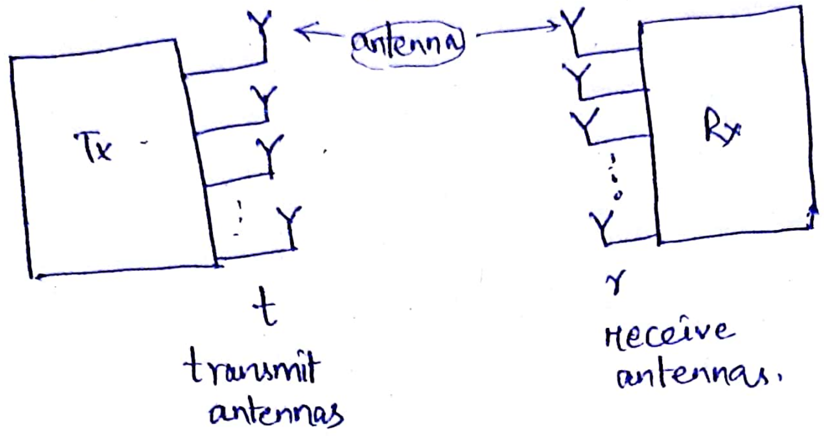
Lecture - 20 (20 mins) Introduction to MIMO

MIMO stands for multiple input multiple output

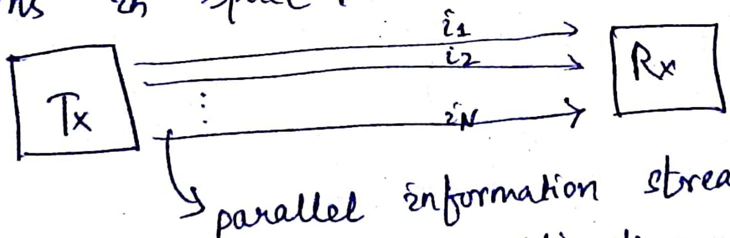
Definition :- MIMO systems have multiple antennas at both transmitter and the receiver



→ As there are multiple antennas, MIMO systems can be employed for diversity gain.

→ MIMO can increase the data rate by transmitting several information streams in parallel at same transmit power.

Spatial multiplexing :- multiplexing parallel information streams in space.

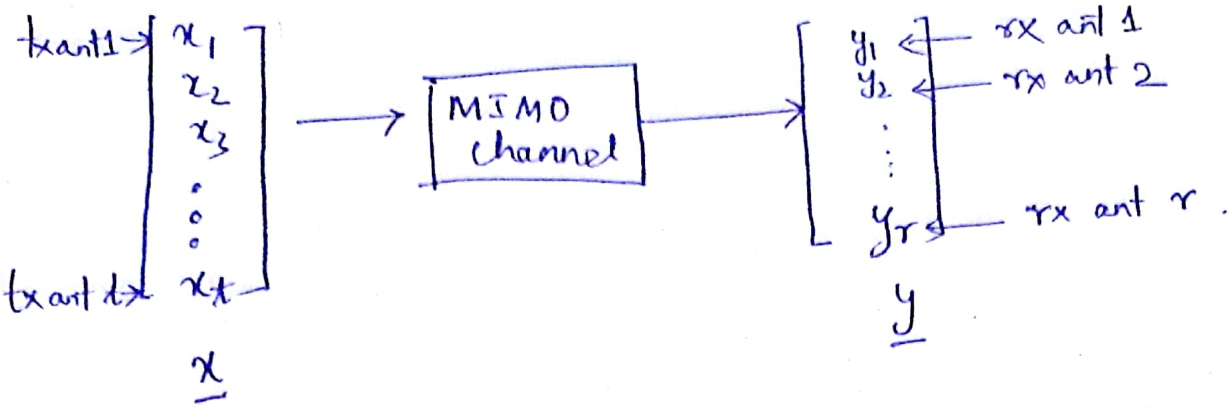


This is possible through multi-dimensional signal processing.

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_t \end{bmatrix}$$

$\underline{x}$

$\underline{x}$ is the vector where each x_i denotes the symbol transmitted from each antenna. where $1 \leq i \leq t$



$\underline{x}$ = transmit vector.
$\underline{y}$ = receive vector

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_r \end{bmatrix} = \underbrace{\begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1t} \\ h_{21} & h_{22} & \cdots & h_{2t} \\ \vdots & & & \\ h_{r1} & h_{r2} & \cdots & h_{rt} \end{bmatrix}_{r\times t}}_{\text{Channel matrix}} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_t \end{bmatrix}_{t\times 1}$$

Each h_{ij} is a flat fading channel coefficient

$$y_1 = h_{11}x_1 + h_{12}x_2 + \cdots + h_{1t}x_t$$

$x_1, x_2, \ldots, x_t$ interfere at y_1.

$$y_2 = h_{21}x_1 + h_{22}x_2 + \cdots + h_{2t}x_t$$

$x_1, x_2, \ldots, x_t$ interfere at y_2.

h_{ij} is the channel coefficient between i^{th} receive antenna and j^{th} transmit antenna.

Hence, there is a total of rt channel coefficients

Hence, MIMO channel matrix

$$H = \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1t} \\ h_{21} & & \cdots & h_{2t} \\ h_{r1} & & \cdots & h_{rt} \end{bmatrix}_{r\times t}$$

MIMO System Model:-

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_r \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1t} \\ h_{21} & \cdots & \cdots & \cdot \\ h_{r1} & h_{r2} & \cdots & h_{rt} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_t \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_t \end{bmatrix}$$

$\underline{y} = H\underline{x} + \underline{n}$ → $\underline{n}$ is r-dimensional receive vector.

$\underline{y}$ → r dimensional receive vector

H → $r \times t$ Channel matrix

$\underline{x}$ → $\underline{x}$ is t dimensional transmit vector

For special case, $t = 1$

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_r \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_r \end{bmatrix} x + \begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_r \end{bmatrix} \qquad \underline{y} = \underline{h}x + \underline{n}$$

This is a receive diversity system. Hence, this is a single input multiple output system.

For MISO system,

$$y = \begin{bmatrix} h_1 & h_2 & \cdots & h_t \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_t \end{bmatrix} + n$$

$$y = \underline{h}^T \underline{x} + n \qquad \text{(Transmit diversity system)}$$

→ when $r = t = 1$,

$$y = hx + n$$

Noise $\underline{n}$

$$\underline{n} = \begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_r \end{bmatrix} \leftarrow \text{r dimensional vector}$$

$$E(|n_i|^2) = \sigma_n^2$$

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$$E[\underline{n}\,\underline{n}^H] = E\left(\begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ \vdots \\ n_r \end{bmatrix}\begin{bmatrix} n_1^* & n_2^* & n_3^* & \cdots & n_r^* \end{bmatrix}\right)$$

↑ Covariance matrix

$$= E\left(\begin{bmatrix} |n_1|^2 & n_1 n_2^* & \cdots & n_1 n_r^* \\ n_2 n_1^* & |n_2|^2 & & \\ & & \ddots & \\ n_r n_1^* & & & |n_r|^2 \end{bmatrix}\right)$$

$$\therefore \quad E[\underline{n}\,\underline{n}^H] = \begin{bmatrix} \sigma_n^2 & 0 & \cdots & 0 \\ \vdots & \sigma_n^2 & & \vdots \\ & & \ddots & \\ 0 & \cdots & & \sigma_n^2 \end{bmatrix}$$

$$\Rightarrow R_n = E[\underline{n}\,\underline{n}^H] = \sigma_n^2 I.$$

This is said to be spatially white noise.

Spatio temporally white noise:- means that noise is uncorrelated across antennas and time.

$$\therefore \quad \boxed{R_n = E[\underline{n}\,\underline{n}^H] = \sigma_n^2 I}$$

Lecture-21 MIMO System Model and Zero Forcing Receiver.

We will discuss linear receiver.

$$\underline{y} = H\underline{x} + \underline{n}$$

$$H^{-1}\underline{y} = H^{-1}H\underline{x} + H^{-1}\underline{n}$$

1. inverse only exists for $r = t$
2. Even for square matrices, inverse need not exists if it is rank deficient.

We define a generalised inverse for $r \geq t$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_r \end{bmatrix} = \begin{bmatrix} h_{11} & \cdots & h_{1t} \\ h_{21} & \cdot & \vdots \\ \vdots & \ddots & \vdots \\ h_{r1} & \cdots & h_{rt} \end{bmatrix}_{r \times t} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_t \end{bmatrix} + \underline{n}$$

This is a thin matrix

This is a case where there are more equations than unknowns. (Overdetermined system) i.e. Number of equations is r and no. of unknowns is t.

Hence, this system might not have an exact solution.

Amongst all possible solutions or transmit vectors $\underline{x}$ choose the minimum error vector.

$$\text{error} = \|\underbrace{\underline{y}}_{\text{measurement}} - H\underbrace{\underline{x}}_{\text{unknown}}\|^2$$

This is known as least squares solution.

Brief discussion of vector differentiation :-

Let $f(\underline{x})$:

$$\frac{d(f(\underline{x}))}{d\underline{x}}$$

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_t \end{bmatrix} \qquad \frac{df}{d\underline{x}} = \begin{bmatrix} \frac{df}{dx_1} \\ \frac{df}{dx_2} \\ \vdots \\ \frac{df}{dx_t} \end{bmatrix}_{t \times 1}$$

Let $f(\underline{x}) = \underline{c}^T \underline{x} = \underline{x}^T \underline{c}$

$$= c_1 x_1 + c_2 x_2 + \cdots + c_t x_t$$

$$\frac{d(\underline{c}^T \underline{x})}{d\underline{x}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_t \end{bmatrix} = \bar{c} \quad \therefore \boxed{\frac{d(\underline{c}^T \underline{x})}{d\underline{x}} = \frac{d(\underline{x}^T \underline{c})}{d\underline{x}} = \underline{c}}$$

Now, we want to minimise $\|\underline{y} - H\underline{x}\|^2$

$$= (\underline{y} - H\underline{x})^T (\underline{y} - H\underline{x})$$

$$= (\underline{y}^T - \underline{x}^T H^T)(\underline{y} - H\underline{x})$$

$$= \underline{y}^T \underline{y} - \underline{x}^T H^T \underline{y} - \underline{y}^T H \underline{x} + \underline{x}^T H^T H \underline{x}$$

Now, lets differentiate w.r.t. $\underline{x}$

$$\frac{d}{d\underline{x}}\left\{ \underline{y}^T \underline{y} - \underline{x}^T H^T \underline{y} - \underline{y}^T H \underline{x} + \underline{x}^T H^T H \underline{x} \right\}$$

$$= 0 - H^T \underline{y} - H^T \underline{y} + H^T H \underline{x} + H^T H \underline{x}$$

$$= -2H^T \underline{y} + 2H^T H \underline{x}$$

$$\frac{d\|\underline{y} - H\underline{x}\|^2}{d\underline{x}} = -2H^T \underline{y} + 2H^T H \underline{x} = 0$$

$$\Rightarrow H^T H \underline{x} = H^T \underline{y}$$

$$\Rightarrow \boxed{\underline{x} = (H^T H)^{-1} H^T \underline{y}}$$

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This is the approximate solⁿ that minimizes Least Square Error.
This is known as the Zero forcing Receiver.

For complex matrices,

$$\boxed{\hat{\underline{x}} = (H^H H)^{-1} H^H \underline{y}}$$

For complex channel matrix H

This $(H^H H)^{-1} H^H = H^{+}$ is called pseudo inverse of H.

$$(H^H H)^{-1} H^H \times H = (H^H H)^{-1} (H^H H) = I$$

This is left inverse because

1. $B = A^{-1}$ only if $AB = BA = I$
2. Inverse is unique, however pseudo-inverse is not unique.

If H^{-1} exists, the ~~$H^H H^H$~~ $(H^H H)^{-1} H^H \times H = I$

If H^{-1} exists, then the pseudo-inverse reduces to the conventional true inverse.

Diversity Order of ZF receiver:

The diversity order of ZF receiver $= r+1-t$
$= r-t+1$

Diversity order $= r-t+1$
$= 4-2+1 = 3$

If $r=t$, then diversity $=$ ~~0~~ $r-t+1 = 0+1 = 1$

The diversity order of this system is poor because it results in noise amplication.

Hence, disadvantage of ZF receiver.

① It results in Noise amplification.

MIMO Example:- $H = \begin{bmatrix} 2 & 3 \\ 1 & 3 \\ 4 & 2 \end{bmatrix}$

$r = 3$ = receive antennas
$t = 2$ = number of transmit antennas.

System model.

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

$$\underline{y} = H\underline{x} + \underline{n}$$

$$y_1 = 2x_1 + 3x_2 + n_1$$
$$y_2 = x_1 + 3x_2 + n_2$$
$$y_3 = 4x_1 + 2x_2 + n_3$$

$$\text{pinv}(H) = H^{+} = (H^H H)^{-1} H^H$$

$$H^H H = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 3 & 2 \end{bmatrix}_{2\times3} \begin{bmatrix} 2 & 3 \\ 1 & 3 \\ 4 & 2 \end{bmatrix}_{3\times2} = \begin{bmatrix} 21 & 17 \\ 17 & 22 \end{bmatrix}$$

$$H^H H = \begin{bmatrix} 21 & 17 \\ 17 & 22 \end{bmatrix}$$

$$(H^H H)^{-1} = \frac{1}{21\cdot22 - 17\cdot17} \begin{bmatrix} 22 & -17 \\ -17 & 21 \end{bmatrix} = \begin{bmatrix} 0.1272 & -0.098 \\ -0.0983 & 0.121 \end{bmatrix}$$

$$H^{+} = (H^H H)^{-1} H^H = \begin{bmatrix} 0.1272 & -0.0983 \\ -0.0983 & 0.1214 \end{bmatrix} \begin{bmatrix} 2 & 1 & 4 \\ 3 & 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -0.0405 & -0.1676 & 0.3121 \\ 0.1676 & 0.2659 & -0.1503 \end{bmatrix}$$

$$\underset{2\times3}{H^{\dagger}}\ \underset{3\times1}{\underline{y}} = \underset{2\times1}{\hat{\underline{x}}}$$

$\boxed{\text{The LS suffere from the problem of noise-enhancement.}}$

Hence, we need a better solution.

So, we go for a minimum mean square error.

MIMO - MMSE receiver.

A MMSE receiver is a Bayesian approach.

In MMSE, the problem is to estimate a scalar x given

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_r \end{bmatrix}}_{\underline{y}}$$

Linear estimator:- $\hat{x} = \underline{c}^T \underline{y}$

I have $\underline{y}$ measurements and I want a linear combiner $\underline{c}$ so that I get an estimate of $\underline{x}$.

Choose $\underline{c}$ such that $E\|\hat{x} - x\|^2$ is minimised.

$$\min E\left\{\|\hat{x} - x\|^2\right\}$$

$$= \min E\left\{\|\underline{c}^T\underline{y} - \underline{x}\|^2\right\}$$

The squared error (considering x is a scalar quantity

$$E\left\{(\underline{c}^T\underline{y} - x)^T(\underline{c}^T\underline{y} - x)\right\}$$

$$= E\left\{(\underline{c}^T\underline{y} - x)(\underline{c}^T\underline{y} - x)^T\right\}$$

$$= E\left\{(\underline{c}^T\underline{y} - x)(\underline{y}^T\underline{c} - x)\right\}$$

$$= E\left\{\underline{c}^T\underline{y}\,\underline{y}^T\underline{c} - \underline{c}^T\underline{y}\,x - x\,\underline{y}^T\underline{c} + x^2\right\}$$

$E[\underline{y}\,\underline{y}^T] = R_{yy}$ = Covariance matrix of $\underline{y}$.

$E[x\,\underline{y}^T] = R_{xy}$ = Cross- covariance

$E[y\,x^T] = R_{xy}^T = R_{yx}$

Lec-22 MIMO-MM
SE & Intro to SVD

$$= \underline{c}^T R_{yy}\,\underline{c} - R_{xy}\,\underline{c} - \underline{c}^T R_{yx} + R_{xx}$$

I want to minimise above.

$$\min \underbrace{\underline{c}^T R_{yy}\,\underline{c} - 2\,\underline{c}^T R_{yx} + R_{xx}}_{\text{This is a function of } \underline{c}}$$

$\frac{\partial F}{\partial \underline{c}} = 0$ for minimum

$$2 R_{yy}\,\underline{c} - 2 R_{yx} = 0$$

$$\Rightarrow R_{yy}\,\underline{c} = R_{yx}$$

$$\Rightarrow \boxed{\underline{c} = R_{yy}^{-1} R_{yx}}$$

This is the Linear Minimum mean Square Error estimator for the quantity $\underline{x}$.

$\therefore\ \hat{x} = \underline{c}^T \underline{y}$ is my LMMSE estimate for real.

$\boxed{\hat{x} = \underline{c}^H \underline{y}}$ for complex cases.

$\Rightarrow \boxed{\hat{x} = R_{yx} R_{yy}^{-1}\,\underline{y}}$ vector cases,

Now, we will expand this to

$$\underline{y} = H\underline{x} + \underline{n}$$

$E[\underline{x}\,\underline{x}^H]$ = covariance of transmitted symbols. also called transmit covariance.

$$E\left\{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_t \end{bmatrix}\begin{bmatrix} x_1^* & x_2^* & \cdots & x_t^* \end{bmatrix}\right\}$$

$$\Rightarrow R_{xx} = E\left\{\begin{bmatrix} |x_1|^2 & x_1 x_2^* & \cdots & x_1 x_t^* \\ x_2 x_1^* & |x_2|^2 & & \vdots \\ & & \ddots & \vdots \\ x_t x_1^* & \cdots & \cdots & |x_t|^2 \end{bmatrix}\right\}$$

Diagonal terms are powers and off-diagonal elements are correlations

We can assume un-correlated symbols.

$$= \begin{bmatrix} P_d & 0 & \cdots & 0 \\ 0 & P_d & & \vdots \\ 0 & & \ddots & \vdots \\ 0 & \cdots & \cdots & P_d \end{bmatrix} = P_d I_t$$

= Transmit power times the Identity matrix.

$$R_{yy} = E\left\{\left[\underline{y}\,\underline{y}^H\right]\right\}$$

$$= E\left\{(H\underline{x}+\underline{n})(H\underline{x}+\underline{n})^H\right\}$$

$$= E\left\{(H\underline{x}+\underline{n})(\underline{x}^H H^H + \underline{n}^H)\right\}$$

$$= E\left\{H\underline{x}\,\underline{x}^H H^H + H\underline{x}\,\underline{n}^H + \underline{n}\,\underline{x}^H \underline{H}^H + \underline{n}\,\underline{n}^H\right\}$$

$E\left[\underline{x}\,\underline{n}^H\right] = 0$, also, $E\left[\underline{n}\,\underline{x}^H\right] = 0$

$$\Rightarrow R_{yy} = H R_{xx} H^H + E\left[\underline{n}\,\underline{n}^H\right]$$

$$\Rightarrow R_{yy} = H R_{xx} H^H + \sigma_n^2 I$$

$$\Rightarrow \boxed{R_{yy} = P_d H H^H + \sigma_n^2 I}$$

This is covariance matrix of received symbol vectors $\underline{y}$.

$$R_{yx} = E[\underline{y}\,\underline{x}^H]$$
$$= E[(H\underline{x} + \underline{n})\,\underline{x}^H]$$
$$= E[H\underline{x}\,\underline{x}^H + \underline{n}\,\underline{x}^H]$$
$$= H \cdot P_d + 0$$
$$\Rightarrow R_{yx} = H \cdot P_d.$$

Now, we know $\underline{C} = R_{yy}^{-1} R_{yx}$

$$\Rightarrow \underline{C} = (P_d HH^H + \sigma_n^2 I)^{-1} P_d H$$
$$= P_d (P_d H \cdot H^H + \sigma_n^2 I)^{-1} H$$

$$\hat{x} = \underline{C}^H \underline{y}$$
$$= \{P_d (P_d H \cdot H^H + \sigma_n^2 I)^{-1} H\}^H \underline{y}$$
$$\boxed{\hat{x} = P_d \cdot H^H \cdot (P_d H \cdot H^H + \sigma_n^2 I)^{-1} \underline{y}}$$

Linear Minimum Mean Square Error Estimator for MIMO system.

The quantity $H^H (P_d H \cdot H^H + \sigma_n^2 I)^{-1}$
$$= (P_d H^H H + \sigma_n^2 I)^{-1} H$$

$$\Leftrightarrow (P_d H^H H + \sigma_n^2 I) \cdot H^H = H^H (P_d H \cdot H^H + \sigma_n^2 I)$$
$$\Leftrightarrow P_d H^H H H^H + \sigma_n^2 H^H = P_d H^H H H^H + \sigma_n^2 H^H$$

So, $\hat{x} = P_d H^H \left(P_d H \cdot H^H + \sigma_n^2 I \right)^{-1} y$

$= P_d \left(P_d H^H H + \sigma_n^2 I \right)^{-1} H^H y$

$$\boxed{\hat{x} = P_d \left(P_d H^H H + \sigma_n^2 I \right)^{-1} H^H y}$$

Linear Minimum Mean Squared Error estimator for MIMO system

For SISO, $H = h$

$$\hat{x} = P_d \left(\frac{h^*}{P_d |h|^2 + \sigma_n^2} \right) y$$

$$\Rightarrow \hat{x} = P_d \left(\frac{h^*}{\sigma_n^2} \right) y \quad \left(\because \text{As, } h \to 0 \right)$$

This formula doesnot let estimate $\hat{x}$ blow up as $h \to 0$ because it is bounded by σ_n^2.
Hence, for SISO case, MIMO MMSE estimator doesnot result in noise enhancement. Therefore, it is superior to LS estimator or ZF receiver.

$$\hat{x} = P_d \left(P_d H^H H + \sigma_n^2 I \right)^{-1} H^H y$$

At high SNR, $\sigma_n^2 \to 0$

$$\hat{x} \approx P_d \left(P_d H^H H \right)^{-1} H^H y$$

$$\hat{x} \approx \left(H^H H \right)^{-1} H^H y$$

$\therefore$ At high SNR, MMSE tends to LS or ZF receiver.

At low SNR, $\hat{x} = P_d (\sigma_n^2 I)^{-1} H^H \underline{y}$

$$\hat{x} = \frac{P_d}{\sigma_n^2} H^H y$$

At low SNR, ~~hence~~ the MMSE reduces to a matched filter.

MIMO LMMSE → At low SNR → Matched Filter or an MRC

MIMO LMMSE → At high SNR → Zero Forcing Receiver

Next → We will discuss decomposition of a MIMO channel. A key aspect of MIMO system is the SVD of a MIMO system.

Decomposition of the MIMO channel, H

$\boxed{\text{Singular Value Decomposition}}$:- We assume $r \geq t$

$$H = U \Sigma V^H$$

$$= \begin{bmatrix} u_1 & u_2 & \cdots & u_t \end{bmatrix} \underbrace{\begin{bmatrix} \sigma_1 & & & 0 \\ & \sigma_2 & & \\ & & \ddots & \\ 0 & & & \sigma_t \end{bmatrix}}_{t \text{ columns}} \begin{bmatrix} v_1^H \\ v_2^H \\ \vdots \\ v_t^H \end{bmatrix} \; t \text{ rows}$$

The columns u_i are orthonormal.

$\|u_i\|^2 = 1$ and $\langle u_i, u_j \rangle = 0$ for $i \neq j$

$u_i^H u_j = 0$

$\|v_i\|^2 = 1$ and $v_i^H v_j = 0$ if $i \neq j$

$V V^H = V^H V = I$ (Here V is a unitary matrix)

$U^H U = I$

$\sigma_1, \sigma_2, \ldots, \sigma_t$ are known as singular values.

$\sigma_1, \sigma_2, \ldots \sigma_t \geq 0$

$\underbrace{\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_t \geq 0}_{\text{Singular values are ordered}}$

A notable difference between matrices decomposition using eigenvalue & SVD is that EVD exists only for square matrices. But SVD exists for any dimension matrix.

Examples:- $H = \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{2\times1}$

$$= \underbrace{\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}}_{U} \underbrace{[\sqrt{2}]}_{\Sigma} \underbrace{[1]}_{V}$$

$\sigma_1 = \sqrt{2} > 0$

$$u_1 = \left\| \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 1/\sqrt{2} \end{bmatrix} \right\|^2 = \frac{1}{2} + \frac{1}{2} = 1$$

$V = 1$ unitary matrix.

Ex: 2

$$\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{5} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{5} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{Not a Valid SVD}}$$

because $\sigma_1 = 1, \sigma_2 = \sqrt{5}$ and so, $\sigma_1 \leq \sigma_2$

Singular values are not ordered.

The valid SVD is

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \sqrt{5} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Ex: 3

$$H = \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\boxed{\sigma_1 = 2\sqrt{2} \geq \sigma_2 = \sqrt{2} \geq 0}$$ ordered sigular values.

Note:
The number of non-zero singular values is equal to the rank of the matrix.

SVD Based Optimal MIMO transmission and Capacity.

Lec-23

Any MIMO channel matrix can be written in SVD form as

$$H = \underline{U} \Sigma \underline{V}^H$$

$$\underline{y} = H\underline{x} + \underline{n}$$
$$= (\underline{U}\Sigma\underline{V}^H)\underline{x} + \underline{n}$$

At the receiver, multiply $\underline{y}$ with $\underline{U}^H$.

$$\underline{U}^H \underline{y} = \underline{\tilde{y}} = \underline{U}^H(\underline{U}\Sigma\underline{V}^H\underline{x} + \underline{n})$$

$$\Rightarrow \tilde{y} = \Sigma \underline{V}^H \underline{x} + \underline{U}^H \underline{n}$$
$$\Rightarrow \tilde{y} = \Sigma \underline{V}^H \underline{x} + \underline{\tilde{n}}$$

We will now denote $U^H n$ by $\tilde{n}$ which is noise vector.

Now, we will do some manipulation on transmitter side, this is done before transmitting, hence it is called pre-coding.

Let denote transmit vector $\underline{x} = V\tilde{x}$

So, $\tilde{y} = \Sigma V^H V \tilde{x} + \tilde{n}$

$$V^H V = I$$

So, $\tilde{y} = \Sigma \tilde{x} + \tilde{n}$

In the receiver, we multiplied by $\underline{U}^H$ and in the transmitter we precode using V. So, the system model is

$$\tilde{y} = \Sigma \tilde{x} + \tilde{n}$$

But Σ is a diagonal matrix.

$$\begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \vdots \\ \tilde{y}_t \end{bmatrix} = \begin{bmatrix} \sigma_1 & & & 0 \\ & \sigma_2 & & \\ & & \ddots & \\ 0 & & & \sigma_t \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_t \end{bmatrix} + \begin{bmatrix} \tilde{n}_1 \\ \tilde{n}_2 \\ \vdots \\ \tilde{n}_t \end{bmatrix}$$

Earlier, we said all symbols interfere at every rx antenna. All symbols transmitted cause interference at every receive antenna. However, now, we have every transmitted $\tilde{x}_i$ is received at the receiver $\tilde{y}_i$.

This is known as Decoupling of MIMO channels

This can also be ~~an~~ said as Parallelization of MIMO System.

Hence,
$$\tilde{y}_1 = \sigma_1 \tilde{x}_1 + \tilde{n}_1$$
$$\tilde{y}_2 = \sigma_2 \tilde{x}_2 + \tilde{n}_2$$
$$\vdots$$
$$\tilde{y}_t = \sigma_t \tilde{x}_t + \tilde{n}_t$$

Looks like a collection of t parallel channels. gain of channel i is σ_i

Hence, this happens due to precoding at transmitter and beam forming at the receiver.

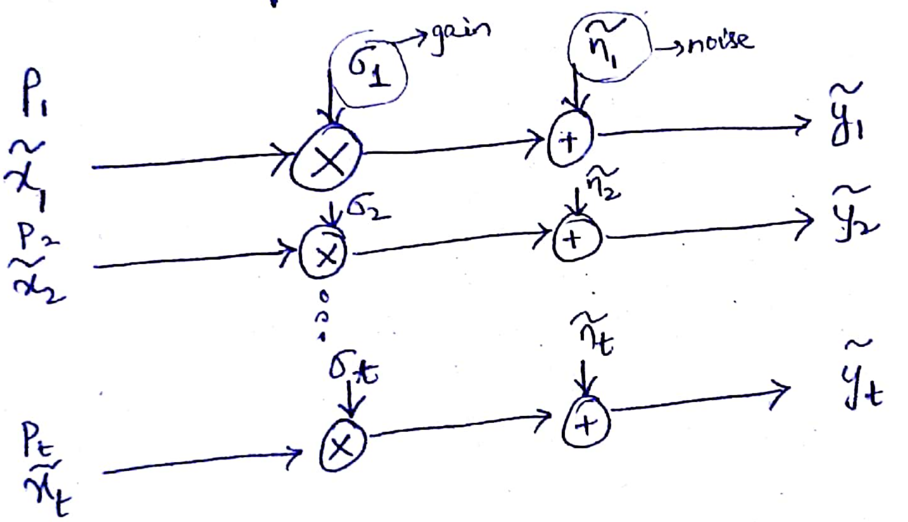
→ We transmit t information symbols in parallel. Hence, this is also called spatial multiplexing.

$$\underline{\tilde{n}} = U^H \underline{n}$$

$$E[\underline{\tilde{n}}\,\underline{\tilde{n}}^H] = E\{\underline{U}^H \underline{n}\,\underline{n}^H U\}$$
$$= \underline{U}^H E\{\underline{n}\,\underline{n}^H\} \underline{U}$$
$$= \sigma_n^2 \underline{U}^H \underline{U} = \sigma_n^2 I$$

$$\boxed{\sigma_{\tilde{n}}^2 = \sigma_n^2}$$ ⇒ Power of noise before beamforming is same as power of noise before beamforming.

The SNR of i^{th} parallel channel $= \dfrac{\sigma_i^2 P}{\sigma_n^2}$



Schematic diagram of t parallel channels so called spatial multiplexing

SNR of the i^{th} stream is $SNR_i = \dfrac{P_i \sigma_i^2}{\sigma_n^2}$

Maximum rate = Shannon capacity
$$= B \log_2(1 + SNR)$$

Capacity of the i^{th} parallel channel channel
$$= B \log_2\left(1 + \frac{P_i \sigma_i^2}{\sigma_n^2}\right)$$

Scanned by CamScanner

$C_1 = \log_2\left(1 + \frac{P_1 \sigma_1^2}{\sigma_n^2}\right) \leftarrow$ Capacity of 1^{st} channel.

$C_2 = \log_2\left(1 + \frac{P_2 \sigma_2^2}{\sigma_n^2}\right) \leftarrow$ Capacity of 2^{nd} channel.

Hence, net capacity is the sum of capacity of all parallel channels.

Hence, $\boxed{\text{Total MIMO capacity} = \sum_{i=1}^{t} \log_2\left(1 + \frac{P_i \sigma_i^2}{\sigma_n^2}\right)}$ Sum of individual capacities of each of the t information streams

So fare, we have said capacity of a parallel channel depends on power, however we have not said how to allocate power to different information streams. How to allocate power optimally to thae info. streams.

Given a transmit power P, how to optimally allocate P to all the transmit streams?

we know, $P_1 + P_2 + \dots + P_t \leq P$ to achieve maximum

optimally allocating is necessary capacity.

Optimal MIMO power Allocation

I want to maximise capacity

$$\max \sum_{i=1}^{t} \log_2\left(1 + \frac{P_i \sigma_i^2}{\sigma_n^2}\right)$$

subject to constraint that

$$\sum_{i=1}^{t} P_i = P_t$$

This is a constrained maximization problem.

For this kind of problem, we have to use Lagrange multipliers.

$$f = \sum_{i=1}^{t} \log_2\left(1 + \frac{P_i \sigma_i^2}{\sigma_n^2}\right) + \lambda\left(P - \sum P_i\right)$$

Hence, the constrained maximization problem can be represented as

$$f = \sum_{i=1}^{t} \log_2\left(1 + \frac{P_i \sigma_i^2}{\sigma_n^2}\right) + \underbrace{\lambda}_{\text{Lagrange multiplier}}\left(P - \sum P_i\right)$$

For maximization. $\frac{df}{dP_i} = 0$ for e.g. $\frac{df}{dP_1} = 0$

$$\Rightarrow \frac{\sigma_1^2/\sigma_n^2}{1 + \frac{P_1 \sigma_1^2}{\sigma_n^2}} + \lambda(-1) = 0$$

$$\Rightarrow \frac{\sigma_1^2/\sigma_n^2}{1 + \frac{P_1 \sigma_1^2}{\sigma_n^2}} = \lambda$$

$$\Rightarrow \frac{\sigma_1^2}{\sigma_n^2} \cdot \frac{1}{\lambda} = 1 + \frac{P_1 \sigma_1^2}{\sigma_n^2}$$

$$\Rightarrow \frac{1}{\lambda} = \frac{\sigma_n^2}{\sigma_1^2} + P_1$$

$$\Rightarrow P_1 = \left(\frac{1}{\lambda} - \frac{\sigma_n^2}{\sigma_1^2}\right)^+$$

Similarly, $P_2 = \left(\frac{1}{\lambda} - \frac{\sigma_n^2}{\sigma_2^2}\right)^+$

$\vdots$

$$P_t = \left(\frac{1}{\lambda} - \frac{\sigma_n^2}{\sigma_t^2}\right)^+$$

Scanned by CamScanner

$$x^+ = \begin{cases} x & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

$$\sum_{i=1}^{t} P_i = P$$

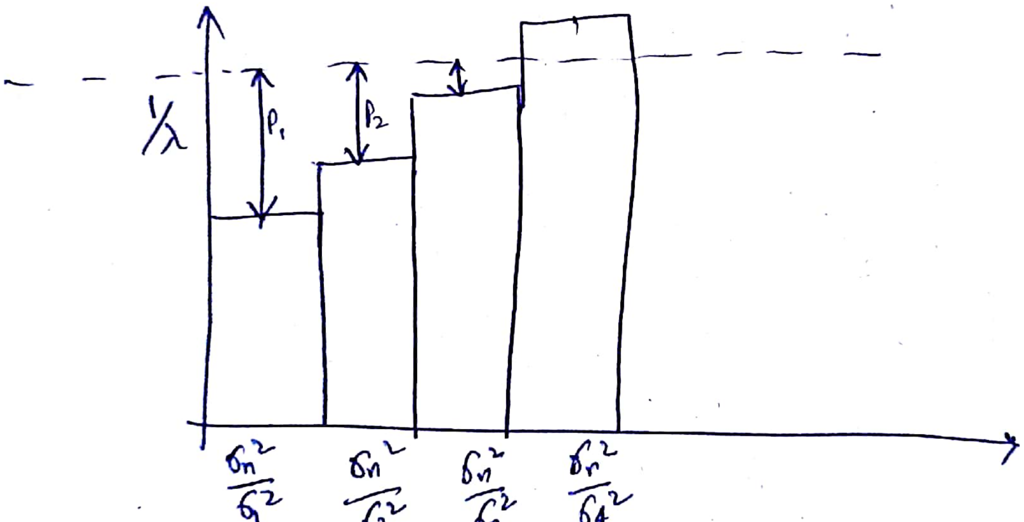
$$\Rightarrow \sum_{i=1}^{t} \left(\frac{1}{\lambda} - \frac{\sigma_n^2}{\sigma_i^2} \right)^+ = P$$

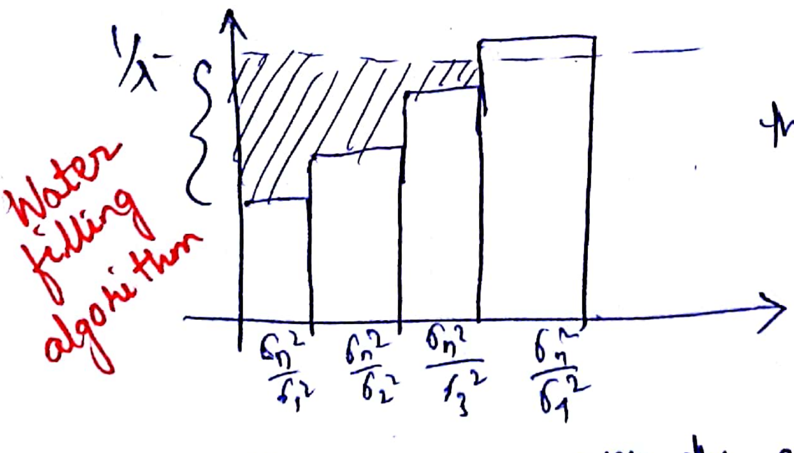
* Let us look some more into this power constraint.

$$P_i = \frac{1}{\lambda} - \frac{\sigma_n^2}{\sigma_i^2}$$

Since, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_t$, therefore

$$\frac{\sigma_n^2}{\sigma_1^2} \leq \frac{\sigma_n^2}{\sigma_2^2} \cdots\cdots$$





We can consider this as there are bars and water is filled in the bars

The + sign makes the system non-linear.

Assume $N = t$ channels have non-zero or positive power. We start with assumption that.

$$\frac{1}{\lambda} \geq \frac{\sigma_n^2}{\sigma_i^2} \quad i = 1, 2, 3, \ldots, N$$

$$\sum_{i=1}^{t} \left(\frac{1}{\lambda} - \frac{\sigma_n^2}{\sigma_i^2} \right) = P$$

$$P_N = \frac{1}{\lambda} - \frac{\sigma_n^2}{\sigma_N^2}$$

$P_N < 0$ $\quad$ $P_N > 0$

$P_N = 0$ $\quad$ Done optimal power allocation.

Substitute $N = t-1$

repeat

Example of MIMO power allocation

$$H = \begin{bmatrix} 2 & -6 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}_{3\times 3}$$

No. of txmit antennas = number of receive antennas $= 3$

Compute maximum capacity or optimal power allocation at total power $= -1.25\,dB$; $\sigma_n^2 = 3\,dB$.

$$\sigma_n^2 = 3\,dB = 2$$

$$P = -1.25\,dB = 0.75$$

$$c_1^T c_2 = [2 \ \ 3 \ \ 0] \begin{bmatrix} -6 \\ 4 \\ 0 \end{bmatrix} = 0$$

$$H = \begin{bmatrix} 2 & -6 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{13} & -6/\sqrt{52} & 0 \\ 3/\sqrt{13} & 4/\sqrt{52} & 0 \\ 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} \sqrt{13} & 0 & 0 \\ 0 & \sqrt{52} & 0 \\ 0 & 0 & 2 \end{bmatrix}}_{\text{These are not ordered}}$$

$$= \begin{bmatrix} -6/\sqrt{52} & 2/\sqrt{13} & 0 \\ 4/\sqrt{52} & 3/\sqrt{13} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{52} & 0 & 0 \\ 0 & \sqrt{13} & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$U \quad \Sigma \quad V^H$

$\sigma_1 = \sqrt{52} \Rightarrow \sigma_1^2 = 52$

$\sigma_2 = \sqrt{13} \Rightarrow \sigma_2^2 = 13$

$\sigma_3 = 2 \Rightarrow \sigma_3^2 = 4$

3 non-zero singular values, hence rank of channel matrix is 3.

SVD Based MIMO optimal transmission and capacity — Lec-24

$$C = \log_2\left(1 + \frac{P_1 \sigma_1^2}{\sigma_n^2}\right) + \log_2\left(1 + \frac{P_2 \sigma_2^2}{\sigma_n^2}\right) + \log_2\left(1 + \frac{P_3 \sigma_3^2}{\sigma_n^2}\right)$$

$$= \log_2\left(1 + \frac{P_1 \times 52}{2}\right) + \log_2\left(1 + \frac{P_2 \times 13}{2}\right) + \log_2\left(1 + \frac{P_3 \times 4}{2}\right)$$

$P_1 + P_2 + P_3 \leq 0.75$

$N = t = 3$

$$\left(\frac{1}{\lambda} - \frac{1}{26}\right) + \left(\frac{1}{\lambda} - \frac{2}{13}\right) + \left(\frac{1}{\lambda} - \frac{1}{2}\right) = 0.75$$

$$\Rightarrow \frac{1}{\lambda} = \frac{0.75 + \frac{1}{26} + \frac{2}{13} + \frac{1}{2}}{0.48} = 0.48$$

$P_3 = 0.48 - 0.5 = -0.02 < 0$

$\therefore P_3 = 0 \quad N = t - 1 = 3 - 1 = 2$

Bzoo

$$\left(\frac{1}{\lambda} - \frac{1}{26}\right) + \left(\frac{1}{\lambda} - \frac{2}{13}\right) = 0.75$$

$$\Rightarrow \frac{1}{\lambda} = \frac{0.75 + \frac{1}{26} + \frac{2}{13}}{2}$$

$$\Rightarrow 1/\lambda = 0.4712$$

$P_1 = 0.4712 - \frac{1}{26} = 0.4327$ which is > 0.

$P_2 = 0.4712 - \frac{2}{13} = 0.3174 > 0$.

$P_3 = 0$

So, optimal powers are $P_1 = 0.4327 = -3.63$ dB $\quad P_2 = 0.3174 = -4.98$ dB

$P_3 = 0$

The maximum capacity corresponding to this is

$$C_{max} = \log_2\left(1 + \frac{52 \times 0.4327}{2}\right) + \log_2\left(1 + \frac{13 \times 0.3174}{2}\right)$$

$$+ 0$$

$= 5.23$ bits per sec per Hz

$SVD = U \Sigma V^H$

$$V^H = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} v_1^H \\ v_2^H \\ v_3^H \end{matrix}$$

The actual transmit vector $\underline{x}$ is given as.

$$\underline{x} = \begin{array}{c} \left[\, v_1 \mid v_2 \mid v_3 \,\right] \\ \uparrow \quad \uparrow \quad \uparrow \\ \text{col1} \quad \text{col2} \quad \text{col3} \end{array} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix} = v_1 \tilde{x}_1 + v_2 \tilde{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tilde{x}_1 + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tilde{x}_2$$

Now, $\tilde{x}_1 = \sqrt{P_1}\, b_1$ and $\tilde{x}_2 = \sqrt{P_2}\, b_2$

$$\Rightarrow \underline{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} 0.66\, b_1 + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} 0.56\, b_2$$

b_1 and b_2 are unit power constellations symbols ;

$\underline{x}$ is the actual transmit vector to maximise capacity.

Asymptotic Capacity of a MIMO system :-

$$C = \log_2 \left| I + \frac{1}{\sigma_n^2} H R_x H^H \right|$$

H = channel matrix

R_x = is the transmit covariance, i.e. $E\{\underline{x}\, \underline{x}^H\}$

If power is distributed uniformly, then transmit covariance

$$R_x = \frac{P_t}{t} I$$

$P_t \rightarrow$ Total transmit power.

$$C = \log_2 \left| I + \frac{P_t}{t\sigma_n^2} H H^H \right|$$

If $t >> r$, $HH^H = \begin{bmatrix} - & \underline{h}_1^H & - \\ - & \underline{h}_2^H & - \\ & \vdots & \\ - & \underline{h}_t^H & - \end{bmatrix} \begin{bmatrix} | & | & & | \\ \underline{h}_1 & \underline{h}_2 & \cdots & \underline{h}_r \\ | & | & & | \end{bmatrix}$

$$= \begin{bmatrix} \underline{h}_1^H \underline{h}_1 & & & \\ & \underline{h}_2^H \underline{h}_2 & & \\ & & \ddots & \\ & & & \underline{h}_t^H \underline{h}_r \end{bmatrix}$$

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The diagonal terms $\underline{h}_i^H \underline{h}_i = \|h_i\|^2 \longrightarrow$

$\|h_i\|^2 \longrightarrow t$ as $t \to \infty$

$h_i^H h_j = 0$ when $i \neq j$

$$\therefore HH^H = \begin{bmatrix} t & & 0 \\ & t & \\ 0 & & \ddots & \\ & & & t \end{bmatrix} = t[I]_{r\times r}$$

Asymptotic capacity is $C_{asymptotic} = \log_2 \left| I + \frac{1}{\sigma_n^2} \frac{P_t}{t} \cdot tI \right|$

$$= \log_2 \left| I + \frac{P_t}{\sigma_n^2} I \right|$$

$$= r \log_2 \left(1 + \frac{P_t}{\sigma_n^2}\right)$$

because $\boxed{I_{r\times r} + \frac{P_t}{\sigma_n^2} I_{r\times r} = \left(1 + \frac{P_t}{\sigma_n^2}\right)^r}$

Hence $\boxed{C_{asymp} = r \log_2 \left(1 + \frac{P_t}{\sigma_n^2}\right)}$ Total transmit power.

Asymptotically, capacity increases linearly w.r.t. r given a constant transmit power P_t.

Hence,

$$\boxed{C_{asymp} = \underbrace{\min(r,t)}_{\text{MIMO increase in}} \underbrace{\log_2 \left(1 + \frac{P_t}{\sigma_n^2}\right)}_{\text{SISO capacity}}}$$

SPACE TIME CODING

Alamouti Code :-

Lets consider a 1×2 system. Single receive antenna and multiple transmit antenna.

$$H = [h_1 \quad h_2]$$

1X2 System Model

$$y = [h_1 \quad h_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + n$$

Symbol txmitted from tx ant 2.

Symbol transmitted from tx ant 1.

Consider a symbol x I transmit.

$$x_1 = \frac{h_1^*}{\|h\|} x \qquad \text{from ant 1.}$$

$$x_2 = \frac{h_2^*}{\|h\|} x \qquad \text{from ant 2.}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} h_1^*/\|h\| \\ h_2^*/\|h\| \end{bmatrix} x$$

Transmit beamforming

$$y = [h_1 \quad h_2] \begin{bmatrix} h_1^*/\|h\| \\ h_2^*/\|h\| \end{bmatrix} x + n$$

$$h_1 h_1^* + h_2 h_2^* = \|h_1\|^2 + \|h_2\|^2 = \|h\|^2$$

$$= \|h\| x + n$$

$$\boxed{SNR = \frac{\|h\|^2 P}{\sigma_n^2}}$$

which is exactly same as MRC. This means if we can do conversion of x as given above, then it doesnot matter whether how many receivers or transmitters we have, we can get the same performance. Diversity order is 2 here.

Hence, Tx diversity is similar to Rx diversity.

Is Tx div exactly same as Rx diversity?

Lets examine txmit vector = $\begin{bmatrix} \frac{h_1^*}{\|h\|} \\ \frac{h_2^*}{\|h\|} \end{bmatrix} x$

Can the transmit vector operation be done at the transmitter?

For that we need to know h_1 & h_2. This is known as channel state information (CSI). h_1 & h_2 are channel coefficients.

Transmit beam-forming is only possible when Channel State Information is available at the transmitter.

Hence, obtaining Tx diversity is challenging compared to obtaining Rx diversity.

Alamouti Code:-

1. It is a $\boxed{\text{space-time code}}$ proposed for 1x2 wireless systems
2. It achieves a diversity order of 2 $\boxed{\text{without CSI}}$ at the transmitter.

Lets consider a 1x2 system

1 Rx antenna

2 Tx antennas

Consider 2 symbols $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

At 1st instant, we transmit $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ ← From Tx Ant 1. ← From Tx ant 2.

$$y(1) = \begin{bmatrix} h_1 & h_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + n(1)$$ (For the 1st transmit instant)

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For the second transmit instant.

Transmit vector = $\begin{bmatrix} -x_2^* \\ x_1^* \end{bmatrix}$ ← From 1st transmit antenna
← From 2nd transmit antenna.

$$y(2) = \begin{bmatrix} h_1 & h_2 \end{bmatrix} \begin{bmatrix} -x_2^* \\ x_1^* \end{bmatrix} + n(2)$$

Now, we take $y^*(2)$

$$y^*(2) = \begin{bmatrix} h_1^* & h_2^* \end{bmatrix} \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} + n^*(2)$$

$$= \begin{bmatrix} -h_1^* & h_2^* \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} + n^*(2)$$

$$\Rightarrow y^*(2) = \begin{bmatrix} h_2^* & -h_1^* \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + n^*(2)$$

Stack $y(1)$ and $y^*(2)$ as below

$$\begin{bmatrix} y(1) \\ y^*(2) \end{bmatrix} = \underbrace{\begin{bmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{bmatrix}}_{} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underline{n}$$

Effectively we converted a 1x2 system into a 2x2 system.

Col 1 and col 2 of the H matrix are orthogonal.

because $c_1^H c_2 = h_1^* h_2 + h_2(-h_1^*)$

$$= 0$$

There is an interesting thing here because a 1x2 system is converted into a 2x2 system whose column vectors are orthogonal.

Every column can be used as receive beamformer after normalizing it, i.e., $\frac{c_1}{\|c_1\|}$ can be used as receive beamformer

$$w_1 = \begin{bmatrix} \frac{h_1}{\|h\|} \\ \frac{h_2^*}{\|h\|} \end{bmatrix}$$

I will employ this w_1 as the receive beamformer

$$\underline{w_1}^H \underline{y} = \begin{bmatrix} \frac{h_1}{\|h\|} \\ \frac{h_2^*}{\|h\|} \end{bmatrix}^H \begin{bmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\underline{w_1}^H \underline{n}}_{\text{denote by } \tilde{n}_1}$$

$$= \begin{bmatrix} \frac{h_1^*}{\|h\|} & \frac{h_2}{\|h\|} \end{bmatrix} \begin{bmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \tilde{n}_1$$

$$= \begin{bmatrix} \|h\| & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \tilde{n}_1$$

$$= \|h\| x_1 + \tilde{n}_1$$

Thus $w_1 = \frac{1}{\|h\|} \begin{bmatrix} h_1 \\ h_2^* \end{bmatrix}$

Lec-25 OSTBCs and Introduction to V-BLAST Receiver

Now, SNR at receiver is

$$= \frac{\|h\|^2 P_1}{\sigma_n^2}$$

Since, there is $\|h\|^2$ in the denominator hence diversity order is 2. $\|h\| = \sqrt{|h_1|^2 + |h_2|^2}$

Since, the $\underline{w_1}^H \underline{y} = \|h\| x_1 + \tilde{n}_1$ is independent of x_2, so, we can decode x_1 using this.

Similarly, to decode x_2, the beamformer $\underline{w_2}$ is given as

$$\underline{w_2} = \frac{\underline{c_2}}{\|\underline{c_2}\|} = \frac{1}{\|\underline{h}\|}\begin{bmatrix} h_2 \\ -h_1^* \end{bmatrix}$$

SNR of Alamouti $= \dfrac{\|\underline{h}\|^2 P_1}{\sigma_n^2}$

where P_1 = power allocated to x_1.

Total transmit power $= P_t$ is fixed!

Remember, our txmit vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Hence transmit power has to be split between symbols x_1 & x_2.

$$P_1 = P_2 = \frac{P}{2}$$

$\therefore$ SNR of Alamouti $= \dfrac{P}{2}\dfrac{\|\underline{h}\|^2}{\sigma_n^2}$

$$= \frac{1}{2}\,\frac{\|\underline{h}\|^2 P}{\sigma_n^2}$$

This value of $\frac{1}{2}$ results in 3 dB loss of SNR as compared to MRC.

Columns $\underline{c}_1$ and $\underline{c}_2$ are orthogonal. (by design of Alamouti code)

Alamouti code belongs to a special class of codes termed as $\underline{\text{Orthogonal Space Time Block Codes}}$.

Orthogonal → $\langle \underline{c}_1, \underline{c}_2 \rangle = 0$

Space Time Block →

$$\text{Space}\downarrow \begin{matrix} x_1 & -x_2^* \\ x_2 & +x_1^* \end{matrix} \quad \xrightarrow{\text{time}}$$

→ This is a block of symbols.

Alamouti codes transmits a set of 2 symbols in 2 time instants. Hence, the effectively transmission is 1 symbol per time instant.

Hence, it is termed as a rate-1 code.

This is also called as a Full-rate code

* Alamouti is therefore as full-rate OSTBC.

Example :- 1×2 system. $\longrightarrow$ 1 rx ant, 2 txmit ant

$$y = \begin{bmatrix} 1+j & 3+4j \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$y(1) = \begin{bmatrix} 1+j & 3+4j \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (1^{st} \text{ time instant})$$

$$y(2) \begin{bmatrix} 1+j & 3+4j \end{bmatrix} \begin{bmatrix} -x_2^* \\ x_1^* \end{bmatrix} \quad (2^{nd} \text{ time instant})$$

At receiver,

$$y^*(2) = \begin{bmatrix} 1-j & 3-4j \end{bmatrix} \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$$

$$y^*(2) = \begin{bmatrix} -1+j & 3-4j \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$$

$$= \begin{bmatrix} 3-4j & -1+j \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} y(1) \\ y^*(2) \end{bmatrix} = \begin{bmatrix} 1+j & 3+4j \\ 3-4j & -1+j \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 1+j \\ 3-4j \end{bmatrix} \qquad C_2 = \begin{bmatrix} 3+4j \\ -1+j \end{bmatrix}$$

Consider $C_1^H C_2 = (1+j)^* (3+4j) + (3-4j)^* (-1+j)$

$= 0$

Hence, the beamformer at receiver to detect x_1 is given as $w_1 = \frac{C_1}{\|C_1\|} = \frac{1}{\sqrt{27}} \begin{bmatrix} 1+j \\ 3-4j \end{bmatrix}$

For detection of x_1, we perform

$\underline{W}_1^H \underline{y}$

$$W_2 = \frac{C_2}{\|C_2\|} = \frac{1}{\sqrt{27}}\begin{bmatrix} 3+4j \\ -1+j \end{bmatrix}$$

To detect x_2, we perform $\underline{W}_2^H \underline{y}$

Example of Another OSTBC :- 1X3 Mimo system.

for 1 Rx ant and 3 Tx Ant.

The channel coefficients are $[h_1 \ h_2 \ h_3]$. Symbols are x_1, x_2, x_3, x_4

The STBC structure

$$\text{Space} \downarrow \begin{bmatrix} x_1 & -x_2 & -x_3 & -x_4 & x_1^* & -x_2^* & -x_3^* & -x_4^* \\ x_2 & x_1 & x_4 & -x_3 & x_2^* & x_1^* & x_4^* & -x_3^* \\ x_3 & -x_4 & x_1 & x_2 & x_3^* & -x_4^* & x_1^* & x_2^* \end{bmatrix}$$

$\xrightarrow{\text{time}}$ 8 time instants

Observe 4 symbols over 8 time instants. Hence

net transmission rate $= \frac{4}{8} = \frac{1}{2}$

$$y(1) = [h_1 \ h_2 \ h_3]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= h_1 x_1 + h_2 x_2 + h_3 x_3 + 0 x_4$$

$$= [h_1 \ h_2 \ h_3 \ 0]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

$$y(2) = [h_1 \ h_2 \ h_3]\begin{bmatrix} -x_2 \\ x_1 \\ -x_4 \end{bmatrix}$$

$$= -h_1 x_2 + h_2 x_1 + h_3 x_4 + 0 x_3$$

$$= h_2 x_1 - h_1 x_2 + 0 x_3 + h_3 x_4$$

$$= [h_2 \ -h_1 \ 0 \ -h_3]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

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$$\begin{bmatrix} y(1) \\ y(2) \\ y(3) \\ y(4) \\ y^*(5) \\ y^*(6) \\ y^*(7) \\ y^*(8) \end{bmatrix} = \begin{bmatrix} h_1 & h_2 & h_3 & 0 \\ h_2 & -h_1 & 0 & -h_3 \\ h_3 & 0 & -h_1 & h_2 \\ 0 & h_3 & -h_2 & -h_1 \\ h_1^* & h_2^* & h_3^* & 0 \\ h_2^* & -h_1^* & 0 & -h_3^* \\ h_3^* & 0 & -h_1^* & h_2^* \\ 0 & h_3^* & -h_2^* & -h_1^* \end{bmatrix}_{8\times 4} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}_{4\times 1}$$

$$C_1^H C_2 = [h_1^* \;\; h_2^* \;\; h_3^* \;\; 0 \;\; h_1 \;\; h_2 \;\; h_3 \;\; 0] \begin{bmatrix} h_2 \\ -h_1 \\ 0 \\ h_3 \\ h_2^* \\ -h_1^* \\ 0 \\ h_3^* \end{bmatrix}$$

$$= h_1^* h_2 - h_2^* h_1 + 0 + 0 + h_1 h_2^* - h_2 h_1^* + 0 + 0$$

$$= 0$$

Hence, columns are orthogonal. Therefore this is OSTBC.

The rate of this OSTBC is $\frac{1}{2}$.

Non-Linear MIMO Receiver

V-BLAST Technique

$\boxed{\text{Vertical - Bell Labs Layered Space Time Architecture}}$

This is a non-linear receiver which employs successive interference cancellation (SIC).

Here, the impact of each estimated symbol is cancelled. Further, because it employs SIC, it is non-linear in nature.

$\underline{y} = H\underline{x} + \underline{n}$

H is $r \times t$ channel matrix, $r \geq t$

$$= \begin{bmatrix} \underline{h}_1 & \underline{h}_2 & \cdots & \underline{h}_t \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_t \end{bmatrix} + \underline{n}$$

$$\Rightarrow \underline{y} = \underline{h}_1 x_1 + \underline{h}_2 x_2 + \cdots + \underline{h}_t x_t + \underline{n}$$

Consider the pseudo-inverse or the left-inverse of H.
Let this matrix is given as Q.

$$Q = H^{\dagger}$$

By definition, $Q = \begin{bmatrix} \underline{q}_1^H \\ \underline{q}_2^H \\ \vdots \\ \underline{q}_t^H \end{bmatrix}$

We know $QH = I$

$$\Rightarrow \begin{bmatrix} \underline{q}_1^H \\ \underline{q}_2^H \\ \vdots \\ \underline{q}_t^H \end{bmatrix} \begin{bmatrix} \underline{h}_1 & \underline{h}_2 & \cdots & \underline{h}_t \end{bmatrix} = I$$

$$\Rightarrow \begin{bmatrix} \underline{q}_1^H \underline{h}_1 & \underline{q}_1^H \underline{h}_2 & \cdots \\ \underline{q}_2^H \underline{h}_1 & \underline{q}_2^H \underline{h}_2 & \cdots \\ \vdots & \ddots & \\ \vdots & & \ddots \end{bmatrix} = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \end{bmatrix}$$

$$\Rightarrow \underline{q}_1^H \underline{h}_1 = \underline{q}_2^H \underline{h}_2 = \cdots = 1$$

$$\underline{q}_1^H \underline{h}_2 = \underline{q}_2^H \underline{h}_1 = \cdots = 0$$

$$\underline{q}_i^H \underline{h}_j = \begin{cases} 1, & \text{if } i=j \\ 0, & i \neq j \end{cases}$$

~~Reas~~ : We can use this advantage for detection scheme $\underline{q}_1^H$ is orthogonal to $h_2, h_3 \cdots h_t$, so I can use this to cancel interference from $x_2, x_3, \ldots, x_t$.

$$\underline{y} = \underline{h}_1 x_1 + \underline{h}_2 x_2 + \cdots + \underline{h}_t \underline{x}_t + \underline{n}$$

$$\tilde{\underline{y}}_1 = \underline{q}_1^H \underline{y}$$

$$= \underline{q}_1^H (\underline{h}_1 x_1 + \underline{h}_2 x_2 + \cdots + \underline{h}_t x_t) + \underbrace{\underline{q}_1^H \underline{n}}_{\tilde{n}}$$

$$= x_1 + 0 + \cdots + 0 + \tilde{n}$$

$$\begin{cases} \underline{q}_1^H \underline{h}_1 = 1 \\ \underline{q}_1^H h_2 = 0 \\ \vdots \end{cases}$$

$$\tilde{y}_1 = x_1 + \tilde{n}$$

This can now be employed to decode x_1.

Now, we remove the effect of x_1 from the received vector

$$\tilde{y}_2 = \underline{y} - \underline{h}_1 \underline{x}_1$$

$$= (\underline{h}_1 x_1 + \underline{h}_2 x_2 + \cdots + \underline{h}_t \underline{x}_t) - \underline{h}_1 \underline{x}_1 + \underline{n}$$

$$= \underline{h}_2 x_2 + \underline{h}_3 x_3 + \cdots + \underline{h}_t x_t + \underline{n}$$

$$\Rightarrow \hat{y}_2 = \underbrace{\begin{bmatrix} \underline{h}_2 & \underline{h}_3 & \cdots & \underline{h}_t \end{bmatrix}}_{t-1 \text{ columns}} \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ x_t \end{bmatrix} + \underline{n}$$

So, this is $r \times (t-1)$ matrix

By cancelling x_1, we reduced it to a $r \times (t-1)$ MIMO system.

$$\hat{y}_2 = H_1^1 \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_t \end{bmatrix} + \bar{n}$$

Consider $Q^1 = (H^1)^\dagger$

Now, we repeat the process by decoding x_2 and so on.

Advantage of this scheme is diversity order progressively increases as procedure proceeds through the scheme. At the end,

$$\hat{y}_t = \underline{h}_t x_t + n_t$$

This is effectively an $r \times 1$

This is r^{th} order diversity.

Streams that are decoded later experience progressively higher diversity

Example of VBLAST receiver

$$\underline{y} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underline{n}$$

$$H = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \quad Q = H^\dagger = \begin{array}{c} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \underline{q}_1^H \\ \underline{q}_2^H \end{array}$$

$$\underline{q}_1^H \underline{y} = \begin{bmatrix} 3 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underline{q}_1^H \underline{n}$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \tilde{n}_1$$

$\Rightarrow \tilde{y}_1 = x_1 + \tilde{n}_1$ From $\tilde{y}_1$, we can detect x_1

$$\hat{y}_2 = \underline{y} - \underline{h}_1 x_1$$

$$= \begin{bmatrix}1 & 2\\ 1 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} - \begin{bmatrix}1\\ 1\end{bmatrix}x_1 + \underline{n}$$

$$= \begin{bmatrix}2\\ 3\end{bmatrix}x_2 + \underline{n}$$

$$\Rightarrow \hat{y}_2 = \begin{bmatrix}2\\ 3\end{bmatrix}x_2 + \underline{n}$$

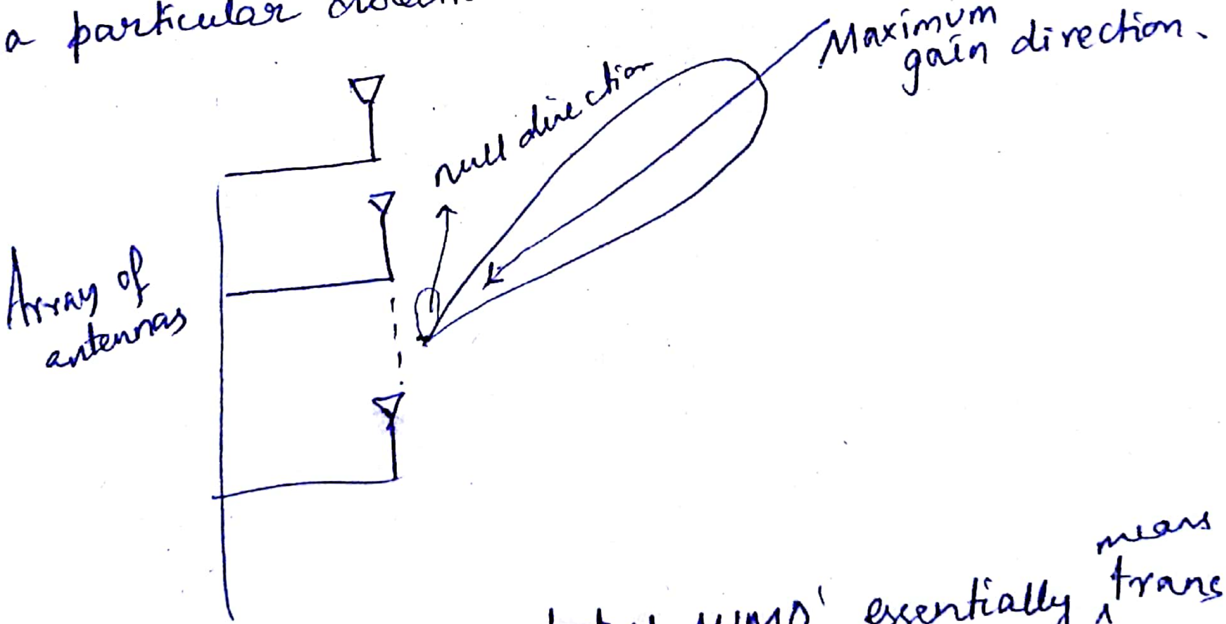
Now we can decode x_2 using $\hat{y}_2$

Infact, one can employ MRC since this is last symbol since this experiences a diversity order of 2.

→ Diversity order increases because div. order $= r - t + 1$
As t decreases in SIC, so, diversity order increases.

MIMO Beamforming :-

In traditional systems, beamforming refers to an antenna array at receiver to steer the beam in a particular direction.



Beamforming in the context of MIMO essentially means transmission in one spatial dimension.

$$\underline{y} = H\underline{x} + \underline{n}$$

$$= U \Sigma V^H \underline{x} + \underline{n}$$

$$= \begin{bmatrix} \underline{u}_1 & \underline{u}_2 & \cdots & \underline{u}_t \end{bmatrix} \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_t \end{bmatrix} \begin{bmatrix} v_1^H \\ v_2^H \\ \vdots \\ v_t^H \end{bmatrix} x + \underline{n}$$

Let my transmit vector $\underline{x}$ be having one symbol, so

$$\underline{x} = \underline{V_1}\, \tilde{x}_1$$

transmitting one symbol $\tilde{x}_1$

Dominant transmission mode of the MIMO channel.

$\underline{V_1}$ is an unit vector in n-dimensional space.

$\underline{V_1}$ is hence, the abstract direction in n-dimensional space along which $\underline{x}$ is being transmitted.

$$\underline{y} = \begin{bmatrix} \underline{u}_1 & \underline{u}_2 & \cdots & \underline{u}_t \end{bmatrix} \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_t \end{bmatrix} \begin{bmatrix} v_1^H \\ v_2^H \\ \vdots \\ v_t^H \end{bmatrix} V_1 \tilde{x}_1 + \underline{n}$$

$$= \begin{bmatrix} \underline{u}_1 & \underline{u}_2 & \cdots & \underline{u}_t \end{bmatrix} \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_t \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tilde{x}_1 + \underline{n}$$

$$= \begin{bmatrix} \underline{u}_1 & \underline{u}_2 & \cdots & \underline{u}_t \end{bmatrix} \begin{bmatrix} \sigma_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tilde{x}_1 + \underline{n}$$

$$= \underline{u}_1\, \sigma_1\, \tilde{x}_1 + \underline{n}$$

So, we transmitted along the direction $\underline{v}_1$ and we are receiving along the direction $\underline{u}_1$, which is the dominant receive direction.

Gain of channel is σ_1 because we transmit along dominant mode.

We perform MRC at receiver.

$\underline{y} = \sigma_1 \underline{u}_1 \tilde{x}_1 + \underline{n}$

$\underline{u}_1$ can be employed for MRC

$$\underline{u}_1^H \underline{y} = \underline{u}_1^H (\sigma_1 \underline{u}_1 \tilde{x}_1 + \underline{n})$$

$$= \sigma_1 \tilde{x}_1 + \underbrace{\underline{u}_1^H \underline{n}}_{\tilde{n}_1}$$

$$\Rightarrow \tilde{y}_1 = \sigma_1 \tilde{x}_1 + \tilde{n}_1.$$

$$SNR = \sigma_1^2 \frac{P}{\sigma_n^2}$$

σ_1 is the largest singular value

This is the gain associated with the dominant mode.

This effectively becomes a SISO channel with gain σ_1 & $\tilde{x}_1$ is transmitted.

This scheme of MIMO Beamforming is termed as Maximal Ratio Transmission (MRT)

Advantage of MRT is that it results in simplistic transmission and reception scheme for MIMO by effectively converting into SISO channel model.

This is simplistic as compared to MIMO-ZF, MIMO-MMSE and MIMO-VBLAST.