Finite difference scheme for Poisson equation

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Poisson equation

$$
\begin{aligned}
-\Delta u &= f \quad \text{in } \Omega \\
u &= 0 \quad \text{in } \partial\Omega
\end{aligned}
$$

$$\Omega = (0,1) \times (0,1)$$

$$\Omega_h = \{(x_i, y_j) : \quad 0 < x_i, y_j < 1\} \qquad \|u\|_{\Omega_h} = \max_{(x_i,y_j)\in\Omega_h} |u(x_i, y_j)|$$

$$\bar{\Omega}_h = \{(x_i, y_j) : \quad 0 \le x_i, y_j \le 1\} \qquad \|u\|_{\bar{\Omega}_h} = \max_{(x_i,y_j)\in\bar{\Omega}_h} |u(x_i, y_j)|$$

$$\Gamma_h = \{(x_i, y_j) : \quad x_i y_j = 0 \text{ or } 1\} \qquad \|u\|_{\Gamma_h} = \max_{(x_i,y_j)\in\Gamma_h} |u(x_i, y_j)|$$

We define $u_i^j := u(x_i, y_j), \quad f_i^j := f(x_i, y_j)$

We use the grid function v to denote the approximate solution.

$$-\left(\frac{v_{i+1}^j - 2v_i^j + v_{i-1}^j}{(\Delta x)^2} + \frac{(v_i^{j+1} - 2v_i^j + v_i^{j-1})}{(\Delta y)^2}\right) = f_i^j$$

$$P_{\Delta x,\Delta y} v_i^j = \frac{v_{i+1}^j - 2v_i^j + v_{i-1}^j}{(\Delta x)^2} + \frac{(v_i^{j+1} - 2v_i^j + v_i^{j-1})}{(\Delta y)^2}$$

Finite difference equation ($h = \Delta x = \Delta y$)

$$-P_h v_i^j = f_i^j$$

Truncation error:

$$-P_h u = \mathcal{O}(h^2)$$

$$4v_i^j = v_{i+1}^j + v_{i-1}^j + v_i^{j+1} + v_i^{j-1} + h^2 f_i^j, \quad 1 \leq i, j \leq M-1$$

$$x_0 = y_0 = 0 \qquad x_i = i\Delta x = ih, i = 0, \ldots, n+1$$
$$x_{n+1} = y_{n+1} = 1 \qquad y_i = i\Delta y = ih, i = 0, \ldots, n+1$$

Let $n = 2$

$$\begin{aligned} 4v_1^1 &- (v_2^1 + v_0^1 + v_1^2 + v_1^0) = h^2 f_1^1 \\ 4v_2^1 &- (v_3^1 + v_1^1 + v_2^2 + v_2^0) = h^2 f_2^1 \\ 4v_1^2 &- (v_2^2 + v_0^2 + v_1^3 + v_1^1) = h^2 f_1^2 \\ 4v_2^2 &- (v_3^2 + v_1^2 + v_2^3 + v_2^1) = h^2 f_2^2 \end{aligned}$$

As on boundary $v_i^j = g_i^j$,this can be written as a linear system

$$Av = b_{g,f}$$

$$Av = \begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix} \begin{bmatrix} v_1^1 \\ v_2^1 \\ v_1^2 \\ v_2^2 \end{bmatrix} = \begin{bmatrix} g_0^1 & + & g_1^0 - h^2 f_1^1 \\ g_3^1 & + & g_2^0 - h^2 f_2^1 \\ g_0^2 & + & g_1^3 - h^2 f_1^2 \\ g_3^2 & + & g_2^3 - h^2 f_2^2 \end{bmatrix} = bg, f.$$

$n = 3$

$$
\begin{aligned}
4v_1^1 - v_2^1 - v_1^2 &= v_1^0 + v_0^1 + h^2 f_1^1 \\
4v_2^1 - v_3^1 - v_2^1 - v_1^1 &= v_2^0 + h^2 f_2^1 \\
4v_3^1 - v_3^2 - v_2^2 &= v_3^0 + v_4^1 + h^2 f_3^1 \\
4v_1^2 - v_2^2 - v_1^3 - v_1^1 &= v_0^2 + h^2 f_1^2 \\
4v_2^2 - v_3^2 - v_1^2 - v_2^3 - v_2^1 &= 0 + h^2 f_2^2 \\
4v_3^2 - v_2^2 - v_3^1 - v_3^3 &= v_4^2 + h^2 f_3^2 \\
4v_1^3 - v_2^3 - v_1^2 &= v_0^3 + v_1^4 + h^2 f_1^3 \\
4v_2^3 - v_3^3 - v_2^2 - v_1^3 &= v_2^4 + h^2 f_2^3 \\
4v_3^3 - v_2^3 - v_3^2 &= v_4^3 + v_3^4 + h^2 f_3^3
\end{aligned}
$$

$$Av = \begin{bmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} v_1^1 \\ v_2^1 \\ v_3^1 \\ v_1^2 \\ v_2^2 \\ v_3^2 \\ v_1^3 \\ v_2^3 \\ v_3^3 \end{bmatrix} = b_{g,f}$$

Lemma

The matrix A in the discretization above is invertible.

Lemma

If v is such that $-P_h v \leq 0(-P_h v \geq 0)$ in Ω_h, then v attains its maximu (minimum) for some $(x_i, y_j) \in \Gamma_h$.

Lemma

For any mesh function v, the following estimate holds

$$\|v\|_{\bar{\Omega}_h} \leq \|v\|_{\Gamma_h} + C\,\|P_h v\|_{\Omega}.$$

Theorem

Let v and u be the solutions of $-P_h v = f$ and $-\Delta u = f$, respectively (with $v_i^j = 0$ on Γ_h and $u = 0$ on $\partial\Omega$), then

$$\|v - u\|_{\Omega_h} \leq Ch^2 \|u\|_{C^4(\bar{\Omega})}$$

References I