Finite difference schemes for linear PDE

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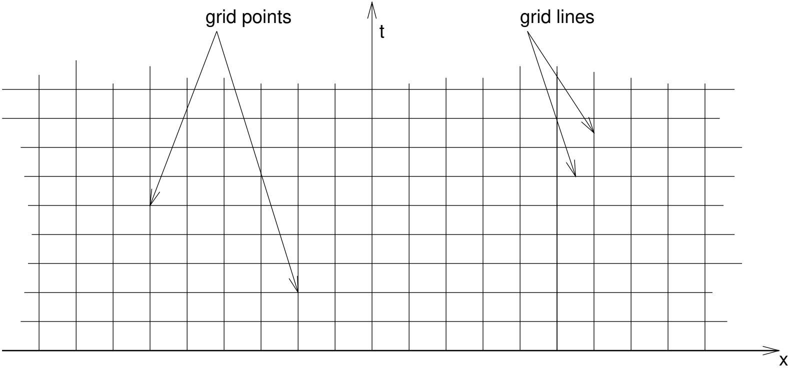
We consider

$$Pu = u_t + au_x = 0, \quad -\infty < x < \infty, t > 0, \quad u = u(x,t)$$

with initial condition

$$u(x,0) = u_0(x)$$

First we define the grid points in the (x, t) plane by drawing vertical and horizontal lines through the points (x_i, t_n)



$$
\begin{aligned}
t_n &= n\Delta t, \quad n = 0, 1, 2, \ldots \\
x_j &= ih, \quad j = 0, \pm 1, \pm 2, \ldots
\end{aligned}
$$

The lines $x = x_i$ and $t = t_n$ are called grid lines and their intersections are called mesh points of the grid.

The basic idea of **finite difference method is to replace derivatives by finite differences.** This can be done in many ways;

By replacing the derivatives by finite differences and neglecting the error terms we have list of difference equations. For example

$$P_{\Delta t,h}v = \frac{v_j^{n+1} - v_j^n}{\Delta t} + a\ \frac{(v_{j+1}^n - v_j^n)}{h} = 0 \text{ (forward time – forward space)}$$

$$P_{\Delta t,h}v = \frac{v_i^{n+1} - v_j^n}{\Delta t} + a\frac{(v_j^n - v_{j-1}^n)}{h} = 0 \text{ (forward time – backward space)}$$

$$P_{\Delta t,h}v = \frac{v_j^{n+1} - v_j^n}{\Delta t} + a\frac{(v_{j+1}^n - v_{j-1}^n)}{2h} = 0 \text{ (forward time – central space)}$$

$$P_{\Delta t,h}v = \frac{v_j^n - v_j^{n-1}}{\Delta t} + a\frac{(v_{j+1}^n - v_j^n)}{h} = 0 \text{ (backward time – forward space)}$$

$$P_{\Delta t,h}v = \frac{v_j^n - v_j^{n-1}}{\Delta t} + a\frac{(v_j^n - v_{j-1}^n)}{h} = 0 \text{ (backward time – backward space)}$$

Cosistency, Stability and **convergence**

Solution need not converge always!

$$u_t + u_x = 0$$

$$u(x,0) = u_0(x) = \begin{cases} 1 & \text{if} \quad x \le 0 \\ 2x^3 - 3x^2 + 1 & \text{if} \quad 0 \le x \le 1 \\ 0 & \quad x \ge 1 \end{cases}$$

$$u(x,t) = u_0(x-t) = \begin{cases} 1 & x \le t \\ 2(x-t)^3 - 3(x-t)^2 + 1 & 0 \le x - t \le 1 \\ 0 & x \ge t+1 \end{cases}$$

$$v_i^{n+1} = v_j^n - \frac{\Delta t}{h}(v_{j+1}^n - v_j^n) \text{ (forward time} - \text{forward space).}$$

Definition

$$||u||_2 = \left(h \sum_j |u(x_j)|^2 \right)^{1/2} = \left(h \sum_j |u_j|^2 \right)^{1/2}$$

$$||u||_\infty = \sup_j |u(x_j)| = \sup_j |u_j|.$$

Definition : Given a partial differential equation $Pu = 0$ and a finite difference scheme $P_{\Delta t,h} v = 0$ we say that the finite difference scheme is **consistent** with the partial differential equation in norm $|| \cdot ||$, if for the actual solution u of $Pu = 0$,

$$||P_{\Delta t,h} u|| \to 0 \quad \text{as} \;\; \Delta t, h \to 0.$$

Definition : The finite difference method is accurate of order (p, q) in $||\cdot||$ if for the actual solution u of $Pu = 0$,

$$||P_{\Delta t,h}u|| = O(h^p) + O(\Delta t^q).$$

We can write

$$v_j^{n+1} = \sum_{l=-k}^{k} \alpha_l v_{j+l}^n = \alpha_{-k} v_{j-k}^n + \ldots \alpha_0 v_j^n + \ldots + \alpha_k v_{j+k}^n \,.$$

Shift operator: S_+ and S_- $\qquad S_+ v_j = v_{j+1}, \quad S_- v_j = v_{j-1}$

$$v_j^{n+1} = \alpha_{-k} S_-^k v_j^n + \ldots \alpha_{-1} S_- v_j^n + \alpha_0 v_j^n + \alpha_1 S_+ v_j^n + \ldots + \alpha_k S_+^k v_j^n$$

Definition : The finite difference method is called **stable** in $||\cdot||$, if there exist constants K and β such that

$$||v^n|| \leq Ke^{\beta t}||v^0||$$

where $t = n\Delta t, K$ and β are independent of h and Δt.

Definition : The finite difference method is called **strongly stable** in $||\cdot||$, if $||v^n|| \leq ||v^{n-1}||$.

Definition: A finite difference method is **unconditionally stable** if it is stable for any time step Δt and space step h.

Example: $v_j^{n+1} = v_j^n - a\frac{\Delta t}{h}(v_j^n - v_{j-1}^n)$ (**FTBS**) for $u_t + au_x = 0$

$$v_j^{n+1} = v_j^n(1 - a\lambda) + a\lambda v_{j-1}^n, \qquad \lambda = \frac{\Delta t}{h}$$

$$\begin{aligned} ||v^{n+1}||_\infty &= \sup_j |v_j^{n+1}| = \sup_j |v_j^n(1 - a\lambda) + a\lambda v_{j-1}^n| \\ &\le \sup_j \{|1 - a\lambda|\ |v_j^n| + a\lambda\ |v_{j-1}^n|\} \ \le |1 - a\lambda|||v^n||_\infty + |a\lambda|||v^n \end{aligned}$$

If $0 < a\lambda \le 1$, then $||v^{n+1}||_\infty \le ||v^n||_\infty \le \ldots \le ||v^0||_\infty$. Therefore this scheme is l_∞-stable if $a\lambda \le 1$ (conditionally stable).

Solution of $u_t + u_x = 0, \qquad u(x,0) = \sin(2\pi x)$

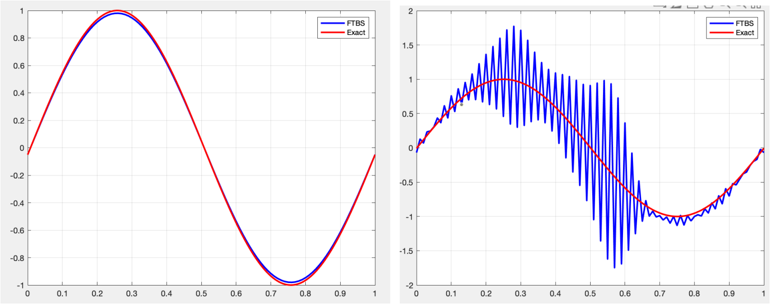


Figure: (Left)solution at $t = 1$ with $\Delta t = 0.9h$. (Right) solution at $t = 1$ with $\Delta t = 1.3h$.

Example: $v_j^n = v_j^{n-1} - \lambda(v_j^n - v_{j-1}^n) \;\; n \geq 1$ **(BTBS)**

$$-\lambda v_{j-1}^n + v_j^n(1+\lambda) = v_j^{n-1}$$

This can be written in the matrix form, $Av^n = v^{n-1}$ where $A = (a_{ij})$ with $a_{ii} = (1+\lambda)$ and $a_{ii-1} = -\lambda$ and $a_{ij} = 0$ if $j \neq i, i-1$.

Definition: A difference scheme $\boldsymbol{v}^{n+1} = Q\boldsymbol{v}^n$ approximating the PDE $Pu = 0$ is a convergent scheme at time t in $\|\cdot\|$ if, as $(n+1)\Delta t$ converges to t

$$\left\| \boldsymbol{u}^{n+1} - \boldsymbol{v}^{n+1} \right\| \to 0$$

as h and Δt converges to 0.

Definition: A difference scheme $\boldsymbol{v}^{n+1} = Q\boldsymbol{v}^n$ approximating the PDE $Pu = 0$ is a convergent scheme of order (p, q) in $\|\cdot\|$ if for any time t, as as $(n+1)\Delta t$ converges to t

$$\left\|\boldsymbol{u}^{n+1} - \boldsymbol{v}^{n+1}\right\| = \mathcal{O}(h^p) + \mathcal{O}(\Delta t^q)$$

as h and Δt converges to 0.

Definition : A finite difference method is said to be **linear** if it is of the form

$$v_j^{n+1} = \sum_{l=-m_1}^{m_2} c_l v_{l+j}^n \text{ where } c_l\,'s \text{ are constants}$$

m_1, m_2 are non-negative integers.

Theorem (Lax): If a finite difference method is linear, stable and accurate of order (p, q) in $||\cdot||$, then it is convergent of order (p, q) in $||\cdot||$.
Proof:

Von-Neumann Analysis

Define the discrete Fourier transform of $v = (v_j)_{j=-\infty}^{\infty}$ be a sequence by

$$\widehat{v}(\xi) = \sum_{j=-\infty}^{\infty} v_j e^{ij\xi} \qquad i = \sqrt{-1}, \quad \xi \in [0, 2\pi].$$

$$\begin{aligned} \widehat{S_+ v} &= \sum_j (S_+ v_j) e^{ij\xi} = \sum_j v_{j+1} e^{ij\xi} \\ &= \sum_j v_j e^{i(j-1)\xi} = e^{-i\xi} \sum_j v_j e^{ij\xi} = e^{-i\xi} \widehat{v}(\xi). \end{aligned}$$

Similarly $\quad \widehat{S_- v} = e^{i\xi} \widehat{v}(\xi).$

Example: (FTBS)

$$\begin{aligned} v_j^{n+1} &= v_j^n - \frac{\Delta t}{h}(v_j^n - v_{j-1}^n) \\ &= (1-\lambda)v_j^n + \lambda v_{j-1}^n, \qquad \lambda = \frac{\Delta t}{h} \\ &= (1-\lambda)v_j^n + \lambda S_- v_j^n \\ &= ((1-\lambda) + \lambda S_-)v_j^n \end{aligned}$$

$$\begin{aligned} \Rightarrow \qquad v^{n+1} &= Q(S_+, S_-)v^n, \quad Q(S_+, S_-) = (1-\lambda)I + \lambda S_- \\ \Rightarrow \qquad \widehat{v}^{n+1} &= (1-\lambda)\hat{v}^n + \lambda\widehat{S_- v}^n \\ &= (1-\lambda+\lambda e^{i\xi})\widehat{v}^n \end{aligned}$$

In general

$$\begin{aligned} \widehat{v}^{n+1} &= Q(e^{-i\xi}, e^{i\xi})\widehat{v}^n \\ \rho(\xi) &= Q(e^{-i\xi}, e^{i\xi}) \text{ is called } \textbf{amplification factor} \end{aligned}$$

Definition: A symbol $\rho(\xi)$ is said to satisfy the Von Neumann condition if there exists a constant $C > 0$ (independent of $\Delta t, h, n$ and ξ) such that

$$|\rho(\xi)| \leq 1 + C\Delta t \qquad \text{for} \quad \xi \in [0, 2\pi]$$

Necessary and **sufficient** condition for stability

Theorem : A finite difference method $v^{n+1} = Qv^n$ is stable in the l_2 norm iff the Von-Neumann condition is satisfied.

Proof:

Examples: $(u_t + au_x = 0)$

1. Godunov Scheme :

$$\begin{aligned} v_j^{n+1} &= v_j^n - \lambda \frac{(1 + sgn\, a)}{2} a(v_j^n - v_{j-1}^n) - \lambda \frac{(1 - sgn\, a)}{2} a(v_{j+1}^n - v_j^n) \\ &= \max(0, -a\lambda) v_{j+1}^n + (1 - |\lambda a|) v_j^n + \max(0, a\lambda) v_{j-1}^n \end{aligned}$$

2. FTCS

$$
\begin{aligned}
v_j^{n+1} &= v_j^n - \frac{a\Delta t}{2h}(v_{j+1}^n - v_{j-1}^n) \\
v^{n+1} &= (1 - a\frac{\lambda}{2}S_+ + a\frac{\lambda}{2}S_-)v^n \\
\rho(\xi) &= 1 - a\frac{\lambda}{2}(e^{-i\xi} - e^{i\xi}) = 1 + ia\lambda\sin\xi \\
|\rho(\xi)|^2 &= 1 + a^2\lambda^2\sin^2\xi > 1 \text{ if } \xi \neq 0, \pi
\end{aligned}
$$

This scheme is not strongly stable in $\|\cdot\|_2$ It is accurate of order $(p,q) = (2,1)$

This scheme is stable in $\|\cdot\|_2$ with restriction $\Delta t \leq \frac{h^2}{a^2}$ (which is more restrictive than CFL)

2. Lax-Friedrichs Scheme :

$$
\begin{aligned}
v_j^{n+1} &= \frac{v_{j+1}^n + v_{j-1}^n}{2} - \frac{\Delta t a}{2h}(v_{j+1}^n - v_{j-1}^n) \\
v^{n+1} &= \left(\frac{1}{2}(1 - a\lambda)S_+ + \frac{1}{2}(1 + a\lambda)S_-\right) v^n \\
\rho(\xi) &= \frac{1}{2}(1 - a\lambda)e^{-i\xi} + \frac{1}{2}(1 + a\lambda)e^{i\xi} \\
|\rho(\xi)| &\leq \frac{1}{2}|1 - a\lambda| + \frac{1}{2}|1 + a\lambda| \leq 1 \quad \text{if} |a\lambda| \leq 1
\end{aligned}
$$

The **LF** scheme is $\|\cdot\|_2$ if $|a\lambda| \leq 1.$

Note:

The **LF** scheme is convergent in $\|\cdot\|_2$ and $\|\cdot\|_\infty$ if $|a\lambda| \leq 1$

The **LF** Scheme is first order accurate i.e. $p = 1, q = 1$

3. Laxw-Wendroff scheme

$$u(x,t+\Delta t) = u(x,t) + \Delta t u_t(x,t) + \frac{(\Delta t)^2}{2} u_{tt}(x,t) + O(\Delta t^3)$$
$$= u(x,t) - \Delta t a u_x + \frac{(\Delta t)^2}{2} a^2 u_{xx} + O(\Delta t)^3$$
$$v_j^{n+1} = v_j^n - \frac{\Delta t a}{2h}(v_{j+1}^n - v_{j-1}^n) + \frac{a^2(\Delta t)^2}{2h^2}(v_{j+1}^n - 2v_j^n + v_{j-1}^n)$$

This scheme is second order accurate i.e, $p = q = 2$

$$
\begin{aligned}
\rho(\xi) &= 1 + \frac{\lambda a}{2}(e^{i\xi} - e^{-i\xi}) + \frac{a^2\lambda^2}{2}(e^{i\xi} + e^{-i\xi} - 2) \\
&= 1 - a^2\lambda^2(1 - \cos\xi) + i\lambda a \sin\xi \\
&= 1 - 2a^2\lambda^2 \sin^2\frac{\xi}{2} + i\lambda a \sin\xi \;\; (\text{because } \; 1 - \cos\xi = 2\sin^2\frac{\xi}{2})
\end{aligned}
$$

$$
|\rho(\xi)|^2 = 1 - 4a^2\lambda^2(1 - a^2\lambda^2)\sin^4\frac{\xi}{2}
$$

$$
|\rho(\xi)| \leq 1 \quad \text{if} \quad |a\lambda| \leq 1
$$

Scheme is $\|\cdot\|_2$ stable. Hence converges in $\|\cdot\|_2$ norm

4. Crank-Nicolson Scheme

$$
\begin{aligned}
\frac{u(x,t+\Delta t)-u(x,t)}{\Delta t} &= \frac{1}{\Delta t}\int_{t}^{t+\Delta t} u_t(x,\xi)d\xi \\
&= \frac{u_t(x,t+\Delta t)+u_t(x,t)}{2}+O(\Delta t^2) \\
&= -a\frac{u_x(x,t+\Delta t)+u_x(x,t)}{2}+O(\Delta t^2) \\
&= -\frac{a}{2}\frac{(u(x+h,t+\Delta t)-u(x-h,t+\Delta t))}{2h} \\
&\quad -\frac{a}{2}\frac{(u(x+h,t)-u(x-h,t))}{2h}+O(\Delta t^2)+O(h^2)
\end{aligned}
$$

The CN scheme is given by

$$
\frac{v_i^{n+1}-v_i^n}{\Delta t} = -\frac{a}{2}\frac{(v_{i+1}^{n+1}-v_{i-1}^{n+1})}{2h}-\frac{a}{2}\frac{(v_{i+1}^{n}-v_{i-1}^{n})}{2h}
$$

This scheme is second order accurate i.e., $p=q=2$

$$
\begin{aligned}
-\frac{a\lambda}{4} v_{j-1}^{n+1} + v_j^{n+1} + \frac{a\lambda}{4} v_{j+1}^{n+1} &= v_j^n - \frac{a\lambda}{4}(v_{j+1}^n - v_{j-1}^n) \\
\left(\frac{-a\lambda}{4} S_- + I + \frac{a\lambda}{4} S_+\right) v^{n+1} &= \left(I - \frac{a\lambda}{4}(S_+ - S_-)\right) v^n \\
\left(-\frac{a\lambda}{4} e^{i\xi} + 1 + \frac{a\lambda}{4} e^{-i\xi}\right) \widehat{v}^{n+1} &= \left(1 - \frac{a\lambda}{4}(e^{-i\xi} - e^{i\xi})\right) \widehat{v}^n \\
\widehat{v}^{n+1} &= \left(\frac{1 + \frac{a\lambda}{2} i \sin\xi}{1 - \frac{a\lambda}{2} i \sin\xi}\right) \widehat{v}^n \\
\rho(\xi) &= \frac{1 + \frac{a\lambda}{2} i \sin\xi}{1 - \frac{a\lambda}{2} i \sin\xi} = \frac{z}{\bar{z}} \\
|\rho(\xi)| &= 1
\end{aligned}
$$

The CN scheme is unconditionally stable in $\|\cdot\|_2$ and hence converges in the same norm.

Scheme	Stable	Strongly stable	CFL
FTCS	Yes	No	$\Delta t \leq \frac{h^2}{a^2}$
BTBS* $(a > 0)$	Yes	Yes	-
Upwind	Yes	Yes	$\|a\|\lambda \leq 1$
Lax-Friedrichs	Yes	Yes	$\|a\|\lambda \leq 1$
Lax-Wendroff	Yes	Yes	$\|a\|\lambda \leq 1$

Parabolic equation

$$
\begin{aligned}
u_t &= b u_{xx}, \qquad -\infty < x < \infty, \;\; t > 0, \quad b > 0 \\
u(x,0) &= u_0(x)
\end{aligned}
$$

$$
\begin{aligned}
u_t &= \frac{u(x, t+\Delta t) - u(x,t)}{\Delta t} + O(\Delta t) \\
u_{xx} &= \frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2} + O(h^2)
\end{aligned}
$$

$$
\begin{aligned}
\frac{v_i^{n+1} - v_i^n}{\Delta t} &= \frac{b}{h^2}(v_{i+1}^n - 2v_i^n + v_{i-1}^n) \text{ is of order } (p,q) = (2,1) \\
v_i^{n+1} &= v_i^n(1 - 2\lambda b) + \lambda b v_{i+1}^n + \lambda b v_{i-1}^n
\end{aligned}
$$

Crank-Nicolson scheme

$$v_i^{n+1} = v^n + \frac{b\Delta t}{2h^2}(v_{i+1}^{n+1} - 2v_i^{n+1} + v_{i-1}^{n+1}) + \frac{b\Delta t}{2h^2}(v_{i+1}^n - 2v_i^n + v_{i-1}^n)$$

This scheme is second order accurate i.e. $p = q = 2$.

θ -**Scheme:** $(0 \leq \theta \leq 1)$

$$v_i^{n+1} = v_i^n + \theta b\lambda(v_{i+1}^{n+1} - 2v_i^{n+1} + v_{i-1}^{n+1}) + (1-\theta)\lambda b(v_{i+1}^n - 2v_i^n + v_{i-1}^n)$$

- If $\theta = 1/2, \quad \theta$ - scheme is nothing but Crank-Nicolson Scheme.
- The order of accuracy is $(p, q) = (2, 1)$ if $\theta \neq 1/2$.

l_2 stability: θ -Scheme

$$
\begin{aligned}
-\theta b\lambda v_{i+1}^{n+1} + (1+2\theta b\lambda)v_i^{n+1} - \theta b\lambda v_{i-1}^{n+1} &= (1-\theta)\lambda b v_{i-1}^{n} + (1-2(1-\theta)\lambda b)v_i^n \\
&\quad + (1-\theta)\lambda b v_{i+1}^{n}.
\end{aligned}
$$

$$
\begin{aligned}
(-\theta b\lambda e^{-i\xi} + (1+2\theta b\lambda) - \theta b\lambda e^{i\xi})\widehat{v}^{n+1} &= \Big((1-\theta)\lambda b e^{i\xi} + (1-2(1-\theta)\lambda b) \\
&\quad +(1-\theta)\lambda b e^{-i\xi}\Big)\,\widehat{v}^{n}
\end{aligned}
$$

$$
\rho(\xi) = \frac{1-4(1-\theta)\lambda b\sin^2\frac{\xi}{2}}{1+4\theta b\lambda\sin^2\frac{\xi}{2}}
$$

$$|\rho(\xi)| \leq 1 \quad \text{if} \quad (1-2\theta)w \leq 2, \quad w = 4\lambda b \sin^2\frac{\xi}{2}$$

- Unconditionally stable in $\|\cdot\|_2$ if $\theta \geq 1/2$
- Conditionally stable $\|\cdot\|_2$ if $\theta < 1/2, \qquad \lambda b \leq \frac{1}{2(1-2\theta)}$

l_∞ **stability:** θ -Scheme

$$Av^{n+1} = Bv^n$$

$$a_{ij} = \begin{cases} 1 + 2\theta\lambda b & \text{if } j = i \\ -\theta\lambda b & \text{if } j = i-1 \\ & \text{or } j = i+1 \\ 0 & \text{otherwise,} \end{cases} \qquad b_{ij} = \begin{cases} 1 - 2(1-\theta)b\lambda & \text{if } j = i \\ (1-\theta)\lambda b & \text{if } j = i-1 \\ & \text{or } j = i+1 \\ 0 & \text{otherwise} \end{cases}$$

References I