UNIVERZITA KOMENSKÉHO V BRATISLAVE
FAKULTA MATEMATIKY, FYZIKY A INFORMATIKY

COMPUTER ASSISTED SEARCH FOR GRAPH WITH PRESCRIBED DEGREES AND CYCLE STRUCTURE

Diplomová práca

2016 Bc. Viktor Sklenáčár

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I, Viktor Sklenčár, hereby certify that this thesis is a result of my independent work with a guidance of my supervisor, except where otherwise stated.

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Preface

Dedication

I dedicate this thesis and give special thanks to my supervisor RNDr. Tatiana Jacayová, PhD. for her encouragement and guidance during the whole process of this work.

Abstract

While the central problem of this area, the so called Cage Problem – the problem of finding the smallest graph of given degree and girth – is a famously hard problem, recent generalizations of the problem have opened new possibilities for finding specific families of graphs for a prescribed set of degrees and cycles. Among the best known generalizations are the problem of biregular cages (graphs with two degrees and given girth) and the even/odd girth problem (graphs where both the smallest even and odd cycle lengths are prescribed). Ultimately, the problem leads the the Generalized Cage Problem – the problem of finding the smallest graphs with a given degree set and given set of smallest cycle lengths. Especially this last problem is wide open and appears to promise a number of possible computer assisted insights. The problem is also closely related to the problem of the classification of the cycle spectra of graphs and hamiltonicity. Generalized cages are interesting in that they allow for ingenious combination of graph theory results with the use of computers, and also due to the fact that their study allows for a sufficiently wide range of problems to allow for many directions of research.

Keywords: graph, degree girth, cage, Hamiltonian cycle, cycle spectrum of a graph, computer assisted graph

Abstrakt

Aj keď základný problém danej oblasti tzv. Problém klietok – problém, v ktorom sa hľadá najmenší graf daného stupňa a obvodu – je všeobecne známy ťažký problém, zovšeobecnenia tohto problému, ktoré sa v poslednej dobe objavili, otvorili nové možnosti na nájdenie špecifických tried grafov s predpísanou množinou stupňov a cyklov. Medzi najznámejšie zovšeobecnenia patria problém bi-regularnych klietok (grafov s dvoma predpísanými stupňami a daným obvodom) a problém párneho/nepárneho obvodu (grafy, pre ktoré sú dané dve dĺžky najkratšieho párneho a nepárneho cyklu). V konečnom dôsledku tieto alternácie vedú ku Zovšeobecnenému problému klietok – problému, v ktorom sa hľadá najmenší graf s danou predpísanou množinou stupňov a danými predpísanými dĺžkami najmensích cyklov. Na tomto probléme sa intenzívne pracuje, a je stále otvorený. Vyzerá, že počítačmi podporované vyhľadávania a konštrukcie by nám mohli poskytnúť hlbšie porozumenie problému. Daný problém úzko súvisí aj s problémom klasifikácie cyklového spektra grafov a s hamiltonicitou. Zovšeobecnené klietky zaujímavým spôsobom kombinujú výsledky teórie grafov s použitím počítačového prehľadávania a počítačových konštrukcií.

Kľúčové slová: graf, stupeň, Hamiltonovská cesta

Contents

Chapter 1

Introduction

1.1 Graph Theory

Graph is represented by number of vertexes and edges among them. Purely mathematically, graph is a pair $G = (V, E)$ of sets such that $E \subseteq [V]^2$ and $V \cap E = \emptyset$. Set V contains all vertices (node or points) of graph G. Elements of set E are its edges (lines).

Depending on number of edges and vertexes, we can distinguish a sparse and a dense graph. The former usually have much less edges in compare with square power of vertexes, mathematically $|E| << |V|^2$, whereas the latter holds condition that number of edges $|E|$ is close to $|V|^2$. [THC09]

Two standard representations of graph are known, the one as a collection of adjacency lists or the other one as an adjacency matrix, both can be used for directed and undirected graphs. Each method has its own pros and cons depending on type and purpose of the graph.

1.1.1 The degree of a vertex

Let G be a non-empty graph defines as $G = (V, E)$. Then let's take an arbitrary vertex v in G. Set of all neighbours of the vertex v is denoted by $N_G(v)$. Degree of (or valency) $d(v)$ of a vertex v is the number of $|E|(v)$ of edges at v. [Die05]

1.1.2 Paths, walks, and cycles

A walk (of length k) in a graph G is a non-empty alternating sequence $v_0e_0v_1e_1...e_k - 1v_k$ of vertices and edges such that $e_1 = v_i, v_i + 1$ for all $i \leqslant k$. In a walk, vertices and edges may repeat. A walk is considered to be *closed* if $v_0 = v_k$, otherwise is *open*. A special case of walk with no repeated edges is called *trail*. In addition, if a trail is closed, we refer to *circuit*.

We often refer to a path by the natural sequence of its vertices starting from x_0 to x_k. Formally, a path is a a non-empty graph $P = (V, E)$ where $V = x_0, x_1, ..., x_k$, $E = x_0x_1, x_1x_2, ..., x_kx_1$ and all x_i are distinct (no repeated vertices and edges). Vertexes x_0 and x_k are called ends and $x_1, ..., x_k - 1$ are *inner* vertices of P. According definition of P holds that x_i and $x_i + 1$ are *linked* $\forall i, 0 \leqslant i \leqslant k$. The cardinality of E of path P is its length and path of length k where $k \geqslant 0$ is denoted by P^k. [Die05]

Let a path P be a sequence of vertices $x_0...x_k - 1$, where $k \geqslant 0$. Let introduce an edge $x_k - 1x_0$ to a path P. We close the path P by adding the edge to it. Then graph $C = P + x_k - 1x_0$ is a *cycle*. Similarly to paths, we also denote a cycle by its (cyclic) sequence of vertices. According that, graph C can be written as $C = x_0...x_k - 1x_0$. The *length* of a cycle is a number of edges. Given that a cycle is a closed walk without repeated vertices and edges, its length is equal to number of vertices.

1.1.3 Hamilton Cycle

A cycle containing all the vertices of a graph G is said to be a Hamilton cycle of G. Then G is said to be Hamiltonian. The origin of this term is a game invented in 1857 by Sir William Rowan Hamilton based on the construction of cycles containing all the vertices in the graph of the dodecahedron (TODO see Fig+ 1.11). [Bol02]

Since all vertexes are included and cannot repeat in Hamilton cycle H, it holds that $d(v) = 2$ for all v in H. Hamilton cycle does not exist for all graph. But, there is no specific way to find whether Hamilton cycle exists or not.

The so-called travelling salesman problem is well now problem of finding a least expensive tour in the graph. A salesman makes a tour of n cities and he has to end up at the same place he started. The cost of the journey between any two cities is known. Though a considerable amount of work has been done on this problem since its solution would have important practical applications, it is not known whether or not there is an efficient algorithm for finding a least expensive route.

1.2 Graph Algorithms

Existence of graphs in the field of computer science is important. They allow to preserve data in certain ways, which are convenient for certain purpose. Depending on a problem, a graph representation is chosen. There are two standard way how to represent graphs - as a collection of adjacency list or adjacency matrix. The former is more suitable for sparse graphs, where $|E|$ is much less then $|V|^2$, whereas the latter is more convenient when working with dense graphs, which $|E|$ is close to $|V|^2$ and we need to quickly tell if two vertices are connected or not.

1.2.1 Adjacency list representation

A graph $G = (V, E)$ is represented by an array of Adj $|V|$ lists, one for each vertex of V. Each list $Adj[u]$ contains all vertices adjacent to u in G. Thus, for all v in $Adj[u]$ there is an edge $(u, v) \in E$. The amount of memory used for such representation of the graph requires is $\Theta(V + E)$. If a graph is undirected, the reduction of required memory can be done by a simple method. Redundant information about connected vertices in a one of both direction can be omitted. The possible problem could arise while using an algorithm regardless index (order) of vertices. [THC09]

1.2.2 Adjacency matrix representation

The other representation of a graph $G = (V, E)$ consists of $|V| \times |V|$ matrix $A = (a_ij)$ such that if $(i, j) \in E$ then $(a_ij) = 1$, otherwise $(a_ij) = 0$. The adjacency matrix of a graph requires $\Theta(V^2)$ memory, independent on the graph of the number of edges in the graph since we keep also an information about not connected vertices. However, the memory usage can be cut about a half in a case if an undirected graph. Since edges (u, v) and (v, u) represent the same edge, the transpose matrix of the adjacency matrix A of an undirected graph is the same: $A = A^T$. Therefore to store only the diagonal and the part above should be enough.

1.2.3 ONLY REFERENCES

In this paragraph, we refer to other references which are currently not used in the document, but will be. The purpose is to show them on the literature list. [IR78] [GE11] [Sac63]

Chapter 2

Motivation

Chapter 3

Overview

3.1 Graph as a data structure

As is mentioned above, two main exist to represent a graph. It is really important to decide well for one of them at the very beginning. Both have some advantages and limitations one should be aware of while working with them. So far, we mentioned only memory requirements, but even more important is performance.

3.1.1 Choosing a right representation

There is no problem with a small graphs and any representation of them. When we are talking only about several vertices and edges, the execution time is rather small for most of the graph algorithms. However, our construction can yield huge graphs and trivial small problems suddenly become unreasonably huge.

3.2 Sachs

3.2.1 Theorem I

Corresponding to any three integers $v \geqslant 3, \gamma \geqslant 2$ and $\omega \geqslant 1$ there is a graph Γ with the following properties:

I. Γ is regular and has valency v.

II. Γ *has girth* γ.

III. Γ *has a Hamiltonian circuit.*

IV. The waist-cicruits of Γ *are mutually disjoint and together constitute a* $2 - factor\ \Theta\ of\ \Gamma$.

V. Γ *is the union of* Θ *and* $v - 2\ 1 - factors\ of\ \Gamma$.

VI. The number of waist-circuits contained in Γ *is divisible by* δ.

3.2.2 Remark I by Sachs

The case $v = 2$. A circuit of length γ is the only type of connected graph which has Properties I and II with $v = 2$; it has Properties III, IV and V as well.

The reason why the circuit γ for the case $v = 2$ is the only one which holds Properties I and II is that the circuit is minimum, so by definition it is girt γ . If the valency of the subgraph of Γ would be decreased, the result subgraph would not hold regular property of a graph.

3.2.3 Regular graphs with some prescribed circuits

therefore suppose that $a_2, ..., a_e$ are not all zero. Let y denote the least value of i ($2 \leq i \leq e$) for which. By Theorem 1 there may be constructed a graph F having Properties I-IV of Theorem 1 and containing more than $a_2 + a3... + a_e$ waist-circuits

3.2.4 Generalized truncation

Let H be a graph of order k, V (H) = TODO. The generalized truncation of a k-regular graph G with a vertex-neighbourhood labelling p by the graph H (of order k) is the graph T(G; p;H) obtained from G by replacing the vertices of G by copies of H as follows: each vertex v of G is replaced by the graph H attached to the dangling darts originally emanating from v according to the rule that u_i is attached to the dart labelled by i. If no confusion is possible, we will omit the reference to p, and simply talk about the graph T(G;H).

Theorem 2.2: Let G be a finite (k; g)-graph with a vertex-neighbourhood labelling p, and let H be a (k'; g')-graph of order k. The generalized truncation graph T(G; p;H) is a (k' + 1)-regular graph of girth not smaller than min (2g; g'), and if g' <= 2g, then g' is the exact girth of T(G; p;H).

Proof: In T(G; p;H), color the edges of H red, and the remnants of old edges G blue. The blue edges form a 1-factor of T(G; p;H), and each vertex v of T(G; p;H) is incident with one blue and k' red edges of T(G; p;H). Thus, T(G; p;H) is a (k' +1)- regular graph. The cycles of T(G; p;H) have no two adjacent blue edges. If a cycle C contains no blue edges, it must be contained in a copy of H, and hence its length must be p g'. If C contains both red and blue edges, and we remove the red edges, the sequence of blue edges is non-repeating. Moreover, any two blue edges connected via a path of red edges in T(G; p;H) must have been originally adjacent in G. Thus, the blue edges constitute a walk in the original graph G that contains at least one cycle of G, and therefore the red/blue cycle is of length at least 2g. Finally, T(G; p;H) obviously contains a cycle of length g'.

Chapter 4

Issue

Chapter 5

Proposal

Chapter 6

Implementation

Chapter 7

Results

Chapter 8

Conclusion

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