ODE.

* Wellposedness theory of $\begin{cases} y' = f(t, y) \\ y(t_0) = y_0 \end{cases}$

existence
uniqueness
Continuous dependence on the data

(Picard's theorem) Suppose that the realvalued function $(x, y) \to f(x, y)$ is continuous in the rectangular region $D := \{(x, y) : x_0 \leq x \leq X_M,\ y_0 - C \leq y \leq y_0 + C\}$;

that $|f(x, y_0)| \leq K$ when $x_0 \leq x \leq X_M$ and that f satisfies Lipschitz condition: there exists $L > 0$, such that

$$|f(x, u) - f(x, v)| \leq L |u - v| \quad \forall (x, u), (x, v) \in D.$$

Assume further that $C \geq \frac{K}{L}\left(e^{L(X_M - x_0)} - 1\right)$.

Then there exists a unique function

$y \in C^1[x_0, X_M]$ such that $y(x_0) = y_0$

and $y' = f(x, y)$ for $x \in [x_0, X_M]$;

moreover $|y(x) - y_0| \leq C$, $x_0 \leq x \leq X_M$.

Note: it is only a sufficient condition

Eg: $\begin{cases} y' = x + \sin y \\ y(0) = 1 \end{cases}$

$\begin{cases} y' = y/t, \quad y(0) = 0 \\ f \text{ is not continuous but solution exists.} \end{cases}$

Continuous dependence of the solution on initial data and dynamics.

<u>theorem</u> Let D be as in the previous Picard's theorem. Suppose $f, \tilde{f} \in C(D)$ and be Lipschitz continuous w.r.to y on D with Lipschitz constants L & $\tilde{L}$, respectively. Let y & $\tilde{y}$ be, respectively the solutions of IVP

$$y' = f(x, y), \quad y(x_0) = y_0$$

and

$$\tilde{y}' = \tilde{f}(x, \tilde{y}), \quad \tilde{y}(\tilde{x}_0) = \tilde{y}_0$$

In some closed intervals I_1 & I_2 containing x_0 & $\tilde{x}_0$. For small $|x_0 - \tilde{x}_0|$ let I be any finite interval containing both x_0 & $\tilde{x}_0$, where both y & $\tilde{y}$ are defined. Then,

$$\max_{x \in I} |y(x) - \tilde{y}(x)| \leq \left(|y_0 - \tilde{y}_0| + |I| \max_D |f(x,y) - \tilde{f}(x,y)| + M|x_0 - \tilde{x}_0| \right) e^{L_0 |I|}.$$

where $|I|$ is the length of the interval I, $M = \max\left(\max_D f, \max_D \tilde{f} \right)$ and $L_0 = \min(L, \tilde{L})$

<u>Proof</u> we give a proof when $x_0 = \tilde{x}_0$.

It is easy to see that the solutions y & $\tilde{y}$ satisfy the following integral equations:

$$y(x) = y_0 + \int_{x_0}^{x} f(\xi, y(\xi))\, d\xi$$

$$\tilde{y}(x) = \tilde{y}_0 + \int_{x_0}^{x} \tilde{f}(\xi, \tilde{y}(\xi))\, d\xi \qquad \forall\, x \in I$$

Subtracting yields

$$y(x) - \tilde{y}(x) = y_0 - \tilde{y}_0 + \int_{x_0}^{x} (f(\xi, y(\xi)) - \tilde{f}(\xi, \tilde{y}(\xi)))\, d\xi$$

Up on adding and subtracting the term $f(\xi, \tilde{y}(\xi))$ we get

$$|y(x)-\tilde{y}(x)| \leq |y_0-\tilde{y}_0| + \int_{x_0}^{x} |f(\xi, y(\xi)) - f(\xi, \tilde{y}(\xi))| \, d\xi$$

$$+ \int_{x_0}^{x} |f(\xi, \tilde{y}(\xi)) - \tilde{f}(\xi, \tilde{y}(\xi))| \, d\xi$$

$$\leq |y_0 - \tilde{y}_0| + L\int_{x_0}^{x} |y(\xi) - \tilde{y}(\xi)| \, d\xi$$

$$+ |I| \max_{D} |f(x,y) - \tilde{f}(x,\tilde{y})|$$

$$\therefore \ |y(x) - \tilde{y}(x)| \leq C + L\int_{x_0}^{x} |y(\xi) - \tilde{y}(\xi)| \, d\xi$$

Applying Gronwall's inequality

$$|y(x) - \tilde{y}(x)| \leq C \exp\left(L \int_{x_0}^{x} d\xi\right)$$

$$= C \exp(L(x - x_0))$$

$$\leq C \exp L|I|$$

If we add & subtract $\tilde{f}(\xi, y(\xi))$ we obtain the estimate

$$|y(x) - \tilde{y}(x)| \leq C \exp(\tilde{L}|I|)$$

Combining both we get

$$|y(x) - \tilde{y}(x)| \leq \left(|y_0 - \tilde{y}_0| + |I| \max_D |f(\xi, y) - \tilde{f}(\xi, y)|\right) e^{L_0 |I|}$$

This completes the proof.

One-step Methods

Consider the IVP

$$\frac{dy}{dx} = f(x, y)$$
$$y(x_0) = y_0$$

in the domain $[x_0, X_M]$

We divide the domain $[x_0, X_M]$ into N points

$$x_0, x_1, x_2, \ldots, x_{N-1} = X_M$$

and $\quad x_n - x_{n-1} = h. \quad \forall\, n = 1, 2, \cdots, N-1$

If $y(x)$ is an exact solution of the IVP then

$$y(x_n + h) = y(x_n) + h\, y'(x_n) + O(h^2)$$

$$\Rightarrow \quad y'(x_n) = \frac{y(x_{n+1}) - y(x_n)}{h} + O(h) \quad \text{——} (1)$$

Now consider the differential equation at x_n

$$y'(x_n) - f(x_n, y(x_n)) = 0$$

using (1) we write

$$\frac{y(x_{n+1}) - y(x_n)}{h} - f(x_n, y(x_n) + O(h) = 0$$

ie
$$\frac{y(x_{n+1}) - y(x_n)}{h} - f(x_n, y(x_n)) \approx 0$$

Now moving from "$\approx$" to "$=$" we obtain the difference formula

$$\frac{y_{n+1} - y_n}{h} - f(x_n, y_n) = 0$$

or $\quad y_{n+1} = y_n + h f(x_n, y_n)$ ——— (2)

where $\quad y_n \approx y(x_n) \qquad n = 0, 1, 2, \ldots, N-1$

the difference equation (2) is called Euler method.

We see that for smooth solution

$$\frac{y(x_{n+1}) - y(x_n)}{h} - f(x_n, y(x_n))$$

$$= \underbrace{y'(x_n) - f(x_n, y(x_n))}_{=0} + O(h)$$

Difference equation = Differential equation + T.E

For this reason we define Truncation Error at x_n as

$$T_n = \frac{y(x_{n+1}) - y(x_n)}{h} - f(x_n, y(x_n))$$

for the difference scheme (2).

More generally a one step method may be written in the form

(3) ——— $y_{n+1} = y_n + h\,\Phi(x_n, y_n; h)$, $n = 0, 1, 2, \ldots, N-1$

$$y_0 = y(x_0)$$

where Φ is a continuous function of its variables.

For the Euler method $\Phi(x_n, y_n; h) = h f(x_n, y_n)$

We also define the global error by

$$e_n = y(x_n) - y_n .$$

and $$T_n = \frac{y(x_{n+1}) - y(x_n)}{h} - \Phi(x_n, y(x_n); h)$$

<u>Theorem</u> Consider the one-step method (3) where, in addition to being a continuous function of its arguments, Φ is assumed to satisfy a Lipschitz condition w.r. to its second argument, that is there exists a positive constant L_Φ such that for $0 \leq h \leq h_0$ and for all $(x,u), (x,v) \in D$

$D = \{ x_0 \leq x \leq X_M, |y - y_0| \leq C \}$

we have that

$$|\Phi(x, u; h) - \Phi(x, v; h)| \leq L_\phi |u - v|$$

Then assuming that $|y_n - y_0| \leq C, \forall n = 1, 2, N$

It follows that

$$|e_n| \le \frac{T}{L_\Phi}\left(e^{L_\Phi(x_n - x_0)} - 1\right), n = 1, 2, \ldots, N$$

where $T = \max\limits_{0 \le n \le N-1} T_n$

Proof:

We have

$$y_{n+1} = y_n + h\,\Phi(x_n, y_n; h)$$

& $$y(x_{n+1}) = y(x_n) + h\,\Phi(x_n, y(x_n); h) + h T_n$$

Subtracting we get

$$e_{n+1} = e_n + h\left(\Phi(x_n, y(x_n); h) - \Phi(x_n, y_n; h)\right) + h T_n$$

since $(x_n, y(x_n)), (x_n, y_n) \in D$ the Lipschitz condition gives

$$|e_{n+1}| \le |e_n| + h L_\Phi |e_n| + h T_n$$

$$= (1 + h L_\Phi)|e_n| + h T_n$$

$$\le (1 + h L_\Phi)\left[(1 + h L_\Phi)|e_{n-1}| + h T_{n-1}\right] + h T_n$$

$$= (1 + h L_\Phi)^2 |e_{n-1}| + h(1 + h L_\Phi) T_{n-1} + h T_n$$

$$\vdots$$

$$\le (1+hL_\Phi)^3 |e_{n-2}| + h(1+hL_\Phi)^2 T_{n-2} + h(1+hL_\Phi)T_{n-1} + hT_n$$

$$\le (1+hL_\Phi)^{n+1} \underset{\overset{\shortparallel}{0}}{|e_0|} + h\left[(1+hL_\Phi)^n T_0 + (1+hL_\Phi)^{n-1} T_1 + \cdots + T_n\right]$$

or

$$|e_n| \le Th\left((1+hL_\Phi)^{n-1} + (1+hL_\Phi)^{n-2} + \cdots + 1\right)$$

$$= Th\left(\frac{(1+hL_\Phi)^n - 1}{(1+hL_\Phi) - 1}\right)$$

$$= \frac{T}{L_\Phi}\left((1+hL_\Phi)^n - 1\right)$$

But $1+hL_\Phi \le e^{hL_\Phi}$

$$\therefore \quad |e_n| \le \frac{T}{L_\Phi}\left(e^{nhL_\Phi} - 1\right)$$

$$= \frac{T}{L_\Phi}\left(e^{(x_n-x_0)L_\Phi} - 1\right), \quad n = 0, 1, \ldots N$$

This completes the proof.

Exercise: obtain the T.E for Euler method.

Exercise: Determine the error bound for the problem

$$y' = \tan^{-1} y$$

$$y(0) = y \quad \text{in } [0, 1]$$

Consistency and Convergence.

Definition (consistency) the numerical method

$y_{n+1} = y_n + h\,\Phi(x_n, y_n; h)$ is consistent with the differential equation $y' = f(x, y)$ if the T.E, is such that for any $\epsilon > 0$ there exists a $h(\epsilon) > 0$

$|T_n| < \epsilon \quad \forall \; 0 < h < h(\epsilon)$ for any point $(x_n, y(x_n))$ on any solution curve. □

ie $T_n \to 0$ as $h \to 0$, $n \to \infty$ $\qquad h = \dfrac{X_M - x_0}{N}$

Now for $x \in [x_0, X_M]$ such that

$$\lim_{\substack{n\to\infty \\ h\to 0}} x_n = x$$

$$\lim_{\substack{n\to\infty \\ h\to 0}} T_n = \lim_{\substack{n\to\infty \\ h\to 0}} \left(y'(x_n) - \Phi(x_n, y(x_n); h) \right)$$

$$= y'(x) - \Phi(x, y(x); 0)$$

$\therefore$ the numerical method is consistent iff

$$\Phi(x, y; 0) = f(x, y) \qquad \text{——— (c)}$$

Theorem

Convergence theorem: Suppose that the IVP satisfies the conditions of the Picard's theorem and also that its approximation generated from

$$y_{n+1} = y_n + h\,\Phi(x_n, y_n; h) \quad \text{when} \quad h \le h_0$$

lies in the domain D. Assume further that the function $\Phi(\cdot,\cdot;\cdot)$ is continuous on $D \times [0, h_0]$ and satisfies the consistency condition (c) above and Lipschitz condition

$$|\Phi(x, u; h) - \Phi(x, v; h)| \le L_\Phi |u - v|$$

on $D \times [0, h_0]$

then we have convergence of the numerical solution to solution of the IVP in the sense that

$$\lim_{n \to \infty} y_n = y(x) \quad \text{as} \quad x_n \to x \in [x_0, X_M]$$

when $n \to \infty$ and $h \to 0$

Proof: Exercise.

Definition The numerical method described above is said to have order of accuracy p, if p is the largest positive integer such that for any sufficiently smooth solution curve $(x, y(x))$ in D of the IVP, there exists constants K and h_0 such that $|T_n| \leq K h^p$ for $0 < h \leq h_0$

Runge-Kutta methods

Consider the IVP
$$\left.\begin{aligned} y'(x) &= f(x, y) \\ y(x_0) &= y_0 \end{aligned}\right\}$$

By fundamental theorem of I.C we can write

$$\int_{x_n}^{x_{n+1}} y'(x) = \int_{x_n}^{x_{n+1}} f(x(\xi), y(\xi))\, d\xi$$

$$\Rightarrow \quad y(x_{n+1}) = y(x_n) + \int_{x_n}^{x_{n+1}} f(x(\xi), y(\xi))\, d\xi$$

If we use mid-point rule quadrature to to find the integral approximately, we obtain

$$y(x_{n+1}) = y(x_n) + h f\left(x_n + \frac{h}{2}, y\left(x_n + \frac{h}{2}\right)\right) + O(h^3)$$

$$= y(x_n) + h f\left(x_n + \frac{h}{2}, y(x_n) + \frac{h}{2} y'(x_n) + O(h^2)\right) + O(h^3)$$

$$= y(x_n) + h\left(f\left(x_n + \frac{h}{2}, y(x_n) + \frac{h}{2} y'(x_n)\right) + \frac{\partial f}{\partial y}\left(x_n + h/2, \tilde{\xi}\right) O(h^2)\right) + O(h^3)$$

$$= y(x_n) + h\left(f\left(x_n + \frac{h}{2}, y(x_n) + \frac{h}{2} f(x_n, y_n)\right)\right) + O(h^3)$$

Finally we obtain a numerical scheme

$$y_{n+1} = y_n + h\left(f\left(x_n + \frac{h}{2},\ y_n + \frac{h}{2} f(x_n, y_n)\right)\right)$$

Called modified Euler method and

T.E $\quad T_n = O(h^2)$

Now if we use trapezoidal rule for the quadrature we obtain

$$y(x_{n+1}) = y(x_n) + \frac{h}{2}\left(f(x_n, y(x_n)) + f(x_n + h, y(x_n + h)\right) + O(h^3)$$

$$= y(x_n) + \frac{h}{2}\left(f(x_n, y(x_n)) + f(x_n + h, y(x_n) + h y'(x_n) + O(h^2))\right)$$

$$+ O(h^3)$$

$$= y(x_n) + \frac{h}{2}\left(f(x_n, y(x_n)) + f(x_n + h, y(x_n) + h f(x_n, y(x_n))\right)$$
$$+ \frac{\partial f}{\partial y}(x_n + h, \xi)\, O(h^2)\Big) + O(h^3)$$

and we obtain a numerical scheme

$$y_{n+1} = y_n + \frac{h}{2}\left(f(x_n, y_n) + f(x_n+h, y_n+hf(x_n, y_n))\right)$$

this is a second-order method called Improved Euler method.

Both the second-order schemes described above are called Runge-Kutta method.

In-general the Runge-Kutta method of order two is a class of schemes written in the form

$$y_{n+1} = y_n + h(ak_1 + bk_2)$$

where $k_1 = f(x_n, y_n)$

$$k_2 = f(x_n + \alpha h, y_n + \beta h k_1)$$

where a, b, α & β are to be determined.

Note: For the above described two schemes determine a, b, α & β.

We write $y_{n+1} = y_n + h\,\Phi(x_n, y_n; h)$

where $\Phi(x_n, y_n; h) = a\,f(x_n, y_n) + b\,f(x_n + \alpha h, y_n + \beta h f(x_n, y_n))$

The T.E is given by

$$T_n = \frac{y(x_{n+1}) - y(x_n)}{h} - \Phi(x_n, y(x_n); h)$$

Noting that

$$y'(x_n) = f$$

$$y''(x_n) = f_x + f_y\, y'(x_n)$$

$$= f_x + f_y f$$

all evaluated at $(x_n, y(x_n))$

$$y'''(x_n) = f_{xx} + f_{xy} f + f_y(f_x + f_y f) + f(f_{yx} + f_{yy} f)$$

$$= f_{xx} + f_{xy} f + f_y f_x + f_y^2 f + f f_{yx} + f^2 f_{yy}.$$

Using Taylor expansion we obtain

$$\Phi(x_n, y(x_n); h) = af + b\Big(f + \alpha h \frac{\partial f}{\partial x} + \beta h \frac{\partial f}{\partial y} f$$

$$+ \frac{1}{2}(\alpha h)^2 \frac{\partial^2 f}{\partial x^2} + \alpha h \beta h \frac{\partial f}{\partial x y} f$$

$$+ \frac{1}{2}(\beta h)^2 f^2 \frac{\partial^2 f}{\partial y^2} + O(h^3)\Big)$$

Thus we obtain the truncation error in the form

$$T_n = \frac{y(x_n+h) - y(x_n)}{h} - \Phi(x_n, y(x_n); h)$$

$$= y'(x_n) + \frac{h}{2} y''(x_n) + \frac{h^2}{6} y'''(x_n) - \Phi(x_n, y(x_n); h) + O(h^3)$$

$$= f + \frac{h}{2}\left(f_x + f_y f\right) + \frac{h^2}{6}\Big(f_{xx} + f_{xy} f + (f_{xy} + f_{yy} f) f + f_y (f_x + f_y f)\Big)$$

$$- \Big(a f + b\big(f + \alpha h f_x + \beta h f_y f + \frac{\alpha^2 h^2}{2} f_{xx} + \alpha\beta h^2 f_{xy} f + f^2 \frac{\beta^2 h^2}{2} f_{yy}\big) + O(h^3)\Big)$$

$$= (1 - a - b) f + \frac{h}{2}\left(f_x + f_y f - b\alpha f_x - b\beta f_y\right)$$

$$+ h^2\Big[\frac{1}{6} f_{xx} - \frac{b\alpha^2}{2} f_{xx} + \frac{f_{xy} f}{6} - b\alpha\beta f_{xy} f$$

$$+\frac{1}{6}(f_{xy}+f_{yy}f)f - \frac{\beta^2}{2}b f_{yy}f + \frac{1}{6}f_y(f_x+f_yf)\Big]$$

$$+ O(h^3)$$

$$=(1-a-b)f + \frac{h}{2}(f_x + f_yf - b\alpha f_x - b\beta f_y)$$

$$+ h^2\Big[\left(\frac{1}{6} - \frac{b\alpha^2}{2}\right)f_{xx} + f^2 f_{yy}\left(\frac{1}{6} - \frac{\beta^2 b}{2}\right)$$

$$+ f f_{xy}\left(\frac{1}{3} - b\alpha\beta\right)$$

$$+ \frac{1}{6}f_y(f_x + f_y f)\Big] + O(h^3)$$

For the consistency of the scheme we require that

$$(1-a-b)f = 0$$

ie $\quad a+b = 1 \quad$ ——— (C1)

For second-order scheme we require that the second term to be zero

ie if $\frac{1}{2}(f_x + f_y f) - b\alpha f_x - b\beta f_y f = 0$

for all functions f

ie if $b\alpha = b\beta = \frac{1}{2}$

The method is second-order accurate if

$$\alpha = \beta \quad , \quad b = \frac{1}{2\alpha}$$

$$a = 1 - \frac{1}{2\alpha} \qquad \alpha \neq 0$$

a, b, β are all determined by α

$\therefore$ We get a one parameter family of second-order schemes with truncation error

$$T_h = h^2 \left(\left(\tfrac{1}{6} - \frac{\alpha}{4} \right) (f_{xx} + f^2 f_{yy}) + \left(\tfrac{1}{3} - \frac{\alpha}{2} \right) f f_{xy} + \tfrac{1}{6} (f_x f_y + f f_y^2) \right) + O(h^3)$$

Schemes are given by

$$
\begin{cases}
y_{n+1} = y_n + \left(1-\frac{1}{2\alpha}\right) k_1 + \frac{1}{2\alpha} K_2 \\
k_1 = f(x_n, y_n) \\
K_2 = f(x_n + \alpha h,\ y_n + \alpha h f(x_n, y_n))
\end{cases}
$$

In a similar way we can derive R-K method of order 3, 4 etc.

One of the most frequently used method of the R-K family is the classical fourth-order methods:

$$y_{n+1} = y_n + \frac{1}{6} h \left(K_1 + 2K_2 + 2K_3 + K_4\right)$$

$$k_1 = f(x_n, y_n)$$

$$k_2 = f\left(x_n + \tfrac{1}{2} h,\ y_n + \tfrac{1}{2} h k_1\right)$$

$$k_3 = f(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hk_2)$$

$$k_4 = f(x_n + h, y_n + hk_3)$$

In general an s- stage R-K method of order s can be written as

$$y_{n+1} = y_n + h \sum_{i=1}^{s} b_i k_i$$

where $k_i = f(x_n + c_i h, y_n + h \sum_{j=1}^{s} a_{ij} k_j), 1 \le i \le s$

Butcher Tableau

$$\begin{array}{c|cccc}
c_1 & a_{11} & a_{12} & \cdots & a_{1s} \\
c_2 & \vdots & & & \vdots \\
\vdots & \vdots & & & \vdots \\
c_s & a_{s1} & a_{s2} & \cdots & a_{ss} \\
\hline
 & b_1 & b_2 & & b_s
\end{array}$$

For the consistency we require $\sum_{j=1}^{s} b_j = 1$

It is explicit if $A = (a_{ij})$ is lower triangular matrix. Otherwise implicit.

The Butcher tableau for classical four-stage

$$
\begin{array}{c|cccc}
0 & & & & \\
\frac{1}{2} & \frac{1}{2} & & & \\
\frac{1}{2} & 0 & \frac{1}{2} & & \\
1 & 0 & 0 & 1 & \\
\hline
 & \frac{1}{6} & \frac{2}{3} & \frac{2}{3} & \frac{1}{6}
\end{array}
$$

* Linear multi-steps methods.

Consider the IVP $y' = f(x, y)$

$$y(x_0) = y_0$$

By Fundemental theorem of I.C we have

$$y(x_{n+1}) = y(x_{n-1}) + \int_{x_{n-1}}^{x_{n+1}} f(\xi, y(\xi))\, d\xi$$

If we apply Simpson's rule to integrate we obtain:

$$y(x_{n+1}) = y(x_{n-1}) + \frac{h}{3}\Big(f(x_{n-1}, y(x_{n-1})) + 4 f(x_n, y(x_n)) + f(x_{n+1}, y(x_{n+1}))\Big) + O(h^5)$$

leads to a method of the form

$$y_{n+1} = y_{n-1} + \frac{h}{3}\left(f(x_{n-1}, y_{n-1}) + 4f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right)$$

which is a method of order four.

Given an equially spaced mesh points (x_n) with step size h, we consider the general linear k-step method

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f(x_{n+j}, y_{n+j})$$

where α_i, β_j are real constant.

We shall assume that $\alpha_k \neq 0$ and α_0 & β_0 are not both equal to zero.

$\beta_k = 0 \Rightarrow$ Explicit method

$\beta_k \neq 0 \Rightarrow$ Implicit method.

Examples:

(1) Explicit and implicit Euler methods

$$y_{n+1} = y_n + h f(x_n, y_n)$$

$$y_{n+1} = y_n + h f(x_{n+1}, y_{n+1})$$

(2) Trapezium Rule method (implicit)

$$y_{n+1} = y_n + \frac{1}{2} h \left(f(x_n, y_n) + f(x_{n+1}, y_{n+1}) \right)$$

Using this notation $f_n := f(x_n, y_n)$

(3) Adams-Bashforth method (Explicit)

$$y_{n+4} = y_{n+3} + \frac{h}{24}\left(55 f_{n+3} - 59 f_{n+2} + 37 f_{n+1} - 9 f_n \right)$$

(4) Adams-Moulton method (implicit)

$$y_{n+3} = y_{n+2} - \frac{1}{24} h \left(9 f_{n+3} + 19 f_{n+2} - 5 f_{n+1} - 9 f_n \right)$$

Zero-Stability

It is clear that for a multi-step method of steps k, we need to determine $y_0, y_1, \ldots, y_{k-1}$ before we can apply this method.

$y_0, y_1, \ldots, y_{k-1}$ can be computed using a suitable one-step method.

Therefore the starting values will contain numerical error subject to the one-step method used. So, we expect that the error generated from "small perturbations" in the starting conditions should not grow exponentially. This leads to the concept of "zero stability".

Definition A linear k-step method for the IVP is said to be "zero stable" if there exists a constant K such that, for any two sequences (y_n) & (z_n) that have been generated by the same formulae but different starting values $z_0, z_1, \ldots, z_{k-1}$, respectively, we have

$$|y_n - z_n| \le K \max\{|y_0 - z_0|, \ldots, |y_{k-1} - z_{k-1}|\}$$

for $x_n \le X_M$ and as $h \to 0$.

Given the linear k-step method

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} f_{n+j} ,$$

we consider its first and second characteristic polynomials, respectively

$$\rho(z) = \sum_{j=0}^{k} \alpha_j z^j$$

$$\sigma(z) = \sum_{j=0}^{k} \beta_j z^j$$

and assume that $\alpha_k \ne 0$, $\alpha_0^2 + \beta_0^2 \ne 0$

We consider the case where $f(x,y) \equiv 0$, as we merely required to consider the behavior when the scheme is applied to $y' = 0$.

We have the following lemma.

<u>Lemma</u> Consider the k^{th}-order homogeneous linear recursive relation

$$(\gamma) \text{ --- } \alpha_k y_{n+k} + \alpha_{k-1} y_{n+k-1} + \cdots + \alpha_0 y_n = 0$$

$$n = 0, 1, 2, \ldots$$

with $\alpha_k \neq 0$, $\alpha_0 \neq 0$, $\alpha_j \in \mathbb{R}$, $j = 0, 1, \ldots, k$

and corresponding cha. polynomial

$$P(z) = \alpha_k z^k + \cdots + \alpha_1 z + \alpha_0 .$$

Let z_r, $1 \leq r \leq l$, $l \leq k$, be the distinct roots of polynomial P, and $m_r \geq 1$ denote the multiplicity of z_r, with $m_1 + m_2 + \cdots + m_l = k$. If a sequence y_n of Complex numbers satisfies (γ) above, then

$$y_n = \sum_{r=1}^{l} P_r(n) z_r^n \quad \text{for all } n \geq 0$$

where $P_r(\cdot)$ is a polynomial in n of degree $m_r - 1$, $1 \le r \le l$.

Proof:

Assume that all roots z_r are distinct.

We observe that the sequences

(z_r^n), $r = 1, 2, \dots, k$ satisfies the recurrence relation.

$$\alpha_k z_1^k + \alpha_{k-1} z_1^{k-1} + \cdots + \alpha_1 z_1 + \alpha_0 = 0$$

$$\Rightarrow \alpha_k z_1^{k+1} + \alpha_{k-1} z_1^k + \cdots + \alpha_1 z_1^2 + \alpha_0 z = 0$$

$$\alpha_k z_1^{k+2} + \alpha_{k-1} z_1^{k+1} + \cdots + \alpha_1 z_1^3 + \alpha_0 z^2 = 0$$

Proceeding like this

$$\alpha_k z_1^{k+n} + \alpha_{k-1} z_1^{k-1+n} + \cdots + \alpha_1 z_1^{n-1} + \alpha_0 z^n = 0$$

$\Rightarrow$ (z_1^n) satisfies the recurrence relation.

for $n = 0, 1, 2, 3, \ldots, K-1$,

we consider the expressions

$$C_1 z_1^0 + C_2 z_2^0 + \cdots + C_K z_K^0 = y_0$$

$$C_1 z_1^1 + C_2 z_2^1 + \cdots + C_K z_K^1 = y_1$$

$$\vdots$$

$$C_1 z_1^{K-1} + C_2 z_2^{K-1} + \cdots + C_K z_K^{K-1} = y_{K-1}$$

which can be written in matrix form

as

$$DC = Y \qquad D = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ z_1 & z_2 & z_2 & & z_K \\ z_1^2 & z_2^2 & z_1^2 & \cdots & z_K^2 \\ \vdots & & & & \\ z_1^{K-1} & z_2^{K-1} & \cdots & & z_K^{K-1} \end{pmatrix}$$

$$C = (C_1, C_2, \ldots, C_K)^T$$

which is known as Vandermode matrix

and $\det D = \prod_{r<s} (z_s - z_r)$. Since the roots are distinct $\det D \neq 0$ and have

$DC = Y$ has a unique solution.

$$C = (c_1, c_2 \ldots, c_K)^T$$

And we can write

$$y_m = c_1 z_1^m + c_2 z_2^m + \cdots + c_K z_K^m \quad , m = 0, 1, 2, \ldots, K-1$$

But

$$\alpha_K y_K + \alpha_{K-1} y_{K-1} + \cdots + \alpha_1 y_1 + \alpha_0 y_0 = 0$$

gives $y_K = \frac{-1}{\alpha_K} (\alpha_{K-1} y_{K-1} + \cdots + \alpha_1 y_1 + \alpha_0 y_0)$

Without loss of generality we assume that $\alpha_K = 1$ and we write

$$y_K = -\left(\alpha_{K-1}\left(\sum_{r=1}^{K} c_r z_r^{K-1}\right) + \alpha_{K-2}\left(\sum_{r=1}^{K} c_r z_r^{K-2}\right)\right.$$

$$+ \cdots + \alpha_1 \sum_{r=1}^{k} c_r z_r + \alpha_0 \sum_{r=1}^{k} c_r \Big)$$

Combining the similar terms we obtain

$$y_k = -\Big[c_1 \left(\alpha_0 + \alpha_1 z_1 + \cdots + \alpha_{k-1} z_1^{k-1}\right)$$

$$+ c_2 \left(\alpha_0 + \alpha_1 z_2 + \alpha_3 z_2^2 + \cdots + \alpha_{k-1} z_2^{k-1}\right)$$

$$+ \cdots + c_k \left(\alpha_0 + \alpha_1 z_k + \alpha_3 z_k^2 + \cdots + \alpha_{k-1} z_k^{k-1}\right)$$

$$= -\left(-c_1 z_1^k + -c_2 z_2^k + \cdots - c_k z_k^k\right)$$

since $P(z_1) = z_1^k + \alpha_{k-1} z_1^{k-1} + \cdots + \alpha_1 z_1 + \alpha_0 = 0$

$$\therefore \; y_k = \sum_{r=1}^{k} c_r z_r^k .$$

Assuming that the result is true for $n \leq k+1$ we can show that the result is true for $n \leq k+2$

$\therefore$ By principle of mathematical induction it follows that $y_n = \sum_{r=1}^{k} c_r z_r^k \quad \forall n.$

Now if z_{r_0} is a root of multiplicity $m_{r_0} = 2$ and all others are of multiplicity one we consider the expression.

$$c_1 + c_2 \times + c_3 + \cdots + c_k = y_0$$

$$c_1 z_1^1 + c_2 \times 1 \times z_1^1 + c_3 z_3^1 + \cdots + c_k z_k^1 = y_1$$

$$\vdots$$

$$c_1 z_1^{k-1} + c_2 \left[(k-1) z_1^{k-1} \right] + c_3 z_3^{k-1} + \cdots + c_k z_k^{k-1} = y_{k-1}$$

Exercise.

$$y_n = \sum_{r=1}^{l} P_r(n) z_r^n$$

Root condition A linear multi step method is zero stable iff all roots of the first-characteristic polynomial are inside the closed unit disc in the complex plane, with any which lie on the unit circle bieng simple.

Proof:

We only prove the necessary condition.

Assume that the given linear k-step multistep method is zero stable.

Now, when this method is applied to $\begin{cases} y' = 0 \\ y(0) = 0 \end{cases}$

we have $$\alpha_k y_{n+k} + \alpha_{k-1} y_{n+k-1} + \cdots + \alpha_0 y_n = 0$$

and by previous lemma we can write

$$y_n = \sum_{r=1}^{l} P_r(n) z_r^n \quad \text{for } n = 0, 1, 2, \ldots$$

where z_r is a root of Ch. polynomial ρ with multiplicity $m_r \geq 1$.

Now if $|z_r| > 1$ for some r

We can find starting values $y_0, y_1, y_2, \ldots y_{k-1}$

such that y_n grows like z_r^n

and eventually $|y_n| \to \infty$ as $n \to \infty$, which is a contradiction to the fact that the given linear k-step method is zero stable.

ie we can choose $y_0, y_1, y_2, \ldots y_{k-1}$ such that $P_r(n) \neq 0$ $\therefore$ $y_n \to \infty$ as $n \to \infty$

Similar way if z_r has multiplicity $m_r > 1$

and $|z_r| = 1$ for some r, then also we can choose $y_0, y_1, y_2, \ldots, y_{k-1}$ such that

$P_r(n) \neq 0$ but $P_r(n) = a + bn$ $\quad b \neq 0$

$\therefore$ $y_n \to \infty$ as $n \to \infty$

Example: Determine the zero stability of

$$11 y_{n+3} + 27 y_{n+2} - 27 y_{n+1} - 11 y_n = 3h\,(f_{n+3} + 9 f_{n+2} + 9 f_{n+1} + f_n)$$

$$\rho(z) = 11 z^3 + 27 z^2 - 27 z - 11$$

roots $z_1 = 1$, $z_2 \approx -0.32$ $z_3 = -3.14$

$|z_3| > 1$

Consistency of multistep methods.

The T.E of the multi step method

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j}$$

is defined to be

$$T_n = \frac{\sum_{j=0}^{k} [\alpha_j y(x_{n+j}) - h \beta_j f(x_{n+j}, y(x_{n+j}))]}{h \sum_{j=0}^{k} \beta_j}$$

we require that $\sigma(1) = \sum_{j=0}^{k} \beta_j \neq 0$

Definition The given linear k-step method is consistent with the given ODE if the T.E defined above is such that for any $\epsilon > 0$, there exists $h(\epsilon)$ such that $|T_n| < \epsilon$ for $0 < h < h(\epsilon)$.

Using the fact $f(x_{n+j}, y(x_{n+j})) = y'(x_{n+j})$ and Taylor expansion we get

$$T_n = \frac{1}{h\sigma(1)} \left(C_0\, y(x_n) + C_1 h\, y'(x_n) + C_2 h^2 y''(x_n) + \cdots \right)$$

where

$$C_0 = \sum_{j=0}^{k} \alpha_j$$

$$C_1 = \sum_{j=1}^{k} j\alpha_j - \sum_{j=0}^{k} \beta_j$$

$$C_2 = \sum_{j=1}^{k} \frac{j^2}{2!}\alpha_j - \sum_{j=1}^{k} j\beta_j$$

$$\vdots$$

$$C_q = \sum_{j=1}^{k} \frac{j^q}{q!}\alpha_j - \sum_{j=1}^{k} \frac{j^{q-1}}{(q-1)!}\beta_j$$

Therefore for consistency we require that

$$C_0 = C_1 = 0$$

ie $\rho(1) = 0$ & $\rho'(1) = \sigma(1)$

Order of accuracy

The linear-multistep method is of order P

If $C_0 = C_1 = C_2 = \cdots = C_p = 0$ & $C_{p+1} \neq 0$

In this case $T_n = \frac{C_{p+1}}{\sigma(1)} h^p y^{p+1}(x_n) + O(h^{p+1})$

Example:

Dalquist's Equivalence theorem.

For a linear k-step method that is consistent with the ODE where f is with desirable properties and with consistent starting values, zero-stability is necessary and sufficient condition for convergence.
Also, $e_n = y(x_n) - y_n = O(h^p)$ if the T.E is of order p.

Stiff system

Consider the IVP $y' = \lambda y$

$y(0) = y_0$

where λ is a constant.

Solution of this equation is given by

$$y = y_0 e^{\lambda x}.$$

When $\lambda < 0$ $\quad y \to 0$ as $x \to \infty$

Euler method & implicit Euler method solutions

$$y_n^E = (1+\lambda h)^n y_0 \qquad y_n^I = (1-\lambda h)^{-n} y_0$$

$$y_n = y_{n-1} + h\lambda y_{n-1}$$
$$= (1+h\lambda) y_{n-1}$$
$$\vdots$$
$$= (1+h\lambda)^n y_0$$

$$y_n = y_{n-1} + h\lambda y_n$$
$$(1-\lambda h) y_n = y_{n-1}$$
$$y_n = (1-\lambda h)^{-1} y_{n-1}$$
$$\vdots$$
$$y_n = (1-\lambda h)^{-n} y_0$$

when $\lambda < 0$ we require that

$y_n^E \to 0$, this happens only when

$|1+h\lambda| < 1$ ie $-1 < 1+h\lambda < 1$

$-2 < h\lambda$

ie if $h|\lambda| < 2$ or $h < 2/|\lambda|$

This gives a restriction on the step size.

So when a numerical method is applied to $y' = \lambda y$ with $\lambda < 0$, we require to impose addition condition on the step size.

Here for implicit Euler method we only require $1 - \lambda h > 1$ ie $\lambda h < 0$, which is always true.

So the explicit Euler method, in general explicit methods becomes useless when $\lambda < 0$ & $|\lambda|$ is large.

In this situation we look for implicit linear multi-step methods.

$$\sum_{j=0}^{K} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j}$$

when this method is applied to $y' = y$

we observe

$$\sum_{j=0}^{k} (\alpha_j - h\beta_j \lambda) y_{n+j} = 0 \qquad \text{———————} \quad (\gamma)$$

The ch. poly of this linear recurrence relation is given by $\pi(z ; \lambda h) = \sum_{j=0}^{k} (\alpha_j - \lambda h \beta_j) z^j$

which is called as stability polynomial.

According to the lemma for the above recurrence relation (γ) we can write

$$y_n = \sum_{r=1}^{l} P_r(n) z_r^n$$

P_r has degree $m_r - 1$, $1 \le r \le l$, m_r - multiplicity of z_r.

If $\lambda \in \mathbb{C}$ & $Re\lambda < 0$

Solution of $y' = \lambda y$ $y(0) = y_0$

$y(x) \to 0$ as $x \to \infty$

For this we require that $z_r = z_r(\lambda h)$

such that $|z_r| < 1 \quad \forall\, r$

Definition (Absolutely stable)

A linear multistep method is said to be absolutely stable for a given value of λh if for each root $z_r = z_r(\lambda h)$ of $\pi(\cdot\,; \lambda h)$ satisfies $|z_r(\lambda h)| < 1$.

Definition (Region of Absolute stability)

The region of absolute stability of a linear multistep method is the set of all points λh in the complex plane for which the method is absolutely stable.

Definition (A-stable)

A linear multi-step method is said to be A-stable if its region of absolute stability contains the negative (left) complex half plane.

Negative result: (Dalquist)

(a) No explicit linear multistep method is A-stable.

(b) No A-stable linear multistep method can have order greater that 2.

Ex: Trapezium rule method:

$$y_{n+1} = y_n + \frac{1}{2}h\left(f_{n+1} + f_n\right)$$

$$y_{n+1} - y_n - \tfrac{1}{2}h\left(\lambda y_{n+1} + \lambda y_n\right)$$

$$y_{n+1}\left(1 - \frac{h\lambda}{2}\right) - y_n\left(1 + \frac{h\lambda}{2}\right) = 0$$

$$\pi(\cdot\, ; \lambda h) = \left(1 - \frac{\lambda h}{2}\right)z - \left(1 + \frac{h\lambda}{2}\right)$$

root $\quad z = \left(\dfrac{1 + \frac{h\lambda}{2}}{1 - \frac{\lambda h}{2}}\right)$

Now $|z| < 1$ iff $\mathrm{Re}\left(\frac{\lambda h}{2}\right) < 0$

For $\left|\dfrac{1+\varepsilon}{1-\varepsilon}\right| < 1$ if $|1+\varepsilon|^2 < |1-\varepsilon|^2$

ie if $(1+\varepsilon)(1+\bar{\varepsilon}) < (1-\varepsilon)(1-\bar{\varepsilon})$

ie if $1 + \bar{\varepsilon} + \varepsilon + \cancel{\varepsilon\bar{\varepsilon}} < 1 - \bar{\varepsilon} - \varepsilon + \cancel{\varepsilon\bar{\varepsilon}}$

ie if $\bar{\xi} + \xi + \bar{\xi} + \xi < 0$

ie if $Re\,\xi + Re\,\xi < 0$

ie if $Re\,\xi < 0$

Therefore if $Re\,\lambda h < 0$ ie if $h\,Re\,\lambda < 0$ we have $|z| < 1$

$\therefore$ The Trapezium rule method is A-stable.

<u>Example:</u> Absolute stability region of RK methods.

Third order RK method.

$$y_{n+1} = y_n + \frac{h}{6}(K_1 + 4K_2 + K_3)$$

$$K_1 = f(x_n, y_n)$$

$$K_2 = f(x_n + \tfrac{h}{2}, y_n + \tfrac{h}{2}K_1)$$

$$K_3 = f(x_n + h, y_n + h(2K_2 - K_1))$$

When this method is applied to $y' = \lambda y$, we

we have

$$y_{n+1} = y_n + \frac{h}{6}\left(\lambda y_n + 4\lambda(y_n + \tfrac{h}{2}K_1) + \lambda(y_n + h(2K_2 - K_1))\right)$$

$$= y_n + \frac{h}{6}\Big[\lambda y_n + 4\lambda\left(y_n + \frac{h}{2}\lambda y_n\right) + \lambda\left(y_n + h\left(2\lambda\left(y_n + \frac{h}{2}k_1\right) - k_1\right)\right)\Big]$$

$$= y_n + \frac{h}{6}\Big[\lambda y_n + 4\lambda\left(y_n + \frac{h}{2}\lambda y_n\right) + \lambda\left(y_n + 2\lambda h y_n + \lambda h^2 k_1 - h\lambda y_n\right)\Big]$$

$$= y_n + \frac{h}{6}\Big[\lambda y_n + 4\lambda\left(y_n + \frac{h}{2}\lambda y_n\right) + \lambda y_n + 2\lambda^2 h y_n + \lambda^2 h^2 \lambda y_n - h\lambda^2 y_n\Big]$$

$$= y_n\left(1 + \frac{\lambda h}{6} + \frac{2}{3}\lambda h + \frac{\lambda^2 h^2}{3} + \frac{\lambda h}{6} + \frac{\lambda^2 h^2}{3} + \frac{\lambda^3 h^3}{6} - \frac{\lambda^2 h^2}{6}\right)$$

$$= y_n\left(1 + \lambda h + \frac{\lambda^2 h^2}{2} + \frac{\lambda^3 h^3}{6}\right)$$

$\therefore$ Stability polynomial is given by

$$\pi(\cdot\,;\lambda h) = z - \left(1 + \lambda h + \frac{\lambda^2 h^2}{2} + \frac{\lambda^3 h^3}{6}\right)$$

root $\quad z = 1 + \lambda h + \frac{\lambda^2 h^2}{2} + \frac{\lambda^3 h^3}{6}$

Now find the region λh in complex plane where $|z| < 1$

Setting $\eta = \lambda h$

We need η satisfying $\left|1 + \eta + \frac{\eta^2}{2} + \frac{\eta^3}{6}\right| < 1$

We can plot this in python.