ forms an ideal of $\mathcal{B}(\mathcal{H})$, so a bounded operator multiplied by a compact operator produces a compact operator.
4. We can measure the size of a compact operator by the rate of decay of its sequence of singular values.

For item 5, we need the following lemma,

Lemma 1.5. *Let $T \in \mathcal{K}(\mathcal{H})$. Then for $n \geq 0$,*

$$\mu_n(T^2) \leq \|T\|\mu_n(T). \tag{1.7}$$

Proof. By definition,

$$\mu_n(T^2) = \inf\{\|T^2 - F\| \;:\; \mathrm{rank}(F) \leq n\}. \tag{1.8}$$

Since if F has rank n, TF has rank not exceeding n, we have

$$\begin{aligned} \mu_n(T^2) &\leq \inf\{\|T^2 - TF\| \;:\; \mathrm{rank}(F) \leq n\} && (1.9)\\ &\leq \inf\{\|T\|\|T - F\| \;:\; \mathrm{rank}(F) \leq n\} && (1.10)\\ &= \|T\|\mu_n(T). && (1.11) \end{aligned}$$

$\square$

Hence we have property 5: if T is infinitesimal, then the singular values of T^2 decay more rapidly than those of T.

Now for properties 6, 7 and 8, we need a way of defining df. This is precisely the role played by the quantised differential.

1.4 Quantised Differentials

The following definition may at first glance seem strange and unmotivated,

Definition 1.6. Consider the operator $\mathcal{D} = \frac{1}{i}\frac{d}{dx}$ of differentiation on $\mathbb{R}$. By the Borel functional calculus, we can define $F := \operatorname{sgn}(\mathcal{D})$. F is called the Hilbert transform. For a function $f \in L^1(\mathbb{R})$ we have the pointwise multiplication operator M_f considered as an operator on $L^2(\mathbb{R})$, M_f may be a not-everywhere defined unbounded operator when $f \notin L^\infty(\mathbb{R})$

The operator

$$đf := [F, M_f] \tag{1.12}$$

is an operator on $L^2(\mathbb{R})$ called the *quantised differential* of f.

$đf$ is supposed to play the role of df in 17th century analysis. We use the symbol $đf$ instead of df since in modern mathematics the symbol df is often used to denote the exterior differential of f. It must be emphasised that the exterior differential df and the quantised differential $đf$ are very different objects. The difference between df and $đf$ will be explained in detail in Chapter 5.

Similar to the case of functions $f : \mathbb{R} \to \mathbb{C}$, we can also study functions on the circle. Let

$$\mathbb{T} := \{z \in \mathbb{C} \; : \; |z| = 1\}. \tag{1.13}$$

$\mathbb{T}$ is parametrised by the map $\theta : \mathbb{T} \to [0, 2\pi)$ given by $\exp(i\theta) \mapsto \theta$ for $\theta \in [0, 2\pi)$.

The pull-back of the Lebesgue measure on $[0, 2\pi)$ to $\mathbb{T}$ by θ is denoted $\boldsymbol{m}$, and is normalised to have $\boldsymbol{m}(\mathbb{T}) = 1$.

Then we have a differentiation operator

$$\mathcal{D} := \frac{1}{2\pi i}\frac{d}{d\theta}, \tag{1.14}$$

acting on $L^2(\mathbb{T}, \boldsymbol{m})$. Define $F := \operatorname{sgn}(\mathcal{D})$. For a function $g : \mathbb{T} \to \mathbb{C}$, we can define the quantised differential $đg := [F, M_g]$.

Remark 1.7. We have chosen to call the quantity $[F, M_f]$ a quantised differential, and to denote it $đf$. This terminology is not universal. In particular, the book [6, Ch. 4] calls $[F, M_f]$ a quantised derivative and denotes it df. We shall use the notation $đf$ to prevent confusion with the exterior derivative df, and we call it a differential to be consistent with the idea of an “infinitesimal increment”.

It is not easy to motivate the definition $đf := [F, M_f]$. Instead, we shall show that $đf$ satisfies all the properties anticipated for a differential.

The two questions that we shall attempt to answer are as follows:

1. In what sense is it true that if f is smoother than g, then $đf$ is smaller than $đg$? This will be answered in Chapter 4.
2. In what sense is it true that if φ is a function that is smooth on the range of a function f, then $đ\varphi(f) = \varphi'(f)đf$? This will be answered in Chapter 7.

In order to answer those questions, it is informative to give an explanation of the origin of the definition of $đf$.

1.5 Noncommutative geometry

1.5.1 Introduction to the Noncommutative world

Noncommutative geometry is a relatively new topic in mathematics. Noncommutative geometry is best thought of not as a collection of results, but instead as a perspective on mathematics.

It is difficult to give a completely satisfying general definition of noncommutative geometry, but one can say that noncommutative geometry is the study of noncommutative algebras which are somehow similar to algebras of functions on geometric spaces, using the methods and language of geometry.

The idea that one can approach noncommutative algebras in this way is not new, but the particular philosophy that we explore here is due to Alain Connes, and was first introduced in his 1985 paper "Noncommutative Differential Geometry", [9].

The key idea which underlies most of noncommutative geometry is the duality between geometric spaces and algebras.

Example 1.1. *Let X be a compact Hausdorff space. $C(X)$ is defined to be the algebra of continuous complex valued functions on X. $C(X)$ naturally carries the structure of a commutative unital C^*-algebra. In fact, for any commutative unital C^*-algebra $\mathcal{A}$, there is a compact Hausdorff space K such that $\mathcal{A}$ is isometrically $*$-isomorphic to $C(K)$.*

Given a continuous function $f : X \to Y$ between compact Hausdorff spaces X and Y, there is a pull-back function $f_* : C(Y) \to C(X)$ defined by $f_*(h) = h \circ f$ for $h \in C(Y)$. Since $\mathrm{id}_{X*} = \mathrm{id}_{C(X)}$, and $(f \circ g)_* = g_* \circ f_*$, the mapping $X \mapsto C(X)$ is a functor.

Let **CHTop** *be the category of compact Hausdorff spaces with morphisms as continuous functions, and let* **CUC*Alg** *be the category of commutative unital C^*-algebras with morphisms as continuous $*$-algebra homomorphisms.*

Thus we have a contravariant functor,

$$C : \mathbf{CHTop} \to \mathbf{CUC^*Alg}. \tag{1.15}$$

This effects an equivalence of categories,

$$\mathbf{CUC^*Alg} \cong \mathbf{CHTop}^{Op}. \tag{1.16}$$

Let $\mathbf{UC^*Alg}$ *be the category of unital* C^**-algebras, which are not necessarily commutative. Inspired by the duality between commutative unital* C^**-algebras and compact Hausdorff spaces, we define the category of (potentially) noncommutative compact Hausdorff spaces to be* $\mathbf{UC^*Alg}^{Op}$*.*

For more details on the duality between C^*-algebras and compact Hausdorff spaces, we refer to the book [10].

1.5.2 A brief introduction to Quantum Mechanics

Noncommutative geometry can be thought of simply as the study of noncommutative algebras using geometric language. However, much research in noncommutative geometry is inspired by quantum mechanics. It is therefore instructive to give a brief description of quantum mechanics.

Definition 1.8. A *quantum mechanical system* is a pair $(\mathcal{A}, \mathcal{H})$ where $\mathcal{H}$ is a complex separable Hilbert space and $\mathcal{A}$ is a $*$-algebra of (possibly densely defined and unbounded) operators on $\mathcal{H}$. Denote the inner product on $\mathcal{H}$ by $(\cdot,\cdot)$ and $\|\psi\|^2 := (\psi, \psi)$.

A self-adjoint element of $\mathcal{A}$ is called an *observable*. The elements of $\mathcal{H}$ are called *states*.

Typically, we identify together elements of $\mathcal{H}$ which differ by a nonzero scale factor, and the element $0 \in \mathcal{H}$ is ignored entirely. So technically we work over the *projective Hilbert space* $\mathbb{P}\mathcal{H}$.

To specify *the state of the system* $(\mathcal{A}, \mathcal{H})$ is the same as specifying some $\psi \in \mathcal{H}$.

We think of $(\mathcal{A}, \mathcal{H})$ as encoding a physical system. Typically we think of an observable as a measurable property of a system. The states correspond to potential configurations of the system.

Given an observable A, the potential range of values that can be measured for the corresponding physical quantity is the spectrum $\sigma(A)$. Unlike in classical mechanics, quantum mechanics can only make predictions that are probabilistic. The second difference to classical mechanics is that the act of observation changes the state of the system.

The link between $\sigma(A)$ and A is provided by the spectral theorem, stated and proved in [11, p.369].

Theorem 1.9 (The Spectral Theorem). *Let $(\mathcal{A}, \mathcal{H})$ be a quantum mechanical system. Let $A \in \mathcal{A}$ be an observable. Then there is a projection valued measure E_A on $\sigma(A)$ such that*

$$A = \int_{\sigma(A)} \lambda \, dE_A(\lambda).$$

Using the spectral theorem, we may state our first postulate.

Postulate 1.10. *Let $(\mathcal{A}, \mathcal{H})$ be a quantum mechanical system, in a state ψ.*

Let $A \in \mathcal{A}$ be an observable, with associated spectral measure E_A.

For some Borel set $\Delta \subseteq \sigma(A)$, the probability that the observed value of A lies in Δ is given by

$$P_A(\Delta; \psi) := \frac{(\psi, E_A(\Delta)\psi)}{\|\psi\|^2} = \frac{\|E_A(\Delta)\psi\|^2}{\|\psi\|^2}.$$

(Note that this is the same for any scalar multiple of ψ, and is undefined for $\psi = 0$.)

Suppose that A is now observed, and the value is known to lie in the set $\Delta \subseteq \sigma(A)$. Then the state of the system changes to

$$\frac{E_A(\Delta)\psi}{\|E_A(\Delta)\psi\|}.$$

Remark 1.11. There are two extraordinary features of this postulate.

1. The state of the system changes upon observation.

2. The order of observation is important, since for any observables A and B, and $\Sigma \times \Delta \subseteq \sigma(A) \times \sigma(B)$, in general,

$$\frac{E_A(\Sigma)E_B(\Delta)\psi}{\|E_A(\Sigma)E_B(\Delta)\psi\|} \neq \frac{E_B(\Delta)E_A(\Sigma)\psi}{\|E_B(\Delta)E_A(\Sigma)\psi\|}.$$

Remark 1.12. This discussion of quantum mechanics is far from comprehensive. The treatment here is based on [12].

1.5.3 The Noncommutative Perspective

Inspired by quantum mechanics, quantum mechanics can be thought of as the study of spaces with coordinates such that the order of observation of coordinates is important.

Accordingly, a general noncommutative space can be defined as a pair $(\mathcal{A}, \mathcal{H})$, just as a quantum mechanical system. The elements of $\mathcal{A}$ are "coordinate functions", and their action on the elements of $\mathcal{H}$ quantifies how the "geometry" changes under observation.

We approach the problem of defining infinitesimals in a modern sense with this point of view in mind. That is, we are committed to viewing all objects of interest as being operators on a Hilbert space. The study of functions of one variable can be thought of as a study of operators on the Hilbert space $L^2(\mathbb{R})$. We therefore seek "generalised functions" as being operators on $L^2(\mathbb{R})$. This motivates our definition of $đf$ as an operator. For Chapters 2-4 we shall simply take this definition as given. In Chapter 5 we will return to the problem of the origin of the quantised differential.

In Chapter 6 we will attempt to develop the quantised calculus in a more general setting.

Chapter 8 covers an example of the application of quantised calculus in one variable to the theory of Julia sets.

Chapter 2

Quantised Differentials on $\mathbb{R}$ and $\mathbb{T}$

2.1 The Quantised Differential on $\mathbb{T}$

We shall introduce a Fourier analytic description of the quantised differential of a function on the circle. Let $\mathbb{T} = \{\zeta \in \mathbb{C} \ : \ |\zeta| = 1\}$ be the unit circle in the complex plane, which is a compact group equipped with a normalised Haar measure, which we denote $\boldsymbol{m}$.

When we write function spaces $L^p(\mathbb{T})$, we shall implicitly mean $L^p(\mathbb{T}, \boldsymbol{m})$.

We write $z : \mathbb{T} \to \mathbb{T}$ for the identity function, $z = \mathrm{id}_{\mathbb{T}}$.

For $f \in L^1(\mathbb{T})$ and $n \in \mathbb{Z}$, we write the nth Fourier coefficient as,

$$\hat{f}(n) = \int_{\mathbb{T}} z^{-n} f \, d\boldsymbol{m}. \tag{2.1}$$

A review of elementary Fourier analysis and notation is given in Appendix A.

Recall that we have the operator $\mathcal{D}$ that acts on functions on $\mathbb{T}$, defined by $\mathcal{D}(z^n) = nz^n$. We also have $F := \mathrm{sgn}(\mathcal{D})$, which acts by $F(z^n) = \mathrm{sgn}(n)z^n$.

The important operator $\mathbf{P}_+$, called the Riesz Projection, is given by $\mathbf{P}_+(z^n) := \max\{\mathrm{sgn}(n), 0\}z^n$.

By definition, $H^2(\mathbb{T}) := \mathbf{P}_+ L^2(\mathbb{T})$.

Proposition 2.1. *For $\varphi \in L^1(\mathbb{T})$, the* quantised differential *of φ is the (potentially unbounded, densely defined) linear operator*

$$đ\varphi := 2[\mathbf{P}_+, M_\varphi]. \tag{2.2}$$

Proof. This is a consequence of the observation that $F = 2\mathbf{P}_+ - \mathbf{1}$, where $\mathbf{1}$ is the identity operator. □

In this chapter, we will discuss alternative descriptions of $đ\varphi$.

Since $H^2(\mathbb{T})$ is a closed subspace of $L^2(\mathbb{T})$, being the image of $L^2(\mathbb{T})$ under a bounded projection, it has an orthogonal complement which we denote $H^2_-(\mathbb{T})$.

Hence, we may consider the quantised derivative $đ\varphi$ as an operator on the Hilbert space $H^2(\mathbb{T}) \oplus H^2_-(\mathbb{T})$.

Lemma 2.2. *Let* $\varphi \in L^2(\mathbb{T})$ *and define* $\varphi_+ := \mathbf{P}_+\varphi$ *and* $\varphi_- := \mathbf{P}_-\varphi$. *Then* $đ\varphi : H^2(\mathbb{T}) \oplus H^2_-(\mathbb{T}) \to H^2(\mathbb{T}) \oplus H^2_-(\mathbb{T})$ *may be written as*

$$đ\varphi(f \oplus g) = 2(\mathbf{P}_+ M_{\varphi_+})g \oplus -2(\mathbf{P}_- M_{\varphi_-})f \tag{2.3}$$

for $f \in H^2(\mathbb{T})$ *and* $g \in H^2_-(\mathbb{T})$.

Proof. This is a simple computation. Let $f \in H^2(\mathbb{T})$ and $g \in H^2_-(\mathbb{T})$. Then,

$$đ\varphi(f + g) = 2[\mathbf{P}_+, M_{\varphi_+} + M_{\varphi_-}](f + g). \tag{2.4}$$

Hence,

$$\begin{aligned} đ\varphi(f + g) &= 2[\mathbf{P}_+, M_{\varphi_+}]f + 2[\mathbf{P}_+, M_{\varphi_-}]f + 2[\mathbf{P}_+, M_{\varphi_+}]g + 2[\mathbf{P}_+, M_{\varphi_-}]g \\ &= 2(\mathbf{P}_+ M_{\varphi_-})f + 2(\mathbf{P}_+ M_{\varphi_+})g - 2M_{\varphi_-}f \end{aligned}$$

since $\mathbf{P}_+ f = f$ and $\mathbf{P}_+ g = 0$.

By the identity $\mathbf{P}_+ = \mathbf{1} - \mathbf{P}_-$, we find

$$đ\varphi(f + g) = 2(\mathbf{P}_+ M_{\varphi_+})g - 2(\mathbf{P}_- M_{\varphi_-})f. \tag{2.5}$$

□

The problem of determining the boundedness of $đ\varphi$ is then reduced to the problem of determining the boundedness of operators of the form $\mathbf{P}_+ M_\psi : H^2_-(\mathbb{T}) \to H^2(\mathbb{T})$ and $\mathbf{P}_- M_\psi : H^2(\mathbb{T}) \to H^2_-(\mathbb{T})$ for $\psi \in L^2(\mathbb{T})$. We may simplify this further with the following lemma:

Lemma 2.3. *Let* $\psi \in L^2(\mathbb{T})$. *Then*

$$(\mathbf{P}_+ M_\psi)^* = \mathbf{P}_- M_{\overline{\psi}}. \tag{2.6}$$

and therefore $\mathbf{P}_+ M_\psi$ *is bounded if and only if* $\mathbf{P}_- M_{\overline{\psi}}$ *is.*

Proof. Let $e_k(z) = z^k$.

This is again a simple computation. Let $m, n \in \mathbb{Z}$ with $m \geq 0$ and $n < 0$. Then,

$$\begin{aligned}\langle(\mathbf{P}_+ M_\psi)e_n, e_m\rangle &= \int_{\mathbb{T}} \sum_{k>-n} \hat{\psi}(k)\zeta^{k+n-m}\, d\boldsymbol{m}(\zeta) \\ &= \hat{\psi}(m-n).\end{aligned}$$

Similarly,

$$\begin{aligned}\langle e_n, (\mathbf{P}_- M_{\overline{\psi}})e_m\rangle &= \int_{\mathbb{T}} \sum_{k>m} \hat{\varphi}(k)\zeta^{n-m+k}\, d\boldsymbol{m}(\zeta) \\ &= \hat{\psi}(m-n).\end{aligned}$$

Hence, $(\mathbf{P}_+ M_\psi)^* = \mathbf{P}_- M_{\overline{\psi}}$.

□

For $\psi \in L^2(\mathbb{T})$, define

$$H_\psi := \mathbf{P}_- M_\psi : H^2 \to H^2_-. \tag{2.7}$$

In other words, we may write $đ\varphi : H^2(\mathbb{T}) \oplus H^2_-(\mathbb{T}) \to H^2(\mathbb{T}) \oplus H^2_-(\mathbb{T})$ as a matrix,

$$đ\varphi = 2\begin{pmatrix} 0 & -H^*_{\overline{\varphi_+}} \\ H_{\varphi_-} & 0 \end{pmatrix} \tag{2.8}$$

Therefore, we only need to study operators of the form H_ψ.

We study these operators using the Fourier transform. Use the standard basis $\{z^n\}_{n\geq 0}$ on $H^2(\mathbb{T})$ and the standard basis with negative indices $\{z^{-n}\}_{n\geq 0}$ on $H^2(\mathbb{T})$.

Let $\psi \in L^2(\mathbb{T})$. Then in the bases above, H_ψ has matrix representation with (n,k)th entry $\hat{\psi}(-n-k)$.

This means that H_ψ is represented by a *Hankel matrix*. So we require results on Hankel matrices. This is covered in Chapter 3.

2.2 Integral representation of the quantised differential on $\mathbb{T}$

In the preceding section we have realised the quantised differential as a Hankel operator. Now we present an alternative description of the quantised differential as an integral operator. The representation as an integral operator is well known and was extensively used by A. Connes in [6].

Lemma 2.4.

$$\text{p.v.} \int_{\mathbb{T}} \frac{1}{\tau - 1}\, d\boldsymbol{m}(\tau) = -\frac{1}{2} \tag{2.9}$$

where the principal value is defined to be

$$\lim_{\varepsilon\to 0}\int_{|\tau-1|>\varepsilon}\frac{1}{\tau-1}\,d\boldsymbol{m}(\tau). \tag{2.10}$$

Proof. Note that

$$\text{p.v.}\int_{\mathbb{T}}\frac{1}{\tau-1}\,d\boldsymbol{m}(\tau)=\text{p.v.}\int_{\text{im}(\tau)>0}\frac{1}{\overline{\tau}-1}+\frac{1}{\tau-1}\,d\boldsymbol{m}(\tau). \tag{2.11}$$

Now split up the integral into upper and lower semicircular parts,

$$\text{p.v.}\int_{\mathbb{T}}\frac{1}{\tau-1}\,d\boldsymbol{m}(\tau)=\text{p.v.}\int_{\text{Im}(\tau)>0}2\,\text{Re}\left(\frac{1}{\tau-1}\right)\,d\boldsymbol{m}(\tau). \tag{2.12}$$

However, if $\tau=\exp(i\theta)\neq 1$, then

$$\begin{aligned}\text{Re}\left(\frac{1}{\tau-1}\right)&=\text{Re}\left(\frac{e^{-i\theta/2}}{2i\sin(\theta/2)}\right)\\&=-\frac{1}{2}.\end{aligned}$$

Hence,

$$\text{p.v.}\int_{\mathbb{T}}\frac{1}{\tau-1}\,d\boldsymbol{m}(\tau)=2\text{p.v.}\int_{\text{Im}(\tau)>0}-\frac{1}{2}\,d\boldsymbol{m}(\tau)=-\frac{1}{2}. \tag{2.13}$$

□

Theorem 2.5. *Let $\varphi\in L^2(\mathbb{T})$. Then*

$$\mathbf{P}_+\varphi(\zeta)=\text{p.v.}\int_{\mathbb{T}}\frac{\varphi(\tau)}{1-\overline{\tau}\zeta}\,d\boldsymbol{m}(\tau)+\frac{1}{2}\varphi(\zeta) \tag{2.14}$$

and hence,

$$F\varphi(\zeta)=2\text{p.v.}\int_{\mathbb{T}}\frac{\varphi(\tau)}{1-\overline{\tau}\zeta}\,d\boldsymbol{m}(\tau). \tag{2.15}$$

where in both equations, the principal value means that the integral is to be taken along the set $\{\tau\in\mathbb{T}\ :|\tau-\zeta|>\varepsilon\}$ and then consider the limit $\varepsilon\to 0$.

Proof. First we check this for $\varphi=\zeta^n$ for $n\in\mathbb{Z}$.

First let $n\geq 0$. Then

$$\text{p.v.}\int_{\mathbb{T}}\frac{\tau^n}{1-\overline{\tau}\zeta}\,d\boldsymbol{m}(\tau)=\text{p.v.}\int_{\mathbb{T}}\frac{z^n\tau^n}{1-\overline{\tau}}\,d\boldsymbol{m}(\tau) \tag{2.16}$$

by translation invariance. Hence,

$$\begin{aligned}
\text{p.v.} \int_{\mathbb{T}} \frac{\tau^n}{1-\overline{\tau}\zeta}\, d\boldsymbol{m}(\tau) &= \zeta^n \text{p.v.} \int_{\mathbb{T}} \frac{\tau^{n+1}}{\tau - 1}\, d\boldsymbol{m}(\tau) \\
&= \zeta^n \text{p.v.} \int_{\mathbb{T}} \frac{\tau^{n+1}-1}{\tau-1} + \frac{1}{\tau-1}\, d\boldsymbol{m}(\tau) \\
&= \zeta^n \text{p.v.} \int_{\mathbb{T}} 1 + \tau + \tau^2 + \cdots + \tau^n\, d\boldsymbol{m}(\tau) + \zeta^n \text{p.v.} \int_{\mathbb{T}} \frac{1}{\tau-1}\, d\boldsymbol{m}(\tau) \\
&= \zeta^n + z^n \text{p.v.} \int_{\mathbb{T}} \frac{1}{\tau-1}\, d\boldsymbol{m}(\tau) \\
&= \frac{1}{2}\zeta^n
\end{aligned}$$

where the last step follows from lemma 2.4.

Suppose $n > 0$, then

$$\text{p.v.} \int_{\mathbb{T}} \frac{\tau^{-n}}{1-\overline{\tau}\zeta}\, d\boldsymbol{m}(\tau) = \zeta^{-n} \text{p.v.} \int_{\mathbb{T}} \frac{\tau^{1-n}}{\tau-1}\, d\boldsymbol{m}(\tau) \tag{2.17}$$

by translation invariance. Hence,

$$\begin{aligned}
\text{p.v.} \int_{\mathbb{T}} \frac{\tau^{-n}}{1-\overline{\tau}\zeta}\, d\boldsymbol{m}(\tau) &= z\zeta^{-n} \text{p.v.} \int_{\mathbb{T}} \frac{1}{\tau^n - \tau^{n-1}}\, d\boldsymbol{m}(\tau) \\
&= \zeta^{-n} \overline{\text{p.v.} \int_{\mathbb{T}} \frac{\tau^n}{1-\tau}} \\
&= -\frac{1}{2}\zeta^{-n}.
\end{aligned}$$

Hence,

$$\text{p.v.} \int_{\mathbb{T}} \frac{\tau^n}{1-\overline{\tau}\zeta}\, d\boldsymbol{m}(\tau) = \begin{cases} \frac{1}{2}\zeta^n \text{ if } n \geq 0 \\ -\frac{1}{2}\zeta^n \text{ if } n < 0. \end{cases} \tag{2.18}$$

Hence we have

$$F\varphi(\zeta) = 2\text{p.v.} \int_{\mathbb{T}} \frac{\varphi(\tau)}{1-\overline{\tau}\zeta}\, d\boldsymbol{m}(\tau). \tag{2.19}$$

for $\varphi = z^n$. To extend this to arbitrary $\varphi \in L^2(\mathbb{T})$, we see that $\varphi = \sum_{n\in\mathbb{Z}} \hat{\varphi}(n) z^n$, which converges in the L^2 sense. Since $\sum_{n\in\mathbb{Z}} \hat{\varphi}(n) z^n$ converges in the L^2 sense, it converges in the L^1 sense.

Now fix $\varepsilon > 0$. By the dominated convergence theorem, we have

$$\int_{|\tau-\zeta|>\varepsilon} \frac{\varphi(\tau)}{1-\overline{\tau}\zeta}\, d\boldsymbol{m}(\tau) = \sum_{n\in\mathbb{Z}} \hat{\varphi}(n) \int_{|\tau-\zeta|>\varepsilon} \frac{\tau^n}{1-\overline{\tau}\zeta}\, d\boldsymbol{m}(\tau). \tag{2.20}$$

Now we take the limit $\varepsilon \to 0$. Again by the dominated convergence theorem for sums, the result follows.

$\square$

So we have the following integral form of the quantised derivative. Let $\varphi, f \in L^2(\mathbb{T})$. Then

$$
\begin{aligned}
đ\varphi(f)(\zeta) &= ([F, M_\varphi]f)(\zeta) && (2.21)\\
&= F(\varphi f) - \varphi(F(f)) && (2.22)\\
&= \text{p.v.} \int_{\mathbb{T}} \frac{\varphi(\tau)f(\tau)}{1-\overline{\tau}\zeta} - \frac{\varphi(\zeta)f(\tau)}{1-\overline{\tau}\zeta}\, d\boldsymbol{m}(\tau) && (2.23)\\
&= \text{p.v.} \int_{\mathbb{T}} \frac{\varphi(\tau)-\varphi(\zeta)}{1-\overline{\tau}z} f(\tau)\, d\boldsymbol{m}(\tau). && (2.24)
\end{aligned}
$$

for almost all $\zeta \in \mathbb{T}$.

2.3 The Cayley Transform

2.3.1 Notation

$\mathbb{H} = \{z \in \mathbb{C} \ : \ \Im(z) > 0\}$ denotes the upper half plane, and $\mathbb{D} = \{z \in \mathbb{C} \ : \ |z| < 1\}$ denotes the open unit ball. $\mathbb{T} = \{z \in \mathbb{C} \ : \ |z| = 1\}$.

We use normalised Haar measure on $\mathbb{T}$, denoted $\boldsymbol{m}$. Lebesgue measure on $\mathbb{R}$ is denoted λ.

Throughout these notes, ω denotes the *Cayley transform.* $\omega : \mathbb{C} \setminus \{1\} \to \mathbb{C}$, and

$$\omega(\zeta) = i\frac{1+\zeta}{1-\zeta},\ \zeta \in \mathbb{D}. \tag{2.25}$$

For a Banach space E, and a measure space (X, Σ, μ), we define

$$\|f\|_{L^p(X;E)} = \left(\int_X \|f\|_E^p \, d\mu\right)^{1/p} \tag{2.26}$$

for $p \in (0, \infty)$, and

$$\|f\|_{L^\infty(X;E)} = \inf\{C > 0 \ : \ \mu\{x \in X \ : \ \|f(x)\|_E > C\} = 0\} \tag{2.27}$$

for a weakly measurable $f : X \to E$. We define $L^p(X; E)$ as the set of measurable $f : X \to E$ with $\|f\|_{L^p(X;E)} < \infty$. As usual, we identify together functions on a measure space (X, Σ, μ) which agree μ-almost everywhere.

$L^0(X; E)$ denotes the set of all (μ-almost everywhere equivalence classes of) weakly measurable functions from X to E.

When X is a set with counting measure, we denote $L^p(X; E)$ as $\ell^p(X; E)$.

Suppose $\zeta \in \mathbb{T}$. Provided that $\zeta \neq 1$, we see that $\omega(\zeta)$ is defined, and ω maps $\mathbb{T} \setminus \{1\}$ smoothly to $\mathbb{R}$. Thus for $f \in L^0(\mathbb{R}; E)$, we can define $\tilde{f} \in L^0(\mathbb{T}; E)$ by

$$\tilde{f} := f \circ \omega^{-1}. \tag{2.28}$$

Thus we can define the important operator $U : L^0(\mathbb{T}; E) \to L^0(\mathbb{R}; E)$,

$$(Uf)(x) = \frac{1}{\sqrt{\pi}} \frac{(f \circ \omega^{-1})(x)}{x + i}, \tag{2.29}$$

and $U^{-1} : L^0(\mathbb{R}; E) \to L^0(\mathbb{T}; E)$,

$$(U^{-1}h)(\zeta) = \sqrt{\pi}(\omega(\zeta) + i)(h \circ \omega)(\zeta). \tag{2.30}$$

2.3.2 Images under the Cayley Transform

2.3.3 Initial results

It is obvious from the definition that U is linear.

Now define, for $g \in L^0(\mathbb{R}; E)$,

$$(\mathcal{F})g(\zeta) = \frac{\sqrt{\pi}}{2i} \frac{(g \circ \omega)(\zeta)}{1 - \zeta}, \ \zeta \in \mathbb{T}.$$

Lemma 2.6. *U and $\mathcal{F}$ are inverse functions, hence U is a bijection.*

Proof. Let $g \in L^0(\mathbb{R}; E)$, and let $t \in \mathbb{R}$. Then we simply compute,

$$\begin{aligned}
(U \circ \mathcal{F})(g)(t) &= \frac{\sqrt{\pi}}{2i} \frac{g(t)}{1 - \omega^{-1}(t)} \frac{1}{\sqrt{\pi}} \frac{1}{t + i} \\
&= \frac{1}{2i(1 - \omega^{-1}(t))(t + i)} g(t) \\
&= g(t).
\end{aligned}$$

So $U \circ \mathcal{F}$ is the identity function on $L^0(\mathbb{R})$.

Similarly, let $f \in L^0(\mathbb{R}; E)$, and $\zeta \in \mathbb{T}$, then

$$\begin{aligned}
(\mathcal{F} \circ U)(f)(\zeta) &= \frac{\sqrt{\pi}}{2i} \frac{1}{1 - \zeta} \frac{g(\zeta)}{i + \omega(\zeta)} \\
&= g(\zeta).
\end{aligned}$$

So $\mathcal{F} \circ U$ is the identity function on $L^0(\mathbb{T}; E)$. $\square$

ω may be regarded as a function from $\mathbb{T} \setminus \{1\} \to \mathbb{R}$. If we define $\omega(1)$ to be some arbitrary value, say $\omega(1) = 0$, we have a measureable function $\omega : \mathbb{T} \to \mathbb{R}$.

Thus there is a pushforward of the Haar measure $\boldsymbol{m}$ on $\mathbb{T}$ to $\mathbb{R}$, denoted $\omega_*(\boldsymbol{m})$, defined by

$$\omega_*(\boldsymbol{m})(A) = \boldsymbol{m}(\omega^{-1}(A))$$

for all Lebesgue measurable sets A.

We may describe this with the following result,

Lemma 2.7. *The pushforward measure,* $\omega_*(\boldsymbol{m})$ *has Lebesgue Radon-Nikodym derivative*

$$\frac{d\omega_*(\boldsymbol{m})}{d\lambda} = \frac{1}{\pi|i+t|^2}$$

Proof. Let a be the arc length measure on $\mathbb{T}$, so $\boldsymbol{m} = \frac{a}{2\pi}$, now let $A \subseteq \mathbb{R}$ be Lebesgue measurable, then

$$\begin{aligned} \omega_*(a)(A) &= \int_A \left| \frac{d(\omega^{-1}(t)}{dt} \right| \, d\lambda(t) \\ &= \int_A \frac{2}{|i+t|^2} d\lambda(t). \end{aligned}$$

Hence, the required result follows. $\square$

A less obvious result is the following,

Theorem 2.8. U *is an isometry from* $L^2(\mathbb{T}; E)$ *to* $L^2(\mathbb{R}; E)$*.*

Proof. Let $f \in L^2(\mathbb{T}; E)$. Then

$$\begin{aligned} \|Uf\|^2_{L^2(\mathbb{R};E)} &= \int_{\mathbb{R}} \frac{1}{\pi|i+t|^2} \|(f \circ \omega^{-1})(t)\|^2_E \, d\lambda \\ &= \int_{\mathbb{R}} \|(f \circ \omega^{-1})(t)\|^2_E \, d(\omega_*\boldsymbol{m})(t) \\ &= \int_{\mathbb{T}} \|f\|^2_E d\boldsymbol{m}. \end{aligned}$$

So U embeds $L^2(\mathbb{T}; E)$ into $L^2(\mathbb{R}; E)$ isometrically. We may similarly prove the opposite embedding. $\square$

We also have,

Theorem 2.9. $UL^\infty(\mathbb{T}; E) \subset L^\infty(\mathbb{R}; E)$*. The inclusion here is continuous, and it is not true that* $L^\infty(\mathbb{R}; E) = UL^\infty(\mathbb{T}; E)$*.*

Proof. This is evident from the definition of U. Let $f \in L^\infty(\mathbb{T}; E)$. Then,

$$\begin{aligned}\|Uf\|_{L^\infty(\mathbb{R};E)} &= \sup_{t\in\mathbb{R}} \frac{1}{\sqrt{\pi}|i+t|}\|(f\circ\omega^{-1})(t)\|_E \\ &< \sup_{z\in\mathbb{T}} \|f(z)\|_E \\ &= \|f\|_{L^\infty(\mathbb{T};E)}.\end{aligned}$$

However, consider a constant function $c \in L^\infty(\mathbb{R}; E)$. We see that $U^{-1}c$ is unbounded. □

The following theorem is absolutely crucial for the link between quantised calculus on the circle and the line. This proof is based on one given in an unpublished note by A Carey, F Gesztesy, G Levitina, D Potapov, F Sukochev and D Zanin:

Proposition 2.10. *Let* $\mathcal{D}_\mathbb{T}$ *denote the differentiation operator on the circle, and let* $\mathcal{D}_\mathbb{R}$ *denote differentiation on the line. Then* $\operatorname{sgn}(\mathcal{D}_\mathbb{T})$ *and* $\operatorname{sgn}(\mathcal{D}_\mathbb{R})$ *are unitarily equivalent, with the equivalence being given by operator* U*.*

Proof. It follows from "Lemma about the image of H^2" in [13, p. 253] and the Paley-Wiener theorem in [13, p.254] that U maps the Hardy space $H^2(\mathbb{T})$ into the Hardy space $H^2(\mathbb{R}) := \mathcal{F}^{-1}(\chi_{[0,\infty)}L^2(\mathbb{R}))$ where $\mathcal{F}$ is the Fourier transform.

One notes that $\operatorname{sgn}(\mathcal{D}_\mathbb{T}) = 2\mathbf{P}_+ - \mathbf{1}$ and $\operatorname{sgn}(\mathcal{D}_\mathbb{R}) = 2\mathbb{P}_+ - \mathbf{1}$, where $\mathbb{P}_+$ is the projection onto $H^2(\mathbb{R})$.

Hence since $H^2(\mathbb{T})$ and $H^2(\mathbb{R})$ are subspaces mapped into each other by the unitary U, we conclude that the projections $\mathbf{P}_+$ and $\mathbb{P}$ are unitarily equivalent by U. □

Remark 2.11. Let $f \in L^1(\mathbb{T})$. Then M_f is a linear operator on $L^2(\mathbb{T})$ (potentially not defined on all of $L^2(\mathbb{T})$. M_{Uf} is a linear operator on $L^2(\mathbb{T})$.

Proposition 2.12. *Let* $f \in L^1(\mathbb{T})$*. Then* $UM_fU^{-1} = M_{f\circ\omega^{-1}}$*.*

Proof. Let $h \in L^2(\mathbb{R})$, and $x \in \mathbb{R}$. Then,

$$\begin{aligned}(UM_fU^{-1}h)(x) &= UM_f(\sqrt{\pi}(\omega(\zeta)+i)(h\circ\omega)(\zeta)) && (2.31)\\ &= U(f(\zeta)(\sqrt{pi}(\omega(\zeta)+i)(h\circ\omega)(\zeta)) && (2.32)\\ &= h(x)(f\circ\omega^{-1}(x)) && (2.33)\\ &= M_{f\circ\omega^{-1}}h. && (2.34)\end{aligned}$$

□

Proposition 2.13. *Let* $\varphi \in L^1(\mathbb{T})$*. Then* $U đ\varphi U^{-1} = đ(\varphi\circ\omega^{-1})$*.*

Similarly, if $f \in L^1(\mathbb{R})$*, then* $đf = Uđ(f\circ\omega)U^{-1}$*.*

Proof. This follows from Propositions 2.10 and 2.12. □

Chapter 3

Properties of Hankel Operators

3.1 Definition of a Hankel matrix

A Hankel matrix is an infinite matrix $\{M_{j,k}\}_{j,k\geq 0}$ whose $(j,k)th$ entry depends only on $j+k$. If $a = \{a_j\}_{j\geq 0}$, Let $M_a = \{a_{j+k}\}_{j,k\geq 0}$ be the Hankel matrix with (j,k)th entry a_{j+k}. That is,

$$M_a = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_1 & a_2 & a_3 & a_4 & \cdots \\ a_2 & a_3 & a_4 & a_5 & \cdots \\ a_3 & a_4 & a_5 & a_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{3.1}$$

An infinite matrix does not necessarily define an operator on $\ell^2(\mathbb{N})$, however any infinite matrix can be identified with a linear operator on the dense subset $c_{00}(\mathbb{N}) \subset \ell^2(\mathbb{N})$ of sequences of finite support.

For a sequence $a \in c_{00}(\mathbb{N})$, and an infinite matrix $M = (M_{j,k})_{j,k\geq 0}$, we define $Ma \in c_{00}(\mathbb{N})$ as $Ma = \{\sum_{k=0}^{\infty} M_{j,k}a_k\}_{j=0}^{\infty}$.

Hence we shall interchangeably talk about infinite matrices and linear operators $c_{00}(\mathbb{N}) \to c_{00}(\mathbb{N})$.

Since $c_{00}(\mathbb{N})$ is dense in $\ell^2(\mathbb{N})$, if an infinite matrix considered as an operator on $c_{00}(\mathbb{N})$ is bounded on $c_{00}(\mathbb{N})$ in the ℓ^2-norm, then the matrix extends uniquely to an operator on $\ell^2(\mathbb{N})$. Conversely, any operator on $c_{00}(\mathbb{N})$ which extends to a bounded operator on $\ell^2(\mathbb{N})$ is bounded on $c_{00}(\mathbb{N})$ in the ℓ^2-norm, and the extension to $\ell^2(\mathbb{N})$ is unique.

Denote the inner product on $\ell^2(\mathbb{N})$ as $(a,b) := \sum_{n=0}^{\infty} \overline{a_n} b_n$, which we note is linear in the second argument.

We shall be chiefly concerned with determining conditions on the sequence $\{a_n\}_{n=0}^{\infty}$ such that the associated Hankel matrix $\{a_{j+k}\}_{j,k\geq 0}$ lies in some ideal of compact operators. According to the "Calkin correspondence" philosophy, (explained in detail in Appendix E), an ideal of compact operators can be determined by an associated rate of decay of singular values.

For a (complex separable) Hilbert space $\mathcal{H}$, and an operator $T \in \mathcal{B}(\mathcal{H})$, we say that $T \in \mathcal{L}^{p,q}$ if $\{\mu_n\}_{n=0}^{\infty} \in \ell^{p,q}(\mathbb{N})$. The main goal of this chapter is to determine sufficient conditions on $\{a_n\}_{n=0}^{\infty}$ such that the infinite matrix $\{a_{j+k}\}_{j,k\geq 0}$ defines an operator on $\ell^2(\mathbb{N})$ that is $\mathcal{L}^{p,q}$.

The presentation here is based on the book *Hankel Operators and their Applications*, by V.V. Peller, [14], but we will only prove the results necessary for a basic introduction to quantised calculus.

See Appendix A for the elementary properties of the Fourier transform.

A *polynomial* on $\mathbb{T}$ is a finite linear combination of the *monomials* $\{z^n\}_{n\in\mathbb{Z}}$. We call the space of polynomials $P(\mathbb{T})$. An *analytic polynomial* is a polynomial consisting only of non-negative powers of z, we denote $P_A(\mathbb{T})$ for the space of analytic polynomials.

It is easy to see that the Fourier transform gives a vector space isomorphism between $P(\mathbb{T})$ and $c_{00}(\mathbb{Z})$, and $P_A(\mathbb{T})$ and $c_{00}(\mathbb{N})$.

3.2 Bounded Hankel operators

It if of interest to determine when a Hankel operator defines a bounded linear operator on $\ell^2(\mathbb{N})$. This is answered completely by the *Nehari theorem*, which we cover now.

Theorem 3.1 (Nehari)**.** *Let $a = \{a_j\}_{j=0}^{\infty}$ be a sequence. Then the associated Hankel matrix M_a defines a bounded linear operator on $\ell^2(\mathbb{N})$ if and only if there exists $\psi \in L^{\infty}(\mathbb{T}, \boldsymbol{m})$ such that*

$$\hat{\psi}(m) = a_m \tag{3.2}$$

for $m \geq 0$.

Proof. Suppose first that there is $\psi \in L^{\infty}(\mathbb{T})$ such that $\hat{\psi}(m) = a_m$ for $m \geq 0$.

Choose $f, h \in P_A(\mathbb{T})$, so that $\hat{f}, \hat{h} \in c_{00}(\mathbb{N})$.

Let $g \in P_A(\mathbb{T})$ be given by $g = \sum_{n=0}^{\infty} \overline{\hat{g}(n)} z^n$. Let $q = fg$.

Then we compute,

$$(\hat{h}, M_a\hat{f}) = \sum_{j,k\geq 0} \overline{\hat{h}(j)} a_{j+k}\hat{f}(k) \tag{3.3}$$

$$= \sum_{j,k\geq 0} \overline{\hat{h}(j)}\hat{\psi}(j+k)\hat{f}(k) \tag{3.4}$$

$$= \sum_{j\geq 0} \hat{\psi}(j) \sum_{k=0}^{j} \hat{g}(j)\hat{f}(j-k) \tag{3.5}$$

$$= \sum_{j\geq 0} \hat{\psi}(j)\hat{q}(j) \tag{3.6}$$

$$= \int_{\mathbb{T}} \psi(\zeta)q(\overline{\zeta})\, d\boldsymbol{m}(\zeta). \tag{3.7}$$

Hence,

$$|(\hat{h}, M_a\hat{f})| \leq \|\psi\|_\infty \|q\|_1 \tag{3.8}$$

$$\leq \|\psi\|_\infty \|f\|_2 \|h\|_2 \tag{3.9}$$

$$= \|\psi\|_\infty \|\hat{f}\|_2 \|\hat{h}\|_2. \tag{3.10}$$

And thus M_a is bounded on $\ell^2(\mathbb{N})$.

Conversely, suppose that M_a is bounded on $\ell^2(\mathbb{N})$.

Let $\mathcal{L}$ be the linear functional on $P(\mathbb{T})$ defined by

$$\mathcal{L}(q) := \sum_{n\geq 0} a_n\hat{q}(n). \tag{3.11}$$

If $a \in \ell^1(\mathbb{N})$, then $\mathcal{L}$ is bounded on $H^1(\mathbb{T})$, since the inverse Fourier transform of a is in $L^\infty(\mathbb{T})$. Now let us prove in this case that $\|\mathcal{L}\| \leq \|M_a\|$.

Let $q \in H^1(\mathbb{T})$, with $\|q\|_1 \leq 1$. Then $q = fg$ for some $f, g \in H^2(\mathbb{T})$ with $\|f\|_2, \|g\|_2 \leq 1$.

Then we can compute,

$$|\mathcal{L}(q)| = \left|\sum_{n=0}^{\infty} a_n \sum_{m=0}^{n} \hat{f}(m)\hat{g}(n-m)\right| \tag{3.12}$$

$$= \sum_{n,m\geq 0} a_{n+m}\hat{f}(n)\hat{g}(m) \tag{3.13}$$

$$= (M_a\hat{f}, \overline{\hat{g}}). \tag{3.14}$$

And hence,

$$|\mathcal{L}(q)| \leq \|M_a\|\|f\|_2\|g\|_2 \leq \|M_a\|. \tag{3.15}$$

So $\mathcal{L}(q)$ is bounded on $H^1(\mathbb{T})$ whenever $a \in \ell^1(\mathbb{N})$.

Now we consider a to be an arbitrary sequence such that M_a is bounded. Let $r \in (0,1)$ and

$$a^{(r)} = \{r^j a_j\}_{j\geq 0}. \tag{3.16}$$

Then $a^{(r)} \in \ell^1(\mathbb{N})$,

Now we can see that $M_{a^{(r)}} = D_r M_a D_r$, where D_r is multiplication by the sequence $\{r^j\}_{j\geq 0}$. Since $\|D_r\| \leq 1$, we must have $\|M_{a^{(r)}}\| \leq \|M_a\|$. Since $a^{(r)} \in \ell^1(\mathbb{N})$, we have that the linear functional

$$\mathcal{L}_r(q) := \sum_{n=0}^{\infty} a_n r^n \hat{q}(n) \tag{3.17}$$

is bounded on $H^1(\mathbb{T})$, and the functionals $\{\mathcal{L}_r\}_{r\in(0,1)}$ converge strongly to $\mathcal{L}$, and are uniformly bounded. Hence $\mathcal{L}$ is continuous on $H^1(\mathbb{T})$.

Now by the Hahn-Banach theorem, since $\mathcal{L}$ is a linear functional on the subspace $H^1(\mathbb{T})$ which has norm bounded by $\|M_a\|$, it is the restriction of a linear functional on $L^1(\mathbb{T})$, with norm bounded by $\|M_a\|$. Hence $\mathcal{L}(q) = (\psi, q)$ for some $\psi \in L^\infty(\mathbb{T})$ with $\|\psi\|_\infty \leq \|M_a\|$. This proves the result. $\square$

The key idea of this theorem is that it relates the sequence defining a Hankel operator with a function on $\mathbb{T}$. We can state this in a slightly more elegant way using the following result of Fefferman; which can be found in the book *Bounded Analytic functions* [15, Cor 4.5, p.240];

Proposition 3.2. *The space* BMO($\mathbb{T}$) *is defined as the set of measurable functions f on $\mathbb{T}$ (modulo almost-everywhere equivalence) such that*

$$\sup_I \int_I \frac{1}{\boldsymbol{m}(I)} \left| f - \frac{1}{\boldsymbol{m}(I)} \int_I f \, d\boldsymbol{m} \right| \, d\boldsymbol{m} < \infty \tag{3.18}$$

where I is taken over all arcs in $\mathbb{T}$.

BMO($\mathbb{T}$) *is a Banach space when equipped with the norm,*

$$\|f\|_{\mathrm{BMO}(\mathbb{T})} = \sup_I \frac{1}{\boldsymbol{m}(I)} \int_I \left| f - \frac{1}{\boldsymbol{m}(I)} \int_I f \, d\boldsymbol{m} \right| \, d\boldsymbol{m} + |\hat{f}(0)| \tag{3.19}$$

Then

$$\mathrm{BMO}(\mathbb{T}) = L^\infty(\mathbb{T}) + \mathbf{P}_+ L^\infty(\mathbb{T}) = \{f + \mathbf{P}_+ g \ : \ f, g \in L^\infty(\mathbb{T})\}. \tag{3.20}$$

Using this description of BMO($\mathbb{T}$), we can prove the following, which is an equivalent statement of the Nehari theorem, 3.1.

Corollary 3.3. *Let $a = \{a_j\}_{j=0}^\infty$ be a sequence. Then M_a defines a bounded operator on $\ell^2(\mathbb{N})$ if and only if*

$$\varphi := \sum_{n=0}^{\infty} a_n z^n \in \mathrm{BMO}(\mathbb{T}) \cap H^1(\mathbb{T}). \tag{3.21}$$

Proof. By theorem 3.1, M_a is bounded if and only if $\varphi = \mathbf{P}_+\psi$ for some $\psi \in L^\infty(\mathbb{T})$. Hence if M_a is bounded, then $\varphi \in \mathrm{BMO}(\mathbb{T}) \cap H^1(\mathbb{T})$.

Conversely, if $\varphi \in \mathrm{BMO}(\mathbb{T}) \cap H^1(\mathbb{T})$, then $\varphi = f + \mathbf{P}_+ g$ for $f, g \in L^\infty$. Thus $\varphi = \mathbf{P}_+\varphi = \mathbf{P}_+(f+g)$, so M_a is bounded. $\square$

From now on, we are no longer interested in Hankel matrices M_a defined by an arbitrary sequence a, we are only interested in those matrices M_a such that a arises from the Fourier transform of a function.

Definition 3.4. Let $\varphi \in H^1(\mathbb{T})$. Let Γ_φ be the Hankel matrix with (i,j)th entry $\hat{\varphi}(i+j)$.

The properties of Hankel operators, as we shall now demonstrate in part, can be summarised in the following table:

φ	Γ_φ
Rational	Finite rank
VMO	Compact
$B_{pp}^{1/p}$	$\mathcal{L}^p$
BMO	Bounded

The left hand column of the table is a list of function spaces, and the right hand side is a list of operator spaces. This is a table of necessary and sufficient conditions on φ so that Γ_φ lies in some operator space.

At the moment not all of the terms in the table have been defined. The Besov classes $B_{pp}^{1/p}$ will be introduced in Definition 3.12, and the class VMO is defined in Proposition 3.8.

We will not prove every relation implied by the above table. In particular, we will only prove that $\varphi \in \mathrm{VMO}(\mathbb{T})$ implies that Γ_φ is compact, and that $\varphi \in B_{pp}^{1/p}$ implies that $\Gamma_\varphi \in \mathcal{L}^p$. Using these sufficient conditions, we will be able to find sufficient conditions on φ so that Γ_φ is in $\mathcal{L}^{p,q}$ for $p \in [1,\infty]$ and $q \in (0,\infty]$.

More detailed results are proved in the book by V.V. Peller, [14], where both the necessity and sufficiency of all the results in the above table are proved.

3.3 Finite Rank Hankel operators

The strongest condition that we can put on a Hankel matrix Γ_φ is that it is a finite rank operator on $\ell^2(\mathbb{N})$. The problem of determining φ such that Γ_φ is finite rank was solved by Kronecker, as follows.

Theorem 3.5 (Kronecker)**.** *Let $\varphi \in H^1(\mathbb{T})$, and Γ_φ be the associated Hankel matrix. Then Γ_φ defines a finite rank operator on $\ell^2(\mathbb{N})$ if and only if φ is a rational function.*

Proof. Suppose that $\operatorname{rank}(\Gamma_\varphi) = n$. Then the first $n+1$ columns of the matrix of Γ_φ are linearly dependent. Let B denote the backward shift operator, $B(a_0, a_1, \ldots) := (a_1, a_2, \ldots)$ and let F be the forward shift operator, $F(a_0, a_1, \ldots) := (0, a_0, a_1, \ldots)$. Let $a = \hat{\varphi}$.

Hence there exist complex scalars $\{c_0, c_1, \ldots, c_n\}$ not all equal to zero such that

$$c_0 a + c_1 B a + \cdots + c_n B^n a = 0. \tag{3.22}$$

Now let $n, k \geq 0$. It is elementary that

$$F^n B^k a = F^{n-k} a - F^{n-k}(a_0, a_1, \ldots, a_{k-1}, 0, 0, \ldots). \tag{3.23}$$

So hence we have,

$$0 = F^n \sum_{k=0}^{n} c_k B^k a \tag{3.24}$$

$$= \sum_{k=0}^{n} c_k F^n B^k a \tag{3.25}$$

$$= \sum_{k=0}^{n} c_k F^{n-k} a - p \tag{3.26}$$

where p is a finitely supported sequence.

Let $q = (c_n, c_{n-1}, \ldots, c_0, 0, 0, \ldots)$. Then we have,

$$0 = q * a - p. \tag{3.27}$$

Where the $*$ is convolution. Therefore, if we take the inverse Fourier transform, denotes $\check{a} = \sum_{n\geq 0} a_n z^n$ and $\check{p} = \sum_{n \geq 0} p_n z^n$, (the operation $s \mapsto \check{s}$ being the inverse Fourier transform), we get,

$$\varphi \check{a} = \check{p}. \tag{3.28}$$

And hence φ is a quotient of two polynomials.

Conversely, suppose that φ is a rational function. Suppose that $\varphi = p/q$, where $p, q \in P(\mathbb{T})$. Let $n = \max\{\deg p, \deg q\}$. If

$$q = \sum_{k=0}^{n} c_{n-k} z^k \tag{3.29}$$

then since $\varphi q = p$, we have

$$\sum_{k=0}^{n} c_k F^{n-k} a = p. \tag{3.30}$$

Now multiply by B^n,

$$B^n \sum_{k=0}^{n} c_k F^{n-k} a = \sum_{k=0}^{n} c_k B^k a \tag{3.31}$$

$$= 0. \tag{3.32}$$

Let $m \leq n$ be the largest number for which $c_m \neq 0$. Then $B^m a$ is a linear combination of the $B^k a$ with $k \leq m-1$,

$$B^m a = \sum_{k=0}^{m-1} d_k B^k a \tag{3.33}$$

for some coefficients d_k.

We now proceed by induction to show that any row is a linear combination of the first n rows.

Let $k > m$. Then we have,

$$B^k a = B^{k-m} B^m a \tag{3.34}$$

$$= \sum_{j=0}^{m-1} d_j B^{k-m+1} a. \tag{3.35}$$

Since $k - m + j < k$, we have that the terms on the right hand side are linear combinations of the first m rows by the inductive hypothesis. Hence $\text{rank}(\Gamma_\varphi) \leq m$. □

3.4 Compactness of Hankel Operators

If $\varphi \in L^1(\mathbb{T})$, we are interested in conditions on φ such that Γ_φ is compact.

Our first result shows that Hankel matrices are continuous in their symbol.

Proposition 3.6. *Let $\varphi \in L^\infty(\mathbb{T})$, then*

$$\|\Gamma_\varphi\| \leq \|\varphi\|_\infty. \tag{3.36}$$

Proof. It was shown in the proof of theorem 3.1 that if g, f are sequences of finite support, then

$$|(g, \Gamma_\varphi f)| \leq \|\varphi\|_\infty \|g\|_2 \|f\|_2. \tag{3.37}$$

Hence $\|\Gamma_\varphi\| \leq \|\varphi\|_\infty$. □

Corollary 3.7. *If $\varphi \in C(\mathbb{T})$, then Γ_φ is compact.*

Proof. Note that Γ_φ is finite rank for φ a polynomial, and polynomials are dense in $C(\mathbb{T})$ with the $\|\cdot\|_\infty$ norm. Since we have $\|\Gamma_\varphi\| \leq \|\varphi\|_\infty$, the map $\varphi \mapsto \Gamma_\varphi$ is continuous from

$C(\mathbb{T})$ to $\mathcal{B}(\ell^2(\mathbb{N}))$. Since the closure of the finite rank operators in $\mathcal{B}(\mathcal{H})$ is the set of compact operators, we conclude that for $\varphi \in C(\mathbb{T})$, Γ_φ is compact. □

To complete our characterisation of compact Hankel operators, we require the following result of Fefferman, found in [15]:

Proposition 3.8. *The class* VMO($\mathbb{T}$) *is the set of measurable functions* $f : \mathbb{T} \to \mathbb{C}$ *such that*

$$\lim_{\boldsymbol{m}(I)\to 0} \frac{1}{\boldsymbol{m}(I)} \int_I \left| f - \frac{1}{\boldsymbol{m}(I)} \int_I f \, d\boldsymbol{m} \right| \, d\boldsymbol{m} = 0 \tag{3.38}$$

where the limit is over all arcs $I \subseteq \mathbb{T}$.

VMO($\mathbb{T}$) *is alternatively viewed as the closure of the trigonometric polynomials in* BMO($\mathbb{T}$). *It is a result of Fefferman [15] that,*

$$\mathrm{VMO}(\mathbb{T}) = C(\mathbb{T}) + \mathbf{P}_+ C(\mathbb{T}). \tag{3.39}$$

So we have the following:

Proposition 3.9. *Let* $\varphi \in \mathrm{VMO}(\mathbb{T})$. *Then* Γ_φ *is compact.*

Proof. if $\varphi \in \mathrm{VMO}(\mathbb{T})$, then by proposition 3.8, $\varphi = f + \mathbf{P}_+ g$ for $f, g \in C(\mathbb{T})$. Hence, $\mathbf{P}_+\varphi = \mathbf{P}_+(f + g)$. Hence, there exists $h \in C(\mathbb{T})$ with $\Gamma_\varphi = \Gamma_h$, simply by choosing $h = f + g$. This is because Γ_φ depends only on the Fourier coefficients $\hat{\varphi}(n)$ for $n \geq 0$, so $\Gamma_\varphi = \Gamma_{\mathbf{P}_+\varphi}$.

Hence by corollary 3.7, we see that Γ_φ is compact. □

3.5 Hankel Operators of Trace Class

We recall the definition of the $\mathcal{L}^1$ norm on $\mathcal{B}(\mathcal{H})$,

Definition 3.10. Let $\mathcal{H}$ be a separable Hilbert space, and let $T \in \mathcal{B}(\mathcal{H})$. For a non-negative integer n, we define the nth singular value,

$$\mu_n(T) := \inf\{\|T - F\| \ : \ F \in \mathcal{B}(\mathcal{H}), \mathrm{rank}(F) \leq n\}. \tag{3.40}$$

For $p \in (0, \infty)$, we define the $\mathcal{L}^p$ norm of T as

$$\|T\|_p = \left(\sum_{n=0}^{\infty} |\mu_n(T)|^p \right)^{1/p} \tag{3.41}$$

with the convention that $\|T\|_p = \infty$ if the sum does not converge. The space $\mathcal{L}^p$ is the set of $T \in \mathcal{B}(\mathcal{H})$ such that $\|T\|_p < \infty$.

In particular we are interested in the case $p = 1$. We are interested in finding conditions on a function φ holomorphic in the unit disc such that Γ_φ is in $\mathcal{L}^1$.

Lemma 3.11. *Let $\mathcal{H}$ be a separable Hilbert space. Suppose that $x, y \in \mathcal{H}$, and let T be the rank one operator defined by $T(\zeta) = \langle x, \zeta\rangle y$. Then $\|T\|_1 = \|x\|\|y\|$.*

The answer is provided by the *Besov classes.* Many different definitions of Besov spaces can be found, but the one of most relevance to us is given below.

Definition 3.12. We define a sequence of polynomials $\{W_n\}_{n\in\mathbb{Z}} \subset P(\mathbb{T})$ as follows. First,

$$W_0 = z^{-1} + 1 + z. \tag{3.42}$$

And now for $n > 0$, we define W_n by asserting that $\widehat{W}(2^n) = 1$, $\widehat{W}(2^{n-1}) = 0$, $\widehat{W}(2^{n+1}) = 0$, and $\widehat{W}$ is a linear increasing function between 2^{n-1} and 2^n, and a linear decreasing function between 2^n and 2^{n+1}. We assert that $\widehat{W}(n)$ is symmetric in n, and is zero for all values not already defined.

Now, for $p, q > 0$, and $s \geq 0$, we define the *Besov class* $B^s_{pq}(\mathbb{T})$ to be the space of distributions f on $\mathbb{T}$ such that

$$\sum_{n\geq 0} 2^{nsq}\|W_n * f\|_p^q < \infty. \tag{3.43}$$

We shall denote B^s_{pp} as B^s_p.

In particular, we are going to prove that if $\varphi \in B^1_1(\mathbb{T})$, then $\Gamma_\varphi \in \mathcal{L}^1$. First, we need a lemma.

Lemma 3.13. *Let $f \in P_A(\mathbb{T})$ be an analytic polynomial of degree at most m. Then,*

$$\|\Gamma_f\|_1 \leq (m+1)\|f\|_1. \tag{3.44}$$

Proof. Let $\zeta \in \mathbb{T}$, now define the following elements of $\ell^2(\mathbb{N})$,

$$x_\zeta(j) = \begin{cases} \zeta^j, \ 0 \leq j \leq m, \\ 0, \ j > m \end{cases} \tag{3.45}$$

$$y_\zeta(j) = \begin{cases} f(\zeta)\zeta^{-k}, \ 0 \leq k \leq m, \\ 0, \ k > m. \end{cases} \tag{3.46}$$

That is, $x_\zeta = (1, \zeta, \zeta^2, \ldots, \zeta^m, 0, 0, \ldots)$ and $y_\zeta = f(\zeta)\overline{x_\zeta}$.

Let A_ζ be the rank one operator, $A_\zeta(x) = (x_\zeta, x)y_\zeta$, so that $\|A_\zeta\|_1 = \|x_\zeta\|_2\|y_\zeta\|2 = (m + 1)|f(\zeta)|$

Then we have a component-wise equality of infinite matrices,

$$\Gamma_f = \int_{\mathbb{T}} A_\zeta \, d\boldsymbol{m}(\zeta). \tag{3.47}$$

Hence, $\|\Gamma_f\|_1 \leq (m+1)\|f\|_1$ by the triangle inequality. $\square$

Theorem 3.14. *Let $\varphi \in B_1^1$. Then $\Gamma_\varphi \in \mathcal{L}^1$.*

Proof. We have the following L^∞-convergent sequence,

$$\varphi = \sum_{n \geq 0} W_n * \varphi. \tag{3.48}$$

Hence,

$$\Gamma_\varphi = \sum_{n \geq 0} \Gamma_{W_n * \varphi}. \tag{3.49}$$

So since the degree of W_n is 2^{n+1}, we have

$$\|\Gamma_\varphi\|_1 \leq \sum_{n \geq 0} 2^{n+1} \|W_n * \varphi\|_1. \tag{3.50}$$

$\square$

This proves the sufficiency of the condition $\varphi \in B_1^1$ so that Γ_φ is trace class. The proof of the necessity of this condition is more difficult,

Theorem 3.15. *Let φ be a function holomorphic in the unit disc. Then if $\Gamma_\varphi \in \mathcal{L}^1$, then $\varphi \in B_1^1$.*

Proof. Define a pair of sequences of polynomials $\{Q_n\}_{n=0}^\infty$ as follows,

$$\widehat{Q}_n(k) = \begin{cases} 0, \ k \leq 2^{n-1} \\ 1 - \frac{|k-2^n|}{2^{n-1}}, \ 2^{n-1} \leq k \leq 2^n + 2^{n-1}, \\ 0 \ k \geq 2^n + 2^{n-1}. \end{cases} \tag{3.51}$$

and a sequence $\{R_n\}_{n=0}^\infty$,

$$\widehat{R}_n(k) = \begin{cases} 0, \ k \leq 2^n \\ 1 - \frac{|k-2^n-2^{n-1}|}{2^{n-1}}, \ 2^n \leq 2^{n+1} \\ 0, \ k \geq 2^{n+1}. \end{cases} \tag{3.52}$$

This is a decomposition of the sequence $\{W_n\}_{n=0}^\infty$, given by $W_n = Q_n + \frac{1}{2}R_n$.

First we prove that

$$\sum_{n\geq 0} 2^{2n+1}\|Q_{2n+1}*\varphi\|_1 < \infty. \tag{3.53}$$

To this end, we wish to construct an operator B such that

$$\langle \Gamma_\varphi, B\rangle = \sum_{n\geq 0} 2^{2n}\|Q_{2n+1}*\varphi\|_1. \tag{3.54}$$

Now define the sequence of squares, for $n \geq 1$,

$$S_n = [2^{2n-1}, 2^{2n-1}+2^{2n}-1] \times [2^{2n-1}+1, 2^{2n-1}+2^{2n}]. \tag{3.55}$$

Note that this sequence is pairwise disjoint.

Let $\{\psi_n\}_{n=0}^{\infty}$ be a sequence in $L^\infty(\mathbb{T})$, yet to be defined with $\|\psi_n\|_\infty \leq 1$. Now we define the matrix $B = \{B_{j,k}\}_{j,k\geq 0}$ by

$$B_{j,k} = \begin{cases} \widehat{\psi}_n(j+k), (j,k) \in S_n, n \geq 1, \\ 0, (j,k) \notin \bigcup_{n\geq 1} S_n. \end{cases} \tag{3.56}$$

We wish to prove that B is bounded, and in fact $\|B\| \leq 1$. Let $\{e_n\}_{n\geq 0}$ be the standard basis for $\ell^2(\mathbb{N})$, with $e_n(m) = \delta_{n,m}$. Define the subspaces

$$\mathcal{H}_n = \{e_j \,:\, 2^{2n-1} \leq j \leq 2^{2n-1}+2^{2n}-1\}, \tag{3.57}$$
$$\mathcal{H}'_n = \{e_j \,:\, 2^{2n-1}+1 \leq j \leq 2^{2n-1}+2^{2n}\}. \tag{3.58}$$

Let P_n and P'_n be the orthogonal projection onto $\mathcal{H}_n$ and $\mathcal{H}'_n$ respectively. So that

$$B = \sum_{n\geq 1} P'_n \Gamma_{\psi_n} P_n, \tag{3.59}$$

where P_n and P'_n are the orthogonal projections onto $\mathcal{H}_n$ and $\mathcal{H}'_n$ respectively.

Now since the spaces $\{\mathcal{H}_n\}_{n\geq 1}$ are pairwise orthogonal, as are the spaces $\{\mathcal{H}'_n\}_{n\geq 1}$, we have

$$\begin{aligned} \|B\| &\leq \sup_{n\geq 1} \|P'_n \Gamma_{\psi_n} P_n\| && (3.60)\\ &\leq \sup_n \|\Gamma_{\psi_n}\| && (3.61)\\ &\leq \sup_n \|\psi_n\|_\infty && (3.62)\\ &\leq 1. && (3.63) \end{aligned}$$

Now, we compute

$$\langle \Gamma_\varphi, B\rangle = \sum_{n\geq 1}\langle \Gamma_\varphi, P_n'\Gamma_{\psi_n}P_n\rangle \tag{3.64}$$

$$= \sum_{n\geq 1}\sum_{j=2^{2n}}^{2^{2n}+2^{2n+1}} (2^{2n} - |j - 2^{2n+1}|)\overline{\widehat{\varphi}}(j)\widehat{\psi_n}(j) \tag{3.65}$$

$$= \sum_{n\geq 1} 2^{2n}(Q_{2n+1} * \varphi, \psi_n). \tag{3.66}$$

Now, using the sharpness of Hölder's inequality, we can choose a sequence $\{\psi_n\}_{n=0}^{\infty}$ so that $\langle Q_{2n+1} * \varphi, \psi_n\rangle$ is arbitrarily close to $\|Q_{2n+1} * \varphi\|_1$. Hence,

$$\sum_{n\geq 1} 2^{2n+1}\|Q_{2n+1} * \varphi\|_1 = 2\langle \Gamma_\varphi, B\rangle \leq 2\|\Gamma_\varphi\|_1. \tag{3.67}$$

In exactly the same way, we may prove that

$$\sum_{n\geq 1} 2^{2n}\|Q_{2n} * \varphi\|_1 < \infty \tag{3.68}$$

that

$$\sum_{n\geq 0} 2^{2n+1}\|R_{2n+1} * \varphi\|_1 < \infty \tag{3.69}$$

and

$$\sum_{n\geq 1} 2^{2n}\|R_{2n} * \varphi\|_1 < \infty \tag{3.70}$$

and therefore that $\varphi \in B_1^1$. □

Remark 3.16. We thus conclude that

$$\frac{1}{6}\sum_{n\geq 1} 2^n\|W_n * \varphi\|_1 \leq \|\Gamma_\varphi\|_1 \leq 2\sum_{n\geq 0} 2^n\|W_n * \varphi\|_1. \tag{3.71}$$

3.6 Interpolation

For this section, $\mathcal{L}^\infty$ denotes the space of compact operators.

Up to now we have proved necessary and sufficient conditions on φ so that Γ_φ lies in a certain ideal one at a time. In particular, we have with great effort proved that $\Gamma_\varphi \in \mathcal{L}^1$ if and only if $\varphi \in B_{11}^1(\mathbb{T})$. This amounts to a continuous injection:

$$B_{11}^1(\mathbb{T}) \to \mathcal{L}^1 \tag{3.72}$$

with equivalence of norms. We denote $\Gamma\mathcal{L}^\infty$ as the class of compact Hankel operators.

We also know that we have an injection

$$\mathrm{VMO}(\mathbb{T}) \to \mathcal{L}^{\infty}. \tag{3.73}$$

It is a remarkable fact that using the embeddings 3.72 and 3.73 alone we can find sufficient conditions on φ so that $\Gamma_\varphi \in \mathcal{L}^{p,q}$ for $p \in [1,\infty]$,$q \in (0,\infty]$.

We accomplish this via interpolation theory. The basic material is covered in Appendix C. The pair $(B^1_{11}(\mathbb{T}), \mathrm{VMO}(\mathbb{T}))$. The map $\varphi \mapsto \Gamma_\varphi$ takes the pair $(B^1_{11}(\mathbb{T}), \mathrm{VMO}(\mathbb{T}))$ to $(\mathcal{L}^1, \mathcal{L}^\infty)$. Hence for any interpolation functor $F : \mathbf{NLS}_1 \to \mathbf{NLS}$ (using the notation of Appendix C, we have that if $\varphi \in F(B^1_{11}(\mathbb{T}), \mathrm{VMO}(\mathbb{T}))$, then $\Gamma_\varphi \in F(\mathcal{L}^1, \mathcal{L}^\infty)$.

In order to therefore find sufficient conditions on φ so that $\Gamma_\varphi \in \mathcal{L}^{p,q}$, we simply need to find an interpolation functor $F_{p,q}$ such that $F_{p,q}(\mathcal{L}^1, \mathcal{L}^\infty) = \mathcal{L}^{p,q}$.

It turns out that there is indeed such a functor, the K functor from real interpolation theory. For a definition see Appendix C. We can choose $F_{p,q} = K(\cdot,\cdot)_{\theta,q}$ where $\theta = 1 - p^{-1}$. This is due to the following result:

Proposition 3.17. *Let $p \in (0,\infty)$ and $q \in (0,\infty]$. If $\theta = 1-p^{-1}$, then $K(\mathcal{L}^1, \mathcal{L}^\infty)_{\theta,q} = \mathcal{L}^{p,q}$.*

Proof. We include only a sketch proof. The details of this interpolation result are included in [16].

The key estimate is as follows. Let $T \in \mathcal{L}^1 + \mathcal{L}^\infty = \mathcal{L}^\infty$. Using a result from G.E. Karadzhov [17], the K functional, $K(t, T, \mathcal{L}^1, \mathcal{L}^\infty)$, given in definition C.6, can be estimated by

$$c_1 \left(\sum_{j=0}^{n} \mu_j(T)^p \right)^{1/p} \leq K(t, T, \mathcal{L}^1, \mathcal{L}^\infty) \leq c_2 \left(\sum_{j=0}^{n} \mu_j(T)^p \right)^{1/p}, \tag{3.74}$$

for $n^{1/p} \leq t \leq (n+1)^{1/p}$, for some positive constants c_1, c_2. This estimate can be found in [17]. $\square$

So we have shown that if $\varphi \in K(B^1_{11}(\mathbb{T}), \mathrm{VMO}(\mathbb{T}))_{1-1/p,q}$, then $\Gamma_\varphi \in \mathcal{L}^{p,q}$. Peller [14, Theorem 4.4, p. 256] shows that this condition is not just sufficient but also necessary. Of particular interest is the case $p = q$. This is solved by the following result, which is [14, Theorem 4.3, p.255]:

Proposition 3.18. *Let $p \in (1,\infty)$. Then $K(B^1_{11}(\mathbb{T}), \mathrm{VMO}(\mathbb{T})) = B^{1/p}_{pp}(\mathbb{T})$.*

3.7 Extrapolation

We have determined sufficient conditions for $\Gamma_\varphi \in \mathcal{L}^{p,q}$, for $p \in [1,\infty]$ and $q \in (0,\infty]$. We now turn our attention to more exotic ideals of $\mathcal{B}(\ell^2(\mathbb{N}))$. The Macaev-Dixmier ideal $\mathcal{M}_{1,\infty}$ is described in detail Appendix E.

Most important for our purposes is the equivalence of norms, proved in Theorem 4.5 of [18],

$$\|x\|_{\mathcal{M}_{1,\infty}} = \limsup_{s\downarrow 0} s\|x\|_{\mathcal{L}_{s+1}}. \tag{3.75}$$

First we need the following result, proved in the proof of [14, Theorem 3.1,p.250].

Proposition 3.19. *Let $\varphi \in H^1(\mathbb{T})$, and let $p \in (1,2)$. Then there is a constant C not depending on p such that*

$$\|\Gamma_\varphi\|_{\mathcal{L}^p}^p \le 2^{1-p} \sum_{n\ge 0} 2^n \|W_n * \varphi\|_p^p \tag{3.76}$$

Where the right hand side is defined to be $\|\varphi\|_{B^{1/p}_{pp}}$.

Hence, we have

Proposition 3.20. *Let φ be a function on $\mathbb{T}$ such that $\limsup_{s\downarrow 0} s\|\varphi\|_{B^{1/(s+1)}_{s+1,s+1}} < \infty$. Then $\Gamma_\varphi \in \mathcal{M}_{1,\infty}$.*

Proof. By Proposition 3.19, we see that

$$\|\Gamma_\varphi\|_{\mathcal{M}_{1,\infty}} = \limsup_{s\downarrow 0} s\|\Gamma_\varphi\|_{s+1} \le \limsup_{s\downarrow 0} s\|\varphi\|_{B^{1/(s+1)}_{s+1,s+1}}. \tag{3.77}$$

□

There is an alternative characterisation of Besov classes on the circle, which we refer to here,

Proposition 3.21. *Let $\zeta, \tau \in \mathbb{T}$. For a function $f : \mathbb{T} \to \mathbb{C}$, define*

$$\Delta_\tau f(\zeta) = f(\tau\zeta) - f(\zeta). \tag{3.78}$$

Then, let $s > 0$, $p, q \in [1,\infty]$ and n be an integer $n > s$,

$$B^s_{pq}(\mathbb{T}) = \left\{ f \in L^p \ : \ \int_{\mathbb{T}} \frac{\|\Delta^n_\tau f\|_p^q}{|1-\tau|^{1+sq}} \, d\boldsymbol{m}(\tau) < \infty \right\} \tag{3.79}$$

and the choice of n is irrelevant.

Chapter 4

Operator Ideal Membership of Quantised Differentials

4.1 Transference from Hankel Operators to Quantised differentials

Let $\varphi \in L^2(\mathbb{T})$. We consider $đ\varphi$ as an operator on $H^2(\mathbb{T}) \oplus H^2_-(\mathbb{T})$. Then we have the description,

$$đ\varphi = 2\begin{pmatrix} 0 & -H^*_{\overline{\varphi_+}} \\ H_{\varphi_-} & 0 \end{pmatrix} \tag{4.1}$$

as proved in lemma 2.2.

Hence, to determine when $đ\varphi$ falls into some ideal of operators, it suffices to check $H_{\overline{\varphi_+}}$ and H_{φ_-}.

4.1.1 Relation between H_φ and Γ_φ

One can compute that H_ψ has matrix representation $\{\psi(-j-k)\}_{j\geq 0, k>0}$.

Hence it is equivalent to study H_ψ and Γ_φ, where $\varphi \in H^1(\mathbb{T})$ and

$$\hat{\varphi}(n) = \hat{\psi}(-n). \tag{4.2}$$

Hence,

$$\varphi = A - \sum_{n\geq 0} \hat{\psi}(-n) z^n \tag{4.3}$$

where the A- means that this is an Abel converging sum, see Appendix A for details. Hence we have that $\varphi \in L^1(\mathbb{T})$, with $\hat{\varphi(n)} = 0$ for $n < 0$ and $\hat{\varphi(n)} = \hat{\psi}(-n)$ for $n \geq 0$.

For $\zeta \in \mathbb{T}$, let $\tilde{\psi}(\zeta) = \psi(\zeta^{-1})$.

Thus, $\varphi = \mathbf{P}_+\tilde{(}\zeta)$.

4.1.2 Bounded Quantised Differentials

The weakest condition that we can place on an operator is that it be bounded. The Nehari theorem 3.1 gives us necessary and sufficient conditions for a quantised differential to be bounded.

Proposition 4.1. *Let $\varphi \in L^1(\mathbb{T})$. Then $\text{đ}\varphi$ defines a bounded linear operator on $L^2(\mathbb{T})$ if and only if $\varphi \in \mathrm{BMO}(\mathbb{T})$.*

Proof. The operator H_f is a Hankel operator, with (j,k)th entry $\hat{f}(-j-k)$, so by theorem 3.1, we have that $\text{đ}\varphi$ is bounded if and only if $\overline{\varphi_-}, \varphi_+ \in \mathrm{BMO}(\mathbb{T})$.

By the description of $\mathrm{BMO}(\mathbb{T})$ as the set of all $\varphi \in L^1(\mathbb{T})$ such that

$$\sup_I \frac{1}{\boldsymbol{m}(I)} \int_I |f - \frac{1}{\boldsymbol{m}(I)} \int_I f \, d\boldsymbol{m}| \, d\boldsymbol{m} < \infty, \tag{4.4}$$

it is clear that $f \in \mathrm{BMO}(\mathbb{T})$ if and only if $\overline{f} \in \mathrm{BMO}(\mathbb{T})$.

By the Fefferman decomposition, given in [15],

$$\mathrm{BMO}(\mathbb{T}) = L^\infty(\mathbb{T}) + \mathbf{P}_+ L^\infty(\mathbb{T}) \tag{4.5}$$

it follows that $f \in \mathrm{BMO}(\mathbb{T})$ if and only if $\mathbf{P}_+ f, \mathbf{P}_- f \in \mathrm{BMO}(\mathbb{T})$.

Thus, $\text{đ}\varphi$ is bounded if and only if $\varphi \in \mathrm{BMO}(\mathbb{T})$. □

4.1.3 Finite Rank Quantised Differentials

On the other hand, the *strongest* condition that one can place on an operator is that it be finite rank. Kronecker's theorem 3.5 gives us conditions for a quantised differential to be finite rank.

Proposition 4.2. *Let $\varphi \in L^1(\mathbb{T})$. Then the operator $\text{đ}\varphi$ on $L^2(\mathbb{T})$ is finite rank if and only if φ is a rational function.*

Proof. By theorem 3.5, it is necessary and sufficient that $\overline{\varphi_+}$ and φ_- are rational functions. Hence it is necessary and sufficient that φ is rational. □

4.1.4 Compact Quantised Differentials

Recall from chapter 1, that we wished to have some justification of the claim that if f is continuous, then $\text{đ}f$ is infinitesimal. This is totally justified by the following proposition:

Proposition 4.3. *If $\varphi \in \mathrm{VMO}(\mathbb{T})$, then $\text{đ}\varphi \in \mathcal{K}(\mathcal{H})$.*

Proof. By equation 4.1, we need to consider operators of the form H_{φ_-} and $H_{\overline{\varphi_+}}$. Hence we have $\text{đ}\varphi$ is compact if $\overline{\varphi_-}$ and φ_+ in $\mathrm{VMO}(\mathbb{T})$. But since $\mathrm{VMO}(\mathbb{T})$ is closed under the image of $\mathbf{P}_+$ and conjugation, we see that this is equivalent to $\varphi \in \mathrm{VMO}(\mathbb{T})$. □

4.1.5 Trace Class Quantised Differentials

To determine which quantised differentials are trace class, we need the following:

Lemma 4.4. *Let $\varphi \in L^1(\mathbb{T})$. Then $\varphi \in K(B^1_{11}(\mathbb{T}), \mathrm{VMO}(\mathbb{T}))_{\theta,q}$ if and only if $\overline{\varphi_+}, \varphi_- \in K(B^1_{11}(\mathbb{T}), \mathrm{VMO}(\mathbb{T}))_{\theta,q}$, See Appendix C, Definition C.6 for details on the K-method.*

Proof. First we check that $K(B^1_{11}(\mathbb{T}), \mathrm{VMO}(\mathbb{T}))_{\theta,q}$ is closed under the conjugation map.

Recall that for a Banach pair (X_0, X_1) with norms $\|\cdot\|_0$ and $\|\cdot\|_1$ respectively, the K-functional is

$$K(x, t, X_0, X_1) = \inf\{\|x_0\|_0 + t\|x_1\|_1 \ : \ x_0 \in X_0, x_1 \in X_1, x = x_0 + x_1\}.$$

Hence it is sufficient to check that $\|\varphi\|_{\mathrm{VMO}} = \|\overline{\varphi}\|_{\mathrm{VMO}}$ and $\|\varphi\|_{B^1_{11}} = \|\overline{\varphi}\|_{B^1_{11}}$.

We see that by proposition 3.8, $\mathrm{VMO}(\mathbb{T})$ is the closure of the rational functions in the $\mathrm{BMO}(\mathbb{T})$ norm. Since the $\mathrm{BMO}(\mathbb{T})$ norm is invariant under conjugation, as discussed in the proof of proposition 4.1, we see that $\|\varphi\|_{\mathrm{VMO}} = \|\overline{\varphi}\|_{\mathrm{VMO}}$.

So prove that $\|\varphi\|_{B^1_{11}} = \|\overline{\varphi}\|_{B^1_{11}}$ we simply observe the equivalent $B^1_{11}(\mathbb{T})$ norm described in proposition 3.21.

Now we must check that $\varphi \in K(B^1_{11}(\mathbb{T}), \mathrm{VMO}(\mathbb{T}))_{\theta,q}$ if and only if both $\varphi_+ := \mathbf{P}_+\varphi$ and $\varphi_- := \mathbf{P}_-\varphi$ are in $K(B^1_{11}(\mathbb{T}), \mathrm{VMO}(\mathbb{T}))_{\theta,q}$.

To prove this, see that if $\varphi \in \mathrm{VMO}(\mathbb{T})$, then by proposition 3.8, there exist continuous functions f and g such that $\varphi = f + \mathbf{P}_+ g$. Hence $\mathbf{P}_+\varphi = \mathbf{P}_+(f + g) \in \mathrm{VMO}(\mathbb{T})$. Conversely, if φ_+ and φ_- are in $\mathrm{VMO}(\mathbb{T})$, then $\varphi = \varphi_+ + \varphi_- \in \mathrm{VMO}(\mathbb{T})$ simply because $\mathrm{VMO}(\mathbb{T})$ is closed under addition.

Now we prove that $\varphi \in B^1_{11}(\mathbb{T})$ if and only if both φ_+ and φ_- are in $B^1_{11}(\mathbb{T})$. It is clear that if φ_+ and φ_- are in $B^1_{11}(\mathbb{T})$ then $\varphi \in B^1_{11}(\mathbb{T})$ since $B^1_{11}(\mathbb{T})$ is a vector space.

Recall that $\varphi \in B^1_{11}(\mathbb{T})$ if and only if

$$\sum_{n \in \mathbb{Z}} 2^{|n|} \|W_n * \varphi\|_1 \tag{4.6}$$

where the polynomials W_n are defined in definition 3.12.

See that $W_n * (\mathbf{P}_+\varphi) = 0$ for $n < 0$, since W_n is a polynomial in negative powers of z for $n > 0$. Also, for $n > 0$ we have $W_n * (\mathbf{P}_+\varphi) = W_n * \varphi$ since W_n is a polynomial in positive powers of z for $n > 0$. Hence,

$$\sum_{n\in\mathbb{Z}} 2^{|n|}\|W_n * (\mathbf{P}_+\varphi)\|_1 = \sum_{n\geq 0} 2^{|n|}\|W_n * \varphi\|_1 \tag{4.7}$$

$$\leq \sum_{n\in\mathbb{Z}} 2^{|n|}\|W_n * \varphi\|_1. \tag{4.8}$$

Hence if $\varphi \in B^1_{11}(\mathbb{T})$, we have that $\varphi_+ \in B^1_{11}(\mathbb{T})$. Since $\varphi_- = \varphi - \varphi_+$, we conclude also that $\varphi_- \in B^1_{11}(\mathbb{T})$. □

So we get the following:

Corollary 4.5. *Let* $\varphi \in K(B^1_{11}(\mathbb{T}), \mathrm{VMO}(\mathbb{T}))_{(1-p)^{-1},q}$. *Then* $đ\varphi \in \mathcal{L}_{p,q}$.

Proof. Since we have $\varphi \in K(B^1_{11}(\mathbb{T}), \mathrm{VMO}(\mathbb{T}))_{\theta,q}$, we conclude that both $\overline{\varphi_+}$ and φ_- are in $K(B^1_{11}(\mathbb{T}), \mathrm{VMO}(\mathbb{T}))_{\theta,q}$ by lemma 4.4. Thus by proposition 3.17, we see that both H_{φ_-} and $H_{\overline{\varphi_+}}$ are in $\mathcal{L}^{p,q}$. Hence, $đ\varphi \in \mathcal{L}^{p,q}$. □

Proposition 4.6. *Let* $\varphi \in L^1(\mathbb{T})$ *be such that* $\limsup_{s\downarrow 1} s\|\varphi\|_{B^{1/(s+1)}_{s+1,s+1}}$ *is finite, Then* $đ\varphi \in \mathcal{M}_{1,\infty}$.

Proof. From the proof of proposition 4.4, we see that $\|\overline{\varphi_+}\|_{B^{1/(s+1)}_{s+1,s+1}} \leq \|\varphi\|_{B^{1/(s+1)}_{s+1,s+1}}$, and similarly for φ_-.

Hence, H_{φ_-} and $H_{\overline{\varphi_+}}$ are in $\mathcal{M}_{1,\infty}$ by proposition 3.20, and consequently $đ\varphi \in \mathcal{M}_{1,\infty}$. □

4.2 Quantised Differentials on $\mathbb{R}$

Using proposition 2.13, we can transfer our results about quantised differentials on $\mathbb{T}$ to differentials on $\mathbb{R}$.

Definition 4.7. The spaces $\mathrm{BMO}(\mathbb{R})$ and $\mathrm{VMO}(\mathbb{R})$ are defined very similarly to the corresponding spaces on $\mathbb{T}$.

In particular, for a function $f \in L^1_{loc}(\mathbb{R})$, for an interval I (sufficiently small so that the integral exists),

$$f_I := \frac{1}{\lambda(I)}\int_I f\, d\lambda. \tag{4.9}$$

(Recall that λ denotes one dimensional Lebesgue measure).

The $\mathrm{BMO}(\mathbb{R})$ semi-norm is

$$\|f\|_{\mathrm{BMO}(\mathbb{R})} = \sup_I \frac{1}{\lambda(I)}\int_I |f - f_I|\, d\lambda. \tag{4.10}$$

The chief importance of the BMO spaces in harmonic analysis lies in their conformal invariance. This is clarified in the following proposition, proved in [15, Cor 1.3, p.129],

Proposition 4.8. *We have* $\varphi \in \mathrm{BMO}(\mathbb{T})$ *if and only if* $\varphi \circ \omega^{-1} \in \mathrm{BMO}(\mathbb{R})$.

Proposition 4.9. *Let* $\varphi \in L^\infty(\mathbb{R})$. *Then* $đ\varphi$ *is compact if and only if* $\varphi \circ \omega^{-1} \in \mathrm{VMO}(\mathbb{T})$.

Using the Fefferman decomposition, this means that there exist functions $g, f \in C(\mathbb{T})$ *so that* $\varphi = g \circ \omega + (\mathbf{P}_+ f) \circ \omega$.

In general, we can say the following:

Proposition 4.10. *Let* $\mathcal{E} \subseteq \mathcal{B}(L^2(\mathbb{R}))$. *The mapping in equation 2.29 is a unitary map from* $L^2(\mathbb{R})$ *to* $L^2(\mathbb{T})$. *For an operator* $E \in \mathcal{E}$, *define* $\mathcal{U}E = UEU^{-1} \in \mathcal{B}(L^2(\mathbb{T}))$.

Denote the image of $\mathcal{E}$ *under* $\mathcal{U}$ *as* $\mathcal{E}_{\mathbb{T}}$.

Suppose that $X \subseteq L^0(\mathbb{T})$ *is such that* $f \in X$ *implies that* $đf \in \mathcal{E}_{\mathbb{T}}$.

For a function $\varphi \in \mathrm{BMO}(\mathbb{R})$, *we have that* $đ\varphi \in \mathcal{E}$ *if* $\varphi \circ \omega^{-1} \in X$.

Proof. This immediately follows from proposition 2.13, which states that $\mathcal{U}đ\varphi = đ(\varphi \circ \omega^{-1})$. $\square$

Chapter 5

Abstract Differential Algebra and Spectral Triples

5.1 Introduction

We now return to the discussion of the origin of quantised calculus. Noncommutative geometry, as described in Chapter 1, is envisaged as the study of pairs $(\mathcal{A}, \mathcal{H})$, where $\mathcal{A}$ is an algebra of operators on a Hilbert space $\mathcal{H}$.

A classical example is $\mathcal{H}_1 = L^2([0,1])$, and $\mathcal{A}_1 = L^\infty([0,1])$, encoded as pointwise multiplication operators. Since we have an isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2 = L^2(\mathbb{T})$, and this induces an algebra isomorphism between $\mathcal{A}_1$ and $\mathcal{A}_2 = L^\infty(\mathbb{T})$.

Therefore at the level of operator algebras, the pair $(\mathcal{A}_1, \mathcal{H}_1)$ is indistinguishable from $(\mathcal{A}_2, \mathcal{H}_2)$. However, topologically, $[0,1]$ is very distinct from $\mathbb{T}$. Therefore, to encode the "geometry" of a noncommutative space $(\mathcal{A}, \mathcal{H})$, we require more information. To attempt to find an solution to this problem, we shall explore how geometry is encoded in algebra in differential geometry.

We refer to Chapter 8 of the book [8] for more details on this topic.

5.2 Classical Differential Algebra

Let M be an n dimensional manifold. For an encyclopaedic reference on differential geometry, see the text [19]. The cotangent bundle $\Omega^1(M)$ is a rank n vector bundle on M. We build higher bundles by wedge products,

$$\Omega^p(M) := \bigwedge_p \Omega^1(M).$$

and define $\Omega^0(M) := C^\infty(M)$. See that $\Omega^p(M)$ is a rank $\binom{n}{p}$ vector bundle.

The exterior algebra bundle is the direct sum of all the $\Omega^p(M)$,

$$\Omega(M) := \bigoplus_{p=0}^{\infty} \Omega^p(M).$$

$\Omega(M)$ is a *graded algebra*.

In general, if A is an algebra over a ring R, we say that A is $\mathbb{N}$-graded if there exists a decomposition into submodules $A^{(p)}$,

$$A = \bigoplus_{p=0}^{\infty} A^{(p)}$$

such that $A^{(n)}A^{(m)} \subseteq A^{(n+m)}$.

In the case $A = \Omega(M)$, we have $R = \mathbb{R}$, and $A^{(p)} = \Omega^p(M)$.

The exterior derivative, $d : \Omega(M) \to \Omega(M)$ acts on the grading by,

$$d : \Omega^p(M) \to \Omega^{p+1}(M).$$

and $d^2 = 0$. Hence we have a sequence,

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \xrightarrow{d} 0.$$

Denote d_p as the restriction of d to $\Omega^p(M)$. Then we have the *de Rham cohomology* spaces,

$$H^p_{dR}(M) = \frac{\ker(d_p)}{\mathrm{im}(d_{p-1})}$$

This is a sequence of real vector spaces, and their dimensions, called Betti numbers, are well known to be topological invariants of M.

The maps d_p satisfy a graded version of Leibniz's rule, for $a \in \Omega^n(M)$ and $b \in \Omega^m(M)$, we have:

$$d_{n+m}(ab) = d_n(a)b + (-1)^n a d_m(b)$$

5.3 Abstract Differential Algebra

5.3.1 Graded Differential Algebras

We now take the ideas of the previous section and move them to a more abstract setting. Let R be a commutative ring, and let A be an $\mathbb{N}$-graded algebra over R, with decomposition

$$A = \bigoplus_{p=0}^{\infty} A^{(p)}$$

There is also an R-linear map $d : A \to A$ such that,

$$d : A^{(p)} \to A^{(p+1)}.$$

and $d^2 = 0$. If we denote the restriction of d to $A^{(p)}$ as d_p, we require that the maps d_p satisfy a graded Leibniz rule, for $a \in A^{(n)}$ and $b \in A^{(m)}$,

$$d_{n+m}(ab) = d_n(a)b + (-1)^n a d_m(b).$$

A pair (A, d) satisfying these conditions is called a *differential graded algebra.*

Thus we have a sequence,

$$0 \to A^{(0)} \xrightarrow{d} A^{(1)} \xrightarrow{d} \cdots \xrightarrow{d} A^{(n)} \xrightarrow{d} \cdots$$

The quotient R-modules,

$$H^p_{dR}(M) := \frac{\ker(d_p)}{\operatorname{im}(d_{p-1})}.$$

are the de Rham cohomology modules for the graded differential algebra (A, d).

5.3.2 Kähler Differentials

Given an R-algebra A, we would like to be able to build an algebra of differential forms over A, in a manner analogous to how $\Omega^1(M)$ is constructed from $C^\infty(M)$. It turns out that there is a good way of doing this, called the algebra of *Kähler differentials.* This is simplest in the commutative case, which we briefly outline here.

Let R be a commutative ring, and let A be a unital commutative R-algebra. The module $\Omega^1_{\text{com}}(A)$ of Kähler differentials is defined as

$$\Omega^1_{\text{com}}(A) := \frac{A \otimes_R A}{\langle c \otimes (ab) - (ca) \otimes b - (bc) \otimes a \rangle}$$

The idea here is that $\Omega^1_{\text{com}}(A)$ is the left A-module spanned by all symbols of the form adb, where $d(ab) = adb + bda$. We think of $a \otimes b$ as adb.

More precisely, we let $d : A \to \Omega^1_{\text{com}}(A)$ be given by

$$da := 1_A \otimes a$$

Where 1_A is the unit in A.

The utility of $\Omega^1_{\text{com}}(A)$ is that it allows us to study all derivations on A.

In full abstraction, a derivation on A is a map $\theta : A \to M$, where M is some left A-module, such that θ satisfies the Leibniz rule,

$$\theta(ab) = a\theta(b) + b\theta(a).$$

We see that d is a derivation on A to the A-module $\Omega^1_{\text{com}}(A)$. It is in fact universal with this property,

Theorem 5.1. *Let A be a unital commutative R-algebra, and let $\theta : A \to M$ be a derivation to some left A-module M. There exists a unique R-linear map $\Omega(\theta)$ such that the following diagram commutes,*

$$\begin{tikzcd} A \arrow[r, "d"] \arrow[dr, "\theta"'] & \Omega^1_{\text{com}}(A) \arrow[d, dotted, "\Omega(\theta)"] \\ & M \end{tikzcd}$$

In other words, there is an isomorphism of R-modules,

$$D(A, M) \cong \text{Hom}_R(\Omega^1_{\text{com}}(A), M).$$

Where $D(A, M)$ is the set of derivations from A to M. Note that this universal property defines $\Omega^1_{\text{com}}(A)$ up to unique isomorphism.

Proof. Basically, $\Omega(\theta)$ maps adb to $a\theta(b)$. Checking the universal property is routine. $\square$

We would now like to create a similar algebra of differentials for a noncommutative associative algebra A over R. In the noncommutative case, we must restrict attention to derivations that take values in A-bimodules, rather than left A modules.

Definition 5.2. Let A be an associative unital algebra over a commutative ring R. Let $m : A \otimes A \to A$ be the multiplication map. We define

$$\Omega^1(A) = \ker(m)$$

This is an A bimodule.

This the motivation behind this definition is not at all clear. However, this does agree with the commutative case and this provides the appropriate definition for noncommutative Kähler differentials. To see this, we define the map $d : A \to \Omega^1(A)$, by

$$d(a) = 1_A \otimes a - a \otimes 1_A.$$

We see that d is a derivation. In fact, $\Omega^1(A)$ should be thought of as the space of all linear combinations of terms of the form $ad(b)$.

$\Omega^1(A)$ satisfies the same universal property as Ω^1_{com}. Namely, if M is an A-bimodule, and $\theta : A \to M$ is a derivation, then there exists a unique R-linear map $\Omega(\theta)$ such that the following diagram commutes,

$$\begin{tikzcd}
A \arrow[r, "d"] \arrow[dr, "\theta"] & \Omega^1(A) \arrow[d, dotted, "\Omega(\theta)"] \\
& M
\end{tikzcd}$$

5.3.3 Universal Differential Algebra

Given an associative unital algebra R over a commutative ring R, we define

$$\Omega^p(A) := \bigotimes_{A,p} \Omega^1(A).$$

And the algebra,

$$\Omega A = \bigoplus_p \Omega^p(A).$$

We extend the function $d : A \to \Omega^1(A)$ to ΩA by

$$d(a \otimes da_1 \otimes da_2 \otimes \cdots \otimes da_n) = da \otimes da_1 \otimes da_2 \otimes \cdots \otimes da_n.$$

Theorem 5.3. *ΩA is the "largest" graded differential algebra generated by A.*

If (Γ, Δ) is a graded differential algebra, with grading $\Gamma = \bigoplus_n \Gamma^{(n)}$, and $\rho : A \to \Gamma^{(0)}$ is an algebra homomorphism, then ρ extends uniquely to a morphism $\Omega A \to \Gamma$ such that the following diagram commutes,

$$\begin{tikzcd}
\Omega^p(A) \arrow[r, "\rho"] \arrow[d, "d"] & \Gamma^{(p)} \arrow[d, "\Delta"] \\
\Omega^{p+1}(A) \arrow[r, "\rho"] & \Gamma^{(p+1)}
\end{tikzcd}$$

5.3.4 The Insufficiency of Kähler Differentials

The ostensible purpose of Kähler differentials is to provide a generalisation of the construction of the exterior algebra bundle of a manifold to the noncommutative setting.

However the graded algebra of Kähler differentials is in general considerably bigger than necessary for this purpose.

For a manifold M, the classical exterior algebra bundle $\Omega(M)$ has the property that $\Omega^k(M)$ is zero dimensional when k exceeds the dimension of M. However for an algebra A, the dimensions of the spaces $\Omega^k(A)$ of Kähler differentials never become zero unless A is already zero dimensional.

Therefore in order to create a definition of the exterior algebra bundle over a (potentially noncommutative) algebra A which more closely resembles the exterior algebra bundle over a manifold.

The solution is to give more information about A, and construct an algebra of differential forms as a specific quotient of $\Omega(A)$.

5.4 Noncommutative Geometry

5.4.1 Spectral Triples

Recall from Chapter 1 that a general noncommutative space should be thought of as a pair $(\mathcal{A}, \mathcal{H})$, where $\mathcal{A}$ is an algebra of operators on a Hilbert space $\mathcal{H}$.

We shall consider noncommutative spaces equipped with a "Dirac Operator", $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ such that $\mathcal{A}$ is contained in $\mathcal{B}(\mathcal{H})$.

A (commutative) example of this is as follows: Let (M, g) be a compact Riemannian manifold, and let $\mathcal{A} = C^\infty(M)$, and $\mathcal{H} = L^2(M, g)$. However this is not enough information to recover the geometry of M. A convenient way to recover the geometry of M from algebraic data is to give a Dirac operator. The definition of a general Dirac operator affiliated to a manifold is given in Appendix D.

Definition 5.4 (Spectral Triple)**.** A spectral triple is a triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, where $\mathcal{H}$ is a Hilbert space with $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ a $*$-algebra of operators.

$\mathcal{D}$ is a densely defined unbounded operator on $\mathcal{H}$ satisfying the following two properties:

1. $[\mathcal{D}, a]$ is densely defined and extends to a bounded operator for all $a \in \mathcal{A}$.

2. $a(\lambda - \mathcal{D})^{-1}$ is compact for all $a \in \mathcal{A}$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

A spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ can be either *even* or *odd*:

- We say that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is even if there exists a $\mathbb{Z}/2\mathbb{Z}$ grading on the linear operators on $\mathcal{H}$ such that $\mathcal{A}$ is even and $\mathcal{D}$ is odd. Equivalently, there is an operator Γ on $\mathcal{H}$ with $\Gamma^2 = 1$ and $\Gamma^* = \Gamma$ such that $a\Gamma = \Gamma a$ for all $a \in \mathcal{A}$ and $\mathcal{D}\Gamma = -\Gamma\mathcal{D}$.
- If no such operator Γ exists, then we say that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is odd.

5.4.2 Connes Differentials

Given a spectral triple, $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, we would like to construct an "exterior algebra" on $\mathcal{A}$. Connes does this by identifying the 1-form da with $[D, a]$.

Since $[D, a]$ is a derivation on $\mathcal{A}$, by the universal property we have a map, $\pi : \Omega\mathcal{A} \to \mathcal{B}(\mathcal{H})$ given by $\pi(ada_1da_2 \cdots da_n) = a[D, a_1][D, a_2] \cdots [D, a_n]$.

One may then naïvely define the algebra of differential forms as $\pi(\Omega\mathcal{A})$, but this does not work since there exists $a \in \Omega\mathcal{A}$ such that $\pi(a) = 0$ but $\pi(da) \neq 0$. These are called "junk forms" and we must factor them out to get a good differential algebra. Hence, define

Definition 5.5. Let J_0 be the graded ideal of $\Omega\mathcal{A}$ defined by

$$J_0^{(p)} = \{a \in \Omega^p(\mathcal{A}) \ : \ \pi(a) = 0\}$$

And define $J^{(p)} = J_0^{(p)} + dJ_0^{(p)}$. Then $J = \bigoplus_p J^{(p)}$.

Now we can define the algebra of Connes' forms,

$$\Omega_{\mathcal{D}}\mathcal{A} = \frac{\Omega\mathcal{A}}{J} \cong \frac{\pi(\Omega\mathcal{A})}{\pi(dJ_0)}$$

$\Omega_{\mathcal{D}}\mathcal{A}$ is naturally graded by the gradings on $\Omega\mathcal{A}$ and J, with the space of p-forms being $\Omega_{\mathcal{D}}^p\mathcal{A} = \Omega^p(\mathcal{A})/J^{(p)}$.

Since J is a differential ideal, the operator d on $\Omega\mathcal{A}$ extends to $\Omega_{\mathcal{D}}\mathcal{A}$.

Definition 5.6 (Quantum Differentiability)**.** A spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is called QC^k for $k \geq 0$ if $\mathcal{A}$ is contained in the domain of the operator δ^k, where $\delta(a) = [|\mathcal{D}|, a]$.

Definition 5.7 (Summability)**.** A spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is called (p, ∞) summable for $p > 0$ if $(1 + \mathcal{D}^2)^{-1/2} \in \mathcal{L}^{p,\infty}$.

5.5 Quantised Differentials in Noncommutative Geometry

We have already given a thorough description of the quantised differentials $\text{đ}f$ when f is a function on the circle. It so happens that we can find analogous, although weaker, results in far greater generality.

It is in fact easier to move to a setting that is extremely general.

Definition 5.8. Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple, either even or odd. Since $\mathcal{D}$ is a self adjoint operator, by the Borel functional calculus we can define the operator $F := \operatorname{sgn}(\mathcal{D})$. For $a \in \mathcal{A}$, we define

$$đa := [F, a]. \tag{5.1}$$

It is here that we see the distinction between *derivatives* and *differentials*. In a spectral triple, the algebra of Connes forms plays the role of derivatives of functions, and the quantised differentials are a different object entirely.

Theorem 5.9. *Let $p \geq 1$, and let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a (p, ∞) summable QC^1 spectral triple such that $\mathcal{D}$ is invertible. Then, for all $a \in \mathcal{A}$*

$$đa \in \mathcal{L}^{p,\infty}. \tag{5.2}$$

Proof. Since $\mathcal{D} = |\mathcal{D}|F$, we can compute

$$[\mathcal{D}, a] = |\mathcal{D}|[F, a] + [|\mathcal{D}|, a]F. \tag{5.3}$$

Or in other words,

$$da = |\mathcal{D}|đa + \delta(a)F. \tag{5.4}$$

Since $\mathcal{D}$ is invertible, $|\mathcal{D}|$ is invertible. Thus,

$$đa = |\mathcal{D}|^{-1}da + |\mathcal{D}|^{-1}\delta(a)F.$$

By assumption, $da, \delta(a) \in \mathcal{B}(\mathcal{H})$. Thus, since $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is (p, ∞)-summable, $|\mathcal{D}|^{-1} \in \mathcal{L}^{p,\infty}$. Hence, $đa \in \mathcal{L}^{p,\infty}$. $\square$

Chapter 6

Higher Dimensions

6.1 Introduction

We have explored the concept of a quantised derivatives on $\mathbb{T}$ and on $\mathbb{R}$. It is natural to consider generalisations to other spectral triples.

$\mathbb{R}$ and $\mathbb{T}$ are important as settings for analysis since they are spaces possessing some symmetry. In other words, there is a group action on $\mathbb{R}$ (namely, the action of $\mathbb{R}$ on itself by translation) that is faithful and transitive. Similarly the group $\mathbb{T}$ acts on itself by rotation.

We consider now more general noncommutative spaces with a group action. There is a beautiful generalisation of the Fourier transform in the theory of von Neumann algebras relating to the action of compact groups. In the next section, we explore this idea.

For an introduction to the theory of von Neumann algebras, see the book [10].

6.2 Compact Group actions

First we recall the definition of the σ-weak topology.

Definition 6.1. Let $\mathcal{H}$ be a Hilbert space, and let $\{\xi_j\}_{j=0}^{\infty}$ and $\{\eta_j\}_{j=0}^{\infty}$ be sequences of elements of $\mathcal{H}$ such that

$$\sum_{j=0}^{\infty} \|\xi_j\|^2 < \infty \tag{6.1}$$

$$\sum_{j=0}^{\infty} \|\eta_j\|^2 < \infty. \tag{6.2}$$

Then for $a \in \mathcal{B}(\mathcal{H})$, define the semi-norm,

$$\left| \sum_{j=0}^{\infty} \langle \xi_j, a\eta_j \rangle \right|. \tag{6.3}$$

This system of semi-norms defines the σ-weak topology on $\mathcal{B}(\mathcal{H})$.

For more details, see [10].

Let G be a compact abelian group equipped with normalised Haar measure μ, and let $\mathcal{M}$ be a von Neumann algebra.

Recall that a group action on a von Neumann algebra is a mapping $\alpha : G \times \mathcal{M} \to \mathcal{M}$ such that for all $g \in G$ the function $a \mapsto \alpha(g, a)$ is an algebra homomorphism. We assert that all group actions are σ-weakly continuous, meaning that the map $g \mapsto \alpha(g, x)$ is continuous from G to $\mathcal{M}$ when $\mathcal{M}$ is equipped with the σ-weak topology.

Suppose that G acts on M in a way that is

1. *ergodic*: the only projections in $\mathcal{M}$ fixed by G are 0 and 1.
2. *free*: there is no nontrivial projection $p \in \mathcal{M}$ such that some $g \in G$ not equal to the identity fixes all of $p\mathcal{M}p$.

We will need to compute integrals of von Neumann algebra valued functions. To do this, will will use the *Pettis integral*. See the paper [20] for details. The important feature of the Pettis integral that we will need is that it commutes with a group action. That is, if G is a compact abelian group with Haar measure μ, and $f : G \to \mathcal{M}$ is a function that is continuous in the σ-weak topology, and $g \in G$, and α is a G-action on $\mathcal{M}$, we have

$$\alpha(g, \int_G f(s)\, d\mu(s)) = \int_G \alpha(g, f(s))\, d\mu(s). \tag{6.4}$$

We then have the following result:

Proposition 6.2. *Suppose that G is a compact abelian group that acts freely and ergodically on the von Neumann algebra $\mathcal{M}$. We denote the action as $\alpha : G \times \mathcal{M} \to \mathcal{M}$. Then there is a set*

$$\{u(p) \ : \ p \in \widehat{G}\} \tag{6.5}$$

of unitary eigenoperators for the action indexed by the dual group $\widehat{G}$.

Proof. Let $x \in \mathcal{M}$ and $p \in \widehat{G}$. Define

$$\hat{x}(p) := \int_G \alpha(s, x) p(s)^{-1}\, d\mu(s). \tag{6.6}$$

Let $g \in G$, then we can compute,

$$\alpha(g, \hat{x}(p)) = \int_G \alpha(g, \alpha(s, x))\, d\mu(s) \tag{6.7}$$

$$= \int_G \alpha(sg, x) p(s)^{-1}\, d\mu(s) \tag{6.8}$$

$$= p(g)\hat{x}(p). \tag{6.9}$$

Hence we have that for each $g \in G$, the linear map $x \mapsto \alpha(g, x)$ has eigenvector $\hat{x}(p)$ with eigenvalue $p(g)$. This is to say that $\hat{x}(p)$ is an eigenoperator for the action of G. Hence, for $x, y \in \mathcal{M}$, we have that $\hat{x}(p)\hat{y}(p)^*$ is a fixed point of G, hence a scalar multiple of 1 by the ergodicity of the action.

Then define

$$u(p) := \frac{\hat{x}(p)}{\|\hat{x}(p)\|}. \tag{6.10}$$

□

Example 6.1. *The prototypical example of this decomposition is for $\mathcal{M} = L^\infty(\mathbb{T})$, and $G = \mathbb{T}$ with Haar measure acting on $\mathcal{M}$ by rotation. Then $\widehat{G} = \mathbb{Z}$, and for $f \in L^\infty(\mathbb{T})$ and $n \in \mathbb{Z}$, we have*

$$\hat{f}(n)(\zeta) = \int_{\mathbb{T}} f(\tau\zeta)\tau^{-n} d\boldsymbol{m}(\tau) \tag{6.11}$$

This is simply the nth Fourier coefficient multiplied by ζ^n. Hence we have (up to a unimodular scaling factor)

$$u(n)(\zeta) = \zeta^n. \tag{6.12}$$

So the system of unitaries is the set of monomials on $\mathbb{T}$.

The important feature of this system of unitaries is that it spans $\mathcal{M}$ in the following sense, proved in [21, Thm 2.3]:

Proposition 6.3. *Let G be a compact abelian group acting freely and ergodically on a von Neumann algebra $\mathcal{M}$. Let*

$$\mathcal{P} = \{u(p) \ : \ p \in \widehat{G}\} \tag{6.13}$$

be the corresponding unitary eigenoperators of G. Then the span of $\mathcal{P}$ is dense in the σ-weak topology on $\mathcal{M}$.

6.2.1 The space $\mathcal{L}^2(\mathcal{M}, \tau)$

Suppose that τ is a faithful trace on $\mathcal{M}$ such that $\tau(1) = 1$, and τ is invariant under the action of G, that is for all $g \in G$ and $x \in \mathcal{M}$ we have $\tau(\alpha(g, x)) = \tau(x)$. Then the map $(x, y) \mapsto \tau(x^*y)$ is an inner product on $\mathcal{M}$, and the completion of $\mathcal{M}$ in this inner product is denoted $\mathcal{L}^2(\mathcal{M}, \tau)$. See the book [22, Ch. 1] for introduction to the theory of traces on von Neumann algebras.

We require the following lemma from Pedersen [21], Theorem 3.6.5,

Lemma 6.4. *A state φ on a von Neumann algebra is normal if and only if it is σ-weakly continuous.*

We now prove that if $\mathcal{M}$ carries a free and ergodic group action, such a trace exists.

Lemma 6.5. *Let $\mathcal{M}$ be a von Neumann algebra, and let G act on $\mathcal{M}$ freely and ergodically. Then there is a G-invariant state on $\mathcal{M}$, and it is a faithful normal trace.*

Proof. Define for $x \in \mathcal{M}$,

$$\tau(x) = \int_G \alpha(s, x)\, d\mu(s) = \hat{x}(0). \tag{6.14}$$

See that $\alpha(g, \tau(x)) = \tau(x)$ since the Haar measure is G-invariant, so $\tau(x)$ is G-invariant so by ergodicity must be in $\mathbb{C}1$. If we identify $\mathbb{C}1$ with $\mathbb{C}$, we can think of $\tau(x)$ as a scalar, so $\tau(x)$ is a linear functional. We see that τ is a state since $\tau(1) = \int_G 1\, d\mu = 1$.

If $\tau(x^*x) = 0$, we must have $\alpha(s, x)^*\alpha(s, x) = 0$ for all $s \in G$, so $x = 0$. Hence τ is faithful.

Recall that a state φ is normal if for any sequence of pairwise orthogonal projections $\{p_n\}_{n\in\mathbb{N}}$, we have that $\varphi(\sum_n p_n) = \sum_n \tau(p_n)$. We have that the series $\sum_n p_n$ converges strongly, hence σ-weakly. Thus τ is normal since $\alpha(s, -)$ is σ-weakly continuous.

Since τ is normal, by lemma 6.4 it is σ-weakly continuous.

Hence to prove that τ is a trace it is sufficient to prove that

$$\tau(u(p)u(q)) = \tau(u(q)u(p)) \tag{6.15}$$

for all $p, q \in \widehat{G}$ since the system $\{u(p) \,:\, p \in \widehat{G}\}$ is σ-weakly dense in $\mathcal{M}$ by proposition 6.3.

Let $g \in G$, since τ is G-invariant then we have

$$\begin{aligned} \tau(u(p)u(q)) &= \tau(\alpha(g, u(p)u(q))) && (6.16)\\ &= p(g)q(g)\tau(u(p)u(q)). && (6.17) \end{aligned}$$

If $p \neq q^{-1}$, we can find g such that $p(g)q(g) \neq 1$. Thus, $\tau(u(p)u(q)) = 0$.

Otherwise, if $p = q^{-1}$ there is a scalar λ_p with $|\lambda_p| = 1$ such that

$$\tau(u(p)u(q)) = \lambda_p \tau(u(p)u(p)^*) = \lambda_p \tau(1) = \lambda_p. \tag{6.18}$$

Similarly we compute $\tau(u(q)u(p)) = \lambda_p$, so $\tau(u(p)u(q)) = \tau(u(q)u(p))$.

Since the set $\{u(p) \ : \ p \in \widehat{G}\}$ is σ-weakly dense in $\mathcal{M}$, we have that $\tau(xy) = \tau(yx)$ for all $x, y \in \mathcal{M}$.

□

Proposition 6.6. *Let $\mathcal{P} := \{u(p) \ : \ p \in \widehat{G}\}$ be the set of unitary eigenoperators corresponding to the action of G. Then $\{u(p) \ : \ p \in \widehat{G}\}$ is an orthonormal basis for $\mathcal{L}^2(\mathcal{M}, \tau)$.*

Proof. By standard Hilbert space theory, it is sufficient to prove that $\mathcal{P}$ is orthonormal and has dense span (see, e.g., [11] for details on the theory of Hilbert spaces).

First we prove ortho-normality. Let $p, q \in \widehat{G}$. Then for all $g \in G$,

$$\tau(u(p)^* u(q)) = \tau(p(g)^{-1} q(g) u(p)^* u(q)) \tag{6.19}$$

$$= p(g)^{-1} q(g) \tau(u(p)^* u(q)). \tag{6.20}$$

If $p \neq q$, there is some $g \in G$ such that $p(g)^{-1} q(p) \neq 1$, so we conclude that $\tau(u(p)^* u(q)) = 0$. If $p = q$, then by unitarity we have

$$\tau(u(p)^* u(q)) = \tau(1) = 1. \tag{6.21}$$

Hence $\mathcal{P}$ is orthonormal.

To prove that the span of $\mathcal{P}$ is dense in $\mathcal{L}^2(\mathcal{M}, \tau)$, it is sufficient to prove that it is dense in $\mathcal{M}$ in the norm $\|x\|_2 := \sqrt{\tau(x^* x)}$. This follows from the σ-weak continuity of τ. □

Remark 6.7. In fact, for any $p \geq 1$, we have that the norm $\|x\|_p = (\tau(|x|^p))^{1/p}$ is σ-weakly continuous, so $\{u(p) \ : \ p \in \widehat{G}\}$ is dense in $\mathcal{L}^p(M, \tau)$, defined as the completion of $\mathcal{M}$ in the norm $\| \cdot \|_p$.

6.2.2 Noncommutative Harmonic Analysis

We now fix $G = \mathbb{T}^d$, for $d \geq 1$, so that $\widehat{G} = \mathbb{Z}^d$. It is of interest to prove certain results that are analogous to classical results of harmonic analysis. In particular, one known classical result is that if $f \in L^1(\mathbb{T}^d, \boldsymbol{m})$, then the Fourier coefficients $\hat{f}(n)$ vanish as $\|n\| \to \infty$. We now prove an analogy of this result.

Let $\boldsymbol{m}$ denote the normalised Haar measure on $\mathbb{T}^d$.

From now on, denote the action of $t \in \mathbb{T}^d$ on $x \in \mathcal{M}$ as $\alpha_t(x)$.

Fix $\mathcal{M}$ a von Neumann Algebra, and let $\mathbb{T}^d$ act on $\mathcal{M}$ freely and ergodically. Let τ be a normalised faithful $\mathbb{T}^d$-invariant trace on $\mathcal{M}$. Let

$$\mathcal{P} := \{u(n) \ : \ n \in \mathbb{Z}^d\} \tag{6.22}$$

be the spanning system of unitary eigenoperators of the group action.

For $a \in \mathcal{L}^1(\mathcal{M}, \tau)$ and $n \in \mathbb{Z}^d$, define

$$\hat{a}(n) := \tau(au(n)^*) \tag{6.23}$$

For any $a \in \mathcal{L}^2(\mathcal{M}, \tau)$, we have that

$$a = \sum_{n \in \mathbb{Z}^d} \hat{a}(n)u(n) \tag{6.24}$$

where the convergence is in the $\mathcal{L}^2$ norm. This is an isomorphism with $\ell^2(\mathbb{Z}^d)$. In general, for $N = (N_1, \ldots, N_d) \in \mathbb{Z}^d$, define

$$S_N a := \sum_{n=-N}^{N} \hat{a}(n)u(n) \tag{6.25}$$

where the summation runs over all multi-indices $n = (n_1, \ldots, n_d)$ such that for each $1 \leq j \leq d$, we have $-N_j \leq n_j \leq N_J$. S_N is called the Cèsaro sum. and

$$\sigma_N a := \frac{1}{N_1 N_2 \cdots N_d} \sum_{n=(1,\ldots,1)}^{N} S_n a. \tag{6.26}$$

and the summation runs over multi-indices $n = (n_1, \ldots, n_d)$ such that for each j, $1 \leq n_j \leq N_j$.

Now we define the subspaces of $\mathcal{M}$ analogous to "*continuous*" and "*uniformly continuous*" functions.

Definition 6.8. Define $\mathcal{C}(\mathcal{M})$ to be the closure of the span of $\mathcal{P}$ in the norm topology of $\mathcal{M}$.

Define $\mathcal{U}(\mathcal{M})$ to be the set of elements of $\mathcal{M}$ such that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sup_{|t|<\delta} \|\alpha_t(x) - x\| < \varepsilon. \tag{6.27}$$

Where, for $t \in \mathbb{T}^d$, If $t = (e^{2\pi i\varphi_1}, \ldots, e^{2\pi i\varphi_d})$, we denote

$$|t| = \max\{|\varphi_1|, \ldots, |\varphi_d|\}. \tag{6.28}$$

Definition 6.9. Let $n = (n_1, \ldots, n_d) \in \mathbb{Z}^d$, and $t = (t_1, \ldots, t_d) \in \mathbb{T}^d$. Then we introduce the notation

$$t^d := t_1^{n_1} \cdots t_d^{n_d}. \tag{6.29}$$

Definition 6.10. Suppose that $\varphi \in L^1(\mathbb{T}^d, \boldsymbol{m})$, and $x \in \mathcal{M}$. Define the *convolution*

$$\varphi * x = \int_{\mathbb{T}^d} \varphi(t)\alpha_t(x)\, d\boldsymbol{m}(t). \tag{6.30}$$

It can be proved that

$$S_N x = D_N * x \tag{6.31}$$

where

$$D_N(t) = \prod_{j=1}^{d} \frac{t_j^{N_j+1/2} - t_j^{-N_j-1/2}}{t_j^{1/2} - t_j^{-1/2}} \tag{6.32}$$

and

$$\sigma_N x = F_N * x \tag{6.33}$$

where

$$F_N(t) = \prod_{j=1}^{d} \frac{t_j^{N_j} - 2 + t_j^{-N_j}}{N(t_j^{1/2} - t_j^{-1/2})^2} \tag{6.34}$$

where the fractional powers in the formulae for D_N and F_N are defined in a principal value sense. D_N is called the Dirichlet kernel, and F_N is the Féjer kernel. See [23] for proofs.

Now we recall the definition approximate identities,

Definition 6.11. Let Λ be a directed set. A net $\{\Phi_\lambda\}_{\lambda\in\Lambda} \subset L^1(\mathbb{T}^d, \boldsymbol{m})$ is called an approximation to the identity if,

1. For all λ, we have $\int_{\mathbb{T}^d} \Phi_\lambda \, d\boldsymbol{m} = 1$.
2. We have $\sup_\lambda \int_{\mathbb{T}^d} |\Phi_\lambda| \, d\boldsymbol{m} < \infty$
3. For any $\delta > 0$, we have $\lim_{\lambda\in\Lambda} \int_{|t|>\delta} |\Phi_\lambda| \, d\boldsymbol{m} = 0$.

It can be proved that the sequence $\{F_n\}_{n\in\mathbb{N}^d}$ is an approximate identity (see [24] for a detailed discussion of the kernels $\{F_n\}_{n=1}^{\infty}$).

Approximate identities are so named because of the following two results:

Proposition 6.12. *Let $x \in \mathcal{U}(\mathcal{M})$, and let $\{\Phi_\lambda\}_{\lambda\in\Lambda}$ be an approximate identity. Then we have*

$$\lim_{\lambda\in\Lambda} \|\Phi_\lambda * x - x\| = 0. \tag{6.35}$$

Proof. Using the fact that $\int_{\mathbb{T}} \Phi_\lambda \, d\mu = 1$, we compute,

$$\Phi_n * x - x = \int_{\mathbb{T}} \Phi_\lambda(t)(\alpha_t(x) - x) \, d\mu(t). \tag{6.36}$$

Let $\varepsilon > 0$. Choose δ small enough such that

$$\sup_{|t|<\delta} \|\alpha_t(x) - x\| < \varepsilon. \tag{6.37}$$

Now estimate,

$$\|\Phi_\lambda * x - x\| \leq \int_{|t|<\delta} |\Phi_\lambda(t)| \|\alpha_t(x) - x\| \, d\mu(t) + (C+1) \int_{|t|\geq\delta} |\Phi_\lambda(t)| \|x\| d\mu(t) \tag{6.38}$$

where C is a constant such that $\|\alpha_t(x)\| < C\|x\|$. So taking the limit over λ, we have

$$\lim_{\lambda\in\Lambda} \|\Phi_\lambda * x - x\| < \varepsilon \sup_{\lambda\in\Lambda} \int_{\mathbb{T}} |\Phi_n| \, d\boldsymbol{m}. \tag{6.39}$$

But ε is arbitrary, so the result follows. □

Proposition 6.13. *We have that*

$$\mathcal{U}(\mathcal{M}) \subseteq \mathcal{C}(\mathcal{M}). \tag{6.40}$$

Proof. Let $a \in \mathcal{U}(\mathcal{M})$. Since $\{F_n\}_{n\in\mathbb{N}^d}$ is an approximate identity, we have that $F_n * a \to a$ in the norm topology. But $F_n * a \in \operatorname{span}(\mathcal{P})$. Hence $a \in \mathcal{C}(\mathcal{M})$. Hence we have that $\mathcal{U}(\mathcal{M}) \subseteq \mathcal{C}(\mathcal{M})$.

□

Proposition 6.14. *Let $\mathcal{M}$ be a von Neumann algebra with a free and ergodic group action α of $\mathbb{T}^d$. Let $\mathcal{P} = \{u(n) \, : \, n \in \mathbb{Z}^d\}$ be the corresponding system of eigenoperators and trace τ. Then $\mathcal{P} \subseteq \mathcal{U}(\mathcal{M})$, and hence $\mathcal{U}(\mathcal{M})$ is dense in $\mathcal{L}^p(\mathcal{M}, \tau)$*

Proof. First let $u(n) \in \mathcal{P}$. Then for $t \in \mathbb{T}^d$, we have

$$\alpha_t(u(n)) = t^n u(n) \tag{6.41}$$

Hence

$$\|\alpha_t(u(n)) - u(n)\| = |t^n - 1| \|u(n)\|. \tag{6.42}$$

So since $\lim_{t\to 1} |t^n - 1| = 0$, we have that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sup_{|t|<\delta} \|\alpha_t(u(n)) - u(n)\| < \varepsilon. \tag{6.43}$$

Now let a be in the linear span of $\mathcal{P}$. Since a is only a finite linear combination of terms of the form $u(n)$, $n \in \mathbb{Z}$, we have that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\sup_{|t|<\delta} \|\alpha_t(a) - a\| < \varepsilon. \tag{6.44}$$

Hence the linear span of $\mathcal{P}$ is contained in $\mathcal{U}(\mathcal{M})$.

Since the span of $\mathcal{P}$ is dense in $\mathcal{L}^p(\mathcal{M}, \tau)$, we conclude that $\mathcal{U}(\mathcal{M})$ is dense in $\mathcal{L}^p(\mathcal{M}, \tau)$. □

As usual, define $\|x\|_p = \tau(|x|^p)^{1/p}$. We require the following lemma:

Lemma 6.15. *Let $p \geq 1$. Suppose $\varphi \in L^1(\mathbb{T}^d, \boldsymbol{m})$, and $x \in \mathcal{L}^p(\mathcal{M}, \tau)$. Then*

$$\|\varphi * x\|_p \leq \|\varphi\|_1 \|x\|_p. \tag{6.45}$$

Proof. First we establish the case $p = 1$. This is a computation,

$$\|\varphi * x\|_1 \leq \int_{\mathbb{T}^d} |\varphi(t)| \|\alpha_t(a)\|_1 \, d\boldsymbol{m}(t). \tag{6.46}$$

Similarly, we define $\|x\|_\infty = \|x\|$. Thus the $p = \infty$ case,

$$\|\varphi * x\|_\infty \leq \|\varphi\|_1 \|x\|. \tag{6.47}$$

Consider the function $T(x) = \varphi * x$. We have shown that $T : \mathcal{L}^1(\mathcal{M}, \tau) \to \mathcal{L}^1(\mathcal{M}, \tau)$ with norm $\|\varphi\|_1$, and $T : \mathcal{M} \to \mathcal{M}$ with norm $\|\varphi\|_1$.

We see that since $\tau(1) = 1$, we have $\mathcal{M} \subseteq \mathcal{L}^1(\mathcal{M}, \tau)$. Thus $(\mathcal{L}^1(\mathcal{M}, \tau), \mathcal{M})$ is a compatible pair (as in definition C.2).

Hence for any exact interpolation functor F (see Appendix C for details on interpolation theory), we have that $T : F(\mathcal{L}^1(\mathcal{M}, \tau), \mathcal{M}) \to F(\mathcal{L}^1(\mathcal{M}, \tau), \mathcal{M})$ with norm $\|\varphi\|_1$. In Theorem 4.1 of [16] is it shown that there exists an exactly interpolation functor functor F with $F(\mathcal{L}^1(\mathcal{M}, \tau), \mathcal{M}) = \mathcal{L}^p(\mathcal{M}, \tau)$.

Hence by interpolation, the result follows. □

Proposition 6.16. *If $\{\Phi_\lambda\}_{\lambda \in \Lambda}$ is an approximate identity, and $x \in \mathcal{L}^p(\mathcal{M}, \tau)$ and $p \geq 1$ then*

$$\lim_{\lambda \in \Lambda} \|\Phi_\lambda * x - x\|_p = 0. \tag{6.48}$$

Proof. Let $\varepsilon > 0$. Since $\mathcal{U}(\mathcal{M})$ contains the linear span of $\mathcal{P}$, and the linear span of $\mathcal{P}$ is dense in $\mathcal{L}^p(\mathcal{M}, \tau)$ in the $\|.\|_p$ norm, we can find $y \in \mathcal{U}(\mathcal{M})$ such that $\|x - y\|_p < \varepsilon$. Hence,

$$\begin{aligned} \|\Phi_\lambda * x - x\|_p &\leq \|\Phi_\lambda * x - \Phi_\lambda * y\|_p + \|\Phi_\lambda * y - y\|_p + \|y - x\|_p & (6.49) \\ &\leq \sup_{\mu \in \Lambda} \|\Phi_\mu\|_1 \varepsilon + \|\Phi_\lambda * y - y\|_p + \varepsilon. & (6.50) \end{aligned}$$

Now take the limit over λ, and thus we obtain the result. □

At last we can prove the following result:

Proposition 6.17. *Let $x \in \mathcal{L}^1(\mathcal{M}, \tau)$. Then we have $\hat{x}(n) \to 0$ as $\|n\| \to \infty$.*

Proof. For $n = (n_1, \ldots, n_d) \in \mathbb{Z}^d$, denote $|n| := (n_1, \ldots, n_d)$. We have that $\tau((\sigma_{|N|-1}x)u(N)^*) = 0$, since $\sigma_{|N|-1}x$ is a linear combination of $\{u(n) \ : \ |n| < |N|\}$. Hence,

$$|\hat{x}(n)| = |\tau(xu(n)^*)| \tag{6.51}$$
$$\leq |\tau((x - \sigma_{|n|-1}x)u(n)^*)| \tag{6.52}$$
$$\leq \|u(n)^*\|\|x - \sigma_{|n|-1}x\|_1 \tag{6.53}$$
$$= \|x - F_{|n|-1} * x\|_1. \tag{6.54}$$

But the right hand side vanishes as $\|n\| \to \infty$. □

6.2.3 The Operators δ_j

Let $\mathcal{M}$ be a von Neumann Algebra, and let $\mathbb{T}^d$ act freely and ergodically on $\mathcal{M}$.

The generalised differentiation operator δ_j is defined as the *infinitesimal generator* of the action of $\mathbb{T}^d$, as follows,

Definition 6.18. Let $j \in \{1, \ldots, d\}$.

For $t \in \mathbb{T}$, we have an action α_t^j on $\mathcal{M}$ which is the action of the jth coordinate of $\mathbb{T}^d$ on $\mathcal{M}$

For $x \in \mathcal{M}$, we define

$$\delta_j(x) = \lim_{t \to 1} \frac{\alpha_t^j(x) - x}{|t|} \tag{6.55}$$

where $|t|$ is the minimal normalised arc length between t and 1.

The limit is in the sense of the norm topology on $\mathcal{M}$.

This limit may not exist for all x. Let $\mathrm{Dom}(\delta_j)$ be the set of all x such that $\delta_j(x)$ exists.

Note that $\mathrm{Dom}(\delta_j)$ is automatically a vector space.

Lemma 6.19. *We have $\mathcal{P} \subset \mathrm{Dom}(\delta_j)$.*

Proof. For $t \in \mathbb{T}$, and $u(n) \in \mathcal{P}$ with $n = (n_1, \ldots, n_d)$, we have $\alpha_t^j u(n) = t_j^n u(n)$.

Hence,

$$\frac{\alpha_t^j(u(n)) - u(n)}{|t|} = \frac{t^{n_j} - 1}{|t|} u(n). \tag{6.56}$$

We parametrise $\mathbb{T}$ by $t \mapsto \exp(i\theta)$, for $\theta \in [0, 2\pi)$. Then we have

$$\frac{t^{n_j} - 1}{|t|} = \frac{\exp(in_j\theta) - 1}{\theta}. \tag{6.57}$$

Hence,

$$\delta_j(u(n)) = \lim_{t\to 1} \frac{t^{n_j}-1}{|t|}u(n) \tag{6.58}$$

$$= \lim_{\theta\to 0} \frac{\exp(in_j\theta)-1}{\theta}u(n) \tag{6.59}$$

$$= in_j u(n). \tag{6.60}$$

Hence, $u(n) \in \mathrm{Dom}(\delta_j)$. $\square$

Proposition 6.20. *Suppose that* $x \in \mathrm{Dom}(\delta_j)$. *Then*

$$\widehat{\delta_j(x)}(n) = in_j\widehat{x}(n). \tag{6.61}$$

Proof. By definition, $\mathrm{Dom}(\delta_j) \subseteq \mathcal{M} \subseteq \mathcal{L}^1(\mathcal{M},\tau)$.

Let $x \in \mathrm{Dom}(\delta_j)$.

Hence, we have $F_n*x \to x$ in the $\mathcal{L}^1$ sense, and since $\delta_j(x) \in \mathcal{L}^1(\mathbb{T})$, we have $F_n*\mathcal{D}(x) \to \mathcal{D}(x)$ in the $\mathcal{L}^1$ sense.

Note that since $F_n * x$ is in the linear span of $\mathcal{P}$, we have $F_n * x \in \mathrm{Dom}(\delta_j)$. See that

$$\delta_j(F_n * x) = \frac{1}{n_1 n_2 \cdots n_d}\sum_{k=0}^{n} \delta_j(D_k * x). \tag{6.62}$$

where the sum is over multi-indices $k = (k_1,\ldots,k_d)$ with each $1 \le k_m \le n_m$. We also have,

$$\delta_j(D_k * x) = \sum_{j=-k}^{k} \hat{x}(j)\mathcal{D}(u(j)) \tag{6.63}$$

$$= \sum_{j=-k}^{k} ij\hat{x}(j)u(j). \tag{6.64}$$

Now we have,

$$\widehat{\delta_j(x)}(n) = \lim_{k\to\infty} \tau(F_k * xu(-n)) \tag{6.65}$$

$$= \lim_{k\to\infty} in_j\hat{x}(n). \tag{6.66}$$

$\square$

Now define $\mathcal{D}_j : \mathcal{L}^2(\mathcal{M},\tau) \to \mathcal{L}^2(\mathcal{M},\tau)$ as a (potentially not everywhere defined) linear operator,

$$\mathcal{D}_j := \frac{1}{i}\delta_j. \tag{6.67}$$

We define $\mathrm{Dom}(\mathcal{D}_j) = \mathrm{Dom}(\delta_j)$.

Hence we have the formula,

$$\widehat{\mathcal{D}_j x}(n) = n_j \hat{x}(n). \tag{6.68}$$

6.2.4 Noncommutative Tori

So far we have discussed abstract von Neumann algebras possessing a free and ergodic action by the group $\mathbb{T}^d$. It turns out that all such von Neumann algebras have a convenient description.

Definition 6.21. Let $d > 0$ be a positive integer, and let θ be a real $d \times d$ anti-symmetric matrix. Let $\mathcal{H} = \ell^2(\mathbb{Z}^d)$. $\mathcal{H}$ has orthonormal basis $\{e_n\}_{n\in\mathbb{Z}^d}$ where e_n is the element of $\ell^2(\mathbb{Z}^d)$ equal to 1 in the nth position and 0 elsewhere, for $n \in \mathbb{Z}^d$.

Define the operators $\{U_1, U_2, \dots, U_d\}$ on $\mathcal{H}$ as follows.

Let $k > j \in \{1, 2, \dots, d\}$. Let $b_j \in \mathbb{Z}^d$ be the vector equal to 0 everywhere except the jth position where it is 1. Define,

$$U_k e_{b_j} := \exp(2\pi i \theta_{j,k}) e_{b_j+b_k} \tag{6.69}$$

and for $k \le j$,

$$U_k e_{b_j} := e_{b_k+b_j}. \tag{6.70}$$

We also assume that, for $k < j$,

$$U_k^* e_{b_j} := e_{b_j-b_k}. \tag{6.71}$$

This uniquely defines $U_k : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$, since for any $n \in \mathbb{Z}^d$, we can obtain e_n by successively applying U_k for certain choices of k to some e_{b_j}, also using the relation that,

$$e_{b_j} = U_k^* e_{b_j+b_k}. \tag{6.72}$$

Then each U_k is unitary, and the von Neumann algebra generated by $\{U_1, \dots, U_d\}$ is denoted $\mathcal{N}_\theta$.

Proposition 6.22. *Let $\mathcal{M}$ be a von Neumann algebra such that there is an action $\alpha : \mathbb{T}^d \times \mathcal{M} \to \mathcal{M}$ that is free and ergodic. Then $\mathcal{M}$ is isometrically isomorphic to $\mathcal{N}_\theta$ for some θ.*

Proof. By proposition 6.3, $\mathcal{M}$ is generated by a family of unitaries $\{u(n) : n \in \mathbb{Z}^d\}$ in the σ-weak topology.

Define the mapping,

$$\Theta : \ell^2(\mathbb{Z}^d) \to \mathcal{L}^2(\mathcal{M}, \tau) \tag{6.73}$$

given by $\Theta(c) = \sum_{n\in\mathbb{Z}^d} c_n u(n)$. It was proved in proposition 6.6 that this is an isometric isomorphism of Hilbert spaces.

Let $k \in \mathbb{Z}^d$. We define an action π of $u(k)$ on $\ell^2(\mathbb{Z}^d)$ as follows. Let $c \in \ell^2(\mathbb{Z}^d)$. Now define,

$$\pi(u(k))c := \Theta^{-1}(u(k)\Theta(c)). \tag{6.74}$$

Let e_n be as in Definition 6.21. We shall compute $\pi(u(k))(e_n)$. First, see that since $\Theta(e_n) = u(n)$,

$$\pi(u(k))(e_n) = \Theta^{-1}(u(k)u(n)). \tag{6.75}$$

Now consider the element $u(k)u(n)u(k+n)^*$. This is fixed by the action of $\mathbb{T}^d$, hence by ergodicity is a scalar. Define

$$\omega_{k,n}1 := u(k)u(n)u(k+n)^*. \tag{6.76}$$

Since the right hand side of equation 6.76 is unitary, we conclude that $|\omega_{k,n}| = 1$.

We can see that since $u(k)u(k)^* = 1 = u(n)u(n)^*$, we have $\omega_{0,n} = \omega_{k,0} = 1$.

Hence,

$$\pi(u(k))(e_n) = \Theta^{-1}(\omega_{k,n}u(k+n)) \tag{6.77}$$

$$= \omega_{k,n}e_{k+n}. \tag{6.78}$$

Let $\{b_1, \dots, b_d\} \subset \mathbb{Z}^d$ be the multi-indices with b_j equal to 0 everywhere except the jth position where it is equal to 1.

Define, for $j, l \in \{1, \dots, d\}$, with $j \leq l$, the operator $U_j : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$,

$$U_j e_{b_l} := \omega_{b_j,b_l}^{-1}\pi(u(b_j))e_{b_l} \tag{6.79}$$

$$= e_{b_j+b_l}. \tag{6.80}$$

and for $j > l$,

$$U_j e_{b_l} := \pi(u(b_j))e_{b_l} \tag{6.81}$$

$$= \omega_{b_j,b_l}e_{b_j+b_l}. \tag{6.82}$$

We also define that,

$$U_j^* e_{b_l} := e_{b_l-b_j}. \tag{6.83}$$

Since it is a composition of isometries, U_j is unitary. We shall show that $\{U_1, \dots, U_d\}$ satisfies the relation of a noncommutative torus $\mathcal{N}_\theta$ for some θ.

Since $|\omega_{e_j,e_l}| = 1$, we can find $\theta_{j,l} \in \mathbb{R}$ such that $\omega_{e_j,e_l} = \exp(2\pi i\theta_{j,l})$.

Now define $\theta_{l,j} = -\theta_{j,l}$, and $\theta_{j,j} = 0$.

Hence we have defined a $d \times d$ antisymmetric real matrix θ, with (j,l)th entry $\theta_{j,l}$. We now show that the unitaries $\{U_1, \ldots, U_d\}$ satisfy the relations of the algebra $\mathcal{N}_\theta$.

If $j \leq l$, we have

$$U_j e_{b_l} = e_{b_j+b_l}. \tag{6.84}$$

We also see that,

$$U_j^* e_{b_l} := e_{b_l-b_j}. \tag{6.85}$$

For $j > l$,

$$U_j e_{b_l} = \exp(2\pi i\theta_{j,i}) e_{b_j+b_l}. \tag{6.86}$$

Now π extends by linearity to a map from $\operatorname{span}\{u(n) \ : \ n \in \mathbb{Z}^d\}$ to $\mathcal{B}(\mathcal{H})$. Since π is a composition of isometries, π must be σ-weakly continuous. Hence π extends by continuity to a map

$$\pi : \mathcal{M} \to \mathcal{B}(\mathcal{H}). \tag{6.87}$$

Now any $T \in \operatorname{im}\pi$ is generated by a σ-weak converging sequence of polynomials in $\{u(n) \ : \ n \in \mathbb{Z}^d\}$. Hence since σ-weak convergence clearly implies weak convergences, we see that $\pi(\mathcal{M}) \subseteq \mathcal{N}_\theta$.

Since π is an isometry, we see that the map $\pi : \mathcal{M} \to \mathcal{N}_\theta$ is injective. We must now prove that it is surjective.

Let $T \in \mathcal{N}_\theta$. Then there is a sequence p_n of polynomials in $\{U_1, \ldots, U_d\}$ that converges in the weak operator topology to T. We see that $\Theta^{-1}(p_n)$ must also converge in the weak operator topology, hence it converges to some $x \in \mathcal{M}$. Hence by continuity, $\Theta(x) = T$.

Thus π is surjective, so is an isometric isomorphism $\mathcal{M} \cong \mathcal{N}_\theta$.

$\square$

6.3 Quantised Differentials on Noncommutative Tori

Let $d > 0$ be a positive integer. Let $\mathcal{N}_\theta$ be a noncommutative torus associated to the $d \times d$ anti-symmetric real matrix θ. $\mathcal{N}_\theta$ has a normalised trace τ. The Hilbert space $\mathcal{L}^2(\mathcal{N}_\theta, \tau)$ has orthonormal basis $\{u(n)\}_{n \in \mathbb{Z}^d}$ where $u(n) = U_1^{n_1} U_2^{n_2} \cdots U_d^{n_d}$.

Define $\omega_{j,k}$ for $j,k \in \mathbb{Z}^d$ to be such that $u(j)u(k) = \omega_{j,k} u(k+j)$.

Given $f \in \mathcal{L}^2(\mathcal{N}_\theta, \tau)$, we define

$$\hat{f}(n) := \tau(f u(n)^*). \tag{6.88}$$

Then we have

$$f = \sum_{k\in\mathbb{Z}^d} \hat{f}(k)u(k), \tag{6.89}$$

which converges in the $\mathcal{L}^2$-sense. Note that $\tau(f) = \hat{f}(0)$.

For a function $F : \mathbb{Z}^d \to \mathbb{R}$, define $F(D)u(k) = F(k)u(k)$ as a densely defined linear operator on $\mathcal{L}^2(\mathcal{N}_\theta, \tau)$.

It is easy to see that $F(D)$ is bounded on $\mathcal{L}^2(\mathcal{N}_\theta, \tau)$ if and only if $F \in \ell^\infty(\mathbb{Z}^d)$, with operator norm $\|F\|_\infty$.

We shall identify $u(n) \in \mathcal{N}_\theta$ with the multiplication operator $M_{u(n)}f = u(n)f$ for $f \in \mathcal{L}^2(\mathcal{N}_\theta, \tau)$

6.3.1 Commutators

Let $F : \mathbb{Z}^d \to \mathbb{R}$. Let $n \in \mathbb{Z}^d$. We are interested in $[F(D), u(n)]$.

Let $f \in \mathcal{L}^2(\mathcal{N}_\theta, \tau)$.

Then,

$$u(n)f = \sum_{k\in\mathbb{Z}^d} \hat{f}(k)u(n)u(k) \tag{6.90}$$

$$= \sum_{k\in\mathbb{Z}^d} \hat{f}(k)\omega_{n,k}u(n+k). \tag{6.91}$$

Hence,

$$F(D)u(n)f = \sum_{k\in\mathbb{Z}^d} \hat{f}(k)\omega_{n,k}F(n+k)u(n+k). \tag{6.92}$$

Now we compute,

$$u(n)F(D)f = \sum_{k\in\mathbb{Z}^d} \hat{f}(k)F(k)\omega_{n,k}u(n+k). \tag{6.93}$$

Thus,

$$[F(D), u(n)]f = \sum_{k\in\mathbb{Z}^d} \hat{f}(k)(F(n+k) - F(k))\omega_{n,k}u(n+k). \tag{6.94}$$

We can simplify this further,

$$[F(D), u(n)]f = \sum_{k\in\mathbb{Z}^d} \hat{f}(k)(F(n+k) - F(k))u(n)u(k) \tag{6.95}$$

$$= u(n)\sum_{k\in\mathbb{Z}^d} \hat{f}(k)F(n+k)u(k) - u(n)\sum_{k\in\mathbb{Z}^d} \hat{f}(k)F(k)u(k) \tag{6.96}$$

$$= u(n)F(n+D)f - u(n)F(D)f \tag{6.97}$$

$$= u(n)F(n+D)f - u(n)F(D)f \tag{6.98}$$

So $[F(D), u(n)] = u(n)(F(n+D) - F(D))$.

6.3.2 Membership of $\mathcal{L}^p$

Proposition 6.23. *For any $p > 0$, we have $[F(D), u(n)] \in \mathcal{L}^p$ if and only if $\{F(k+n) - F(k)\} \in \ell^p(\mathbb{Z}^d)$ and the (quasi) norm $\|F(D), u(n)\|_p$ is equal to $\|\{F(k+n) - F(k)\}_{k\in\mathbb{Z}^d}\|_p$.*

We can see that since $F(D)$ and $F(n+D)$ are self adjoint,

$$[F(D), u(n)]^* = (F(n+D) - F(D))u(n)^* \tag{6.99}$$

Hence,

$$[F(D), u(n)]^*[F(D), u(n)] = (F(n+D) - F(D))^2. \tag{6.100}$$

Hence,

$$|[F(D), u(n)]| = |F(n+D) - F(D)|. \tag{6.101}$$

Now,

$$\begin{aligned} \mathrm{Tr}(|[F(D), u(n)]|^p) &= \sum_{k\in\mathbb{Z}^d} \tau(u(k)^*|F(D+n) - F(D)|^p u(k)) && (6.102)\\ &= \sum_{k\in\mathbb{Z}^d} \tau(u(k)^*|F(k+n) - F(k)|^p u(k)) && (6.103)\\ &= \sum_{k\in\mathbb{Z}^d} |F(k+n) - F(k)|^p. && (6.104) \end{aligned}$$

This is exactly the equality of $\|[F(D), u(n)]\|_p$ and $\|\{|F(k+n) - F(k)|\}_{k\in\mathbb{Z}^d}\|_p$.

6.3.3 Double commutators

We are also interested in the properties of the double commutators,

$$[[F(D), u(n)], u(n)]$$

for $n \in \mathbb{Z}^d$. By equation 6.98, this is

$$\begin{aligned} [[F(D), u(n)], u(n)] &= [F(D), u(n)]u(n) - u(n)[F(D), u(n)] && (6.105)\\ &= u(n)(F(n+D) - F(D))u(n) - u(n)u(n)(F(D+n) - F(D)) && (6.106)\\ &= (F(D) - F(D-n))u(n)^2 - (F(D-n) - F(D-2n))u(n)^2 && (6.107)\\ &= (F(D-2n) - 2F(D-n) + F(D))u(n)^2 && (6.108)\\ &= u(n)(F(D+n) - 2F(D) + F(D-n))u(n). && (6.109) \end{aligned}$$

Hence,

$$|[[F(D), u(n)], u(n)]| = |F(D + 2n) - 2F(D + n) + F(D)|.$$

So we have $[[F(D), u(n)], u(n)] \in \mathcal{L}^p$ if and only if $\{F(k+n)-2F(k)+F(k-n)\}_{k\in\mathbb{Z}^d} \in \ell^p(\mathbb{Z}^d)$.

6.3.4 Application to quantised differentials

Recall that for functions on the circle $\mathbb{T}$, we have the result that φ is a rational function if and only if $\text{đ}\varphi$ is finite rank. In the case of functions on a higher dimensional torus, or more generally a noncommutative torus, there is no analogous result.

However, analogous to proposition 4.3, we can show the following:

Proposition 6.24. *Let $\mathcal{N}_\theta$ be a noncommutative torus associated to the $d \times d$ matrix θ. We define $\mathcal{A}_\theta$ to be the closure of the subalgebra generated by $\{U_1, \dots, U_d\}$ in the norm topology. Let $F \in \ell^\infty(\mathbb{Z}^d)$ be such that $\{F(k + n) - F(k)\}_{k\in\mathbb{Z}^d} \in c_0(\mathbb{Z}^d)$ for all n.*

If $a \in \mathcal{A}_\theta$, then $[F(D), a]$ is compact.

Proof. Since $F \in \ell^\infty(\mathbb{Z}^d)$, we conclude that $\|F(D)\| = \|F\|_\infty < \infty$, so $\|[F(D), a]\| \leq 2\|F\|_\infty\|a\|$.

It was shown in section 6.3.1 that

$$[F(D), u(n)] = u(n)(F(n + D) - F(D)). \tag{6.110}$$

We now define a finite rank approximation to $F(D)$. Let $k \in \mathcal{N}$, then define

$$F_k(n) := \begin{cases} F(n), & \text{if } \|n\| < k \\ 0 & \text{otherwise.} \end{cases} \tag{6.111}$$

Hence $F_k(n + D) - F_k(D)$ is a finite rank operator.

We have assumed that $\{F(n + k) - F(k)\}_{k\in\mathbb{Z}^d} \in c_0(\mathbb{Z}^d)$. Thus, for every $\varepsilon > 0$, there is a $K > 0$ such that for $k > K$,

$$\|F(n + D) - F(D) - F_k(n + D) + F_k(D)\| < \varepsilon. \tag{6.112}$$

Thus we have that $[F_k(D), u(n)]$ converges in the operator norm to $[F(D), u(n)]$. But $[F_k(D), u(n)]$ is finite rank, hence $[F(D), u(n)]$ is compact.

Thus for any linear combination a of the terms $\{u(n)\}_{n\in\mathbb{Z}^d}$, we have that $[F(D), a]$ is compact. Since $\|[F(D), a]\| \leq 2\|F\|_\infty\|a\|$, we have that the map $a \mapsto [F(D), a]$ is continuous. Hence, for any a in the norm closure of the set $\text{span}\{u(n)\}_{n\in\mathbb{Z}^d}$, $[F(D), a]$ is compact. However the norm closure of the set $\text{span}\{u(n)\}_{n\in\mathbb{Z}^d}$ is exactly $\mathcal{A}_\theta$. □

Remark 6.25. The commutative equivalent of this is as follows. We select $d = 1$, then $\mathcal{N}_\theta = L^\infty(\mathbb{T})$, and $\mathcal{A}_\theta = C(\mathbb{T})$. If we choose $F(k) = \mathrm{sgn}(k)$, then for $f \in C(\mathbb{T})$, we have that $[F(D), M_f]$ is exactly the quantised differential of f. We see that since $\{\mathrm{sgn}(n+k) - \mathrm{sgn}(k)\}_{k\in\mathbb{Z}} \in c_0(\mathbb{Z})$ since it is a sequence of finite support, we have that $[\mathrm{sgn}(D), M_f]$ is compact when $f \in C(\mathbb{T})$. This is just a weaker version of proposition 4.3.

To demonstrate that we do not have analogues for most of the results of chapter 4, we must clarify the meaning of a quantised differential on a higher dimensional torus. We shall consider the two dimensional commutative case only.

Definition 6.26. Let σ_1 and σ_2 be the first and second Pauli matrices, that is:

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Let u and v be the first and second coordinate functions on the two-torus $\mathbb{T}^2 := \{(z, w) \in \mathbb{C}^2 : |z| = |w| = 1\}$, given by $u(z, w) = z$ and $v(z, w) = w$. Then we have the differentiation operators $\mathcal{D}_1$ and $\mathcal{D}_2$, given by $\mathcal{D}_1(u^n v^m) = n u^n v^m$ and $\mathcal{D}_2(u^n v^m) = m u^n v^m$.

Then the Dirac operator on the torus is defined to be:

$$\begin{aligned} \mathcal{D} &:= \sigma_1 \otimes \mathcal{D}_1 + \sigma_2 \otimes \mathcal{D}_2 \\ &= \begin{pmatrix} 0 & \mathcal{D}_1 - i\mathcal{D}_2 \\ \mathcal{D}_1 + i\mathcal{D}_2 & 0 \end{pmatrix}. \end{aligned}$$

which is an operator on the Hilbert space $\mathbb{C}^2 \otimes L^2(\mathbb{T}^2)$.

We now define the operators $\mathcal{D}_1/\sqrt{\mathcal{D}_1^2 + \mathcal{D}_2^2}$ and $\mathcal{D}_2/\sqrt{\mathcal{D}_1^2 + \mathcal{D}_2^2}$, by

$$\frac{\mathcal{D}_1}{\sqrt{\mathcal{D}_1^2 + \mathcal{D}_2^2}}(u^n v^m) := \begin{cases} \frac{n}{\sqrt{n^2+m^2}} u^n v^m \ (n, m) \neq (0, 0) \\ 0, (n, m) = 0. \end{cases} \tag{6.113}$$

The Hilbert transform on the two torus is then defined to be:

$$\mathrm{sgn}(\mathcal{D}) := \sigma_1 \otimes \frac{\mathcal{D}_1}{\sqrt{\mathcal{D}_1^2 + \mathcal{D}_2^2}} + \sigma_2 \otimes \frac{\mathcal{D}_2}{\sqrt{\mathcal{D}_1^2 + \mathcal{D}_2^2}} \tag{6.114}$$

For $f \in L^1(\mathbb{T}^2)$, we have the potentially not everywhere defined or bounded pointwise multiplication operator $1 \otimes M_f$ on $\mathbb{C}^2 \otimes L^2(\mathbb{T})$. Define,

$$đf := [\mathrm{sgn}(\mathcal{D}), M_f] \tag{6.115}$$

as the quantised differential on the two-torus.

Note that for any $f \in C(\mathbb{T}^d)$, we have $đf$ is compact by proposition 6.24.

Proposition 6.27. *Let v be the second coordinate function with quantised differential $đv$ as in definition 6.26. Then $đv \notin \mathcal{L}^1$.*

Proof. It is sufficient to look at the commutator,

$$\left[\frac{\mathcal{D}_1}{\sqrt{\mathcal{D}_1^2+\mathcal{D}_2^2}}, M_v\right]. \tag{6.116}$$

By proposition 6.23, we have that this operator is $\mathcal{L}^1$ only if

$$S := \sum_{(k_1,k_2)\in\mathbb{Z}^2\setminus\{(0,0),(0,-1)\}} \left|\frac{k_1}{\sqrt{k_1^2+(k_2+1)^2}} - \frac{k_1}{\sqrt{k_1^2+k_2^2}}\right| < \infty. \tag{6.117}$$

We specialise this to $k_1, k_2 > 0$,

$$S > \sum_{k_1,k_2>0} \left|\frac{k_1}{\sqrt{k_1^2+(k_2+1)^2}} - \frac{k_1}{\sqrt{k_1^2+k_2^2}}\right| \tag{6.118}$$

$$= \sum_{k_1>0}\sum_{k_2>0} \frac{k_1}{\sqrt{k_1^2+k_2^2}} - \frac{k_1}{\sqrt{k_1^2+(k_2+1)^2}}. \tag{6.119}$$

However the inner sum telescopes, so we have

$$S > \sum_{k_1>0} \frac{k_1}{\sqrt{k_1^2+1^2}} \tag{6.120}$$

$$= \infty. \tag{6.121}$$

Hence $đv \notin \mathcal{L}^1$. □

What this proposition demonstrates is that the results of Chapter 4 appear to have no analogue in the two dimensional case: even for a function that is extremely smooth (such as v), the quantised differential ($đv$) does not lie even within the ideal $\mathcal{L}^1$, so there is evidently no hope of finding a class of smooth functions whose quantised differentials lie in $\mathcal{L}^1$.

Remark 6.28. The computation in Proposition 6.27 applies equally well to noncommutative tori of any dimension.

Chapter 7

The Chain Rule in Quantised Calculus

7.1 Introduction

It is desirable to find a noncommutative generalisation of the chain rule. In classical analysis, as practiced by Newton and Leibniz, the chain rule could be stated as:

$$df(x) = f'(x)dx + o(dx) \tag{7.1}$$

where $f : \mathbb{R} \to \mathbb{R}$, and dx denotes some infinitesimal increment. The term $o(dx)$ denotes an infinitesimal quantity much smaller than dx. One could then "ignore sufficiently small infinitesimals" to get,

$$df(x) = f'(x)dx. \tag{7.2}$$

In the setting of quantised calculus, "ignoring sufficiently small infinitesimals" means working modulo some ideal of compact operators whose singular values decay sufficiently rapidly.

7.2 The setting

Let $\mathcal{H}$ be, as usual, a complex separable Hilbert space. We define symmetrically normed ideals of $\mathcal{B}(\mathcal{H})$.

Definition 7.1. Let $\mathcal{E}$ be a two-sided ideal of $\mathcal{B}(\mathcal{H})$ equipped with a norm $\|\cdot\|_{\mathcal{E}}$ which satisfies the inequality

$$\|ACB\|_{\mathcal{E}} \leq \|A\|\|C\|_{\mathcal{E}}\|B\| \tag{7.3}$$

for $A, B \in \mathcal{B}(\mathcal{H})$ and $C \in \mathcal{E}$.

Definition 7.2. If $\mathcal{E}$ is an ideal of operators containing the ideal of finite rank operators, we denote the closure of the finite rank operators in the $\|\cdot\|_{\mathcal{E}}$ norm as $\mathcal{E}_0$.

Definition 7.3 (Almost circular spectral triples)**.** Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple. We say that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is *almost circular* with respect to the symmetrically normed ideal $\mathcal{E}$ if the following properties are satisfied:

1. For all $a \in \mathcal{A}$, $\text{đ}a \in \mathcal{E}$.
2. $\mathcal{A}$ is closed under the holomorphic functional calculus.
3. Let $\mathcal{A}_0 \subseteq \mathcal{B}(\mathcal{H})$ be the collection of all T such that $[\mathrm{sgn}(\mathcal{D}), T]$ is finite rank. Then $\mathcal{A}$ is contained within the norm-closure of $\mathcal{A}_0$.
4. $\mathcal{A}$ is commutative.
5. $\mathcal{E}$ contains the finite rank operators.

7.3 The Commutator Lemma

Lemma 7.4. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be almost circular with respect to the ideal $\mathcal{E}$, and $a, b \in \mathcal{A}$. Then*

$$[\text{đ}a, b] \in \mathcal{E}_0.$$

Proof. Consider the map $T : \mathcal{A} \to \mathcal{E}$,

$$T(x) = [\text{đ}a, x].$$

Since $T(x)$ is a polynomial in x, T is continuous from $\mathcal{A}$ to $\mathcal{E}$ since $\mathcal{L}^{p,\infty}$ is a symmetrically normed ideal. Now since $\mathcal{A}$ is commutative,

$$[\text{đ}a, x] = [a, \text{đ}x].$$

Hence, for $x \in \mathcal{A}_0$, we have $T(x)$ is finite rank. Since $\mathcal{A}_0$ is dense in $\mathcal{A}$, we conclude that $T(x) \in \mathcal{E}_0$ for $x \in \mathcal{A}$. Now set $x = b$ and the claim is proved. □

7.4 The Chain rule

Lemma 7.5. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be almost circular with respect to the ideal $\mathcal{E}$. Let $p \in \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial, and $a_1, \ldots, a_n \in \mathcal{A}$. Then,*

$$\text{đ}p(a_1, \ldots, a_n) \equiv \sum_{k=1}^{n} \frac{\partial p}{\partial x_k}(a_1, \ldots, a_n)\, \text{đ}a_k \mod \mathcal{E}_0 \tag{7.4}$$

Proof. First we consider the case where p is a monomial of a single variable, that is $p(x_1,\dots,x_n) = x_j^k$ for some j,k. We can compute,

$$đp(a_1,\dots,a_n) = [F, a_j^k] \tag{7.5}$$

$$= \sum_{m=1}^{k} a_j^{m-1}[F,a_j]a_j^{k-m} \tag{7.6}$$

By lemma 7.4, this implies

$$đp(a_1,\dots,a_n) \equiv k a_j^{k-1}\, đa_j \mod \mathcal{E}_0. \tag{7.7}$$

Hence the claim is proved for $p(x_1,\dots,x_n) = x_j^k$. The general claim follows from the following Leibniz rule:

$$đ(ab) = (đa)b + b(đa) \tag{7.8}$$

$$\equiv b(đa) + a(đb) \mod \mathcal{E}_0 \tag{7.9}$$

and linearity. □

Proposition 7.6. *Let $(\mathcal{A},\mathcal{H},\mathcal{D})$ be an almost circular spectral triple with respect to the ideal $\mathcal{E}$. Let $a \in \mathcal{A}$, and let φ be holomorphic on a simply connected neighbourhood of $\sigma(a)$. Then define $\varphi(a)$ by the holomorphic functional calculus. Then,*

$$đ\varphi(a) \equiv \varphi'(a)đa \mod \mathcal{E}_0 \tag{7.10}$$

Proof. By lemma 7.5, the result is true when φ is a polynomial. Since φ is holomorphic on a simply connected set, there is a sequence $\{p_n\}_{n=1}^{\infty}$ that converges to φ uniformly on $\sigma(a)$, and $\{p_n'\}_{n=1}^{\infty}$ converges uniformly to φ. Thus the result follows. □

Our proposition 7.6 is inspired by Theorem 8b of Connes' book [6]. Connes' proves this result for $(\mathcal{A},\mathcal{H},\mathcal{D}) = (C^\infty(\mathbb{T}), L^2(\mathbb{T}), \mathcal{D}_{\mathbb{T}})$. In that book, the equivalent of 7.6 is proved for $\varphi \in C^\infty(\sigma(a))$, and $a = a^*$. So our result is neither more or less general than Connes', but it applies in a more abstract situation of an almost circular spectral triple.

Example 7.1. *An example of an almost circular spectral triple is $(C^\infty(\mathbb{T}), L^2(\mathbb{T}), \mathcal{D}_{\mathbb{T}})$. The ideal $\mathcal{E}$ should be chosen to be $\mathcal{L}^{1,\infty}$. Then the class of elements $a \in C^\infty(\mathbb{T})$ such that $đa$ is finite rank is exactly the rational functions.*

Chapter 8

An Application to Fractal Geometry

8.1 Introduction

So far we have only covered quantised calculus on its own, with the aim of proving analogies of classical facts. It is natural to wonder whether there are any uses of quantised calculus. This chapter very briefly, and without proofs, states some results about the application of quantised calculus to the computation of integrals over Julia sets. The following is based on the discussion in Chapter 4 of [6].

8.2 The Dixmier Trace

We have not yet discussed integration in quantised calculus. We give a very brief overview here. The book [6] goes into further detail, and the book [22] covers more technical topics.

Definition 8.1. A linear functional $\omega \in \ell^\infty(\mathbb{N})^*$ is called an extended limit if it satisfies the following properties:

1. $\|\omega\| \leq 1$.
2. $\omega(a) \geq 0$ for $a \geq 0$.
3. ω is shift invariant: meaning that if $B(a_0, a_1, \ldots) = (a_1, a_2, \ldots)$ is the shift operator, then $\omega = \omega \circ B$.

The existence of extended limits is a consequence of the Hahn-Banach theorem, and is proved in [22, Thm 6.2.5]. The use of extended limits is in the definition of a Dixmier trace.

Proposition 8.2. *Let ω be an extended limit, and let $T > 0$ be an operator in $\mathcal{M}_{1,\infty}$. Define,*

$$\mathrm{Tr}_\omega(T) := \omega \left\{ \frac{1}{\log(2^N+1)} \sum_{k=0}^{2^N} \mu_k(T) \right\}_{N>0}. \tag{8.1}$$

Then Tr_ω is additive, and extends to a linear functional on all of $\mathcal{M}_{1,\infty}$. Tr_ω is called a Dixmier trace.

Proposition 8.2 is proved in [22, Ch. 2].

8.3 Julia Sets

Definition 8.3. Let $c \in \mathbb{C}$, and let $\varphi(z) = z^2 + c$. Define

$$B := \{z \in \mathbb{C} \ : \ \sup_{n \geq 0}\{|\varphi^n(z)|\} < \infty\}. \tag{8.2}$$

Let $J := \partial B$. J is called the Julia set associated to φ.

When c is small, it can be shown that J is a Jordan curve whose complement has two connected components, an interior and exterior. See [25] for the theory of Julia sets.

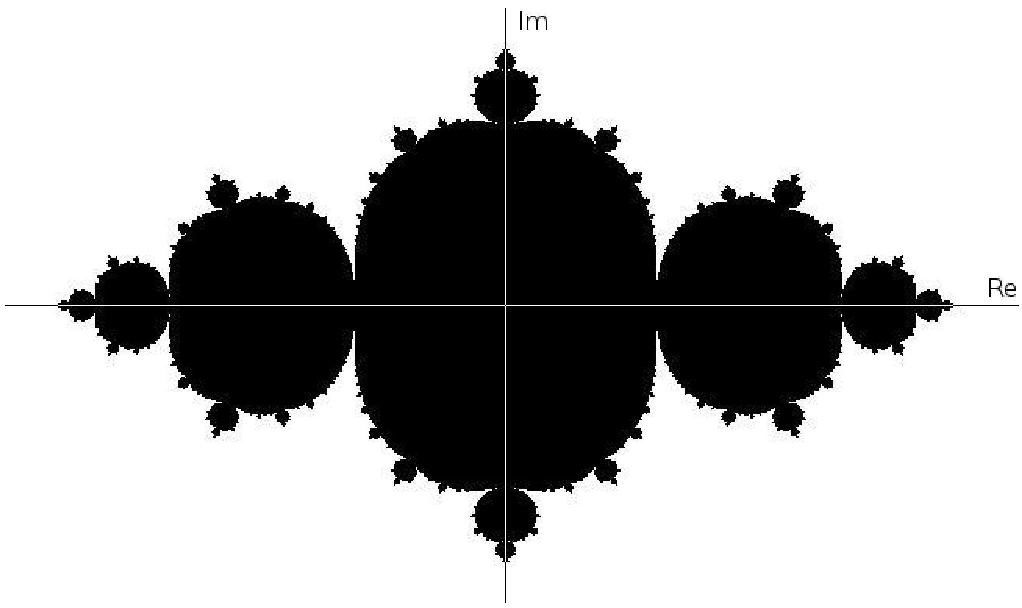

Figure 8.1: B for $c = -3/4$. © easyfractalgenerator.com

The conformal mapping theorem [26, Prop. 2.8.1] states that there is a holomorphic bijection Z between the open unit disc $\{z \in \mathbb{C} \ : \ |z| < 1\}$ and the interior of B. Then Carathéodory's theorem [26, Prop. 2.8.7] implies that Z extends to a homeomorphism from $\mathbb{T}$ to J.

Let p be the Hausdorff dimension of J.

There is a p-dimensional Hausdorff measure, Λ_p on J. See [27] for the theory of Hausdorff measures. Let $f \in C_0(\mathbb{C})$ and let Tr_ω be a Dixmier trace.

We have that $f \circ Z : \mathbb{T} \to \mathbb{C}$, and we consider $M_{f(Z)} \in \mathcal{B}(L^2(\mathbb{T}))$, and $đZ \in \mathcal{B}(L^2(\mathbb{T}))$.

We consider the quantity

$$\mathrm{Tr}_\omega(M_{f(Z)}|đZ|^p). \tag{8.3}$$

Remark 8.4. In Chapter 4, we found sufficient conditions that the map Z must satisfy so that $đZ \in \mathcal{L}^{p,\infty}$. These conditions are useful because the expression 8.3 is well defined when $đZ \in \mathcal{L}^{p,\infty}$. Connes [6, Ch. 4, Prop. 7] used a different method to show that $đZ \in \mathcal{L}^{p,\infty}$.

Herein lies the utility of quantised calculus. It is proved by Connes in [6, Ch. 4, Thm 17] that there is a non-zero constant λ, which does not depend on f, such that

$$\mathrm{Tr}_\omega(M_{f(Z)}|đZ|^p) = \lambda \int_J f \, d\Lambda_p. \tag{8.4}$$

Hence the quantised calculus provides a means of computing integrals with respect to Hausdorff measure on a Julia set.

Appendix A

Classical Harmonic Analysis

A.1 Introduction

Given a function $f : \mathbb{T} \to \mathbb{C}$, we have an associated Fourier series,

$$f \sim \sum_{n\in\mathbb{Z}} \hat{f}(n) z^n \tag{A.1}$$

where $z : \mathbb{T} \to \mathbb{T}$ is the identity function, and

$$\hat{f}(n) = \int_{\mathbb{T}} z^{-n} f \; d\boldsymbol{m} \tag{A.2}$$

where $\boldsymbol{m}$ is the normalised Haar (or arc length) measure on $\mathbb{T}$. Implicitly f is sufficiently regular so that these Fourier coefficients exist.

I have used the symbol "$\sim$" rather than an equals sign, since in general we do not have equality. In general, for $f \in L^1(\mathbb{T}, \boldsymbol{m})$, the Fourier series might diverge almost everywhere if it is interpreted as a sum.

However, if one turns to alternative methods of summation, it is possible to interpret $\sim$ as an equality.

Here we consider Abel summation.

A.2 Abel summation

Abel summation is inspired by Abel's theorem, which we prove now.

Proposition A.1. *Suppose that*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is a power series converging in $\{z \in \mathbb{C} \ : \ |z| < 1\}$ *such that the coefficients come from a Banach space* $(X, \|\cdot\|)$*. Suppose further that the sum*

$$\sum_{n=0}^{\infty} a_n$$

converges. Then,

$$\lim_{z \to 1^-} f(z) = \sum_{n=0}^{\infty} a_n.$$

where by $z \to 1^-$*, we mean that* z *is restricted to the subset of the unit disc where* $|1-z| \leq M(1-|z|)$ *for some constant* M*.*

Proof. Assume without loss of generality that

$$\sum_{n=0}^{\infty} a_n = 0.$$

Now define

$$s_k = \sum_{n=0}^{k} a_n$$

and $s_{-1} = 0$. Then we have

$$f(z) = \sum_{n=0}^{\infty} (s_n - s_{n-1}) z^n.$$

so

$$f(z) = (1-z) \sum_{k=0}^{\infty} s_k z^k.$$

Let $\varepsilon > 0$, and choose n large enough such that $\|s_k\| < \varepsilon$ for $k > n$. Then we have

$$\left\| (1-z) \sum_{k=n}^{\infty} s_k z^k \right\| \leq \varepsilon |1-z| \sum_{k=n}^{\infty} |z|^k = \varepsilon |1-z| \frac{|z|^n}{1-|z|} \leq M\varepsilon.$$

When z is sufficiently close to 1, we have

$$\left\| (1-z) \sum_{k=0}^{n-1} s_k z^k \right\| < \varepsilon.$$

Hence, for z sufficiently close to 1, we have

$$\|f(z)\| < (M+1)\varepsilon.$$

□

With this in mind, we define the Abel summation method.

Definition A.2. Let $\{a_k\}_{k=0}^{\infty} \subset X$ be a sequence in a Banach space $(X, \|\cdot\|)$. Suppose that for all $r \in (1-\varepsilon, 1)$, for some $\varepsilon > 0$ we have

$$\sum_{k=0}^{\infty} a_k r^k$$

exists. Then we define the Abel sum, denoted by,

$$A - \sum_{k=0}^{\infty} a_k$$

as

$$A - \sum_{k=0}^{\infty} a_k := \lim_{r \to 1^-} \sum_{k=0}^{\infty} a_k r^k.$$

if this limit exists.

Abel's theorem automatically implies that the Abel sum of a series agrees with the usual sum, however there are series which are summable in the Abel sense but not the classical sense. It is easy to see that,

$$A - \sum_{n=0}^{\infty} (-1)^n = \lim_{r \to 1^-} \sum_{n=0}^{\infty} (-r)^n = \lim_{r \to 1^-} \frac{1}{1+r} = \frac{1}{2}.$$

A.3 The Poisson Kernel

To sum a Fourier series in the Abel sense, an important technical tool is the Poisson Kernel, which we introduce in this section.

Given $f \in L^1(\mathbb{T}, \boldsymbol{m})$, define

$$A_r f := \sum_{n \in \mathbb{Z}} r^{|n|} \hat{f}(n) z^n.$$

with $r \in (0,1)$. $A_r f$ exists since the Fourier coefficients of f are bounded. By definition,

$$A - \sum_{n \in \mathbb{Z}} \hat{f}(n) z^n = \lim_{r \to 1^-} A_r f.$$

where the limit is taken in an appropriate Banach space.

Like with classical and Cesàro sums, Abel sums can be constructed with a convolution.

Proposition A.3. *We can write,*

$$A_r f = P_r * f.$$

where

$$P_r = 1 + \frac{rz}{1-rz} + \frac{r}{z-r}$$

Proof. The Fourier coefficients of $A_r f$ are the Fourier coefficients of f multiplied by the coefficients of

$$\sum_{n\in\mathbb{Z}} r^{|n|} z^n.$$

Hence define

$$P_r = \sum_{n\in\mathbb{Z}} r^{|n|} z^n.$$

The result follows from summing this geometric series. $\square$

The reason for the superiority of Abel summation over classical summation is that the Poisson kernels form an approximate identity.

Proposition A.4. *The Poisson kernels $\{P_r\}_{r\in(0,1)}$ form an approximate identity.*

Proof. By the formula,

$$P_r = \sum_{n\in\mathbb{Z}} r^{|n|} z^n,$$

we have

$$\int_{\mathbb{T}} P_r \, d\boldsymbol{m} = 1.$$

To complete the proof, we convert to coordinates. Let $z = \exp(2\pi i\theta)$, for $\theta \in (-1/2, 1/2)$, and we can regard P_r as a function of θ.

Then we have

$$\begin{aligned} P_r(\theta) &= 1 + \frac{re^{2\pi i\theta}}{1 - re^{2\pi i\theta}} + \frac{re^{-2\pi i\theta}}{1 - re^{-2\pi i\theta}} \\ &= 1 + \frac{re^{2\pi i\theta}(1 - re^{-2\pi i\theta}) + re^{-2\pi i\theta}(1 - re^{2\pi i\theta})}{(1 - re^{2\pi i\theta})(1 - re^{-2\pi i\theta})} \\ &= 1 + \frac{2r\cos(2\pi\theta) - 2r^2}{1 + r^2 - 2r\cos(2\pi\theta)} \\ &= \frac{1 - r^2}{1 - 2r\cos(2\pi\theta) + r^2} \end{aligned}$$

Let $\delta > 0$. Now we estimate,

$$\int_{\delta}^{1/2} P_r(\theta)\, d\theta \le \int_{\delta}^{1/2} \frac{1 - r^2}{1 - 2r\cos(2\pi\delta) + r^2}\, d\theta \le \frac{1 - r^2}{1 - 2r\cos(2\pi\delta) + r^2}$$

Hence this integral vanishes as $r \to \infty$.

Moreover, we have

$$1 - 2r\cos(2\pi\theta) + r^2 \ge (1 - r)^2.$$

Hence $P_r \ge 0$. Therefore, $\|P_r\|_1 = 1$.

Thus the Poisson kernels form an approximate identity. $\square$

As a consequence of this, we have

1. If $f \in C(\mathbb{T})$, then $A_r f \to f$ uniformly.

2. If $f \in L^p(\mathbb{T}, \boldsymbol{m})$, for $1 \leq p < \infty$, we have $A_r f \to f$ in the L^p sense.

A.4 Harmonic functions on $\mathbb{D}$

The reason that Abel summation is so important in harmonic analysis is that there is a close connection between the Abel sums $A_r f$ for a function f on $\mathbb{T}$, and functions on $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ which are *holomorphic*.

Given $f \in L^1(\mathbb{T})$, define $\tilde{f} : \mathbb{D} \to \mathbb{C}$ by

$$\tilde{f}(r\zeta) = A_r f(\zeta) = (P_r * f)(\zeta)$$

for $r \in [0, 1]$ and $\zeta \in \mathbb{T}$.

Note that, by Young's convolution inequality,

$$\left(\int_{\mathbb{T}} |\tilde{f}(r\zeta)|^p \, d\boldsymbol{m}(\zeta) \right)^{1/p} \leq \|P_r\|_1 \|f\|_p.$$

Hence,

$$\sup_{r \in [0,1)} \left(\int_{\mathbb{T}} |\tilde{f}(r\zeta)|^p \, d\boldsymbol{m}(\zeta) \right)^{1/p} \leq \|f\|_p.$$

This motivates the definition of the *Hardy spaces*:

Definition A.5 (Hardy Spaces)**.** Let $p \in (0, \infty]$. For $p < \infty$, let $H^p(\mathbb{T})$ denote the space of complex valued functions f which are complex differentiable in the open unit disc such that

$$\|f\|_{H^p} := \sup_{r \in [0,1)} \left(\int_{\mathbb{T}} |f(r\zeta)|^p \, d\boldsymbol{m}(\zeta) \right)^{1/p} < \infty. \tag{A.3}$$

For $p = \infty$, instead we require

$$\|f\|_{H^\infty} := \sup_{\zeta \in \mathbb{D}} |f(\zeta)| < \infty.$$

The link between Abel summation and Hardy spaces is provided by the following theorems, proved in [23]:

Theorem A.6. *Let $f \in L^p(\mathbb{T})$, for $p \in [1, \infty]$. Then $\tilde{f} \in H^p(\mathbb{T})$ if and only if $\hat{f}(n) = 0$ for $n < 0$.*

Theorem A.7. *Let $f \in H^p(\mathbb{T})$, for $p \in [1, \infty]$. Then, for almost all $\zeta \in \mathbb{T}$, the limit*

$$\lim_{r \to 1^-} f(r\zeta)$$

exists, and defines a function in $L^p(\mathbb{T})$.

Combining theorems A.6 and A.7, we have the following result, also proved in [23]

Theorem A.8. *Let $p \in [1, \infty]$. The space $H^p(\mathbb{T})$ embeds isometrically into the subspace of $L^p(\mathbb{T})$ consisting of those f with $\hat{f}(n) = 0$ for $n < 0$. The embedding ι is given by, for $f \in H^p(\mathbb{T})$*

$$\iota(f)(\zeta) = \lim_{r \to 1^-} f(r\zeta) \tag{A.4}$$

for almost all $\zeta \in \mathbb{T}$.

The map $f \mapsto \tilde{f}$, when restricted to the subspace of f with $\hat{f}(n) = 0$ for $n < 0$, is the inverse of ι.

Appendix B

Lorentz Spaces

B.1 Introduction

Lorentz spaces form a broad generalisation of L^p spaces and weak L^p spaces. These notes cover the basic definitions.

B.2 Weak L^p-spaces

Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space. Given a measurable function $f : X \to \mathbb{C}$, we may define the distribution function

$$d_f(\alpha) = \mu\{x \in X \ : \ |f(x)| \geq \alpha\}.$$

for $\alpha \geq 0$.

By Markov's inequality, for $p > 0$,

$$d_f(\alpha) \leq \frac{\|f\|_p^p}{\alpha^p}.$$

The linear span of the class of functions f for which

$$d_f(\alpha) \leq \frac{C^p}{\alpha^p}$$

for some constant C is called the weak L^p space or $L^{p,w}(X)$. Given $f \in L^{p,w}(X)$, define

$$\|f\|_{p,w} := \sup_{\alpha > 0} \alpha d_f(\alpha)^{1/p}.$$

B.3 Non-increasing Rearrangements

Again let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space, and $f : X \to \mathbb{C}$ is measurable. Then define, for $t \geq 0$,

$$f^*(t) = \inf\{s \geq 0 \ : \ d_f(s) \leq t\}.$$

f^* is called the non-increasing rearrangement of f.

For the following two lemmas, see [24, Prop 1.1.4].

Lemma B.1.

$$\left(\int_X |f|^p \ d\mu\right)^{1/p} = \left(\int_0^\infty (f^*)^p \ d\mu\right)^{1/p}$$

Lemma B.2.

$$\sup_{t>0} t^{1/p} f^*(t) = \sup_{\alpha>0} \alpha d_f(\alpha)^{1/p}.$$

Inspired by the above two results is the following definition,

Definition B.3 (Lorentz spaces)**.** Let $p, q > 0$. For a σ-finite measure space $(X, \mathcal{A}, \mu)$, and a measurable function $f : X \to \mathbb{C}$, define

$$\|f\|_{p,q} = \left(\int_0^\infty (t^{1/p} f^*(t))^q \frac{dt}{t}\right)^{1/q}$$

and

$$\|f\|_{p,\infty} = \sup_{t\geq 0} t^{1/p} f^*(t).$$

When X is a set with counting measure, the $L^{p,q}$ spaces are denoted $\ell^{p,q}(X)$.

Appendix C

Interpolation

C.1 Introduction

The method of interpolation is a very powerful one in analysis, and it allows many results to be obtained from "edge" cases.

A comprehensive account of interpolation theory is the book by Bergh and Löfström [28].

Historically, interpolation is motivated by the Riesz-Thorin theorem, which we state here. See [28, Section 1.1] for proof.

Theorem C.1. *Let $p_0, q_0 \in [1, \infty]$ and $p_1, q_1 \in [1, \infty]$ with $p_0 \neq p_1$ and $q_0 \neq q_1$. Suppose that (U, μ) and (V, ν) are measure spaces. Let T be an operator such that*

$$T : L^{p_0}(U) \to L^{q_0}(V)$$

with norm M_0 and

$$T : L^{p_1}(U) \to L^{q_1}(V)$$

with norm M_1.

Let $\theta \in (0, 1)$, and $p_\theta^{-1} = \theta p_0^{-1} + (1 - \theta)p_1^{-1}$, and $q_\theta^{-1} = \theta q_0^{-1} + (1 - \theta)q_1^{-1}$.

Then,

$$T : L^{p_\theta}(U) \to L^{q_\theta}(V)$$

with norm $M \leq M_0^\theta M_1^{1-\theta}$.

This theorem can be proved directly, however it is more insightful to prove it using abstract interpolation theory. The purpose of this appendix is to briefly introduce this theory.

C.2 Abstract Interpolation Theory

Let **NLS** be the category of normed linear spaces with morphisms given by bounded linear maps.

Definition C.2. A pair $X, Y \in \mathbf{NLS}$ is called a compatible pair if X and Y are both subspaces of a topological vector space U.

Definition C.3. The category $\mathbf{NLS}_1$ is the category of compatible pairs (X, Y) of normed spaces, where a morphism $T : (X_1, Y_1) \to (X_2, Y_2)$ is a linear map $T : X_1 + Y_1 \to X_2 + Y_2$ such that $T : X_1 \to X_2$ and $T : Y_1 \to Y_2$ is bounded.

Proposition C.4. *Let $\Delta : \mathbf{NLS}_1 \to \mathbf{NLS}$ be the function that maps (X, Y) to $X \cap Y$, where $X \cap Y$ is given the norm,*

$$\|x\|_{X\cap Y} = \max\{\|x\|_X, \|x\|_Y\}.$$

Let $\Sigma : \mathbf{NLS}_1 \to \mathbf{NLS}$ be given by $\Sigma((X, Y)) = X + Y$, where $X + Y$ is given the norm,

$$\|x\|_{X+Y} = \inf\{\|x_1\|_X + \|x_2\|_Y \ : \ x = x_1 + x_2, x_1 \in X, x_2 \in Y\}.$$

Then Δ and Σ are functors.

Definition C.5. An interpolation functor is a functor $\mathcal{F} : \mathbf{NLS}_1 \to \mathbf{NLS}$ such that for $(X, Y) \in \mathbf{NLS}_1$, we have

$$\Delta((X, Y)) \subseteq \mathcal{F}((X, Y)) \subseteq \Sigma((X, Y))$$

We say that an interpolation functor $\mathcal{F}$ is *uniform* if for any morphism $T : (X_1, Y_1) \to (X_2, Y_2)$ in $\mathbf{NLS}_1$, we have

$$\|\mathcal{F}(T)\| \leq C \max\{\|T\|_{X_1\to X_2}, \|T\|_{Y_1\to Y_2}\}.$$

for some constant $C > 0$. If $C = 1$, we say that $\mathcal{F}$ is *exact.*

We say that an interpolation functor $\mathcal{F}$ is of exponent $\theta \in (0, 1)$ if

$$\|\mathcal{F}(T)\| \leq C\|T\|^{\theta}_{X_1\to X_2}\|T\|^{1-\theta}_{Y_1\to Y_2}.$$

for some constant $C > 0$. If $C = 1$, we say that $\mathcal{F}$ is *exact of exponent θ.*

C.3 Real Interpolation: The K Method

Definition C.6. Let $(X_0, X_1) \in \mathbf{NLS}_1$. For $x \in X_0 + X_1$ and $t > 0$, define

$$K(x, t; X_0, X_1) = \inf\{\|x_0\| + t\|x_1\| \ : \ x_0 \in X_0, x_1 \in X_1, x = x_0 + x_1\}.$$

Then for $\theta \in (0, 1)$ and $q \in [1, \infty)$, define

$$\|x\|_{\theta,q;K} := \left(\int_0^\infty (t^{-\theta} K(x, t; X_0, X_1))^q \, \frac{dt}{t} \right)^{1/q}$$

and for $\theta \in [0, 1]$,

$$\|x\|_{\theta,\infty;K} := \sup_{t>0} t^{-\theta} K(x, t; X_0, X_1).$$

The space $K(X_0, X_1)_{\theta,q}$ is the set of $x \in X_0 + X_1$ such that $\|x\|_{\theta,q;K} < \infty$.

Proposition C.7. *The function $(X, Y) \mapsto K(X, Y)_{\theta,q}$ is an exact interpolation functor of order θ.*

C.4 Complex Interpolation

Definition C.8. Let (X_0, X_1) be a compatible pair of Banach spaces. Let $\mathcal{S} := \{z \in \mathbb{C} \ : \ \Re(z) \in (0, 1)\}$.

Define the set $\mathcal{F}(X_0, X_1)$ to be the space of functions $f : \overline{\mathcal{S}} \to X_0 + X_1$ which are complex differentiable in $\mathcal{S}$, continuous on $\overline{\mathcal{S}}$ and bounded on $\partial\mathcal{S}$.

It is true that $\mathcal{F}(X_0, X_1)$ is a Banach space under the norm,

$$\|f\|_{\mathcal{F}(X_0,X_1)} = \max\{\sup_{t\in\mathbb{R}} \|f(it)\|, \sup_{t\in\mathbb{R}} \|f(1 + it)\|\}.$$

Let $\theta \in (0, 1)$. Then, define

$$(X_0, X_1)_\theta = \{f(\theta) \ : \ f \in \mathcal{F}(X_0, X_1)\}.$$

We define the norm,

$$\|x\|_{(X_0,X_1)_\theta} = \inf\{\|f\|_{\mathcal{F}(X_0,X_1)} \ : \ x = f(\theta)\}.$$

Proposition C.9. *The mapping $(X_0, X_1) \to (X_0, X_1)_\theta$ is an interpolation functor, which is exact of exponent θ.*

C.5 Using interpolation

Once we know how to describe certain interpolation spaces, results such as the Riesz-Thorin theorem become immediate. The following is Theorem 5.1.1 of [28].

Proposition C.10. *Let X be a measure space. Let $p_0, p_1 \in [1, \infty]$, and $p_\theta^{-1} = \theta p_0^{-1} + (1 - \theta)p_1^{-1}$ for $\theta \in (0,1)$. Then*

$$(L^{p_0}(X), L^{p_1}(X))_\theta = L^{p_\theta}(X).$$

Another immediate corollary:

Proposition C.11. *Let G be a locally compact group abelian group equipped with bivariant Haar measure μ, and $p \in [1, \infty]$. For $\varphi \in L^1(G, \mu)$, the mpa $Tf = \varphi * f$ is bounded on $L^p(G, \mu)$, with norm less than or equal to $\|\varphi\|_1$.*

Proof. To prove the case $p = 1$, we let $f \in L^p(G)$, and compute,

$$\begin{aligned} \|\varphi * f\|_1 &= \int_G \left| \int_G \varphi(x-y) f(y) \, d\mu(y) \right| \, d\mu(x) \\ &\leq \int_G \int_G |\varphi(x-y)||f(y)| \, d\mu(y) \, d\mu(x) \\ &= \|\varphi\|_1 \|f\|_1 \end{aligned}$$

where the interchange of integrals is justified by Tonelli's theorem.

Now for $p = \infty$, we compute,

$$\|\varphi * f\|_\infty = \text{ess-sup}_{x \in G} \left| \int_G \varphi(x-y) f(y) \, d\mu(y) \right|$$

By Hölder's inequality, this can be bounded by $\|\varphi\|_1 \|f\|_\infty$.

The rest of the cases follow from complex interpolation of the pair $(L^1(G, \mu), L^\infty(G, \mu))$. $\square$

Appendix D

Dirac Operators

D.1 Introduction

This chapter is intended to give an introduction to the relationship between the Dirac operator and the exterior algebra. The general philosophy is that any expressions involving coordinates are to be avoided.

Throughout these notes, (M, g) is a Riemannian manifold. For an introduction to Riemannian geometry, see [19].

D.2 Music, Clifford bundles and Modules

A metric g on a manifold M gives us a canonical isomorphism between T^*M and TM, called $\sharp$, pronounced "sharp". For $x \in M$, given a linear functional $\omega \in T_x^*M$ we define $\sharp\omega$ to be the unique vector such that $\omega(v) = g(\sharp\omega, v)$ for all $v \in T_xM$. This is called the "musical isomorphism".

The Clifford bundle of (M, g) is a vector bundle on M defined as follows.

Definition D.1. Let $x \in M$. The Clifford algebra at x, $\mathrm{Cliff}_x(M, g)$ is defined as the free associative unital algebra generated by T_xM modulo the relation

$$uv + vu = -2g(u, v)1 \tag{D.1}$$

where $u, v \in \mathrm{Cliff}_x(M, g)$, and 1 is the identity in $\mathrm{Cliff}_x(M, g)$.

The Clifford bundle $\mathrm{Cliff}(M, g)$ is the vector bundle on M whose fibres are $\mathrm{Cliff}_x(M, g)$.

There is an embedding $TM \hookrightarrow \mathrm{Cliff}(M, g)$.

Now for a vector bundle V on M, we say that V is a Clifford module if there is a right multiplication map $\gamma : \mathrm{Cliff}(M, g) \otimes V \to V$.

A *connection* on V is a linear map

$$\nabla : \Gamma(V) \to \Omega^1(M) \otimes \Gamma(V). \tag{D.2}$$

satisfying the Leibniz rule, for $f \in C^\infty(M)$ and $v \in \Gamma(V)$,

$$\nabla(fv) = df \otimes v + f\nabla(v). \tag{D.3}$$

Now we may define a *Dirac operator*. Suppose V is a Clifford bundle with connection ∇. Then the composition of linear maps,

$$\Gamma(V) \xrightarrow{\nabla} \Omega^1(M) \otimes \Gamma(V) \xrightarrow{\sharp\otimes I} \mathcal{X}(M) \otimes \Gamma(V) \xrightarrow{\gamma} \Gamma(V) \tag{D.4}$$

is called the Dirac operator associated with V and ∇.

D.3 Relationship with differentials

Let V be a Clifford module with connection ∇.

Via the musical isomorphism, we may regard any differential form $\omega \in \Gamma(T^*M)$ as an operator on $\Gamma(V)$, since $\sharp(\omega)$ is an element of $\Gamma(\mathrm{Cliff}(M, g))$, it may act on V.

Similarly, by pointwise multiplication, any $f \in C^\infty(M)$ is an operator on $\Gamma(V)$.

Theorem D.2. *We have an equality of operators on* $\Gamma(V)$,

$$[D, f] = df. \tag{D.5}$$

for any $f \in C^\infty(M)$.

Proof. Let $f \in C^\infty(M)$ and $v \in \Gamma(v)$. Let us compute $D(fv)$.

By definition,

$$D(fv) = (\gamma \circ (\sharp \otimes I) \circ \nabla)(fv). \tag{D.6}$$

By the Leibniz rule,

$$(\sharp \otimes I)\nabla(fv) = \sharp(df) \otimes v + (I \otimes f)(\sharp \otimes 1)\nabla(v). \tag{D.7}$$

Hence,

$$D(fv) = \gamma(\sharp(df))v + fD(v). \tag{D.8}$$

Therefore, $[D, f]v = df(v)$. □

Appendix E

Ideals Of Compact Operators

E.1 Two-sided Ideals

In what follows, let $\mathcal{H}$ be a separable complex infinite dimensional Hilbert space.

Definition E.1. A linear subspace $\mathcal{J}$ of $\mathcal{B}(\mathcal{H})$ is called a two sided ideal if for any $A \in \mathcal{J}$ and $B \in \mathcal{B}(\mathcal{H})$, we have $AB, BA \in \mathcal{J}$.

In this thesis we have been primarily concerned with ideals of compact operators. Recall that for a compact operator $T \in \mathcal{K}(\mathcal{H})$, the sequence $\{\mu_n(T)\}_{n=0}^{\infty}$ of singular values is a sequence of positive numbers vanishing towards zero. Let $\mu : \mathcal{K}(\mathcal{H}) \to c_0(\mathbb{N})$ denote the mapping $T \mapsto \{\mu_n(T)\}$.

If we regard compact operators as infinitesimals, we determine the "size" of an infinitesimal T (i.e. a compact operator) in terms of the rate of decay of $\mu(T)$.

It turns out that there is an extremely useful description of ideals of compact operators in terms of sequences of singular values.

E.2 The Calkin Correspondence

Definition E.2. For a sequence $x \in c_0(\mathbb{N})$, let x^* denote the non-increasing rearrangement (see Appendix B for details).

A subspace $J \subseteq c_0(\mathbb{N})$ is called a *Calkin space* if for any positive sequences $x, y \in c_0(\mathbb{N})$, with $x^* \leq y^*$ (component-wise), and $y \in J$, then $x \in J$.

The following is proved is Theorem 1.2.3 of [22].

Proposition E.3 (The Calkin Correspondence)**.** *Let $\mathcal{J}$ be a two-sided ideal of compact operators on $\mathcal{H}$. Associate to $\mathcal{J}$ the following subset of $c_0(\mathbb{N})$,*

$$J_+ = \{\mu(T) \ : \ T \in \mathcal{J}\}. \tag{E.1}$$

Denote by J the linear subspace of $c_0(\mathbb{N})$ generated by J_+.

Then J is a Calkin space.

Conversely, given a Calkin space J, we construct an ideal $\mathcal{J}$ as follows. For a sequence $x \in c_0(\mathbb{N})$, let $\mathrm{Diag}(x)$ denote the operator on $\mathcal{H}$ that is a diagonal matrix with nth diagonal entry x_n with respect to a given fixed orthonormal basis $\{e_n\}_{n=0}^{\infty}$ of $\mathcal{H}$. Let $\mathcal{S}$ be the subset of $\mathcal{K}(\mathcal{H})$ given by

$$\mathcal{S} = \{\mathrm{Diag}(x) \ : \ x \in J\}. \tag{E.2}$$

Let $\mathcal{J}$ be the ideal generated by $\mathcal{S}$.

The correspondence $J \leftrightarrow \mathcal{J}$ is a bijection between Calkin spaces and ideals of compact operators.

E.3 Lorentz and Macaev Ideals

The Calkin correspondence inspires the definitions of a vast array of ideals:

Definition E.4. The space $\mathcal{L}^{p,q} \subseteq \mathcal{K}(\mathcal{H})$ is the linear span of all positive operators T such that $\mu(T) \in \ell^{p,q}$ (see appendix B for the definition of $\ell^{p,q}$).

Definition E.5. The space $m_{1,\infty} \subseteq c_0(\mathbb{N})$ is defined to be the set formed by the linear span of all positive sequences x such that

$$\sup_{N\geq 0} \left\{ \frac{1}{\log(N+1)} \sum_{n=0}^{N} x_n \right\} < \infty. \tag{E.3}$$

The ideal $\mathcal{M}_{1,\infty}$ is defined to be the linear span of the set of positive operators T such that $\mu(T) \in m_{1,\infty}$.

Since it is easily verified (as is done in [22]) that the spaces $\ell^{p,q}$ and $m_{1,\infty}$ are Calkin spaces, it follows that $\mathcal{L}^{p,q}$ and $\mathcal{M}_{1,\infty}$ are two-sided ideals.

For our purposes, the following description of $\mathcal{M}_{1,\infty}$ will be useful, proved in Theorem 4.5 of [18].

Proposition E.6. *Let $x \in \mathcal{M}_{1,\infty}$. Then there is an equivalence of norms,*

$$\|x\|_{\mathcal{M}_{1,\infty}} \cong \limsup_{s\downarrow 0} s\|x\|_{\mathcal{L}^{s+1}}. \tag{E.4}$$

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