Assignment 4

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Course: *Engineering Mathematics (ENG1005)*
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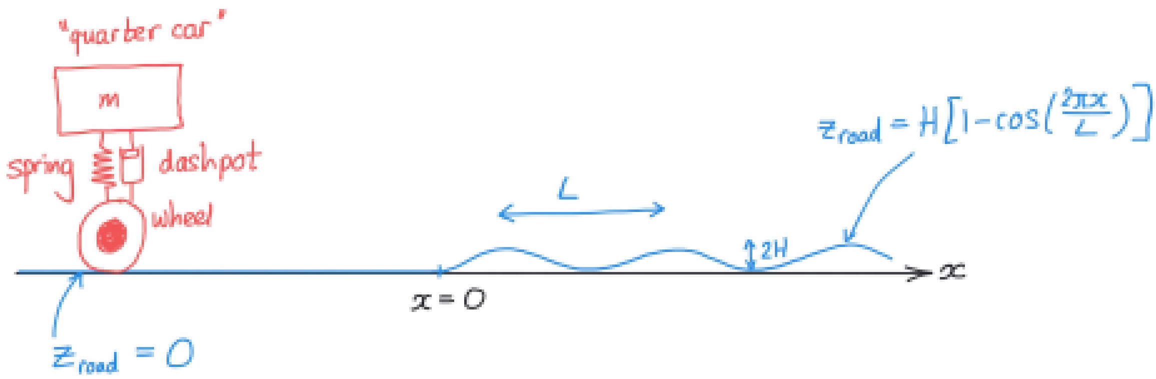
A Bumpy Ride?

Corrugations are periodic ripples that often form on the surfaces of dry dirt roads. Driving on such roads can be unpleasant and tricky. In this assignment, we will explore whether there is any truth to the urban legend that driving faster on such roads leads to a less bumpy ride.

Simple descriptions of a car's suspension consider a single wheel supporting a quarter of the mass of the car. The suspension itself is modelled as a damped spring. This setup is sketched below



We will assume that the car is travelling at a horizontal speed V over the road. This speed is constant, but the driver can choose its value. For $x < 0$, the road is flat $z_{road} = 0$ and for $x > 0$ there are corrugations given by

$$z_{road}(x) = H[1 - \cos(\frac{2\pi x}{L})]$$

where H is the height of each corrugation and L is its length. We will assume that time $t = 0$ occurs when the car just reaches $x = 0$. Then the equation governing the vertical displacement $z(t)$ of the car body is

$$mz'' = -k[z - z_{road}(Vt)] - Dz'$$

where primes indicate differentiation with respect to time, m is the quarter mass of the car, D is the damping factor of the suspension and k is the spring constant. We will take $m = \frac{1}{2}$, $D = 5$, $k = 37$, $H = 0.1$, $L = 0.2$ (these are crudely reasonable numbers if masses are measured in tonnes, lengths in metres and time in seconds). We will explore the response for different speeds V .

Question 1

Briefly explain what each term in the differential equation above represents. Your explanation should include why the argument of z_{road} is Vt.

Answer. What Each Term Represents:

Term	Represents
D	Dampening Factor of Suspension
k	Spring Constant
m	$\frac{1}{4}$ Mass of Car
V	Speed of Car
z	Vertical Displacement of the Car
z'	Vertical Velocity of the Car
z''	Vertical Acceleration of the Car
z_{road}	Hieght of road

- The input to function $z_{road}(x)$, x, is a distance from the start of the corrugations.
- The car drives at speed V the elapsed time sie$t$ is a speed multiplied by time which is a distance.
- Therefore the

Question 2

Solve for the homogeneous solution of the differential equation.

Answer. Rearrange ODE,

$$
\begin{aligned}
mz'' &= -k[z - z_{road}(Vt)] - Dz' && \text{(given)} \\
m\frac{d^2z}{dt^2} &= -kz + k \times z_{road}(Vt) - D\frac{dz}{dt} \\
m\frac{d^2z}{dt^2} + D\frac{dz}{dt} + kz &= k \times z_{road}(Vt)
\end{aligned}
$$

For homogeneous solution, let $z_h(t) = Ae^{\lambda t}$, $\frac{dz_h}{dt} = A\lambda e^{\lambda t}$ and $\frac{d^2z_h}{dt^2} = A\lambda^2 e^{\lambda t}$.

Solve homogenous ODE,

$$
\begin{aligned}
0 &= m\frac{d^2z_h}{dt^2} + D\frac{dz_h}{dt} + kz_h && \text{(homogenous)} \\
&= \frac{1}{2} \times \frac{d^2z_h}{dt^2} + 5\frac{dz_h}{dt} + 37z_h && \text{(substitution)} \\
&= \frac{1}{2} \times A\lambda^2 e^{\lambda t} + 5A\lambda e^{\lambda t} + 37Ae^{\lambda t} && \text{(substitution)} \\
\because Ae^{\lambda t} &\neq 0 \\
\frac{1}{2} \times \lambda^2 + 5\lambda + 37 &= 0 \\
&= \lambda^2 + 10\lambda + 74 \\
\lambda &= -5 \pm \sqrt{5^2 - 74} \\
&= -5 \pm 7i
\end{aligned}
$$

The homogenous solution z_h is,

$$
z_h = \tilde{A}e^{-5t}\cos(7t) + \tilde{B}e^{-5t}sin(7t) \qquad \text{(Real Solution)}
$$

where $\tilde{A}$ and $\tilde{B}$ are real numbers.

Question 3

What is the long-term behaviour of the homogeneous solution as $t \to \infty$?

Answer.

$$\begin{aligned} z_h &= \tilde{A}e^{-5t}\cos(7t) + \tilde{B}e^{-5t}sin(7t) && \text{(Question 2)} \\ \lim_{t\to\infty} z_h &= \lim_{t\to\infty}(\tilde{A}e^{-5t}\cos(7t) + \tilde{B}e^{-5t}sin(7t)) \\ &= \lim_{t\to\infty}(e^{-5t}) \times \lim_{t\to\infty}(\tilde{A}\cos(7t) + \tilde{B}sin(7t)) \end{aligned}$$

$$\because \lim_{t\to\infty}(e^{-5t}) = 0$$

and

$$\lim_{t\to\infty}(\tilde{A}\cos(7t) + \tilde{B}sin(7t)) = DNE \qquad \text{(Does Not Exist)}$$

$$\therefore \lim_{t\to\infty} z_h = 0$$

As the long-term behaviour of the homogeneous solution as $t \to \infty$ is $z_h \to 0$.

Question 4

Based on this long-term behaviour, we will only look at the particular solution to the differential equation from now on. To make the algebra cleaner, we define a value

$$a = \frac{2\pi V}{L}$$

Solve for the particular solution to this differential equation. You should show that the particular solution is given by

$$z(t) = b\cos(at) + c\sin(at) + H$$

where the constants b and c are given by

$$b = \frac{-(k-a^2m)kH}{(k-a^2m)^2+(aD)^2} \quad \text{and} \quad c = \frac{-aDkH}{(k-a^2m)^2+(aD)^2}$$

Answer. For particular solution, $z_p(t)$, guess

$$\begin{aligned} z_p(t) &= b\cos(at) + c\sin(at) + H \\ \frac{dz_p}{dt} &= ac\cos(at) - ab\sin(at) \\ \frac{d^2z_p}{dt^2} &= -ab^2\cos(at) - ac^2\sin(at) \end{aligned}$$

Solve particular ODE,

$$m\frac{d^2z_p}{dt^2} + D\frac{dz_p}{dt} + kz_p = k \times z_{road}(Vt) \qquad \text{(Question 2)}$$

$$\begin{aligned}
LHS &= m(-ab^2\cos(at) - ac^2\sin(at)) \\
&\quad + D(ac\cos(at) - ab\sin(at)) \\
&\quad + k(b\cos(at) + c\sin(at) + H) && \text{(Substitution)} \\
&= -ab^2m\cos(at) - ac^2m\sin(at) \\
&\quad + acD\cos(at) - abD\sin(at) \\
&\quad + bk\cos(at) + ck\sin(at) + kH \\
&= (-ab^2m + acD + bk)\cos(at) \\
&\quad + (-ac^2m - abD + ck)\sin(at) + kH
\end{aligned}$$

$$\begin{aligned}
RHS &= kH[1 - \cos(\frac{2\pi Vt}{L})] && \text{(Substitution)} \\
&= kH[1 - \cos(at)] && \text{(Substitution)} \\
&= kH - kH\cos(at)
\end{aligned}$$

Equating RHS and LHS,

$$\begin{aligned}
kH - kH\cos(at) &= (-ab^2m + acD + bk)\cos(at) + (-ac^2m - abD + ck)\sin(at) + kH \\
-kH\cos(at) &= (-ab^2m + acD + bk)\cos(at) + (-ac^2m - abD + ck)\sin(at)
\end{aligned}$$

$$\therefore -kH = -ab^2m + acD + bk$$

$$\begin{aligned}
\text{and} \quad 0 &= -ac^2m - abD + c \\
b &= \frac{c - ac^2m}{aD}
\end{aligned}$$

$$\begin{aligned}
c &= \frac{-aDkH}{(k - a^2m)^2 + (aD)^2} \\
-kH &= -a(\frac{c - ac^2m}{aD})^2m + acD + (\frac{c - ac^2m}{aD})k \\
&= -\frac{am(c(1 - acm))^2}{a^2D^2} + acD + \frac{kc(1 - acm)}{aD} \\
-aDkH &= -\frac{mc^2(1 - acm)^2}{D} + c + kc(1 - acm) \\
&= c(-\frac{mc(1 - acm)^2}{D} + 1 + k(1 - acm))
\end{aligned}$$

The particular solution z_p is,

Question 5

Plot graphs of the particular solutions of this ODE for a variety of values of a. Be sure to include relatively slow and relatively fast speeds.

Answer.

Question 6

Describe the amplitudes of the solutions plotted in the last question. How does the speed affect the amplitude?

Answer.

Question 7

The comfort of the passengers in the vehicle depends on the vertical acceleration $z''(t)$ of the vehicle. Plot graphs of the acceleration of the vehicle for the speeds used in the earlier plots.

Answer.

Question 8

What do your plots in the previous question say about the experience of driving over these ridges at various speeds?

Answer.