

On the Chebyshev approximation of a function with two variables

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Abstract

There is presented an approach to find an approximation polynomial of a function with two variables based on the two dimensional discrete Fourier transform. The approximation polynomial is expressed through Chebyshev polynomials. An uniform convergence result is given.

Keywords: Chebyshev function, double series, discrete Fourier transform

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1 Introduction

The purpose of the paper is to present some aspects about the construction of an approximation polynomial for a function with two variables. The approximation polynomial is expressed through Chebyshev polynomials.

Constructing an approximation polynomial of a function with the corresponding applications is the subject of the *Chebfun* software, presented in details in [4], [2]. The *Chebfun2* part of the software deals with the construction of an approximation polynomial of a function with two variables. According to [5], [6], to this end it is used a method based on Gaussian elimination as a low rank function approximation.

In *Chebfun* the approximation polynomial of a function with one variable is obtained using an one dimensional discrete Fourier transform. The approach of this paper will use a two dimensional discrete Fourier transform.

In spectral methods the Chebyshev polynomials are often used. The same form of the approximation polynomial are used in [1], [9], too.

After recalling of some formulas on the Fourier series for a function with two variables and on the two dimensional discrete Fourier transform there is presented an algorithm to obtain an approximation polynomial of a function with two variables and a convergence result. A Lagrange type interpolation problem for a function with two variables is studied. Two applications are mentioned: a numerical integration formula on a rectangle and a numerical computation of the partial derivatives.

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2 Two dimensional Fourier series

Let $f : \mathbb{R}^2 \rightarrow R$ be a continuous periodical function in each variable with the period 2π . The Fourier series attached to the function is [8], t.3

$$f(x, y) \sim \sum_{n,m=0}^{\infty} (a_{n,m} \cos nx \cos my + b_{n,m} \cos nx \sin my + c_{n,m} \sin nx \cos my + d_{n,m} \sin nx \sin my)$$

with the coefficients given by

$$\begin{aligned} a_{0,0} &= \frac{1}{4\pi^2} \iint_{\Omega} f(x, y) dx dy & a_{n,m} &= \frac{1}{\pi^2} \iint_{\Omega} f(x, y) \cos nx \cos my dx dy \\ a_{n,0} &= \frac{1}{2\pi^2} \iint_{\Omega} f(x, y) \cos nx dx dy & b_{n,m} &= \frac{1}{\pi^2} \iint_{\Omega} f(x, y) \cos nx \sin my dx dy \\ a_{0,m} &= \frac{1}{2\pi^2} \iint_{\Omega} f(x, y) \cos my dx dy & c_{n,m} &= \frac{1}{\pi^2} \iint_{\Omega} f(x, y) \sin nx \cos my dx dy \\ b_{0,m} &= \frac{1}{2\pi^2} \iint_{\Omega} f(x, y) \sin my dx dy & d_{n,m} &= \frac{1}{\pi^2} \iint_{\Omega} f(x, y) \sin nx \sin my dx dy \\ c_{n,0} &= \frac{1}{2\pi^2} \iint_{\Omega} f(x, y) \sin nx dx dy \end{aligned}$$

where $\Omega = [0, 2\pi]^2$.

The complex form of the Fourier series is

$$\sum_{n,m \in \mathbb{Z}} \gamma_{n,m} e^{inx+imy}$$

with

$$\begin{aligned} \gamma_{0,0} &= a_{0,0} \\ \gamma_{n,0} &= \frac{1}{2}(a_{n,0} - ic_{n,0}) & \gamma_{-n,0} &= \frac{1}{2}(a_{n,0} + ic_{n,0}) \\ \gamma_{0,m} &= \frac{1}{2}(a_{0,m} - ib_{0,m}) & \gamma_{0,-m} &= \frac{1}{2}(a_{0,m} + ib_{0,m}) \\ \gamma_{n,m} &= \frac{1}{4}(a_{n,m} - ib_{n,m} - ic_{n,m} - d_{n,m}) & \gamma_{-n,m} &= \frac{1}{4}(a_{n,m} - ib_{n,m} + ic_{n,m} + d_{n,m}) \\ \gamma_{n,-m} &= \frac{1}{4}(a_{n,m} + ib_{n,m} - ic_{n,m} + d_{n,m}) & \gamma_{-n,-m} &= \frac{1}{4}(a_{n,m} + ib_{n,m} + ic_{n,m} - d_{n,m}) \end{aligned}$$

or

$$\gamma_{m,n} = \frac{1}{4\pi^2} \iint_{\Omega} f(x, y) e^{-inx-imy} dx dy, \quad \forall n, m \in \mathbb{Z}. \quad (1)$$

If the function is even in any variable then the $b_{n,m}, c_{n,m}, d_{n,m}$ coefficients are all zero.

We shall suppose that the convergence conditions of the Fourier series to $f(x, y)$ are fulfilled (the function has bounded first order partial derivatives in Ω and in a neighborhood of (x, y) there exists $\frac{\partial^2 f}{\partial x \partial y}$, or $\frac{\partial^2 f}{\partial y \partial x}$, which is continuous in (x, y) , cf. [8], t.3, 697).

3 Two dimensional discrete Fourier transform

Let be the infinite matrix $(x_{k,j})_{k,j \in \mathbb{Z}}$ with the periodicity properties $x_{k+p,j} = x_{k,j}$, $x_{k,j+q} = x_{k,j}$, $\forall k, j \in \mathbb{Z}$. The discrete Fourier transform construct another infinite matrix $(y_{r,s})_{r,s \in \mathbb{Z}}$ with an analog periodicity properties defined by

$$y_{r,s} = \sum_{k=0}^{p-1} \sum_{j=0}^{q-1} x_{k,j} e^{-i \frac{2\pi kr}{p}} e^{-i \frac{2\pi js}{q}},$$

for $r \in \{0, 1, \dots, p-1\}$ and $s \in \{0, 1, \dots, q-1\}$.

The complexity to compute the pq numbers with the discrete fast Fourier transform algorithm is $pq \log_2 pq$.

As an application, if the Fourier series coefficients (1) are computed using the trapezoidal rule for each of the iterated integrals then:

$$\begin{aligned} \gamma_{n,m} &= \frac{1}{4\pi^2} \iint_{\Omega} f(x, y) e^{-inx - imy} dx dy = \frac{1}{4\pi^2} \int_0^{2\pi} e^{-inx} \left(\int_0^{2\pi} f(x, y) e^{-imy} dy \right) dx \approx \\ &\approx \frac{1}{4\pi^2} \int_0^{2\pi} e^{-inx} \left(\frac{2\pi}{q} \sum_{j=0}^{q-1} f(x, \frac{2\pi j}{q}) e^{-im \frac{2\pi j}{q}} \right) dx = \\ &= \frac{1}{2\pi q} \sum_{j=0}^{q-1} e^{-im \frac{2\pi j}{q}} \int_0^{2\pi} f(x, \frac{2\pi j}{q}) e^{-inx} dx \approx \frac{1}{pq} \sum_{k=0}^{p-1} \sum_{j=0}^{q-1} f(\frac{2\pi k}{p}, \frac{2\pi j}{q}) e^{-in \frac{2\pi k}{p}} e^{-im \frac{2\pi j}{q}}. \end{aligned}$$

Thus the Fourier coefficients $(\gamma_{n,m})$ may be computed applying the discrete Fourier transform to $\left(f(\frac{2\pi k}{p}, \frac{2\pi j}{q})\right)_{k \in \{0,1,\dots,p-1\}, j \in \{0,1,\dots,q-1\}}$.

4 The Chebyshev series

Considering a continuous two real variables function $f(x, y)$, $x, y \in [-1, 1]$ there is attached the Chebyshev series

$$f(x, y) \sim \sum_{n,m=0}^{\infty} \alpha_{n,m} T_n(x) T_m(y) \quad (2)$$

where

$$\begin{aligned}\alpha_{0,0} &= \frac{1}{\pi^2} \int_{-1}^1 \int_{-1}^1 \frac{f(x,y)}{\sqrt{1-x^2} \sqrt{1-y^2}} dx dy & \alpha_{n,0} &= \frac{2}{\pi^2} \int_{-1}^1 \int_{-1}^1 \frac{f(x,y)T_n(x)}{\sqrt{1-x^2} \sqrt{1-y^2}} dx dy \\ \alpha_{0,m} &= \frac{2}{\pi^2} \int_{-1}^1 \int_{-1}^1 \frac{f(x,y)T_m(y)}{\sqrt{1-x^2} \sqrt{1-y^2}} dx dy & \alpha_{n,m} &= \frac{4}{\pi^2} \int_{-1}^1 \int_{-1}^1 \frac{f(x,y)T_n(x)T_m(y)}{\sqrt{1-x^2} \sqrt{1-y^2}} dx dy\end{aligned}$$

Changing $x = \cos t, y = \cos s$, the coefficient $\alpha_{n,m}$ will be

$$\begin{aligned}\alpha_{n,m} &= \frac{4}{\pi^2} \int_0^\pi \int_0^\pi f(\cos t, \cos s) \cos nt \cos ms dt ds = \\ &= \frac{1}{\pi^2} \iint_\Omega f(\cos t, \cos s) \cos nt \cos ms dt ds.\end{aligned}\tag{3}$$

Analogous formulas may be obtained for $\alpha_{0,0}, \alpha_{n,0}$ and $\alpha_{0,m}$, too. Thus the coefficients of the Chebyshev series are the coefficients of the Fourier coefficients of the function $\varphi(t, s) = f(\cos t, \cos s)$.

If the function f has second order derivatives then the Fourier series attached to φ converges to φ and consequently

$$f(x, y) = \sum_{n,m=0}^{\infty} \alpha_{n,m} T_n(x) T_m(y), \quad x, y \in [-1, 1].\tag{4}$$

The polynomial

$$f_{n,m}(x, y) = \sum_{k=0}^n \sum_{j=0}^m \alpha_{k,j} T_k(x) T_j(y)$$

is called the Chebyshev approximation polynomial of the function $f(x, y)$ in the square $[-1, 1]^2$.

The parameters n, m are determined adaptively to satisfy the inequalities $|\alpha_{k,j}| < tol (= 10^{-15}$, machine precision), for $k > n$ and $j > m$. This is the goal of the algorithm 1. The coefficients whose absolute value are less than tol are eliminated and the remained coefficients are stored as a sparse matrix.

The $f_{n,m}(x, y)$ polynomial may be obtained with the least square method as the solution of the optimization problem

$$\min_{\lambda_{k,j}} \int_{-1}^1 \int_{-1}^1 \frac{1}{\sqrt{1-x^2} \sqrt{1-y^2}} \left(f(x, y) - \sum_{k=0}^n \sum_{j=0}^m \lambda_{k,j} T_k(x) T_j(y) \right)^2 dx dy.$$

Due to the Parseval equality

$$\alpha_{0,0}^2 + \frac{1}{2} \sum_{n=0}^{\infty} \alpha_{n,0}^2 + \frac{1}{2} \sum_{m=0}^{\infty} \alpha_{0,m}^2 + \frac{1}{4} \sum_{n,m=1}^{\infty} \alpha_{n,m}^2 = \frac{1}{\pi^2} \int_{-1}^1 \int_{-1}^1 \frac{f^2(x, y)}{\sqrt{1-x^2} \sqrt{1-y^2}} dx dy$$

the quality of the approximation polynomial may be evaluated by

$$\frac{1}{\pi^2} \int_{-1}^1 \int_{-1}^1 \frac{f^2(x, y)}{\sqrt{1-x^2}\sqrt{1-y^2}} - \left(\alpha_{0,0}^2 + \frac{1}{2} \sum_{k=0}^n \alpha_{k,0}^2 + \frac{1}{2} \sum_{j=0}^m \alpha_{0,j}^2 + \frac{1}{4} \sum_{k=1}^n \sum_{j=1}^m \alpha_{k,j}^2 \right). \quad (5)$$

Algorithm 1 Algorithm to compute the Chebyshev approximation polynomial

```

1: procedure CHEBFUN2(f)
2:    $n \leftarrow 8$ 
3:    $tol \leftarrow 10^{-15}$ 
4:    $sw \leftarrow true$ 
5:   while  $sw$  do            $\triangleright$  The approximation polynomial is determined adaptively
6:      $m \leftarrow 2n$ 
7:      $x, y \leftarrow \cos \frac{2k\pi}{m}, k = 0 : m - 1$ 
8:      $z \leftarrow f(x, y)$ 
9:      $g \leftarrow FFT(z)/m^2$ 
10:     $a \leftarrow 4\Re g(1 : n, 1 : n)$ 
11:     $a(1, 1) \leftarrow a(1, 1)/4$ 
12:     $a(1, 2 : n) \leftarrow a(1, 2 : n)/2$ 
13:     $a(2 : n, 1) \leftarrow a(2 : n, 1)/2$ 
14:    if  $|a(i-1 : i, 1 : n)| < tol \& |a(1 : n, i-1 : i)| < tol$  then
15:       $sw \leftarrow false$ 
16:    else
17:       $n \leftarrow 2n$ 
18:    end if
19:   end while
20:   for  $i = 1 : n$  do            $\triangleright$  Removal of negligible coefficients
21:     for  $j = 1 : n$  do
22:       if  $|a(i, j)| < tol$  then
23:          $a(i, j) \leftarrow 0$ 
24:       end if
25:     end for
26:   end for
27:   return  $a$ 
28: end procedure

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The value of the polynomial $f_{n,m}$ in a point (x, y) may be computed adapting the Clenshaw algorithm, [9], but we find that the evaluation of the expression $f_{n,m}(x, y) = V'_n(x)A_{n,m}V_m(y)$, where $V_\nu(s) = (T_0(s), T_1(s), \dots, T_\nu(s))'$ and $A = (a_{k,j})_{k=0:n, j=0:m}$ is more efficient within a matrix oriented software. V' denotes the transpose of the vector V . The complexity order of both algorithms is $O(nm)$.

5 The Chebyshev series of partial derivatives

We assume that the function $f(x, y)$ has first order continuous partial derivatives and the series (4), there is required to find the coefficients $(b_{n,m})_{n,m \in \mathbb{N}}$ such that

$$\frac{\partial f(x, y)}{\partial x} = \sum_{n,m=0}^{\infty} b_{n,m} T_n(x) T_m(y). \quad (6)$$

Using the equalities

$$T'_1(x) = T_0(x), \quad T'_2(x) = T_1(x)$$

and

$$\frac{1}{2} \left(\frac{T'_{n+1}(x)}{n+1} - \frac{T'_{n-1}(x)}{n-1} \right) = T_n(x), \quad n > 1.$$

(6) may be written as

$$\begin{aligned} \frac{\partial f(x, y)}{\partial x} &= \sum_{m=0}^{\infty} \left(\sum_{n=0}^{\infty} b_{n,m} T_n(x) \right) T_m(y) = \\ &= \sum_{m=0}^{\infty} \left(b_{0,m} T'_1(x) + \frac{b_{1,m}}{2} \frac{T'_2(x)}{2} + \sum_{k=2}^{\infty} \frac{b_{k,m}}{2} \left(\frac{T'_{k+1}(x)}{k+1} - \frac{T'_{k-1}(x)}{k-1} \right) \right) T_m(y) = \\ &= \sum_{m=0}^{\infty} \left(\left(b_{0,m} - \frac{b_{2,m}}{2} \right) T'_1(x) + \sum_{k=2}^{\infty} \frac{1}{2k} (b_{k-1,m} - b_{k+1,m}) T'_k(x) \right) T_m(y) = \\ &= \sum_{m=0}^{\infty} \left(\sum_{k=1}^{\infty} \alpha_{k,m} T'_k(x) \right) T_m(y). \end{aligned}$$

Identifying the coefficients of $T'_k(x)$ there is obtained the linear algebraic system

$$\begin{cases} b_{0,m} - \frac{b_{2,m}}{2} &= \alpha_{1,m} \\ \frac{1}{2k} (b_{k-1,m} - b_{k+1,m}) &= \alpha_{k,m}, \quad k \geq 2, \quad m \in \mathbb{N}. \end{cases} \quad (7)$$

Summing the above equalities for $k = n+1, n+3, n+5, \dots$ it results

$$b_{n,m} = (n+1)a_{n+1,m} + (n+3)a_{n+3,m} + (n+5)a_{n+5,m} + \dots \quad \forall n, m \in \mathbb{N}.$$

In the same way it is deduced that the coefficients of the series

$$\frac{\partial f(x, y)}{\partial y} = \sum_{n,m=0}^{\infty} c_{n,m} T_n(x) T_m(y)$$

satisfies the relations

$$\begin{cases} c_{n,0} - \frac{c_{n,2}}{2} &= \alpha_{n,1} \\ \frac{1}{2j} (c_{n,j-1} - c_{n,j+1}) &= \alpha_{n,j}, \quad j \geq 2, \quad m \in \mathbb{N}. \end{cases} \quad (8)$$

Let be

$$\frac{\partial f^2(x, y)}{\partial x \partial y} = \frac{\partial f^2(x, y)}{\partial y \partial x} = \sum_{n,m=0}^{\infty} d_{n,m} T_n(x) T_m(y). \quad (9)$$

Because $\frac{\partial f^2(x, y)}{\partial x \partial y} = \frac{\partial}{\partial y} \left(\frac{\partial f(x, y)}{\partial x} \right)$ applying (8) it results

$$\frac{1}{2j} (d_{k,j-1} - d_{k,j+1}) = b_{k,j}$$

and consequently, for $k, j > 1$,

$$\frac{1}{4kj} (d_{k-1,j-1} - d_{k-1,j+1} - d_{k+1,j-1} + d_{k+1,j+1}) = \frac{1}{2k} (b_{k-1,j} - b_{k+1,j}) = \alpha_{k,j}. \quad (10)$$

Denoting $\Delta_{k,j} = d_{k-1,j-1} - d_{k-1,j+1} - d_{k+1,j-1} + d_{k+1,j+1}$, from the Parseval equality corresponding to (9) it results that

$$\sum_{k=2}^{\infty} \sum_{j=2}^{\infty} \Delta_{k,j}^2 \leq 16M_{1,1}^2,$$

where $M_{1,1} \geq \max\{|\frac{\partial^2 f}{\partial x \partial y}(x, y)| : x, y \in [-1, 1]\}$.

6 The convergence of the Chebyshev series

Using the techniques presented in [7] and [4], in some hypotheses it may be proven that the convergence in (4) is uniform in $[-1, 1]^2$, for $n, m \rightarrow \infty$. First we state

Theorem 6.1 *If the function f has second order continuous derivatives then*

$$|\alpha_{n,0}| \leq \frac{2M_{2,0}}{(n-1)^2} \quad \text{and} \quad |\alpha_{n,1}| \leq \frac{8M_{2,0}}{\pi(n-1)^2}, \quad n > 1, \quad (11)$$

$$|\alpha_{0,m}| \leq \frac{2M_{0,2}}{(m-1)^2} \quad \text{and} \quad |\alpha_{1,m}| \leq \frac{8M_{0,2}}{\pi(m-1)^2}, \quad m > 1, \quad (12)$$

where $M_{2,0} \geq \max\{|\frac{\partial^2 f}{\partial x^2}(x, y)| : x, y \in [-1, 1]\}$, $M_{0,2} \geq \max\{|\frac{\partial^2 f}{\partial y^2}(x, y)| : x, y \in [-1, 1]\}$.

Proof. The coefficient $\alpha_{n,0}$ may be written as

$$\alpha_{n,0} = \frac{2}{\pi^2} \int_0^\pi \left(\int_0^\pi f(\cos t, \cos s) \cos ns dt \right) ds.$$

Two partial integrations are performed in the internal integral

$$\begin{aligned} & \int_0^\pi f(\cos t, \cos s) \cos n t dt = \\ & = \frac{1}{2n} \int_0^\pi \frac{\partial^2 f}{\partial y^2}(\cos t, \cos s) \sin t \left(\frac{\sin(n-1)t}{n-1} - \frac{\sin(n+1)t}{n+1} \right) dt. \end{aligned}$$

It results that

$$|\int_0^\pi f(\cos t, \cos s) \cos n t dt| \leq \frac{\pi}{2n} M_{2,0} \left(\frac{1}{n-1} + \frac{1}{n+1} \right) \leq \frac{\pi M_{2,0}}{(n-1)^2} \quad (13)$$

and consequently $|\alpha_{n,0}| \leq \frac{2M_{2,0}}{(n-1)^2}$.

Using (13) in $\alpha_{n,1} = \frac{4}{\pi^2} \int_0^\pi (\int_0^\pi f(\cos t, \cos s) \cos n t dt) \cos s ds$ it results

$$|\alpha_{n,1}| \leq \frac{4}{\pi} \frac{M_{2,0}}{(n-1)^2} \int_0^\pi |\cos s| ds = \frac{8M_{2,0}}{\pi(n-1)^2}.$$

The proof of (12) is similar. ■

Theorem 6.2 *If the function f has second order continuous partial derivatives then $\lim_{n,m \rightarrow \infty} f_{n,m} = f$ uniformly in $[-1, 1]^2$.*

Proof. From

$$f(x, y) = \sum_{n,m=0}^{\infty} \alpha_{n,m} T_n(x) T_m(y) \quad \text{and} \quad f_{n,m}(x, y) = \sum_{k=0}^n \sum_{j=0}^m \alpha_{k,j} T_k(x) T_j(y)$$

it results

$$f(x, y) - f_{n,m}(x, y) = \sum_{k=0}^n \sum_{j=m+1}^{\infty} \alpha_{k,j} T_k(x) T_j(y) + \sum_{k=n+1}^{\infty} \sum_{j=0}^{\infty} \alpha_{k,j} T_k(x) T_j(y).$$

Then

$$\begin{aligned} |f(x, y) - f_{n,m}(x, y)| & \leq \sum_{k=0}^n \sum_{j=m+1}^{\infty} |\alpha_{k,j}| + \sum_{k=n+1}^{\infty} \sum_{j=0}^{\infty} |\alpha_{k,j}| = \\ & = \sum_{j=m+1}^{\infty} |\alpha_{0,j}| + \sum_{j=m+1}^{\infty} |\alpha_{1,j}| + \sum_{k=2}^n \sum_{j=m+1}^{\infty} |\alpha_{k,j}| + \\ & + \sum_{k=n+1}^{\infty} |\alpha_{k,0}| + \sum_{k=n+1}^{\infty} |\alpha_{k,1}| + \sum_{k=n+1}^{\infty} \sum_{j=2}^{\infty} |\alpha_{k,j}| \end{aligned} \quad (14)$$

and using the Cauchy-Buniakowsky-Schwarz inequality it follows

$$\begin{aligned}
& (f(x, y) - f_{n,m}(x, y))^2 \leq \\
& \leq 6 \left(\left(\sum_{j=m+1}^{\infty} |\alpha_{0,j}| \right)^2 + \left(\sum_{j=m+1}^{\infty} |\alpha_{1,j}| \right)^2 + \left(\sum_{k=2}^n \sum_{j=m+1}^{\infty} |\alpha_{k,j}| \right)^2 + \right. \\
& \quad \left. + \left(\sum_{k=n+1}^{\infty} |\alpha_{k,0}| \right)^2 + \left(\sum_{k=n+1}^{\infty} |\alpha_{k,1}| \right)^2 + \left(\sum_{k=n+1}^{\infty} \sum_{j=2}^{\infty} |\alpha_{k,j}| \right)^2 \right)
\end{aligned}$$

The following inequality holds $\sum_{i=\nu+1}^{\infty} \frac{1}{i^2} < \int_{\nu}^{\infty} \frac{dx}{x} = \frac{1}{\nu}$.

Using the results of Theorem 6.1, the first, second, fourth and the fifth expression are increased by

$$\begin{aligned}
\sum_{j=m+1}^{\infty} |\alpha_{0,j}| & \leq 2M_{0,2} \sum_{j=m}^{\infty} \frac{1}{j^2} < \frac{2M_{0,2}}{m-1} \\
\sum_{j=m+1}^{\infty} |\alpha_{1,j}| & \leq \frac{8M_{0,2}}{\pi} \sum_{j=m}^{\infty} \frac{1}{j^2} < \frac{8M_{0,2}}{\pi(m-1)} < \frac{4M_{0,2}}{m-1} \\
\sum_{k=n+1}^{\infty} |\alpha_{k,0}| & \leq 2M_{2,0} \sum_{k=n}^{\infty} \frac{1}{k^2} < \frac{2M_{2,0}}{n-1} \\
\sum_{k=n+1}^{\infty} |\alpha_{k,1}| & \leq \frac{8M_{2,0}}{\pi} \sum_{k=n}^{\infty} \frac{1}{k^2} < \frac{8M_{2,0}}{\pi(n-1)} < \frac{4M_{2,0}}{n-1}.
\end{aligned}$$

For the third and sixth expression we shall use (10) and then the Cauchy-Buniakowsky-Schwarz' inequality

$$\begin{aligned}
\left(\sum_{k=2}^n \sum_{j=m+1}^{\infty} |\alpha_{k,j}| \right)^2 & = \left(\sum_{k=2}^n \sum_{j=m+1}^{\infty} \frac{|\Delta_{k,j}|}{4kj} \right)^2 \leq \frac{1}{16} \left(\sum_{k=2}^n \sum_{j=m+1}^{\infty} \Delta_{k,j}^2 \right) \left(\sum_{k=2}^n \sum_{j=m+1}^{\infty} \frac{1}{k^2 j^2} \right) \leq \\
& \leq M_{1,1}^2 \sum_{k=2}^n \frac{1}{k^2} \sum_{j=m+1}^{\infty} \frac{1}{j^2} \leq \frac{\pi^2 M_{1,1}^2}{6m},
\end{aligned}$$

and respectively

$$\left(\sum_{k=n+1}^{\infty} \sum_{j=2}^{\infty} |\alpha_{k,j}| \right)^2 \leq \left(\sum_{k=n+1}^{\infty} \sum_{j=2}^{\infty} \frac{|\Delta_{k,j}|}{4kj} \right)^2 \leq M_{1,1}^2 \left(\sum_{k=n+1}^{\infty} \frac{1}{k^2} \right) \left(\sum_{j=2}^{\infty} \frac{1}{j^2} \right) \leq \frac{\pi^2 M_{1,1}^2}{6n}$$

Consequently

$$|f(x, y) - f_{n,m}(x, y)| \leq \sqrt{6} \left(\frac{20M_{0,2}^2}{(m-1)^2} + \frac{20M_{2,0}^2}{(n-1)^2} + \frac{\pi^2 M_{1,1}^2}{6m} + \frac{\pi^2 M_{1,1}^2}{6n} \right)^{\frac{1}{2}} \rightarrow 0,$$

when $m, n \rightarrow \infty$. ■

7 The Lagrange interpolation polynomial

For any grids $-1 \leq x_0 < x_1 < \dots < x_n \leq 1$, $-1 \leq y_0 < y_1 < \dots < y_m \leq 1$ and any $f : [-1, 1]^2 \rightarrow \mathbb{R}$ the expression of the Lagrange interpolation polynomial is

$$L_{n,m}(x, y) = \sum_{i=0}^n \sum_{j=0}^m f(x_i, y_j) l_{x_i}(x) l_{y_j}(y)$$

where

$$l_{x_i}(x) = \prod_{k=0, k \neq i}^n \frac{x - x_k}{x_i - x_k}, \quad \text{and} \quad l_{y_j}(x) = \prod_{l=0, l \neq j}^m \frac{y - y_l}{y_j - y_l}.$$

This polynomial satisfies the interpolation restrictions

$$L_{n,m}(x_k, y_l) = f(x_k, y_l), \quad \forall k \in \{0, 1, \dots, n\}, \text{ and } \forall l \in \{0, 1, \dots, m\}.$$

In the set of (n, m) degree polynomials there exists a unique interpolation polynomial.

If $x_i = \cos \frac{i\pi}{n}$, $i \in \{0, 1, \dots, n\}$ and $y_j = \cos \frac{j\pi}{m}$, $j \in \{0, 1, \dots, m\}$ then using the *discrete orthogonality* relations, [9],

$$\sum_{k=0}^n \gamma_k T_p(x_k) T_q(x_k) = \begin{cases} 0 & \text{if } p \neq q \\ \frac{n}{2} & \text{if } p = q \in \{1, 2, \dots, n-1\} \\ n & \text{if } p = q \in \{0, n\} \end{cases} = n \alpha_p \delta_{p,q},$$

where

$$\gamma_{n,k} = \begin{cases} \frac{1}{2} & \text{if } k \in \{0, n\} \\ 1 & \text{if } k \in \{1, 2, \dots, n-1\} \end{cases} \quad \text{and} \quad \alpha_{n,i} = \begin{cases} \frac{1}{2} & \text{if } i \in \{1, 2, \dots, n-1\} \\ 1 & \text{if } i \in \{0, n\} \end{cases}$$

the Lagrange interpolation polynomial may be written as

$$L_{n,m}(x, y) = \sum_{i=0}^n \sum_{j=0}^m c_{i,j} T_i(x) T_j(y),$$

where $c_{i,j} = \frac{4}{nm} \gamma_{n,i} \gamma_{m,j} \sum_{k=0}^n \sum_{l=0}^m \gamma_{n,k} \gamma_{m,l} f(x_k, y_l) T_i(x_k) T_j(y_l)$.

This polynomial will be called the Lagrange-Chebyshev interpolating polynomial.

As in [4], the following statements occur:

Theorem 7.1 (*Aliasing of Chebyshev polynomials, [4]*) For any $n \geq 1$ and $0 \leq m \leq n$ the polynomials $T_m, T_{n \pm m}, T_{2n \pm m}, \dots$ take the same values on the grid $(\cos \frac{k\pi}{n})_{0 \leq k \leq n}$.

Theorem 7.2 (*Aliasing formula of Chebyshev coefficients*) Let

$$f(x, y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \alpha_{i,j} T_i(x) T_j(y)$$

and let $L_{n,m}(x, y) = \sum_{i=0}^n \sum_{j=0}^m c_{i,j} T_i(x) T_j(y)$ be its Lagrange-Chebyshev interpolant. Then

$$c_{i,j} = \sum_{p,q=0}^{\infty} \alpha_{2pn+i, 2qm+j} + \sum_{p,q=1}^{\infty} \alpha_{2pn-i, 2qm-j} \quad (15)$$

Proof. Supposing that $(c_{i,j})_{0 \leq i \leq m, 0 \leq j \leq m}$ are given by (15) and $\varphi(x, y) = \sum_{i=0}^n \sum_{j=0}^m c_{i,j} T_i(x) T_j(y)$. For any $(k, l) \in \{0, 1, \dots, n\} \times \{0, 1, \dots, m\}$

$$\begin{aligned} \varphi(x_k, y_l) &= \sum_{i=0}^n \sum_{j=0}^m c_{i,j} T_i(x_k) T_j(y_l) = \\ &= \sum_{i=0}^n \sum_{j=0}^m \left(\sum_{p,q=0}^{\infty} \alpha_{2pn+i, 2qm+j} + \sum_{p,q=1}^{\infty} \alpha_{2pn-i, 2qm-j} \right) T_i(x_k) T_j(y_l). \end{aligned}$$

It is observed that when the indexes i, j, p, q go through their values then $(2pn \pm i, 2qm \pm j)$ go through $\mathbb{N} \times \mathbb{N}$ and any two pairs are distinct. Thus, with Theorem 7.1,

$$\varphi(x_k, y_l) = \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \alpha_{s,t} T_s(x_k) T_t(y_l) = f(x_k, y_l).$$

The unicity of the interpolating polynomial in the set of (n, m) degree polynomials implies $L_{n,m} = \varphi$. ■

A consequence of (15) is a relation between $L_{n,m}(x, y)$ and the approximation polynomial $f_{n,m}(x, y) = \sum_{i=0}^n \sum_{j=0}^m \alpha_{i,j} T_i(x) T_j(y)$:

$$\begin{aligned} L_{n,m}(x, y) &= \sum_{i=0}^n \sum_{j=0}^m c_{i,j} T_i(x) T_j(y) = \\ &= \sum_{i=0}^n \sum_{j=0}^m \left(\alpha_{i,j} + \sum_{q=1}^{\infty} \alpha_{i, 2qm \pm j} + \sum_{p=1}^{\infty} \alpha_{2pn \pm i, j} + \sum_{p,q=1}^{\infty} \alpha_{2pn \pm i, 2qm \pm j} \right) T_i(x) T_j(y) = \\ &= f_{n,m}(x, y) + \sum_{i=0}^n \sum_{j=m+1}^{\infty} \alpha_{i,j} T_i(x) T_{\mu_j}(y) + \sum_{i=n+1}^{\infty} \sum_{j=0}^m \alpha_{i,j} T_{\nu_i}(x) T_j(y) + \end{aligned}$$

$$+ \sum_{i=n+1}^{\infty} \sum_{j=m+1}^{\infty} \alpha_{i,j} T_{\nu_i}(x) T_{\mu_j}(y).$$

Then

$$|L_{n,m} - f_{n,m}(x, y)| \leq \sum_{i=0}^n \sum_{j=m+1}^{\infty} |a_{i,j}| + \sum_{i=n+1}^{\infty} \sum_{j=0}^{\infty} |a_{i,j}|. \quad (16)$$

Know, it can prove the uniform convergence of the Lagrange-Chebyshev interpolation polynomials:

Theorem 7.3 *If the function f has second order continuous partial derivatives then $\lim_{n,m \rightarrow \infty} L_{n,m} = f$ uniformly in $[-1, 1]^2$.*

Proof. Using (14) and (16) we obtain

$$\begin{aligned} |f(x, y) - L_{n,m}(x, y)| &\leq |f(x, y) - f_{n,m}| + |f_{n,m}(x, y) - L_{n,m}(x, y)| \leq \\ &\leq 2 \left(\sum_{i=0}^n \sum_{j=m+1}^{\infty} |a_{i,j}| + \sum_{i=n+1}^{\infty} \sum_{j=0}^{\infty} |a_{i,j}| \right). \end{aligned}$$

The rest of the proof follows the proof of Theorem 6.2. ■

8 Applications

1. Integrating $f_{n,m}(x, y)$ on Ω there is obtained

$$\iint_{\Omega} f(x, y) dx dy \approx \iint_{\Omega} f_{n,m}(x, y) dx dy = 4 \sum_{k=0, \text{ even}}^n \sum_{j=0, \text{ even}}^m \frac{\alpha_{n,m}}{(1-k^2)(1-j^2)}. \quad (17)$$

2. Computation of the first order partial derivatives. Practically, knowing the Chebyshev approximation polynomial $f_{n,m}(x, y) = \sum_{k=0}^n \sum_{j=0}^m \alpha_{k,j} T_k(x) T_j(y)$, and with the assumption that $\alpha_{k,j} \approx 0$ for $k > n > 4$, and for any $j \in \{0, 1, \dots, m\}$ the first n equations of the system (7) will be

$$\begin{cases} b_{0,j} - \frac{b_{2,j}}{2} &= \alpha_{1,j} \\ \frac{1}{2k}(b_{k-1,j} - b_{k+1,j}) &= \alpha_{k,j}, \quad k \in \{2, 3, \dots, n-2\} \\ \frac{1}{2(n-1)}b_{n-2,j} &= \alpha_{n-1,j} \\ \frac{1}{2n}b_{n-1,j} &= \alpha_{n,j} \end{cases}$$

with the solution

$$\begin{aligned} b_{n-1,j} &= 2n\alpha_{n,j} \\ b_{n-2,j} &= 2(n-1)\alpha_{n-1,j} \\ b_{k,j} &= 2(k+1)\alpha_{k+1,j} + b_{k+2,j}, \quad k \in \{n-3, n-4, \dots, 2, 1\} \\ b_{0,j} &= \alpha_{1,j} + \frac{b_{2,j}}{2} \end{aligned} .$$

Then $\frac{\partial f(x,y)}{\partial x} \approx \sum_{k=0}^{n-1} \sum_{j=0}^m b_{k,j} T_k(x) T_j(y)$.

The partial derivative $\frac{\partial f(x,y)}{\partial y}$ may be computed similarly.

Due to the truncation of the Chebyshev series the numerical result is influenced by the truncation error as well as by rounding errors. The automatic differentiation [3] is a method which eliminates the truncation error but it requires a specific computational environment related to the definition of the *elementary* functions (e.g. *apache commons-math3* v. 3.4).

9 Examples

Using a *Scilab* implementation the following results are obtained

1. $f(x,y) = \cos xy$ [2], Ch. 11, p. 2. The matrix of the coefficients is

$$\begin{array}{rrrrr} 0.880725579 & 0. & - & 0.117388011 & 0. & 0.001873213 \\ 0. & 0. & 0. & 0. & 0. & \\ -0.117388011 & 0. & - & 0.114883808 & 0. & 0.002484444 \\ 0. & 0. & 0. & 0. & 0. & \\ 0.001873213 & 0. & 0.002484444 & 0. & 0.000603385 & \end{array}$$

The value of the indicator given by (5) is $3.97247 \cdot 10^{-10}$.

On an equidistant grid of size 50×50 in $[0, 1]^2$ the maximum absolute error is 0.000082141.

The integral given by (17) is 3.784330902, while *Mathematica* gives 4SinIntegral[1] \approx 3.78433228147.

2. $g(x) = \cos 10xy^2 + e^{-x^2}$ [2], Ch. 12, p. 6. The size of matrix of coefficients is 33×43 .

The value of the indicator given by (5) is 0.

On an equidistant grid of size 50×50 in $[0, 1]^2$ the maximum absolute error is $2.98594 \cdot 10^{-13}$.

The integral given by (17) is 4.590369905, which is equal to that given by *Mathematica*.

10 Conclusions

There is presented an alternative to the Gaussian elimination method used in *Chebfun* software in order to construct an approximation polynomial of a function with two variables.

Because the discrete Fourier transform is a common tool for the usual mathematical softwares, this approach has a relative simple implementation, but as a drawback, if the tolerance is the machine precision then it may require a large amount of memory.

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