Finite difference schemes for linear PDE

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We consider

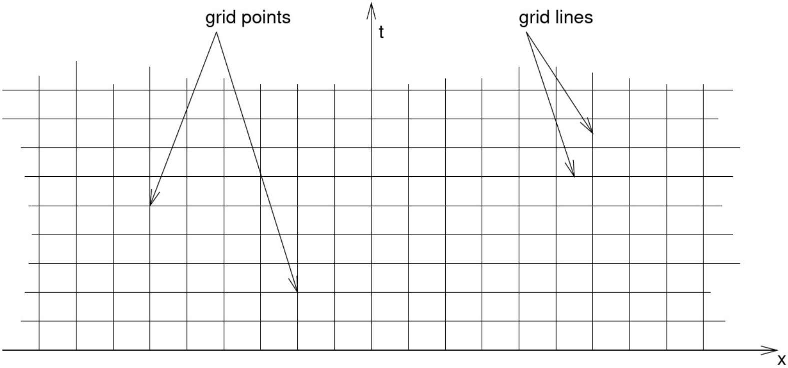
$$Pu = u_t + au_x = 0, \quad -\infty < x < \infty, t > 0, \quad u = u(x,t)$$

with initial condition

$$u(x,0) = u_0(x)$$

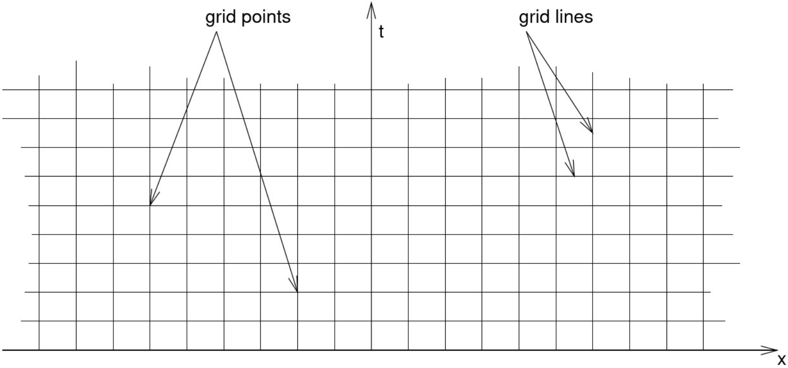
First we define the grid points in the (x, t) plane by drawing vertical and horizontal lines through the points (x_i, t_n)

$$\int_a^b f(x)\,dx.$$



$$
\begin{aligned}
t_n &= n\Delta t, \quad n = 0, 1, 2, \ldots \\
x_j &= ih, \quad j = 0, \pm 1, \pm 2, \ldots
\end{aligned}
$$

The lines $x = x_i$ and $t = t_n$ are called grid lines and their intersections are called mesh points of the grid.



$$t_n = n\Delta t, \quad n = 0, 1, 2, \ldots$$
$$x_j = ih, \quad j = 0, \pm 1, \pm 2, \ldots$$

The lines $x = x_i$ and $t = t_n$ are called grid lines and their intersections are called mesh points of the grid.

The basic idea of **finite difference method is to replace derivatives by finite differences.** This can be done in many ways;

By replacing the derivatives by finite differences and neglecting the error terms we have list of difference equations. For example

$$P_{\Delta t,h}v = \frac{v_j^{n+1} - v_j^n}{\Delta t} + a\,\frac{(v_{j+1}^n - v_j^n)}{h} = 0 \text{ (forward time – forward space)}$$

$$P_{\Delta t,h}v = \frac{v_i^{n+1} - v_j^n}{\Delta t} + a\frac{(v_j^n - v_{j-1}^n)}{h} = 0 \text{ (forward time – backward space)}$$

$$P_{\Delta t,h}v = \frac{v_j^{n+1} - v_j^n}{\Delta t} + a\frac{(v_{j+1}^n - v_{j-1}^n)}{2h} = 0 \text{ (forward time – central space)}$$

$$P_{\Delta t,h}v = \frac{v_j^n - v_j^{n-1}}{\Delta t} + a\frac{(v_{j+1}^n - v_j^n)}{h} = 0 \text{ (backward time – forward space)}$$

$$P_{\Delta t,h}v = \frac{v_j^n - v_j^{n-1}}{\Delta t} + a\frac{(v_j^n - v_{j-1}^n)}{h} = 0 \text{ (backward time – backward space)}$$

Cosistency, Stability and **convergence**

Solution need not converge always!

$$u_t + u_x = 0$$

$$u(x,0) = u_0(x) = \begin{cases} 1 & \text{if} \quad x \leq 0 \\ 2x^3 - 3x^2 + 1 & \text{if} \quad 0 \leq x \leq 1 \\ 0 & \quad\ \ x \geq 1 \end{cases}$$

$$u(x,t) = u_0(x-t) = \begin{cases} 1 & x \leq t \\ 2(x-t)^3 - 3(x-t)^2 + 1 & 0 \leq x - t \leq 1 \\ 0 & x \geq t+1 \end{cases}$$

$$v_i^{n+1} = v_j^n - \frac{\Delta t}{h}(v_{j+1}^n - v_j^n) \ \text{(forward time} - \text{forward space).}$$

Definition

$$||u||_2 = \left(h \sum_j |u(x_j)|^2 \right)^{1/2} = \left(h \sum_j |u_j|^2 \right)^{1/2}$$

$$||u||_\infty = \sup_j |u(x_j)| = \sup_j |u_j|.$$

Definition : Given a partial differential equation $Pu = 0$ and a finite difference scheme $P_{\Delta t,h}v = 0$ we say that the finite difference scheme is **consistent** with the partial differential equation in norm $||\cdot||$, if for the actual solution u of $Pu = 0$,

$$||P_{\Delta t,h}u|| \to 0 \quad \text{as} \ \ \Delta t, h \to 0.$$

Definition : The finite difference method is accurate of order (p, q) in $||\cdot||$ if for the actual solution u of $Pu = 0$,

$$||P_{\Delta t,h}u|| = O(h^p) + O(\Delta t^q).$$

We can write

$$v_j^{n+1} = \sum_{l=-k}^{k} \alpha_l v_{j+l}^n = \alpha_{-k} v_{j-k}^n + \ldots \alpha_0 v_j^n + \ldots + \alpha_k v_{j+k}^n \,.$$

Shift operator: S_+ and S_- $\qquad S_+ v_j = v_{j+1}, \quad S_- v_j = v_{j-1}$

$$v_j^{n+1} = \alpha_{-k} S_-^k v_j^n + \ldots \alpha_{-1} S_- v_j^n + \alpha_0 v_j^n + \alpha_1 S_+ v_j^n + \ldots + \alpha_k S_+^k v_j^n$$

Definition : The finite difference method is called **stable** in $||\cdot||$, if there exist constants K and β such that

$$||v^n|| \leq K e^{\beta t} ||v^0||$$

where $t = n\Delta t, K$ and β are independent of h and Δt.

Definition : The finite difference method is called **strongly stable** in $||\cdot||$, if $||v^n|| \leq ||v^{n-1}||$.

Definition: A finite difference method is **unconditionally stable** if it is stable for any time step Δt and space step h.

Example: $v_j^{n+1} = v_j^n - a\frac{\Delta t}{h}(v_j^n - v_{j-1}^n)$ **(FTBS)** for $u_t + au_x = 0$

$$v_j^{n+1} = v_j^n(1 - a\lambda) + a\lambda v_{j-1}^n, \qquad \lambda = \frac{\Delta t}{h}$$

$$\begin{aligned} ||v^{n+1}||_\infty &= \sup_j |v_j^{n+1}| = \sup_j |v_j^n(1 - a\lambda) + a\lambda v_{j-1}^n| \\ &\leq \sup_j \{|1 - a\lambda| \; |v_j^n| + a\lambda \; |v_{j-1}^n|\} \; \leq |1 - a\lambda| ||v^n||_\infty + |a\lambda| ||v^n \end{aligned}$$

If $0 < a\lambda \leq 1$, then $||v^{n+1}||_\infty \leq ||v^n||_\infty \leq \ldots \leq ||v^0||_\infty$. Therefore this scheme is l_∞-stable if $a\lambda \leq 1$ (conditionally stable).

Solution of $u_t + u_x = 0, \qquad u(x,0) = \sin(2\pi x)$

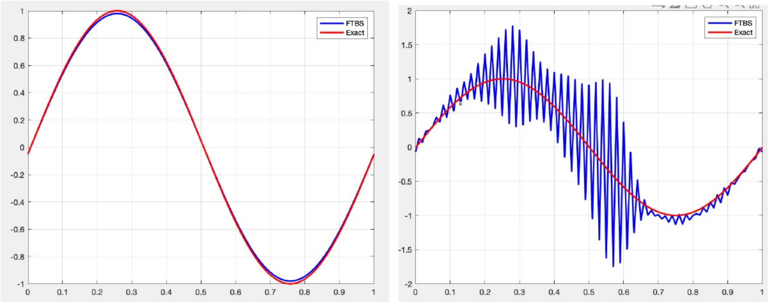


Figure: (Left)solution at $t = 1$ with $\Delta t = 0.9h$. (Right) solution at $t = 1$ with $\Delta t = 1.3h$.

Example: $v_j^n = v_j^{n-1} - \lambda(v_j^n - v_{j-1}^n) \quad n \geq 1$ **(BTBS)**

$$-\lambda v_{j-1}^n + v_j^n(1+\lambda) = v_j^{n-1}$$

This can be written in the matrix form, $Av^n = v^{n-1}$ where $A = (a_{ij})$ with $a_{ii} = (1+\lambda)$ and $a_{ii-1} = -\lambda$ and $a_{ij} = 0$ if $j \neq i, i-1$.

Definition: A difference scheme $\boldsymbol{v}^{n+1} = Q\boldsymbol{v}^n$ approximating the PDE $Pu = 0$ is a convergent scheme at time t in $\|\cdot\|$ if, as $(n+1)\Delta t$ converges to t

$$\left\|\boldsymbol{u}^{n+1} - \boldsymbol{v}^{n+1}\right\| \to 0$$

as h and Δt converges to 0.

Definition: A difference scheme $\boldsymbol{v}^{n+1} = Q\boldsymbol{v}^n$ approximating the PDE $Pu = 0$ is a convergent scheme of order (p, q) in $\|\cdot\|$ if for any time t, as as $(n+1)\Delta t$ converges to t

$$\left\|\boldsymbol{u}^{n+1} - \boldsymbol{v}^{n+1}\right\| = \mathcal{O}(h^p) + \mathcal{O}(\Delta t^q)$$

as h and Δt converges to 0.

Definition : A finite difference method is said to be **linear** if it is of the form

$$v_j^{n+1} = \sum_{l=-m_1}^{m_2} c_l v_{l+j}^n \text{ where } c_l\,'s \text{ are constants}$$

m_1, m_2 are non-negative integers.

Theorem (Lax): If a finite difference method is linear, stable and accurate of order (p, q) in $||\cdot||$, then it is convergent of order (p, q) in $||\cdot||$.

Proof:

$u_0(x)$

$u(x,t) = u_0(x-at)$

Von-Neumann Analysis

Define the discrete Fourier transform of $v = (v_j)_{j=-\infty}^{\infty}$ be a sequence by

$$\widehat{v}(\xi) = \sum_{j=-\infty}^{\infty} v_j e^{ij\xi} \qquad i = \sqrt{-1}, \quad \xi \in [0, 2\pi].$$

$$\begin{aligned}
\widehat{S_+ v} &= \sum_j (S_+ v_j) e^{ij\xi} = \sum_j v_{j+1} e^{ij\xi} \\
&= \sum_j v_j e^{i(j-1)\xi} = e^{-i\xi} \sum_j v_j e^{ij\xi} = e^{-i\xi}\widehat{v}(\xi).
\end{aligned}$$

Similarly $\widehat{S_- v} = e^{i\xi}\widehat{v}(\xi)$.

$\| \ \|_2$

Stability

$\|u^n\|_2 \le K e^{\beta t} \|u^0\|_2$

$\wedge : l_2 \longrightarrow L_2[0, 2\pi]$

$u \longrightarrow \hat{u}$

$u \longrightarrow \hat{u}$

$\widehat{(S_+ u)}(\xi) =$

$\widehat{S_+ u}(\xi) = e^{-i\xi} \hat{v}(\xi)$

$\widehat{S_- u}(\xi) = e^{i\xi} \hat{v}(\xi)$

Example: (FTBS)

$u_t + u_x = 0$

$$
\begin{aligned}
v_j^{n+1} &= v_j^n - \frac{\Delta t}{h}(v_j^n - v_{j-1}^n) \\
&= (1-\lambda)v_j^n + \lambda v_{j-1}^n, \qquad \lambda = \frac{\Delta t}{h} \\
&= (1-\lambda)v_j^n + \lambda S_- v_j^n \\
&= ((1-\lambda)I + \lambda S_-)v_j^n
\end{aligned}
$$

$|\lambda| \leq$

$\lambda \leq 1$

$\Delta t \leq h$

$$
\begin{aligned}
\Rightarrow \quad v^{n+1} &= Q(S_+, S_-)v^n, \quad Q(S_+, S_-) = (1-\lambda)I + \lambda S_- \\
\Rightarrow \quad \hat{v}^{n+1}(\xi) &= (1-\lambda)\hat{v}^n + \lambda \widehat{S_- v}^n \\
&= (1-\lambda + \lambda e^{i\xi})\hat{v}^n(\xi) \\
&= \rho(\xi)\hat{v}^n(\xi)
\end{aligned}
$$

$\hat{v}^{n+1} = \widehat{((1-\lambda)I + \lambda S_-)v}^n$

$\widehat{(1-\lambda)v}^n + \widehat{\lambda S_- v}^n$

$= (1-\lambda)\hat{v}^n(\xi) + \lambda e^{+i\xi}\hat{v}^n(\xi)$

In general

$$
\begin{aligned}
\hat{v}^{n+1} &= Q(e^{-i\xi}, e^{i\xi})\hat{v}^n \\
\rho(\xi) &= Q(e^{-i\xi}, e^{i\xi}) \text{ is called } \textbf{amplification factor}
\end{aligned}
$$

$v^{n+1} = Q v^n \Longrightarrow \hat{v}^{n+1}(\xi) = \rho(\xi)\hat{v}^n(\xi)$

$$|u^{n+1}(\xi)| = |\rho(\xi)||u^n(\xi)|$$

$$|\rho(\xi)| \le 1 + C\Delta t$$

$$|u^{n+1}(\xi)| \le |u^n(\xi)|$$

$$u_t = a^2 u_{xx} + bu$$

$$|u^{n+1}(\xi)| \le |\rho(\xi)|^{n+1} |u^0(\xi)|$$

$$\le (1 + C\Delta t)^{n+1} |u^0(\xi)|$$

$$\|u^{n+1}\| \le k e^{\beta t} \|u^0\|$$

$$\le e^{C\Delta t (n+1)} |u^0(\xi)| = e^{Ct} \|u^0\|$$

$$|u^{n+1}(\xi)| = |\rho(\xi)|\,|u^n(\xi)|$$

$$|\rho(\xi)| \leq 1 + C\Delta t$$

$$|u^{n+1}(\xi)| \leq |u^n(\xi)|$$

$$u_t = a^2 u_{xx} + bu$$

$$|u^{n+1}(\xi)| \leq |\rho(\xi)|^{n+1} |u^0(\xi)|$$

$$\|u^{n+1}\| \leq K e^{\beta t} \|u^0\|$$

$$\leq (1 + C\Delta t)^{n+1} |u^0(\xi)|$$

$$\leq e^{C\Delta t(n+1)} |u^0(\xi)| = e^{Ct} \|u^0\|$$

Definition: A symbol $\rho(\xi)$ is said to satisfy the Von Neumann condition if there exists a constant $C > 0$ (independent of $\Delta t, h, n$ and ξ) such that

$$|\rho(\xi)| \leq 1 + C\Delta t \qquad \text{for} \quad \xi \in [0, 2\pi]$$

Necessary and **sufficient** condition for stability

Theorem : A finite difference method $v^{n+1} = Qv^n$ is stable in the ~~l_2~~ $\|\cdot\|_2$ norm iff the Von-Neumann condition is satisfied.

Proof:

$|\rho(\xi)| \leq 1 + c\Delta t, \quad c > 0$

Von-Neuman $\Longrightarrow$ Stability

$$\sum_j (v_j^{n+1})^2 = \frac{1}{2\pi}\int_0^{2\pi} |\hat{v}^{n+1}(\xi)|^2 d\xi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} |\rho(\xi)|^2 |\hat{v}^n(\xi)|^2 d\xi$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} (1+c\Delta t)^2 |\hat{v}^n(\xi)|^2 d\xi$$

$$\left(\|v^{n+1}\|_2 \leq Ke^{\beta t}\|v^0\| \right)$$

$f \in l_2$

$\|f\|_{l^2} = \|\hat{f}\|_{L^2}$

Parseval's relation.

$$\le \frac{1}{2\pi}\int_0^{2\pi} \left(1+C\Delta t\right)^{2(n+1)} \hat{u}^0(\xi)^2 \, d\xi$$

$$= \left(1+C\Delta t\right)^{2(n+1)} \frac{1}{2\pi}\int_0^{2\pi} \left(\hat{u}^0\right)^2(\xi)\, d\xi$$

$$h\sum_j \left(u_j^{n+1}\right)^2 \le \left(1+C\Delta t\right)^{2(n+1)} h\sum_j \left(u_j^0\right)^2$$

$$\| u^{n+1}\|_2^2 \le \left(1+C\Delta t\right)^{2(n+1)} \| u^0\|_2^2$$

$$\| u^{n+1}\|_2 \le \left(1+C\Delta t\right)^{(n+1)} \| u^0\|_2$$

$$\le e^{Ct} \| u^0\|_2 \, ,$$

$K = 1$

$\beta = C$

Von-Neumann is not satisfied $\Longrightarrow$ the scheme is not stable.

$\cancel{|\rho(\xi)| \leq 1 + C\Delta t \quad \forall \xi \in [0, 2\pi]}$

$\forall\, C_k > 0 \quad \exists\, \xi_k \in [0, 2\pi] \Rightarrow |\rho(\xi_k)| > 1 + C_k \Delta t$

We get a sequence $C_k \longrightarrow \infty$

and $\{\xi_k\} \Rightarrow \rho(\xi_k) \longrightarrow \infty$

$$\hat{u}^{n+1}(\xi_k) = \rho(\xi_k)^{n+1}\, \hat{u}^{0}(\xi_k)$$

Examples: $(u_t + au_x = 0)$

1. Godunov Scheme : (Upwind Scheme)

$$
\begin{aligned}
v_j^{n+1} &= v_j^n - \lambda \frac{(1+sgn\ a)}{2} a(v_j^n - v_{j-1}^n) - \lambda \frac{(1-sgn\ a)}{2} a(v_{j+1}^n - v_j^n) \\
&= \max(0, -a\lambda) v_{j+1}^n + (1 - |\lambda a|) v_j^n + \max(0, a\lambda) v_{j-1}^n
\end{aligned}
$$

$$\operatorname{Sgn} a = \begin{cases} 1 & \text{if } a > 0 \\ -1 & \text{if } a \leq 0 \end{cases}$$

$$\|v^{n+1}\|_\infty \leq \left(|a\lambda| + |1-|a\lambda||\right) \|v^n\|_\infty$$

$$|a\lambda| \leq 1$$

$$\leq \|v^n\|_\infty$$

$$v^{n+1} = \left(\max(0,-a\lambda) S_+ + (1-|a\lambda|) + \max(0,a\lambda) S_-\right) v^n$$

$$\hat{v}^{n+1} = G(S_+, S_-)\, \hat{v}$$

$$|\rho(\xi)| = \left|\max(0,-a\lambda) e^{-i\xi} + (1-|a\lambda|) + \max(0,a\lambda) e^{i\xi}\right|$$

$$|\varphi(\xi)| \le \big|1-\lambda a\big| + |a\lambda| \le 1 \quad \text{if} \quad |\lambda a| \le 1$$

$$\max(0,a) = \frac{|a|-a}{2}$$

$$\max\{0,-a\} = \frac{|a|+a}{2}$$

2. FTCS

$$\begin{aligned} v_j^{n+1} &= v_j^n - \frac{a\Delta t}{2h}(v_{j+1}^n - v_{j-1}^n) \\ v^{n+1} &= (1 - a\frac{\lambda}{2}S_+ + a\frac{\lambda}{2}S_-)v^n \\ \rho(\xi) &= 1 - a\frac{\lambda}{2}(e^{-i\xi} - e^{i\xi}) = 1 + ia\lambda\sin\xi \\ |\rho(\xi)|^2 &= 1 + a^2\lambda^2\sin^2\xi > 1 \text{ if } \xi \neq 0, \pi \end{aligned}$$

This scheme is not strongly stable in $\|\cdot\|_2$ It is accurate of order $(p, q) = (2, 1)$

This scheme is stable in $\|\cdot\|_2$ with restriction $\Delta t \leq \frac{h^2}{a^2}$ (which is more restrictive than CFL)

2. FTCS

$$
\begin{aligned}
v_j^{n+1} &= v_j^n - \frac{a\Delta t}{2h}(v_{j+1}^n - v_{j-1}^n) \\
v^{n+1} &= (1 - a\frac{\lambda}{2}S_+ + a\frac{\lambda}{2}S_-)v^n \\
\rho(\xi) &= 1 - a\frac{\lambda}{2}(e^{-i\xi} - e^{i\xi}) = 1 + ia\lambda\sin\xi \\
|\rho(\xi)|^2 &= 1 + a^2\lambda^2\sin^2\xi > 1 \text{ if } \xi \neq 0, \pi
\end{aligned}
$$

$Q(S_+, S_-)$

This scheme is not strongly stable in $\|\cdot\|_2$ It is accurate of order $(p, q) = (2, 1)$

$|\rho(\xi)| \leq 1 + c\Delta t$

$|\Delta t| \leq \frac{h^2}{a^2}$

$h = \frac{1}{50}$

This scheme is stable in $\|\cdot\|_2$ with restriction $\Delta t \leq \frac{h^2}{a^2}$ (which is more restrictive than CFL)

2. Lax-Friedrichs Scheme :

$$
\begin{aligned}
v_j^{n+1} &= \frac{v_{j+1}^n + v_{j-1}^n}{2} - \frac{\Delta t a}{2h}(v_{j+1}^n - v_{j-1}^n) \\
v^{n+1} &= \left(\frac{1}{2}(1 - a\lambda)S_+ + \frac{1}{2}(1 + a\lambda)S_-\right) v^n \\
\rho(\xi) &= \frac{1}{2}(1 - a\lambda)e^{-i\xi} + \frac{1}{2}(1 + a\lambda)e^{i\xi} \\
|\rho(\xi)| &\leq \frac{1}{2}|1 - a\lambda| + \frac{1}{2}|1 + a\lambda| \leq 1 \quad \text{if} |a\lambda| \leq 1
\end{aligned}
$$

The **LF** scheme is $\|\cdot\|_2$ if $|a\lambda| \leq 1$.

Note:

The **LF** scheme is convergent in $\|\cdot\|_2$ and $\|\cdot\|_\infty$ if $|a\lambda| \leq 1$

The **LF** Scheme is first order accurate i.e. $p = 1, q = 1$

3. Laxw-Wendroff scheme

$u_t = -au_x$

$$
\begin{aligned}
u(x,t+\Delta t) &= u(x,t) + \Delta t u_t(x,t) + \frac{(\Delta t)^2}{2} u_{tt}(x,t) + O(\Delta t^3) \\
&= u(x,t) - \Delta t a u_x + \frac{(\Delta t)^2}{2} a^2 u_{xx} + O(\Delta t)^3
\end{aligned}
$$

$$
v_j^{n+1} = v_j^n - \frac{\Delta t a}{2h}(v_{j+1}^n - v_{j-1}^n) + \frac{a^2(\Delta t)^2}{2h^2}(v_{j+1}^n - 2v_j^n + v_{j-1}^n)
$$

This scheme is second order accurate i.e, $p = q = 2$

$v^{n+1} = Q v^n$

$|\rho(\xi)|^2 = 1 - 4a^2\lambda^2(1-a^2\lambda^2)\sin^4 \xi/2 \qquad \xi \in [0, 2\pi]$

≤ 1 if $|a\lambda| \leq 1$

$\|\cdot\|_\infty$ not stable.

$$
\begin{aligned}
\rho(\xi) &= 1 + \frac{\lambda a}{2}(e^{i\xi} - e^{-i\xi}) + \frac{a^2\lambda^2}{2}(e^{i\xi} + e^{-i\xi} - 2) \\
&= 1 - a^2\lambda^2(1 - \cos\xi) + i\lambda a \sin\xi \\
&= 1 - 2a^2\lambda^2 \sin^2\frac{\xi}{2} + i\lambda a \sin\xi \ \ (\text{because } \ 1 - \cos\xi = 2\sin^2\frac{\xi}{2})
\end{aligned}
$$

$$
|\rho(\xi)|^2 = 1 - 4a^2\lambda^2(1 - a^2\lambda^2)\sin^4\frac{\xi}{2}
$$

$$
|\rho(\xi)| \leq 1 \quad \text{if} \quad |a\lambda| \leq 1
$$

Scheme is $\|\cdot\|_2$ stable. Hence converges in $\|\cdot\|_2$ norm

4. Crank-Nicolson Scheme

$$
\begin{aligned}
\frac{u(x,t+\Delta t)-u(x,t)}{\Delta t} &= \frac{1}{\Delta t}\int_t^{t+\Delta t} u_t(x,\xi)\,d\xi \\
&= \frac{u_t(x,t+\Delta t)+u_t(x,t)}{2} + O(\Delta t^2) \\
&= -a\frac{u_x(x,t+\Delta t)+u_x(x,t)}{2} + O(\Delta t^2) \\
&= -\frac{a}{2}\frac{(u(x+h,t+\Delta t)-u(x-h,t+\Delta t))}{2h} \\
&\quad -\frac{a}{2}\frac{(u(x+h,t)-u(x-h,t))}{2h} + O(\Delta t^2) + O(h^2)
\end{aligned}
$$

$P_h u = Pu + O(\Delta t^2 + h^2)$

The CN scheme is given by

$$
\frac{v_i^{n+1}-v_i^n}{\Delta t} = -\frac{a}{2}\frac{(v_{i+1}^{n+1}-v_{i-1}^{n+1})}{2h} - \frac{a}{2}\frac{(v_{i+1}^{n}-v_{i-1}^{n})}{2h}
$$

$P_h v$

$P_h u = O(\Delta t^2) + O(h^2)$

This scheme is second order accurate i.e., $p = q = 2$

$$-\frac{a\lambda}{4}v_{j-1}^{n+1} + v_j^{n+1} + \frac{a\lambda}{4}v_{j+1}^{n+1} = v_j^n - \frac{a\lambda}{4}(v_{j+1}^n - v_{j-1}^n)$$

$$\left(\frac{-a\lambda}{4}S_- + I + \frac{a\lambda}{4}S_+\right)v^{n+1} = \left(I - \frac{a\lambda}{4}(S_+ - S_-)\right)v^n$$

$$\left(-\frac{a\lambda}{4}e^{i\xi} + 1 + \frac{a\lambda}{4}e^{-i\xi}\right)\widehat{v}^{n+1} = \left(1 - \frac{a\lambda}{4}(e^{-i\xi} - e^{i\xi})\right)\widehat{v}^n$$

$$\widehat{v}^{n+1} = \left(\frac{1 + \frac{a\lambda}{2}i\sin\xi}{1 - \frac{a\lambda}{2}i\sin\xi}\right)\widehat{v}^n$$

$$\rho(\xi) = \frac{1 + \frac{a\lambda}{2}i\sin\xi}{1 - \frac{a\lambda}{2}i\sin\xi} = \frac{z}{\bar{z}}$$

$$|\rho(\xi)| = 1$$

The CN scheme is unconditionally stable in $\|\cdot\|_2$ and hence converges in the same norm.

Scheme	Stable	Strongly stable	CFL
FTCS	Yes	No	$\Delta t \leq \frac{h^2}{a^2}$. *
BTBS* $(a > 0)$	Yes	Yes	-
Upwind	Yes	Yes	$\|a\|\lambda \leq 1$
Lax-Friedrichs	Yes	Yes	$\|a\|\lambda \leq 1$
Lax-Wendroff	Yes	Yes	$\|a\|\lambda \leq 1$

Parabolic equation

$$u_t = bu_{xx}, \qquad -\infty < x < \infty, \;\; t > 0, \;\; b > 0$$
$$u(x,0) = u_0(x)$$

$$u_t = \frac{u(x,t+\Delta t) - u(x,t)}{\Delta t} + O(\Delta t)$$
$$u_{xx} = \frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2} + O(h^2)$$

$$\frac{v_i^{n+1} - v_i^n}{\Delta t} = \frac{b}{h^2}(v_{i+1}^n - 2v_i^n + v_{i-1}^n) \text{ is of order } (p,q) = (2,1)$$
$$v_i^{n+1} = v_i^n(1 - 2\lambda b) + \lambda b v_{i+1}^n + \lambda b v_{i-1}^n$$

Crank-Nicolson scheme

$$v_i^{n+1} = v^n + \frac{b\Delta t}{2h^2}(v_{i+1}^{n+1} - 2v_i^{n+1} + v_{i-1}^{n+1}) + \frac{b\Delta t}{2h^2}(v_{i+1}^n - 2v_i^n + v_{i-1}^n)$$

This scheme is second order accurate i.e. $p = q = 2$.

$$= \int_t^{t+\Delta t} u(x,\xi)\, d\xi$$

θ -Scheme: $(0 \leq \theta \leq 1)$

$$v_i^{n+1} = v_i^n + \theta b\lambda(v_{i+1}^{n+1} - 2v_i^{n+1} + v_{i-1}^{n+1}) + (1-\theta)\lambda b(v_{i+1}^n - 2v_i^n + v_{i-1}^n)$$

- If $\theta = 1/2$, θ - scheme is nothing but Crank-Nicolson Scheme.
- The order of accuracy is $(p, q) = (2, 1)$ if $\theta \neq 1/2$.

l_2 **stability:** θ -Scheme

$$-\theta b\lambda v_{i+1}^{n+1} + (1+2\theta b\lambda)v_i^{n+1} - \theta b\lambda v_{i-1}^{n+1} = (1-\theta)\lambda b v_{i-1}^n + (1-2(1-\theta)\lambda b)v_i^n + (1-\theta)\lambda b v_{i+1}^n.$$

$$(-\theta b\lambda e^{-i\xi} + (1+2\theta b\lambda) - \theta b\lambda e^{i\xi})\widehat{v}^{n+1} = \big((1-\theta)\lambda b e^{i\xi} + (1-2(1-\theta)\lambda b) + (1-\theta)\lambda b e^{-i\xi}\big)\,\widehat{v}^n$$

$$\rho(\xi) = \frac{1-4(1-\theta)\lambda b\sin^2\frac{\xi}{2}}{1+4\theta b\lambda\sin^2\frac{\xi}{2}}$$

$$|\rho(\xi)| \leq 1 \quad \text{if} \quad (1-2\theta)w \leq 2, \qquad w = 4\lambda b \sin^2 \frac{\xi}{2}$$

- Unconditionally stable in $\|\cdot\|_2$ if $\theta \geq 1/2$
- Conditionally stable $\|\cdot\|_2$ if $\theta < 1/2, \qquad \lambda b \leq \frac{1}{2(1-2\theta)}$

l_∞ **stability:** θ -Scheme

$$Av^{n+1} = Bv^n$$

$$a_{ij} = \begin{cases} 1 + 2\theta\lambda b & \text{if } j = i \\ -\theta\lambda b & \text{if } j = i-1 \\ & \text{or } j = i+1 \\ 0 & \text{otherwise,} \end{cases} \quad b_{ij} = \begin{cases} 1 - 2(1-\theta)b\lambda & \text{if } j = i \\ (1-\theta)\lambda b & \text{if } j = i-1 \\ & \text{or } j = i+1 \\ 0 & \text{otherwise} \end{cases}$$

References I