From the equation to the code: aggregation and autoconversion

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1 Autoconversion (zraut)

Lohmann and Roeckner (1996) give the autoconversion rate Q_{aut} (kg/(kg s)) as

$$Q_{\text{aut}} = \frac{\gamma_1 \cdot 6 \cdot 10^{28} \cdot n^{-1.7} \left(10^{-6} N_l\right)^{-3.3} \left(10^{-3} \rho_{air} q_{cl}\right)^{4.7}}{\rho_{air}} \tag{1}$$

This can be generalized as:

$$Q_{\text{aut}} = \alpha q_{cl}^{4.7} \tag{2}$$

$$\frac{\partial q_x}{\partial t} = -\alpha q_x^{\beta} \tag{3}$$

with $q_x = q_{cl}$, $\beta = 4.7$, and

$$\begin{aligned}
\alpha &= \frac{\gamma_1 \cdot 6 \cdot 10^{28} \cdot n^{-1.7} \left(10^{-6} N_l\right)^{-3.3} \left(10^{-3} \rho_{air}\right)^{4.7}}{\rho_{air}} \\
&= \frac{\gamma_1 \cdot 6 \cdot 10^{28} \cdot 10^{-1.7} \left(10^{-6} N_l\right)^{-3.3} \left(10^{-3} \rho_{air}\right)^{4.7}}{\rho_{air}} \\
&= \frac{\gamma_1 \cdot 1.2 \cdot 10^{27} \left(10^{-6} N_l\right)^{-3.3} \left(10^{-3} \rho_{air}\right)^{4.7}}{\rho_{air}}
\end{aligned}$$

. The integration and discretization then proceeds as follows:

$$q_x^{-\beta} \partial q_x = -\alpha \partial t \tag{4}$$

$$\frac{1}{1-\beta} q_{x,t+1}^{-\beta+1} - \frac{1}{1-\beta} q_{x,t}^{-\beta+1} = -\alpha \Delta t \tag{5}$$

$$q_{x,t+1}^{1-\beta} = q_{x,t}^{1-\beta} - \alpha \Delta t \cdot (1-\beta) \tag{6}$$

$$q_{x,t+1} = \left(q_{x,t}^{1-\beta} - \alpha \Delta t \cdot (1-\beta)\right)^{\frac{1}{1-\beta}} \tag{7}$$

$$= \left(q_{x,t}^{1-\beta} \cdot \left(1 - \alpha \Delta t \cdot (1-\beta) q_{x,t}^{\beta-1}\right)\right)^{\frac{1}{1-\beta}} \tag{8}$$

$$= q_{x,t} \cdot \left(1 - \alpha \Delta t \cdot (1-\beta) q_{x,t}^{\beta-1}\right)^{\frac{1}{1-\beta}} \tag{9}$$

$$q_{x,t+1} - q_{x,t} = -q_{x,t} + q_{x,t} \cdot \left(1 + \alpha \Delta t \cdot (\beta-1) q_{x,t}^{\beta-1}\right)^{-\frac{1}{\beta-1}} \tag{10}$$

The code reads as follows:

```
    ztmp1(1:kproma) = ccraut*1.2e27_dp * prho_rcp(1:kproma) &
* (pcdnc(1:kproma)*1.e-6_dp)**(-3.3_dp) &
* (prho(1:kproma)*1.e-3_dp)**4.7_dp
zraut(1:kproma) = MERGE(ztmp1(1:kproma), 0._dp, ld_prcp_warm(1:kproma))
zraut(1:kproma) = pxlb(1:kproma) * (  1._dp &
- (1._dp + ztmst*exm1_2*zraut(1:kproma)*pxlb(1:kproma)**exm1_2)**exp_2)
```

With

$$\begin{aligned} \text{ztmp1} &= \alpha \\ \text{ccraut} &= \gamma_1 \\ \text{prho\_rcp} &= \frac{1}{\rho_{\text{air}}} \\ \text{pcdnc} &= N_l \\ \text{prho} &= \rho_{\text{air}} \\ \text{pxlb} &= q_x \\ \text{ztmst} &= \Delta t \\ \text{exm1\_2} &= 4.7 - 1 = \beta - 1 \\ \text{exp\_2} &= -\frac{1}{4.7-1} = -\frac{1}{\beta - 1} \end{aligned}$$

(ccraut is a tuning parameter in the simulations, while $\gamma_1 = 15$ in Lohmann and Roeckner (1996)) this translates to

$$\begin{aligned} \text{zraut} &= q_{x,t} \cdot \left(1 - \left(1 + \Delta t \cdot (\beta - 1) \cdot \alpha \cdot q_x^{\beta-1}\right)^{-\frac{1}{\beta-1}}\right) && (11) \\ &= q_{x,t} - q_{x,t} \cdot \left(1 + \alpha \Delta t \cdot (\beta - 1) \cdot q_x^{\beta-1}\right)^{-\frac{1}{\beta-1}} && (12) \\ &= -\left(q_{x,t+1} - q_{x,t}\right) && (13) \end{aligned}$$

2 Aggregation

Lohmann and Roeckner (1996) give the conversion rate from cloud ice to snow by aggregation of ice crystals as:

$$\begin{aligned} Q_{\text{agg}} &= \frac{\gamma_2 q_{ci}}{\Delta t_1} && (14) \\ &= \frac{\gamma_2 q_{ci} \cdot c_1}{-2 \log \frac{r_{iv}}{r_{so}}^3} && (15) \\ &= \frac{\gamma_2 q_{ci} \cdot q_{ci} \cdot \rho_{\text{air}} \cdot a_3 E_{ii} X \left(\frac{\rho_\sigma}{\rho_{\text{air}}}\right)^{0.33}}{-2 \cdot \rho_i \cdot \log \frac{r_{iv}}{r_{so}}^3} && (16) \\ &= -\frac{\gamma_2 q_{ci}^2 \cdot \rho_{\text{air}} \cdot a_3 E_{ii} X \left(\frac{\rho_\sigma}{\rho_{\text{air}}}\right)^{0.33}}{6 \cdot \rho_i \cdot \log \frac{r_{iv}}{r_{so}}} && (17) \\ &= -\frac{\gamma_2 q_{ci}^2 \cdot \rho_{\text{air}} \cdot 17.5 \cdot \left(\frac{\rho_\sigma}{\rho_{\text{air}}}\right)^{0.33}}{6 \cdot \rho_i \cdot \log \frac{r_{iv}}{r_{so}}} && (18) \end{aligned}$$

Again, this can be generalized as:

$$\begin{aligned} Q_{\text{agg}} &= -\alpha q_{ci}^2 && (19) \\ \frac{\partial q_x}{\partial t} &= -\alpha q_x^\beta && (20) \end{aligned}$$

The integration and discretization then proceeds as follows:

$$q_x^{-\beta}\partial q_x = -\alpha\partial t \tag{21}$$

$$\frac{1}{1-\beta}q_{x,t+1}^{-\beta+1} - \frac{1}{1-\beta}q_{x,t}^{-\beta+1} = -\alpha\Delta t \tag{22}$$

$$q_{x,t+1}^{1-\beta} = q_{x,t}^{1-\beta} - \alpha\Delta t\cdot(1-\beta) \tag{23}$$

$$q_{x,t+1} = \left(q_{x,t}^{1-\beta} - \alpha\Delta t\cdot(1-\beta)\right)^{\frac{1}{1-\beta}} \tag{24}$$

$$= \left(q_{x,t}^{1-\beta}\cdot\left(1-\alpha\Delta t\cdot(1-\beta)q_{x,t}^{\beta-1}\right)\right)^{\frac{1}{1-\beta}} \tag{25}$$

$$= q_{x,t}\cdot\left(1-\alpha\Delta t\cdot(1-\beta)q_{x,t}^{\beta-1}\right)^{\frac{1}{1-\beta}} \tag{26}$$

$$q_{x,t+1} - q_{x,t} = -q_{x,t} + q_{x,t}\cdot\left(1-\alpha\Delta t\cdot(1-\beta)q_{x,t}^{\beta-1}\right)^{-\frac{1}{\beta-1}} \tag{27}$$

$$= q_{x,t}\cdot\left(-1+\left(1-\alpha\Delta t\cdot(1-\beta)q_{x,t}^{\beta-1}\right)^{-\frac{1}{\beta-1}}\right) \tag{28}$$

With $\beta = 2$:

$$q_{x,t+1} - q_{x,t} = q_{x,t}\cdot\left(-1+\left(1-\alpha\Delta t\cdot(-1)q_{x,t}^{1}\right)^{-\frac{1}{1}}\right) \tag{29}$$

$$= q_{x,t}\cdot\left(-1+(1+\alpha\Delta t\cdot q_{x,t})^{-1}\right) \tag{30}$$

The code reads as follows:

```
zsaut(1:kproma) = pxib(1:kproma) * (1._dp - 1._dp/(1._dp+ztmp1(1:kproma)*ztmst*pxib(1:kproma)))
```

with

```
  zc1(1:kproma) = 17.5_dp / crhoi * prho(1:kproma) * pqrho(1:kproma)**0.33_dp

ztmp1(1:kproma) = -6._dp / zc1(1:kproma) * LOG10(1.e4_dp*zris(1:kproma)) !SF zdt2
!SF Note: 1.e-4 = minimum size of snow flake
ztmp1(1:kproma) = ccsaut / ztmp1(1:kproma)

ztmp1(1:kproma) = MERGE(ztmp1(1:kproma), 0._dp, ll1(1:kproma))
```

Using

$$\mathrm{pqrho} = \frac{1.3}{\rho_{\mathrm{air}}} \tag{31}$$

this reads

$$\mathrm{ztmp1} = \frac{\mathrm{ccsaut}\cdot 17.5\cdot\rho_{\mathrm{air}}\left(\frac{1.3}{\rho_{\mathrm{air}}}\right)^{0.33}}{-6\cdot\log\frac{zris}{1\cdot 10^{-4}}\cdot\rho_i} \tag{32}$$

$$= -\alpha \tag{33}$$

$$\mathrm{zsaut} = q_{xi}\cdot\left(1-(1+\alpha\Delta t\cdot p_{xi})^{-1}\right) \tag{34}$$

$$= -\left(q_{x,t+1} - q_{x,t}\right) \tag{35}$$

References

Lohmann, U and E Roeckner (1996). “Design and Performance of a New Cloud Microphysics Scheme Developed for the ECHAM General Circulation Model”. In: p. 16.