Finite difference schemes for linear PDE

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September 19, 2022

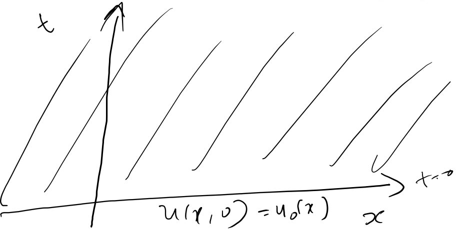
We consider

$$Pu = u_t + au_x = 0, \quad -\infty < x < \infty, t > 0, \quad u = u(x,t)$$

with initial condition

$$u(x,0) = u_0(x)$$

$a \in \mathbb{R}$



$u_t + a u_x = 0$

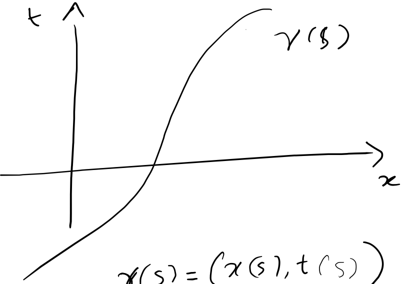
$u = u(x,t)$

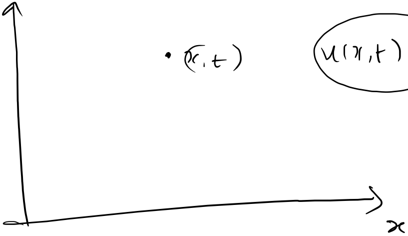
$u(x,t) = u_0(x - at)$

$u(x,10)$

$u_t + a u_x = 0$

$u(x,0) = 0$





$u(x,t) = ?$

$\gamma(s) = (x(s), t(s))$

$z(s) = u(\gamma(s))$

$= u(x(s), t(s))$

$z'(s) = u_x \times x'(s) + u_t \times t'(s)$

$z'(s) = 0$ if $x'(s) = a$

$t'(s) = 1$

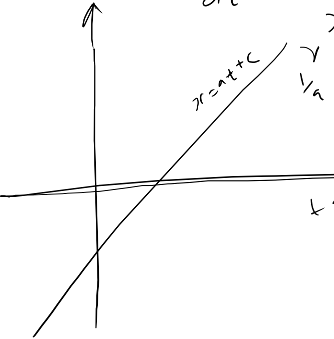
If $\dfrac{dx}{ds} = a$, $\dfrac{dt}{ds} = 1$ then $z'(s) = 0$

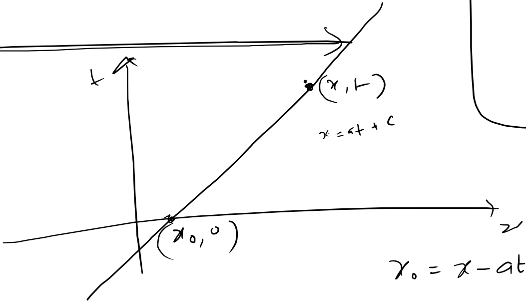
$z(s) = \text{constant}$

$u(x(s), t(s)) = \text{constant}$

$$\frac{dx}{dt} = a$$

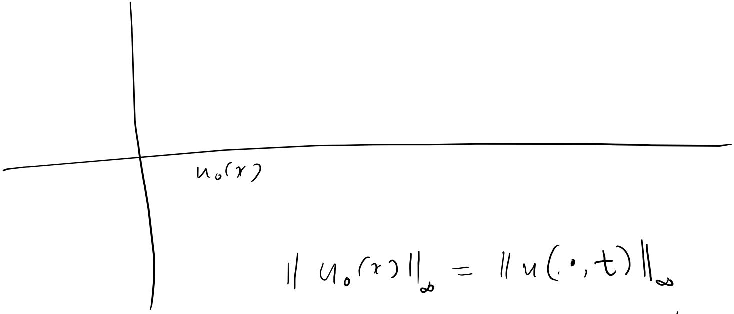
$$x = at + c$$





$$\boxed{\begin{aligned} u(x,t) &= u(x_0, 0) \\ &= u_0(x_0) \\ &= u_0(x-at) \end{aligned}}$$

$x_0 = x - at$

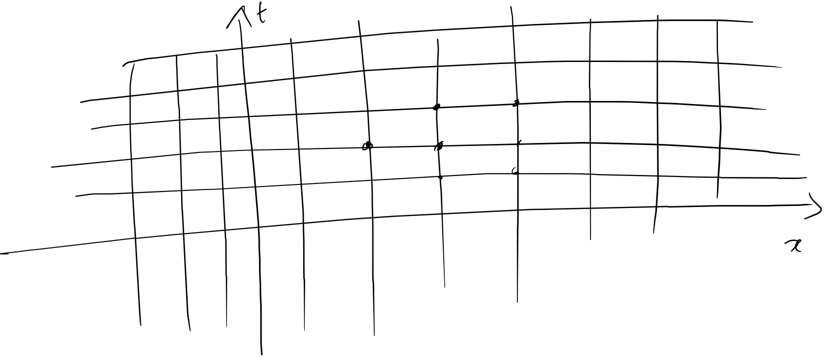


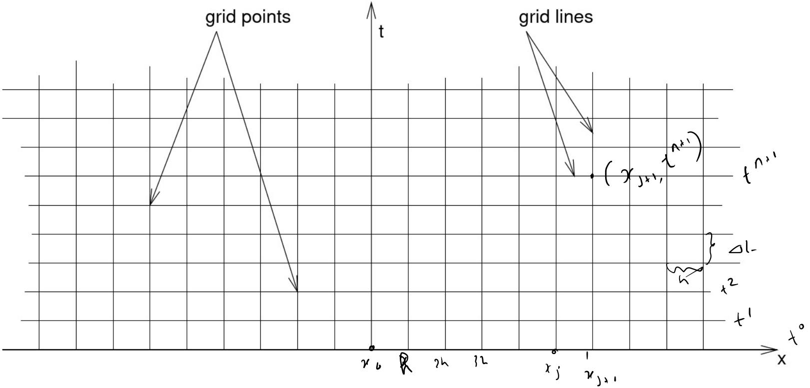
$$\| u_0(x) \|_\infty = \| u(\cdot, t) \|_\infty$$

$$\| \quad \|_2 = \| \quad \|_2$$

First we define the grid points in the (x, t) plane by drawing vertical and horizontal lines through the points (x_i, t_n)

$u_t + a u_x = 0$





$$
\begin{aligned}
t_n &= n\Delta t, \quad n = 0, 1, 2, \ldots \\
x_j &= jh, \quad j = 0, \pm 1, \pm 2, \ldots
\end{aligned}
$$

The lines $x = x_i$ and $t = t_n$ are called grid lines and their intersections are called mesh points of the grid.

The basic idea of **finite difference method is to replace derivatives by finite differences.** This can be done in many ways;

$\frac{\partial u}{\partial t} := u_t$, $\frac{\partial}{\partial x} u := u_x$

$u \to$ exact solution

(x_j, t^n)

$$u(x_j, t^{n+1}) = u(x_j, t^n) + \Delta t\, u_t(x_j, t^n) + O(\Delta t^2)$$

$$u_t(x_j, t^n) = \frac{u(x_j, t^{n+1}) - u(x_j, t^n)}{\Delta t} + O(\Delta t)$$

$$u(x_{j-1}, t^n) = u(x_j, t^n) - \Delta x\, u_x(x_j, t^n) + O(\Delta x^2)$$

$$u_x(x_j, t^n) = \frac{u(x_j, t^n) - u(x_{j-1}, t^n)}{\Delta x} + O(\Delta x)$$

$u(x_j, t^n) = u_j^n$

$$Pu(x_j, t^n) = \frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_j^n - u_{j-1}^n}{\Delta x} + O(\Delta t) + O(\Delta x) = 0$$

$$\to \frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_j^n - u_{j-1}^n}{\Delta x} \approx 0 \quad =$$

By replacing the derivatives by finite differences and neglecting the error terms we have list of difference equations. For example

$$P_{\Delta t,h}v = \frac{v_j^{n+1} - v_j^n}{\Delta t} + a\,\frac{(v_{j+1}^n - v_j^n)}{h} = 0 \text{ (forward time -- forward space)}$$

FTFS

$$P_{\Delta t,h}v = \frac{v_i^{n+1} - v_j^n}{\Delta t} + a\frac{(v_j^n - v_{j-1}^n)}{h} = 0 \text{ (forward time -- backward space)}$$

$$P_{\Delta t,h}v = \frac{v_j^{n+1} - v_j^n}{\Delta t} + a\frac{(v_{j+1}^n - v_{j-1}^n)}{2h} = 0 \text{ (forward time -- central space)}$$

$$P_{\Delta t,h}v = \frac{v_j^n - v_j^{n-1}}{\Delta t} + a\frac{(v_{j+1}^n - v_j^n)}{h} = 0 \text{ (backward time -- forward space)}$$

$$P_{\Delta t,h}v = \frac{v_j^n - v_j^{n-1}}{\Delta t} + a\frac{(v_j^n - v_{j-1}^n)}{h} = 0 \text{ (backward time -- backward space)}$$

$P_{\Delta t,h}$

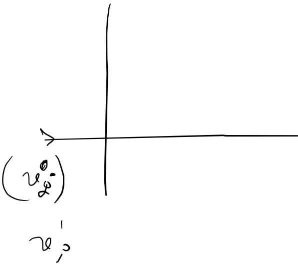
Cosistency, Stability and **convergence**

Pu

Replace u_j^n by v_j^n approximation u

$$\frac{v_j^{n+1} - v_j^n}{\Delta t} + a\,\frac{v_j^n - v_{j-1}^n}{\Delta x} = 0 \quad \text{—— FD scheme.}$$

$$v_j^{n+1} = v_j^n - a\frac{\Delta t}{\Delta x}\left(v_j^n - v_{j-1}^n\right)$$



Difference equation $=$ differential equ $+ \tau$

$$\left.\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\left(\frac{u_j^n - u_{j-1}^n}{\Delta x}\right)\right|_{(x_j, t^n)} = \underline{u_t + a u_x}_{(x_j, t^n)} + (\tau_j^n)$$

$$\frac{v_j^{n+1} - v_j^n}{\Delta t} + \frac{a}{\Delta x}(v_j^n - v_{j-1}^n) = 0$$

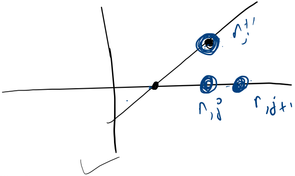
$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_j^n - u_{j-1}^n}{\Delta x} = u_t + O(\Delta t) + a u_x + O(\Delta x)$$

$$= u_t + a u_x + \underline{O(\Delta t) + O(\Delta x)}$$

$$= 0 + \tau_j^n$$

$$P_{\Delta t, h} u(x_j, t^n) = \tau_j^n$$

Solution need not converge always!



$$u_t + u_x = 0$$

$$u(x,0) = u_0(x) = \begin{cases} 1 & \text{if} \quad x \le 0 \\ 2x^3 - 3x^2 + 1 & \text{if} \quad 0 \le x \le 1 \\ 0 & \quad x \ge 1 \end{cases}$$

$$u(x,t) = u_0(x-t) = \begin{cases} 1 & x \le t \\ 2(x-t)^3 - 3(x-t)^2 + 1 & 0 \le x - t \le 1 \\ 0 & x \ge t+1 \end{cases}$$

$v^1_{j_0} = 0$

$u(x_{j_0}, 1) \neq 0$

$$v_j^{n+1} = v_j^n - \frac{\Delta t}{h}(v_{j+1}^n - v_j^n) \ (\text{forward time} - \text{forward space}).$$

x_{j_0}

$x_{j_0} = 1$

$v^1_{j_0} = v^0_{j_0} - \frac{\Delta t}{h}\left(v^0_{j_0+1} - v^0_{j_0}\right) = 0$

$v^0_{j_0} = u_0(x_{j_0})$

Definition

$$||u||_2 = \left(h \sum_j |u(x_j)|^2 \right)^{1/2} = \left(h \sum_j |u_j|^2 \right)^{1/2}$$

l_2

L_2

$$||u||_\infty = \sup_j |u(x_j)| = \sup_j |u_j|.$$

$(u_j^n)_{j=-\infty}^{\infty}$

$||v^n||_\infty$

$t = 10$

Definition : Given a partial differential equation $Pu = 0$ and a finite difference scheme $P_{\Delta t,h}v = 0$ we say that the finite difference scheme is **consistent** with the partial differential equation in norm $||\cdot||$, if for the actual solution u of $Pu = 0$,

$$||P_{\Delta t,h}u|| \to 0 \quad \text{as} \quad \Delta t, h \to 0.$$

$P_{\Delta t, h} u(x_j, t^n) = \tau_j^n$

$P_{\Delta t, h} v = 0$

Definition : Given a partial differential equation $Pu = 0$ and a finite difference scheme $P_{\Delta t,h}v = 0$ we say that the finite difference scheme is **consistent** with the partial differential equation in norm $||\cdot||$, if for the actual solution u of $Pu = 0$,

$$||P_{\Delta t,h}u|| \to 0 \quad \text{as} \quad \Delta t, h \to 0.$$

Definition : The finite difference method is accurate of order (p, q) in $||\cdot||$ if for the actual solution u of $Pu = 0$,

$$||P_{\Delta t,h}u|| = O(h^p) + O(\Delta t^q).$$

(p, q)

We can write

$$v_j^{n+1} = \sum_{l=-k}^{k} \alpha_l v_{j+l}^n = \alpha_{-k} v_{j-k}^n + \ldots \alpha_0 v_j^n + \ldots + \alpha_k v_{j+k}^n .$$

$S_+ : l_2 \to l_2$

Shift operator: S_+ and S_- $\qquad S_+ v_j = v_{j+1}, \quad S_- v_j = v_{j-1}$

$$v_j^{n+1} = \alpha_{-k} S_-^k v_j^n + \ldots \alpha_{-1} S_- v_j^n + \alpha_0 v_j^n + \alpha_1 S_+ v_j^n + \ldots + \alpha_k S_+^k v_j^n$$

(a_{ij}), $a_{i\,i+1} = 1$

$S_-^k = S_- \circ S_- \circ S_- \circ S \ldots S_-$

for n, $(v_j^n)_{j=-\infty}^{\infty} \in l_2$

$$v^{n+1} = \underline{Q(S_+, S_-)} v^n$$

(v_j)

$(S_+ v)_j = v_{j+1}$

$$v^{n+1} = Q v^n$$

$u^{n+1} = Q u^n$

Definition : The finite difference method is called **stable** in $|| \cdot ||$, if there exist constants K and β such that

$$||v^n|| \leq K e^{\beta t} ||v^0||$$

where $t = n\Delta t$, K and β are independent of h and Δt.

$$||u^n|| \leq K e^{\beta t} ||u^0|| \quad \forall$$

Definition : The finite difference method is called **strongly stable** in $||\cdot||$, if $||v^n|| \leq ||v^{n-1}||$.

$\|u^n\| \le K e^{\beta t} \|u^0\|$

$\|v^n\| \le \|v^{n-1}\|$

Δt & h

Definition: A finite difference method is **unconditionally stable** if it is stable for any time step Δt and space step h.

Example: $v_j^{n+1} = v_j^n - a\frac{\Delta t}{h}(v_j^n - v_{j-1}^n)$ **(FTBS)** for $u_t + au_x = 0$

$\lambda = \frac{\Delta t}{h}$

$$v_j^{n+1} = v_j^n(1 - a\lambda) + a\lambda v_{j-1}^n, \quad \lambda = \frac{\Delta t}{h}$$

$$\begin{aligned} ||v^{n+1}||_\infty &= \sup_j |v_j^{n+1}| = \sup_j |v_j^n(1-a\lambda) + a\lambda v_{j-1}^n| \\ &\leq \sup_j \{|1-a\lambda|\ |v_j^n| + a\lambda\ |v_{j-1}^n|\} \leq |1-a\lambda|||v^n||_\infty + |a\lambda|||v^n||_\infty \leq 1 \times ||v^n||_\infty \end{aligned}$$

If $0 < a\lambda \leq 1$

$0 < a\lambda \leq 1$

If $0 < a\lambda \leq 1$, then $||v^{n+1}||_\infty \leq ||v^n||_\infty \leq \ldots \leq ||v^0||_\infty$. Therefore this scheme is $\cancel{l_\infty}$-stable if $a\lambda \leq 1$ (conditionally stable).

$\| \; \|_\infty$

$a > 0$

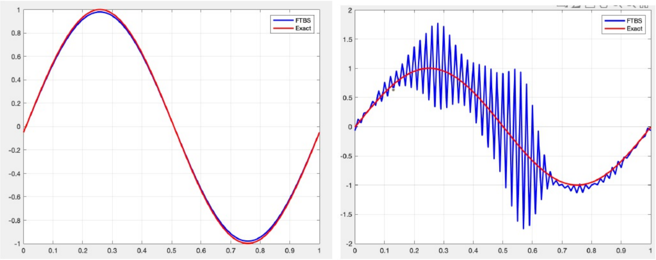
$a\lambda \leq 1$

$\boxed{\Delta t \leq \frac{h}{a}}$

Solution of $u_t + u_x = 0, \qquad u(x,0) = \sin(2\pi x)$

$a = 1$

$\Delta t \le \frac{h}{1}$



$t = 1$

Figure: (Left)solution at $t = 1$ with $\Delta t = 0.9h$. (Right) solution at $t = 1$ with $\Delta t = 1.3h$.

$\Delta t = 1.1h$

CFL

Example: $v_j^n = v_j^{n-1} - \lambda(v_j^n - v_{j-1}^n) \;\; n \geq 1$ **(BTBS)**

$$-\lambda v_{j-1}^n + v_j^n(1+\lambda) = v_j^{n-1} \qquad \forall j$$

This can be written in the matrix form, $Av^n = v^{n-1}$ where $A = (a_{ij})$ with $a_{ii} = (1+\lambda)$ and $a_{ii-1} = -\lambda$ and $a_{ij} = 0$ if $j \neq i, i-1$.

$$A v^n = v^{n-1}$$

$$v^n = A^{-1} v^{n-1}$$

$$A = (I + \lambda I)(I + C)$$

$$c_{i,i-1} = \frac{-\lambda}{1+\lambda}$$

$$A v^n = v^{n-1}$$

$$\begin{pmatrix} 1+\lambda & 0 & 0 \\ -\lambda & 1+\lambda & 0 \\ 0 & -\lambda & 1+\lambda \end{pmatrix}$$

$$\|-C\|_\infty = \sup_i \sum_j |c_{ij}| \leq \frac{\lambda}{1+\lambda} \leq 1$$

$$\|-C\|_\infty \leq 1$$

$$\|(I - -C)^{-1}\| = \|(I + C)^{-1}\| \leq \frac{1}{1 - \|C\|_\infty}$$

$$(I - -C)^{-1} = I + -C + -C^2 + (-C)^3 + \cdots$$

$$\|(I + C)^{-1}\|_\infty \le I + \|C\| + \|C\|^2 + \|C\|^3 + \cdots$$

$$= \frac{1}{1 - \|C\|_\infty}$$

$$A^{-1} = \frac{1}{(1+\lambda)} \times (I + C)^{-1}$$

$$\|A^{-1}\|_\infty \le \|(I + C)^{-1}\| \frac{1}{1+\lambda} \le \frac{1}{1 - \|C\|_\infty} \times \frac{1}{(1+\lambda)}$$

$$\le \frac{1}{1+\lambda} \times \frac{1}{\left(1 - \frac{\lambda}{(1+\lambda)}\right)} \le 1$$

$$\|u^n\|_\infty = \|A^{-1} u^{n-1}\|_\infty \le \|A^{-1}\|_\infty \|u^{n-1}\|_\infty$$

$$\le \|u^{n-1}\|_\infty$$

Definition: A difference scheme $\boldsymbol{v}^{n+1} = Q\boldsymbol{v}^n$ approximating the PDE $Pu = 0$ is a convergent scheme at time t in $\|\cdot\|$ if, as $(n+1)\Delta t$ converges to t

$$\left\| \boldsymbol{u}^{n+1} - \boldsymbol{v}^{n+1} \right\| \to 0$$

$t = (n+1)\Delta t$

as h and Δt converges to 0.

Definition: A difference scheme $\boldsymbol{v}^{n+1} = Q\boldsymbol{v}^n$ approximating the PDE $Pu = 0$ is a convergent scheme of order (p, q) in $\|\cdot\|$ if for any time t, as as $(n+1)\Delta t$ converges to t

$$\left\| \boldsymbol{u}^{n+1} - \boldsymbol{v}^{n+1} \right\| = \mathcal{O}(h^p) + \mathcal{O}(\Delta t^q)$$

as h and Δt converges to 0.

Definition : A finite difference method is said to be **linear** if it is of the form

$$v_j^{n+1} = \sum_{l=-m_1}^{m_2} c_l v_{l+j}^n \text{ where } c_l\,'s \text{ are constants}$$

m_1, m_2 are non-negative integers.

Theorem (Lax): If a finite difference method is linear, stable and accurate of order (p, q) in $||\cdot||$, then it is convergent of order (p, q) in $||\cdot||$.

Proof:

$$v^n = Q v^{n-1} \qquad Q = Q(S_+, S_-)$$

$$= Q(Q v^{n-1}) = Q^2 v^{n-2}$$

$$\vdots$$

$$= Q^n v^0$$

$$\|v^n\| = \|Q^n v^0\| \leq K e^{\beta t} \|v^0\|$$

$$\Rightarrow \frac{\|Q^n v^0\|}{\|v^0\|} \leq K e^{\beta t} < \infty$$

$$\Rightarrow \|Q^n\|_\infty \leq K e^{\beta t}$$

$$\|v^n - u^n\| = O(\Delta t^q) + O(h^p)$$

$$\|v^n\| \leq K e^{\beta t} \|v^0\|$$

$$\|Q^n\|_\infty = \sup_{\|v^0 \neq 0\|} \frac{\|Q^n v^0\|}{\|v^0\|}$$

$$u^n = Q u^{n-1} + \Delta t \left(O(\Delta t^q) + O(h^p) \right)$$

$\|Q^n\| \le K e^{\beta t}$

by consistency

$$e^n = v^n - u^n$$

$$= Q v^{n-1} - \left(Q u^{n-1} + \Delta t \left(O(\Delta t^q) + O(h^p) \right) \right)$$

$$= Q \left(v^{n-1} - u^{n-1} \right) + \Delta t \left(O(\Delta t^q) + O(h^p) \right)$$

$$= Q e^{n-1} + \Delta t \left(O(\Delta t^q) + O(h^p) \right)$$

$$= Q^n e^0 + \Delta t \sum_{j=0}^{n-1} Q^j \left(O(h^p) + O(\Delta t^q) \right)$$

$$= \Delta t \sum_{j=0}^{n-1} Q^j \left(O(h^p) + O(\Delta t^q) \right)$$

$$\|e^n\| \le \Delta t \sum_{j=0}^{n-1} \|Q^j\| \left(\quad \right) \le K \Delta t\, e^{\beta t} \times n \left(O(\Delta t^q) + O(h^p) \right)$$

$e^0 = v^0 - u^0$

$= 0$

$n \Delta t = t$

$$\|e^n\| \leq K \cancel{t}\, e^{\beta t} \ast \left(O(\Delta t^q) + O(h^p) \right)$$

$$\leq K e^{(\beta+1)t} \left(O(\Delta t^q) + O(h^p) \right)$$

$t < e^t$

$h \to 0$
$\Delta t \to 0$

$$\lim_{\substack{\Delta t \to 0 \\ h \to 0}} \|e^n\| = 0$$

Von-Neumann Analysis

Define the discrete Fourier transform of $v = (v_j)_{j=-\infty}^{\infty}$ be a sequence by

$$\widehat{v}(\xi) = \sum_{j=-\infty}^{\infty} v_j e^{ij\xi} \qquad i = \sqrt{-1}, \;\; \xi \in [0, 2\pi].$$

$$\begin{aligned} \widehat{S_+ v} &= \sum_j (S_+ v_j) e^{ij\xi} = \sum_j v_{j+1} e^{ij\xi} \\ &= \sum_j v_j e^{i(j-1)\xi} = e^{-i\xi} \sum_j v_j e^{ij\xi} = e^{-i\xi} \widehat{v}(\xi). \end{aligned}$$

Similarly $\quad \widehat{S_- v} = e^{i\xi} \widehat{v}(\xi).$

Example: (FTBS)

$$\begin{aligned} v_j^{n+1} &= v_j^n - \frac{\Delta t}{h}(v_j^n - v_{j-1}^n) \\ &= (1-\lambda)v_j^n + \lambda v_{j-1}^n, \qquad \lambda = \frac{\Delta t}{h} \\ &= (1-\lambda)v_j^n + \lambda S_- v_j^n \\ &= ((1-\lambda) + \lambda S_-)v_j^n \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad v^{n+1} &= Q(S_+, S_-)v^n, \quad Q(S_+, S_-) = (1-\lambda)I + \lambda S_- \\ \Rightarrow \quad \widehat{v}^{n+1} &= (1-\lambda)\hat{v}^n + \lambda \widehat{S_- v}^n \\ &= (1-\lambda+\lambda e^{i\xi})\widehat{v}^n \end{aligned}$$

In general

$$\begin{aligned} \widehat{v}^{n+1} &= Q(e^{-i\xi}, e^{i\xi})\widehat{v}^n \\ \rho(\xi) &= Q(e^{-i\xi}, e^{i\xi}) \text{ is called } \textbf{amplification factor} \end{aligned}$$

Definition: A symbol $\rho(\xi)$ is said to satisfy the Von Neumann condition if there exists a constant $C > 0$ (independent of $\Delta t, h, n$ and ξ) such that

$$|\rho(\xi)| \leq 1 + C\Delta t \qquad \text{for} \quad \xi \in [0, 2\pi]$$

Necessary and **sufficient** condition for stability

Theorem : A finite difference method $v^{n+1} = Qv^n$ is stable in the l_2 norm iff the Von-Neumann condition is satisfied.

Proof:

Examples: $(u_t + au_x = 0)$

1. Godunov Scheme :

$$\begin{aligned} v_j^{n+1} &= v_j^n - \lambda\frac{(1+sgn\ a)}{2}a(v_j^n - v_{j-1}^n) - \lambda\frac{(1-sgn\ a)}{2}a(v_{j+1}^n - v_j^n) \\ &= \max(0, -a\lambda)v_{j+1}^n + (1-|\lambda a|)v_j^n + \max(0, a\lambda)v_{j-1}^n \end{aligned}$$

2. FTCS

$$
\begin{aligned}
v_j^{n+1} &= v_j^n - \frac{a\Delta t}{2h}(v_{j+1}^n - v_{j-1}^n) \\
v^{n+1} &= (1 - a\frac{\lambda}{2}S_+ + a\frac{\lambda}{2}S_-)v^n \\
\rho(\xi) &= 1 - a\frac{\lambda}{2}(e^{-i\xi} - e^{i\xi}) = 1 + ia\lambda\sin\xi \\
|\rho(\xi)|^2 &= 1 + a^2\lambda^2\sin^2\xi > 1 \text{ if } \xi \neq 0, \pi
\end{aligned}
$$

This scheme is not strongly stable in $\|\cdot\|_2$ It is accurate of order $(p, q) = (2, 1)$

This scheme is stable in $\|\cdot\|_2$ with restriction $\Delta t \leq \frac{h^2}{a^2}$ (which is more restrictive than CFL)

2. Lax-Friedrichs Scheme :

$$v_j^{n+1} = \frac{v_{j+1}^n + v_{j-1}^n}{2} - \frac{\Delta t a}{2h}(v_{j+1}^n - v_{j-1}^n)$$
$$v^{n+1} = \left(\frac{1}{2}(1 - a\lambda)S_+ + \frac{1}{2}(1 + a\lambda)S_-\right) v^n$$
$$\rho(\xi) = \frac{1}{2}(1 - a\lambda)e^{-i\xi} + \frac{1}{2}(1 + a\lambda)e^{i\xi}$$
$$|\rho(\xi)| \leq \frac{1}{2}|1 - a\lambda| + \frac{1}{2}|1 + a\lambda| \leq 1 \quad \text{if} |a\lambda| \leq 1$$

The **LF** scheme is $\|\cdot\|_2$ if $|a\lambda| \leq 1$.

Note:

The **LF** scheme is convergent in $\|\cdot\|_2$ and $\|\cdot\|_\infty$ if $|a\lambda| \leq 1$

The **LF** Scheme is first order accurate i.e. $p = 1, q = 1$

3. Laxw-Wendroff scheme

$$
\begin{aligned}
u(x,t+\Delta t) &= u(x,t) + \Delta t u_t(x,t) + \frac{(\Delta t)^2}{2} u_{tt}(x,t) + O(\Delta t^3) \\
&= u(x,t) - \Delta t a u_x + \frac{(\Delta t)^2}{2} a^2 u_{xx} + O(\Delta t)^3 \\
v_j^{n+1} &= v_j^n - \frac{\Delta t a}{2h}(v_{j+1}^n - v_{j-1}^n) + \frac{a^2(\Delta t)^2}{2h^2}(v_{j+1}^n - 2v_j^n + v_{j-1}^n)
\end{aligned}
$$

This scheme is second order accurate i.e, $p = q = 2$

$$
\begin{aligned}
\rho(\xi) &= 1 + \frac{\lambda a}{2}(e^{i\xi} - e^{-i\xi}) + \frac{a^2\lambda^2}{2}(e^{i\xi} + e^{-i\xi} - 2) \\
&= 1 - a^2\lambda^2(1 - \cos\xi) + i\lambda a \sin\xi \\
&= 1 - 2a^2\lambda^2 \sin^2\frac{\xi}{2} + i\lambda a \sin\xi \ \ (\text{because } \ 1 - \cos\xi = 2\sin^2\frac{\xi}{2})
\end{aligned}
$$

$$
|\rho(\xi)|^2 = 1 - 4a^2\lambda^2(1 - a^2\lambda^2)\sin^4\frac{\xi}{2}
$$

$$
|\rho(\xi)| \le 1 \quad \text{if} \quad |a\lambda| \le 1
$$

Scheme is $\|\cdot\|_2$ stable. Hence converges in $\|\cdot\|_2$ norm

4. Crank-Nicolson Scheme

$$\begin{aligned}
\frac{u(x,t+\Delta t)-u(x,t)}{\Delta t} &= \frac{1}{\Delta t}\int_t^{t+\Delta t} u_t(x,\xi)d\xi \\
&= \frac{u_t(x,t+\Delta t)+u_t(x,t)}{2} + O(\Delta t^2) \\
&= -a\frac{u_x(x,t+\Delta t)+u_x(x,t)}{2} + O(\Delta t^2) \\
&= -\frac{a}{2}\frac{(u(x+h,t+\Delta t)-u(x-h,t+\Delta t))}{2h} \\
&\quad -\frac{a}{2}\frac{(u(x+h,t)-u(x-h,t))}{2h} + O(\Delta t^2) + O(h^2)
\end{aligned}$$

The CN scheme is given by

$$\frac{v_i^{n+1}-v_i^n}{\Delta t} = -\frac{a}{2}\frac{(v_{i+1}^{n+1}-v_{i-1}^{n+1})}{2h} - \frac{a}{2}\frac{(v_{i+1}^n-v_{i-1}^n)}{2h}$$

This scheme is second order accurate i.e., $p = q = 2$

$$
\begin{aligned}
-\frac{a\lambda}{4}v_{j-1}^{n+1} + v_j^{n+1} + \frac{a\lambda}{4}v_{j+1}^{n+1} &= v_j^n - \frac{a\lambda}{4}(v_{j+1}^n - v_{j-1}^n) \\
\left(\frac{-a\lambda}{4}S_- + I + \frac{a\lambda}{4}S_+\right)v^{n+1} &= \left(I - \frac{a\lambda}{4}(S_+ - S_-)\right)v^n \\
\left(-\frac{a\lambda}{4}e^{i\xi} + 1 + \frac{a\lambda}{4}e^{-i\xi}\right)\widehat{v}^{n+1} &= \left(1 - \frac{a\lambda}{4}(e^{-i\xi} - e^{i\xi})\right)\widehat{v}^n \\
\widehat{v}^{n+1} &= \left(\frac{1 + \frac{a\lambda}{2}i\sin\xi}{1 - \frac{a\lambda}{2}i\sin\xi}\right)\widehat{v}^n \\
\rho(\xi) &= \frac{1 + \frac{a\lambda}{2}i\sin\xi}{1 - \frac{a\lambda}{2}i\sin\xi} = \frac{z}{\bar{z}} \\
|\rho(\xi)| &= 1
\end{aligned}
$$

The CN scheme is unconditionally stable in $\|\cdot\|_2$ and hence converges in the same norm.

Scheme	Stable	Strongly stable	CFL
FTCS	Yes	No	$\Delta t \leq \frac{h^2}{a^2}$
BTBS* $(a > 0)$	Yes	Yes	-
Upwind	Yes	Yes	$\|a\|\lambda \leq 1$
Lax-Friedrichs	Yes	Yes	$\|a\|\lambda \leq 1$
Lax-Wendroff	Yes	Yes	$\|a\|\lambda \leq 1$

Parabolic equation

$$u_t = bu_{xx}, \quad -\infty < x < \infty,\ t > 0,\ \ b > 0$$
$$u(x,0) = u_0(x)$$

$$u_t = \frac{u(x,t+\Delta t) - u(x,t)}{\Delta t} + O(\Delta t)$$
$$u_{xx} = \frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2} + O(h^2)$$

$$\frac{v_i^{n+1} - v_i^n}{\Delta t} = \frac{b}{h^2}(v_{i+1}^n - 2v_i^n + v_{i-1}^n) \text{ is of order } (p,q) = (2,1)$$
$$v_i^{n+1} = v_i^n(1 - 2\lambda b) + \lambda b v_{i+1}^n + \lambda b v_{i-1}^n$$

Crank-Nicolson scheme

$$v_i^{n+1} = v^n + \frac{b\Delta t}{2h^2}(v_{i+1}^{n+1} - 2v_i^{n+1} + v_{i-1}^{n+1}) + \frac{b\Delta t}{2h^2}(v_{i+1}^n - 2v_i^n + v_{i-1}^n)$$

This scheme is second order accurate i.e. $p = q = 2$.

θ -Scheme: $(0 \leq \theta \leq 1)$

$$v_i^{n+1} = v_i^n + \theta b\lambda(v_{i+1}^{n+1} - 2v_i^{n+1} + v_{i-1}^{n+1}) + (1-\theta)\lambda b(v_{i+1}^n - 2v_i^n + v_{i-1}^n)$$

- If $\theta = 1/2$, θ - scheme is nothing but Crank-Nicolson Scheme.
- The order of accuracy is $(p, q) = (2, 1)$ if $\theta \neq 1/2$.

l_2 stability: θ -Scheme

$$-\theta b\lambda v_{i+1}^{n+1} + (1+2\theta b\lambda)v_i^{n+1} - \theta b\lambda v_{i-1}^{n+1} = (1-\theta)\lambda b v_{i-1}^n + (1-2(1-\theta)\lambda b)v_i^n + (1-\theta)\lambda b v_{i+1}^n.$$

$$(-\theta b\lambda e^{-i\xi} + (1+2\theta b\lambda) - \theta b\lambda e^{i\xi})\widehat{v}^{n+1} = \big((1-\theta)\lambda b e^{i\xi} + (1-2(1-\theta)\lambda b) + (1-\theta)\lambda b e^{-i\xi}\big)\,\widehat{v}^n$$

$$\rho(\xi) = \frac{1-4(1-\theta)\lambda b\sin^2\frac{\xi}{2}}{1+4\theta b\lambda\sin^2\frac{\xi}{2}}$$

$$|\rho(\xi)| \leq 1 \quad \text{if} \quad (1-2\theta)w \leq 2, \qquad w = 4\lambda b \sin^2 \frac{\xi}{2}$$

- Unconditionally stable in $\|\cdot\|_2$ if $\theta \geq 1/2$
- Conditionally stable $\|\cdot\|_2$ if $\theta < 1/2, \qquad \lambda b \leq \frac{1}{2(1-2\theta)}$

l_∞ **stability:** θ -Scheme

$$Av^{n+1} = Bv^n$$

$$a_{ij} = \begin{cases} 1 + 2\theta\lambda b & \text{if } j = i \\ -\theta\lambda b & \text{if } j = i-1 \\ & \text{or } j = i+1 \\ 0 & \text{otherwise,} \end{cases} \quad b_{ij} = \begin{cases} 1 - 2(1-\theta)b\lambda & \text{if } j = i \\ (1-\theta)\lambda b & \text{if } j = i-1 \\ & \text{or } j = i+1 \\ 0 & \text{otherwise} \end{cases}$$

References I