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THERMO-FLUID DYNAMICS OF SEPARATED

TWO-PHASE FLOW

A THESIS

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In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy in the

School of Mechanical Engineering

Georgia Institute of Technology

December, 1971

THERMO-FLUID DYNAMICS OF SEPARATED

TWO-PHASE FLOW

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To Gonul and Turkay

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NOMENCLATURE

A   area, and coefficient defined by Equation (11.51)

a   amplitude of interfacial waves

$\overline{\overline{a}}$ metric tensor of the surface

B   coefficient defined by Equation (11.52)

C   static mass concentration weighted by area, and comlex wave velocity in Part II

C_i a factor which determines the amplification of interfacial waves

$\overline{C}_i$ a factor related to C_i by Equation (11.54)

C_r propagation velocity of interfacial waves

$\overline{C}_r$ related to C_r by Equation (11.53)

$\text{Cov}(\psi, \varphi)$ covariance of any two quantity, ψ and φ

D_e pipe diameter

D_i central core diameter in annular and/or jet flow

D   diffusion coefficient based on diffusion mass flux with respect to the baricenter of the mixture

D_v drift coefficient based on drift flux with respect to the center of volume of the mixture

Ec   Eckert number

$\hat{e}$ unit vector defined in cylindrical coordinates, $\hat{e}_x$, $\hat{e}_r$, $\hat{e}_\phi$

Fr   Froude number

G   mass flow rate per unit area of the total cross sectional plane

$\vec{g}$ gravitational acceleration

H   half distance between two plates

h   implicit surface function

i   specific enthalpy, and $i = \sqrt{-1}$ used in Part II

$\tilde{\imath}$	specific total enthalpy defined by Equation (7.42)
$\hat{\imath}$	unit vector in the x-direction
$\vec{j}_K$	volumetric flux of the K-th phase weighed by the total cross sectional area
$\vec{j}$	volumetric flux of the mixture weighed by the total cross sectional area
$\hat{\jmath}$	unit vector in the y-direction
$\hat{k}$	unit vector in the z-direction
k	wave number
k_K	conduction coefficient of the K-th phase, $K = 1,2$
L	length
L_b	break-up length
$\dot{m}_{K\ell}$	mass flow rate per unit are of the surface, $K = 1,2$; $\ell = e,i$
$N_{cov(mom\ K)}$	flow regime group, $K = 1,2$
N_D	drift number
N_{pch}	phase change number
N_{sup}	supply number
N_ρ	density ratio group
$\hat{n}$	unit normal vector
P	pressure
Pe	Peclet number
Q	volumetric flow rate
$\vec{q}$	heat flux
$\vec{qs}$	entropy flux
q^+_{1e}	dimensionless heat flux number
Re	Reynolds number
r	radius in cylindrical coordinates

$\vec{r}$ — vector coordinate of a point in space

$\vec{r}_1, \vec{r}_2$ — surface tangential vectors to the curve $u_2 = $ const. and $u_1 = $ const., respectively

s — specific entropy

T — temperature

t — time

$\hat{t}$ — unit tangent vector

u — specific internal energy

$\bar{u}$ — specific total internal energy defined by Equation (7.37)

u_1, u_2 — a pair of surface coordinates identifying a surface point

$\vec{V}_{Km}$ — diffusion velocity of the K-th phase, (K = 1,2), with respect to the baricenter of the mixture

$\vec{V}_{Kj}$ — drift velocity of the K-th phase, (K = 1,2), with respect to the center of volume of the mixture

$\vec{V}_r$ — relative velocity between two phases

$\vec{v}$ — local velocity vector

$\vec{v}_i$ — interfacial surface velocity

$\vec{v}_m$ — mixture velocity as applied to the center of gravity of the mixture

W — mass flow rate

We — Weber number

X — coefficient defined by Equation (10.36)

x — spatial coordinate in the direction of motion

Y — coefficient defined by Equation (10.46)

y — spatial coordinate

z — spatial coordinate

α — area concentration weighed by area

α_{ν} — volumetric concentration

β — kinematic concentration weighed by volumetric flow rate

$\Delta\psi$ differences of ψ, $\Delta\psi = \psi_1 - \psi_2$

Δi_{12} heat of vaporization

$\overline{\overline{\delta}}$ Kronecker delta

λ wave length of interfacial wave

ξ perimeter, and property group used in Part II

η in plane flow thickness of phase 1

χ quality

$\varkappa$ mean curvature at a point

$\varkappa_1, \varkappa_2$ principal curvatures at a point on the interface

θ angle to flow direction

$\overline{\overline{\theta}}$ surface flux tensor inwardly directed through the interfacial boundary curve

μ dynamic viscosity

μ' dilatational or bulk viscosity

ν kinematic viscosity

$\overline{\overline{\pi}}$ stress tensor

$\overline{\overline{\pi}}_D$ diffusion stress tensor

$\overline{\overline{\pi}}^T$ total stress tensor defined by Equation (6.52)

$\overline{\overline{\tau}}$ viscous stress tensor

ρ density

ρ_β flowing mixture density

σ surface tension

$\vec{\phi}$ generalized flux of any quantity ψ, with respect to the baricenter of the fluid

ϕ angular direction in cylindrical coordinates

$\vec{\phi}$ generalized mixture flux with respect to the baricenter of the mixture

$\vec{\phi}_R$ reversible increase of thermal energy

$\dot{\phi}_{IR}$ irreversible increase of thermal energy

Ω reaction or phase change frequency

ψ generalized function

$\dot{\psi}$ generation rate of unit mass of ψ

φ generalized function

$\Gamma_{K\ell}$ mass formation of the K-th phase weighed by the total mixture volume, $K = 1,2$; $\ell = 3, i$

Subscripts

1   component or phase 1

2   component or phase 2

ad   adiabatic

cr   critical

di   diabatic

e   external, and/or fixed solid surface

i   internal, and/or interfacial surface

K   damivariable taking on the values 1 and 2

ℓ damivariable taking on the values e and i

m   mixture quantities and the most unstable wave in Part II

T   total

ξ identifies perimetrical quantity

TP   two phase

SP   single phase

x,y,z   identifying the directions

Superscripts

'   disturbed variables

−   undisturbed variables

+   dimensionless variables

Averages

< >   mass-weighed average

≪ ≫   average with respect to cross sectional area of the fluid

≪≪ ≫≫ average with respect to volume of the fluid

Covariances

Cov(mech 1)	covariance of mechanical energy defined by Equation (7.3a)
Cov(en 1)	covariance of energy defined by Equation (7.3b)
Cov(conv 1)	covariance of convective terms defined by Equation (7.4a)
Cov(int 1)	covariance of internal energy defined by Equation (7.4b)
Cov(work 1)	covariance of work terms defined by Equation (7.4c)
Cov(mom 1)	covariance of convective momentum term defined by Equation (6.3)

SUMMARY

The overall objectives of this dissertation are two-fold. First, to provide a general thermo-fluid dynamic formulation of separated two-phase flow, and second, to use this formulation for analyzing various dynamic aspects of the film flow regime. Consequently, the thesis is divided into two parts.

In Part I, the conservation equations which are expressed both in terms of the _two-fluid_ model and of the _diffusional_, i.e., drift, model are derived. These equations, derived from the space averaging procedure, take into account the effects of surface phenome   ich as surface tension and surface shear and of surface processes such as momentum, heat and mass transfer at the interface. Finally these conservation equations are used to establish the similarity groups appropriate to separated flows.

In Part II, the two-fluid model formulation of Part I, is used to develop a stability theory of separated plane flows. The analysis takes into account the effects of surface tension and liquid viscosity, of mass and heat transfer at the interface as well as the dynamic effects of the vapor on the liquid film. The most unstable growth factor, derived from the analysis, is used to calculate the onset of liquid entrainment and of flow plugging. The results show that evaporation has a destabilizing effect on liquid films whereas it stabilizes vapor films, i.e., it has a destabilizing effect on the Kelvin-Helmholtz instability

whereas it has a stabilizing effect on the Rayleigh-Taylor instability. A comparison of predicted results with available experimental data shows a satisfactory agreement.

PART I

THERMO-FLUID DYNAMIC FORMULATION

OF SEPARATED TWO-PHASE FLOW

CHAPTER I

INTRODUCTION

1.1 Significance of the Problem

The simultaneous flow of two or three phases and/or several components occurs in a large number of engineering systems as well as in many natural phenomena. Some examples are listed below.

a) Energetics: Power and Propulsion Systems and Components

Boiling water and pressurized water nuclear reactors, liquid metal fast breeder reactors, Rankine cycle liquid metal space power plants, MHD generators, liquid propellant rockets (cooling, heat exchanger and combustor), metalized solid propellant rockets, heat pipes, boilers and condensers for power stations, two-phase propulsors for under water and surface applications, drag-reduction devices, etc.

b) Process Systems and Components

Extraction and distillation units, spray and cooling towers, fluidized beds, evaporators, desalination systems, emulsifiers, air-solid separators, steam-water separators, cyclones, cryogenic heat exchangers and pumps, electronic cooling systems, ejectors, atomizers, dryers, absorbers, combustion devices and chemical reactors, etc.

c) Transport Systems and Devices

Pneumatic conveyors; air-lift pumps; cavitating pumps and hydrofoils; pumps and piping for transport of oil and gas mixtures, of slurries and/or of fibers; hydraulic conveyors of wheat, pulvarized coal, ores and other solids; highway traffic flow and control; etc.

d) Environmental Control Systems and Devices

Pollutant separators and purifiers, sewage treatment plants (flow, fermentation and settling), air pollution control devices, refrigerators, coolers, dust collectors, life support equipment for space application, etc.

e) Information Systems and Devices

Superfluidity of liquid helium, conducting and/or charged liquid films, liquid crystals, etc.

f) Biological Systems and Devices

Flow of blood; distribution of fish eggs by current; transport, chemical reaction and diffusion through capillary networks, etc.

g) Geo-Meteorological Phenomena

Sedimentation, i.e., transport of river sediments; soil erosion and transport by wind, sea and rivers; snow drifts; sand dune formation; nucleation and motion of rain drops inside clouds; icing phenomena in the atmosphere; etc.

It can be concluded from the foregoing that the simultaneous flow of two phases or of two or more immiscible liquids characterizes the operation of many important engineering systems of interest to various branches of technology and science.

1.2 Topographical, Structural Classifications

At first glance it might appear that the various systems, components and phenomena listed above have very little in common. Actually, the contrary is true. If we recall that the singular characteristic of two phase or of two immiscible mixtures is the presence of one or several

interfaces, between the phases or components, it can be noticed that many of the systems listed above have a common structure, i.e., a common topography of the interface. Furthermore, whereas single phase flows can be classified according to the geometry of the flow in laminar, transitional and turbulent flow, the flow of two phases or of a mixture of immiscible liquids can be classified according to the geometry of the interface into three classes, i.e., in separated flows, transitional or mixed flows and dispersed flows. These three classes of structured flows are shown in Figure 1.

Depending upon the type of the interface, the class of stratified flows can be divided into plane flows and quasi-axisymmetric flows each of which can be subdivided into two regimes. Thus, the plane flow includes film and stratified flows, whereas the quasi-axisymmetric flow consists of the annular and the jet flow regimes. The various arrangements, i.e., configurations of the two phases and of the immiscible liquids, are shown in Figure 1.

The class of dispersed flows can also be divided into several types. Thus, depending upon the geometry of the interface, one can consider spherical, elliptical, granular particles, etc. However, it is more convenient to subdivide the class of dispersed flows by considering the phase of the dispersion. Accordingly, we can distinguish three regimes: bubbly, droplet or mist and particulate solid flow. In each regime the geometry of the dispersion can be spherical, spheroidal, elliptical etc. The various configurations between the phases and mixture components are shown in Figure 1.

Finally, the third class is characterized by the presence of both

	Phase		Single Phase	Two Phase		Geometry
	Component		Two or Multicom.	Single Component	Two or Multicom.	
Class	Type	Regime	Configuration	Configuration	Configuration	
Separated Flows	Plane Flow	Film flow	Liquid A & B	a) Liquid film in vapor b) Vapor film in liquid (boiling)	a) Liquid film in gas b) Gas film in liquid	
		Stratified flow (layered)	Liquid A,B,C etc.	—	Liquid A,C & gas B, etc.	
		Annular flow	Core: liquid A Annulus: liquid B	a) Core: liquid annulus:vapor b) Core: vapor annulus:liquid	a) Core: liquid annulus: gas b) Core: gas annulus:liquid	
		Jet flow	Liquid A jet in liquid B	a) Liquid jet in vapor b) Vapor jet in liquid	a) Liquid jet in gas b) Gas jet in liquid	
Mixed or Transitional Flows	Quasi-axisymmetric Flow	Slug flow	Slug: liquid A Continum:liquid B	Slug: vapor Continum: liquid	Slug: gas Continum: liquid	
		Bubbly annular flow		Core: vapor Annulus: liquid with vapor bubbles	Core: gas Annulus: liquid with gas bubbles	
		Droplet annular flow	Core: liquid A Annulus:liquid B Droplets:liquid B	Core: vapor with liquid droplets Annulus: liquid	Core: gas with liquid droplets Annulus: liquid	
		Bubbly droplet annular flow		Core: vapor with liquid droplets Annulus: liquid with vapor bubbles	Core: gas with liquid droplets Annulus: liquid with vapor bubbles	
		Film flow with entrainment	Liquid A film in liquid B with droplets A, etc.	Liquid film in vapor with droplets, etc.	Liquid film in gas with droplets, etc.	
Dispersed Flows	Spherical Spheroidal Granular, Etc. Dispersion	Bubbly flow		Bubbles: vapor Continum: liquid	Bubbles: gas Continum: liquid	
		Droplet flow	Droplets:liquid A Continum:liquid B	Droplets: liquid Continum: vapor	Droplets: liquid Continum: gas	
		Particulate flow		a)Particles:solid continum: liquid b)Particles:solid continum: vapor (ice formation)	a)Particles: solid continum:liquid b)Particles: solid Continum: gas	

Figure 1. Classification of Structured Flows

separated and dispersed flows, whence it can be named the class of mixed flows. However, it could be referred to also as the class of transitional flows since in practice it occurs as a transition from dispersed to separated flows and vice versa. For example, in a boiling liquid the slug flow regime occurs as a transition from bubbly to annular flow. Here too, it is more convenient to subdivide the class of mixed flows according to the phase of dispersion. Consequently, we can distinguish five regimes, i.e., slug flow, bubbly-annular flow, droplet-annular flow, bubbly annular-droplet core flow and film flow with entrainment. The various configurations between the phases and mixture components are shown in Figure 1.

The various systems, components, and phenomena listed in the previous section were grouped according to applications and technologies. However, in view of the topographical, i.e., structural classification shown in Figure 1, these systems and components could be grouped according to the type of flow, i.e., flow regime that characterizes their operation. Consequently, they can be grouped as follows.

A. <u>Separated, Two-Phase (or Component) Systems</u>

a) Gas (or Vapor) Core-Liquid Film, Plane and Axisymmetric Flow

Liquid metal boilers and condensers, evaporators, distillation units, desalination apparatus, charged and conducting liquid films, conventional boilers and condensers, boiling water nuclear reactors, liquid propellant rockets (film cooling of nozzles), heat pipes, etc.

b) Liquid Jet-Gas (or Vapor) Annulus Flow Regime

Cryogenic heat exchangers, ejectors, atomizers, etc.

B. Mixed Flows

Most of the power and propulsion systems and components such as: boiling water nuclear reactors, Rankine cycle liquid metal space power plants, MHD generators, two-phase propulsors, distillation systems, steam-water separators, evaporators, condensers, etc.

C. Dispersed Flows

a) Gas-Solid Particles System

Cyclones, separators, combustors, heterogeneous reactors, pneu-matic conveyors, solid propellant rockets, dust collectors, fluidized beds, air pollution and fallout separation and control devices, soil erosion and transport by wind, sand dune formation, icing phenomena in the atmosphere, snow drifts, etc.

b) Gas-Liquid Droplet System

MHD generators, liquid propellant rocket combustors, wet steam turbines, two-phase propulsors, wet steam separators, atomizers, dryers, absorbers, gas coolers, nucleation and motion of rain drops, cryogenic heat exchangers, etc.

c) Liquid-Liquid Droplet System

Emulsifiers, homogenizers and extraction units, flow of blood, polymer flow, etc.

d) Liquid-Solid Particle System

Fluidized beds, hydraulic conveyors, erosion and sediment transport by rivers and sea, sedimentation, sewage plants, separators, etc.

e) Liquid-Gas (or Vapor) Bubbly System

Boiling water and pressurized water nuclear reactors, liquid metal fast breeder reactors, Rankine cycle liquid metal boilers,

conventional boilers and evaporators, absorbers, air lift pumps, cavi-
tating pumps, refrigerators; distillation, flotation and aeration units;
pumps and piping for transport of oil and gas mixtures, electronic cool-
ing system ejectors, etc.

In view of this topographical, i.e., structural classification in
separated, mixed and dispersed flows, it could be expected that many of
the systems enumerated above should exhibit a large number of steady
state and dynamic similarities. This indeed is the case. For example,
it was shown in references [1-7]* that the theory of kinematic waves,
which was developed by Lighthill and Whitham [8] to analyze the flow of
cars on roads, can be extended to analyze and predict transient response
of dispersed two-phase systems with applications to fluidized beds,
boiling water nuclear reactors, and cryogenic heat exchangers.

It is evident that if a firm understanding is attained of the
thermo-fluid dynamic characteristics of any of the above two-phase flow
regime, then these results could be applied to predict the operational
performance of systems and components in a variety of technologies.
Indeed, such a general method of analysis has been attained and used in
the field of single phase flows. There, studies of the thermo-fluid
dynamic characteristics of laminar and of turbulent flows have been
first carried out and then, the results have been applied to various
technologies.

However, as it will be seen below, in the field of two-phase flow

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*Numbers in brackets refer to reference listed in Bibliography,

the opposite approach has been followed most often. Here, the tendency has been to analyze the thermal and/or fluid dynamic problems of a particular system, component or process, say of a nuclear reactor, refrigerator, pollutant separator, liquid propellant rocket or open channel sediment transport. Consequently, a broad understanding of the thermofluid dynamic behavior of two-phase systems has not been attained yet, nor is a generalized method available at the present time to analyze and predict the performance of these systems. In Chapter II we shall substantiate this conclusion.

1.3 Requirements

The design of engineering systems and the ability to predict their performance depend on the availability of experimental data and of conceptual models which can be used to describe a physical process with a required degree of accuracy.

From a scientific, as well as from a practical point of view, it is essential that the various characteristics and properties of such conceptual models and processes should be formulated clearly, on a rational basis, and supported by experimental data. For this purpose specially designed experiments are required which must be conducted in conjunction with and in support of analytical investigations.

It is well established in continuum mechanics that the conceptual models for single phase flow of a gas or of a liquid, are formulated in terms of field equations which describe the conservation laws of mass, momentum, energy, charge, etc. These field equations are then complemented by appropriate constitutive equations such as the constitutive equations of state, stress, chemical reactions, etc., which specify the

thermodynamic, transport and chemical properties of a given constituent material, i.e., of a specified solid, liquid or gas.

It is to be expected, therefore, that the conceptual models which describe the steady state and dynamic characteristics of structured multiphase or multi-component media should be also formulated in terms of the appropriate field and constitutive equations. However, the derivation of such equations for the flow of structured media is considerably more complicated than for strictly continuous, i.e., homogeneous media, i.e., for single phase flow.

In order to appreciate the difficulties in deriving balance equations for structured, i.e., inhomogeneous media, we recall that in continuum mechanics the field theories are constructed on integral balances of mass, momentum and energy. Thus, if the variables in the region of integration are continuously differentiable and the Jakobian transformation between material and spatial coordinates exists, then the Euler type differential balance can be obtained by using the Leibnitz's rule or more specifically the Reynolds transport theorem which allow us to interchange differential and integral operations.

In multi-phase or multi-component flows the presence of interfacial surfaces introduces great difficulties in the mathematical and physical formulation of the problem.

From the mathematical point of view, a multi-phase flow can be considered as a field which is subdivided into single phase regions with moving boundaries separating the constituent phases. The differential balance holds for each sub-region, however, it can not be applied to the set of these sub-regions in the normal sense without violating the above

conditions of continuity.

From the point of view of physics, the difficulties which are encountered in deriving the field and constitutive equations appropriate to multi-phase flow systems stem from the presence of the interface and the fact that both the steady and dynamic characteristics of multi-phase flows depend upon the structure of the flow. For example, the steady state and the dynamic characteristics of dispersed two-phase flow systems depend on the collective dynamics of solid particles, bubbles or droplets interacting with each other and with the surrounding continuous phase; whereas, in the case of separated flows these characteristics depend upon the structure and dynamics of the interface.

In order to determine the collective interaction of particles and the dynamics of the interface, it is necessary to describe first the local properties of the flow and then to obtain a macroscopic description by means of appropriate averaging procedures.

For dispersed flows, for example, it is necessary to determine the rates of nucleation, evaporation or condensation, motion and disintegration of single droplets (bubbles) as well as the collisions and coalescence processes of several droplets (or bubbles).

For separated flow, the structure and the dynamics of the interface greatly influence the rates of mass, heat and momentum transfer as well as the stability of the system. For example, the performance and flow stability of a condenser for space application depend on the dynamics of the vapor interface. Similarly, the rate of droplet entrainment from a liquid film, and therefore, the effectiveness of film cooling, depend on the stability of the vapor liquid interface.

It can be concluded from this discussion that in order to derive the field and constitutive equations appropriate to structured, multi-phase media it is necessary to describe the local characteristics of the flow from which the macroscopic properties should be obtained by means of an appropriate averaging procedure. It is evident also that the design, performance and very often the safe operation of a great number of important technological systems, which were enumerated in the preceding sections, depend on the availability of realistic and accurate field and constitutive equations. It will be seen in Chapter 2, that, for two phase flow, these equations have not been established yet.

1.4 Purpose and Outline of the Dissertation

1.4.1 Overall Objectives

The overall objectives of this dissertation are two-fold. First, to provide a general thermo-fluid dynamic formulation of separated two-phase flow, and second, to use this formulation for analyzing various dynamic aspects of the annular and film flow regimes. Consequently, the thesis is divided into two parts.

In Part I, the conservation equations which are expressed both in terms of the two-fluid model and of the diffusional, i.e., drift model are derived. These equations, derived from the appropriate averaging procedure, take into account the effects of surface phenomena such as surface tension and surface shear and of surface processes such as momentum, heat and mass transfer at the interface. Finally, these conservation equations are used to establish the similarity groups appropriate to separated flows.

In Part II, the two-fluid model formulation of Part I, is used to

develop a stability theory of separated plane flows. The analysis takes
into account the effects of surface tension and liquid viscosity, of
mass and heat transfer at the interface as well as the dynamic effects of
the vapor on the liquid film. The most unstable growth factor, derived
from the analysis, is used to calculate the onset of liquid entrainment
and of flow plugging. The results show that evaporation has a destabi-
lizing effect on liquid films whereas it stabilizes vapor films, i.e., it
has a stabilizing effect on Rayleigh-Taylor instability. A comparison
of predicted results with available experimental data shows a satisfactory
agreement.

1.4.2 Outline of Part I

An evaluation of the presently available conservation equations
for two-phase flow is presented in Chapter 2. It is shown there that:
1) the momentum and energy equations proposed by various authors are
incorrect and in disagreement with each other and 2) the set of govern-
ing equations for the mixture used by various authors is incomplete and
incorrect when applied to two-phase mixtures in thermal non-equilibrium.

The two-fluid and the drift models are discussed in Chapter 3,
which deals with basic definitions and relations such as the averaging
procedure, concentration, velocity fields and the fundamental identity.

The local field and constitutive equations together with the
appropriate boundary conditions are given in Chapter 4. These equations
are used then to derive the averaged general balance equations for each
phase as well as for the mixture.

Chapter 5 deals with the continuity equations of the two phases
and of the mixture. For the drift model, the continuity equation of the

vapor is expressed in the form of both the diffusion and the void propagation equations. The appropriate similarity groups are obtained from these two equations by expressing them in a non-dimensional form.

The averaged momentum equations of each phase and of the mixture are derived in Chapter 6 together with the appropriate scaling parameters.

The derivation of the averaged total energy equations for the individual phases as well as for the mixture is presented in Chapter 7, together with equations for the internal and mechanical energies and the enthalpy.

Finally, in Chapter 8, the entropy equations are derived based on the two-fluid and the drift, i.e., diffusional models.

1.4.3 Outline of Part II

Previous investigations concerned with the stability of plane flows are reviewed in Chapter 10. It is noted there that with the exception of two references, the effects of heat and mass transfer at the interface are not taken into account in formulations presently available.

In Chapter 11, the stability theory is formulated based on the linearized form of the two-fluid model. A stability criterion is derived which is used to determine the break-up conditions of liquid film flow.

Chapter 12 considers the stability of falling liquid films. It presents stability criteria which were derived for both adiabatic and diabatic films.

Finally, Chapter 13 deals with the effects of heat and mass transfer on the flow stability of two inviscid fluids. The stability criterion shows that evaporation has a destabilizing effect on liquid films whereas it has a stabilizing effect on Rayleigh-Taylor instability.

CHAPTER II

PREVIOUS WORK

2.1 Introduction

In view of the importance of the two-phase flow system, there have been numerous publications concerning the general area of two-phase flow. Although the first known work in this area was published in 1830, a recent literature survey by Gouse [9] shows that the number of publications for all gas-liquid flow studies excluding atomization and cavitation has increased exponentially after 1940.

In the review that follows only studies which deal directly with different formulations and approaches are discussed. The reader is referred to the recent reviews by Scott [10], Dukler and Wicks [11] and particularly in film flow by Fulford [12] and the three major books by Tong [13], Brodkey [14] and Wallis [15] for discussions and analyses of the proposed correlations as well as for a list of references.

As a consequence of the topographical characteristics of two-phase flow two approaches have been used in deriving the set of conservation equations for the mixture. One was based on the <u>model of interacting continua</u>, whereas the other was based on <u>the separated flow model</u>. Although the first approach is appropriate to dispersed flows, it will be reviewed briefly here for the sake of completeness and to note that interface phenomena were not taken into account in these formulations.

2.2 Dispersed Flows-Interacting Continua

Many authors treated the dispersed two-phase flow mixture as continuum whose thermodynamic and transport properties depend upon the thermodynamic and transport properties of each phase as well as upon the local concentration. In effect, the continuum hypothesis implies implicitly that each point in the mixture flow field is occupied simultaneously by several different particles. The mixture is then represented as a superposition of continuous media, each of which follows its own individual motion as well as the mixture as a whole.

The simplest formulation based on the continuum assumption is the homogeneous model where it is postulated that both phases move with the same velocity, i.e., that the relative velocity between these phases is zero. Since the interaction between the two phases depends upon the relative velocity, it is obvious that these interactions can not be taken into account by this model. It is not surprising therefore, that for applications to two-phase flow systems, the homogeneous model is of limited value. Consequently, we shall not discuss it further, although we note that because of its simplicity, this model has been widely used in the literature as discussed by the numerous references listed in references [13-15] among others.

A more realistic formulation for dispersed flows is provided the model of interacting-continua since it takes into account the effect of relative velocity and, therefore, the interaction between the two phases. This approach has been the basis for the formulation of the problems involving (solid or liquid), mixture, Panton [16], and Van Deemter and Der Laan [17] for the thermomechanical formulation of the diffusion

processes, Truesdell [18], Bowen [19] and Müller [20], and the formulations of Delhaye [24, 25], Teletov [26, 27], Frankl [28] and Diunin [29] which were based on time and space averaging procedures.

The mechanical theory of diffusion, which was put forth by Maxwell and Stefan, was improved by Truesdell [18] (see also Truesdell and Toupin [22, sections 158, 159, 215, 243] and Truesdell [23, pp. 81-98]). The diffusional model used by the author is based on three "metaphysical" principles concerning 1) the mixture properties, 2) interfacial transport and 3) the motion of the mixture as a single body. Using these principles, Truesdell postulated equations of balance of mass, momentum and energy for each constituent and derived the necessary and sufficient conditions so that the balance of mass, momentum and energy for the mixture can be satisfied. Later, in an effort to unite the purely mechanical and thermodynamical theories of diffusion, Truesdell's formulation has been generalized in various ways by the authors of references [19 and 20]. However, their equations of balance of mass and linear momentum are essentially the same as Truesdell's. The only difference appears in the formulation of the entropy equation. We note that these analyses do not take into account the effect of the interfacial source. In fact, the surface tension does not appear in any of these formulations. We note also that the time and space averaging procedures appropriate to mixtures were not considered in references [18-20].

A local averaging procedure was introduced by Van Deemter and Der Laan, reference [17], who averaged the field equations over a small volume element. However, the equations derived in this reference are of a purely phenomenological nature. The kinetics of particles motion

and the nature of fluid particle interaction were introduced formally as a stress tensor and a drag force. It is evident that further evaluation of these functions should be based on a theory of particle kinetics. In application of Van Deemter's equations to the fluidized beds problem Carrier [21] and Zuber [1] provided the physical insight to the phenomenological coefficients by introducing the single particle dynamic equations.

A time-space averaging procedure was used by the authors of references [16, 24-29] who focused their attention to a given volume element which is occupied alternately by one of the two phases. Consequently, in order to define the local occurrence of a particular phase it was necessary to consider statistical averages.

Among the numerous and very important contributions made by Soviet researchers toward establishing the field equations for two-phase mixtures, we shall note only those of Teletov [26, 27], Frankl [28] and Diunin [29]. In these formulations a four-dimensional space time cylinder was considered, and the flow properties were averaged over space and time. One important aspect of the work of Teletov, Frankl and Diunin is the fact that they were apparently the first to a) investigate various methods for obtaining the appropriate expressions for time and space averages and b) express the field equations in terms of these averages.

Panton in reference [16], formulated the mixture equation by integrating the time averaged local conservation equations over a small volume element and then limiting the volume element to zero size which implies implicitly the concept of continuum for each phase as well as for the mixture.

In references [24, 25], Delhaye used a statistical averaging

approach to formulate the bubbly flow regime in order to bring a non-continuum two phase flow field into the continuum framework in terms of statistically averaged equations. In his work the formulation was done in terms of two continuity equations, i.e., one for gas phase and one for the mixture, two momentum equations, i.e., one for each phase, and one energy equation for the mixture, where gas phase was assumed to be in thermal equilibrium in itself, i.e., no temperature variations in the vapor phase.

We note in closing that the effects of surface phenomena and processes were ignored in most of these references and furthermore, when they were considered, they were not taken properly into account.

2.3 Separated Flow Models

The numerous analyses based on the separated flow model can be divided into two groups. In the first group are analyses formulated by considering one dimensional slip flow, whereas in the second one are those which are based on the area-averaged field equations. Of the numerous publications we shall review in what follows only those which are indicative of the various approaches that have been used. We shall also note the significant differences which exist between the resulting equations.

2.3.1 Slip Flow Model Field Equations

Analysis belonging to this group considered the flow of the individual phases to be one-dimensional and the interface between them to be smooth. The assumption commonly made was that the pressure is constant at a given cross-sectional area of the pipe, and both fluids were assumed to be incompressible. To be more specific, according to this

model the two phases or constituents are completely separated by the interfaces, and the variations in the cross-sectional planes are completely ignored as are the effects of the interface, i.e., of surface tension, surface shear and of heat and mass transfer at the interface. Evidently this model does not permit any variations in velocity, density, and properties within the cross-sectional areas of individual phases (such as would be present, for example, with a parabolic velocity profile). The effect of zero velocity at the wall is assumed to extent over an extremely small portions of the fluid and is neglected. One can see that in the laminar flow regime error caused by this assumption will be large.

In order to make a comparison between the conservation equations based on the slip flow model, proposed by various researches we shall denote by ρ, v, P and W, the density, the velocity, the pressure and the flow rate of the fluid whereas the subscripts 1 and 2, will be used to differentiate the phases or constituents. The volumetric concentration of the second phase will be noted by α.

Martinelli and Nelson [30] derived the slip flow model mixture momentum equation. In their analysis momentum balances were made by considering each phase separately as a control volume in the differential element. For steady flow in a vertical pipe, they are given by Equations (2.1) and (2.2) for phase 1 and 2, respectively

$$\frac{d}{dx}\left(W_1 v_1\right) = -A_1 \frac{dP}{dx} - \frac{dF_1}{dx} - A_1 \rho_1 g + v_1 \frac{dW_1}{dx} \quad (2.1)$$

$$\frac{d}{dx}\left(W_2 v_2\right) = -A_2 \frac{dP}{dx} - \frac{dF_2}{dx} - A_2 \rho_2 g + v_1 \frac{dW_2}{dx} \quad (2.2)$$

where A_1 and A_2 are the cross-sectional area of phase 1 and 2, respectively. F_1 and F_2 were defined as the net forces expanded by phase 1

and 2, respectively, in overcoming frictional resistances.

We note that these forces, i.e., F_1 and F_2, which were not <u>de-fined</u> explicitly include the wall friction as well as the interfacial forces exerted from one phase upon the other. We note also that in deriviating Equations (2.1) and (2.2) the pressure over cross-sectional planes is assumed to be <u>constant</u>. Furthermore, no interfacial effects other than the interfacial mass transport which appears as the last term in Equations (2.1) and (2.2) was included.

The <u>mixture</u> momentum equation was obtained by adding Equations (2.1) and (2.2) and using the mass balance

$$d W_1 + d W_2 = 0 \qquad\qquad (2.3)$$

The mixture momentum balance thus became

$$\frac{d}{dx} \left(W_1 V_1 + W_2 V_2 \right) = -A_T \frac{dP}{dx} - \frac{dP}{dx}\Big)_{Fr} - \left(A_1 \varrho_1 + A_2 \varrho_2 \right) g \qquad (2.4)$$

where

$$\frac{dP}{dx}\Big)_{Fr} = \frac{d}{dx} \left(F_1 + F_2 \right) \qquad\qquad (2.5)$$

Although no detailed analysis has been made on the nature of the frictional forces F_1 and F_2, (2.4) has been used for most if not all correlations of two-phase frictional pressure drop. For example, in the absence of gravity terms Martinelli and Nelson integrated Equation (2.4) along the tube length obtaining thereby the total static pressure drop in terms of frictional and acceleration multipliers which had to be determined from experimental correlations.

Since the authors of reference [30] assumed that the pressures of

the two phase are equal, then in the absence of gravitational forces, i.e., horizontal flow, Equation (2.1) together with Equation (2.2) assert that the sum of the frictional pressure drop and momentum drop for each phase should be equal. In flows with vaporization, the momentum pressure drop of the vapor phase is apt to be appreciable, whereas the momentum presure drop of the liquid phase is generally small. Thus, the criteria of equal pressure drops which is used in this model becomes questionable.

Following the same method, i.e., by making momentum balances over the individual phases, Kutaleladze [31] modified the Martinelli-Nelson's mixture momentum equation in order to include the time dependent effects. The final result of his analysis was given by the following mixture momentum equation

$$(1-\alpha) \; \rho_1 \; \frac{\partial v_1}{\partial t} + \alpha \; \rho_2 \; \frac{\partial v_2}{\partial t} + (1-\alpha) \; \rho_1 v_1 \; \frac{\partial v_1}{\partial x} + \alpha \; \rho_2 v_2 \; \frac{\partial v_2}{\partial x} = \tag{2.6}$$

$$= - \; \frac{\partial P}{\partial x} - \frac{\partial P}{\partial x} \Big)_{Tr} - \Big[(1-\alpha)\rho_1 + \alpha \rho_2 \Big] \; g - (v_2 - v_1) \; \frac{\partial}{\partial x} \; \alpha \; \rho_1 \, v_1$$

In an analysis similar to reference [30], Brodkey [14] proposed steady state individual phase momentum equations for an annular two-phase flow as follows:

$$\frac{d}{dx} \left(W_1 \, v_1 \right) = - A_1 \frac{dP}{dx} - \zeta_{w_1} \frac{dA_w}{dx} - \zeta_i \frac{dA_i}{dx} - A_1 \rho_1 g + v_1 \frac{dW_1}{dx} \tag{2.7}$$

$$\frac{d}{dx} \left(W_2 \, v_2 \right) = - A_2 \frac{dP}{dx} + \zeta_i \frac{dA_i}{dx} - A_2 \rho_2 g + v_1 \frac{dW_2}{dx} \tag{2.8}$$

whereas for the mixture, the same equation as Kutaleladze's was obtained if

$$\frac{dP}{dx}\bigg)_{Fr} = \mathcal{C}_{wi}\,\frac{dA_w}{dx} \qquad (2.9)$$

Later, these momentum equations were rederived by many other investigators. For example, by taking momentum balances over each phase of a steady annular flow in a vertical, constant area tube, Levy [32] and Isbin, Moen and Mosher [33] derived the equations for each phase; they are given by

$$\rho_1 v_1 \frac{dv_1}{dx} = -\frac{dP}{dx} + \frac{dP}{dx}\bigg)_{1TP} - \rho_1 g \qquad (2.10)$$

$$\frac{1}{A_2}\frac{d}{dx}(A_2 \rho_2 v_2^2) = -\frac{dP}{dx} + \frac{dP}{dx}\bigg)_{2TP} - \rho_2 g - \frac{v_1}{A_2}\frac{d}{dx}(A_1 \rho_1 v_1) \qquad (2.11)$$

where $\frac{dP}{dx})_{1TP}$ and $\frac{dP}{dx})_{2TP}$ were characterized as the frictional pressure drops in phase 1 and 2, respectively due to the wall friction only. However, these interpretations were not correct because these terms included also the interfacial effects in general.

The addition of Equations (2.10) and (2.11) yields the mixture momentum equation

$$\frac{d}{dx}\left[(1-\alpha)\rho_1 v_1^2 + \alpha \rho_2 v_2^2\right] = -\frac{dP}{dx} - \frac{dP}{dx}\bigg)_{TP} - \left[(1-\alpha)\rho_1 + \alpha \rho_2\right]g \qquad (2.12)$$

The term $\frac{dP}{dx})_{TP}$ represents the two-phase frictional pressure drop and is equal to

$$\frac{dP}{dx}\bigg)_{TP} = (1-\alpha)\frac{dP}{dx}\bigg)_{1TP} + \alpha\frac{dP}{dx}\bigg)_{2TP} \qquad (2.13)$$

Recently in 1969 Wallis [15] proposed one-dimensional momentum equations. In his analysis Wallis expressed the momentum equation for phase 1 by

$$(1-\alpha)\,\rho_1\,v_1\,\frac{dv_1}{dx} = -(1-\alpha)\,\frac{dP}{dx} - (1-\alpha)\,\rho_1\,g + (F_{12} - F_{w1}) - \qquad (2.14)$$

$$(1-\eta)\,(v_2 - v_1)\,\left[(1-\alpha)\,\rho_1\,v_1 + \alpha\,\rho_2\,v_2\right]\,\frac{dX}{dx}$$

whereas the momentum equation for the second phase by

$$\alpha\,\rho_2\,v_2\,\frac{dv_2}{dx} = -\alpha\,\frac{dP}{dx} - \alpha\,\rho_2\,g - (F_{12} + F_{w2}) - \qquad (2.15)$$

$$\eta\,(v_2 - v_1)\,\left[(1-\alpha)\,\rho_1\,v_1 + \alpha\,\rho_2\,v_2\right]\,\frac{dX}{dx}$$

where F_{W1} and F_{W2} are the drag forces from the duct walls on the first and second phases per unit volume of the flow, respectively.

$F_{12} = -F_{21}$ are the drag forces between faces, η is the ratio which is charged to second phase due to phase change per unit volume of the duct..

X is the quality defined as the ratio of the vapor flow rate to the total mixture flow rate.

The mixture momentum equation was obtained by adding equations (2.14) and (2.15) and resulted in

$$\frac{d}{dx}\left[(1-\alpha)\,\rho_1\,v_1^2 + \alpha\,\rho_2\,v_2^2\right] = -\frac{dP}{dx} - \left[(1-\alpha)\,\rho_1 + \alpha\,\rho_2\right]g - \quad (2.16)$$

$$(F_{w1} + F_{w2})$$

The introduction of η into Equations (2.14) and (2.15) is questionable because there is no justification for this correction factor. Although, there should be a force due to the phase change, it must be entirely charged to one of the phases depending on whether we have condensation or evaporation.

Furthermore, Wallis attempted to determine η from the mixture

entropy equation under the <u>isentropic flow condition</u>. However, if we consider the entropy production due to the relative velocity between phases, we conclude that a two-phase flow can <u>never</u> be isentropic. Thus for separated flow in particular, i.e., the slip flow model always has entropy production, and therefore to make an isentropic flow analysis is incorrect. We note also that only homogeneous flow can be isentropic since the relative velocity is equal to zero. However, in this case the term

$$\eta (v_2 - v_1) \left[(1-\alpha) e_1 v_1 + \alpha e_2 v_2 \right] \frac{d\chi}{dx}$$

would be always zero, since the relative velocity $(v_2 - v_1)$ is zero, therefore, there is no need to introduce η into the analysis as Wallis did.

We shall not list the additional expressions of momentum equations which have been proposed in the literature as they are too numerous. Those listed above are indicative of the approaches used and of the significant differences which exist between them. We note in particular, that the momentum equation for the <u>individual</u> phase proposed by various authors, are <u>not</u> in agreement with each other. This statement can be verified by comparing Equations (2.1) and (2.2) with Equations (2.7) and (2.8) and/or with Equations (2.10) and (2.11) and/or with Equations (2.14) and (2.15). These differences stem from the different assumptions and approaches used by the various authors in order to account for the inter-actions between the two phases. Since these interaction effects dis-appear when the momentum equations for the individual phases are added

it is not surprising that the resulting momentum equations for the mixture are in agreement with each other (compare for example Equations (2.4), (2.6), (2.12) and (2.16).

Turning our attention now to the <u>energy equations</u> which have been reported in the literature we observe more arguments and disagreements. This is natural, however, because if the momentum equations are not in agreement with each other then the energy equations can not be either. As it will be seen the basic problem in energy equation formulations arises from weighing the energy equation for each phase.

Isbin and Su [34] considered the problem of mechanical energy equation for a two-phase flow system. They wrote the energy equation for a single-phase, constant density, time independent flow system as

$$\frac{1}{\rho}\frac{dP}{dx} + v\frac{dv}{dx} + g + \frac{dF}{dx} = 0 \qquad (2.17)$$

where F is the frictional energy loss per unit mass. We note that the equation, Equation (2.17) is apparently not the mechanical energy equation but it is merely the momentum equation expressed per unit mass.

Noting that Equation (2.17) is applicable to a two-phase system only if no interfacial mass transfer occurs, the authors argued that the proper way to apply Equation (2.17) to a two-phase flow system is to weigh the energy equation per pound mass of each phase by the mass of the phase contained in the differential section ($(1-\alpha)\rho_1 A_T dx$ for the liquid, $\alpha\rho_2 A_T dx$ for the vapor phase). With this weighing method, and after summing the equations, the following mechanical energy equation for the mixture was obtained.

$$\left[(1-\alpha)\, e_1\, v_1 dv_1 + \alpha\, e_2\, v_2\, dv_2 \right] + dP + g \left[(1-\alpha)\, e_1 + \alpha\, e_2 \right] dx + \qquad (2.18)$$

$$\left[(1-\alpha)\, e_1\, dF_1 + \alpha\, e_2\, dF_2 \right] = 0$$

Other investigators, however, multiplied Equation (2.17) alter-
nately by the appropriate mass flow rate (W_1 and W_2) of each phase.
Using this weighing procedure and after adding the two energy equations,
Vohr [35] proposed the following energy equation for the mixture:

$$\left[(1-\alpha)\, e_1\, v_1^2\, \frac{dv_1}{dx} + \alpha\, e_2\, v_2^2\, \frac{dv_2}{dx} \right] + (1-\alpha)\, e_1 \frac{du_1}{dx} + \alpha e_2 \frac{du_2}{dx} + \quad (2.19)$$

$$\left[(1-\alpha)\, v_1 + \alpha\, v_2 \right] \frac{dP}{dx} + \left[(1-\alpha)\, e_1 v_1 + \alpha\, e_2\, v_2 \right] g = 0$$

where u_1 and u_2 are the specific internal energy for phase 1 and 2,
respectively.

The mixture mechanical energy equation proposed by Brodkey [14]
and Lamb and White [36], is given by

$$\frac{d}{dx} \left[(1-\alpha)\, e_1\, \frac{v_1^3}{2} + \alpha\, e_2\, \frac{v_2^3}{2} \right] + \left[(1-\alpha)\, v_1 + \alpha\, v_2 \right] \frac{dP}{dx} + \quad (2.20)$$

$$\left[(1-\alpha)\, e_1 v_1 + \alpha\, e_2\, v_2 \right] g = - \frac{d}{dx} \left(\phi_1 + \phi_2 \right)$$

where ϕ_1 and ϕ_2 are the viscous dissipation terms for phase 1 and 2
respectively.

Comparing Equations (2.18), (2.19) and (2.20), one can see that
they can not be converted to each other i.e., that they are not in agree-
ment with each other. The difference is due to the weighing of the
individual phase equations. The manner in which the contributions of
each phase were weighed in these equations were different in each case.

Another inconsistency between the equations is due to the interpreta-
tion of the various terms. Finally, the interfacial energy transfer
mechanism was not accounted correctly in these formulations.

2.3.2 Field Equations Based on Area Averages

In derivations belonging to this group, various authors tried to
integrate the single phase differential conservation equations over the
entire cross-sectional plane of the pipe. Following this method Meyer
[37] proposed a set of equations expressing the one-dimensional mixture
continuity, momentum and energy equations. They are given by the mixture
continuity equation:

$$\frac{\partial}{\partial t} \left(\rho_{(mass)} A_T \right) + \frac{\partial}{\partial x} \left(G_T A_T \right) = 0 \qquad (2.21)$$

the mixture momentum equation:

$$\frac{\partial}{\partial t} \left(G_T A_T \right) + \frac{\partial}{\partial x} \left(\frac{1}{\rho_{(mom)}} G_T^2 A_T \right) = - A_T \frac{\partial P}{\partial x} - \int_{S_w} \tau_w \, dS - \rho_{(mass)} A_T g \qquad (2.22)$$

the mixture energy equation:

$$\frac{\partial}{\partial t} \left(\rho_{(mass)} \, i_{(mass)} \, A_T \right) + \frac{\partial}{\partial x} \left(G_T \, \mathring{i}_{(flow)} A_T \right) = \frac{q_w + q_A}{S_1} A_T \qquad (2.23)$$

where

$$\rho_{(mass)} = \frac{1}{A_T} \int_{A_T} \rho \, dA \quad ; \quad i_{(mass)} = \frac{1}{\rho_{(mass)} A_T} \int_{A_T} \rho \, i \, dA \qquad (2.24)$$

$$\frac{1}{\rho_{(mom)}} = \frac{1}{G_T^2 A_T} \int_{A_T} \rho \, v^2 \, dA ; \quad i_{(flow)} = \frac{1}{G_T A_T} \int_{A_T} \rho \, v \, i \, dA$$

$$G_T = \frac{1}{A_T} \int_{A_T} \rho \, v \, dA \quad ; \qquad q_w = \frac{S_1}{A_T} \int_{A_T} q_f \, dA$$

$$q_A = \frac{\xi}{A_T} \int_{A_T} q'' \, dA$$

It can be seen from Equations (2.21),(2.22) and (2.23) that the mixture equations have been expressed in terms of flowing quantities, such as G_T defined as the mass flow rate per unit area of the pipe. Consequently, this improper formulation forced Meyer to introduce two mixture densities from two conservation equations, i.e., an area averaged density $\rho_{(mass)}$, from the mixture continuity equation, and a momentum averaged density $\rho_{(mom)}$, from the mixture momentum equation. Also he found it necessary to define two different mixture enthalpies, i.e., an area weighted mean enthalpy $i_{(mass)}$, and a flow weighted enthalpy $i_{(flow)}$. Although it does not appear in the above set of equations later in his paper Meyer had to define one more density $\rho_{(en)}$, i.e., an energy averaged density from the energy equation defined by

$$\rho_{(en)} = \rho_{(mass)} \frac{d\, i_{(flow)}}{d\, i_{(mass)}} + \left(i_{(mass)} - i \right) \frac{d\, \rho_{(mass)}}{d\, i_{(flow)}} \qquad (2.25)$$

Thus three mixture densities and two enthalpies were introduced.

In applying Meyer's mixture conservation equations to the critical flow of vapor-liquid mixtures Cruver and Moulton [38] defined one more density weighted by velocity

$$\frac{1}{\rho_{(flow)}} = \frac{1}{w_T} \int_{A_T} v \, dA \qquad (2.26)$$

As a result of references [37] and [38] four mixture densities were defined, i.e., area averaged, momentum averaged, energy averaged and velocity averaged.

If we recall here that according to the basic principles of thermo-dynamic the properties such as density, enthalpy, etc. (see, for example, reference [22]) are:

    1) additive,

    2) invariant under change of motion,

we conclude that the formulations above are incorrect.

It is of interest to note here yet another incorrect definition of mixture properties which has been widely used in the literature, it is based on the quality X defined by

$$ X = \frac{W_2}{W_1 + W_2} \tag{2.27}$$

where W_1 and W_2 are the mass flow rate of the vapor and of the liquid respectively. Thus for example in reference [15], the mixture entropy was defined by

$$ s_m = (1 - X) \, s_1 + X \, s_2 \tag{2.28a}$$

the mixture enthalpy by

$$ i_m = (1 - X) \, i_1 + X \, i_2 \tag{2.28b}$$

and mixture density by

$$ \frac{1}{\varrho_m} = \frac{1 - X}{\varrho_1} + \frac{X}{\varrho_2} \tag{2.28c}$$

It will be made clear in Chapter 3, that mixture quantities should be expressed in terms of static parameters, such as the void fraction or mass concentration and not in terms of mass flow rates, such as the quality. To illustrate this point and demonstrate the error which arises

when Equations (2.28a, b and c) are used we pose the following problem:

consider a closed system where no motions take place, i.e., $W_1 = W_2 = 0$,

what would be the mixture entropy or mixture enthalpy? The answer

according to Equation (2.28) would be undeterminate in view of the defini-

tion of quality because the quality defined by $X = \dfrac{W_1}{W_1 + W_2} = \dfrac{0}{0}$. Whereas

the static mass concentration C will have a definite value. This con-

tradiction stems from the improper formulation of two-phase flow properties,

i.e., when they are not expressed in terms of static parameters.

It should be observed here, that Meyer treated the two-phase flow

field as a single-phase flow where the flow field is <u>continuous</u>, i.e.,

<u>simply connected</u>. However, as it was discussed in Section 1.3, in two-

phase flow the over-all flow field is <u>multiply</u> connected, consequently,

the effects due to the presence of interfaces should be taken into con-

sideration in a proper integration process. An analysis which does not

take into account the effects of the interface in formulating the problem

is incorrect because it does <u>not</u> account, properly, for the limits of

integration.

In the literature dealing with fluid dynamics of single phase

flows in particular with lubricating films, open channel flows etc.,

the method of deriving the two-dimensional momentum and energy equation

by area-averaging the local field equations, is well established.

Similarly, the "jump" conditions at the interface, i.e., the limits of

integration to be used in the averaging procedure are well known. In

the literature dealing with two-phase flow this method of area-averaging

has been used by Linehan [39] and Delhaye [24, 25, 40] together with the

"jump" conditions at the interface in order to derive the one and two-

dimensional field equations. In reference [39], Linehan used the thin film approximation to derive the field equations, however, in the "jump" conditions, he neglected the effects of surface tension and surface shear. These two effects were taken into account, however, by Delhaye, who derived the continuity momentum and energy equations by using the "jump" conditions proposed by Slattery reference [41].

We note that in these references a clear distinction between the two-fluid and the diffusional models has not been made. Furthermore, none of these references presents thermodynamic relations and equations which take into account the interface phenomena. Thus, the two-phase flow equations presently available, do not reduce to the well known equations of thermodynamics derived by Gibbs. Finally, similarity criteria for separated two-phase flows, have not been established yet.

2.4 Traditional Formulations

Thus far in this Chapter, we have discussed the various approaches which have been used in deriving the conservation equations for two-phase flows. Now, we shall review briefly how these equations have been used to formulate and analyze two-phase flow problems.

It was pointed out by the authors of reference [42], that with exception of reference [40], it has been traditionally a common procedure to formulate a two-phase flow problem by means of three field equations and three constitutive equations (see for example the formulations of references [13, 15, 37, 38, 43 among others]). The three field equations were those for the mixture, i.e., the mixture continuity equation, the mixture momentum equation and the mixture energy equation (which usually did not include the mechanical and potential energy terms). The three

constitutive equations which were used were the empirical correlations for the void fraction, i.e., vapor volumetric concentration, for the two-phase friction factor and the two-phase heat transfer coefficient. It was noted in reference [42], that these six equations result in ten dependent variables consequently, the traditional formulations were incomplete in the mathematical sense. Thus, additional assumption had to be introduced leading to additional differences and disagreements between the various analyses.

In the chapter that follows we shall discuss in more detail the two-fluid and the diffusional models in fluid dynamics. Here we note, that formulations of two-phase flows systems expressed in terms of the three field equations for the mixture, are similar to well established formulations of chemically reacting binary systems based on the diffusional model. In view of this similarity it was stressed in references [6, 42], that any analysis of two-phase flows, based on the diffusional model, must include one additional field equation, i.e., the continuity equation for one of the two phases. It was shown in reference [42], that the significance of omitting the second continuity equation in traditional formulations of two-phase flow systems, was that the effects of thermal non-equilibrium could not be accounted for. Above, the three conservation equation for the mixture, can not account for the effect of thermal non-equilibrium.

Finally, it was noted in references [6, 42], that although the traditional formulations of two-phase flow were based on the three conservation equations for the mixture, these equations were not expressed in terms of the center of mass of the mixture (see for example Equations

(2.4, 2.6, 2.12, 2.16, 2.18, 2.19, 2.20)). This circumstance had two important consequences. First, as noted in references [6, 42], the conventional formulations are <u>inadequate</u> to analyze the dynamic behavior of two-phase flow systems. Second, the thermodynamic properties of the mixture were not defined properly. In fact, as we have discussed already in the preceding section, various authors were forced to introduce <u>four</u> definitions for the density of the mixture, three definitions for the enthalpy of the mixture, etc. Such a sorry state of this branch of thermo-fluid dynamics is rather surprising since it was discussed by Prigogine and Mazur [44] that if a diffusional model is used to describe the behavior of a mixture of gases then the field equations must be expressed in terms of the center of mass of the mixture. If this is not done, then one can not express properly the thermodynamic equations and relations.

2.5 Conclusions

The preceding review of the literature concerned with the conservation equations for separated two-phase flows indicates that:

1) The momentum equations for the individual phases which were derived by considering the slip flow model are in disagreement with each other.

2) Similarly, the energy equations are in disagreement with each other.

3) The conservation equations for the mixture based on the slip flow model are not in agreement with each other.

4) With the exception of the work of Delhaye, interfacial phenomena and processes have not been taken into account in analyses of diabatic separated two-phase flows.

5) The thermodynamic properties of the mixture have not been defined properly because the field equations for the mixture were not expressed in terms of the center of mass of the mixture. Therefore, four definitions of mixture density, three definitions of mixture enthalpy, etc., have been introduced in analyses of separated two-phase flow systems.

6) The conservation equation for the mixture which have been used in the literature are inadequate for analyses of dynamic aspects because they have not been expressed in terms of the center of mass of the mixture.

7) Most formulations and analyses of two-phase flow systems presently available are based on an incomplete set of field equations. In particular, they are expressed only in terms of the continuity, momentum and energy equations for the mixture. They omit the second continuity equation for one of the two phases. Therefore, these formulations can not take into account the effects of thermal non-equilibrium.

8) A distinction has not been made in the literature between formulations based on the two-fluid model and the diffusional model.

9) Similarity criteria appropriate to separated two-phase flows have not been established yet.

The analysis that follows is directed at removing these difficulties thereby improving the present state of knowledge.

CHAPTER III

MODELS, BASIC DEFINITIONS AND RELATIONS

3.1 Two-Fluid and Diffusion (Drift) Models

3.1.1 Analytical Methods and Models

It was discussed in Section 1.2 and illustrated in Figure 1 that the most important characteristic of two-phase flow systems is the presence and the effect of interface between the phases. The topography of the interface serves not only to divide structured flows into three classes, i.e., separated, mixed and dispersed flows, but it gives also rise to processes and problems specific to each class. Furthermore, this topography compels the application of a different method of analysis to each class. This differentiation of analytical methods stems from the fact that a two-phase region is not simply connected, i.e., the variables in the region of integration are not continuously differentiable. In fact, a two-phase region can be considered as a field which is subdivided into simply connected, i.e., single phase regions with moving boundaries separating the constituent phases.

In theory, the problem could be formulated in terms of field equations applicable to each continuous subregion with matching boundary conditions at the moving interface. Such a formulation would result in a multiboundary problem with the position and the condition at the boundary being unknown. It is evident that unless the topography of the interface is simple, as it is for example in separated flows of Figure 1, such an approach would encounter unsurmountable mathematical difficulties.

Consequently, different methods of analysis must be applied to the different classes of structured flows.

For separated two-phase flows, the problem can be formulated by considering <u>two-continua</u> coupled by the appropriate "jump" conditions at the boundary, i.e., at the interface. It is evident that these "jump" conditions will play a most important role in such a formulation.

However, for dispersed flows in order to eliminate the mathematical difficulties caused by the discontinuities of the variables, it is useful to transform the entire field to a continuum. This can be accomplished by a time averaging procedure or by means of Boltzman's equation applied to the dispersed phase. In two-phase flow systems the velocities of the two phases are never equal, i.e., there is always a relative velocity $\vec{V}_r$, between the phases. Consequently, there is always a dynamic interaction between the phases. For this reason a two-phase flow problem should be formulated always in terms of two velocity fields.

However, depending on the magnitude of the difference between these two velocities as well as the difference between the two densities, both separated flows and dispersed flows can be formulated in terms of a <u>two-fluid model</u> and of a <u>diffusion model</u>, both of which have specific advantages as well as disadvantages which are discussed below.

3.1.2 The Two-Fluid Model

The two-fluid model is formulated by considering each phase separately. Therefore, the formulation is expressed in terms of two sets (one for each phase) of conservation equations, i.e., in terms of <u>six</u> field equations: two continuity equations, two momentum equations and two energy equations.

In effect, the two-fluid model follows the model proposed by Landau [45], for analyzing the superfluidity of helium II. In a discussion of this phenomenon Prigogine and Mazur noted in reference [44] that the two-fluid model will yield satisfactory results whenever the two components of a mixture are weakly coupled so that the equilization of velocities does not take place. This can be expected whenever there is a large difference between the densities and velocities of the two components.

In two-phase flow it can be expected, therefore, that the two-fluid model will be applicable to separated flows. Furthermore, this model will be most useful in problems concerned with the dynamics of the interface and other interactions between the two phases. Since a formulation based on this model is represented in terms of two equations of continuity, two momentum equations and two energy equations, an analysis based on this model may encounter mathematical difficulties. The model is therefore not well suited for analyses of system dynamics. Neither can it be used effectively to determine the mixture properties (in particular the entropy) and the similarity groups.

3.1.3 The Diffusion or Drift Model

In contrast to the two-fluid model, the diffusion model is formulated by considering the entire mixture. Therefore, the formulation is expressed in terms of four field equations: three for the mixture (continuity, momentum and energy) plus the diffusion (or void propagation equation references [4, 6, 7]) for one of the phases.

It is evident that the diffusional model follows the approach used to analyze the dynamic behavior of a mixture of gases or of miscible

liquids reference [49]. It is applicable whenever the two components of the mixture are closely coupled, i.e., whenever they interact so that the difference between the velocities and other properties of the two components are small.

In two-phase flow it can be expected therefore, the diffusional model will be applicable to dispersed flows and in particular to the bubbly and slug flow regimes.

In the diffusion model the interactions between the phases are specified by appropriate constitutive equations; for example, an appropriate <u>diffusion or drift velocity</u> is specified for a particular flow regime, reference [6]. This drift velocity in turn specifies a drift stress in the momentum equation for the mixture as well as an energy transport term by drift in the energy equation for the mixture (reference [6]).

When expressed in terms of the center of mass of the mixture, the diffusion model is most useful for analyses of system dynamics as shown in references [6, 7]. Furthermore, as the results of this investigation show, the diffusion or drift model can be used effectively for determining the set of similarity groups appropriate to two-phase flow systems.

We note that since the diffusion model is expressed in terms of the conservation equations for the mixture, it is of utmost importance to correctly define the properties of the mixture.

Since both the two-fluid and the diffusion models have their own specific advantages, disadvantages and applications, both will be used in formulating the thermo-fluid dynamic equations for separated two-phase flows.

3.2 Physical Model

The general separated two-phase flow system we shall investigate in the following sections is illustrated in Figure 2, where the two-phases* are distinguished by subscript 1 and 2**.
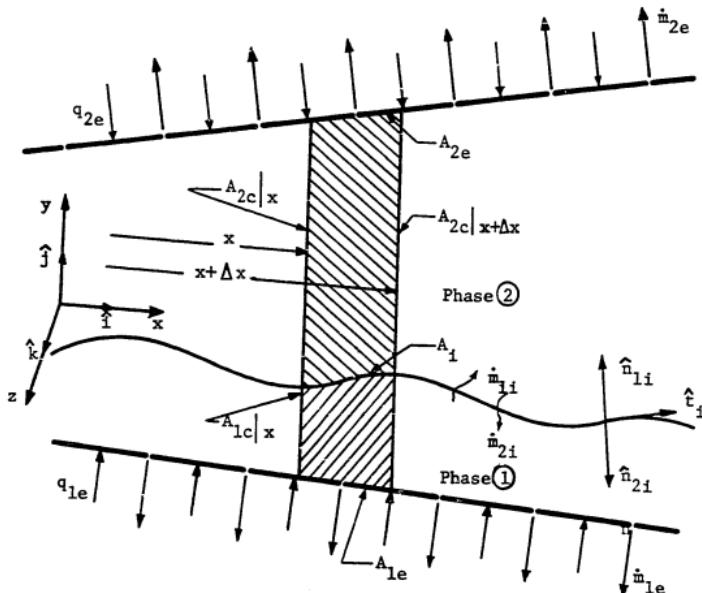
Figure 2. The General Separated Flow Model

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*Since the mathematical description of the two-phase one-component, two-phase two-component or single-phase two-component flows are identical, we shall use the expression two-phase flow throughout the development of the field and constitutive equations.
**It is a usual practice in the literature to identify the heavy phase by subscript 1, whereas the lighter phase by subscript 2.

Phases identified by 1 and 2 are flowing separately in a duct in the main flow direction which is taken to be x. A mass transfer at the external boundaries $\dot{m}_{Ke}$, leaves the K-th phase through the porous wall, and a mass flux $\dot{m}_{Ki}$ leaves it at the interface. Here K takes on the value 1 and 2 identifying phase 1 and 2, respectively.

We shall frequently refer to Figure 2 and the notation employed on it, therefore it becomes necessary to define rigorously a few terms appearing on it.

A_{Ke} is the external surface of the K-th phase contact with the external i.e., fixed boundaries.

A_i is the interfacial, i.e., contact area between two-phases.

A_{Kc} is the internal cross-sectional area of the K-th phase perpendicular to the main flow direction.

A_{Tc} is the total cross-sectional area of the duct bounded by solid boundaries. A_{Tc} is not necessarily constant. From Figure 2 it is evident that

$$A_{Tc} = \sum_{k=1}^{2} A_{kc}$$

ξ_{Ke} is the intersection curve of A_{Kc} and A_{Ke}.

ξ_i is the intersection curve of A_{Kc} and A_i.

$\hat{n}_{Ki}$ is the unit normal vector at the internal surfaces bounding the K-th phase and directed always outward from the K-th phase.

$\hat{n}_{Ke}$ is the unit normal vector at the external surfaces bounding the K-th phase and directed outward from the same phase.

3.3 Integral Relations

It was seen in the preceding chapter that the conservation equations

for two phase flow were usually derived in the literature by applying the conservation concept to a control volume. These equations were used in turn to define terms which represent various effects, interactions or mixture properties. These definitions were based most often on spurious arguments rather than on fundamental principles. It was seen how this procedure lead to incorrect definitions of mixture properties and other two phase flow parameters.

In order to obtain a rational formulation it is necessary to return to the basic laws of dynamics and thermodynamics and to make clear the various fundamental aspects of the phenomenon through consistent definitions of averages, variables and mixture properties. Indeed, the formulation must be turned around: the basic parameters which characterize two-phase flow should be first properly defined, and then, after correctly averaging the field equations, the variables which appear in these equations must be defined explicitly in terms of these basic parameters.

We stress again, that the proper definition of mixture properties is of particular importance to the diffusion model because, as discussed above, this model is formulated in terms of the three conservation equations for the mixture and the diffusion equation.

In view of the foregoing we shall start by considering the averaging procedure and the definition of mixture properties appropriate to separated two phase flow.

3.3.1 Averages

In order to derive the field and constitutive equations appropriate to structured, multi-phase media it is necessary to describe the local macroscopic properties by means of an appropriate averaging

procedure. In separated flows, the two phases can be considered as two continua connected by an interface across which the properties change discontinuously. The appropriate averaging procedure consists therefore of area-averaging the two, local continua and coupling them by means of the appropriate "jump" condition at the gas-liquid interface. This approach used in references [24, 25, 39] is well known approach used in deriving the hydraulic equations for open channel flow, the Reynolds lubrication equation etc.

Let us consider therefore a quantity $\psi_K(x,y,z,t)$-scalar, vector or higher order tensor- of the K-th phase, (K = 1,2), and also consider a volume $\mathcal{V}_K$ enclosing the K-th phase cross-sectional plane. Then we define the <u>volume-averaged</u> value of the quantity ψ_K, by

$$\ll \psi_K \gg (t) = \frac{1}{\mathcal{V}_K(t)} \iiint_{\mathcal{V}_K} \psi_K \, d\mathcal{V} \tag{3.1}$$

By expressing the volume as $\mathcal{V}_K(t) = A_{Kc} \Delta x$ and by considering the limits of Equation (3.1) as $\Delta x \to 0$, we obtain the definition of the <u>area-averaged</u> value of the quantity ψ_K, i.e.,

$$\ll \psi_K \gg (x,t) = \frac{1}{A_{Kc}} \iint_{A_{Kc}(x,t)} \psi_K(x,y,z,t) \, dA \tag{3.2}$$

Two observations can be made with respect to Equation (3.2). First, we note that since the integration is performed over the cross-sectional plane perpendicular to the main flow direction designated by x, the resulting area-averaged quantity $\ll \psi_K \gg$ must be function of x and time, t. We note, further, that Equation (3.2) is an averaged quantity

of the K-th phase, obtained by weighing the quantity by the cross-sectional area of the K-th phase. Consequently, the averaged value $\langle\langle\psi_K\rangle\rangle$ applies to the <u>center of area</u>, or more meaningfully it applies to the <u>center of volume</u> of the K-th phase.

Although the definition given by Equation (3.2) may be useful in some two-phase flow analyses, sometimes it may be more advantageous to formulate the problem with respect to <u>center of mass</u> instead of <u>center of volume</u>. Ultimately, we are led to consider the <u>mass-weighted, area-averaged</u> value of the quantity ψ_K, defined by

$$\langle \psi_K \rangle (x,t) = \iint\limits_{A_{Kc}} \rho_K \psi_K \, dA \;\Big/\; \iint\limits_{A_{Kc}} \rho_K \, dA \qquad (3.3)$$

where upon applying the area-averaged values, i.e., Equation (3.2), we obtain

$$\langle \psi_K \rangle (x,t) = \langle\langle \rho_K \psi_K \rangle\rangle \big/ \langle\langle \rho_K \rangle\rangle \qquad (3.4)$$

where $\langle\psi_K\rangle$ applies to the <u>center of mass</u> of the K-th phase and <u>not</u> to the center of volume. In effect, comparison of the two definitions, $\langle\langle\psi_K\rangle\rangle$ and $\langle\psi_K\rangle$ reveals that unless $\langle\langle\rho_K\rangle\rangle = \rho_K$, the area-averaged value $\langle\langle\psi_K\rangle\rangle$, and the mass-weighted area-averaged value $\langle\psi_K\rangle$, of the quantity ψ_K, are different. In the analysis of incompressible fluids or in those analyses where the mass density variation over the cross-sectional plane is negligible, i.e., $\langle\langle\rho_K\rangle\rangle \approx \rho_K$, only then

$$\langle \psi_K \rangle = \langle\langle \psi_K \rangle\rangle \qquad (3.5)$$

3.3.2 The Covariance

The introduction of averaged system variables into the non-linear

equation of continuity momentum and energy increases the analytic
problem complexity because the average of a product is not the same as
the product of the averages of the variables such as ψ_K and φ_K. That is

$$\langle \psi_k \cdot \varphi_k \rangle = \frac{\iint\limits_{A_{Kc}} \rho_k \psi_k \varphi_k \, dA}{\iint\limits_{A_{Kc}} \rho_k \, dA} \neq \langle \psi_k \rangle \cdot \langle \varphi_k \rangle \qquad (3.6)$$

In particular

$$\langle \psi_k^2 \rangle \neq \langle \psi_k \rangle^2 \qquad (3.7)$$

unless ψ_K is constant over the cross-sectional plane over which averag-
ing is taken.

However, such a relation between the average of a product and the
product of the averages of the variables ψ_K and φ_K can be accomplished
by recalling the definition of covariance, see for example reference
[46, pp. 215-222], given by

$$Cov(\psi_k \cdot \varphi_k) = \langle \psi_k \cdot \varphi_k \rangle - \langle \psi_k \rangle \cdot \langle \varphi_k \rangle \qquad (3.8)$$

We note that in Equation (3.8), ψ_K may differ from, or equal to
φ_K, as well. If they are equal, then Equation (3.7) expresses the average
value of $\psi_K (= \varphi_K)$ squared, i.e., of $\langle \psi_K^2 \rangle$, in terms of the square of the
area average value of ψ, i.e., of $\langle \psi_K \rangle^2$. Hence, we recover the difference
between the right and the left hand sides of inequality (3.7).

For a laminar flow of an incompressible fluid in a circular duct
of radius R, for example, one has the parabolic velocity distribution

$$v(r) = \frac{\Delta P}{4 \mu L} R^2 \left[1 - \left(\frac{r}{R} \right)^2 \right]$$

In this particular example,

$$\ll v \gg^2 = \langle v \rangle^2 = \frac{1}{4} \left(\frac{\Delta P}{4 \mu L} \right)^2 R^4$$

whereas

$$\ll v^2 \gg = \langle v^2 \rangle = \frac{1}{3} \left(\frac{\Delta P}{4 \mu L} \right)^2 R^4 = \frac{4}{3} \langle v \rangle^2$$

Difference between them indicates that the covariance

$$Cov(v \cdot v) = \langle v^2 \rangle - \langle v \rangle^2 = \frac{1}{3} \langle v \rangle^2$$

can be important as much as 33 percent of $\langle v \rangle^2$. In turbulent flow, however, it is expected that Cov (v·v) would not be as important as is in laminar flow because in turbulent flow velocity profile is almost flat.

3.3.3. Integral Transformations

In derivations that follow we shall integrate the basic conservation equations in the local forms over the respective cross-sectional planes A_{Kc}, (K = 1,2). It is necessary, therefore, to consider certain integral transformation theorems which will be encountered frequently.

<u>The Divergence Theorem Over Surfaces</u>. Let us consider a volume element $\mathcal{V}_K$ of the K-th phase contained within two parallel planes $A_{Kc}|_x$ and $A_{Kc}|_{x+\Delta x}$ infinitely close together, with distance Δx, perpendicular to x - axis. According to Figure 2, the volume element cuts the internal and external boundaries with the surfaces designated by A_{Ki} and A_{Ke}, respectively.

The divergence theorem over the volume element $\mathcal{V}_K$ is

$$\iiint\limits_{\mathcal{V}_K} \nabla \cdot \vec{F}_K \, d\mathcal{V} = \iint\limits_{A_K} \vec{F}_K \cdot \hat{n}_K \, dA \tag{3.9}$$

Here, F_K is a first or higher order tensorial quantity associated with the K-th phase. $\hat{n}_K$ is the unit normal vector directed outward from the surface, A_K. Finally, A_K is the total area enclosing the volume element $\mathcal{V}_K$. According to Figure 2 it is evident that

$$A_K = A_{Kc}|_x + A_{Kc}|_{x+\Delta x} + A_{Ke} + A_{Ki} \tag{3.10}$$

In view of Equation (3.10), the right hand side of Equation (3.9) may be expanded to yield

$$\iiint\limits_{\mathcal{V}_K} \nabla \cdot \vec{F}_K \, dA = \iint\limits_{A_{Kc}|_x} \vec{F}_K \cdot \hat{n}_K \, dA + \iint\limits_{A_{Kc}|_{x+\Delta x}} \vec{F}_K \cdot \hat{n}_K \, dA + \sum_{\ell = e,i} \iint\limits_{A_{K\ell}} \vec{F}_K \cdot \hat{n}_K \, dA \tag{3.11}$$

where the summation is taken over the internal boundary, i.e., interfacial area which is identified by the subscript i, and the external boundary, i.e., solid boundaries which is identified by subscript e.

Area elements at the boundaries can be expressed by

$$dA = \left(\hat{n}_K \cdot \hat{n}_{K\xi} \right)^{-1} d\xi_K \, dx \tag{3.12}$$

where ξ is the wetted perimeter, and $\hat{n}_{K\xi}$ the unit normal vector to the

perimeter located in the cross-sectional plane, $A_{Kc}\big|_x$.

If the surface were given by

$$\mathfrak{z} = \mathfrak{z}(x, y, t) \tag{3.13}$$

then the area element would be expressed by

$$dA = \left[1 + \left(\frac{\partial \mathfrak{z}}{\partial x}\right)^2 + \left(\frac{\partial \mathfrak{z}}{\partial y}\right)^2\right]^{1/2} dx\, dy \tag{3.14}$$

Finally, from Figure 2 it is easy to see that

$$\hat{n}_k\big|_{x+\Delta x} = -\hat{n}_k\big|_x = \hat{\imath} \tag{3.15}$$

Using Equations (3.12) and (3.15) in Equation (3.11), we obtain

$$\int_x^{x+\Delta x} \iint_{A_{Kc}} \nabla \cdot \vec{F}_k \, dA \, dx = \iint_{A_{Kc}\big|_{x+\Delta x}} \vec{F}_k \cdot \hat{n}_k \, dA - \iint_{A_{Kc}\big|_x} \vec{F}_k \hat{n}_k \, dA + \tag{3.16}$$

$$\sum_{\ell=e,i} \int_x^{x+\Delta x} \int_{\mathfrak{z}_\ell} (\vec{F}_k \cdot \hat{n}_k)(\hat{n}_k \cdot \hat{n}_{k\mathfrak{z}})^{-1} \, d\mathfrak{z} \, dx$$

This equation can be simplified in the following manner. The volume integral on the left side and the last surface integral on the right side are first estimated by the mean-value theorem for integrals. Thus

$$\int_x^{x+\Delta x} \iint_{A_{Kc}(x,t)} \nabla \cdot \vec{F}_k \, dA \, dx = \Delta x \iint_{A_{Kc}(x_0,t)} \nabla \cdot \vec{F}_k \, dA \tag{3.17}$$

$$\int_x^{x+\Delta x} \int_{\mathfrak{z}_\ell(x,t)} (\vec{F}_k \cdot \hat{n}_k)(\hat{n}_k \cdot \hat{n}_{k\mathfrak{z}})^{-1} \, d\mathfrak{z} \, dx = \Delta x \int_{\mathfrak{z}_\ell(x_1,t)} (\vec{F}_k \cdot \hat{n}_k)(\hat{n}_k \cdot \hat{n}_{k\mathfrak{z}})^{-1} d\mathfrak{z} \tag{3.18}$$

Here, x_0 and x_1 must lie between x and $x + \Delta x$. Next, Equation (3.16) is divided by Δx and the limit $\Delta x \to 0$ is taken. During this process x_0 and x_1 takes on the values x. Then

$$\iint\limits_{A_{Kc}(x,t)} \nabla \cdot \vec{F}_K \, dA = \frac{\partial}{\partial x} \iint\limits_{A_{Kc}(x,t)} \vec{F}_K \cdot \hat{n}_K \, dA + \sum_{\ell=e,i} \int\limits_{S_\ell} (\vec{F}_K \cdot \hat{n}_K)(\hat{n}_K \cdot \hat{n}_{K\ell})^{-1} dS \quad (3.19)$$

An alternative form of Equation (3.17) can be written in view of Equation (3.12), and of the area averaged variables defined by Equation (3.2). Hence, the divergence theorem valid for surfaces becomes:

$$\iint\limits_{A_{Kc}} \nabla \cdot \vec{F}_K \, dA = \frac{\partial}{\partial x} \left(A_{Kc} \langle\!\langle F_{Kx} \rangle\!\rangle \right) + \sum_{\ell=e,i} \int\limits_{S_\ell} \vec{F}_K \cdot \hat{n}_K \, \frac{dA}{dx} \quad (3.20)$$

In which $\vec{F}_K \cdot \hat{n}_K = F_{Kx}$ was used since, by definition, $\hat{n}_K = \hat{i}$.

The Leibnitz's Theorem Over Surfaces. The generalized Leibnitz's theorem for a volume element $\mathcal{V}_K$ is

$$\iiint\limits_{\mathcal{V}_K(t)} \frac{\partial \Psi_K}{\partial t} \, dV = \frac{d}{dt} \iiint\limits_{\mathcal{V}_K(t)} \Psi_K \, dV - \iint\limits_{A_K(t)} \Psi_K (\vec{v}_\ell \cdot \hat{n}_K) \, dA \quad (3.21)$$

where $\vec{v}_\ell$ is the velocity of the surface surrounding the volume element $\mathcal{V}_K$.

Use of Equation (3.8) in (3.21) yields:

$$\int\limits_x^{x+\Delta x} \iint\limits_{A_{Kc}(x,t)} \frac{\partial \Psi_K}{\partial t} \, dA \, dx = \frac{d}{dt} \int\limits_x^{x+\Delta x} \iint\limits_{A_{Kc}(x,t)} \Psi_K \, dA \, dx - \iint\limits_{A_{Kc}|_x} (\vec{v}_\ell \cdot \hat{n}_K) \, \Psi_K \, dA \quad (3.22)$$

$$- \iint\limits_{A_{Kc}|_{x+\Delta x}} (\vec{v}_\ell \cdot \hat{n}_K) \, \Psi_K \, dA - \sum_{\ell=e,i} \int\limits_x^{x+\Delta x} \int\limits_{S_\ell} (\vec{v}_\ell \cdot \hat{n}_K) \, \Psi_K \, dA$$

On the right-hand side the time derivative is taken inside the

integration over x and the integral is then estimated by means of the mean-value theorem for integrals. Thus

$$\frac{d}{dt} \int_{x}^{x+\Delta x} \iint_{A_{kc}} \Psi_k \, dA \, dx = \Delta x \frac{\partial}{\partial t} \iint_{A_{kc}(x_0,t)} \Psi_k \, dA \tag{3.23}$$

For the last integral in Equation (3.22), Equation (3.18) is used. After substitution Equations (3.18) and (3.23) in Equation (3.22) the resulting equation is, then, divided by Δx and the limit $\Delta x \to 0$ is taken. During this process x_0 and x_1 take on the value x. Thus

$$\iint_{A_{kc}(x,t)} \frac{\partial \Psi_k}{\partial t} \, dA = \frac{\partial}{\partial t} \iint_{A_{kc}(x,t)} \Psi_k \, dA - \sum_{l=e,i} \int_{J_l} (\vec{v}_l \cdot \hat{n}_k) \Psi_k \frac{dA}{dx} \tag{3.24}$$

which is the desired Leibnitz's theorem valid for surfaces. Finally, in terms of area averages it can be expressed as:

$$\iint_{A_{kc}(x,t)} \frac{\partial \Psi_k}{\partial t} \, dA = \frac{\partial}{\partial t} \left(A_{kc} \ll \Psi_k \gg \right) - \sum_{l=e,i} \int_{J_l} (\vec{v}_l \cdot \hat{n}_k) \Psi_k \frac{dA}{dx} \tag{3.25}$$

It should be noted that in arriving at this equation it has been assumed that the external boundary can move with time. If it is fixed, i.e., $\vec{v}_l = 0$, then the Leibnitz's theorem over surfaces becomes:

$$\iint_{A_{kc}(x,t)} \frac{\partial \Psi_k}{\partial t} \, dA = \frac{\partial}{\partial t} \left(A_{kc} \ll \Psi_k \gg \right) - \int_{J_i} (\vec{v}_i \cdot \hat{n}_k) \Psi_k \frac{dA}{dx} \tag{3.26}$$

3.4 Characteristics of Two-Phase Flow Field

As it was noted in Section 3.1, there are certain parameters which characterize the two-phase flow field. Therefore, it is necessary to define these parameters carefully for a separated flow structure

and to discuss their interrelations. In what follows we shall present

these parameters under the following topics:

1. Concentrations in two-phase flow,

2. Mixture properties in two-phase flow,

3. Velocity fields in two-phase flow,

4. Flux fields in two-phase flow.

3.4.1 Concentrations in Two-Phase Flow

Concentrations can be defined properly <u>only</u> in terms of static

parameters, although one finds in the literature on two-phase flows

definitions of concentrations defined in terms of kinematic*, i.e.,

flowing parameters. In the former case one restrict the attention to a

region in the flow field and then observes the fraction of a static

element of volume (or mass) which is occupied at any instant by a given

phase. Whereas in the latter case, one directs his attention to a sur-

face in the flow field and then measures the fraction of the total <u>flow</u>,

by mass or by volume, across the given surface area which is composed

of two phases.

Usually in the literature, the kinematic and static concentrations

have been used interchangeably or simultaneously without proper defini-

tions. Such a carelessness leads to errors and difficulties which are

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*The kinematic parameters, i.e., the "flowing" volumetric concen-
tration β, and the quality X, are defined in Appendix A where we show
their relation to the static parameters, i.e., the volumetric concentra-
tion α, and the mass concentration C. The errors which stem from such
improper formulation in terms of the "kinematic" flowing volumetric con-
centration β, and of the flowing mass concentration X, (quality), are
discussed in Appendix A.

discussed further in Appendix A.

   Static Concentrations. As noted above, in order to define properly
static concentrations one directs his attention to a static volume
element, however small, in the mixture. For example, one can select a
volume element in such a way that it may be occupied by both phases in
variable proportions or that it may be occupied alternately by one and
only one phase. In the latter case it would be necessary to use time
averages in order to define the local occurrence of a phase whereas in
the former case, it is possible to define a local area-average.

   In view of the geometrical appearance of the separated flow, a
definition of concentration in terms of local occurrences is not meaning-
ful. however, a meaningful definition of concentration can be made by
considering a volume element which includes the total cross-sectional
area of the channel. Following this point of view we can define the
volumetric concentration of phase 2, α_{v}^{*} , i.e., the void fraction as the
volume of phase 2 per unit volume of the mixture. Thus

$$\alpha_{v} = V_{2} / V_{T}$$
(3.27)

Here, V_{T} is the volume element of the mixture consisting of two volumes,
V_{1} and V_{2}, occupied by phase 1 and 2, respectively.

   As noted in previous section we shall use the area averages
rather than the volume averages; therefore, it is necessary to define
the area void fraction. For this purpose, let us consider the volume V_{T}

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   *In the literature the symbol α is used loosely to represent the
volumetric void fraction as well as the area void fraction. In order to
avoid possible confusion between these two definitions we distinguished
the former one by subscript v representing the void based on volume.

contained between two parallel planes cutting along two-cross sections $A_{Tc}|_x$ and $A_{Tc}|_{x+\Delta x}$ infinitesimally close together, distant from Δx, perpendicular to x axis, see Figure 2. Hence from Equation (3.27) it can be written that

$$\alpha_{\mathcal{V}} = \frac{\int_x^{x+\Delta x} A_{2c}(x,t)\,dx}{\int_x^{x+\Delta x} A_{Tc}(x,t)\,dx} \qquad (3.28)$$

in which the integrals can be estimated by means of the mean-value theorem for integrals. Thus

$$\alpha_{\mathcal{V}}(x_o,x,t) = \frac{A_{2c}(x_2,t)}{A_{Tc}(x_o)} \qquad (3.29)$$

where x_0 and x_2 must lie between x and $x+\Delta x$.

By taking the limit $\Delta x \to 0$, i.e., $x_0, x_2 \to x$, we obtain the area-averaged void fraction, α, as:

$$\alpha(x,t) = \lim_{\Delta x \to 0} \alpha_{\mathcal{V}} = \frac{A_{2c}(x,t)}{A_{Tc}(x)} \qquad (3.30a)$$

From this equation it is evident that

$$(1-\alpha)[x,t] = \frac{A_{1c}(x,t)}{A_{Tc}(x)} \qquad (3.30b)$$

which represents the area concentration of the second phase.

The mass concentration of phase 2, C, is defined as the mass of phase 2 per unit mass of the mixture. Thus,

$$C_{\mathcal{V}} = \frac{M_2}{M_T} = \frac{\iiint\limits_{\mathcal{V}_2} \varrho_2\,d\mathcal{V}}{\iiint\limits_{\mathcal{V}_1} \varrho_1\,d\mathcal{V} + \iiint\limits_{\mathcal{V}_2} \varrho_2\,d\mathcal{V}} \qquad (3.31)$$

where M_T is the total mass of the mixture while M_K, $(K = 1,2)$, is the mass of the K-th phase in volume V_T, and ρ_K is the mass density of the K-th phase.

The derivation of the area mass fraction follows closely the method used in derivating the area void fraction thus the <u>area-weighed mass concentration</u>, C, becomes

$$C(x,t) = \frac{\alpha \langle\!\langle \rho_2 \rangle\!\rangle}{(1-\alpha) \langle\!\langle \rho_1 \rangle\!\rangle + \alpha \langle\!\langle \rho_2 \rangle\!\rangle} \tag{3.32a}$$

and consequently

$$(1-C)[x,t] = \frac{(1-\alpha) \langle\!\langle \rho_1 \rangle\!\rangle}{(1-\alpha) \langle\!\langle \rho_1 \rangle\!\rangle + \alpha \langle\!\langle \rho_2 \rangle\!\rangle} \tag{3.32b}$$

which represents the area-weighted mass concentration of the second phase.

It should be noted that both α and C are defined in terms of the static parameters of the mixture and, therefore, do not depend on the kinematics of the flow field and constitutive equations for the two-phase system.

Having defined the static concentrations we can define now the <u>correct mixture density</u> in terms of static parameters. Again, by taking a volume element and isolating it from the rest of the mixture we can write the instantaneous mixture mass as:

$$M_T = \sum_{K=1}^{2} M_K \tag{3.33}$$

which can be expanded in terms of volumes and mass densities to yield

$$\rho_m = \frac{1}{\mathcal{V}_T} \sum_{k=1}^{2} \rho_k \mathcal{V}_k \qquad (3.34)$$

where ρ_m is defined as the <u>mixture density</u>.

Using the relation between area and volume void fractions, in the limit as $\Delta x \to 0$, the mixture density can be expressed in terms of the volumetric concentration α, thus

$$\rho_m = (1-\alpha) \langle\!\langle \rho_1 \rangle\!\rangle + \alpha \langle\!\langle \rho_2 \rangle\!\rangle \qquad (3.35)$$

in which $(1-\alpha)\langle\!\langle\rho_1\rangle\!\rangle$ and $\alpha\langle\!\langle\rho_2\rangle\!\rangle$ are nothing but <u>partial</u> densities or peculiar densities, used by some authors, of phase 1 and 2, respectively.

Substitution of Equation (3.35) in (3.32), gives compact expression for the mass concentration as:

$$C = \alpha \langle\!\langle \rho_2 \rangle\!\rangle \big/ \rho_m \qquad (3.36a)$$

$$1 - C = (1-\alpha) \langle\!\langle \rho_1 \rangle\!\rangle \big/ \rho_m \qquad (3.36b)$$

Dividing Equation (3.36a) by $\langle\!\langle\rho_2\rangle\!\rangle$ and (3.36b) by $\langle\!\langle\rho_1\rangle\!\rangle$ and then adding, we obtain the mixture density expressed in terms of the mass concentration, C, thus

$$\frac{1}{\rho_m} = \frac{1-C}{\langle\!\langle \rho_1 \rangle\!\rangle} + \frac{C}{\langle\!\langle \rho_2 \rangle\!\rangle} \qquad (3.37)$$

3.4.2 <u>Mixture Properties</u>

For analyzing a two-phase flow problem as a mixture, it is essential to express the results in terms of the center of mass. This is required, as Prigogine pointed out in connection with the thermodynamics of gas mixtures, in order to express correctly the properties of the mixture.

If this is not done, one can not express properly the thermodynamic relations for the mixture.

For this purpose let ψ_m be a mixture quantity associated to a unit mass of the mixture and ψ_K be a quantity related to a unit mass of the K-th phase. Applying the principle of an additive set of functions to express the properties, we obtain

$$\iiint_{V_T(t)} \rho_m \, \psi_m \, dV = \sum_{K=1}^{2} \iiint_{V_k(t)} \rho_k \, \psi_k \, dV \tag{3.38}$$

Using the mean-value theorem for integrals and then taking the limit as $\Delta x \to 0$, we can express the volume integrals in terms of area integrals, thus

$$\psi_m = (1-\alpha) \frac{\langle\!\langle \rho_1 \psi_1 \rangle\!\rangle}{\rho_m} + \alpha \frac{\langle\!\langle \rho_2 \psi_2 \rangle\!\rangle}{\rho_m} \tag{3.39a}$$

which can be expressed also in terms of the mass concentrations, C, thus

$$\psi_m = (1-C) \frac{\langle\!\langle \rho_1 \psi_1 \rangle\!\rangle}{\langle\!\langle \rho_1 \rangle\!\rangle} + C \frac{\langle\!\langle \rho_2 \psi_2 \rangle\!\rangle}{\langle\!\langle \rho_2 \rangle\!\rangle} \tag{3.39b}$$

Introducing Equation (3.4), we can express the mixture property ψ_m, in terms of mass-weighted averages, thus

$$\psi_m = (1-\alpha) \frac{\langle\!\langle \rho_1 \rangle\!\rangle}{\rho_m} \langle \psi_1 \rangle + \alpha \frac{\langle\!\langle \rho_2 \rangle\!\rangle}{\rho_m} \langle \psi_2 \rangle \tag{3.40a}$$

$$\psi_m = (1-C) \langle \psi_1 \rangle + C \langle \psi_2 \rangle \tag{3.40b}$$

Equation (3.40) expresses any mixture property applied to the center of gravity in terms of individual phase properties and of static

concentrations. By substituting proper identifications for ψ, we obtain the mixture density, internal energy, enthalpy, entropy, etc., which are summarized in Table 1.

Table 1. Mixture Properties

Mixture Properties	
Density	$\rho_m = (1-\alpha) \ll \rho_1 \gg + \alpha \ll \rho_2 \gg$ $\dfrac{1}{\rho_m} = (1-C) / \ll \rho_1 \gg + C / \ll \rho_2 \gg$
Internal Energy	$u_m = \left[(1-\alpha) \ll \rho_1 \gg <u_1> + \alpha \ll \rho_2 \gg <u_2> \right] \dfrac{1}{\rho_m}$ $= (1-C) <u_1> + C <u_2>$
Enthalpy	$i_m = \left[(1-\alpha) \ll \rho_1 \gg <i_1> + \alpha \ll \rho_2 \gg <i_2> \right] \dfrac{1}{\rho_m}$ $= (1-C) <i_1> + C <i_2>$
Entropy	$s_m = \left[(1-\alpha) \ll \rho_1 \gg <s_1> + \alpha \ll \rho_2 \gg <s_2> \right] \dfrac{1}{\rho_m}$ $= (1-C) <s_1> + C <s_2>$

3.4.3 Velocity Fields in Two-Phase Flow

Since in a two-phase flow system, the velocities of the two phases are never equal, a two-phase flow problem always must be formulated in terms of two velocity fields. However, there are several velocity fields that are useful in analyzing various aspects of a two-phase flow system. Depending upon the particular aspect, one can select a frame of reference and formulate the problem in terms of the velocity fields that are most

representative of, and appropriate to, the solution of that particular problem. In the following we shall present these velocity fields which can be used in a two-phase flow problem analysis.

The flow field in two-phase system can be formulated either by considering each phase separately or by considering the entire mixture. In the first case, the velocity fields are expressed in terms of the velocities of the individual phases. This formulation leads therefore to the two-fluid model. In the second case, the velocity fields are expressed in terms of the relevant velocities of the mixture (the velocities of the center of mass and the center of volume) and of the relative, i.e., diffusional velocities of the two phase with respect to these mixture velocities. Therefore, the latter formulation is more suitable for application to the diffusional model.

In order to derive the velocities for the individual phases let $\vec{V}_K = \vec{V}_K(x,y,z,t)$ be the local velocity of the K-th phase, then substituting for in Equation (3.4) we obtain the velocity of the center of mass of the K-th phase $\langle \vec{v}_K \rangle$, i.e., mass-weighed, area-averaged velocity of the K-th phase, thus

$$\langle \vec{v}_k \rangle (x,t) = \langle\!\langle \varrho_k \vec{v}_k \rangle\!\rangle \big/ \langle\!\langle \varrho_k \rangle\!\rangle \quad (K = 1,2) \quad (3.41)$$

In order to derive the expression for the velocity of the mixture we consider the additivity principle for set of functions which, in the case of linear momentum states that the total linear momentum of the mixture in any region is the sum of the linear momenta of the constituents presently occupying that region. This principle can be formulated as:

$$\iiint_{\mathcal{V}_T} \varrho_m \vec{v}_m \, d\mathcal{V} = \sum_{k=1}^{2} \iiint_{\mathcal{V}_k} \varrho_k \vec{v}_k \, d\mathcal{V} \qquad (3.42)$$

We note that this expression gives rise to the definition of <u>the</u>
<u>local mixture velocity</u> $\vec{v}_m$, acting on the center of gravity of the volume
element $\mathcal{V}_T$. Considering a volume element contained between two parallel
planes $A_{Tc}\big|_x$ and $A_{Tc}\big|_{x+\Delta x}$, infinitesimally close together, we can express
the mixture velocity in terms of area averages. As before reduce the
volumetric averages to area averages, by considering the limits as
$\Delta x \rightarrow 0$, thus

$$\vec{v}_m(x,t) = (1-\alpha)\,\frac{\langle\!\langle \varrho_1 \vec{v}_1 \rangle\!\rangle}{\varrho_m} + \alpha\,\frac{\langle\!\langle \varrho_2 \vec{v}_2 \rangle\!\rangle}{\varrho_m} \qquad (3.43a)$$

$$= (1-C)\,\frac{\langle\!\langle \varrho_1 \vec{v}_1 \rangle\!\rangle}{\langle\!\langle \varrho_1 \rangle\!\rangle} + C\,\frac{\langle\!\langle \varrho_2 \vec{v}_2 \rangle\!\rangle}{\langle\!\langle \varrho_2 \rangle\!\rangle} \qquad (3.43b)$$

Introducing the definitions of the mass-weighed average velocity, we can
express the <u>mixture velocity</u>, thus

$$\vec{v}_m = (1-\alpha)\,\frac{\langle\!\langle \varrho_1 \rangle\!\rangle}{\varrho_m} \langle \vec{v}_1 \rangle + \alpha\,\frac{\langle\!\langle \varrho_2 \rangle\!\rangle}{\varrho_m} \langle \vec{v}_2 \rangle^{*} \qquad (3.44a)$$

$$= (1-C)\,\langle \vec{v}_1 \rangle + C\,\langle \vec{v}_2 \rangle \qquad (3.44b)$$

The following observation can be made with respect to Equation
(3.44). We note that Equation (3.44b) is an averaged velocity of the
mixture, obtained by weighing the respective velocities of the two phases

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*For the purpose of simplicity we have dropped the symbols (x,t)
representing the dependence of x and t. The dependence on x, t is assumed
to be understood.

by the weight factors (1-C) and C, which are proportional to the masses occupied by phase 1 and 2, respectively. Consequently, Equation (3.44) can be interpreted as <u>the velocity of the center of mass of the mixture</u>.

In multiphase or multi-component systems one is frequently interested in the velocity of a given phase with respect to the center of gravity of the mixture. This leads to the definition of the <u>diffusion velocity</u> $\vec{V}_{Km}$, of the K-th phase with respect to the center of mass, given by

$$\vec{V}_{km} = \langle \vec{v}_k \rangle - \vec{v}_m \, , \, K = 1, 2 \qquad (3.45)$$

By means of Equations (3.44) and (3.45), it can be easily shown that the diffusion velocities of phase 1 $\vec{v}_{1m}$, and phase 2 $\vec{v}_{2m}$, can be expressed also by

$$\vec{V}_{1m} = - C \, \vec{V}_r = - \frac{\langle\!\langle \ell_2 \rangle\!\rangle}{\ell_m} \propto \vec{V}_r \qquad (3.46a)$$

$$\vec{V}_{2m} = (1-C)\vec{V}_r = \frac{\langle\!\langle \ell_1 \rangle\!\rangle}{\ell_m} (1-\alpha)\vec{V}_r \qquad (3.46b)$$

where $\vec{V}_r$ is the relative velocity, defined by

$$\vec{V}_r = \langle \vec{v}_2 \rangle - \langle \vec{v}_1 \rangle \qquad (3.47)$$

From Equation (3.46) it can be, further, shown that the following relations hold:

$$\vec{V}_{2m} = - \frac{1-C}{C} \, \vec{V}_{1m} \qquad (3.48)$$

$$\vec{V}_r = \vec{V}_{2m} - \vec{V}_{1m} \qquad (3.49)$$

It can be seen from Equations (3.46), (3.47) and (3.49) that, when the effect of the relative velocity is neglected as it is in the case of homogenous flow model, then

$$\vec{V}_{1m} = \vec{V}_{2m} = 0 \qquad (3.50)$$

and it follows from Equation (3.42) that

$$\langle \vec{v}_1 \rangle = \langle \vec{v}_2 \rangle = \vec{v}_m \qquad (3.51)$$

thus, the two phases have the same velocity, which is equal to the velocity of the center of gravity of the mixture.

If we are interested, for example, in determining the response of the volumetric concentration to changes of pressure, power, etc., it may be advantageous to formulate the problem in terms of the velocity of the center of volume $\vec{j}$, and of the drift velocities $\vec{V}_{1j}$ and $\vec{V}_{2j}$ of phase 1 and 2 with respect to $\vec{j}$.

We define the volumetric flux densities of phase 1 $\vec{j}_1$, and of phase 2 $\vec{j}_2$ by

$$\vec{j}_1 = \frac{A_{1c}}{A_{Tc}} \frac{\iint_{A_{1c}} \rho_1 \vec{v}_1 \, dA}{\iint_{A_{1c}} \rho_1 \, dA} = (1-\alpha)\langle \vec{v}_1 \rangle \qquad (3.52a)$$

$$\vec{j}_2 = \frac{A_{2c}}{A_{Tc}} \frac{\iint_{A_{2c}} \rho_2 \vec{v}_2 \, dA}{\iint_{A_{2c}} \rho_2 \, dA} = \alpha \langle \vec{v}_2 \rangle \qquad (3.52b)$$

and the volumetric flux density of the mixture by

$$\vec{j} = \vec{j}_1 + \vec{j}_2 \qquad (3.53)$$

or in view of Equation (3.52), by

$$\vec{j} = (1-\alpha) \langle \vec{v_1} \rangle + \alpha \langle \vec{v_2} \rangle \qquad (3.54)$$

Two observations can be made with respect to Equations (3.52) and (3.54). First, it should be noted that these equations correspond mathematically to the definitions of the number velocities in the kinetic theory of gases. We note, further, that Equation (3.54) is an average velocity of the mixture, obtained by weighing the respective velocities of the two phases by the weight factors (1-α) and α, which are proportional to the two volumes occupied by the liquid and the vapor phase, respectively. Consequently, Equation (3.54) can be interpreted either as the volumetric flux density of the mixture or the velocity of the center of volume of the mixture.

In analogy with the kinetic theory of gases and of mixtures, we define the drift velocity $\vec{V}_{Kj}$, of the K-th phase with respect to the center of volume of the mixture by

$$\vec{V}_{kj} = \langle \vec{v}_k \rangle - \vec{j} \qquad (3.55)$$

By means of Equations (3.52) through (3.55) it can be shown that the drift velocities of phase 1 and 2 can be expressed by

$$\vec{V}_{1j} = -\alpha \vec{V}_r \qquad (3.56a)$$

$$\vec{V}_{2j} = (1-\alpha) \vec{V}_r \qquad (3.56b)$$

$$\vec{V}_r = \vec{V}_{2j} - \vec{V}_{1j} \qquad (3.57)$$

It can be seen from Equations (3.56) and (3.57) that, if $\vec{V}_r = 0$,

then

$$\vec{V}_{ij} = \vec{V}_{2i} = 0 \qquad (3.58)$$

and it follows from Equation (3.56) that

$$\langle \vec{V_1} \rangle = \langle \vec{V_2} \rangle = \vec{j} \qquad (3.59)$$

thus, the two phases have the same velocity, which is equal to the volumetric flux density of the mixture.

It can be seen, further, from Equations (3.51) and (3.59) that

$$\vec{V}_m = \vec{j} \qquad (3.60)$$

which shows that in case of homogenous flow velocities of the center of mass and of volume of the mixture are equal.

However, when the effect of relative velocity is taken into account, then from Equations (3.44) and (3.54) it is easy to derive that

$$\vec{j} - \vec{V}_m = \alpha (1-\alpha) \frac{\Delta \rho}{\rho_m} \vec{V}_r = C(1-C) \frac{\Delta \rho \, \rho_m}{\langle\langle \rho_1 \rangle\rangle \langle\langle \rho_2 \rangle\rangle} \vec{V}_r \qquad (3.61)$$

which states that, the center of mass and the center of volume of the mixture move with <u>different</u> velocities. For example, it can be expected that in separated flow systems the center of mass will move with a velocity close to that of the heavy phase (which accounts for most of the mass), whereas the velocity of the center of volume will move with a velocity that is closer to that of the vapor (which accounts for most of the volume).

It can be expected, further, that this difference between the velocities of the center of mass and of the volume will effect the dynamic

characteristics of a two-phase mixture.

Relations Between the Velocity Fields. After having defined

different velocity fields in two-phase flow system, we shall discuss

briefly how these fields can be used in a formulating of a two-phase

flow problem.

Mathematically speaking, any two of the preceding velocity fields

are adequate for all two-phase flow problems but each has certain advan-

tages. For example, one could formulate the problem in terms of area

averaged velocities, $\langle \vec{v}_1 \rangle$ and $\langle \vec{v}_2 \rangle$. As another alternative, however,

one could eliminate $\langle \vec{v}_2 \rangle$, for example, by means of the relative velocity

$\vec{V}_r$ and formulate the problem in terms of $\langle \vec{v}_1 \rangle$ and of $\vec{V}_r$, etc.

It is advantageous, however, to follow the well-established method

developed in the kinetic theory of gas mixtures, and formulate the

problem in terms of the velocity of the center of mass $\vec{v}_m$ and of the

velocities relative to this moving frame of reference, i.e., the

diffusion velocities. However, since the expressions for the drift

velocities have been studied and established for different two-phase

flow regimes, references [47 and 48], it may be more advantageous to use

the drift velocities instead of the diffusion velocities.

In Table 2 we summarized the velocity fields for six independent

sets of velocities. Two independent velocity fields are $\langle \vec{v}_1 \rangle$ and $\langle \vec{v}_2 \rangle$

in the second column, $\vec{v}_m$ and $\vec{V}_{1m}$ in the third column, $\vec{v}_m$ and $\vec{V}_{2m}$ in the

fourth column, $\vec{v}_m$ and $\vec{V}_{2j}$ in the fifth column, $\vec{j}_1$ and $\vec{j}_2$ in the sixth

column, $\vec{j}$ and $\vec{V}_{1j}$ in the seventh column and $\vec{j}$ and $\vec{V}_{2j}$ in the eighth

column.

Table 2. Velocities in Terms of Different Velocity Fields

Dependent Velocity	Independent Velocity Fields		
	$\langle \vec{V_1} \rangle$ and $\langle \vec{V_2} \rangle$	$\vec{V}_m$ and $\vec{V}_{1m}$	$\vec{V}_m$ and $\vec{V}_{2m}$
$\langle \vec{V_1} \rangle$	$\langle \vec{V_1} \rangle$	$\vec{V}_m + \vec{V}_{1m}$	$\vec{V}_m - \frac{c}{1-c} \vec{V}_{2m}$ $= \vec{V}_m - \frac{\alpha}{1-\alpha} \frac{\langle\langle \rho_2 \rangle\rangle}{\langle\langle \rho_1 \rangle\rangle} \vec{V}_{2m}$
$\langle \vec{V_2} \rangle$	$\langle \vec{V_2} \rangle$	$\vec{V}_m - \frac{1-c}{c} \vec{V}_{1m}$ $= \vec{V}_m - \frac{1-\alpha}{\alpha} \frac{\langle\langle \rho_1 \rangle\rangle}{\langle\langle \rho_2 \rangle\rangle} \vec{V}_{1m}$	$\vec{V}_m + \vec{V}_{2m}$
$\vec{V}_m$	$(1-c)\langle \vec{V_1} \rangle + c \langle \vec{V_2} \rangle$ $= (1-\alpha) \frac{\langle\langle \rho_1 \rangle\rangle}{\rho_m} \langle \vec{V_1} \rangle + \alpha \frac{\langle\langle \rho_2 \rangle\rangle}{\rho_m} \langle \vec{V_2} \rangle$	$\vec{V}_m$	$\vec{V}_m$
$\vec{V}_{1m}$	$-c(\langle \vec{V_2} \rangle - \langle \vec{V_1} \rangle)$ $= -\alpha \frac{\langle\langle \rho_2 \rangle\rangle}{\rho_m} (\langle \vec{V_2} \rangle - \langle \vec{V_1} \rangle)$	$\vec{V}_{1m}$	$-\frac{c}{1-c} \vec{V}_{2m}$ $= -\frac{\alpha}{1-\alpha} \frac{\langle\langle \rho_2 \rangle\rangle}{\langle\langle \rho_1 \rangle\rangle} \vec{V}_{2m}$
$\vec{V}_{2m}$	$(1-c)(\langle \vec{V_2} \rangle - \langle \vec{V_1} \rangle)$ $= (1-\alpha) \frac{\langle\langle \rho_1 \rangle\rangle}{\rho_m} (\langle \vec{V_2} \rangle - \langle \vec{V_1} \rangle)$	$-\frac{1-c}{c} \vec{V}_{1m}$ $= -\frac{1-\alpha}{\alpha} \frac{\langle\langle \rho_1 \rangle\rangle}{\langle\langle \rho_2 \rangle\rangle} \vec{V}_{1m}$	$\vec{V}_{2m}$
$\vec{J_1}$	$(1-\alpha)\langle \vec{V_1} \rangle$	$(1-\alpha)(\vec{V}_m + \vec{V}_{1m})$	$(1-\alpha)\vec{V}_m - \alpha \frac{\langle\langle \rho_2 \rangle\rangle}{\langle\langle \rho_1 \rangle\rangle} \vec{V}_{2m}$
$\vec{J_2}$	$\alpha \langle \vec{V_2} \rangle$	$\alpha \vec{V}_m - \frac{(1-\alpha)\langle\langle \rho_1 \rangle\rangle}{\langle\langle \rho_2 \rangle\rangle} \vec{V}_{1m}$	$\alpha(\vec{V}_m + \vec{V}_{2m})$
$\vec{J}$	$(1-\alpha)\langle \vec{V_1} \rangle + \alpha \langle \vec{V_2} \rangle$	$\vec{V}_m - (1-\alpha) \frac{\Delta \rho}{\langle\langle \rho_1 \rangle\rangle} \vec{V}_{1m}$	$\vec{V}_m + \alpha \frac{\Delta \rho}{\langle\langle \rho_2 \rangle\rangle} \vec{V}_{2m}$
$\vec{V}_{1j}$	$-\alpha(\langle \vec{V_2} \rangle - \langle \vec{V_1} \rangle)$	$\frac{\rho_m}{\langle\langle \rho_1 \rangle\rangle} \vec{V}_{1m}$	$-\frac{\alpha}{1-\alpha} \frac{\rho_m}{\langle\langle \rho_2 \rangle\rangle} \vec{V}_{2m}$
$\vec{V}_{2j}$	$(1-\alpha)(\langle \vec{V_2} \rangle - \langle \vec{V_1} \rangle)$	$-\frac{1-\alpha}{\alpha} \frac{\rho_m}{\langle\langle \rho_1 \rangle\rangle} \vec{V}_{1m}$	$\frac{\rho_m}{\langle\langle \rho_2 \rangle\rangle} \vec{V}_{2m}$
$\vec{V}_r$	$\langle \vec{V_2} \rangle - \langle \vec{V_1} \rangle$	$-\frac{\vec{V}_{1m}}{c} \vec{V}_{1m}$ $= -\frac{\rho_m}{\alpha\langle\langle \rho_2 \rangle\rangle} \vec{V}_{1m}$	$\frac{\vec{V}_{2m}}{1-c}$ $= \frac{\rho_m}{(1-\alpha)\langle\langle \rho_1 \rangle\rangle} \vec{V}_{2m}$

Table 2. Velocities in Terms of Different Velocity Fields (Continued)

Independent Velocity Fields			
$\vec{V}_m$ and $\vec{V}_{2j}$	$\vec{J}_1$ and $\vec{J}_2$	$\vec{J}$ and $\vec{V}_{1j}$	$\vec{J}$ and $\vec{V}_{2j}$
$\vec{V}_m - \frac{\alpha}{1-\alpha}\frac{\langle\langle\rho_2\rangle\rangle}{\rho_m}\vec{V}_{2j}$	$\frac{\vec{J}_1}{1-\alpha}$	$\vec{J}+\vec{V}_{1j}$	$\vec{J}-\frac{\alpha}{1-\alpha}\vec{V}_{2j}$
$\vec{V}_m + \frac{\langle\langle\rho_2\rangle\rangle}{\rho_m}\vec{V}_{2j}$	$\frac{\vec{J}_2}{\alpha}$	$\vec{J}-\frac{1-\alpha}{\alpha}\vec{V}_{1j}$	$\vec{J}+\vec{V}_{2j}$
$\vec{V}_m$	$\frac{1}{\rho_m}(\langle\langle\rho_1\rangle\rangle\vec{J}_1 + \langle\langle\rho_2\rangle\rangle\vec{J}_2)$	$\vec{J}+(1-\alpha)\frac{\Delta\rho}{\rho_m}\vec{V}_{1j}$	$\vec{J}-\alpha\frac{\Delta\rho}{\rho_m}\vec{V}_{2j}$
$-\frac{\alpha}{1-\alpha}\frac{\langle\langle\rho_2\rangle\rangle}{\rho_m}\vec{V}_{2j}$	$\frac{\langle\langle\rho_2\rangle\rangle}{(1-\alpha)\rho_m}[\alpha\vec{J}_1 - (1-\alpha)\vec{J}_2]$	$\frac{\langle\langle\rho_2\rangle\rangle}{\rho_m}\vec{V}_{1j}$	$-\frac{\alpha}{1-\alpha}\frac{\langle\langle\rho_2\rangle\rangle}{\rho_m}\vec{V}_{2j}$
$\frac{\langle\langle\rho_2\rangle\rangle}{\rho_m}\vec{V}_{2j}$	$\frac{\langle\langle\rho_2\rangle\rangle}{\alpha\rho_m}[(1-\alpha)\vec{J}_2 - \alpha\vec{J}_1]$	$-\frac{1-\alpha}{\alpha}\frac{\langle\langle\rho_2\rangle\rangle}{\rho_m}\vec{V}_{1j}$	$\frac{\langle\langle\rho_2\rangle\rangle}{\rho_m}\vec{V}_{2j}$
$(1-\alpha)\{\vec{V}_m - \frac{\alpha}{1-\alpha}\frac{\langle\langle\rho_2\rangle\rangle}{\rho_m}\vec{V}_{2j}\}$	$\vec{J}_1$	$(1-\alpha)(\vec{J}+\vec{V}_{1j})$	$(1-\alpha)\vec{J} - \alpha\vec{V}_{2j}$
$\alpha\{\vec{V}_m + \frac{\langle\langle\rho_2\rangle\rangle}{\rho_m}\vec{V}_{2j}\}$	$\vec{J}_2$	$\alpha\vec{J} - (1-\alpha)\vec{V}_{1j}$	$\alpha(\vec{J}+\vec{V}_{2j})$
$\vec{V}_m + \alpha\frac{\Delta\rho}{\rho_m}\vec{V}_{2j}$	$\vec{J}_1 + \vec{J}_2$	$\vec{J}$	$\vec{J}$
$-\frac{\alpha}{1-\alpha}\vec{V}_{2j}$	$\frac{\alpha\vec{J}_1 - (1-\alpha)\vec{J}_2}{1-\alpha}$	$\vec{V}_{1j}$	$-\frac{\alpha}{1-\alpha}\vec{V}_{2j}$
$\vec{V}_{2j}$	$\frac{(1-\alpha)\vec{J}_2 - \alpha\vec{J}_1}{\alpha}$	$-\frac{1-\alpha}{\alpha}\vec{V}_{1j}$	$\vec{V}_{2j}$
$\frac{1}{1-\alpha}\vec{V}_{2j}$	$\frac{(1-\alpha)\vec{J}_2 - \alpha\vec{J}_1}{\alpha(1-\alpha)}$	$-\frac{\vec{V}_{1j}}{\alpha}$	$\frac{\vec{V}_{2j}}{1-\alpha}$

3.4.4 Flux Fields

a) <u>With Respect to Fixed Frame of Reference</u>

The <u>mass flow rates</u> for the two phases are

$$W_1 = \iint\limits_{A_{1c}} \rho_1\, v_{1x}\, dA = A_{1c} \ll \rho_1 v_{1x} \gg \qquad (3.62a)$$

$$W_2 = \iint\limits_{A_{2c}} \rho_2\, v_{2x}\, dA = A_{2c} \ll \rho_2 v_{2x} \gg \qquad (3.62b)$$

and combining Equation (3.41) with Equation (3.62) we obtain

$$W_1 = A_{1c} \ll \rho_1 \gg \langle v_{1x} \rangle \qquad (3.63a)$$

$$W_2 = A_{2c} \ll \rho_2 \gg \langle v_{2x} \rangle \qquad (3.63b)$$

We define the <u>mass flux</u> of each phase by

$$G_1 = W_1 \,/\, A_{T_c} \qquad (3.64a)$$

$$G_2 = W_2 \,/\, A_{T_c} \qquad (3.64b)$$

Substituting Equation (3.63) in Equation (3.64)

$$G_1 = (1-\alpha) \ll \rho_1 \gg \langle v_{1x} \rangle \qquad (3.65a)$$

$$G_2 = \alpha \ll \rho_2 \gg \langle v_{2x} \rangle \qquad (3.65b)$$

For the mixture we have thus

$$W_T = W_1 + W_2 = A_{T_c}\, \rho_m\, v_{mx} \qquad (3.66)$$

and

$$G_T = G_1 + G_2 = \rho_M \, V_{mx} \qquad (3.67)$$

Thus the latter mass flux G_T is the sum of the two linear momenta of the phases.

The <u>volumetric flow rates</u> for the two phases are

$$Q_1 = \frac{\iint_{A_{1c}} \rho_1 \, v_{1x} \, dA}{\iint_{A_{1c}} \rho_1 \, dA} \qquad (3.68a)$$

$$Q_2 = \frac{\iint_{A_{2c}} \rho_2 \, v_{2x} \, dA}{\iint_{A_{2c}} \rho_2 \, dA} \qquad (3.68b)$$

which, in view of Equation (3.41) becomes

$$Q_1 = A_{1c} \, \langle v_{1x} \rangle \qquad (3.69a)$$

$$Q_2 = A_{2c} \, \langle v_{2x} \rangle \qquad (3.69b)$$

The total flow rate for the mixture is:

$$Q_T = A_{Tc} \left[(1-\alpha) \, \langle v_{1x} \rangle + \alpha \, \langle v_{2x} \rangle \right] \qquad (3.70)$$

In view of Equations (3.69) and (3.52), the <u>volumetric flux densi-ties</u>, which were already defined in relation with the velocity fields, become

$$j_{1x} = \frac{Q_1}{A_{Tc}} = (1-\alpha) \, \langle v_{1x} \rangle \qquad (3.71a)$$

$$j_{2x} = \frac{Q_2}{A_{T_2}} = \alpha \langle v_{2x} \rangle \qquad (3.71b)$$

Finally, the <u>volumetric flux density of the mixture</u> become

$$j_x = \frac{Q_T}{A_{T_x}} = j_{1x} + j_{2x} = (1-\alpha) \langle v_{1x} \rangle + \alpha \langle v_{2x} \rangle \qquad (3.72)$$

b) <u>Volume Flux with Respect to the Center of Volume</u>

As j_1 and j_2 represent volumetric fluxes with respect to a fixed frame of reference, we can define also a volumetric flux with respect to a moving frame of reference say with a frame which moves with velocity $\vec{j}$, i.e., with the velocity of the center of volume, thus

$$j_{Kj} = j_K - j \qquad , K = 1,2 \qquad (3.73)$$

which, in view of Equations (3.71 and 72) and of the relations summarized in Table 2 becomes

$$j_{1j} = (1-\alpha) \ V_{1j} = -\alpha (1-\alpha) \ V_r \qquad (3.74a)$$

and

$$j_{2j} = \alpha \ V_{2j} = \alpha (1-\alpha) \ V_r \qquad (3.74b)$$

These two volume fluxes have the important property that their sum is zero which follows directly from Equation (3.74)

$$j_{1j} + j_{2j} = 0 \qquad (3.75)$$

c) <u>Mass Flux with Respect to the Center of Mass</u>

As G_1 and G_2 represent the mass fluxes with respect also to a fixed frame of reference, we can define two mass fluxes with respect to the center of mass, i.e., with respect to a reference frame moving with

velocity v_m, thus

$$G_{km} = G_k - v_m \qquad , K = 1,2 \qquad (3.76)$$

However, in view of ...tions (3.64 and 3.67) and the relations summarized on Table 2, mass fluxes with respect to the center of mass can be expressed as:

$$G_{1m} = (1-\alpha) \ll \rho_1 \gg V_{1m} = \rho_m (1-c) V_{1m} = -\rho_m c(1-c) V_r \qquad (3.77a)$$

and

$$G_{2m} = \alpha \ll \rho_2 \gg V_{2m} = \rho_m c V_{2m} = \rho_m c(1-c) V_r \qquad (3.77b)$$

It can be seen that the sum of these two mass fluxes is zero, that is

$$G_{1m} + G_{2m} = 0 \qquad (3.78)$$

3.5 Fundamental Identity

It will be seen in what follows that after integrating the field equations over the cross-sectional plane of each phase we obtain from the integration of the convective flux terms two area-averaged fluxes $(1-\alpha) \ll \rho_1 \vec{v}_1 \psi_1 \gg$ and $\alpha \ll \rho_2 \vec{v}_2 \psi_2 \gg$. Note that these two terms are fluxes with respect to the center of mass of phase 1 and phase 2, respectively. The fundamental question which arises when we consider the mixture is to determine the relation which exists between the mean convective flux $\rho_m \vec{v}_m \psi_m$, with respect to the center of mass of the mixture and the area averaged fluxes $(1-\alpha) \ll \rho_1 \vec{v}_1 \psi_1 \gg$ and $\alpha \ll \rho_2 \vec{v}_2 \psi_2 \gg$ with respect to the centers of mass of the individual phase.

In order to derive this relation, let us consider the mean flux of the mixture $\rho_m \vec{v}_m \psi_m$. From the derivations in the previous section it is obvious that

$$\rho_m \vec{v}_m \psi_m = \left[(1-\alpha) \langle\!\langle \rho_1 \rangle\!\rangle \langle \vec{v}_1 \rangle + \alpha \langle\!\langle \rho_2 \rangle\!\rangle \langle \vec{v}_2 \rangle \right] \left[(1-\alpha) \frac{\langle\!\langle \rho_1 \rangle\!\rangle}{\rho_m} \langle \psi_1 \rangle + \alpha \frac{\langle\!\langle \rho_2 \rangle\!\rangle}{\rho_m} \langle \psi_2 \rangle \right] \quad (3.79)$$

By expanding the right side of Equation (3.79), and then using the mixture density definition, we obtain

$$\rho_m \vec{v}_m \psi_m = (1-\alpha) \langle\!\langle \rho_1 \rangle\!\rangle \langle \vec{v}_1 \rangle \langle \psi_1 \rangle + \alpha \langle\!\langle \rho_2 \rangle\!\rangle \langle \vec{v}_2 \rangle \langle \psi_2 \rangle - \qquad (3.80)$$

$$\frac{1}{\rho_m} \alpha (1-\alpha) \langle\!\langle \rho_1 \rangle\!\rangle \langle\!\langle \rho_2 \rangle\!\rangle \left[\left(\langle \psi_2 \rangle - \langle \psi_1 \rangle \right) \left(\langle \vec{v}_2 \rangle - \langle \vec{v}_1 \rangle \right) \right]$$

In view of Equations (3.4), (3.8), and (3.47), it can also be expressed by

$$\rho_m \vec{v}_m \psi_m = (1-\alpha) \langle\!\langle \rho_1 \vec{v}_1 \psi_1 \rangle\!\rangle + \alpha \langle\!\langle \rho_2 \vec{v}_2 \psi_2 \rangle\!\rangle - \frac{1}{\rho_m} \alpha (1-\alpha) \langle\!\langle \rho_1 \rangle\!\rangle \langle\!\langle \rho_2 \rangle\!\rangle \vec{V}_r \Delta\langle\psi\rangle - (3.81)$$

$$(1-\alpha) \langle\!\langle \rho_1 \rangle\!\rangle \, \text{Cov}(\vec{v}_1 \cdot \psi_1) - \alpha \langle\!\langle \rho_2 \rangle\!\rangle \, \text{Cov}(\vec{v}_2 \cdot \psi_2)$$

where we have defined the difference in mass-weighted quantities $\Delta\langle\psi\rangle$ by

$$\Delta\langle\psi\rangle = \langle\psi_1\rangle - \langle\psi_2\rangle \qquad (3.82)$$

By means of Equation (3.36), it can be shown that the relation derived above can be also expressed in terms of the mass concentration C, thus

$$\rho_m \vec{v}_m \psi_m = (1-c) \rho_m \langle \vec{v}_1 \psi_1 \rangle + c \rho_m \langle \vec{v}_2 \psi_2 \rangle - c(1-c) \rho_m \vec{V}_r \Delta\langle\psi\rangle - (3.83)$$

$$(1-c) \rho_m \, \text{Cov}(\vec{v}_1 \cdot \psi_1) - c \rho_m \, \text{Cov}(\vec{v}_2 \cdot \psi_2)$$

Equation (3.81) or (3.83) is the fundamental identity which is of

great importance because all field equations for two-phase mixture are based on this identity.

The second term on the right hand side shows the effect of the relative velocity, can be interpreted easily by using the diffusion velocities $\vec{V}_{Km}$, (K = 1,2). In effect it follows from Equations (3.48) and (3.49) that

$$(1-c)\,\varrho_m \langle \vec{v}_1 \cdot \vec{\psi}_1 \rangle + c\,\varrho_m \langle \vec{v}_2 \cdot \vec{\psi}_2 \rangle = \varrho_m \vec{v}_m \cdot \vec{\psi}_m + \left[(1-c)\,\varrho_m \vec{V}_{1m} \langle \vec{\psi}_1 \rangle + c\,\varrho_m \vec{V}_{2m} \langle \vec{\psi}_2 \rangle \right] + \quad (3.84)$$

$$(1-c)\,\varrho_m \, Cov\,(\vec{v}_1 \cdot \vec{\psi}_1) + c\,\varrho_m \, Cov\,(\vec{v}_2 \cdot \vec{\psi}_2)$$

which states that, the total convective flux per unit area of the total channel cross-sectional area is equal to the mixture flux with respect to stationary coordinate system plus the diffusional fluxes with respect to the center of the gravity of the mixture.

Actually, the diffusive transport at the velocity $\vec{V}_{Km}$, (K = 1,2), of the quantities associated with the K-th phase takes into account the actions on the mixture if the mixture is conceived as a body in motion with the velocity $\vec{v}_m$.

The result expressed in Equation (3.84) is consistent with the conceptual basis of the kinetic theory of gas mixtures. Finally, the covariance terms shows the effect of the velocity and of the quantity distribution across the channel cross section.

An alternative expression for the fundamental identity was given by Truesdell, reference [23, pp. 87], in which the material derivative of the mean value is expressed in terms of the mean value of the material derivative.

3.6 Summary

In this chapter we have discussed the general characteristics and aspects of separated two-phase flow and furthermore we have presented the basic relations and parameters relevant to this flow regime. In particular:

1) The analytical methods and models applicable to separated two-phase flows were discussed. The formulations based on the two-fluid and on the diffusion models were described. The characteristic of each model as well as the specific advantages of each model were enumerated.

2) The applicability of a time average and of a space average was discussed. For separated flow two expressions for averaged properties were derived, one based on the area-averaged and the other on the mass-weighed area-averaged value.

3) The difference between the concentrations based on static and on kinematic parameters was noted. It was stressed that only the first one can be used to describe correctly the mixture properties of a two-phase system.

4) The importance was stressed of expressing the mixture properties in terms of the baricenter of the mixture. The relevant thermodynamic properties have been derived and expressed therefore in terms of the static volume (or mass) concentration and of the baricenter.

5) Various velocity fields which can be used to describe a two-phase system were presented together with relations that exist between them. The significance of these fields was discussed also.

6) Expressions for the flux fields associated with these velocity fields were derived and discussed.

7) The fundamental identity of two-phase flow was derived. It relates the mean convective flux with respect to the center of mass of the mixture to the area averaged fluxes with respect to the centers of mass of the individual phases.

CHAPTER IV

DERIVATION OF AREA AVERAGED GENERAL BALANCE EQUATIONS

4.1 Introduction

It is well known in continuum mechanics reference [22], that the conservation equations can be derived from the general balance equation.

It was discussed in Section 1.2 and illustrated in Figure 2 that in the separated two-phase flow, the flow field of each phase is simply connected, i.e., the variables in each region of integration are continuously differentiable. It is to be expected, therefore, that in separated two phase flow, the conservation equations of each phase could be derived from such a balance ·quation, provided that we account, properly, for the interaction between the phases. It is evident that these interfacial effects expressed by jump conditions, will play a most important role in such a formulation.

In this chapter we shall list first the local field and constitutive equations for each phase together with the local expression for the general balance equation. From these and the appropriate "jump" conditions we shall derive the general one-dimensionalized balance equations appropriate to the two-fluid and to the diffusion models.

4.2 Local Formulation

4.2.1 Field and Constitutive Equations

As usual, each phase flow field is formulated in terms of three field equations. These three field equations are conservation equations

for the K-th phase, (K = 1,2)*:

1. The continuity equation

$$\frac{\partial \rho_k}{\partial t} + \nabla \cdot \rho_k \vec{v}_k = 0 \qquad , \; K = 1,2 \qquad (4.1)$$

where ρ represents the mass density and $\vec{v}$ the velocity vector.

2. The linear momentum equation

$$\frac{\partial \rho_k \vec{v}_k}{\partial t} + \nabla \cdot \rho_k \vec{v}_k \vec{v}_k = \nabla \cdot \bar{\bar{\pi}}_k + \rho_k \vec{g}_k \qquad , \; K = 1,2 \qquad (4.2)$$

where $\bar{\bar{\pi}}_K$ is the stress tensor, $\vec{g}_K$ is the external body force per unit mass. It is customary to separate the stress tensor, $\bar{\bar{\pi}}$, into thermodynamic pressure P, and a viscous stress tensor $\bar{\bar{\tau}}$, according to

$$\bar{\bar{\pi}}_k = -P_k \bar{\bar{\delta}} + \bar{\bar{\tau}}_k \qquad , \; K = 1,2 \qquad (4.3)$$

where $\bar{\bar{\delta}}$ is the Kronecker delta.

3. The energy equation

$$\frac{\partial}{\partial t} \rho_k (u_k + \frac{v_k^2}{2}) + \nabla \cdot \rho_k \vec{v}_k (u_k + \frac{v_k^2}{2}) = -\nabla \cdot \vec{q}_k + \qquad (4.4)$$

$$\nabla \cdot (\bar{\bar{\pi}}_k \cdot \vec{v}_k) + \rho_k \vec{g}_k \cdot \vec{v}_k \qquad , \; K = 1,2$$

where u_K denotes the specific internal energy and $\vec{q}_K$ the heat flux.

These three field equations can be summarized in a single equation known as the general balance or general conservation law, reference [22]. According to this law a quantity ψ, (representing scalar or higher order tensorial quantity), is balanced as:

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*In the two-phase flow literature it is a common practice to identify the heavier phase by 1 and the lighter one by 2.

$$\frac{\partial}{\partial t} \, \rho_k \psi_k + \nabla \cdot \rho_k \vec{v}_k \psi_k = -\nabla \cdot \vec{\phi}_k + \rho_k \dot{\psi}_k \qquad , \; K = 1,2 \qquad (4.5)$$

where $\vec{\phi}$ indicates influx of the quantity ψ, and $\dot{\psi}$ the rate of generation of the quantity per unit mass at each point throughout the field.

In order to obtain three field equations from Equation (4.5), the proper identifications for ψ, ϕ and ψ are summarized in Table 3.

Table 3. Proper Identifications for the Generalized Functions

Notation	ψ	ϕ	$\dot{\psi}$
Continuity	1	0	0
Momentum	$\vec{v}$	$-\overline{\overline{\pi}}$	$\vec{g}$
Energy	$u + v^2/2$	$\vec{q} - \overline{\overline{\pi}} \cdot \vec{v}$	$\vec{g} \cdot \vec{v}$

The additional equations used in the formulation are the constitutive equations. These are:

1. The thermal constitutive equation of state

$$\rho_k = \rho_k \left(P_k, T_k \right) \qquad , \; K = 1,2 \qquad (4.6)$$

where T is the temperature.

2. The caloric constitutive equation of state

$$u_k = u_k \left(\rho_k, T_k \right) \qquad , \; K = 1,2 \qquad (4.7)$$

3. The mechanical constitutive equation: By assuming the fluids are Newtonian, one can express the stress tensor as

$$\overline{\overline{\tau}}_k = \left(\mu_k' - \frac{2}{3} \, \mu_k \right) \nabla \cdot \vec{v}_k \, \overline{\overline{\delta}} + \mu_k \left[\nabla \vec{v}_k + (\nabla \vec{v}_k)^\dagger \right] \qquad , \; K = 1,2 \qquad (4.8)$$

where μ is the dynamic viscosity, and μ' the dilatational or bulk visco-
sity. The bulk viscosity is identically zero for low density monatomic
gases and is probably not too important in dense gases and liquids, and
therefore, in a great majority of engineering analysis it is simply
ignored.

4. The energetic constitutive equation: By using the Fourier
law of heat conduction for isotropic media, one can express the heat
flux vector by

$$\vec{q}_K = -k_K \nabla T_K \qquad , K = 1,2 \qquad (4.9)$$

This system of equations constitues the field and constitutive
equations involving thirty variables in thirty equations. To complete
the system, it is necessary to consider end values, i.e., the initial
and boundary conditions of the flow field.

4.2.2 Boundary Conditions

In two-phase flow systems the boundary conditions must be specified
not only at the external boundaries, i.e., at the fixed or solid bound-
aries, but also at the internal boundaries, i.e., at the moving boundaries
or interfaces. The external boundary conditions can be specified as in
single phase flow problems, i.e., by specifying the velocity and the heat
flux, etc., at the boundaries.

However, the specification of the boundary conditions at the in-
ternal surfaces, which are moving with time, should be formulated properly.
Particularly, for the separated flow regime, the structure and the dynamics
of the interface greatly influence the rates of mass, heat and momentum
transfer as well as the stability of the system.

In Appendix B, we discuss the motion of interfaces and the existing theories which have been constructed to analyze the interfacial mass, momentum and energy transports. For a singular surface* with the distributed source, the general balance equation at the interface was derived by Slattery [41] and used by Delhaye [40]; it is expressed by means of Equation (B.10) in Appendix B, i.e., by

$$\sum_{K=1}^{2} \hat{n}_K \cdot \left[\rho_K (\vec{v}_K - \vec{v}_i) \, \psi_K + \vec{\phi}_K \right] = - \nabla_s \cdot \overline{\overline{\theta}} \tag{4.10}$$

where the various variables, which appeared before, have their usual meanings. The new variables are defined as follows: $\hat{n}_K$ is the unit normal vector at the interface directed outwardly from the K-th phase. $\vec{v}_i$ is the interfacial velocity vector defined by Equation (B.7) and ∇_s is defined as the divergence operator on the surface in order to differentiate it from the divergence operator ∇, in the space. Finally, $\overline{\overline{\theta}}$ is the surface flux tensor inwardly directed through the interfacial boundary curve. Values of $\overline{\overline{\theta}}$ for mass, momentum and energy are discussed in more detail in Appendix B**.

In order to obtain mass, momentum and energy balances at the interface from Equation (4.10), the expressions for ψ and $\vec{\phi}$ should be taken from Table 3 whereas the $\nabla_s \cdot \overline{\overline{\theta}}$ term should be taken from Appendix B. If this is done, one finds Equation (B.13) for mass balance

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*Singular surface is defined as a surface of a limiting value of a quantity at a point on a path restricted to one side of the surface differs from that obtained by approaching this point from the other side of the surface.

**We note that if the surface source term $\overline{\overline{\theta}}$, is neglected in Equation (4.10), then this equation reduces to the well known Kotchine's "jump" conditions given in reference [22].

$$\sum_{k=1}^{2} \hat{n}_{ki} \cdot \varrho_k (\vec{v}_k - \vec{v}_i) = 0 \tag{4.11}$$

Equation (B.25) for <u>momentum balance</u>,

$$\sum_{k=1}^{2} \hat{n}_{ki} \cdot \left[\varrho_k (\vec{v}_k - \vec{v}_i) \vec{v}_k + P_k \overline{\overline{s}} - \overline{\overline{\varepsilon}}_k \right] = 2 \varkappa \sigma \hat{n}_2 +$$
$$\frac{1}{H^2} \left[\hat{r}_1 \left(G \frac{\partial}{\partial u_1} - F \frac{\partial}{\partial u_2} \right) + \hat{r}_2 \left(E \frac{\partial}{\partial u_2} - F \frac{\partial}{\partial u_1} \right) \right] \sigma \tag{4.12}$$

and Equation (B.37) for <u>energy balance</u>,

$$\sum_{k=1}^{2} \hat{n}_{ki} \cdot \left[\varrho_k (\vec{v}_k - \vec{v}_i) \left(u_k + \frac{v_k^2}{2} \right) + \left(\vec{q}_k + P_k \vec{v}_k - \overline{\overline{\varepsilon}}_k \cdot \vec{v}_k \right) \right] = \tag{4.13}$$
$$2 \varkappa \sigma \vec{v}_i \cdot \hat{n}_i + \frac{1}{H^2} \left[\hat{r}_1 \left(G \frac{\partial}{\partial u_1} - F \frac{\partial}{\partial u_2} \right) + \hat{r}_2 \left(E \frac{\partial}{\partial u_2} - F \frac{\partial}{\partial u_1} \right) \right] \cdot \sigma \vec{v}_i$$

The terms appearing in these equations as well as their implica-
tions are discussed in detail in Section B.2.2 of Appendix B.

The field and constitutive equations listed in Section 4.2.1
together with the interfacial balance equations, i.e., Equations (4.11)-
(4.13) complete the formulation in terms of local parameters, of a
separated two-phase flow system. Of course, boundary conditions at the
external surfaces and the initial conditions must be given in order to be
able to solve the problem.

4.3 Averaged General Balance Equation for Individual Phases

In the previous paragraph the partial differential equations
which describe the local conservation of mass, linear momentum and
energy were set forth. In order to close this system of field equations
we also listed the necessary constitutive equations. In effect, we
described the field of the K-th phase by the general balance equation,
Equation (4.5), and accounted for the actions of one phase upon the

other by general balance equation at the interface, Equation (4.10).

These equations are exact and valid throughout the flow fields of the two phases, however, it is not feasible to seek a detailed solution for this system of equations. The difficulty stems from the fact that they involve three space coordinates, a time variable (in problems involving transients) and a large number of dependent variables. Therefore, even for most simple cases, these equations must be solved numerically.

However, for many engineering applications these equations can be simplified by means of proper averages. The advantage of such an approach is two-fold. First, the variables appearing in the final equations will have explicit definitions in terms of averaged values. Consequently, it will be easy to compare predicted results with experimental data (which in two-phase flow are most often presented in terms of average values). Secondly, by means of space averages it will be possible to reduce the number of space variables and to treat the problem as an one-dimensional flow of problem.

For the purpose of averaging, let us consider the cross sectional plane A_{Kc}, of the K-th phase, (see Figure 2), which cuts the interfacial surface along the curve ξ_i, and the solid boundaries along the curve ξ_{Ke}. Let us integrate the general balance equation (4.5) over A_{Kc}, thus

$$\iint\limits_{A_{Kc}} \frac{\partial}{\partial t}\, \rho_k \psi_k \, dA + \iint\limits_{A_{Kc}} \nabla \cdot \left(\rho_k \vec{v}_k \psi_k + \overline{\mathcal{F}}_k \right) dA = \iint\limits_{A_{Kc}} \rho_k \dot{\psi}_k \, dA, \quad K = 1,2 \quad (4.14)$$

following equation expressing the <u>general balance equation</u> for the indi-
vidual phases, i.e., for K = 1 and K = 2, in terms of <u>mass-weighed averages</u>:

$$\frac{\partial}{\partial t}\left(\frac{A_{KC}}{A_{TC}} \ll \rho_{\kappa} \gg \langle \psi_{\kappa} \rangle\right) + \frac{\partial}{\partial x}\left(\frac{A_{KC}}{A_{TC}} \ll \rho_{\kappa} \gg \langle \nu_{\kappa z} \rangle \langle \psi_{\kappa} \rangle\right) = \qquad (4.19)$$

$$= -\frac{\partial}{\partial x}\left(\frac{A_{KC}}{A_{TC}} \ll \phi_{\kappa z} \gg\right) + \frac{A_{KC}}{A_{TC}} \ll \rho_{\kappa} \gg \langle \dot{\psi}_{\kappa} \rangle - \frac{A_{KC}}{A_{TC}}\left[\ll \rho_{\kappa} \gg \langle \nu_{\kappa z} \rangle \langle \psi_{\kappa} \rangle + \right.$$

$$\ll \phi_{\kappa z} \gg + \ll \rho_{\kappa} \gg Cov(\nu_{\kappa z} \cdot \psi_{\kappa})\left]\frac{d \ell_n A_{TC}}{dx} - \frac{1}{A_{TC}}\sum_{\ell = e, i}\int_{S_{\kappa\ell}} \hat{n}_{\kappa} \cdot \left[\rho_{\kappa}(\vec{\nu}_{\kappa} - \vec{\nu}_{\ell})\psi_{\kappa} + \right.$$

$$\left. \vec{\phi}_{\kappa}\right]\frac{dA}{dx} - \frac{\partial}{\partial x}\left[\frac{A_{KC}}{A_{TC}} \ll \rho_{\kappa} \gg Cov(\nu_{\kappa z}, \psi_{\kappa})\right]$$

where the covariance defined by Equation (3.8) has been used in order to
express average of product in terms of product of averages.

Equation (4.19) which is applicable to each phases, i.e., K = 1,2
represents the <u>one-dimensionalized area-averaged general balance equation</u>
for a quantity ψ. The various terms which appear in this equation have
the following meaning: The terms on the left side represent, respectively,
the time rate of change of quantity ψ_K-mass, momentum or energy-per unit
volume of the channel, and the convective flux of the quantity with respect
to the <u>fixed</u> frame of reference. The right side terms are interpreted,
respectively, as the flux of the quantity with respect to the <u>baricenter</u>
of the <u>K-th phase</u>, the generation of the quantity, the effect of area
change of the channel, interaction at the internal and external boundaries,
and finally effect of the distribution of the velocity as well as the
quantity ψ_K.

The general balance equations for each phase can be obtained from
Equation (4.19) by assigning to the K the values 1 and 2. Thus by taking

into account the void fraction definition, Equation (3.30), we obtain from Equation (4.19) the following averaged general balance equations <u>for the phase 1, K = 1</u>:

$$\frac{\partial}{\partial t}(1-\alpha)\lll \rho_1 \ggg \langle \Psi_1 \rangle + \frac{\partial}{\partial x}(1-\alpha)\lll \rho_1 \ggg \langle v_{1x} \rangle \langle \Psi_1 \rangle = -\frac{\partial}{\partial x}(1-\alpha)\lll \Phi_{1x} \ggg + \quad (4.20a)$$

$$(1-\alpha)\lll \rho_1 \ggg \langle \dot{\Psi}_1 \rangle - (1-\alpha)\left[\lll \rho_1 \ggg \langle v_{1x}\rangle \langle \Psi_1 \rangle + \lll \Phi_{1x} \ggg + \right.$$

$$\left.\lll \rho_1 \ggg \, Cov(v_{1x}\cdot\Psi_1)\right]\frac{d\ell n \, A_{Tc}}{dx} - \frac{1}{A_{Tc}}\sum_{\ell=e,i}\int_{\mathcal{I}_{1\ell}}\hat{n}_{1\ell}\cdot\left[\rho_{1\ell}(\vec{v}_{1\ell}-\vec{v}_i)\,\Psi_{1\ell}+\vec{\Phi}_{1\ell}\right]\frac{dA}{dx} -$$

$$\frac{\partial}{\partial x}\left[(1-\alpha)\lll \rho_1 \ggg \, Cov(v_{1x}\cdot\Psi_1)\right]$$

and for the phase 2, K = 2:

$$\frac{\partial}{\partial t}\,\alpha\lll \rho_2 \ggg\langle \Psi_2 \rangle + \frac{\partial}{\partial x}\,\alpha\lll \rho_2 \ggg\langle v_{2x}\rangle\langle \Psi_2\rangle = -\frac{\partial}{\partial x}\,\alpha\lll \Phi_{2x}\ggg + \quad (4.20b)$$

$$\alpha\lll \rho_2 \ggg\langle \dot{\Psi}_2\rangle - \alpha\left[\lll \rho_2 \ggg\langle v_{2x}\rangle\langle \Psi_2\rangle + \lll \Phi_{2x}\ggg + \lll \rho_2 \ggg\,Cov(v_{2x}\cdot\Psi_2)\right]\frac{d\ell n \, A_{Tc}}{dx} -$$

$$\frac{1}{A_{Tc}}\sum_{\ell=e,i}\int_{\mathcal{I}_{2\ell}}\hat{n}_{2\ell}\cdot\left[\rho_{2\ell}(\vec{v}_{2\ell}-\vec{v}_i)\,\Psi_{2\ell}+\vec{\Phi}_{2\ell}\right]\frac{dA}{dx} - \frac{\partial}{\partial x}\left[\alpha\lll \rho_2 \ggg\,Cov(v_{2x}\cdot\Psi_2)\right]$$

We note that in order to express the averaged general balance equations in terms of mass concentration C, rather than area concentration α, one can use the relationship between C and α given by Equation (3.36).

4.4 General Balance Equation for the Mixture

In the previous section we considered each phase, i.e., fluid separately. The term, which accounts for the actions of one phase upon the other, emerged naturally as a result of the proper averaging process. Besides this "two-fluid" point of view, we may consider the diffusion model in which the mixture is regarded as being in motion as a "<u>single</u>"

fluid. In order to obtain the general balance equation which governs the total mixture flow field, we shall add two general balance equations derived for each phase, i.e., Equations (4.20a and b), thus

$$\frac{\partial}{\partial t}\left[(1-\alpha)\ll\rho_1\gg\langle\psi_1\rangle + \alpha\ll\rho_2\gg\langle\psi_2\rangle\right] + \frac{\partial}{\partial x}\left[(1-\alpha)\ll\rho_1\gg\langle v_{1x}\rangle\langle\psi_1\rangle + \right. \tag{4.21}$$

$$\left.\alpha\ll\rho_2\gg\langle v_{2x}\rangle\langle\psi_2\rangle\right] = -\frac{\partial}{\partial x}\left[(1-\alpha)\ll\phi_{1x}\gg + \alpha\ll\phi_{2x}\gg\right] + \left[(1-\alpha)\ll\rho_1\gg\langle\psi_1\rangle + \alpha\ll\rho_2\gg\langle\psi_2\rangle\right] -$$

$$\left[(1-\alpha)\ll\rho_1\gg\langle v_{1x}\rangle\langle\psi_1\rangle + \alpha\ll\rho_2\gg\langle v_{2x}\rangle\langle\psi_2\rangle + (1-\alpha)\ll\phi_{1x}\gg + \alpha\ll\phi_{2x}\gg + \right.$$

$$\left. \text{Cov}(v\cdot\psi)\right]\frac{d\ell_m A T_c}{dx} - \frac{1}{A T_c}\sum_{\ell=e,i}\sum_{K=1}^{2}\int_{J_{K\ell}}\hat{n}_{K\ell}\cdot\left[\rho_{K\ell}(\hat{v}_{K\ell}-\vec{v}_\ell)\psi_{K\ell} + \vec{\phi}_{K\ell}\right]\frac{dA}{dx} -$$

$$\frac{\partial}{\partial x}\text{Cov}(v\cdot\psi)$$

where the covariance term Cov (v·ψ) is defined by

$$\text{Cov}(v\cdot\psi) = (1-\alpha)\ll\rho_1\gg\text{Cov}(v_{1x}\cdot\psi_1) + \alpha\ll\rho_2\gg\text{Cov}(v_{2x}\cdot\psi_2) \tag{4.22}$$

Substituting the mixture property definition, Equation (3.40), the fundamental identity, Equation (3.81) and the general balance equation at the interface, Equation (4.10) in (4.21) we obtain

$$\frac{\partial}{\partial t}\rho_m\psi_m + \frac{\partial}{\partial x}\rho_m v_{mx}\psi_m = -\frac{\partial}{\partial x}\phi_{mx} + \rho_m\dot{\psi}_m - \frac{\partial}{\partial x}\left[\alpha(1-\alpha)\frac{\ll\rho_1\gg\ll\rho_2\gg}{\rho_m}V_{rx}\Delta\langle\psi\rangle\right] + \tag{4.23}$$

$$\left[\rho_m v_{mx}\psi_m + \alpha(1-\alpha)\frac{\ll\rho_1\gg\ll\rho_2\gg}{\rho_m}V_{rx}\Delta\langle\psi\rangle + \phi_{mx} + \text{Cov}(v\cdot\psi)\right]\frac{d\ell_m A T_c}{dx} +$$

$$\frac{1}{A T_c}\int_{J_i}\nabla_o\cdot\bar{\bar{\Theta}}\frac{dA}{dx} - \frac{1}{A T_c}\sum_{K=1}^{2}\int_{J_{Ka}}\hat{n}_{K\ell}\cdot\left[\rho_{K\ell}(\vec{v}_{K\ell}-\vec{v}_\ell)\psi_{K\ell} + \vec{\phi}_{K\ell}\right]\frac{dA}{dx} - \frac{\partial}{\partial x}\text{Cov}(v\cdot\psi)$$

where we defined ϕ_{mx} as the mixture flux given by

$$\phi_{mx} = (1-\alpha)\ll\phi_{1x}\gg + \alpha\ll\phi_{2x}\gg \tag{4.24}$$

We note that since the mixture is conceived as a single fluid in motion with the velocity v_m, the diffusive transport V_{Km} of the quantities associated with the K-th phase is taken into account in the term

$$\alpha(1-\alpha)\frac{\ll\rho_1\gg\ll\rho_2\gg}{\rho_m}V_{rx}\Delta\langle\psi\rangle = (1-c)\rho_m V_{1mx}\langle\psi_1\rangle + c\,\rho_m V_{2mx}\langle\psi_2\rangle \tag{4.25a}$$

$$\alpha(1-\alpha)\frac{\langle\langle e_1\rangle\rangle\langle\langle e_2\rangle\rangle}{\bar{e}_m}\,V_{r_x}\,\Delta\langle\Psi\rangle = (1-\alpha)\langle\langle e_1\rangle\rangle\,V_{1m_x}\langle\Psi_1\rangle + \alpha\langle\langle e_2\rangle\rangle\,V_{2m_x}\langle\Psi_2\rangle \quad (4.25b)$$

in which $(1-C)\ \rho_m\vec{V}_{1m}$ and $C\ \rho_m\vec{V}_{2m}$ represent the diffusion mass fluxes with respect to the center of gravity of the mixture for phase 1 and 2, respectively. Therefore, by expressing the mixture general balance equation with respect to center of mass of the mixture, an additional term arises in the equation due to the effect of the _relative_ velocity, between the two phases. As will be seen in the further chapters this term will give rise to the diffusion mass flux, diffusion momentum flux, (drift stress), diffusion energy flux in the continuity, momentum and energy equations, respectively.

Now we impose the requirement, reference [23] that the mixture as a whole i.e., as a "single" fluid model does not know whether it is simple or heterogeneous. To do so, we shall modify the mixture flux term $\vec{\phi}_m$, by introducing the total flux $\vec{\Phi}_m$ defined as

$$\vec{\Phi}_m = \vec{\phi}_m + (1-C)\,e_m\,\vec{V}_{1m}\langle\Psi_1\rangle + C\,e_m\,\vec{V}_{2m}\langle\Psi_2\rangle \qquad (4.26)$$

Substituting Equation (4.26) in (4.23) yields the one-dimension-alized _mixture balance equation_, thus

$$\frac{\partial}{\partial t}\,e_m\Psi_m + \frac{\partial}{\partial x}\,e_m\,v_{mx}\Psi_m = -\frac{\partial}{\partial x}\,\Phi_{mx} + e_m\dot{\Psi}_m -$$

$$\left[e_m\,v_{mx}\Psi_m + \Phi_{mx} + Cov(\Psi\cdot\Psi)\right]\frac{d}{dx}\ln A_{Tc} + \frac{1}{A_{Tc}}\int_{S_i}\nabla_s\cdot\bar{\bar{\theta}}\,\frac{dA}{dx} -$$

$$\frac{1}{A_{Tc}}\sum_{k=1}^{2}\int_{\Gamma_{ke}}\hat{n}_{ke}\cdot\left[e_{ke}(\vec{v}_{ke}-\vec{v}_e)\,\Psi_{ke} + \vec{\Phi}_{ke}\right]\frac{dA}{dx} - \frac{\partial}{\partial x}\,Cov(\Psi\cdot\Psi)$$

The first term on the right hand side accounts for the effect of the relative velocity between phases; the second is the source term;

the third accounts for the effect of changes of the cross sectional area (as in nozzles, etc.); the fourth term accounts for the effect of the interfacial source, i.e., surface tension; the fifth accounts for the effect of transport of quantity at solid boundary (as in porous film cooling); the last term expresses the covariance terms.

If the one-dimensionalized general mixture equation, Equation (4.27) is compared with that of the single phase equation, we obtain additional term due to the interfacial source, otherwise the general balance equation of the mixture will have the same form as that of the one-dimensional single phase flow in a conduit.

In reference [23] the mixture balance equations were derived from the continuum point of view. For the reason of comparison we can put them into our notation and express as a general balance equation as follows:

$$\frac{\partial}{\partial t} \varrho_m \psi_m + \nabla \cdot \varrho_m \vec{v}_m \psi_m = - \nabla \cdot \vec{\Phi}_m + \varrho_m \dot{\psi}_m \qquad (4.28)$$

Comparison of Equation (4.27) with Equation (4.28) reveals that Equation (4.28) does not account for the interfacial source which is taken into account by Equation (4.27). The reason for the disagreement emanates from the fact that in the derivation of Equation (4.28), the author postulated the Kotchine's "jump" condition

$$\sum_{k=1}^{2} \varrho_{ki} (\vec{v}_{ki} - \vec{v}_i) \cdot \hat{n}_{ki} \psi_{ki} - \vec{\Phi}_{ki} \cdot \hat{n}_{ki} = 0 \qquad (4.29)$$

as a constitutive equation to be satisfied at the interfaces. By comparing Equation (4.29) to Equation (4.10) it is seen that Equation (4.29) does not account for the surface effects.

4.5 Two-Dimensional Flow Field

It was discussed in Section 1.2 and illustrated on Figure 1, that the class of separated flows can be divided into plane flows and quasi-axisymmetric flows. If the flow can be approximated by two-dimensional field, then the line integrals appearing in the balance equations, i.e., Equations (4.20), (4.21) and (4.27), can be estimated immediately. This will be done for two-dimensional plane and axisymmetric cylindrical flow systems.

4.5.1 Plane Flow

Two-dimensional plane flow is illustrated in Figure 15 and the essential geometric relations appropriate to the plane flow are derived in Appendix C.1. By taking the width as unity, i.e., $\xi_i = \xi_{1e} = \xi_{2e} = 1$; $A_{Tc} = H$, and substituting Equations (C.11) and (C.13) in Equations (4.20), (4.21) and (4.27), we obtain:

Plane flow general balance equation for phase 1,

$$\frac{\partial}{\partial t} \left(\eta \ll \rho_1 \gg <\psi_1> \right) + \frac{\partial}{\partial x} \left(\eta \ll \rho_1 \gg <v_{1x}><\psi_1> \right) = -\frac{\partial}{\partial x} \left(\eta \ll \phi_{1x} \gg \right) + \qquad (4.30)$$

$$\eta \ll \rho_1 \gg <\dot{\psi}_1> - \hat{n}_{1i} \cdot \left[\rho_{1i} (\vec{v}_{1i} - \vec{v}_i) \psi_{1i} + \vec{\phi}_{1i} \right] \left[1 + \left(\frac{\partial \eta}{\partial x} \right)^2 \right]^{1/2} -$$

$$\hat{n}_{1e} \cdot \left[\rho_{1e} (\vec{v}_{1e} - \vec{v}_e) \psi_{1e} + \vec{\phi}_{1e} \right] - \frac{\partial}{\partial x} \eta \ll \rho_1 \gg Cov(v_{1x} \cdot \psi_1)$$

Plane flow general balance equation for phase 2,

$$\frac{\partial}{\partial t} \left[(H-\eta) \ll \rho_2 \gg <\psi_2> \right] + \frac{\partial}{\partial x} \left[(H-\eta) \ll \rho_2 \gg <v_{2x}><\psi_2> \right] = -\frac{\partial}{\partial x} \left[(H-\eta) \ll \phi_{2x} \gg \right] + (4.31)$$

$$(H-\eta) \ll \rho_2 \gg <\dot{\psi}_2> - \hat{n}_{2i} \cdot \left[\rho_{2i} (\vec{v}_{2i} - \vec{v}_i) \psi_{2i} + \vec{\phi}_{2i} \right] \left[1 + \left(\frac{\partial \eta}{\partial x} \right)^2 \right]^{1/2} -$$

$$\frac{\partial}{\partial x} (H-\eta) \ll \rho_2 \gg Cov(v_{2x} \cdot \psi_2)$$

Plane flow general balance equation for the mixture,

$$\frac{\partial}{\partial t} \, \rho_m \, \Psi_m + \frac{\partial}{\partial x} \, \rho_m \, v_{mx} \, \Psi_m = -\frac{\partial}{\partial x} \, \Phi_{mx} + \rho_m \, \dot{\Psi}_m + \tag{4.32}$$

$$\frac{1}{H} \, (\nabla_s \cdot \bar{\bar{\theta}}) \left[1 + \left(\frac{\partial \eta}{\partial x}\right)^2 \right]^{1/2} - \frac{1}{H} \, \hat{n}_{1e} \cdot \left[\rho_{1e} (\vec{v}_{1e} - \vec{v}_e) \, \Psi_{1e} + \vec{\Phi}_{1e} \right] - \frac{\partial}{\partial x} \, Cov \, (v \cdot \Psi)$$

In these equations η is the thickness of phase 1, and H is the half

distance between two parallel plates, see Figure 15.

4.5.2 Annular Flow

Axisymmetric annular flow, i.e., $\xi_i = \pi D_i$, illustrated in Figure

16 and the essential geometric relations are derived in Appendix C.2.

Substituting Equations (C.22), (C.26) and (C.27) in Equations (4.20),

(4.21) and (4.27), we obtain:

annular flow general balance equation for phase 1,

$$\frac{\partial}{\partial t} \, (1-\alpha) \, \langle\!\langle \rho_1 \rangle\!\rangle \, \langle \Psi_1 \rangle + \frac{\partial}{\partial x} \, (1-\alpha) \, \langle\!\langle \rho_1 \rangle\!\rangle \, \langle v_{1x} \rangle \langle \Psi_1 \rangle = -\frac{\partial}{\partial x} \, (1-\alpha) \, \langle\!\langle \Phi_{1x} \rangle\!\rangle + \tag{4.33}$$

$$(1-\alpha) \, \langle\!\langle \rho_1 \rangle\!\rangle \langle \dot{\Psi}_1 \rangle - (1-\alpha) \left[\langle\!\langle \rho_1 \rangle\!\rangle \langle v_{1x} \rangle \langle \Psi_1 \rangle + \langle\!\langle \Phi_{1x} \rangle\!\rangle + \langle\!\langle \rho_1 \rangle\!\rangle \, Cov(v_{1x} \cdot \Psi_1) \right] \frac{d \ln D_e^2}{dx} -$$

$$\frac{\partial}{\partial x} \, (1-\alpha) \, \langle\!\langle \rho_1 \rangle\!\rangle \, Cov \, (v_{1x} \cdot \Psi_1) - \hat{n}_{1i} \cdot \left[\rho_{1i} (\vec{v}_{1i} - \vec{v}_i) \, \Psi_{1i} + \vec{\Phi}_{1i} \right] \left\{ 1 + \left[\frac{\partial}{\partial x} \left(\frac{D_e \sqrt{\alpha}}{2} \right) \right]^2 \right\}^{1/2} \frac{4\sqrt{\alpha}}{D_e} -$$

$$\hat{n}_{1e} \cdot \left[\rho_{1e} (\vec{v}_{1e} - \vec{v}_e) \, \Psi_{1e} + \vec{\Phi}_{1e} \right] \left[1 + \left(\frac{d}{dx} \, \frac{D_e}{2} \right)^2 \right]^{1/2} \frac{4}{D_e}$$

annular flow general balance equation for phase 2,

$$\frac{\partial}{\partial t} \, \alpha \, \langle\!\langle \rho_2 \rangle\!\rangle \langle \Psi_2 \rangle + \frac{\partial}{\partial x} \, \alpha \, \langle\!\langle \rho_2 \rangle\!\rangle \langle v_{2x} \rangle \langle \Psi_2 \rangle = -\frac{\partial}{\partial x} \, \alpha \, \langle\!\langle \Phi_{2x} \rangle\!\rangle + \alpha \, \langle\!\langle \rho_2 \rangle\!\rangle \langle \dot{\Psi}_2 \rangle - \tag{4.34}$$

$$\alpha \left[\langle\!\langle \rho_2 \rangle\!\rangle \langle v_{2x} \rangle \langle \Psi_2 \rangle + \langle\!\langle \Phi_{2x} \rangle\!\rangle + \langle\!\langle \rho_2 \rangle\!\rangle \, Cov(v_{2x} \cdot \Psi_2) \right] \frac{d \ln D_e^2}{dx} -$$

$$\frac{\partial}{\partial x} \, \alpha \, \langle\!\langle \rho_2 \rangle\!\rangle Cov(v_{2x} \cdot \Psi_2) - \hat{n}_{2i} \cdot \left[\rho_{2i} (\vec{v}_{2i} - \vec{v}_i) \, \Psi_{2i} + \vec{\Phi}_{2i} \right] \left\{ 1 + \left[\frac{\partial}{\partial x} \left(\frac{D_e \sqrt{\alpha}}{2} \right) \right]^2 \right\}^{1/2} \frac{4\sqrt{\alpha}}{D_e}$$

annular flow mixture general balance equation

$$\frac{\partial}{\partial t} \, \rho_m \, \Psi_m + \frac{\partial}{\partial x} \, \rho_m \, v_{mx} \, \Psi_m = -\frac{\partial}{\partial x} \, \Phi_{mx} + \rho_m \, \dot{\Psi}_m - \left[\rho_m \, v_{mx} \, \Psi_m + \Phi_{mx} + \tag{4.35}$$

$$Cov \, (v \cdot \Psi) \right] \frac{d \ln D_e^2}{dx} + \frac{4\sqrt{\alpha}}{D_e} \, (\nabla_s \cdot \bar{\bar{\theta}}) \left\{ 1 + \left[\frac{\partial}{\partial x} \left(\frac{D_e \sqrt{\alpha}}{2} \right) \right]^2 \right\}^{1/2} -$$

$$\hat{n}_{1e} \cdot \left[\rho_{1e} (\vec{v}_{1e} - \vec{v}_e) \, \Psi_{1e} + \vec{\Phi}_{1e} \right] \left[1 + \left(\frac{d}{dx} \, \frac{D_e}{2} \right)^2 \right]^{1/2} \frac{4}{D_e} - \frac{\partial}{\partial x} \, Cov \, (v \cdot \Psi)$$

In these equations D_e is the variable pipe diameter, see Figure 16.

4.7 Summary

1. By considering the flow field of each phase as a continuum, the time dependent, separated two-phase flow problem was formulated in terms of a) the fundamental conservation equations applicable to the flow field of each phase, b) the constitutive equations describing the behavior of each constituent, and c) of the internal boundary conditions prescribing the interfacial balance. The difficulty of seeking a solution of such set of equations was noted and discussed.

2. Using this formulation the mass-weighed area-averaged general balance equations for each phase were derived. They describe the fields of each phase in terms of area-averaged variables and take into account the effects of interactions at the internal boundaries between the phases and of the transport of the quantities at the external boundaries.

3. By adding the area-averaged general balance equations for each phase, the mixture general balance equation was obtained. It was noted that this equation differs from those available in the literature (applicable to single phase flow of mixtures) by the presence of an interfacial source term. It was shown further that expressing the mixture general balance equation with respect to the baricenter of the mixture gives rise to diffusive fluxes with respect to the baricenter of the mixture.

CHAPTER V

CONTINUITY, DIFFUSION AND VOID PROPAGATION EQUATIONS

5.1 Introduction

It was seen in Chapter 2 that the conventional formulation of a two-phase flow problem has been expressed in terms of only one continuity equation. Furthermore, it was discussed briefly in there that a two-phase flow problem with a phase change should be formulated in terms of two continuity equations. In order to substantiate this statement let us look at the well-established field of transport phenomena.

It is well known that in the analyses of multi-component chemically reacting mixtures, the number of continuity equations, is equal to the number of components. In addition to these two continuity equations, two constitutive equations - one for the rate of chemical reaction and the other for the diffusion coefficient - are required to completely define the kinematic requirements of the reacting systems.

If, instead of a single phase, two component chemically reacting system, we consider now the present problem, namely a single component, two-phase flow system in thermodynamic non-equilibrium, we could expect that a similar number of equations will be required to describe the kinematic field of the latter system regardless of what model we have chosen.

In this chapter we shall formulate the kinematic field of a two-phase flow system in two-fluid and the diffusion models. It will be

seen that although forms of the field and constitutive equations in both models are slightly different from each other, the number of the field and constitutive equations are the same. Finally, from the differential equations we shall derive the relevant similarity groups which can be used for purposes of scaling.

5.2 Two-Fluid Model Formulation

A formulation based on the two-fluid model should be expressed in terms of two continuity equations for the individual phases. Therefore, in what follows we shall derive the continuity equation for each phase and discuss the additional requirements which are necessary in order to complete the kinematic description of a separated two-phase flow field.

5.2.1 Phase Continuity Equations

A. <u>Derivation</u>

To obtain the <u>mass weighted area-averaged continuity equations</u> from the averaged general balance equation, Equation (4.20), we use the proper identifications for ψ, $\overset{\bullet}{\psi}$ and $\vec{\phi}$. According to Table 3, for mass transfer they are given by

$$\psi = 1, \text{ mass per unit mass}, \tag{5.1}$$

$$\overset{\bullet}{\psi} = 0, \text{ no distributed mass generation}$$

$$\vec{\phi} = 0, \text{ no mass flux with respect to the baricenter of gravity}$$
$$\text{of each phase.}$$

Substituting Equation (5.1) in Equations (4.20a and b), we obtain the mass-weighted area-averaged continuity equations for <u>phase 1</u>,

$$\frac{\partial}{\partial t} \left[(1-\alpha) \ll \rho_1 \gg \right] + \frac{\partial}{\partial x} \left[(1-\alpha) \ll \rho_1 \gg \langle v_{1x} \rangle \right] = -(1-\alpha) \ll \rho_1 \gg \langle v_{1x} \rangle \frac{d}{dx} \ell n A_{T_c} - \quad (5.2a)$$

$$\frac{1}{A_{T_c}} \sum_{\ell = e, i} \int_{\mathcal{I}_{1\ell}} \hat{n}_{1\ell} \cdot \rho_{1\ell} (\vec{v}_{1\ell} - \vec{v}_\ell) \; \frac{dA}{dx}$$

and for phase 2,

$$\frac{\partial}{\partial t} \left(\alpha \ll \rho_2 \gg \right) + \frac{\partial}{\partial x} \left(\alpha \ll \rho_2 \gg \langle v_{2x} \rangle \right) = - \alpha \ll \rho_2 \gg \langle v_{2x} \rangle \frac{d}{dx} \ell n A_{T_c} - \quad (5.2b)$$

$$\frac{1}{A_{T_c}} \sum_{\ell = e, i} \int_{\mathcal{I}_{2\ell}} \hat{n}_{2\ell} \cdot \rho_{2\ell} (\vec{v}_{2\ell} - \vec{v}_\ell) \; \frac{dA}{dx}$$

The first term on tne right hand side accounts for the effect of changes of cross sectional area (as in nozzles, etc.), the second term accounts for the effect of mass injection at the external boundaries (as in porous film cooling) as well as the mass transfer at the internal surfaces (as evaporation or condensation). Defining by $\dot{m}_{Ke}$, and $\dot{m}_{Ki}$, the mass flux outward from the K-th phase at the external and internal surfaces, respectively, it follows that

$$\dot{m}_{K\ell} = \hat{n}_{K\ell} \cdot \rho_{K\ell} (\vec{v}_{K\ell} - \vec{v}_\ell) ; \; K = 1, 2; \; \ell = e, i \quad (5.3)$$

Furthermore, defining by Γ_{Ki}, and Γ_{Ke}, the mass formation of the K-th phase per unit volume of the mixture from internal and external surfaces, respectively, the line integrals appearing in Equation (5.2) can be interpreted as

$$- \frac{1}{A_{T_c}} \int_{\mathcal{I}_{K\ell}} \hat{n}_{K\ell} \cdot \rho_{K\ell} (\vec{v}_{K\ell} - \vec{v}_\ell) \frac{dA}{dx} = - \frac{1}{A_{T_c}} \int_{\mathcal{I}_{K\ell}} \dot{m}_{K\ell} \frac{dA}{dx} = \Gamma_{K\ell} \quad (5.4)$$

in which K takes the values 1 and 2 whereas ℓ assumes the values e and i. The conservation of mass at the interface. i.e., Equations (4.9)

together with Equation (4.10), require that

$$\sum_{k=1}^{2} \dot{m}_{ki} = \sum_{k=1}^{2} \Gamma_{ki} = 0 \qquad (5.5)$$

Using Equation (5.4), we obtain from Equation (5.2a and b) alternative forms of the mass-weighted area-averaged continuity equations for phase 1 and 2, thus

$$\frac{\partial}{\partial t}(1-\alpha)\langle\langle\rho_1\rangle\rangle + \frac{\partial}{\partial x}(1-\alpha)\langle\langle\rho_1\rangle\rangle\langle v_{1x}\rangle = -(1-\alpha)\langle\langle\rho_1\rangle\rangle\langle v_{1x}\rangle\frac{d}{dx}\ln A_{Tc} + \sum_{\ell=e,i}\Gamma_{1\ell} \quad (5.6a)$$

$$\frac{\partial}{\partial t}\alpha\langle\langle\rho_2\rangle\rangle + \frac{\partial}{\partial x}\alpha\langle\langle\rho_2\rangle\rangle\langle v_{2x}\rangle = -\alpha\langle\langle\rho_2\rangle\rangle\langle v_{2x}\rangle\frac{d}{dx}\ln A_{Tc} + \sum_{\ell=e,i}\Gamma_{2\ell} \quad (5.6b)$$

which can be expressed in terms of mass concentration C, rather than void fraction α. To do so, we shall simply use the relation between α and C, which are given by Equation (3.36); thus from Equations (5.5) and (3.36) we obtain

$$\frac{\partial}{\partial t}(1-C)\rho_m + \frac{\partial}{\partial x}(1-C)\rho_m\langle v_{1x}\rangle = -(1-C)\rho_m\langle v_{1x}\rangle\frac{d}{dx}\ln A_{Tc} + \sum_{\ell=e,i}\Gamma_{1\ell} \qquad (5.7a)$$

$$\frac{\partial}{\partial t}C\rho_m + \frac{\partial}{\partial x}C\rho_m\langle v_{2x}\rangle = -C\rho_m\langle v_{2x}\rangle\frac{d}{dx}\ln A_{Tc} + \sum_{\ell=e,i}\Gamma_{2\ell} \qquad (5.7b)$$

In the above set of equations, i.e., Equations (5.6) and (5.7), we need to discuss only the significance of the source term Γ_K, because all other terms appearing in these equations are self-explanatory. This will be done in the section which follows.

B. Discussion

We note here that in a two or three dimensional analysis of

separated flow regimes such as annular or the jet flow where each phase can be considered separately, i.e., as a single phase flow problem, the source term Γ_K, does not appear explicitly in the continuity equation i.e., Equation (4.1), but will appear as a boundary condition at the internal and external boundaries. In the one-dimensionalized formulation, however, the source term appears, see Equation (5.7), as a consequence of the boundary conditions and of the area averaging procedure.

By comparing the averaged continuity equations, Equations (5.6a and b) with one-dimensional continuity equations for a given species undergoing chemical reaction, reference [50], it can be recognized that the source term Γ_{Ki}, corresponds to the rate of production of a K-th constituent of a single phase multi-component chemically reacting flow. It is well-known that in analysis of multi-component flow systems, the source term is specified by an appropriate constitutive equation for chemical reaction. Consequently, even if the mass injection Γ_{Ke}, at the external surfaces is known a priori, in order to specify the vaporization or condensation rate Γ_{Ki}, at the internal surfaces, it will be still necessary to specify the <u>constitutive equation for net vaporization</u> (or of net condensation), which in thermal non-equilibrium two-phase systems, depends on the structure of the vapor-liquid interface, reference [51].

5.2.2 Two-Dimensional Flow

Although above formulation is general and applicable for a general flow field, it may be of interest to express the formulation in the practical separated flow problems, i.e., plane and annular flow regimes.

A. Plane Flow

Plane flow continuity equations can be obtained either from plane flow general balance equation, i.e., Equations (4.29) and (4.30) with appropriate identifications for ψ, $\dot{\psi}$ and $\vec{\phi}$, i.e., Equation (5.1) or from Equations (5.6a and b) together with the relations derived in Appendix B. Here we shall obtain them directly from the general balance equations. Therefore, in view of Equations (4.29), (4.30) and (5.1) we can express the continuity equations for __phase 1__ as

$$\frac{\partial}{\partial t}\, \eta \, \langle\!\langle \varrho_1 \rangle\!\rangle + \frac{\partial}{\partial x}\, \eta \, \langle\!\langle \varrho_1 \rangle\!\rangle \langle v_{1x} \rangle = - \dot{m}_{1i} \left[1 + \left(\frac{\partial \eta}{\partial x} \right)^2 \right]^{1/2} - \dot{m}_{1e} \qquad (5.8a)$$

whereas for __phase 2__ as

$$\frac{\partial}{\partial t}\, (H-\eta)\, \langle\!\langle \varrho_2 \rangle\!\rangle + \frac{\partial}{\partial x}\, (H-\eta)\, \langle\!\langle \varrho_2 \rangle\!\rangle \langle v_{2x} \rangle = - \dot{m}_{2i} \left[1 + \left(\frac{\partial \eta}{\partial x} \right)^2 \right]^{1/2} \qquad (5.8b)$$

where we have used Equation (5.3) for $\dot{m}_{K\ell}$

In the presence of long interfacial waves

$$\frac{\partial \eta}{\partial x} \ll 1.0 \qquad (5.9)$$

which in turn implies that

$$\left(\frac{\partial \eta}{\partial x} \right)^2 \simeq 0 \qquad (5.10)$$

and it can be neglected in Equations (5.8a and b).

B. Annular Flow

Similarly, to obtain the annular flow equations. We shall use the general balance equations appropriate to two dimensional axially symmetric flow, i.e., Equations (4.31) and (4.32), together with Equation (5.1). In this case we find for __phase 1__

$$\frac{\partial}{\partial t} (1-\alpha) \langle\langle \rho_1 \rangle\rangle + \frac{\partial}{\partial x} (1-\alpha) \langle\langle \rho_1 \rangle\rangle \langle v_{1x} \rangle = \qquad (5.11a)$$

$$-(1-\alpha) \langle\langle \rho_1 \rangle\rangle \langle v_{1x} \rangle \frac{d}{dx} \ln D_e^2 + \Gamma_{1i} + \Gamma_{1e}$$

whereas for <u>phase 2</u>

$$\frac{\partial}{\partial t} \alpha \langle\langle \rho_2 \rangle\rangle + \frac{\partial}{\partial x} \alpha \langle\langle \rho_2 \rangle\rangle \langle v_{2x} \rangle = -\alpha \langle\langle \rho_2 \rangle\rangle \langle v_{2x} \rangle \frac{d}{dx} \ln D_e^2 + \Gamma_{2i} \qquad (5.11b)$$

where

$$\Gamma_{1i} = -\Gamma_{2i} = -\frac{4\sqrt{\alpha}}{D_e} \dot{m}_{1i} \left[1 + \left(\frac{\partial}{\partial x} \frac{D_e \sqrt{\alpha}}{2} \right)^2 \right]^{1/2} \qquad (5.12a)$$

and

$$\Gamma_{1e} = -\frac{4}{D_e} \dot{m}_{1e} \left[1 + \left(\frac{d}{dx} \frac{D_e}{2} \right)^2 \right]^{1/2} \qquad (5.12b)$$

Again if the interfacial waves are long compared with a character-istic length in longitudional direction then a similar approximation to Equation (5.10) can be done, thus

$$\left(\frac{\partial}{\partial x} \frac{D_e \sqrt{\alpha}}{2} \right)^2 \simeq 0 \qquad (5.13)$$

5.3 Diffusion Model Formulation

A kinematic formulation based on the diffusion model is represented in terms of again two continuity equations. However, in analogy with the chemically reacting binary mixture the continuity equations appro-priate to the diffusion model are: the mixture continuity equation and the diffusion equation or the void propagation equation for one of the phases. In the following sections we shall derive these equations in order to describe the kinematics of a two-phase flow in terms of the diffusion model.

5.3.1 Derivation of Mixture Continuity Equation

In order to derive the mixture continuity equation we shall use the one-dimensionalized general balance equation for the mixture, i.e., Equation (4.27), together with the proper identifications expressed by Equation (5.1). In view of these identifications for ψ, $\dot{\psi}$ and for $\vec{\phi}$, Equation (4.23) yields the mixture continuity equation, thus

$$\frac{\partial \rho_m}{\partial t} + \frac{\partial}{\partial x} \rho_m v_{mx} = -\rho_m v_{mx} \frac{d}{dx} \ln A_{Tc} + \sum_{k=1}^{2} \Gamma_{ke} \qquad (5.14)$$

where we have made use of the conservation of mass requirement at the interface, which is mathematically expressed by Equation (5.5).

We note that in the absence of the external mass injection term Γ_{Ke}, and of the change in channel area, Equation (5.14) reduces to one-dimensional continuity equation for a single phase flow in a pipe. By substituting the definition of the material derivative following the motion of the baricenter of the mixture given by

$$\frac{D_m}{Dt} = \frac{\partial}{\partial t} + v_{mx} \frac{\partial}{\partial x} \qquad (5.15)$$

into Equation (5.8), the mixture continuity equation can be expressed also as

$$\frac{D_m \rho_m}{Dt} + \rho_m \frac{\partial v_{mx}}{\partial x} = -\rho_m v_{mx} \frac{d}{dx} \ln A_{Tc} + \sum_{k=1}^{2} \Gamma_{ke} \qquad (5.16)$$

In the case of two-dimensional plane flow where $A_{Tc} = H \times 1$, and $\Gamma_{2e} = 0$, (see Figure 15), the mixture continuity equation reduces to

$$\frac{\partial \rho_m}{\partial t} + \frac{\partial}{\partial x} \rho_m v_{mx} = \Gamma_{1e} \qquad (5.17)$$

whereas for the <u>annular flow</u> where $A_{Tc} = \frac{\pi D_e^2}{4}$, and $\Gamma_{2e} = 0$, (see Figure 16), the mixture continuity equation becomes

$$\frac{\partial \rho_m}{\partial t} + \frac{\partial}{\partial x} \rho_m v_{mx} = \Gamma_{1e} - \rho_m v_{mx} \frac{d \ln D_e^2}{dx} \qquad (5.18)$$

$$= -\dot{m}_{1e} \frac{4}{D_e} \left[1 + \left(\frac{d}{dx} \frac{D_e}{2} \right)^2 \right]^{1/2} - \rho_m v_{mx} \frac{d \ln D_e^2}{dx}$$

We shall proceed now with the derivation of the diffusion equation. However, since this equation is expressed in terms of mixture properties, i.e., in terms of either the velocity of the center of mass or of the center of volume we must first derive the appropriate expression for the latter velocity.

5.3.2 Volumetric Flux of the Mixture

The kinematic aspects of a mixture can be analyzed by considering either the velocity of the center of mass or of the center of volume. Since these two velocities are related by Equation (3.58), it is necessary to consider only one of the two.

If we are interested in determining the response of the volumetric concentration to various perturbations, then it may be advantageous to formulate the problem in terms of velocity of the center of volume of the mixture $\vec{j}$, and of the drift velocities $\vec{V}_{1j}$ and $\vec{V}_{2j}$ of phase 1 and 2 with respect to $\vec{j}$. Expressions for these velocities are given by Equations (3.54), (3.56a), and (3.56b). In order to derive this relation we shall start with Equations (5.6a and b). Carrying out the differentiations on these two equations and eliminating the $\frac{\partial \alpha}{\partial t}$ term between the resulting equations and in view of the conservation of mass at the interface, i.e., Equation (5.5), we obtain

$$\frac{\partial \dot{j}_x}{\partial x} = \Gamma_{ai} \frac{\Delta e}{\langle\!\langle e_1 \rangle\!\rangle \langle\!\langle e_2 \rangle\!\rangle} - \frac{1-\alpha}{\langle\!\langle e_1 \rangle\!\rangle} \frac{D_1 \langle\!\langle e_1 \rangle\!\rangle}{Dt} -$$

$$\frac{\alpha}{\langle\!\langle e_2 \rangle\!\rangle} \frac{D_2 \langle\!\langle e_2 \rangle\!\rangle}{Dt} + \sum_{k=1}^{2} \frac{\Gamma_{ke}}{\langle\!\langle e_k \rangle\!\rangle} - \dot{j}_x \frac{d \ln A_{Tc}}{dx} \qquad (5.19)$$

We note here that in arriving at Equation (5.19), we have used expressions for the volumetric flux j_x given by Equation (3.64), and of the material derivative given by

$$\frac{D_k}{Dt} = \frac{\partial}{\partial t} + \langle v_{kx} \rangle \frac{\partial}{\partial x} \qquad , K = 1,2 \qquad (5.20)$$

Equation (5.19), implies that, in general, the change of the volumetric flux density in the x-direction is proportional to the rate of vapor mass formation per unit volume of the system, to the compressibilities of the phases and to the total cross-sectional area change along the channel. Under certain conditions Equation (5.19) can be integrated easily to obtain the axial distribution of the volumetric flux density. For instance, in the case of adiabatic two-phase flow of the incompressible fluids without mass injection at the external boundaries, from Equation (5.19), after integration we obtain

$$\frac{\dot{j}_x(x,t)}{\dot{j}_x(x_o,t)} = \frac{A_{Tc}(x_o)}{A_{Tc}(x)} \qquad (5.21)$$

Consequently, j_x is a known function of x and t because $j_x(x_o,t)$ and $A_{Tc}(x_o)$ are known functions denoting the pipe inlet conditions. It is important to recognize that for constant area pipe, j_x becomes only a function of time, which may be very useful relation in the analysis of adiabatic flows.

In general case, however, the volumetric flux density is obtained by integrating Equation (5.19), thus

$$\dot{j}_x(x,t) = \dot{j}_x(x_o,t) \frac{A_{T_c}(x_o)}{A_{T_c}(x)} + \frac{1}{A_{T_c}(x)} \int_{x_o}^{x} A_{T_c}(x) \Big[\Gamma_{2i} \frac{\Delta \rho}{\langle\!\langle \rho_i \rangle\!\rangle \langle\!\langle \rho_2 \rangle\!\rangle} - \qquad (5.22)$$

$$\frac{1-\alpha}{\langle\!\langle \rho_i \rangle\!\rangle} \frac{D_i \langle\!\langle \rho_i \rangle\!\rangle}{Dt} - \frac{\alpha}{\langle\!\langle \rho_2 \rangle\!\rangle} \frac{D_2 \langle\!\langle \rho_2 \rangle\!\rangle}{Dt} + \sum_{k=1}^{2} \frac{\Gamma_{k\rho}}{\langle\!\langle \rho_k \rangle\!\rangle} \Big] dx$$

From Equation (5.22) two observations can be made: First, the volumetric flux density j_x, of the mixture, i.e., the velocities of the center of volume at a given point in the system, depends on the inlet velocity, and on the <u>integrated</u> effect of the mass generation (due to mass injection and phase change) decreased by the effects of the compressibilities of the two phases. Second, in order to evaluate j_x it is necessary to specify the constitutive equation for interfacial source term Γ_{Ki}, the significance of which was discussed in Section 5.2.1.

5.3.3 Diffusion Equations

A. Derivations

In order to derive the diffusion equation, we follow the standard procedure used in the multi-component systems, namely we replace the convective fluxes by the appropriate expressions involving diffusions.

We start with the continuity equation for phase 2 and express the velocity $\langle \vec{v}_2 \rangle$, of phase 2 in terms of the diffusion velocity $\vec{V}_{2m}$, and of the baricenter velocity $\vec{v}_m$, of the mixture; thus from Equations (3.45) and (5.7b) we obtain after arrangement

$$\frac{\partial}{\partial t} C \rho_m + \frac{\partial}{\partial x} C \rho_m v_{mx} = -\frac{\partial}{\partial x} C \rho_m V_{2mx} + \qquad (5.23a)$$

$$\sum_{\ell=e,i} \Gamma_{2\ell} - C \rho_m (v_{mx} + V_{2mx}) \frac{d \ell_n A_{T_c}}{dx}$$

or using Equation (3.36), above equation can be equally expressed in

terms of α, thus

$$\frac{\partial}{\partial t} \ll \ell_2 \gg + \frac{\partial}{\partial x} \ll \ell_2 \gg v_{m \varkappa} = -\frac{\partial}{\partial x} \ll \ell_2 \gg V_{2m \varkappa} + \qquad (5.23b)$$

$$\sum_{\ell = e, i} \Gamma_{2\ell} - \ll \ell_2 \gg (v_{m \varkappa} + V_{2m \varkappa}) \frac{d \ln h_{\tau_2}}{dx}$$

The various terms of Equation (5.23) account for the change of
the mass of the second phase with time, and for the effects due to con-
vection, diffusion, production at the internal and external surfaces and
finally for the effect of a variable area of the pipe.

In the literature, dealing with mass transfer (see, for example,
reference [50]), it is conventional procedure to express the diffusional
mean flux by means of the Fick's first law of diffusion, which is given
by

$$c \ell_m V_{2m \varkappa} = - \ell_m D \frac{\partial c}{\partial x} \qquad (5.24)$$

where D is the diffusion coefficient. Furthermore, mass concentrations
of components of a mixture are stated with reference to unit of volume,
namely partial densities of components. Partial density of the K-th
constituent is defined as the mass of the K-th constituent per unit
volume of the mixture. Consequently, in two-phase flow system partial
densities ρ_K^*, can be expressed by

$$\ell_1^* = (1-c) \ell_m = (1-\alpha) \ll \ell_1 \gg \qquad (5.25a)$$

$$\ell_2^* = c \ell_m = \alpha \ll \ell_2 \gg \qquad (5.25b)$$

Inserting Equation (5.23) and (5.24) in Equation (5.22) we get
the following two-phase flow diffusion equation expressed in terms of

partial density thus

$$\frac{\partial \ell_2^*}{\partial t} + \frac{\partial}{\partial x} \ell_2^* v_{mx} = \frac{\partial}{\partial x} \left(e_m D \frac{\partial c}{\partial x} \right) + \sum_{\ell = e, i} \Gamma_{2\ell} - \qquad (5.26)$$

$$\left(\ell_2^* v_{mx} - e_m D \frac{\partial c}{\partial x} \right) \frac{d \ln A T_c}{dx}$$

which is completely equivalent to binary diffusion equation used in the analysis of the binary mixtures. In effect, in the absence of last two terms, which account for the mass production at the internal and external boundaries of the second phase, and for the area change of the duct, respectively, one can recover the equivalent Fick's second law of diffusion written for a one-dimensional binary mixtures of non-reacting systems.

As in binary diffusion, the two phase flow diffusion equation can be expressed in a variety of forms. In the following paragraphs we shall express it in terms of concentration diffusions, i.e., in terms the mass concentration and the void diffusion equations.

In order to obtain the diffusion equation in terms of mass concentration we use the mixture continuity equation in Equation (5.26); thus from Equations (5.14) and (5.26) we obtain, after some arrangement:

$$\frac{\partial c}{\partial t} + v_{mx} \frac{\partial c}{\partial x} = \frac{1}{e_m} \frac{\partial}{\partial x} \left(e_m D \frac{\partial c}{\partial x} \right) - e_m D \frac{\partial c}{\partial x} \frac{d \ln A T_c}{dx} + \quad (5.27)$$

$$\frac{1}{e_m} \left[\Gamma_{2i} + (1-c) \Gamma_{2e} - c \Gamma_{1e} \right]$$

Similarly we can derive the void diffusion equation. To do so, we shall follow the basic pattern which was used in the derivation of the diffusion equation. We start with the continuity equation for phase 2 and express the velocity $\langle \vec{v_2} \rangle$, of the second phase in terms of the

drift velocity $\vec{V}_{2j}$, and of the velocity $\vec{j}$, of the center of volume, i.e., the volumetric flux density of the mixture; thus from Equations (3.55) and (5.6b), we obtain

$$\frac{\partial}{\partial t} \propto \langle\langle \rho_2 \rangle\rangle + \frac{\partial}{\partial x} \propto \langle\langle \rho_2 \rangle\rangle \, j_x = -\frac{\partial}{\partial x} \propto \langle\langle \rho_2 \rangle\rangle V_{2jx} + \qquad (5.28a)$$

$$\sum_{\ell=e,i} \Gamma_{2\ell} - \propto \langle\langle \rho_2 \rangle\rangle \left(j_x + V_{2jx} \right) \frac{d \ln \hbar T_c}{dx}$$

which can be also expressed in terms of the mixture velocity v_{mK}, thus

$$\frac{\partial}{\partial t} \propto \langle\langle \rho_2 \rangle\rangle + \frac{\partial}{\partial x} \propto \langle\langle \rho_2 \rangle\rangle \, v_{mx} = -\frac{\partial}{\partial x} \frac{\propto \langle\langle \rho_2 \rangle\rangle \langle\langle \rho_2 \rangle\rangle}{\rho_m} V_{2jx} + \qquad (5.28b)$$

$$\sum_{\ell=e,i} \Gamma_{2\ell} - \propto \langle\langle \rho_2 \rangle\rangle \left(v_{mx} + \frac{\langle\langle \rho_1 \rangle\rangle}{\rho_m} V_{2jx} \right) \frac{d \ln \hbar T_c}{dx}$$

which corresponds to Equation (5.23).

Differentiating Equation (5.28a), and using Equation (5.19), results in

$$\frac{\partial \propto}{\partial t} + j_x \frac{\partial \propto}{\partial x} = -\frac{\partial}{\partial x} \propto V_{2jx} + \propto (1-\propto) \left[\frac{1}{\langle\langle \rho_1 \rangle\rangle} \frac{D_1 \langle\langle \rho_1 \rangle\rangle}{Dt} - \frac{1}{\langle\langle \rho_2 \rangle\rangle} \frac{D_2 \langle\langle \rho_2 \rangle\rangle}{Dt} \right] + (5.29)$$

$$\frac{\rho_m}{\langle\langle \rho_1 \rangle\rangle \langle\langle \rho_2 \rangle\rangle} \Gamma_{2i} + \frac{(1-\propto)\langle\langle \rho_1 \rangle\rangle \Gamma_{2e} - \propto \Gamma_{1e}}{\langle\langle \rho_1 \rangle\rangle \langle\langle \rho_2 \rangle\rangle} - \propto V_{2jx} \frac{d \ln \hbar T_c}{dx}$$

where the material derivatives used in the second term on the right side are defined by Equation (5.20).

Expressing the drift flux in terms of the diffusion coefficient $D_{\mathcal{S}}$ - the subscript $\mathcal{S}$ is used in order differentiate the drift flux from the diffusion flux which is expressed in terms of D- the Fick's first law can be expressed by

$$\propto V_{2jx} = -D_{\mathcal{S}} \frac{\partial \propto}{\partial x} \qquad (5.30)$$

Inserting Equation (5.30) in (5.29), we obtain the <u>void diffusion</u> equation, thus

$$\frac{\partial \alpha}{\partial t} + j_x \frac{\partial \alpha}{\partial x} = \frac{\partial}{\partial x}\left(D_\nu \frac{\partial \alpha}{\partial x}\right) + \alpha(1-\alpha)\left(\frac{1}{\langle\!\langle\rho_1\rangle\!\rangle}\frac{D_1\langle\!\langle\rho_1\rangle\!\rangle}{Dt} - \frac{1}{\langle\!\langle\rho_2\rangle\!\rangle}\frac{D_2\langle\!\langle\rho_2\rangle\!\rangle}{Dt}\right) + \quad (5.31)$$

$$\frac{\rho_m}{\langle\!\langle\rho_1\rangle\!\rangle\langle\!\langle\rho_2\rangle\!\rangle}\ \Gamma_{ii} + \frac{(1-\alpha)\langle\!\langle\rho_1\rangle\!\rangle\Gamma_{2e} - \alpha\langle\!\langle\rho_2\rangle\!\rangle\Gamma_{1e}}{\langle\!\langle\rho_1\rangle\!\rangle\langle\!\langle\rho_2\rangle\!\rangle} + D_\nu \frac{\partial\alpha}{\partial x}\frac{d\ln\tau_c}{dx}$$

If we neglect the source term and the effects of the area change and of compressibility we obtain again Fick's second law of diffusion

$$\frac{\partial \alpha}{\partial t} + j_x \frac{\partial \alpha}{\partial x} = \frac{\partial}{\partial x}\left(D_\nu \frac{\partial \alpha}{\partial x}\right) \qquad (5.32)$$

B. <u>Discussions</u>

In the proceding section we have derived various forms of the diffusion equation, i.e., Equations (5.23), (5.26), (5.27) and (5.29) or (5.31), appropriate to a two-phase flow system. Although each of these equations describes the concentration profile in a two-phase mixture, from the appearance of these equations it is obvious that those which are expressed in terms of α will be advantageous for analyzing two-phase flow of <u>incompressible</u> fluids, whereas those which are expressed in terms of C will be more useful to two-phase flow of <u>compressible</u> fluids. Otherwise these equations are fairly general and valid for systems with variable mixture density and concentrations and variable diffusivity.

In order to use these equations expressed in the various forms, some information must be available on the <u>diffusion flux</u> $\alpha \ll \rho_2 \gg \vec{V}_{2m}$. This brings us to the specification of the diffusion coefficient D, or drift coefficient D_ν, by a constitutive equation.

It was noted in the preceding sections that we can describe the kinematics of a two-phase flow field in terms of two field equations

(one for the mixture and the other for a phase, namely the diffusion or drift equations), together with two constitutive equations: one for the vapor generation Γ_{Ki}, and the other for the diffusion flux. This formulation is, therefore, completely in agreement with the formulation of a multi-component chemically reacting mixtures.

However, the present lack of a constitutive equation for diffusion of the type given by Equation (5.24) restricts the usefulness of the diffusion coefficient because experimental data on the diffusion coefficient for two-phase mixture are almost non-existent. Furthermore, an additional difficulty may arise when the diffusion coefficient depends on the concentration; such a case Equations (5.27) and (5.29) become a non-linear partial differential equations for which solutions are not usually available.

For many two-phase flow regimes (bubbly, churn turbulent, slug flow etc.) the vapor drift velocity is known, reference [51]. Consequently, it is advantageous to formulate the problem in terms of the void diffusion equation or in terms of the void propagation equation derived in the section that follows.

5.3.4 The Void Propagation Equations

A. <u>Derivation</u>

It was shown in reference [51] that for most flow regimes of practical interest the second phase drift velocity $\overrightarrow{V}_{2j}$, is either a constant or a function of the void fraction α. For such cases it is advantageous to transform the continuity equation for the second phase into void propagation equation rather than into diffusion equation and analyze it in terms of kinematic waves. Consequently, we can express

the first term on the right hand side of Equation (5.29) as

$$\frac{\partial}{\partial x} \propto V_{2jx} = \left(V_{2jx} + \propto \frac{dV_{2jx}}{dx} \right) \frac{\partial \propto}{\partial x} \tag{5.33}$$

By substituting Equation (5.33) in Equation (5.29), we obtain the general form of the <u>void propagation equation</u> for a two-phase flow system with a change of phase, thus

$$\frac{\partial \propto}{\partial t} + C_{K} \frac{\partial \propto}{\partial x} = \propto (1-\propto) \left[\frac{1}{\ll \rho_{1} \gg} \frac{D_{1} \ll \rho_{1} \gg}{Dt} - \frac{1}{\ll \rho_{2} \gg} \frac{D_{2} \ll \rho_{2} \gg}{Dt} \right] + \tag{5.34}$$

$$\frac{\rho_{m}}{\ll \rho_{1} \gg \ll \rho_{2} \gg} \Gamma_{2i} + \frac{(1-\propto) \ll \rho_{1} \gg \Gamma_{2e} - \propto \Gamma_{1e}}{\ll \rho_{1} \gg \ll \rho_{2} \gg} - \propto V_{2jx} \frac{d\ell_{m} A_{Tc}}{dx}$$

where we have defined the velocity of the kinematic wave C_{K}, by

$$C_{K} = j_{x} + \frac{\partial}{\partial x} (\propto V_{2jx}) = j_{x} + V_{2jx} + \propto \frac{dV_{2jx}}{d\propto} \tag{5.35}$$

which in view of Equation (3.53b), can be also expressed by

$$C_{K} = j_{x} + \frac{\partial}{\partial x} \propto (1-\propto) V_{rx} = j_{x} + V_{2jx} + \propto \frac{\partial}{\partial x} (1-\propto) V_{rx} \tag{5.36}$$

Expressing the volumetric flux density j_{x} in Equation (5.35) by means of Equation (5.22), we obtain the complete form of the kinematic wave velocity C_{K}, thus

$$C_{K} = j_{x}(x_{0},t) \frac{A_{Tc}(x_{0})}{A_{Tc}(x)} + V_{2jx} + \propto \frac{dV_{2jx}}{d\propto} + \int_{x_{0}}^{x} A_{Tc} \left[\Gamma_{2i} \frac{\Delta \rho}{\ll \rho_{1} \gg \ll \rho_{2} \gg} - \right. \tag{5.37}$$

$$\left. \frac{(1-\propto)}{\ll \rho_{1} \gg} \frac{D_{1} \ll \rho_{1} \gg}{Dt} - \frac{\propto}{\ll \rho_{2} \gg} \frac{D_{2} \ll \rho_{2} \gg}{Dt} + \sum_{k=1}^{2} \frac{\Gamma_{ke}}{\ll \rho_{k} \gg} \right] dx$$

Finally, we obtain from Equations (5.34) and (5.37) the following expression for the <u>void propagation equation</u> in a two-phase flow system with an internal and external mass transfers, thus

$$\frac{\partial \alpha}{\partial t} + \left\{ j_x(x_0,t) \frac{A_{T_c}(x_0)}{A_{T_c}(x)} + V_{\ast jx} + \alpha \frac{d V_{\ast jx}}{dx} + \int_{x_0}^{x} A_{T_c} \left[\Gamma_{\ast i} \frac{\Delta e}{\ll e_1 \gg \ll e_2 \gg} - \right. \right. \tag{5.38}$$

$$\frac{(1-\alpha)}{\ll e_1 \gg} \frac{D_1 \ll e_1 \gg}{Dt} - \frac{\alpha}{\ll e_2 \gg} \frac{D_2 \ll e_2 \gg}{Dt} + \sum_{k=1}^{2} \frac{\Gamma_{ke}}{\ll e_k \gg} \left] dx \right\} \frac{\partial \alpha}{\partial x} =$$

$$\alpha(1-\alpha) \left(\frac{1}{\ll e_1 \gg} \frac{D_1 \ll e_1 \gg}{Dt} - \frac{1}{\ll e_2 \gg} \frac{D_2 \ll e_2 \gg}{Dt} \right) + \frac{e_m}{\ll e_1 \gg \ll e_2 \gg} \Gamma_{\ast i} +$$

$$\frac{(1-\alpha) \ll e_1 \gg \Gamma_{2e} - \alpha \Gamma_{1e}}{\ll e_1 \gg \ll e_2 \gg} - \alpha V_{\ast jx} \frac{d \ln A_{T_c}}{dx}$$

The rate of propagation of the voids as well as the change, i.e., the distortion of the void as it propagates along the duct can be determined by means of this equation.

B. <u>Discussions</u>

Following observations can be made with respect to Equation (5.34), (5.37) and (5.38).

First, from Equation (5.34) it can be seen that variations in volumetric concentration α, are propagated through the flowing mixture by the kinematic waves whose velocity C_K, is given by Equation (5.37). These variations in α are attenuated or amplified as they flow upstream depending on the local effects of the mass generation at the internal and external surfaces, the compressibilities of the two phases and on the cross-sectional area of the conduit.

Secondly, it can be observed from Equation (5.37) that the velocity of the kinematic waves depends on the inlet conditions and on the intergrated effect of the mass generation at the boundaries decreased by the effect of the compressibilities of the two phases. Dependency on the flow regime, however, is reflected by the effect which a flow regime has on the drift velocity.

Finally, it can be seen from Equation (5.38) that the void propagation equation predicts the void response as function of time and of space to variations of inlet volumetric flux density $j(x_o,t)$, flow regime, compressibility of the each phase, the constitutive equation for the vaporization Γ_{Ki}, and of the cross-sectional area change of the duct.

If we neglect the effect of compressibilities and of the area change, and if we set the source terms equal to zero, for such a system the void propagation equation reduces to

$$\frac{\partial \alpha}{\partial t} + C_K \frac{\partial \alpha}{\partial x} = 0 \qquad (5.39)$$

which indicates that changes of volumetric concentration α, propagate with respect to a fixed frame of reference with velocity C_K given by Equation (5.35). In other words, the void α, is constant on waves which propagate with velocity C_K.

In this form theory of kinematic waves was developed by Kynch [52], for analyzing the problem of sedimentation and, independently by Lighthill and Whitham [8], for analyzing flood waves and traffic flow on highways. In the former reference, the waves propagating with the velocity C_K, were called <u>continuity waves</u> because they were generated by the equation of continuity whereas in the latter, they were called <u>kinematic waves</u> in order to differentiate them from dynamic waves which depend on the second law of motion. As Lighthill and Whitham pointed out, in reference [8] the important differences between these two systems of waves is that kinematic waves have only one velocity whereas dynamic waves have at least two (forwards and backwards). Consequently, the kinematic waves can propagate in one direction.

5.3.5 Scaling Criteria

Thus far in this chapter we have derived and discussed the set of equations which describe the kinematics of a two-phase flow field. Next we shall consider the dimensionless forms of these equations, so that the dimensionless groups obtained from the differential equations can be used for the purposes of the order of magnitude and of similarity analyses in separated two-phase flow systems.

In order to obtain the dimensionless forms of the mixture continuity and the diffusion equations, it is convenient to define the following variables:

$$X^+ = X/L = \text{dimensionless coordinate,} \qquad (5.40)$$

$$v_m^+ = v_m/v_{mo} = \text{dimensionless mixture velocity,}$$

$$V_{2j}^+ = V_{2j}/V_{2jo} = \text{dimensionless drift velocity,}$$

$$t^+ = t\, v_{mo}/L = \text{dimensionless time}$$

$$\rho_m^+ = \rho_m/\rho_{mo} = \text{dimensionless mixture density}$$

$$\rho_K^+ = <<\rho_K>>/\rho_{Ko} = \text{dimensionless K-th phase density, K = 1,2}$$

$$\Gamma_{K\ell}^+ = \Gamma_{K\ell}/\Gamma_{K\ell o} = \text{dimensionless mass transfer, K = 1,2; l = e,i}$$

in which L, v_{mo}, ρ_{mo}, ρ_{Ko}, and $\Gamma_{K\ell o}$ represent, respectively, any convenient characteristic length, velocity, mixture density, density of the K-th phase, and a characteristic mass generation. We note that depending upon the purpose, the characteristic variables can be defined in a slightly different way. Turning our attention now to the scaling the concentration, it can be observed that $\alpha = \dfrac{A_{2c}}{A_{Tc}}$, can be scaled in different ways. For instance, the characteristic scaling parameter for it can be difference of concentration between inlet and exit of the pipe if heat transfer problem is considered to be important or it may be scaled,

in terms of an average film thickness if the dynamic of interfaces is important.

Since both the plane flow and the annular flow can be described with the equations of the same type, in the following we shall use the general equation which, with proper substitution, can be applied to both cases.

Introducing the characteristic scaling parameters given by Equation (5.40) into the mixture continuity equation, Equation (5.14), we obtain on rearrangement

$$\frac{\partial \varrho_m^+}{\partial t^+} + \frac{\partial}{\partial x^+} \varrho_m^+ \, v_m^+ = - \varrho_m^+ \, v_m^+ \, \frac{d \ln A_{Tc}}{d x^+} + \sum_{k=1}^{2} N \varrho_k \, N_{sup_k} \Gamma_{ke}^+ \quad (5.41)$$

Since the diffusion model is formulated in terms of two continuity equations, to be consistent one more continuity equation is needed for scaling purposes. In view of the discussion made in Section 5.3.3, however, the void diffusion equation in the form of Equation (5.28b) will be used for the purpose of scaling because it does not introduce an unknown diffusion coefficient. Hence, introducing the scaling parameters in Equation (5.28b) we obtain in dimensionless form of the void propagation equation.

$$\frac{\partial}{\partial t^+} \alpha^+ \varrho_2^+ + \frac{\partial}{\partial x^+} \alpha^+ \varrho_2^+ \, v_{mx}^+ = - N\varrho_1 \, N_D \, \frac{\partial}{\partial x^+} \, \alpha^+ \, \frac{\varrho_1^+ \varrho_2^+}{\varrho_m^+} \, V_{2jx}^+ - \quad (5.42)$$

$$\alpha^+ \varrho_2^+ \left(v_{mx}^+ + \frac{\varrho_1^+}{\varrho_2^+} \, N\varrho_1 \, N_D \, V_{2jx}^+ \right) \frac{d \ln A_{Tc}}{d x} + \frac{1}{\alpha_o} \left(N_{pch} \, \Gamma_{2i}^+ + N_{sup2} \Gamma_{2a}^+ \right)$$

In these equations we have defined

The <u>Phase Change</u> number by

$$N_{pch} = \left(\frac{\Gamma_{2io}}{\varrho_{2o}} \right) \left(\frac{L}{v_{mo}} \right) \quad (5.43)$$

The <u>Drift</u> number by

$$N_D = \frac{V_{2jo}}{V_{mo}} \qquad (5.44)$$

The <u>Supply</u> numbers by

$$N_{supK} = \left(\frac{\Gamma_{Keo}}{\ell_{Ko}}\right)\left(\frac{L}{V_{mo}}\right) \quad , \; K = 1,2 \qquad (5.45)$$

The <u>Density Ratio</u> groups by

$$N_{\ell K} = \frac{\ell_{Ko}}{\ell_{mo}} \qquad , \; K = 1,2 \qquad (5.46)$$

We note that the "frequency of phase change" Ω , defined by

$$\Omega = \frac{\Gamma_{2io}}{\ell_{2o}} \qquad (5.47)$$

scales the rate of phase change. Indeed, it has the same meaning as the reaction frequency in chemical kinetics. The ratio L/v_{mxo}, scales the residence time of a particle in the mixture. We can express therefore, the <u>Phase Change</u> number as:

$$N_{pch} = \Omega \left(\frac{L}{V_{mo}}\right) \qquad (5.48)$$

Expressed in this form, i.e., as the product of the character-istic frequency and of the residence time, the Phase Change number is of a similar form to the Damköhler first group, reference [53], which is one of the most important similarity group used in scaling chemical reactors and rocket engines. It can be expected therefore, that for two-phase flow with a change of phase, the phase change number will play the same role as Damköhler first group in chemically reacting systems. In fact, the equality of the Phase Change number in two different systems

ensures that the phase change has progressed equally in both. If this is not satisfied, the dynamic conditions in the two systems will not be similar, since the phase change in one would have progressed further than in the other.

The Drift number N_D, plays the same role in two phase flow as Damköhler's second group in chemical kinetic. It scales the effect of the relative velocity between the phases. Since the drift velocity V_{2jx}, or its equivalent the diffusion velocity V_{2n}, depends on the flow regime, this group characterize the flow pattern.

It is to be noted here that the characteristic concentration α_o, comes as an independent group from the boundary conditions. Actually, the equality of this number between two different systems ensures that the geometric similarity is accomplished in both.

The significance of the Supply number given by Equation (5.45) is similar to that of the Phase Change number whereas the significance of the density ratio groups are self explanatory. In fact, they scale the relative importance of the densities. However, we note that these two density groups are not independent from each other. Recalling the definition of the mixture density it can be shown that

$$(1-\alpha)_o \; \frac{\rho_{1o}}{\rho_{mo}} + \alpha_o \; \frac{\rho_{2o}}{\rho_{mo}} = 1 \qquad (5.49)$$

It is evident that if N_{ρ_1} is chosen as independent density ratio group, then the other density ratio group N_{ρ_2}, is specified by means of Equation (5.49), or vice versa.

5.4 Summary

1) The kinematic aspects of a two-phase flow field were formulated

by considering a two-fluid model and a diffusion model. The first was expressed in terms of the continuity equations of the two phases, whereas the diffusion model was described by means of the continuity equation for the mixture and the diffusion or void propagation equation for one of the two phases.

2) From these formulations it was concluded that any formulation based on these two models should be expressed in terms of two field equations and two constitutive equations. In the two-fluid model these two constitutive equations were given by the equation for net vaporization and the interfacial kinematic coupling equation, i.e., the interfacial mass balance equation. In the diffusion model, the constitutive equations are the equation for net evaporation, and the equation for the diffusion flux or the drift flux.

3) The equations in their dimensionless form were used to obtain similarity groups. The significance of these groups was discussed and in particular, that of the Phase Change number, the Drift number and the Supply number. It was noted that the first two numbers are of great importance as scaling parameters.

4) Finally, it was seen that, in the absence of data on diffusion coefficient, an alternate formulation in terms of the kinematic waves is preferable. Such a formulation leads to the void propagation equation which should be used in place of the diffusion equation.

CHAPTER VI

MOMENTUM EQUATIONS

6.1 Introduction

It was discussed in detail in Chapter 2, that the momentum equations which have been proposed in the literature were not complete and are inappropriate for the dynamic system analysis of a separated two-phase flow. In view of these deficiencies it is desirable to derive here the momentum equations from the general balance equations derived in Chapter 4.

In a development similar to that of the previous chapter, the formulation will be expressed in terms of the two-fluid model and of the diffusion model. As before the two-fluid model formulation will be expressed in terms of two field equations, however, the diffusion model will be in terms of one field equation, i.e., that of the mixture. Furthermore, the effects of diffusion or drift stress and of interfacial dynamic interactions on the mixture flow will be demonstrated. Finally, similarity groups and scaling criteria will be derived and their significance will be discussed in detail.

6.2 Two-Fluid Model Formulation

6.2.1 Derivation of Phase Equations

It was noted in Section 3.1, that two-fluid model formulation is always expressed in terms of two field equations for each field. Following this principle we shall derive the averaged phase momentum

equations.

To obtain the mass-weighted area-averaged momentum equations from the general balance equation, i.e., Equation (4.20), we use the proper identifications for ψ, $\overline{\overline{\psi}}$ and $\vec{\phi}$. According to Table 3, for momentum transport they are given by

$$\psi = \vec{v}, \text{ momentum per unit mass,} \qquad (6.1)$$

$$\overline{\overline{\psi}} = \vec{g}, \text{ distributed momentum generation or sink per unit mass,}$$

$$\vec{\phi} = - \overline{\overline{\pi}} = -(-P\overline{\overline{\delta}} + \overline{\overline{\tau}}), \text{ molecular momentum flux with respect to center of mass of the fluid.}$$

Introducing Equation (6.1) in Equations (4.20a and b), we obtain the mass-weighted area-averaged momentum equations for phase 1:

$$\frac{\partial}{\partial t}(1-\alpha)\langle\!\langle\varrho_i\rangle\!\rangle\langle\vec{v}_i\rangle + \frac{\partial}{\partial x}(1-\alpha)\langle\!\langle\varrho_i\rangle\!\rangle\langle v_{ix}\rangle\langle\vec{v}_i\rangle = \frac{\partial}{\partial x}(1-\alpha)\left(-\langle\!\langle P_i\rangle\!\rangle\overline{\overline{\delta}} + \alpha\overline{\overline{\tau}}_i\rangle\!\rangle\right)\cdot\hat{1} + \quad (6.2a)$$

$$(1-\alpha)\langle\!\langle\varrho_i\rangle\!\rangle\vec{g} - \left\{(1-\alpha)\left[\langle\!\langle\varrho_i\rangle\!\rangle\langle v_{ix}\rangle\langle\vec{v}_i\rangle - \left(-\langle\!\langle P_i\rangle\!\rangle\overline{\overline{\delta}} + \langle\!\langle\overline{\overline{\tau}}_i\rangle\!\rangle\right)\cdot\hat{1}\right] + Cov(mom 1)\right\}\frac{d\ell_i A_{T_c}}{dx}$$

$$\frac{1}{A_{T_c}}\sum_{\ell=e,i}\int_{J_{i\ell}}\left[\dot{m}_{i\ell}\vec{v}_{i\ell} - \left(-P_{i\ell}\overline{\overline{\delta}} + \overline{\overline{\tau}}_{i\ell}\right)\cdot\hat{n}_{i\ell}\right]\frac{dA}{dx} - \frac{\partial}{\partial x}Cov(mom 1)$$

and for phase 2:

$$\frac{\partial}{\partial t}\alpha\langle\!\langle\varrho_2\rangle\!\rangle\langle\vec{v}_2\rangle + \frac{\partial}{\partial x}\alpha\langle\!\langle\varrho_2\rangle\!\rangle\langle v_{2x}\rangle\langle\vec{v}_2\rangle = \frac{\partial}{\partial x}\alpha\left(-\langle\!\langle P_2\rangle\!\rangle\overline{\overline{\delta}} + \langle\!\langle\overline{\overline{\tau}}_2\rangle\!\rangle\right)\cdot\hat{1} + \quad (6.2b)$$

$$\alpha\langle\!\langle\varrho_2\rangle\!\rangle\vec{g} - \left\{\alpha\left[\langle\!\langle\varrho_2\rangle\!\rangle\langle v_{2x}\rangle\langle\vec{v}_2\rangle - \left(-\langle\!\langle P_2\rangle\!\rangle\overline{\overline{\delta}} + \langle\!\langle\overline{\overline{\tau}}_2\rangle\!\rangle\right)\cdot\hat{1}\right] + Cov(mom 2)\right\}\frac{d\ell_2 A_{T_c}}{dx}$$

$$\frac{1}{A_{T_c}}\sum_{\ell=e,i}\int_{J_{2\ell}}\left[\dot{m}_{2\ell}\vec{v}_{2\ell} - \left(-P_{2\ell}\overline{\overline{\delta}} + \overline{\overline{\tau}}_{2\ell}\right)\cdot\hat{n}_{2\ell}\right]\frac{dA}{dx} - \frac{\partial}{\partial x}Cov(mom 2)$$

where

$$Cov(mom 1) = (1-\alpha)\langle\!\langle\varrho_i\rangle\!\rangle Cov(v_{ix}\vec{v}_i) ; \quad Cov(mom 2) = \alpha\langle\!\langle\varrho_2\rangle\!\rangle Cov(v_{2x}\vec{v}_2) \quad (6.3)$$

Equations (6.2a and b) which represent the averaged momentum equations for phase 1 and 2, respectively, are fairly general. The first term on the right hand side accounts for the effects of the pressure and viscous stresses; the second is the source term, i.e., gravitational force; the third accounts for the effect of changes of cross sectional area; the fourth term accounts for the effect of momentum transfers at the internal and external surfaces due to the mass transfer and the molecular action, the last term expresses the effect of velocity distribution over the cross sectional areas.

These momentum equations should be compared to those proposed in the literature and reviewed in Chapter 2. It will be seen that these relations do not account for most of the effects which are accounted for by the various terms on the right hand side of Equations (6.2a and b). We note also that if the total cross sectional area is constant and no mass injection occurs at the solid wall, then these equations can be reduced to those reported by Delhaye [24].

By expanding the left hand side of Equations (6.2a and b) and using the continuity equations, Equations (5.2a and b) we get the equation of motion for phase 1:

$$(1-\alpha)\langle\!\langle \varrho_1 \rangle\!\rangle \left(\frac{\partial \langle \vec{v_1} \rangle}{\partial t} + \langle \vec{v_{1x}} \rangle \frac{\partial \langle \vec{v_1} \rangle}{\partial x} \right) = \frac{\partial}{\partial x}\left[(1-\alpha)\left(-\langle\!\langle P_1 \rangle\!\rangle \bar{\bar{\delta}} + \langle\!\langle \bar{\bar{\tau}}_1 \rangle\!\rangle \right)\cdot\hat{x} \right] + \tag{6.4a}$$

$$(1-\alpha)\langle\!\langle \varrho_1 \rangle\!\rangle \vec{g} - \left\{ (1-\alpha)\left[(\langle\!\langle P_1 \rangle\!\rangle \bar{\bar{\delta}} - \langle\!\langle \bar{\bar{\tau}}_1 \rangle\!\rangle)\cdot\hat{x} \right] + Cov(mom 1) \right\} \frac{d \ln A_{Tc}}{dx} -$$

$$\frac{1}{A_{Tc}} \sum_{\ell=e,i} \int_{S_{i\ell}} \left[\dot{m}_{i\ell}(\vec{v}_{i\ell} - \langle \vec{v_1} \rangle) - (-P_{i\ell}\bar{\bar{\delta}} + \bar{\bar{\tau}}_{i\ell})\cdot\hat{n}_{i\ell} \right] dA - \frac{\partial}{\partial x} Cov(mom 1)$$

and the equation of motion for phase 2:

$$\alpha \langle\!\langle \rho_2 \rangle\!\rangle \left(\frac{\partial \langle \vec{v}_2 \rangle}{\partial t} + \langle v_2 \rangle \frac{\partial \langle \vec{v}_2 \rangle}{\partial x} \right) = \frac{\partial}{\partial x} \left[\alpha \left(- \langle\!\langle P_2 \rangle\!\rangle \bar{\bar{\varepsilon}} + \langle\!\langle \bar{\bar{\tau}}_2 \rangle\!\rangle \right) \cdot \hat{1} \right] + \quad (6.4b)$$

$$\alpha \langle\!\langle \rho_2 \rangle\!\rangle \vec{g} - \left\{ \alpha \left[\left(\langle\!\langle P_2 \rangle\!\rangle \bar{\bar{\varepsilon}} - \langle\!\langle \bar{\bar{\tau}}_2 \rangle\!\rangle \right) \cdot \hat{1} \right] + Cov(mom\,2) \right\} \frac{d \ln A_{T_c}}{dx} -$$

$$\frac{1}{A_{T_c}} \sum_{\ell = e, i} \int_{\vec{I}_{2\ell}} \left[\dot{m}_{2\ell} \left(\vec{v}_{2\ell} - \langle \vec{v}_2 \rangle \right) - \left(- P_{2\ell} \bar{\bar{\varepsilon}} + \bar{\bar{\tau}}_{2\ell} \right) \cdot \hat{n}_{2\ell} \right] \frac{dA}{dx} - \frac{\partial}{\partial x} Cov(mom\,2)$$

As it is evident from the above derivations that the momentum balances are vectorial equations, therefore, for each phase we have essentially three components x, y and z. Furthermore, these field equations must be supplemented by the interfacial balance of momentum transfer which couples the dynamic fields of the two-phases. Such a relation was derived in Appendix B and given by Equation (4.12). Further discussion about the constitutive equations will be made in relation to the two-dimensional flow problems.

6.2.2 Two-Dimensional Flow

Our purpose here is to reduce the general momentum balance equations to the simplest form possible. At the same time, however, we shall retain the terms which account for the essential physical processes. As a first step of the simplifications, we restrict the developments to two-dimensional flow fields which is equivalent to the assumption that all the components of vectors following one of the three directions are negligibly small in comparison with the other two directions. With this restriction in mind we shall simplify the momentum equations for two major flow fields, i.e., plane and annular flows.

A. Plane Flow

Plane flow momentum equations can be obtained from the general balance equations appropriate to the plane flow geometry i.e., Equations

(4.30) and (4.31) with proper identifications for ψ, $\hat{\psi}$ and $\vec{\phi}$, i.e.,
Equation (6.1). Therefore, substituting Equation (6.1) in Equation
(4.30) we obtain the momentum equation for <u>phase 1</u> as:

$$\frac{\partial}{\partial t}\left(\eta \langle\!\langle \rho_1 \rangle\!\rangle \langle\vec{v}_1\rangle\right) + \frac{\partial}{\partial x}\left(\eta \langle\!\langle \rho_1 \rangle\!\rangle \langle v_{1x}\rangle \langle\vec{v}_1\rangle\right) = -\frac{\partial}{\partial x}\eta \langle\!\langle P_1 \rangle\!\rangle \overline{\overline{S}} \cdot \hat{1} + \frac{\partial}{\partial x}\eta \langle\!\langle \overline{\overline{E}}_1 \rangle\!\rangle \cdot \hat{1} + \quad (6.5)$$

$$\eta \langle\!\langle \rho_1 \rangle\!\rangle \vec{g} - \left[\dot{m}_{1i}\vec{v}_{1i} + \left(P_{1i}\overline{\overline{S}} - \overline{\overline{E}}_{1i}\right)\cdot\hat{n}_{1i}\right]\left[1 + \left(\frac{\partial\eta}{\partial x}\right)^2\right]^{1/2} -$$

$$\dot{m}_{1e}\vec{v}_{1e} + \left(P_{1e}\overline{\overline{S}} - \overline{\overline{E}}_{1e}\right)\cdot\hat{n}_{1e} - H\frac{\partial}{\partial x}\,\text{Cov(mom 1)}$$

and substituting Equation (6.1) in Equation (4.30) we obtain the momentum equation for <u>phase 2</u> as:

$$\frac{\partial}{\partial t}\left[(H-\eta)\langle\!\langle \rho_2 \rangle\!\rangle \langle\vec{v}_2\rangle\right] + \frac{\partial}{\partial x}\left[(H-\eta)\langle\!\langle \rho_2 \rangle\!\rangle \langle v_{2x}\rangle \langle\vec{v}_2\rangle\right] = -\frac{\partial}{\partial x}(H-\eta)\langle\!\langle P_2 \rangle\!\rangle \overline{\overline{S}} \cdot \hat{1} + (6.6)$$

$$\frac{\partial}{\partial x}(H-\eta)\langle\!\langle \overline{\overline{E}}_2 \rangle\!\rangle \cdot \hat{1} + (H-\eta)\langle\!\langle \rho_2 \rangle\!\rangle \vec{g} - \left[\dot{m}_{2i}\vec{v}_{2i} + \left(P_{2i}\overline{\overline{S}} - \overline{\overline{E}}_{2i}\right)\cdot\hat{n}_{2i}\right]\left[1 + \left(\frac{\partial\eta}{\partial x}\right)^2\right]^{1/2} -$$

$$H\frac{\partial}{\partial x}\,\text{Cov(mom 2)}$$

In view of the geometric relations derived in Appendix B.1, vector multiplications appearing in Equation (6.5) and (6.6) can be carried out. For example, it can be shown that at the phase boundaries following relations hold:

$$\left[1 + \left(\frac{\partial\eta}{\partial x}\right)^2\right]^{1/2}\left(-P_{1i}\overline{\overline{S}} + \overline{\overline{E}}_{1i}\right)\cdot\hat{n}_{1i} = \left[-\left(-P_{1i} + \tau_{1xx}|_i\right)\frac{\partial\eta}{\partial x} + \tau_{1xy}|_i\right]\hat{1} + \quad (6.7)$$

$$\left[-\tau_{1xy}|_i\frac{\partial\eta}{\partial x} + \left(-P_{1i} + \tau_{1yy}|_i\right)\right]\hat{j}$$

$$\left[1+\left(\tfrac{\partial \eta}{\partial x}\right)^2\right]^{1/2}\left(-P_{2i}\,\bar{\bar{\delta}}+\bar{\bar{\tau}}_{2i}\right)\cdot\hat{n}_{2i}=\left[\left(-P_{2i}+\tau_{2xx}|_i\right)\tfrac{\partial \eta}{\partial x}-\tau_{2xy}|_i\right]\hat{1}+ \qquad (6.8)$$

$$\left[\tau_{2xy}|_i\,\tfrac{\partial \eta}{\partial x}-\left(-P_{2i}+\tau_{2yy}|_i\right)\right]\hat{\jmath}$$

$$\left(-P_{1e}\,\bar{\bar{\delta}}+\bar{\bar{\tau}}_{1e}\right)\cdot\hat{n}_{1e}=-\tau_{1xy}|_e\,\hat{1}-\left(-P_{1e}+\tau_{1yy}|_e\right)\hat{\jmath} \qquad (6.9)$$

$$-\tfrac{\dot{m}_{1i}}{H}\left[1+\left(\tfrac{\partial \eta}{\partial x}\right)^2\right]^{1/2}=\Gamma_{1i} \qquad (6.10)$$

$$-\tfrac{\dot{m}_{2i}}{H}\left[1+\left(\tfrac{\partial \eta}{\partial x}\right)^2\right]^{1/2}=\Gamma_{2i} \qquad (6.11)$$

Furthermore, no-slip condition at the external surfaces requires that

$$v_{1e_x}=0 \qquad (6.12)$$

When we substitute Equations (6.7) through (6.12) resulting equations will be too complicated for the purpose of practical applications. Therefore, further simplifications are desirable in order to use the momentum equations which we have derived.

For this purpose we shall utilize the Prandtl's boundary layer theory according to which the Navier-Stokes equations can be simplified to yield approximate solutions. As it is well-known Prandtl's approximation is based on the order of magnitude analysis through which the order of magnitude of the terms of the equations can be determined with repsect to the thickness $\delta \ll 1$, known as the dimensionless boundary layer thickness, reference [54, 55].

Motivated by the Prandtl's boundary layer analysis, we shall carry

out now a similar order of magnitude analysis for the two-dimensional flow of a compressible viscous fluid. For a two-dimensional fluid the continuity equation for the K-th phase (K = 1,2) is

$$\frac{\partial \rho_k}{\partial t} + \frac{\partial}{\partial x}(\rho_k v_{kx}) + \frac{\partial}{\partial y}(\rho_k v_{ky}) = 0 \qquad (6.13)$$

whereas the x and y components of the equation of motion are given respectively by:

$$\rho_k \left(\frac{\partial v_{kx}}{\partial t} + v_{kx}\frac{\partial v_{kx}}{\partial x} + v_{ky}\frac{\partial v_{kx}}{\partial y} \right) = -\frac{\partial P_k}{\partial x} + \frac{\partial}{\partial x}\left\{ \mu_k \left[2\frac{\partial v_{kx}}{\partial x} - \right. \right. \qquad (6.14)$$

$$\left. \left. \frac{2}{3}\left(\frac{\partial v_{kx}}{\partial x} + \frac{\partial v_{ky}}{\partial y} \right) \right] \right\} + \frac{\partial}{\partial y}\left[\mu_k \left(\frac{\partial v_{kx}}{\partial y} + \frac{\partial v_{ky}}{\partial x} \right) \right] + \rho_k g_x$$

and

$$\rho_k \left(\frac{\partial v_{ky}}{\partial t} + v_{kx}\frac{\partial v_{ky}}{\partial x} + v_{ky}\frac{\partial v_{ky}}{\partial y} \right) = -\frac{\partial P_k}{\partial y} + \frac{\partial}{\partial y}\left\{ \mu_k \left[2\frac{\partial v_{ky}}{\partial y} - \right. \right. \qquad (6.15)$$

$$\left. \left. \frac{2}{3}\left(\frac{\partial v_{kx}}{\partial x} + \frac{\partial v_{ky}}{\partial y} \right) \right] \right\} + \frac{\partial}{\partial x}\left[\mu_k \left(\frac{\partial v_{kx}}{\partial y} + \frac{\partial v_{ky}}{\partial x} \right) \right] + \rho_k g_y$$

Since for most practical separated two-phase flow systems liquid flows in a film adjacent to the heated or cooled walls we can make the "thin film" formulation which is based on the assumption that the film thickness is small compared to the longitudinal dimension. Thus, if η_{1o} denotes a typical value of the film thickness $\eta_1(x,t)$, and L denotes a typical longitudinal dimension, we introduce

$$\varepsilon_1 = \frac{\eta_{1o}}{L} \ll 1 \qquad (6.16)$$

as a perturbation parameter which is similar to the dimensionless bound-
ary layer thickness δ used in the Prandtl's argument. Of course, we could
have identified L with a typical wave-length of interfacial waves, then
this would lead to the __long-wave__ approximation.

Turning our attention now to the vapor or the gas phase, that
is to phase 2, we observe that the second phase is moving over the
liquid which is in motion. However, we can treat the second phase as
the flow over a rough surface. It is well known even if the second
phase is mostly in turbulent motion, then in the immediate neighborhood
of a wall the apparent turbulent stresses are small compared with the
viscous stresses. It follows therefore, that there exists a very thin
layer next to the interface which, in essence, behaves like a boundary
layer. For the purpose of scaling we can choose a typical laminar
boundary layer thickness η_{2o}, which satisfies

$$\varepsilon_2 = \frac{\eta_{2o}}{L} \ll 1 \qquad (6.17)$$

We note that this particular choice will be very useful for the non-
dimensionalization of the interfacial momentum balance and for the
determination of the various terms appearing on it.

We want to compare the magnitudes of the various terms in Equa-
tion (6.13)-(6.15). For this purpose let us first introduce the follow-
ing dimensionless coordinates defined by

$$x^+ = \frac{x}{L} \quad ; \quad y_k^+ = \frac{y}{\eta_{k_o}} \qquad (6.18)$$

Introducing a reference velocity U_K, and a reference density ρ_{Ko}, the
dimensionless velocity in the mean flow direction and the density become

$$v^+_{k_x} = \frac{v_{k_x}}{U_k} \quad ; \quad \ell^*_k = \frac{\ell_k}{\ell_{k_o}} \qquad (6.19)$$

Upon measuring the time in units of L/U_K, the scale for the y-component of the velocity is obtained by substituting Equations (6.16)-(6.19), in the continuity equation, Equation (6.13), and requiring the terms in the resulting dimensionless form of the continuity equation be order of unity we obtain

$$v^+_{k_y} = \frac{v_{k_y}}{\varepsilon_k U_k} \qquad (6.20)$$

which is in agreement with the boundary layer approximation.

Furthermore, measuring the pressure in unit of $\rho_{Ko} U^2_K$, we can reduce Equations (6.14) and (6.15) to dimensionless form. It is found that (6.14) remains of the same form, i.e.,

$$\ell^*_k \left(\frac{\partial v^+_{k_x}}{\partial t^+} + v^+_{k_x} \frac{\partial v^+_{k_x}}{\partial x^+} + v^+_{k_y} \frac{\partial v^+_{k_x}}{\partial x^+} \right) = -\frac{\partial P^+_k}{\partial x^+} + \frac{1}{Re_k} \left\{ \frac{\partial}{\partial x^+} \mu^+_k \left[2 \frac{\partial v^+_{k_x}}{\partial x^+} - \right. \right. \qquad (6.21)$$

$$\left. \left. \frac{2}{3} \left(\frac{\partial v^+_{k_x}}{\partial x^+} + \frac{\partial v^+_{k_y}}{\partial y^+} \right) \right] \right\} + \frac{1}{\varepsilon^2_k Re_k} \left[\frac{\partial}{\partial y^+} \mu^+_k \left(\frac{\partial v^+_{k_x}}{\partial y^+} + \varepsilon^2_k \frac{\partial v^+_{k_y}}{\partial x^+} \right) \right] + \frac{\ell^*_k g^+_x}{Fr_k}$$

whereas Equation (6.15) in dimensionless form becomes

$$\varepsilon^2_k \ell^*_k \left(\frac{\partial v^+_{k_y}}{\partial t^+} + v^+_{k_x} \frac{\partial v^+_{k_y}}{\partial x^+} + v^+_{k_y} \frac{\partial v^+_{k_y}}{\partial y^+} \right) = -\frac{\partial P^+_k}{\partial y^+} + \frac{1}{Re_k} \left\{ \frac{\partial}{\partial y^+} \mu^+_k \left[2 \frac{\partial v^+_{k_y}}{\partial y^+} - \right. \right. \qquad (6.22)$$

$$\left. \left. \frac{2}{3} \left(\frac{\partial v^+_{k_x}}{\partial x^+} + \frac{\partial v^+_{k_y}}{\partial y^+} \right) \right] \right\} + \frac{1}{Re_k} \left[\frac{\partial}{\partial x^+} \mu^+_k \left(\frac{\partial v^+_{k_x}}{\partial x^+} + \varepsilon^2_k \frac{\partial v^+_{k_y}}{\partial x^+} \right) \right] + \varepsilon_k \frac{\ell^*_k g^+_y}{Fr_k}$$

where

$$Re_K = \frac{U_K L}{\nu_{K_o}} \qquad (6.23)$$

is the <u>Reynolds number</u> and

$$Fr_K = \frac{U_K}{\sqrt{g L}} \qquad (6.24)$$

is the <u>Froude number</u>.

The boundary layer theory is an asymptotic approximation for a large Reynolds number whose magnitude is $O(1/\varepsilon_K^2)$. Furthermore, if we assume that the Froude number is $O(\varepsilon_K)$, the zero-th order approximation in ε_K is obtained from Equations (6.21) and (6.22) by neglecting the terms of order ε_K and higher, and reverting to the unstarred original variables, thus

$$\rho_K \left(\frac{\partial v_{Kx}}{\partial t} + v_{Kx} \frac{\partial v_{Kx}}{\partial x} + v_{Ky} \frac{\partial v_{Kx}}{\partial y} \right) = - \frac{\partial P_K}{\partial x} + \frac{\partial \mathcal{T}_{Kxy}}{\partial y} + \rho_K g_x \qquad (6.25a)$$

and

$$0 = - \frac{\partial P_K}{\partial y} + \rho_K g_y \qquad (6.25b)$$

It should be noted here that Equations (6.25a and b) are equivalent to the hydraulic approximation.

Furthermore, it can be shown from Equation (6.21) that the magnitude of the stress terms are:

$$\frac{\partial}{\partial x} \mathcal{T}_{Kxx} \approx O(\varepsilon_K^2) \qquad (6.26)$$

$$\frac{\partial}{\partial y} \mathcal{T}_{Kxy} \approx O(1 + \varepsilon_K^2) \qquad (6.27)$$

$$\frac{\partial}{\partial x} \eta_K \approx O(\varepsilon_K) \qquad (6.28)$$

From these equations it is evident that the normal stress τ_{Kxx}, can be dropped from the averaged momentum equations in comparison with the shear stress τ_{Kxx}, similarly, $(\frac{\partial \eta}{\partial x})^2$ can be neglected when compared with unity.

Using the approximations expressed by Equations (6.26) through (6.29), we obtain the averaged momentum equations for phase 1 and 2 in the x-direction, thus

$$\frac{\partial}{\partial t}\left(\eta \ll \rho_{1x} \gg \langle v_{1x} \rangle \right) + \frac{\partial}{\partial x}\left(\eta \ll \rho_{1} \gg \langle v_{1x}^2 \rangle \right) = -\frac{\partial}{\partial x} \, \eta \ll P_{1} \gg + \qquad (6.29a)$$

$$\ll \rho_{1} \gg \eta \, g_x - \dot{m}_{1i} \, v_{1ix} + \tau_{1i} - \tau_{1e} + P_{1i} \frac{\partial \eta}{\partial x} - H \frac{\partial}{\partial x} \, \text{Cor(mom 1)}$$

$$\frac{\partial}{\partial t}\left[(H-\eta) \ll \rho_{2} \gg \langle v_{2x} \rangle \right] + \frac{\partial}{\partial x}\left[(H-\eta) \ll \rho_{2} \gg \langle v_{2x}^2 \rangle \right] = -\frac{\partial}{\partial x} (H-\eta) \ll P_{2} \gg \quad (6.29b)$$

$$\ll \rho_{2} \gg (H-\eta) \, g_x - \dot{m}_{2i} \, v_{2ix} - \tau_{2i} - P_{2i} \frac{\partial \eta}{\partial x} - H \frac{\partial}{\partial x} \, \text{Cor(mom 2)}$$

where τ_{1i} and τ_{1e} are, respectively, the shear stresses at the interfacial and the external surfaces.

We note that the averaged pressure $\ll \rho_{K} \gg$ and the interfacial pressure P_{Ki}, $(K = 1,2)$, appearing in the averaged momentum equations can be related by integrating Equation (6.26), thus

$$\ll P_{1} \gg = P_{1i} - \frac{1}{2} \ll \rho_{1} \gg \eta \, g_y^* \qquad (6.30a)$$

$$\ll P_{2} \gg = P_{2i} + \frac{1}{2} \ll \rho_{2} \gg (H-\eta) \, g_y \qquad (6.30b)$$

Finally, using Equation (6.30a) for P_{1i} and (6.30b) for P_{2i}, we

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*η for η_1 and $(H - \eta)$ for η_2 were used.

obtain from Equation (6.29) the averaged momentum equations expressed in terms of the averaged pressures only, thus

$$\frac{\partial}{\partial t}\left(\eta \ll \rho_1 \gg \langle v_{1x}\rangle\right) + \frac{\partial}{\partial x}\left(\eta \ll \rho_1 \gg \langle v_{1x}\rangle^2\right) = -\eta \frac{\partial \ll \bar{P}_1 \gg}{\partial x} + \quad (6.31a)$$

$$\frac{1}{2} \ll \rho_1 \gg \eta \, g_y \frac{\partial \eta}{\partial x} + \ll \rho_1 \gg \eta \, g_x - \dot{m}_{1i} \, v_{1ix} + \tau_{1i} - \tau_{1e} - H \frac{\partial}{\partial x} \, \text{Cov (mom 1)}$$

and

$$\frac{\partial}{\partial t}\left[(H-\eta) \ll \rho_2 \gg \langle v_{2x}\rangle\right] + \frac{\partial}{\partial x}\left[(H-\eta) \ll \rho_2 \gg \langle v_{2x}\rangle^2\right] = -(H-\eta) \frac{\partial \ll \bar{P}_2 \gg}{\partial x} + \quad (6.31b)$$

$$\frac{1}{2} \ll \rho_2 \gg g_y (H-\eta) \frac{\partial \eta}{\partial x} + \ll \rho_2 \gg (H-\eta) \, g - \dot{m}_{2i} \, v_{2ix} - \tau_{2i} - H \frac{\partial}{\partial x} \, \text{Cov (mom 2)}$$

By expanding the left hand side of Equations (6.31a and b) and using the continuity equations, Equations (5.8a and b) we obtain the equation of motion for phase 1:

$$\eta \ll \rho_1 \gg \frac{D_1 \langle v_{1x}\rangle}{Dt} = -\eta \, \frac{\partial \ll \bar{P}_1 \gg}{\partial x} + \frac{1}{2} \ll \rho_1 \gg \eta \, g_y \frac{\partial \eta}{\partial x} + \quad (6.32a)$$

$$\eta \ll \rho_1 \gg g_x - \dot{m}_{1i}\left(v_{1ix} - \langle v_{1x}\rangle\right) + \tau_{1i} - \tau_{1e} - H \frac{\partial}{\partial x} \, \text{Cov (mom 1)}$$

and for phase 2:

$$(H-\eta) \ll \rho_2 \gg \frac{D_2 \langle v_{2x}\rangle}{Dt} = -(H-\eta) \frac{\partial \ll \bar{P}_2 \gg}{\partial t} + \frac{1}{2} \ll \rho_2 \gg (H-\eta) g_y \frac{\partial \eta}{\partial x} + \quad (6.32b)$$

$$(H-\eta) \ll \rho_2 \gg g_x - \dot{m}_{2i}\left(v_{2ix} - \langle v_{2x}\rangle\right) - \tau_{2i} - H \frac{\partial}{\partial x} \, \text{Cov (mom 2)}$$

where the material derivatives are defined by Equation (5.18).

In order to complete the dynamic description of the field, these two field equations must be supplemented by expressions for the wall stress τ_{1e} and either one of the interfacial stresses τ_{1i} or τ_{2i}. In addition to these two constitutive equations dynamic coupling, i.e., interfacial interactions between the two phases should be specified. As usual coupling can be specified by means of the "jump" conditions which are derived in Appendix A for general three dimensional flow field, Equation (B.25) or (4.12). In what follows we shall perform an order of magnitude analysis on the "jump" equations.

In Appendix C.1.4, the interfacial balance appropriate to the two-dimensional plane flow is derived and given by Equation (C.17). As expected, it is a vector expression. In our one-dimensionalized analysis we shall use the components of it in x- and y-directions. Therefore, it will be necessary to obtain the x- and y-components of the interfacial momentum balance equation, Equation (C.17). However, for the purpose of simplicity we shall derive the normal and tangential components instead of x- and y-components.

To obtain the normal and tangential components, respectively, we shall multiply Equation (C.17) throughout by unit vectors $\hat{n}_{1i}$ and $\hat{t}_i$, and using the resulting relations together with Equations (C.4), (C.5) and (C.6) it can be easily shown that the normal and tangential components of the interfacial balance are given by:

$$- \dot{m}_{1i}^2 \frac{\Delta \rho_i}{\rho_{1i}\, \rho_{2i}} + (P_{1i} - P_{2i}) + \left[(\tau_{2xx} - \tau_{1xx})_i \left(\frac{\partial \eta}{\partial x}\right)^2 + \right. \tag{6.33a}$$

$$\left. (\tau_{2yy} - \tau_{1yy})_i - (\tau_{2xy} - \tau_{1xy})_i \frac{\partial \eta}{\partial x} \right] \left[1 + \left(\frac{\partial \eta}{\partial x}\right)^2 \right]^{-1} = 2 \varkappa \sigma$$

$$\left\{-(\tau_{2xx} - \tau_{1xx})_i \frac{\partial \eta}{\partial x} + (\tau_{2y\bar{y}} - \tau_{1\bar{y}\bar{y}})_i + (\tau_{2x\bar{y}} \tau_{1x\bar{y}})_i \left[1 - \left(\frac{\partial \eta}{\partial x}\right)^2\right]\right\}\left\{1 + \left(\frac{\partial \eta}{\partial x}\right)^2\right\}^{-1} \quad (6.33b)$$

$$= \frac{\partial \sigma}{\partial s} = \frac{\partial \sigma}{\partial x}\left[1 + \left(\frac{\partial \eta}{\partial x}\right)^2\right]^{-1/2}$$

where we made use of Equation (B.27) for the velocity differences in the normal direction.

The surface conditions that are consistent with Equations (6.25) and (6.26) will be obtained by applying the boundary layer approximation to Equation (6.32). Using the scaling parameters, we can reduce Equations (6.33a) and (6.33b) to dimensionless form. It is found that Equation (6.33a) is identical to its dimensionless form

$$-\left(\frac{\dot{m}_{10}}{\rho_{20}U_1}\right)^2 \frac{\dot{m}_1^*}{N\rho} \left(\frac{1}{\rho_2^*} - \frac{1}{\rho_1^* N_\rho}\right) + \left(P_i^* - \frac{\rho_{20}U_2^2}{\rho_{10}U_1^2} P_i^*\right) + \quad (6.34a)$$

$$\frac{\rho_{20}U_2^2}{\rho_{10}U_1^2} \frac{r_2^*}{Re_2}\left\{\left[2\frac{\partial v_{2x}^*}{\partial x^*} - \frac{2}{3}\left(\frac{\partial v_{2x}^*}{\partial x^*} + \frac{\partial v_{2y}^*}{\partial y^*}\right)\right]\varepsilon_1^2\left(\frac{\partial \eta_1}{\partial x}\right)^2 - \right.$$

$$2\left(\frac{1}{\varepsilon_2}\frac{\partial v_{2x}^*}{\partial y^*} + \varepsilon_2\frac{\partial v_{2y}^*}{\partial x^*}\right)\varepsilon_1\left(\frac{\partial \eta_1}{\partial x^*}\right) +$$

$$\left[2\frac{\partial v_{2y}^*}{\partial y^*} - \frac{2}{3}\left(\frac{\partial v_{2x}^*}{\partial x^*} + \frac{\partial v_{2y}^*}{\partial y^*}\right)\right]\right\}\left[1 + \varepsilon_1^2\left(\frac{\partial \eta_1}{\partial x^*}\right)^2\right]^{-1/2} -$$

$$\frac{r_1^*}{Re_1}\left\{\left[2\frac{\partial v_{1x}^*}{\partial x^*} - \frac{2}{3}\left(\frac{\partial v_{1x}^*}{\partial x^*} + \frac{\partial v_{1y}^*}{\partial y^*}\right)\right]\varepsilon_1^2\left(\frac{\partial \eta_1^*}{\partial x^*}\right)^2 - 2\left(\frac{1}{\varepsilon_1}\frac{\partial v_{1x}^*}{\partial y^*} + \varepsilon_1\frac{\partial v_{1y}^*}{\partial x^*}\right)\varepsilon_1\left(\frac{\partial \eta_1^*}{\partial x^*}\right) + \right.$$

$$\left.\left[2\frac{\partial v_{1y}^*}{\partial y^*} - \frac{2}{3}\left(\frac{\partial v_{1x}^*}{\partial x^*} + \frac{\partial v_{1y}^*}{\partial y^*}\right)\right]\right\}\left[1 + \varepsilon_1^2\left(\frac{\partial \eta_1^*}{\partial x^*}\right)^2\right]^{-1/2} = -\Gamma^+ \frac{\varepsilon_1}{W_c}\frac{\partial \eta_1^-}{\partial x^-}$$

whereas Equation (6.35) in dimensionless form becomes

$$\left(\frac{\rho_{2o}\, U_2^{*2}}{\rho_{1o}\, U_1^{*2}}\right)\frac{M_2^{*}}{Re_2}\left\{-\left[2\frac{\partial v_{2x}^{*}}{\partial x^{+}}-\frac{2}{3}\left(\frac{\partial v_{2x}^{*}}{\partial x^{+}}+\frac{\partial v_{2y}^{*}}{\partial y^{+}}\right)\right]\varepsilon_1\frac{\partial \eta_1^{+}}{\partial x^{+}}+\right.$$

$$\left[2\frac{\partial v_{2y}^{*}}{\partial y^{+}}-\frac{2}{3}\left(\frac{\partial v_{2x}^{*}}{\partial x^{+}}+\frac{\partial v_{2y}^{*}}{\partial y^{+}}\right)\right]\varepsilon_1^{*}\frac{\partial \eta_1^{+}}{\partial x^{+}}+$$

$$\left(\frac{1}{\varepsilon_2}\frac{\partial v_{2x}^{*}}{\partial y^{+}}+\varepsilon_2\frac{\partial v_{2y}^{*}}{\partial x^{+}}\right)\left[1-\left(\frac{\partial \eta_1^{+}}{\partial x^{+}}\right)^2\varepsilon_1^2\right]\right\}-$$

$$\frac{M_1^{*}}{Re_1}\left\{-\left[2\frac{\partial v_{1x}^{*}}{\partial x^{+}}-\frac{2}{3}\left(\frac{\partial v_{1x}^{*}}{\partial x^{+}}+\frac{\partial v_{1y}^{*}}{\partial y^{+}}\right)\right]\varepsilon_1\frac{\partial \eta_1^{+}}{\partial x^{+}}+\left[2\frac{\partial v_{1y}^{*}}{\partial y^{+}}-\right.\right.$$

$$\frac{2}{3}\left(\frac{\partial v_{1x}^{*}}{\partial x^{+}}+\frac{\partial v_{1y}^{*}}{\partial y^{+}}\right)\varepsilon_1^{*}\frac{\partial \eta_1^{+}}{\partial x^{+}}+\left(\frac{1}{\varepsilon_1}\frac{\partial v_{1x}^{*}}{\partial y^{+}}+\varepsilon_1\frac{\partial v_{1y}^{*}}{\partial x^{+}}\right)\left[1-\left(\frac{\partial \eta_1^{+}}{\partial x^{+}}\right)^2\varepsilon_1^2\right]\right\}=$$

$$=\frac{1}{We}\frac{\partial \sigma^{+}}{\partial x^{+}}\left[1+\varepsilon_1^2\left(\frac{\partial \eta_1^{+}}{\partial x^{+}}\right)^2\right]^{1/2}$$

<div align="right">(6.34b)</div>

In these equations N_ρ is the Density Ratio group defined by Equation (5.50) whereas We is the Weber number defined by

$$We=\frac{\rho_{1o}\, L\, U_1^{*}}{\sigma_o}$$

<div align="right">(6.35a)</div>

Before assessing the magnitude of the various terms, we can make several observations with respect to Equation (6.34). First it may be pointed out that the pressure is non-dimensionalized so as render the pressure gradient and the inertia effects of equal magnitude, a fact that is well known in the boundary layer analyses. Furthermore, noting that the pressure drops in both phases are the same it can be seen that

$$\rho_{1o}\, U_1^{*2}\simeq \rho_{2o}\, U_2^{*2}$$

<div align="right">(6.35b)</div>

Second, we note that the Weber numbers encountered in the experiment with water and alcohol film flows are quite small. Therefore, we can suppose that We is of $0(\varepsilon_1)$.

Finally, since in the separated two-phase flow systems the center of mass of the mixture moves closely with the center of the liquid from the definition of the phase change number given by Equation (5.47) it is evident that the first term in Equation (6.34a) is closely related to the phase change number

$$N_{pch} \simeq \frac{\dot{m}_{1o}}{\rho_{2o} u_1}$$ (6.36)

which can be assessed from the diffusion equation to be of the order of unity i.e., $O(1)$.

If we use Equations (6.35b) and (6.36) with $W_e \approx O(\varepsilon_1)$, $N_{pch} \approx O(1)$ and $Re_K \approx O(\frac{1}{\varepsilon_K^2})$, then the terms of Equation (6.34a) representing the contribution of $O(1)$ yields in the dimensional form

$$P_{1i} - P_{2i} = \dot{m}_{1i} \frac{\Delta \rho_i}{\rho_{1i} \rho_{2i}} + 2 \varkappa \sigma$$ (6.37a)

whereas for the major contribution of the tangential shear stress jump condition we have from Equation (6.34b) the following result in dimensional form

$$(\tau_{2xy} - \tau_{1xy})_i = \frac{\partial \sigma}{\partial x}$$ (6.37b)

In Equation (6.37) $\varkappa$ is the mean curvature of the surface. Under the boundary layer approximation it is given by

$$2 \varkappa = - \frac{\partial^2 \eta}{\partial x^\nu}$$ (6.38)

Equations (6.37a and b) specify the conditions which the shear stresses and the pressures must satisfy at the interface.

Several observations can be made with respect to Equations

(6.37a and b): First, we note that the temperature of the interface may not be uniform, giving rise to a non-uniform surface tension and consequently, according to Equation (6.37b) to a shear at the interface. This _thermo-capillary_ effect induces a flow from the region of high temperature to the region of low temperature, i.e., from the region of low surface tension to the region of high surface tension. It can be seen that in the case of film flow, as the film becomes thinner the continuous flow from the hollow to the peaks may lead to serious problems, such as a destruction of the film. Secondly, as the mass transfer, either in the form of evaporation or condensation, takes place at the interface, according to Equation (6.37a) there is always a _thrust_ exerted by the vapor towards the liquid phase. Again in the case of film flow this vapor-thrust may lead to the destruction of the film. Finally, we note that in absence of mass transfer and of the temperature gradient along the interfacial surface, the two equations, Equations (6.37a and b) reduce to standard expressions, reference [56, 57].

Before closing this section it is appropriate to recall here the conventional formulation of momentum equations which have been reviewed in detail in Chapter 2. These formulations have not included the right hand sides of the last two equations. The significance of omitting these terms now becomes clear, that the equations available in the literature cannot account for the correct interfacial interactions, such as for the effects due to the interfacial _surface curvature_, _surface shear_ and due to the _vapor thrust_. Consequently, effect of interface dynamics on the two-phase flow system can not be determined by means of conventional heretofore available formulations.

As a conclusion of this section we can summarize that the dynamic formulation based on the two-fluid model can be expressed in terms of two field equations, Equations (6.31a and b) or Equations (6.32a and b), and four additional equations, one for the specification of the wall shear τ_{1e}, one for the specification of either phase interfacial shear τ_{1i} or τ_{2i} and the other two for the dynamic coupling of the two phases, i.e., Equations (6.37a and b).

B. <u>Annular Flow</u>

Annular flow momentum equations can be obtained from the general balance equation appropriate to the annular flow structure, i.e., from Equation (4.33) and (4.34), together with the identifications for ψ, $\dot{\psi}$ and $\vec{\phi}$ which are given by Equation (6.1). However, for the reason of brevity we shall not repeat the developments leading to final results we shall write only the final equations.

The order of magnitude analysis which is based on the assumption of $\frac{\partial n}{\partial x} \ll 1.0$ can be carried out equally for the purpose of simplication of the annular flow model. It is an easy matter to show that the first order approximation yields the x-component momentum equation similar to the type of Equation (6.25) meanwhile the radial component for a vertical annular flow results in

$$\frac{\partial P_k}{\partial r} = 0 \qquad (6.39a)$$

which in turn yields

$$\langle\!\langle P_k \rangle\!\rangle = P_k = P_{ki} \qquad , K = 1,2 \qquad (6.39b)$$

In view of these approximations and in view of the general momentum

balance equation given by Equation (4.33) and (4.34) together with Equation (6.1), it can be shown that the appropriate annular flow momentum equations for phase 1 and 2 are, respectively, given by:

$$\frac{\partial}{\partial t}(1-\alpha) \ll \rho_1 \gg \langle v_{1x} \rangle + \frac{\partial}{\partial x}(1-\alpha) \ll \rho_1 \gg \langle v_{1x} \rangle^2 = -(1-\alpha)\frac{\partial P_1}{\partial x} + \qquad (6.40a)$$

$$(1-\alpha) \ll \rho_1 \gg g_x - \left[(1-\alpha) \ll \rho_1 \gg \langle v_{1x} \rangle^2 + Cov\,(mom\,1)\right]\frac{d \ln D_e^2}{dx} - \dot{m}_{1i}v_{1ix}\frac{4\sqrt{\alpha}}{D_e} - $$

$$\dot{m}_{1e}\,v_{1ex}\left[1 + \left(\frac{d}{dx}\frac{D_e}{2}\right)^2\right]^{1/2}\frac{4}{D_e} + \frac{4\sqrt{\alpha}}{D_e}\,\tau_{ii} - \frac{4}{D_e}\,\tau_{1e} - \frac{\partial}{\partial x}\,Cov\,(mom\,1)$$

$$\frac{\partial}{\partial t}\alpha \ll \rho_2 \gg \langle v_{2x} \rangle + \frac{\partial}{\partial x}\alpha \ll \rho_2 \gg \langle v_{2x} \rangle^2 = -\alpha\frac{\partial P_2}{\partial x} + \alpha \ll \rho_2 \gg g_x - \qquad (6.40b)$$

$$\left[\alpha \ll \rho_2 \gg \langle v_{2x} \rangle^2 + Cov\,(mom\,2)\right]\frac{d \ln D_e^2}{dx} - \dot{m}_{2i}\,v_{2ix}\frac{4\sqrt{\alpha}}{D_e} - \frac{4\sqrt{\alpha}}{D_e}\,\tau_{2i} - \frac{\partial}{\partial x}Cov\,(mom\,2)$$

Again, by expanding the left hand side of Equations (6.3a and b) and using the continuity equations, Equations (5.11a and b) equation of motion appropriate for annular flow structure can be derived, thus for phase 1

$$(1-\alpha) \ll \rho_1 \gg \frac{D_1 \langle v_{1x} \rangle}{Dt} = -(1-\alpha)\frac{\partial P_1}{\partial x} + (1-\alpha) \ll \rho_1 \gg g_x + \qquad (6.41a)$$

$$\Gamma_{1i}\left(v_{1ix} - \langle v_{1x} \rangle\right) + \Gamma_{1e}\left(v_{1ex} - \langle v_{1x} \rangle\right) + \tau_{1i}\frac{4\sqrt{\alpha}}{D_e} - $$

$$\tau_{1e}\frac{4}{D_e} - \frac{1}{D_e^2}\frac{\partial}{\partial x}D_e^2\,Cov\,(mom\,1)$$

and for phase 2

$$\alpha \ll \rho_2 \gg \frac{D_2 \langle v_{2x} \rangle}{Dt} = -\alpha \frac{\partial P_2}{\partial x} + \alpha \ll \rho_2 \gg g_x + \qquad \text{(6.41b)}$$

$$\cdot \; \Gamma_{2i} (v_{2ix} - \langle v_{2x} \rangle) - \tau_{2i} \frac{4 \sqrt{\alpha}}{D_e} - \frac{1}{D_e^2} \frac{\partial}{\partial x} D_e^2 \; C_v \; \text{(mom 2)}$$

where we used Equation (5.4) in order to express m_{ki} in terms of Γ_{ki}.

To complete the description of the dynamic field of the two-phase flow problem, the same comments made in the preceding section apply to the annular flow case. In this case, however, the appropriate interfacial momentum balance is given by Equation (C.36). From this equation it can be shown that the interfacial momentum balance in the normal and tangential direction are given, respectively, by

$$-\overset{\cdot}{m}_{ii}^2 \frac{\Delta \rho_i}{\rho_{ii} \rho_{2i}} + (P_{ii} - P_{2i}) + \left\{ (\tau_{2xx} - \tau_{1xx})_i \left[\frac{\partial}{\partial x} \left(\frac{D_e \sqrt{\alpha}}{2} \right) \right]^2 - \right. \qquad \text{(6.42a)}$$

$$2 (\tau_{2xr} - \tau_{1xr})_i \frac{\partial}{\partial x} \left(\frac{D_e \sqrt{\alpha}}{2} \right) + (\tau_{2rr} - \tau_{1rr})_i \left. \left[1 + \left(\frac{\partial}{\partial x} \frac{D_e \sqrt{\alpha}}{2} \right)^2 \right]^{-1} \right\}$$

$$= 2 \kappa \sigma$$

$$(\tau_{2xx} - \tau_{1xx})_i \frac{\partial}{\partial x} \left(\frac{D_e \sqrt{\alpha}}{2} \right) - (\tau_{2rr} - \tau_{1rr})_i \frac{\partial}{\partial x} \left(\frac{D_e \sqrt{\alpha}}{2} \right) + \qquad \text{(6.42b)}$$

$$(\tau_{2xr} - \tau_{1xr})_i \left[-1 + \left(\frac{\partial}{\partial x} \frac{D_e \sqrt{\alpha}}{2} \right)^2 \right] = \frac{\partial \sigma}{\partial x} \left[1 + \left(\frac{\partial}{\partial x} \frac{D_e \sqrt{\alpha}}{2} \right)^2 \right]^{1/2}$$

Introducing the long wave or boundary layer type of approximation into Equations (6.42a and b), yields the interfacial pressure condition as

$$P_{1i} - P_{2i} = -2\varkappa\sigma + \dot{m}_{1i}^2 \frac{\Delta e_i}{e_{1i}\, e_{2i}} \qquad (6.43a)$$

and the interfacial condition as

$$(\tau_{2\varkappa r} - \tau_{1\varkappa r})_i = -\frac{\partial\sigma}{\partial x} \qquad (6.43b)$$

which must be satisfied at the interface. In these equations $\varkappa$ is given by Equation (C.37) which in view of the approximation is simplified to

$$2\varkappa = \frac{2}{D_e\sqrt{\alpha}} - \frac{\partial^2}{\partial x^2}\left(\frac{D_e\sqrt{\alpha}}{2}\right) \qquad (6.44)$$

The "jump" conditons given by either Equation (6.42) or (6.43), together with the field equations given by Equation (6.41) and two constitutive equations for wall and interfacial shear formulate two-fluid model annular flow problem consistent with the approximations introduced in the previous section.

C. Film-Flow Discussion

Thus far, the dynamic field description based on two-fluid model was formulated in terms of two momentum equations - one for each phase - and of two interfacial coupling equations representing the influence of one phase upon the other. Furthermore, to complete the system specification of two additional constitutive equations - one for external surface shear and the other for interfacial shear - was pointed out. However, in order to see the effects of various parameters upon the liquid film flow dynamics, we can express the formulation in terms of single field equation. For example, substituting Equations (6.37a and b) in Equation (6.32a) we obtain the equation of motion applicable to a liquid film flowing on a plane surface, thus

$$\eta \ll e_i \gg \frac{D_i \langle v_{ix} \rangle}{Dt} = -\eta \frac{\partial}{\partial x} \left(P_{2i} + \dot{m}_{ii}^2 \frac{\Delta \rho_i}{C_{ii} \rho_{2i}} - \sigma \frac{\partial^2 \eta}{\partial x^2} \right) - \qquad (6.45)$$

$$\dot{m}_{ii} \left(v_{iix} - \langle v_{ix} \rangle \right) + \dot{m}_{ie} \langle v_{ix} \rangle + \ll e_i \gg g_y \eta \frac{\partial \eta}{\partial x} + \ll e_i \gg \eta g_x +$$

$$\tau_{2i} - \tau_{ie} - \frac{\partial \sigma}{\partial s} - H \frac{\partial}{\partial x} C_{v \,(mom\,i)}$$

in which τ_{21} and τ_{1e} should be specified by means of constitutive equations.

This equation can be used to evaluate the effects of various parameters on the dynamics of the thin liquid film flowing over a plate.

1) The Effect of Flow Regimes

Thus, the effect of the liquid flow regime is reflected in the wall shear stress τ_{1e}, the covariant term and two momentum sink terms in Equation (6.45). While the importance of the velocity distribution in determining the values of τ_{1e}, and of the covariance term are well known, reference [58], it can be seen from Equation (6.45) that the effects of mass transfer on the film flow dynamics depend also on the velocity profile.

For example, for plane flow in a laminar flow regime, the effect of the momentum sink amounts to approximately

$$\left(v_{iix} - \langle v_{ix} \rangle \right) \dot{m}_{ii} \simeq \frac{1}{2} \langle v_{ix} \rangle \dot{m}_{ii} \qquad (6.46)$$

whereas for a turbulent, i.e., a flat profile the contribution of this term is almost nil.

2) Thermo-Capillary Effects

The effects of surface tension are accounted for by two terms in Equation (6.45), one affects the pressure distribution the other the shear.

The importance of these two terms will depend on the particular applica-
tion. For example, it is well known that for heat pipe application, the
thermo-capillary effects are most important.

3) The Effect of the Vapor Phase

It can be seen from Equation (6.45) that the motion of the vapor
phase influences the motion of the liquid through two terms: the pressure
gradient and the vapor shear stress at the interface. In fact, for an
upward flowing film the liquid flow is due to the action of the vapor
only.

4) The Effects of Heat and Mass Transfer

Of particular interest is to evaluate the effects of heat and
mass transfer on the liquid film. It was noted already that when $\dot{m}_{li}$
is positive as in evaporation, the effect of mass transfer is to reduce
the pressure distribution as well as to reduce the momentum, i.e., it has
an effect of a momentum sink. In order to determine more precisely this
effect, it is necessary to consider in more detail the mass flow $\dot{m}_{li}$,
in Equation (6.45).

By examining the continuity equations it can be seen that in the
case of evaporation the mass flux $\dot{m}_{li}$, acts as a sink, indeed, as it
was noted in Chapter 5, it plays the same role as the sink (or source)
terms in the continuity equations of chemically reacting mixtures.
Whereas in chemically reacting mixtures, the sinks (or sources) are
specified by appropriate constitutive equations of chemical kinetics, in
two phase flow they are specified the appropriate constitutive equations
of phase change, i.e., of evaporation or condensation, references
[4, 42, 51]. Indeed, it was shown that the constitutive equation of

evaporation a) depends on the flow regime and b) determines the thermodynamic non-equilibrium of the two-phase mixture.

For example in the present problem, the expression for $\dot{m}_{1i}$, will depend on the particular application. Thus, it will have a different form depending on whether the evaporation from the liquid film is effected by exposure to hot gases (as in rocket engines), or by decreasing the system pressure (as in flash evaporators), or by heat transfer through the liquid film (as in boilers, evaporators or nuclear reactors). Let us consider briefly this last application.

The simplest expression for the constitutive equation of evaporation can be obtained by assuming that the two phase mixture is in thermodynamic equilibrium. For such a two-phase system $\dot{m}_{1i}$, can be obtained from an energy balance, see, for example, Chapter 7, thus

$$\dot{m}_{1i} \; \Delta i_{12} = q_{1e} \tag{6.47}$$

where Δi_{12} is the latent heat of vaporization and q_{1e} is the heat flux at the solid wall. Assuming for simplicity, that heat flux through the liquid film is by conduction only, we obtain from Equation (6.47) the expression for $\dot{m}_{1i}$, thus

$$\dot{m}_{1i} = \frac{k_1}{\Delta i_{12}} \; \frac{T_{1e} - T_s}{\eta} \tag{6.48}$$

where T_{1e} and T_s are the wall and saturation temperatures, and k_1 is the coefficient of thermal conductivity.

Substituting Equation (6.48) in Equation (6.45) we obtain the following expression for the <u>equation of motion of a liquid film evaporating from a heated plane surface</u>

$$\eta \ll \rho_i \gg \frac{D_i \langle v_{ix} \rangle}{Dt} = -\eta \frac{\partial}{\partial x} \left[P_{2i} - \sigma \frac{\partial^2 \eta}{\partial x^2} + \left(\frac{k_i}{\Delta i_{12}} \Delta T \right)^2 \frac{\Delta \ell_i}{\ell_i \rho_{xi}} - \right. \quad (6.49)$$

$$\frac{k_i}{\Delta i_{12}} \frac{\Delta T}{\eta} (v_{iix} - \langle v_{ix} \rangle) + \dot{m}_{ie} \langle v_{ix} \rangle + \ll \rho_i \gg \eta g_y \frac{\partial \eta}{\partial x} +$$

$$\tau_{2i} - \tau_{ie} - H \frac{\partial \sigma}{\partial x} - H \frac{\partial}{\partial x} C_{\nu (\text{mom}\, i)}$$

With reference to this equation, we can make several important observations. First, we note that as for evaporating films the film thickness decreases, the effects of inertia and gravitational forces become less and less important, whereas those due to evaporation become dominant. Second, since the gradient of the film thickness, is negative for evaporating films, the evaporation thrust acts in the opposite direction to the vapor shear τ_{2i}. Consequently whether in upward or downward flow, the evaporation thrust can induce an _interruption of the liquid flow_ which results in a local _dryout_. Note, that this dryout is _not_ brought about by the _total_ evaporation of the film, but it can occur with a finite film thickness because of the dynamic effects in the film and at its interface.

We close by observing that because the liquid dryout can impose an operating limit on nuclear reactors, evaporators desalination plants etc., it is one of the most important (unresolved) problems in the nuclear reactor and chemical process industries. This and other dynamic problems, based on these equations, will be treated in Part II of the thesis.

6.3 Diffusional Model Formulation

6.3.1 Mixture Equation

In order to derive the mixture momentum equation we shall use

the one-dimensionalized general balance equation for the mixture to-gether with the identifications for ψ, $\dot{\psi}$ and $\vec{\phi}$ which for the case of momentum transport assume values given by Equation (6.1). Hence, ex-pressing ψ, $\dot{\psi}$ and $\vec{\phi}$ in Equation (4.27) by means of Equation (6.1) we obtain from Equation (4.27) the following expression for the <u>mixture momentum equation</u> in a two-phase flow system with an interfacial inter-action, thus

$$\frac{\partial}{\partial t}\left(\rho_m v_{mx}\right) + \frac{\partial}{\partial x}\left(\rho_m v_{mx}^2\right) = \frac{\partial}{\partial x}\pi_{mxx}^T + \rho_m g_x - \tag{6.51}$$

$$\left[\rho_m \dot{v}_{mx}^+ - \pi_{mxx}^T + Cov(momT)\right]\frac{d\ell_n A_{Tc}}{dx} + \frac{1}{A_{Tc}}\int_{S_i}\nabla_s\cdot\bar{\bar{\theta}}_{(mom)}\frac{dA}{dx}$$

$$\frac{1}{A_{Tc}}\sum_{k=1}^{2}\int_{S_{ke}}\left[\dot{m}_{ke}\,v_{kix} + (\bar{\bar{\pi}}_{ke}\,\hat{n}_{ke})\cdot\hat{1}\right]\frac{dA}{dx} - \frac{\partial}{\partial x}Cov(momT)$$

where the terms are explained as follows:

$\nabla_s\cdot\theta_{(mom)}$ is the surface contribution to the over-all mixture momentum conservation and given by Equation (B.19).

$\bar{\bar{\pi}}_m^T$ is the total momentum flux with respect to the baricenter of the mixture, i.e., total stress tensor. Using the definition of the generalized total flux $\bar{\phi}_m^T$ of the mixture given by Equation (4.26) we obtain from Equations (6.1) and (4.26) the following expression for the total momentum flux $\bar{\bar{\pi}}_m^T$, with respect to the baricenter of the mixture, thus

$$\bar{\bar{\pi}}_m^T = \bar{\bar{\pi}}_m - \bar{\bar{\pi}}_D \tag{6.52}$$

where $\bar{\bar{\pi}}_m$ is the mixture stress tensor whereas $\bar{\bar{\pi}}_D$ is the diffusion stress tensor.

Expressing $\vec{\phi}$ by means of Equation (6.1), Equation (4.24) defines the mixture stress tensor $\bar{\bar{\pi}}_m$ as

$$\bar{\bar{\pi}}_m = (1-\alpha)\left(-\ll P_1 \gg \bar{\bar{S}} + \ll \bar{\bar{\tau}}_1 \gg\right) + \alpha\left(-\ll P_2 \gg \bar{\bar{S}} + \ll \bar{\bar{\tau}}_2 \gg\right) \quad (6.53)$$

Defining by P_m, and $\bar{\bar{\tau}}_m$, the <u>mixture pressure</u> and the <u>mixture viscous stress tensor</u>, it follows that

$$P_m = (1-\alpha) \ll P_1 \gg + \alpha \ll P_2 \gg \quad (6.54)$$

$$\bar{\bar{\tau}}_m = (1-\alpha) \ll \bar{\bar{\tau}}_1 \gg + \alpha \ll \bar{\bar{\tau}}_2 \gg \quad (6.55)$$

A brief comment is appropriate here with respect to these equations. Several researchers have proposed a stress tensor for the mixture, which was obtained by multiplying the stress tensors of the individual phases by the mass concentration instead of volume concentration as in Equation (6.55). Since these mixture stress tensors were defined by these authors and not derived as the one given by Equation (6.55) we conclude that these arbitrary definitions are incorrect.

The analysis above clearly shows that all terms in the mixture which depend upon density are weighed by mass concentration, whereas those which depend upon surfaces are weighed by volume concentration. In fact, if we recall the thermodynamic analysis of gas mixtures, we realize that $(1-\alpha) \ll P_1 \gg$ and $\alpha \ll P_2 \gg$ are nothing but the <u>partial pressure</u> of phase 1 and 2, respectively, whereas $(1-\alpha) \ll \bar{\bar{\pi}}_1 \gg$ and $\alpha \ll \bar{\bar{\pi}}_2 \gg$ are the <u>partial stresses</u>. In view of Equations (6.53), (6.54) and (6.55) <u>mixture stress tensor</u> can be also expressed by

$$\bar{\bar{\pi}}_m = -P_m \bar{\bar{S}} + \bar{\bar{\tau}}_m \quad (6.56)$$

Whereas from Equations (6.1) and (4.26), the <u>diffusion stress</u> can be expressed by

$$\bar{\bar{\pi}}_D = (1-c)\, \rho_m\, \vec{V}_{1m} \otimes \vec{V}_{1m} + c\, \rho_m\, \vec{V}_{2m} \otimes \vec{V}_{2m} \qquad (6.57a)$$

which in view of the relation between $\vec{V}_{1m}$ and $\vec{V}_{2m}$, i.e., Equation (3.59) can be also expressed as

$$\bar{\bar{\pi}}_D = \frac{c}{1-c}\, \rho_m\, \vec{V}_{2m} \otimes \vec{V}_{2m} \qquad (6.57b)$$

where $\otimes$ stands for the dyadic product. Since $(1 - c)\rho_m \vec{V}_{1m}$ and $c\rho_m \vec{V}_{2m}$ represent, respectively, the diffusion mass fluxes of phase 1 and 2 with respect to the center of gravity of the mixture, Equation (6.57) expresses the momentum transfer by diffusion with respect to the bari-center of the mixture. Therefore, by expressing the mixture momentum equation with respect to the center of mass, an additional term arises due to the relative velocity between the two phases.

Instead of expressing the stress due to relative velocity in terms of a diffusion velocity with respect to center of mass, as it is usually done in analysis of single phase mixture, it is advantageous in two-phase flow to express it relative to the center of volume because as it was shown in reference [51] the drift velocity results in simple expressions which, for a given flow regime, is either constant or function of α only. In terms of the drift velocities $\vec{V}_{1j}$ and $\vec{V}_{2j}$, the momentum transfer by diffusion can be expressed in various forms. Using the relations given on Table 1, it can be shown that

$$\bar{\bar{\pi}}_D = \alpha\,(1-\alpha)\,\frac{\langle\!\langle e_i \rangle\!\rangle \langle\!\langle e_2 \rangle\!\rangle}{\rho_m}\,(\vec{V}_{1j} - \vec{V}_{ij}) \otimes (\vec{V}_{2j} - \vec{V}_{ij}) \qquad (6.58)$$

$$= \alpha\,(1-\alpha)\,\frac{\langle\!\langle e_i \rangle\!\rangle \langle\!\langle e_2 \rangle\!\rangle}{\rho_m}\,\vec{V}_r \otimes \vec{V}_r$$

$$= \frac{\alpha}{1-\alpha}\,\frac{\langle\!\langle e_i \rangle\!\rangle \langle\!\langle e_2 \rangle\!\rangle}{\rho_m}\,\vec{V}_{1j} \otimes \vec{V}_{2j}$$

The covariance term appearing in Equation (6.54) is defined by means of Equation (4.22). Substituting Equation (6.1) in Equation (4.22) we obtain Cov (mom) term, thus

$$(6.59)$$

$$Cov\,(mom\,T) = Cov\,(mom\,1) + Cov\,(mom\,2)$$

$$= Cov\left[\,(1-\alpha)\,\langle\!\langle e_i \rangle\!\rangle\,(v_{1k} \cdot v_{1k}) + \alpha\,\langle\!\langle e_2 \rangle\!\rangle\,(v_{2k} \cdot v_{2k})\right]$$

Finally, the external line integral over ξ_e in Equation (6.51) accounts for the effects of the momentum transfer at the external boundaries which may be due to convective as well as molecular momentum transfer.

In view of the mixture continuity equation, i.e., Equation (5.14) the momentum equation can be also expressed as the mixture equation of motion, thus

$$\rho_m \frac{D_m \upsilon_{mx}}{Dt} = -\frac{\partial P_m}{\partial x} + \frac{\partial}{\partial x} \pi_{Dxx} + \rho_m g_x - \tag{6.60}$$

$$\left[P_m - \tau_{mxx} + \pi_{Dxx} + Cov_{(momT)} \right] \frac{d}{dx} \frac{\ell n A_{Tc}}{dx} + \frac{1}{A_{Tc}} \int_{\mathcal{S}_i} \nabla_s \cdot \bar{\bar{\theta}}_{(mom)} \frac{dA}{dx} -$$

$$\frac{1}{A_{Tc}} \sum_{k=1}^{2} \int_{\mathcal{S}_{ke}} \left\{ \dot{m}_{ke} \left(\upsilon_{kex} - \upsilon_{mx} \right) + \left[P_{ke} \bar{\bar{\delta}} - \tau_{ke} \right) \cdot \hat{n}_{ke} \right] \cdot \hat{\imath} \right\} \frac{dA}{dx} -$$

$$\frac{\partial}{\partial x} Cov_{(momT)}$$

where the material derivative $\frac{D_m}{Dt}$ is defined by Equation (5.15).

The constitutive equations appropriate to the diffusional dynamic formulation of the problem will be given in the following section where we shall also simplify Equation (6.51). We note in closing that the mixture momentum equation can not be equivalent to the ordinary momentum equation in continuum mechanics as postulated in reference [23]. If there were no surface source term, i.e., if $\nabla_s \cdot \bar{\bar{\theta}}_{(mom)} = 0$, only then Equation (6.51) would become equivalent to the ordinary single phase momentum equation with the proper definitions of the relative momentum flux $\bar{\bar{\pi}}_m^T$ with respect to the baricenter of the mixture.

6.3.2 Two-Dimensional Flow

A. Plane Flow

Mixture momentum equation appropriate to the plane flow geometry can be obtained either from the general balance equation, Equation (4.30) or from the momentum equation given by Equation (6.51). For the reason of brevity, in derivations we shall obtain it directly from Equation (4.30) by means of the proper identifications for ψ, $\dot{\psi}$ and $\vec{\phi}$. Therefore, substituting Equation (6.1) in Equation (4.30) and in view of the discussions made in Section 6.2.2 retaining only the x-component we obtain

the mixture momentum balance, thus

$$\frac{\partial}{\partial t}(\rho_m v_{mx}) + \frac{\partial}{\partial x}(\rho_m v_{mx}^2) = -\frac{\partial P_m}{\partial x} + \frac{\partial \tau_{mxx}}{\partial x} - \frac{\partial \pi_{0xx}}{\partial x} + \tag{6.61}$$

$$\rho_m g_x - \frac{1}{H}\left(2\kappa\sigma \, n_{lix} + \frac{\partial \sigma}{\partial s} \, t_{ix}\right)\left[1 + \left(\frac{\partial \eta}{\partial x}\right)^2\right]^{1/2} -$$

$$\frac{1}{H}\left\{\dot{m}_{le} v_{lex} + \left[(P_{le}\bar{\bar{s}} - \bar{\bar{\Sigma}}_{le})\cdot\hat{n}_{le}\right]\cdot\hat{1}\right\} - \frac{\partial}{\partial x} Cov(momT)$$

By using Equation (6.12) for v_{lex}, Equation (C.5) for $\hat{n}_{li}$, Equation (C.7) for $\hat{n}_{le}$, Equation (C.4) for $\hat{t}_i$, and Equation (C.16) together with Equation (6.38) for $\nabla_s\cdot\theta_{(mom)}$, and finally using the Prandtl's type of approximation given by Equations (6.29) through Equation (6.31), we obtain from Equation (6.61) simplified mixture momentum equation applicable for plane flow geometry, thus

$$\frac{\partial}{\partial t}(\rho_m v_{mx}) + \frac{\partial}{\partial x}(\rho_m v_{mx}^2) = -\frac{\partial P_m}{\partial x} - \tag{6.62}$$

$$\frac{\partial}{\partial x}\left(\frac{\alpha}{1-\alpha}\frac{\ll\rho_1\gg\ll\rho_2\gg}{\rho_m}\right) + \rho_m g_x - \frac{1}{H}\left[-\sigma\left(\frac{\partial^2\eta}{\partial x^2}\right)\frac{\partial\eta}{\partial x} + \frac{\partial\sigma}{\partial s}\right] -$$

$$\frac{\tau_{le}}{H} - \frac{\partial}{\partial x} Cov(momT)$$

where Equation (6.58) has been used for expressing the drift stress in terms of the drift velocity of the second phase. The reason for expressing it here with respect to the center of volume stems from the fact that this definition of the drift velocity results in simple expressions, reference [51].

In view of the mixture continuity equation, Equation (5.18) one can obtain also the equation of motion for the mixture, thus

$$\rho_m \frac{D_m \bar{v}_{mx}}{Dt} = -\frac{\partial \bar{P}_m}{\partial x} - \frac{\partial}{\partial x}\left[\frac{\alpha}{1-\alpha}\frac{\langle\langle\rho_1\rangle\rangle\langle\langle\rho_2\rangle\rangle}{\rho_m}V_{2jx}^2\right] + \qquad (6.63)$$

$$\rho_m g_x - \Gamma_{1e}\bar{v}_{mx} - \frac{1}{H}\left(-\sigma\frac{\partial^2\eta}{\partial x^2}\frac{\partial\eta}{\partial x} + \frac{\partial\sigma}{\partial s}\right) -$$

$$\frac{\tau_{1e}}{H} - \frac{\partial}{\partial x}\text{Cov}(\text{mom}_T)$$

In order to complete the description of the two-phase flow field based on the diffusion model, <u>one</u> additional constitutive equation is necessary to determine the shear at the wall. However, as it will be seen in Chapter 9, where the over-all formulation in terms of the kinematic, dynamic and the thermal fields of the general problem will be given, we shall need one more equation relating two pressure fields P_1 and P_2. In order to justify for such a need let us investigate the implication of Equation (6.62). First of all, this equation is written in the x-direction without taking into account the y-direction momentum transport. In the two-fluid model formulation such an equation is given by Equation (6.32). Integration of this equation for both phases yielded the pressure differences between the two phases. Therefore, a relation which involves the pressure differences must be supplemented in order to complete the stratified two-phase flow field where there exists a pressure differences tween two phases. In effect, we have already derived such a relation in Section 6.2.2. It is given by Equation (6.37a).

Finally, in closing we note the difference in formulations. By comparing the formulations based on two different models, i.e., two-fluid and diffusion models, we observe that in the case of the two-fluid model two constitutive equations - one for τ_{1e} and one for either τ_{1i} or τ_{2i} - and two additional jump conditions summing up four additional

relations are needed for closing the system of equation. On the other hand, in the case of diffusional model only one constitutive equation for τ_{1e} and one additional equation for interfacial pressure summing up two additional relations are needed for closing the dynamic field of the mixture flow.

B. Annular Flow

Momentum equation of the annular flow can be developed in a manner similar to that of the plane flow. Therefore, we shall not repeat the calculations here but we shall give only the final equations.

In view of Equation (6.1) and of the geometric relations derived in Appendix C.2, from Equation (4.33) the mixture momentum equation of the annular flow can be obtained thus,

$$\frac{\partial}{\partial t}\left(\rho_m \, v_{mx}\right) + \frac{\partial}{\partial x}\left(\rho_m \, v_{mx}^2\right) = -\frac{\partial P_m}{\partial x} - \frac{\partial}{\partial x}\left(\frac{\alpha}{1-\alpha} \frac{\langle\langle \rho_c \rangle\rangle \langle\langle \rho_a \rangle\rangle}{\rho_m} \, v_{zjx}^2\right)_+ \quad (6.64)$$

$$\rho_m \, g_x - \left[\rho_m \, v_{mx}^2 + P_m + \frac{\alpha}{1-\alpha} \frac{\langle\langle \rho_c \rangle\rangle \langle\langle \rho_a \rangle\rangle}{\rho_m} \, v_{zjx}^2 + Cov(mom\tau)\right] \frac{2}{De} \frac{dDe}{dx} -$$

$$\frac{4\sqrt{\alpha}}{De}\left(-2\alpha\, \Gamma \frac{\partial}{\partial x} \frac{De\sqrt{\alpha}}{2} + \frac{\partial \Gamma}{\partial s}\right) + \Gamma_{1e} \, v_{1ex} + \frac{2}{De} \, P_i \frac{dDe}{dx} +$$

$$\frac{4}{De} \, \tau_{1e} - \frac{\partial}{\partial x} \, Cov(mom\tau)$$

which in view of Equation (6.54) and Equation (6.44) can be also expressed as

$$\frac{\partial}{\partial t}(\ell_m v_{mx}) + \frac{\partial}{\partial x}(\ell_m v_{mx}^2) = -\frac{\partial P_m}{\partial x} - \frac{\partial}{\partial x}\left(\frac{\alpha}{1-\alpha}\frac{\ll \ell_1 \gg \ll \ell_2 \gg}{\ell_m} v_{2jx}^2\right) + \quad (6.65)$$

$$\ell_m g_x - \left[\ell_m v_{mx}^2 + \frac{\alpha}{1-\alpha}\frac{\ll \ell_1 \gg \ll \ell_2 \gg}{\ell_m} v_{2jx}^2 + Cov\,(mom\,T)\right]\frac{2}{D_e}\frac{dD_e}{dx} -$$

$$\frac{4\sqrt{\alpha}}{D_e}\left[\left(-\frac{2}{D_e\sqrt{\alpha}} + \frac{v^2}{\partial x^2}\frac{D_e\sqrt{\alpha}}{2}\right)\sigma - \frac{\partial}{\partial x}\frac{D_e\sqrt{\alpha}}{2} + \frac{\partial \sigma}{\partial s}\right] +$$

$$\frac{2}{D_e}(P_1 - P_2)\frac{dD_e}{dx} - \frac{4}{D_e}\tau_{1e} - \frac{\partial}{\partial x}Cov\,(mom\,T)$$

from which one can obtain also the equation of motion of the mixture, thus

$$\ell_m \frac{D_m v_{mx}}{Dt} = -\frac{\partial P_m}{\partial x} - \frac{\partial}{\partial x}\left(\frac{\alpha}{1-\alpha}\frac{\ll \ell_1 \gg \ll \ell_2 \gg}{\ell_m} v_{2jx}^2\right) + \ell_m g_x - \quad (6.66)$$

$$\left[\frac{\alpha}{1-\alpha}\frac{\ll \ell_1 \gg \ll \ell_2 \gg}{\ell_m} v_{2jx}^2 + Cov\,(mom\,T)\right]\frac{2}{D_e}\frac{dD_e}{dx} -$$

$$\frac{4\sqrt{\alpha}}{D_e}\left[\left(-\frac{2}{D_e\sqrt{\alpha}} + \frac{\partial^2}{\partial x^2}\frac{D_e\sqrt{\alpha}}{2}\right)\sigma - \frac{\partial}{\partial x}\left(\frac{D_e\sqrt{\alpha}}{2}\right) + \frac{\partial \sigma}{\partial s}\right] + \Gamma_{1e}(v_{1ex} v_{mx}) +$$

$$\frac{2}{D_e}(P_1 - P_2)\frac{dD_e}{dx} - \frac{4}{D_e}\tau_{1e} - \frac{\partial}{\partial x}Cov\,(mom\,T)$$

We note that the discussion made concerning the <u>additional relations</u> for the plane flow case exactly applies to the annular flow case. In fact, the need for the specification of the pressure difference now becomes clear from Equation (6.65).

6.3.4 Scaling Criteria

As in Chapter 5 where the dimensionless groups necessary for the kinematic similarity of the system were derived, in this section we shall derive the similarity groups arising from the dynamic field of a separated two-phase flow mixture. However, for the simplicity plane flow and annular flow momentum equations will be brought into a single equation so that the general similarity groups can be derived from it.

It can be seen from Equations (6.63) and (6.66), equation of motion for a constant area duct can be expressed as

$$\rho_m \frac{\partial V_{mx}}{\partial t} + \rho_m V_{mx} \frac{\partial V_{mx}}{\partial x} = - \frac{\partial P_m}{\partial x} - \frac{\partial}{\partial x}\left(\frac{\alpha}{1-\alpha} \frac{\langle\langle\rho_1\rangle\rangle\langle\langle\rho_2\rangle\rangle}{\rho_m} V_{2jx}^2\right) + \quad (6.67)$$

$$\rho_m g_x - F_\sigma \frac{\xi_i}{A_{Tc}} - \Gamma_{ie} V_{mx} - \frac{\xi_{ie}}{A_{Tc}} \tau_{ie} - \frac{\partial}{\partial x} \text{Cov}(m \cup u \cup T)$$

where F_σ is the capillary force per unit interfacial area.

For the plane flow

$$\frac{\xi_{ie}}{A_{Tc}} = \frac{\xi_{ii}}{A_{Tc}} = \frac{1}{H} \qquad (6.68)$$

$$F_\sigma = -\sigma \frac{\partial^2 \eta}{\partial x^2} \frac{\partial \eta}{\partial x} + \frac{\partial \sigma}{\partial x} \qquad (6.69)$$

For the annular flow

$$\frac{\xi_i}{A_{Tc}} = \frac{4\sqrt{\alpha}}{D_e} \qquad (6.70)$$

$$\frac{\xi_{ie}}{A_{Tc}} = \frac{4}{D_e} \qquad (6.71)$$

$$F_\sigma = \left(-\frac{2}{D_e\sqrt{\alpha}} + \frac{\partial^2}{\partial x^2} D_e\sqrt{\alpha}\right)\sigma \frac{\partial}{\partial x}\left(\frac{D_e\sqrt{\alpha}}{2}\right) + \frac{\partial \sigma}{\partial x} \qquad (6.72)$$

In addition to the dimensionless scaling parameters introduced in connection with the kinematic field of the system, i.e., Equation (5.40), by measuring the system pressure in units of $\rho_{mo}V_{mo}^2$, the external surface shear in units of τ_{1eo}, the gravity in units of g, and the capillary forces in units of F_{σ_o}, we can reduce Equation (6.67) to dimensionless form. It is found that Equation (6.67) is identical to its dimensionless form

$$e_m^+ \frac{\partial v_{mx}^+}{\partial x^+} + e_m^+ v_{mx}^+ \frac{\partial v_{mx}^+}{\partial x^+} = - \frac{\partial P_m^+}{\partial x^+} - \tag{6.73}$$

$$\frac{\alpha_0}{(1-\alpha)_0} N_{\rho_1} N_{\rho_2} N_D^2 \left[\frac{\alpha^+}{(1-\alpha)^+} \frac{e_1^+ e_2^+}{e_m^+} V_{2jx}^+ \right] + \frac{1}{Fr_m} e_m^+ g_x^+ - N_\sigma F_\sigma^+ -$$

$$N_{\rho_1} N_{sup_1} v_{mx}^+ - f_m \tau_{ie}^+ - (1-\alpha)_0 N_{\rho_1} \frac{\partial}{\partial x^+}[(1-\alpha)^+ e_1^+ N_{cov(mom1)}] -$$

$$\alpha_0 N_{\rho_2} \frac{\partial}{\partial x^+}[\alpha^+ e_2^+ N_{cov(mom2)}]$$

It can be seen that in addition to the drift number $N_D = V_{gjo}/v_{mo}$, the density ratio groups $N_{\rho K} = \rho_{Ko}/\rho_{mo}$, $K = 1,2$, the supply number groups $N_{sup\,K} = (\frac{\Gamma_{Keo}}{\rho_{mo}})(\frac{L}{v_{mo}})$, and the geometric similarity group α_0, which were introduced previously in connection with the kinematic field of the system, i.e., with the dimensionless form of the mixture continuity equation, the dynamic field expressed by Equation (6.73) introduces additional dimensionless groups given below:

$$Fr_m = \frac{v_{mo}}{\sqrt{gL}} \text{, the \underline{Froude} number for the mixture} \tag{6.74}$$

$$f_m = \frac{S_{ie} L \tau_{ieo}}{A_{Tc} \rho_{mo} v_{mo}^2} \text{, the \underline{Friction Factor} for the mixture} \tag{6.75}$$

$$\left. \begin{aligned} N_{cov(mom1)} &= \frac{(1-\alpha)_0 \, Cov\,(v_{1x} \cdot v_{1x})}{v_{mo}^2} \\ N_{cov(mom2)} &= \frac{\alpha_0 \, Cov\,(v_{2x} \cdot v_{2x})}{v_{mo}^2} \end{aligned} \right\} \quad \text{the \underline{Flow Regime} groups} \tag{6.76}$$

These four similarity groups are in common in both the plane flow and the annular flow. However, other groups appearing in Equation (6.73) take different forms for the different cases. It is an easy matter to show that for the plane flow

$$\frac{\mathfrak{J}_{ie}L}{A_{T_c}} = \frac{\mathfrak{J}_i L}{A_{T_c}} = \frac{L}{H} = \frac{1}{\varepsilon_H} \tag{6.77}$$

$$\frac{F_{\sigma_o} F_{\sigma}^+}{\varrho_{mo} v_{mo}^2} = \frac{1}{We_m} \left[\varepsilon_1^2 \left(\frac{\partial^+ \eta_i^+}{\partial x^+} \right) \frac{\partial \eta_i^+}{\partial x^+} + \frac{\partial \sigma^+}{\partial x^+} \right] \tag{6.78}$$

On the other hand, for the <u>annular flow</u>

$$\frac{\mathfrak{J}_i L}{A_{T_c}} = 4\frac{\sqrt{\alpha}L}{D_e} = 4\sqrt{\alpha_o} \frac{\sqrt{\alpha^+}}{\varepsilon_D} \tag{6.79}$$

$$\frac{\mathfrak{J}_{ie}L}{A_{T_c}} = 4\frac{L}{D_e} = \frac{4}{\varepsilon_D} \tag{6.80}$$

$$\frac{F_{\sigma_o} F_{\sigma}^+}{\varrho_{mo} v_{mo}^2} = \frac{1}{We_m} \left[\left(-\frac{2}{\sqrt{\alpha^+}} + \alpha_o \varepsilon_D^2 \frac{\partial^2 \sqrt{\alpha^+}}{\partial x^{+2}} \right) \sigma^+ \frac{\partial \sqrt{\alpha^+}}{\partial x^+} + \frac{\partial \sigma^+}{\partial x^+} \right] \tag{6.81}$$

where we defined

$$We_m = \rho_{mo}\, v_{mo}L/\sigma_o, \text{ the } \underline{Weber} \text{ number for the mixture.} \tag{6.82}$$

$$\varepsilon_1 = \eta_{1o}/L; \ \varepsilon_H = H/L; \ = \varepsilon_D/L, \text{ the } \underline{Kinematic} \text{ groups,} \tag{6.83}$$

In section 5.3.5 it was shown that the Density Ratio group N_{ρ_K},
($K = 1,2$), is not independent. Once the groups N_{ρ_1} and α_o are specified
N_{ρ_2} can not be changed freely. A similar argument holds for the kinematic
group ε_H. It can be shown from Equation (6.83) and the definition of
that

$$\varepsilon_H = \frac{H}{L} = \frac{\varepsilon_1}{1-\alpha_o} \quad \text{and} \quad \varepsilon_D = \frac{D_e}{L} = \frac{\varepsilon_i}{\sqrt{\alpha_o}} \tag{6.8}$$

therefore, they are not independent from α_o and ε_1 and ε_i of which property
was discussed in Section 6.2.2 as a perturbation parameter.

Since $C_{ov}(mom\ K) \sim C_{ov}(v_{Kx} \cdot v_{Kx})$ depends on the K-th phase flow
regime it becomes evident that this group reflects the effects of the
flow regimes. For example, it can be expected that for turbulent flow

this number may be order of zero whereas for laminar flow can be large

enough to be effective on the mixture flow behavior.

It is evident from Equations (6.73) through (6.84) that in view

of so many similarity groups, an exact scaling of two systems can not

be achieved in practice. However, for different applications not all

groups will have the same importance. Consequently, Equation (6.73)

can be used to determine the importance of the various groups and select

the appropriate scaling parameters.

6.4 Summary

1) The momentum equations for separated two-phase flow problem

were formulated based on the two-fluid and the diffusional models.

2) From the simplified formulations it was concluded that the

formulation based on the two-fluid model should be expressed in terms of

two-field equations - one for each phase and four additional equations -

two constitutive equations for τ_{1i} and τ_{1e} and two interfacial coupling

equations; whereas the formulation based on the diffusion model should be

formulated in terms of one field equation for the mixture and two

additional equations - one constitutive equation for τ_{1e} and one for

interfacial pressure coupling.

3) From the general formulations a Prandtl type, i.e., a boundary

layer approximation was used, and the appropriate forms for the momentum

equation were derived for both the plane flow and the annular flow.

4) From the equations for plane flow using the "thin film"

approximation the momentum equation for liquid films was derived which

describes the dynamics of flowing films and take into account the effects

of a) body and thermo-capillary forces, b) shear and pressure forces of

the flowing vapor phase, c) mass addition and/or removal at the external as well as the internal boundaries, d) heat flux at the boundaries, and 3) flow regimes.

    5) The importance of various processes and parameters was discussed in relation to the liquid films. It was shown in particular that the evaporation thrust can _interrupt_ the flow of the liquid and induce a local dryout.

    6) The mixture equations for the plane flow as well as the annular flow were expressed in dimensionless form and used to obtain similarity groups.

CHAPTER VII

ENERGY EQUATIONS

7.1 Introduction

It was discussed in Chapter 2, that there is great disagreement about the correct form of the energy equations. Some authors weighed the energy equations for each phase by the appropriate mass flow rates whereas others weighed them by means of the static weight which would exist if there were no flow.

In view of the controversy concerning the correct form of the energy equations for two-phase flow systems and in view of the importance of such systems, it seems desirable to present the exact general macroscopic energy balance equations for these systems. The equations will be derived in a manner similar to that developed in the previous chapters, i.e., the energy formulation will be based on both models: the two-fluid and the diffusional model.

It will be observed that when effects of diffusion and of motion are neglected the mixture total energy equation will be reduced to those derived by Gibbs. To the writer's knowledge, this is the first time that the field equations for two-phase flow have been reduced to those of classical thermodynamics of mixtures.

Furthermore, auxiliary energy equations, such as the internal energy, enthalpy and the mechanical energy equations will be derived together with the appropriate jump conditions.

7.2 Two-Fluid Model Formulation of the Energy Equation

Since the total energy equation is a conservation law, it can be obtained from the general balance equations developed in Chapter 4.

7.2.1 Derivation of Phase Equations

As usual the two-fluid model will be formulated in terms of two energy equations, one for each phase. To obtain the mass-weighted area-averaged total energy equations from the general balance equation, we use the proper identifications for ψ, $\dot{\psi}$ and $\vec{\phi}$. From Table 3 it can be seen that, for total energy transport they are given by:

$$\psi = u + \frac{v^2}{2} \quad , \text{ energy per unit mass} \qquad (7.1)$$

$$\dot{\psi} = \vec{g} \cdot \vec{v} \quad , \text{ distributed energy source or sink per unit mass}$$

$$\vec{\phi} = \vec{q} - \bar{\bar{\pi}} \cdot \vec{v} \, , \text{ total energy flux with respect to the center of mass of the fluid.}$$

where u and $\vec{q}$ are, respectively, the specific internal energy and the heat flux vector. Other terms have their usual meanings.

Introducing Equation (7.1) in Equations (4.20a) and using the co-variance definition given by Equation (3.8) for the second time for $<v^2>$ term, we obtain the mass-weighted area-averaged energy equation for phase 1:

$$\frac{\partial}{\partial t} (1-\alpha) \ll \rho_1 \gg \left(\langle u_1 \rangle + \frac{\langle v_1^2 \rangle}{2} \right) + \frac{\partial}{\partial x} (1-\alpha) \ll \rho_1 \gg \langle v_{1x} \rangle \left(\langle u_1 \rangle + \frac{\langle v_1^2 \rangle}{2} \right) = \quad (7.2)$$

$$= -\frac{\partial}{\partial x} (1-\alpha) \ll q_{1x} \gg + \frac{\partial}{\partial x} (1-\alpha) \{ \ll \overline{\overline{\pi}}_1 \cdot \hat{1} \gg \} \cdot \langle \vec{v}_1 \rangle + (1-\alpha) \ll \rho_1 \gg \vec{g} \cdot \langle \vec{v}_1 \rangle -$$

$$\left[(1-\alpha) \ll \rho_1 \gg \langle v_{1x} \rangle \left(\langle u_1 \rangle + \frac{\langle v_1^2 \rangle}{2} \right) + (1-\alpha) \ll q_{1x} \gg - (1-\alpha) \ll \overline{\overline{\pi}}_1 \cdot \hat{1} \gg \cdot \langle \vec{v}_1 \rangle + \right.$$

$$\left. \text{Cov}(\text{en } 1) \right] \frac{d \ln A T_c}{dx} - \frac{1}{A T_c} \sum_{l=e,i} \int_{\mathcal{J}_{1l}} \left[\dot{m}_{1l} \left(u_{1l} + \frac{v_{1l}^2}{2} \right) + (\vec{q}_{1l} - \overline{\overline{\pi}}_{1l} \vec{v}_{1l}) \cdot \dot{m}_{1l} \right] \frac{dA}{dx} -$$

$$\frac{\partial}{\partial t} \text{Cov}(\text{mech } 1) - \frac{\partial}{\partial x} \text{Cov}(\text{en } 1)$$

in which Cov(mech 1) and Cov(en 1) are defined as follows:

$$\text{Cov}(\text{mech } 1) = (1-\alpha) \ll \rho_1 \gg \text{Cov}\left(\frac{\vec{v}_1 \cdot \vec{v}_1}{2} \right) \quad (7.3a)$$

$$\text{Cov}(\text{en } 1) = \text{Cov}(\text{conv } 1) + \text{Cov}(\text{int } 1) - \text{Cov}(\text{work } 1) \quad (7.3b)$$

where

$$\text{Cov}(\text{conv } 1) = (1-\alpha) \ll \rho_1 \gg \text{Cov}\left(v_{1x} \frac{v_1^2}{2} \right) + \langle v_{1x} \rangle \text{Cov}(\text{mech } 1) \quad (7.4a)$$

$$\text{Cov}(\text{int } 1) = (1-\alpha) \ll \rho_1 \gg \text{Cov}(v_{1x} \cdot u_1) \quad (7.4b)$$

$$\text{Cov}(\text{work } 1) = (1-\alpha) \text{Cov}\left[(\overline{\overline{\pi}}_1 \cdot \hat{1}) \cdot \vec{v}_1 \right] \quad (7.4c)$$

The terms on the left hand side of Equation (7.2) stand for the
rate of change of energy and the rate of energy input by convection
whereas the first, second and the third terms on the right side are,
respectively, the rate of energy input by conduction, rate of work done
by viscous and pressure and gravitational forces. The fourth and sixth
terms account for the effect of area change and of the energy distribu-
tion whereas the line integral gives the energy contribution of the

processes occurring on the internal and external phase boundaries. It is seen from the integrand that these contributions are due to the convective mass flux, the heat flux by conduction, the surface work by pressure and viscous forces.

It should be noted here that the energy equation can be expressed also in terms of the enthalpy i, instead of the internal energy u. In order to do this we simply expand the stress tensor $\overline{\overline{\pi}}_1$ by means of Equation (4.3) and then use the thermodynamic definition of the enthalpy

$$i_K = u_K + \frac{P_K}{\rho_K} \qquad , \; K = 1,2 \qquad (7.5)$$

Averaging of Equation (7.5) yields

$$\langle i_K \rangle = \langle u_K \rangle + \frac{\langle\!\langle P_K \rangle\!\rangle}{\langle\!\langle \rho_K \rangle\!\rangle} \qquad (7.6)$$

Substituting Equation (7.6) in Equation (7.2), and rearranging one obtains the energy equation expressed in terms of the enthalpy rather than the internal energy, thus

$$\frac{\partial}{\partial t}\,(1-\alpha)\,\langle\!\langle \rho_i \rangle\!\rangle \left(\langle i_i \rangle + \frac{\langle v_i \rangle^2}{2} \right) + \frac{\partial}{\partial x}\,(1-\alpha)\,\langle\!\langle \rho_i \rangle\!\rangle\,\langle v_{i,x} \rangle \left(\langle i_i \rangle + \frac{\langle v_i \rangle^2}{2} \right) = \quad (7.7a)$$

$$= \frac{\partial}{\partial t}\,(1-\alpha)\,\langle\!\langle P_i \rangle\!\rangle - \frac{\partial}{\partial x}\,(1-\alpha)\,\langle\!\langle q_{i,x} \rangle\!\rangle + \frac{\partial}{\partial x}\,(1-\alpha)\,\langle\!\langle \overline{\overline{E}}_i \cdot \hat{\imath} \rangle\!\rangle \cdot \langle \vec{v}_i \rangle +$$

$$(1-\alpha)\,\langle\!\langle \rho_i \rangle\!\rangle\,\vec{g} \cdot \langle \vec{v}_i \rangle - \left[(1-\alpha)\,\langle\!\langle \rho_i \rangle\!\rangle\,\langle v_{i,x} \rangle \left(\langle i_i \rangle + \frac{\langle v_i \rangle^2}{2} \right) + (1-\alpha)\,\langle\!\langle q_{i,x} \rangle\!\rangle - $$

$$(1-\alpha)\,\langle\!\langle \overline{\overline{E}}_i \cdot \hat{\imath} \rangle\!\rangle \cdot \langle \vec{v}_i \rangle + \text{Cov}(en1) \right] \frac{d\ell n\, A_{Tc}}{dx} - $$

$$\frac{1}{A_{Tc}} \sum_{\ell=e,i} \int_{S_{i\ell}} \left[\dot{m}_{i\ell} \left(i_{i\ell} + \frac{v_{i\ell}^2}{2} \right) + \left(\vec{q}_{i\ell} - \overline{\overline{E}}_{i\ell} \cdot \vec{v}_{i\ell} + P_{i\ell}\, v_\ell \right) \cdot \hat{n}_{i\ell} \right] \frac{dA}{dx} - $$

$$\frac{\partial}{\partial t}\,\text{Cov}(mech1) - \frac{\partial}{\partial x}\,\text{Cov}(en1)$$

Similarly application of Equations (7.1), (4.20) and of (7.6) yields the energy equation for the second phase

$$\frac{\partial}{\partial t} \alpha \ll \rho_2 \gg \left(\langle i_2 \rangle + \frac{\langle v_2 \rangle^2}{2} \right) + \frac{\partial}{\partial x} \alpha \ll \rho_2 \gg \langle v_{2x} \rangle \left(\langle i_2 \rangle + \frac{\langle v_2 \rangle^2}{2} \right) = \qquad (7.7b)$$

$$= \frac{\partial}{\partial t} \alpha \ll P_2 \gg - \frac{\partial}{\partial x} \alpha \ll q_{2x} \gg + \frac{\partial}{\partial x} \alpha \ll \bar{\bar{\tau}}_2 \cdot \hat{\imath} \gg \cdot \langle \vec{v}_2 \rangle + \alpha \ll \rho_2 \gg \vec{g} \cdot \langle \vec{v}_2 \rangle -$$

$$\left[\alpha \ll \rho_2 \gg \langle v_{2x} \rangle \left(\langle i_2 \rangle + \frac{\langle v_2 \rangle^2}{2} \right) + \alpha \ll q_{2x} \gg - \alpha \ll \bar{\bar{\tau}}_2 \cdot \hat{\imath} \gg \cdot \langle \vec{v}_2 \rangle + Cov(en\,2) \right] \frac{d \ln A_{T_c}}{dx} -$$

$$\frac{1}{A_{T_c}} \sum_{\ell \cdot e \cdot i} \left[\dot{m}_{2\ell} \left(i_{2\ell} + \frac{v_{2\ell}^2}{2} \right) + \left(\vec{q}_{2\ell} - \bar{\bar{\tau}}_{2\ell} \cdot \vec{v}_{2\ell} + P_{2\ell} \, v_\ell \right) \cdot \hat{n}_{2\ell} \right] \frac{dA}{dx} -$$

$$\frac{\partial}{\partial t} Cov(mech\,2) - \frac{\partial}{\partial x} Cov(en\,2)$$

where covariance terms have the same form as Equations (7.3) and (7.4) except $(1 - \alpha)$ and 1 are replaced by α and 2.

In order to complete the energetic description of the two-phase flow field based on two-fluid model, an additional equation representing the interfacial coupling of the two fields, should be added to the field equations, Equations (7.7a and b). The interfacial coupling effects are expressed by the appropriate energy "jump" condition given by Equation (B.34) in which the surface energy contribution is expressed in a general sense. With the surface stress expressed only in terms of the surface tension σ, the "jump" condition for the energy is given by Equation (B.37).

As it is seen from this equation it is necessary to specify the surface velocity $\vec{v}_i$ by an additional relation. This can be accomplished

by using Equation (B.6) and the no-slip requirement at the interface. Equation (B.6) specifies the normal component $\vec{v}_{in}$ of the interfacial velocity

$$v_{in} = \vec{v}_i \cdot \hat{n}_i = -\frac{\frac{\partial h}{\partial t}}{|\nabla h|} \qquad (7.8)$$

where $h(x,y,z,t) = 0$ describes the surface in implicit form, whereas the tangential component v_{it} is given by no-slip condition

$$(v_{1t} - v_{it}) = (v_{2t} - v_{it}) = 0 \qquad (7.9)$$

Consequently, two-fluid model field equations together with the energy "jump" condition at the interface and the interfacial velocity $\vec{v}_i$, describe the energy transfer aspects of the two-phase flow systems. Further discussion about the constitutive equations will be given in conjunction with the two-dimensional flow formulations.

Energy equations derived so far are quite general and can be simplified to apply to engineering systems in the manner developed in the previous chapter which will be done in the following section.

7.2.2 Two-Dimensional Flow

Two-dimensional flow simplifications appropriate to the plane flow as well as the annular flow case can be made in view of the geometrical relations derived in Appendix C.

A. Plane Flow

Using the relations derived in Appendix C.1, it can be shown from Equation (7.1) and the general balance equation, Equation (4.28), that the plane flow energy equation for phase 1 can be expressed as:

$$\frac{\partial}{\partial t} \eta \langle\!\langle \rho_1 \rangle\!\rangle \left(\langle i_1 \rangle + \frac{\langle V_1 \rangle^2}{2} \right) + \frac{\partial}{\partial x} \eta \langle\!\langle \rho_1 \rangle\!\rangle \langle V_{1x} \rangle \left(\langle i_1 \rangle + \frac{\langle V_1 \rangle^2}{2} \right) = \qquad (7.10a)$$

$$\frac{\partial}{\partial t} \eta \langle\!\langle P_1 \rangle\!\rangle - \frac{\partial}{\partial x} \eta \langle\!\langle q_{1x} \rangle\!\rangle + \frac{\partial}{\partial x} \eta \langle\!\langle \overline{\overline{\Sigma}}_1 \cdot \hat{\iota} \rangle\!\rangle \cdot \langle \vec{V}_1 \rangle + \eta \langle\!\langle \rho_1 \rangle\!\rangle \, \vec{g} \cdot \langle \vec{V}_1 \rangle -$$

$$\left[\dot{m}_{1i} \left(i_{1i} + \frac{V_{1i}^2}{2} \right) + \left(\vec{q}_{1i} + P_{1i} \vec{V}_i - \overline{\overline{\Sigma}}_{1i} \cdot \vec{V}_{1i} \right) \cdot \hat{n}_{1i} \right] \left[1 + \left(\frac{\partial \eta}{\partial x} \right)^2 \right]^{1/2} -$$

$$\left[\dot{m}_{1e} \left(i_{1e} + \frac{V_{1e}^2}{2} \right) - \left(\vec{q}_1 - \overline{\overline{\Sigma}}_{1e} \cdot \vec{V}_{1e} \right) \hat{\jmath} \right] - \frac{\partial}{\partial t} \text{Cov (mech 1)} - \frac{\partial}{\partial x} \text{Cov (en 1)}$$

and <u>for phase 2</u> can be expressed as:

$$\frac{\partial}{\partial t} (H-\eta) \langle\!\langle \rho_2 \rangle\!\rangle \left(\langle i_2 \rangle + \frac{\langle V_2 \rangle^2}{2} \right) + \frac{\partial}{\partial x} (H-\eta) \langle V_{2x} \rangle \left(\langle i_2 \rangle + \frac{\langle V_2 \rangle^2}{2} \right) = \qquad (7.10b)$$

$$\frac{\partial}{\partial t} (H-\eta) \langle\!\langle P_2 \rangle\!\rangle - \frac{\partial}{\partial x} (H-\eta) \langle\!\langle q_{2x} \rangle\!\rangle + \frac{\partial}{\partial x} (H-\eta) \langle\!\langle \overline{\overline{\Sigma}}_2 \cdot \hat{\iota} \rangle\!\rangle \cdot \langle \vec{V}_2 \rangle + (H-\eta) \langle\!\langle \rho_2 \rangle\!\rangle \vec{g} \cdot \langle \vec{V}_2 \rangle -$$

$$\left[\dot{m}_{2i} \left(i_{2i} + \frac{V_{2i}^2}{2} \right) + \left(\vec{q}_{2i} + P_{2i} \vec{V}_i - \overline{\overline{\Sigma}}_{2i} \cdot \vec{V}_{2i} \right) \cdot \hat{n}_{2i} \right] \left[1 + \left(\frac{\partial}{\partial x} \right)^2 \right]^{1/2} -$$

$$\frac{\partial}{\partial t} \text{Cov (mech 2)} - \frac{\partial}{\partial x} \text{Cov (en 2)}$$

These two equations describe the energy conservation of a separated two-phase flow system. The only restriction is the assumption of two dimensionality. Equations (7.10a and b) are valid for systems with variable density and variable η.

Because Equations (7.10 and b) are fairly general, they are rather complex. However, by using the Prandtl's boundary layer simplification

which is valid for a single-phase viscous fluid, we can eliminate certain terms, such as certain derivatives of the heat flux as well as the rate of work done by viscous forces. The procedure used here is essentially the same as that used in Section 6.2.2 in connection with the Navier Stokes equations.

Before starting it is proper to note here that in the theory of boundary layer a distinction has been made between the <u>hydrodynamic</u> and <u>thermal</u> boundary layers. It is well-known that their orders depend upon the property group, i.e., upon the Prandtl number. However, in our analogy to the boundary layer theory, for the liquid film we do not make such a distinction. Since we are concerned with liquid films which are thin when compared to a characteristic longitudional dimension, we shall assume that the both layers are scaled with a characteristic fluid thickness η_{10}. On the other hand, since for the gas phase the Prandtl number is order of unity, both the thermal and hydrodynamic boundary layers are order of unity. Therefore, it is possible to scale both layers with the hydrodynamic boundary layer thickness η_{20}.

Now considering the two-dimensional flow of a compressible viscous fluid over a flat surface, one can write the energy equation in the following form.

$$\rho_k\, c_{P_k} \left(\frac{\partial T_k}{\partial t} + v_{k_x} \frac{\partial T_k}{\partial x} + v_{k_y} \frac{\partial T_k}{\partial y} \right) +$$

$$\rho_k \left(\frac{\partial v_k^2/2}{\partial t} + v_{k_x} \frac{\partial v_k^2/2}{\partial x} + v_{k_y} \frac{\partial v_k^2/2}{\partial y} \right) =$$

$$= \frac{\partial}{\partial x} \left(k_k \frac{\partial T_k}{\partial x} \right) + \frac{\partial}{\partial y} \left(k_k \frac{\partial T_k}{\partial y} \right) + \beta_k T_k \frac{\partial P_k}{\partial t} -$$

$$(1 - \beta_k T_k) \left(v_{k_x} \frac{\partial P_k}{\partial x} + v_{k_y} \frac{\partial P_k}{\partial y} \right) + \rho_k\, \vec{g} \cdot \vec{v}_k +$$

$$\frac{\partial}{\partial x} \left[2\mu_k\, v_{k_x} \frac{\partial v_{k_x}}{\partial x} - \frac{2}{3} \mu_k v_{k_x} \left(\frac{\partial v_{k_x}}{\partial x} + \frac{\partial v_{k_y}}{\partial y} \right) + \right.$$

$$\left. \mu_k\, v_{k_y} \left(\frac{\partial v_{k_x}}{\partial y} + \frac{\partial v_{k_y}}{\partial x} \right) \right] +$$

$$\frac{\partial}{\partial y} \left[2\mu_k\, v_{k_y} \frac{\partial v_{k_y}}{\partial y} - \frac{2}{3} \mu_k v_{k_y} \left(\frac{\partial v_{k_x}}{\partial x} + \frac{\partial v_{k_y}}{\partial y} \right) + \right.$$

$$\left. \mu_k\, v_{k_x} \left(\frac{\partial v_{k_x}}{\partial y} + \frac{\partial v_{k_y}}{\partial x} \right) \right)$$

<div align="right">(7.11)</div>

where β is the coefficient of volume expansivity, and k is the conduction coefficient.

We shall scale the temperature by the temperature difference ΔT, between the external surface temperature T_e and the interfacial surface temperature T_i, thus

$$T^+ = T / \Delta T \qquad (7.12)$$

Furthermore, since $v^2 = v_x^2 + v_y^2$, and since v_y is scaled by εU, then the scale for the kinetic energy becomes $U^2 (1 + \varepsilon^2)$. However, in Chapter 6, we neglected the second order effects in ε, then we shall scale the kinetic energy with $U^2/2$.

Introducing the scaling parameters defined in Section 6.2.2 together with Equation (7.12), it can be shown that the dimensionless form

of Equation (7.11) becomes

$$\rho_k^+ \left[C_{Pk} \left(\frac{\partial T_k^+}{\partial t^+} + v_{kx}^+ \frac{\partial T_k^+}{\partial x^+} + v_{ky}^+ \frac{\partial T_k^+}{\partial y^+} \right) + \right. \tag{7.13}$$

$$\left. Ec_k \left(\frac{\partial v_k^{+2}/2}{\partial t^+} + v_{kx}^+ \frac{\partial v_k^{+2}/2}{\partial x^+} + v_{ky}^+ \frac{\partial v_k^{+2}/2}{\partial y^+} \right) \right] =$$

$$= \frac{1}{Pe_k} \left[\frac{\partial}{\partial x^+} \left(k_k^+ \frac{\partial T_k^+}{\partial x^+} \right) + \frac{1}{\varepsilon_k^2} \frac{\partial}{\partial y^+} \left(k_k^+ \frac{\partial T_k^+}{\partial y^+} \right) \right] + Ec_k \left[\beta_k^+ T_k^+ \frac{\partial P_k^+}{\partial t^+} - \right.$$

$$\left. (1 - \beta_k^+ T_k^+) \left(v_{kx}^+ \frac{\partial P_k^+}{\partial x^+} + v_{ky}^+ \frac{\partial P_k^+}{\partial y^+} \right) \right] + \frac{Ec_k}{Fr_k} \rho_k^+ (g_x^+ v^+ + g_y^+ v_{ky}^+) +$$

$$\frac{Ec_k}{Re_k} \left\{ \frac{\partial}{\partial x^+} \mu_k^+ \left[2 v_{kx}^+ \frac{\partial v_{kx}^+}{\partial x^+} - \frac{2}{3} v_{kx}^+ \left(\frac{\partial v_{kx}^+}{\partial x^+} + \frac{\partial v_{ky}^+}{\partial y^+} \right) + \right. \right.$$

$$\left. v_y^+ \left(\frac{\partial v_{kx}^+}{\partial y^+} + \varepsilon_k^2 \frac{\partial v_{ky}^+}{\partial x^+} \right) \right] +$$

$$\frac{\partial}{\partial y^+} \mu_k^+ \left[2 v_{ky}^+ \frac{\partial v_{ky}^+}{\partial y^+} - \frac{2}{3} v_{ky}^+ \left(\frac{\partial v_{kx}^+}{\partial x^+} + \frac{\partial v_{ky}^+}{\partial y^+} \right) + \right.$$

$$\left. \left. v_y^+ \left(\frac{\partial v_{kx}^+}{\varepsilon_k^2 \partial y^+} + \frac{\partial v_{ky}^+}{\partial x^+} \right) \right] \right\}$$

where

$$Pe_k = Pr_k \cdot Re_k = \frac{\nu_{ko}}{k_{ko}/\rho_{ko} C_{Po}} \frac{L U_k}{\nu_{ko}}, \underline{Peclet} \text{ number} \tag{7.14}$$

$$Ec_k = \frac{U_k^2}{C_{Po} \Delta T} \qquad , \underline{Eckert} \text{ number} \tag{7.15}$$

which scales the relative importance of the compression and frictional heating against the thermal energy.

From Equation (7.13) it can be seen that

$$\frac{\partial q_x}{\partial x} \simeq \varepsilon^2 \qquad ; \quad \frac{\partial q_y}{\partial y} \simeq 1 \tag{7.16}$$

$$\frac{\partial}{\partial t}\, \eta \ll \rho_1 \gg \left(\langle i_1 \rangle + \frac{\langle v_1 \rangle^2}{2}\right) + \frac{\partial}{\partial x}\, \eta \ll \rho_1 \gg \langle v_{1x} \rangle \left(\langle i_1 \rangle + \frac{\langle v_1 \rangle^2}{2}\right) = \qquad (7.19a)$$

$$= \frac{\partial}{\partial t}\, \eta \ll P_1 \gg + \eta \ll \rho_1 \gg g_x \langle v_{1x} \rangle -$$

$$\left[\dot{m}_{1i} \left(i_{1i} + \frac{v_{1i}^2}{2} \right) + g_{1ig} n_{1ig} + P_{1i}\, v_{in} - \tau_{1xy}|_i\, v_{1ix}\, n_{1ig} \right] -$$

$$\left[\dot{m}_{1e} \left(i_{1e} + \frac{v_{1e}^2}{2} \right) + g_{1eg} n_{1eg} + P_{1e}\, v_{en} - \tau_{1xy}|_e\, v_{1ex}\, n_{1eg} \right] -$$

$$\frac{\partial}{\partial t}\, \text{Cov} (\text{mech } 1) - \frac{\partial}{\partial x}\, \text{Cov} (\text{en } 1)$$

$$\frac{\partial}{\partial t}\, (1-\eta) \ll \rho_2 \gg \left(\langle i_2 \rangle + \frac{\langle v_2 \rangle^2}{2}\right) + \frac{\partial}{\partial x}\, (1-\eta) \ll \rho_2 \gg \langle v_{2x} \rangle \left(\langle i_2 \rangle + \frac{\langle v_2 \rangle^2}{2}\right) = (7.19b)$$

$$= \frac{\partial}{\partial t}\, (1-\eta) \ll P_2 \gg + (1-\eta) \ll \rho_2 \gg g_x \langle v_{2x} \rangle -$$

$$\left[\dot{m}_{2i} \left(i_{2i} + \frac{v_{2i}^2}{2} \right) + g_{2ig} n_{2ig} + P_{2i}\, v_{in} - \tau_{2xy}|_i\, v_{2ix}\, n_{2ig} \right] -$$

$$\frac{\partial}{\partial t}\, \text{Cov} (\text{mech } 2) - \frac{\partial}{\partial x}\, \text{Cov} (\text{en } 2)$$

In view of the approximations general expression for Cov(en 1), i.e., Equation (7.3b) reduces to

$$\text{Cov} (\text{en } K) = \text{Cov} (\text{conv } K) + \text{Cov} (\text{int } K) \qquad (7.20)$$

whereas Cov(mech K) has the same form as Equation (7.3a).

To complete the formulation based on the two-fluid model, the external heat flux q_{1e} and either q_{1y} or q_{2yi} must be given by means of an energetic constitutive equation. Whereas the energy "jump" condition can be expressed by Equation (C.20). Using Equations (C.20) and (7.4) in Equation (C.20) and neglecting the second order terms we obtain interfacial "jump" condition appropriate for the plane flow analysis, thus

$$\dot{m}_{1i} \left[(i_{1i} - i_{2i}) + \left(\frac{v_{1i}^2}{2} - \frac{v_{2i}^2}{2} \right) \right] + \left(\vec{q}_{1ij} - \vec{q}_{2ij} \right) n_{1ij} + (P_{1i} - P_{2i}) v_{1in} \quad (7.21)$$

$$\left(\tau_{1xy} |_i \, v_{1ix} - \tau_{2xy} |_i \, v_{2ix} \right) n_{1iy} = -\sigma \frac{\partial^2 \eta}{\partial x^2} \quad v_{in} + \hat{t}_i \cdot \frac{\partial \sigma}{\partial s} \vec{v}_i$$

where v_{in} is specified by Equation (B.6) which in the case of two-dimensional plane flow reduces to

$$v_{in} = \vec{v}_i \cdot \hat{n}_{1i} = \frac{\partial \eta}{\partial t} \left[1 + \left(\frac{\partial \eta}{\partial x} \right)^2 \right]^{-1/2} \simeq \frac{\partial \eta}{\partial t} \quad (7.22)$$

and the normal vectors are given by Equations (B.5), (B.6) and (B.8) reduce to

$$\hat{n}_{1i} = -\hat{n}_{2i} \simeq -\frac{\partial \eta}{\partial x} \, \hat{c} + \hat{j} \quad ; \quad \hat{n}_{1i\underline{x}} = -\hat{n}_{2i\underline{x}} = -\hat{n}_{1e\underline{x}} = -\hat{n}_R = \hat{j} \quad (7.23)$$

It is interesting to note that the first term in Equation (7.21) accounts for the net convective energy transport; the second represents the net heat flux by conduction whereas the third and last terms on the left hand side stand for the net work done by the pressure and the viscous shear forces. The first term on the right accounts for the reversible part of the surface energy, i.e., surface dilatational work whereas the second accounts for the irreversible part of the surface energy.

B. Annular Flow

In view of Equation (7.1) from the general balance equations, Equations (4.31) and (4.32) we obtain the annular flow energy equation for each phase, thus for phase 1:

$$\frac{\partial}{\partial t} (1-\alpha) \langle\!\langle \rho_1 \rangle\!\rangle \left(\langle i_1 \rangle + \frac{\langle v_1^2 \rangle}{2} \right) + \frac{\partial}{\partial x} (1-\alpha) \langle\!\langle \rho_1 \rangle\!\rangle \langle v_{1x} \rangle \left(\langle i_1 \rangle + \frac{\langle v_1^2 \rangle}{2} \right) = \qquad (7.24a)$$

$$= -\frac{\partial}{\partial x} (1-\alpha) \langle\!\langle q_{1x} \rangle\!\rangle + \frac{\partial}{\partial t} (1-\alpha) \langle\!\langle P_1 \rangle\!\rangle + \frac{\partial}{\partial x} (1-\alpha) \left(\langle\!\langle \bar{\bar{\tau}}_1 \cdot \hat{\imath} \rangle\!\rangle \right) \cdot \langle \vec{v}_1 \rangle +$$

$$(1-\alpha) \langle\!\langle \rho_1 \rangle\!\rangle \, g_x \, \langle v_{1x} \rangle - \Big[(1-\alpha) \langle\!\langle \rho_1 \rangle\!\rangle \langle v_{1x} \rangle \left(\langle i_1 \rangle + \frac{\langle v_1^2 \rangle}{2} \right) + (1-\alpha) \langle\!\langle q_{1x} \rangle\!\rangle -$$

$$(1-\alpha) \langle\!\langle \bar{\bar{\tau}}_1 \cdot \hat{\imath} \rangle\!\rangle \cdot \langle \vec{v}_1 \rangle + \text{Cov}(\text{en}1) \Big] \frac{d \ln D_e^2}{dx} -$$

$$\Big[\dot{m}_{1i} \left(i_{1i} + \frac{v_{1i}^2}{2} \right) + \left(\vec{q}_{1i} - \bar{\bar{\tau}}_{1i} \cdot \vec{v}_{1i} + P_{1i} \, \vec{v}_{1i} \right) \cdot \hat{n}_{1i} \Big] \left\{ 1 + \left[\frac{\partial}{\partial x} \left(\frac{D_e \sqrt{\alpha}}{2} \right) \right]^2 \right\}^{1/2} \frac{4 \sqrt{\alpha}}{D_e} -$$

$$\Big[\dot{m}_{1e} \left(i_{1e} + \frac{v_{1e}^2}{2} \right) + \left(\vec{q}_{1e} - \bar{\bar{\tau}}_{1e} \cdot \vec{v}_{1e} \right) \cdot \hat{n}_{1e} \Big] \left\{ 1 + \left[\frac{d}{dx} \left(\frac{D_e}{2} \right) \right]^2 \right\}^{1/2} \frac{4}{D_e} -$$

$$\frac{\partial}{\partial t} \text{Cov}(\text{mech}1) - \frac{\partial}{\partial x} \text{Cov}(\text{en}1)$$

and for <u>phase 2</u>:

$$\frac{\partial}{\partial t} \alpha \langle\!\langle \rho_2 \rangle\!\rangle \left(\langle i_2 \rangle + \frac{\langle v_2^2 \rangle}{2} \right) + \frac{\partial}{\partial x} \alpha \langle\!\langle \rho_2 \rangle\!\rangle \langle v_{2x} \rangle \left(\langle i_2 \rangle + \frac{\langle v_2^2 \rangle}{2} \right) = \qquad (7.24b)$$

$$= -\frac{\partial}{\partial x} \alpha \langle\!\langle q_{2x} \rangle\!\rangle + \frac{\partial}{\partial x} \alpha \langle\!\langle P_2 \rangle\!\rangle + \frac{\partial}{\partial x} \alpha \left(\langle\!\langle \bar{\bar{\tau}}_2 \cdot \hat{\imath} \rangle\!\rangle \right) \cdot \langle \vec{v}_2 \rangle +$$

$$\alpha \langle\!\langle \rho_2 \rangle\!\rangle \, g_x \, \langle v_{2x} \rangle - \Big[\alpha \langle\!\langle \rho_2 \rangle\!\rangle \langle v_{2x} \rangle \left(\langle i_2 \rangle + \frac{\langle v_2^2 \rangle}{2} \right) + \alpha \langle\!\langle q_{2x} \rangle\!\rangle -$$

$$\alpha \langle\!\langle \bar{\bar{\tau}}_2 \cdot \hat{\imath} \rangle\!\rangle \cdot \langle \vec{v}_2 \rangle + \text{Cov}(\text{en}2) \Big] \frac{d \ln D_e^2}{dx} -$$

$$\Big[\dot{m}_{2i} \left(i_{2i} + \frac{v_{2i}^2}{2} \right) + \left(\vec{q}_{2i} - \bar{\bar{\tau}}_{2i} \cdot \vec{v}_{2i} + P_{2i} \, \vec{v}_{2i} \right) \cdot \hat{n}_{2i} \Big] \left\{ 1 + \left[\frac{\partial}{\partial x} \left(\frac{D_e \sqrt{\alpha}}{2} \right) \right]^2 \right\}^{1/2} \frac{4 \sqrt{\alpha}}{D_e} -$$

$$\frac{\partial}{\partial t} \text{Cov}(\text{mech}2) - \frac{\partial}{\partial x} \text{Cov}(\text{en}2)$$

where Cov(mech 1) and Cov(en 1) are defined by Equations (7.3a) and

(7.3b) respectively.

These equations are valid for a general two-dimensional flow of an arbitrary interfacial structure. However, by introducing the boundary layer approximations we can simplify these equations to obtain

$$\frac{\partial}{\partial t}(1-\alpha)\ll\rho_1\gg\left(\langle i_1\rangle + \frac{\langle v_1\rangle^2}{2}\right) + \frac{\partial}{\partial x}(1-\alpha)\ll\rho_1\gg\langle v_{1x}\rangle\left(\langle i_1\rangle + \frac{\langle v_1\rangle^2}{2}\right) = \quad (7.25a)$$

$$= \frac{\partial}{\partial t}(1-\alpha)\ll P_1\gg + (1-\alpha)\ll\rho_1\gg g_x\langle v_{1x}\rangle -$$

$$\left[(1-\alpha)\ll\rho_1\gg\langle v_{1x}\rangle\left(\langle i_1\rangle + \frac{\langle v_1\rangle^2}{2}\right) + \text{Cov (en 1)}\right]\frac{d\ln D_e^{\frac{1}{2}}}{dx} -$$

$$\left[\dot{m}_{1i}\left(i_{1i} + \frac{v_{1i}^2}{2}\right) + \left(q_{1ir}\,n_{1ir} + P_i\,\bar{v}_i\cdot\hat{n}_{1i} - \tau_{1xr|i}\,v_{1ix}\,n_{1ir}\right)\right]\frac{4\sqrt{\alpha}}{D_e} -$$

$$\left[\dot{m}_{1e}\left(i_{1e} + \frac{v_{1e}^2}{2}\right) + \left(q_{1er}\,n_{1er} - \tau_{1xr|e}\,v_{1ex}\,n_{1er}\right)\right]\frac{4\sqrt{\alpha}}{D_e}\left[1 + \left(\frac{d}{dx}\frac{D_e}{2}\right)^2\right]^{\frac{1}{2}} -$$

$$\frac{\partial}{\partial t}\text{Cov (mech 1)} - \frac{\partial}{\partial x}\text{Cov (en 1)}$$

$$\frac{\partial}{\partial t}\alpha\ll\rho_2\gg\left(\langle i_2\rangle + \frac{\langle v_2\rangle^2}{2}\right) + \frac{\partial}{\partial x}\alpha\ll\rho_2\gg\langle v_{2x}\rangle\left(\langle i_2\rangle + \frac{\langle v_2\rangle^2}{2}\right) = \quad (7.25b)$$

$$= \frac{\partial}{\partial t}\alpha\ll P_2\gg + \alpha\ll\rho_2\gg g_x\langle v_{2x}\rangle -$$

$$\left[\alpha\ll\rho_2\gg\langle v_{2x}\rangle\left(\langle i_2\rangle + \frac{\langle v_2\rangle^2}{2}\right) + \text{Cov (en 2)}\right]\frac{d\ln D_e^{\frac{1}{2}}}{dx} -$$

$$\left[\dot{m}_{2i}\left(i_{2i} + \frac{v_{2i}^2}{2}\right) + \left(q_{2ir}\,n_{2ir} + P_i\,\bar{v}_i\cdot\hat{n}_{2i} - \tau_{2xr|i}\,v_{2ix}\,n_{2ir}\right)\right]\frac{4\sqrt{\alpha}}{D_e} -$$

$$\frac{\partial}{\partial t}\text{Cov (mec 2)} - \frac{\partial}{\partial x}\text{Cov (en 2)}$$

where Cov(mech 1) is still defined by Equation (7.3a) whereas Cov(en 1) is defined by Equation (7.20).

To complete the system the energy balance at the interface should
be given. For two-dimensional flow fields it is given by Equation
(C.39) which in view of the boundary layer approximation, can be re-
duced to

$$\dot{m}_{ii}\left[\left(i_{1i}+\frac{v_{1i}^2}{2}\right)-\left(i_{2i}+\frac{v_{2i}^2}{2}\right)\right]+\left(q_{1ii}-q_{2ii}\right)n_{1ii}-\left(P_{1i}-P_{2i}\right)v_{in}- \quad (7.26)$$

$$\left(\tau_{1x+li}\,v_{1ix}-\tau_{2x+li}\,v_{2ix}\right)=\left(\frac{2}{D_e\sqrt{x}}-\frac{x^2}{\partial x^2}\frac{D_e\sqrt{x}}{2}\right)\sigma\,v_{in}+\hat{t}_i\cdot\frac{\partial\sigma\vec{v}_i}{\partial s}$$

where v_{in} is specified by Equation (B.6). As a consequence of the
boundary layer approximation this latter equation can be simplified to

$$(7.27)$$

whereas the normal vectors can be approximated by

$$v_{in}=\vec{v}_i\cdot\hat{n}_{2i}=\frac{\partial}{\partial t}\frac{D_e\sqrt{x}}{2} \qquad (7.28a)$$

$$\hat{n}_{1i}=-\hat{n}_{2i}=\frac{\partial}{\partial x}\frac{D_e\sqrt{x}}{2}\hat{e}_x-\hat{e}_r \qquad (7.28b)$$

The field equations, Equations (7.25a and b) together with equa-
tions (7.26) through (7.28) and with the additional constitutive equa-
tions for q_{1e} and for either q_{1i} or q_{2i} constitute the complete descrip-
tion of the energy field for an annular flow.

7.3 Diffusion-Model Formulation of the Energy Equation
7.3.1 General Mixture Energy Equation

Formulation based on the diffusion model is expressed in terms of
only one energy equation, i.e., of the mixture energy equation. To

obtain it we use the identifications for ψ, $\vec{\psi}$ and $\vec{\phi}$ given by Equation (7.1) in conjunction with the general balance equation, Equation (4.27), thus

$$\frac{\partial}{\partial t}\left[(1-\alpha)\,\langle\!\langle\rho_1\rangle\!\rangle\,\langle u_1 + \frac{v_1^2}{2}\rangle + \alpha\,\langle\!\langle\rho_2\rangle\!\rangle\,\langle u_2 + \frac{v_2^2}{2}\rangle\right]_+ \qquad (7.29)$$

$$\frac{\partial}{\partial x}\left[(1-\alpha)\,\langle\!\langle\rho_1\rangle\!\rangle\,\langle v_{1x}\,(u_1 + \frac{v_1^2}{2})\rangle + \alpha\,\langle\!\langle\rho_2\rangle\!\rangle\,\langle v_{2x}(u_2 + \frac{v_2^2}{2})\rangle\right] =$$

$$= -\frac{\partial}{\partial x}\left[(1-\alpha)\,\langle\!\langle q_{1x}\rangle\!\rangle + \alpha\,\langle\!\langle q_{2x}\rangle\!\rangle\right] + \frac{\partial}{\partial x}\left[(1-\alpha)\,\langle\!\langle(\bar{\pi}_1\cdot\hat{\iota})\cdot\vec{v}_1\rangle\!\rangle + \langle\!\langle(\bar{\pi}_2\cdot\hat{\iota})\cdot\vec{v}_2\rangle\!\rangle\right]_+$$

$$\rho_m\,\vec{g}\cdot\vec{v}_m + \frac{1}{A_{T_c}}\int_{J_i} \nabla_s\cdot\bar{\bar{\Theta}}_{(en)}\,\frac{dA}{dx} -$$

$$\frac{1}{A_{T_c}}\sum_{k=1}^{2}\int_{J_{ke}}\left[\dot{m}_{ke}\,(u_{ke} + \frac{v_{ke}^2}{2}) + (q_{ke} - \bar{\pi}_{ke}\cdot\vec{v}_{ke})\cdot\hat{n}_{ke}\right]\frac{dA}{dx}$$

Equation (7.29) is not suitable for the purpose of the diffusion model formulation because it is not expressed with respect to the center of mass of the mixture. Since the basic assumption of this model is to conceive the mixture as a single fluid in motion with the velocity $\vec{v}_m$, the diffusive transport at the velocity $\vec{v}_{km}$, of the quantities associated with the K-th phase must be taken into account with the proper definitions.

In order to see the effects of the diffusive transport let us evaluate various terms in Equation (7.29):

$$\frac{\partial}{\partial t}\left[(1-\alpha)\,\langle\!\langle\rho_1\rangle\!\rangle\,\langle u_1 + \frac{v_1^2}{2}\rangle + \alpha\,\langle\!\langle\rho_2\rangle\!\rangle\,\langle u_2 + \frac{v_2^2}{2}\rangle\right] = \frac{\partial}{\partial t}\,\rho_m\,(u_m + \frac{v_m^2}{2})_+ \quad (7.30)$$

$$\frac{\partial}{\partial t}\left[(1-\alpha)\,\langle\!\langle\rho_1\rangle\!\rangle\,\frac{v_{1m}^2}{2} + \alpha\,\langle\!\langle\rho_2\rangle\!\rangle\,\frac{v_{2m}^2}{2}\right] + \frac{\partial}{\partial t}\,C_{o\phi}\,(mech\,T)$$

where Cov(mech T) is defined by

$$Cov(mechT) = (1-\alpha) \langle\langle\rho_1\rangle\rangle Cov\left(\frac{\vec{v_1}\cdot\vec{v_1}}{2}\right) + \alpha \langle\langle\rho_2\rangle\rangle Cov\left(\frac{\vec{v_2}\cdot\vec{v_2}}{2}\right) \quad (7.31)$$

In Equation (7.30), the first two terms appearing on the right side are, respectively, the rate of change of the total energy of the mixture, and of the <u>net diffusional mechanical energy</u> of the mixture. The last term accounts for the effect of the velocity distribution which depends on the flow regimes.

Consider now the second term on the left hand side of Equation (7.29) it can be expressed as

$$\frac{\partial}{\partial x}\left[(1-\alpha)\langle\langle\rho_1\rangle\rangle\langle\vec{v}_{1x}(u_1+\frac{v_1^2}{2})\rangle + \alpha\langle\langle\rho_2\rangle\rangle\langle v_{2x}(u_2+\frac{v_2^2}{2})\rangle\right]= \quad (7.32)$$

$$\frac{\partial}{\partial x}\rho_m v_{mx}(u_m+\frac{v_m^2}{2}) + \frac{\partial}{\partial x}\left\{\left[(1-\alpha)\langle\langle\rho_1\rangle\rangle\frac{v_{1m}^2}{2}+\alpha\langle\langle\rho_2\rangle\rangle\frac{v_{2m}^2}{2}\right]v_{mx}\right\}+$$

$$\frac{\partial}{\partial x}\left\{\left[(1-\alpha)\langle\langle\rho_1\rangle\rangle V_{1mx}\vec{V}_{1m}+\alpha\langle\langle\rho_2\rangle\rangle V_{2mx}\vec{V}_{2m}\right]\cdot\vec{v}_m\right\}+\frac{\partial}{\partial x}\left[Cov(convT)+\right.$$

$$\left.Cov(intT)\right]\frac{\partial}{\partial x}\left[(1-\alpha)\langle\langle\rho_1\rangle\rangle\langle u_1\rangle+\frac{v_1^2}{2})V_{1mx}+\alpha\langle\langle\rho_2\rangle\rangle\langle u_2\rangle+\frac{v_2^2}{2})V_{2mx}\right]$$

where Cov(conv T) and Cov(int T) are defined by

$$Cov(convT) = v_{mx} Cov(mechT) + (1-\alpha)\langle\langle\rho_1\rangle\rangle Cov(v_{1x}\cdot\frac{v_1^2}{2})+ \quad (7.33a)$$

$$\alpha\langle\langle\rho_2\rangle\rangle Cov(v_{2x}\cdot\frac{v_2^2}{2}) + \frac{\alpha}{1-\alpha}\frac{\langle\langle\rho_2\rangle\rangle}{\langle\langle\rho_1\rangle\rangle}\rho_m V_{2mx} Cov\left[\frac{\vec{v_2}\cdot\vec{v_2}}{2}-\frac{\vec{v_1}\cdot\vec{v_1}}{2}\right]$$

$$Cov(int\tau) = (1-\alpha) \langle\!\langle \rho_1 \rangle\!\rangle \, Cov(v_1 \cdot u_1) + \alpha \langle\!\langle \rho_2 \rangle\!\rangle \, Cov(v_{2} \cdot u_{2}) \qquad (7.33b)$$

In Equation (7.32), the first two terms represent the convective total energy and the diffusional mechanical energy with respect to a fixed frame of reference. The third term denotes the work done by the diffusional stresses, $\bar{\bar{\pi}}_D$. The fourth term gives the energy flux with respect to the center of mass of the mixture. Finally, the last term accounts for the effect of the specific internal energy and of the velocity distributions.

Finally, work terms in Equation (7.29) can be combined to give

$$\frac{\partial}{\partial x}\left[(1-\alpha) \langle\!\langle \bar{\bar{\pi}}_1 \cdot \hat{\imath} \cdot \vec{v}_1 \rangle\!\rangle + \alpha \langle\!\langle \bar{\bar{\pi}}_2 \cdot \hat{\imath} \cdot \vec{v}_2 \rangle\!\rangle\right] = \frac{\partial}{\partial x}(\bar{\bar{\pi}}_m \cdot \hat{\imath}) \cdot \vec{v}_m + \qquad (7.34)$$

$$\frac{\partial}{\partial x}\left[(1-\alpha) \langle\!\langle \bar{\bar{\pi}}_1 \cdot \hat{\imath} \rangle\!\rangle \cdot \vec{V}_{1m} + \alpha \langle\!\langle \bar{\bar{\pi}}_2 \cdot \hat{\imath} \rangle\!\rangle \cdot \vec{V}_{2m}\right] + \frac{\partial}{\partial x} Cov(work\,\tau)$$

where Cov(work T) is defined by

$$Cov(work\,\tau) = (1-\alpha) \, Cov\left[(\bar{\bar{\pi}}_1 \cdot \hat{\imath}) \cdot \vec{v}_1\right] + \alpha \, Cov\left[(\bar{\bar{\pi}}_2 \cdot \hat{\imath}) \cdot \vec{v}_2\right] \qquad (7.35)$$

In Equation (7.34), the first two terms on the right side account for the energy fluxes due to the mixture stress $\bar{\bar{\pi}}_m$ and due to the relative velocity. Finally the last term accounts for the distribution of the velocity and of the stresses. Substituting Equations (7.30) through (7.34) in Equation (7.29) results in

$$\frac{\partial}{\partial t}\left[\varrho_m\left(u_m + \frac{V_m^2}{2}\right) + (1-\alpha)\ll\varrho_1\gg\frac{V_{1m}^2}{2} + \alpha\ll\varrho_2\gg\frac{V_{2m}^2}{2}\right] + \qquad (7.36)$$

$$\frac{\partial}{\partial x}\left\{\varrho_m\, v_{mx}\left(u_m + \frac{V_m^2}{2}\right) + v_{mx}\left[(1-\alpha)\ll\varrho_1\gg\frac{V_{1m}^2}{2} + \alpha\ll\varrho_2\gg\frac{V_{2m}^2}{2}\right]\right\} =$$

$$= -\frac{\partial q_{mx}}{\partial x} + \frac{\partial}{\partial x}\left[\left(\overline{\overline{\pi}}_m\cdot\hat{\imath}\right)\cdot\vec{v}_m\right] + \varrho_m\,\vec{g}\cdot\vec{v}_m -$$

$$\frac{\partial}{\partial x}\left\{\left[(1-\alpha)\ll\varrho_1\gg V_{1mx}\vec{V}_{1m} + \alpha\ll\varrho_2\gg V_{2mx}\vec{V}_{2m}\right]\cdot\vec{v}_m\right\} -$$

$$\frac{\partial}{\partial x}\left[(1-\alpha)\ll\varrho_1\gg\left(\langle u_1\rangle + \frac{V_{1m}^2}{2}\right)V_{1mx} + \alpha\ll\varrho_2\gg\left(\langle u_2\rangle + \frac{V_{2m}^2}{2}\right)V_{2mx}\right] +$$

$$\frac{\partial}{\partial x}\left[(1-\alpha)\ll\overline{\overline{\pi}}_1\cdot\hat{\imath}\gg\cdot\vec{V}_{1m} + \alpha\ll\overline{\overline{\pi}}_2\cdot\hat{\imath}\gg\cdot\vec{V}_{2m}\right] - \left\{\varrho_m\left(u_m + \frac{V_m^2}{2}\right) + \right.$$

$$\left[(1-\alpha)\ll\varrho_1\gg\frac{V_{1m}^2}{2} + \alpha\ll\varrho_2\gg\frac{V_{2m}^2}{2}\right]v_{mx} + q_{mx} + \left(\overline{\overline{\pi}}_m\hat{\imath}\right)\cdot\vec{v}_m +$$

$$\left[(1-\alpha)\ll\varrho_1\gg V_{1mx}\vec{V}_{1m} + \alpha\ll\varrho_2\gg V_{2mx}\vec{V}_{2m}\right]\cdot\vec{v}_m +$$

$$\left[(1-\alpha)\ll\varrho_1\gg\left(\langle u_1\rangle + \frac{V_{1m}^2}{2}\right)V_{1mx} + \alpha\ll\varrho_2\gg\left(\langle u_2\rangle + \frac{V_{2m}^2}{2}\right)V_{2mx}\right] +$$

$$\left[(1-\alpha)\ll\overline{\overline{\pi}}_1\cdot\hat{\imath}\gg\cdot\vec{V}_{1m} + \alpha\ll\overline{\pi}\cdot\hat{\imath}\gg\cdot\vec{V}_{2m}\right]\right\}\frac{d\ln A_{\overline{c}}}{dx} + \frac{1}{A_{\overline{c}}}\int_{\Sigma_i}\nabla_s\cdot\overline{\overline{\theta}}_{(con)}\frac{dA}{dx} -$$

$$\frac{1}{A_{\overline{c}}}\sum_{k=1}^{2}\int_{\Sigma_{ke}}\left[\dot{m}_{ke}\left(u_{ke} + \frac{V_{ke}^2}{2}\right) + \left(\vec{q}_{ke} - \overline{\overline{\pi}}_{ke}\vec{v}_{ke}\right)\cdot\hat{n}_{ke}\right]\frac{dA}{dx} -$$

$$\frac{\partial}{\partial t}\,\text{Cov}\,(\text{mech}\,T) - \frac{\partial}{\partial x}\,\text{Cov}\,(\text{en}\,T)$$

However, in accordance with the kinetic theory of mixtures, we can define new variables as follows:

$$\langle\bar{u}_k\rangle = \langle u_k\rangle + 1/2\,V_{km}^2 \qquad , \qquad \text{(7.37)}$$

, total specific internal energy of the K-th phase,

$$\ll\vec{q}_k^{\,T}\gg = \ll\vec{q}_k\gg - \ll\overline{\overline{\pi}}_k\gg\cdot\vec{V}_{km} + \ll\rho_k\gg\langle\bar{u}_k\rangle\,\vec{V}_{km}$$

, total heat flux of the K-th phase with respect to the center of gravity of the same phase. $\qquad$ (7.38)

Since the total internal energy is a mass weighted quantity we can use Equation (3.74) to define the corresponding <u>total specific internal energy $\bar{u}_m$, of the mixture</u>, thus

$$\rho_m\,\bar{u}_m = (1-\alpha)\,\ll \rho_1 \gg \langle \bar{u}_1 \rangle + \alpha \ll \rho_2 \gg \langle \bar{u}_2 \rangle \qquad (7.39)$$

Since the heat flux is an area-weighted quantity, we can define corresponding total heat flux of the <u>mixture</u> similar to Equation (6.57), thus

$$\vec{q}^{\,T}_m = (1-\alpha)\,\ll \vec{q}^{\,T}_1 \gg + \alpha \ll \vec{q}^{\,T}_2 \gg \qquad (7.40)$$

Using these expressions together with the total mixture stress tensor $\bar{\bar{\pi}}^{\,T}_m$, defined by Equation (6.55) in Equation (7.36), we obtain the <u>mixture energy equation</u> in the desired form, thus

$$\frac{\partial}{\partial t}\,\rho_m\Big(\bar{u}_m + \frac{v_m^2}{2}\Big) + \frac{\partial}{\partial x}\,\rho_m\,v_{mx}\Big(\bar{u}_m + \frac{v_m^2}{2}\Big) = -\frac{\partial \vec{q}^{\,T}_{mx}}{\partial x} - \qquad (7.41)$$

$$\frac{\partial}{\partial x}\Big[(\bar{\bar{\pi}}^{\,T}_m \cdot \hat{\imath})\cdot \vec{v}_m\Big] + \rho_m\,\vec{g}\cdot\vec{v}_m - \Big\{\rho_m v_{mx}\Big(\bar{u}_m + \frac{v_m^2}{2}\Big) + \vec{q}^{\,T}_{mx} +$$

$$\Big[(\bar{\bar{\pi}}^{\,T}_m \cdot \hat{\imath})\cdot \vec{v}_m\Big] + Cov(e_nT) + Cov\Big[(\bar{\bar{\pi}}\cdot\hat{\imath})\cdot\vec{v}\Big]\Big\}\frac{d\ell_n A_{Tc}}{dx} +$$

$$\frac{1}{A_{Tc}}\int_{\xi_i} \nabla_s\cdot\bar{\bar{\theta}}_{(e_n)}\,\frac{dA}{dx} - \frac{1}{A_{Tc}}\sum_{\kappa=1}^{2}\int_{J_{ke}}\Big[\dot{m}_{ke}\Big(u_{ke} + \frac{v_{ke}^2}{2}\Big) + (\vec{q}_{ke} - \bar{\bar{\pi}}_{ke}\cdot\vec{v}_{ke})\cdot\hat{n}_{ke}\Big]\frac{dA}{dx} -$$

$$\frac{\partial}{\partial t}\,Cov(mech\,T) - \frac{\partial}{\partial x}\,Cov(e_nT)$$

The energy equation expressed in the form of Equation (7.41) is easy to interpret. The left side members give the rate of change of the mixture energy and of input of energy by convection. The first term on the right is the mixture energy flux with respect to the baricenter of

the mixture, the second is the rate of work done by the pressure, viscous and the diffusive forces, the third is the rate of work done by gravitational forces; the fourth term accounts for the effect of the area changes, the sixth gives the energy contribution at the external boundaries, the last two terms account for the distribution of the various variables. Finally, the fifth term accounts for the contribution by the internal surfaces.

Now defining the total specific enthalpy $\bar{i}_k$ of the K-th phase by

$$\bar{i}_k = \bar{u}_k + \frac{P_k}{\rho_k} = i_k + \frac{1}{2} V_{km}^2 \qquad (7.42)$$

we can also express the mixture energy equation in terms of the total enthalpy rather than the total energy. Therefore, from Equation (7.35) we obtain

$$\frac{\partial}{\partial t}\, \rho_m \left(\bar{i}_m + \frac{v_m^2}{2}\right) + \frac{\partial}{\partial x}\, \rho_m \, v_{mx} \left(\bar{i}_m + \frac{v_m^2}{2}\right) = -\frac{\partial\, q_{mx}^T}{\partial x} + \qquad (7.43)$$

$$\frac{\partial}{\partial x}\left[(\bar{\bar{\varepsilon}}_m^T \cdot \hat{\imath})\cdot \vec{v}_m\right] + \rho_m\, \vec{g}\cdot\vec{v}_m + \frac{\partial}{\partial t}\, P_m - \left\{\rho_m\, v_{mx}\left(\bar{i} + \frac{v_m^2}{2}\right) + \right.$$

$$\left. q_{mx}^T - \left[(\bar{\bar{\varepsilon}}_m^T \cdot \hat{\imath})\cdot \vec{v}_m\right] + Cov\,(e_m T) + Cov\left[(\bar{\bar{\varepsilon}}\cdot\hat{\imath})\cdot\vec{v}_m\right]\right\}\frac{d\,\ell_n A_{T_c}}{dx} +$$

$$\frac{1}{A_{T_c}} \int_{S_i} \nabla_s \cdot \bar{\theta}_{(en)}\, \frac{dA}{dx} -$$

$$\frac{1}{A_{T_c}} \sum_{k=1}^{2} \int_{S_{ke}} \left[\dot{m}_{ke}\left(i_{ke} + \frac{v_{ke}^2}{2}\right) + \left(\vec{q}_{ke} - \bar{\bar{\varepsilon}}_{ke}\cdot v_{ke}\right)\cdot \hat{n}_{ke}\right]\frac{dA}{dx} -$$

$$\frac{\partial}{\partial t}\, Cov\,(mech\,T) - \frac{\partial}{\partial x}\, Cov\,(en\,T)$$

where the mixture pressure P_m is defined by Equation (6.57) whereas the total mixture viscous stress tensor is defined as follows

$$\bar{\bar{\Sigma}}_m^T = \bar{\bar{\pi}}_m^T + P_m \bar{\bar{\delta}} \tag{7.44}$$

$$\bar{\bar{C}}_m^T = \bar{\bar{\Sigma}}_m - \bar{\bar{\pi}}_D \tag{7.45}$$

in which the mixture viscous stress τ_m, and the mixture diffusional stress tensor $\bar{\bar{\pi}}_D$ are given by Equations (6.58) and (6.60), respectively.

In order to express the mixture energy equation in terms of the material derivative we simply expand the left hand side of Equation (7.43) or (7.41) and then use the mixture continuity equation, Equation (5.14). For example, from Equation (7.43) we obtain

$$\rho_m \frac{D_m}{Dt}\left(\tau_m + \frac{v_m^2}{2}\right) = -\frac{\partial q_m^T}{\partial x} + \frac{\partial}{\partial x}\left[(\bar{\bar{\Sigma}}_m^T \cdot \hat{\iota})\cdot \vec{v}_m\right] + \tag{7.46}$$

$$\rho_m \vec{g}\cdot\vec{v}_m + \frac{\partial P_m}{\partial t} - \left\{ q_{mx}^T - \left[(\bar{\bar{\Sigma}}^T\cdot\hat{\iota})\cdot\vec{v}_m\right] + \right.$$

$$Cov(enT) + Cov\left[(\bar{\bar{\Sigma}}\cdot\hat{\iota})\cdot\vec{v}_m\right]\frac{d\ln A_{Tc}}{dx} + \frac{1}{A_{Tc}}\int_{S_i}\vec{v}_s\cdot\bar{\bar{\theta}}_{(en)}\frac{dA}{dx} -$$

$$\frac{1}{A_{Tc}}\sum_{k=1}^{2}\int_{S_{ke}}\left[\dot{m}_{ke}\left(i_{ke} + \frac{v_{ke}^2}{2}\right) + \left(\vec{q}_{ke} - \bar{\bar{\Sigma}}_{ke}\cdot\vec{v}_{ke}\right)\cdot\hat{n}_{ke}\right]\frac{dA}{dx} -$$

$$\frac{\partial}{\partial t}Cov(mechT) \quad \frac{\partial}{\partial x}Cov(enT)$$

In terms of the total internal energy $\bar{u}_m$, we can express Equation (7.46) as

$$\rho_m \frac{D_m}{Dt}\left(\bar{u}_m + \frac{v_m^2}{2}\right) = -\frac{\partial q_m^T}{\partial x} + \frac{\partial}{\partial x}\left[(\bar{\bar{\pi}}_m^T\cdot\hat{\iota})\cdot\vec{v}_m\right] + \rho_m\vec{g}\cdot\vec{v}_m - \tag{7.47}$$

$$\left[q_{mx}^T - (\bar{\bar{\pi}}_m^T\cdot\hat{\iota})\cdot\vec{v}_m + Cov(enT)\right]\frac{d\ln A_{Tc}}{dx} + \frac{1}{A_{Tc}}\int_{S_i}\vec{v}_s\cdot\bar{\bar{\theta}}_{(en)}\frac{dA}{dx} -$$

$$\frac{1}{A_{Tc}}\sum_{k=1}^{2}\int_{S_{ke}}\left[\dot{m}_{ke}\left(u_{ke} + \frac{v_{ke}^2}{2}\right) + \left(\vec{q}_{ke} - \bar{\bar{\pi}}_{ke}\cdot\vec{v}_{ke}\right)\cdot\hat{n}_{ke}\right]\frac{dA}{dx} -$$

$$\frac{\partial}{\partial t}Cov(mechT) - \frac{\partial}{\partial x}Cov(enT)$$

Either equation, Equations (7.41), (7.43), (7.46) or (7.47) describes the conservation of total energy for the diffusional model. We note that only one additional equation is required, i.e., the constitutive equation for the external heat flux q_{1e}. If we compare the formulation based on the diffusional model with the two-fluid model we recognize that in the two-fluid formulation two constitutive equation-one for q_{1e} and other for either q_{1i} or q_{2i}- together with the jump condition are required. Since in the diffusional model the balance between q_{1i} and q_{2i} is accounted for by means of the energy jump condition, there is no need to specify either q_{1i} or q_{2i}.

7.3.2 Discussion:

Several observations can be made with respect to the final form of the mixture energy equation:

1) It can be noticed that the mixture energy equation can not be expressed in the form identical to that of the single phase energy equation, as it was shown in reference [23]. As it is demonstrated by (7.41) or (7.43), due to the <u>internal surface</u> effects the mixture as a total, knows that it is heterogeneous rather than single continuum. However, if interfacial effects are neglected, i.e., if $\nabla_s \cdot \vec{\theta}_{(an)} = 0$ only then the mixture energy equation becomes equivalent to the ordinary laws of balance in one-dimensional continuum mechanics.

2) If we neglect the effect of interface, we recover from Equation (7.41) or (7.43) the energy balance of continuum mechanics.

3) It should be noted that when the effect of <u>diffusion</u> and of <u>motion</u> are neglected the energy equation reduces to that derived by Gibbs for the <u>heterogenous binary mixtures</u>.

In order to demonstrate this fact we shall dot product the vector form of the mixture equation of motion, i.e., vector form of Equation (6.63) by the mixture velocity $\vec{v}_m$ and then substract it from the mixture energy equation, Equation (7.47). This procedure yields in the absence of external mass transport the following <u>mixture total internal energy</u> $\bar{u}_m$, equation

$$\rho_m \, A_{T_c} \, \frac{D_m \bar{u}_m}{Dt} = - \frac{\partial}{\partial x} \, A_{T_c} \, q^T_{m x} - P_m \, A_{T_c} \, \frac{\partial v_{m x}}{\partial x} + \qquad (7.48)$$

$$A_{T_c} \left(\overline{\overline{\Sigma}}_m \cdot \hat{c} \right) \cdot \frac{\partial \vec{v}_m}{\partial x} + \left[\left(\nabla_s \cdot \overline{\overline{\theta}}_{(en)} \right) - \vec{v}_m \cdot \left(\nabla_s \cdot \overline{\overline{\theta}}_{(mom)} \right) \right] \frac{\vec{r}_i}{(\hat{n}_i \cdot \hat{n}_{if})} -$$

$$\sum_{k=1}^{2} \left[\vec{q}_{ke} + \overline{\overline{\Pi}}_{ke} \cdot \left(\vec{v}_m - \vec{v}_{ke} \right) \right] \cdot \hat{n}_{ke} \frac{s_{ke}}{(\hat{n}_{ke} \cdot \hat{n}_{kef})} -$$

$$\frac{\partial}{\partial x} \, A_{T_c} \, \text{Cov} \, (en T) + \vec{v}_m \cdot \frac{\partial}{\partial x} \, A_{T_c} \, \text{Cov} \, (mom T)$$

where we used Equation (3.12) in order to express $\frac{dA}{dx}$. In view of Equations (B.36) and (B.19)

$$\nabla_s \cdot \overline{\overline{\theta}}_{(en)} - \vec{v}_m \cdot \left(\nabla_s \cdot \overline{\overline{\theta}}_{(mom)} \right) = 2 \, x \, \sigma \, \left(\vec{v}_i - \vec{v}_m \right) \cdot \hat{n}_i + \qquad (7.49)$$

$$\left(\vec{v}_i - \vec{v}_m \right) \cdot \frac{1}{\mu^2} \left[\vec{r}_i \left(G \frac{\partial}{\partial u_1} - F \frac{\partial}{\partial u_2} \right) + \vec{r}_2 \left(E \frac{\partial}{\partial u_2} - F \frac{\partial}{\partial u_1} \right) \right] \sigma +$$

$$\frac{\sigma}{\mu^2} \left[\vec{r}_i \left(G \frac{\partial}{\partial u_1} - F \frac{\partial}{\partial u_2} \right) + \vec{r}_2 \left(E \frac{\partial}{\partial u_2} - F \frac{\partial}{\partial u_1} \right) \right] \cdot \vec{v}_i$$

Furthermore, we recall the physical meaning of the divergence of the velocity. From the fluid mechanics considerations it is well-known that the divergence of the velocity field physically corresponds to the relative rate of change of the dilatation following a particle path. In our one-dimensionalized motion in the x-direction, as a result of a variation in the mixture velocity in the direction of motion an element of initial length dx is stretched in the time increment dt to the new length

dx + d(dx) where d(dx) = $dx(\frac{\partial v_m}{\partial x})dt$. Therefore, since we are comparing

the initial length dx, then

$$\frac{\partial V_{mx}}{\partial x} = \frac{1}{dx} \frac{D_m (dx)}{Dt} \qquad (7.50)$$

Similarly from the surface dilatation definition it is also well-known

that, references [62,63]

$$\frac{1}{H^2}\left[\vec{r}_1\left(G\frac{\partial}{\partial u_1} - F\frac{\partial}{\partial u_2}\right) + \vec{r}_2\left(E\frac{\partial}{\partial u_2} - F\frac{\partial}{\partial u_1}\right)\right] \cdot \vec{V}_i = \qquad (7.51)$$

$$= \frac{1}{dA_i} \frac{D_m (dA_i)}{Dt} = \frac{1}{\frac{z_i\, dx}{(\hat{n}_{ii} \cdot \hat{n}_{i\,\bar{3}})}} \frac{D_m (dA_i)}{Dt}$$

Substituting Equation (7.50) and (7.51) in Equation (7.47), and then

integrating over the initial length x_o, we obtain

$$\frac{D_m}{Dt}\int_{x_o} \rho_m \bar{u}_m\, A_{Tc}\, dx = -\int_{x_o} P_m\, A_{Tc}\, \frac{D_m (dx)}{Dt} +$$

$$\int_{x_o} A_{Tc}\left(\bar{\bar{\Sigma}}_m \cdot \hat{t}\right) \frac{\partial \vec{V}_m}{\partial x}\, dx - \int_{x_o} \frac{\partial\, g_{mx}}{\partial x}\, dx +$$

$$\int_{x_o}\left\{ 2\,\kappa\,\sigma\cdot(\vec{V}_i - \vec{V}_m)\cdot\hat{n}_{ii} + (\vec{V}_i - \vec{V}_m)\cdot\frac{1}{H^2}\left[\vec{r}_1\left(G\frac{\partial}{\partial u_1} - F\frac{\partial}{\partial u_2}\right) +\right.\right.$$

$$\vec{r}_2\left(E\frac{\partial}{\partial u_2} - F\frac{\partial}{\partial u_1}\right)\right]\Gamma + \frac{\sigma}{H^2}\left[\vec{r}_1\left(G\frac{\partial}{\partial u_1} - F\frac{\partial}{\partial u_2}\right) +\right.$$

$$\left.\left.\vec{r}_2\left(E\frac{\partial}{\partial u_2} - F\frac{\partial}{\partial u_1}\right)\right]\cdot\vec{V}_i\right\}\frac{z_i}{(\hat{n}_{ii} \cdot \hat{n}_{i\,\bar{3}})}\quad \frac{dx}{dA}\quad \frac{D_m (dA)}{Dt} -$$

$$\int_{x_o}\sum_{k=1}^{2}\int_{z_{ke}}\left(\vec{q}_{ke} + \bar{\pi}_{ke}\cdot(\vec{V}_m - \vec{V}_{ke})\cdot\hat{u}_{ke}\right)\frac{z_{ke}}{(\hat{n}_{ke}\cdot\hat{n}_{ke\,\bar{3}})}\, dx -$$

$$\int_{x_o}\frac{\partial}{\partial x}\, A_{Tc}\, Cov(exT)\, dx + \int_{x_o}\vec{V}_m\cdot\frac{\partial}{\partial x}\, A_{Tc}\, Cov(momT)\, dx$$

We note that from Equation (3.12)

$$\frac{\xi_i}{(\hat{n}_{ii} \cdot \hat{n}_{ii_Y})} dx = dA_i \quad ; \quad \frac{\xi_{ke}}{(\hat{n}_{ke} \cdot \hat{n}_{ke_Y})} dx = dA_{ke} \qquad (7.53)$$

Furthermore, we observe that since the relative motion and gradients in the system do not appear in the equations of thermostatics, then

$$\bar{u}_m = u_m \qquad (7.54)$$

$$\bar{\bar{\varepsilon}}_m = 0$$

$$\vec{v}_i - \vec{v}_m = 0$$

similarly all the covariance terms disappear in Equation (7.50).

In view of Equations (7.53) and (7.54), Equation (7.52) reduces to

$$\frac{D_m U_m}{Dt} = - \int_{x_o} P_m \, A_{Tc} \frac{D_m (dx)}{Dt} + \int_{x_o} q_{ie} \frac{dA}{dx} dx + \iint \tau \frac{D_m (dA)}{Dt} \quad (7.55)$$

where U_m is the total internal energy contained in the initial volume. In view of Equation (6.57) we have $A_{Tc} P_m = A_{1c} P_1 + A_{2c} P_2$ and since the pressures P_1 and P_2, are uniform at any time, then the integrations in Equation (7.55) can be performed immediately to yield

$$d U_m = -P_1 \, dV_1 - P_2 \, dV_2 + \tau \, dA + dQ \qquad (7.56)$$

where we used

$$A_{1c} \, dx = dV_1 \quad ; \quad A_{2c} dx = dV_2 \; ; \left\{ \iint q_{ic} dA_c \right\} dt = dQ \quad (7.57)$$

We note that Equation (7.56) is the well-known Gibbs equation

which was derived originally for analyzing the classical thermodynamics of mixtures. We note that the reduction above was made possible by expressing the formulation in terms of the center of <u>mass</u> of the mixture. This confirms a statement made by Prigogine regarding the importance of formulating a thermodynamic problem in terms of the center of mass.

7.3.3 Two-Dimensional Flow

In this section we shall simplify the mixture total energy equation for two-dimensional flow geometries.

A. Plane Flow

By introducing the geometric relations defined in Appendix C.1, we obtain from Equation (7.43), the plane flow mixture total energy equation

$$\frac{\partial}{\partial t} \rho_m \left(\bar{\imath}_m + \frac{V_m^2}{2} \right) + \frac{\partial}{\partial x} \rho_m V_{mx} \left(\bar{\imath}_m + \frac{V_m^2}{2} \right) = -\frac{\partial q^T_{mx}}{\partial x} + \qquad (7.58)$$

$$\frac{\partial}{\partial x} \left[(\bar{\bar{\Sigma}}_m \cdot \bar{\imath}) \cdot \vec{V}_m \right] + \rho_m g_x V_{mx} + \frac{\partial P_m}{\partial t} + \frac{1}{H} q_s \cdot \bar{\theta}_{(en)} \left[1 + \left(\frac{\partial \eta}{\partial x} \right)^2 \right]^{1/2} -$$

$$\frac{1}{H} \left[\dot{m}_{le} \left(i_{le} + \frac{V_{le}^2}{2} \right) + \left(\vec{q}_{le} - \bar{\bar{\Sigma}}_{le} \cdot \vec{V}_{le} \right) \cdot \hat{n}_{le} \right] -$$

$$\frac{\partial}{\partial t} Cov(mech \, i) - \frac{\partial}{\partial x} Cov(en \, T)$$

Now the various terms can be simplified by means of the boundary layer approximations introduced in Section 7.2.2. It can be shown from Equations (7.16), (7.38) and (7.40) that

$$\frac{\partial q^T_{mx}}{\partial x} \simeq \frac{\partial}{\partial x} \left[(1-\alpha) \ll \rho_1 \gg \langle \bar{\imath}_1 \rangle V_{1mx} + \alpha \ll \rho_2 \gg \langle \bar{\imath}_2 \rangle V_{2mx} \right] (7.59)$$

whereas from Equation (7.17) and (7.45)

$$\frac{\partial}{\partial x}\left[\left(\overline{\overline{\Sigma}}_i \cdot \hat{\imath}\right) \cdot \vec{v}_m\right] \simeq 0 \qquad (7.60)$$

In view of Equations (C.17) and (C.18) it can be shown that under boundary layer assumption

$$\left[\nabla_s \cdot \overline{\overline{\Theta}}_{(en)}\right]\left[1+\left(\frac{\partial \eta}{\partial x}\right)^2\right]^{1/2} \simeq \frac{\partial^2 \eta}{\partial x^2} v_{in} \cdot \hat{\imath} \cdot \frac{\partial \vec{G} \vec{v}_i}{\partial s} \qquad (7.61)$$

where v_{in} is given by Equation (7.23).

Introducing Equations (7.59) through (7.61) in Equation (7.58) and using the no-slip condition at the external boundaries, i.e., $\vec{v}_{1e} = 0$, we obtain the plane flow mixture total energy equation, thus

$$\frac{\partial}{\partial t}\,\varrho_m\left(\overline{\imath}_m+\frac{v_m^2}{2}\right)+\frac{\partial}{\partial x}\,\varrho_m v_{mx}\left(\overline{\imath}_m+\frac{v_m^2}{2}\right)=\frac{\partial P_m}{\partial t}- \qquad (7.62)$$

$$\frac{\partial}{\partial x}\left[(1-\alpha)\langle\!\langle\varrho_1\rangle\!\rangle\langle\overline{\imath}_1\rangle\,v_{imx}+\alpha\langle\!\langle\varrho_2\rangle\!\rangle\langle\overline{\imath}_2\rangle\,v_{2mx}\right]-$$

$$\frac{\partial}{\partial x}\left(\pi_{Dxx}\,v_{mx}\right)+\varrho_m\,g_x\,v_{mx}-\frac{1}{H}\left\{-\frac{\partial^2\eta}{\partial x^2}\,v_{in}+\hat{\imath}\cdot\frac{\partial\vec{G}\vec{v}_i}{\partial s}\right\}-$$

$$\frac{1}{H^2}\left[\dot{m}_{1e}\left(i_{1e}+\frac{v_{1e}^2}{2}\right)-\mathcal{E}_{1e g}\right]-\frac{\partial}{\partial t}\,\mathrm{Cov}(\mathrm{mech\ T})-\frac{\partial}{\partial x}\,\mathrm{Cov}\,(\mathrm{en\,T})$$

where Cov(mech T) is still defined by Equation (7.3a), whereas under the approximation of boundary layer Cov(en T) reduces to

$$\mathrm{Cov}\,(\mathrm{en\,T}) = \mathrm{Cov}\,(\mathrm{int\,T})+\mathrm{Cov}\,(\mathrm{conv\text{-}T}) \qquad (7.63)$$

The second term in Equation (7.62) accounts for the mixture total enthalpy transport by diffusion, the third accounts for the effect of work by diffusion stresses $\overline{\overline{\pi}}_D$, the fifth and the sixth terms account for the effects of the interfacial energy source and the energy transport at the external surfaces.

Equation (7.62) together with the constitutive equation for q_{1e} specifies the conservation of total energy for plane flow.

To obtain the energy equation in terms of the total mixture internal energy $\bar{u}_m$ rather than the enthalpy $\bar{I}_m$, Equation (7.42) can be used in Equation (7.62).

B. Annular Flow

Following the same procedure, annular flow mixture total energy equation results in

$$\frac{\partial}{\partial t} \rho_m \left(\bar{I}_m + \frac{v_m^2}{2}\right) + \frac{\partial}{\partial x} \rho_m v_{mx} \left(\bar{I}_m + \frac{v_m^2}{2}\right) = \frac{\partial P_m}{\partial t} - \qquad (7.64)$$

$$\frac{\partial}{\partial x} \left[(1-\alpha) \langle\langle \rho_1 \rangle\rangle \langle \bar{I}_1 \rangle \, v_{1mx} + \alpha \langle\langle \rho_2 \rangle\rangle \langle \bar{I}_2 \rangle \, v_{2mx} \right] -$$

$$\frac{\partial}{\partial x} \, \Pi'_{Dxx} v_{mx} + \rho_m g_x v_{mx} - \frac{4\sqrt{\alpha}}{D_e} \left\{ \left[\frac{2}{D_e \sqrt{\alpha}} - \frac{\partial^2}{\partial x^2} \left(\frac{D_e \sqrt{\alpha}}{2} \right) \right] \sigma \, v_{in} + \right.$$

$$\left. \hat{t}_i \cdot \frac{\partial \sigma \vec{v}_i}{\partial s} \right\} - \frac{4}{D_e} \left[\dot{m}_{1e} \left(i_{1e} + \frac{v_{1e}^2}{2} \right) + q_{1e} - \tau_{2xx} v_{1ex} \right] \left[1 + \left(\frac{d D_e}{d x} \right)^2 \right]^{1/2} -$$

$$\frac{\partial}{\partial t} \, Cov(mech \, T) - \frac{\partial}{\partial x} \, Cov(en \, T)$$

where v_{in} is given by Equation (7.27).

Equation (7.64) together with an additional constitutive equation formulates the energy field of an annular flow.

7.4 Auxiliary Energy Equations

7.4.1 Local and Interfacial Balance Relations

The energy conservation equations derived in the previous sections are mathematically rigorous but not very convenient for application to several problems of practical interest. In fact, the total energy equation includes both terms that account for the mechanical and the thermal

energy transfers. For example, mechanical energy transfer may not be important in some applications where the thermal effects dominate or vice versa. Consequently, it is often desirable to use either mechanical or thermal energy transfer equation alone rather than the complete equation of energy. Since these forms of energy transfer effects are implicitly included in the complete equation of energy they are called auxiliary energy equations.

For the derivations of the auxiliary energy equations, we can still use the area-averaged mass-weighted general balance equation provided that we use the proper identifications for the quantity ψ, the influx $\vec{\phi}$ of ψ and the rate of generation $\dot{\psi}$ of ψ. By comparing the local form of the general balance equation given by Equation (4.5) with the mechanical energy, internal energy and the enthalpy equations we observe that the identifications given in Table 4 hold.

Table 4. Proper Identifications for the Auxiliary Energy Equations

Notation	ψ	$\vec{\phi}$	$\dot{\psi}$
Mechanical Energy	$\dfrac{v^2}{2}$	$-\bar{\bar{\pi}} \cdot \vec{v}$	$-\bar{\bar{\pi}} : \nabla \vec{v} + \vec{g} \cdot \vec{v}$
Internal Energy	u	$\vec{q}$	$\bar{\bar{\pi}} : \nabla \vec{v}$
Enthalpy	$i - \dfrac{P}{\rho}$	$\vec{q}$	$\bar{\bar{\pi}} : \nabla \vec{v}$

The identifications summarized in Table 4 can be also used to obtain the interfacial balance equations by using general interfacial balance equation given by Equation (B.10). It is evident from this equation that we need to separate the surface total energy contribution into the mechanical and the thermal energy parts. In the bulk fluids, the mechanical effects can be easily separated from the thermal effects by substracting the mechanical energy equation from the total energy equation obtaining thereby the thermal energy equation which is applicable locally in the continuum. To obtain, however, thermal and mechanical surface contributions we can follow the single phase flow analogy.

We note that in the single phase flow the mechanical energy equation is obtained by taking the dot product of the equation of motion with the local fluid velocity. It is expected that the same will be true for the surface contribution to the mechanical energy. Thus, multiplying the momentum surface term given by Equation (B.19) and using it for the mechanical energy surface source term in Equation (B.10), we obtain in conjunction with the first line of Table 4 the following <u>interfacial mechanical energy balance</u> equation

$$\sum_{k=1}^{2} \rho_{ki} \left(\vec{v}_{ki} - \vec{v}_i \right) \cdot \hat{n}_{ki} \frac{v_{ki}^2}{2} - \left(\overline{\overline{\pi}}_{ki} \cdot \hat{n}_{ki} \right) \cdot \vec{v}_{ki} = (M.E)_s \qquad (7.65a)$$

where $(M.E.)_s$ is the surface mechanical energy contribution and given by

$$(M.E)_s = 2 \kappa \sigma \ \vec{v}_i \cdot \hat{n}_{1i} + \frac{\vec{v}_i}{H^2} - \left[\vec{r}_1 \left(G_1 \frac{\partial}{\partial u_1} - F \frac{\partial}{\partial u_1} \right) + \vec{r}_2 \left(E \frac{\partial}{\partial u_2} - F \frac{\partial}{\partial u_1} \right) \right] \sigma \qquad (7.65b)$$

By substracting Equation (7.65a) from the total energy inter-facial balance equation given by Equation (B.37) we obtain the <u>inter-facial internal energy balance</u> equation, thus

$$\sum_{k=1}^{2} \rho_{ki}\,(\vec{v}_{ki} - \vec{v}_i)\cdot\hat{n}_{ki}\,u_{ki} + \vec{q}_{ki}\cdot\hat{n}_{ki} = (I.E)_s \qquad (7.66a)$$

where $(I.E.)_s$ is the surface thermal energy contribution and given by

$$(I.E)_s = \frac{\sigma}{H^2}\left[\vec{r_1}\left(G\frac{\partial}{\partial u_1} - F\frac{\partial}{\partial u_2}\right) + \vec{r_2}\left(E\frac{\partial}{\partial u_2} - F\frac{\partial}{\partial u_1}\right)\right]\cdot\vec{v}_i \qquad (7.66b)$$

It is an easy matter to obtain the <u>enthalpy balance</u> from the internal energy balance. For this, we simply use the thermodynamic relation between the internal energy and the enthalpy. Therefore, using Equation (7.5) in Equation (7.66) we obtain

$$\sum_{k=1}^{2} \rho_{ki}\,(\vec{v}_{ki} - \vec{v}_i)\cdot\hat{n}_{ki}\,i_{ki} + \vec{q}_{ki}\cdot\hat{n}_{ki} - P_{ki}(\vec{v}_{ki} - \vec{v}_i) = (I.E)_s \qquad (7.67)$$

For two-dimensional flow case the <u>interfacial mechanical energy balance</u> reduces to

$$\sum_{k=1}^{2} \rho_{ki}\,(\vec{v}_{ki} - \vec{v}_i)\cdot\hat{n}_{ki}\,\frac{v_{ki}^2}{2} - (\bar{\bar{\pi}}_{ki}\cdot\hat{n}_{ki})\cdot\vec{v}_{ki} = 2\kappa\,\sigma\,v_{in} + v_{it}\,\frac{\partial\sigma}{\partial s} \qquad (7.68)$$

whereas for the <u>internal energy</u> and the <u>enthalpy</u>, respectively, reduce to

$$\sum_{k=1}^{2} \rho_{ki}\,(\vec{v}_{ki} - \vec{v}_i)\cdot\hat{n}_{ki}\,u_{ki} + \vec{q}_{ki}\cdot\hat{n}_{ki}\quad \sigma\,\hat{t}_i\cdot\frac{\partial\vec{v}_i}{\partial s} \qquad (7.69)$$

and

$$\sum_{k=1}^{2} e_{ki} (\vec{v}_{ki} - \vec{v}_i) \cdot \hat{n}_{ki} \, i_{ki} + \vec{q}_{ki} \cdot \hat{n}_{ki} - P_{ki} (\vec{v}_{ki} - \vec{v}_i) \cdot \hat{n}_{ki} = \tau \hat{t}_i \cdot \frac{\partial \vec{v}_i}{\partial s} \quad (7.70)$$

After obtaining the proper identifications for ψ, $\dot{\psi}$, $\vec{\phi}$ and the interfacial balance relations, auxiliary energy equations will be derived in the sections which follow.

7.4.2 Enthalpy Equations

A. Two-Fluid Model Formulation

To obtain the mass-weighed area-averaged enthalpy equation from the general balance equation, i.e., Equation (4.20), we use the proper identifications for ψ, $\dot{\psi}$ and $\vec{\phi}$ which are given in Table 4. Introducing them in Equations (4.20a and b), we obtain the mass-weighed area-averaged enthalpy equations for each phase, thus for phase 1:

$$\frac{\partial}{\partial t} (1-\alpha) \ll \rho_1 \gg \langle i_1 \rangle + \frac{\partial}{\partial x} (1-\alpha) \ll \rho_1 \gg \langle v_{1x} \rangle \langle i_1 \rangle = \quad (7.71a)$$

$$-\frac{\partial}{\partial x} (1-\alpha) \ll q_{1x} \gg + \frac{\partial}{\partial t} (1-\alpha) \ll P_1 \gg + \frac{\partial}{\partial x} (1-\alpha) \langle v_{1x} \rangle \ll P_1 \gg +$$

$$(1-\alpha) \left(\dot{\Phi}_{1R} + \dot{\Phi}_{1IR} \right) - \Big[(1-\alpha) \ll \rho_1 \gg \langle v_{1x} \rangle \langle i_1 \rangle + (1-\alpha) \ll q_{1x} \gg -$$

$$(1-\alpha) \langle v_{1x} \rangle \ll P_1 \gg + \text{Cov (ent 1)} - (1-\alpha) \text{Cov} (P_1 \cdot v_{1x}) \Big] \frac{d \ell n \, A_{Tc}}{dx} -$$

$$\frac{1}{A_{Tc}} \sum_{\ell=e,i} \int_{S_{1\ell}} \Big[\dot{m}_{1\ell} \, i_{1\ell} + \vec{q}_{1\ell} \cdot \hat{n}_{1\ell} - P_{1\ell} (\vec{v}_{1\ell} - \vec{v}_\ell) \Big] \frac{dA}{dx} -$$

$$\frac{\partial}{\partial x} \text{Cov (ent 1)} + \frac{\partial}{\partial x} (1-\alpha) \text{Cov} (P_1 \cdot v_{1x})$$

and for <u>phase 2</u>

$$\frac{\partial}{\partial t} \propto \ll \rho_2 \gg \langle i_2 \rangle + \frac{\partial}{\partial x} \propto \ll \rho_2 \gg \langle v_{2x} \rangle \langle i_2 \rangle = \qquad (7.71b)$$

$$= - \frac{\partial}{\partial x} \propto \ll q_{2x} \gg + \frac{\partial}{\partial t} \propto \ll P_2 \gg + \frac{\partial}{\partial x} \propto \langle v_{2x} \rangle \ll P_2 \gg +$$

$$\propto \left(\dot{\Phi}_{2R} + \dot{\Phi}_{2IR} \right) - \left[\propto \ll \rho_2 \gg \langle v_{2x} \rangle \langle i_2 \rangle + \propto \ll q_{2x} \gg - \right.$$

$$\left. \propto \langle v_{2x} \rangle \ll P_2 \gg + Cov\,(ent\,2) - \propto Cov\,(P_2 \cdot v_{2x}) \right] \frac{d\,\alpha_2\,A_{Tc}}{dx} -$$

$$\frac{1}{A_{Tc}} \sum_{\ell = e, i} \oint_{\zeta_{2\ell}} \left[\dot{m}_{2\ell}\, i_{2\ell} + \vec{q}_{2\ell} \cdot \hat{m}_{2\ell} - P_{2\ell}\, (\vec{v}_{2\ell} - \vec{v}_\ell) \right] \frac{d\,A}{dx} -$$

$$\frac{\partial}{\partial x}\,Cov\,(ent\,2) + \frac{\partial}{\partial x} \propto Cov\,(P_2 \cdot v_{2x})$$

where

$$\dot{\Phi}_{KIR} = \ll \bar{\bar{\Sigma}}_K : \nabla \vec{v}_K \gg = \frac{1}{A_{Kc}} \iint_{A_{Kc}} \left(\bar{\bar{\Sigma}}_K : \nabla \vec{v}_K \right) dA \quad , K = 1, 2 \qquad (7.72)$$

represents the <u>irreversible</u> rate of the thermal energy increase per unit

volume of the K-th phase by viscous dissipation,

$$\dot{\Phi}_{KR} = - \ll P_K\, \nabla \cdot \vec{v}_K \gg = - \frac{1}{A_{Kc}} \iint_{A_{Kc}} \left(P_K\, \nabla \cdot \vec{v}_K \right) dA \quad , K = 1, 2 \qquad (7.73)$$

represents the <u>reversible</u> rate of the thermal energy increase per unit
volume of the K-th phase by volume expansion,

$$C_{ov}(ent\ 1) = (1-\alpha) \ll \rho_1 \gg C_{ov}(v_{1x} \cdot \dot{\iota}_1) \qquad (7.74a)$$

$$C_{ov}(ent\ 2) = \alpha \ll \rho_2 \gg C_{ov}(v_{2x} \cdot \dot{\iota}_2) \qquad (7.74b)$$

Let us consider the reversible increase of the thermal energy.
Recalling the definition of the covariance, from Equation (7.73)

$$\bar{\bar{\Phi}}_{1R} = - \ll P_1 \nabla \cdot \vec{v}_1 \gg = - \ll P_1 \gg \langle \nabla \cdot \vec{v}_1 \rangle - C_{ov}(P_1 \nabla \cdot \vec{v}_1) \qquad (7.75)$$

By using the divergence theorem over surfaces given by Equation (3.20),
it can be evaluated as

$$(1-\alpha) \bar{\bar{\Phi}}_{1R} = - \ll P_1 \gg \left[\frac{\partial}{\partial x}(1-\alpha)\langle v_{1x} \rangle + \sum_{\ell,e,i} \int_{\mathcal{F}_{1\ell}} \vec{v}_{1\ell} \cdot \hat{n}_{1\ell} \frac{dA}{dx} + \right. \qquad (7.76)$$

$$\left. (1-\alpha)\langle v_{1x} \rangle \frac{d \ell_n A_{TC}}{dx} \right] - (1-\alpha) C_{ov} \left[P_1 (\nabla \cdot \vec{v}_1) \right]$$

The corresponding relation for the second phase can be similarly
derived. Substituting Equation (7.75) in Equation (7.71a), and its
corresponding form valid for the second phase in Equation (7.71b), and
then expressing the resulting equations in terms of the convective
derivative we obtain

$$(1-\alpha) \ll \rho_1 \gg \frac{D_1 \langle i_1 \rangle}{Dt} = -\frac{\partial}{\partial x} \; (1-\alpha) \ll q_{1x} \gg + \frac{D_1}{Dt} (1-\alpha) \ll P_1 \gg + \quad (7.77a)$$

$$(1-\alpha) \dot{\Phi}_1 - \left[(1-\alpha) \ll q_{1x} \gg - (1-\alpha) \ll P_1 \gg \langle v_{1x} \rangle + \text{Cov (ent 1)} \right] \frac{d \ln A T_c}{dx} -$$

$$\frac{1}{A T_c} \sum_{l=e,i} \int_{S_{1l}} \left[\dot{m}_{1l} \left(i_{1l} - \langle i_1 \rangle \right) + \vec{q}_{1l} \cdot \hat{n}_{1l} - P_{1l} \langle \vec{v}_1 \rangle - \vec{v}_l \rangle \cdot \hat{n}_{1l} \right] \frac{dA}{dx} -$$

$$\frac{\partial}{\partial x} \; \text{Cov (ent 1)} - (1-\alpha) \; \text{Cov} \left(\vec{v}_1 \cdot \nabla P_1 \right)$$

$$\alpha \ll \rho_2 \gg \frac{D_2 \langle i_2 \rangle}{Dt} = -\frac{\partial}{\partial x} \; \alpha \ll q_{2x} \gg + \frac{D_2}{Dt} \; \alpha \ll P_2 \gg + \quad (7.77b)$$

$$\alpha \dot{\Phi}_2 - \left[\alpha \ll q_{2x} \gg - \alpha \ll P_2 \gg \langle v_{2x} \rangle + \text{Cov (ent 2)} \right] \frac{d \ln A T_c}{dx} -$$

$$\frac{1}{A T_c} \sum_{l=e,i} \int_{S_{2l}} \left[\dot{m}_{2l} \left(i_{2l} - \langle i_2 \rangle \right) + \vec{q}_{2l} \cdot \hat{n}_{2l} - P_{2l} \langle \vec{v}_2 \rangle - \vec{v}_l \rangle \cdot \hat{n}_{2l} \right] \frac{dA}{dx} -$$

$$\frac{\partial}{\partial x} \; \text{Cov (ent 2)} - \alpha \; \text{Cov} \left(\vec{v}_2 \cdot \nabla P_2 \right)$$

where $\frac{D_K}{D_t}$, (K = 1,2) is the convective derivative following the K-th phase

center of mass, i.e., Equation (5.18).

Equation (7.77a) for the first phase and Equation (7.77b) for the

second phase describe the thermal field description of a separated two-

phase flow based on the two-fluid model. For the completeness these

two field equations must be supplemented by an additional relation ex-

pressing the interfacial enthalpy balance i.e., Equation (7.67) and by

two constitutive equations - one for the external heat flux q_{1e}, and one

for the conduction heat flux of either one phase, i.e., q_{1i} or q_{2i}, and

one for the conduction heat flux of either one phase, i.e., q_{1i} or q_{2i},

In order to express the enthalpy equation appropriate for the

two-dimensional <u>plane flow</u>, the geometric relations derived in Appendix C.1 will be used. Consequently from Equation (7.77a) for the <u>first phase</u> we obtain

$$\eta \ll \rho_1 \gg \frac{D_1 \langle i_1 \rangle}{Dt} = -\frac{\partial}{\partial x} \eta \ll q_{1x} \gg + \frac{D_1}{Dt} \eta \ll P_1 \gg + \qquad (7.78a)$$

$$\eta \dot{\Phi}_1 - \left[\dot{m}_{1i} (i_{1i} - \langle i_1 \rangle) + \vec{q}_{1i} \cdot \hat{n}_{1i} - P_{1i} (\langle \vec{v}_1 \rangle - \vec{v}_i) \cdot \hat{n}_{1i} \right] \left[1 + \left(\frac{\partial \eta}{\partial x} \right)^2 \right]^{1/2} -$$

$$\left[\dot{m}_{1e} (i_{1e} - \langle i_1 \rangle) + \vec{q}_{1e} \cdot \hat{n}_{1e} - P_{1e} (\langle \vec{v}_1 \rangle - \vec{v}_e) \cdot \hat{n}_{1e} \right] -$$

$$H \frac{\partial}{\partial x} Cov(ent 1) - \eta Cov(\vec{v}_1 \cdot \nabla P_1)$$

whereas from Equation (7.77b) for the second phase we obtain

$$(H-\eta) \ll \rho_2 \gg \frac{D_2 \langle i_2 \rangle}{Dt} = -\frac{\partial}{\partial x} (H-\eta) \ll q_{2x} \gg + \frac{D_2}{Dt} (H-\eta) \ll P_2 \gg + (7.78b)$$

$$(H-\eta) \dot{\Phi}_2 - \left[\dot{m}_{2i} (i_{2i} - \langle i_2 \rangle) + \vec{q}_{2i} \cdot \hat{n}_{2i} - P_{2i} (\langle \vec{v}_2 \rangle - \vec{v}_e) \cdot \hat{n}_{2i} \right] \left[1 + \left(\frac{\partial \eta}{\partial x} \right)^2 \right]^{1/2} -$$

$$H \frac{\partial}{\partial x} Cov(ent 2) - (H-\eta) Cov(\vec{v}_2 \cdot \nabla P_2)$$

In view of the boundary layer approximations further simplifications are possible. Let us consider the last covariance term. For two dimensional flow it can be expressed as

$$Cov \left[\vec{v}_k \cdot (\nabla P_k) \right] = \ll v_{kx} \frac{\partial P_k}{\partial x} + v_{ky} \frac{\partial P_k}{\partial y} \gg - \qquad (7.79)$$

$$\langle v_{kx} \hat{\imath} + v_{ky} \hat{\jmath} \rangle \cdot \ll \frac{\partial P_k}{\partial x} \hat{\imath} + \frac{\partial P_k}{\partial y} \hat{\jmath} \gg$$

However, it was seen in Chapter 6 that

$$\frac{\partial P}{\partial y} \approx O(1) \qquad ; \quad \mathcal{V}_y \approx O(\varepsilon) \qquad (7.80a)$$

whereas

$$\frac{\partial P}{\partial x} \approx O(1) \qquad ; \quad \mathcal{V}_x \simeq O(1) \qquad (7.80b)$$

Consequently, to be consistent with the boundary layer analysis the terms having order of ε and higher can be neglected from Equation (7.79), and we obtain

$$Cov\left[\vec{\mathcal{V}}_k \cdot (\nabla P)\right] = \ll \mathcal{V}_{kx} \frac{\partial P_k}{\partial x} \gg - <\mathcal{V}_{kx}> \cdot \ll \frac{\partial P_k}{\partial x} \gg \qquad (7.81)$$

Furthermore, we note that in view of the hydraulic approximation, i.e., of Equation (6.30), we have:

$$\frac{\partial P_k}{\partial x} = \frac{\partial P_{ki}}{\partial x} - g_y \frac{\partial}{\partial x}\left(\ll \rho_k \gg \eta\right) = f(x,t) \qquad (7.82)$$

which is only function of x and t <u>not</u> of y. Therefore, from Equation (7.81)

$$Cov\left[\vec{\mathcal{V}}_k \cdot (\nabla P_k)\right] = 0 \qquad (7.83)$$

By expanding the material derivative of the pressure and using Equation (7.22) it can be shown that

$$\frac{D_i}{Dt}\eta \ll P_i \gg + \sum_{\ell=e,i} P_{i\ell}\left(<\vec{v}_i> - \vec{v}_\ell\right) \cdot \hat{n}_{i\ell} = \qquad (7.84a)$$

$$= \eta \frac{D_i \ll P_i \gg}{Dt} + \left(\ll P_i \gg - P_{ii}\right)\frac{D_i \eta}{Dt}$$

$$\frac{D_2}{Dt}(H-\eta)\ll P_2\gg + \sum_{\ell=e,i} P_{2\ell}(\langle \vec{V}_2\rangle - \vec{V}_\ell)\cdot \hat{n}_{2\ell} = \qquad (7.84b)$$

$$= (H-\eta)\frac{D_2\ll P_2\gg}{Dt} - \left(\ll P_2\gg - P_{2i}\right)\frac{D_2\eta}{Dt}$$

By using Equation (7.16) for $\dfrac{\partial q_x}{\partial x}$, Equation (7.83) for the covariance and Equation (7.84a) for the material derivative from Equation (7.78a) we obtain

$$\eta \ll \rho_1 \gg \frac{D_1\langle i_1\rangle}{Dt} = \eta \frac{D_1\ll P_1\gg}{Dt} + \eta \dot{\Phi}_{11R} - \qquad (7.85a)$$

$$\left[\dot{m}_{1i}\left(i_{1i}-\langle i_1\rangle\right) + q_{1i}\right] - \left[\dot{m}_{1e}\left(i_{1e}-\langle i_1\rangle\right) - q_{1e}\right] +$$

$$\left(\ll P_1\gg - P_{1i}\right)\frac{D_1\eta}{Dt} - H\frac{\partial}{\partial x} \text{ Cov } (e_1 t 1)$$

Similarly, we obtain from Equation (7.78a)

$$(H-\eta)\ll \rho_2 \gg \frac{D_2\langle i_2\rangle}{Dt} = (H-\eta)\frac{D_2\ll P_2\gg}{Dt} + (H-\eta)\dot{\Phi}_{21R} \qquad (7.85b)$$

$$\left[\dot{m}_{2i}\left(i_{2i}-\langle i_2\rangle\right) - q_{2i}\right] -$$

$$\left(\ll P_2\gg - P_{2i}\right)\frac{D_2\eta}{Dt} - H\frac{\partial}{\partial x} \text{ Cov } (e_1 t 2)$$

Equation (7.85a) for the first phase, and Equation (7.85b) for the second phase together with the interfacial balance condition and with the two constitutive equations for q_{1e} and either for q_{1i} and q_{2i} constitute the thermal formulation for a plane flow structure consistent with the boundary layer approximations.

For the annular flow similar equations can be obtained. However, in this case by Equation (6.42) pressure distribution becomes uniform in the r-direction. Using $\frac{\partial P}{\partial r} = 0$, and applying the boundary layer simplifications it can be shown that the enthalpy equations appropriate for the annular flow can be given by

$$(1-\alpha) \ll \rho_1 \gg \frac{D_1 \langle i_1 \rangle}{Dt} = (1-\alpha) \frac{D_1 P_1}{Dt} + (1-\alpha) \dot{\Phi}_{1IR} - \tag{7.86a}$$

$$\frac{4\sqrt{\alpha}}{D_e} \left[\dot{m}_{1i} \left(i_{1i} - \langle i_1 \rangle \right) - q_{1i} \right] - \frac{4}{D_e} \left[\dot{m}_{1e} \left(i_{1e} - \langle i_1 \rangle \right) - \right.$$

$$\left. q_{1e} \right] \left[1 + \left(\frac{d}{dx} \frac{D_e}{2} \right)^2 \right]^{1/2} - \frac{\partial}{\partial x} C_{ov} \text{ (ent 1)}$$

$$\alpha \ll \rho_2 \gg \frac{D_2 \langle i_2 \rangle}{Dt} = \alpha \frac{D_2 P_2}{Dt} + \alpha \dot{\Phi}_{2IR} - \tag{7.86b}$$

$$\frac{4\sqrt{\alpha}}{D_e} \left[\dot{m}_{2i} \left(i_{2i} - \langle i_2 \rangle \right) + q_{2i} \right] - \frac{\partial}{\partial x} C_{ov} \text{ (ent 2)}$$

These two equations describe thermal field of an annular flow. Of course, as in the case of the plane flow q_{1e} and either q_{1i} or q_{2i} should be given by a constitutive equation, and the interfacial balance given by Equation (7.70) should be also used.

As a conclusion it can be stated that a two-fluid formulation of the thermal field of a two-phase flow can be described two field and two additional constitutive equation.

B. Diffusion Model Formulation

To formulate the problem based on the diffusional model we need to consider the mixture enthalpy equation which can be obtained by

adding the individual phase equations. Consequently, by summing Equations (7.71a and b) and using the enthalpy interfacial balance, the mixture enthalpy equation can be given by

$$(1-\alpha) \langle\!\langle \rho_1 \rangle\!\rangle \frac{D_1 \langle i_1 \rangle}{Dt} + \alpha \langle\!\langle \rho_2 \rangle\!\rangle \frac{D_2 \langle i_2 \rangle}{Dt} + \Gamma_{2i}(\langle i_2 \rangle - \langle i_1 \rangle) = \quad (7.87)$$

$$= -\frac{\partial}{\partial x}\left[(1-\alpha) \langle\!\langle q_{1x} \rangle\!\rangle + \alpha \langle\!\langle q_{2x} \rangle\!\rangle \right] + \left\{ \frac{\partial}{\partial t}\left[(1-\alpha) \langle\!\langle P_1 \rangle\!\rangle + \alpha \langle\!\langle P_2 \rangle\!\rangle \right] + \right.$$

$$\left. \frac{\partial}{\partial x}\left[(1-\alpha) \langle v_{1x} \rangle \langle\!\langle P_1 \rangle\!\rangle + \alpha \langle v_{2x} \rangle \langle\!\langle P_2 \rangle\!\rangle \right] \right\} +$$

$$\left\{ \left[(1-\alpha) \dot{\Phi}_{1IR} + \alpha \dot{\Phi}_{2IR} \right] + \left[(1-\alpha) \dot{\Phi}_{1R} + \alpha \dot{\Phi}_{2R} \right] \right\} -$$

$$\left\{ \left[(1-\alpha) \langle\!\langle q_{1x} \rangle\!\rangle + \alpha \langle\!\langle q_{2x} \rangle\!\rangle \right] - \left[(1-\alpha) \langle v_{1x} \rangle \langle\!\langle P_1 \rangle\!\rangle + \alpha \langle v_{2x} \rangle \langle\!\langle P_2 \rangle\!\rangle \right] + \right.$$

$$\left[\text{Cov}(\text{ent }1) + \text{Cov}(\text{ent }2) \right] +$$

$$\left. \left[(1-\alpha)\, \text{Cov}(P_1 \cdot v_{1x}) + \alpha\, \text{Cov}(P_2 \cdot v_{2x}) \right] \right\} \frac{d \ln A_{Tc}}{dx} -$$

$$\frac{1}{A_{Tc}} \int_{\xi_i} (\text{I.E})_s \frac{dA}{dx} - \frac{1}{A_{Tc}} \sum_{k=1}^{2} \left[\dot{m}_{ke}(i_{ke} - \langle i_k \rangle) - \right.$$

$$P_{ke}(\vec{v}_{ke} - \vec{v}_e)\cdot \hat{n}_{ke} + \vec{q}_{ke}\cdot \hat{n}_{ke} \right] \frac{dA}{dx} -$$

$$\frac{\partial}{\partial x}\left[\text{Cov}(\text{ent }1) + \text{Cov}(\text{ent }2) \right] +$$

$$\frac{\partial}{\partial x}\left[(1-\alpha)\, \text{Cov}(P_1 \cdot v_{1x}) + \alpha\, \text{Cov}(P_2 \cdot v_{2x}) \right]$$

$$\frac{\partial}{\partial t} \varrho_m i_m + \frac{\partial}{\partial x} \varrho_m v_{mx} i_m = -\frac{\partial q_{mx}}{\partial x} + \frac{\partial P_m}{\partial t} + \frac{\partial}{\partial x} P_m v_{mx} + \quad (7.88)$$

$$\dot{\Phi}_{mR} + \dot{\Phi}_{mIR} - \frac{\partial}{\partial x} \left\{ \left[(1-\alpha) \ll \varrho_1 \gg \langle i_1 \rangle \, v_{1mx} + \alpha \ll \varrho_2 \gg \langle i_2 \rangle \, v_{2mx} \right] - \right.$$

$$\left[(1-\alpha) \ll P_1 \gg v_{1mx} + \alpha \ll P_2 \gg v_{2mx} \right] \right\} -$$

$$\left\{ \varrho_m v_{mx} i_m + \left[(1-\alpha) \ll \varrho_1 \gg \langle i_1 \rangle \, v_{1mx} + \alpha \ll \varrho_2 \gg \quad v_{2mx} \right] - \right.$$

$$P_m v_{mx} - \left[(1-\alpha) \ll P_1 \gg v_{1mx} + \alpha \ll P_2 \gg v_{2mx} \right] + Cov \, (e_m t \, T) -$$

$$\left[(1-\alpha) \, Cov \, (P_1 \cdot v_{1x}) + \alpha \, Cov \, (P_2 \cdot v_{2x}) \right] \right\} \frac{1}{A_{Tc}} \frac{d Q_m A_{Tc}}{dx} -$$

$$\frac{1}{A_{Tc}} \int_{\mathcal{J}_i} (I \cdot E)_s \, \frac{dA}{dx} -$$

$$\frac{1}{A_{Tc}} \sum_{k=1}^{2} \int_{\mathcal{J}_{ke}} \left[\dot{m}_{ke} \, i_{ke} - P_{ke} (\vec{v}_{ke} - \vec{v}_e) \cdot \hat{n}_{ke} + q_{ke} \cdot \hat{n}_{ke} \right] \frac{dA}{dx} -$$

$$\frac{\partial}{\partial x} Cov \, (e_m t \, T) + \frac{\partial}{\partial x} \left[(1-\alpha) Cov \, (P_1 \cdot v_{1x}) + \alpha \, Cov \, (P_2 \cdot v_{2x}) \right]$$

In Equation (7.88), i_m, q_{mx} and P_m are respectively, the mixture enthalpy, mixture conduction heat flux and the mixture pressure, and are given by Table 2, Equation (7.40) and by (6.54). $\dot{\Phi}_{mIR}$ is the total viscous dissipation per unit volume of the mixture and given by

$$\dot{\Phi}_{m\Sigma R} = (1-\alpha)\, \dot{\Phi}_{1\Sigma R} + \alpha\, \dot{\Phi}_{2\Sigma R} \qquad (7.89a)$$

whereas $\dot{\Phi}_{mR}$ is the reversible increase of the thermal energy and given by

$$\dot{\Phi}_{mR} = (1-\alpha)\, \dot{\Phi}_{1R} + \alpha\, \dot{\Phi}_{2R} \qquad (7.89b)$$

The fourth term on the right hand side is the enthalpy transport by diffusion with respect to the baricenter of the mixture. It can be also expressed as

$$(1-\alpha)\, \ll\rho_1\gg \langle i_1\rangle\, V_{1mx} + \alpha \ll\rho_2\gg \langle i_2\rangle\, V_{2mx} = \alpha \ll\rho_2\gg V_{2mx}\, (\langle i_2\rangle - \langle i_1\rangle) \qquad (7.90)$$

$$= \alpha(1-\alpha)\, \frac{\ll\rho_1\gg \ll\rho_2\gg}{\rho_m}\, (V_{2jx} - V_{1jx})\, (\langle i_2\rangle - \langle i_1\rangle)$$

$$= \alpha\, \frac{\ll\rho_1\gg \ll\rho_2\gg}{\rho_m}\, V_{2jx}\, (\langle i_2\rangle - \langle i_1\rangle)$$

the remaining terms in Equation (7.88) accounts for the diffusional effects of the pressure variations on the energy content of the system, the total area change, the enthalpy transport at the surfaces, and for the non-uniform distribution of the variables.

It is important to note further that in the absence of diffusion and of the interfacial surface effects the mixture enthalpy equation expressed by Equation (7.88) reduces to the one-dimensional single phase flow equation.

For the purpose of the practical application, this equation can be simplified for two-dimensional plane and annular flow regimes.

For the plane flow model, see Figure 15,

$$A_{T_C} = H \times \ell \quad ; \quad (1-\alpha) = \eta / H \quad ; \quad \hat{n}_{1e} = -\hat{j} \qquad (7.91)$$

$$\frac{d A_{T_C}}{d x} = 0 \quad ; \quad \alpha = \frac{H-\eta}{H} \quad ; \quad \hat{n}_{1i} = -\hat{n}_{2i} = \frac{\partial \eta}{\partial x} \hat{\imath} + \hat{j} \qquad (7.92)$$

Inserting Equations (7.91) and (7.92) in Equation (7.88) and using Equation (7.16), mixture enthalpy balance for the <u>plane</u> flow can be expressed as

$$\frac{\partial}{\partial t} \varrho_m i_m + \frac{\partial}{\partial x} \varrho_m v_{mx} i_m = \frac{\partial P_m}{\partial t} + \frac{\partial}{\partial x} v_{mx} P_m + \dot{\Phi}_{mR} + \dot{\Phi}_{m I \bar{e}} \qquad (7.93)$$

$$\frac{\partial}{\partial x} \left\{ \left[(1-\alpha) \ll \varrho_1 \gg \langle i_1 \rangle \, V_{1mx} + \alpha \ll \varrho_2 \gg \langle i_2 \rangle \, V_{2mx} \right] - \right.$$

$$\left[(1-\alpha) \ll P_1 \gg V_{1mx} + \alpha \ll P_2 \gg V_{2mx} \right] \right\} - \frac{1}{H} \, \sigma \cdot \vec{\tau}_i \cdot \frac{\partial \vec{v}_i}{\partial x} -$$

$$\frac{1}{H} \left[\dot{m}_{1e} \, i_{1e} - P_{1e} \, \vec{v}_{1e} \cdot \hat{n}_{1e} + \vec{q}_{1e} \cdot \hat{n}_{1e} \right] -$$

$$\frac{\partial}{\partial x} \, \text{Cov} \, (\varrho_m t T) - \frac{1}{H} \frac{\partial}{\partial x} \left[(1-\alpha) \, \text{Cov} \, (P_1 \cdot v_{1x}) + \alpha \, \text{Cov} \, (P_2 \cdot v_{2x}) \right]$$

If $P_K \simeq \ll P_K \gg$ then the last covariance terms drop, and the $\dot{\Phi}_{mR}$ term can be evaluated very easily from Equation (7.76) to obtain

$$\dot{\Phi}_{mR} = (1-\alpha) \dot{\Phi}_{1R} + \alpha \dot{\Phi}_{2R} = - P_1 \frac{\partial}{\partial x} (1-\alpha) \langle v_{1x} \rangle - P_2 \frac{\partial}{\partial x} \alpha \langle v_{2x} \rangle - \quad (7.94)$$

$$\sum_{\ell=e,i} \int_{\mathcal{F}_{1\ell}} (\vec{v}_{1\ell} \cdot \hat{n}_{1e}) \, P_1 \, \frac{dA}{dx}$$

Substituting Equation (7.94) in Equation (7.93) and dropping the last covariance terms, we obtain

$$\frac{\partial}{\partial t} \rho_m i_m + \frac{\partial}{\partial x} \rho_m v_{mx} i_m = \frac{\partial P_m}{\partial t} + (1-\alpha)\langle v_{1x}\rangle \frac{\partial P_1}{\partial x} + \alpha \langle v_{2x}\rangle \frac{\partial P_2}{\partial x} + \quad (7.95)$$

$$\dot{\Phi}_{mIR} - \frac{\partial}{\partial x} \left[(1-\alpha)\langle\!\langle \rho_1 \rangle\!\rangle V_{1mx} \langle i_1 \rangle + \alpha \langle\!\langle \rho_2 \rangle\!\rangle V_{2mx} \langle i_2 \rangle \right] -$$

$$\frac{1}{H} \sigma \,\hat{t}_i \cdot \frac{\partial \vec{v}_i}{\partial x} - \frac{1}{H} \left[\dot{m}_{1e} i_{1e} + \vec{q}_{1e} \cdot \hat{n}_{1=} \right] -$$

$$\frac{1}{H} \left[P_1 \,\vec{v}_{1i} \cdot \hat{n}_{1i} + P_2 \,\vec{v}_{2i} \cdot \hat{n}_{2i} \right] - \frac{\partial}{\partial x} \,\text{Cov}\,(\rho v_b T)$$

Furthermore if we assume $P_1 \simeq P_2 \simeq P_m$, then we obtain

$$\frac{\partial}{\partial t} \rho_m i_m + \frac{\partial}{\partial x} \rho_m v_{mx} i_m = \frac{D_m P_m}{Dt} + \dot{\Phi}_{mIR} - \qquad\qquad (7.96)$$

$$\frac{\partial}{\partial x} \left[(1-\alpha)\langle\!\langle \rho_1 \rangle\!\rangle V_{1mx} \langle i_1 \rangle + \alpha \langle\!\langle \rho_2 \rangle\!\rangle V_{2mx} \langle i_2 \rangle + \left[(1-\alpha) V_{1mx} + \alpha V_{2mx} \right] \frac{\partial P_m}{\partial x} - \right.$$

$$\frac{1}{H} \sigma \,\hat{t}_i \cdot \frac{\partial \vec{v}_i}{\partial x} - \frac{1}{H} \left(\dot{m}_{1e} i_{1e} + \vec{q}_{1e} \cdot \hat{n}_{1e} - \frac{1}{H} \,P_m (\vec{v}_{1i} \cdot \hat{n}_{1i} + \vec{v}_{2i} \cdot \hat{n}_{2i}) - \right.$$

$$\frac{\partial}{\partial x} \,\text{Cov}\,(\rho v_b T)$$

Using Table 2 for expressing the diffusion velocities V_{1mx} and V_{2mx} in terms of the vapor drift velocity V_{2jx}, and the mixture continuity equation for the left hand side we obtain from Equation (7.96)

$$\rho_{ei} \frac{D_m i_m}{Dt} = \frac{D_m P_m}{\partial t} + \dot{\Phi}_{mIR} - \frac{\partial}{\partial x} \left(\alpha \frac{\langle\langle \rho_i \rangle\rangle \langle\langle \rho_s \rangle\rangle}{\rho_m} V_{2jx} \Delta \langle i \rangle \right) - \quad (7.97)$$

$$\alpha \frac{\Delta \rho}{\rho_m} V_{2jx} \frac{\partial P_m}{\partial x} - \frac{1}{H} \sigma \hat{t}_i \cdot \frac{\partial \vec{v}_i}{\partial x} + \Gamma_{1e} (i_{1e} - i_m) -$$

$$\frac{1}{H} \vec{q}_{1e} \cdot \hat{n}_{1e} + P_m \Gamma_{2i} \frac{\Delta \rho_i}{\rho_{1i} \rho_{2i}} - \frac{\partial}{\partial x} Cov (\rho_m i_T)$$

where

$$\Delta \langle i \rangle = \langle i_2 \rangle - \langle i_1 \rangle \qquad (7.98)$$

Equation (7.97) is the plane flow mixture enthalpy equation derived under the condition of equal pressures, i.e., $P_1 \simeq P_2 \simeq P_m$.

Similarly for the annular flow the mixture enthalpy equation can be expressed by

$$\rho_m \frac{D_m i_m}{Dt} = \frac{D_m P_m}{Dt} + \dot{\Phi}_{mIR} - \frac{\partial}{\partial x} \left(\alpha \frac{\langle\langle \rho_i \rangle\rangle \langle\langle \rho_s \rangle\rangle}{\rho_m} V_{2jx} \Delta \langle i \rangle \right) - \quad (7.99)$$

$$\alpha \frac{\Delta \rho}{\rho_m} V_{2jx} \frac{\partial P_m}{\partial x} - \frac{4\sqrt{\alpha}}{De} \sigma \hat{t}_i \cdot \frac{\partial \vec{v}_i}{\partial x} + \Gamma_{1e} (i_{1e} - i_m) -$$

$$\frac{4}{De} \vec{q}_{1e} \cdot \hat{n}_{1e} + P_m \Gamma_{2i} \frac{\Delta \rho_i}{\rho_i \rho_{2i}} - \frac{\partial}{\partial x} Cov (\rho_m i_T)$$

We note that in order to use the enthalpy equation, it is necessary to know the power input q_{1e}. It is the only constitutive equation required for the formulation based on the diffusion model.

7.4.3 Internal Energy Equations

For the analysis of the incompressible fluids, the specific internal energy is often a more convenient variable to use than the specific enthalpy, since it involves the temperature directly.

We will now derive the forms of the internal energy balance equations required for the both two-fluid as well as the diffusion model formulations.

A. Two-Fluid Model Formulation

No new calculation is necessary; it is sufficient to merely transpose the results obtained for the enthalpy equations. By using the thermodynamic relation expressed by Equation (7.5) in Equation (7.71a and b) the <u>mass-weighted are-averaged internal energy equations</u> for each phase can be expressed as:

$$\frac{\partial}{\partial t} (1-\alpha) \langle\!\langle \rho_1 \rangle\!\rangle \langle u_1 \rangle + \frac{\partial}{\partial x} (1-\alpha) \langle\!\langle \rho_1 \rangle\!\rangle \langle v_{1x} \rangle \langle u_1 \rangle = \qquad (7.100a)$$

$$= -\frac{\partial}{\partial x} (1-\alpha) \langle\!\langle q_{1x} \rangle\!\rangle + (1-\alpha) (\dot{\bar{\Phi}}_{1R} + \dot{\bar{\Phi}}_{1IR}) -$$

$$\left[(1-\alpha) \langle\!\langle \rho_1 \rangle\!\rangle \langle v_{1x} \rangle \langle u_1 \rangle + (1-\alpha) \langle\!\langle q_{1x} \rangle\!\rangle + Cov (int 1) \right] \frac{d \ln A_{T_c}}{dx}$$

$$\frac{1}{A_{T_c}} \sum_{\ell=e,i} \int_{\mathfrak{I}_{i\ell}} \left[\dot{m}_{1\ell} u_{1\ell} + \vec{q}_{1\ell} \cdot \hat{n}_{1\ell} \right] \frac{dA}{dx} - \frac{\partial}{\partial x} Cov (int 1)$$

$$\frac{\partial}{\partial t} \propto \langle\!\langle \rho_2 \rangle\!\rangle \langle u_2 \rangle + \frac{\partial}{\partial x} \propto \langle\!\langle \rho_2 \rangle\!\rangle \langle v_{2x} \rangle \langle u_2 \rangle = \tag{7.100b}$$

$$= -\frac{\partial}{\partial x} \propto \langle\!\langle q_{2x} \rangle\!\rangle + \propto (\dot{\Phi}_{2R} + \dot{\Phi}_{2IR}) -$$

$$\left[\propto \langle\!\langle \rho_2 \rangle\!\rangle \langle v_{2x} \rangle \langle u_2 \rangle + \propto \langle\!\langle q_{2x} \rangle\!\rangle + Cov\,(int\,2) \right] \frac{d\ln A_{T_c}}{dx} -$$

$$\frac{1}{A_{T_c}} \sum_{\ell=e,i} \int_{S_{2\ell}} \left[\dot{m}_{2\ell}\, u_{2\ell} + \vec{q}_{2\ell} \cdot \hat{n}_{2\ell} \right] \frac{dA}{dx} - \frac{\partial}{\partial x} Cov\,(int\,2)$$

where Cov(int K) is given by Equation (7.4b).

It is important to note that, if we neglect the effects of compressibility

$$\dot{\Phi}_{2R} = \dot{\Phi}_{1R} = 0 \tag{7.101}$$

It is obvious that in this case it is more convenient to use the internal energy equation than the enthalpy equation. In fact, as in the case of the single phase flow thermal field can be treated independently from the dynamic flow field.

However, in general we can use Equation (7.76) for $\dot{\Phi}_{KR}$ in Equation (7.100b), and expressing the left hand side in terms of the material derivative for phase 1 we obtain

$$(1-\alpha) \langle\!\langle \rho_1 \rangle\!\rangle \frac{D_1 \langle u_1 \rangle}{Dt} = -\frac{\partial}{\partial x} \langle\!\langle q_{1x} \rangle\!\rangle - \langle\!\langle P_1 \rangle\!\rangle \frac{\partial}{\partial x} (1-\alpha) \langle v_{1x} \rangle + \tag{7.102a}$$

$$(1-\alpha)\, \dot{\Phi}_{1R} - \left[(1-\alpha) \langle\!\langle q_{1x} \rangle\!\rangle + (1-\alpha) \langle\!\langle P_1 \rangle\!\rangle \langle v_{1x} \rangle + Cov\,(int\,1) \right] \frac{d\ln A_{T_c}}{dx} -$$

$$\frac{1}{A_{T_c}} \sum_{\ell=e,i} \int_{S_{1\ell}} \left[\dot{m}_{1\ell}\,(u_{1\ell} - \langle u_1 \rangle) + \vec{q}_{1\ell} \cdot \hat{n}_{1\ell} + \langle\!\langle P_1 \rangle\!\rangle \vec{v}_{1\ell} \cdot \hat{n}_{1\ell} \right] \frac{dA}{dx} -$$

$$\frac{\partial}{\partial x} Cov\,(int\,1) - (1-\alpha)\, Cov\left[P_1 \cdot (\nabla \cdot \vec{v}_1) \right]$$

whereas for phase 2 we obtain

$$\propto \ll \rho_2 \gg \frac{D_2 \langle u_2 \rangle}{Dt} = -\frac{\partial}{\partial x} \propto \ll q_{2x} \gg - \ll P_2 \gg \frac{\partial}{\partial x} \propto \langle v_{2x} \rangle + \tag{7.102b}$$

$$\propto \dot{\Phi}_{2IR} - \left[\propto \ll q_{2x} \gg + \propto \ll P_2 \gg \langle v_{2x} \rangle + Cov\ (\dot{m} + 2) \right] \frac{d \ell_n A_{Tc}}{dx} -$$

$$\frac{1}{A_{Tc}} \sum_{\ell = e,i} \int_{S_{2\ell}} \left[\dot{m}_{2\ell} (u_{2\ell} - \langle u_2 \rangle) + \vec{q}_{2\ell} \cdot \hat{n}_{2\ell} + \ll P_2 \gg \vec{v}_{2\ell} \cdot \hat{n}_{2\ell} \right] \frac{dA}{dx} -$$

$$\frac{\partial}{\partial x} Cov\ (int\ 2) - \propto Cov \left[P_2 \cdot (\nabla \cdot \vec{v}_2) \right]$$

These equations can be used to derive the energy equations applicable to the separated flow of compressible fluids for the plane and annular flow structures.

In view of the Prandtl's type of approximations it can be shown that the plane flow internal energy equations can be expressed as

$$\ll \rho_1 \gg \eta \frac{D_1 \langle u_1 \rangle}{Dt} = - \ll P_1 \gg \frac{\partial}{\partial x} \eta \langle v_{1x} \rangle + \dot{\Phi}_{1IR} - \tag{7.103a}$$

$$\sum_{\ell = e,i} \left[\dot{m}_{1\ell} (u_{1\ell} - \langle u_1 \rangle) + q_{1i} + P_{1i} \vec{v}_{1\ell} \cdot \hat{n}_{1\ell} \right] + \frac{1}{2} \ll \rho_1 \gg g_y \eta (\vec{v}_{1i} \cdot \hat{n}_{1i} - \vec{v}_{1e} \cdot \hat{n}_{1e}) -$$

$$H \frac{\partial}{\partial x} Cov\ (int\ 1) - \eta\ Cov \left[P_1 \cdot (\nabla \cdot \vec{v}_1) \right]$$

$$\ll \rho_2 \gg (H-\eta) \frac{D_2 \langle u_2 \rangle}{Dt} = - \ll P_2 \gg \frac{\partial}{\partial x} (H-\eta) \langle v_{2x} \rangle + \dot{\Phi}_{2IR} -$$

$$\left[\dot{m}_{2i} (u_{2i} - \langle u_2 \rangle) - q_{2i} + P_{2i} \vec{v}_{2i} \cdot \hat{n}_{2i} \right] - \frac{1}{2} (H-\eta) g_y \ll \rho_2 \gg \vec{v}_{2i} \cdot \hat{n}_{2i} -$$

$$\langle\!\langle \rho_2 \rangle\!\rangle (H-\eta) \frac{D_2 \langle u_2 \rangle}{Dt} = -\langle\!\langle P_2 \rangle\!\rangle \frac{\partial}{\partial x} (H-\eta) \langle v_{2x} \rangle + \dot{\Phi}_{2IR} - \tag{7.103b}$$

$$\left[\dot{m}_{2i} (u_{2i} - \langle u_2 \rangle) - q_{2i} + P_{2i} \, \vec{v}_{2i} \cdot \hat{n}_{2i} \right] - \frac{1}{2} \langle\!\langle \rho_2 \rangle\!\rangle (H-\eta) g_y \, \vec{v}_{2i} \cdot \hat{n}_{2i} -$$

$$H \frac{\partial}{\partial x} \, Cov \, (int \, i) - (H-\eta) \quad Cov \left[P_2 \cdot (\nabla \cdot \vec{v}_2) \right]$$

Equations (7.103a and b) are the internal energy equations for two-phase systems with a change of phase. They are of the same form as the equations of the single phase flow. For the complete thermal field description these two field equations must be supplemented by two constitutive equations - one for the external surface heating and the other for the internal surfaces. In relation to the internal surface constitutive equation, if we assume vapor is in _itself_ in thermodynamic equilibrium, i.e., if the vapor follows through the saturation line, then q_{2i} can be taken zero, which specifies one of the constitutive equations.

Similar equations can be obtained for thecase of _annular flow_ by using the two-dimensional flow approximations. Consequently, we obtain for the annular flow

$$(1-\alpha) \langle\!\langle \rho_1 \rangle\!\rangle \frac{D_1 \langle u_1 \rangle}{Dt} = -P_1 \frac{\partial}{\partial x} (1-\alpha) \langle v_{1x} \rangle + \dot{\Phi}_{1IR} - \tag{7.104a}$$

$$\left[(1-\alpha) P_1 \langle v_{1x} \rangle + Cov \, (int \, i) \right] \frac{2}{De} \frac{dDe}{dx} -$$

$$\frac{4\sqrt{\alpha}}{De} \left[\dot{m}_{1i} (u_{1i} - \langle u_1 \rangle) - q_{1i} + P_{1i} \, \vec{v}_{1i} \cdot \hat{n}_{1i} \right] -$$

$$\frac{4}{De} \left[\dot{m}_{1e} (u_{1e} - \langle u_1 \rangle) + q_{1e} + P_{1e} \, \vec{v}_{1e} \cdot \hat{n}_{1e} \right] - \frac{\partial}{\partial x} \, Cov \, (int \, i)$$

$$\alpha \ll \rho_2 \gg \frac{D_2 \langle u_2 \rangle}{Dt} = -P_2 \frac{\partial}{\partial x} \alpha \langle v_{2x} \rangle + \dot{\Phi}_{2 \Sigma R} - \qquad (7.104b)$$

$$\left[\alpha P_2 \langle v_{2x} \rangle + Cov\ (int\ 2) \right] \frac{2}{De} \frac{d De}{dx} -$$

$$\frac{4\sqrt{\pi}}{De} \left[\dot{m}_{2i}\ u_{2i} + \vec{q}_{2i} + P_{2i}\ \vec{v}_{2i} \cdot \hat{n}_{2i} \right] - \frac{\partial}{\partial x} Cov\ (int\ 2)$$

The same comments concerning the additional constitutive equations which made with respect to the plane flow case, apply also for the annular flow case.

B. Diffusion Model Formulation

As it is usual, the diffusion model formulation is based on the mixture equation. To obtain it, we can use

$$i_m = u_m - \frac{P_m}{\rho_m} \qquad (7.105)$$

in the mixture enthalpy equation, Equation (7.88), thus

$$\rho_m \frac{D_m u_m}{Dt} = -\frac{\partial q_{mx}}{\partial x} + \dot{\Phi}_{mR} + \dot{\Phi}_{m\Sigma R} - \qquad (7.106)$$

$$\frac{\partial}{\partial x} \left[\alpha \frac{\ll \rho_1 \gg \ll \rho_2 \gg}{\rho_m} V_{2jx} \Delta \langle u \rangle \right] - \left[\alpha \frac{\ll \rho_1 \gg \ll \rho_2 \gg}{\rho_m} V_{2jx} \Delta \langle u \rangle + \right.$$

$$\left. Cov\ (int\ T) \right] \frac{d \ln A_c}{dx} - \frac{1}{A_c} \int_{\Gamma_i} (I.E)_c \frac{dA}{dx} -$$

$$\frac{1}{A_c} \sum_{k=1}^{2} \int_{\Sigma_{ke}} \left[\dot{m}_{ke}\ u_{ke} + \vec{q}_{ke} \cdot \hat{n}_{ke} \right] \frac{dA}{dx} - \frac{\partial}{\partial x} Cov\ (int\ T)$$

The significance of the various terms appearing in Equation (7.106) are as follows. The term on the left side accounts for the rate of

change of the mixture internal energy as following the center of mass of the mixture. The first term on the right hand side accounts for the effects of the axial heat flux, and the following two terms account, respectively, for the reversible and irreversible rate of increase of the thermal energy. The fourth term represents the internal energy transfer by diffusion, remaining terms account, respectively, the effects of the change of area, transport of the thermal energy at the internal and external boundaries, and finally, effects of the non-uniform distribution of velocity and internal energy.

Simplification of the mixture equation can be achieved in a way similar to that used in connection with the enthalpy equation, therefore, we shall not go through the derivations here.

7.4.4 Mechanical Energy Equations

A. Two-Fluid Model Formulation

Mechanical energy equations appropriate to the two-fluid model formulation can be obtained either by using the general balance equations derived in Chapter 4 in conjunction with the identifications for ψ, $\dot{\psi}$ and $\vec{\phi}$ given in Table 4 or by substracting the averaged internal energy equation from the total energy equation. Although both methods give the same result, we shall use the former method.

From Table 4 it can be seen that

$$\psi = \frac{v^2}{2} \tag{7.107}$$

$$\dot{\psi} = - \left(\overline{\overline{\pi}} : \nabla \vec{v} \right) + \vec{g} \cdot \vec{v}$$

$$\vec{\phi} = \overline{\overline{\pi}} \cdot \vec{v}$$

Using these identifications for τ, $\dot{\Upsilon}$ and $\vec{\Phi}$ in the mass-weighed area-averaged general balance equation, Equation (4.20a and b), averaged mechanical energy equations can be expressed for phase 1 as follows:

$$\frac{\partial}{\partial t} \, (1-\alpha) \, \ll \rho_1 \gg \, \frac{<v_1>^2}{2} + \frac{\partial}{\partial x} \, (1-\alpha) \, \ll \rho_1 \gg \, <v_{1x}> \, \frac{<v_1>^2}{2} = \qquad (7.108a)$$

$$= \frac{\partial}{\partial x} \Big[(1-\alpha) \, \ll \bar{\bar{\pi}}_1 \cdot \hat{\imath} \gg \cdot <\vec{v}_1> \Big] - (1-\alpha) \, \ll \bar{\bar{\pi}}_1 : \nabla \vec{v}_1 \gg + (1-\alpha) \ll \rho_1 \gg \vec{g} \cdot <\vec{v}_1> -$$

$$\Big[(1-\alpha) \, \ll \rho_1 \gg <v_{1x}> \, \frac{<v_1>^2}{2} - (1-\alpha) \, \ll \bar{\bar{\pi}}_1 \cdot \hat{\imath} \gg \cdot <\vec{v}_1> + \text{Cov (conv 1)} -$$

$$\text{Cov (work 1)} \Big] \frac{d \ell n \, A T_c}{dx} - \frac{1}{A T_c} \sum_{\ell=e,i} \int_{S_{1\ell}} \Big[\dot{m}_{1\ell} \, \frac{v_{1\ell}^2}{2} - (\bar{\bar{\pi}}_{1\ell} \cdot \vec{v}_{1\ell}) \cdot \hat{n}_{1\ell} \Big] \frac{dA}{dx} -$$

$$\frac{\partial}{\partial t} \, \text{Cov (mech 1)} - \frac{\partial}{\partial x} \, \text{Cov (conv 1)} + \frac{\partial}{\partial x} \, \text{Cov (work 1)}$$

whereas for the second phase we obtain

$$\frac{\partial}{\partial t} \, \alpha \ll \rho_2 \gg \, \frac{<v_2>^2}{2} + \frac{\partial}{\partial x} \, \alpha \ll \rho_2 \gg \, <v_{2x}> \, \frac{<v_2>^2}{2} = \qquad (7.108b)$$

$$= \frac{\partial}{\partial x} \Big(\alpha \ll \bar{\bar{\pi}}_2 \cdot \hat{\imath} \gg \cdot <\vec{v}_2> \Big) - \alpha \ll \bar{\bar{\pi}}_2 : \nabla \vec{v}_2 \gg + \alpha \ll \rho_2 \gg \vec{g} \cdot <\vec{v}_2> -$$

$$\Big[\alpha \ll \rho_2 \gg <v_{2x}> \, \frac{<v_2>^2}{2} - \alpha \ll \bar{\bar{\pi}}_2 \cdot \hat{\imath} \gg \cdot <\vec{v}_2> + \text{Cov (conv 2)} -$$

$$\text{Cov (work 2)} \Big] \frac{d \ell n \, A T_c}{dx} - \frac{1}{A T_c} \sum_{\ell=e,i} \int_{S_{2\ell}} \Big[\dot{m}_{2\ell} \, \frac{v_{2\ell}^2}{2} - (\bar{\bar{\pi}}_{2\ell} \cdot \vec{v}_{2\ell}) \cdot \hat{n}_{2\ell} \Big] \frac{dA}{dx} -$$

$$\frac{\partial}{\partial t} \, \text{Cov (mech 2)} - \frac{\partial}{\partial x} \, \text{Cov (conv 2)} + \frac{\partial}{\partial x} \, \text{Cov (work 2)}$$

where Cov(mech 1), Cov(conv 1), and Cov(work 1) are defined by Equations (7.3a), (7.4a) and (7.4c), respectively. Corresponding second phase covariance terms can be obtained by using the definitions given for the first phase by replacing $(1 - \alpha)$ by α and the subscript 1 by 2.

Equations (7.108a and b) together with the interfacial balance relation, Equation (7.66) formulate the mechanical energy equation based on the two-fluid model.

It is important to note that for the case of two-dimensional flow if the boundary layer analysis is used, the first term on the right hand side reduces to

$$\ll \bar{\bar{\pi}}_{k} \cdot \vec{t} \gg \cdot \langle \vec{v}_{k} \rangle = \ll P_{k} \gg \langle v_{k_{x}} \rangle \qquad (7.109)$$

whereas the second term becomes

$$\ll \bar{\bar{\pi}}_{1} : \nabla \vec{v}_{1} \gg = \dot{\Phi}_{1R} + \dot{\Phi}_{1IR} \qquad (7.110)$$

and as a result of (7.109) Cov(work K) term becomes

$$Cov \,(work \, 1) = (1-\alpha) \, Cov \, (P_{1} \cdot v_{1x}) \qquad (7.111a)$$

$$Cov \,(work \, 2) = \alpha \, Cov \, (P_{2} \cdot v_{2x}) \qquad (7.111b)$$

Finally, from the order of magnitude analysis on Equation (7.108) the gravitational work term becomes

$$\ll P_{k} \gg \vec{g}_{k} \cdot \langle \vec{v}_{k} \rangle = \ll P_{k} \gg g_{x} \langle v_{k_{x}} \rangle \qquad (7.112)$$

Substituting Equations (7.109) through (7.112) in Equation (7.108)

and expressing the left hand side in terms of the material derivative we obtain for two-dimensional flow

$$(1-\alpha) \ll \rho_1 \gg \frac{D_1}{Dt} \frac{\langle v_1 \rangle^2}{2} = -\frac{\partial}{\partial x} (1-\alpha) \ll P_1 \gg \langle v_{1x} \rangle + \tag{7.113a}$$

$$(1-\alpha)(\overline{\dot{\Phi}}_{1R} + \overline{\dot{\Phi}}_{1IR}) + (1-\alpha) \ll \rho_1 \gg g_x \langle v_{1x} \rangle - \Big[(1-\alpha) \ll P_1 \gg \langle v_{1x} \rangle +$$

$$\text{Cov (conv 1)} - \text{Cov (work 1)} \Big] \frac{d \ell_n A_{T_c}}{dx} -$$

$$\sum_{\ell = e, i} \frac{\mathfrak{z}_{1\ell}}{A_{T_c}} \Big[\dot{m}_{1\ell} \frac{v_{1\ell}^2}{2} - (\overline{\overline{\pi}}_{1\ell} \cdot \vec{v}_{1\ell}) \cdot \hat{n}_{1\ell} \Big] -$$

$$\frac{\partial}{\partial t} \text{Cov (mech 1)} - \frac{\partial}{\partial x} \text{Cov (conv 1)} + \frac{\partial}{\partial x} \text{Cov (work 1)}$$

and

$$\alpha \ll \rho_2 \gg \frac{D_2}{Dt} \frac{\langle v_2 \rangle^2}{2} = -\frac{\partial}{\partial x} \alpha \ll P_2 \gg \langle v_{2x} \rangle + \tag{7.113b}$$

$$\alpha(\overline{\dot{\Phi}}_{2R} + \overline{\dot{\Phi}}_{2IR}) + \alpha \ll \rho_2 \gg g_x \langle v_{2x} \rangle - \Big[\alpha \ll P_2 \gg \langle v_{2x} \rangle +$$

$$\text{Cov (conv 2)} - \text{Cov (work 2)} \Big] \frac{d \ell_n A_{T_c}}{dx} -$$

$$\sum_{\ell = e, i} \frac{\mathfrak{z}_{2\ell}}{A_{T_c}} \Big[\dot{m}_{2\ell} \frac{v_{2\ell}^2}{2} - (\overline{\overline{\pi}}_{2\ell} \cdot \vec{v}_{2\ell}) \cdot \hat{n}_{2\ell} \Big] -$$

$$\frac{\partial}{\partial t} \text{Cov (mech 2)} - \frac{\partial}{\partial x} \text{Cov (conv 2)} + \frac{\partial}{\partial x} \text{Cov (work 2)}$$

where $\mathfrak{z}_{K\ell}$ represents the perimeter of the K-th phase on the ℓ-th surface, ($\ell = e, i$).

Further, simplifications on the each phase mechanical energy equations are possible for the particular geometries. For example, if the cross-sectional area is constant then $\dfrac{dA_{Tc}}{dx} = 0$. If the fluids are incompressible, $\dot{\Phi}_{KR} = 0$. If any of the phase is not contact with the solid wall, as in the case of the annular flow, one of the terms in the summation sign disappear, so on.

For the simplifications for the particular flow structure one can use the geometrical relations derived in Appendix C. However, we shall not go through these derivations.

B. Diffusion Model Formulation

Summing up the individual mechanical energy equations, Equations (7.108a and b), and using the fundamental identity Equation (3.62) together with the interfacial mechanical energy balance relation, Equation (7.66), we obtain for the mixture mechanical energy equation

$$\frac{\partial}{\partial t}\left[\varrho_m\left(\frac{\langle v_1\rangle^2}{2}+\frac{\langle v_2\rangle^2}{2}\right)\right]+\frac{\partial}{\partial x}\left[\varrho_m v_{mx}\left(\frac{\langle v_1\rangle^2}{2}+\frac{\langle v_2\rangle^2}{2}\right)\right]= \quad (7.114)$$

$$\frac{\partial}{\partial x}\left[(\overline{\overline{\pi}}_m\cdot\hat{\imath})\cdot\vec{v}_m\right]-(\vec{\Phi}_{mR}+\vec{\Phi}_{mIR})+\varrho_m\,\vec{g}\cdot\vec{v}_m-$$

$$\frac{\partial}{\partial x}\left\{\left[(1-\alpha)\langle\!\langle\varrho_1\rangle\!\rangle V_{1mx}\frac{\langle v_1\rangle^2}{2}+\alpha\langle\!\langle\varrho_2\rangle\!\rangle V_{2mx}\frac{\langle v_2\rangle^2}{2}\right]-\right.$$

$$\left[(1-\alpha)\,\vec{V}_{1m}\cdot\langle\!\langle\overline{\overline{\pi}}_1\cdot\hat{\imath}\rangle\!\rangle+\alpha\,\vec{V}_{2m}\cdot\langle\!\langle\overline{\overline{\pi}}_2\cdot\hat{\imath}\rangle\!\rangle\right]\right\}-$$

$$\left\{\varrho_m v_{mx}\left(\frac{\langle v_1\rangle^2}{2}+\frac{\langle v_2\rangle^2}{2}\right)-(\overline{\overline{\pi}}_m\cdot\hat{\imath})\cdot\vec{v}_m+\right.$$

$$\left[(1-\alpha)\langle\!\langle\varrho_1\rangle\!\rangle V_{1mx}\frac{\langle v_1\rangle^2}{2}+\alpha\langle\!\langle\varrho_2\rangle\!\rangle V_{2mx}\frac{\langle v_2\rangle^2}{2}\right]-$$

$$\left[(1-\alpha)\,\vec{V}_{1m}\cdot\langle\!\langle\overline{\overline{\pi}}_1\cdot\hat{\imath}\rangle\!\rangle+\alpha\,\vec{V}_{2m}\cdot\langle\!\langle\overline{\overline{\pi}}_2\cdot\hat{\imath}\rangle\!\rangle\right]+$$

$$\left[(1-\alpha)\langle\!\langle\varrho_1\rangle\!\rangle Cov\left(V_{1x}\cdot\frac{v_1^2}{2}\right)+\alpha\langle\!\langle\varrho_2\rangle\!\rangle Cov\left(V_{2x}\cdot\frac{v_2^2}{2}\right)\right]-$$

$$\left.Cov(work\,T)\right\}\frac{d\ln A_{Tc}}{dx}-\frac{1}{A_{Tc}}\int_{J_i}(M.E)_s\frac{dA}{dx}-$$

$$\frac{1}{A_{Tc}}\sum_{k=1}^{2}\int_{S_{ke}}\left[\dot m_{ke}\frac{v_{ke}^2}{2}-(\overline{\overline{\pi}}_{ke}\cdot\vec{v}_{ke})\cdot\hat n_{ke}\right]\frac{dA}{dx}-\frac{\partial}{\partial t}Cov(mech\,T)-$$

$$\frac{\partial}{\partial x}\left[(1-\alpha)\langle\varrho_1\rangle Cov\left(V_{1x}\cdot\frac{v_1^2}{2}\right)+\alpha\langle\varrho_2\rangle Cov\left(V_{2x}\frac{v_2^2}{2}\right)\right]+\frac{\partial}{\partial x}Cov(work\,T)$$

where $(ME)_s$ is the surface contribution to the mechanical energy balance and expressed by Equation (7.65b).

Although this equation expresses the mechanical energy balance for the mixture it is not consistent with the diffusion model in which the independent velocity fields are the mixture velocity and one of the diffusion velocities or one of the drift velocities. In order to express Equation (7.114) in terms of the diffusion velocities we apply the fundamental identity once more and use the relations tabulated in

Table 2. As a result we obtain the following mixture mechanical energy equation expressed with respect to the center of mass of the mixture

$$\frac{\partial}{\partial t}\left[\rho_m \frac{v_m^2}{2} + (1-\alpha) \ll\rho_1\gg \frac{V_{1m}^2}{2} + \alpha \ll\rho_2\gg \frac{V_{2m}^2}{2}\right] + \tag{7.115}$$

$$\frac{\partial}{\partial x}\left\{ V_{mx}\left[\rho_m \frac{v_m^2}{2} + (1-\alpha) \ll\rho_1\gg \frac{V_{1m}^2}{2} + \alpha \ll\rho_2\gg \frac{V_{2m}^2}{2}\right]\right\} =$$

$$= \frac{\partial}{\partial x}\left[(\bar{\bar{\pi}}_m^T \cdot \hat{\iota}) \cdot \vec{V}_m \right] - \left[(1-\alpha) \ll\bar{\bar{\pi}}_1 : \nabla\vec{V}_1\gg + \alpha \ll\bar{\bar{\pi}}_2 : \nabla\vec{V}_2\gg\right] +$$

$$\rho_m \vec{g} \cdot \vec{V}_m - \frac{\partial}{\partial x}\left\{ (1-\alpha)\left[-\ll\bar{\bar{\pi}}_1 \cdot \hat{\iota}\gg \cdot \vec{V}_{1m} + \ll\rho_1\gg \frac{V_{1m}^2}{2} V_{1mx}\right] +\right.$$

$$\alpha\left[-\ll\bar{\bar{\pi}}_2 \cdot \hat{\iota}\gg \cdot \vec{V}_{2m} + \ll\rho_2\gg \frac{V_{2m}^2}{2} V_{2mx}\right]\Big\} -$$

$$\left\{ V_{mx}\left[\rho_m \frac{v_m^2}{2} + (1-\alpha) \ll\rho_1\gg \frac{V_{1m}^2}{2} + \alpha\ll\rho_2\gg \frac{V_{2m}^2}{2}\right] - (\bar{\bar{\pi}}_m^T \cdot \hat{\iota}) \cdot \vec{V}_m +\right.$$

$$(1-\alpha)\left[-\ll\bar{\bar{\pi}}_1\cdot\hat{\iota}\gg \cdot \vec{V}_{1m} + \ll\rho_1\gg \frac{V_{1m}^2}{2} V_{1mx}\right] +$$

$$\alpha\left[-\ll\bar{\bar{\pi}}_2\cdot\hat{\iota}\gg \cdot \vec{V}_{2m} + \ll\rho_2\gg \frac{V_{2m}^2}{2} V_{2mx}\right] + Cov(conv\ T) -$$

$$Cov(work\ T)\Big\} \frac{d\ln A_c}{dx} - \frac{1}{A_{Tc}} \int_{\dot{s}_i} (M \cdot \bar{\sigma})_s \frac{dA}{dx} -$$

$$\frac{1}{A_{Tc}} \sum_{k=1}^{2} \int_{\dot{s}_{ke}}\left[\dot{m}_{ke} \frac{v_{ke}^2}{2} - (\bar{\bar{\pi}}_{ke} \cdot \vec{v}_{ke}) \cdot \hat{n}_{ke}\right] \frac{dA}{dx} - \frac{\partial}{\partial t} Cov(mech\ T) -$$

$$\frac{\partial}{\partial x} Cov(conv\ T) + \frac{\partial}{\partial x} Cov(work\ T)$$

where $\bar{\bar{\pi}}_m^T$ is the total stress tensor including the drift stresses. Expression for it is given by Equation (7.44).

The terms appearing in Equation (7.115) are easy to interpret. The first term on the left hand side accounts for the mechanical energy changes of the center of mass whereas the second term accounts for the effect of the drift between the phases which arises as a result of expressing the mixture quantities with respect to the center of gravity

of the mixture. The first term on the right hand side accounts for the total work done by viscous as well as drift stresses whereas the second term represents the part of the mechanical energy dissipated into internal energy. Third term is the work done by the gravitational forces. Fourth accounts for the diffusional transport of the energy. Fifth term shows the effects of the total area change on the mixture mechanical energy balance. The seventh and the eighth term represents, respectively, effects of the mechanical energy transfer at the internal and the external surfaces. Finally, covariance terms represent the effect of the flow regimes, i.e., effect of the velocity profiles in the each phase.

Simple cases of the mixture mechanical energy equation can be obtained by using the geometric relations derived in Appendices B and C. However, for the purpose of brevity we shall not go through the derivations here.

7.5 Scaling Criteria

As in Chapter 5 and Chapter 6 where the dimensionless groups necessary for the kinematic and dynamic similarity of the system were derived, in this section we shall derive the similarity groups arising from the thermal field of a separated two-phase flow mixture. However, for the simplicity plane flow and annular flow enthalpy equations will be brought into a single equation so that the general similarity groups can be derived from it.

It can be seen from Equations (7.96) and (7.99), the mixture enthalpy equation for a constant area duct can be expressed as

$$\rho_m \frac{\partial i_m}{\partial t} + \rho_m v_{mx} \frac{\partial i_m}{\partial x} = \frac{\partial P_m}{\partial t} + v_{mx} \frac{\partial P_m}{\partial x} + \dot{\Phi}_{mIR} - \qquad (7.116)$$

$$\frac{\partial}{\partial x}\left(\alpha \frac{\langle\langle\rho_1\rangle\rangle \langle\langle\rho_2\rangle\rangle}{\rho_m} V_{2jx} \Delta\langle i\rangle \right) + \alpha \frac{\Delta\langle\langle\rho\rangle\rangle}{\rho_m} \frac{\partial P_m}{\partial x} + \frac{f_{1e}}{A T_c} q_{1e} -$$

$$\frac{f_i}{A T_c} \sigma \cdot \hat{t}_i \cdot \frac{\partial \vec{v}_i}{\partial x} + \Gamma_{2i} P_m \frac{\Delta \rho_i}{\rho_{1i} \rho_{2i}} - \frac{\partial}{\partial x} \text{Cov}(\text{ent } T)$$

where we have used $P_1 \simeq P_2 \simeq P_m$. It means we have neglected the effects
of pressure jump across the interface on the mixture enthalpy balance.

Defining additional dimensionless parameter $i_m^+ = i_m/i_{mo}$, and
introducing it together with the dimensionless groups defined in Sections
5.3.5 and 6.3.4, we can reduce Equation (7.116) to dimensionless form.
It is found that Equation (7.116) is identical to its dimensionless
form

$$\rho_m^+ \frac{\partial i_m^+}{\partial t^+} + \rho_m^+ v_{mx}^+ \frac{\partial i_m^+}{\partial x^+} = E_c \left(\frac{\partial P_m^+}{\partial t^+} + v_{mx}^+ \frac{\partial P_m^+}{\partial x^+} \right) + \qquad (7.117)$$

$$N_\nu \dot{\Phi}_{mIR}^+ - \alpha_o N_{\rho_1} N_{\rho_2} N_D \frac{\partial}{\partial x^+}\left(\alpha^+ \frac{\rho_1^+ \rho_2^+}{\rho_m^+} V_{2jx}^+ \Delta\langle i\rangle^+ \right) +$$

$$\alpha_o (N_{\rho_1} - \dot{N}_{\rho_2}) N_D E_c \frac{\partial P_m^+}{\partial x^+} + N_{St} \, q_{1e}^+ -$$

$$N_{\sigma e} \sigma^+ \hat{t}_i \cdot \frac{\partial \vec{v}_i^+}{\partial x^+} + E_c N_{Peh} \Gamma_{2i} P_m \left(\frac{1}{\rho_2^+} - \frac{1}{\rho_1^+} \frac{N_{\rho_2}}{N_{\rho_1}} \right) -$$

$$N_{\rho_1} \frac{\partial}{\partial x^+}\left[(1-\alpha)^+ \rho_1^+ N_{cov}(\text{ent}1) \right] - N_{\rho_2} \frac{\partial}{\partial x^+}\left[\alpha^+ \rho_2^+ N_{cov}(\text{ent}2) \right]$$

It can be seen that in addition to the dimensionless groups obtained
from the kinematic similarity (see Section 5.3.5) and from the dynamic

similarity (see Section 6.3.4), the thermal field expressed by Equation (6.117) introduces additional dimensionless groups   below:

$$E_c = \frac{v_{mo}^2}{i_{mo}} \qquad \text{Eckert number} \qquad (7.118)$$

$$N_{St} = \frac{\zeta_{ie}}{A_{Tc}} \frac{q_{ie}}{\rho_{mo} \, v_{mo} \, i_{mo}} \qquad \text{Stanton number} \qquad (7.119)$$

$$N_{\mathcal{D}} = \frac{\dot{\Phi}_{mIR})_o}{\rho_{mo} \, v_{mo} \, i_{mo} / L} \qquad \text{Dissipation group} \qquad (7.120)$$

$$N_{\sigma e} = \frac{\zeta_i}{A_{Tc}} \frac{\sigma_o}{\rho_{mo} \, v_{mo}} \qquad \text{Surface Energy group} \qquad (7.121)$$

$$\left.\begin{array}{l} N_{cov(ent1)} = (1-\alpha)_o \, \dfrac{Cov(v_{1x} \cdot i_1)}{v_{mo} \, i_{mo}} \\[2ex] N_{cov(ent2)} = \alpha_o \, \dfrac{Cov(v_{2x} \cdot i_2)}{v_{mo} \, i_{mo}} \end{array}\right\} \quad \text{Flow Regime groups} \qquad (7.122)$$

The dimensionless Eckert number scales the relative importance of the heat generation due to compression against the thermal energy. As in the case of single phase flow, it is possible to conclude that heat due to compression is important for the calculation of the thermal field when the mixture velocity as seen at the center of mass of the mixture is so large that the thermal energy increase is of the same order of magnitude as the prescribed thermal energy increase.

Expressed in the form of Equation (7.110), i.e., as the ratio of the heat transfer at the solid boundaries to the convective energy flux, the dimensionless Stanton number has the same meaning as the single phase flow.

The Dissipation group expressed by Equation (7.120) scales the relative importance of the dissipation of mechanical energy, i.e., frictional heat, into the thermal energy. It is well-known that in the case of single phase flow this group is related to Eckert and Reynolds numbers. From

this analogy it is now possible to conclude that the frictional heat is important for the calculation of the thermal field when the velocity of the center of mass is so large that the thermal energy increase is of the same order of magnitude as the prescribed thermal energy increase.

The Surface Energy group, expressed by Equation (7.121) compares the relative importance of the interfacial surface energy with the thermal energy. We note that one can express it in the following form

$$N_{\Gamma e} = \left(\frac{v_{m_o}^2}{i_{m_o}} \right) \left(\frac{\gamma_i}{A_{\Gamma e}} \frac{\sigma}{e_{m_o} v_{m_o}^2} \right) \qquad (7.123)$$

where the first term is Eckert number while the second term is related to Weber number which was obtained from the dynamic similarity requirements in Section 6.3.4. It is evident, therefore, that the Surface Energy group does not introduce any additional dimensionless group.

Finally, since Cov(ent K) $\sim$ Cov($v_{Kx} \cdot i_K$) depends on the K-th phase flow regime it becomes evident that the Flow Regime group reflects the effects of the flow regimes.

In accordance with the preceding argument as well as the arguments made in Section 5.3.5 and 6.3.4 it is to be expected that one needs so many similarity groups to obtain an exact scaling of two systems. However, for different applications not all groups will have the same importance. Consequently, Equations (5.41) and (5.42) for the kinematic similarity, Equation (6.73) for the dynamic similarity and finally Equation (7.117) for the thermal similarity can be used to determine the importance of the various groups and to select the appropriate scaling parameters.

7.6 Summary

1) The total energy equations for separated two-phase flow problem were formulated by considering a two-fluid model and a diffusional model. The first was expressed in terms of two field equations, one for each phase, whereas the diffusional model was described by means of the mixture energy equation.

2) From the general formulation a Prandtl type, i.e., a boundary layer approximation was used, and the appropriate forms for the total energy equation were derived for both the plane flow and the annular flow.

3) When the effects of diffusion and of motion are neglected the mixture total energy equation reduces to that derived by Gibbs for the heterogenous binary mixtures.

4) For the purpose of practical applications, the auxiliary energy balance equations, i.e., internal energy, mechanical energy, and enthalpy balance equations, were derived together with the appropriate "jump" conditions.

5) The mixture enthalpy equation in its dimensionless form was used to obtain similarity groups arising from the thermal field of a two-phase mixture.

CHAPTER VIII

ENTROPY EQUATIONS

8.1 Introduction

Thus far all the balance equations which are necessary for the solution of a separated two-phase flow system were derived from the single-phase flow equations - by matching the conditions at the interface. As it is well-known in single-phase flow analysis these balance equations are supplemented by the entropy equation or the second law of thermodynamics which specifies the direction of a particular transport process.

Following the same principles of the single-phase flow equations, it is the purpose of this chapter to derive the entropy equations based on the both two-fluid and diffusion models.

8.2 Two-Fluid Model Formulation

8.2.1 Derivation of Phase Equations

Defining by S, and T, the local specific entropy and the local temperature, respectively, the local differential form of the entropy balance for the K-th phase, (K = 1,2), can be expressed as

$$\rho_K T_K \frac{D_K s_K}{Dt} = - \nabla \cdot \vec{q}_K + \overline{\overline{\tau}}_K : \nabla \vec{v}_K \qquad (8.1)$$

which gives an equation for the rate of change of entropy. Dividing through Equation (8.1) by the temperature T, and rearranging yields

$$\frac{\partial}{\partial t} \varrho_k s_k + \nabla \cdot (\varrho_k \vec{v}_k s_k) = -\nabla \cdot \left(\frac{\vec{q}_k}{T_k}\right) + \vec{q}_k \cdot \nabla \left(\frac{1}{T_k}\right) + \frac{1}{T_k} \bar{\bar{\varepsilon}}_k : \nabla \vec{v}_k \quad (8.2)$$

It is customary to express Equation (8.2) as

$$\frac{\partial}{\partial t} \varrho_k s_k + \nabla \cdot (\varrho_k \vec{v}_k s_k) = -\nabla \cdot \vec{q}_k^s + \varrho_k \dot{s}_k \qquad (8.3)$$

where $\vec{q}^s$ is the entropy flux with respect to the center of gravity of the fluid and defined by

$$\vec{q}^s = \frac{\vec{q}_k}{T_k} \qquad (8.4)$$

whereas $\dot{s}_K$ is the rate of entropy production per unit mass associated with the energy transport and with the momentum transport and defined by

$$\dot{s}_k = \frac{1}{\varrho_k} \left(\vec{q}_k \cdot \nabla \frac{1}{T_k} + \frac{1}{T_k} \bar{\bar{\varepsilon}}_k : \nabla \vec{v}_k \right) \qquad (8.5)$$

If the proper identifications

$$\psi = s \qquad (8.6)$$

$$\dot{\psi} = \dot{s} = \frac{1}{\varrho} \left(\vec{q} \cdot \nabla \frac{1}{T} + \frac{1}{T} \bar{\bar{\varepsilon}} : \nabla \vec{v} \right)$$

$$\vec{\phi} = \vec{q}^s = \frac{\vec{q}}{T}$$

are used, it can be seen that Equation (8.4) has the same form as the local form of the general balance equation, Equation (4.5). Consequently, Equation (4.20) can be used in conjunction with Equation (8.6) to obtain the mass-weighed area-averaged general balance equations applicable for each phase. In view of Equation (8.6), Equation (4.20) yields for phase 1:

$$\frac{\partial}{\partial t}(1-\alpha) \ll \rho_1 \gg \langle s_1 \rangle + \frac{\partial}{\partial x}(1-\alpha) \ll \rho_1 \gg \langle v_{1x} \rangle \langle s_1 \rangle = \qquad (8.7a)$$

$$= -\frac{\partial}{\partial x}(1-\alpha) \ll q_{1x}^s \gg + (1-\alpha) \ll \rho_1 \gg \langle \dot{s}_1 \rangle -$$

$$\left[(1-\alpha) \ll \rho_1 \gg \langle v_{1x} \rangle \langle s_1 \rangle + (1-\alpha) \ll q_{1x}^s \gg + Cov(entr\, 1)\right]\frac{d\ln A_{Tc}}{dx} -$$

$$\frac{1}{A_{Tc}}\sum_{\ell=e,i}\int_{S_{i\ell}}\left(\dot{m}_{1\ell}\, s_{1\ell} + \vec{q}_{1\ell}^{\,s}\cdot\hat{n}_{1\ell}\right)\frac{dA}{dx} - \frac{\partial}{\partial x}Cov(entr\, 1)$$

whereas Equation (4.20b) yields for phase 2:

$$\frac{\partial}{\partial t}\alpha \ll \rho_2 \gg \langle s_2 \rangle + \frac{\partial}{\partial x}\alpha \ll \rho_2 \gg \langle v_{2x} \rangle \langle s_2 \rangle = \qquad (8.7b)$$

$$= -\frac{\partial}{\partial x}\alpha \ll q_{2x}^s \gg + \alpha \ll \rho_2 \gg \langle \dot{s}_2 \rangle -$$

$$\left[\alpha \ll \rho_2 \gg \langle v_{2x} \rangle \langle s_2 \rangle + \alpha \ll q_{2x}^s \gg + Cov(entr\, 2)\right]\frac{d\ln A_{Tc}}{dx} -$$

$$\frac{1}{A_{Tc}}\sum_{\ell=e,i}\int_{S_{2\ell}}\left(\dot{m}_{2\ell}\, s_{2\ell} + \vec{q}_{2\ell}^{\,s}\cdot\hat{n}_{2\ell}\right)\frac{dA}{dx} - \frac{\partial}{\partial x}Cov(entr\, 2)$$

where

$$Cov(entr\, 1) = (1-\alpha) \ll \rho_1 \gg Cov(v_{1x}, s_1); \quad Cov(entr\, 2) = \alpha \ll \rho_2 \gg Cov(v_{2x}, s_2) \quad (8.8)$$

Expanding the left hand side of Equations (8.7a and b), and using the continuity equations, Equation (5.2a and b) entropy equations can be also expressed in terms of the material derivatives defined by Equation (5.18), thus

$$(1-\alpha) \ll \rho_1 \gg \frac{D_1\langle s_1 \rangle}{Dt} = -\frac{\partial}{\partial x}(1-\alpha) \ll q_{1x}^s \gg + (1-\alpha) \ll \rho_1 \gg \langle \dot{s}_1 \rangle - \quad (8.9a)$$

$$\left[(1-\alpha) \ll q_{1x}^s \gg + Cov(entr\, 1)\right]\frac{d\ln A_{Tc}}{dx} -$$

$$\frac{1}{A_{Tc}}\sum_{\ell=e,i}\int_{S_{1\ell}}\left[\dot{m}_{1\ell}(s_{1\ell} - \langle s_1 \rangle) + \vec{q}_{1\ell}\cdot\hat{n}_{1\ell}\right]\frac{dA}{dx} - \frac{\partial}{\partial x}Cov(entr\, 1)$$

$$\alpha \langle\langle \rho_2 \rangle\rangle \frac{D_2 \langle s_2 \rangle}{Dt} = -\frac{\partial}{\partial x} \alpha \langle\langle \frac{q_{2x}^s}{} \rangle\rangle + \alpha \langle\langle \rho_2 \rangle\rangle \langle \dot{s}_2 \rangle - \left[\alpha \langle\langle q_{2x}^s \rangle\rangle + \right. \qquad (8.9b)$$

$$\left. \text{Cov(entr 2)} \right] \frac{d \ln A_{\tau_c}}{dx} - \frac{1}{A_{\tau_c}} \sum_{\ell = e, i} \int_{S_{2\ell}} \left[\dot{m}_{2\ell} \left(s_{2\ell} - \langle s_2 \rangle \right) + \vec{q}_{2\ell}^s \cdot \hat{n}_{2\ell} \right] dA - \frac{\partial}{\partial x} \text{Cov(entr 2)}$$

Averaged equations, Equation (8.9a) for phase 1, and Equation (8.9b) for phase 2, describe the entropy field of a separated two-phase flow field. These field equations must be supplemented by the initial and the boundary conditions.

Boundary conditions at the external surfaces are relatively easy to specify and basically follows the lines frequently used in the single phase flow analysis. Condition to be specified at the interfaces is the interfacial entropy coupling of the two phases, which is also called the entropy jump condition. It can be obtained from the internal energy jump condition Equation (7.66a) by replacing the internal energy by means of the thermodynamic Euler equation. Consequently, we obtain from Equation (7.66) the following relation

$$\sum_{k=1}^{2} \rho_{ki} \left(\vec{v}_{ki} - \vec{v}_i \right) \cdot \hat{n}_{ki} \, s_{ki} + P_{ki} \left(\vec{v}_{ki} - \vec{v}_i \right) \cdot \hat{n}_{ki} + \vec{q}_{jk}^s \cdot \hat{n}_k = (I.E)_s \qquad (8.10)$$

where $(I.E)_s$ is specified by Equation (7.66b).

Now it will be assumed that the local thermal equilibrium between the two phases, that is, $T_{1i} = T_{2i} = T_i$ in the immediate neighborhood of the interface. It is proper to stress here, however, that the equality of temperatures at the interface does not imply the thermal equilibrium of the two-phase mixture. For example, for heated ducts first phase adjacent to the external surfaces can be superheated in the bulk with a temperature gradient allowing the no-temperature slip at the interface.

The same may be true for the second phase which may be subcooled in the bulk with the temperature gradient so that the temperatures match at the interface.

Therefore, dividing through Equation (8.10) by the common interfacial temperature the following relation can be obtained for the interfacial entropy balance

$$\sum_{k=1}^{2} \dot{m}_{ki}\, s_{ki} + \vec{q}_{ki}^{\,s} \cdot \hat{n}_{ki} = (\mathcal{I}.\mathcal{E})_s - \sum_{k=1}^{2} \frac{P_{ki}}{T_i}\,(\vec{V}_{ki} - \vec{V}_i) \cdot \hat{n}_{ki} = (\text{ENT})_s \quad (8.11)$$

where $(\text{ENT})_s$ can be evaluated by means of Equation (7.66b), thus

$$(\text{ENT})_s = \frac{\bar{v}}{T_i\, H^2} \left[\vec{T}_i \left(G \frac{\partial}{\partial u_1} - F \frac{\partial}{\partial u_2} \right) + \vec{T}_2 \left(E \frac{\partial}{\partial u_2} - F \frac{\partial}{\partial u_1} \right) \right] \cdot \vec{V}_i - \quad (8.12)$$

$$\sum_{k=1}^{2} \frac{P_{ki}}{T_i}\,(\vec{V}_{ki} - \vec{V}_i) \cdot \hat{n}_{ki}$$

8.2.2 Two-Dimensional Flow

A. Plane Flow

To be consistent with the Prandtl's type of approximations which have been used for the simplifications of the momentum and of the energy equations, we shall neglect the effect of the axial entropy flux changes because this terms directly proportional to the axial heat flux. As it is demonstrated by Equation (7.16) it is proportional to the second order in ε which was neglected in the derivation of the two-dimensional energy equations. Furthermore, using the geometric simplifications, i.e., $\left(\frac{\partial \eta}{\partial x} \right) \ll 1$, it can be shown that the entropy equations appropriate for the two-fluid model formulation of a two-dimensional plane flow problem can be given by

$$\eta \ll \varrho_1 \gg \frac{D_1 \langle s_1 \rangle}{Dt} = \eta \ll \varrho_1 \gg \langle \dot{s}_1 \rangle - \left[\dot{m}_{1i} \left(s_{1i} - \langle s_1 \rangle \right) + \right. \quad (8.13a)$$

$$\left. \vec{q}_{1i}^{\,s} \cdot \hat{n}_{1i} \right] - \left[\dot{m}_{1e} \left(s_{1e} - \langle s_1 \rangle \right) + \vec{q}_{1e}^{\,s} \cdot \hat{n}_{1e} \right] - H \frac{\partial}{\partial x} Cov \left(entr \, 1 \right)$$

$$(H-\eta) \ll \varrho_2 \gg \frac{D_2 \langle s_2 \rangle}{Dt} = (H-\eta) \ll \varrho_2 \gg \langle \dot{s}_2 \rangle - \left[\dot{m}_{2i} \left(s_{2i} - \langle s_2 \rangle \right) + \right. \quad (8.13b)$$

$$\left. \vec{q}_{2i}^{\,s} \cdot \hat{n}_{2i} \right] - H \frac{\partial}{\partial x} Cov \left(entr \, 2 \right)$$

The significance of the terms which appear in these equations are as follows: The term on the left hand side represents the growth of entropy following the center of the mass of the phase. The first term on the right hand side accounts for the effects of the internal heating on the entropy growth, whereas the following terms account for the effects of the entropy transport at the external and the internal surfaces, and finally of the entropy distribution.

B. Annular Flow

In view of the geometric relations and of the approximations introduced in the previous chapters, it can be shown that the annular flow entropy balance gives for each phase the following relations:

$$(1-\alpha) \ll \varrho_1 \gg \frac{D_1 \langle s_1 \rangle}{Dt} = (1-\alpha) \ll \varrho_1 \gg \langle \dot{s}_1 \rangle - \frac{4\sqrt{\alpha}}{De} \left[\dot{m}_{1i} \left(s_{1i} - \langle s_1 \rangle \right) + \right. \quad (8.14a)$$

$$\left. \vec{q}_{1i}^{\,s} \cdot \hat{n}_{1i} \right] - \frac{4}{De} \left[\dot{m}_{1e} \left(s_{1e} - \langle s_1 \rangle \right) + \vec{q}_{1e}^{\,s} \cdot \hat{n}_{1e} \right] - \frac{1}{De^2} \frac{\partial}{\partial x} De^2 \, Cov \left(entr \, 1 \right)$$

$$\alpha \ll \varrho_2 \gg \frac{D_2 \langle s_2 \rangle}{Dt} = \alpha \ll \varrho_2 \gg \langle \dot{s}_2 \rangle - \frac{4\sqrt{\alpha}}{De} \left[\dot{m}_{2i} \left(s_{2i} - \langle s_2 \rangle \right) + \right. \quad (8.14b)$$

$$\left. \vec{q}_{2i}^{\,s} \cdot \hat{n}_{2i} \right] - \frac{1}{De^2} \frac{\partial}{\partial x} De^2 \, Cov \left(entr \, 2 \right)$$

Significance of the terms can be interpreted by analogy to the plane flow case.

8.3 Diffusion Model Formulation

From the diffusional model concept underlined in the previous chapters, formulation will be made in terms of the mixture entropy equation which can be obtained by adding the individual phase equations. Consequently, by using the interfacial entropy balance, entropy equation for the mixture can be given by

$$(1-\alpha) \ll \rho_1 \gg \frac{D_1 \langle s_1 \rangle}{Dt} + \alpha \ll \rho_2 \gg \frac{D_2 \langle s_2 \rangle}{Dt} - \Gamma_{2i} (\langle s_2 \rangle - \langle s_1 \rangle) = \quad (8.15)$$

$$= -\frac{\partial q^s_{mx}}{\partial x} + \rho_m \dot{s}_m - \left[q^s_{mx} + Cov(entr T) \right] \frac{d \ln A_{Tc}}{dx} - \frac{1}{A_{Tc}} \int_{S_{ii}} (ENT)_s \frac{dA}{dx} -$$

$$\frac{1}{A_{Tc}} \sum_{k=1}^{2} \int_{S_{ke}} (\dot{m}_{ke} s_{ke} + \vec{q}^s_{ke} \cdot \hat{n}_{ke}) \frac{dA}{dx} - \frac{\partial}{\partial x} Cov(entr T)$$

which can be expressed also in terms of the mixture properties. For this purpose the fundamental identity with $\psi = s$, can be used in Equation (8.15), thus we obtain

$$\rho_m \frac{D_m s_m}{Dt} = -\frac{\partial q^s_{mx}}{\partial x} + \rho_m \dot{s}_m - \frac{\partial}{\partial x} \left[(1-\alpha) \ll \rho_1 \gg \langle s_1 \rangle V_{1mx} + \quad (8.16) \right.$$

$$\alpha \ll \rho_2 \gg \langle s_2 \rangle V_{2mx} \right] - \left\{ q^s_{mx} + \left[(1-\alpha) \ll \rho_1 \gg \langle s_1 \rangle V_{1mx} + \alpha \ll \rho_2 \gg \langle s_2 \rangle V_{2mx} \right] + \right.$$

$$Cov(entr T) \right\} \frac{d \ln A_{Tc}}{dx} - \frac{1}{A_{Tc}} \int_{S_i} (ENT)_s \frac{dA}{dx} -$$

$$\frac{1}{A_{Tc}} \sum_{k=1}^{2} \int_{S_{ke}} \left[\dot{m}_{ke} (s_{ke} - \langle s_k \rangle) + \vec{q}^s_{ke} \cdot \hat{n}_{ke} \right] \frac{dA}{dx} - \frac{\partial}{\partial x} Cov(entr T)$$

where $\vec{q}_m^s$ is the mixture entropy flux and defined by

$$\vec{q}_m^s = (1-\alpha) \ll \vec{q}_1^s \gg + \alpha \ll \vec{q}_2^s \gg \qquad (8.17)$$

$\dot{S}_m$ is the generation of the entropy per unit mass of the mixture due to the internal heating and defined by

$$\rho_m \dot{S}_m = (1-\alpha) \ll \rho_1 \gg \langle \dot{s}_1 \rangle + \alpha \ll \rho_2 \gg \langle \dot{s}_2 \rangle \qquad (8.18)$$

The third term accounts for the entropy flux by diffusion. The fourth term gives for the effects of the variable duct area. The fifth term accounts for the entropy generation due to the interfacial dissipation of energy. Finally, the sixth and the seventh terms account respectively, for the entropy transport at the external surfaces and for the entropy distribution. It is important to note that due to the existence of the interfaces there is another mode of the entropy generation.

It is important to note that the growth of entropy $\dfrac{D_K \langle S_K \rangle}{Dt}$, of the K-th phase, K = 1,2, can be any amount so long as the total growth of entropy for the mixture remains non-negative. Since locally at every point in the bulk fluids $\dot{S}_K \geqslant 0$, by averaging resulting term $\dot{S}_m$ for the mixture must be positive. Consequently from Equation (8.16) it follows that

$$A_{Tc} \rho_m \frac{D_m S_m}{Dt} \geqslant -\frac{\partial}{\partial x}\left(A_{Tc} q_{mx}^s\right) - \int_{S_i} (ENT)_s \frac{dA}{dx} - \qquad (8.19)$$

$$\frac{\partial}{\partial x} A_{Tc}\left[(1-\alpha) \ll \rho_1 \gg \langle s_1 \rangle V_{1mx} + \alpha \ll \rho_2 \gg \langle s_2 \rangle V_{2mx}\right] -$$

$$\sum_{K=1}^{2} \int_{S_{Ke}} \left[\dot{m}_{Ke}(s_{Ke} - s_m) + \vec{q}_{Ke}^s \cdot \hat{n}_{Ke}\right] \frac{dA}{dx} - \frac{\partial}{\partial x} A_{Tc} Cov(ent, T)$$

If the mixture reduces to a single phase or a single constituent, that is, if no diffusion and no interfaces, then we recover the Clausius-Duhem inequality, reference [23] with the difference of covariance term. Furthermore, in the absence of the external mass flow Equation (8.19) does not reduce to the simple form which was found in reference [23]. Difference eminates from the fact that the second term appearing in Equation (8.19) does not appear in the derivations in reference [23].

To obtain the two-dimensional plane flow we shall use Equation (8.16) with the usual approximations. Consequently, we get

$$\rho_m \frac{D_m s_m}{Dt} = \rho_m \dot{s}_m - \frac{\partial}{\partial x}\Big[(1-\alpha) \langle\!\langle \rho_1 \rangle\!\rangle \langle s_1 \rangle V_{1mx} + \quad (8.20)$$

$$\alpha \langle\!\langle \rho_2 \rangle\!\rangle \langle s_2 \rangle V_{2mx} \Big] - \frac{1}{H}(ENT)_s - \frac{1}{H}\Big[\dot{m}_{1e}(s_{1e}-s_m) +$$

$$\vec{q}^s_{1e} \cdot \hat{n}_{1e} \Big] - \frac{\partial}{\partial x} Cov(ent_r T)$$

Similarly for the annular flow we obtain

$$\rho_m \frac{D_m s_m}{Dt} = \rho_m \dot{s}_m - \frac{\partial}{\partial x}\Big[(1-\alpha) \langle\!\langle \rho_1 \rangle\!\rangle \langle s_1 \rangle V_{1mx} + \quad (8.21)$$

$$\alpha \langle\!\langle \rho_2 \rangle\!\rangle \langle s_2 \rangle V_{2mx} \Big] - \Big[(1-\alpha) \langle\!\langle \rho_1 \rangle\!\rangle \langle s_1 \rangle V_{1mx} + \alpha \langle\!\langle \rho_2 \rangle\!\rangle \langle s_2 \rangle V_{2mx} \Big] +$$

$$Cov(ent_r T)\Big] \frac{2}{De}\frac{dDe}{dx} - \frac{4\sqrt{\alpha}}{De}(ENT)_s -$$

$$\frac{4}{De}\Big[\dot{m}_{1e}(s_{1e}-s_m) + \vec{q}^s_{1e} \cdot \hat{n}_{1e} - \frac{\partial}{\partial x}Cov(ent_r T)$$

Equation (8.20) for the plane flow and Equation (8.21) for the annular flow formulates the entropy equations based on the diffusion model.

8.4 Summary

1. The local entropy balance was put into the form of local general balance equation by defining the entropy flux and the entropy generation.

2. In view of the proper definitions, the entropy equations for separated two-phase flow problem were formulated based on the two-fluid model and the diffusional models.

3. Using the thermodynamic relation between the internal energy and the entropy, the entropy "jump" condition was derived from the internal energy "jump" condition.

4. From the general formulations, the appropriate forms for the entropy equation were derived for both the plane flow and the annular flow.

CHAPTER IX

SUMMARY TO PART I

The analysis which was presented in preceding chapters was directed at providing a general thermo-fluid dynamic formulation of separated two-phase flow. The results can be summarized in three major topics:

1. Models, Basic Definitions and Relations,

2. Two-Fluid Model Formulation,

3. Diffusion Model Formulation.

9.1 Models, Basic Definitions and Relations

In Chapter 3, we have discussed the general characteristics and aspects of separated two-phase flow together with the basic relations and parameters relevant to this regime. In particular:

1. The formulations based on the two-fluid and on the diffusion models were described. It was concluded that depending on the magnitude of the relative velocity as well as the difference between the two densities, both separated flows and dispersed flows can be formulated in terms of a two-fluid model and of a diffusion model.

2. In discussing statistical and spatial averages, it was concluded that for separated flows only the latter was applicable. In view of this conclusion, two expressions for spatial averaged properties were derived one based on the volume, i.e., <u>area-averaged</u> value

$$\langle\langle +_k \rangle\rangle (x,t) = \frac{1}{A_{kc}} \iint_{A_{kc}} +_k (x,y,z,t)\, dA\ , \quad K=1,2 \quad (3.3)$$

the other on the <u>mass-weighed area-averaged</u> value

$$\langle +_k \rangle (x,t) = \frac{\displaystyle\iint_{A_{kc}} \rho_k(x,y,z,t)\, +_k(x,y,z,t)\, dA}{\displaystyle\iint_{A_{kc}} \rho_k(x,y,z,t)\, dA} \qquad (3.4)$$

If $\langle\langle \rho_K \rangle\rangle \simeq \rho_K$, only then the two averages expressed by Equations (3.3) and (3.4) become identical.

3. The difference between the concentrations based on static and on kinematic parameters was noted. It was stressed that those which are defined in terms of static parameters instead of kinematic parameters should be used to describe correctly the mixture properties of a two-phase system.

4. The importance of expressing the mixture properties in terms of the baricenter of the mixture was emphasized. The relevant thermodynamic properties have been derived and expressed therefore, in terms of the static volume (or mass) concentration and of the baricenter.

5. Various velocity fields which can be used to derive a two-phase flow system were noted. The significance of these fields was discussed and the relations that exist between them were derived and summarized on Table 2. Furthermore, expressions for the flux fields associated with these velocity fields were derived and discussed.

6. The fundamental identity which relates the mean flux of the mixture property to the mean fluxes of the phases was derived. In

particular, the transport by diffusion (or by drift) was observed.

In Chapter 4, the flow field of each phase was considered as a continuum, and the separated two-phase flow problem was formulated in terms of a) the <u>local</u> fundamental conservation equations applicable to the flow field of each phase, b) the constitutive equations describing the behavior of each constituent, and c) of the internal boundary conditions prescribing the interfacial balance.

However, in view of the difficulty of seeking a solution of such a set of equations, the mass-weighed area-averaged general balance equations for each phase as well as for the mixture were derived. These balance equations were later used for the formulations based on the two-fluid and on the diffusion models. In what follows we shall summarize the formulations based on the two-fluid and on the diffusion models.

<h2 style="text-align:center">9.2 Two-Fluid Model Formulation</h2>

9.2.1 General Formulation

The mass-weighed area-averaged general balance equations for each phase are given by

$$\frac{\partial}{\partial t} (1-\alpha) \lll \rho_1 \ggg \langle \Psi_1 \rangle + \frac{\partial}{\partial x} (1-\alpha) \lll \rho_1 \ggg \langle v_{1x} \rangle \langle \Psi_1 \rangle = . \tag{4.20a}$$

$$= -\frac{\partial}{\partial x} (1-\alpha) \lll \phi_{1x} \ggg + (1-\alpha) \lll \rho_1 \ggg \langle \Phi_1 \rangle -$$

$$(1-\alpha) \left[\lll \rho_1 \ggg \langle v_{1x} \rangle \langle \Psi_1 \rangle + \lll \Phi_{1x} \ggg + \lll \rho_1 \ggg Cov(v_{1x}, \Psi_1) \right] \frac{d \, \ln A_{T_c}}{dx} -$$

$$\frac{1}{A_{T_c}} \sum_{l=e,i} \int_{J_{il}} \hat{n}_{il} \cdot \left[\rho_{il} (\vec{v}_{il} - \vec{v}_l) \Psi_{il} + \vec{\Phi}_{il} \right] \frac{dA}{dx} -$$

$$\frac{\partial}{\partial x} (1-\alpha) \lll \rho_1 \ggg Cov(v_{1x}, \Psi_1)$$

$$\frac{\partial}{\partial t} \propto \langle\langle \rho_2 \rangle\rangle \langle \psi_2 \rangle + \frac{\partial}{\partial x} \propto \langle\langle \rho_2 \rangle\rangle \langle v_{2x} \rangle \langle \psi_2 \rangle = -\frac{\partial}{\partial x} \propto \langle\langle \Phi_{2x} \rangle\rangle + \quad (4.20b)$$

$$\propto \langle\langle \rho_2 \rangle\rangle \langle \dot{\psi}_2 \rangle - \propto \left[\langle\langle \rho_2 \rangle\rangle \langle v_{2x} \rangle \langle \psi_2 \rangle + \langle\langle \Phi_{2x} \rangle\rangle + \right.$$

$$\langle\langle \rho_2 \rangle\rangle \, Cov \, (v_{2x} \cdot \psi_2) \left] \frac{d \ln A_{Tc}}{dx} - \frac{1}{A_{Tc}} \sum_{\ell = e,i} \int_{S_{2\ell}} \hat{n}_{2\ell} \cdot \left[\rho_{2\ell} \, (\vec{v}_{2\ell} - \vec{v}_i) \, \psi_2\ell + \right. \right.$$

$$\left. + \vec{\Phi}_{2\ell} \right] \frac{dA}{dx} - \frac{\partial}{\partial x} \propto \langle\langle \rho_2 \rangle\rangle \, Cov \, (v_{2x} \cdot \psi_2)$$

The above two equations describe the field of each phase in terms of mass-weighed area-averaged variables and take into account the effects of the interactions at the internal boundaries between the phases and of the transport of quantities at the external boundaries.

These two general balance equations were supplemented by the boundary conditions at internal surfaces, i.e., by the "jump" conditions. The general "jump" condition which can be used to derive mass, momentum and energy interactions between the two phases are given in Appendix B and expressed by

$$\sum_{k=1}^{2} \hat{n}_{ki} \cdot \left[\rho_{ki} \, (\vec{v}_{ki} - \vec{v}_i) \, \psi_{ki} + \vec{\Phi}_{ki} \right] = -\nabla_s \cdot \bar{\bar{\theta}} \qquad (B.10)$$

With the proper identifications for ψ, $\dot{\psi}$, $\vec{\phi}$ and $\bar{\bar{\theta}}$, these general balance equations were used to derive: a) the continuity equations for the individual phases as well as for the interfacial mass balance, b) the momentum equations for the individual phases as well as for the interfacial momentum balance, c) the energy equations for the individual phases as well as for the interfacial energy balance, and d) the entropy equations for the individual phases as well as for the interfacial entropy balance.

In their general form, the six field equations, three for each phase, and the three interfacial "jump" conditions are:

The equation of continuity of each phase:

$$\frac{\partial}{\partial t}\,(1-\alpha)\,\langle\!\langle \rho_1\rangle\!\rangle + \frac{\partial}{\partial x}\,(1-\alpha)\,\langle\!\langle \rho_1\rangle\!\rangle\,\langle v_{1x}\rangle = \tag{5.6a}$$

$$= -\,(1-\alpha)\,\langle\!\langle \rho_1\rangle\!\rangle\,\langle v_{1x}\rangle\,\frac{d\ln A_{Tc}}{dx} + \sum_{\ell=e,i}\Gamma_{1\ell}$$

$$\frac{\partial}{\partial t}\,\alpha\,\langle\!\langle \rho_2\rangle\!\rangle + \frac{\partial}{\partial x}\,\alpha\,\langle\!\langle \rho_2\rangle\!\rangle\,\langle v_{2x}\rangle = \tag{5.6b}$$

$$= -\,\alpha\,\langle\!\langle \rho_2\rangle\!\rangle\,\langle v_{2x}\rangle\,\frac{d\ln A_{Tc}}{dx} + \sum_{\ell=e,i}\Gamma_{2\ell}$$

The interfacial mass balance:

$$\sum_{k=1}^{2}\Gamma_{ki} = 0 \tag{5.5}$$

In these equations $\Gamma_{K\ell}$ is defined by

$$\Gamma_{k\ell} = -\frac{1}{A_{Tc}}\int_{S_{k\ell}}\hat{n}_{k\ell}\cdot\rho_{k\ell}\,(\vec{v}_{k\ell}-\vec{v}_\ell)\,\frac{dA}{dx} = -\frac{1}{A_{Tc}}\int_{S_{k\ell}}\dot{m}_{k\ell}\,\frac{dA}{dx} \tag{5.4}$$

The interfacial source term Γ_{Ki}, must be specified by an appropriate constitutive equation for net vaporization, (or of net condensation). Functional form of this term depends on the structure of the interface as well as the temperature fields in the phases.

The momentum equation of each phase:

$$\frac{\partial}{\partial t}(1-\alpha)\langle\!\langle \rho_1\rangle\!\rangle\langle\vec{v}_1\rangle + \frac{\partial}{\partial x}(1-\alpha)\langle\!\langle \rho_1\rangle\!\rangle\langle v_{1x}\rangle\langle\vec{v}_1\rangle = \qquad (6.2a)$$

$$= \frac{\partial}{\partial x}\Big[(1-\alpha)\big(-\langle\!\langle P_1\rangle\!\rangle\,\overline{\overline{\delta}} + \langle\!\langle \overline{\overline{\Sigma}}_1\rangle\!\rangle\big)\cdot\hat{1}\Big] + (1-\alpha)\langle\!\langle \rho_1\rangle\!\rangle\,\vec{g} -$$

$$\Big\{(1-\alpha)\Big[\langle\!\langle \rho_1\rangle\!\rangle\langle v_{1x}\rangle\langle\vec{v}_1\rangle - \big(-\langle\!\langle P_1\rangle\!\rangle\,\overline{\overline{\delta}} + \langle\!\langle \overline{\overline{\Sigma}}_1\rangle\!\rangle\big)\cdot\hat{1}\Big] + \mathrm{Cov(mom1)}\Big\}\frac{d\ln A_{T_c}}{dx} -$$

$$\frac{1}{A_{T_c}}\sum_{\ell=e,i}\int_{\mathfrak{z}_{1\ell}}\Big[\dot{m}_{1\ell}\,\vec{v}_{1\ell} - \big(-P_{1\ell}\,\overline{\overline{\delta}} + \overline{\overline{\Sigma}}_{1\ell}\big)\cdot\hat{n}_{1\ell}\Big]\frac{dA}{dx} -$$

$$\frac{\partial}{\partial x}\,\mathrm{Cov(mom1)}$$

$$\frac{\partial}{\partial t}\,\alpha\,\langle\!\langle \rho_2\rangle\!\rangle\langle\vec{v}_2\rangle + \frac{\partial}{\partial x}\,\alpha\,\langle\!\langle \rho_2\rangle\!\rangle\langle v_{2x}\rangle\langle\vec{v}_2\rangle = \qquad (6.2b)$$

$$= \frac{\partial}{\partial x}\Big[\alpha\big(-\langle\!\langle P_2\rangle\!\rangle\,\overline{\overline{\delta}} + \langle\!\langle \overline{\overline{\Sigma}}_2\rangle\!\rangle\big)\cdot\hat{1}\Big] + \alpha\,\langle\!\langle \rho_2\rangle\!\rangle\,\vec{g} -$$

$$\Big\{\alpha\Big[\langle\!\langle \rho_2\rangle\!\rangle\langle v_{2x}\rangle\langle\vec{v}_2\rangle - \big(-\langle\!\langle P_2\rangle\!\rangle\,\overline{\overline{\delta}} + \langle\!\langle \overline{\overline{\Sigma}}_2\rangle\!\rangle\big)\cdot\hat{1}\Big] + \mathrm{Cov(mom2)}\Big\}\frac{d\ln A_{T_c}}{dx} -$$

$$\frac{1}{A_{T_c}}\sum_{\ell=e,i}\int_{\mathfrak{z}_{2\ell}}\Big[\dot{m}_{2\ell}\,\vec{v}_{2\ell} - \big(-P_{2\ell}\,\overline{\overline{\delta}} + \overline{\overline{\Sigma}}_{2\ell}\big)\cdot\hat{n}_{2\ell}\Big]\frac{dA}{dx} -$$

$$\frac{\partial}{\partial x}\,\mathrm{Cov(mom2)}$$

The interfacial momentum balance:

$$\sum_{k=1}^{2}\dot{m}_{ki}\,\vec{v}_{ki} + \big(P_{ki}\,\overline{\overline{\delta}} - \overline{\overline{\Sigma}}_{ki}\big)\cdot\hat{n}_{ki} = 2\kappa\sigma\,\hat{n}_{1i} + \qquad (B.25)$$

$$\frac{1}{H^2}\Big[\vec{r}_1\Big(G\frac{\partial}{\partial u_1} - F\frac{\partial}{\partial u_2}\Big) + \vec{r}_2\Big(E\frac{\partial}{\partial u_2} - F\frac{\partial}{\partial u_1}\Big)\Big]\sigma$$

The total energy equation of each phase:

$$\frac{\partial}{\partial t} (1-\alpha) \ll \ell_1 \gg \left(\langle i_1 \rangle + \frac{\langle v_1 \rangle^2}{2} \right) + \frac{\partial}{\partial x} (1-\alpha) \ll \ell_1 \gg \langle v_{1x} \rangle \left(\langle i_1 \rangle + \frac{\langle v_1 \rangle^2}{2} \right) = \quad (7.7a)$$

$$= \frac{\partial}{\partial t} (1-\alpha) \ll P_1 \gg - \frac{\partial}{\partial x} (1-\alpha) \ll q_{1x} \gg + \frac{\partial}{\partial x} (1-\alpha) \ll \bar{\bar{\Sigma}}_1 \cdot \hat{1} \gg \cdot \langle \vec{v}_1 \rangle + (1-\alpha) \vec{g} \cdot \langle \vec{v}_1 \rangle -$$

$$\left\{ (1-\alpha) \left[\ll \ell_1 \gg \langle v_{1x} \rangle \left(\langle i_1 \rangle + \frac{\langle v_1 \rangle^2}{2} \right) + \ll q_{1x} \gg - \ll \bar{\bar{\Sigma}}_1 \cdot \hat{1} \gg \cdot \langle \vec{v}_1 \rangle \right] + \right.$$

$$\left. \text{Cov}(\text{en 1}) \right\} \frac{d \ln A_{Tc}}{dx} -$$

$$\frac{1}{A_{Tc}} \sum_{\ell=e,i} \int_{\mathcal{I}_{1\ell}} \left[\dot{m}_{1\ell} \left(i_{1\ell} + \frac{v_{1\ell}^2}{2} \right) + \left(\vec{q}_{1\ell} - \bar{\bar{\Sigma}}_{1\ell} \cdot \vec{v}_{1\ell} + P_{1\ell} \vec{v}_{1\ell} \right) \cdot \hat{n}_{1\ell} \right] \frac{dA}{dx} -$$

$$\frac{\partial}{\partial t} \text{Cov}(\text{mech 1}) - \frac{\partial}{\partial x} \text{Cov}(\text{en 1})$$

$$\frac{\partial}{\partial t} \alpha \ll \ell_2 \gg \left(\langle i_2 \rangle + \frac{\langle v_2 \rangle^2}{2} \right) + \frac{\partial}{\partial x} \alpha \ll \ell_2 \gg \langle v_{2x} \rangle \left(\langle i_2 \rangle + \frac{\langle v_2 \rangle^2}{2} \right) = \quad (7.7b)$$

$$= \frac{\partial}{\partial t} \alpha \ll P_2 \gg - \frac{\partial}{\partial x} \alpha \ll q_{2x} \gg + \frac{\partial}{\partial x} \alpha \ll \bar{\bar{\Sigma}}_2 \cdot \hat{1} \gg \cdot \langle \vec{v}_2 \rangle + \alpha \vec{g} \cdot \langle \vec{v}_2 \rangle -$$

$$\left\{ \alpha \left[\ll \ell_2 \gg \langle v_{2x} \rangle \left(\langle i_2 \rangle + \frac{\langle v_2 \rangle^2}{2} \right) + \ll q_{2x} \gg - \ll \bar{\bar{\Sigma}}_2 \cdot \hat{1} \gg \cdot \langle \vec{v}_2 \rangle \right] + \right.$$

$$\left. \text{Cov}(\text{en 2}) \right\} \frac{d \ln A_{Tc}}{dx} -$$

$$\frac{1}{A_{Tc}} \sum_{\ell=e,i} \int_{\mathcal{I}_{2\ell}} \left[\dot{m}_{2\ell} \left(i_{2\ell} + \frac{v_{2\ell}^2}{2} \right) + \left(\vec{q}_{2\ell} - \bar{\bar{\Sigma}}_{2\ell} \cdot \vec{v}_{2\ell} + P_{2\ell} \vec{v}_{2\ell} \right) \cdot \hat{n}_{2\ell} \right] \frac{dA}{dx} -$$

$$\frac{\partial}{\partial t} \text{Cov}(\text{mech i}) - \frac{\partial}{\partial x} \text{Cov}(\text{en 1})$$

The interfacial total energy balance:

$$\sum_{k=1}^{2} \dot{m}_{ki} \left(i_{ki} + \frac{v_{ki}^2}{2} \right) + \left(\vec{q}_{ki} + P_{ki} \vec{v}_i - \bar{\bar{\Sigma}}_{ki} \cdot \vec{v}_{ki} \right) \cdot \hat{n}_{ki} = \quad (B.37)$$

$$= 2 \varkappa \sigma \; \vec{v}_i \cdot \hat{n}_{ii} + \frac{1}{H^2} \left[\vec{r}_1 \left(G \frac{\partial}{\partial u_1} - F \frac{\partial}{\partial u_2} \right) + \vec{r}_2 \left(E \frac{\partial}{\partial u_2} - F \frac{\partial}{\partial u_1} \right) \right] \cdot \sigma \; \vec{v}_i$$

These equations listed above are the basic field equations and the interfacial "jump" conditions. The additional equations which were derived are the mechanical energy, the internal energy, the enthalpy equations and the entropy equation together with proper "jump" conditions. We shall give here only the enthalpy and the entropy equations:

The enthalpy equation of each phase:

$$\frac{\partial}{\partial t}\left[(1-\alpha) \ll \rho_1 \gg \langle i_1 \rangle\right] + \frac{\partial}{\partial x}\left[(1-\alpha) \ll \rho_1 \gg \langle i_1 \rangle\right] = -\frac{\partial}{\partial x}(1-\alpha) \ll q_{1x} \gg + \quad (7.71a)$$

$$\frac{D_1}{Dt}(1-\alpha) \ll P_1 \gg + (1-\alpha)\dot{\bar{\Phi}}_{12R} - \left[(1-\alpha) \ll q_{1x} \gg - (1-\alpha) \ll P_1 \gg \langle v_{1x} \rangle + \right.$$

$$Cov(ent 1)\left]\frac{d \ln A_{Tc}}{dx} - \frac{1}{A_{Tc}}\sum_{\ell=e,i}\int_{S_{1\ell}}\left[\dot{m}_{1\ell}\, i_{1\ell} - \vec{q}_{1\ell}\cdot\hat{n}_{1\ell} - \right.$$

$$P_{1\ell}(\langle \vec{v}_1 \rangle - \vec{v}_\ell)\cdot\hat{n}_{1\ell}\left]\frac{dA}{dx} - \frac{\partial}{\partial x}Cov(ent 1) - (1-\alpha)Cov(\vec{v}_1\cdot\nabla P_1)\right.$$

$$\frac{\partial}{\partial t}\left[\alpha \ll \rho_2 \gg \langle i_2 \rangle\right] + \frac{\partial}{\partial x}\left[\alpha \ll \rho_2 \gg \langle i_2 \rangle\right] = -\frac{\partial}{\partial x}\alpha \ll q_{2x} \gg + \quad (7.71b)$$

$$\frac{D_2}{Dt}\alpha \ll P_2 \gg + \alpha\dot{\bar{\Phi}}_{21R} - \left[\alpha \ll q_{2x} \gg - \alpha \ll P_2 \gg \langle v_{2x} \rangle + \right.$$

$$Cov(ent 2)\left]\frac{d \ln A_{Tc}}{dx} - \frac{1}{A_{Tc}}\sum_{\ell=a,i}\int_{S_{2\ell}}\left[\dot{m}_{2\ell}\, i_{2\ell} - \vec{q}_{2\ell}\cdot\hat{n}_{2\ell} - \right.$$

$$P_{2\ell}(\langle \vec{v}_2 \rangle - \vec{v}_\ell)\cdot\hat{n}_{2\ell}\left]\frac{dA}{dx} - \frac{\partial}{\partial x}Cov(ent 2) - \alpha Cov(\vec{v}_2\cdot\nabla P_2)\right.$$

The interfacial enthalpy conditions:

$$\sum_{k=1}^{2}\dot{m}_{ki}\, i_{ki} + \vec{q}_{ki}\cdot\hat{n}_{ki} - P_{ki}(\vec{v}_{ki} - \vec{v}_i) = \quad (7.67)$$

$$= \frac{\sigma}{H^2}\left[\vec{r}_1\left(G\frac{\partial}{\partial u_1} - F\frac{\partial}{\partial u_2}\right) + \vec{r}_2\left(E\frac{\partial}{\partial u_2} - F\frac{\partial}{\partial u_1}\right)\right]\cdot\vec{v}_i$$

The entropy equation of each phase:

$$\frac{\partial}{\partial t} (1-\alpha) \langle\!\langle \rho_1 \rangle\!\rangle \langle s_1 \rangle + \frac{\partial}{\partial x} (1-\alpha) \langle\!\langle \rho_1 \rangle\!\rangle \langle V_{1x} \rangle \langle s_1 \rangle = \tag{8.7a}$$

$$= -\frac{\partial}{\partial x} (1-\alpha) \langle\!\langle q^s_{1x} \rangle\!\rangle + (1-\alpha) \langle\!\langle \rho_1 \rangle\!\rangle \langle \dot{s}_1 \rangle -$$

$$\left[(1-\alpha) \langle\!\langle \rho_1 \rangle\!\rangle \langle V_{1x} \rangle \langle s_1 \rangle + (1-\alpha) \langle\!\langle q^s_{1x} \rangle\!\rangle + Cov \,(entr 1) \right] \frac{d \ln A_{Tc}}{dx} -$$

$$\frac{1}{A_{Tc}} \sum_{\ell=e,i} \int_{J_{1\ell}} \left(\dot{m}_{1\ell} \, s_{1\ell} + \vec{q}^s_{1\ell} \cdot \hat{n}_{1\ell} \right) \frac{dA}{dx} - \frac{\partial}{\partial x} \, Cov \,(entr 1)$$

$$\frac{\partial}{\partial t} \alpha \langle\!\langle \rho_2 \rangle\!\rangle \langle s_2 \rangle + \frac{\partial}{\partial x} \alpha \langle\!\langle \rho_2 \rangle\!\rangle \langle V_{2x} \rangle \langle s_2 \rangle = \tag{8.7b}$$

$$= -\frac{\partial}{\partial x} \alpha \langle\!\langle q^s_{2x} \rangle\!\rangle + \alpha \langle\!\langle \rho_2 \rangle\!\rangle \langle \dot{s}_2 \rangle -$$

$$\left[\alpha \langle\!\langle \rho_2 \rangle\!\rangle \langle V_{2x} \rangle \langle s_2 \rangle + \alpha \langle\!\langle q^s_{2x} \rangle\!\rangle + Cov \,(entr 2) \right] \frac{d \ln A_{Tc}}{dx} -$$

$$\frac{1}{A_{Tc}} \sum_{\ell=e,i} \int_{J_{2\ell}} \left(\dot{m}_{2\ell} \, s_{2\ell} + \vec{q}^s_{2\ell} \cdot \hat{n}_{2\ell} \right) \frac{dA}{dx} - \frac{\partial}{\partial x} \, Cov \,(entr 2)$$

The entropy "jump" condition

$$\sum_{k=1}^{2} \dot{m}_{ki} s_{ki} + q^s_{ki} = \frac{\nabla}{T_i} \frac{1}{H^2} \left[\vec{r}_1 \left(G \frac{\partial}{\partial u_1} - F \frac{\partial}{\partial u_2} \right) + \right. \tag{8.12}$$

$$\left. \vec{r}_2 \left(E \frac{\partial}{\partial u_2} - F \frac{\partial}{\partial u_1} \right) \right] \cdot \vec{V}_i - \sum_{k=1}^{2} \frac{P_{ki}}{T_i} \, (\vec{U}_{ki} - \vec{V}_i) \cdot \hat{n}_{ki}$$

Equations listed above are quite general. For the purpose of practical applications these equations were reduced to the simplest form possible. At the same time, however, the terms which account for the essential physical processes were retained. As a first step of the simplifications, we restricted the developments to two-dimensional flow fields. With this approximation the general equations can be simplified to plane and annular flows geometries which are summarized below.

9.2.2   Plane Flow Formulation

For this particular flow geometry, the field and interfacial "jump" conditions are:

The equation of continuity of each phase:

$$\frac{\partial}{\partial t}\, \eta \ll \rho_1 \gg + \frac{\partial}{\partial x}\, \eta \ll \rho_1 \gg \langle v_{1x} \rangle = -\dot{m}_{1i}\left[1+\left(\frac{\partial \eta}{\partial x}\right)^2\right]^{1/2} - \dot{m}_{1e} \quad (5.8a)$$

$$\frac{\partial}{\partial t}\, (H-\eta) \ll \rho_2 \gg + \frac{\partial}{\partial x}\, (H-\eta) \ll \rho_2 \gg \langle v_{2x} \rangle = -\dot{m}_{2i}\left[1+\left(\frac{\partial \eta}{\partial x}\right)^2\right]^{1/2} \quad (5.8b)$$

If $\left(\frac{\partial \eta}{\partial x}\right) \ll 1.0$, then $\left(\frac{\partial \eta}{\partial x}\right)^2$ can be neglected.

The interfacial mass balance:

$$\dot{m}_{1i} + \dot{m}_{2i} = 0 \quad (5.5)$$

The momentum equation of each phase:

$$\frac{\partial}{\partial t}\, \eta \ll \rho_1 \gg \langle v_{1x} \rangle + \frac{\partial}{\partial x}\, \eta \ll \rho_1 \gg \langle v_{1x} \rangle^2 = -\frac{\partial}{\partial x}\, \eta \ll P_1 \gg + \quad (6.31a)$$

$$\ll \rho_1 \gg \eta\, g_x - \dot{m}_{1i}\, v_{1ix} + \tau_{1i} - \tau_{1e} - \frac{\partial}{\partial x}\, \eta \ll \rho_1 \gg \text{Cov}\,(v_{1x}\, v_{1x})$$

$$\frac{\partial}{\partial t}\, (H-\eta) \ll \rho_2 \gg \langle v_{2x} \rangle + \frac{\partial}{\partial x}\, (H-\eta) \ll \rho_2 \gg \langle v_{2x} \rangle^2 = -\frac{\partial}{\partial x}\, (H-\eta) \ll P_2 \gg + \quad (6.31b)$$

$$\ll \rho_2 \gg (H-\eta)\, g_x - \dot{m}_{2i}\, v_{2ix} - \tau_{2i} - \frac{\partial}{\partial x}\, (H-\eta) \ll \rho_2 \gg \text{Cov}\,(v_{2x}\, v_{2x})$$

The interfacial momentum balance: in normal direction

$$P_{1i} - P_{2i} = \dot{m}_{1i}^2\, \frac{\Delta \rho}{\rho_{1i}\, \rho_{2i}} - \sigma\, \frac{\partial^2 \eta}{\partial x^2} \quad (6.37a)$$

in tangential direction:

$$\mathcal{E}_{2i} - \mathcal{E}_{1i} = \frac{\partial \sigma}{\partial x} \qquad (6.37b)$$

The total energy equation of each phase:

$$\frac{\partial}{\partial t}\, \eta \ll \rho_1 \gg \left(\langle i_1\rangle + \frac{\langle v_1\rangle^2}{2}\right) + \frac{\partial}{\partial x}\, \eta \ll \rho_1 \gg \langle v_{1x}\rangle \left(\langle i_1\rangle + \frac{\langle v_1\rangle^2}{2}\right) = \qquad (7.19a)$$

$$= \eta \ll \rho_1 \gg g_x \langle v_{1x}\rangle + \frac{\partial}{\partial t}\, \eta \ll P_1 \gg - \left[\dot{m}_{1i}\left(i_{1i} + \frac{v_{1i}^2}{2}\right) + g_{1i} + P_{1i}\, v_{in} - \mathcal{E}_{1xy})_i\, v_{1ix}\right] -$$

$$\left[\dot{m}_{1e}\left(i_{1e} + \frac{v_{1e}^2}{2}\right) - g_{1e}\right] - \frac{\partial}{\partial t}\, Cov\,(mech\,1) - \frac{\partial}{\partial x}\, Cov\,(en\,1)$$

$$\frac{\partial}{\partial t}\,(H-\eta) \ll \rho_2 \gg \left(\langle i_2\rangle + \frac{\langle v_2\rangle^2}{2}\right) + \frac{\partial}{\partial x}\,(H-\eta) \ll \rho_2 \gg \langle v_{2x}\rangle \left(\langle i_2\rangle + \frac{\langle v_2\rangle^2}{2}\right) = (7.19b)$$

$$= (H-\eta) \ll \rho_2 \gg g_x \langle v_{2x}\rangle + \frac{\partial}{\partial t}\,(H-\eta) \ll P_2 \gg - \left[\dot{m}_{2i}\left(i_{2i} + \frac{v_{2i}^2}{2}\right) - \right.$$

$$\left. g_{2i} - P_{2i}\, v_{in} + \mathcal{E}_{2xy})_i\, v_{2ix}\right] - \frac{\partial}{\partial t}\, Cov\,(mech\,2) - \frac{\partial}{\partial x}\, Cov\,(en\,2)$$

The interfacial total energy balance:

$$\dot{m}_{1i}\left[(i_{1i} - i_{2i}) + \left(\frac{v_{1i}^2}{2} - \frac{v_{2i}^2}{2}\right) + (g_{1i} - g_{2i}) + (P_{1i} - P_{2i})\, v_{in} - (7.21)\right.$$

$$(\mathcal{E}_{1xy})_i\, v_{1ix} - \mathcal{E}_{2xy})_i\, v_{2ix}) = -\sigma \frac{\partial^2 \eta}{\partial x^2} + \hat{t}_i \cdot \frac{\partial \sigma \vec{v}_i}{\partial x}$$

It can be seen that the plane flow is expressed in terms of six field equations (three for each phase), and four interfacial conditions, (one for mass, one for energy, and two for momentum transport). However, counting the number of variables we observe that the mixture is described in terms of nine main variables which are α, $\ll \rho_1 \gg$, $\ll \rho_2 \gg$, $\ll P_1 \gg$, $\ll P_2 \gg$ $\langle v_{1K}\rangle$, $\langle v_{2x}\rangle$, $\langle i_1\rangle$ and $\langle i_2\rangle$, and eighteen boundary values of various variables which are ρ_{1i}, ρ_{2i}, P_{1i}, P_{2i}, v_{1ix}, v_{2ix}, v_{in}, i_{1i}, i_{2i}, τ_{1i}, τ_{2i}, τ_{1e}, q_{1i}, q_{1e}, m_{1i}, m_{2i} and σ. Therefore, additional seventeen

relations must be specififed for the complete description of the system based on the two-fluid model.

In order to make the problem tractable it was assumed that the density variations over the cross sectional planes are negligible, i.e., $\ll \rho_k \gg \, \simeq \, \rho_k$, and that the vapor phase is at saturation temperature. With these assumptions the additional relations that are needed to completely describe the system reduces to fifteen and they are:

The thermal constitutive equation of state for the liquid

$$\ll \rho_1 \gg (x,t) = \ll \rho_1 \gg (\, \langle i_1 \rangle \, , \, \ll P_1 \gg) \qquad (4.6)$$

which relates the density of the liquid to the averaged enthalpy of the liquid and to the pressure.

The thermal constitutive equation of state for the vapor:

$$\ll \rho_2 \gg (x,t) = \ll \rho_2 \gg (\, \ll P_2 \gg) \qquad (4.6)$$

which relates the vapor density to the local saturation pressure.

The caloric equation of state for the vapor:

$$\ll i_2 \gg (x,t) = \ll i_2 \gg (\, \ll P_1 \gg) \qquad (4.7)$$

which specifies the vapor enthalpy in terms of the vapor pressure.

The constitutive equation of net vaporization:

$$\dot{m}_{1i} = - \dot{m}_{2i} = \dot{m}_{1i} (\, \langle i_1 \rangle \, , \, \alpha \, , \, \ll P_1 \gg) \qquad (9.1)$$

which specifies the net rate of vapor formulation per unit of interfacial area as function of the liquid enthalpy, concentration and pressure.

Mechanical constitutive equation:

$$\tau_{ie} = f_{ie} \; \frac{1}{2} \; \langle\langle \rho_i \rangle\rangle \; \langle v_{ix} \rangle^2 \qquad (9.2)$$

which relates the shear stress in terms of the liquid density and velocity.
The energetic constitutive equation:

$$h_{ie} = h \; (\langle v_{ix} \rangle , \alpha , \langle i_i \rangle , \langle\langle \rho_i \rangle\rangle) \qquad (9.3)$$

which specifies the heat transfer coefficient as a function of liquid
phase main variables.
Constitutive equation for surface tension:

$$\sigma = \sigma \; (\langle i_2 \rangle) \qquad (9.4)$$

which specifies the surface tension as a function of the vapor enthalpy.
Interfacial motion:

$$\nu_\cdot = \frac{\partial \eta}{\partial t} \qquad (7.22)$$

which relates the normal component of the interfacial velocity, $\vec{v}_i$, to
the volumetric concentration $(1 - \alpha) = \eta/H$.
Interfacial Pressures:

$$\langle\langle P_1 \rangle\rangle = P_{1i} - \frac{1}{2} \; \langle\langle \rho_1 \rangle\rangle \; \eta \; g_y \qquad (6.30a)$$

$$\langle\langle P_2 \rangle\rangle = P_{2i} + \frac{1}{2} \; \langle\langle \rho_2 \rangle\rangle \; (H-\eta) \; g_y \qquad (6.30b)$$

which relate the interfacial pressure to the area-averaged pressures.
Interfacial velocities:

Velocity profile depends on the flow regime, i.e., laminar or
turbulent regimes, experienced by an individual phase. Therefore, de-
pending on the flow regime interfacial value of the velocity can be

related to the averaged value and to the concentration, i.e.,

$$v_{1ix} = v_{1ix} \left(\alpha , \langle v_{1x} \rangle \right) \tag{9.5}$$

and

$$v_{2ix} = v_{2ix} \left(\alpha , \langle v_{2x} \rangle \right) \tag{9.6}$$

Interfacial shear:

Either τ_{1i} or τ_{2i} must be also supplemented for the completeness of the formulation based on the two-fluid model. This can be expressed as

$$\tau_{1i} = \frac{1}{2} f_{1i} \langle\langle \rho_1 \rangle\rangle V_r^2 \tag{9.7}$$

Finally, since the vapor is at the saturation condition, then

$$\langle i_2 \rangle \simeq i_2 \simeq i_{2i} \tag{9.8}$$

and

$$\vec{q}_{2i} \simeq 0 \tag{9.9}$$

It concludes that the system becomes complete with the six field equations, four internal boundary conditions, and fifteen additional constitutive equations.

9.2.3 Annular Flow

For this flow geometry, the field and the interfacial "jump" conditions are:

The equation of continuity of each phase:

$$\frac{\partial}{\partial t}\,(1-\alpha)\,\langle\!\langle \rho_1 \rangle\!\rangle + \frac{\partial}{\partial x}\,(1-\alpha)\,\langle\!\langle \rho_1 \rangle\!\rangle\,\langle v_{1x} \rangle = \qquad (5.11a)$$

$$= -\,(1-\alpha)\,\langle\!\langle \rho_1 \rangle\!\rangle\,\langle v_{1x} \rangle\,\frac{d\,\ln D_e^{\pm}}{dx} + \Gamma_{1i} + \Gamma_{1e}$$

$$\frac{\partial}{\partial t}\,\alpha\,\langle\!\langle \rho_2 \rangle\!\rangle + \frac{\partial}{\partial x}\,\alpha\,\langle\!\langle \rho_2 \rangle\!\rangle\,\langle v_{2x} \rangle = -\,\alpha\,\langle\!\langle \rho_2 \rangle\!\rangle\,\langle v_{2x} \rangle\,\frac{d\,\ln D_e^{\pm}}{dx} + \Gamma_{2i} \qquad (5.11b)$$

The interfacial mass balance:

$$\Gamma_{1i} + \Gamma_{2i} = 0 \qquad (5.5)$$

The momentum equation of each phase:

$$\frac{\partial}{\partial t}\,(1-\alpha)\,\langle\!\langle \rho_1 \rangle\!\rangle\,\langle v_{1x} \rangle + \frac{\partial}{\partial x}\,(1-\alpha)\,\langle\!\langle \rho_1 \rangle\!\rangle\,\langle v_{1x} \rangle^2 = -\,(1-\alpha)\,\frac{\partial P_1}{\partial x} + \qquad (6.40a)$$

$$(1-\alpha)\,\langle\!\langle \rho_1 \rangle\!\rangle\,g_x - \left[\,(1-\alpha)\,\langle\!\langle \rho_1 \rangle\!\rangle\,\langle v_{1x} \rangle + \mathrm{Cov}\,(\mathrm{mom}\,1)\right]\frac{d\,\ln D_e^{\pm}}{dx} +$$

$$\Gamma_{1i}\,v_{1i_x} + \Gamma_{1e}\,v_{1e_x} + \frac{4\sqrt{\alpha}}{D_e}\,\tau_{1i} - \frac{4}{D_e}\,\tau_{1e} - \frac{\partial}{\partial x}\,\mathrm{Cov}\,(\mathrm{mom}\,1)$$

$$\frac{\partial}{\partial t}\,\alpha\,\langle\!\langle \rho_2 \rangle\!\rangle\,\langle v_{2x} \rangle + \frac{\partial}{\partial x}\,\alpha\,\langle\!\langle \rho_2 \rangle\!\rangle\,\langle v_{2x} \rangle^2 = -\,\alpha\,\frac{\partial P_2}{\partial x} + \alpha\,\langle\!\langle \rho_2 \rangle\!\rangle\,g_x - \qquad (6.40b)$$

$$\left[\,\alpha\,\langle\!\langle \rho_2 \rangle\!\rangle\,\langle v_{2x} \rangle^2 + \mathrm{Cov}\,(\mathrm{mom}\,2)\right]\frac{d\,\ln D_e^{\pm}}{dx} + \Gamma_{2i}\,v_{2i_x} - \frac{4\sqrt{\alpha}}{D_e}\,\tau_{2i} - \frac{\partial}{\partial x}\,\mathrm{Cov}\,(\mathrm{mom}\,2)$$

The interfacial momentum balance: in normal direction

$$P_{1i} - P_{2i} = -\left[\frac{2}{D_e\sqrt{\alpha}} - \frac{\partial^2}{\partial x^2}\,\frac{D_e\sqrt{\alpha}}{2}\right]\sigma + \dot{m}_{1i}^2\,\frac{\Delta\rho}{\rho_{1i}\,\rho_{2i}} \qquad (6.43a)$$

in tangential direction:

$$\tau_{2i} - \tau_{1i} = \frac{\partial\sigma}{\partial x} \qquad (6.43b)$$

The energy equation of each phase:

$$\frac{\partial}{\partial t}\,(1-\alpha)\,\ll\rho_1\gg\,(\langle i_1\rangle+\frac{\langle v_1\rangle^2}{2}) + \frac{\partial}{\partial x}\,(1-\alpha)\,\ll\rho_1\gg\,(\langle i_1\rangle+\frac{\langle v_1\rangle^2}{2})\langle v_{1x}\rangle = (7.25a)$$

$$= \frac{\partial}{\partial t}\,(1-\alpha)\,P_1 + (1-\alpha)\ll\rho_1\gg g_x\langle v_{1x}\rangle -$$

$$\left[(1-\alpha)\ll\rho_1\gg\langle v_{1x}\rangle\,(\langle i_1\rangle+\frac{\langle v_1\rangle^2}{2}) + Cov(en\,1)\right]\frac{d\ln D_e^2}{dx} -$$

$$\Gamma_{1i}\,(i_{1i}+\frac{v_{1i}^2}{2}) - [-g_{1i}+P_{1i}v_{in}+\tau_{1xr}|_i\,v_{1x}]\,\frac{4\sqrt{\alpha}}{D_e} -$$

$$\Gamma_{1e}\,(i_{1e}+\frac{v_{1e}^2}{2}) - \frac{4}{D_e}\left[1+\left(\frac{d}{dx}\,\frac{D_e}{2}\right)^2\right]^{1/2} g_{1e} -$$

$$\frac{\partial}{\partial t}\,Cov(mech\,1) - \frac{\partial}{\partial x}\,Cov(en\,1)$$

$$\frac{\partial}{\partial t}\,\alpha\,\ll\rho_2\gg\,(\langle i_2\rangle+\frac{\langle v_2\rangle^2}{2}) + \frac{\partial}{\partial x}\,\alpha\ll\rho_2\gg(\langle i_2\rangle+\frac{\langle v_2\rangle^2}{2})\langle v_{2x}\rangle = (7.25b)$$

$$= \frac{\partial}{\partial t}\,\alpha\,P_2 + \alpha\ll\rho_2\gg g_x\langle v_{2x}\rangle -$$

$$\left[\alpha\ll\rho_2\gg\langle v_{2x}\rangle\,(\langle i_2\rangle+\frac{\langle v_2\rangle^2}{2}) + Cov(en\,2)\right]\frac{d\ln D_e^2}{dx} -$$

$$\Gamma_{2i}\,(i_{2i}+\frac{v_{2i}^2}{2}) - [g_{2i}-P_{2i}v_{in}-\tau_{2xr}|_i\,v_{2ix}]\,\frac{4\sqrt{\alpha}}{D_e} -$$

$$\frac{\partial}{\partial t}\,Cov(mech\,2) - \frac{\partial}{\partial x}\,Cov(en\,2)$$

The interfacial energy balance:

$$\dot{m}_{1i}\left[(i_{1i}-i_{2i}) + \frac{1}{2}(v_{1i}^2-v_{2i}^2)\right] - (g_{1i}-g_{2i}) - \qquad (7.26)$$

$$(P_{1i}-P_{2i})\,v_{in} - (\tau_{1xr}|_i\,v_{1ix}-\tau_{2xr}|_i\,v_{2ix}) =$$

$$= \left(\frac{2}{D_e\sqrt{\alpha}} - \frac{\partial^2}{\partial x^2}\,\frac{D_e\sqrt{\alpha}}{2}\right)\sigma\,v_{in} + \hat{t}_i\cdot\frac{\partial}{\partial x}\,\sigma\,\tilde{v}_i$$

Similar to the plane flow case, consistency of the system can be accomplished by counting the variables and equations. It can be shown that the same kind of constitutive equations are needed for completeness. However, annular flow geometry does not allow the pressure variations over the cross sectional planes, i.e.,

$$\langle\!\langle P_k \rangle\!\rangle = P_k = P_{ki} \qquad , \; K = 1,2 \qquad (6.39b)$$

Hence, the number of variables reduces by two so does the number of additional equations.

In view of the complexity of the problem, for practical applications it is desirable to reduce the system to the diffusion model.

9.3 Diffusion Model Formulation

9.3.1 General Formulation

The general balance equation of the mixture is given by -

$$
\frac{\partial}{\partial t}\, \rho_m \Psi_m + \frac{\partial}{\partial x}\, \rho_m v_{mx} \Psi_m = -\frac{\partial}{\partial x}\, \mathbb{F}_{mx} + \rho_m \dot{\Psi}_m - \tag{4.27}
$$

$$
\left[\rho_m v_{mx} \Psi_m + \mathbb{F}_{mx} + C_{ov}(v \cdot \Psi) \right] \frac{d \ell_n A_{T_c}}{dx} +
$$

$$
\frac{1}{A_{T_c}} \int_{\Sigma_i} \nabla_s \cdot \bar{\bar{\theta}}\; \frac{dA}{dx} - \frac{1}{A_{T_c}} \sum_{k=1}^{2} \int_{\Sigma_{ke}} \hat{n}_{ke}\left[\rho_{ke}(\vec{v}_{ke} \vec{v}_e)\, \Psi_{ke} + \right.
$$

$$
\left. \Phi_{ke} \right] \; \frac{dA}{dx} - \frac{\partial}{\partial x}\, C_{ov}(v \cdot \Psi)
$$

where $\mathbb{F}_m$ is the total mixture flux of a particular property ψ, which composed of two parts, molecular and diffusion (or drift) parts. It is defined by Equation (4.26).

Equation (4.27) describes the mixture field as a whole and takes into account the effects of the interactions at the internal boundaries between the phases and of the transport of quantities at the external boundaries.

This general balance equation was used to derive: a) the continuity equation of the mixture, b) the momentum equation of the mixture, c) the energy equations of the mixture, and d) the entropy equation of the mixture. In their general form they are:

The equation of continuity of the mixture:

$$\frac{\partial \rho_m}{\partial t} + \frac{\partial}{\partial x} \rho_m v_{m,x} = - \rho_m v_{m,x} \frac{d \ln A_{T_c}}{dx} + \sum_{k=1}^{2} \Gamma_{ke} \qquad (5.14)$$

We note that the kinematic field of a two-phase mixture is described in terms of two continuity equations. As discussed in Chapter 5, the additional continuity equation can be given in form of the diffusion equation or in the form of the void propagation equation.

The diffusional type of continuity equation was expressed in various ways. In terms of the drift velocity V_{2j} it was shown that diffusion equation can be expressed by

$$\frac{\partial}{\partial t} \alpha \ll \rho_2 \gg + \frac{\partial}{\partial x} \alpha \ll \rho_2 \gg j_x = - \frac{\partial}{\partial x} \alpha \ll \rho_2 \gg V_{2j_x} + \qquad (5.28a)$$

$$\sum_{L=e,i} \Gamma_{2\ell} - \alpha \ll \rho_2 \gg (j_x + V_{2j_x}) \frac{d \ln A_{T_c}}{dx}$$

Another alternative was derived by expressing the diffusion (or drift) flux in terms of the diffusion coefficient. However, it was shown that this method of representation was not useful in two-phase flow systems

because the diffusion coefficient has not been available.

The void propagation equation was obtained by making use of the kinematic wave velocity C_K, and in general it was given by

$$\frac{\partial \alpha}{\partial t} + C_K \frac{\partial \alpha}{\partial x} = \alpha (1-\alpha) \left[\frac{1}{\alpha \langle\!\langle \rho_1 \rangle\!\rangle} \frac{D_1 \langle\!\langle \rho_1 \rangle\!\rangle}{Dt} - \frac{1}{\langle\!\langle \rho_2 \rangle\!\rangle} \frac{D_2 \langle\!\langle \rho_2 \rangle\!\rangle}{Dt} \right] + \quad (5.34)$$

$$\frac{\rho_m}{\langle\!\langle \rho_1 \rangle\!\rangle \langle\!\langle \rho_2 \rangle\!\rangle} \Gamma_{2i} + \frac{(1-\alpha)\langle\!\langle \rho_1 \rangle\!\rangle \Gamma_{2a} - \alpha}{\langle\!\langle \rho_1 \rangle\!\rangle \langle\!\langle \rho_2 \rangle\!\rangle} - \alpha V_{2jx} - \alpha V_{2jx} \frac{d\ell_n A_{Tc}}{dx}$$

where the velocity of kinematic waves C_K, is given by

$$C_K = j_x + \frac{\partial}{\partial x} (\alpha V_{2jx}) = j_x + V_{2jx} + \alpha \frac{d V_{2jx}}{dx} \qquad (5.35)$$

It is evident that for the flow of the incompressible fluids this equation can be reduced greatly.

The momentum equation of the mixture:

$$\frac{\partial}{\partial t} \rho_m V_{mx} + \frac{\partial}{\partial x} \rho_m V_{mx}^2 = \frac{\partial}{\partial x} \pi_{mxx}^T + \rho_m g_x - \qquad (6.51)$$

$$\left[\rho_m V_{mx}^2 - \pi_{mxx}^T + Cov(momT) \right] \frac{d\ell_n A_{Tc}}{dx} + \frac{1}{A_{Tc}} \oint_{S_i} \nabla_S \cdot \bar{\bar{\theta}}_{(mom)} \frac{dA}{dx} -$$

$$\frac{1}{A_{Tc}} \sum_{k=1}^{2} \oint_{S_{ke}} \left[\dot{m}_{ke} V_{kex} + (\bar{\bar{\pi}}_{ke} \cdot \hat{n}_{ke}) \cdot \hat{i} \right] \frac{dA}{dx} - \frac{\partial}{\partial x} Cov(momT)$$

The total energy equation of the mixture:

$$\frac{\partial}{\partial t}\,\rho_m\left(\bar{\imath}_m + \frac{v_m^2}{2}\right) + \frac{\partial}{\partial x}\,\rho_m v_{mx}\left(\bar{\imath}_m + \frac{v_m^2}{2}\right) = -\frac{\partial \bar{q}_{mx}^c}{\partial x} + \qquad (7.43)$$

$$\frac{\partial}{\partial x}\left[\left(\bar{\bar{\Sigma}}_m \cdot \hat{\imath}\right)\cdot\bar{v}_m\right] + \rho_m\,\bar{q}\cdot\bar{v}_m + \frac{\partial \bar{P}_m}{\partial t} - \left\{\rho_m v_{mx}\left(\bar{\imath}_m + \frac{v_m^2}{2}\right) + \bar{q}_{mx}^c - \right.$$

$$\left.\left[\left(\bar{\bar{\Sigma}}_m \cdot \hat{\imath}\right)\cdot\bar{v}_m\right] + Cov\,(c_m T) + Cov\left[\left(\bar{\bar{\epsilon}}\cdot\hat{\imath}\right)\cdot\bar{v}_m\right]\right\}\frac{d\,\ell_m A_{T_c}}{dx} +$$

$$\frac{1}{A_{T_c}}\int_{\mathfrak{F}_i}\nabla_s\cdot\bar{\theta}_{(c_s)}\,\frac{dA}{dx} -$$

$$\frac{1}{A_{T_c}}\sum_{k=1}^{2}\int_{\mathfrak{F}_{ke}}\left[\dot{m}_{ke}\left(i_{ke}+\frac{v_{ke}^2}{2}\right)+\left(\vec{q}_{ke}-\bar{\bar{\Sigma}}_{ke}\cdot\vec{v}_{ke}\right)\cdot\hat{n}_{ke}\right]\frac{dA}{dx} -$$

$$\frac{\partial}{\partial x}\,Cov\,(me_m kT) - \frac{\partial}{\partial x}\,Cov\,(c_m T)$$

In addition to the total energy equation, auxiliary energy equations together with the entropy equations were derived. In what follows we shall only list the enthalpy and the entropy equations. These are given respectively by:

$$\frac{\partial}{\partial t}\, \rho_m\, i_m + \frac{\partial}{\partial x}\, \rho_m\, \upsilon_{mx}\, i_m = -\frac{\partial q_{mx}}{\partial x} + \frac{\partial P_m}{\partial t} + \frac{\partial}{\partial x}\,(P_m\, \upsilon_{mx}) + \quad (7.88)$$

$$\dot{\mathcal{E}}_{mR} + \dot{\mathcal{E}}_{mIR} - \frac{\partial}{\partial x}\left\{\left[(1-\alpha)\,\langle\!\langle\rho_1\rangle\!\rangle\langle i_1\rangle\, V_{1mx} + \alpha\,\langle\!\langle\rho_2\rangle\!\rangle\langle i_2\rangle\, V_{2mx}\right] - \right.$$

$$\left[(1-\alpha)\,\langle\!\langle P_1\rangle\!\rangle\, V_{1mx} + \alpha\,\langle\!\langle P_2\rangle\!\rangle\, V_{2mx}\right]\right\} - \left\{\rho_m\, \upsilon_{mx}\, i_m - \left[(1-\alpha)\,\langle\!\langle\rho_1\rangle\!\rangle\langle i_1\rangle\, V_{1mx} + \right.\right.$$

$$\alpha\,\langle\!\langle\rho_2\rangle\!\rangle\langle i_2\rangle\, V_{2mx}\right] - P_m\, \upsilon_{mx} - \left[(1-\alpha)\,\langle\!\langle P_1\rangle\!\rangle\, V_{1mx} + \alpha\,\langle\!\langle P_2\rangle\!\rangle\, V_{2mx}\right] +$$

$$Cov\,(ent\,T) - \left[(1-\alpha)\,Cov\,(P_1 \cdot V_{1x}) + \alpha\,Cov\,(P_2 \cdot V_{2x})\right]\right\}\,\frac{d\ln A_{T_c}}{dx} +$$

$$\frac{1}{A_{T_c}}\int_{S_i}(I.E)_S\,\frac{dA}{dx} - \frac{1}{A_{T_c}}\sum_{k=1}^{2}\int_{S_{ke}}\left\{\dot{m}_{ke}\, i_{ke} + \left[P_{ke}(\vec{V}_{ke} - \vec{v}_e) + q_{ke}\right]\cdot\hat{n}_{ke}\right\}\frac{dA} {dx} -$$

$$\frac{\partial}{\partial x}\cdot Cov\,(ent\,T) + \frac{\partial}{\partial x}\left[(1-\alpha)\,Cov\,(P_1 \cdot V_{1x}) + \alpha\,Cov\,(P_2 \cdot V_{2x})\right]$$

$$\frac{\partial}{\partial t}\, \rho_m\, s_m + \frac{\partial}{\partial x}\, \rho_m\, \upsilon_{mx}\, s_m = -\frac{\partial q_{mx}^s}{\partial x} + \rho_m\, \dot{s}_m - \quad (8.16)$$

$$\frac{\partial}{\partial x}\left[(1-\alpha)\,\langle\!\langle\rho_1\rangle\!\rangle\langle s_1\rangle\, V_{1mx} + \alpha\,\langle\!\langle\rho_2\rangle\!\rangle\langle s_2\rangle\, V_{2mx}\right] - \left\{\rho_m\, \upsilon_{mx}\, s_m + q_{mx}^s + \right.$$

$$\left[(1-\alpha)\,\langle\!\langle\rho_1\rangle\!\rangle\langle s_1\rangle\, V_{1mx} + \alpha\,\langle\!\langle\rho_2\rangle\!\rangle\langle s_2\rangle\, V_{2mx}\right] + Cov\,(ent\,T)\right\}\,\frac{d\ln A_{T_c}}{dx} -$$

$$\frac{1}{A_{T_c}}\int_{S_i}(ENT)_S\,\frac{dA}{dx} - \frac{1}{A_{T_c}}\sum_{k=1}^{2}\int_{S_{ke}}\left(\dot{m}_{ke}\rangle s_{ke} + \vec{q}_{ke}^s\cdot\hat{n}_{ke}\right)\frac{dA}{dx} - \frac{\partial}{\partial x}\,Cov\,(ent\,T)$$

Furthermore, it was shown that when the effect of <u>diffusion</u> and of <u>motion</u> are neglected the energy equation reduces to that derived by Gibbs for the <u>heterogeneous binary</u> mixtures.

9.3.2 Plane Flow Formulation

For plane flows geometry, the four field equations reduce to

The equation of continuity of the mixture:

$$\frac{\partial \rho_m}{\partial t} + \frac{\partial}{\partial x} \rho_m v_{mx} = \Gamma_{ie} \tag{5.16}$$

The equation of continuity of the vapor phase can be expressed as the diffusional type equation, that is

$$\frac{\partial}{\partial t} \alpha \langle\langle \rho_x \rangle\rangle + \frac{\partial}{\partial x} \alpha \langle\langle \iota_x \rangle\rangle j_x = -\frac{\partial}{\partial x} \alpha \langle\langle \rho_x \rangle\rangle V_{xjx} + \Gamma_{xi} \tag{5.29}$$

or as the void propagation equation:

$$\frac{\partial \alpha}{\partial t} + C_x \frac{\partial \alpha}{\partial x} = \alpha (1-\alpha) \left[\frac{1}{\langle \rho_x \rangle} \frac{D_1 \langle\langle \rho_1 \rangle\rangle}{Dt} - \frac{1}{\langle \rho_x \rangle} \frac{D_x \langle\langle \rho_x \rangle\rangle}{Dt} \right] + \tag{5.34}$$

$$\frac{\rho_m}{\langle \rho_x \rangle \langle \rho_x \rangle} \Gamma_{xi} - \frac{\alpha \Gamma_{iu}}{\langle \rho_x \rangle \langle \iota_x \rangle}$$

The momentum equation of the mixture:

$$\frac{\partial}{\partial t} \rho_m v_{mx} + \frac{\partial}{\partial x} \rho_m v_{mx}^2 = -\frac{\partial P_m}{\partial t} - \frac{\partial}{\partial x} \left[\frac{\alpha}{1-\alpha} \frac{\langle\langle \rho_x \rangle\rangle \langle \rho_x \rangle}{\rho_m} V_{xjx} \right] \tag{6.62}$$

$$\rho_m g_x - \frac{1}{H} \left[-\tau \left(\frac{\partial^2 \eta}{\partial x^2} \right) \left(\frac{\partial \eta}{\partial x} \right) + \frac{\partial \tau}{\partial x} \right] -$$

$$\frac{\tau_{iu}}{H} - \frac{\partial}{\partial x} Cov(mom\,\tau)$$

The total energy equation of the mixture:

$$\frac{\partial}{\partial t} \, \rho_m \left(\bar{\imath}_m + \frac{v_m^2}{2} \right) + \frac{\partial}{\partial x} \, \rho_m \, v_{mx} \left(\bar{\imath}_m + \frac{v_m^2}{2} \right) = \frac{\partial \bar{P}_m}{\partial t} - \tag{7.62}$$

$$\frac{\partial}{\partial x} \left[\propto \frac{\ll \rho_1 \gg \ll \rho_2 \gg}{\rho_m} \, V_{2jx} \, \Delta \langle \bar{\imath} \rangle \right] - \frac{\partial}{\partial x} \left(\pi_{Dxx} \, v_{mx} \right) +$$

$$\rho_m \, g_x \, v_{mx} - \frac{1}{H} \left\{ - \frac{\partial^2 \eta}{\partial x^2} \, v_{in} + \hat{t}_i \cdot \frac{\partial \bar{\sigma} \, \bar{v}_i}{\partial s} \right\} -$$

$$\frac{1}{H} \left[\dot{m}_{1L} \left(i_{1L} + \frac{v_{1L}^2}{2} \right) - g_{1Lg} \right] - \frac{\partial}{\partial t} \, \text{Cov} \, (\text{mech} \, T) - \frac{\partial}{\partial x} \, \text{Cov} \, (x - T)$$

It can be seen that the plane flow is expressed in terms of four field equations involving <u>eighteen</u> variables which are α, $\ll \rho_1 \gg$, $\ll \rho_2 \gg$, ρ_m $\ll P_1 \gg$, $\ll P_2 \gg$, P_m, $\langle i_1 \rangle$, $\langle i_2 \rangle$, i_m, v_{mx}, v_{2jx}, v_{in}, Γ_{2i}, q_{1e}, τ_{1e}, σ and j_x. Therefore, additional <u>fourteen</u> relations must be supplemented for the complete description of the system.

In what follows we shall assume that the density variations over the cross sectional planes are negligible, i.e., $\ll \rho_k \gg \simeq \rho_k$, and that the vapor phase is at saturation temperature. With these assumptions the additional relations that are necessary to completely describe the system are:

The thermal constitutive equation of state for the liquid:

$$\ll \rho_1 \gg (x, t) = \ll \rho_1 \gg (\langle i_1 \rangle, \ll P_1 \gg) \tag{4.6}$$

which relates the density of the liquid to the averaged enthalpy of the liquid and to the pressure.

Thermal constitutive equation of state for the vapor:

$$\langle\!\langle \rho_{\text{z}} \rangle\!\rangle (x,t) = \langle\!\langle \rho_{\text{z}} \rangle\!\rangle \, (\langle\!\langle P_{\text{z}} \rangle\!\rangle) \qquad\qquad (4.6)$$

which relates the vapor density to the local saturation pressure.

The thermal constitutive equation of state for the mixture:

$$\rho_{\text{m}}(x,t) = (1-\alpha) \langle\!\langle \rho_{\text{1}} \rangle\!\rangle + \alpha \langle\!\langle \rho_{\text{2}} \rangle\!\rangle \qquad\qquad (3.55)$$

which relates the mixture density ρ_m, to the volumetric concentration and to the two densities $\langle\!\langle\rho_1\rangle\!\rangle$ and $\langle\!\langle\rho_2\rangle\!\rangle$.

The caloric constitutive equation of state for the vapor:

$$i_{\text{2}}(x,t) = i_{\text{2}} \, (\langle\!\langle P_{\text{2}} \rangle\!\rangle) \qquad\qquad (4.7)$$

which specifies the vapor enthalpy in terms of the vapor pressure.

The caloric constitutive equation of state for the mixture:

$$i_{\text{m}}(x,t) = (1-\alpha) \frac{\langle\!\langle \rho_{\text{1}} \rangle\!\rangle}{\rho_{\text{m}}} \langle i_{\text{1}} \rangle + \alpha \frac{\langle\!\langle \rho_{\text{2}} \rangle\!\rangle}{\rho_{\text{m}}} \langle i_{\text{2}} \rangle \qquad \text{(Table 3.1)}$$

which expresses the enthalpy of the mixture in terms of concentration and enthalpies of both phases.

The constitutive equation for the mean pressure:

$$P_{\text{m}}(x,t) = (1-\alpha) \langle\!\langle P_{\text{1}} \rangle\!\rangle + \alpha \langle\!\langle P_{\text{2}} \rangle\!\rangle \qquad\qquad (6.54)$$

which expresses the mean pressure in terms of concentration and of pressures of both phases.

The equation for the pressure difference:

$$\langle\!\langle P_{\text{1}} \rangle\!\rangle - \langle\!\langle P_{\text{2}} \rangle\!\rangle = H^{\text{2}} \, \Gamma_{\text{2i}}^{\text{2}} \, \frac{\Delta \rho_{\text{i}}}{\rho_{\text{1i}} \rho_{\text{2i}}} - \frac{1}{2} \rho_{\text{m}} \, H \, g_{y} - \sigma \, \frac{\partial^{\text{2}} \eta}{\partial x^{\text{2}}} \qquad\qquad (9.10)$$

which is obtained by combining Equations (9.28) and (9.19a).

The kinematic constitutive equation:

$$V_{2jx} = V_{2jx} (v_{mx}, \alpha, P_m)$$ (9.11)

which specifies the vapor drift velocity in terms of the baricentric

velocity v_{mx}, concentration and pressure.

The constitutive equation of net vaporization:

$$\Gamma_{2i} = \Gamma_{2i} (<i_i>, \alpha, P_m)$$ (9.12)

which specifies the net rate of vapor mass formation per unit volume as

function of the liquid enthalpy, concentration and pressure.

The mechanical constitutive equation:

$$\tau_m = f_m \frac{1}{2} \rho_m v_{m}^2$$ (9.13)

which expresses the shear stress in terms of the local mixture density

and baricentric velocity.

The energetic constitutive equation:

Defining by h, the heat transfer coefficient, it follows that, in

general

$$h = h (v_{mx}, \alpha, <i_i>, P_m)$$ (9.14)

which specifies the heat transfer coefficient as a function of bari-

centric velocity, concentration, liquid enthalpy and pressure.

The constitutive equation for the surface tension:

$$\sigma = \sigma (<i_2>)$$ (9.15)

which specifies the surface tension as a function of the vapor enthalpy.

The interfacial motion:

$$v_{in} = \frac{\partial \eta}{\partial t}$$ (7.22)

which relates the normal component of the interfacial velocity V_i, to the volumetric concentration, $(1 - \alpha) = \eta/H$.

The velocity relation for j_x:

$$j_x = V_{mx} + \alpha \frac{\Delta \rho}{\rho_m} V_{xjx} \qquad \text{(Table 3.2)}$$

The system is therefore, complete and consists of four field equations, and fourteen additional constitutive equations involving eighteen variables.

9.3.3 Annular Flow

For this flow geometry, the four field equations are:

The equation of continuity of the mixture:

$$\frac{\partial \rho_m}{\partial t} + \frac{\partial}{\partial x} \rho_m V_{mx} = - \rho_m V_{mx} \frac{d \ln D_c^2}{dx} + \Gamma_{ic} \qquad (5.18)$$

The equation of continuity of the vapor phase:

$$\frac{\partial}{\partial t} (\alpha \ll \rho_x \gg) + \frac{\partial}{\partial x} (\alpha \ll \rho_x \gg j_x) = - \frac{\partial}{\partial x} (\alpha \ll \rho_x \gg V_{xjx}) + \qquad (5.28b)$$

$$\Gamma_{xi} - \alpha \ll \rho_x \gg (j_x + V_{xjx}) \frac{d \ln D_c^2}{dx}$$

which can be also expressed as the void propagation equation:

$$\frac{\partial \alpha}{\partial t} + C_k \frac{\partial \alpha}{\partial x} = \alpha (1-\alpha) \left[\frac{1}{\ll \rho_x \gg} \frac{D_x \ll \rho_x \gg}{Dt} - \frac{1}{\ll \rho_x \gg} \frac{D_x \ll \rho_x \gg}{Dt} \right] + \qquad (5.34)$$

$$\frac{\rho_m}{\ll \rho_x \gg \ll \rho_x \gg} \Gamma_{xi} - \frac{\alpha \Gamma_{ic}}{\ll \rho_x \gg} - \alpha V_{xjx} \frac{d \ln D_c^2}{dx}$$

The momentum equation of the mixture:

$$\frac{\partial}{\partial t}\,\varrho_m\,v_{mx} + \frac{\partial}{\partial x}\,\varrho_m\,v_{mx}^2 = -\frac{\partial P_m}{\partial x} - \frac{\partial}{\partial x}\left[\frac{\alpha}{1-\alpha}\,\frac{\langle\!\langle \varrho_1 \rangle\!\rangle \,\langle\!\langle \varrho_2 \rangle\!\rangle}{\varrho_m}\,V_{2j x}^2\right] + \quad (6.65)$$

$$\varrho_m\,g_x - \left[\varrho_m\,v_{mx}^2 + \frac{\alpha}{1-\alpha}\,\frac{\langle\!\langle \varrho_1 \rangle\!\rangle \,\langle\!\langle \varrho_2 \rangle\!\rangle}{\varrho_m}\,V_{2j x}^2 + \text{Cov}\,(\text{mom}\Gamma)\right]\frac{d\ln D_c^2}{dx} -$$

$$\frac{4\sqrt{\pi}}{D_c}\left\{\left[-\frac{2}{D_c\sqrt{\alpha}} + \frac{\partial^2}{\partial x^2}\left(\frac{D_c\sqrt{\alpha}}{2}\right)\right]\sigma\,\frac{\partial}{\partial x}\left(\frac{D_c\sqrt{\alpha}}{2}\right) + \frac{\partial C}{\partial x}\right\} +$$

$$\Gamma_{1c}\,v_{1cx} + \frac{2}{D_c}\,(P_1 - P_2)\,\frac{dD_c}{dx} - \frac{4}{D_c}\,\tau_{1c} - \frac{\partial}{\partial x}\,\text{Cov}\,(\text{mom}\,\Gamma)$$

The total energy equation of the mixture:

$$\frac{\partial}{\partial t}\,\varrho_m\left(\bar{\iota}_m + \frac{v_m^2}{2}\right) + \frac{\partial}{\partial x}\,\varrho_m\,v_{mx}\left(\bar{\iota}_m + \frac{v_m^2}{2}\right) = \frac{\partial P_m}{\partial t} - \quad (7.64)$$

$$\frac{\partial}{\partial x}\left[\alpha\,\frac{\langle\!\langle \varrho_1 \rangle\!\rangle \,\langle\!\langle \iota_{\iota_2} \rangle\!\rangle}{\varrho_m}\,V_{2j x}\,\Delta\langle\bar{\iota}\rangle\right] - \frac{\partial}{\partial x}\,(\pi_{Dx}\,v_{mx}) + \varrho_m\,g_x\,v_{mx} -$$

$$\frac{4\sqrt{\pi}}{D_c}\left\{\left[\frac{2}{D_c\sqrt{\alpha}} - \frac{\partial^2}{\partial x^2}\left(\frac{D_c\sqrt{\alpha}}{2}\right)\right]\sigma\,v_{in} + \hat{\imath}_i\cdot\frac{\partial C\,\vec{v}_i}{\partial s}\right\} -$$

$$\frac{4}{D_c}\left[\dot{m}_{1c}\left(i_{1c} + \frac{v_{1c}^2}{2}\right) + q_{1cn} - \tau_{xx}\,v_{1cx}\right]\left[1 + \left(\frac{dD_c}{dx}\right)^2\right]^{1/2} -$$

$$\frac{\partial}{\partial t}\,\text{Cov}\,(\text{mech}\,\Gamma) - \frac{\partial}{\partial x}\,\text{Cov}\,(\text{en}\,\Gamma)$$

As for the case of plane flow, it can be shown that the same kind of constitutive equations are needed for completing the total system of equations.

PART II

STABILITY ANALYSIS

CHAPTER X

INTRODUCTION

10.1 Significance of the Problem

The flow of liquid films is a separated two-phase flow pattern of great interest to various technologies because many engineering operations and systems are greatly affected by the behavior of such films. In the aerospace technology, for example, the performance of a rocket engine cooled by a liquid film, depends on the protective effectiveness of the film. In the chemical process, nuclear reactor and power generating industries the performance of distillation and absorption plants; of condensers, boilers and evaporators, of desalination plants; of nuclear reactors, etc., are greatly affected by the film because the process of mass and heat transfer which occur in these systems, are intimately connected to fluid motion.

One of the most important problems in this area is concerned with the stability and the destruction of liquid films flowing over heated surfaces. The appearance of large dry patches in liquid films flowing over surfaces heated or unheated have been recently observed and reported by several researchers, references [64-66]. The appearance and behavior of these vapor patches in two-phase systems with heat addition is important for several reasons. First, the destruction of a liquid film which covers a heated surface results in abrupt decrease of heat transfer coefficient and attendant overheating of the surface which may cause the burn-out of the heated duct. In addition to this possibility, the

destruction of the liquid marks a change in the flow regime from the annular to the drop flow regime. Since it is known that the steady state parameters (the pressure drop, the heat transfer coefficient, etc.) as well as the transient response of a system depend on the flow regime, it is of interest to obtain criteria which could be used to predict such a change of flow pattern.

10.2 Previous Work

Two kinds of stabilities are involved in the liquid film flow. One is related to the stability of flow within the liquid film, known as _flow stability_ whereas the other is related to the disturbances on the interface of the liquid film, known as _interface stability_.

Flow stability is similar to that involved in pipe or channel flows, Tollmien-Schlichting instability. It occurs when a fluid is undergoing transition from laminar to turbulent flow as a result of amplification by viscosity of infinitesimal disturbances in the fluid. These disturbances originate either inside or outside the film as has been demonstrated by changing the turbulence level of the flow external to the boundary layer or film. This type of instability in boundary layers of a homogenous fluid has been extensively studied by Lin [67], see also reference [54]. The criterion for the onset of this stability is that the Reynolds number exceed a certain value called the minimum critical Reynolds number.

However, the interface stability is peculiar to film flow and can be subdivided into inception of interfacial disturbances and inception of liquid entrainment leading to the breakdown of the liquid film. There is no doubt that these instabilities are interrelated and have important

effects on the flow pattern, pressure drop and heat transfer of the flow system. The interfacial instability which governs the liquid film flow is our basic concern, and we shall now review the existing work which has been carried out in the past.

Falling film flow in which the velocity of the gas phase is negligible, can be considered as a special case of two-phase flow. In fact, many important contributions have resulted from the study of such films. Theoretical work on this subject was primarily concerned with the inception of the interfacial waves. Although there has been many publications, mainly two approaches have been used to analyze the behavior of such liquid film:

1. Periodic Solutions of Steady State

2. Linear Stability Analyses.

We shall review them briefly below.

10.2.1 Periodic Solutions of Steady State

The former theory was developed first by Kapitza [68], and known as the Kapitza's theory of film flow description. Kapitza considered equation of motion and boundary conditions for thin films and he took into account surface tension effects. The author developed as a first approximation, a linearized treatment yielding the differential equation,

$$\frac{\sigma \bar{\eta}}{\rho_1} \frac{d^3 \phi}{dx^3} + \left(C_r - \langle v \rangle \right) \left(C_r - \frac{9}{10} \langle v \rangle \right) \frac{d\phi}{dx} - \quad (10.1)$$

$$\frac{3\nu}{\bar{\eta}^2} \left(C_r - 3 \langle v \rangle \right) \phi + g - \frac{3\nu \langle v \rangle}{\bar{\eta}^2} = 0$$

where C_r is the velocity of the wave propagation, $\langle v \rangle$ is the average

velocity, and $\emptyset$ is the free surface deformation function defined by

$$\eta = \bar{\eta} \, (1 + \phi) \tag{10.2}$$

In order for an undamped periodic solution to exist it is necessary that the constant term in Equation (10.1) and the coefficient of $\emptyset$ be equal to zero, so that, to the first approximation

$$\langle v \rangle = \frac{\bar{\eta} \, g}{3 \nu} \tag{10.3}$$

$$C_r = 3 \langle v \rangle \tag{10.4}$$

Using Equations (10.3 and 10.4), Equation (10.1) becomes

$$\frac{\sigma \bar{\eta}}{\rho} \, \frac{d^3 \phi}{dx^3} + 4.2 \langle v \rangle^2 \frac{d\phi}{dx} = 0 \tag{10.5}$$

from which the function $\emptyset$ defining the surface profile is

$$\phi = \alpha \, \sin \left[\langle v \rangle \left(4.2 \, \rho \, / \, \sigma \bar{\eta} \right)^{\frac{1}{2}} (x - C_r t) \right] \tag{10.6}$$

where α is the dimensionless wave amplitude.

Kapitza has also attempted to predict the equilibrium wave amplitude and the critical Reynolds number by minimizing the viscous energy dissipation, and further has provided some experimental data for qualitative verification of his theory. It is fair to say that his theory still lacks consistency and rigor, although many attempts have been made by others, references [56, 69, 70 among others], to rectify such objections. First of all, the third order equation, Equation (10.1), is not consistent because some of the terms are left out from the original derivation, and because the variation of the film thickness η with the

longitudional coordinate x, was neglected in most of the derivations. Second of all, what is even more disconcerting is the fact that the equilibrium wave amplitude and the critical Reynolds number are not the deductive consequences of the third order equation, Equation (10.1). Rather, they were deduced from a physical argument.

Recently, Lee [71] presented the derivation of Kapitza's third order equation by applying the boundary layer approximation to the Navier-Stokes equations, kinematic surface condition, and tangential and normal shear stress continuity conditions. Finally, the equilibrium wave amplitude was deduced entirely from the dynamic equations by investigating the non-linear effect on the isolated class of basic wave motions pertaining to the linear system. The wave speed and amplitude so obtained as the first-order approximation are shown to provide better comparison with the experimental data than those of references [56, 68, 69 and 70].

10.2.2 Linear Stability Analyses

The analyses which belong to this group, i.e., to the linearized stability analysis, attempted to investigate the stability of Nusselt's solution in terms of the linearized theory of small perturbations, references [72, 73 and 74]. The general method of dealing with the problem is to set up the main equations of flow, i.e., the Navier-Stokes equations, on which small perturbations are imposed, leading to an equation of the Orr-Sommerfeld type, which is then solved by various approximate means to determine the conditions for stability to exist.

In his early paper, Yih [72] carried out numerical calculations for stability in the case of flow on a vertical wall with no surface

tension effects, from which it appears that there exist a critical

Reynolds number, Re=1.5.

Benjamin [73] has given a detailed treatment of the onset of two-dimensional instability in film flow, taking capillary effects into account. It was deduced that surface tension has a stabilizing effect, especially at short wavelengths, but instability can not be converted to stability merely by increasing the surface tension. It was found that, for a vertical slope the critical Reynolds number found to be zero. Obviously this result contradicts Yih's result.

In order to resolve the controversy, recently, Yih [74] has given a detailed treatment of the stability of film flow on an inclined plane. Three cases are considered in detail: small wave number, i.e., large wave-length, small Reynolds numbers, and large wave numbers. In the first case the results are in agreement with the results of Benjamin, but large wave numbers and zero surface tension, Benjamin's conclusions are shown to be invalid.

Recently Bankoff [75] extended the Yih-Benjamin analysis of the stability to small surface perturbations of a thin liquid film falling down an inclined plane to take into account the effects of evaporation or condensation. In his formulation the interfacial kinematic condition was not used correctly. Furthermore, it was found there that evaporation destabilizes the film, whereas condensation stabilizes. This conclusion, however, as it will be demonstrated in Chapter 12, is not correct. Phase change at the interface, either in the form of condensation or in the form of evaporation, always destabilizes the liquid film.

It is important to point out here that there is a kind of reciprocal

relation between the conventional linearized stability analysis and Kapitza's method of analysis. In the former, a sinusoidal traveling wave is imposed along the mean flow direction and then the eigen value problem across the film is formulated to solve for the dispersion relation, i.e., the imaginary wave speed expressed in terms of all other parameters involved in the formulation. On the other hand, in the latter, i.e., in the Kapitza's method, the boundary condition for the laminar film flow are fulfilled approximately by the factor $(y-y^2/2\eta)$ and the periodicity conditions along the flow direction were sought.

In addition to the above analyses, the stability of flow in open channels has been investigated theoretically from a more macroscopic or hydraulic point of view by several authors. Most of these stability criteria are expressed in the form of a numerical value for the critical Froude number. Most of these treatments refer to flow in channels of very small slope, and under these circumstances, surface instability usually comes into existence in the turbulent regime. Hence, the results, which are based mainly on the Chezy or Manning coefficient for turbulent flow, are not directly applicable in the case of thin film flow on steep surfaces, where the instability of laminar flow is usually in question.

10.2.3 Experimental Investigations

In spite of its importance to various technologies, unfortunately, published literature on the subject of the inception of interfacial disturbance for a film flow adjoining to a gas or vapor flow is scarce. This is probably due to the fact that, in most cases of such flows, the interface is easily disturbed by the existence of the relative velocity.

Thus, it may be very difficult to define the point of instability. Some of the experimental results will be discussed here.

Among the experimental studies, Gazley [76] indicates that the formation of the interfacial disturbances for a given pair of fluids depends on the relative velocity of the two fluid streams, while the depth of the liquid film has only slight effect. A relative velocity of the order of 10-15 ft/min. was required for the formation of the interfacial disturbances.

The extensive experimental observations made by Dukler [77] in a vertical tube with downward annular flow led him to conclude that the liquid Reynolds number was not a significant parameter that determines wave motion or wave height, although he noted an increase in amplitude of the interface waves with an increase in the liquid velocity.

Kinney et al. [78] observed visually the annular liquid flow with the concurrent turbulent air flow in horizontal transparent tubes. The surface of the liquid film was observed to be relatively smooth at low liquid-flow rates. However, the liquid film becomes wavy when a critical flow rate was exceeded. Furthermore, for more viscous liquid films this critical flow rate was higher. Now, it can be inferred that in contradiction to Dukler's result the liquid Reynolds number is indeed an important parameter.

In a similar development Knuth [79] observed that small disturbances with wave lengths ten times the film thickness were present for all flow rates investigated. These observations are certainly contradicts the observations of reference [78] where a critical liquid flow rate was ascertained. Furthermore, according to Knuth's observations

for liquid flow rates larger than some critical value a second type of surface wave, one with a long wave length, appeared. Knuth, however, agreed with Kinney that the instability depended primarily on the liquid Reynolds number. We recall that this disagrees with Dukler's conclusion. Furthermore, Kinney and Knuth are in agreement that the gas Reynolds number has a negligible effect on interfacial stability.

Although existing work seemingly contains contradictions and disagreements, it does indicate that flow characteristics change and that objectionable phenomena can occur in two-phase flows when the gas-liquid interface becomes wavy.

10.2.4 Conclusions

The survey presented here is by no means complete. However, it is intended to serve as an indication of the scope of the research accomplished thus far, and to draw attention to the deficiency of our knowledge on the subject.

As it was observed the majority of research efforts thus far have been experimental. On the other hand, only qualitative trends have been indicated by the analytical approach to the falling film flow where the effects of the adjoining phase is neglected. Particularly, the effects of heat and mass transfer at the film boundaries were not considered at all. Thus, most investigations which were concerned with dynamic aspects of the interface, were formulated for adiabatic flows and neglected the effects of mass transfer. Furthermore, only the incipient point for the appearance for the waves are concerned. They did not consider the instability of the interfacial waves, i.e., when they will be effective to change the flow regime.

10.3 Effect of Heat and Mass Transfer

It is important here to stress the effects of heat and mass transfer on the stability of the liquid films.

The effects of the vapor shear and pressure forces on the entrainment of drops and on the subsequent destruction of the liquid film are present in flows with heat addition as well as in adiabatic flows. However, in nonisothermal systems additional mechanisms of liquid film destruction must be considered. It becomes of interest, therefore, to investigate the mechanism of film destruction by thermal effects.

Referring to Zuber and Staub [80], where the thermal effects were discussed in a concise way and the numerical evaluation of the vapor thrust on the Kelvin-Helmholtz instability were analyzed by expanding the method of Lamb [57], there are three mechanisms arising from thermal effects that can lead to the destruction of a liquid film flowing over a heated surface:

First, the temperature of the liquid-vapor interface may not be uniform, giving rise to a non-uniform surface tension and consequently, to a shear stress at the interface. This thermo-capillary effect, expressed by Equation (6.37b), and shown on Figure 3.a, induces a flow from the region of high temperature to the region of low temperature, i.e., from the region of low surface tension to the region of high surface tension. It can be seen that as the film becomes thinner the continuous flow from the furrows to the peaks will lead to the destruction of the film.

Second, as the vaporization takes place at the interface there is a thrust exerted by the vapor in the direction perpendicular, i.e., toward the heating surface. This vapor thrust mechanism, expressed by
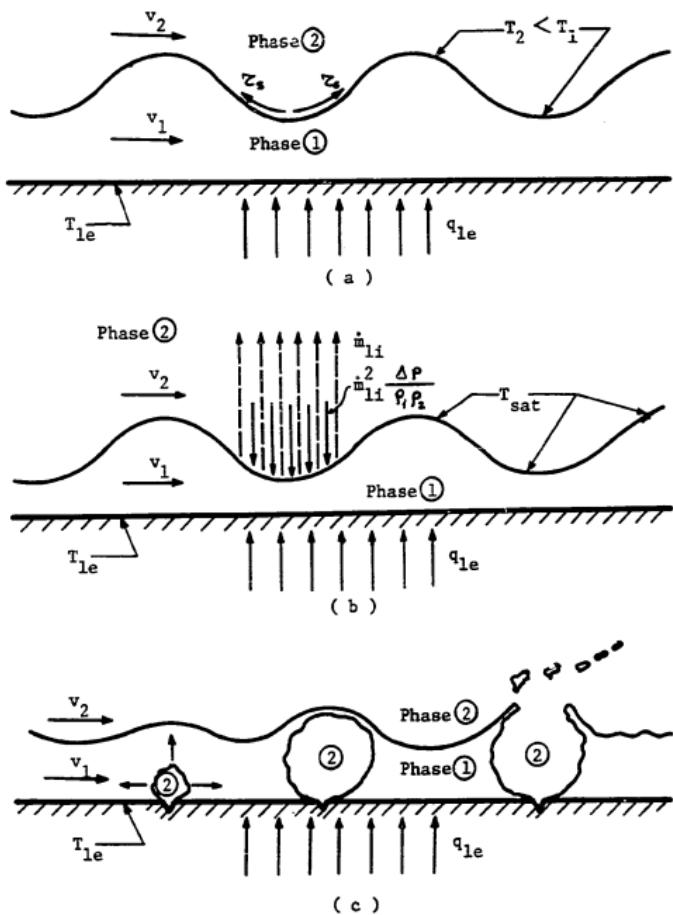
Figure 3. Destruction of Thin Liquid Film Caused by Thermal Effects:

a) Thermo-Capillary Effect
b) Vapor-Thrust Effect
c) Bubble Growth Effect

Equation (6.37a), and shown on Figure 3.b can lead to the destruction of the film because the vaporization and, consequently, the destabilizing effect of the vapor thrust increases as the liquid film becomes thinner.

Finally, bubble nucleation may lead to the destruction of the film as shown on Figure 3.c. Whether or not the bubble nucleation will lead to the destruction of the film will depend on the relative dimensions of the bubble and of the film as well as on the heat flux density and the liquid flow.

We note that for a liquid film in the turbulent flow regime the variations of the temperature of the liquid-vapor interface may become too small and short lived to generate a dry patch. It can be added here that the process of vaporization at the interface together with the flow of a turbulent vapor at saturation temperature will also tend to minimize the temperature variations at the interface. Consequently, under these conditions whether or not the thermo-capillary effect is important remains to be determined from experiments because any formulations of this problem must be based on assumptions that need verification.

In contrast to the thermo-capillary force, the vapor thrust due to vaporization does not depend on temperature variations along the interface. Consequently, the destabilizing effect of the vapor thrust can be present when either the liquid or the vapor or both are in turbulent flow. It is important to emphasize here that, whereas the vaporization at the interface has a destabilizing effect on a liquid film, it has a stabilizing effect on a vapor film, see Equation (6.37) where it changes its sign from the liquid film to vapor film. This difference is a consequence of the change in the direction of the vapor thrust: toward the heating surface

in liquid films and away from it in vapor films. It is expected, there-fore, in the Rayleigh-Taylor instability vapor thrust will have a stabilizing effect whereas in Kelvin-Helmholtz instability vapor thrust will have a destabilizing effect.

It can be concluded from the preceding discussion that any one of the three mechanism which arise from thermal effects may lead to the formation of dry patches in liquid films flowing over a heated surface.

10.4 Purpose of Part II

It appears from the foregoing that for liquid film flows, in addition to dynamic effects there are thermal effects which can lead to the destruction of liquid films. In Part I a formulation which takes into account these effects as well as dynamic effects were developed with respect to the two-fluid and the diffusion models.

Based on the two-fluid model formulation it is the purpose of Part II of thesis:

1. To develop a stability theory so that the conditions under which disturbances will grow can be calculated by determining the neutral stability condition,

2. To calculate the most unstable growth rate factor so that a criterion can be derived for the point where interfacial waves lead to unfavorable phenomena such as liquid entrainment leading to the break-up and flow plugging leading to change of flow regime,

3. To apply the criterion to analyze two specific cases such as the stability of the free film flow where the second phase velocity is neglected and the stability of two inviscid fluids.

CHAPTER XI

GENERALIZED STABILITY THEORY

11.1 Introduction

The equations of hydrodynamics, in spite of their complexity,
allow some simple patterns of flow as stationary solutions. These
patterns of flow can, however, be realized only for certain ranges of the
parameters characterizing them. Outside these ranges, they can not be
realized. The reason for this lies in their inherent instability, i.e.,
in their instability to sustain themselves against small perturbations
to which any physical system is subject. The question then may be
asked, for a given flow: is it stable relative to infinitesimaly small
disturbances? This is the problem of hydrodynamic stability.

The mathematical treatment of such a problem generally proceeds
along the following lines.

We start from an initial flow which represents a stationary state
of the system. By supposing that the various physical variables describ-
ing the flow suffer infinitesimal increments, we first obtain the equa-
tionsgoverning these increments. To solve the problem of hydrodynamic
stability, one must, therefore, obtain the solution of a system of non-
linear partial differential equations, in general a very difficult task.
In order to render the problem manageable, the mathematical formulation
is cast in a different way. We assume that, for small disturbances,
the equations may be linearized; that is, we shall neglect terms quadratic
or higher in the disturbances and their derivatives. The theory derived

on the basis of such linearized equations is called the linear stability theory in contrast to non-linear theories which attempt to allow for finite amplitudes of the perturbations.

It is the purpose of this chapter to use the linearized perturbation theory on the <u>two-fluid model balance equations</u> describing a separated two-phase flow system. An infinitesimal disturbance will be applied to the balance equatons, and the conditions under which these disturbances will grow will be calculated by determining the neutral stability condition. Furthermore, a criterion for determining the break-up length of a liquid film into drops will be suggested in terms of most rapidly growing waves.

11.2 Generalized Stability Criterion

11.2.1 Two-Fluid Model Conservation Equations

Consider the generalized separated two-phase flow system shown in Figure 2. The motion is assumed to be two-dimensional flow of incompressible fluids. The x-axis is taken in the main flow direction.

For the two-fluid model formulation, the equations governing the flow system were derived in the preceding chapters. For compressible fluids, the field equations appropriate for a two-phase flow in a constant area duct are given as follows:

The equation of continuity of each phase:

$$\frac{\partial (1-\alpha)}{\partial t} + \frac{\partial (1-\alpha) \langle v_{1x} \rangle}{\partial x} = \frac{\Gamma_{1i}}{\rho_1} \qquad (11.1a)$$

$$\frac{\partial \alpha}{\partial t} + \frac{\partial \alpha \langle v_{2x} \rangle}{\partial x} = \frac{\Gamma_{2i}}{\rho_2} \qquad (11.1b)$$

The interfacial mass balance:

$$\Gamma_{1i} + \Gamma_{2i} = 0 \qquad (11.2)$$

The equation of motion of each phase:

$$\rho_1 \left(\frac{\partial \langle v_1 \rangle}{\partial t} + \langle v_1 \rangle \frac{\partial \langle v_1 \rangle}{\partial x} \right) = -\frac{\partial \langle P_1 \rangle}{\partial x} + F_1 - \frac{1}{1-\alpha} \frac{\partial}{\partial x} Cov_{(mom 1)} \qquad (11.3a)$$

$$\rho_2 \left(\frac{\partial \langle v_2 \rangle}{\partial t} + \langle v_2 \rangle \frac{\partial \langle v_2 \rangle}{\partial x} \right) = -\frac{\partial \langle P_2 \rangle}{\partial x} + F_2 - \frac{1}{\alpha} \frac{\partial}{\partial x} Cov_{(mom 2)} \qquad (11.3b)$$

where F_1 and F_2 stand for the generalized forces including the effect of one phase on the other.

In view of Equations (11.3) and (6.32) for the plane flow configuration without external mass transfer, $\dot{m}_{1e} = 0$, F_1 and F_2 can be given by:

$$F_1 = \rho_1 H g_y \frac{\partial (1-\alpha)}{\partial x} + \rho_1 g_x + \frac{\Gamma_{1i}}{1-\alpha} (v_{1ix} - \langle v_{1x} \rangle) + \frac{\tau_{1i}}{H(1-\alpha)} - \frac{\tau_{1e}}{H(1-\alpha)} \qquad (11.4a)$$

and

$$F_2 = -\rho_2 H g_y \frac{\partial \alpha}{\partial x} + \rho_2 g_x + \frac{\Gamma_{2i}}{\alpha} (v_{2ix} - \langle v_{2x} \rangle) - \frac{\tau_{2i}}{H\alpha} - \frac{\tau_{2e}}{H\alpha} \qquad (11.4b)$$

whereas for the annular flow configuration in view of Equations (11.3) and (6.41) F_1 and F_2 can be given by:

$$F_1 = \rho_1 g + \Gamma_{1i} (v_{1ix} - \langle v_{1x} \rangle) + \frac{\tau_{1i}}{1-\alpha} \frac{4\sqrt{\alpha}}{D_e} - \frac{\tau_{1e}}{1-\alpha} \frac{4}{D_e} \qquad (11.5a)$$

and

$$F_2 = \rho_2 g + \Gamma_{2i} (v_{2ix} - \langle v_{2x} \rangle) + \frac{\tau_{2i}}{\alpha} \frac{4\sqrt{\alpha}}{D_e} \qquad (11.5b)$$

Finally, the covariance terms are defined by

$$Cov_{(mom 1)} = (1-\alpha) \rho_1 Cov(v_{1x} v_{1x}) = (1-\alpha) \rho_1 \left(\langle v_{1x} v_{1x} \rangle - \langle v_{1x} \rangle \langle v_{1x} \rangle \right) \qquad (11.6a)$$

and

$$Cov_{(mom\,2)} = \alpha \rho_2 \; Cov(v_{2i}\, v_{2x}) = \alpha \rho_2 \left(\langle v_{2i}\, v_{2x} \rangle - \langle v_{2x} \rangle \cdot \langle v_{2x} \rangle \right) \quad (11.6b)$$

For the plane flow, the momentum "jump" conditions in the normal and the tangential directions are, respectively given by:

$$P_{1i} - P_{2i} = \dot{m}_{1i}^2 \; \frac{\Delta e}{\rho_1 \rho_2} - \sigma \; \frac{\partial^2 \eta}{\partial x^2} \quad (11.7a)$$

$$\tau_{2i} - \tau_{1i} = \frac{\partial \sigma}{\partial x} \quad (11.7b)$$

whereas for the annular flow, the momentum "jump" conditions can be given by:

$$P_{1i} - P_{2i} = \dot{m}_{1i}^2 \; \frac{\Delta e}{\rho_1 \rho_2} - \sigma \left(\frac{2}{D_e \sqrt{\alpha}} - \frac{\partial^2}{\partial x^2} \; \frac{D_e \sqrt{\alpha}}{2} \right) \quad (11.8a)$$

$$\tau_{2i} - \tau_{1i} = \frac{\partial \sigma}{\partial x} \quad (11.8b)$$

It will be assumed that the vapor phase is at the saturation condition so that the temperature variation along the interface can be neglected which implies, in turn, the surface tension variation along the interface can be neglected. Under this condition Equations (11.9) and (11.7) reduce to

$$\tau_{1i} = \tau_{2i} \quad (11.9)$$

The enthalpy equation of each phase:

$$(1-\alpha)\, \rho_1 \; \frac{D_1 \langle i_1 \rangle}{Dt} + \Gamma_{1i} \langle i_1 \rangle = (1-\alpha) \; \frac{D_1 \langle\!\langle P_1 \rangle\!\rangle}{Dt} + (1-\alpha)\, \dot{\Phi}_{1,z\overline{z}} \quad (11.10a)$$

$$\frac{\Im_i}{A_{T_c}} \left(\dot{m}_{1i}\, i_{1i} + q_{1i} \right) - \left(\langle\!\langle P_i \rangle\!\rangle - P_{1i} \right) \frac{D_1 \alpha}{Dt} + \frac{\Im_{1e}}{A_{T_c}}\, q_{1e} - \frac{1}{\partial x} \; Cov_{(ent\,1)}$$

$$\alpha \rho_2 \frac{D_2 \langle i_2 \rangle}{Dt} + \Gamma_{2i} \langle i_2 \rangle = \alpha \frac{D_2 \langle\!\langle P_2 \rangle\!\rangle}{Dt} + \alpha \dot{\Phi}_{2IR} - \qquad (11.10b)$$

$$\frac{\xi_i}{A_{Tc}} \left(\dot{m}_{2i} \, i_{2i} - q_{2i} \right) + \left(\langle\!\langle P_2 \rangle\!\rangle - P_{2i} \right) \frac{D_2 \alpha}{Dt} - \frac{\partial}{\partial x} \text{Cov (ent 2)}$$

We note that for the plane flow

$$\frac{\xi_i}{A_{Tc}} = \frac{1}{H} \; ; \; \frac{\xi_{ie}}{A_{Tc}} = \frac{1}{H} \qquad (11.11a)$$

and

$$\langle\!\langle P_1 \rangle\!\rangle - P_{1i} = -\frac{1}{2} \rho_1 (1-\alpha) \, H \, g_y \; ; \; \langle\!\langle P_2 \rangle\!\rangle - P_{2i} = \frac{1}{2} \rho_2 \alpha H g_y \qquad (11.11b)$$

whereas for the annular flow

$$\frac{\xi_i}{A_{Tc}} = \frac{4\sqrt{\alpha}}{D_e} \; ; \; \frac{\xi_{ie}}{A_{Tc}} = \frac{4}{D_e} \qquad (11.12a)$$

and

$$\langle\!\langle P_1 \rangle\!\rangle - P_{1i} = \langle\!\langle P_2 \rangle\!\rangle - P_{2i} = 0 \qquad (11.12b)$$

The interfacial enthalpy "jump" condition

$$\sum_{k=1}^{2} \dot{m}_{ki} \, i_{ki} + \vec{q}_{ki} \cdot \hat{n}_{ki} - P_{ki} (\vec{v}_{ki} - \vec{v}_i) \cdot \hat{n}_{ki} = \sigma \hat{t}_i \cdot \frac{\partial \vec{v}_i}{\partial s} \qquad (11.13)$$

The set of equations listed above are the basic field and the interfacial coupling equations which describes the complete separated flow formulation of incompressible fluids based on the two-fluid model.

11.2.2 Interfacial Mass Transfer

By examining the continuity equations it can be seen that in the case of evaporation the mass flux $\Gamma_{2i} = -\Gamma_{1i}$, acts as a sink for the liquid film, indeed, as it was noted in Chapter 5, it plays the same role as the sink (or source) terms in the continuity equations

of chemically reacting mixtures. Whereas in chemically reacting mixtures, the sinks (or sources) are specified by appropriate constitutive equations of chemical kinetics, in two-phase flow they are specified by the appropriate constitutive equations of phase change, i.e., of evaporation or condensation, references [4, 42, 51]. Indeed, it was shown that the constitutive equation of evaporation a) depends on the flow regime and b) determines the thermodynamic non-equilibrium of the two-phase mixture.

For example in the present problem, the expression for Γ_{Ki}, will depend on the particular application. Thus, it will have a different form depending on whether the evaporation from the liquid film is effected by exposure to hot gases (as in rocket engines), or by decreasing the system pressure (as in flash evaporators), or by heat transfer through the liquid film (as in boilers, evaporators or nuclear reactors). Let us consider briefly this last application.

Under certain conditions, the interfacial mass transfer Γ_{Ki}, can be calculated directly from the energy balance. In order to demonstrate the implications of such an approach, it is instructive to consider the addition of the enthalpy equations expressed as

$$(1-\alpha)\, \rho_1 \, \frac{D_1 \langle i_1 \rangle}{Dt} + \alpha\, \rho_2 \, \frac{D_2 \langle i_2 \rangle}{Dt} + \Gamma_{2i}\, (\langle i_2 \rangle - \langle i_1 \rangle) = \qquad (11.14)$$

$$= \frac{\bar{\xi}_{ie}}{A_{T_c}}\, q_{ie} + \left[(1-\alpha)\, \frac{D_1 \langle\!\langle P_1 \rangle\!\rangle}{Dt} + \alpha\, \frac{D_2 \langle\!\langle P_2 \rangle\!\rangle}{Dt} \right] +$$

$$\left[(1-\alpha)\, \dot{\Phi}_{1IR} + \alpha\, \dot{\Phi}_{2IR} \right] - \frac{\bar{\xi}_i}{A_{T_c}} \sum_{k=1}^{2} P_{ki}\, (\vec{v}_{ki} - \vec{v}_i) + \sigma\, \hat{t}_i \cdot \frac{\partial \vec{v}_i}{\partial s} +$$

$$\left[-\left(\langle\!\langle P_1 \rangle\!\rangle - P_{1i} \right) \frac{D_1 \alpha}{Dt} + \left(\langle\!\langle P_2 \rangle\!\rangle - P_{2i} \right) \frac{D_2 \alpha}{Dt} \right] - \frac{\partial}{\partial x}\, C_{op} (ext\,T)$$

where Equation (11.13) and Equation (11.2) were used for the interfacial energy balance and for the interfacial mass balance.

The significance of the various terms in Equation (11.14) as follows: The first two terms on the left-hand side account for the thermodynamic non-equilibrium of the liquid and of the vapor, respectively. The third term is the energy required to generate a mass, Γ_{2i} of vapor per unit volume. The first term on the right-hand side of Equation (11.16) represents the power input per unit volume. The other terms, respectively, account for the effects of system pressure variations on the energy content, viscous dissipation, interfacial energy transfer, pressure variations over the cross-sectional planes and for the enthalpy variation over the cross sectional planes.

In practice, the right-hand side of Equation (11.14) is known from the kinematic and the dynamic fields and as well as from the constitutive equations whereas the three terms on the left-hand side are not. In other words, for a given energy input to the mixture, i.e., for a given value of the right-hand side, we do not know how much of that energy is used to superheat the liquid, to superheat the vapor or to generate an unknown amount, Γ_{2i} of vapor.

For a mixture where the entire energy transferred to the mixture is used, immediately, i.e., without any time delay to vaporize the liquid, the convective thermal energy transport can be neglected. The rate of vapor formation Γ_{2i}, therefore, can be computed directly from Equation (11.14), thus

$$\Gamma_{2i} = \frac{\xi_{ie}}{A_{T_c}} \frac{q_{ie}}{\langle i_2 \rangle - \langle i_1 \rangle} \tag{11.15}$$

where the effects of the pressure variations and of the surface tension were neglected in comparison with thermal effects.

We note that neglecting convection within the liquid film and thus assuming heat transfer by conduction only, it is possible to write the heat balance within the liquid film as follows:

$$q_{1e} = k_1 \frac{T_{1e} - T_s}{\eta} = k_1 \frac{\Delta T}{\eta} \tag{11.16}$$

where k_1 is the conduction coefficient of the liquid, and T_{1e} and T_s are, respectively, temperature of the solid boundary and of the interfacial surface.

Combining Equation (11.16) with Equation (11.15) we obtain

$$\Gamma_{2i} = \frac{S_{1e}}{A_{T_c}} \frac{q_{1e}}{\langle i_2 \rangle - \langle i_1 \rangle} = \frac{S_{1e}}{A_{T_c}} \frac{k_1 \Delta T}{\eta (\langle i_2 \rangle - \langle i_1 \rangle)} \tag{11.17}$$

Furthermore, since the vapor enthalpy is determined at the local saturation condition, and since $\langle i_1 \rangle \simeq i_1$, then the enthalpy difference can be taken as the heat of vaporization, Δi_{12}, and Equation (11.17) becomes

$$\Gamma_{2i} = \frac{S_{1e}}{A_{T_c}} \frac{q_{1e}}{\Delta i_{12}} = \frac{S_{1e}}{A_{T_c}} \frac{k_1 \Delta T}{\eta \Delta i_{12}} \tag{11.18}$$

which can be used as a constitutive equation for the rate of interfacial vaporization per unit mixture volume.

11.2.3 Disturbed Equations

In order to determine under what conditions waves appearing on the interface lead to instability the behavior of a very small disturbance in a uniform flow will be examined. An equation for such a disturbance may be obtained by substituting the following into the basic equations

derived in the preceding section:

$$\alpha = \bar{\alpha} + \alpha' \qquad (11.19a)$$

$$\langle v_k \rangle = \langle \bar{v_k} \rangle + v_k' \qquad K = 1,2 \qquad (11.19b)$$

$$\Gamma_{ki} = \bar{\Gamma}_{ki} + \Gamma_{ki}' \qquad K = 1,2 \qquad (11.19c)$$

$$\langle P_k \rangle = \langle \bar{P_k} \rangle + P_k' \qquad K = 1,2 \qquad (11.19d)$$

$$F_k = \bar{F}_k + F_k' \qquad K = 1,2 \qquad (11.19e)$$

$$Cov(mom\,k) = \overline{Cov}(mom\,k) + Cov'(mom\,k) \atop ,K=1,2 \qquad (11.19f)$$

The barred quantities are time averages, and the primed quantities are
the disturbances on the uniform flow. The disturbances will be assumed
to be small enough that second order terms in the primed quantities may
be neglected. If it is considered that the mean flow itself satisfies
the basic field and the constitutive equations, after substituting Equa-
tions (11.19a-f) in Equations (11.1 and 2), for the disturbed continuity
equations we obtain

$$-\frac{\partial \alpha'}{\partial t} - \langle \bar{v_1} \rangle \frac{\partial \alpha'}{\partial x} + (1-\bar{\alpha})\frac{\partial v_1'}{\partial x} - \frac{d\bar{\alpha}}{dx} v_1' - \frac{d\langle \bar{v_1} \rangle}{dx}\alpha' = \frac{1}{\rho_1}\Gamma_1' \quad (11.20a)$$

$$\frac{\partial \alpha'}{\partial t} + \langle \bar{v_2} \rangle \frac{\partial \alpha'}{\partial t} + \bar{\alpha}\frac{\partial v_2'}{\partial x} + \frac{d\bar{\alpha}}{dx} v_2' + \frac{d\langle \bar{v_2} \rangle}{dx}\alpha' = \frac{1}{\rho_2}\Gamma_2' \quad (11.20b)$$

whereas for the equations of motions we obtain

$$\rho_1\left(\frac{\partial v_1'}{\partial t} + \langle \bar{v_1} \rangle \frac{\partial v_1'}{\partial x} + \frac{d\langle \bar{v_1} \rangle}{dx} v_1'\right) = -\frac{\partial P_1'}{\partial x} + F_1' - \qquad (11.21a)$$

$$\frac{1}{1-\bar{\alpha}}\frac{\partial}{\partial x} Cov'(mom\,1)$$

$$\rho_2 \left(\frac{\partial V_2'}{\partial t} + \overline{\langle V_2 \rangle} \frac{\partial V_2'}{\partial x} + \frac{d \overline{\langle V_2 \rangle}}{dx} V_2' \right) = -\frac{\partial P_2'}{\partial x} + F_2' - \qquad (11.21b)$$

$$\frac{1}{\alpha} \frac{\partial}{\partial x} C_{ov(mom2)}'$$

It is demonstrated in Appendix D that this thinning effect of the phase change in the x-direction is very small. Therefore, one can consider the mean flow to be quasi-fully developed flow in which the time averaged quantities are not varying appreciably in the x-direction so that the multiplication of the primed quantities with the x-derivatives of the mean quantities can be assumed to be second order. To be consistent with the linearization these terms can be neglected in comparison with the first order effects, and we obtain from Equations (11.20 and 11.21), following simplified expressions:

$$-\frac{\partial \alpha'}{\partial t} - \langle V_1 \rangle \frac{\partial \alpha'}{\partial x} + (1-\alpha) \frac{\partial V_1'}{\partial x} = \frac{1}{\rho_1} \Gamma_{1i}'^{*} \qquad (11.22a)$$

$$\frac{\partial \alpha'}{\partial t} + \langle V_2 \rangle \frac{\partial \alpha'}{\partial x} + \alpha \frac{\partial V_2'}{\partial x} = \frac{1}{\rho_2} \Gamma_{2i}' \qquad (11.22b)$$

$$\rho_1 \left(\frac{\partial V_1'}{\partial t} + \langle V_1 \rangle \frac{\partial V_1'}{\partial x} \right) = -\frac{\partial P_1'}{\partial x} + F_1' - \frac{1}{1-\alpha} \frac{\partial}{\partial x} C_{ov(mom1)}' \qquad (11.23a)$$

$$\rho_2 \left(\frac{\partial V_2'}{\partial t} + \langle V_2 \rangle \frac{\partial V_2'}{\partial x} \right) = -\frac{\partial P_2'}{\partial x} + F_2' - \frac{1}{\alpha} \frac{\partial}{\partial x} C_{ov(mom2)}' \qquad (11.23b)$$

Differentiating Equations (11.22a and b) with respect to x and then substracting the resulting equations we obtain

$$\rho_1 \frac{\partial^2 V_1'}{\partial x \partial t} - \rho_2 \frac{\partial^2 V_2'}{\partial x \partial t} + \rho_1 \langle V_1 \rangle \frac{\partial^2 V_1'}{\partial x^2} - \rho_2 \langle V_2 \rangle \frac{\partial^2 V_2'}{\partial x^2} = \qquad (11.24)$$

$$= -\frac{\partial^2 \Delta P'}{\partial x^2} + \frac{\partial \Delta F'}{\partial x} - \frac{1}{1-\alpha} \frac{\partial}{\partial x} C_{ov(mom1)}' - \frac{1}{\alpha} C_{ov(mom2)}'$$

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*For the purpose of simplicity bar over quantities was dropped. It is understood that unprimed quantities belong to the mean flow.

where

$$\Delta P' = P_1' - P_2' \tag{11.25}$$

$$\Delta F' = F_1' - F_2' \tag{11.26}$$

Depending upon the particular application, ΔF, $\Delta \langle P \rangle$ and the covariance terms can be expressed in terms of the mean flow variables. However, in view of Equations (11.4, 11.5, 11.7, 11.8, 11.11 and 11.12), we can assume that

$$\Delta F = \Delta F \left(\alpha, \langle v_1 \rangle, \langle v_2 \rangle, \frac{\partial \alpha}{\partial x} \right) \tag{11.27}$$

$$\Delta \langle P \rangle = \Delta \langle P \rangle \left(\alpha, \frac{\partial \alpha}{\partial x}, \frac{\partial^2 \alpha}{\partial x^2} \right) \tag{11.28}$$

$$\text{Cov (mom K)} = \text{Cov (mom K)} \left(\alpha, \langle v_1 \rangle, \langle v_2 \rangle \right) \qquad K = 1,2 \tag{11.29}$$

and in view of Equation (11.17)

$$\Gamma_{ki} = \Gamma_{ki} (\alpha) \qquad K = 1,2 \tag{11.30}$$

Expanding Equations (11.27-11.30) into Taylor expansion, and keeping only the first order terms, it can be shown that

$$\Delta F' = \frac{\partial \Delta F}{\partial \alpha} \alpha' + \frac{\partial \Delta F}{\partial \langle v_1 \rangle} v_1' + \frac{\partial \Delta F}{\partial \langle v_2 \rangle} v_2' + \frac{\partial \Delta F}{\partial \left(\frac{\partial \alpha}{\partial x} \right)} \frac{\partial \alpha'}{\partial x} \tag{11.31}$$

$$\Delta P' = \frac{\partial \Delta \langle P \rangle}{\partial \alpha} \alpha' + \frac{\partial \Delta \langle P \rangle}{\partial \left(\frac{\partial \alpha}{\partial x} \right)} \frac{\partial \alpha'}{\partial x} + \frac{\partial \Delta \langle P \rangle}{\partial \left(\frac{\partial^2 \alpha}{\partial x^2} \right)} \frac{\partial^2 \alpha'}{\partial x^2} \tag{11.32}$$

$$Cov(mom\,K)' = \frac{\partial}{\partial \alpha} Cov(mom\,K)\,\alpha' + \frac{\partial}{\partial \langle v_1\rangle} Cov(mom\,K)\,v_1' + \quad (11.33)$$

$$\frac{\partial}{\partial \langle v_2\rangle} Cov(mom\,K)\,v_2' \qquad\qquad K = 1,2$$

and

$$\Gamma_{ki}' = \frac{d\Gamma_{ki}}{d\alpha}\,\alpha' \qquad\qquad K = 1,2 \quad (11.34)$$

Introducing Equations (11.31-11.34) in Equation (11.24), making use of the interfacial balance $\Gamma_{2i} = \Gamma_{1i} = \Gamma$, and rearranging the resulting equation, we obtain

$$\rho_1 \frac{\partial^2 v_1'}{\partial x\,\partial t} - \rho_2 \frac{\partial^2 v_2'}{\partial x\,\partial t} + \rho_1 \langle v_1\rangle \frac{\partial^2 v_1'}{\partial x^2} - \rho_2 \langle v_2\rangle \frac{\partial^2 v_2'}{\partial x^2} = \quad (11.35)$$

$$= -\frac{\partial \Delta\langle P\rangle}{\partial\left(\frac{\partial^2\alpha}{\partial x^2}\right)}\frac{\partial^4\alpha'}{\partial x^4} - \frac{\partial \Delta\langle P\rangle}{\partial\left(\frac{\partial\alpha}{\partial x}\right)}\frac{\partial^3\alpha'}{\partial x^3} + \left[-\frac{\partial \Delta\langle P\rangle}{\partial\alpha} + \frac{\partial \Delta F}{\partial\left(\frac{\partial\alpha}{\partial x}\right)} - \right.$$

$$\left.\frac{1}{1-\alpha}\frac{\partial}{\partial\alpha}Cov(mom\,1) - \frac{1}{\alpha}\frac{\partial}{\partial\alpha}Cov(mom\,2)\right]\frac{\partial^2\alpha'}{\partial x^2} -$$

$$\frac{1}{1-\alpha}\frac{\partial}{\partial\langle v_1\rangle}Cov(mom\,1)\,\frac{\partial^2 v_1'}{\partial x^2} - \frac{1}{\alpha}\frac{\partial}{\partial\langle v_2\rangle}Cov(mom\,2)\,\frac{\partial^2 v_2'}{\partial x^2} +$$

In view of the disturbed continuity equations, i.e., Equations (11.22a and b), we can form the derivatives of v_1' and v_2' in terms of those of α'. Therefore, differentiating Equations (11.22a and b) with respect to time as well as x, we can form

$$\frac{\partial v_1'}{\partial t} = \frac{1}{1-\alpha}\left(\frac{\partial\alpha'}{\partial t} + \langle v_1\rangle\frac{\partial\alpha'}{\partial x} - \frac{1}{\rho_1}\frac{d\Gamma}{d\alpha}\alpha'\right) \qquad (11.36a)$$

$$\frac{\partial v_2'}{\partial x} = \frac{1}{\alpha}\left(-\frac{\partial \alpha'}{\partial t} - \langle v_2 \rangle \frac{\partial \alpha'}{\partial x} + \frac{1}{\rho_2}\frac{d\Gamma}{d\alpha}\alpha'\right)$$ (11.36b)

$$\frac{\partial^2 v_1'}{\partial x^2} = \frac{1}{1-\alpha}\left(\frac{\partial^2 \alpha'}{\partial t \partial x} + \langle v_1 \rangle \frac{\partial^2 \alpha'}{\partial x^2} - \frac{1}{\rho_1}\frac{d\Gamma}{d\alpha}\frac{\partial \alpha'}{\partial x}\right)$$ (11.36c)

$$\frac{\partial^2 v_2'}{\partial x^2} = \frac{1}{\alpha}\left(-\frac{\partial^2 \alpha'}{\partial t \partial x} - \langle v_2 \rangle \frac{\partial^2 \alpha'}{\partial x^2} + \frac{1}{\rho_2}\frac{d\Gamma}{d\alpha}\frac{\partial \alpha'}{\partial x}\right)$$ (11.36d)

$$\frac{\partial^2 v_1'}{\partial t \partial x} = \frac{1}{1-\alpha}\left(\frac{\partial^2 \alpha'}{\partial t^2} + \langle v_1 \rangle \frac{\partial^2 \alpha'}{\partial t \partial x} - \frac{1}{\rho_1}\frac{d\Gamma}{d\alpha}\frac{\partial \alpha'}{\partial t}\right)$$ (11.36e)

$$\frac{\partial^2 v_2'}{\partial t \partial x} = \frac{1}{\alpha}\left(-\frac{\partial^2 \alpha'}{\partial t^2} - \langle v_2 \rangle \frac{\partial^2 \alpha'}{\partial t \partial x} + \frac{1}{\rho_2}\frac{d\Gamma}{d\alpha}\frac{\partial \alpha'}{\partial t}\right)$$ (11.36f)

Introducing Equations (11.36a-f) in Equation (11.35), and rearranging, we obtain

$$X_1\frac{\partial^4 \alpha'}{\partial x^4} + X_2\frac{\partial^3 \alpha'}{\partial x^3} + X_3\frac{\partial^2 \alpha'}{\partial x^2} + X_4\frac{\partial^2 \alpha'}{\partial x \partial t} + X_5\frac{\partial^2 \alpha'}{\partial t^2} +$$ (11.37)

$$X_6\frac{\partial \alpha'}{\partial x} + X_7\frac{\partial \alpha'}{\partial t} + X_8\,\alpha' = 0$$

where X's are defined as follows:

$$X_1 = \frac{\partial \Delta \langle P \rangle}{\partial\left(\frac{\partial^3 \alpha}{\partial x^3}\right)}$$ (11.38a)

$$X_2 = \frac{\partial \Delta \langle P \rangle}{\partial\left(\frac{\partial^2 \alpha}{\partial x^2}\right)}$$ (11.38b)

$$X_3 = \left(\frac{\rho_1\langle v_1\rangle^2}{1-\alpha} + \frac{\rho_2\langle v_2\rangle^2}{\alpha}\right) + \frac{\partial \Delta\langle P\rangle}{\partial \alpha} - \frac{\partial \Delta F}{\partial\left(\frac{\partial \alpha}{\partial x}\right)} +$$ (11.38c)

$$\frac{1}{1-\alpha}\frac{\partial}{\partial x}Cov(mom1) + \frac{1}{\alpha}\frac{\partial}{\partial x}Cov(mom2) + \frac{\langle v_1\rangle}{(1-\alpha)^2}\frac{\partial}{\partial\langle v_1\rangle}Cov(mom1) -$$

$$\frac{\langle v_2\rangle}{\alpha^2}\frac{\partial}{\partial\langle v_2\rangle}Cov(mom2) - \frac{\langle v_2\rangle}{\alpha(1-\alpha)}\frac{\partial}{\partial\langle v_1\rangle}Cov(mom1) + \frac{\langle v_1\rangle}{\alpha(1-\alpha)}\frac{\partial}{\partial\langle v_1\rangle}Cov(mom2)$$

$$X_4 = 2 \left(\frac{\rho_1 \langle v_1 \rangle}{1-\alpha} + \frac{\rho_2 \langle v_2 \rangle}{\alpha} + \frac{1}{(1-\alpha)^2} \frac{\partial}{\partial \langle v_1 \rangle} \text{Cov}(\text{mom}1) + \frac{1}{\alpha^2} \frac{\partial}{\partial \langle v_2 \rangle} \text{Cov}(\text{mom}2) \right. \quad (11.38\text{d})$$

$$\left. \frac{1}{\alpha(1-\alpha)} \frac{\partial}{\partial \langle v_2 \rangle} \text{Cov}(\text{mom }1) + \frac{1}{\alpha(1-\alpha)} \frac{\partial}{\partial \langle v_1 \rangle} \text{Cov}(\text{mom }2) \right)$$

$$X_5 = \frac{1}{1-\alpha} + \frac{\rho_2}{\alpha} \qquad (11.38\text{e})$$

$$X_6 = -\left(\frac{1}{1-\alpha} + \frac{\langle v_2 \rangle}{\alpha} \right) \frac{d\Gamma}{d\alpha} - \frac{\partial \Delta F}{\partial \alpha} - \frac{\langle v_1 \rangle}{1-\alpha} \frac{\partial \Delta F}{\partial \langle v_1 \rangle} + \frac{\langle v_2 \rangle}{\alpha} \frac{\partial \Delta F}{\partial \langle v_2 \rangle} + \quad (11.38\text{f})$$

$$\frac{1}{\rho_2 \alpha(1-\alpha)} \frac{\partial}{\partial \langle v_2 \rangle} \text{Cov}(\text{mom}1) \frac{d\Gamma}{d\alpha} - \frac{1}{\rho_1 \alpha(1-\alpha)} \frac{\partial}{\partial \langle v_1 \rangle} \text{Cov}(\text{mom}2) \frac{d\Gamma}{d\alpha} -$$

$$\frac{1}{\rho_1 (1-\alpha)^2} \frac{\partial}{\partial \langle v_1 \rangle} \text{Cov}(\text{mom}1) \frac{d\Gamma}{d\alpha} + \frac{1}{\rho_2 \alpha^2} \frac{\partial}{\partial \langle v_2 \rangle} \text{Cov}(\text{mom}2) \frac{d\Gamma}{d\alpha}$$

$$X_7 = -\left(\frac{1}{1-\alpha} + \frac{1}{\alpha} \right) \frac{d\Gamma}{d\alpha} - \frac{1}{1-\alpha} \frac{\partial \Delta F}{\partial \langle v_1 \rangle} + \frac{1}{\alpha} \frac{\partial \Delta F}{\partial \langle v_2 \rangle} \qquad (11.38\text{g})$$

$$X_8 = \frac{1}{\rho_1(1-\alpha)} \frac{\partial \Delta F}{\partial \langle v_1 \rangle} \frac{d\Gamma}{d\alpha} - \frac{1}{\rho_2 \alpha} \frac{\partial \Delta F}{\partial \langle v_2 \rangle} \frac{d\Gamma}{d\alpha} \qquad (11.38\text{h})$$

Equation (11.37) is the required <u>characteristic</u> equation from which the stability of the system under consideration can be determined. Now, the task of the stability theory consists in determining whether the disturbance is amplified or decays for a given time averaged mean motion; the flow is considered stable or unstable depending on whether the former or the latter is the case.

<u>11.2.4 Mathematical Form of the Disturbance</u>

The mean flow with a concentration α is assumed to be influenced by a disturbance which is composed of a number of discrete partial fluctuations, each of which is said to consist of a wave which is propagated in the x-direction. Therefore, analyzing the disturbance into normal modes, we seek solutions whose dependence on x and t is given by

$$\alpha' = a_i \exp\left[i\left(kx - Ckt\right)\right] \qquad (11.39)$$

where any arbitrary disturbance is assumed expanded in a Fourier series, each of whose terms represents a partial oscillation. In Equation (11.39) k is a real quantity which is called the wave number and

$$\lambda = 2\pi / k \qquad (11.40)$$

is the wavelength of the disturbance. The quantity C is complex

$$C = C_r + i C_i \qquad (11.41)$$

where C_r denotes the velocity of propagation of the wave in the x-direction whereas $C_i k$ is the growth factor which determines the degree of amplification or damping. The disturbances are damped if

$$C_i k < 0 \qquad (11.42a)$$

and the mean flow is <u>stable</u>, the disturbances are amplified if

$$C_i k > 0 \qquad (11.42b)$$

and the mean flow is <u>unstable</u>. Finally, the mean flow is said to be <u>neutrally stable</u>, in which the disturbances are neither damped nor amplified, if

$$C_i k = 0 \qquad (11.42c)$$

The meaning of the various terms can be seen by considering the real part of Equation (11.39):

$$\alpha' = a_i \exp\left(C_i kt\right) \cos\left[k\left(x - C_r t\right)\right] \qquad (11.43a)$$

which describes a sinisoidal wave of amplitude

$$a = a_i \, exp \, (C_i \, k \, t) \qquad (11.43b)$$

of velocity C_r, and of wave length λ given by Equation (11.40).

Introducing Equation (11.39) in the characteristic equation given by Equation (11.37), the following algebraic relation is obtained between the governing variables:

$$-C^2 + \left(\frac{X_4}{X_5} - \frac{X_7}{X_5 k} i \right) C + \frac{X_1}{X_5} k^2 - \frac{X_3}{X_5} + \frac{X_8}{X_5 k^2} + \qquad (11.44)$$

$$\left(-\frac{X_2}{X_5} k + \frac{X_6}{X_5 k} \right) i = 0$$

which can be also expressed as

$$-C^2 + 2 \left(Y_1 + Y_2 \, i \right) C + Y_3 + Y_4 = 0 \qquad (11.45)$$

where we defined

$$Y_1 = \frac{X_4}{2 X_5} \quad ; \quad Y_3 = \frac{X_1}{X_5} k^2 - \frac{X_3}{X_5} + \frac{X_8}{X_5 k^2} \qquad (11.46)$$

$$Y_2 = -\frac{X_7}{2 X_5 k} \quad ; \quad Y_4 = -\frac{X_2}{X_5} k + \frac{X_6}{X_5 k}$$

Substituting Equation (11.41) in Equation (11.44), and separating the real and imaginary parts, we obtain two algebraic equation for C_r and C_i, thus

$$(C_i - Y_2)^2 - (C_r - Y_1)^2 + Y_1^2 - Y_2^2 + Y_3 = 0 \qquad (11.47)$$

$$-2 (C_r - Y_1) (C_i - Y_2) + 2 Y_1 Y_2 + Y_4 = 0 \qquad (11.48)$$

which in a condensed notation can be expressed as:

$$\overline{C}_i^2 - \overline{C}_r^2 + A = 0 \tag{11.49}$$

$$-2\,\overline{C}_i\,\overline{C}_r + B = 0 \tag{11.50}$$

where we defined

$$A = Y_1^2 - Y_2^2 + Y_3 \tag{11.51}$$

$$B = 2\,Y_1\,Y_2 + Y_4 \tag{11.52}$$

$$\overline{C}_r = C_r - Y_1 \tag{11.53}$$

$$\overline{C}_i = C_i - Y_2 \tag{11.54}$$

In these equations the parameters A and B depend on the particular flow system. Therefore, as soon as system is specified A and B can be calculated in terms of the mean flow variables. After this one can proceed to solve the important variables C_r and C_i. The former yields the velocity of propagation of the wave in the x-direction whereas C_i determines the degree of amplification.

11.2.5 Stability Criterion

Since the stability of the interfacial waves depends on the sign of C_i, we should evaluate $\overline{C}_i$ from Equations (11.49 and 50) and then analyze the conditions under which the disturbances grow or decay. Solving Equation (11.50) for $\overline{C}_r$ and then using this value, Equation (11.49) yields

$$\overline{C}_i = \mp \frac{1}{\sqrt{2}}\left(-A \mp \sqrt{A^2 + B^2}\,\right)^{1/2} \tag{11.55}$$

which in view of Equation (11.54) gives

$$C_i = Y_2 \mp \frac{1}{\sqrt{2}} \left(-A \mp \sqrt{A^2 + B^2} \right)^{1/2} \qquad (11.56)$$

By definition C_i is a real quantity and since $\left| \sqrt{A^2 + B^2} \right| \geqslant |A|$, then the minus sign in front of the square root can be eliminated leaving

$$C_i = Y_2 \mp \frac{1}{\sqrt{2}} \left(-A + \sqrt{A^2 + B^2} \right)^{1/2} \qquad (11.57)$$

Substituting $\bar{C}_i$ back either in Equation (11.49) or in (11.50) gives $\bar{C}_r$, which, in turn, in view of Equation (11.53) yields the velocity C_r, of propagation of the wave.

Although the parameters A and B, in general, depend on the particular application, we can interpret Equations (11.49 and 50) geometrically. If we consider the case in which $A > 0$, $\bar{C}_r$ and $\bar{C}_i$ behave as they are shown on Figure 4.

It can be seen from the figure that for each value of the parameters A and B, there exist a pair of solutions for $\bar{C}_r$ and $\bar{C}_i$. Although these solutions are equal in numerical value they are different in sign. This situation is not surprising because we are analyzing the dynamic waves which propagate the downstream as well as the upstream directions. Positive value of $\bar{C}_r$, therefore corresponds to the wave train propagating in + x - direction with respect to Y_1, see Equation (11.53) for the definition of C_r, whereas negative value of $\bar{C}_r$ corresponds to the wave propagating - x - direction, i.e., propagating upstream. As a result it can be concluded that the upper half portion of Figure 4 corresponds to downstream moving waves whereas lower half corresponds to upstream moving waves.

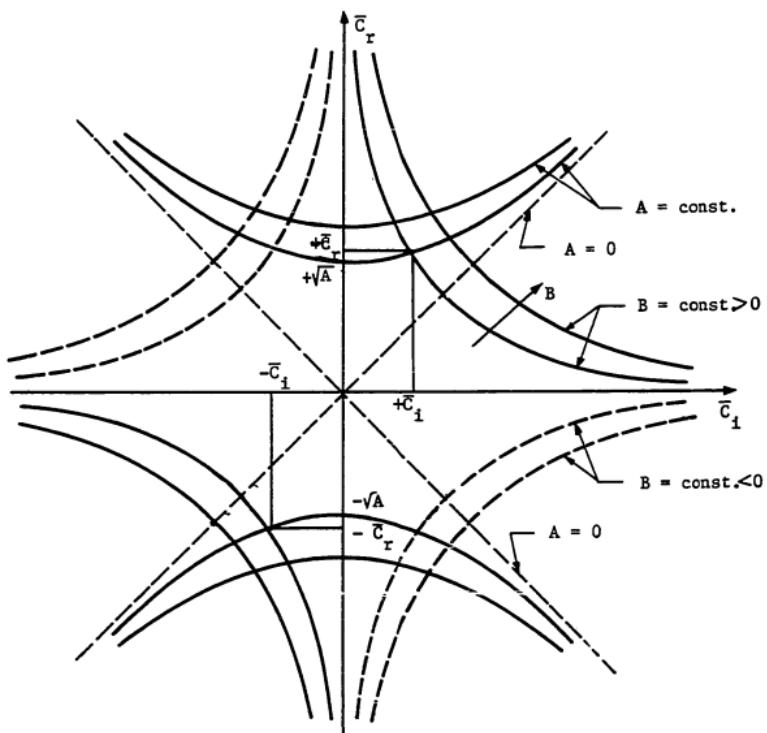According to Equation (11.55), C_i can be expressed as
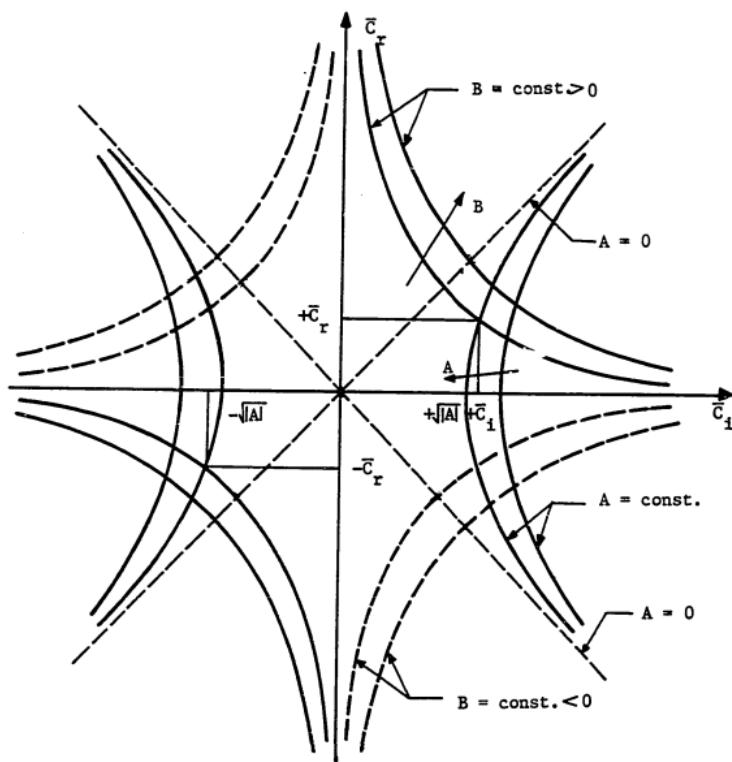
Figure 4. Stability Curves for A>0

Figure 5. Stability Curves for A < 0

$$C_i = \overline{C}_i + Y_2 \qquad (11.58)$$

Therefore, the requirement for stability becomes

$$C_i = \overline{C}_i + Y_2 < 0 \qquad (11.59)$$

It is evident from Figure 4 and Equation (11.59) that the positive values of $\overline{C}_i$ is more dangerous then the negative values of it. Therefore, the discussion can be confined for the case where $\overline{C}_i > 0$. In this case, however, in order to be able to satisfy Equation (11.59), Y_2 must be negative. Now we can state the <u>necessary</u> condition for the stability of the system as

$$Y_2 < 0 \qquad (11.60)$$

Of course, the <u>sufficient</u> condition follows immediately from Equation (11.40) as

$$C_i < 0 \qquad (11.61)$$

If we consider the case in which $A < 0$, according to Equations (11.49 and 50) $\overline{C}_r$ and $\overline{C}_i$ behave as they are shown on Figure 5.

In general appearance of Figure 4 and Figure 5 are similar. Only difference is that the A = Constant hyperbola changes axis. However, the same discussions which was done for the previous case applies to this case. The necessary and the sufficient conditions expressed by Equations (11.60 and 61) respectively, remains the same.

Finally, the neutral stability condition can be obtained either from Equations (11.49) and (11.50) or from Equation (11.57) by simply substituting $C_i = 0$, thus

$$Y_2^4 + A Y_2^2 - \frac{B^2}{4} = 0 \qquad (11.62)$$

which, upon substituting Equations (11.51) and (11.52), gives for the neutral stability condition:

$$Y_3 - \frac{1}{4}\left(\frac{Y_4}{Y_2}\right)^2 - Y_1\left(\frac{Y_4}{Y_2}\right) = 0 \qquad (11.63)$$

This is the condition under which disturbances propagate without amplifying or decaying.

11.3 Break-Up of Liquid Films

It should be emphasized here that developing valid stability criteria and growth rates represents only the first step in identifying and eliminating undesirable phenomena associated with two-phase flows. This information indicates the conditions under which interfacial waves occur and what their growth rates are. However, the appearance of the wave on the interface does not necessarily mean that harmful phenomena will occur. To answer completely the question of whether the waves are harmful or not, it will be necessary to predict either the time or the length which is required for these waves to grow enough so that liquid is entrained in the vapor or the gas phase or, in the annular flow, the liquid waves sometimes becomes so large that they essentially join together from opposite sides and form liquid plugs. Although, the analysis which has been carried out in the preceding sections is applicable for both cases we shall be primarily concerned with the former one, i.e., with liquid entrainment which essentially leads to the break-up the thin liquid film.

A mathematical model is proposed here to predict the point at

which undesirable conditions such as film break-up will be attained under given conditions.

If an interface of two fluids is perturbed by disturbances of all wave lengths, then the disturbances having the maximum positive growth rate dominates the interface. However, since the maximum growth rate disturbance can originate at any point along the interface, it will be assumed that only those formed at the furthest upstream distance will dominate. This situation is illustrated on Figure 6 where the continuous formation of maximum growth rate waves are originated at x = 0.
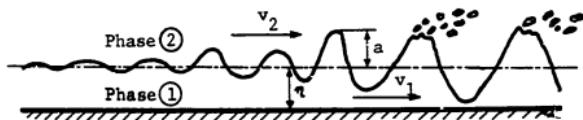

Figure 6. Maximum Growth on a Liquid Film

In the preceding section we have calculated C_i from which the growth rate $C_i k$ for an individual wave can be formed. Since the growth rate will depend on the wave length λ, or equally on the wave number $k = \frac{2\pi}{\lambda}$, the wave number k_m, or the corresponding wave length λ_m, which makes the growth rate maximum can be calculated by differentiating the growth rate factor with respect to k and then forming

$$\frac{dC_i k}{dk} = 0 \qquad (11.64)$$

k_m found in such a way can be substituted in the growth rate expression to find the maximum growth rate $C_i k_m$, in terms of other flow variables. This will be the rate at which amplitude of the disturbances will grow fastest.

For most unstable waves the amplitude becomes

$$a = a_i \exp(C_i k_m t) \qquad (11.65)$$

logarithm of which results in

$$C_i k_m t = \ln\frac{a}{a_i} \qquad (11.66)$$

If t_b denotes the time at which the waves lead to unfavorable phenomena such as liquid entrainment and flow plugging, the from Equation (11.66), t_b becomes

$$t_b = \frac{1}{C_i k_m} \ln\frac{a_b}{a_i} \qquad (11.67)$$

where a_b is the amplitude at which either entrainment or plugging occurs. To determine a_b, the amplitude for tearing of the interfacial waves can be obtained from experiments, whereas for plugging it is clear that the amplitude plus the film thickness must be of the order of one-half the flow passage diameter.

Since the waves travel with the wave speed C_r, a more practical quantity can be obtained by considering the length L_b instead the time t_b, thus from Equation (11.67)

$$L_b = \frac{C_r}{C_i k_m} \ln\frac{a_b}{a_i} \qquad (11.68)$$

where L_b denotes the point where the waves lead to unfavorable phenomena such as liquid entrainment leading to the break-up and flow plugging changing the flow regime from annular to plug flow. One can estimate thereby, the length for formation of dry spots in the liquid film.

In the remaining two chapters of this dissertation the stability criterion derived above will be applied to the particular problems so that the predicted results can be compared to experimental data.

11.4 Summary

The stability problem of a general separated two-phase flow system was analyzed. Instability and wave formation criterion at the interface were established in terms of the generalized parameters which depend on the particular flow system. Furthermore, assuming that the most unstable wave mode is responsible for the break-up of a thin liquid film a method for determining the break-up length was developed. Results obtained in this chapter are general so that one can use them for a particular separated flow configuration provided the coefficients such as F's, X's, Y's, A and B are calculated properly.

CHAPTER XII

STABILITY OF LIQUID FLOW DOWN A HEATED INCLINED PLANE

12.1 Introduction

The free film flow, where the velocity of adjacent gas or vapor phase is negligible, has been studied as a purely fluid-mechanical problem. In the absence of the heat and mass transfer, work done by Benjamin [73] and Yih [72, 74] are well-known and were discussed briefly in Chapter 10. The latter obtained analytical neutral-stability curves, which established that free surface flow down a vertical plane is unstable for all Reynolds numbers.

For the purpose of the stability analysis of such liquid films, we shall follow here the general framework provided in Chapter 11 in which the analysis was extended to take into account the effects of evaporation or condensation.

12.2 Undisturbed Flow

The laminar flow whose stability is to be examined is a uniform two-dimensional stream bounded on one side by a free surface and on the other by a fixed wall which is inclined at an angle θ with the horizontal. A liquid film of thickness η_b draining steadily down on the heat plane is illustrated on Figure 7.

Let x be the distance in the direction of mean flow, and y the distance from the heated wall. The fluid is assumed to have constant density and viscosity.
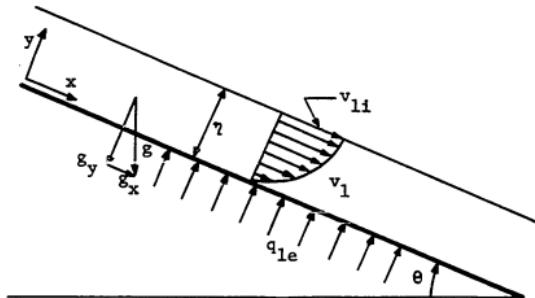
Figure 7. Sketch of the Undisturbed low

For a uniform laminar flow, the boundary layer equations, i.e.,
Equations (6.25) and (6.26), reduce to

$$\nu_1 \frac{d^2 v_1}{dx^2} + g_x = 0 \tag{12.1}$$

$$- \frac{dP_1}{dy} + \rho_1 g_y = 0 \tag{12.2}$$

where g_x and g_y are the components of the gravity in the x and y direc-
tions, respectively. From Figure 7 it is evident that

$$g_x = g \sin\theta \quad \text{and} \quad g_y = -g \cos\theta \tag{12.3}$$

Equations (12.1) and (12.2) are subject to the following boundary condi-
tions: at the solid wall, (y = 0),

$$v_1 = 0 \text{ ; no-slip} \tag{12.4}$$

at the free surface, (y = η),

$$\tau_{1i} = \mu_1 \frac{dv_1}{dy} = 0 \text{ ; zero interfacial shear} \tag{12.5}$$

$$P_i = P_{ii} \quad ; \text{interfacial pressure} \qquad (12.6)$$

It can be shown on integration of Equations (12.1) and (12.2) that:

$$v_i(y) = \frac{g_x \eta^2}{\nu_i} \left[\left(\frac{y}{\eta} \right) - \frac{1}{2} \left(\frac{y}{\eta} \right)^2 \right] \qquad (12.7)$$

and

$$P_i(y) = P_{ii} - (\eta - y) \, \rho_i \, g_y \qquad (12.8)$$

Averaging over the film thickness η, the averaged velocity $\langle v_1 \rangle$ can be calculated as:

$$\langle v_i \rangle = \frac{g_x \eta^2}{3 \nu_i} \qquad (12.9)$$

whereas the averaged pressure can be given as:

$$\langle P_i \rangle = P_{ii} - \frac{1}{2} \, \rho_i \, \eta \, g_y \qquad (12.10)$$

By substituting $y = \eta$ in Equation (12.7), the interfacial velocity v_{1i} can be given as

$$v_{ii} = \frac{g_x \eta^2}{2 \nu_i} \qquad (12.11)$$

which in view of Equation (12.9) can be also expressed in terms of the averaged velocity, thus

$$v_{ii} = \frac{3}{2} \langle v_i \rangle \qquad (12.12)$$

Finally, from Equations (12.7) and (12.9) it can be shown that the wall shear stress τ_{1e}, can be expressed as:

$$\tau_{1e} = \frac{3\, \ell_1\, \gamma_1\, \langle v_1 \rangle}{\gamma} \qquad (12.13)$$

12.3 Calculation of ΔF, $\Delta \langle P \rangle$ and Covariance Terms

After establishing the undisturbed flow variables we can calculate

now ΔF, $\Delta \langle P \rangle$ and Cov(mom K), (K = 1,2), terms which are necessary to

proceed with the stability analysis which was developed in Chapter 11.

In order to be consistent with the general development of the

preceding chapter, the variables should be expressed in terms of the di-

mensionless concentration α rather than the film thickness η. To do so

we shall use the plane flow notation because the problem on hand is a

special case of the general plane flow configuration, thus

$$\langle v_1 \rangle = \frac{g_x}{3\, \gamma_1}\ (1-\alpha)^2\ H^2 \qquad (12.14)$$

$$\langle P_1 \rangle = P_{1i} - \frac{1}{2}\, \ell_1\, (1-\alpha)\, H\, g_y \qquad (12.15)$$

$$\Sigma_{1e} = \frac{3\, \ell_1\, \gamma_1\, \langle v_1 \rangle}{(1-\alpha)\, H} \qquad (12.16)$$

Substituting Equations (12.12, 14, 15, 16 and 11.11) in the plane

flow force function F_1 which is given by Equation (11.3a), we obtain

after some rearrangement

$$F_1 = -\frac{1}{2}\, \ell_1\, H\, g_y\, \frac{\partial \alpha}{\partial y} + \ell_1\, g_x - \frac{3\, \ell_1\, \gamma_1\, \langle v_1 \rangle}{(1-\alpha)^2\, H^2} - \frac{k_1\, \Delta T}{2\, \Delta i_{12}\, (1-\alpha)^2\, H^2} \qquad (12.17)$$

Since by assumption effects of the vapor or gas phase notion on the

liquid film flow is negligible, then it can be shown from Equation (11.3b)

that

$$F_2 = 0 \qquad (12.18)$$

Combining Equation (12.17) with Equation (12.18) yields

$$\Delta F = F_i \qquad (12.19)$$

Since $\langle P_2 \rangle = P_{2i} = P_2$, then in view of Equation (12.15) the averaged pressure difference can be expressed in terms of the interfacial pressure, thus

$$\Delta \langle P \rangle = \langle P_1 \rangle - \langle P_2 \rangle = -\frac{1}{2} \rho_1 (1-\alpha) H g_y + (P_{1i} - P_{2i}) \qquad (12.20)$$

Using the interfacial pressure difference expressed by Equation (11.6) in Equation (12.20) we obtain

$$\Delta \langle P \rangle = \sigma H \frac{\partial^2 \alpha}{\partial x^2} - \frac{1}{2} \rho_1 (1-\alpha) H g_y + \left(\frac{k_1 \Delta T}{\Delta i_{12}} \right)^2 \frac{\Delta \rho}{\rho_1 \rho_2} \frac{1}{(1-\alpha)^2 H^2} \qquad (12.21)$$

Finally, recalling the definition of Cov(mom K), (K = 1,2), given by Equation (6.3), we obtain

$$Cov(mom\ 1) = \frac{1}{5} \rho_1 (1-\alpha) \langle v_1 \rangle^2 \qquad (12.22)$$

Since by assumption the vapor phase is not in motion, it is evident that

$$Cov(mom\ 2) = 0 \qquad (12.23)$$

12.4 Stability Criterion

By introducing the expressions for $\Delta \langle P \rangle$, ΔF and for Cov(mom K), (K = 1,2), in Equation (11.36) we have calculated the X's in Appendix E, from which in view of Equation (11.46) one can obtain Y's, thus

$$Y_1 = \frac{6}{5} \langle v_1 \rangle \qquad (12.24a)$$

$$Y_2 = \frac{1}{4} \frac{k_1 \Delta T}{\ell_1 \eta^2 \Delta i_{12} k} - \frac{3}{2} \frac{\eta_i}{\ell_1 \eta^2 k} \qquad (12.24b)$$

$$Y_3 = \frac{\sigma \eta}{\ell_1} k^2 - \frac{6}{5} \langle v_1 \rangle^2 - \eta g_y - \frac{2}{\ell_1} \left(\frac{k_1 \Delta T}{\eta \Delta i_{12}} \right)^2 \frac{\Delta \ell}{\ell_1 \ell_2} - \qquad (12.24c)$$

$$\frac{k_1 \Delta T}{\Delta i_{12} \eta^2 k} \left(\frac{3 \nu}{\eta^2 k} + \frac{1}{2} \frac{k_1 \Delta T}{\ell_1 \eta^2 k} \right)$$

$$Y_4 = \frac{9 \eta_i \langle v_1 \rangle^2}{\ell_1 \eta^2 k} + \frac{1}{10} \frac{k_1 \Delta T}{\Delta i_{12} \ell_1 \eta^2 k} \qquad (12.24c)$$

In Chapter 11, the necessary and the sufficient conditions for the stability of the interfacial disturbances were derived. Therefore, utilizing Equation (12.24b) in Equation (11.60) we obtain for the necessary condition the following inequality:

$$Y_2 = \frac{1}{4} \frac{k_1 \Delta T}{\ell_1 \eta^2 \Delta i_{12} k} - \frac{3}{2} \frac{\eta_i}{\ell_1 \eta^2 k} \leqslant 0 \qquad (12.25)$$

which can be also expressed as

$$\frac{1}{6} \frac{k_1 \Delta T}{\Delta i_{12}} - \eta_i \leqslant 0 \qquad (12.26)$$

We note that for the adiabatic free film flow, where the first term in Equation (12.26) drops out, the necessary condition is automatically satisfied.

As a sufficient condition for the stability we obtain from Equations (12.24), (11.51), (11.52), (11.57), (11.61) the following condition

$$\frac{\sigma \eta k^2}{\rho_1} - \frac{6}{5}\langle v_1 \rangle^2 - \eta g_y - \frac{2}{\rho_1}\left(\frac{k_1 \Delta T}{\eta \Delta i_{12}}\right)^2 \frac{\Delta \rho}{\rho_1 \rho_2} - \tag{12.27}$$

$$\frac{3\eta}{\rho_1 \eta^2 k}\frac{k_1 \Delta T}{\rho_1 \eta^2 \Delta i_{12} k}\left[1 + \frac{1}{6}\left(\frac{k_1 \Delta T}{\Delta i_{12}}\right)\frac{1}{\eta_1}\right] +$$

$$\frac{36}{5}\left(\frac{1 + \frac{1}{9c}\frac{k_1 \Delta T}{\eta_1 \Delta i_{12}}}{1 - \frac{1}{6}\frac{k_1 \Delta T}{\eta_1 \Delta i_{12}}}\right)\left(1 - \frac{5}{4}\frac{1 + \frac{k_1 \Delta T}{\eta_1 \Delta i_{12}}}{1 - \frac{1}{6}\frac{k_1 \Delta T}{\eta_1 \Delta i_{12}}}\right)\langle v_1 \rangle \geq 0$$

Dividing through by $\langle v_1 \rangle^2$, the stability condition can be put into dimensionless form, thus

$$\frac{3^{1/3}\zeta}{Re^{5/3}}k^{+2} - \frac{1}{Fr^2}g_y^+ - 2\frac{q_{1c}^{+2}}{Re^{4/3}}\frac{\Delta \rho}{\rho_2} - 3\frac{q_{1c}^+}{Re^{5/3}k^{+2}}\left(1 + \frac{1}{6}Re^{1/3}q_{1c}^+\right) + \tag{12.28}$$

$$\frac{36}{5}\left(\frac{1 + \frac{1}{90}q_{1c}^+ Re^{1/3}}{1 - \frac{1}{6}q_{1c}^+ Re^{1/3}}\right)\left(1 - \frac{5}{4}\frac{1 + \frac{1}{90}q_{1c}^+ Re^{1/3}}{1 - \frac{1}{6}q_{1c}^+ Re^{1/3}}\right) \geq \frac{6}{5}$$

In this relation, we have defined:

The Reynolds number by

$$Re = \frac{\ell \langle v_1 \rangle}{\nu_1} \tag{12.29}$$

The Froude number by

$$Fr = \frac{\langle v_1 \rangle}{(\ell g)^{1/2}} \tag{12.30}$$

The property group by

$$\zeta = \sigma \bar{\rho}^{-1} g_x^{-1/3} \nu^{-4/3} \tag{12.31}$$

The dimensionless heat flux group by

$$q_{1e}^{+} = \frac{k_1 \Delta T}{\eta \, \Delta i_{12}} \; \iota^{1/3} \, \rho_1^{-1} \, g_x^{-1/3} \, \nu^{-1/3} = \frac{q_{1e}}{\Delta i_{12}} \; \iota^{1/3} \, \rho_1^{-1} \, g_x^{-1/3} \, \nu^{1/3} \qquad (12.32)$$

Furthermore, k^{+} and g_y^{+} are, respectively, are dimensionless wave number and the dimensionless gravity. They are defined by

$$k^{+} = k \, \eta \qquad (12.33)$$

and

$$g_y^{+} = \frac{g_y}{g} \qquad (12.34)$$

In view of Equations (12.9, 29 and 30), it is evident that

$$Fr^2 = g_x^{+} \; \frac{Re}{3} \qquad (12.35)$$

so that only the Reynolds number can be chosen independently.

If we express Equation (12.26) in terms of the dimensionless groups defined above, the necessary condition can be expressed by

$$\frac{1}{6} \, q_{1e}^{+} \, Re^{1/3} \leq 1 \qquad (12.36)$$

It is of interest to find out the range of the Reynolds number which satisfy this condition. For example, taking the values appropriate to water at 100°C: $\nu_l = 0.002944$ Cm^2/Sec; $\rho_l = 1$ Gram/Cm^3; $g = 981$ Cm^2/Sec; $\Delta i_{fg} = 560$ Calorie/Gram; $\sigma = 59$ Dyne/Cm. and the wall heat flux $q_{1e} = 7.53$ Calorie/Cm^2 Sec(= 10^5 Btu/hr.ft^2), we get from Equation (12.32) $q_{1e}^{+} = 0.01362$. Substituting this numerical value in Equation (12.36), for the limit of Reynolds number we obtain

$$Re^{1/3} \leq 440.5 \qquad (12.37)$$

This value is well above the Reynolds numbers that are encountered in the free liquid film flow analysis. Consequently, it can be said that the necessary condition for the stability of the film is always satisfied even with high heat transfer flux.

We note that since Equation (12.36) is true for every case, we can neglect the terms which are order of $1/6q_{1e}{}^{+}Re^{1/3}$, and from Equation (12.28) we obtain for the sufficient condition

$$3^{1/3} \frac{\xi}{Re^{5/3}} k^{+2} - 3 \frac{g_y^+}{g_x^+} \frac{1}{Re} - 2 \frac{q_{1e}^{+2}}{Re^{4/3}} \frac{\Delta e}{e_2} - 3 \frac{q_{1e}^+}{Re^{5/3} k^{+2}} \geq 3 \quad (12.38)$$

In arriving to Equation (12.38), we made use of Equation (12.35) in order to express the Froude number in terms of the Reynolds number.

It is evident from Equation (12.3) that

$$\frac{g_y^+}{g_x^+} = - \cot \theta \qquad (12.39)$$

Substituting Equation (12.39) in Equation (12.38), and rearranging one can show that

$$3^{1/3} \xi k^{+4} + (3 \cot \theta \ Re^{2/3} - 2 \ q_{1e}^{+2} \frac{\Delta e}{e_2} Re^{1/3} - \qquad (12.40)$$
$$3 \ Re^{5/3}) \ k^{+2} - 3 \ q_{1e}^+ \geq 0$$

This is the stability criterion which can be used to investigate the stability of the free surface of the liquid film flow.

Before going into numerical evaluation of this equation, several observations can be made with respect to this inequality:

1. Since by definition ξ is proportional to the surface tension, see Equation (11.31), due to the first term in Equation (12.40) it is

obvious that the surface tension has stabilizing effect,

    2. Consiering now the second term it is evident that if $\theta < 90°$, the gravity has stabilizing effect, whereas if $\theta > 90°$, the gravity has destabilizing effect,

    3. Since by definition, $q_{1e}^+ > 0$ for evaporation then it can be seen that due to the heat flux terms in Equation (12.40) the evaporation has destabilizing effect,

    4. Since for condensation heat flux changes its sign, i.e., $q_{1e}^+ < 0$, the condensation has dual effect on the stability. Due to the third term it has destabilizing effect whereas due to the fifth term it has stabilizing effect. Relative order of stabilizing and destabilizing effects can be determined by comparing the third and the fifth terms. Hence, if

$$q_{1e}^+ \, \frac{\Delta \rho}{\rho_2} \, Re^{5/3} < \frac{3}{2} \qquad (12.41)$$

then it is evident that the stabilizing effect overcomes the destabilizing effect. If we trace back to the origin of these two terms it can be seen that the term which contributes to the stability arises from the kinematic field whereas the term which promotes the instability originates from the dynamic field, i.e., from the interfacial vapor thrust term in the two phase pressure difference.

    The critical condition, i.e., the neutral stability condition, can be obtained from Equation (12.40) by setting it to zero and analyzing the relation between various mean flow parameters, thus

$$3^{1/3} \, \sum \, k_{cr}^{+4} + (3 \, \cot \theta \, Re^{2/3} - 2 \, q_{1e}^{+2} \, \frac{\Delta \rho}{\rho_2} \, Re^{1/3} - 3 Re^{5/3}) k_{cr}^{+2} \qquad (12.42)$$

$$3 \, q_{1e}^+ = 0$$

Under this condition, i.e., when $C_i = 0$, the velocity of propagation of the wave in the x-direction can be obtained from Equation (11.42c) by using the growth factor $C_i k = 0$, thus

$$C_r = -\frac{Y_4}{2 Y_2} \tag{12.43}$$

Substituting the expressions Y_2 and Y_4, and making non-dimensional by dividing through the averaged velocity $<v_1>$, we obtain for the dimensionless velocity of propagation as

$$C_r^+ = \frac{C_r}{<v_1>} = 3 \frac{1 + \frac{1}{90} q_{1e}^+ Re^{1/3}}{1 - \frac{1}{6} q_{1e}^+ Re^{1/3}} \tag{12.44}$$

which in view of the approximation introduced by Equation (12.36), we obtain

$$C_r^+ = 3 \tag{12.45}$$

which agrees with the result found by Kapitza who used a different form of argument applicable only to very long waves. This also agrees with the result of Lighthill and Whitham's kinematic wave propagation which can be demonstrated as follows: from Equation (12.9) the volumetric flow rate can be expressed as

$$Q = \frac{1}{3} \frac{g_x}{\nu_1} z^3 \tag{12.46}$$

Hence, the existence of an explicit relation between flow rate and the depth of the fluid suggests that any gradual or long-wave disturbance from the uniform state would be propagated downstream with a dimensionless velocity.

$$C_r = \frac{dQ}{d\eta} \qquad (12.47)$$

which leads to the same result as that given by Equation (12.45). This simple result provides an interesting check on the analysis which by a totally different method has shown that the very long periodic waves are indeed propagated with a velocity equal to $3 \langle v_1 \rangle$.

Returning now to the stability criterion, we can analyze the effects of various parameters such as Reynolds number, heat flux and the property group by taking into consideration some special cases. Major contribution can be achieved by differentiating two cases, the adiabatic and the diadiabatic cases, separately.

12.4.1 Adiabatic Film Flow

Adiabatic case where there is no heat flow in or out of the liquid film can be obtained by simply equating q_{1e}^+ to zero in Equation (12.42). Hence, for an adiabatic flow of a liquid film over an inclined plane we obtain

$$3^{1/3} \left\{ k^+ \right\}_{ad}^2 + 3 \left(C_{ot} \theta \; Re^{2/3} - Re^{7/3} \right) = 0 \qquad (12.48)$$

where $k_{cr}^+)_{ad}$ stands for the critical wave number for adiabatic case.

Some special cases following from Equation (12.48) deserve to be noted. In the absence of the surface tension that is to say for very long wave-length λ, i.e., for very small wave number k, we can neglect the quadratic term in k^+ in determining the critical Re

$$Re)_{cr} = C_{ot} \theta \qquad (12.49)$$

Hence, for plane flow with inclination θ there is a critical Re given by Equation (12.49), above which flow will be unstable. Consequently, the

stabilizing effect of the gravity is confirmed.

We note that this result compare favorably with the result obtained by Yih [74] from a solution of the Orr-Sommerfeld equation for small values of the wave number k^+, thereby, giving support to the simple analysis with integral equations. Restating his criteria for the critical condition in nomenclature used in this work one gets

$$Re)_{cr} = \frac{5}{6} \cot \theta \qquad (12.50)$$

It can be seen that Yih's result can be expressed in an exactly similar form, apart from replacing the coefficient 1 in Equation (12.49) by 5/6.

Let us now consider the case of a vertical plane, i.e., $\theta = 90°$. The term involving $\cot \theta$ in Equation (12.48) vanishes, and the critical wave number can be expressed as

$$k_{cr}^+)_{ad} = 3^{1/3} \frac{Re^{5/6}}{\xi^{1/2}} \qquad (12.51)$$

According to this equation, for various values of the property group ξ, Figure 8 shows the relation between the Reynolds number and the dimensionless wave length $\lambda_{cr}^+ = 1/k_{cr}^+$. The curves shown in the figure are curves of neutral stability; and it can be inferred that, for a particular ξ, the region lying below the respective curve represents stability, while that above it represents instability. The following properties of Figure 8 may be particularly noted:

Every curve for $\xi > 0$, goes to ∞ as $Re \to 0$. Therefore, for all finite values of Re there is a finite range of unstable λ^+, and therefore, the flow is never completely stable, i.e., there is no absolute stability region, although as it was shown above there was an absolute stability region for $\theta < 90°$.
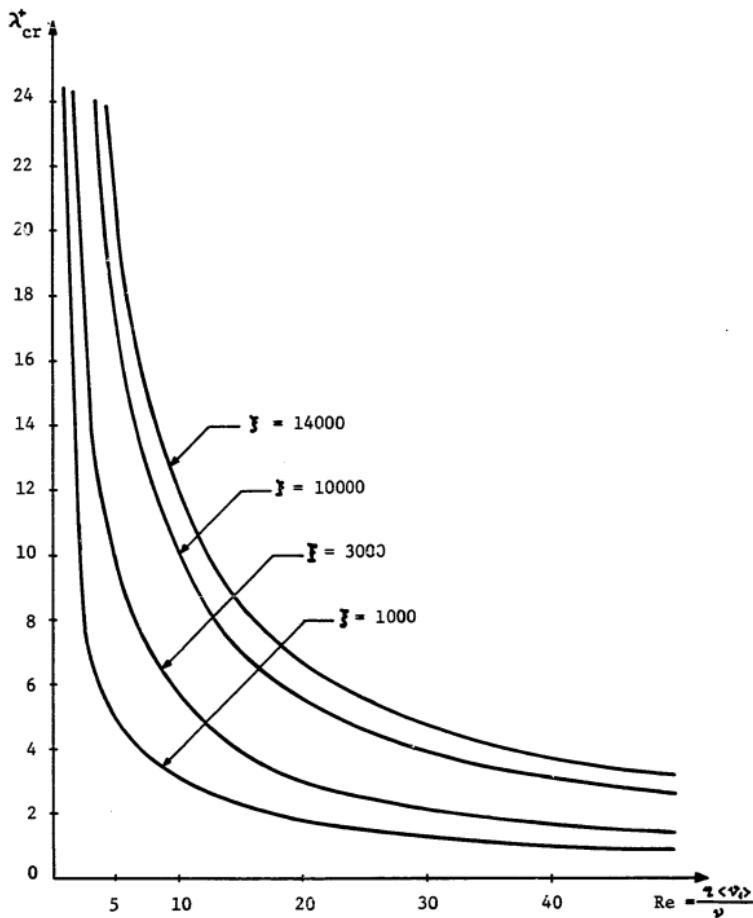
Figure 8. Curves of Neutral Stability for Laminar
Flow Down a Vertical Plane with Various
Values of the Parameter Ϝ.

The stabilizing effect of the surface tension is demonstrated by the fact that the stable region below the curves increases as Γ increases. However, consistent with Benjamin's results surface tension can not induce a complete stability, although it may stabilize waves whose wave-length is less than a certain critical value.

Before concluding this section, we can look at another special case for which $\theta > 90°$, that is liquid film stream runs down underside of the plane; therefore, it is expected that the action of gravity promotes the instability. This effect can be confined from Equation (12.48) in which gravitational terms becomes negative when $\theta > 90°$. A special case can be investigated by letting $\theta = 180°$. From Equation (12.9) $<v_1> = 0$ yielding the case of stability of a stationary liquid film underside of a horizontal plane, known as Rayleigh-Taylor instability. From the dimensional stability criterion with $\theta = 180°$, $<v_1> = 0$, and $q_{1e} = 0$, the stability condition becomes

$$k^2 \geqslant \frac{\rho_1 g}{\sigma}$$

(12.52)

or in terms of more practical parameter the wave-length the criterion becomes

$$\lambda \leqslant \frac{2\pi}{\sqrt{\rho_1 g / \sigma}}$$

(12.53)

This is seen to check with the result obtained by Chandrasekhar [81, Equation 51, pp. 435], who calculated the effect of surface tension on the Rayleigh-Taylor instability of the boundary between two fluids.

12.4.2 Diabatic Film Flow

Including heat transfer effects, and solving Equation (12.42) for

k^+_{cr})di, we obtain

$$k^+_{cr})_{di} = \frac{3^{1/3}}{2^{4/3}\xi^{4/3}} \left\{ \left(\frac{2}{3} q^{+2}_{1e} \frac{\Delta\rho}{\rho_2} Re^{1/3} + Re^{5/3} \cot\theta \, Re^{4/3} \right)_+ \right.$$ (12.54)

$$\left. \left[\left(\frac{2}{3} q^{+2}_{1e} \frac{\Delta\rho}{\rho_2} Re^{1/3} + Re^{5/3} \cot\theta \, Re^{2/3} \right)^2 + \frac{4}{3^{2/3}} \xi \, q^+_{1e} \right]^{1/2} \right\}^{1/2}$$

which reduces to the adiabatic case critical wave number when $q^+_{1e} = 0$.

For a vertical flow, i.e., $\theta = 90°$, by comparing Equations (12.54) with (12.51), it can be seen that for a given Reynolds number and the property group given by Equation (12.31), following relation holds

$$k^+_{cr})_{di} > k^+_{cr})_{ad}$$ (12.55)

or

$$\lambda^+_{cr})_{di} < \lambda^+_{cr})_{ad}$$ (12.56)

This indicates that the critical wave-length appropriate for the adiabatic flow is less than that of the diadiabatic flow. Since in Figure 8, the stable region corresponds to the area below the curves then it can be seen from Equation (12.56) that the mass transfer reduces the stable region.

For an illustrative example let us consider a water film flowing steadily down a vertical wall, i.e., $\theta = 90°$, at the atmospheric pressure, with the free liquid surface being at the saturation temperature as the result of a uniform wall heat flux, q^+_{1e}. The following table, based upon Equations (12.54) and (12.51), compares the behavior of the water

film for different Reynolds numbers.

Figure 5. Effect of the Heat Transfer on the Dimensionless Critical
          Wave Length,

q_{1e} Cal/Cm²Sec	Re			
	1	5	25	50
	λ^+_{cr}			
0	82.625	22.609	5.662	3.071
0.0753	65.605	21.562	5.563	2.916
0.753	43.600	20.165	5.541	2.914
7.53	25.791	18.623	5.528	2.913

From Table 5, one can infer that the effect of the interfacial mass

transfer, which is proportional to the heat flux, on the critical wave-

length are larger at low Reynolds number, however, as the Reynolds number

increases it decreases and at high enough Reynolds numbers effect is

almost nil.

12.5 Growth Factor

We consider now the growth rate factor which was derived in

Chapter 11. In view of Equation (11.47), the dimensionless growth rate

can be given by

$$C_i^+ k^+ = Y_2^+ k^+ \mp \frac{1}{\sqrt{2}} \left\{ -A^+ k^{+2} + \left[(A^+ k^+)^2 + (B^+ k^+)^2 \right]^{1/2} \right\}^{1/2} \qquad (12.57)$$

in which A^+, B^+ and Y_2^+ are determined in terms of the coefficients cal-

culated in Section (11.2). From Equation (12.24b) and in view of the

approximation introduced by Equation (12.26) Y_2^+ can be given by

$$Y_2^+ = -\frac{3}{2} \frac{1}{Re\, k^{+2}} \qquad (12.58a)$$

and

$$Y_2^+ k^{+2} = -\frac{3}{2} \frac{1}{Re} \qquad (12.58b)$$

From Equations (11.24a, b, c, and 11.41) and in view of Equation (12.26) A^+ can be expressed by the following relation

$$A^+ = \frac{6}{5} - \frac{9}{4} \frac{1}{Re\, k^{+2}} + \frac{3^{4/3}\, \xi\, k^{+4}}{Re^{5/3}} - \frac{g_y^+}{Fr^2} - 2\frac{q_{ie}^{+2}}{Re^{5/3}} \qquad (12.59a)$$

and

$$A^+ k^{+2} = \frac{3^{4/3}\, \xi\, k^{+4}}{Re^{5/3}} + \left(\frac{6}{5} - \frac{g_y^+}{Fr^2} - \frac{q_{ie}^{+2}}{Re^{5/3}}\right) k^{+2} - \frac{9}{4} \frac{1}{Re} \qquad (12.59b)$$

Finally, from Equations (11.24a, b, d, and 11.42) for B^+ we get

$$B^+ = \frac{27}{5} \frac{1}{Re\, k^+} \qquad (12.60a)$$

and

$$B^+ k^{+2} = \frac{27}{5} \frac{k^+}{Re} \qquad (12.60b)$$

According to Equations (12.57-12.60), $c_i^+ k^+ \to 0$ as $k^+ \to 0$ and $c_i^+ k^+ < 0$ as $k^+ \to \infty$. There exists, therefore, a mode of maximum instability that is to say there exists a maximum growth rate at which the wave amplifies most rapidly. To determine this most dangerous wave number km^+, which makes $c_i k^+$ maximum, Equation (12.57) can be differentiated and then set to zero, thus

$$k_m^{+6} + \frac{A_2}{2A_1} k_m^{+4} - \frac{A_3}{A_1} k_m^{+2} - \left[\frac{1}{8} \left(\frac{B_1}{A_1} \right)^2 + \frac{1}{2} \frac{A_2}{A_1} \frac{A_3}{A_1} \right] = 0 \qquad (12.61)$$

where we defined

$$A_1 = 3^{1/3} \frac{\overline{\delta}}{Re^{5/3}} \qquad (12.62a)$$

$$A_2 = \frac{6}{5} - \frac{\overline{g}_{\overline{y}}^+}{Fr^2} - \frac{2 q_{1e}^{+2}}{Re^{4/3}} \qquad (12.62b)$$

$$A_3 = - \left(\frac{9}{4} \frac{1}{Re} + 3 \frac{q_{1e}^+}{Re^{5/3}} \right) \qquad (12.62c)$$

$$B_1 = \frac{27}{5} \frac{1}{Re} \qquad (12.62d)$$

Equation (12.61) can be solved by comparing it with

$$a x^3 + b x^2 + c x + d = 0 \qquad (12.63)$$

whose real root is given by

$$x = - \frac{b}{3} + \left\{ -q + \left(q^2 + p^3 \right)^{1/2} \right\}^{1/3} + \left\{ -q - \left(q^2 + p^2 \right)^{1/2} \right\}^{1/3} \qquad (12.64)$$

where q and p are defined as

$$q = \left(\frac{b}{3} \right)^3 - \left(\frac{b}{3} \right) \left(\frac{c}{2} \right) + \frac{d}{2} \qquad (12.65)$$

$$p = \frac{c}{3} - \left(\frac{b}{3} \right)^2 \qquad (12.66)$$

In view of Equations (12.64-66), Equation (12.64) can be solved for k_m^+, and upon substituting k_m^+ in Equation (12.57) one obtains the maximum growth rate factor $C_i^+ k_m^+$ as functions of $\overline{\xi}$, Re and q_{1e}^+.

In order to compare and evaluate the destabilizing effect caused by <u>local</u> evaporation, it is advantageous to calculate the maximum growth rate for adiabatic film and $C_i^+ k_m^+)_{di}$ for diabatic films. The ratio gives the relative increase of instability caused by evaporation at the

liquid film surface.

For this purpose consider the motion of a water film on a
vertical surface at atmospheric pressure with the interface at the
saturation temperature. The following table is based upon Equations
(12.57-12.63) compares the behavior of the film for different heat
fluxes.

Table 6. Relative Effect of Evaporation on the Stability of a Vertical
Water Film

$\dfrac{\text{Calorie}}{\text{Cm}^2 \ \text{Sec}}$	Reynolds Number				
	10	25	50	100	200
	$(c_i^+ k_m^+)_{di} / (c_i^+ k_m^+)_{ad}$				
0.753	1.031	1.0082	1.0015	1.0002353	1.00009
7.53	1.204	1.0743	1.0090	1.0038755	1.00187
75.3	3.601	1.7480	1.3785	1.2225135	1.11243

It can be seen from Table 6 that the ratio is always greater than
one, which implies that the evaporation enhances the instability. How-
ever, it can be seen also that this effect even becomes significant at
high Re numbers provided the heat flux is high enough.

12.6 Film Break-Up Criterion

The liquid film break-up length L_b was derived in Chapter 11, and
given by Equation (11.68). This equation can be made dimensionless by
dividing through by $\eta \langle v_1 \rangle$. Hence, for the dimensionless break-up length
L_b we obtain the following expression:

$$L_b^+ = \frac{L_b}{\eta} = \frac{C_r^+}{C_i^+ \, k_m^+} \, \ell_m \frac{a_m}{a_i} \tag{12.67}$$

where C_r^+ is the dimensionless velocity of propagation and given by

$$C_r^+ = \frac{C_r}{\langle v_i \rangle} \tag{12.68}$$

whereas $\frac{a}{ai}$ is the ratio of the amplitude at the length L_b to the initial

amplitude. In view of Equations (12.24a, b, and d), (11.52), (11.50) and

the approximation introduced by Equation (12.36), it can be shown that the

wave propagation velocity can be expressed as

$$C_r = \frac{\frac{27}{5} \frac{\Gamma_i \langle v_i \rangle}{\rho_i \eta^2 k}}{2 \left(C_i + \frac{3}{2} \frac{\Gamma_i}{\rho_i \eta^2 k} \right)} + \frac{6}{5} \langle v_i \rangle \tag{12.69}$$

which in dimensionless form becomes

$$C_r^+ = \frac{27}{10 \left(\frac{3}{2} + C_i^+ k_m^+ \right)} + \frac{6}{5} \tag{12.70}$$

By substituting Equation (12.70) in Equation (11.67) yields for the

break-up length:

$$L_b^+ = \frac{\frac{6}{5} + \frac{27}{10 \left(3/2 + C_i^+ k_m^+ \right)}}{C_i^+ k_m^+} \, \ell_m \frac{a_b}{a_i} \tag{12.71}$$

It is evident from the above equation that in order to determine

the break-up length one needs to know $C_i^+ k_m^+$, a_b and a_i. By the method

presented in the previous chapter $C_i^+ k_m^+$, can be calculated for given

conditions. However, in order to determine explicitly a_b, the wave

amplitude leading to break-up of the liquid film, it is necessary to resort

to experiments. For the case of liquid entrainment it has shown in

reference [82] that the wave heights approach the magnitude of the average

liquid film thickness. Therefore, the waves may break-up when the wave amplitude become comparable to the film thickness. Finally, it is also necessary to know the initial wave amplitude a_i, in order to determine L_b. Weber [83] states that $\ln \dfrac{a_b}{a_i}$ reaches a constant value for the break-up of moving liquid sheets. We note that this configuration is not the same as the one of interest here. The dependence upon the initial dimensionless wave amplitude is, however, logarithmic, so that for large values of the argument of the logarithmic function, changes in the argument result in very small changes in the functional value. Therefore, in the absence of any data fitting for the case investigated here we shall use the value given by Weber, which is

$$\ln \frac{a_b}{a_i} = 12 \qquad (12.72)$$

to illustrate orders of magnitude from existing data. Hence substituting Equation (12.69) in Equation (12.68), finally we obtain

$$L_b^+ = \frac{12 \left[\dfrac{6}{5} + \dfrac{27}{5} \dfrac{1}{(3/2 + C_i^+ \, k_m^+)} \right]}{C_i^+ \, k_m^+} \qquad (12.73)$$

Once the most dangerous growth rate $C_i^+ \, k_m^+$, is calculated from the preceding section's results, one can determine from Equation (12.73) the dimensionless length L_b.

For the isothermal flow of water on vertical plane, dimensionless break-up length versus Reynolds number is plotted on Figure 9 with ξ as parameter. It is evident from the figure that the dimensionless length is very strong function of the Reynolds number at the lower values of Re. As ξ increases for a given Re, the length increases delaying the
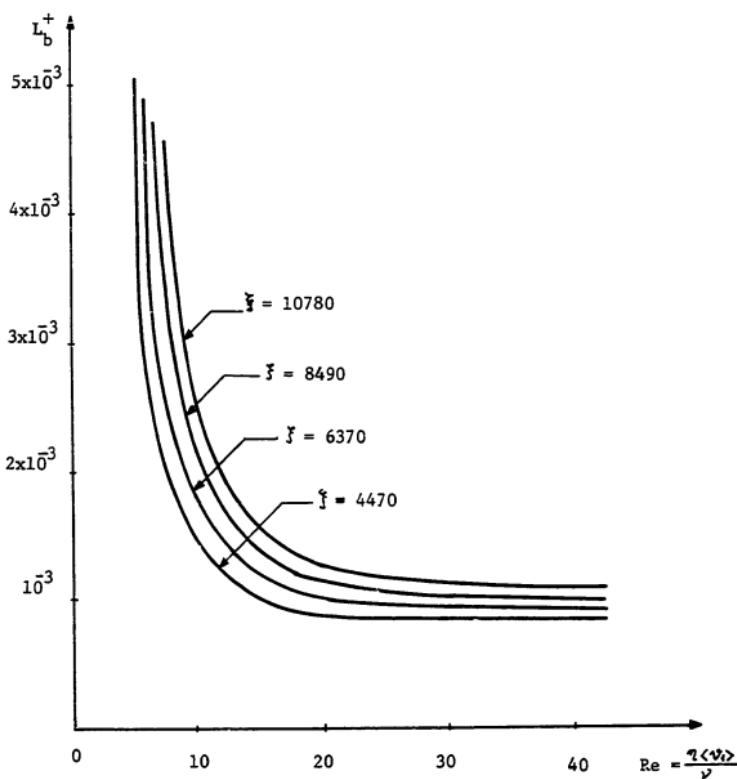
Figure 9. Break-Up Length Versus Reynolds Number; $q_{1e} = 0$

formation of dry spots.

Experimental data are available in reference [84] where the minimum stable film thickness for an annular film of water on the inside of a 1-inch diameter smooth copper pipe was measured. The test section used in this study was 6 inches, and the break-up was always observed at the end of the test section. For different fluid temperatures with no interfacial mass transfer, the minimum stable thickness was measured.

On Table 7, experimental results of reference [84] is compared with Figure 9.

Table 7. Film Thickness of Water versus Break-Up Length

	Properties				Results of Ref[84,Fig.4]			Fig.9
Temp. (°C)	Surface Tension (Dyn/Cm)	Viscosity (Centipoise)	Density (Gram/Cm³)	Property Group ξ	Film Thickness (Cm)	Re	$L_b^+ = \frac{L_b}{\eta}$	$L_b^+ = \frac{L_b}{\eta}$
30	71.2	0.801	0.996	4470	0.0148	12.8	1027	1119
45	68.8	0.599	0.990	6360	0.0115	12.5	1323	1379
60	66.2	0.469	0.983	8490	0.0098	12.5	1546	1612
75	63.5	0.380	0.975	10780	0.0083	12.5	1826	1850

It can be seen from Table 7 that the agreement is quite good even in view of the obvious limitations of linearized theory in this type of investigations. Most important of all, the comparison with experiment justifies that the break-up of thin liquid films is governed by the growth of the most unstable surface waves. Of course, the analysis will cease

to apply as soon as thin dry spots are formed on solid surfaces. Beyond this point, however, the contact angle which is the measure of the wetability of the liquid plays an important role for the stability of the dry spots.

In order to see the dynamic effect of the interfacial phase change on the break-up length we can form the ratio of two lengths, one without mass transfer and the other with mass transfer. Hence, from Equation (12.73) for the ratio we obtain

$$\frac{L_b^+)_{di}}{L_b^+)_{ad}} = \frac{\frac{6}{5} + \frac{27}{10} \frac{1}{1.5 + C_i^+ k_m^+)_{di}}}{\frac{6}{5} + \frac{27}{10} \frac{1}{1.5 + C_i^+ k_m^+)_{ad}}} \frac{C_i^+ k_m^+)_{ad}}{C_i^+ k_m^+)_{di}} \qquad (12.74)$$

Since $C_i^+ k_m^+ < 1.0$, then the propagation velocity can be taken equal in both cases, and from Equation (12.74) we obtain

$$\frac{L_b)_{di}}{L_b)_{ad}} = \frac{C_i^+ k_m^+)_{ad}}{C_i^+ k_m^+)_{di}} \qquad (12.75)$$

Now we note that according to Table 6

$$\frac{C_i^+ k_m^+)_{di}}{C_i^+ k_m^+)_{ad}} > 1.0 \qquad (12.76)$$

Therefore, in view of this inequality it can be seen from Equation (12.75) that

$$L_b^+)_{di} = \frac{C_i^+ k_m^+)_{ad}}{C_i^+ k_m^+)_{di}} \quad L_b)_{ad} < L_b)_{ad} \qquad (12.77)$$

which states that due to the destabilizing effect of the interfacial mass transfer the break-up length of the film with mass transfer shorter than the corresponding break-up lengths in a system where interfacial mass

transfer does not occur.

12.7 Conclusions

1. An evaluation of the importance of various parameters on
the instability of free liquid films has been made.

2. The results indicate that, a) the surface tension has
stabilizing effect irrespective of the configuration of the plate, b)
the gravity has a stabilizing effect when $\theta < 90°$, and a destabilizing
effect when $\theta > 90°$, c) evaporation from the surface of a liquid film
has a destabilizing effect, while condensation has dual effects-one
destabilizes and the other stabilizes. Relative importance of these
two opposite effects was given by Equation (12.41).

3. It was shown that, as the Reynolds number decreases, i.e., as
the film thickness decreases, and the heat flux increases, the thermal
effects become important and may become the dominant destabilizing
factor.

4. The effect of evaporation is to decrease both the critical
and the most dangerous wave length and to increase therefore the growth
rate of disturbances.

5. An immediate result of the previous conclusion is that as the
growth rate increases, the break-up length of the most dangerous wave
decreases.

CHAPTER XIII

STABILITY OF TWO INVISCID FLUIDS

13.1 Introduction

The interaction of two fluids moving along a continuous inter-
face is encountered in many industrial applications, see Chapter 1.
A velocity difference between the two fluids encourages the growth of
waves until eventually one fluid mass may disintegrate and be swept
away in the other. This kind of instability that is the instability
set up by the relative velocity between two phases is called the Kelvin-
Helmholtz instability, and has been studied entirely from the fluid-
mechanical point of view, see for example, Lamb [57].

In this chapter we shall consider the stability of two inviscid
fluids flowing parallel to each other in which the effect of vaporiza-
tion or condensation will be accounted for. Following the general frame-
work described in Chapter 11, we shall investigate effects of inter-
facial mass transfer on the stability of interfacial waves.

13.2 Undisturbed Flow

Consider two incompressible, inviscid fluids confined between two
parallel planes which are located a distance of H apart from each other.
The liquid film, i.e., phase 1, has a thickness of η whereas the vapor
or gas, i.e., phase 2, has a thickness of $H-\eta$ as described on Figure 10.

We shall consider a heat conduction through the thin liquid film
and assume that the velocities v_1 and v_2 of the liquid and of the vapor

are uniform permitting a vortex sheet at the interface. This assumption renders the analysis approximate; however, they make it also tractable and simple.
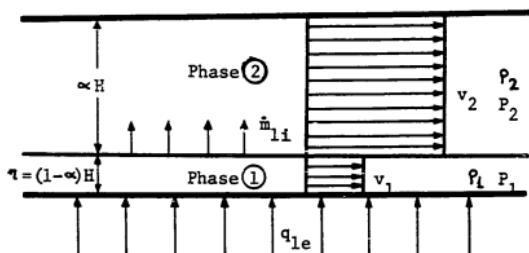

Figure 10. Undisturbed Flow of Two Inviscid Fluids

13.3 Stability Criterion

After having calculated X's and Y's in Appendix F, now we can proceed with the investigation of the stability criterion within the general framework of Chapter 11.

Using Equations (F.15c, 15d, 16a and 16b) in Equations (11.51, and 52) for A we obtain

$$A = \frac{\left(\frac{\varrho_1 v_1}{1-\alpha} + \frac{\varrho_2 v_2}{\alpha} \right)^2}{\left(\frac{\varrho_1}{1-\alpha} + \frac{\varrho_2}{\alpha} \right)^2} + \frac{1}{\frac{\varrho_1}{1-\alpha} + \frac{\varrho_2}{\alpha}} \left[\sigma H k^2 + H \Delta \varrho \, g - \right. \tag{13.1}$$

$$\left. 2 \left(\frac{k_1 \Delta T}{\Delta i_{12} (1-\alpha) H} \right)^2 \frac{1}{1-\alpha} \frac{\Delta \varrho}{\varrho_1 \varrho_2} - \left(\frac{\varrho_1 v_1^2}{1-\alpha} + \frac{\varrho_2 v_2^2}{\alpha} \right) \right]$$

which upon re-arranging yields

$$A = \frac{1}{\left(\frac{\varrho_1}{1-\alpha} + \frac{\varrho_2}{\alpha} \right)^2} \left[\left(\sigma H k^2 + H \Delta \varrho \, g \right) \left(\frac{\varrho_1}{1-\alpha} + \frac{\varrho_2}{\alpha} \right) - \right. \tag{13.2}$$

$$\left. 2 \left(\frac{k_1 \Delta T}{\Delta i_{12} (1-\alpha) H} \right)^2 \frac{1}{1-\alpha} \frac{\Delta \varrho}{\varrho_1 \varrho_2} \left(\frac{\varrho_1}{1-\alpha} + \frac{\varrho_2}{\alpha} \right) \right]$$

whereas for B we obtain

$$B = 0 \tag{13.3}$$

In Equation (13.2), V_r is the relative velocity defined by

$$V_r = V_2 - V_1 \tag{13.4}$$

In view of Equation (13.3), Equation (11.50) becomes

$$\overline{C}_i \overline{C}_r = 0 \tag{13.5}$$

Hence, Equation (11.49) can be expressed as

$$\left(\overline{C}_r + i \, \overline{C}_i \right)^2 - A = 0 \tag{13.6}$$

square root of which yields

$$\overline{C}_r + i\,\overline{C}_i = \mp\sqrt{A} \qquad (13.7)$$

In view of Equations (F.15a) and (11.53)

$$\overline{C}_r = C_r - \frac{\frac{\varrho_1 v_1}{1-\alpha} + \frac{\varrho_2 v_2}{\alpha}}{\frac{\varrho_1}{1-\alpha} + \frac{\varrho_2}{\alpha}} \qquad (13.8)$$

whereas in view of Equations (F.16a) and (11.54)

$$\overline{C}_i = C_i \qquad (13.9)$$

Therefore, substituting (13.8) and (13.9) in Equation (13.7) we obtain

$$C_r + iC_i = \frac{\frac{\varrho_1 v_1}{1-\alpha} + \frac{\varrho_2 v_2}{\alpha}}{\frac{\varrho_1}{1-\alpha} + \frac{\varrho_2}{\alpha}} \mp \sqrt{A} \qquad (13.10)$$

As it is discussed in Chapter 11, C_r denotes the velocity of propagation of the wave in the x-direction, whereas C_i determines the degree of amplification of disturbances on the interface.

There are several points to be noticed with respect to Equation (13.10):

1. If

$$A > 0 \qquad (13.11)$$

that is, if

$$\left(\sigma H k^2 + H \Delta \varrho\, g\right)\left(\frac{\varrho_1}{1-\alpha} + \frac{\varrho_2}{\alpha}\right) - \qquad (13.12)$$

$$2\left(\frac{k_1 \Delta T}{\Delta i_{12}\,(1-\alpha)H}\right)^2 \frac{1}{1-\alpha}\frac{\Delta \varrho}{\varrho_1 \varrho_2}\left(\frac{\varrho_1}{1-\alpha} + \frac{\varrho_2}{\alpha}\right) - \frac{\varrho_1 \varrho_2}{\alpha(1-\alpha)}V_r^2 > 0$$

then the right hand side of Equation (13.10) becomes pure real number, thus

$$C_i = 0 \qquad (13.13)$$

and the waves propagate with the real velocity

$$C_r = \frac{1}{\frac{\varrho_1}{1-\alpha} + \frac{\varrho_2}{\alpha}} \left\{ \left(\frac{\varrho_1 v_1}{1-\alpha} + \frac{\varrho_2 v_2}{\alpha} \right) + \left[\left(\sigma H k^2 + H \Delta \varrho \, g \right) \left(\frac{\varrho_1}{1-\alpha} + \frac{\varrho_2}{\alpha} \right) - \right. \right. \qquad (13.14)$$
$$\left. \left. 2 \left(\frac{k_1 \Delta T}{\Delta i_{12} (1-\alpha) H} \right)^2 \frac{1}{1-\alpha} \frac{\Delta \varrho}{\varrho_1 \varrho_2} \left(\frac{\varrho_1}{1-\alpha} + \frac{\varrho_2}{\alpha} \right) - \frac{\varrho_1 \varrho_2}{\alpha (1-\alpha)} V_r^2 \right]^{1/2} \right\}$$

We note that the interfacial waves propagate downstream as well as upstream with respect to a mean velocity V defined by

$$V = \frac{\frac{\varrho_1 v_1}{1-\alpha} + \frac{\varrho_2 v_2}{\alpha}}{\frac{\varrho_1}{1-\alpha} + \frac{\varrho_2}{\alpha}} \qquad (13.15)$$

which shows that the two fluids are coupled in parallel not in series. We note, furthermore, that for sufficiently small values of the wave number k, that is to say, for sufficiently large values of the wave-length λ the second term in the square root of Equation (13.14) is large compared with the first term; the force governing the motion of the waves is mainly that of gravity, which is the case for shallow water waves. On the other hand, when k is very large, i.e., λ is very small, the first term dominates, and the motion is mainly governed by the surface tension; the capillary waves. Finally, we note that in the absence of the surface tension and of the interfacial mass transfer

Equation (13.14) reduces to the result of Long [85] where the method of characteristics was used to obtain the wave speed.

2. If

$$A < 0 \tag{13.16}$$

then the right hand side of Equation (13.10) will be complex, thus

$$C_i = \sqrt{-A} \tag{13.17}$$

which, in view of Equation (13.1) yields a positive growth factor $C_i k$

$$C_i k = \frac{k}{\frac{\rho_1}{1-\alpha} + \frac{\rho_2}{\alpha}} \left[-\left(\sigma H k^2 + H \Delta \rho g \right) \left(\frac{\rho_1}{1-\alpha} + \frac{\rho_2}{\alpha} \right) + \tag{13.18} \right.$$

$$\left. 2 \left(\frac{k_1 \Delta T}{\Delta i_{12} (1-\alpha) H} \right)^2 \frac{1}{1-\alpha} \frac{\Delta \rho}{\rho_1 \rho_2} \left(\frac{\rho_1}{1-\alpha} + \frac{\rho_2}{\alpha} \right) + \frac{\rho_1 \rho_2}{\alpha (1-\alpha)} V_r^2 \right]^{1/2}$$

which in turn gives rise to the instability of interfacial waves. Furthermore, note that the interfacial mass transfer whether in form of evaporation, $\Delta T > 0$ or in the form of condensation $\Delta T < 0$ always promotes the growth factor which in turn helps instability. As it is well known from the Kelvin-Helmholtz instability the relative velocity enhances the instability whereas the surface tension and the gravity act in the direction of stabilization.

3. The critical condition is obtained from Equation (13.10) by setting

$$A = 0 \tag{13.19}$$

which implies the neutral stability, that is

$$C_i = 0 \qquad (13.20)$$

Hence, in view of Equations (13.19) and (13.1) the critical condition can be given by

$$-\left(\sigma H k^2 + H \Delta \varrho \, g\right) + 2\left(\frac{k_1 \Delta T}{\Delta i_{12}\,(1-\alpha)H}\right)^2 \frac{1}{1-\alpha} + \frac{\varrho_1 \varrho_2}{\frac{\varrho_1}{1-\alpha} + \frac{\varrho_2}{\alpha}} \frac{V_r^2}{\alpha(1-\alpha)} = 0 \quad (13.21)$$

We note that the critical condition expressed above can be put into dimensionless form, thus

$$\left(\frac{k_{cr}^{+2}}{We} + \frac{1}{Fr^2}\frac{\Delta\varrho}{\varrho_2} - 2\, q_{1e}^{+2}\frac{\Delta\varrho}{\varrho_1}\right)\left(\frac{\alpha}{1-\alpha} + \frac{\varrho_2}{\varrho_1}\right) - 1 = 0 \qquad (13.22)$$

where we have defined:
the <u>Weber</u> number:

$$We = \frac{\eta\,\varrho_2\,V_r^2}{\sigma} \qquad (13.23)$$

the <u>Froude</u> number:

$$Fr = \frac{V_r}{\sqrt{\eta\,g}} \qquad (13.24)$$

the <u>heat flux</u> number:

$$q_{1e}^{+} = \frac{k_1 \Delta T}{(1-\alpha)H\,\Delta i_{12}}\frac{1}{\varrho_2 V_r} \qquad (13.25)$$

which in view of Equation (11.16) can be expressed in terms of q_{1e} rather than ΔT, thus

$$q_{1e}^{+} = \frac{q_{1e}}{\Delta i_{12}\,\varrho_2\,V_r} \qquad (13.26)$$

and the dimensionless wave number k^+ is defined by

$$k^+ = k \, (1-\alpha) \, H = 2\pi \, \frac{(1-\alpha) H}{\lambda} \qquad (13.27)$$

We note that since for low system pressure operations

$$\frac{\rho_2}{\rho_1} \ll 1.0 \qquad (13.28)$$

therefore, neglecting terms which are order of $\dfrac{\rho_2}{\rho_1}$ and solving Equation (13.22) for the critical wave number k^+_{cr}, we obtain

$$k^+_{cr} = We^{1/2} \left(-\frac{1}{F_r^2} \, \frac{\rho_1}{\rho_2} + \frac{1-\alpha}{\alpha} + 2 \, q^{+2}_{lie} \right)^{1/2} \qquad (13.29)$$

This equation gives the critical wave number. It can be seen from Equation (13.28) that, in adiabatic system, the critical wave number is smaller than the corresponding wave number in a system where evaporation takes place. This statement can be equally expressed as the critical wave length, in adiabatic systems, are longer than the corresponding wave lengths in a system where phase change occurs.

13.4 Growth Rate

Growth rate expressed by Equation (13.10) can be put in dimensionless form, thus

$$C_i^+ k^+ = \left(\frac{\alpha}{1-\alpha} \right)^{1/2} \left(\frac{\rho_2}{\rho_1} \right)^{1/2} \frac{k^+}{\left(\frac{\alpha}{1-\alpha} + \frac{\rho_2}{\rho_1} \right)} \left[\left(-\frac{k^{+2}}{We} - \frac{1}{F_r^2} \, \frac{\Delta\rho}{\rho_2} + \right. \right. \qquad (13.30)$$

$$\left. \left. 2 \, q^{+2}_{lie} \, \frac{\Delta\rho}{\rho_1} \right) \left(\frac{\alpha}{1-\alpha} + \frac{\rho_2}{\rho_1} \right) + 1 \right]^{1/2}$$

As it is seen from this equation there are seven dimensionless groups rendering

$$C_i^+ k^+ = f \left(\frac{\alpha}{1-\alpha} \; ; Fr \; ; We \; ; q_{l1e}^+ \; ; \frac{\rho_2}{\rho_1} \; ; k^+ \right) \qquad (13.31)$$

However, some simplifications can be accomplished by making some approximations. First of all, as noted above $\frac{\rho_2}{\rho_1} \ll 1$, so that $\frac{\rho_2}{\rho_1}$ can be neglected. Of course, this approximation will cease to apply when we approach to the critical pressure. Secondly, since we are dealing with very thin liquid films with high speed vapor phase, i.e., with large relative velocity it can be seen from Equation (13.24) that the Froude number will be so large that the inverse of it can be neglected with respect to surface tension effects. Under these two assumptions Equation (13.30) becomes

$$C_i^+ k^+ = \left(\frac{\rho_2}{\rho_1} \right)^{1/2} k^+ \left(- \frac{k^{+2}}{We} + 2\, q_{l1e}^{+2} + \frac{1-\alpha}{\alpha} \right)^{1/2} \qquad (13.32)$$

whereas Equation (13.31) yields

$$C_i^+ k^+ = f \left(\frac{\alpha}{1-\alpha} \; ; We \; ; q_{l1e}^+ \; ; \frac{\rho_2}{\rho_1} \; ; k^+ \right) \qquad (13.33)$$

There are still five groups which $C_i^+ k^+$ depends on. However, the most convenient graphical presentation of the dimensionless growth rate can be accomplished by taking, for example, adiabatic flow of the air-water system. For this case

$$q_{l1e}^+ = 0 \qquad \text{and} \qquad \frac{\rho_2}{\rho_1} = 0.00112 \qquad (13.34)$$

and Equation (13.32) can be presented with $C_i^+ k^+$ versus $\lambda^+ = \frac{1}{k^+}$ by treating We and $\frac{\alpha}{1-\alpha}$ as parameters, as in Figures 11 and 12. Figure 11
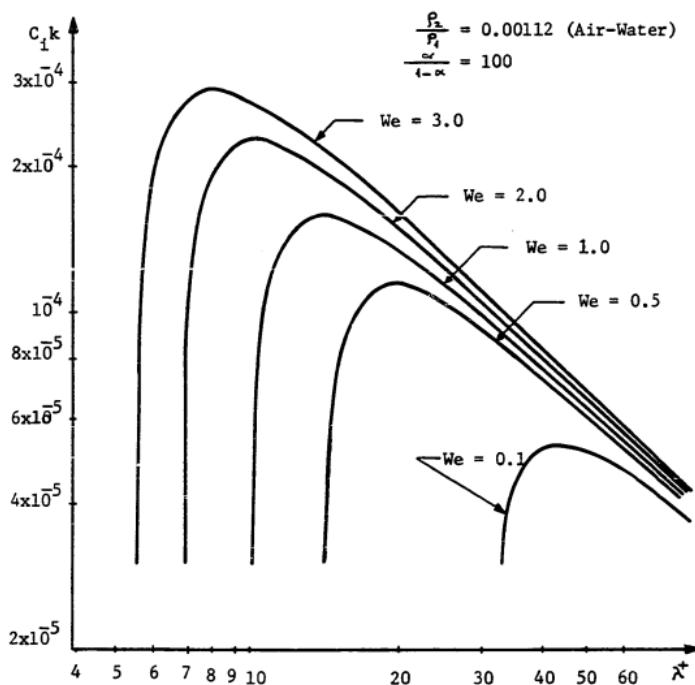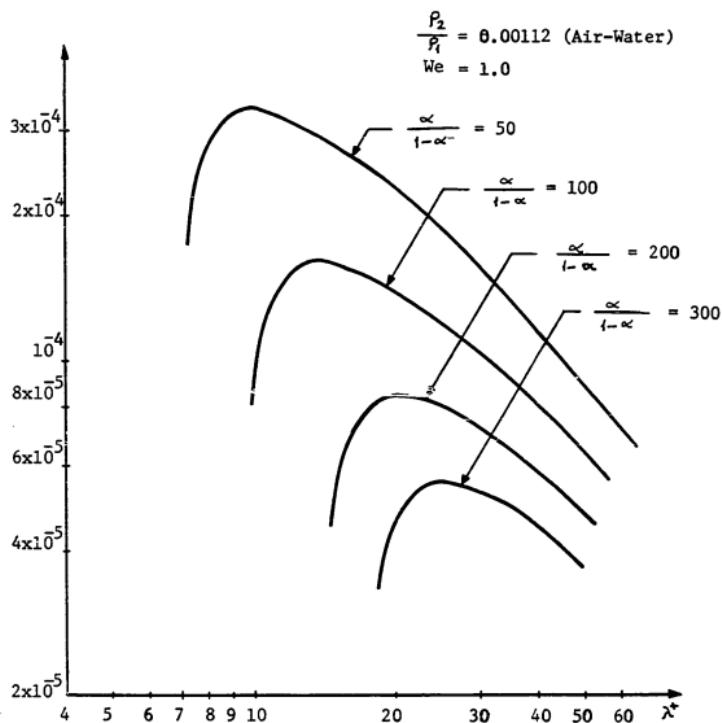
Figure 11. Effect of Weber Number
on Growth Rate

Figure 12. Effect of Concentration Ratio
on Growth Rate

shows the effect of the Weber number We, on the growth rate when the two fluids are air and water of which density ratio is given by Equation (13.34) Figure 12 shows the effect of volumetric concentration ratio on the growth rate when the Weber number is 2.

Figure 11 shows that the growth rate has a decided maximum for a given Weber number, particularly at higher Weber numbers. A disturbance of that wave-length will dominate the interface and probably will lead to the break-up of the liquid film before any other disturbances could. Figure 11 shows that a definite lower limit exists for λ^+, below which the interface is stable. This means that short wave lengths of disturbance on thick sheets are stable unless the Weber numbers are high. Decreasing the interfacial tension increases the Weber number and increases the growth rate correspondingly. For disturbances of long wave length, the Weber number becomes less significant and the growth rates are smaller. Figure 12 shows the effect of the volumetric concentration ratio $\frac{\alpha}{1-\alpha}$, and increase in this ratio yields smaller peaks which means less dangerous growth rates.

After seeing various effects on the growth rate $C^+_i k^+$, we can proceed now with obtaining the most unstable growth factor. The rate of growth of the disturbance depends on the growth factor which is a function of the wave number k^+. It can be expected that the motion of the interface will be determined by the disturbances whose growth is the fastest. Since $C^+_i k^+$ is a function of λ^+ the wave length which maximizes the growth factor $C^+_i k^+$ given by Equation (13.30), will be, therefore, the dominant disturbance. Maximizing Equation (13.30) with respect to k^+, and solving for k^+ we obtain

$$k_m = \left(\frac{We}{2} \right)^{1/2} \left[\frac{1}{\frac{\alpha}{1-\alpha} + \frac{\rho_2}{\rho_1}} - \frac{1}{Fr} \frac{\Delta\rho}{\rho_2} + 2 \, q^{+2} \iota \rho \frac{\Delta\rho}{\rho_1} \right]^{1/2} \quad (13.35)$$

The first two terms on the right hand side of Equation (13.35) is the contribution of the adiabatic flow instability whereas the third term is due to the destabilizing effect of evaporation. It can be seen, there-fore, from Equations (13.29) and (13.35) that, in adiabatic systems, both the critical wave length λ_{cr}^{+} and the most dangerous wave length λ_{m}^{+}, are longer than the corresponding wave lengths in a system where evaporation takes place.

Before concluding this section, we can consider an interesting special case. In the absence of the relative velocity, $V_r = 0$, consider now the stability of an incompressible, inviscid fluid layer of depth αH and density ρ_2 on top of another layer of incompressible, inviscid fluid of depth $(1 - \alpha)H$ and density ρ_1. The stability condition of the interfacial disturbances can be obtained from Equation (13.18) with $V_r = 0$, thus

$$- (1-\alpha) H \sigma k^2 - (1-\alpha) H \Delta\rho \, g + 2 \left(\frac{k_1 \Delta T}{(1-\alpha) H \Delta i_{12}} \right)^2 \frac{\Delta\rho}{\rho_1 \rho_2} \leqslant 0 \quad (13.36)$$

Thus when the liquid on top is heavier than the one below, we have well known Raleigh-Taylor instability. The effects of surface tension and gravity have been well documented in Chandrasekhar's book, reference [81]. Here we address ourselves to the effect of interfacial mass transfer on the stability of such a configuration.

Since the heavy fluid is situated on top of the light fluid then

$$\rho_1 - \rho_2 = \Delta\rho < 0 \quad (13.37)$$

and the second and the third terms in Equation (13.36) changes their sign. The stability condition, therefore, can be expressed

$$(1-\alpha) H (\rho_2 - \rho_1) g \leq (1-\alpha) H \sigma k^2 + 2 \left(\frac{k_1 \Delta T}{(1-\alpha) H \Delta \dot{\iota}_{12}} \right)^2 \frac{\rho_2 - \rho_1}{\rho_1 \rho_2} \quad (13.38)$$

It is evident from Equation (13.38) that the surface tension together with the interfacial mass transfer <u>stabilizes</u> the interfacial waves whereas the gravity destabilizes. We note particularly, that the effect of interfacial mass transfer. Although its effect, in the case of the light fluid on top, is in the direction of discouraging the stability, it encourages the stability of a system where the heavy fluid is on top of the light fluid. Therefore, when there is an interfacial mass transfer, the growth rate of the Rayleigh-Taylor instability may be greatly reduced whereas the growth rate of the Kelvin-Helmholtz instability, i.e., light fluid on top, is increased.

If one lets $k \rightarrow 0$ in Equation (13.38), i.e., $\lambda \rightarrow \infty$, as for very long waves, then the instability condition becomes

$$(1-\alpha) H g \leq 2 \left(\frac{k_1 \Delta T}{(1-\alpha) H \Delta \dot{\iota}_{12}} \right)^2 \frac{1}{\rho_1 \rho_2} \quad (13.39)$$

Thus, when Equation (13.39) is just satisfied, very long waves ($k \rightarrow 0$) become stable.

These results may be relevant to the problem of boiling heat transfer. It is well known that the state of nucleate boiling will go over to <u>stable film boiling</u> when the temperature gradient in the film exceeds certain critical value. For the film boiling, a fairly stable

layer of vapor, i.e., light fluid, lies underneath the heavier liquid contrary to the criterion of Rayleigh-Taylor instability. Although, our result, Equation (13.38) shows the instability will still persist at large wave numbers, yet the growth rate can be greatly reduced.

13.5 Comparison of the Destabilizing Effects

In order to compare and evaluate the destabilizing effects caused by the adiabatic flow instability and by the evaporation, it is advantageous to calculate the ratio of the most unstable growth factor of the diadiabatic system to that valid for an adiabatic system.

Substituting Equation (13.35) in Equation (13.30) and re-arranging we obtain for the most unstable growth factor

$$C_i^* k_m)_{di} = \frac{\left(\frac{\alpha}{1-\alpha}\right)^{1/2} \left(\frac{\rho_2}{\rho_1}\right)^{1/2}}{\frac{\alpha}{1-\alpha} + \frac{\rho_2}{\rho_1}} \frac{\sqrt{We}}{2} \left[\frac{1}{\frac{\alpha}{1-\alpha} + \frac{\rho_2}{\rho_1}} - \frac{1}{Fr^2} \frac{\Delta\rho}{\rho_2} + 2 q_{1e}^{+} \frac{\Delta\rho}{\rho_1} \right] \quad (13.40)$$

For adiabatic flow, by substituting $q^+_{1e} = 0$, we obtain from Equation (13.40)

$$C_i^* k_m)_{ad} = \frac{\left(\frac{\alpha}{1-\alpha}\right)^{1/2} \left(\frac{\rho_2}{\rho_1}\right)^{1/2}}{\frac{\alpha}{1-\alpha} + \frac{\rho_2}{\rho_1}} \frac{\sqrt{We}}{2} \left[\frac{1}{\frac{\alpha}{1-\alpha} + \frac{\rho_2}{\rho_1}} - \frac{1}{Fr^2} \frac{\Delta\rho}{\rho_2} \right] \quad (13.41)$$

Dividing Equation (13.40) by Equation (13.41) we obtain

$$\frac{C_i^r k_m^+)_{di}}{C_i^r k_m^+)_{ad}} = 1 + \frac{2 \, q_{1e}^{+2} \, \frac{\Delta \rho}{\rho_1}}{\frac{1}{\frac{\alpha}{1-\alpha} + \frac{\rho_2}{\rho_1}} - \frac{1}{F_r^2} \frac{\Delta \rho}{\rho_2}} \qquad (13.42)$$

Finally, introducing the approximation involving the Froude number and the density ratio the growth rate, ratio becomes

$$\frac{C_i^r k_m^+)_{di}}{C_i^r k_m^+)_{ad}} = 1 + 2 \, \frac{\alpha}{1-\alpha} \, q_{1e}^{+2} \qquad (13.43)$$

The values of this ratio are given in Table 8 for several typical cases employing water and liquid metals.

Table 8. Relative Effect of Evaporation on the Stability

Liquid	Saturation Temperature F°	q_{1e} Btu/hr ft^2	$\frac{\alpha}{1-\alpha}$	V_r ft/Sec	$\frac{C_i^r k_m^+)_{di}}{C_i^r k_m^+)_{ad}}$
Water	212	10^5	100	20.0	1.295
Water	212	10^5	100	100.0	1.012
Water	495	10^5	100	3.6	1.012
Sodium	1200	10	100	500.0	1.657
Sodium	1200	10	100	100.0	1.164
Sodium	1700	10	100	25.0	3.630
Potassium	1500	10^5	100	10.0	2.127
Potassium	1500	5×10^5	100	50.0	1.782

It can be seen from Table 8 that for water the effect of evaporation can become significant only at relatively low vapor velocities. For

sodium and potassium, however, this effect is significant even at high vapor velocities and becomes dominant at lower velocities.

13.6 Break-Up Length

The liquid film break-up length L_b, was derived in Chapter 11, and given by Equation (11.66). By expressing it in dimensionless form, we obtain

$$L_b^+ = \frac{L_b}{\ell} = \frac{C_r^+}{C_i^+ k_m^+} \ell_n \frac{a_b}{a_i} \qquad (13.44)$$

where C_r^+ is the dimensionless velocity of propagation of the most unstable wave, and defined by

$$C_r^+ = \frac{C_r}{V_r} \qquad (13.45)$$

Since the concern here is with the maximum growth rate wave, the corresponding values of C_r^+ and $C_i^+ k_m^+$ should be inserted in Equation (13.44). $C_i^+ k_m^+$ was evaluated in the preceding section, and expressed by Equation (13.41). In accordance with Equation (13.77) and (13.7), however,

$$\overline{C}_r = 0 \qquad (13.46)$$

In view of Equation (13.8) this equation yields

$$C_r = \frac{\frac{\ell_i V_i}{1-\alpha} + \frac{\ell_b V_b}{\alpha}}{\frac{\ell_i}{1-\alpha} + \frac{\ell_b}{\alpha}} \qquad (13.47)$$

which, upon non-dimensionalization, gives for the wave velocity:

$$C_r^+ = \frac{\frac{\alpha}{1-\alpha}\frac{v_1}{V_r} + \frac{\rho_2}{\rho_1}\frac{v_2}{V_r}}{\frac{\alpha}{1-\alpha} - \frac{\rho_2}{\rho_1}} \qquad (13.48)$$

Substituting Equations (13.48) and (13.41) in Equation (13.44) yields

$$L_b^+ = \frac{\frac{\alpha}{1-\alpha}\frac{v_1}{V_r} + \frac{\rho_2}{\rho_1}\frac{v_2}{V_r}}{\left(\frac{\alpha}{1-\alpha} + \frac{\rho_2}{\rho_1}\right)^{1/2}} \frac{\left(-\frac{1}{F_r^2}\frac{\Delta\rho}{\rho_2} - \frac{1}{\frac{\rho_2}{\rho_1} + \frac{\alpha}{1-\alpha}} + 2\,q_{\rho ic}^{+\,2}\frac{\Delta\rho}{\rho_1}\right)^{-1}}{\left(\frac{\alpha}{1-\alpha}\right)^{1/2}\left(\frac{\rho_2}{\rho_1}\right)^{1/2}\frac{We^{1/2}}{2}} \ln\frac{a_b}{a_i} \quad (13.49)$$

This is the general expression for the length at which the unfavorable phenomenon can occur. We note that existence of many dimensionless groups appearing in Equation (13.49) makes it difficult to present Equation (13.49) in graphical form. However, approximations introduced in the preceding sections involving the density ratio and the Froude number, and furthermore noting that

$$\frac{v_2}{V_r} \simeq 1 \qquad (13.50)$$

we obtain from Equation (13.49) following simple form

$$L_b^+ = 2\left(\frac{v_1}{V_r}\right)\left(\frac{\rho_1}{\rho_2}\right)^{1/2} We^{1/2}\left(\frac{1-\alpha}{\alpha} + 2\,q_{\rho ic}^{+\,2}\right)^{-1}\ln\frac{a_b}{a_i} \qquad (13.51)$$

$\ln\frac{ab}{ai}$ = 12 which was found in reference [83] is more suitable for this problem than that for the free falling·film where we used

this value, and obtained satisfactory result. Therefore, using this value in Equation (13.51), we obtain finally

$$L_b^* = 24 \left(\frac{v_1}{V_r} \right) \left(\frac{\rho_1}{\rho_L} \right)^{1/2} We^{1/2} \left(\frac{1-\alpha}{\alpha} + 2 q_{1c}^{r^2} \right)^{-1} \tag{13.52}$$

This equation gives the break-up length of a liquid film flowing under the action of an adjacent vapor phase. There are five basic dimensionless groups which affect on the occurrance of undesirable phenomena in a given flow length. It is evident also from Equation (13.51) that, in adiabatic systems, the break-up length is larger than the corresponding length in a system where evaporation takes place. Once again this shows the unfavorable dynamic effect of the interfacial mass transfer on the destruction of the liquid films.

The break-up of liquid films inside closed conduits by wave formation was the basic idea leading to the derivation of Equation (13.52). However, the experimental information available is not sufficient to enable a comparison with the theory. Available experimental measurements have been carried out for the purpose of either the rate of entrainment or the stability of dry spots which were formed already. Neither of these concerns the mechanism which lead to the break-up. Therefore, Equation (13.52) is open for experimental confirmation.

13.7 Conclusions

1. A mathematical analysis based on the instability theory has led to dimensionless equation which explains the mechanism of break-up of thin liquid layers flowing under the action of an adjacent vapor or

gas flow. This analysis is presented algebraically and graphically which facilitate its use.

2. Particularly, an evaluation of the importance of evaporation on the instability of thin films has been made.

3. The results indicate that, the interfacial phase change whether in the form of evaporation or condensation enhances the Kelvin-Helmholtz instability whereas it affects the Rayleigh-Taylor instability in the direction of stabilization.

4. As an immediate result of the previous conclusion, for the case of the light fluid on top of a heavy one, the effect of evaporation is to decrease both the critical and most dangerous wave length and to increase therefore the growth rate of disturbance. In contrast, for the case of the heavy fluid on top configuration the effect of evaporation is to increase the wave length and to decrease therefore the growth rate of disturbance.

5. For water, the destabilizing effect of evaporation can be significant only at very low velocities. For liquid metals, however, this effect is significant at high vapor velocities and becomes dominant at low velocities.

6. In view of the experimental observations which have shown that the rupture of liquid jets occurred through the instability of a wave at the interface, an analysis was carried out, which predicts the break-up length of thin liquid layers flowing under the influence of an adjacent vapor or gas flow. This result predicted by the analysis, however, has not been tested against experiments because quantitative data are not available yet.

CHAPTER XIV

SUMMARY AND CONCLUSIONS

Part I: Thermo-Fluid Dynamic Formulation of Separated Two-Phase Flow

     1) The separated two-phase flow problem was formulated by considering two continua coupled by the appropriate "jump" conditions at the internal boundary, i.e., at the interface.

     2) From this formulation the mass-weighted area-averaged field equations which were expressed both in terms of the two-fluid and of the diffusion, i.e., drift model were derived.

     3) The two-fluid model was formulated by considering each phase separately. Therefore, the formulation was expressed in terms of two sets (one for each phase) of conservation equations, i.e., in terms of six field equations: two continuity equations, two momentum equations and two energy equations.

     4) In contrast to the two-fluid model, the diffusion model was formulated by considering the entire mixture. · Therefore, the formulation was expressed in terms of four field equations: three for the mixture (continuity, momentum and energy) plus the diffusion (or void propagation equation) for one of the phases.

     5) The conservation equations, derived from the appropriate averaging procedure, take into account the effects of surface phenomena such as surface tension and surface shear and of surface processes such as momentum, heat and mass transfer at the interface.

     6) The diffusional model field equations in their dimensionless

form were used to establish the similarity groups appropriate to separated flows. The significance of these groups was discussed.

7) In addition to the continuity, momentum and the total energy equations, auxiliary energy equations, such as the internal energy, enthalpy and the mechanical energy equations were derived together with the appropriate "jump" conditions.

8) It was observed that when the effects of diffusion and of motion are neglected the mixture total energy equation reduces to those derived by Gibbs for thermostatics.

Part II: Stability Analysis

1) The two fluid model formulation of Part I was used to develop a stability theory for a general separated two-phase flow system.

2) The analysis takes into account the effects of surface tension and liquid viscosity, of mass and heat transfer at the interface as well as the dynamic effects of the vapor on the liquid film.

3) Instability and wave formation criteria at the interface were established in terms of the generalized parameters which depend on the particular flow system.

4) Furthermore, assuming that the most unstable mode is responsible for break-up of a thin liquid film a method for determining the break-up length was developed.

5) Results obtained are general so that one can use them for a particular flow configuration provided the coefficients are calculated properly.

6) These results were applied to stability of

      a) viscous liquid flow down a heated inclined plane, and

          b) two inviscid fluids over a heated surface.

     7) Viscous liquid film in free flow over a heated surface results:

          a) the surface tension has stabilizing effect irrespective of the configuration of the plate,

          b) the gravity has a stabilizing effect when $\theta < 90°$, and a destabilizing effect when $\theta < 90°$,

          c) evaporation from the free surface of the liquid film has a destabilizing effect, while condensation has dual effects-one destabilizes the other stabilizes, and the relative importance of these effects was determined,

          d) as the Reynolds number decreases, i.e., as the film thickness decreases, and the heat flux increases, the thermal effects become important and may become the dominant destabilizing factor,

          e) the effect of evaporation is to decrease both the critical and the most dangerous wave length and to increase therefore the growth rate of disturbances,

     7) Two-Inviscid liquids in parallel flows over a heated surface results:

          a) the interfacial phase change whether in the form of evaporation or condensation enhances the Kelvin-Helmholtz instability whereas it affects the Rayleigh-Taylor instability in the direction of stabilization,

          b) as an immediate result of the previous conclusion, for the case of the light fluid on top of a heavy one, the effect of evaporation is to decrease both the critical and most dangerous wave length and to increase therefore the growth rate of disturbance. In contrast, for the case of the heavy fluid on top configuration the effect of evaporation is to increase the wave length and to decrease the growth rate of disturbance.

APPENDICES

APPENDIX A

KINEMATIC CONCENTRATIONS

The volumetric flow concentration of phase 2, β , is defined as the volumetric flow rate of phase 2 per unit mixture volumetric flow rate. Hence, by definition

$$\beta = \frac{Q_2}{Q_T} = \frac{\iint_{A_{2c}} \vec{v}_2 \cdot \hat{n}_2 \, dA}{\sum_{K=1}^{2} \iint_{A_{Kc}} \vec{v}_K \cdot \hat{n}_K \, dA} \qquad (A.1)$$

where Q_T and Q_K, (K = 1,2), are the total and K-th phase volumetric flow rates, respectively, while $\vec{v}_K$ is the K-th phase velocity vector and $\hat{n}_K$ is the unit normal vector to the cross-sectional plane, A_{Kc}.

Introduction of the area averages into Equation (A.1) results in

$$\beta = \frac{\alpha \langle\langle v_{2n}\rangle\rangle}{(1-\alpha) \langle\langle v_{1n}\rangle\rangle + \alpha \langle\langle v_{2n}\rangle\rangle} \qquad (A.2)$$

Similarly the mass flow concentration of phase 2, X , the quality, is defined as the mass flow rate of phase 2, $\dot{w}_2$, per unit mixture mass flow rate, W_T, and can be expressed by:

$$X = \frac{W_2}{W_T} = \frac{\alpha \langle\langle \rho_2 \rangle\rangle \langle v_{2n}\rangle}{(1-\alpha) \langle\langle \rho_1 \rangle\rangle \langle v_{1n}\rangle + \alpha \langle\langle \rho_2 \rangle\rangle \langle v_{2n}\rangle} \qquad (A.3)$$

In Section 3.4.2, the mixture velocity $\vec{v}_m$ of the center of gravity of the mixture is defined in terms of the total linear momentum

considerations. Substituting that expression in Equation (A.3) we can express the quality in terms of $\vec{v}_m$. Thus,

$$\chi = \propto \frac{\langle\langle\rho_2\rangle\rangle}{\rho_m} \frac{\langle v_{2x}\rangle}{v_m} \tag{A.4}$$

In the literature flowing quantities, though wrong, have been used very often to define the mixture properties. For example, the mixture density is defined by:

$$\rho_\beta = \frac{W_T}{Q_T} \tag{A.5}$$

which in terms of averages can be expressed as:

$$\rho_\beta = \frac{(1-\propto)\langle\langle\rho_1\rangle\rangle\langle v_{1x}\rangle + \propto\langle\langle\rho_2\rangle\rangle\langle v_{2x}\rangle}{(1-\propto)\langle\langle v_{1x}\rangle\rangle + \propto\langle\langle v_{2x}\rangle\rangle} \tag{A.6}$$

When $\langle\langle\rho_K\rangle\rangle \simeq \rho_K$, (K = 1,2) then the density defined by Equation (3.42) becomes:

$$\rho_\beta = (1-\beta)\rho_1 + \beta\rho_2 \tag{A.7}$$

or in terms of the quality χ, ρ_β becomes

$$\frac{1}{\rho_\beta} = \frac{1-\chi}{\rho_1} + \frac{\chi}{\rho_2} \tag{A.8}$$

Static Parameters vs. Kinematic Parameters:

From Equation (3.30, 32, 35) and (A.2, 4, 7), it is easy to see that in general

$$\propto \neq \beta \ ; \ C \neq \chi \ ; \ \rho_m = \rho_\beta \tag{A.9}$$

In effect after some algebraic manipulations it is not difficult to show

that

$$\alpha = \frac{\beta}{\beta + (1-\beta)\ \frac{\langle V_{2k}\rangle}{\langle V_{1k}\rangle}} \tag{A.10a}$$

$$C = \frac{X}{X + (1-X)\ \frac{\langle\!\langle V_{2k}\rangle\!\rangle}{\langle\!\langle V_{1k}\rangle\!\rangle}} \tag{A.10b}$$

$$\rho_m = \rho_\beta + (\beta - \alpha)\ \Delta\rho \tag{A.10c}$$

where $\Delta\rho = \rho_1 - \rho_2$.

From Equation (A.10), it is evident that if and only if

$$\langle V_{1k}\rangle = \langle V_{2k}\rangle \tag{A.11}$$

and

$$\langle\!\langle \rho_k \rangle\!\rangle = \rho_k \quad , \ K = 1,2 \tag{A.12}$$

which imply that the flow is homogeneous i.e., the relative velocity between phases is zero, and that the mass density variations over cross-sectional plane is negligible, only then

$$\alpha = \beta \ ; \ C = X \ ; \ \rho_m = \rho_\beta \tag{A.13}$$

Although the latter condition, i.e., Equation (A.12), can be satisfied for incompressible fluids, the former one, i.e., Equation (A.11), never holds in two-phase flow systems. Therefore, the kinematic parameters β, X and ρ_β can <u>never</u> be used to represent the static parameters α, C and ρ_m as it has been done in the literature.

Furthermore, since α and C are expressed in terms of the local

parameters only irrespective of the past history of the fluid particles, they are only sensitive to local variations. However, β and χ depend upon the past history of the fluid particles, and even under certain conditions β and χ are entirely insensitive to the local variations but the channel or pipe input variations. This aspect can be best demonstrated in the adiabatic two-phase flow of incompressible fluids. In view of the continuity requirements and in view of the definitions given by Equations (A.1) and (A.3), it is evident that at every section of the channel β and χ will be constant no matter what the local static concentration variations are. Consequently β and χ will not be affected by local disturbances such as interfacial surface waves. However, since α and C depend on only on local parameters, they will be affected by local disturbances.

From this brief discussion it can be concluded that unless we are not interested in over-all flow field rather than the local one, to use the kinematic parameters in two-phase flow dynamic analyses will be incorrect.

APPENDIX B

INTERFACIAL BALANCE EQUATIONS

B.1 Interfacial Motion

In general we can represent the motion of a surface by

$$\vec{r} = \vec{r}\,(u_1, u_2, t) \qquad\qquad (B.1)$$

in which $\vec{r}$ is the position vector whose components x,y,z define a point on the surface, and (u_1, u_2) stands for a pair of surface coordinates identifying a surface point. It is important to note that in the case of interfacial mass transfer the surface point (u_1, u_2) is <u>not</u> necessarily identified with a material surface.

The velocity of the surface point having the surface coordinates u_1 and u_2 is defined by

$$\vec{v}_i = \left.\frac{\partial \vec{r}}{\partial t}\right)_{u_1, u_2 = const.} \qquad\qquad (B.2)$$

By eliminating the surface parameters, u_1 and u_2 in Equation (B.1), we can express the surface in the implicit form

$$h(x, y, z, t) = 0 \qquad\qquad (B.3)$$

Differentiating Equation (B.3) with respect to time while keeping u_1 and u_2 constant, we obtain

$$\frac{\partial h}{\partial t} + \vec{v}_i \cdot \nabla h = 0 \qquad\qquad (B.4)$$

in which we have used Equation (B.2) for $\vec{v}_i$. In view of the unit normal vector defined by

$$\hat{n} = \frac{\nabla h}{|\nabla h|} \tag{B.5}$$

Equation (B.4) can be rearranged to yield the normal component of the interfacial velocity vector, thus

$$\vec{v}_i \cdot \hat{n} = - \frac{\frac{\partial h}{\partial t}}{|\nabla h|} \tag{B.6}$$

It is interesting to note that since the right hand side of Equation (B.6) is determined from Equation (B.3) alone, it is independent of our choice of the surface coordinates, (u_1, u_2). Hence, for all possible surface coordinates, (u_1, u_2), normal component of the velocity of the moving surface is the same.

B.2 General Balance Equation at the Interface

The purpose of the balance equations for mass, momentum and energy at an interface is to provide connecting conditions between the solutions to the mass, momentum and energy balances applicable in the separate phases i.e., fields. A number of theories have been proposed for deriving the general balance equation at the interface. However, depending upon the description of the interfaces, the balance equation has been developed along four principal lines.

1. Some researchers, such as Gibbs [59] and Buff [60] viewed the interface from the continuum stand point. Interface was considered as a three-dimensional region in which all quantities are continuously changing and matching the bulk flow variables at the edges of this region.

Later this point of view is extended by Slattery [41] to the more practical case, the dynamic situation. In his analysis Slattery assumed the interface as a three-dimensional region which separates the bulk portions of two phases, and in which the constitutive equations specifying the material behavior differ from those applicable in the bulk portions of each phase. Evidently, this point of view requires the exact limits of the interfacial region and the knowledge of the constitutive equations applicable in this region. However, lack of experiments confirming the limits of the interfacial region and as well as the lack of experimental data on the constitutive equations in this region at the present time limit the applicability of this theory to practical problems.

2. In the second approach, which was developed by Scriven [61], (see also Aris [62, pp. 232-244]), the phase interface is regarded as a two-dimensional surface (possibly) containing mass. For this deforming surface Scriven wrote mass and momentum balances under the assumption that there was no mass interaction between the two-dimensional surface and the surrounding bulk phases. Although this approach was rigorous from the mathematical point of view, its full impact on the technology has not been realized yet because the rheological constitutive equations for the interface have not been established yet. Furthermore, the interfacial mass transport which was not accounted for in this approach can not be neglected in two-phase flow systems with a phase change.

3. The third approach, which was initiated by Kotchine and later rederived by Truesdel and Toupin [22, pp. 525-529], differs from the others in that the interface was visualized as singular surface with

respect to the quantities which are transported from one phase to another
via interface. However, the derived general balance equation which is
known as Kotchine's theorem <u>does not</u> take into account for the surface
source term which contribute the over-all balance equations of the
mixture.

4. In an effort to unite these approaches Slattery [41] and Del-
haye [40] have considered the phase interface as a <u>singular surface</u>
containing a <u>distributed source</u> term.   The derivation of the interfacial
general balance which follows is an <u>alternative</u> approach to that given
by Slattery and Delhaye, the discussion for this derivation is based on
a similar development presented by Truesdell and Toupin [22].

Consider a material volume $V(t)$ which contains a singular surface
$A_i(t)$, Figure 13; the general balance equation for a quantity ψ can be
written as

$$\frac{d}{dt} \iiint_{V(t)} \rho\psi \, dV = - \iint_{A(t)} \vec{\Phi} \cdot \hat{n} \, dA + \iiint_{V(t)} \rho\dot{\psi} \, dV - \oint_{\xi_i(t)} \bar{\bar{\theta}} \cdot \hat{n}_\xi \, ds \quad (B.7)$$

where $\vec{\Phi}$ indicates influx of the quantity ψ through the bounding surface
$A(t)$ of $V(t)$, $\dot{\psi}$ the rate of generation of the quantity per unit mass at
each point throughout the material volume $V(t)$, and $\hat{n}$ the unit normal
vector directed outwardly with respect to the boundary surface $A(t)$.
Finally, $\hat{n}_\xi$ is the outward unit normal vector to the boundary curve, $\xi_i(t)$,
located in the plane tangent to the interfacial surface, $A_i(t)$, and $\bar{\bar{\theta}}$ is
the surface flux tensor inwardly directed through the interfacial bound-
ary curve, $\xi_i(t)$. Indeed $\bar{\bar{\theta}}$ takes into account for the possible

distributed surface source term which Kotchine's theorem does <u>not</u> contain.

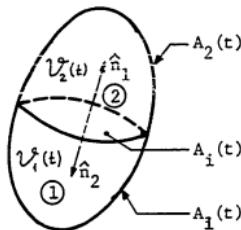Figure 13. Material Volume Element $\hat{\mathcal{V}}(t)$
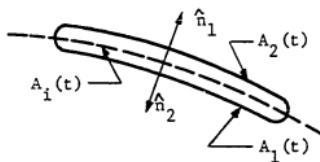

Figure 14. Element as $\hat{\mathcal{V}}(t) \rightarrow 0$

Using the divergence theorem for surfaces on the line integral and the generalized Reynolds transport theorem on the left side of

Equation (B.7), we obtain

$$\iiint_{\mathcal{V}(t)} \frac{\partial \varrho \psi}{\partial t} \, d\mathcal{V} = \iint_{A(t)} \varrho \psi \vec{V} \cdot \hat{n} \, dA + \iint_{A_i(t)} \sum_{k=1}^{2} (\varrho_k \psi_k \vec{V}_i \cdot \hat{n}_k) dA = \quad \text{(B.8)}$$

$$= \iint_{A(t)} \vec{\phi} \cdot \hat{n} \, dA + \iiint_{\mathcal{V}(t)} \varrho \dot{\psi} \, d\mathcal{V} - \iint_{A_i(t)} \nabla_s \cdot \bar{\bar{\theta}} \, dA$$

where ∇_s is the divergence operator on the surfaces.

We now assume that in the neighborhood of $A_i(t)$, the quantities in the volume integrals are bounded while on each side of $A_i(t)$ the quantities in the surface integrals approach limits that are continuous functions of position. These assumptions are not arbitrarily brought but they are entirely consistent with the singular surface model. Under these conditions, we let the areas $A_1(t)$ and $A_2(t)$ shrink down to $A_i(t)$, so that the total volume $\mathcal{V}(t) \to 0$ while the area of the interface, $A_i(t)$, remains finite in the limit, Figure 14. The volume integrals vanish in the limit, and Equation (B.8) becomes

$$\iint_{A_i(t)} \left[\sum_{k=1}^{2} \varrho_{ki} \psi_{ki} (\vec{V}_{ki} - \dot{V}_i) \cdot \hat{n}_{ki} + \vec{\phi}_{ki} \cdot \hat{n}_{ki} + \nabla_s \cdot \bar{\bar{\theta}} \right] dA = 0 \quad \text{(B.9)}$$

Since this relation holds for any area $A_i(t)$, and since the integrand, by assumption, is continuous, it must vanish identically to yield the general balance equation at the interface. Thus

$$\sum_{k=1}^{2} \varrho_k \left(\vec{v}_k - \vec{v}_i \right) \cdot \hat{n}_k \, \psi_k + \vec{f}_k \cdot \hat{n}_k = - \nabla_s \cdot \bar{\bar{\theta}} \quad \text{(B.10)}$$

This is the extended Kotchine's theorem including the distributed surface source term, $\bar{\bar{\theta}}$. From Equation (B.10) the balances concerning mass, momentum and energy can be obtained with the proper identifications for $\psi, \vec{\phi}$ and $\bar{\bar{\theta}}$.

B.2.1 The Mass Balance at the Interface

To obtain the overall mass balance from Equation (B.10), we use the identifications for ψ and $\vec{\phi}$, which are given in Table 3, and Equation (B.10) takes the form

$$\sum_{k=1}^{2} \varrho_k \left(\vec{v}_k - \vec{v}_i \right) \cdot \hat{n}_k = - \nabla_s \cdot \bar{\bar{\theta}}_{(mass)} \quad \text{(B.11)}$$

in which $\nabla \cdot \bar{\bar{\theta}}_{(mass)}$ stands for the surface source term concerning with mass capacity of the surface. However, by hypothesis, surface is singular, i.e., surface has no mass capacity, then it is evident that

$$\bar{\bar{\theta}}_{(mass)} = 0 \quad \text{(B.12)}$$

Under this condition the interfacial balance follows as:

$$\sum_{k=1}^{2} \varrho_k \left(\vec{v}_k - \vec{v}_i \right) \cdot \hat{n}_k = 0 \quad \text{(B.13)}$$

_____

*Since this section involves primarily with surfaces, we dropped the subscript i indicating the variable evaluated at the interface. Nomenclature is assumed to be understood.

Defining by $\dot{m}_K$, the mass flux outward from the K-th phase, it follows that

$$\dot{m}_k = \varrho_k \, (\vec{v}_k - \vec{v}_i) \cdot \hat{n}_k \qquad , \quad K = 1, 2 \qquad (B.14)$$

With this notation Equation (B.13) requires that

$$\dot{m}_1 + \dot{m}_2 = 0 \qquad (B.15)$$

which simply states that the mass of phase 1 which leaves the phase 1 across the interfacial area element is equal to the mass of the phase 2 which enters the phase 2 across the same interfacial area element. It should be noted, further, that Equation (B.13) is the same as the Stokes-Christoffel condition, which has been used for the condition across the shock waves moving with the speed, $\vec{v}_i \cdot \hat{n}_i$.

B.2.2 The Momentum Balance at the Interface

To obtain the general momentum balance at the interface, we use the identifications for ψ and $\vec{\phi}$ given in Table 3, together with the general balance equation, i.e., Equation (B.10). Thus

$$\sum_{k=1}^{2} \varrho_k (\vec{v}_k - \vec{v}_i) \cdot \hat{n}_k \, \vec{v}_k - \bar{\bar{\pi}}_k \cdot \hat{n}_k = -\nabla_s \cdot \bar{\bar{\Theta}}_{(mom)} \qquad (B.16)$$

which in view of Equations (B.14) and (2.3), can be expressed as:

$$\sum_{k=1}^{2} \dot{m}_k \, \vec{v}_k + P_k \, \bar{\bar{\delta}} \cdot \hat{n}_k - \bar{\bar{z}}_k \cdot \hat{n}_k = -\nabla_s \cdot \bar{\bar{\theta}}_{(mom)} \qquad (B.17)$$

in which the terms appearing on the left side represents, respectively, the momentum exchange between two phases due to the interfacial mass

transfer and due to the thermodynamic pressure and the viscous stresses while the term on the right side stands for the interfacial momentum source term having the dimension of force per unit area.

In reference [61], see also references [41 and 62, pp. 230-244], the argument, which has been used in constructing a constitutive equation for the Newtonian fluid, was modified to the moving surfaces in order to express the effects of the surface stress tensor, $\bar{\bar{\theta}}_{(mom)}$. For a Newtonian interface, the appropriate expressions for the interfacial stress have been derived. For example, Equation (10.23.3) of reference [62] expresses the surface stress tensor in terms of the most general linear function of the surface rate-of-deformation tensor with the proportionality coefficients of the surface tension and of the surface dilatational and shear viscosity. Basically these three coefficients are the surface transport properties and should be supplied as functions of the thermodynamic state variables. However, from practical application standpoint the general expression for surface stress tensor is limited because experimental data are not available on surface shear and dilatational viscosities. Thus, although the effects of these two surface viscosities can be formally considered their effect at present, cannot be evaluated quantitatively. Consequently, in this work we shall consider only the static effects, i.e., the effects of the surface tension, σ, since the values of σ are available.

With the surface stress expressed only in terms of the surface tension σ, the stress tensor becomes

$$\bar{\bar{\theta}}_{(mom)} = - \sigma \, \bar{\bar{a}} \tag{B.18}$$

where $\bar{\bar{a}}$ is the metric tensor of the surface, a characteristic term of which is given by

$$\alpha_{x,\beta} = \frac{\partial x}{\partial u_\alpha}\frac{\partial x}{\partial u_\beta} + \frac{\partial y}{\partial u_\alpha}\frac{\partial y}{\partial u_\beta} + \frac{\partial z}{\partial u_\alpha}\frac{\partial z}{\partial u_\beta}$$

where α, $\beta = 1$ and 2.

Taking the components of the surface stress tensor in the space and then evaluating the surface divergence of it, one can express the surface momentum source term as:

$$\nabla_s \cdot \bar{\bar{\theta}}_{(mom)} = -\left\{ 2\kappa\sigma\,\hat{n}_i + \frac{1}{H^2}\left[\vec{r}_1\left(G\frac{\partial}{\partial u_1} - F\frac{\partial}{\partial u_2}\right) + \vec{r}_2\left(E\frac{\partial}{\partial u_2} - F\frac{\partial}{\partial u_1}\right)\right]\sigma\right\}^* \quad \text{(B.19)}$$

Here $\vec{r}$ is the position vector defining the surface and is defined by

$$\vec{r} = x(u_1,u_2,t)\,\hat{\imath} + y(u_1,u_2,t)\,\hat{\jmath} + z(u_1,u_2,t)\,\hat{k} \quad \text{(B.20)}$$

The vector $\vec{r}_1$ which is defined by

$$\vec{r}_1 = \frac{\partial \vec{r}}{\partial u_1}\Big)_{u_2 = const.} \quad \text{(B.21)}$$

is tangential to the curve u_2 = constant at the point $\vec{r}$. Similarly $\vec{r}_2$ is tangential to the curve u_1 = constant and given by

$$\vec{r}_2 = \frac{\partial \vec{r}}{\partial u_2}\Big)_{u_1 = const.} \quad \text{(B.22)}$$

The quantities denoted by E,F,G,H are called the fundamental magnitudes of the first order, and they are given by

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*Detailed discussions of surfaces may be found in Weatherburn [63] while use of the surface tensors in fluid dynamics may be found in Aris [62].

$$E = \vec{r_1} \cdot \vec{r_1} \quad ; \quad F = \vec{r_1} \cdot \vec{r_2} \quad ; \quad G = \vec{r_2} \cdot \vec{r_2} \quad ; \quad H = |\vec{r_1} \times \vec{r_2}| \qquad (B.23)$$

$\varkappa$ is the mean curvature of the surface at the point, which may be defined in terms of the principal curvatures, $\varkappa_1$ and $\varkappa_2$, as:

$$k = \frac{1}{2}\left(k_1 + k_2 \right) \qquad (B.24)$$

Referring to the surface momentum source term it is easy to see that Equation (B.19) expresses the resolution of the surface resistance into tangential and normal components. Although the surface stress is defined on the interfacial surface, because of the curvature it gives rise to the normal interfacial shear. Finally, to obtain the momentum balance at the interface we substitute Equation (B.19) in (B.17). Thus

$$\sum_{k=1}^{2} \dot{m}_k \vec{v}_k + \mu_k \vec{\bar{\sigma}} \cdot \hat{n}_k - \bar{\Sigma}_k \cdot \hat{n}_k = 2\varkappa\sigma \ \hat{n}_i + \qquad (B.25)$$
$$\frac{1}{H^2}\left[\vec{r_1}\left(G\frac{\partial}{\partial u_1} - F\frac{\partial}{\partial u_2} \right) + \vec{r_2}\left(E\frac{\partial}{\partial u_2} - F\frac{\partial}{\partial u_1} \right) \right]\sigma$$

Since the momentum balance equation is a vector expression, it is interesting to see the normal and tangential components of it.

Normal Component. To obtain the normal component we multiply Equation (B.25) throughout by $\hat{n}_1$, the second term on the right hand side disappears since $\hat{n}_1 \cdot \vec{r_1} = \hat{n}_1 \cdot \vec{r_2} = 0$, and using the resulting relations together with Equations (2.3), B.14) and (B.15) it can be easily shown that the normal component of the interfacial balance is given by:

$$\dot{m}_1\left(\vec{v_1} - \vec{v_2} \right) \cdot \hat{n}_1 + \left(\mu_1 - \mu_2 \right) + \left(\Sigma_{1nn} - \Sigma_{2nn} \right) = 2\varkappa\sigma \qquad (B.26)$$

Further application of Equation (A.14) results in

$$(\vec{v}_1 - \vec{v}_2) \cdot \hat{n}_1 = -\dot{m}_1 \frac{\Delta\rho}{\rho_1\rho_2}$$ (B.27)

where $\Delta\rho = \rho_1 - \rho_2$, which is by definition, see footnote on page 40 is positive.

Finally, substitution of Equation (B.27) in (B.26) and re-arrangement yields the pressure difference at the interface, thus,

$$P_1 - P_2 = (\tau_{1nn} - \tau_{2nn}) + \dot{m}_1^2 \frac{\Delta\rho}{\rho_1\rho_2} + 2\kappa\sigma$$ (B.28)

which states that for the mechanical equilibrium at the interface pressure difference in the bulk phases must be balanced by forces due to the normal contribution of the viscous shears, the mass transfer and due to the capillary. It is interesting to note that whether $\dot{m}_1 > 0$, i.e. evaporation, or $\dot{m}_1 < 0$, i.e. condensation, force due to the interfacial mass transfer acts only in one direction. This force is called the "vapor thrust" at the interface. Furthermore, we note that the sign of the capillary force depends upon the curvature of the interface, which becomes very important for those flow geometries where the area to the volume ratio is large, i.e., thin liquid film flow. In effect, for a static film with no interfacial mass transfer, $\dot{m}_1 = \dot{m}_2 = 0$, the first and second terms on the right side drop; thus we obtain from Equation (B.28)

$$P_1 - P_2 = 2\kappa\sigma$$

which means there must be a difference in normal force (that is, pressure) on either side of a film equal to 2σ times the mean curvature if the

film is to be kept stationary.

    Tangential Components. To obtain the tangential component of the momentum balance at the interface we shall multiply Equation (B.25) throughout by the tangential vector at the interface. However, on the interface one can define two principal tangential vectors, one in the u_1 direction whereas the other in the u_2 direction. For the present purpose this difference is immaterial because we can always define a tangent unit vector $\hat{t}$, in the tangent plane as a linear combination of $\vec{r}_1$ and $\vec{r}_2$. Therefore, multiplying Equation (B.25) through by $\hat{t}$, the first term on the right side disappears since $\hat{t} \cdot \hat{n}_\kappa = 0$, and we obtain

$$\dot{m}_1 (\vec{v}_1 - \vec{v}_2) \cdot \hat{t} + (\mu_1 - \mu_2)(\bar{\bar{\delta}} \cdot \hat{n}_1) \cdot \hat{t} - [(\bar{\bar{E}}_1 - \bar{\bar{E}}_2) \cdot \hat{n}_1] \cdot \hat{t} = \quad \text{(B.29)}$$

$$= \frac{1}{H^\perp} \left[\vec{r}_1 \cdot \hat{t} \left(\mu \frac{\partial}{\partial u_1} - \tau \frac{\partial}{\partial u_2} \right) + \vec{r}_2 \cdot \hat{t} \left(\Sigma \frac{\partial}{\partial u_2} - \tau \frac{\partial}{\partial u_1} \right) \right] \upsilon$$

Since

$$\delta_{ij} = 0 \quad \text{if} \quad i \neq j \qquad \qquad \text{(B.30)}$$
$$= 1 \quad \text{if} \quad i = j$$

then

$$(\bar{\bar{\delta}} \cdot \hat{n}) \cdot \hat{t} = 0 \qquad \qquad \text{(B.31)}$$

    Furthermore, no-slip condition for the tangential components of the velocity vectors at the interface requires the first term on the left side of Equation (B.30) to vanish. Hence, from Equation (B.30) we obtain the tangential momentum balance as:

$$\mathcal{Z}_{1nt} - \mathcal{Z}_{2nt} = \frac{1}{H^2} \left[\vec{r}_1 \cdot \hat{t}_1 \left(G \frac{\partial}{\partial u_1} - F \frac{\partial}{\partial u_2} \right) + \vec{r}_2 \cdot \hat{t} \left(E \frac{\partial}{\partial u_2} - F \frac{\partial}{\partial u_1} \right) \right] \sigma \quad \text{(B.32)}$$

which states that due to the surface tension gradient there must be a stress difference between the bulk phases. Therefore, the variation in surface tension from one point to another leads to the appearance of tangential stresses on the interface as described by Equation (B.28). The effect of surface motion due to the temperature gradient at the surface (thermo-capillary action) and due to the surface active substances is discussed in detail by Levich [56, pp. 384-393].

B.2.3 The Energy Balance at the Interface

Using the proper identifications for ψ and $\vec{\phi}$ given in Table 3 together with the general balance equation given by Equation (B.10), the general energy balance at the interface is obtained as:

$$\sum_{K=1}^{2} \left[\varrho_k (\vec{v}_k - \vec{v}_i)(u_k + \frac{v_k^2}{2}) + \vec{q}_k - \bar{\pi}_k \cdot \vec{v}_k \right] \cdot \hat{n}_k = - \nabla_s \cdot \vec{\theta}_{(en)} \quad \text{(B.33)}$$

which, in view of Equations (B.14) and (4.3), can be expressed as:

$$\sum_{k=1}^{2} \dot{m}_k \left(u_k + \frac{v_k^2}{2} \right) + \vec{q}_k \cdot \hat{n}_k + P_k \vec{v}_k \cdot \hat{n}_k - (\bar{\mathcal{Z}}_k \cdot \vec{v}_k) \cdot \hat{n}_k = - \nabla_s \cdot \vec{\theta}_{(en)} \quad \text{(B.34)}$$

Here, the term on the left side represents the energy exchange at the interface due to the mass transfer, mechanical work and finally due to the conductive heat transfer while the terms on the right side accounts for the interfacial source term.

Parallel to the single-phase flow theory, where the rate of work per unit volume of the fluid is expressed by divergence of the term $(\bar{\mathcal{Z}} \cdot \vec{v})$, we can express the energy source term on the interface by:

$$\nabla_s \cdot \bar{\bar{\theta}}_{(e_n)} = \nabla_{s'} \cdot \left[\dot{\bar{\theta}}_{(mom)} \cdot \vec{v}_i \right] \tag{B.35}$$

which in view of Equation (B.18) can be expressed as

$$\nabla_s \cdot \bar{\bar{\theta}}_{(e_n)} = -\left\{ i k \sigma \cdot \vec{v}_i \cdot \hat{n}_1 + \frac{1}{H^s} \left[\vec{r}_1 \left(G \frac{\partial}{\partial u_1} - F \frac{\partial}{\partial u_2} \right) + \vec{r}_2 \left(E \frac{\partial}{\partial u_2} - F \frac{\partial}{\partial u_1} \right) \right] \cdot \sigma \vec{v}_i \right\} \tag{B.36}$$

Combining Equations (B.30) and (B.32) we obtain the energy balance at the interface. Thus

$$\sum_{k=1}^{2} \left[\rho_k \left(\vec{v}_k - \vec{v}_i \right) \left(u_k + \frac{v_k^2}{2} \right) + \vec{q}_k + \Pi_k \vec{v}_k - \bar{\bar{\Sigma}}_k \cdot \vec{v}_k \right] \cdot \hat{n}_k = \tag{B.37}$$

$$= i k \sigma \cdot \vec{v}_i \cdot \hat{n}_1 + \frac{1}{H^s} \left[\vec{r}_1 \left(G \frac{\partial}{\partial u_1} - F \frac{\partial}{\partial u_2} \right) + \vec{r}_2 \left(E \frac{\partial}{\partial u_2} - F \frac{\partial}{\partial u_1} \right) \right] \cdot \sigma \vec{v}_i$$

$$= i k \sigma \cdot \vec{v}_i \cdot \hat{n}_1 + \frac{\sigma}{H^s} \left[\vec{r}_1 \left(G \frac{\partial}{\partial u_1} - F \frac{\partial}{\partial u_2} \right) + \vec{r}_2 \left(E \frac{\partial}{\partial u_2} - F \frac{\partial}{\partial u_1} \right) \right] \cdot \vec{v}_i +$$

$$+ \frac{\vec{v}_i}{H^s} \cdot \left[\vec{r}_1 \left(G \frac{\partial}{\partial u_1} - F \frac{\partial}{\partial u_2} \right) + \vec{r}_2 \left(E \frac{\partial}{\partial u_2} - F \frac{\partial}{\partial u_1} \right) \right] \sigma$$

The terms on the left hand side of Equation (7.2) stand for the convective energy transport, the energy transport by conduction and the work done by the surface forces. Whereas the terms on the right hand side stand for the part of the energy used in increasing the internal energy, and part of the energy used in setting up the motion.

APPENDIX C

TWO-DIMENSIONAL GEOMETRIC RELATIONS

Here we shall derive two-dimensional geometric relations in Cartesian and cylindrical coordinates so that they can be used in the analysis of two-dimensional plane flow and the axially symmetric annular flow. As it was discussed in Section 1.2 and illustrated on Figure 1, these two flow regimes are separated flow patterns of great interest to various technologies.

C.1 Cartesian Coordinates, i.e., Plane Flow

Physical system appropriate to plane flow analysis is illustrated on Figure 15.
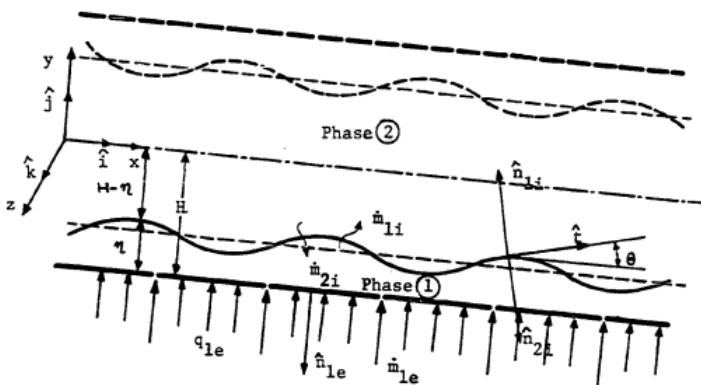

Figure 15. Typical Plane Flow Model

A liquid, i.e., phase 1, flows downward on a plate and concurrently* with the vapor phase, i.e., phase 2. A liquid mass flux $\dot{m}_{1e}$, leaves the liquid film through the porous wall, and a mass flux $\dot{m}_{1i}$, leaves it at the interface, i.e., evaporation process, or a vapor mass flux $\dot{m}_{2i}$ leaves the vapor phase at the interface, i.e., condensation process.

Plane flow illustrated in Figure 15 is general and, therefore, applicable to external as well as internal flow systems. In internal plane flow situation Figure 15 represents a typical element of a system which consists of a series of parallel plates, such as packed-tower systems used very frequently in chemical processes. Therefore, symmetry axis between two plates located on distance H, from the plates are justifiable. In external plane flow situations, however, distance 2H, between two plates, can be taken as infinite such as film cooling of external surfaces.

Considering a typical plate, we shall derive the following geometrical relations appropriate to two-dimensional plane flow, i.e., any variations in the z-direction will be neglected.

C.1.1 Unit Vectors

From Figure 15 it is easy to see that the interfacial unit tangent vector $\hat{t}_i$, can be written as

$$\hat{t}_i = \cos\theta \ \hat{i} + \sin\theta \ \hat{j} = \cos\theta \ (\hat{i} + \tan\theta \ \hat{j}) \qquad (C.1)$$

where θ is the angle between tangential direction and the x-axis.

Defining by η, the liquid thickness measured from the plate, it

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*We note that since the formulation is general, the flow of liquid downward or upward and the flow of vapor concurrently or countercurrently are immaterial.

follows that

$$\tan \theta = \frac{\partial \eta}{\partial x} \tag{C.2}$$

$$\cos \theta = \left[1 + \left(\frac{\partial \eta}{\partial x} \right)^2 \right]^{-1/2} \tag{C.3}$$

Using these expressions in Equation (C.1), interfacial unit tangent expression becomes

$$\hat{t}_i = \left[1 + \left(\frac{\partial \eta}{\partial x} \right)^2 \right]^{-1/2} \left[\hat{\imath} + \left(\frac{\partial \eta}{\partial x} \right) \hat{\jmath} \right] \tag{C.4}$$

From Figure 15 and view of Equation (C.4) the <u>interfacial unit normal vector</u> $\hat{n}_{1i}$, directed outward from the liquid, can be expressed as

$$\hat{n}_{1i} = \left[1 + \left(\frac{\partial \eta}{\partial x} \right)^2 \right]^{-1/2} \left[-\left(\frac{\partial \eta}{\partial x} \right) \hat{\imath} + \hat{\jmath} \right] \tag{C.5}$$

Since $\hat{n}_{2i} = -\hat{n}_{1i}$, then

$$\hat{n}_{2i} = \left[1 + \left(\frac{\partial \eta}{\partial x} \right)^2 \right]^{-1/2} \left[\left(\frac{\partial \eta}{\partial x} \right) \hat{\imath} - \hat{\jmath} \right] \tag{C.6}$$

Unit normal vector $\hat{n}_{1e}$ at the external surfaces, i.e., at the bottom plate is simply given by

$$\hat{n}_{1e} = -\hat{\jmath} \tag{C.7}$$

Similarly <u>perimetrical unit normal vector</u> $\hat{n}_{K\ell\xi}$, (K = 1, 2; ℓ = e, i), defined as the unit normal vector to the perimeter $\xi_{K\ell}$, directed outward from the K-th phase and located in the cross-sectional plane A_{Kc}, can be expressed from Figure 15 as

$$\hat{n}_{1i_J} = -\hat{n}_{2i_J} = \hat{J} \quad ; \quad \hat{n}_{1e_J} = \hat{n}_{1e} = -\hat{J} \tag{C.8}$$

C.1.2 Concentrations

Using the __area fractional concentration__ expression, i.e., Equation (4.30), from Figure 15 we obtain

$$\alpha = \frac{H - \eta}{H} = 1 - \frac{\eta}{H} \tag{C.9a}$$

and

$$1 - \alpha = \frac{\eta}{H} \tag{C.9b}$$

C.1.3 Area Elements

General expression for an area element at the boundaries was given by Equation (4.12). Using Equations (C.5) and (C.8) in Equation (4.12) we obtain the __interfacial area element__, thus

$$dA|_i = \left[1 + \left(\frac{\partial \eta}{\partial x} \right)^2 \right]^{1/2} d\zeta \, dx \tag{C.10}$$

and

$$\frac{dA}{dx}\Big|_i = \left[1 + \left(\frac{\partial \eta}{\partial x} \right)^2 \right]^{1/2} d\zeta \tag{C.11}$$

We note that in two-dimensional plane flow

$$d\zeta_i = d\zeta \tag{C.12}$$

Similarly, using Equation (C.7) in Equation (4.12), it can be shown that the area element at the external boundaries are given by

$$dA_1\big|_e = d\,\Upsilon_{1e}\,dx \qquad\qquad \text{(C.13a)}$$

$$dA_2\big|_e = 0 \qquad\qquad \text{(C.13b)}$$

Equation (C.13b) simply states that the second phase, does not come in contact with the external surfaces which can be seen from Figure 15.

C.1.4 Surface Balance Equations

From Figure 15 it can be seen that interfacial surface coordinates (u_1, u_2), appropriate to two-dimensional Cartesian coordinates can be taken as $u_1 = s$ and $u_2 = z$, where s is defined as the arc-length along the interface. With this notation and in view of Equation (B.20), interfacial tangent vectors $\vec{r}_1$ and $\vec{r}_2$, defined by Equations (B.21) and (B.22) become

$$\hat{r}_1 = \frac{\partial x}{\partial s}\,\hat{\imath} + \frac{\partial y}{\partial s}\,\hat{\jmath} = \hat{t}_i \qquad\qquad \text{(C.14a)}$$

$$\hat{r}_2 = \hat{k} \qquad\qquad \text{(C.14b)}$$

By means of these equations, the fundamental magnitudes of the first order defined by Equation (B.23), can be written as

$$E = \left(\frac{\partial x}{\partial s}\right)^2 + \left(\frac{\partial y}{\partial s}\right)^2 = 1 \quad ; \quad F = 0 \qquad\qquad \text{(C.15)}$$

$$H = \left[\left(\frac{\partial x}{\partial s}\right)^2 + \left(\frac{\partial y}{\partial s}\right)^2\right]^{1/2} = 1 \quad ; \quad G = 1$$

In view of Equations (C.14) and (C.15), from Equation (B.19) the interfacial momentum source becomes

$$\nabla_s \cdot \bar{\bar{\theta}}_{(mom)} = -\left(2\,\varkappa\,\sigma\,\hat{n}_{ii} + \frac{\partial \sigma}{\partial s}\,\hat{t}_i\right) \qquad\qquad \text{(C.16)}$$

Finally, substituting Equation (C.16) in (B.25) we obtain the interfacial momentum balance equation, i.e., momentum "jump" condition appropriate to two-dimensional Cartesian coordinates, thus

$$\sum_{k=1}^{2} \dot{m}_{ki} \, \vec{v}_{ki} + \textbf{P}_{ki} \cdot \hat{n}_{ki} - \bar{\bar{\Sigma}}_{ki} \cdot \hat{n}_{ki} = 2 \, \varkappa \, \sigma \, \hat{n}_{ii} + \frac{\partial \sigma}{\partial s} \, \hat{t}_{i} \qquad (C.17)$$

where $\varkappa$ is the mean curvature of the surface at a point of interest and given in reference [56, pp. 379]

$$2 \, \varkappa = - \left(\frac{\partial^{2} \eta}{\partial x^{2}} \right) \left[1 + \left(\frac{\partial \eta}{\partial x} \right)^{2} \right]^{-3/2} \qquad (C.18)$$

Similarly, from Equation (B.36) together with Equations (C.14) and (C.15) the interfacial energy source becomes

$$\nabla_{a} \cdot \bar{\bar{\Theta}}_{(en)} = - \left(2 \, \varkappa \, \sigma \, \vec{v}_{i} \cdot \hat{n}_{ii} + \hat{t}_{i} \cdot \frac{\partial \sigma \, \vec{v}_{i}}{\partial s} \right) \qquad (C.19)$$

and the interfacial energy balance equation becomes

$$\sum_{k=1}^{2} \dot{m}_{ki} \left(u_{ki} + \frac{v_{ki}^{2}}{2} \right) + \left(\vec{q}_{ki} + \textbf{P}_{ki} \, \vec{v}_{ki} - \bar{\bar{\Sigma}}_{ki} \cdot \vec{v}_{ki} \right) \cdot \hat{n}_{ki} = \qquad (C.20)$$

$$= 2 \, \varkappa \, \sigma \, \vec{v}_{i} \cdot \hat{n}_{ii} + \hat{t}_{i} \cdot \frac{\partial \sigma \, \vec{v}_{i}}{\partial s}$$

where $\varkappa$ is given by Equation (C.18).

C.2   Cylindrical Coordinates, i.e. Annular Flow

Physical system appropriate to annular flow analysis is illustrated in Figure 16.
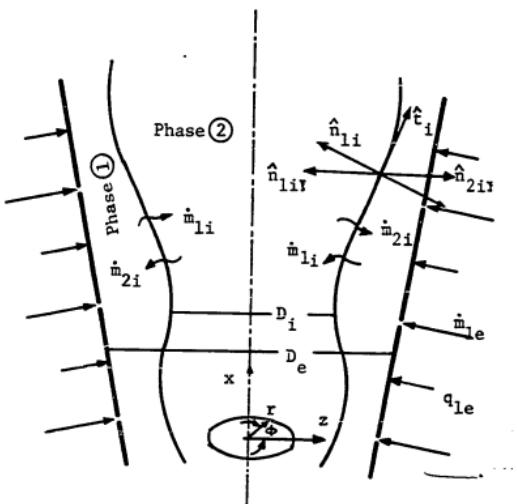
Figure 16. Typical Annular Flow Model

Considering the annular flow model illustrated on Figure 16, we shall derive the necessary geometrical relations to be used in the annular flow analysis. It is assumed that flow is axisymmetric, i.e., nothing changes in the circumferential direction ϕ.

C.2.1 Concentrations

Using the area fractional concentration expression, i.e., Equation (4.30), from Figure 16 we obtain

$$(1-\alpha) = \frac{(D_e - D_i)^2}{D_e^2} = \left(1 - \frac{D_i}{D_e}\right)^2 \tag{C.21a}$$

and

$$\alpha = \frac{D_i^2}{D_e^2} \tag{C.21b}$$

where D_i is the internal, i.e., interfacial diameter whereas D_e is the external, i.e., pipe diameter.

<u>C.2.2 Unit Vectors</u>

Following the same path taken in Section C.1.1 we obtain:

<u>interfacial unit tangent vector $\hat{t}_i$,</u>

$$\hat{t}_i = \left[1 + \left(\frac{\partial}{\partial x} \frac{D_i}{2}\right)^2\right]^{-1/2} \left[\hat{e}_x + \left(\frac{\partial}{\partial x} \frac{D_i}{2}\right)\hat{e}_r\right] \tag{C.22}$$

where $\hat{e}_x$ and $\hat{e}_r$ represent the unit vectors in the axial and radial directions, respectively. By means of Equation (C.21b), Equation (C.22) can be expressed in terms of α, thus

$$\hat{t}_i = \left[1 + \left(\frac{\partial}{\partial x} \frac{D_e \sqrt{\alpha}}{2}\right)^2\right]^{-1/2} \left[\hat{e}_x + \left(\frac{\partial}{\partial x} \frac{D_e \sqrt{\alpha}}{2}\right)\hat{e}_r\right] \tag{C.23}$$

<u>interfacial unit normal vectors, $\hat{n}_{1i}$ and $\hat{n}_{2i}$,</u>

$$\hat{n}_{1i} = -\hat{n}_{2i} = \left[1 + \left(\frac{\partial}{\partial x} \frac{D_e \sqrt{\alpha}}{2}\right)^2\right]^{-1/2} \left[\left(\frac{\partial}{\partial x} \frac{D_e \sqrt{\alpha}}{2}\right)\hat{e}_x - \hat{e}_r\right] \tag{C.24}$$

<u>external unit normal vector,</u>

$$\hat{n}_{1e} = \left[1 + \left(\frac{d}{dx} \frac{b_e}{2} \right)^2 \right]^{-1/2} \left[-\left(\frac{d}{dx} \frac{b_e}{2} \right) \hat{e}_x + \hat{e}_r \right] \qquad (C.25)$$

perimetrical unit normal vectors, $\hat{n}_{K\ell\xi}$

$$\hat{n}_{1i\xi} = -\hat{n}_{2i\xi} = \hat{e}_r \quad ; \quad \hat{n}_{1-\xi} = \hat{e}_r \qquad (C.26)$$

C.2.3 Area Elements

Using the general area element expression given by Equation (4.12), we obtain:

interfacial area element, dA/i

$$dA|_i = \left[1 + \left(\frac{\partial}{\partial x} \frac{b_e \sqrt{x}}{2} \right)^2 \right]^{1/2} d\xi\, dx \qquad (C.27)$$

we note that in annular flow geometry ξ_i is given by

$$d\xi_i = \frac{b_i}{2} d\phi = \frac{D_e \sqrt{x}}{2} d\phi \qquad (C.28)$$

In view of Equation (C.28), Equation (C.27)

$$dA|_i = \left[1 + \left(\frac{\partial}{\partial x} \frac{D_e \sqrt{x}}{2} \right)^2 \right]^{1/2} \frac{D_e \sqrt{x}}{2} d\phi\, dx \qquad (C.29)$$

consequently,

$$\frac{dA}{dx}\Big|_i = \left[1 + \left(\frac{\partial}{\partial x} \frac{D_e \sqrt{x}}{2} \right)^2 \right]^{1/2} \frac{D_e \sqrt{x}}{2} d\phi \qquad (C.30)$$

External area element dA_1/e

$$dA_1\big|_e = \left[1 + \left(\frac{d}{dx}\, \frac{D_e}{2} \right)^2 \right]^{1/2} d\zeta_e\, dx \qquad \text{(C.31)}$$

But

$$d\zeta_e = \frac{D_e}{2}\, d\phi \qquad \text{(C.32)}$$

Substituting Equation (C.32) in Equation (C.31) we obtain

$$\frac{dA_1}{dx}\bigg|_e = \left[1 + \left(\frac{d}{dx}\, \frac{D_e}{2} \right)^2 \right]^{1/2} \frac{D_e}{2}\, d\phi \qquad \text{(C.33)}$$

Since the second phase has no external surfaces, then

$$\frac{dA_2}{dx}\bigg|_e = 0 \qquad \text{(C.34)}$$

C.2.4. Surface Balance Equations

Similar developments made in Section C.1.4 can be followed to derive the axially symmetric cylindrical coordinate representation of the interfacial balance equations. As a result it can be shown that the interfacial momentum source term becomes

$$\nabla_s \cdot \bar{\bar{\theta}}_{(mom)} = -\left(2\kappa\sigma\, \hat{n}_{2i} + \frac{\partial\sigma}{\partial s}\, \hat{t}_i \right) \qquad \text{(C.35)}$$

and the interfacial momentum balance equation becomes

$$\sum_{k=1}^{2} \dot{m}_{ki}\, \vec{v}_{ki} + (P_{ki}\, \bar{\bar{\delta}} - \bar{\bar{\tau}}_{ki}) \cdot \hat{n}_{ki} = 2\kappa\sigma\, \hat{n}_{2i} + \frac{\partial\sigma}{\partial s}\, \hat{t}_i \qquad \text{(C.36)}$$

where the mean curvature κ, is given in reference [56, pp. 337]

$$2k = \frac{2}{D_e \sqrt{z}} - \frac{\frac{\partial^2}{\partial x^2}\left(\frac{D_e \sqrt{z}}{2}\right)}{\left\{1 + \left[\frac{\partial}{\partial x}\left(\frac{D_e \sqrt{z}}{2}\right)\right]^2\right\}^{3/2}} \tag{C.37}$$

Finally, the <u>interfacial energy source</u> and the <u>energy balance</u> are given, respectively, by

$$\nabla_s \cdot \bar{\bar{\theta}}_{(en)} = -\left(2k\sigma\, \vec{v}_i \cdot \hat{n}_{2i} + \hat{t}_i \cdot \frac{\partial \sigma \vec{v}_i}{\partial s}\right) \tag{C.38}$$

$$\sum_{k=1}^{2} \dot{m}_{ki}\left(u_{ki} + \frac{v_{ki}^2}{2}\right) + \left(\vec{q}_{ki} + P_{ki}\vec{v}_{ki} - \bar{\bar{z}}_{ki}\cdot\vec{v}_{ki}\right)\cdot\hat{n}_{ki} = \tag{C.39}$$

$$= 2k\sigma\, \vec{v}_i \cdot \hat{n}_{2i} + \hat{t}_i \cdot \frac{\partial \sigma \vec{v}_i}{\partial s}$$

APPENDIX D

ESTIMATE OF INTERFACIAL EVAPORATION

In order to estimate the thinning of the liquid film along the flow length L, we use the steady state liquid film continuity equation, thus from Equation (5.8a) and Equation (11.18) for a liquid film flowing on a flat plate we can write

$$\frac{d}{dx}\left(\eta \, \rho_l \langle v_{lx} \rangle\right) = \frac{dG}{dx} = - H \, \Gamma_{li} = - \frac{q_{li}}{\Delta i_{lv}} \qquad (D.1)$$

where G is the flow rate per unit length. Equation (D.1) expresses the decrease in the flow rate per unit length that results from vaporization at the interface. Integration of Equation (D.1) results in the following expression

$$G_o - G_L = \frac{q_{li}}{\Delta i_{lv}} \, L \qquad (D.2)$$

In order to estimate the percentage change of the initial flow rate G_o, Equation (D.2) is written

$$\frac{G_o - G_L}{G_o} = \frac{q_{li}}{\Delta i_{lv}} \, \frac{L}{G_o} \qquad (D.3)$$

For the purpose of having some idea about the thinning effect of vaporization, we can make a representative calculation. Using following values:

Wall heat flux, q_{1e}, Cal/cm^{2}Sec   0.753

Heating length, L, Cm   50

Initial liquid flow rate, G_o, Gram/Cm Sec   10

Heat of vaporization, Δi_{12}, Cal/Gram   560

Substituting these numerical values in Equation (D.3), one can find the fractional change of initial flow rate G_o

$$\frac{G_o - G_L}{G_o} = 0.0068 \qquad \text{(D.4)}$$

Since $G = \eta \rho_1 \langle v_1 \rangle$, then approximating $\langle v_1 \rangle_o \simeq \langle v_{1L} \rangle$ it can be seen that, in terms of the initial film thickness, the change due to vaporization can be approximated by

$$\frac{\eta_o - \eta_L}{\eta_o} \simeq 0.0068 \qquad \text{(D.5)}$$

This calculation demonstrates that the thinning effect of vaporization is small enough to be neglected.

APPENDIX E

DETERMINATION OF X's FOR THE INSTABILITY OF THE FREE FILM FLOW

In Section 12.3, ΔF, $\Delta \ll P \gg$ and Cov(mom K), K = 1,2, are calculated. Here, we shall use them in Equation (11.36) in order to evaluate X's.

Performing the differentiations which are necessary for the determination of X's, and expressing $(1 - \alpha)$ in terms of the film thickness η we obtain from Equation (12.19)

$$\frac{\partial \Delta F}{\partial \alpha} = \frac{6 \eta \langle v_1 \rangle}{\eta^2} - \frac{k_1 \Delta T}{\Delta i_{12}} \frac{\langle v_1 \rangle}{\eta^2} \tag{E.1}$$

$$\frac{\partial \Delta F}{\partial \langle v_1 \rangle} = - 3 \eta \tag{E.2}$$

$$\frac{\partial \Delta F}{\partial \langle v_2 \rangle} = 0 \tag{E.3}$$

$$\frac{\partial \Delta F}{\partial \left(\frac{\partial \alpha}{\partial x} \right)} = - \frac{1}{2} \, \rho_1 \, \eta \, g_y \tag{E.4}$$

whereas from Equation (12.21) we obtain

$$\frac{\partial \Delta \langle P \rangle}{\partial \alpha} = \frac{1}{2} \, \rho_1 \, \eta \, g_y \tag{E.5}$$

$$\frac{\partial \Delta \langle P \rangle}{\partial \left(\frac{\partial \alpha}{\partial x} \right)} = 0 \tag{E.6}$$

$$\frac{\partial \Delta \langle P \rangle}{\partial \left(\frac{\partial^2 x}{\partial x^2} \right)} = \bar{\bar{\sigma}} \cdot \eta \tag{E.7}$$

and finally from Equation (12.22)

$$\frac{\partial}{\partial x} \text{Cov}(\text{mom} \perp) = - \rho_1 \frac{\langle v_1 \rangle^2}{5} \tag{E.8}$$

$$\frac{\partial}{\partial \langle v_1 \rangle} \text{Cov}(\text{mom} \perp) = \frac{2}{5} \rho_1 \langle v_1 \rangle \tag{E.9}$$

$$\frac{\partial}{\partial \langle v_2 \rangle} \text{Cov}(\text{mom} \perp) = 0 \tag{E.10}$$

Substituting Equations (E.1-10) and (11.23) in Equation (11.36) one can show that X's are given by

$$X_1 = \bar{\bar{\sigma}} \cdot \eta \tag{E.11a}$$

$$X_2 = 0 \tag{E.11b}$$

$$X_3 = \frac{6}{5} \rho_1 \langle v_1 \rangle^2 + \rho_1 \eta \, g_y + 2 \left(\frac{k_1 \Delta T}{\eta \Delta i_{12}} \right)^2 \frac{\Delta \varepsilon}{\rho_1 \tilde{c}_2} \tag{E.11c}$$

$$X_4 = \frac{12}{5} \rho_1 \langle v_1 \rangle \tag{E.11d}$$

$$X_5 = \rho_1 \tag{E.11e}$$

$$X_6 = 9 \frac{k_1 \langle v_1 \rangle}{\eta^2} + \frac{1}{10} \frac{k_1 \Delta T}{\eta \Delta i_{12}} \langle v_1 \rangle \tag{E.11f}$$

$$X_7 = -\frac{1}{2} \frac{k_1 \Delta T}{\eta^2 \Delta i_{12}} - \frac{3 \, r_1}{\eta^2} \qquad \text{(E.11g)}$$

$$X_8 = -3 \frac{r_1}{e_1^* \eta^2} \frac{k_1 \Delta T}{\eta^2 \Delta i_{12}} + \frac{1}{2 e_1^*} \left(\frac{k_1 \Delta T}{\eta^2 \Delta i_{12}} \right)^2 \qquad \text{(E.11h)}$$

APPENDIX F

DETERMINATION OF X's AND Y's FOR THE INSTABILITY OF TWO INVISCID FLUIDS

Since the flow configuration fits to the horizontal plane flow model, see, Figure 10, under the assumption stated in Section (13.1) it can be shown from Equations (11.3a and b) that the generalized force functions F_1 and F_2 are given by

$$F_1 = -\frac{1}{2} \rho_1 H g \frac{\partial \alpha}{\partial x} \tag{F.1}$$

$$F_2 = -\frac{1}{2} \rho_2 H g \frac{\partial \alpha}{\partial x} \tag{F.2}$$

whereas averaged pressures can be given by

$$\langle P_1 \rangle = P_{1i} - \frac{1}{2} \rho_1 H (1-\alpha) g \tag{F.3}$$

$$\langle P_2 \rangle = P_{2i} + \frac{1}{2} \rho_2 H \alpha g \tag{F.4}$$

Furthermore, since the velocities are uniform within both phases, in view of Equation (6.3), it is evident that

$$Cov(mom\ k) = 0 \qquad , \ K = 1,2 \tag{F.5}$$

Forming differences of generalized force functions and pressures, we obtain

$$\Delta F = -\frac{1}{2} H g \Delta \rho \frac{\partial \alpha}{\partial x} \tag{F.6}$$

$$\Delta \langle P \rangle = (P_{1i} - P_{2i}) - \frac{1}{2} \rho_1 H g \left[\rho_1 (1-\alpha) + \rho_2 \alpha \right] \tag{F.7}$$

In view of Equation (11.6), $\Delta \langle P \rangle$ becomes

$$\Delta \langle P \rangle = \sigma H \frac{\partial^2 x}{\partial x^2} + \left[-\frac{k_1 \Delta \Gamma}{(1-x) H \Delta i_{1_2}} \right]^2 \frac{\Delta \epsilon}{\rho_1 \rho_2} - \frac{1}{2} H \frac{\partial}{\partial r} \left[(1-x) \epsilon + x \hat{\epsilon}_2 \right] \quad (F.8)$$

Performing the necessary differentiations which will be used in the derivation of X's, we obtain from Equation (F.6)

$$\frac{\partial \Delta F}{\partial \left(\frac{\partial x}{\partial x} \right)} = -\frac{1}{2} H g \Delta \rho \quad (F.9)$$

and

$$\frac{\partial \Delta F}{\partial x} = \frac{\partial \Delta F}{\partial \langle v_1 \rangle} = \frac{\partial \Delta F}{\partial \langle v_2 \rangle} = 0 \quad (F.10)$$

whereas from Equation (F.8) we obtain

$$\frac{\partial \Delta \langle P \rangle}{\partial x} = 2 \left(\frac{k_1 \Delta \Gamma}{\Delta i_{1_2}} \right)^2 \frac{1}{(1-x)^3 H} \frac{\Delta \rho}{\rho_1 \rho_2} + \frac{1}{2} H g \Delta \rho \quad (F.11)$$

$$\frac{\partial \Delta \langle P \rangle}{\partial \left(\frac{\partial^2 x}{\partial x^2} \right)} = \sigma H \quad (F.12)$$

$$\frac{\partial \Delta \langle P \rangle}{\partial \left(\frac{\partial x}{\partial x} \right)} = 0 \quad (F.13)$$

Substituting Equations (F.9-F.13), (F.5) and (11.11) in Equation (11.36) one can express X's as follows:

$$X_1 = \sigma H \quad (F.14a)$$

$$X_2 = 0 \quad (F.14b)$$

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$$X_3 = \left(\frac{e_1 v_1^2}{1-\alpha} + \frac{e_2 v_2^2}{\alpha} \right) + 2 \left(\frac{k_1 \Delta T}{\Delta i_{12}} \right)^2 \frac{1}{(1-\alpha)^2 H^2} \frac{\Delta e}{e_1 e_2} + H \Delta e g \quad \text{(F.14c)}$$

$$X_4 = \alpha \left(\frac{e_1 v_1}{1-\alpha} + \frac{e_2 v_2}{\alpha} \right) \quad \text{(F.14d)}$$

$$X_5 = \frac{e_1}{1-\alpha} + \frac{e_2}{\alpha} \quad \text{(F.14e)}$$

$$X_6 = -\left(\frac{v_1}{1-\alpha} + \frac{v_2}{\alpha} \right) \frac{k_1 \Delta T}{\Delta i_{12}} \frac{1}{(1-\alpha)^2 H^2} \quad \text{(F.14f)}$$

$$X_7 = -\left(\frac{1}{1-\alpha} + \frac{1}{\alpha} \right) \frac{k_1 \Delta T}{\Delta i_{12}} \frac{1}{(1-\alpha)^2 H^2} \quad \text{(F.14g)}$$

$$X_8 = 0 \quad \text{(F.14h)}$$

After having calculated X's above, now we can proceed with the calculation of Y's. In view of Equation (11.46) and X's derived above one can show the Y's can be given as follows:

$$Y_1 = \frac{\dfrac{e_1 v_1}{1-\alpha} + \dfrac{e_2 v_2}{\alpha}}{\dfrac{e_1}{1-\alpha} + \dfrac{e_2}{\alpha}} \quad \text{(F.15a)}$$

$$Y_2 = \frac{\dfrac{k_1 \Delta T}{\Delta i_{12}}}{2(1-\alpha)^3 H^2 K} \frac{1}{\alpha(1-\alpha)} \frac{1}{\dfrac{e_1}{1-\alpha} + \dfrac{e_2}{\alpha}} \quad \text{(F.15b)}$$

$$Y_3 = \frac{v^2 H k^2 - \left(\dfrac{e_1 v_1^2}{1-\alpha} + \dfrac{e_2 v_2^2}{\alpha} \right) - 2 \left[\dfrac{k_1 \Delta T}{(1-\alpha) H \Delta i_{12}} \right]^2 \dfrac{\Delta e}{(1-\alpha) e_1 e_2} - H \Delta e g}{\dfrac{e_1}{1-\alpha} + \dfrac{e_2}{\alpha}} \quad \text{(F.15c)}$$

$$Y_4 = - \frac{\dfrac{v_1}{1-\alpha} + \dfrac{v_2}{\alpha}}{\dfrac{\rho_1}{1-\alpha} + \dfrac{\rho_2}{\alpha}} \cdot \frac{k_1 \Delta T}{(1-\alpha)^2 H^2 k} \qquad \text{(F.15d)}$$

We note that in the absence of the interfacial mass transfer, that is, when $q_{1e} = 0$, then Y_2 and Y_4 become zero. However, with the introduction of even very small heat transfer these two terms become different from zero, and $Y_2 > 0$ for evaporation. Hence, according to the necessary stability condition the interface will be unstable no matter what effects of other variables are. This is inconsistent with our first model, the free film flow, where it was seen that viscous effects dominates Y_2 and Y_4, and according to Chapter 12 Y_2 is always negative whereas Y_4 was positive. Furthermore, we note that the effects of the interface mass transfer on Y_2 and Y_4 were so small that they could be neglected in comparison with the viscous effects.

In order to be consistent within the present model we shall assume that

$$Y_2 = 0 \qquad \text{(F.16a)}$$

and

$$Y_4 = 0 \qquad \text{(F.16b)}$$

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VITA

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Following graduation Mr. Gunol Kocamustafaogullari was employed by the Cellulose and Paper Company, SEKA, at Izmit, Turkey. During this time his primary duty was the engineering design work for the conversion of boilers from a coal burning system to a fuel-oil system.

In Summer, 1966 he entered the Graduate School of New York University, New York. He received his Master of Science in Mechanical Engineering in October, 1967. He continued to study toward the Doctorate degree in Mechanical Engineering at New York University.

In September of 1969, he and his thesis advisor transferred from New York University to the Georgia Institute of Technology and continued his study toward the Doctor of Philosophy. During his time in this institution, he served as a Graduate Research Assistant in the School of Mechanical Engineering.

In 1965, he was married to the former Gonul Erenkus of Istanbul, Turkey. They now have a wonderful son, Turkay, born on February 5, 1966.