Identifying Optimal Gaussian Filter for Gaussian Noise Removal

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***Abstract*—In this paper we show that the knowledge of noise statistics contaminating a signal can be effectively used to choose an optimal Gaussian filter to eliminate noise. Very specifically, we show that the additive white Gaussian noise (AWGN) contaminating a signal can be filtered best by using a Gaussian filter with specific characteristics. The design of the Gaussian filter bears relationship with the noise statistics in addition to some basic information about the signal. We first derive a relationship between the properties of the Gaussian filter, the noise statistics and the signal and later show through experiments that this relationship can be used effectively to identify the optimal Gaussian filter that can effectively filter noise.**

I. Introduction

Signal smoothing or noise filtering or denoising has been an area of active research and continues to hold the attention of researchers in various fields, for example, [1], [2], [3], [4], [5]. Noise is inherent in signals [6], [7] and a necessary first step is noise removal, often called preprocessing, before any other processing can take place. A successful preprocessing step to remove noise improves the performance of the *core processing* on the signal [8]. There are essentially two ways of taking care of noise inherently present in the signal, namely, (a) preprocessing of the signal to remove noise or (b) use a set of noise robust algorithms that can compensate for the inherent noise. In signal processing literature preprocessing of the signal is the preferred approach (for example see [9]).

A. Problem

Let

$$X = [x_1, x_2, \cdots, x_N]$$

be a band limited ($\mathcal{B}$) digitized signal which is sampled at a frequency of f_s (sampling frequency) and let

$$I\!N = [n_1, n_2, \cdots, n_N]$$

be the noise sequence. Further assume that $\{n_i\}_{i=1}^{N}$ is Gaussian distributed with mean $\mu_{I\!N}$ and variance $\sigma^2_{I\!N}$. Let $X_{I\!N}$,

$$X_{I\!N} = X + I\!N \tag{1}$$

represent the signal X contaminated by additive white Gaussian noise (AWGN) $I\!N$. Now the problem can be stated as, given $X_{I\!N}$ estimate $\hat{X}$ such that the error in the estimate is minimum in the mean squared sense, namely

$$minarg_{\hat{X}} ||X - \hat{X}||^2 \tag{2}$$

Typically the process of estimating $\hat{X}$ given $X_{I\!N}$ is called noise filtering or denoising. We will restrict our discussion, in this paper, to the use of a Gaussian smoothing filter for noise removal. The rest of the paper is arranged as follows. We describe Gaussian filtering in Section II which is characterized by σ_f which determines the amount of smoothing. We build theory in Section III which allows identification of an optimal σ_f^{opt}. We show experimentally how the identification of the actual Gaussian filter can be found in Section IV and conclude in Section V.

II. Gaussian Smoothing

A Gaussian filter in the time domain is parameterized by its means μ_f and variance σ_f^2 and is represented by

$$\mathcal{G}_f(\mu_f, \sigma_f^2, t) = \frac{1}{\sqrt{2\pi\sigma_f^2}} \exp^{-\left\{\frac{(t-\mu_f)^2}{2\sigma_f^2}\right\}} \tag{3}$$

Note 1: Given μ_f and σ_f^2 one can construct a Gaussian filter (3) with t running between $[-\infty, \infty]$. Also, it is well known that spanning t between $\mu_f - 3\sigma_f$ and $\mu_f + 3\sigma_f$ covers 99.7 % of the total area under the Gaussian.
So we can approximate $\mathcal{G}_f(\mu_f, \sigma_f^2, t)$ from $t = -\infty$ to ∞ as $\mathcal{G}_f(\mu_f, \sigma_f^2, t)$ from $t = \mu_f - 3\sigma_f$ to $\mu_f + 3\sigma_f$ for the purpose of discussion and subsequent experimentation. Additionally we assume that the mean of filter (μ_f) is zero because the magnitude response is independent of the mean.

Note 2: A non-zero mean will add a linear phase term to the frequency response of the filter.
Let the discrete version of zero mean $\mathcal{G}_f(\sigma_f^2, t)$ from $t = -3\sigma_f$ to $3\sigma_f$ be represented by $G_f[\sigma_f^2,\ m]$ for $m = -\lceil 3\sigma_f \rceil$ to $\lceil 3\sigma_f \rceil$, where $\lceil \bullet \rceil$ represents the ceil of $\bullet$. Let $X_{I\!N}$ smoothed with $G_f[\sigma_f^2, \cdot]$ result in $\hat{X}^{\sigma_f^2}$, namely,

$$\begin{aligned} \hat{X}_k^{\sigma_f^2} &= \sum_{i=-\lceil 3\sigma_f^2 \rceil + k}^{\lceil 3\sigma_f^2 \rceil + k} X_{I\!N_i} G_f[\sigma_f^2, k-i] \\ &= \sum_{i=-\lceil 3\sigma_f^2 \rceil + k}^{3\lceil \sigma_f^2 \rceil + k} (x_i + n_i) G_f[\sigma_f^2, k-i] \end{aligned} \tag{4}$$

for $k = 1, 2, \cdots, N$. Note that in (4) the index i and k represent the samples of a Gaussian filter given in (3) at time instants corresponding to i/f_s and k/f_s respectively, where f_s

denotes the sampling frequency. Let the mean squared error in the estimate be, denoted by $\hat{X}_{I\!N}$ is

$$E_{\sigma_f^2} = \frac{1}{N} \sum_{k=1}^{N} \left(X_k - \hat{X}_k^{\sigma_f^2} \right)^2 \tag{5}$$

We hypothesize that one can achieve an optimal estimate $\hat{X}_k^{\sigma_f^2}$ for some σ_f^2 such that $E_{\sigma_f^2}$ is minimized. We further hypothesize that σ_f^2 is based on the variance of the noise affecting the signal and some properties of the signal. Note that the construction of estimating σ_f^2 which minimizes the mean square error ($E_{\sigma_f^2}$) is similar to the well known Wiener filter [10], however, the main difference in our method and Wiener filter is in the use of the properties of the signal. While Wiener filtering uses the *entire* frequency spectrum of signal and noise we make use of *only* the signal power, signal bandwidth and the noise variance. Meaning we just need the knowledge of the *few* properties of the signal and noise and not the *complete* knowledge of the signal.

III. Our Approach

In the frequency domain we can write (1) as

$$X_{I\!N}(k_\omega) = X(k_\omega) + I\!N(k_\omega) \tag{6}$$

and the Gaussian filter as

$$G(k_\omega) = \exp\left(\frac{-{k_\omega}^2 \sigma_f^2}{2} \right) \tag{7}$$

The estimate of the signal $\hat{X}_{I\!N}(k_\omega)$ due to filtering by Gaussian filter (7) can be written as

$$\hat{X}_{I\!N}(k_\omega) = X(k_\omega) G(k_\omega) + I\!N(k_\omega) G(k_\omega) \tag{8}$$

The error in the filtered output is given by

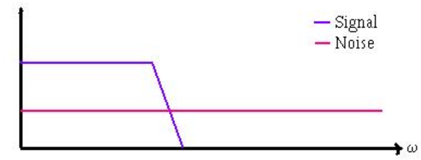
$$\begin{aligned} E(k_\omega) &= X(k_\omega) - \hat{X}_{I\!N}(k_\omega) \\ &= \underbrace{X(k_\omega)\left[1 - G(k_\omega)\right]}_{\text{Signal Distortion}} + \underbrace{N(k_\omega) G(k_\omega)}_{\text{Noise Smoothing}} \end{aligned} \tag{9}$$

As seen in (9) the error in the estimate, $E(k_\omega)$, due to filtering has two components namely, (a) one due to distortion of the signal (the $X(k_\omega)\left[1 - G(k_\omega)\right]$ term in (9)) and (b) two due to the reminiscent noise (the $N(k_\omega)G(k_\omega)$ term in (9)) in the signal after filtering. Let $P_\bullet$ denote the power in the signal $\bullet$, then input and output signal to noise ration ($\mathcal{S}$) are given by

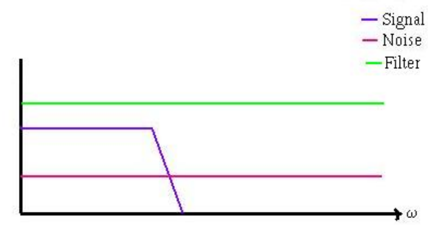
$$\begin{aligned} \mathcal{S}_i &= \frac{P_X}{P_{I\!N}} \\ \mathcal{S}_o &= \frac{P_X}{P_X - P_{\hat{X}}} = \frac{P_X}{P_E} \end{aligned} \tag{10}$$

Note 3: For a certain σ_f^2, the Gaussian filter is able to filter the signal such that $\mathcal{S}_o > \mathcal{S}_i$. Namely, simultaneously remove the noise and not distort the signal.

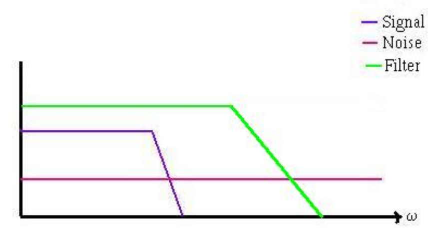
If we increase σ_f^2 then the cutoff frequency and the bandwidth of Gaussian filter will decrease as seen in (7) and subsequently this will lead to more noise removal but on same account the



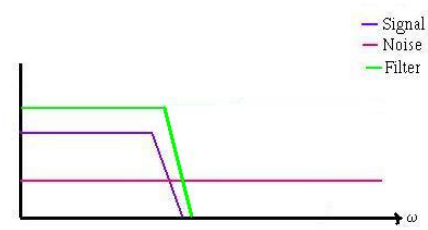
(a) Spectrum of Signal and Noise



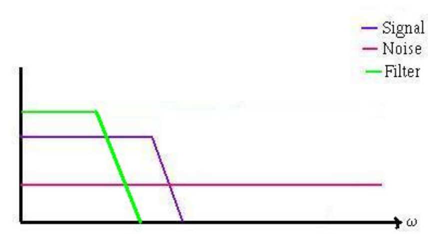
(b) Filter with $\sigma_f \to 0$, $\mathcal{S}_o = \mathcal{S}_i$



(c) Filter with finite σ_f, $\mathcal{S}_o > \mathcal{S}_i$



(d) Filter with σ_f such that S_o is maximum



(e) Filter with large σ_f, $S_o < \mathcal{S}_i$

Fig. 1. The effect of filtering signal with different σ_f

signal distortion will also increase. The effect of σ_f^2 of the Gaussian filter on the signal is shown in Fig. 1. For the sake of explanation, we have assumed an arbitrary spectral shape of the signal and the filter. Fig. 1(a) shows the spectrum of a band-limited signal and AWGN. Fig. 1(b) shows the limiting case when $\sigma_f^2 \to 0$, we have an *all pass* filter which neither affects the signal nor the noise and hence $\mathcal{S}_o = \mathcal{S}_i$. As we increase σ_f^2 the *out of band* noise is filtered without any distortion to the signal (see Figs. 1(c), 1(d)) until the value of σ_f^2 is such that it not only removes the *out of band* noise but also distorts the signal (see Fig. 1(e)). A typical $\mathcal{S}_o$ as a function of σ_f^2 is shown in Fig. 2.

Note 4: Observe that there exists a value of σ_f^2 at which $\mathcal{S}_o$ changes from the state $\mathcal{S}_o > \mathcal{S}_i$ to $\mathcal{S}_o < \mathcal{S}_i$ (zero crossing). We expect that for some $\sigma_f^2 = \sigma_{fR}^2$ such that $\sigma_{fR}^2 \neq 0$, $\mathcal{S}_o = \mathcal{S}_i$.

Now one can hypothesize that for all values of σ_f^2 in the range $[0, \sigma_{fR}^2]$, $\mathcal{S}_o > \mathcal{S}_i$. We further hypothesize that there exists a $\sigma_{f,opt}^2$ (again in the range $[0, \sigma_{fR}^2]$) for which $\mathcal{S}_o$ peaks to achieve $\mathcal{S}_o^{max}$. We now show through curve or model fitting that we can indeed determine an optimal $\sigma_{f,opt}^2$ of the Gaussian filter such that $\mathcal{S}_o$ is maximized.

A. Determining $\sigma_{f,opt}^2$

With an aim to identify the optimal $\sigma_{f,opt}^2$, of the Gaussian filter, to remove noise, we constructed three different signals (X), each with a different bandwidth ($\mathcal{B}$). We constructed the noisy signal ($X_{I\!N}$) by appending X with $I\!N$ with varying $\sigma_{I\!N}^2$. Each of these noise signal combinations were filtered with a Gaussian filter of varying σ_f^2. For all combination of noise and σ_f^2, $\mathcal{S}_o$ is computed. Specifically, the band limited X is constructed by first generating a random sequence of length $\mathcal{N}$ having a normal distribution with mean zero and variance one. This random signal is smoothed using a filter of length $\mathcal{M}(<< \mathcal{N})$ where the impulse response is given by

$$\begin{aligned} h(m) &= 1 \text{ for } 0 \leq m \leq \mathcal{M}-1 \\ &= 0 \text{ otherwise} \end{aligned} \tag{11}$$

Note that if we take a $\mathcal{N}$ point DFT of this smoothed signal, then most of the energy is limited to $f_s/\mathcal{M}$ Hz or $\mathcal{N}/\mathcal{M}$ points. We cut off the high frequency region of the signal, by setting the points from $\mathcal{N}/\mathcal{M}$ to $(\mathcal{N} - \mathcal{N}/\mathcal{M})$ to zero. The inverse DFT of this low-pass filtered signal is a signal with maximum frequency $f_{max} = f_s/\mathcal{M}$ Hz. Note that different values of $\mathcal{M}$ produce a filtered signal with different f_{max} and hence different bandwidth ($\mathcal{B}$). In this manner we constructed three different signals, each of length $\mathcal{N} = 1024$ with $\mathcal{M} = 5, 7$ and 10. We denote these three signals as X_5, X_7 and X_{10} having f_{max} of $\frac{f_s}{5}$, $\frac{f_s}{7}$, $\frac{f_s}{10}$ Hz respectively. An additive white Gaussian noise with $\sigma_{I\!N} = 30, 35$ and 40 denoted by $I\!N_{30}, I\!N_{35}, I\!N_{40}$ is generated. In all we had 9 $X_{I\!N}$'s generated, namely, $X_{I\!N}^1 = X_5 + I\!N_{30}$, $X_{I\!N}^2 = X_5 + I\!N_{35}$, $X_{I\!N}^3 = X_5 + I\!N_{40}$, $X_{I\!N}^4 = X_7 + I\!N_{30}$, $X_{I\!N}^5 = X_7 + I\!N_{35}$, $X_{I\!N}^6 = X_7 + I\!N_{40}$, $X_{I\!N}^7 = X_{10} + I\!N_{30}$, $X_{I\!N}^8 = X_{10} + I\!N_{35}$, $X_{I\!N}^9 = X_{10} + I\!N_{40}$. These signals $\{X_{I\!N}^k\}_{k=1}^9$ are denoised using a Gaussian filter (7) with different σ_f^2. We varied σ_f^2 from 0.3 to 3.5 in steps of 0.01 (320 different values of σ_f^2). For every filtered output (320 outputs for each of the 9 signal noise combinations) signal, $\mathcal{S}_o$ is calculated. Fig. 2 shows the $\mathcal{S}_o$ of the filtered $X_{I\!N}^2$ for different values of σ_f^2. The x-axis shows the different values of σ_f^2 and the bell shaped curve is the $\mathcal{S}_o$; also $\mathcal{S}_i$ (23 dB) is seen as a horizontal line. We now try to fit a curve so as to relate $\mathcal{S}_o$ in terms of $\mathcal{B}$, $\mathcal{S}_i$ and σ_f^2. We did this in two steps using [11].

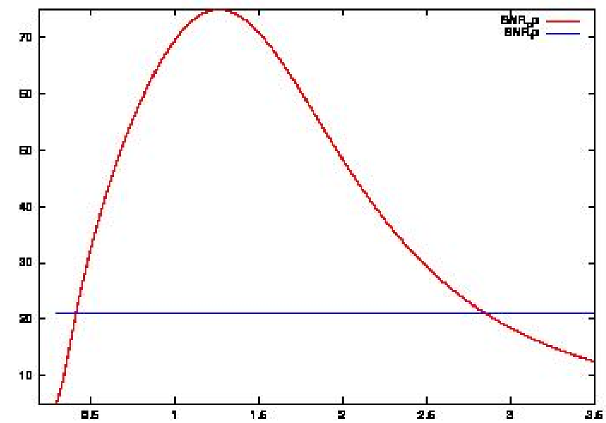


Fig. 2. The $\mathcal{S}_o$ of filtered $X_{I\!N}^2$ for different values of σ_f^2.

Step 1 : For a fixed $\mathcal{B}$, we fit a 3-D curve to relate $\mathcal{S}_o$, $\mathcal{S}_i$ and σ_f^2 using the reciprocal full quadratic function[1], namely,

$$\begin{aligned} \mathcal{S}_o &= \left\{ a_{\mathcal{B}} + b_{\mathcal{B}}\sigma_f + c_{\mathcal{B}}\mathcal{S}_i + d_{\mathcal{B}}{\sigma_f}^2 \right. \\ &\quad \left. + f_{\mathcal{B}}{\mathcal{S}_i}^2 + g_{\mathcal{B}}\sigma_f\mathcal{S}_i \right\}^{-1} \end{aligned} \tag{12}$$

by minimizing the sum of squared absolute error criteria. For each $\mathcal{B} = 5, 7, 10$ we obtained a set of coefficients a, b, c, d, f and g, that best fit (12). In all we had 18 coefficients, namely, $A' = [a_5, a_7, a_{10}]$, $B' = [b_5, b_7, b_{10}]$, $C' = [c_5, c_7, c_{10}]$, $D' = [d_5, d_7, d_{10}]$, $F' = [f_5, f_7, f_{10}]$ and $G' = [g_5, g_7, g_{10}]$

Step 2 : We then fit a quadratic curve for each coefficient set, namely, A', B', C', D', F', G' and $\mathcal{B}$ separately. Using A' we found that a in (12) is related to $\mathcal{B}$ as $a = \alpha_1 + \alpha_2\mathcal{B} + \alpha_3\mathcal{B}^2$. Similarly coefficients b, c, d, f, g can be written in terms of $\mathcal{B}$. Namely,

$$\begin{aligned} a_{\mathcal{B}} &= (0.8364 - 1.504\mathcal{B} + 4.017\mathcal{B}^2) \times 10^{-1} \\ b_{\mathcal{B}} &= (-0.1790 - 2.572\mathcal{B} - 4.164\mathcal{B}^2) \times 10^{-1} \\ c_{\mathcal{B}} &= (-0.4596 + 3.313\mathcal{B} - 7.653\mathcal{B}^2) \times 10^{-2} \\ d_{\mathcal{B}} &= (0.7983 - 8.658\mathcal{B} + 1.575\mathcal{B}^2) \times 10^{-2} \\ f_{\mathcal{B}} &= (0.7481 - 6.817\mathcal{B} + 17.04\mathcal{B}^2) \times 10^{-4} \\ g_{\mathcal{B}} &= (0.5562 - 2.510\mathcal{B} + 6.352\mathcal{B}^2) \times 10^{-3} \end{aligned} \tag{13}$$

[1]We experimented with several functions before converging onto the reciprocal full quadratic function

$\mathcal{B}$	$\sigma_{I\!N}$	$\mathcal{S}_i$	$\sigma_{f,opt}$ (14)	$\sigma_{f,opt}$	$\mathcal{S}_o^{max}$ (15)	$\mathcal{S}_o^{max}$
10	30	28.6	1.23	1.18	96.2	95.5
10	35	21.0	1.33	1.26	74.5	75.0
10	40	16.1	1.39	1.34	59.8	60.9
7	30	39.2	0.90	0.87	95.8	96.3
7	35	28.8	0.97	0.92	74.3	75.6
7	40	22.0	1.00	0.98	59.9	61.3
5	30	52.2	0.65	0.65	90.2	91.4
5	35	38.4	0.69	0.69	70.5	72.2
5	40	29.4	0.72	0.72	57.2	58.9

TABLE I
COMPARISON OF ACTUAL σ_f, $\mathcal{S}_o^{max}$ WITH DERIVED σ_f USING (14), $\mathcal{S}_o^{max}$ USING (15).

$\mathcal{B}$	$\sigma_{I\!N}$	$\mathcal{S}_i$	$\sigma_{f,opt}$ (14)	$\sigma_{f,opt}$	$\mathcal{S}_o^{max}$ (15)	$\mathcal{S}_o^{max}$
8	30	34.9	1.02	0.98	96.3	96.4
8	35	25.7	1.09	1.04	75.7	75.6
8	40	19.7	1.14	1.11	60.9	61.3
4	35	46.8	0.55	0.57	55.0	71.2
4	40	35.8	0.57	0.60	54.6	57.8
12	30	24.1	1.40	1.42	89.3	97.4
12	35	17.7	1.51	1.52	68.0	79.6
12	40	13.6	1.58	1.61	55.0	62.2

TABLE II
COMPARISON OF ACTUAL σ_f, $\mathcal{S}_o^{max}$ WITH DERIVED σ_f USING (14), $\mathcal{S}_o^{max}$ USING (15) FOR TEST SIGNALS.

Substituting (13) in (12) and differentiating it with respect to σ_f and setting

$$\frac{\partial \mathcal{S}_o}{\partial \sigma_f} = 0$$

we get $\sigma_{f,opt}$, namely,

$$\sigma_{f,opt} = -\frac{g_{\mathcal{B}}\mathcal{S}_i + b_{\mathcal{B}}}{2d_{\mathcal{B}}} \tag{14}$$

where $g_{\mathcal{B}}$, $b_{\mathcal{B}}$, $d_{\mathcal{B}}$ are given in (13). We get $\mathcal{S}_o^{max}$ by substituting the value of $\sigma_{f,opt}$ in (12), namely,

$$\mathcal{S}_o^{max} = \left\{a_{\mathcal{B}} + b_{\mathcal{B}}\sigma_{f,opt} + c_{\mathcal{B}}\mathcal{S}_i + d_{\mathcal{B}}{\sigma_{f,opt}}^2 + f_{\mathcal{B}}{\mathcal{S}_i}^2 + g_{\mathcal{B}}\sigma_{f,opt}\mathcal{S}_i\right\}^{-1} \tag{15}$$

IV. EXPERIMENTAL RESULTS

We conducted a number of experiments to verify the correctness of (14) and (15) in identifying $\sigma_{f,opt}$ and $\mathcal{S}_o^{max}$ respectively, these results are shown in Table I and Table II. Table I tries to assess the goodness of the curve fit, namely, the choice of the curve and the construction of (14) and (15) from the data. For this we took the same data that was used to estimate the parameters a, b, c, d, f in (14). As can be seen, the column four ($\sigma_{f,opt}$ calculated using (14)) and column five (actual $\sigma_{f,opt}$ computed from the data) are very close to each other. This is to be expected when the choice of the curve to fit the data is good. However to verify the validity of our approach to identify $\sigma_{f,opt}$ we conducted another set of experiments. We generated several *test* signals with different $\mathcal{M}$ and $I\!N$ with different $\sigma_{I\!N}$, such that these test signals were not part of the signals used to construct (14) using curve fitting. We first determined $\sigma_{f,opt}$ using (14) (column four in Table II) and then identified the actual $\sigma_{f,opt}$ (column five in Table II) by exhaustive search. As can be seen in Table II, the estimation of $\sigma_{f,opt}$ is very close to the actual $\sigma_{f,opt}$ for all signals. As expected, a similar match is seen for $\mathcal{S}_o^{max}$ obtained using (15) and actual $\mathcal{S}_o^{max}$.

V. CONCLUSIONS

Noise removal is a mandatory preprocessing step in many signal processing applications. In this paper, we have shown that it is possible to identify an optimal Gaussian filter that best filters noise, under the assumption that the noise is AWGN. We believe that the major contribution of this paper is in the presentation of a method to determine the best Gaussian filter which depends only on the knowledge of a few properties of the signal; this is unlike other filtering methods, like Wiener, which require the complete knowledge of the signal to determine an optimal filter. We have further shown, experimentally, that the proposed method works well for signals whose bandwidth and the input signal to noise ratio is know.

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