

ODE

* Wellposedness theory of $\begin{cases} y' = f(t, y) \\ y(t_0) = y_0 \end{cases}$

existence

uniqueness

continuous dependence on the data

• (Picard's theorem) Suppose that the real valued function $(x, y) \rightarrow f(x, y)$ is continuous in the rectangular region $D := \{(x, y) : x_0 \leq x \leq x_M, y_0 - C \leq y \leq y_0 + C\}$;

that $|f(x, y_0)| \leq K$ when $x_0 \leq x \leq x_M$

and that f satisfies Lipschitz condition:
there exists $L > 0$, such that

$$|f(x, u) - f(x, v)| \leq L |u - v| \quad \forall (x, u), (x, v) \in D.$$

Assume further that $C \geq \frac{K}{L} (e^{L(x_M - x_0)} - 1)$.

Then there exists a unique function

$y \in C^1[x_0, x_M]$ such that $y(x_0) = y_0$

and $y' = f(x, y)$ for $x \in [x_0, x_M]$;

more over $|y(x) - y_0| \leq C$, $x_0 \leq x \leq x_M$.

Note: it is only a sufficient condition

$$\text{Eq: } \begin{cases} y' = x + \sin y \\ y(0) = 1 \end{cases}$$

$$\begin{cases} y' = y/t & , y(0) = 0 \\ f \text{ is not continuous} & \text{but} \\ \text{solution exists} \end{cases}$$

continuous dependence of the solution on initial data and dynamics.

Theorem Let D be as in the previous Picard's theorem. Suppose $f, \tilde{f} \in C(D)$ and be Lipschitz continuous w.r.t y on D with Lipschitz constants L & \tilde{L} , respectively. Let y & \tilde{y} be, respectively the solutions of IVP

$$y' = f(x, y), \quad y(x_0) = y_0$$

$$\text{and} \quad \tilde{y}' = \tilde{f}(x, \tilde{y}), \quad \tilde{y}(\tilde{x}_0) = \tilde{y}_0$$

In some closed intervals I_1 & I_2 containing x_0 & \tilde{x}_0 . For small $|x_0 - \tilde{x}_0|$ let I be any finite interval containing both x_0 & \tilde{x}_0 , where both y & \tilde{y} are defined. Then,

$$\max_{x \in I} |y(x) - \tilde{y}(x)| \leq \left(|y_0 - \tilde{y}_0| + |I| \max_D |f(x, y) - \tilde{f}(x, y)| + M |x_0 - \tilde{x}_0| \right) e^{L_0 |I|}.$$

where $|I|$ is the length of the interval I ,

$$M = \max_D (\max f, \max \tilde{f}) \quad \text{and} \quad L_0 = \min(L, \tilde{L})$$

Proof we give a proof when $x_0 = \tilde{x}_0$.

It is easy to see that the solutions y & \tilde{y} satisfy the following integral equations:

$$y(x) = y_0 + \int_{x_0}^x f(\xi, y(\xi)) d\xi$$

$$\tilde{y}(x) = \tilde{y}_0 + \int_{x_0}^x \tilde{f}(\xi, \tilde{y}(\xi)) d\xi \quad \forall x \in I$$

Subtracting yields

$$y(x) - \tilde{y}(x) = y_0 - \tilde{y}_0 + \int_{x_0}^x (f(\xi, y(\xi)) - \tilde{f}(\xi, \tilde{y}(\xi))) d\xi$$

Up on adding and subtracting the term
 $f(\xi, \tilde{y}(\xi))$ we get -

$$\begin{aligned}
|y(x) - \tilde{y}(x)| &\leq |y_0 - \tilde{y}_0| + \int_{x_0}^x |f(s, y(s)) - f(s, \tilde{y}(s))| ds \\
&\quad + \int_{x_0}^x |f(s, \tilde{y}(s)) - \tilde{f}(s, \tilde{y}(s))| ds \\
&\leq |y_0 - \tilde{y}_0| + L \int_{x_0}^x |y(s) - \tilde{y}(s)| d\xi \\
&\quad + |I| \max_D |f(x, y) - \tilde{f}(x, \tilde{y})| \\
\therefore |y(x) - \tilde{y}(x)| &\leq C + L \int_{x_0}^x |y(s) - \tilde{y}(s)| d\xi
\end{aligned}$$

Applying Gronwall's inequality

$$\begin{aligned}
|y(x) - \tilde{y}(x)| &\leq C \exp\left(L \int_{x_0}^x d\xi\right) \\
&= C \exp(L(x - x_0)) \\
&\leq C \exp(L|I|)
\end{aligned}$$

If we add & subtract $\tilde{f}(\xi, y(s))$
 we obtain the estimate

$$|y(x) - \tilde{y}(x)| \leq C \exp(\tilde{L}|I|)$$

Combining both we get -

$$|y(x) - \tilde{y}(x)| \leq \left(|y_0 - \tilde{y}_0| + |I| \max_{D} |\tilde{f}(z, y) - \tilde{f}(s, y)| \right) e^{L_0 |I|}$$

This completes the proof.

One-Step Methods

Consider the IVP $\frac{dy}{dx} = f(x, y)$
 $y(x_0) = y_0$

in the domain $[x_0, x_m]$

We divide the domain $[x_0, x_m]$ into N points

$$x_0, x_1, x_2, \dots, x_{N-1} = x_m$$

and $x_n - x_{n-1} = h$. $\forall n = 1, 2, \dots, N-1$

If $y(x)$ is an exact solution of the IVP
then

$$y(x_n + h) = y(x_n) + h y'(x_n) + O(h^2)$$

$$\Rightarrow y'(x_n) = \frac{y(x_{n+1}) - y(x_n)}{h} + O(h) \quad (1)$$

Now consider the differential equation at x_n

$$y'(x_n) - f(x_n, y(x_n)) = 0$$

using (1) we write

$$\frac{y(x_{n+1}) - y(x_n)}{h} - f(x_n, y(x_n)) + O(h) = 0$$

i.e.

$$\frac{y(x_{n+1}) - y(x_n)}{h} - f(x_n, y(x_n)) \approx 0$$

Now moving from " \approx " to " $=$ " we obtain the difference formulae

$$\frac{y_{n+1} - y_n}{h} - f(x_n, y_n) = 0$$

or $y_{n+1} = y_n + h f(x_n, y_n)$ ————— (2)

where $y_n \approx y(x_n)$ $n=0, 1, 2, \dots, N-1$

the difference equation (2) is called Euler method.

We see that for smooth solution

$$\frac{y(x_{n+1}) - y(x_n)}{h} - f(x_n, y(x_n)) \\ = y'(x_n) - f(x_n, y(x_n)) + O(h)$$

Difference equation = Differential equation + T.E

For this reason we define Truncation Error at x_n as

$$T_n = \frac{y(x_{n+1}) - y(x_n)}{h} - f(x_n, y(x_n))$$

for the difference scheme (2).

More generally a one step method may be written in the form

$$(3) \quad y_{n+1} = y_n + h \Phi(x_n, y_n; h), \quad n=0, 1, 2, \dots, N-1$$

$$y_0 = y(x_0)$$

where Φ is a continuous function of its variables.

For the Euler method $\Phi(x_n, y_n; h) = h f(x_n, y_n)$

we also define the global error by

$$\epsilon_n = y(x_n) - y_n .$$

and

$$T_n = \frac{y(x_{n+1}) - y(x_n)}{h} - \Phi(x_n, y(x_n); h)$$

Theorem Consider the one-step method (3) where, in addition to being a continuous function of its arguments, Φ is assumed to satisfy a Lipschitz condition w.r.t to its second argument, that is there exists a positive constant L_Φ such that- $\forall x \quad 0 \leq h \leq h_0$ and for all $(x, u), (x, v) \in D$

we have that-

$$D = \{ x_0 \leq x \leq x_m, |y - y_0| \leq C \}$$

$$|\Phi(x, u; h) - \Phi(x, v; h)| \leq L_\Phi |u - v|$$

Then assuming that $|y_n - y_0| \leq C, \forall n=1, 2, N$

It follows that-

$$|e_n| \leq \frac{1}{L_{\Phi}} (e^{L_{\Phi}(x_n - x_0)}) , n=1, 2, \dots N$$

where $T = \max_{0 \leq n \leq N-1} T_n$

Proof:
we have

$$y_{n+1} = y_n + h \hat{\Phi}(x_n, y_n; h)$$

$$\& y(x_{n+1}) = y(x_n) + h \hat{\Phi}(x_n, y(x_n); h) + h T_n$$

Subtracting we get-

$$e_{n+1} = e_n + h (\hat{\Phi}(x_n, y(x_n); h) - \hat{\Phi}(x_n, y_n; h))$$

since $(x_n, y(x_n)), (x_n, y_n) \in D$ the Lipschitz condition gives $+ h T_n$

$$|e_{n+1}| \leq |e_n| + h L_{\Phi} |e_n| + h T_n$$

$$= (1 + h L_{\Phi}) |e_n| + h T_n$$

$$\leq (1 + h L_{\Phi}) \left[(1 + h L_{\Phi}) |e_{n-1}| + h T_{n-1} \right]$$

$$+ h T_n$$

$$= (1 + h L_{\Phi})^2 |e_{n-1}| + h (1 + h L_{\Phi}) T_{n-1} + h T_n$$

$$\begin{aligned}
&= \left(1 + hL_{\frac{T}{n}}\right)^3 |e_{n-2}| + h \left(1 + hL_{\frac{T}{n}}\right)^2 T_{n-2} \\
&\quad + h \left(1 + hL_{\frac{T}{n}}\right) T_{n-1} \\
&\quad + h T_n \\
&= \left(1 + hL_{\frac{T}{n}}\right)^{n+1} |e_0| + h \left[\left(1 + hL_{\frac{T}{n}}\right)^n T_0 \right. \\
&\quad \left. \begin{array}{c} \parallel \\ 0 \end{array} \right. + \left(1 + hL_{\frac{T}{n}}\right)^{n-1} T_1 \\
&\quad + \cdots + T_n
\end{aligned}$$

or $|e_n| \leq Th \left(\left(1 + hL_{\frac{T}{n}}\right)^{n-1} + \left(1 + hL_{\frac{T}{n}}\right)^{n-2} + \cdots + 1 \right)$

$$= Th \left(\frac{\left(1 + hL_{\frac{T}{n}}\right)^n - 1}{\left(1 + hL_{\frac{T}{n}}\right) - 1} \right)$$

$$= \frac{T}{L_{\frac{T}{n}}} \left(\left(1 + hL_{\frac{T}{n}}\right)^n - 1 \right)$$

But $1 + hL_{\frac{T}{n}} \leq e^{hL_{\frac{T}{n}}}$

$$\begin{aligned}
\therefore |e_n| &\leq \frac{T}{L_{\frac{T}{n}}} \left(e^{nhL_{\frac{T}{n}}} - 1 \right) \\
&= \frac{T}{L_{\frac{T}{n}}} \left(e^{\frac{(x_n - x_0)L_{\frac{T}{n}}}{n}} - 1 \right), n = 0, 1, \dots, N
\end{aligned}$$

This completes the proof.

Exercise: Obtain the T.F. form Euler method.

Exercise: Determine the error bound for the problem

$$y' = \tan^3 y$$

$$y(0) = y \quad \text{in } [0, 1]$$

Consistency and Convergence.

Definition (consistency) the numerical method

$y_{n+1} = y_n + h \Phi(x_n, y_n; h)$ is consistent with the differential equation $y' = f(x, y)$ if the T.E. is such that for any $\epsilon > 0$ there exists a $h(\epsilon) > 0$ such that $|T_n| < \epsilon$ for $0 < h < h(\epsilon)$ for any point $(x_n, y(x_n))$ on any solution curve. \square

$$\text{ie } T_n \rightarrow 0 \text{ as } h \rightarrow 0 \quad h = \frac{x_N - x_0}{N}$$

Now for $x \in [x_0, x_N]$ such that-

$$\lim_{n \rightarrow \infty} x_n = x$$

$$h \rightarrow 0$$

$$\lim_{\substack{n \rightarrow \infty \\ h \rightarrow 0}} T_n = \lim_{\substack{n \rightarrow \infty \\ h \rightarrow 0}} \left(y'(x_n) - \Phi(x_n, y(x_n); h) \right)$$

$$= y'(x) - \Phi(x, y(x); 0)$$

\therefore the numerical method is consistent iff-

$$\Phi(x, y; 0) = f(x, y) \quad \text{--- (c)}$$

Theorem

Convergence theorem: Suppose that the IVP satisfies the conditions of the Picard's theorem and also that its approximation generated from

$$y_{n+1} = y_n + h \Phi(x_n, y_n; h) \quad \text{when } h \leq h_0$$

lies in the domain D . Assume further that the function $\Phi(\cdot, \cdot; \cdot)$ is continuous on $D \times [0, t_0]$ and satisfies the consistency condition (c) above and Lipschitz condition

$$|\Phi(x, u; h) - \Phi(x, v; h)| \leq L_\Phi |u-v| \quad \text{on } D \times [0, t_0]$$

then we have convergence of the numerical solution to solution of the IVP in the sense that-

$$\lim_{n \rightarrow \infty} y_n = y(x) \quad \text{as } x_n \rightarrow x \in [x_0, x_M]$$

when $n \rightarrow \infty$ and $h \rightarrow 0$

Proof: Exercise.

Definition The numerical method described above is said to have order of accuracy p , if p is the largest positive integer such that for any sufficiently smooth solution curve $(x, y(x))$ in D of the TVP, there exists constants K and h_0 such that $|T_n| \leq K h^p$ for $0 < h \leq h_0$.

Runge-Kutta methods

Consider the IVP $\begin{cases} y'(x) = f(x, y) \\ y(x_0) = y_0 \end{cases}$

By fundamental theorem of I.C we can write

$$\int_{x_n}^{x_{n+1}} y'(x) dx = \int_{x_n}^{x_{n+1}} f(x(\xi), y(\xi)) d\xi$$

$$\Rightarrow y(x_{n+1}) = y(x_n) + \int_{x_n}^{x_{n+1}} f(x(s), y(s)) ds$$

If we use mid-point rule quadrature to find the integral approximately, we obtain

$$\begin{aligned}
y(x_{n+1}) &= y(x_n) + h f\left(x_n + \frac{h}{2}, y(x_n + \frac{h}{2})\right) \\
&\quad + O(h^3) \\
&= y(x_n) + h f\left(x_n + \frac{h}{2}, y(x_n) + \frac{h}{2} y'(x_n) + O(h^2)\right) \\
&\quad + O(h^3) \\
&= y(x_n) + h \left(f\left(x_n + \frac{h}{2}, y(x_n) + \frac{h}{2} y'(x_n)\right) \right. \\
&\quad \left. + \frac{\partial f}{\partial y}\left(x_n + \frac{h}{2}, \tilde{s}\right) O(h^2) \right) \\
&\quad + O(h^3) \\
&= y(x_n) + h \left(f\left(x_n + \frac{h}{2}, y(x_n) + \frac{h}{2} f(x_n, y_n)\right) \right. \\
&\quad \left. + O(h^3) \right)
\end{aligned}$$

Finally we obtain a numerical scheme

$$y_{n+1} = y_n + h \left(f(x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)) \right)$$

Called modified Euler method and

$$\text{T.E} \quad T_h = O(h^2)$$

Now if we use trapezoidal rule for the quadrature we obtain

$$\begin{aligned} y(x_{n+1}) &= y(x_n) + \frac{h}{2} \left(f(x_n, y(x_n)) + f(x_n + h, y(x_n + h)) \right) \\ &\quad + O(h^3) \\ &= y(x_n) + \frac{h}{2} \left(f(x_n, y(x_n)) + f(x_n + h, y(x_n) + h y'(x_n)) \right. \\ &\quad \left. + O(h^2) \right) \\ &\quad + O(h^3) \\ &= y(x_n) + \frac{h}{2} \left(f(x_n, y(x_n)) + f(x_n + h, y(x_n) + h f'(x_n, y(x_n)) \right. \\ &\quad \left. + \frac{\partial f}{\partial y}(x_n + h, \xi) O(h^2) \right) + O(h^3) \end{aligned}$$

and we obtain a numerical scheme

$$y_{n+1} = y_n + \frac{h}{2} (f(x_n, y_n) + f(x_n + h, y_n + h f(x_n, y_n)))$$

This is a second-order method called Improved Euler method.

Both the second-order schemes described above are called Runge-Kutta method.

In-general the Runge-Kutta method of order two is a class of schemes written in the form

$$y_{n+1} = y_n + h (a k_1 + b k_2)$$

where $k_1 = f(x_n, y_n)$

$$k_2 = f(x_n + \alpha h, y_n + \beta h k_1)$$

where a, b, α & β are to be determined.

Note: For the above described two schemes determine a, b, α & β .

We write $y_{n+1} = y_n + h \bar{\Phi}(x_n, y_n; h)$

where $\bar{\Phi}(x_n, y_n; h) = a f(x_n, y_n) + b f(x_n + \alpha h, y_n + \beta h f(x_n, y_n))$

The T.E is given by

$$T_h = \frac{y(x_{n+1}) - y(x_n)}{h} - \bar{\Phi}(x_n, y(x_n); h)$$

Noting that $y'(x_n) = f$

$$\begin{aligned} y''(x_n) &= f_x + f_y y'(x_n) \\ &= f_x + f_y f \end{aligned}$$

all evaluated
at $(x_n, y(x_n))$

$$\begin{aligned} y'''(x_n) &= f_{xx} + f_{xy} f + f_y (f_x + f_y f) + f (f_{yx} + f_{yy} f) \\ &= f_{xx} + f_{xy} f + f_y f_x + f_y^2 f + f f_{yx} + f^2 f_{yy} . \end{aligned}$$

Using Taylor expansion we obtain

$$\begin{aligned} \bar{\Phi}(x_n, y(x_n); h) &= a f + b \left(f + \alpha h \frac{\partial f}{\partial x} + \beta h \frac{\partial f}{\partial y} f \right. \\ &\quad \left. + \frac{1}{2} (\alpha h)^2 \frac{\partial^2 f}{\partial x^2} + \alpha h \beta h \frac{\partial^2 f}{\partial x \partial y} f \right. \\ &\quad \left. + \frac{1}{2} (\beta h)^2 f \frac{\partial^2 f}{\partial y^2} + O(h^3) \right) \end{aligned}$$

Thus we obtain the truncation error in the form

$$T_h = \frac{y(x_n+h) - y(x_n)}{h} - \underline{\Phi}(x_n, y(x_n); h)$$

$$= y'(x_n) + \frac{h}{2} y''(x_n) + \frac{h^2}{6} y'''(x_n) - \underline{\Phi}(x_n, y(x_n); h) + O(h^3)$$

$$= f + \frac{h}{2} (f_x + f_y f) + \frac{h^2}{6} (f_{xx} + f_{xy} f + (f_{xy} + f_{yy} f) f + f_y (f_x + f_y f))$$

$$- (a f + b(f + \alpha h f_x + \beta h f_y f + \frac{\alpha^2 h^2}{2} f_{xx} + \alpha \beta h^2 f_{xy} f + f^2 \frac{\beta^2 h^2}{2} f_{yy})) + O(h^3)$$

$$= (1 - a - b)f + \frac{h}{2} (f_x + f_y f - b\alpha f_x - b\beta f_y)$$

$$+ h^2 \left[\frac{1}{6} f_{xx} - \frac{b\alpha^2}{2} f_{xx} + \frac{f_{xy} f}{6} - b\alpha\beta f_{xy} f \right]$$

$$+\frac{1}{6}(f_{xy}+f_{yy}f)f - \frac{\beta^2 b}{2} f_{yy}f + \frac{h}{6}f_y(f_x+f_yf) \Big] \\ + O(h^3)$$

$$= (1-a-b)f + \frac{h}{2} \left(f_x + f_yf - b\alpha f_x - b\beta f_y \right)$$

$$+ h^2 \left[\left(\frac{1}{6} - \frac{b\alpha^2}{2} \right) f_{xx} + f^2 f_{yy} \left(\frac{1}{6} - \frac{\beta^2 b}{2} \right) \right. \\ \left. + f f_{xy} \left(\frac{1}{3} - b\alpha\beta \right) \right. \\ \left. + \frac{1}{6} f_y (f_x + f_y f) \right] + O(h^3)$$

For the consistency of the scheme we require that -

$$(1-a-b)f = 0$$

$$\text{ie } a+b = 1 \quad \text{--- (c1)}$$

For second-order scheme we require

that the second term to be zero

ie if $\frac{1}{2}(f_x + f_y f) - b\alpha f_x - b\beta f_y f = 0$

for all functions f

ie if $b\alpha = b\beta = \frac{1}{2}$

The method is second-order accurate if

$$\alpha = \beta, \quad b = \frac{1}{2\alpha}$$

$$a = 1 - \frac{1}{2\alpha} \quad \alpha \neq 0$$

a, b, β are all determined by α

\therefore we get a one parameter family of
second-order schemes with truncation error

$$T_h = h^2 \left(\left(\frac{1}{6} - \frac{\alpha}{4} \right) (f_{xx} + f^2 f_{yy}) + \left(\frac{1}{3} - \frac{\alpha}{2} \right) f f_{xy} \right. \\ \left. + \frac{1}{6} (f_x f_y + f f_y^2) \right) + O(h^3)$$

Schemes are given by

$$\left\{ \begin{array}{l} y_{n+1} = y_n + \left(1 - \frac{1}{2\alpha}\right) k_1 + \frac{1}{2\alpha} k_2 \\ k_1 = f(x_n, y_n) \\ k_2 = f(x_n + \alpha h, y_n + \alpha h f(x_n, y_n)) \end{array} \right.$$

In a similar way we can derive
R-K method of order 3, 4 etc.

One of the most frequently used method of
the R-K family is the classical fourth-
order methods:

$$y_{n+1} = y_n + \frac{1}{6} h (k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = f(x_n, y_n)$$

$$k_2 = f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}h k_1)$$

$$k_3 = f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}h k_2)$$

$$k_4 = f(x_n + h, y_n + h k_3)$$

In general an s -stage R-K method of order s
can be written as

$$y_{n+1} = y_n + h \sum_{i=1}^s b_i k_i$$

where $k_i = f(x_n + c_i h, y_n + h \sum_{j=1}^s a_{ij} k_j)$, $1 \leq i \leq s$

Butcher Tableau

c_1	a_{11}	a_{12}	\dots	a_{1s}
c_2	.			.
.	:			:
c_s	a_{s1}	a_{s2}	\dots	a_{ss}
	b_1	b_2		b_s

For the consistency we require $\sum_{j=1}^S b_j = 1$

It is explicit if $A = (a_{ij})$ is lower triangular matrix. Otherwise implicit.

The Butcher Tableau for classical four-stage

0				
$\frac{1}{2}$	$\frac{1}{2}$			
$\frac{1}{2}$	0	$\frac{1}{2}$		
1	0	0	1	
	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{6}$

* Linear multi-step methods.

Consider the IVP $y' = f(x, y)$

$$y(x_0) = y_0$$

By Fundamental theorem of I.C we have

$$y(x_{n+1}) = y(x_{n-1}) + \int_{x_{n-1}}^{x_{n+1}} f(\xi, y(\xi)) d\xi$$

If we apply Simpson's rule to integrate we obtain:

$$\begin{aligned} y(x_{n+1}) &= y(x_{n-1}) + \frac{h}{3} \left(f(x_{n-1}, y(x_{n-1})) + \right. \\ &\quad \left. + f(x_n, y(x_n)) + f(x_{n+1}, y(x_{n+1})) \right) \\ &\quad + O(h^5) \end{aligned}$$

leads to a method of the form

$$y_{n+1} = y_{n-1} + \frac{h}{3} (f(x_{n-1}, y_{n-1}) + 4f(x_n, y_n) + f(x_{n+1}, y_{n+1}))$$

which is a method of order four.

Given an equally spaced mesh points (x_n) with step size h , we consider the general linear k-step method

$$\sum_{j=0}^k \alpha_j y_{n+j} = h \sum_{j=0}^k \beta_j f(x_{n+j}, y_{n+j})$$

where α_i, β_j are real constant.

We shall assume that $\alpha_k \neq 0$ and $\alpha_0 \neq \beta_0$ are not both equal to zero.

$\beta_k = 0 \Rightarrow$ Explicit method

$\beta_k \neq 0 \Rightarrow$ Implicit method.

Examples:

(1) Explicit and implicit Euler methods

$$y_{n+1} = y_n + h f(x_n, y_n)$$

$$\hat{y}_{n+1} = y_n + h f(x_{n+1}, y_{n+1})$$

(2) Trapezium Rule method (implicit)

$$y_{n+1} = y_n + \frac{1}{2} h (f(x_n, y_n) + f(x_{n+1}, y_{n+1}))$$

Using this notation $f_n := f(x_n, y_n)$

(3) Adams - Bashforth method (Explicit)

$$y_{n+4} = y_{n+3} + \frac{h}{24} (55 f_{n+3} - 59 f_{n+2} + 37 f_{n+1} - 9 f_n)$$

(4) Adams - Moulton method (Implicit)

$$y_{n+3} = y_{n+2} - \frac{1}{24} h (9 f_{n+3} + 19 f_{n+2} - 5 f_{n+1} - 9 f_n)$$

Zero-Stability

It is clear that for a multi-step method of steps k, we need to determine y_0, y_1, \dots, y_{k-1} before we can apply this method.

y_0, y_1, \dots, y_{k-1} can be computed using a suitable one-step method.

Therefore the starting values will contain numerical error subject to the one-step method used. So, we expect that the error generated from "small perturbations" in the starting conditions should not grow exponentially. This leads to the concept of "zero stability".

Definition A linear k-step method for the IVP is said to be "zero stable" if there exists a constant K such that, for any two sequences (y_n) & (z_n) that have been generated by the

Same formulae but different starting values $\hat{z}_0, \hat{z}_1, \dots, \hat{z}_{k-1}$, respectively, we have

$$|y_n - \hat{z}_n| \leq k \max\{|y_0 - \hat{z}_0|, \dots, |y_{k-1} - \hat{z}_{k-1}|\}$$

for $x_n \leq x_M$ and as $h \rightarrow 0$.

Given the linear k-step method

$$\sum_{j=0}^k \alpha_j y_{n+j} = h \sum_{j=0}^k f_{n+j},$$

we consider its first and second characteristic polynomials, respectively

$$P(z) = \sum_{j=0}^k \alpha_j z^j$$

$$G(z) = \sum_{j=0}^k \beta_j z^j$$

and assume that $\alpha_k \neq 0$, $\alpha_0^2 + \beta_0^2 \neq 0$

We consider the case where $f(x, y) \equiv 0$, as we merely required to consider the behavior

when the scheme is applied to $y=0$.

We have the following lemma.

Lemma Consider the k^{th} -order homogeneous linear recursive relation

$$(x) \quad \alpha_k y_{n+k} + \alpha_{k-1} y_{n+k-1} + \dots + \alpha_0 y_n = 0 \quad n=0, 1, 2, \dots$$

with $\alpha_k \neq 0, \alpha_0 \neq 0, \alpha_j \in \mathbb{R}, j=0, 1, \dots, k$

and corresponding cha. polynomial

$$P(z) = \alpha_k z^k + \dots + \alpha_1 z + \alpha_0$$

Let $z_r, 1 \leq r \leq l, l \leq k$, be the distinct roots of polynomial P , and $m_r \geq 1$ denote the multiplicity of z_r , with $m_1 + m_2 + \dots + m_l = k$. If a sequence y_n of complex numbers satisfies (x) above, then

$$y_n = \sum_{r=1}^l p_r(n) z_r^n \quad \text{for all } n \geq 0$$

where $P_\gamma(\cdot)$ is a polynomial in α of degree $m_\gamma - 1$, $1 \leq \gamma \leq l$.

Proof:

Assume that all roots α_γ are distinct.

We observe that the sequences

(α_γ^n) , $\gamma = 1, 2, \dots, k$ satisfies the recurrence relation.

$$\alpha_k \alpha_1^k + \alpha_{k-1} \alpha_1^{k-1} + \dots + \alpha_1 \alpha_1 + \alpha_0 = 0$$

$$\Rightarrow \alpha_k \alpha_1^{k+1} + \alpha_{k-1} \alpha_1^k + \dots + \alpha_1 \alpha_1^2 + \alpha_0 \alpha_1 = 0$$

$$\alpha_k \alpha_1^{k+2} + \alpha_{k-1} \alpha_1^{k+1} + \dots + \alpha_1 \alpha_1^3 + \alpha_0 \alpha_1^2 = 0$$

Proceeding like this

$$\alpha_k \alpha_1^{k+n} + \alpha_{k-1} \alpha_1^{k-1+n} + \dots + \alpha_1 \alpha_1^{n-1} + \alpha_0 \alpha_1^n = 0$$

$\Rightarrow (\alpha_1^n)$ satisfies the recurrence relation.

for $n=0, 1, 2, 3, \dots, k-1$,

we consider the expressions

$$c_1 z_1^0 + c_2 z_2^0 + \dots + c_k z_k^0 = y_0$$

$$c_1 z_1^1 + c_2 z_2^1 + \dots + c_k z_k^1 = y_1$$

.

.

:

$$c_1 z_1^{k-1} + c_2 z_2^{k-1} + \dots + c_k z_k^{k-1} = y_{k-1}$$

which can be written in matrix form

as

$$DC = Y \quad D = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ z_1 & z_2 & z_3 & \cdots & z_k \\ z_1^2 & z_2^2 & z_3^2 & \cdots & z_k^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_1^{k-1} & z_2^{k-1} & z_3^{k-1} & \cdots & z_k^{k-1} \end{pmatrix}$$

$$C = (c_1, c_2, \dots, c_k)^T$$

which is known as Vandermonde matrix

and $\det D = \prod_{r < s} (\alpha_s - \alpha_r)$. Since the roots are distinct $\det D \neq 0$ and hence

$DC = Y$ has a unique solution.

$$C = (c_1, c_2, \dots, c_k)^T$$

And we can write

$$y_m = c_1 \alpha_1^m + c_2 \alpha_2^m + \cdots + c_k \alpha_k^m, m=0, 1, 2, \dots, k-1$$

But

$$\alpha_k y_k + \alpha_{k-1} y_{k-1} + \cdots + \alpha_1 y_1 + \alpha_0 y_0 = 0$$

gives $y_k = -\frac{1}{\alpha_k} (\alpha_{k-1} y_{k-1} + \cdots + \alpha_1 y_1 + \alpha_0 y_0)$

Without loss of generality we assume that $\alpha_k = 1$ and we write

$$y_k = -(\alpha_{k-1} \left(\sum_{r=1}^k c_r \alpha_r^{k-1} \right) + \alpha_{k-2} \left(\sum_{r=1}^{k-1} c_r \alpha_r^{k-2} \right))$$

$$+ \dots + \alpha_1 \sum_{r=1}^k c_r z_r + \alpha_0 \sum_{r=1}^k c_r \Big)$$

Combining the similar terms we obtain

$$\begin{aligned} y_k &= - \left[c_1 (\alpha_0 + \alpha_1 z_1 + \dots + \alpha_{k-1} z_1^{k-1}) \right. \\ &\quad \left. + c_2 (\alpha_0 + \alpha_1 z_2 + \alpha_3 z_2^2 + \dots + \alpha_{k-1} z_2^{k-1}) \right. \\ &\quad \left. + \dots + c_k (\alpha_0 + \alpha_1 z_k + \alpha_3 z_k^2 + \dots + \alpha_{k-1} z_k^{k-1}) \right] \\ &= - \left(-c_1 z_1^k + -c_2 z_2^k + \dots - c_k z_k^k \right) \end{aligned}$$

$$\text{Since } p(z_1) = z_1^k + \alpha_{k-1} z_1^{k-1} + \dots + \alpha_1 z_1 + \alpha_0 = 0$$

$$\therefore y_k = \sum_{r=1}^k c_r z_r^k .$$

Assuming that the result is true for $n \leq k+1$

we can show that the result is true for
 $n \leq k+2$

\therefore By principle of mathematical induction

it follows that $y_n = \sum_{r=1}^k c_r z_r^n + n$.

Now if z_{r_0} is a root of multiplicity $m_{z_{r_0}} = 2$ and all others are of multiplicity one we consider the expression

$$c_1 + c_2 z + c_3 z^2 + \cdots + c_k z^{k-1} = y_0$$

$$c_1 z_1^1 + c_2 z_1^2 + c_3 z_1^3 + \cdots + c_k z_1^{k-1} = y_1$$

:

:

$$c_1 z_1^{k-1} + c_2 [z_1^{k-1} z_2] + c_3 z_3^{k-1} + \cdots + c_k z_k^{k-1} = y_{k-1}$$

Exercise.

$$y_n = \sum_{r=1}^l p_r(n) z_r^n$$

Root condition A linear multi step method is zero stable iff all roots of the first characteristic polynomial are inside the closed unit disc in the complex plane, with any which lie on the unit circle being simple.

Proof:

We only prove the necessary condition.

Assume that the given linear k-step multistep method is zerostable.

Now, when this method is applied to $\begin{cases} y' = 0 \\ y(0) = 0 \end{cases}$

we have $\alpha_k y_{n+k} + \alpha_{k-1} y_{n+k-1} + \dots + \alpha_0 y_n = 0$

and by previous lemma we can write

$$y_n = \sum_{r=1}^l p_r(n) z_r^n \quad \text{for } n=0, 1, 2, \dots$$

where z_r is a root of ch. polynomial P with multiplicity $m_r \geq 1$.

Now if $|z_r| > 1$ for some r

We can find starting values $y_0, y_1, y_2, \dots, y_{k-1}$

such that y_n grows like λ_r^n

and eventually $|y_n| \rightarrow \infty$ as $n \rightarrow \infty$, which is a contradiction to the fact that the given linear k-step method is zero stable.

i.e we can choose $y_0, y_1, y_2, \dots, y_{k-1}$ such that -
 $p_k(n) \neq 0 \therefore y_n \rightarrow \infty$ as $n \rightarrow \infty$

More precisely, if λ_{r_0} is such that $|\lambda_{r_0}| > 1$

we choose initial data $y_0 = 1, y_1 = \lambda_{r_0}, y_2 = \lambda_{r_0}^2, \dots, y_{k-1} = \lambda_{r_0}^{k-1}$

i.e we find the constants c_1, c_2, \dots, c_k
such that $c_1 + c_2 + \dots + c_{r_0} + \dots + c_k = 1$

$$c_1 z_1 + c_2 z_2 + \dots + c_{r_0} z_{r_0} + \dots + c_k z_k = \lambda_{r_0}$$
$$c_1 \lambda_{r_0}^{k-1} + c_{r_0} \lambda_{r_0}^{k-1} + \dots + c_k \lambda_{r_0}^{k-1} = \lambda_{r_0}^{k-1}$$

Now $c_1 = c_2 = c_{r_0-1} = c_{r_0+1} = \dots = 0 \quad \& \quad c_{r_0} = 1$

Satisfies the equation and hence

$$y_n = \sum_{r=1}^k c_r z_r^n \quad \text{gives}$$

$$y_n = z_{r_0} z_{r_0}^n$$

$$|y_n| = |z_{r_0}|^{n+1} \Rightarrow n \rightarrow \infty$$

$$|y_n| \rightarrow \infty$$

which is a contradiction to the fact that the given scheme is zero stable, i.e.

$$|y_n - z_n| \leq \max\{|1-0|, |z_{r_0}-0|, \dots, |z_{r_0}^{k-1}-0|\}$$

where $z_n = 0$, zero solution.

Similar way if z_r has multiplicity $m_r > 1$

and $|z_r| = 1$ for some r , then also we can choose $y_0, y_1, y_2, \dots, y_{k-1}$ such that

$$p_r(n) \neq 0 \quad \text{but} \quad p_r(n) = a + bn \quad b \neq 0$$

$$\therefore y_n \rightarrow \infty \quad \text{as } n \rightarrow \infty$$

Example: Determine the zero stability of

$$11y_{n+3} + 27y_{n+2} - 27y_{n+1} - 11y_n = 3h(f_{n+3} + 9f_{n+2} + 9f_{n+1} + f_n)$$

$$P(\lambda) = 11\lambda^3 + 27\lambda^2 - 27\lambda - 11$$

$$\text{Roots} \quad \lambda_1 = 1, \quad \lambda_2 \approx -0.32 \quad \lambda_3 = -3-14$$

$$|\lambda_3| > 1$$

Consistency of multistep methods.

The T.E of the multi Step method

$$\sum_{j=0}^k \alpha_j y_{n+j} = h \sum_{j=0}^k \beta_j f_{n+j}$$

is defined to be

$$T_n = \frac{\sum_{j=0}^k [\alpha_j y(x_{n+j}) - h \beta_j f(x_{n+j}, y(x_{n+j}))]}{h \sum_{j=0}^k \beta_j}$$

We require that $\sigma(1) = \sum_{j=0}^k \beta_j \neq 0$

Definition the given linear k-step method is consistent with the given ODE if the T.E defined above is such that for any $\epsilon > 0$, there exists $h(\epsilon)$ such that $|T_n| < \epsilon$ for $0 < h < h(\epsilon)$.

Using the fact $f(x_{n+j}, y(x_{n+j})) = -y'(x_{n+j})$

and Taylor expansion we get-

$$T_n = \frac{1}{h\sigma(1)} \left(c_0 y(x_n) + c_1 h y'(x_n) + c_2 h^2 y''(x_n) + \dots \right)$$

where $c_0 = \sum_{j=0}^k \alpha_j$

$$c_1 = \sum_{j=1}^k j \alpha_j - \sum_{j=0}^k \beta_j$$

$$c_2 = \sum_{j=1}^k \frac{j^2}{2!} \alpha_j - \sum_{j=1}^k j \beta_j$$

.

:

$$c_q = \sum_{j=1}^k \frac{j^q}{q!} \alpha_j - \sum_{j=1}^k \frac{j^{q-1}}{(q-1)!} \beta_j$$

Therefore for consistency we require that-

$$c_0 = c_1 = 0$$

$$\text{i.e. } p(1) = 0 \quad \phi \quad p'(1) = \sigma(1)$$

Order of accuracy

The linear-m multistep method is of order P

If $c_0 = c_1 = c_2 = \dots = c_p = 0$ & $c_{p+1} \neq 0$

$$\text{In this case } T_n = \frac{c_{p+1}}{6(1)} h^p y^{p+1}(x_n) + O(h^{p+1})$$

Example:

Dalquist's Equivalence Theorem.

For a linear k-step method that is consistent with the ODE where f is with desirable properties and with consistent starting values, zero-stability is necessary and sufficient condition for convergence.
Also, $e_n = y(x_n) - y_n = O(h^p)$ if the T.E is of order p.

Stiff systems

Consider the IVP $y' = \lambda y$
 $y(0) = y_0$

where λ is a constant.

Solution of this equation is given by

$$y = y_0 e^{\lambda x}.$$

When $\lambda < 0$ $y \rightarrow 0$ as $x \rightarrow \infty$

Euler method & implicit Euler method solutions

$$y_n^E = (1 + \lambda h)^n y_0 \quad y_n^I = (1 - \lambda h)^{-n} y_0$$

$$y_n = y_{n-1} + h \lambda y_{n-1}$$

$$= (1 + h \lambda) y_{n-1}$$

$$\vdots \\ = (1 + h \lambda)^n y_0$$

$$y_n = y_{n-1} + h \lambda y_n$$

$$(1 - \lambda h) y_n = y_{n-1}$$

$$y_n = (1 - \lambda h)^{-1} y_{n-1}$$

$$\vdots \\ y_n = (1 - \lambda h)^{-n} y_0$$

When $\lambda < 0$ we require that

$y_n^E \rightarrow 0$, this happens only when

$$|1 + h \lambda| < 1 \quad \text{ie} \quad -1 < 1 + h \lambda < 1$$

$$-2 < h \lambda$$

$$\text{i.e. } h |\lambda| < 2 \text{ or } h < 2/|\lambda|$$

This gives a restriction on the step size.

So when a numerical method is applied to $y' = \gamma y$ with $\gamma > 0$, we require to impose additional condition on the step size h .

Here for implicit Euler method we only require $1 - \gamma h > 1$ ie $\gamma h < 0$, which is always true.

So the explicit Euler method, in general explicit methods becomes useless when $\gamma > 0$ & $|f|$ is large.

In this situation we look for implicit linear multi-step methods.

$$\sum_{j=0}^k \alpha_j y_{n+j} = h \sum_{j=0}^k \beta_j f_{n+j}$$

When this method is applied to $y' = y$ we observe

$$\sum_{j=0}^k (\alpha_j - \lambda h \beta_j) y_{n+j} = 0 \quad (8)$$

The ch. poly of this linear sequence relation

is given by $\pi(\lambda; \lambda h) = \sum_{j=0}^k (\alpha_j - \lambda h \beta_j) \lambda^j$

which is called as stability polynomial.

According to the lemma for the above recurrence relation (8) we can write

$$y_n = \sum_{r=1}^l p_r(n) \lambda_r^n$$

p_r has degree $m_r - 1$, $1 \leq r \leq l$, m_r - multiplicity of λ_r .

If $\lambda \in \mathbb{C}$ & $\operatorname{Re} \lambda < 0$

Solution of $y' = \lambda y$ $y(0) = y_0$

$y(x) \rightarrow 0$ as $x \rightarrow \infty$

For this we require that $\lambda_r = \lambda_r(\lambda h)$

such that $|\lambda_r| < 1 + r$

Definition (Absolutely stable)

A linear multi-step method is said to be absolutely stable for a given value of λh if for each root $\lambda_r = \lambda_r(\lambda h)$ of $\Pi(\cdot; \lambda h)$ satisfies $|\lambda_r(\lambda h)| < 1$.

Definition (Region of Absolute stability)

The region of absolute stability of a linear multi-step method is the set of all points λh in the complex plane for which the method is absolutely stable.

Definition (A-stable)

A linear multi-step method is said to be A-stable if its region of absolute stability contains the negative (left) complex half-plane.

Negative result: (Dahlquist)

- (a) No explicit linear multi-step method is A-stable.

(b) No A-stable linear multi step method can have order greater than 2.

Ex: Trapezium rule method:

$$y_{n+1} = y_n + \frac{1}{2} h (f_{n+1} + f_n)$$

$$y_{n+1} - y_n - \frac{1}{2} h (f_{n+1} + f_n) = 0$$

$$y_{n+1} \left(1 - \frac{h\lambda}{2}\right) - y_n \left(1 + \frac{h\lambda}{2}\right) = 0$$

$$\pi(\cdot; \lambda h) = \left(1 - \frac{\lambda h}{2}\right)^2 - \left(1 + \frac{\lambda h}{2}\right)$$

root $\lambda = \left(\frac{1 + \frac{\lambda h}{2}}{1 - \frac{\lambda h}{2}} \right)$

Now $|\lambda| < 1$ if $\operatorname{Re}(\frac{\lambda h}{2}) < 0$

For $\left| \frac{1+\varepsilon}{1-\varepsilon} \right| < 1$ if $|1+\varepsilon|^2 < |1-\varepsilon|^2$

i.e if $(1+\varepsilon)(1-\bar{\varepsilon}) < (1-\varepsilon)(1-\bar{\varepsilon})$

i.e if $1+\bar{\varepsilon} + \varepsilon + \varepsilon\bar{\varepsilon} < 1-\bar{\varepsilon} - \varepsilon + \varepsilon\bar{\varepsilon}$

ie if $\bar{\xi} + \xi + \bar{\xi} + \xi < 0$

ie if $\operatorname{Re}\xi + \operatorname{Re}\bar{\xi} < 0$

ie if $\operatorname{Re}\xi < 0$

Therefore If $\operatorname{Re}\lambda h < 0$ ie if $h\operatorname{Re}\lambda < 0$
we have $|z| < 1$

∴ The Trapezium rule method is A-stable.

Example: Absolute stability region of RK methods.

Third order RK method.

$$y_{n+1} = y_n + \frac{h}{6} (k_1 + 4k_2 + k_3)$$

$$k_1 = f(x_n, y_n)$$

$$k_2 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1)$$

$$k_3 = f(x_n + h, y_n + h(2k_2 - k_1))$$

When this method is applied to $y' = \lambda y$, we

we have

$$y_{n+1} = y_n + \frac{h}{6} \left(2y_n + 4\lambda(y_n + \frac{h}{2}k_1) + \lambda(y_n + h(2k_2 - k_1)) \right)$$

$$= y_n + \frac{h}{6} \left[\lambda y_n + h \lambda \left(y_n + \frac{h}{2} \lambda y_n \right) + \lambda \left(y_n + h (2\lambda (y_n + \frac{h}{2} \lambda y_n) - K_1) \right) \right]$$

$$= y_n + \frac{h}{6} \left[\lambda y_n + h \lambda \left(y_n + \frac{h}{2} \lambda y_n \right) + \lambda \left(y_n + 2\lambda h y_n + \lambda h^2 K_1 - h \lambda y_n \right) \right]$$

$$= y_n + \frac{h}{6} \left[\lambda y_n + h \lambda \left(y_n + \frac{h}{2} \lambda y_n \right) + \lambda y_n + 2\lambda^2 h y_n + \lambda^2 h^2 \lambda y_n - h \lambda^2 y_n \right]$$

$$= y_n \left(1 + \frac{\lambda h}{6} + \frac{2}{3} \lambda h + \frac{\lambda^2 h^2}{3} + \frac{\lambda h}{6} + \frac{\lambda^2 h^2}{3} + \frac{\lambda^3 h^3}{6} - \frac{\lambda^2 h^2}{6} \right)$$

$$= y_n \left(1 + \lambda h + \frac{\lambda^2 h^2}{2} + \frac{\lambda^3 h^3}{6} \right)$$

\therefore Stability polynomial is given by

$$\pi(\cdot; \lambda h) = 1 - \left(1 + \lambda h + \frac{\lambda^2 h^2}{2} + \frac{\lambda^3 h^3}{6} \right)$$

$$\text{Root } z = 1 + \lambda h + \frac{\lambda^2 h^2}{2} + \frac{\lambda^3 h^3}{6}$$

Now find the region λh in complex plane where $|z| < 1$

setting $m = \gamma h$

we need γ satisfying $|1 + \gamma + \frac{\gamma^2}{2} + \frac{\gamma^3}{6}| < 1$

We can plot this in python.