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APPLICATIONS OF THE THEORY OF OPTIMAL CONTROL OF

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FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

Jean-Louis Pierre Armand

August 1971

I certify that I have read this thesis and that in my
opinion it is fully adequate, in scope and quality, as
a dissertation for the degree of Doctor of Philosophy.

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Abstract

After a rapid survey of recent papers dealing with the minimum-mass design of one-dimensional structures under constraints of a dynamic or aeroelastic nature, focusing on the two current trends in the field, an extension of the present methods to two-dimensional structures is outlined. In mathematical form, the problem reduces to that of the minimization of a functional under constraints taking the form of partial differential equations to be satisfied inside the domain covered by the structure. This appears to be a special case of the general theory of optimal control of systems with distributed parameters: a presentation, together with a survey of a few important papers, is made of the subject.

The first part of the present study is an extension of the classical methods of control theory to derive the necessary conditions in the case of two-dimensional systems. An application is then made in the second part for the theoretical case of the minimum-mass design of a simply-supported shear plate for a fixed fundamental frequency of vibration, under the structural mass assumption, for which an optimal thickness distribution in analytical form is found. The case of a minimal constraint placed on the thickness is then investigated. In the third and fourth part respectively, the theory developed in Part A is applied to the minimum-mass design of a sandwich plate and of an elastic plate for a given fundamental frequency of free vibration. Under the most general conditions, the optimization problem reduces to the solution of two simultaneous partial differential equations to be satisfied by the optimal thickness distribution together with the corresponding displacement mode. The first one is the expression of the uniformity of the energy distribution which was pointed out by Ashley and McIntosh for the optimal frequency design of one-dimensional structures, and for various design criteria in one and two dimensions by Prager and Taylor. The second one is the original constraint equation.

These are shown in Part E to be also sufficient conditions for an extremum, using a theorem by Prager applicable to a very general class of structural optimization problems in up to three dimensions, of which the optimal frequency design of a plate with prescribed vanishing displacement along the edge is a particular case. A numerical solution is presented for the case of a simply-supported square sandwich plate.

ACKNOWLEDGMENTS

TABLE OF CONTENTS

INTRODUCTION

Since the recent appearance of the branch of structural optimization which is concerned with constraints of a dynamic (natural frequencies of vibration) or aeroelastic nature*, two trends have developed, which represent two complementary methods of solution.

The first of those makes use of mathematical programming techniques. The structure to be optimized is either a structure with discrete characteristics, such as a stiffened beam, or a continuous one which is first discretized: this is the case of a beam broken into finite elements. This approach, the older, is the one taken by Haug, et al.[2,3], Fox and Kapoor[4], Rubin[5], and Turner[6], to name only a few. Its advantages are obvious: complex structures can be approximated this way with good accuracy, then subject to optimization techniques. The weakness of the method, however, lies in the fact that the optimization is done after the discretization has been made, so that it is impossible to know how close the discrete optimum found is from the true one, and even in some extreme cases it makes no sense at all.[†] In other words, there is no way of getting the optimum within a given accuracy.

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*A small paragraph was devoted to the subject under the heading "dynamic compliance" in the excellent comprehensive review of the field of Structural Optimization done by Sheu and Prager[1] less than three years ago. The number of articles belonging to this classification which have appeared since is impressive, and the subject alone now merits far more extensive review.

[†]The only way to decide where to stop iterating in the optimization process is when it appears to converge; however, it has been frequently proven that, depending on the method chosen to arrive at the optimum, the process might pseudo-converge to a solution which is totally irrelevant and vanishes when another optimization method is tried.

Another approach, more recent, is that of the classical calculus of variations whose foundations date back to the time of Euler and Lagrange and has raised considerable interest ever since: it is applicable to simple, continuous structures. It has been widely used by Turner[7] for his now classical bar, by Keller and Niordson[8] for the problem of the tallest column, by Niordson[9] for the optimization of a beam for a fixed first frequency of vibration, by Olhoff[10] for the optimal design of vibrating circular plates. Variational principles were also used by Prager and Taylor[11] and Taylor[12] and applied to continuous systems.

In the spirit of this approach an extension was made by the use of the techniques of optimal control theory, which are themselves ramifications of the classical calculus of variations; its principles are expressed in a different form, more suitable to the kind of problems encountered in the theory of control. For the structural applications, the constraints are put under a form analogous to that of control constraints, thus imposing the distinction between state and control variables. The necessary conditions are then derived in the form of a system of differential equations, yielding a two-point boundary-value problem. Exact solutions have been found in a few rewarding cases, and numerical methods derived in control theory, requiring discretization but only at this step, have been applied. Accuracy can then be fixed in advance, the limitations of this method now being due to the complexity of the structure. In other words, this is the opposite approach from the one discussed first. This method was first introduced by Ashley and McIntosh[13] for the optimization of structures subject to aeroelastic constraints, and applied by Armand and Vitte[14] to a few simple cases and by Weisshaar[15] to the panel flutter problem started by Turner.

A recent and detailed account of the development of the field of structural — especially dynamic or aeroelastic — optimization will be found in some of the references above, especially[13] and in references[16,17].

The structures optimized up to now were "one-dimensional"; this means that, due to their nature, the problems to be investigated led to constraints

expressed in the form of ordinary differential equations in one independent spatial variable. A beam problem is obviously one-dimensional; so are optimization problems dealing with circular plates when the imposed static or dynamic constraints have the property of axisymmetry.[10] The problems of dynamic structural optimization where the constraints are in the form of partial differential equations have hardly been investigated up-to-date. The only tentative efforts known to the author are those of Johnson[18] and Haug[19] who investigated separately the problem of the optimization of a plate for a fixed first frequency of vibration. Both of them make use of the method of discretization described at the beginning of this chapter, and this is why, unfortunately, it is impossible to give an estimate of the precision which guarantees the validity of their optimal solution. Moreover, Johnson's square plate is divided into only 25 elements, the symmetry of the problem further reducing the accuracy to only 6 elements. Haug, on the other hand, presents an adaptation of the powerful method of steepest descent developed in optimal control theory to two-dimensional problems, which seems very promising and should lead to the numerical solution of number of such problems.

The present work tries to present an adaptation to two-dimensional structures of the methods of optimal control theory already applied to one-dimensional structural optimization, in the spirit of the previous work done in this area[13, 14, 15, 16]. The corresponding branch of Optimal Control Theory is known as optimal control of distributed-parameter systems and is, we might say, a brand new field of investigation. The first ever made, proposing a definition and an evaluation of the task at hand, is contained in a Russian paper dated 1960[20]. The systems to be considered are characterized by either partial differential equations or integral equations to be satisfied on DxT, where D is some spatial domain contained in the Euclidean space E^n for $n \geq 1$ and T is a time domain. In the case of differential equations, there are also boundary conditions to be satisfied. The behavior of the system can be influenced either by mechanisms acting throughout D or at the boundary of D, or both. There is given some functional dependent on the state of the system or its boundary

conditions or on its control, or some combination of these, and two types of questions may be asked:

i) What should the available controls be to minimize the given functional (open-loop control)?

ii) What should the functional relationship between the control and the system state be in order that the functional be minimized (closed-loop control)?

Needless to say, optimal control of distributed-parameter systems is much more difficult than for lumped-parameter systems, and it is still a growing field of investigation. An excellent survey containing an exhaustive list of references is due to Robinson[21]. In his paper, both a historical and technical presentation of the research in this domain is done. Two textbooks have appeared recently, one by the pioneer Butkovskiy[22] presenting a unifying treatment of most of the literature that appeared previously in Russian journals and of great interest although perhaps too broad in scope; the other one is by Lions[23] and is mostly concerned with the mathematical aspect of some particular problems.

The overwhelming majority of research up-to-date has been devoted to linear problems. Although our scope will be more limited than the general goal of optimal control of distributed-parameter systems, we shall never encounter linear problems, and we will have to derive a more general approach applicable to non-linear constraints.

We first have to define the kind of problems that we will encounter for our purpose, which remains structural optimization: the domain DxT will simply be the domain occupied in space by the two-dimensional structure which we consider, referred to a set of orthonormal axes Ox, Oy. The constraint will take the form of a partial differential equation to be satisfied inside the domain, with adequate boundary conditions provided by the physical properties of the structure. For the common case of a minimum-mass problem, the functional to be minimized will be the surface integral of the thickness over the domain covered by the structure. Generally, the deflection (in-plane or out-of-plane) will play the role of a state variable, whereas the thickness will be in some cases the control variable.

A problem of this type is a special case of the more general one stated above, namely a Mayer-Bolza problem for multiple integrals. The most authoritative treatment of the general Mayer-Bolza problem in two dimensions is that of Lur'e[24], who was the first to suspect the applications this theory might have for structural optimization. This paper was later reprinted in a textbook on topics in optimization edited by Leitmann[25] with some additions, in particular an application to magnetohydrodynamics leading to a control of the bang-bang type, itself a reprint of an article published in the Journal of Applied Mathematics and Mechanics[26]. Lur'e derives the necessary conditions for an optimum for a very general Mayer-Bolza type of problem, extending the classical methods of calculus of variations to two dimensions.

In part A of this study, we will extend the methods used by Bryson and Ho[27] in their excellent textbook on optimal control theory to the case of two independent variables. We will be led to the definition of a scalar two-dimensional Hamiltonian, already suggested by Lur'e, and to a set of necessary conditions for an extremal in terms of it analogous to the ones he derived by other means. The case of finite equality or inequality constraints on the control variables will also be considered.

Part B presents an application of the theory to the case of the optimization of a shear plate: we want to find the thickness distribution such as to minimize the mass of this structure, the first frequency of vibration of the homogeneous reference plate with constant thickness being required to keep a constant value. The shear plate was chosen so as to provide a simple, though non-trivial, partial differential equation as a constraint; an analytical solution will be found in the case of simply-supported edges.

Our experience will then be applied to the minimum-mass design of more practical structures, under the same constraint as above. The sandwich plate will be treated in part C, while part D is devoted to the classical elastic plate. Following the steps of the simple shear-plate optimization process, both of these optimization problems will reduce into finding the solution of a system of two simultaneous partial differential equations for the optimal thickness distribution and the displacement mode. The necessary conditions are then in a form showing

that they are nothing but the expression of a very general sufficient optimality condition stated by Prager for a broad class of structural optimization problems to which the considered cases belong; they are therefore necessary and sufficient conditions for an extremal. A numerical treatment of this system of partial differential equations is presented in Part C for a simply-supported sandwich plate. The uniformity of the energy distribution throughout the structure is shown in Part E to be a sufficient condition for an extremal of the mass.

NECESSARY CONDITIONS FOR THE STATIONARY VALUE OF A FUNCTIONAL UNDER CONSTRAINTS EXPRESSED AS PARTIAL DIFFERENTIAL EQUATIONS

1. Statement of the Problem

Let D denote a closed domain in the plane with a piecewise continuo. s boundary ∂D . If the plane is referred to a set of rectangular coordinates (x, y) , let the boundary curve be represented in parametric form by the equations:

$$x = \alpha(\sigma)$$

$$y = \beta(\sigma)$$

where α and β are piecewise continuous functions of the parameter σ possessing a first derivative in the intervals where they are continuous.

In this domain, we consider a system of partial differential equations of the form:

$$\frac{\partial z_i}{\partial x} - X_i(\underset{\sim}{z}, \underset{\sim}{u}; x, y) = 0$$

$$\frac{\partial z_i}{\partial y} - Y_i(\underset{\sim}{z}, \underset{\sim}{u}; x, y) = 0 \qquad (1.1)$$

$$i = 1, 2, \dots n.$$

The X_i and Y_i functions have to be such that they satisfy the compatibility conditions:

$$\frac{DX_i}{Dy} - \frac{DY_i}{Dx} = 0$$

$i = 1, 2, \dots n,$ where the derivatives above are total derivatives taken with respect to all the arguments included, i.e. :

$$\frac{DX_i}{Dy} = \frac{\partial X_i}{\partial \underset{\sim}{z}} \frac{\partial \underset{\sim}{z}}{\partial y} + \frac{\partial X_i}{\partial \underset{\sim}{u}} \frac{\partial \underset{\sim}{u}}{\partial y} + \frac{\partial X_i}{\partial y}$$

$$\frac{DY_i}{Dx} = \frac{\partial Y_i}{\partial \underset{\sim}{z}} \frac{\partial \underset{\sim}{z}}{\partial x} + \frac{\partial Y_i}{\partial \underset{\sim}{u}} \frac{\partial \underset{\sim}{u}}{\partial x} + \frac{\partial Y_i}{\partial x}$$

For a system governed by a set of equations in the form (1.1), the vector function $\underset{\sim}{z} = (z_1, z_2, \ldots, z_n)$ of the arguments x, y describes the mechanical system itself, and therefore the z_i play the role of state variables, whereas the functions $\underset{\sim}{u} = (u_1, u_2, \ldots, u_p)$ of the same arguments represent the distributed controls.

Equations (1.1) represent a standard form of any system of partial differential equations: Lur'e[25] calls it the special case of the Pfaffian system. Any partial differential equation of order superior to one or any system of such equations may be written in the form (1.1), with the number of dependent variables increased as necessary. The technique of decomposition is simply an extension of the one used in the case of the transformation of a high order ordinary differential equation into a system of first order ones by introducing auxiliary dependent variables.[14,27] The only difference resides in the fact that a function in two independent variables possesses $k+1$ derivatives of order k, whereas a function in one independent variable possesses only one: therefore the decomposition will require the introduction of a much considerable number of dependent variables. The compatibility conditions will help reduce this number.

The procedure is best demonstrated on a simple example, treated below. We will then show how the decomposition is done for the general case of a single partial differential equation in two dependent variables which we will always encounter as a constraint in problems of minimum-mass design of structures satisfying conditions on their frequency of vibration or on some aeroelastic eigenvalue. Generally, one of the dependent variable will represent the displacement at a point of the structure, the other representing the thickness distribution.

For the Helmholtz equation, example also treated by Lur'e in the already referenced paper:

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} + uz = 0$$

we find the equivalent system:

$$\frac{\partial z_1}{\partial x} = z_2$$

$$\frac{\partial z_1}{\partial y} = z_3$$

$$\frac{\partial z_2}{\partial x} = u_2$$

$$\frac{\partial z_2}{\partial y} = u_3$$

$$\frac{\partial z_3}{\partial x} = u_3$$

$$\frac{\partial z_3}{\partial y} = -u_2 - u_1 z_1$$

Here we had to introduce the new dependent variables z_2, z_3, u_2, u_3, the original z being denoted z_1 and the original u, u_1. The state variables are now z_1, z_2, z_3, while the controls are u_1, u_2, u_3.

In the light of the example above, we are now able to explain more precisely how the decomposition (1.1) can be achieved in the case of main concern to us, as we will see later, that is of a constraint expressed in the form of one single partial differential equation in the two independent variables x and y, and two dependent variables w and h of order $n_1 \geq 1$ and $n_2 \geq 0$ respectively in those variables. In other words, there is present at least one partial derivative of order n_1 in the first variable called w, and at least one partial derivative of order n_2 in the other variable, called h. The most general form for this type of equation will be:

$$f(w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial^2 w}{\partial x^2}, \ldots, \frac{\partial^{n_1} w}{\partial x^{n_1}}, \frac{\partial^{n_1} w}{\partial x^{n_1-1} \partial y}, \ldots, \frac{\partial^{n_1} w}{\partial x \partial y^{n_1-1}}, \frac{\partial^{n_1} w}{\partial y^{n_1}},$$

$$h, \frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}, \frac{\partial^2 h}{\partial x^2}, \ldots, \frac{\partial^{n_2} h}{\partial x^{n_2}}, \frac{\partial^{n_2} h}{\partial x^{n_2-1} \partial y}, \ldots, \frac{\partial^{n_2} h}{\partial x \partial y^{n_2-1}}, \frac{\partial^{n_2} h}{\partial y^{n_2}}; x, y) = 0 \tag{1.1.a}$$

and we will assume that this equation can be solved for one of the highest derivatives of w appearing, for example, if

$$\frac{\partial^{n_1} w}{\partial y^{n_1}}$$

is present:

$$\frac{\partial^{n_1} w}{\partial y^{n_1}} = \varphi(w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial^2 w}{\partial x^2}, \ldots, \frac{\partial^{n_1} w}{\partial x^{n_1}}, \frac{\partial^{n_1} w}{\partial x^{n_1-1} \partial y}, \ldots, \frac{\partial^{n_1} w}{\partial x \partial y^{n_1-1}},$$

$$h, \frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}, \frac{\partial^2 h}{\partial x^2}, \ldots, \frac{\partial^{n_2} h}{\partial x^{n_2}}, \frac{\partial^{n_2} h}{\partial x^{n_2-1} \partial y}, \ldots, \frac{\partial^{n_2} h}{\partial x \partial y^{n_2-1}}, \frac{\partial^{n_2} h}{\partial y^{n_2}}; x, y) \tag{1.1.b}$$

We will introduce $1 + 2 + \ldots + n_1 = \dfrac{n_1(n_1+1)}{2}$ auxiliary variables z_i as follows:

$$w = z_1$$

$$\frac{\partial z_1}{\partial x} = z_2 \qquad\qquad\qquad\qquad (= \frac{\partial w}{\partial x})$$

$$\frac{\partial z_1}{\partial y} = z_3 \qquad\qquad\qquad\qquad (= \frac{\partial w}{\partial y})$$

$$\frac{\partial z_2}{\partial x} = z_4 \qquad\qquad\qquad\qquad (= \frac{\partial^2 w}{\partial x^2})$$

$$\frac{\partial z_2}{\partial y} = z_5 \qquad\qquad\qquad\qquad (= \frac{\partial^2 w}{\partial x \partial y})$$

(continued)

$$\frac{\partial z_3}{\partial x} = z_5 \qquad\qquad (= \frac{\partial^2 w}{\partial x \partial y})$$

$$\frac{\partial z_3}{\partial y} = z_6 \qquad\qquad (= \frac{\partial^2 w}{\partial y^2})$$

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$$\frac{\partial z_{\frac{(n_1-1)(n_1-2)}{2}+1}}{\partial x} = z_{\frac{n_1(n_1-1)}{2}+1} \qquad\qquad (= \frac{\partial^{n_1-1} w}{\partial x^{n_1-1}})$$

$$\frac{\partial z_{\frac{(n_1-1)(n_1-2)}{2}+1}}{\partial y} = z_{\frac{n_1(n_1-1)}{2}+2} \qquad\qquad (= \frac{\partial^{n_1-1} w}{\partial x^{n_1-2} \partial y})$$

$$\frac{\partial z_{\frac{(n_1-1)(n_1-2)}{2}+2}}{\partial x} = z_{\frac{n_1(n_1-1)}{2}+2} \qquad\qquad (= \frac{\partial^{n_1-1} w}{\partial x^{n_1-2} \partial y})$$

$$\frac{\partial z_{\frac{(n_1-1)(n_1-2)}{2}+2}}{\partial y} = z_{\frac{n_1(n_1-1)}{2}+3} \qquad\qquad (= \frac{\partial^{n_1-1} w}{\partial x^{n_1-3} \partial y^2})$$

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$$\frac{\partial z_{\frac{n_1(n_1-1)}{2}}}{\partial x} = z_{\frac{n_1(n_1+1)}{2}-1} \qquad\qquad (= \frac{\partial^{n_1-1} w}{\partial x \partial y^{n_1-2}})$$

$$\frac{\partial z_{\frac{n_1(n_1-1)}{2}}}{\partial y} = z_{\frac{n_1(n_1+1)}{2}} \qquad\qquad (= \frac{\partial^{n_1-1} w}{\partial y^{n_1-1}})$$

The $(n_1 + 1)\ \underset{\sim}{u}$ variables are now introduced as follows:

$$\frac{\partial z_{\frac{n_1(n_1-1)}{2}+1}}{\partial x} = u_1 \qquad\qquad (= \frac{\partial^{n_1} w}{\partial x^{n_1}})$$

$$\frac{\partial z_{\frac{n_1(n_1-1)}{2}+1}}{\partial y} = u_2 \qquad\qquad (= \frac{\partial^{n_1} w}{\partial x^{n_1-1}\partial y})$$

$$\vdots$$

$$\frac{\partial z_{\frac{n_1(n_1+1)}{2}}}{\partial x} = u_{n_1} \qquad\qquad (= \frac{\partial^{n_1} w}{\partial x\partial y^{n_1-1}})$$

$$\frac{\partial z_{\frac{n_1(n_1+1)}{2}}}{\partial y} = u_{n_1+1} \qquad\qquad (= \frac{\partial^{n_1} w}{\partial y^{n_1}})$$

We now do the same for the dependent variable h, introducing $\frac{n_2(n_2+1)}{2}$ auxiliary state variables h_k and n_2+1 auxiliary control variables v_{ℓ}, following an identical procedure.

Now the given equation will be replaced by the two systems formed above, together with the condition:

$$u_{n_1+1} = \varphi(z_1, z_2, z_3, \ldots, z_{\frac{n_1(n_1+1)}{2}}, u_1, \ldots, u_{n_1}, h_1, h_2, h_3, \ldots,$$

$$h_{\frac{n_2(n_2+1)}{2}} \, , \, v_1, v_2, \ldots, v_{n_2}+1$$

In the case of a constraint of this type, the role of the control variables is played by the $\underset{\sim}{u}$ and $\underset{\sim}{v}$ variables, while the $\underset{\sim}{z}$ and $\underset{\sim}{h}$ define the state of the mechanical system. The $\underset{\sim}{z}$ and $\underset{\sim}{h}$ are the state variables $\underset{\sim}{z}$ of (1.1) whereas the $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are the distributed controls $\underset{\sim}{u}$.

The decomposition of a partial differential equation or of a system of the same into a system of the form (1.1) is of course by no means unique. A question of definition of the controls arises too, as can be seen in the following example.

Consider the partial differential equation, in the two dependent variables w and h, that we will encounter as a constraint in part B:

$$\frac{\partial}{\partial x}(h \frac{\partial w}{\partial x}) + \frac{\partial}{\partial y}(h \frac{\partial w}{\partial y}) + k^2 hw = 0$$

By the process described above, this equation is equivalent to the system:

$$w = z_1$$

$$\frac{\partial z_1}{\partial x} = z_2 \qquad\qquad (= \frac{\partial w}{\partial x})$$

$$\frac{\partial z_1}{\partial y} = z_3 \qquad\qquad (= \frac{\partial w}{\partial y})$$

$$\frac{\partial z_2}{\partial x} = u_1 \qquad\qquad (= \frac{\partial^2 w}{\partial x^2})$$

$$\frac{\partial z_2}{\partial y} = u_2 \qquad\qquad (= \frac{\partial^2 w}{\partial x \partial y})$$

$$\frac{\partial z_3}{\partial x} = u_2 \qquad\qquad (= \frac{\partial^2 w}{\partial x \partial y})$$

$$\frac{\partial z_3}{\partial y} = u_3 \qquad\qquad (= \frac{\partial^2 w}{\partial y^2})$$

(continued)

$$\frac{\partial h}{\partial x} = v_1$$

$$\frac{\partial h}{\partial y} = v_2$$

$$u_3 = -u_1 - k^2 z_1 - \frac{v_1 z_2 + v_2 z_3}{h}$$

where the role of the control $\underset{\sim}{u}$ is actually played by u_1, u_3, v_1, v_2, u_2 bringing no contribution to the system itself as can be seen from the constraint. Because of its special nature, the given equation is also equivalent to the so.... vhat simpler system:

$$w = z_1$$

$$\frac{\partial z_1}{\partial x} = \frac{z_2}{h}$$

$$\frac{\partial z_1}{\partial y} = \frac{z_3}{h}$$

$$\frac{\partial z_2}{\partial x} = u_1$$

$$\frac{\partial z_2}{\partial y} = u_2$$

$$\frac{\partial z_3}{\partial x} = u_3$$

$$\frac{\partial z_3}{\partial y} = -u_1 - k^2 h z_1$$

where the role of the control is now being played by h, u_1, u_2, u_3, the definition of the $\underset{\sim}{u}$ being different than from above.

We will show in Appendix 1 that, although the two systems and the two controls are not the same, the same final result is obtained for the same optimal problem they both describe.

Let us now go back to the statement of the optimal problem. We now have to describe the boundary conditions.

We assume that the first $m \leq n$ functions z_i are prescribed along portions Σ of ∂D:

$$z_i \Big|_{\Sigma} = z_i(\sigma) \qquad i = 1, 2, \ldots m \qquad (1.2)$$

The simplest Mayer–Bolza problem is now formulated as follows: in a suitable class of functions, to which we will come back later, determine the state variables $\underset{\sim}{z}$ and controls $\underset{\sim}{u}$ so as to minimize the functional:

$$J = \int_{\partial D} \ell(\underset{\sim}{z};\sigma)d\sigma + \iint_D L(\underset{\sim}{z};\underset{\sim}{u};x,y)dxdy \qquad (1.3)$$

subject to side conditions (1.1) and (1.2).

The problem may be complicated by considering boundary controls on movable boundaries and/or constraints on the state or control variables. The case of boundary controls and that of constraints on the state variables will not be investigated in the present study; the former can be found, treated in some details, in Refs. 24 and 25. However, we will investigate the case of constraints (finite equalities or inequalities) on the control variables in section 3 of this chapter.

2. The Necessary Conditions

Following the methods of optimal control theory in one single variable, we adjoin the system of partial differential equations (1.1) to J with the help of vector multiplier functions $\underset{\sim}{\lambda}(x,y)$ and $\underset{\sim}{\mu}(x,y)$.

$$J = \int_{\partial D} \ell(\underset{\sim}{z};\sigma)d\sigma + \iint_D \left\{ L(\underset{\sim}{z},\underset{\sim}{u};x,y) + \underset{\sim}{\lambda}^T(x,y)[\underset{\sim}{X}(\underset{\sim}{z},\underset{\sim}{u};x,y) - \frac{\partial \underset{\sim}{z}}{\partial x}] + \right.$$

(continued)

15

$$+\underset{\sim}{\mu}^T(x,y)[\underset{\sim}{Y}(\underset{\sim}{z},\underset{\sim}{u};x,y) - \frac{\partial \underset{\sim}{z}}{\partial y}]\Big\} dxdy \qquad (2.1)$$

We define a scalar function, the Hamiltonian, as

$$H(\underset{\sim}{z},\underset{\sim}{u};x,y) = L(\underset{\sim}{z},\underset{\sim}{u};x,y) + \underset{\sim}{\lambda}^T(x,y)\underset{\sim}{X}(\underset{\sim}{z},\underset{\sim}{u};x,y) + \underset{\sim}{\mu}^T(x,y)\underset{\sim}{Y}(\underset{\sim}{z},\underset{\sim}{u};x,y) \qquad (2.2)$$

J is rewritten then as:

$$J = \int_{\partial D} \ell(\underset{\sim}{z};\sigma)d\sigma + \iint_D \Big\{ H(\underset{\sim}{z},\underset{\sim}{u};x,y) - \underset{\sim}{\lambda}^T(x,y)\frac{\partial \underset{\sim}{z}}{\partial x} - \underset{\sim}{\mu}^T(x,y)\frac{\partial \underset{\sim}{z}}{\partial y} \Big\} dxdy \qquad (2.3)$$

If we add and subtract the quantity:

$$\left(\frac{\partial \underset{\sim}{\lambda}^T}{\partial x} + \frac{\partial \underset{\sim}{\mu}^T}{\partial y} \right) \underset{\sim}{z}$$

to the second integrand, J takes the form:

$$J = \int_{\partial D} \ell(\underset{\sim}{z};\sigma)d\sigma + \iint_D \Big\{ H(\underset{\sim}{z},\underset{\sim}{u};x,y) + \left(\frac{\partial \underset{\sim}{\lambda}^T}{\partial x} + \frac{\partial \underset{\sim}{\mu}^T}{\partial y} \right) \underset{\sim}{z} \Big\} dxdy$$

$$- \iint_D \Big\{ \frac{\partial}{\partial x}(\underset{\sim}{\lambda}^T \underset{\sim}{z}) + \frac{\partial}{\partial y}(\underset{\sim}{\mu}^T \underset{\sim}{z}) \Big\} dxdy$$

Now, by the use of Green's formula (*), the third integral becomes a line integral as follows:

$$\iint_D \Big\{ \frac{\partial}{\partial x}(\underset{\sim}{\lambda}^T \underset{\sim}{z}) + \frac{\partial}{\partial y}(\underset{\sim}{\mu}^T \underset{\sim}{z}) \Big\} dxdy = \int_{\partial D} (\underset{\sim}{\mu}^T dx - \underset{\sim}{\lambda}^T dy)\underset{\sim}{z} \qquad (2.3)$$

Then the integral (2.3) takes the form:

(*) which is the counterpart in two dimensions of the classical integration by parts in one dimension.

16

$$\int_{\partial D} [\mu^T \alpha'(\sigma) - \lambda^T \beta'(\sigma)]z\, d\sigma \tag{2.4}$$

and J becomes:

$$J = \int_{\partial D} \left\{ \ell(z;\sigma) - [\lambda^T \beta'(\sigma) - \mu^T \alpha'(\sigma)]z \right\} d\sigma$$

$$+ \iint_D \left\{ H(z,u;x,y) + \left[\frac{\partial \lambda^T}{\partial x} + \frac{\partial \mu^T}{\partial y} \right] z \right\} dxdy \tag{2.5}$$

Now consider the variation in J due to variations in the control vector $u(x,y)$:

$$\delta J = \int_{\partial D} \left[\frac{\partial \ell}{\partial z} + \lambda^T \beta'(\sigma) - \mu^T \alpha'(\sigma) \right] \delta z\, d\sigma$$

$$+ \iint_D \left\{ \left(\frac{\partial H}{\partial z} + \frac{\partial \lambda^T}{\partial x} + \frac{\partial \mu^T}{\partial y} \right) \delta z + \frac{\partial H}{\partial u} \delta u \right\} dxdy \tag{2.6}$$

We now choose the multiplier functions $\lambda(x,y)$ and $\mu(x,y)$ to cause the coefficients of δz in the surface integral above to vanish, therefore escaping to the task of computing the δz in terms of the variations δu of the controls:

$$\frac{\partial \lambda^T}{\partial x} + \frac{\partial \mu^T}{\partial y} = - \frac{\partial H}{\partial z} \tag{2.7}$$

Along the portions of ∂D where the z_i are prescribed,

$$\delta z_i = 0, \qquad i = 1,2,\ldots m.$$

Along the other portions and for the z_j, $j = m+1,\ldots,n$ which are not at all prescribed, we ask the relations

$$\underset{\sim}{\lambda}^T \beta'(\sigma) - \underset{\sim}{\mu}^T \alpha'(\sigma) = -\frac{\partial \ell}{\partial \underset{\sim}{z}} \tag{2.7a}$$

to hold along ∂D: they serve as boundary conditions for the system (2.7). (2.6) then becomes:

$$\delta J = \iint\limits_{D} \frac{\partial H}{\partial \underset{\sim}{u}} \, \delta \underset{\sim}{u} \, dxdy$$

For an extremum, δJ must be zero for arbitrary $\delta \underset{\sim}{u}(x, y)$; this can only happen if:

$$\frac{\partial H}{\partial \underset{\sim}{u}} = 0 \tag{2.8}$$

Equations (2.8) are analogous to the control equations derived in one-dimensional optimal control theory.

Equations (2.7), (2.7a) and (2.8) are the Euler–Lagrange equations of the classical calculus of variations, for two independent variables. They form a set of necessary conditions for an optimum.

In summary, to find a control vector function $\underset{\sim}{u}(x, y)$ that produces a stationary value of the performance index:

$$J = \int\limits_{\partial D} \ell(\underset{\sim}{z}; \sigma) d\sigma + \iint\limits_{D} L(\underset{\sim}{z}, \underset{\sim}{u}; x, y) dxdy \tag{1.3}$$

we must solve the following system of partial differential equations in D:

$$\frac{\partial \underset{\sim}{z}}{\partial x} = \underset{\sim}{X}(\underset{\sim}{z}, \underset{\sim}{u}; x, y)$$

$$\frac{\partial \underset{\sim}{z}}{\partial y} = \underset{\sim}{Y}(\underset{\sim}{z}, \underset{\sim}{u}; x, y) \tag{1.1}$$

$$\frac{\partial \underset{\sim}{\lambda}}{\partial x} + \frac{\partial \underset{\sim}{\mu}}{\partial y} = -\left(\frac{\partial H}{\partial \underset{\sim}{z}}\right)^T \tag{2.7}$$

18

$$\frac{\partial H}{\partial \underset{\sim}{u}} = 0 \tag{2.8}$$

where the Hamiltonian H is defined as:

$$H(\underset{\sim}{z}, \underset{\sim}{u}; x, y) = L(\underset{\sim}{z}, \underset{\sim}{u}; x, y) + \underset{\sim}{\lambda}^T(x, y)\underset{\sim}{X} + \underset{\sim}{\mu}^T(x, y)\underset{\sim}{Y} \tag{2.2}$$

The boundary conditions are:

$$z_i\Big|_{\Sigma} = z_i(\sigma) \qquad\qquad i = 1, 2, \ldots, m \tag{1.2}$$

(the Σ are the portions of ∂D where the first m variables z_i are prescribed)

$$\lambda_i \beta'(\sigma) - \mu_i \alpha'(\sigma) = -\frac{\partial \ell}{\partial z_i}, \qquad\qquad i = 1, 2, \ldots, m \tag{2.7a}$$

along $\partial D - \Sigma$, and:

$$\lambda_j \beta'(\sigma) - \mu_j \alpha'(\sigma) = -\frac{\partial \ell}{\partial z_j}, \qquad\qquad j = m+1, \ldots, n \tag{2.7a}$$

along ∂D.

We have n+n+n+p = 3n+p equations for the 3n+p unknowns $\underset{\sim}{z}, \underset{\sim}{\lambda}, \underset{\sim}{\mu}$, and $\underset{\sim}{u}$.

3.   Equality and Inequality Constraints on the Control Variables.

For most of the structural optimization problems that we will consider, the role of the control variable will be played by the thickness distribution which we wish to render optimal. For obvious physical reasons, we do not want, in most cases, for this thickness to be zero along a line interior to the structure (leading to the formation of a hinge, subsequently causing the collapse of the structure). Such an inconvenience might be taken care of by the adjunction to equations (1.1) and (1.2) of a minimum thickness constraint, which will appear as an inequality constraint on the control variables.

More generally, suppose we have restrictions imposed on the control functions in the form of r finite equalities:

$$C_k(\underset{\sim}{u};x,y) = 0, \qquad\qquad k = 1,2,\ldots r \qquad\qquad (3.1)$$

and s finite inequalities:

$$D_\ell(\underset{\sim}{u};x,y) \geq 0, \qquad\qquad \ell = 1,2,\ldots,s \qquad\qquad (3.2)$$

We assume that there exists a solution to equations (1.1), (1.2) under the restrictions (3.1), (3.2) imposed on the control functions.

A way to transform the inequality constraints (3.2) into equality constraints is to introduce supplementary artificial controls $\underset{\sim}{u}^* = (u_1^*,u_2^*,\ldots,u_s^*)$, following a traditional technique, by virtue of the following equations:

$$D_\ell^* = D_\ell(\underset{\sim}{u};x,y) - u_\ell^{*2} = 0, \qquad \ell = 1,2,\ldots,s \qquad\qquad (3.3)$$

and to add these controls to the previous set of $\underset{\sim}{u}$, therefore considering only r + s equality constraints with an augmented number of controls.

This point of view is the one adopted in Lur'e's paper[24]. We will present here another method, again directly inspired from the methods of optimal control theory, in particular of Ref. 27, Chapter 3.

Consider the case of equality constraints first. Clearly, the dimension of the control vector $\underset{\sim}{u}$ must be greater or equal than 2 for the problem to be interesting, since for m = 1 the constraint (3.1) determines u(x,y) completely and there is no optimization problem left.

We adjoin (3.1) to the variational-Hamiltonian with a set of Lagrange multipliers $\underset{\sim}{v}(x,y) = (v_1(x,y),\ldots,v_r(x,y))$ as follows:

$$H = L(\underset{\sim}{z},\underset{\sim}{u};x,y) + \underset{\sim}{\lambda}^T(x,y)\underset{\sim}{X}(\underset{\sim}{z},\underset{\sim}{u};x,y) + \underset{\sim}{\mu}^T(x,y)\underset{\sim}{Y}(\underset{\sim}{z},\underset{\sim}{u};x,y)$$

$$+ \underset{\sim}{v}^T(x,y)\underset{\sim}{C}(\underset{\sim}{u};x,y) \qquad\qquad (3.4)$$

The only change this brings about is in the optimality conditions:

$$\frac{\partial H}{\partial \underset{\sim}{u}} = \frac{\partial L}{\partial \underset{\sim}{u}} + \underset{\sim}{\lambda}^T \frac{\partial \underset{\sim}{X}}{\partial \underset{\sim}{u}} + \underset{\sim}{\mu}^T \frac{\partial \underset{\sim}{Y}}{\partial \underset{\sim}{u}} + \underset{\sim}{\nu}^T \frac{\partial \underset{\sim}{C}}{\partial \underset{\sim}{u}} \qquad (3.5)$$

which, together with (3.1), represent $p + r$ conditions to determine the p-component vector $\underset{\sim}{u}(x,y)$ and the r-component vector $\underset{\sim}{v}(x,y)$.

Now consider the inequality constraints (3.2). In the same way as above, we introduce the s-dimensional vector $\underset{\sim}{\xi}(x,y)$ the components of which are the s Lagrange multipliers $\xi_1(x,y), \ldots, \xi_s(x,y)$, and define

$$H = L + \underset{\sim}{\lambda}^T \underset{\sim}{X} + \underset{\sim}{\mu}^T \underset{\sim}{Y} + \underset{\sim}{\xi}^T \underset{\sim}{D} \qquad (3.6)$$

The necessary condition to be satisfied by this augmented Hamiltonian is:

$$\frac{\partial H}{\partial \underset{\sim}{u}} = \frac{\partial L}{\partial \underset{\sim}{u}} + \underset{\sim}{\lambda}^T \frac{\partial \underset{\sim}{X}}{\partial \underset{\sim}{u}} + \underset{\sim}{\mu}^T \frac{\partial \underset{\sim}{Y}}{\partial \underset{\sim}{u}} + \underset{\sim}{\xi}^T \frac{\partial \underset{\sim}{D}}{\partial \underset{\sim}{u}} = 0 \qquad (3.7)$$

which is the same as above with the additional requirement that:

$$\xi_\ell \le 0, \qquad D_\ell = 0$$

or:

$$\xi_\ell = 0, \qquad D_\ell > 0 \qquad \ell = 1, 2, \ldots, s \qquad (3.8)$$

Another approach is the following, again in the case of s inequality constraints:

$$D_\ell(u;x,y) \ge 0 \qquad \ell = 1, 2, \ldots, s \qquad (3.2)$$

If we define H^* as:

$$H^* = L + \underset{\sim}{\lambda}^T \underset{\sim}{X} + \underset{\sim}{\mu}^T \underset{\sim}{Y}$$

(which is nothing but the Hamiltonian defined for the case when there are no constraints on the control variables), then, from (2.6):

$$\delta J = \iint_D \frac{\partial H^*}{\partial \underset{\sim}{u}} \delta \underset{\sim}{u} \, dxdy \qquad (3.9)$$

21

and this is, by definition,

$$\delta J = \iint_D \delta H^*(\underset{\sim}{z}, \underset{\sim}{u}, \underset{\sim}{\lambda}, \underset{\sim}{\mu}; x, y) dx dy$$

where the $\underset{\sim}{\lambda}$ and the $\underset{\sim}{\mu}$ satisfy the set of partial differential equations

$$\frac{\partial \underset{\sim}{\lambda}^T}{\partial x} + \frac{\partial \underset{\sim}{\mu}^T}{\partial y} = - \frac{\partial H}{\partial \underset{\sim}{z}} \tag{2.7}$$

with boundary conditions:

$$\underset{\sim}{\lambda}^T \beta'(\sigma) - \underset{\sim}{\mu}^T \alpha'(\sigma) = - \frac{\partial \underset{\sim}{\ell}}{\partial \underset{\sim}{z}} \tag{2.7a}$$

holding along the portions of ∂D where $\underset{\sim}{z}$ is not defined.

For the control $\underset{\sim}{u}(x, y)$ to be minimizing, we must have:

$$\delta J \geq 0$$

for all admissible $\delta \underset{\sim}{u}(x, y)$. This implies that we have:

$$\delta H^* \geq 0$$

for all admissible $\delta \underset{\sim}{u}(x, y)$ and this all over the domain D.

Hence, at all points on $\underset{\sim}{D} = 0$ the optimal $\underset{\sim}{u}$ has the property that:

$$\delta H^* = \frac{\partial H^*}{\partial \underset{\sim}{u}} \delta \underset{\sim}{u} \geq 0 \tag{3.10}$$

for all $\delta \underset{\sim}{u}$ such that:

$$\delta D_{\ell} = \frac{\partial D_{\ell}}{\partial \underset{\sim}{u}} \delta \underset{\sim}{u} \geq 0, \qquad \ell = 1, 2, \ldots, s$$

which yields (3.7).

Another way of stating (3.10) is to say that δH^* must be non-improving over the set of all possible $\delta \underset{\sim}{u}$.

Actually a much stronger statement, "H* must be minimized over the set of all possible $\underset{\sim}{u}$" holds. This compact statement is the extension to two-dimensional systems of the famous "Minimum (or Maximum, the distinction being due to a different convention in Russian literature and elsewhere to write the Hamiltonian H*) Principle"[28]. A rigorous proof of the above can be found in a paper by Butkovskiy[29], in one by Egorov[30], and in the already referred to paper by Lur'e[25], which presents an extension of the Weierstrass conditions of the classical calculus of variations to this type of problems and is nothing but an alternate way of stating the Minimum Principle.

A remarkable approach using the function-space formulation is found in a recent paper by Neustadt[31], who presents in the general non-linear case an abstract Minimum Principle covering almost all constrained minimization problems ever considered, including lumped and distributed variational problems under a variety of conditions.

4.   Discontinuities in the Distributed Controls

In the definition of the general problem given in section 1, we spoke of a "suitable class of functions". It is now appropriate to explain this notion, and to specify in detail the class of admissible controls and possible behavior of state variables.

The distributed controls will be assumed to belong to a class of functions no less wide than the class of piecewise continuous functions in two independent variables. Possible discontinuities of distributed controls may occur along smooth, closed isolated curves Δ lying entirely inside the domain D.

The state variables z_i are assumed continuous across a curve such that Δ; the variables u_j are in general discontinuous, but their values on both sides of Δ are connected by the requirement that the tangential derivatives $\partial z_i/\partial \sigma$ along Δ be continuous, that is, if we denote by a subscript $-$ the values of the variables attained on the curve Δ from one definite side of it, by $+$ their values attained from the other side,

$$[X_i \gamma'(\sigma) + Y_i \delta'(\sigma)]_- = [X_i \gamma'(\sigma) + Y_i \delta'(\sigma)]_+, \quad i = 1, 2, \ldots, n \qquad (4.1)$$

Recall that the curve Δ has the parametric representation:

$$x = \gamma(\sigma)$$

$$y = \delta(\sigma)$$

γ and δ being two piecewise continuous and differentiable functions of σ.

It can also be shown[25] that along such a line of discontinuity of distributed controls Δ:

$$[\lambda_i \delta'(\sigma) - \mu_i \gamma'(\sigma)]_- = [\lambda_i \delta'(\sigma) - \mu_i \gamma'(\sigma)]_+, \quad i = 1, 2, \ldots, n \qquad (4.2)$$

$$[H - \lambda^T X_- - \mu^T Y_-]_- = [H - \lambda^T X_- - \mu^T Y_-]_+ \qquad (4.3)$$

5.   <u>Remarks and Conclusion</u>

The necessary conditions have thus been established in a very general case of distributed-parameter systems. We are now confronted with a set of partial differential equations to be solved with adequate boundary conditions, which represent the first-order necessary conditions for an extremum.

The original optimization problem is therefore reduced to that of the determination of solutions for this resulting set of partial differential equations. This latter task unfortunately proves to be an extremely difficult one when we try to solve this system for some specific cases. The reason is clear: even for the simplest of all problems, the system formed by the necessary conditions will be greatly complicated. We start from one single partial differential equation as a constraint. This equation has to be broken, as shown in section 1, into a set of first-order equations, having now n variables z_i and p controls u_j. After writing down the necessary conditions, we are left with a system of $3n+p$ partial differential equations in $3n+p$ unknowns. It is therefore understandable that only in very special cases can we find a rigorous analytical solution to a given optimization problem. One such problem of structural interest is presented

and solved in the next part of this study. The approach to the more complex, but at the same time, more realistic problems presented in parts C and D seems to have to be numerical at this stage of their solution.

PART B

MINIMUM-MASS DESIGN OF A SIMPLY-SUPPORTED SHEAR PLATE

FOR A FIXED NATURAL FREQUENCY

A plate-like structure with special properties will be described and analysed.

Given the shape of a homogeneous "shear plate" with constant thickness, we find its fundamental frequency of vibration. The problem considered is then to determine the thickness to be distributed within the same boundary so as to keep the above fundamental frequency constant while achieving the minimum possible mass. Only for simply supported edges will a complete solution be given.

Stating the problem in mathematical form, we are confronted with a situation of the kind investigated in part A, for which an exact analytical solution will be found in the case of a rectangular or a circular boundary.
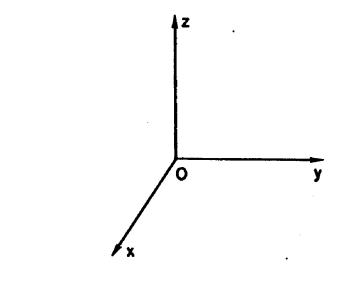
1. The Shear Plate: Definition and Analysis

We define a shear plate as a rather artificial plate-like structure such that the bending rigidity for normal loads is negligible: the running load is thus borne by shear only.

We refer the plate to a set of rectangular Cartesian axes Ox, Oy in the middle plane of the plate; the plate occupies a simply connected domain D with a smooth boundary ∂D.

Now let us consider a rectangular element with sides parallel to the coordinate axes and with lengths dx and dy respectively, around a point P(x,y). Let h(x,y) be the thickness of the plate, $\rho(x,y)$ its density, both at P, and let w(x,y;t) be the normal deflection of P at time t. We choose the axis Oz directly perpendicular to Ox and Oy.

The forces acting on the above element (see figure 1) are all parallel to the z-axis. They are, in projection on Oz:
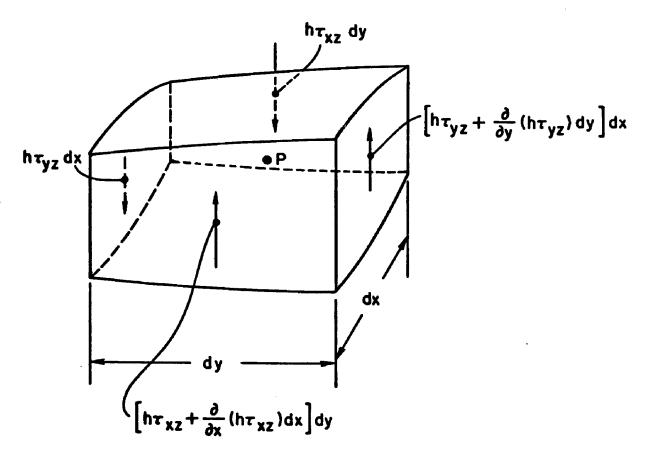
Figure 1. Shear forces acting on the sides of an infinitesimal element of plate.

a) the shear forces:

$$-\tau_{xz} h \, dy \quad \text{and} \quad +\left(\tau_{xz} h + \frac{\partial(\tau_{xz} h)}{\partial x} \, dx\right) dy$$

on the two sides perpendicular to Ox.

$$-\tau_{yz} h \, dx \quad \text{and} \quad +\left(\tau_{yz} h + \frac{\partial(\tau_{yz} h)}{\partial y} \, dy\right) dx$$

on the two sides perpendicular to Oy.

b) the d'Alembert force:

$$-\rho h \frac{\partial^2 w}{\partial t^2} \, dx \, dy$$

From equilibrium,

$$\frac{\partial(h\tau_{xz})}{\partial x} + \frac{\partial(h\tau_{yz})}{\partial y} - \rho h \frac{\partial^2 w}{\partial t^2} = 0$$

But the classical stress-strain relations read

$$\tau_{xz} = G\gamma_{xz} = G \frac{\partial w}{\partial x}$$

$$\tau_{yz} = G\gamma_{yz} = G \frac{\partial w}{\partial y}$$

so that the differential equation of motion takes the form:

$$\frac{\partial}{\partial x}\left(h \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(h \frac{\partial w}{\partial y}\right) - \frac{\rho}{G} h \frac{\partial^2 w}{\partial t^2} = 0 \qquad (1.1)$$

This equation must be satisfied inside the two-dimensional domain D bounded by the frontier ∂D. Let us now examine the boundary conditions.

If the plate is supported at points (x_1, y_1) on segments of the boundary ∂D, we have the boundary condition:

$$w = 0 \quad \text{at} \quad (x_1, y_1) \tag{1.2a}$$

If, on the other hand, the plate is free to deflect transversely at some other points (x_2, y_2) on different segments of the boundary, there cannot be any shear component in the transverse direction. Consequently the boundary condition at such points is:

$$G \frac{\partial w}{\partial n} = 0 \qquad \text{at} \quad (x_2, y_2)$$

or

$$\frac{\partial w}{\partial n} = 0 \qquad \text{at} \quad (x_2, y_2) \tag{1.2b}$$

The boundary curve ∂D contains all points such as (x_1, y_1) and (x_2, y_2). and only one of the boundary conditions (1.2a) or (1.2b) must be satisfied at a given point on the boundary.

We will confine ourselves to the case of a homogeneous plate ($\rho = $ constant). Let us investigate the nature of the free vibrations of the shear plate with constant thickness.

With $h = $ constant, the equation of motion (1.1) becomes:

$$G \nabla^2 w = \rho \frac{\partial^2 w}{\partial t^2} \tag{1.3}$$

which is in the same form that the equation of the free vibration of a membrane where G plays the role of the uniform tension T of the membrane.

Following the classical procedure[32], we write the displacement w as the product of a function W of the spatial coordinates only and a function f depending solely on time:

$$w(x, y; t) = W(x, y) \, f(t)$$

(1.3) becomes:

$$\frac{G}{\rho} \frac{\nabla^2 W}{W} = \frac{\ddot{f}}{f}$$

where the superscript (˙) denotes a derivative with respect to time, and the common value of these two ratios has to be a constant, $-\omega^2$, negative for reasons of stability. f is then harmonic with frequency ω, and W satisfies the partial differential equation:

$$-G\nabla^2 W = \omega^2 \rho W \qquad (1.4)$$

inside the domain D, together with the boundary conditions:

$$W = 0 \qquad \text{at} \qquad (x_1, y_1) \qquad (1.5a)$$

$$\frac{\partial W}{\partial n} = 0 \qquad \text{at} \qquad (x_2, y_2) \qquad (1.5b)$$

The eigenvalue problem represented by equation (1.4) together with the boundary conditions (1.5) is of the classical type:

$$L[W] = \lambda M[W]$$

where the operators L and M are respectively:

$$L = -G\nabla^2$$

$$M = \rho$$

with boundary conditions independent of the eigenvalue λ. The problem is easily shown to be self-adjoint[33]: if W_r and W_s are two eigenfunctions corresponding to two distinct eigenvalues $\lambda_r \neq \lambda_s$, then they satisfy the orthogonality relation:

$$\iint\limits_{D} \rho W_r W_s \, dD = 0, \qquad r \neq s.$$

Also, it can be shown, as in the classical membrane case, that if the boundary condition (1.5a) is satisfied over part of the curve ∂D, the system is positive definite, and that if the boundary condition (1.5b) is satisfied over the entire length of ∂D, the system is only positive.

To pursue the discussion further, we shall confine ourselves to an investigation of the properties of rectangular and circular plates only.

1.1 Vibrations of Homogeneous Rectangular Shear Plate with Constant Thickness

The plate extends over a domain D defined by $0 \le x \le a$, $0 \le y \le b$. The boundary ∂D is made of the straight lines $x = 0, a$ and $y = 0, b$. Let:

$$\frac{\omega^2 \rho}{G} = \alpha^2$$

and write (1.4) as:

$$\nabla^2 w + \alpha^2 w = 0 \qquad\qquad (1.6)$$

where the expression for the Laplacian in rectangular coordinates is:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

(1.6) is solved by separation of variables; to this end, we let the solution have the form:

$$W(x, y) = X(x)\, Y(y)$$

and, after substitution in (1.6) and rearrangement, obtain:

$$\frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} + \frac{1}{Y(y)} \frac{d^2 Y(y)}{dy^2} + \alpha^2 = 0$$

which leads to the equations:

$$\frac{d^2 X(x)}{dx^2} + \beta^2 X(x) = 0$$

$$\frac{d^2 Y(y)}{dy^2} + \gamma^2 Y(y) = 0$$

where

$$\beta^2 + \gamma^2 = \alpha^2$$

31

so that, introducing four constants of integration A_1, A_2, A_3, A_4, the solution W has the form:

$$W(x,y) = A_1 \sin \beta x \sin \gamma y + A_2 \sin \beta x \cos \gamma y + A_3 \cos \beta x \sin \gamma y$$

$$+ A_4 \cos \beta x \cos \gamma y \qquad (1.7)$$

where A_1, A_2, A_3, A_4 as well as β and γ must be determined by means of the boundary conditions.

a) <u>Plate Simply-Supported Along the Edges</u>

The boundary conditions then read:

$$W(0,y) = W(a,y) = W(x,0) = W(x,b) = 0 \qquad (1.8)$$

The first and third one of the above conditions lead to:

$$A_2 = A_3 = A_4 = 0$$

whereas the second and the fourth imply:

$$\sin \beta a = 0$$

$$\sin \gamma b = 0$$

and yield an infinite sequence of discrete roots:

$$\beta_m = \frac{m\pi}{a}, \qquad m = 1, 2, \ldots$$

$$\gamma_n = \frac{n\pi}{b}, \qquad n = 1, 2, \ldots$$

so that we are led to the eigenvalues solution of the problem:

$$\alpha_{mn} = \sqrt{\beta_m^2 + \gamma_n^2} = \pi \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \qquad m, n = 1, 2, \ldots \qquad (1.9)$$

The corresponding modes are:

$$W_{mn} = A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{a} \qquad (1.10)$$

The fundamental frequency of vibration is obtained for $m = n = 1$, and has the value:

$$\omega_{11} = \alpha_{11} \sqrt{\frac{G}{\rho}} = \pi \sqrt{\frac{G}{\rho}(\frac{1}{a^2} + \frac{1}{b^2})} \qquad (1.11)$$

b) <u>Plate Simply-Supported Along Two Parallel Edges and Free Along the Other Two</u>

The boundary conditions now read, assuming the plate to be simply-supported along the parallel edges $x = 0$ and $x = a$:

$$W(0,y) = W(a,y) = 0$$

$$\frac{\partial W}{\partial y}(x,0) = \frac{\partial W}{\partial y}(x,b) = 0 \qquad (1.12)$$

The integration constants in (1.7) are determined then from the first boundary condition:

$$A_3 = A_4 = 0$$

From the third one:

$$A_1 = 0$$

and from the two remaining ones:

$$\sin \beta a = 0$$

$$\sin \gamma b = 0$$

β and γ have to take the discrete values:

$$\beta_m = \frac{m\pi}{a} \qquad m = 1, 2, \ldots$$

$$\gamma_n = \frac{n\pi}{b} \qquad n = 1, 2, \ldots$$

33

The frequencies of vibration are the same as in case a) above. The modes are now being given by:

$$W_{mn} = B_{mn} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} , \qquad\qquad m,n = 1,2,\ldots \qquad (1.13)$$

c) Free-Free Plate

For this idealized case (realizable in practice by assuming the plate to be simply-supported at its center), the boundary conditions read:

$$\frac{\partial W}{\partial y}(x,0) = \frac{\partial W}{\partial y}(x,b) = 0$$

$$\frac{\partial W}{\partial x}(0,y) = \frac{\partial W}{\partial x}(a,y) = 0 \qquad\qquad (1.14)$$

The natural frequencies of vibration are the same as in case a) or b). The natural modes are now given by:

$$W_{mn} = C_{mn} \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} , \qquad\qquad m,n = 1,2,\ldots \qquad (1.15)$$

1.2 Vibrations of a Homogeneous Circular Shear Plate with Constant Thickness

The circular plate extends over a domain D defined by $0 \le r \le a$; the boundary of the domain is the circle ∂D described in polar coordinates by the equation $r = a$.

With polar coordinates r and θ, the differential equation to be satisfied throughout D is:

$$\nabla^2 W(r,\theta) + \alpha^2 W(r,\theta) = 0 \qquad\qquad (1.16)$$

where

$$\alpha^2 = \frac{\omega^2 \rho}{G}$$

Assuming a solution of the form:

$$W(r, \theta) = R(r) \, \Theta(\theta)$$

and recalling that in polar coordinates the Laplacian is given by:

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} .$$

(1.16) reduces to:

$$\left(\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} \right) \Theta + \frac{R}{r^2} \frac{d^2 \Theta}{d\theta^2} + \alpha^2 R \Theta = 0$$

which can be separated into two equations:

$$\frac{d^2 \Theta}{d\theta^2} + m^2 \Theta = 0 \qquad\qquad\qquad (1.17)$$

$$\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} + \left(\alpha^2 - \frac{m^2}{r^2} \right) R = 0 \qquad\qquad (1.18)$$

The constant m^2 is chosen to obtain a harmonic equation for Θ ; further-more, the solution must be periodic in θ with the period 2π ; m has then to be an integer $(m = 0, 1, 2, \ldots)$.

(1.17) yields as solution:

$$\Theta_m(\theta) = C_{1m} \sin m\theta + C_{2m} \cos m\theta, \qquad\qquad m = 0, 1, 2, \ldots$$

(1.18) is a Bessel equation and its solution is:

$$R_m(r) = C_{3m} J_m(\alpha r) + C_{4m} Y_m(\alpha r), \qquad\qquad m = 0, 1, 2, \ldots$$

where $J_m(x)$ and $Y_m(x)$ are the Bessel functions of order m of the first and second kind, respectively.

The general solution of (1.16) can then be written:

$$W_m(r, \theta) = A_{1m} J_m(\alpha r) \sin m\theta + A_{2m} J_m(\alpha r) \cos m\theta + A_{3m} Y_m(\alpha r) \sin m\theta$$

$$+ A_{4m} Y_m(\alpha r) \cos m\theta , \qquad\qquad m = 0, 1, 2, \ldots \qquad (1.19)$$

a) Simply-Supported Plate

The boundary condition reads:

$$W(a, \theta) = 0$$

At every interior point of the plate the displacement must be finite: but Bessel functions of the second kind tend to infinity as the argument approaches zero. It follows that:

$$A_{3m} = A_{4m} = 0$$

so that (1.19) reduces to:

$$W_m(r, \theta) = A_{1m} J_m(\alpha r) \sin m\theta + A_{2m} J_m(\alpha r) \cos m\theta, \quad m = 0, 1, 2, \ldots$$

At $r = a$ we must have:

$$W_m(a, \theta) = 0, \qquad\qquad m = 0, 1, 2, \ldots$$

regardless of the values of θ, which implies:

$$J_m(\alpha a) = 0, \qquad\qquad m = 0, 1, 2, \ldots$$

This equation represents an infinite set of characteristic or frequency equations, and for each m there is an infinite number of discrete solutions α_{mn}. The fundamental mode of vibration is obtained for $m = 0$; as the first zero of $J_0(x)$ occurs at $x = 2.4048$, the fundamental frequency is given by:

$$\omega_{01} = 2.4048 \sqrt{\frac{G}{\rho a^2}} \tag{1.20}$$

and the corresponding mode by:

$$W_{01} = A_{01} J_0 \left(2.4048 \, \frac{r}{a}\right) \tag{1.21}$$

For the upper frequencies ω_{mn} there are two corresponding modes to one particular frequency: the modes are then degenerate.

b) Plate Free Along the Edge

The boundary condition now reads:

$$\frac{\partial}{\partial r} W(r, \theta) \Big|_{r=a} = 0$$

As above, the displacement at the center has to be finite, so that the solution is in the form:

$$W_m(r, \theta) = A_{1m} J_m(\alpha r) \sin m\theta + A_{2m} J_m(\alpha r) \cos m\theta, \quad m = 0, 1, 2, \ldots$$

The boundary condition thus requires:

$$\frac{d}{dr} J_m(\alpha r) \Big|_{r=a} = 0$$

But, by a property of Bessel functions[34]:

$$\frac{d}{dr} J_0(\alpha r) = -\alpha J_1(\alpha r)$$

$$\frac{d}{dr} J_m(\alpha r) = \alpha [J_{m-1}(\alpha r) - \frac{m}{\alpha r} J_m(\alpha r)], \quad m = 1, 2, \ldots$$

the first formula being only a particular case of the second one if we note that:

$$J_{-m}(x) = (-1)^m J_m(x), \quad m = 0, 1, 2, \ldots$$

Thus we must have:

$$J_{m-1}(\alpha a) - \frac{m}{\alpha a} J_m(\alpha a) = 0$$

This equation represents an infinite set of characteristic equations, and for each m there is an infinite number of discrete solutions α_{mn} corresponding to the zeros of this transcendental equation. The lowest frequency corresponds to $m = 0$ and is obtained from the first zero of $J_1(x)$, occuring at $x = 3.8317$; thus,

$$\omega_{01} = 3.8317 \sqrt{\frac{G}{\rho a^2}} \tag{1.22}$$

and the corresponding mode is:

37

$$W_{01}(r, \theta) = A_{01} J_0(3.8317 \frac{r}{a}) \qquad (1.23)$$

Again, for each frequency ω_{mn}, $m \neq 0$, there are two different modes, and it follows that for $m \neq 0$ the natural modes are degenerate.

After this preliminary study of the vibrations of a shear plate, we now are ready to set up the optimization problem. We note that in the case of a circular boundary the optimal thickness distribution will evidently be rotation-invariant (symmetry about 0 and about any axis through 0), i.e., independent of θ, and the optimization problem will then reduce to the classical case of one independent variable, in this case r, encountered in optimal control theory.

2. Optimization of a Simply-Supported Shear Plate

Let us consider a simply-supported shear plate extending over a domain D bounded by the curve ∂D. We assume that the material constituting the plate is homogeneous with density ρ and that the thickness is a constant h_0; the mass of this reference plate is thus:

$$M_0 = \rho h_0 G$$

where G is the area enclosed inside the boundary ∂D. The fundamental frequency of the plate is ω_f, as found above in the cases of a rectangular plate and of a circular one.

We then want to find the optimal thickness distribution $h(x, y)$ relative to an orthonormal set of coordinates such as to yield a minimum of the surface integral:

$$M = \rho \iint\limits_D h(x, y) dx dy$$

or, equivalently, the surface integral:

$$J = \iint_D h(x,y)dxdy \tag{2.1}$$

the constraint the first frequency of vibration be ω_f being expressed under the form of the partial differential equation:

$$\frac{\partial}{\partial x}(h\frac{\partial w}{\partial x}) + \frac{\partial}{\partial y}(h\frac{\partial w}{\partial y}) + \frac{\rho}{G}\omega_f^2 hw = 0 \tag{2.2}$$

to be satisfied inside D, together with the boundary condition:

$$w(x,y) = 0 \qquad\qquad \text{on } \partial D \tag{2.3}$$

2.1 Necessary Conditions for an Optimal

Following the method derived in Part A, we break the partial differential equation (2.2) into a system of first order partial differential equations, with the help of auxiliary variables z_i and u_j.

Let:

$$z_1 = w$$

$$z_2 = h\frac{\partial w}{\partial x}$$

$$z_3 = h\frac{\partial w}{\partial y}$$

The system (A.1.1) then reads:

$$\frac{\partial z_1}{\partial x} = \frac{z_2}{h}$$

$$\frac{\partial z_1}{\partial y} = \frac{z_3}{h}$$

$$\frac{\partial z_2}{\partial x} = u_1$$

$$\frac{\partial z_2}{\partial y} = u_2$$

(continued)

39

$$\frac{\partial z_3}{\partial x} = u_3$$

$$\frac{\partial z_3}{\partial y} = u_1 - \frac{\rho}{G} \omega_f^2 h z_1 \qquad (2.4)$$

The boundary condition is:

$$z_1 = 0 \qquad\qquad \text{on} \quad \partial D \qquad (2.4a)$$

The Hamiltonian is now formed:

$$H = h + \lambda_1 \frac{z_2}{h} + \lambda_2 u_1 + \lambda_3 u_3 + \mu_1 \frac{z_3}{h} + \mu_2 u_2$$

$$+ \mu_3 \left(-u_1 - \frac{\rho}{G} \omega_f^2 h z_1\right) \qquad (2.5)$$

and the necessary conditions for an extremum of J are:

$$\frac{\partial \lambda_1}{\partial x} + \frac{\partial \mu_1}{\partial y} = \frac{\rho}{G} \omega_f^2 \mu_3 h$$

$$\frac{\partial \lambda_2}{\partial x} + \frac{\partial \mu_2}{\partial y} = -\frac{\lambda_1}{h}$$

$$\frac{\partial \lambda_3}{\partial x} + \frac{\partial \mu_3}{\partial y} = -\frac{\mu_1}{h}$$

$$1 - \frac{\lambda_1 z_2}{h^2} - \frac{\mu_1 z_3}{h^2} - \frac{\rho}{G} \omega_f^2 \mu_3 z_1 = 0$$

$$\lambda_2 - \mu_3 = 0$$

$$\mu_2 = 0$$

$$\lambda_3 = 0 \qquad (2.6)$$

The boundary conditions are:

$$z_1 = 0$$

$$\lambda_2 \beta'(\sigma) - \mu_2 \alpha'(\sigma) = 0$$

$$\lambda_3 \beta'(\sigma) - \mu_3 \alpha'(\sigma) = 0 \qquad (2.6a)$$

on ∂D, uniquely determined by the parametric representation:

$$x = \alpha(\sigma)$$

$$y = \beta(\sigma)$$

and they reduce to, considering (2.6):

$$z_1 = 0$$

$$\lambda_2 = \mu_3 = 0 \qquad (2.6a)$$

on ∂D.

Now the first three equations in the system (2.6) are very similar to the first two and the last of the system (2.4). We are thus led to the assumption:

$$\lambda_1 = \frac{z_2}{\alpha}$$

$$\mu_1 = \frac{z_3}{\alpha}$$

$$\lambda_2 = \mu_3 = -\frac{z_1}{\alpha} \qquad (2.7)$$

where α is any constant.

Note that this assumption is compatible with the boundary conditions

$$z_1 = \lambda_2 = \mu_3 = 0$$

on ∂D.

The control equation reduces to:

41

$$1 - \frac{z_2^2}{\alpha h^2} - \frac{z_3^2}{\alpha h^2} + \frac{\rho}{G}\omega_f^2 \frac{z_1^2}{\alpha} = 0$$

or, expressed in terms of the old variable w:

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = \alpha + \frac{\rho}{G}\omega_f^2 w^2 \qquad\qquad (2.8)$$

which is a nonlinear first order partial differential equation to be satisfied by w, together with the boundary condition:

$$w = 0 \qquad\qquad\text{on}\quad \partial D \qquad\qquad (2.8a)$$

(2.8) is a necessary condition for an optimum; we will show in Part C that it is also sufficient, as it is nothing but the expression in the particular case of a shear plate of an extremely general sufficient condition applicable to a broad class of structural optimization problems. Thus the whole optimization problem reduces to solving this partial differential equation: if a solution to (2.8) can be found that satisfies the boundary condition (2.8a), then the solution of the original problem can be immediately derived from the knowledge of w.

Equation (2.8) shows us[+] that α has to be positive: let

$$\alpha = c^2$$

Also, let:

$$\frac{\rho}{G}\omega_f^2 = k^2,$$

and write (2.8) in parametric form, introducing an unknown function $\theta(x,y)$, as:

$$\frac{\partial w}{\partial x} = \sqrt{c^2 + k^2 w^2} \ \cos\theta(x,y)$$

$$\frac{\partial w}{\partial y} = \sqrt{c^2 + k^2 w^2} \ \sin\theta(x,y)$$

From the fact that the value of the second mixed derivative of w is independent of the order of the derivations:

[+] w attains the value 0 on the boundary, which is part of D.

$$\frac{1}{\sqrt{c^2+k^2w^2}} \, k^2 w \, \frac{\partial w}{\partial y} \, \cos \theta - \sqrt{c^2+k^2w^2} \, \frac{\partial \theta}{\partial y} \, \sin \theta$$

$$\equiv \frac{1}{\sqrt{c^2+k^2w^2}} \, k^2 w \, \frac{\partial w}{\partial x} \, \sin \theta + \sqrt{c^2+k^2w^2} \, \frac{\partial \theta}{\partial x} \, \cos \theta$$

or:

$$\cos \theta \, \frac{\partial \theta}{\partial x} + \sin \theta \, \frac{\partial \theta}{\partial x} = 0$$

which is a linear first-order partial differential equation that has to be satisfied by $\theta(x,y)$.

The characteristic lines are given by the system[35]:

$$\frac{dx}{\cos\theta} = \frac{dy}{\sin\theta} = \frac{d\theta}{0}$$

and are therefore defined by

$$\begin{cases} -x \, \sin\theta + y \, \cos\theta = \text{constant} \\ \theta = \theta_0 \end{cases}$$

The characteristics are therefore straight lines, along which θ keeps a constant value.

On the boundary ∂D, w has the constant value 0; therefore,

$$dw \equiv \frac{\partial w}{\partial x} \, dx + \frac{\partial w}{\partial y} \, dy = 0$$

or

$$\cos\theta \, dx + \sin\theta \, dy = 0$$

which shows that $\theta(x,y)$ has a geometric significance: it is the angle that the normal to the contour ∂D, oriented outwards, makes with a direction parallel to the x axis. The characteristics, which are, as we recall, straight lines along which θ keeps a constant value, are therefore the normals to the contour. At any point P interior to the domain, the value of θ is that of the angle the normal to the contour drawn from that point P makes with the direction Ox (Figure 2).
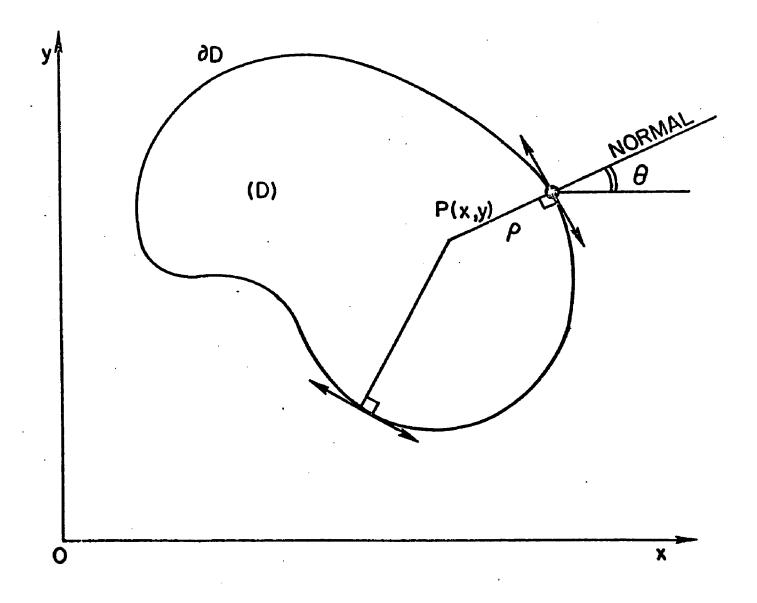
43

Figure 2. Solution of the partial differential equation (2. 8) for a general domain D: definition and interpretation of the different quantities introduced.

44

At P,

$$dw = \sqrt{c^2 + k^2 w^2} \ (\cos\theta \ dx + \sin\theta \ dy)$$

$$= \sqrt{c^2 + k^2 w^2} \ d\rho$$

if ρ is the distance from P to the boundary ∂D, measured along the normal to the boundary pointing outwards.

Thus:

$$d\rho = \frac{dw}{\sqrt{c^2 + k^2 w^2}}$$

or, integrating:

$$\rho - \rho_0 = \frac{L}{k} \log(kw + \sqrt{c^2 + k^2 w^2}) \ ,$$

which is:

$$w = \frac{\epsilon c}{k} \sinh[k(\rho - \rho_0)], \qquad \epsilon = \pm 1$$

For a point on the boundary ($\rho = 0$), then $w = 0$: this shows that the constant ρ_0 has the value zero, and the solution to (2.8) is simply:

$$w = \frac{\epsilon c}{k} \sinh(k\rho) \tag{2.9}$$

where we recall that ρ is the distance from P(coordinates x and y) to the contour ∂D.

For a given contour, there are usually many normals that can be drawn from a point interior to the domain it encloses (figure 2). Thus the question arises to decide which normal to consider to define the distance ρ: it seems logical (and turns out to be the only solution for a rectangular plate) to take for ρ the shortest distance from P to the boundary ∂D.

Another approach to equation (2.8) that reduces it to a form encountered in the theory of plastic torsion of a cylindrical beam is given in Appendix 2.

We now apply the above result, valid for any domain D under the general assumptions outlined at the beginning, to a rectangular, and then to a circular shear plate.

From a point inside a rectangular domain, we may draw four perpendiculars to the boundary. Therefore we have to consider 4 domains labelled (1),(2),(3),(4) as represented in Figure 3, delimited by the boundary, the bissectrices of the 4 right angles at the corners and the segment joining their points of intersection two by two (the latter reducing to a point in the case of a square plate). In each of those domains, the shortest distance of a point to the boundary is readily expressed, and this leads to an analytical expression for w, taking a different form in each domain*.

Let the boundary be formed of the lines $x = 0$, $x = a$, $y = 0$, $y = b$. We may always assume, without any loss of generality, that $a \geq b$. Due to the symmetry of the problem, we need only consider one-fourth of the plate, for instance the portion delimited by the lines $x = 0$, $x = a/2$, $y = 0$, $y = b/2$.

In region (1), w is a function of x only:

$$w = \frac{\epsilon c}{k} \sinh(kx)$$

and the optimal thickness distribution is found from (2.2) which reduces to:

$$\frac{\partial}{\partial x}[h \cosh(kx)] + kh \sinh(kx) = 0$$

The general solution to this partial differential equation of a special type is readily found to be:

$$h = \frac{\overline{\omega}_1(y)}{\cosh^2(kx)}$$

*Note that it is easy to see why the shortest distance to the boundary has to be chosen in this case; another choice for ρ would not allow either the condition $w = 0$ to hold along the boundary ∂D or the displacement w to be continuous inside the domain D.
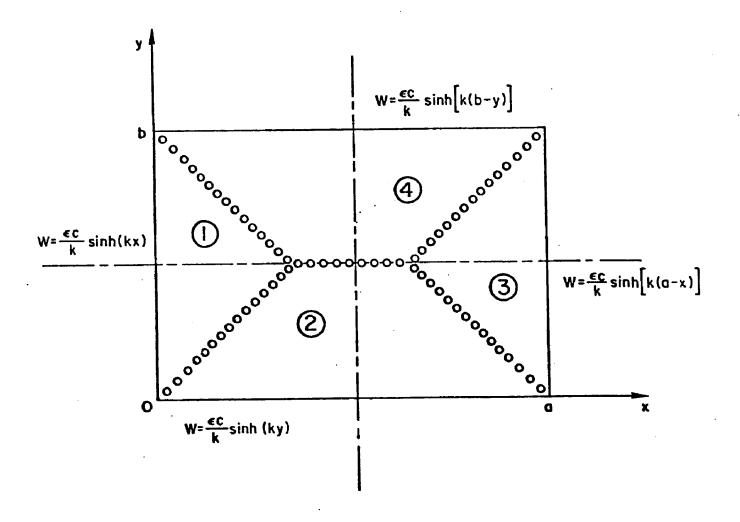
Figure 3. The division of the rectangular plate into four regions, and the corresponding expressions for the optimal displacement w.

where $\bar{\omega}_1(y)$ is an arbitrary function of y.

Similarly, the optimal thickness distribution in (2) is given by:

$$h = \frac{\bar{\omega}_2(x)}{\cosh^2(ky)}$$

Now the shear force has to be continuous when we cross the boundary from region (1) to region (2); this means that the expression:

$$h \frac{\partial w}{\partial n} = h \vec{\nabla}w \cdot \vec{n}$$

has to be continuous across any line with normal $\vec{n}$. In particular, between (1) and (2), we must have:

$$h\left(\frac{\partial w}{\partial x} - \frac{\partial w}{\partial y}\right)\Big|_{(1)} = h\left(\frac{\partial w}{\partial x} - \frac{\partial w}{\partial y}\right)\Big|_{(2)}$$

or:

$$h \frac{\partial w}{\partial x}\Big|_{(1)} = -h \frac{\partial w}{\partial y}\Big|_{(2)}$$

But:

$$\frac{\partial w}{\partial x}\Big|_{(1)} = \frac{\partial w}{\partial y}\Big|_{(2)} = \epsilon c \cosh(kx)$$

as the boundary line is defined by the equation $y = x$.

Thus it is necessary that:

$$h = 0$$

on the segment $y = x$, $0 \leq x \leq b/2$, which implies that

$$\bar{\omega}_1 = \bar{\omega}_2 \equiv 0$$

and

$$h \equiv 0$$

The optimal solution is therefore the trivial one, zero. This result should not surprise us, as we already encountered it in one-dimensional cases[7,13,14]. The trivial solution is a consequence of the constraint equation being homogeneous in h. We also know from past experience[14] that introduction of a minimum thickness constraint will not be of any help.

2.2 The Structural Mass Hypothesis

The way to overcome this difficulty is to assume that the total mass of the plate is made up of two parts with drastically different structural properties: a constant fraction δ_2 is non-structural, whereas the remaining part of the mass, labelled structural, is originally in the proportion δ_1.

The thickness is expressed as:

$$h(x,y) = \delta_1 h^*(x,y) + \delta_2$$

where

$$\delta_1 + \delta_2 = 1$$

h* will often be referred to as the "virtual thickness", by opposition to the actual thickness h.

Note that the non-structural mass is not at the control of the designer, and that we are trying to minimize the total mass by acting on the intermediate variable h*.

The problem is then set up as follows: minimize the total mass

$$M = \rho \iint_D h(x,y)dD = \rho\delta_1 \iint_D h^*(x,y)dD + \rho\delta_2 \Omega$$

or, equivalently, the surface integral:

$$J^* = \iint_D h^*(x,y)dxdy \qquad (2.10)$$

with the constraint:

$$\frac{\partial}{\partial x}(h^* \frac{\partial w}{\partial x}) + \frac{\partial}{\partial y}(h^* \frac{\partial w}{\partial y}) + \frac{\rho}{G} \omega_f^2 (\delta_1 h^* + \delta_2)w = 0$$

$$w = 0 \quad \text{on} \quad \partial D \tag{2.11}$$

The optimization process is similar to the one already performed; under exactly the same assumptions on $\lambda_1, \lambda_2, \mu_1, \mu_3$ we find that the optimal displacement mode w has to satisfy the first-order partial differential equation:

$$(\frac{\partial w}{\partial x})^2 + (\frac{\partial w}{\partial y})^2 = c^2 + \frac{\rho}{G} \delta_1 \omega_f^2 w^2 \tag{2.12}$$

together with the boundary condition:

$$w = 0 \quad \text{on} \quad \partial D$$

which is exactly the same equation as (2.8), with k^2 now having the value

$$\frac{\rho}{G} \delta_1 \omega_f^2.$$

The solution is then, using the same notations as before:

$$w = \frac{\epsilon c}{k} \sinh(k\rho) \tag{2.9}$$

For the rectangular plate, the 4 regions previously defined (figure 3) have to be considered again; the optimal displacement w in each of them is again given by the same expressions, k having the new value

$$\omega_f \sqrt{\frac{\rho}{G}} \delta_1$$

In region (1), from (2.11), the optimal "thickness" h^* has to satisfy the differential equation:

$$\frac{\partial}{\partial x}[h^* \cosh(kx)] + k(h^* + \frac{\delta_2}{\delta_1}) \sinh(kx) = 0 \tag{2.13}$$

or:

$$\frac{\partial h^*}{\partial x} + 2k \tanh(kx)h^* + k \frac{\delta_2}{\delta_1} \tanh(kx) = 0$$

the general solution of which is:

$$h^*(x,y) = -\frac{\delta_2}{2\delta_1} + \frac{\overline{\omega}_1(y)}{\cosh^2(kx)}$$

where $\overline{\omega}_1(y)$ is an arbitrary function of y.

Similarly, in region (2), the optimal "thickness" distribution is given by:

$$h^*(x,y) = -\frac{\delta_2}{2\delta_1} + \frac{\overline{\omega}_2(x)}{\cosh^2(ky)}$$

Now, as before, $h^* \dfrac{\partial w}{\partial n}$ has to be continuous across the line $y = x$, which leads to the condition

$$h^* = 0$$

on the segment $y = x$, $0 \le x \le b/2$.

The expression for h^* in region (3) will be obtained from that in region (1) by changing x into $a-x$; for region (4), from that in (2), y being replaced by $b-y$. The continuity of the shear from (2) to (4) when we cross the segment

$$y = b/2, \qquad b/2 \le x \le a-b/2$$

necessitates:

$$h^* \frac{\partial w}{\partial y} \bigg|_{(2)} = h^* \frac{\partial w}{\partial y} \bigg|_{(4)}$$

or, as:

$$\frac{\partial w}{\partial y} \bigg|_{(2)} = -\frac{\partial w}{\partial y} \bigg|_{(4)} \neq 0$$

on that line,

$$h^* = 0 \quad \text{on} \quad y = b/2, \qquad b/2 \le x \le a-b/2. \tag{2.14}$$

Thus h^* has to be zero on the lines serving to define the 4 regions which are not boundary lines (dotted lines in fig. 3).

Applying this condition, we find the expressions for the undetermined functions $\bar{\omega}_1$ and $\bar{\omega}_2$; due to the condition (2.14), region (2) has to be subdivided into (2) and (2') by the lines $x = b/2$, $x = a-b/2$ as shown in fig. 4. In the same fashion, those lines divide (4) into (4) and (4').

The optimal "thickness" distribution h^* is:

in (1)
$$h^*(x,y) = \frac{\delta_2}{2\delta_1} \left[\frac{\cosh^2(ky)}{\cosh^2(kx)} - 1 \right]$$

in (2)
$$h^*(x,y) = \frac{\delta_2}{2\delta_1} \left[\frac{\cosh^2(kx)}{\cosh^2(ky)} - 1 \right]$$

(2.15)

in (2')
$$h^*(x,y) = \frac{\delta_2}{2\delta_1} \left[\frac{\cosh^2(k\frac{b}{2})}{\cosh^2(ky)} - 1 \right]$$

The expression for h^* in (3) is obtained from that in (1) by changing x into $a-x$; in (4) and (4') respectively by that in (2) and (2') by changing y into $b-y$. We note that h^* is nowhere negative, as it should.

This corresponds to the actual thickness distribution:

in (1)
$$h(x,y) = \frac{\delta_2}{2} \left[1 + \frac{\cosh^2(ky)}{\cosh^2(kx)} \right]$$

in (2)
$$h(x,y) = \frac{\delta_2}{2} \left[1 + \frac{\cosh^2(kx)}{\cosh^2(ky)} \right]$$

(2.16)

in (2')
$$h(x,y) = \frac{\delta_2}{2} \left[1 + \frac{\cosh^2(k\frac{b}{2})}{\cosh^2(ky)} \right]$$

the expressions in regions (3), (4), (4') being derived as above.

The contour lines (lines of constant thickness) corresponding to the case where 40% of the mass is initially structural and thus allowed to vary are
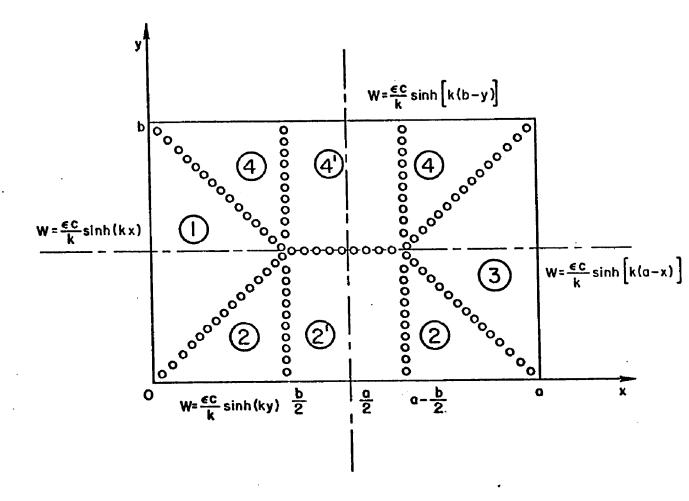
52

Figure 4. Subdivision of the rectangular plate into regions where the optimal thickness has different analytical expressions.

represented in figs. 5 and 6 respectively for a square plate (a/b = 1) and for a rectangular plate for which the ratio of the larger side to the smaller one is equal to 1.5 (a/b = 3/2). We note that the optimal thickness distribution attains its minimum of 0.6 corresponding to h* = 0 along the lines already described in fig. 3, and that a thickness of more than 1.7 times the original one is attained at the four points which are the middle of the sides in the case of the square plate.

The optimal mass of the plate is given by the double integral:

$$M = \rho \iint_D h(x,y)dxdy .$$

We compute it for one-fourth of the plate: due to the symmetry, the total mass will be four times that result.

We obtain, after performing the double integration, the exact expression:

$$M = \rho \delta_2 [\frac{ab}{2} + \frac{2}{k^2} \sinh^2(k\frac{b}{2}) + \frac{a-b}{2k} \sinh(kb)]$$

where:

$$k = \omega_f \sqrt{\frac{\rho}{G} \delta_1} = \pi \sqrt{\delta_1(\frac{1}{a^2} + \frac{1}{b^2})} = \frac{\pi}{ab} \sqrt{\delta_1(a^2 + b^2)}$$

This leads to an optimal mass ratio equal to:

$$m = \delta_2 \left\{ \frac{1}{2} + \frac{2ab}{\pi^2 \delta_1(a^2+b^2)} \sinh^2 \frac{\pi}{2a} \sqrt{\delta_1(a^2+b^2)} \right.$$

$$\left. + \frac{a-b}{2\pi \sqrt{\delta_1(a^2+b^2)}} \sinh \frac{\pi}{a} \sqrt{\delta_1(a^2+b^2)} \right\} \quad (2.17)$$

As could have been expected, this quantity only depends, regarding the geometry, on the ratio a/b = λ of the lengths of the two sides of the rectangular plate, and m is rewritten as:
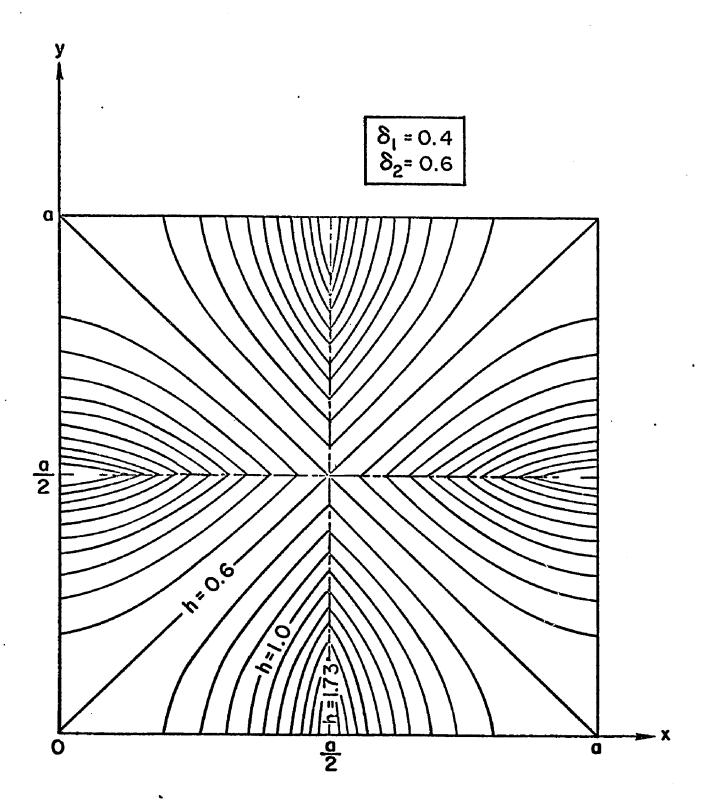
54

Figure 5. Optimal configuration of a square simply-supported shear plate (the thickness of the reference plate is taken equal to unity).
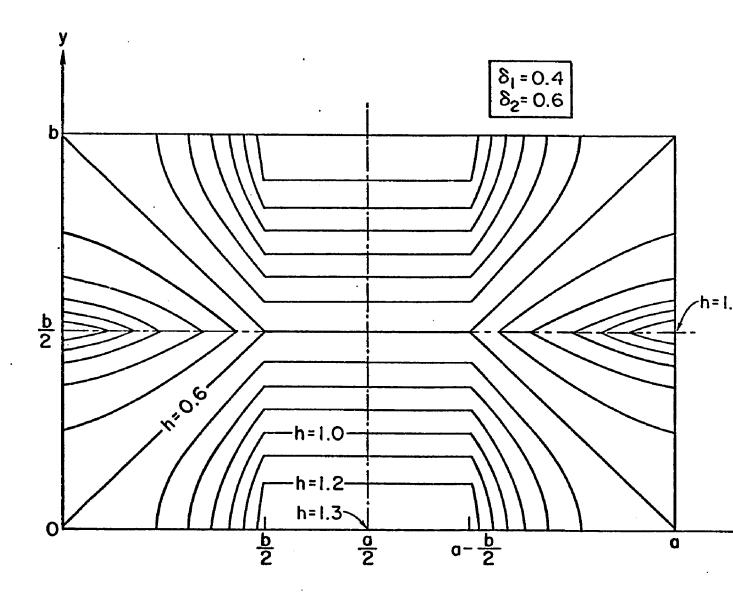
Figure 6. Optimal configuration of a rectangular simply-supported shear plate (a/b = 1.5).

$$m = \delta_2 \left\{ \frac{1}{2} + \frac{2\lambda}{\pi^2 \delta_1 (\lambda^2 + 1)} \sinh^2 \frac{\pi}{2\lambda} \sqrt{\delta_1 (\lambda^2 + 1)} \right.$$

$$\left. + \frac{\lambda - 1}{2\pi \sqrt{\delta_1 (\lambda^2 + 1)}} \sinh \frac{\pi}{\lambda} \sqrt{\delta_1 (\lambda^2 + 1)} \right\} \qquad (2.17)$$

Let us now examine the extreme cases: when $\delta_1 \to 0$, i.e., when all the mass is non-structural and thus impossible to redesign, the optimal mass ratio tends to the value 1, as it should, showing that the result of the optimization process is the original plate itself, as can be seen by noting that in the vicinity of $\delta_1 = 0$,

$$\sinh^2 \frac{\pi}{2\lambda} \sqrt{\delta_1 (\lambda^2 + 1)} \sim \frac{\pi^2 \delta_1 (\lambda^2 + 1)}{4\lambda^2}$$

$$\sinh \frac{\pi}{\lambda} \sqrt{\delta_1 (\lambda^2 + 1)} \sim \frac{\pi}{\lambda} \sqrt{\delta_1 (\lambda^2 + 1)}$$

On the other hand, for $\delta_1 = 1$ and $\delta_2 = 0$ (case examined first when all the mass is structural and allowed to vary), this ratio is equal to zero, due to the fact that the optimal thickness distribution is then identical to zero throughout the domain D.

For a square plate, $a = b$ thus $\lambda = 1$, and the optimal mass ratio is then equal to:

$$m = \delta_2 \left\{ \frac{1}{2} + \frac{1}{\pi^2 \delta_1} \sinh^2 \pi \sqrt{\frac{\delta_1}{2}} \right\} \qquad (2.17a)$$

For the shapes pictured in figs. 5 and 6, the mass savings are found equal to 14.3% and 15.9% respectively, which is an encouraging result if we consider that 60% of the initial mass is not at the control of the designer.

When λ goes to infinity, i.e., when one dimension of the plate becomes extremely large compared to the other one, then the optimal mass ratio tends

to the limit value:

$$m_\infty = \delta_2 \left\{ \frac{1}{2} + \frac{1}{2\pi\sqrt{\delta_1}} \sinh(\pi\sqrt{\delta_1}) \right\}$$

which is identical with the optimal mass ratio found for the minimum-mass design
of the one-dimensional cantilever wing for fixed torsional frequency[14]. The
analogy between that problem and our shear plate one is by the way worth noting.

Also, for a fixed value of δ_1, the mass ratio first decreases when the
ratio a/b increases, reaches a minimum and increases again to the value m_∞
when δ_1 goes to infinity. This variation of m with λ for a δ_1 of 0.3 is plotted
in fig. 7. The asymptote is the straight line $m = 0.901663$, and the maximum
mass saving is obtained in that case for a/b = 3.42. The value of the mass ratio
is then 0.88621, corresponding to a saving of 11.4% as compared to a saving less
than 9% for a square plate made of the same material.

The variation of the mass ratio m with the proportion δ_1 of structural
mass is represented in fig. 8 for different values of the ratio a/b.

2.3 Application to the Circular Plate

We now turn our attention to the optimization of a circular simply-supported
shear plate. The optimal thickness distribution being assumed to be rotation-
invariant, i.e., its expression in polar coordinates to be independent of θ, the
problem might also, as pointed out in the introduction, be viewed as a one-
dimensional one, and solved using the classical methods of optimal control theory,
which will provide two ways of finding the solution.

In polar coordinates (r,θ), an axisymmetric optimal solution $h*^\dagger$ has to
satisfy (2.2) rewritten as:

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†where the superscript (*) has the same significance than before, the actual thick-
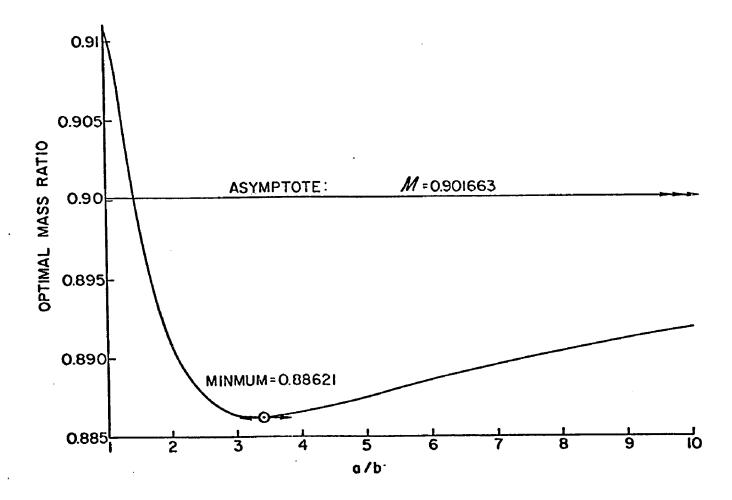ness being $h = \delta_1 h* + \delta_2$, with $\delta_1 + \delta_2 = 1$.

Figure 7. Variation of the optimal mass ratio with a/b for a given value of
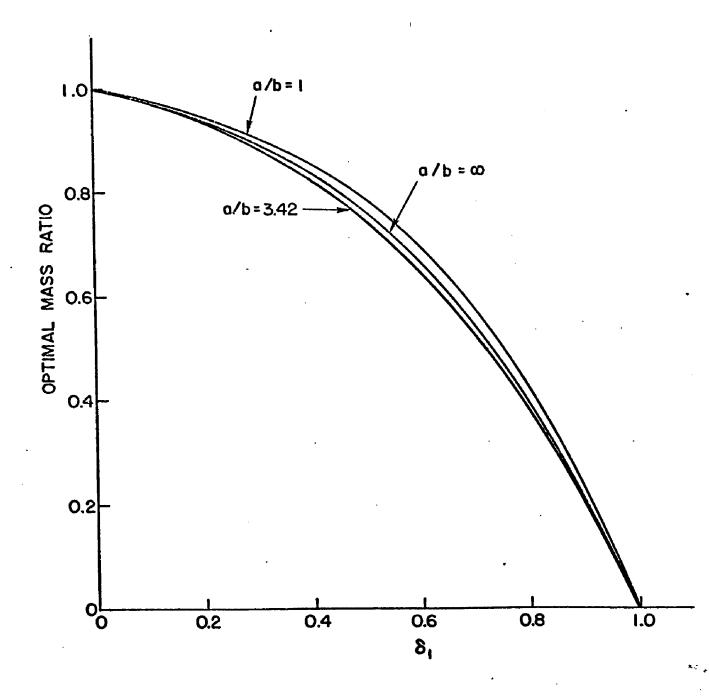δ_1 ($\delta_1 = 0.3$).

Figure 8. Variation of the optimal mass ratio with δ_1 for different values of the ratio a/b.

$$(h^*w')' + \frac{1}{r} h^*w' + k^2(h^* + \frac{\delta_2}{\delta_1}) w = 0 \tag{2.18}$$

where (') denotes the derivative taken with respect to r.

With the value of the optimal displacement given by (2.9) where ρ is taken equal to $a-r$,

$$w = \frac{\epsilon c}{k} \sinh[k(a-r)] \tag{2.19}$$

the optimal virtual thickness h^* has to satisfy the ordinary differential equation:

$$h^{*'} + (\frac{1}{r} - 2k \tanh[k(a-r)]) h^* - k \frac{\delta_2}{\delta_1} \tanh[k(a-r)] = 0 \tag{2.20}$$

For the same reasons than for the rectangular plate (continuity of the shear), the optimal virtual thickness h^* has to vanish at the center of the plate. The solution is thus found to be, after some manipulation:

$$h^* = \frac{\delta_2}{\delta_1} \frac{\sinh(2ka) - \sinh[2k(a-r)] - 2kr \cosh[2k(a-r)]}{8kr \cosh^2[k(a-r)]} \tag{2.21}$$

and the optimal thickness distribution is given by:

$$h = \delta_2 \left\{ \frac{1}{2} + \frac{1}{4 \cosh^2[k(a-r)]} + \frac{\sinh(2ka) - \sinh[2k(a-r)]}{8kr \cosh^2[k(a-r)]} \right\} \tag{2.22}$$

We recall that:

$$k = \omega_f \sqrt{\frac{\rho}{G} \delta_1} = 2.4048 \frac{\sqrt{\delta_1}}{a}$$

The optimal mass ratio is then equal to:

$$m = \frac{\int_o^a 2\pi hrdr}{\pi a^2}$$

61

and, after some easy integrations, found to be:

$$m = \frac{\delta_2}{2} (1 + \frac{\sinh^2(ka)}{k^2 a^2})$$ (2.23)

As expected, the ratio is equal to zero when all the mass is structural $(\delta_1 = 1)$, and tends to the value 1 when all the mass is non-structural $(\delta_1 = 0)$, and therefore not at the control of the designer.

A diametral cross-section of the optimal plate corresponding to the case where 60% of the mass is non-structural is represented in fig. 9. The mass saving then obtained is equal to 8.6%. The variation of the optimal mass ratio with the proportion δ_1 of structural mass in the reference structure is plotted in fig. 10: note that the savings obtained for a circular plate are slightly less important than those obtained in the case of a rectangular plate, as can be seen by comparison of the curves represented in fig. 7 to the one of this figure.

To solve the same problem as a direct application of the methods of optimal control theory, we have to minimize the mass:

$$M = 2\pi\rho \int_0^a hrdr$$

or, which is equivalent, the definite integral:

$$J = \int_0^a hrdr$$ (2.24)

The constraint is given by equation (2.11) rewritten for the case of axisymmetric vibrations, using polar coordinates, as:

$$(h^*w')' + \frac{1}{r} h^*w' + \frac{\rho}{G} \omega_f^2 (\delta_1 h^* + \delta_2) w = 0$$ (2.25)

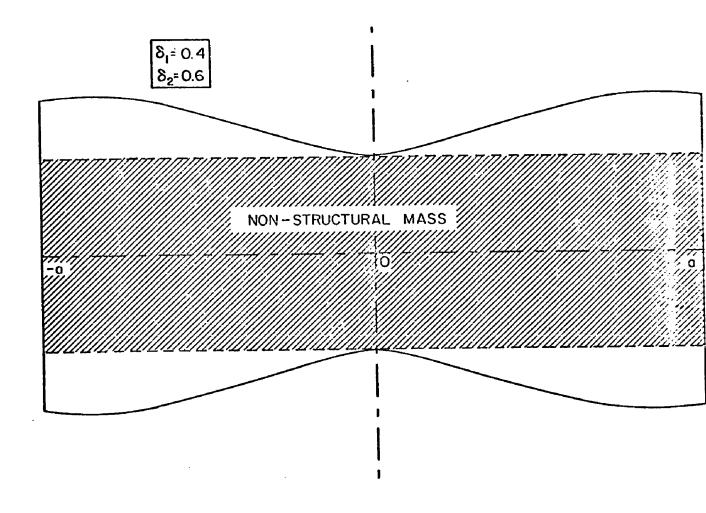(') denoting as before the derivative with respect to r, together with the boundary condition:

Figure 9. Diametral cross-section of an optimal circular simply-supported shear plate for which 40% of the mass is structural.
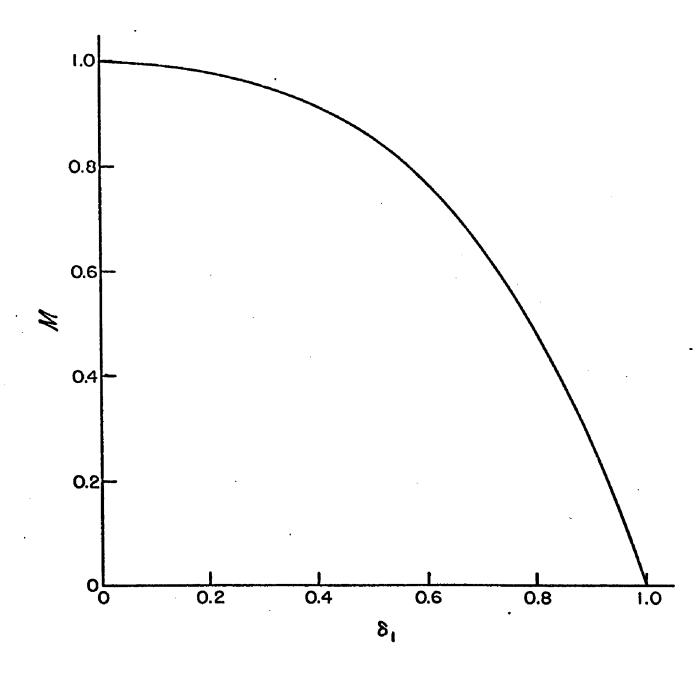
Figure 10. Variation of the optimal mass ratio with the proportion of structural mass δ_1 for a simply-supported circular shear plate.

$w(a) = 0$

We introduce the auxiliary variable:

$s = h*r$

and rewrite (2. 25) as:

$$(sw')' + k^2(s + \frac{\delta_2}{\delta_1} r) w = 0 \qquad\qquad (2.26)$$

$w(a) = 0$

where:

$$k^2 = \frac{\rho}{G} \omega_f^2 \delta_1 = (\frac{2.4048}{a})^2 \delta_1$$

Introducing another auxiliary variable t by the means of:

$t = sw'$,

we rewrite the optimization problem as:

Minimize:

$$J = \int_0^a sdr \qquad\qquad (2.24)$$

subject to the differential constraints:

$$w' = \frac{t}{s} .$$

$$t' = -k^2(s + \frac{\delta_2}{\delta_1} r)w$$

and the boundary conditions

$$w(a) = 0$$

$$t(0) = 0^{\dagger}$$

The Hamiltonian is formed:

$$H = s + \lambda_w \frac{t}{s} - k^2 \lambda_t (s + \frac{\delta_2}{\delta_1} r)w$$

and the necessary conditions for a minimum are given by:

$$\lambda'_w = k^2 \lambda_t (s + \frac{\delta_2}{\delta_1} r)$$

$$\lambda'_t = \frac{\lambda_w}{s} \tag{2.27}$$

$$1 - \frac{\lambda_w t}{s^2} - k^2 \lambda_t w = 0 ,$$

the last equation being the control equation.

The natural boundary conditions are:

$$\lambda_w(0) = 0$$

$$\lambda_t(a) = 0$$

A solution to the problem appears to be such that:

$$\lambda_w = \frac{t}{\alpha}$$

$$\lambda_t = -\frac{w}{\alpha} ,$$

α being a proportionality constant, assumption compatible with the original

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†The second of the conditions results from the fact that $s(0) = 0$ imposes the vanishing of t for $r = 0$.

boundary conditions. The control equation is then rewritten as:

$$\alpha - \frac{t^2}{s^2} + k^2 w^2 = 0 \tag{2.28}$$

or:

$$w'^2 = \alpha + k^2 w^2$$

which shows that α has to be positive: let

$$\alpha = c^2$$

(2.28) is written as:

$$dr = \epsilon \, \frac{dw}{\sqrt{c^2 + k^2 w^2}} \quad , \qquad \epsilon = \pm 1$$

or, integrating:

$$r - r_0 = \frac{\epsilon}{k} \, \text{Log}(kw + \sqrt{c^2 + k^2 w^2})$$

which is also:

$$w = \frac{\epsilon c}{k} \, \sinh[k(r - r_0)] \tag{2.29}$$

and, due to the condition $w(a) = 0$, the optimal displacement mode is finally given by:

$$w = \frac{\epsilon c}{k} \, \sinh[k(a - r)] \tag{2.30}$$

which is exactly the expression of the solution found to the two-dimensional problem, as rewritten for the case of axisymmetrical vibrations in polar coordinates (equation (2.19)). The problem of finding the optimal thickness distribution from now on is the same as treated above.

The solutions to the optimal problem obtained from two different methods — satisfying the necessary conditions for optimality in the classical one-

dimensional case and in the two-dimensional one, as derived in part A — are therefore the same when the two-dimensional domain can be in fact reduced to a one-dimensional one in an image-space as it is the case for the axisymmetric vibrations of a circular plate, therefore proving the validity of the optimality process in two-dimensions for this particular case.

3. Simply-Supported Rectangular Shear Plate with a Minimum-Thickness Constraint

As we have previously seen, the optimal thickness distribution in the case of a rectangular plate is found such that the virtual thickness h^* vanishes along the segments constituted by portions of the bissectrices of the four corner right angles and of the line joining their points of intersection, as represented in fig. 6. This has as a consequence the instability of the structure, and a plate built following this theoretical distribution would immediately collapse if simply-supported. The same situation arises for a square plate, which is a particular case of the above for which h^* is zero along the two diagonals.

It is therefore very desirable to impose a constraint on the virtual thickness h^*; imposing h^* to be at least equal to h_o^* is of course equivalent to impose a minimum actual thickness $h_o = \delta_1 h_o^* + \delta_2$.

The optimization problem is the same as before:

Minimize the surface integral:

$$J^* = \iint_D h^*(x,y)dxdy \qquad (3.1)$$

subject to the constraint:

$$\frac{\partial}{\partial x}(h^* \frac{\partial w}{\partial x}) + \frac{\partial}{\partial y}(h^* \frac{\partial w}{\partial y}) + \frac{\rho}{G}\omega_f^2(\delta_1 h^* + \delta_2)w = 0$$

$$w = 0 \qquad \text{on} \qquad \partial D \qquad (3.2)$$

with the additional inequality constraint on the control variable:

$$h^* - h^*_o \geq 0 \tag{3.3}$$

(3.2) is broken down into a system of the form (A.1.1). Following the method derived in (A.3), we form the augmented Hamiltonian:

$$H = h^* + \lambda_1 \frac{z_2}{h^*} + \lambda_2 u_1 + \lambda_3 u_3 + \mu_1 \frac{z_3}{h^*} + \mu_2 u_2$$

$$- \mu_3 [u_1 + \frac{\rho}{G} \omega_f^2 (\delta_1 h^* + \delta_2) z_1] + \xi (h^* - h^*_o) \tag{3.4}$$

where:

$$\xi(x, y) \leq 0 \qquad \text{if} \qquad h^* = h^*_o$$

$$\xi(x, y) = 0 \qquad \text{if} \qquad h^* \geq h^*_o \tag{3.5}$$

The necessary conditions for an extremal are expressed by the system of partial differential equations:

$$\frac{\partial z_1}{\partial x} = \frac{z_2}{h^*}$$

$$\frac{\partial z_1}{\partial y} = \frac{z_3}{h^*}$$

$$\frac{\partial z_2}{\partial x} = u_1$$

$$\frac{\partial z_2}{\partial y} = u_2$$

$$\frac{\partial z_3}{\partial x} = u_3$$

$$\frac{\partial z_3}{\partial y} = -u_1 - \frac{\rho}{G} \omega_f^2 (\delta_1 h^* + \delta_2) z_1$$

(continued)

$$\frac{\partial \lambda_1}{\partial x} + \frac{\partial \mu_1}{\partial y} = \frac{\rho}{G} \omega_f^2 (\delta_1 h^* + \delta_2) \mu_3$$

$$\frac{\partial \lambda_2}{\partial x} + \frac{\partial \mu_2}{\partial y} = -\frac{\lambda_1}{h^*}$$

$$\frac{\partial \lambda_3}{\partial x} + \frac{\partial \mu_3}{\partial y} = -\frac{\mu_1}{h^*}$$

$$1 - \frac{\lambda_1 z_2}{h^{*2}} - \frac{\mu_1 z_3}{h^{*2}} - \frac{\rho}{G} \omega_f^2 \delta_1 \mu_3 z_1 + \xi = 0$$

$$\lambda_2 - \mu_3 = 0$$

$$\mu_2 = 0$$

$$\lambda_3 = 0 \qquad\qquad\qquad\qquad (3.6)$$

together with the boundary conditions:

$$z_1 = 0$$

$$\lambda_2 \beta'(\sigma) - \mu_2 \alpha'(\sigma) = 0 \qquad\qquad\qquad\qquad (3.6a)$$

$$\lambda_3 \beta'(\sigma) - \mu_3 \alpha'(\sigma) = 0$$

on ∂D, which reduce to:

$$z_1 = 0$$

$$\lambda_2 = \mu_3 = 0$$

Assume, as in (2.7):

$$\lambda_1 = \frac{z_2}{\alpha}$$

$$\mu_1 = \frac{z_3}{\alpha}$$

$$\lambda_2 = \mu_3 = -\frac{z_1}{\alpha} \, ,$$

α being a proportionality constant. The control equation reduces to, going back to the original notations:

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = \alpha + \frac{\rho}{G}\omega_f^2 \delta_1 w^2 + \xi(x,y) \tag{3.7}$$

At every point of the domain D where h* is greater than h_o^*,

$$\xi(x,y) \equiv 0$$

and (3.7) reduces to:

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = \alpha + \frac{\rho}{G}\omega_f^2 \delta_1 w^2$$

Let

$$\alpha = c^2 .$$

The solution to this equation was found to be:

$$w = \frac{\epsilon c}{k} \sinh(k\rho) \tag{3.8}$$

where:

$$k = \omega_f \sqrt{\frac{\rho}{G}\delta_1}$$

and the quantity ρ having been previously defined.

We now focus our attention onto the case of a rectangular plate with sides of lengths a and b (a $\geq$ b). For obvious reasons of symmetry, it is sufficient to consider the portion of the plate which is the left half of the region previously called (2).

Inside this portion, ρ is equal to y, and:

$$w = \frac{\epsilon c}{k} \sinh(ky) \tag{3.8a}$$

The optimal virtual thickness is, as before:

$$h^*(x,y) = -\frac{\delta_2}{2\delta_1} + \frac{\bar{\omega}(x)}{\cosh^2(ky)} \qquad (3.9)$$

where $\bar{\omega}$ is an arbitrary function to be determined.

Now in the regions where the minimum value permissible for h^* is attained, that is:

$$h^* = h_0^* \,,$$

then equation (3.2) reduces to:

$$\nabla^2 w + \frac{\rho}{G}\,\omega_f^2 \left(\delta_1 + \frac{\delta_2}{h_0^*}\right)\, w = 0 \,. \qquad (3.10)$$

This is an equation of the form (1.6), with:

$$\alpha = \omega_f \sqrt{\frac{\rho}{G}\left(\delta_1 + \frac{\delta_2}{h_0^*}\right)} \qquad (3.11)$$

The general solution is, introduce the four constants of integration A_1, A_2, A_3, A_4:

$$w(x,y) = A_1 \sin \beta x \sin \gamma y + A_2 \sin \beta x \cos \gamma y + A_3 \cos \beta x \sin \gamma y$$

$$+ A_4 \cos \beta x \cos \gamma y \qquad (3.12)$$

where:

$$\beta^2 + \gamma^2 = \alpha^2 \,.$$

The problem is now to find the shape of the curve Δ dividing (2) into two regions with different properties: in the first one, which should be attached to the bissectrix of the lower-left corner and to the line defined by

$$y = b/2, \quad b/2 \leq x \leq a/2 \,,$$

h^* keeps the minimum permissible value h_0^*; in the second one, closer to the

side $y = 0$, the distribution of h^* is as given by (3.9).

The shape of this dividing curve Δ can be imagined quite easily, with the help of the shape of the lines of constant thickness already derived for the case where there is no constraint on the thickness, and from previous experience in one-dimensional cases[14]: Δ should consist of a curve starting off somewhere on the lower side of (2) between the points $x = 0$ and $x = a/2$, situated on the same side of the line $y = x$, and prolongated by a nearly straight line parallel to the x-axis after the point with abscissa $b/2$. In any case, in the corridor enclosing the bissectrix $y = x$, the displacement w has to be symmetrical about the same line, and this can happen if and only if:

$$\beta = \gamma = \frac{\alpha}{\sqrt{2}}$$

$$A_2 = A_3 \ .$$

To determine the exact shape of Δ, we note that w and its first derivatives $\partial w/\partial x$ and $\partial w/\partial y$ have to be continuous across it.

A point (ξ, η) of Δ has to be such that:

$$\sinh(k\eta) = a_1 \sin \beta\xi \sin \beta\eta + a_2(\sin \beta\xi \cos \beta\eta + \cos \beta\xi \sin \beta\eta)$$

$$+ a_4 \cos \beta\xi \cos \beta\eta$$

$$0 = a_1 \cos \beta\xi \sin \beta\eta + a_2(\cos \beta\xi \cos \beta\eta - \sin \beta\xi \sin \beta\eta)$$

$$- a_4 \sin \beta\xi \cos \beta\eta$$

$$\frac{k}{\beta} \cosh(k\eta) = a_1 \sin \beta\xi \cos \beta\eta - a_2(\sin \beta\xi \sin \beta\eta - \cos \beta\xi \cos \beta\eta)$$

$$- a_4 \cos \beta\xi \sin \beta\eta \tag{3.13}$$

(the a_1, a_2, a_4 replace the previous constants A_1, A_2, A_4 as the displacement mode w is defined up to a constant multiplicative factor).

The point 0 belongs to the domain in which w is given by (3.12). The condition $w = 0$ at the corner 0 leads to:

$$a_4 = 0 .$$

By subtraction of the last two equations in (3.13), we obtain:

$$a_1 = \frac{k \cosh(k\eta)}{\beta \sin \beta(\xi - \eta)} .$$

From the second equation,

$$a_2 = -\frac{k\cosh(k\eta)\cos \beta\xi \sin \beta\eta}{\beta \sin \beta(\xi - \eta)\cos \beta(\xi + \eta)} .$$

Replacing a_1 and a_2 by their values in the first equation (3.13), we find the relation between ξ and η:

$$\tanh(k\eta) = \frac{k\sin(\beta\eta)}{\beta\sin\beta(\xi - \eta)} [\sin \beta\xi - \cos \beta\xi \tan \beta(\xi + \eta)] .$$

After some trigonometric manipulation, this equation turns out to be solvable in ξ, yielding:

$$\xi = \frac{1}{2\beta} \sin^{-1}\left[\sin(2\beta\eta) - \frac{2k \sin^2(\beta\eta)}{\beta \tanh(k\eta)} \right] \qquad (3.14)$$

where $\sin^{-1}$ is defined by:

$$\sin^{-1} u = \begin{cases} \text{Arcsin}u + 2h\pi \\ \pi - \text{Arcsin}u + 2h\pi , \end{cases} \qquad h = 0, \pm 1, \ldots$$

Arcsinu representing the principal determination of arcsinu, and where the proper choice of the integer h has been made.

Recall that k and β are defined as:

$$k = \omega_f \sqrt{\frac{\rho}{G} \delta_1} = \frac{\pi}{ab} \sqrt{\delta_1(a^2 + b^2)}$$

$$\beta = \omega_f \sqrt{\frac{\rho}{2G}\left(\delta_1 + \frac{\delta_2}{h_0^*}\right)} = \frac{\pi}{ab}\sqrt{\frac{1}{2}\left(\delta_1 + \frac{\delta_2}{h_0^*}\right)(a^2 + b^2)} \quad .$$

The curve Δ is thus defined by the equation:

$$x = \xi(y) \quad .$$

Although we cannot do the inversion in an analytical form, its Cartesian equation is, formally:

$$y = \eta(x) = \xi^{-1}(x) \quad .$$

We are now able to write down the expression for the thickness distribution in region (2): it has to be continuous across Δ; thus, at every point (ξ, η) of Δ:

$$-\frac{\delta_2}{2\delta_1} + \frac{\bar{\omega}(\xi)}{\cosh^2(k\eta)} = h_0^*$$

where the unknown function $\bar{\omega}$ is defined as:

$$\bar{\omega}(x) = \left(h_0^* + \frac{\delta_2}{2\delta_1}\right)\cosh^2[k\eta(x)] \quad .$$

On the right of and below Δ, the optimal "thickness" distribution is given by:

$$h^*(x,y) = -\frac{\delta_2}{2\delta_1} + \left(h_0^* + \frac{\delta_2}{2\delta_1}\right)\frac{\cosh^2[k\eta(x)]}{\cosh^2(ky)} \tag{3.15}$$

(in region (2), the extension of this formula to the other regions being immediate), and this corresponds to a true thickness distribution of:

$$h(x,y) = \frac{\delta_2}{2} + (\delta_1 h_0^* + \frac{\delta_2}{2})\frac{\cosh^2[k\eta(x)]}{\cosh^2(ky)} \quad . \tag{3.16}$$

Of course, in the remaining part of domain (2), h keeps the constant minimum value:

$$h(x,y) = \delta_1 h_0^* + \delta_2 .$$

The distribution in the other 3 regions follows immediately from symmetry.

We cannot here give an explicit expression for the optimal thickness distribution, as the function $\eta(x)$ cannot be put under any analytical form and is only known implicitly.

Of course a computational approach is always possible: contours of constant thickness have been represented in figures 11 and 12 respectively for the case of a square plate and that of a rectangular plate for which the ratio of the lengths of the sides is equal to 3/2, as in the previous unconstrained problem. Again, the initial proportion of structural mass in the reference structure δ_1 was taken equal to 0.4, which means that 60% of the mass is non-structural. The constraint h_0^* was taken equal to 0.5, representing an actual constraint of 0.8 on the true thickness h.

In the case of the square plate, 84.4% of the plate superficy is at the minimum allowed thickness of 0.8. The mass saving then obtained is of 14.1%, slightly inferior to the one obtained in the case where there was no minimum constraint on the thickness.

For the rectangular plate (a/b = 3/2), the minimum allowed thickness is reached on 77.6% of the total superficy. The total mass is found to be of 0.845 that of the original plate, corresponding to a saving of 15.5% that compares well with the 15.9% obtained when there was no constraint on the thickness.

When we compare the corresponding optimal shapes for the cases where there is a minimal constraint on the thickness and the unconstrained ones, the main feature that we notice is the augmentation of the area of the domain formed by the reunion of the points where the thickness was inferior or equal to the given constraint, together with the drastic diminution of the maximal thickness, that was obtained at the four points middle of the sides in the case of the square

$\delta_1 = 0.4$
$\delta_2 = 0.6$
$h_0^* = 0.5$

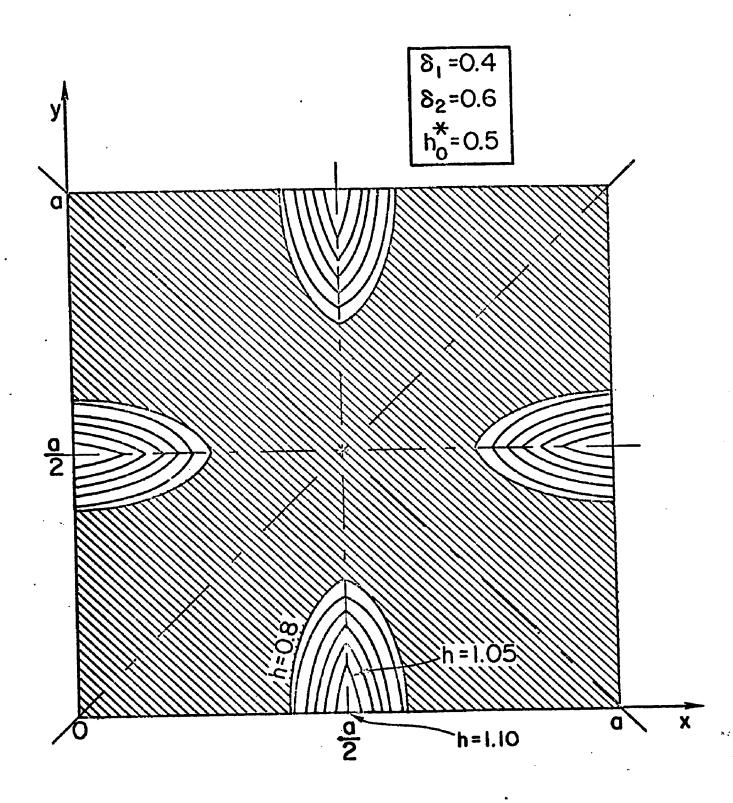$h = 0.8$

$h = 1.05$

$h = 1.10$

Figure 11. Optimal thickness distribution for a simply-supported square shear plate with a minimum-thickness constraint.
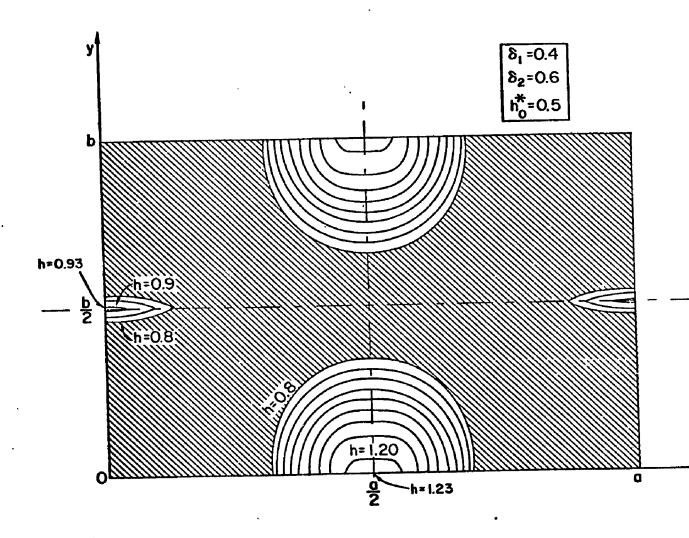
Figure 12. Optimal thickness distribution for a simply-supported rectangular shear plate with a minimum-thickness constraint (a/b = 3/2).

plate, and at the two points middle of the larger sides in the case of the rectangular shape. Physically, this is logical, as in the constrained case, the parts of the plate where the thickness is above the imposed minimum have less to contribute to the rigidity of the structure than in the unconstrained one, where they have to compete with the effect of antagonistic hinges. This can also be viewed in terms of the total potential energy for the first deflection mode of the plate, which has to remain the same for a fixed natural mode, whether or not there is a constraint on the thickness: the contribution of a large thickness at some points of the plate has to balance that of hinges in the unconstrained case, whereas everything gets smoother when a constraint on the thickness is imposed.

This feature (diminution of the maximum thickness attained and augmentation of the area where the thickness is inferior or equal to the given constraint as soon as it is applied) is the exact extension to two-dimensions of properties already recognized for the one-dimensional problems where a minimum constraint was imposed on the thickness[14].

For the practical design, these results are of extreme importance. The shapes obtained when the thickness is subject to a minimal constraint are much easier to realize in practice: the maximum thickness attained in the domain covered by the plate is much smaller than in the unconstrained case, most of the superficy of the plate is at this minimum imposed thickness, and there are no hinges as before. Moreover, together with all these advantages, the striking fact is that the mass saving is only slightly inferior than in the unconstrained case.

Again, these properties were noticed also for one-dimensional cases[14].

PART C

THE MINIMUM-MASS DESIGN OF A SANDWICH PLATE FOR A GIVEN FUNDAMENTAL FREQUENCY OF VIBRATION

A further step in the domain of applications to more realistic structures
is that of the optimization of plate-like structures encountered in practical design
for a given fundamental frequency of vibration. For a sandwich plate, the
necessary conditions lead, when the displacement along the edge is assumed to
vanish, to an optimality condition expressing the uniformity of the optimal
energy distribution in the form of a second order nonlinear partial differential
equation for the optimal displacement mode w that does not involve any design
parameters. The optimal thickness distribution is the solution of a second
order linear partial differential equation.

A numerical solution of these equations is presented for a square aluminum
alloy-aluminum honeycomb panel simply-supported along the edges.

1.   Statement of the Optimization Problem

A sandwich plate is a composite plate consisting of a core layer of
thickness h_0 and of two face layers of thickness t. It is assumed that t is
small compared to h_0 and that the core material is much more flexible than the
face material. Under these assumptions the transverse shears are predominantly
taken by the core plate while the bending stresses are primarily taken by the face
plate (Fig. 13). We will take the elastic modulus in transverse direction of the
core-layer material, G_c, to be infinite.

The core material is generally extremely light, and despite the small
thickness of the face material, most of the weight is concentrated in the faces.
We will therefore act on the face layers with thickness t, and our goal will be
to find the optimal t distribution of the rectangular plate of minimum weight
having the same fundamental frequency of vibration as a uniform reference one.
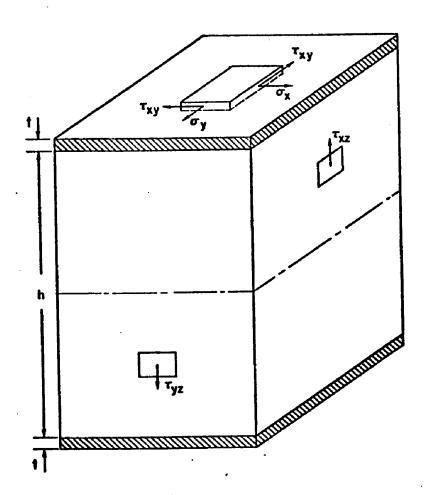
Figure 13. Infinitesimal element of sandwich plate, showing dimensions and relevant components of stress.

Under the small-deflection assumption, the equations of equilibrium for a sandwich plate have been derived by Reissner[36,37] who followed the classical approach of combining the equilibrium equations and the stress-strain relations, and by Hoff[38] who used a variational approach. Assuming that the only external action applied to the plate is the normal force of intensity q, the equilibrium equation reads:

$$\frac{\partial^2 M_x}{\partial x^2} - 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -q \tag{1.1}$$

where M_x, M_y are the bending moments per unit length of sections of the plate perpendicular to the x and y axes respectively, M_{xy} the twisting moment per unit length of sections perpendicular to the x axis, the signs being taken according to the Timoshenko convention[39].

From the stress-strain relations[36],

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)$$

$$M_{xy} = D(1-\nu)\frac{\partial^2 w}{\partial x \partial y} = -M_{yx}$$

$$M_y = -D\left(\nu\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) \tag{1.2}$$

where the bending stiffness factor D is defined as:

$$D = \frac{t(h_0 + t)^2 E_s}{2(1-\nu^2)} \tag{1.3}$$

E_f being the elastic modulus of isotropic face-layer material, ν the Poisson ratio.

Substituting these expressions in the equation of equilibrium (1.1), we obtain:

$$\nabla^2(D\nabla^2 w) + (1-\nu)\left(2\frac{\partial^2 D}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y} - \frac{\partial^2 D}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 D}{\partial y^2}\frac{\partial^2 w}{\partial x^2}\right) = q \qquad (1.4)$$

Under our hypothesis $t \ll h_0$, a very good approximation for D is:

$$D = \frac{h_0^2 E_s}{2(1-\nu^2)}\, t \qquad (1.5)$$

and (1.4) reduces to:

$$\nabla^2(t\nabla^2 w) + (1-\nu)\left(2\frac{\partial^2 t}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y} - \frac{\partial^2 t}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 t}{\partial y^2}\frac{\partial^2 w}{\partial x^2}\right) = \frac{2(1-\nu^2)}{h_0^2 E_s}\, q \qquad (1.6)$$

For a sandwich-plate of uniform face thickness t_0 and uniform core thickness h_0, the differential equation for free vibration is (1.6) where q is replaced by the d'Alembert force $-(h_0\rho_c + t_0\rho_s)(\partial^2 w/\partial t^2)$ and t is a constant t_0:

$$\nabla^4 w = -\frac{2(1-\nu^2)}{h_0^2 E_s}\,\frac{h_0\rho_c + t_0\rho_s}{t_0}\,\frac{\partial^2 w}{\partial t^2} \qquad (1.7)$$

ρ_c is the density of the core-layer material, ρ_f that of the face-layer material.

Denote by n and s the coordinates in the directions normal and tangential to the boundary ∂D. For a clamped edge, we have the geometric boundary conditions:

$$w = 0 \quad \text{and} \quad \frac{\partial w}{\partial n} = 0 \quad \text{along} \quad \partial D. \qquad (1.7a)$$

At a simply-supported edge, the boundary conditions are:

$$w = 0 \quad \text{and} \quad M_n = 0 \quad \text{along} \quad \partial D \qquad (1.7b)$$

where M_n is the bending moment per unit length associated with the cross-section whose normal is n.

In the case of a free edge, the boundary conditions are:

$$M_n = 0 \quad \text{and} \quad V_n = Q_n - \frac{\partial M_{ns}}{\partial s} = 0 \quad \text{along} \quad \partial D \qquad (1.7c)$$

where V_n denotes the vertical force, Q_n is the shearing force and M_{ns} is the twisting moment about the direction n. All three quantities V_n, Q_n and M_{ns} are for one unit length of boundary and are associated with the cross-section whose normal is n.

The relation between the moments and shearing forces and deformations, in terms of normal and tangential coordinates, are[32]:

$$M_n = -D\nabla^2 w + (1-\nu)D\left(\frac{1}{R}\frac{\partial w}{\partial n} + \frac{\partial^2 w}{\partial s^2} \right)$$

$$M_{ns} = (1-\nu)D\left(\frac{\partial^2 w}{\partial n \partial s} - \frac{1}{R}\frac{\partial w}{\partial s} \right)$$

$$Q_n = -D\frac{\partial}{\partial n}\nabla^2 w$$

where the Laplacian has the form:

$$\nabla^2 = \frac{\partial^2}{\partial n^2} + \frac{1}{R}\frac{\partial}{\partial n} + \frac{\partial^2}{\partial s^2}$$

and R denotes the radius of the boundary curve.

To formulate the eigenvalue problem, we let in the classical fashion the displacement w be given by:

$$w(x,y;t) = W(x,y)f(t)$$

where W depends on the spatial coordinates only and f is a time-dependent harmonic function of frequency ω. (1.7) reduces to:

$$\nabla^4 W = \frac{2(1-\nu^2)}{h_0^2 E_s} \frac{h_0 \rho_c + t_0 \rho_f}{t_0} \omega^2 W \qquad (1.8)$$

and the boundary conditions are, depending on the situation at the edge, given by one of the conditions (1.7a), (1.7b) or (1.7c) where w has been replaced by W.

This eigenvalue problem is a classical one[32,33]; it is shown to be self-adjoint, and yields an infinite sequence of eigenvalues, the smaller one being the fundamental frequency of vibration ω_f of the plate under the imposed boundary conditions.

The variable face-layer thickness t of a rectangular plate having for one of its natural frequencies of vibration ω_f — that we can reasonably assume to be also its fundamental frequency if the plate does not differ too drastically from the uniform one — together with the corresponding mode w, have to satisfy the partial differential equation:

$$\nabla^2(t\nabla^2 w) + (1-\nu)\left(2\frac{\partial^2 t}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial^2 t}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 t}{\partial y^2}\frac{\partial^2 w}{\partial x^2}\right)$$

$$- 2\frac{(1-\nu^2)}{h_0^2 E_s}(h_0\rho_c + t\rho_f)\omega_f^2 w = 0 . \tag{1.9}$$

If we introduce the non-dimensional face-layer thickness $\tau = (t/t_0)$, the above equation reduces to:

$$\nabla^2(\tau\nabla^2 w) + (1-\nu)\left(2\frac{\partial^2 \tau}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial^2 \tau}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 \tau}{\partial y^2}\frac{\partial^2 w}{\partial x^2}\right) - \beta^2(\tau+\kappa)W = 0 \tag{1.10}$$

where κ and β^2 are constants depending only on the material properties of the reference structure, defined as:

$$\kappa = \frac{h_0}{t_0}\frac{\rho_c}{\rho_f} \tag{1.11}$$

$$\beta^2 = \frac{2(1-\nu^2)\rho_f}{h_0^2 E_s}\omega_f^2 \tag{1.12}$$

85

We are now able to state the optimization problem: mininize the total mass:

$$M = \iint_D (\rho_c h_0 + \rho_f t)\, dD$$

or, equivalently, the functional:

$$J = \iint_D \tau\, dD \tag{1.13}$$

taken over the domain D covered by the plate, subject to the partial differential equation constraint:

$$\nabla^2(\tau \nabla^2 w) + (1-\nu)\left(2\frac{\partial^2 \tau}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial^2 \tau}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 \tau}{\partial y^2}\frac{\partial^2 w}{\partial x^2} \right) - \beta^2(\tau+\kappa)w = 0 \tag{1.10}$$

together with boundary conditions (1.7a), (1.7b) or (1.7c) along the portions of the boundary ∂D where the edge is respectively clamped, simply-supported or free.

For a rectangular plate with sides of length a and b, a ≥ b, the natural frequencies of vibration are found to be:

$$\omega_{mn} = \pi^2 \left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 \right] \sqrt{\frac{h_0^2 E_s}{2(1-\nu^2)}\frac{t_0}{h_0\rho_c + t_0\rho_f}} \tag{1.14}$$

$$m, n = 1, 2, 3, \ldots \quad .$$

The fundamental frequency is:

$$\omega_f = \omega_{11} = \pi^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right)\sqrt{\frac{h_0^2 E_s}{2(1-\nu^2)}\frac{t_0}{h_0\rho_c + t_0\rho_f}} \tag{1.15}$$

and the value of β^2 in equation (1.10) is, in that particular case:

$$\beta^2 = \frac{\pi^4}{1+\kappa} \left(\frac{1}{a^2} + \frac{1}{b^2} \right)^2 \tag{1.16}$$

The following derivation of the necessary conditions will be valid for the most general shape of the boundary ∂D. For the applications, we will assume the plate to be rectangular and will take the coordinate axes to coincide with two of the sides, the origin being one of the corners of the plate. Only clamped and simply-supported edges will be considered. The boundary conditions will be picked among one of the following:

i.   edges parallel to the x axis.

CLAMPED: $w = 0$, $\partial w / \partial y = 0$ along $y = 0, b$

SIMPLY-SUPPORTED: $w = 0$, $T \dfrac{\partial^2 w}{\partial y^2} = 0$ along $y = 0, b$

ii.   edges parallel to the y axis:

CLAMPED: $w = 0$, $\partial w / \partial x = 0$ along $x = 0, a$

SIMPLY-SUPPORTED: $w = 0$, $T \dfrac{\partial^2 w}{\partial x^2} = 0$ along $x = 0, a$ $\tag{1.10a}$

2.   The Necessary Conditions

The constraint (1.10) is transformed into a set of first order partial differential equations in the form (A.1.1) following step by step the general method outlined in Part A. $\partial^2 T / \partial x \partial y$ is computed from (1.10), leading to the system:

$$z_1 = w$$

$$\frac{\partial z_1}{\partial x} = z_2 \qquad\qquad (= \frac{\partial w}{\partial x})$$

(continued)

87

$$\frac{\partial z_1}{\partial y} = z_3 \qquad\qquad (= \frac{\partial w}{\partial y})$$

$$\frac{\partial z_2}{\partial x} = z_4 \qquad\qquad (= \frac{\partial^2 w}{\partial x^2})$$

$$\frac{\partial z_2}{\partial y} = z_5 \qquad\qquad (= \frac{\partial^2 w}{\partial x \partial y})$$

$$\frac{\partial z_3}{\partial x} = z_5 \qquad\qquad (= \frac{\partial^2 w}{\partial x \partial y})$$

$$\frac{\partial z_3}{\partial y} = z_6 \qquad\qquad (= \frac{\partial^2 w}{\partial y^2})$$

$$\frac{\partial z_4}{\partial x} = z_7 \qquad\qquad (= \frac{\partial^3 w}{\partial x^3})$$

$$\frac{\partial z_4}{\partial y} = z_8 \qquad\qquad (= \frac{\partial^3 w}{\partial x^2 \partial y})$$

$$\frac{\partial z_5}{\partial x} = z_8 \qquad\qquad (= \frac{\partial^3 w}{\partial x^2 xy})$$

$$\frac{\partial z_5}{\partial y} = z_9 \qquad\qquad (= \frac{\partial^3 w}{\partial x \partial y^2})$$

$$\frac{\partial z_6}{\partial x} = z_9 \qquad\qquad (= \frac{\partial^3 w}{\partial x \partial y^2})$$

$$\frac{\partial z_6}{\partial y} = z_{10} \qquad\qquad (= \frac{\partial^3 w}{\partial y^3})$$

$$\frac{\partial z_7}{\partial x} = u_1 \qquad\qquad (= \frac{\partial^4 w}{\partial x^4})$$

$$\frac{\partial z_7}{\partial y} = u_2 \qquad\qquad (= \frac{\partial^4 w}{\partial x^3 \partial y})$$

$$\frac{\partial z_8}{\partial x} = u_2 \qquad\qquad (= \frac{\partial^4 w}{\partial x^3 \partial y})$$

$$\frac{\partial z_8}{\partial y} = u_3 \qquad\qquad (= \frac{\partial^4 w}{\partial x^2 \partial y^2})$$

$$\frac{\partial z_9}{\partial x} = u_3 \qquad\qquad (= \frac{\partial^4 w}{\partial x^2 \partial y^2})$$

$$\frac{\partial z_9}{\partial y} = u_4 \qquad\qquad (= \frac{\partial^4 w}{\partial x \partial y^3})$$

$$\frac{\partial z_{10}}{\partial x} = u_4 \qquad\qquad (= \frac{\partial^4 w}{\partial x \partial y^3})$$

$$\frac{\partial z_{10}}{\partial y} = u_5 \qquad\qquad (= \frac{\partial^4 w}{\partial y^4})$$

$$t_1 = \tau$$

$$\frac{\partial t_1}{\partial x} = t_2 \qquad\qquad (= \frac{\partial \tau}{\partial x})$$

$$\frac{\partial t_1}{\partial y} = t_3 \qquad\qquad (= \frac{\partial \tau}{\partial y})$$

$$\frac{\partial t_2}{\partial x} = v_1 \qquad\qquad (= \frac{\partial^2 \tau}{\partial x^2})$$

$$\frac{\partial t_2}{\partial y} = [\beta^2(t_1 + \kappa)z_1 - t_1(u_1 + 2u_3 + u_5) - 2t_2(z_7 + z_9) - 2t_3(z_8 + z_{10}) - v_1(z_4 + \nu z_6)$$

(continued)

89

$$-v_3(z_6 + \nu z_4)]/2(1-\nu)z_5 \qquad\qquad (= \frac{\partial^2 \tau}{\partial x \partial y})$$

$$\frac{\partial t_3}{\partial x} = \frac{\partial t_2}{\partial y} \qquad\qquad (= \frac{\partial^2 \tau}{\partial x \partial y})$$

$$\frac{\partial t_3}{\partial y} = v_3 \qquad\qquad (= \frac{\partial^2 \tau}{\partial y^2}) \qquad (2.1)$$

The boundary conditions, in the case of a rectangular boundary and simply-supported edges read:

$$z_1 = 0, \quad z_4 = 0 \qquad \text{along} \qquad x = 0,\ a$$

$$z_1 = 0, \quad z_6 = 0 \qquad \text{along} \qquad y = 0,\ b \qquad\qquad (2.1a)$$

We now form the Hamiltonian:

$$H \triangleq t_1 + \lambda_1 z_2 + \lambda_2 z_4 + \lambda_3 z_5 + \lambda_4 z_7 + \lambda_5 z_8 + \lambda_6 z_9 + \lambda_7 u_1 + \lambda_8 u_2 + \lambda_9 u_3 + \lambda_{10} u_4 + \lambda_{11} t_2$$

$$+ \lambda_{12} v_1 + \mu_1 z_3 + \mu_2 z_5 + \mu_3 z_6 + \mu_4 z_8 + \mu_5 z_9 + \mu_6 z_{10} + \mu_7 u_2 + \mu_8 u_3 + \mu_9 u_4 + \mu_{10} u_5$$

$$+ \mu_{11} t_3 + \mu_{13} v_3 + \frac{\lambda_{13} + \mu_{12}}{2(1-\nu)z_5} \left\{ \beta^2(t_1 + \kappa)z_1 - t_1(u_1 + 2u_3 + u_5) - 2t_2(z_7 + z_9) \right.$$

$$\left. -2t_3(z_8 + z_{10}) - v_1(z_4 + \nu z_6) - v_3(z_6 + \nu z_4) \right\} \qquad\qquad (2.2)$$

The necessary conditions read:

$$\frac{\partial \lambda_1}{\partial x} + \frac{\partial \mu_1}{\partial y} = -\frac{\partial H}{\partial z_1} = -\frac{\lambda_{13} + \mu_{12}}{2(1-\nu)z_5}\ \beta^2(t_1 + \kappa)$$

$$\frac{\partial \lambda_2}{\partial x} + \frac{\partial \mu_2}{\partial y} = -\frac{\partial H}{\partial z_2} = -\lambda_1$$

(continued)

$$\frac{\partial \lambda_3}{\partial x} + \frac{\partial \mu_3}{\partial y} = -\frac{\partial H}{\partial z_3} = -\mu_1$$

$$\frac{\partial \lambda_4}{\partial x} + \frac{\partial \mu_4}{\partial y} = -\frac{\partial H}{\partial z_4} = -\lambda_2 + \frac{\lambda_{13}+\mu_{12}}{2(1-\nu)z_5}(v_1 + \nu v_3)$$

$$\frac{\partial \lambda_5}{\partial x} + \frac{\partial \mu_5}{\partial y} = -\frac{\partial H}{\partial z_5} = -\lambda_3 - \mu_2 + \frac{\lambda_{13}+\mu_{12}}{2(1-\nu)z_5^2}\left\{ \beta^2(t_1+\kappa)z_1 \right.$$
$$\left. -t_1(u_1 + 2u_3 + u_5) - 2t_2(z_7 + z_9) - 2t_3(z_8 + z_{10}) - v_1(z_4 + \nu z_6) - v_3(z_6 + \nu z_4) \right\}$$

$$\frac{\partial \lambda_6}{\partial x} + \frac{\partial \mu_6}{\partial y} = -\frac{\partial H}{\partial z_6} = -\mu_3 + \frac{\lambda_{13}+\mu_{12}}{2(1-\nu)z_5}(v_3 + \nu v_1)$$

$$\frac{\partial \lambda_7}{\partial x} + \frac{\partial \mu_7}{\partial y} = -\frac{\partial H}{\partial z_7} = -\lambda_4 + \frac{t_2(\lambda_{13}+\mu_{12})}{(1-\nu)z_5}$$

$$\frac{\partial \lambda_8}{\partial x} + \frac{\partial \mu_8}{\partial y} = -\frac{\partial H}{\partial z_8} = -\lambda_5 - \mu_4 + \frac{t_3(\lambda_{13}+\mu_{12})}{(1-\nu)z_5}$$

$$\frac{\partial \lambda_9}{\partial x} + \frac{\partial \mu_9}{\partial y} = -\frac{\partial H}{\partial z_9} = -\lambda_6 - \mu_5 + \frac{t_2(\lambda_{13}+\mu_{12})}{(1-\nu)z_5}$$

$$\frac{\partial \lambda_{10}}{\partial x} + \frac{\partial \mu_{10}}{\partial y} = -\frac{\partial H}{\partial z_{10}} = -\mu_6 + \frac{t_3(\lambda_{13}+\mu_{12})}{(1-\nu)z_5}$$

$$\frac{\partial \lambda_{11}}{\partial x} + \frac{\partial \mu_{11}}{\partial y} = -\frac{\partial H}{\partial t_1} = -1 - \frac{\lambda_{13}+\mu_{12}}{2(1-\nu)z_5}(\beta^2 z_1 - u_1 - 2u_3 - u_5)$$

$$\frac{\partial \lambda_{12}}{\partial x} + \frac{\partial \mu_{12}}{\partial y} = -\frac{\partial H}{\partial t_2} = -\lambda_{11} + \frac{\lambda_{13}+\mu_{12}}{(1-\nu)z_5}(z_7 + z_9)$$

$$\frac{\partial \lambda_{13}}{\partial x} + \frac{\partial \mu_{13}}{\partial y} = -\frac{\partial H}{\partial t_3} = -\mu_{11} + \frac{\lambda_{13}+\mu_{12}}{(1-\nu)z_5}(z_8 + z_{10})$$

$$\frac{\partial H}{\partial u_1} \equiv \lambda_7 - \frac{t_1(\lambda_{13}+\mu_{12})}{2(1-\nu)z_5} = 0$$

$$\frac{\partial H}{\partial u_2} \equiv \lambda_8 + \mu_7 = 0$$

$$\frac{\partial H}{\partial u_3} \equiv \lambda_9 + \mu_8 - \frac{t_1(\lambda_{13}+\mu_{12})}{(1-\nu)z_5} = 0$$

$$\frac{\partial H}{\partial u_4} \equiv \lambda_{10} + \mu_9 = 0$$

$$\frac{\partial H}{\partial u_5} \equiv \mu_{10} - \frac{t_1(\lambda_{13}+\mu_{12})}{2(1-\nu)z_5} = 0$$

$$\frac{\partial H}{\partial v_1} \equiv \lambda_{12} - \frac{\lambda_{13}+\mu_{12}}{2(1-\nu)z_5}(z_4 + \nu z_6) = 0$$

$$\frac{\partial H}{\partial v_3} \equiv \mu_{13} - \frac{\lambda_{13}+\mu_{12}}{2(1-\nu)z_5}(z_6 + \nu z_4) = 0 \tag{2.3}$$

For a rectangular simply-supported plate, the boundary conditions read:

$$z_1 = \lambda_2 = \lambda_3 = z_4 = \lambda_5 = \lambda_6 = \lambda_7 = \lambda_8 = \lambda_9 = \lambda_{10} = \lambda_{11} = \lambda_{12} = \lambda_{13} = 0$$

$$\text{along } x = 0, \ a$$

$$z_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = z_6 = \mu_7 = \mu_8 = \mu_9 = \mu_{10} = \mu_{11} = \mu_{12} = \mu_{13} = 0$$

$$\text{along } y = 0, \ b \tag{2.3a}$$

We therefore have to solve a system of 46 equations in the 46 unknowns z, u, t, v, λ and μ, which is a formidable task if we consider that the majority of the equations composing it are partial differential equations. A logical and simple assumption, in the line of the one made for the shear plate problem (see Appendix 1), will however yield a simple partial differential equation to be satisfied by the optimal displacement mode w. We verified that this assumption was indeed valid in the shear plate case, and similar ones also have proven to be valid in

one-dimensional cases.[14] We will come back to this assumption in length in Part E.

The top equation of (2.3) is similar in form to the original constraint equation (1.10) which might be written under the form:

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = \beta^2(\tau + \kappa)w$$

with

$$P = \frac{\partial}{\partial x}\left[t\left(\frac{\partial^2 w}{\partial x^2} + \nu\,\frac{\partial^2 w}{\partial y^2}\right)\right] + \frac{\partial}{\partial y}\left[(1-\nu)t\,\frac{\partial^2 w}{\partial x \partial y}\right]$$

$$Q = \frac{\partial}{\partial x}\left[(1-\nu)t\,\frac{\partial^2 w}{\partial x \partial y}\right] + \frac{\partial}{\partial y}\left[t\left(\frac{\partial^2 w}{\partial y^2} + \nu\,\frac{\partial^2 w}{\partial x^2}\right)\right]$$

This leads us to assume

$$\lambda_1 = \frac{P}{\alpha} = -\frac{1}{\alpha}\left\{t_1(z_7 + z_9) + t_2(z_4 + \nu z_6) + (1-\nu)t_3 z_5\right\}$$

$$\mu_1 = \frac{Q}{\alpha} = -\frac{1}{\alpha}\left\{t_1(z_8 + z_{10}) + (1-\nu)t_2 z_5 + t_3(z_6 + \nu z_4)\right\}$$

$$\frac{\lambda_{13} + \mu_{12}}{2(1-\nu)z_5} = +\frac{z_1}{\alpha} \tag{2.5}$$

where α is a proportionality constant. Under this assumption, compatible with the boundary conditions when z_1 is prescribed to vanish along the edge,

$$\lambda_{12} = +\frac{z_1(z_4 + \nu z_6)}{\alpha}$$

$$\mu_{13} = +\frac{z_1(z_6 + \nu z_4)}{\alpha} \tag{2.6}$$

The three equations above the control equations are rewritten as:

93

$$\frac{\partial \lambda_{11}}{\partial x} + \frac{\partial \mu_{11}}{\partial y} = -1 - \frac{z_1(\beta^2 z_1 - u_1 - 2u_3 - u_5)}{\alpha}$$

$$\frac{\partial \lambda_{12}}{\partial x} + \frac{\partial \mu_{12}}{\partial y} = -\lambda_{11} + \frac{2z_1(z_7 + z_9)}{\alpha}$$

$$\frac{\partial \lambda_{13}}{\partial x} + \frac{\partial \mu_{13}}{\partial y} = -\mu_{11} + \frac{2z_1(z_8 + z_{10})}{\alpha} \qquad (2.7)$$

Differentiating the 2nd equation above with respect to x:

$$\frac{\partial^2 \lambda_{12}}{\partial x^2} + \frac{\partial^2 \mu_{12}}{\partial x \partial y} = -\frac{\partial \lambda_{11}}{\partial x} + \frac{2}{\alpha}[z_2(z_7 + z_9) + z_1(u_1 + u_3)],$$

differentiating the last equation above with respect to y:

$$\frac{\partial^2 \lambda_{13}}{\partial x \partial y} + \frac{\partial^2 \mu_{13}}{\partial y^2} = -\frac{\partial \mu_{11}}{\partial y} + \frac{2}{\alpha}[z_3(z_8 + z_{10}) + z_1(u_3 + u_5)]$$

and adding these two expressions, taking account of the first equation (2.7), one obtains:

$$\frac{\partial^2 \lambda_{12}}{\partial x^2} + \frac{\partial^2(\lambda_{13} + \mu_{12})}{\partial x \partial y} + \frac{\partial^2 \mu_{13}}{\partial y^2} = 1 + \frac{z_1}{\alpha}(\beta^2 z_1 - u_1 - 2u_3 - u_5)$$

$$+ \frac{2}{\alpha}[z_1(u_1 + 2u_3 + u_5) + z_2(z_7 + z_9) + z_3(z_8 + z_{10})] \qquad (2.8)$$

Now we rewrite the left hand side replacing $\lambda_{12}, \lambda_{13} + \mu_{12}$ and μ_{13} respectively by their values from (2.5) and (2.6), yielding:

$$z_4(z_4 + \nu z_6) + 2z_2(z_7 + \nu z_9) + z_1(u_1 + \nu u_3) + 2(1 - \nu)(z_5^2 + z_2 z_9 + z_3 z_8 + z_1 u_3)$$

$$+ z_6(z_6 + \nu z_4) + 2z_3(z_{10} + \nu z_8) + z_1(u_5 + \nu u_3)$$

$$= \alpha + z_1(\beta^2 z_1 - u_1 - 2u_3 - u_5) + 2z_1(u_1 + 2u_3 + u_5) + 2z_2(z_7 + z_9) + 2z_3(z_8 + z_{10})$$

$$(2.9)$$

This relation simplifies into:

$$z_4^2 + 2\nu z_4 z_6 + z_6^2 + 2(1-\nu)z_5^2 = \alpha + \beta^2 z_1^2$$

or, in terms of the sole variable w:

$$\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2\nu \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2(1-\nu)\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 = \alpha + \beta^2 w^2$$

that we might rewrite as:

$$(\nabla^2 w)^2 + 2(1-\nu)\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2}\right] = \alpha + \beta^2 w^2 \qquad (2.10)$$

The expression on the left hand side is, up to a constant multiplicative factor, the potential energy of deformation density of the sandwich plate, which is identical to the one given in Ref. 33 for an elastic plate with constant thickness. This potential density might also be expressed very simply as a symmetric quadratic form of the principal radii of curvature ρ_1 and ρ_2 of the deformed plate, and more precisely in terms of the so-called mean and total curvatures (ref. 33, p. 250). For a sandwich plate with varying thickness such as the one we consider for which the bending rigidity is simply proportional to the thickness t, the potential energy density is independent of t. This is actually not the case for a classical elastic plate with varying thickness h: as we will see in the next chapter, where the potential energy density will be encountered again, it is then a function of h, and more precisely is proportional to the square of the varying thickness.

Actually, (2.10) states for the sandwich plate an extremely general optimality principle encountered in a very broad class of structural optimization problems: for a given fundamental frequency of vibration, (2.10) expresses the conservation of the difference between the potential energy and kinetic energy densities. We will come back in Part E to this extremely important feature, which is a necessary and sufficient condition for an extremal.

If we rewrite the left hand side as:

$$\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right)^2 + (1-\nu^2)\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2(1-\nu)\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2$$

we see that it cannot be negative, the Poisson ratio ν of the given material having to be comprised between the extreme values 0 and 1/2. On the boundary ∂D, $w = 0$, and the above expression takes, from (2.10), the value α: this shows that α is actually a positive quantity; let:

$$\alpha = c^2 .$$

If w is a solution of:

$$(\nabla^2 w)^2 + 2(1-\nu) \left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2}\right] = 1 + \beta^2 w^2 \qquad (2.11)$$

then $\epsilon c w$ is a solution of (2.10). As the displacement mode is known up to a multiplicative constant factor only, and if we recall that α was any multiplicative constant, it is sufficient for our purpose to consider (2.11). The latter is a very interesting looking nonlinear partial differential equation of the second order in the dependent variable w, much simpler than what we might have expected from an original system of 46 equations. It is also very general in that it has to be satisfied inside a domain D of any shape, the constant β^2 being determined from the uniform plate, according to (1.12). For a rectangular domain,

$$\beta^2 = \frac{\pi^4}{1+\kappa} \left(\frac{1}{a^2} + \frac{1}{b^2}\right)^2 \qquad (1.16)$$

The different boundary conditions corresponding to different supports (clamped or simply-supported edges) are to be chosen among (1.10a).

Once the optimal displacement mode is found from (2.11), the optimal thickness distribution is given by the original constraint equation:

$$\nabla^2(\tau \nabla^2 w) + (1-\nu)\left(2\frac{\partial^2 \tau}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial^2 \tau}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 \tau}{\partial y^2}\frac{\partial^2 w}{\partial x^2}\right) - \beta^2(\tau + \kappa)w = 0 \tag{1.10}$$

3. A Numerical Application

Equation (2.11) which gives the optimal displacement mode is, as already mentioned, a second order nonlinear partial differential equation. It is unfortunate but true that equations of this type have suscited practically no interest among mathematicians in our century. The main reason, as quoted from Ames[39] seems that nearly no physical situations within the scope of mathematical formulation yields equations of the aforementioned type to be solved as steps towards the solution. It is hoped that optimization problems making use of variational techniques will give a new life to the whole subject: the constraint equation which we started from was a fourth order one, linear in w, whereas the optimality condition (2.11) is a second order nonlinear one. Similarly, for the shear plate, the constraint was a second order partial differential equation linear in w, yielding an optimality condition in the form of a first order nonlinear equation. The characteristic features common to such problems seem that, whereas the constraint is linear, because based on a previous linearization of the system under investigation, the optimality criterion yields a nonlinear equation of order reduced by half. For sandwich structures, we are always led to an equation involving the optimal displacement mode only[11]: it will be nonlinear of the second order if the original constraint is of the fourth order, as is extremely common in the theory of elasticity.

This present disinterest in nonlinear second order partial differential equations is the more blamable because extremely interesting studies of the subject have been made at the end of the XVIIIth and beginning of the XIXth century by some famous mathematicians. People at this time were less confused by all the bushy notations and hermetical rhetoric which seem to flourish under the pens of some so-called modern mathematicians (to which no reference will be made)

when they deal with the subject of partial differential equations. In other words, the goal was then to find a general solution to the partial differential equation, and to impose the boundary conditions to determine the unknown functions or constants appearing in the expression of the general solution. This method of course does not apply to every case, but is worth of more efforts on the part of researchers confronted with a first or second order nonlinear partial differential equation. It worked in the case of equation (B.2.8) for the most general shape of the boundary.

The author wishes to express his gratitude to Ames for referencing in a modern work[39] the admirable work by Forsyth, first published in 1890 and reprinted by Dover in 1955[40], which is a monumental tribute to the theory of differential equations and bears most of the knowledge in this field up to the beginning of the XXth century. In there were we able to find a mention of the original work of Ampère and in particular the exact reference of his two fundamental contributions[41,42] which led us to the discovery of an exact solution of equation (2.11) when the boundary of the plate is an ellipse. However, we have been unable yet to find an exact solution of this equation for a rectangular boundary: the solution cannot in this case be simply a function of the distance to the boundary as it was for equation (B.2.8), due to requirements of continuity for the bending slope θ, the bending moment M_n and the shear V_n along the lines loci of the points from which two different normals with equal length can be drawn to the contour, requirements which cannot be met under such an assumption.

A numerical solution at this stage of research seems the most profitable thing to do, as it also will help orienting future investigations of equation (2.11).

The reference sandwich plate under consideration will be constructed as follows:* the face material is constituted of 7075-T6 aluminum alloy, which has a Young's modulus of 10.4×10^6 psi, a Poisson ratio of 0.33 $(1-\nu^2 = 0.89)$ and a density ρ_f of 0.1007 lb/in^3. Its thickness is $t_0 = 0.012$ in. The core is made

of 3/16-5052 H39-0015 P hexagonal aluminum honeycomb alloy, with a density ρ_c of 0.00254 lb/in^3, and its thickness is $h_0 = 0.3$ in.

For a simply-supported square plate with sides of length $a = 10$ inches, the fundamental frequency of vibration is given by

$$\omega_f = \frac{2\pi^2}{10^2} \sqrt{\frac{(0.3)^2 \times 10.4 \times 10^6}{2 \times 0.89} \frac{0.012}{0.3 \times 0.00254 + 0.012 \times 0.1007}} = 353 \text{ cps}$$

The values of the parameter κ is

$$\kappa = 25 \frac{0.00254}{0.1007} = 0.632$$

and β^2 is given by:

$$\beta^2 = \frac{4\pi^4}{1+\kappa} \frac{1}{a^4} = \frac{239}{a^4} = 0.0239$$

The numerical problem is now reduced to solving simultaneously the partial differential equations

$$(\nabla^2 w)^2 + 1.34(w_{xy}^2 - w_{xx}w_{yy}) = 1 + 0.0239w^2 \tag{3.1}$$

$$\nabla^2(\tau \nabla^2 w) + 0.67(2\tau_{xy}w_{xy} - \tau_{xx}w_{yy} - \tau_{yy}w_{xx}) - 0.0239(\tau + 0.632)w = 0 \tag{3.2}$$

In the square domain limited by the straight lines $x = 0,10$ and $y = 0,10$ under the boundary conditions (simply-supported edges):

$$w = 0, \qquad \tau w_{xx} = 0 \qquad \text{along} \qquad x = 0,10 \tag{3.1a}$$

$$w = 0, \qquad \tau w_{yy} = 0 \qquad \text{along} \qquad y = 0,10 \tag{3.2a}$$

For the computation, 49 × 49 = 2401 interior mesh points were considered, therefore breaking the domain into 2500 small squares with side length of 0.2. The initial start for w was taken to be either $w = 0$ or $w = \sin \frac{\pi x}{10} \sin \frac{\pi y}{10}$, the displacement mode for the original uniform plate. The solution was found from (3.1) after the same number of iterations, 10, using a regular relaxation method. The contour lines for the optimal displacement mode w are drawn in Fig. 14 (interval taken equal to 0.2). The maximum of w is attained at the center of the plate, and its value is equal to 1.23. This pattern is also shown by the computer printout reproduced in Fig. 15. The letters A, B, C, ... correspond to points with a w in the respective ranges 0.1 ± 0.015, 0.2 ± 0.015 0.3 ± 0.015, ...

The above values for w are then used in (3.2) which is now a second order linear partial differential equation for the optimal nondimensional thickness τ.

Two initial guesses for τ were taken, respectively, to be

$$\tau = 1.2 \sin \frac{\pi x}{10} \sin \frac{\pi y}{10} \quad \text{and} \quad \tau = 1.5 \left| \sin \frac{\pi x}{5} \sin \frac{\pi y}{5} \right| .$$

Both vanish along the edges, as should do the optimal thickness, because neither w_{xx} nor w_{yy} are zero along those edges. The convergence was in that case very slow, and the solution was found after 150 iterations. At that stage, the sum of the squares of the differences between the values of τ during two successive iterations, sum taken over the 2401 mesh points, was only 0.08736, thus securing the validity of the solution. Contour lines are represented in Fig. 16, while a computer printout of the optimal configuration, under those same rules governing the representation of w, is reporduced in fig. 17. The optimal thickness distribution along a median axis of symmetry and along a diagonal of the plate are plotted in figs. 18 and 19, respectively. The mass of the optimal face layers is 69.8% that of the original uniform ones, and this corresponds to a total mass saving of 18.5% for the sandwich plate.

A computer program written in the FORTRAN H language to solve the system (3.1), (3.2), using double precision, is presented in Appendix 3.

τ vanishes only along the edges, which does not prove to be practically inconvenient, as was the optimal shear plate. Also, its maximum is 1.47, which is quite reasonable. However, one might be interested in imposing a minimum thickness constraint as an extra inequality constraint. The application of the general method outlined in Part A is straightforward, and the above procedure may be used with only slight modifications. Due to limitations in computer time, numerical computation was not carried out, but it is a simple exercise for the mind to imagine how the optimal thickness distribution will look like when an inequality constraint is applied.

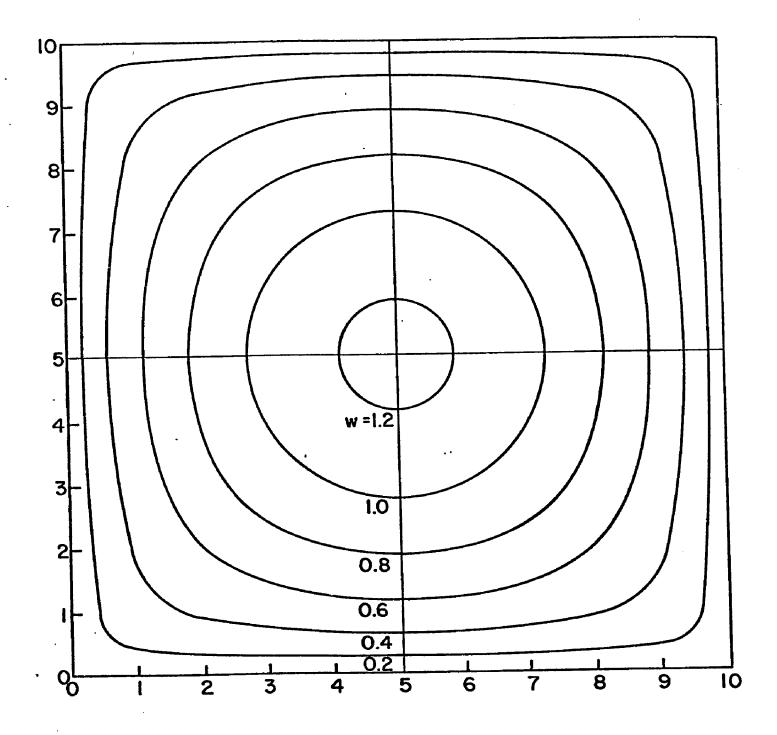Figure 14. Contour lines for the optimal displacement mode w-square simply-supported sandwich plate (the constant c^2 has been set equal to 1).
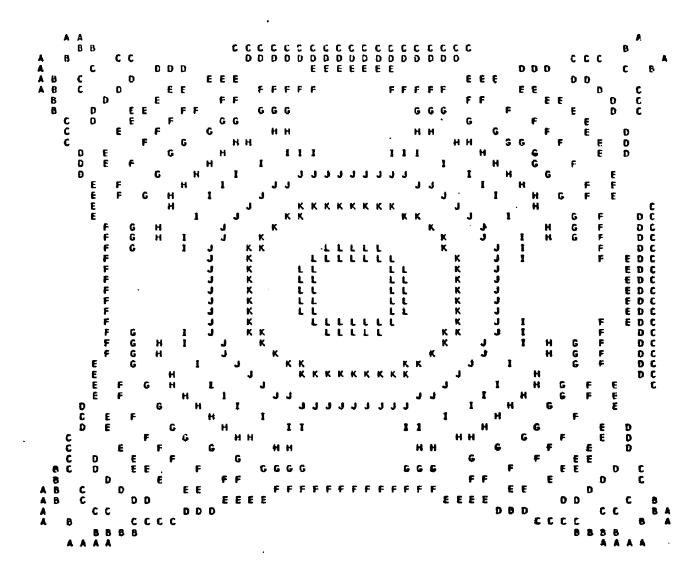
Figure 15. Computer printout for the optimal displacement mode. Letters A, B, C,... correspond to points in the range 0.1 ± 0.015, 0.2 ± 0.015, 0.3 ± 0.015,... .
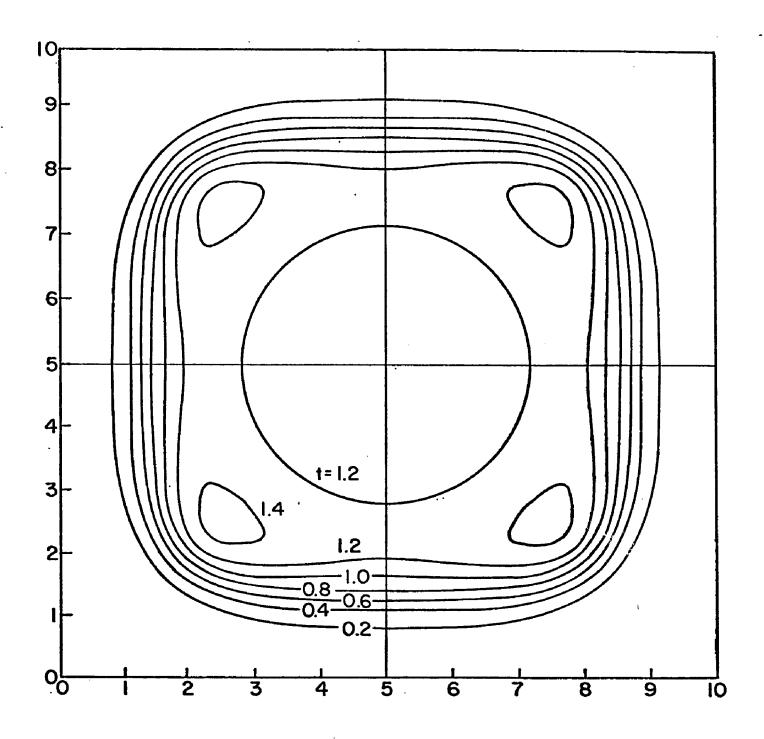
Figure 16. Contour lines for the optimal thickness distribution — square simply-supported sandwich plate.

Figure 17. Computer printout for the optimal thickness distribution. Same notation as in fig. 15.

Figure 18. The optimal thickness distribution along a cross-section following a median axis of symmetry.

Figure 19. The optimal thickness distribution along a cross-section following a diagonal axis of symmetry.

PART D

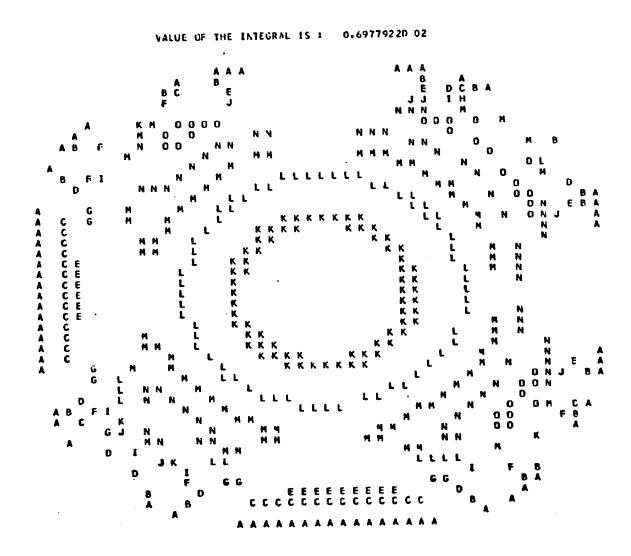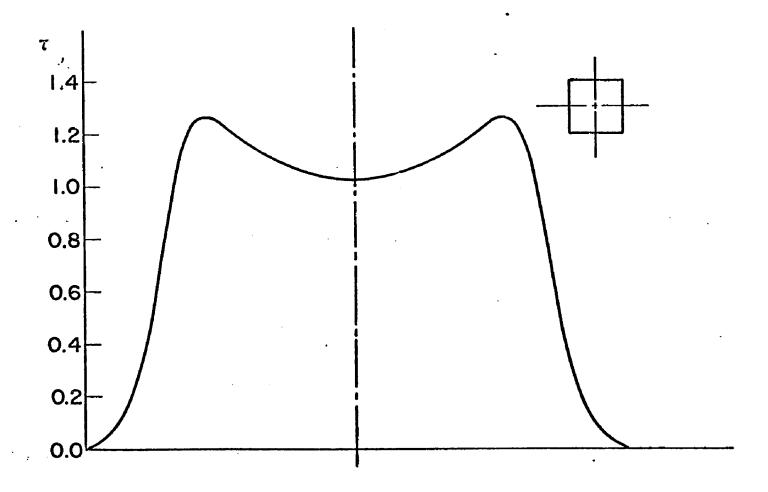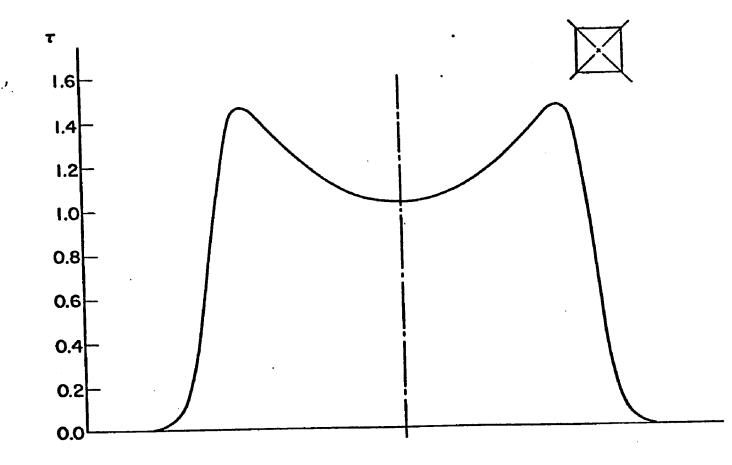## THE MINIMUM-MASS DESIGN OF AN ELASTIC PLATE FOR A GIVEN FUNDAMENTAL FREQUENCY OF VIBRATION

Derivation of the necessary conditions leads for vanishing displacements along the boundary to an optimality criterion expressing again the conservation of the Lagrangian density. It differs from the conditions previously encountered for sandwich structures in the fact that the design parameter h, thickness of the plate, is present. The optimal displacement mode and the optimal thickness distribution are therefore found to be the solutions of a system of two simultaneous nonlinear partial differential equations. Only a numerical approach seems possible in that case.

1.   Statement of the Optimization Problem

The differential equation of equilibrium for a plate of variable thickness h can be found in the classical textbook by Timoshenko,[43] pp. 173–174 or in a paper by Reissner[44]. In rectangular coordinates (x,y) in the plane of the plate, it reads:

$$\nabla^2(D\nabla^2 w)+(1-\nu)\left(2\frac{\partial^2 D}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y}-\frac{\partial^2 D}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\frac{\partial^2 D}{\partial y^2}\frac{\partial^2 w}{\partial x^2}\right)=q \qquad (1.1)$$

in which

w denotes the lateral deflection

$$D = \frac{Eh^3}{12(1-\nu^2)} \quad \text{the flexural rigidity}$$

ν the Poisson's ratio and,

q the intensity of the acting load.

The free vibration of such a plate is governed by equation (1.1) in which q is replaced by the d'Alembert force

$$q = -\rho h \frac{\partial^2 w}{\partial t^2} \qquad (1.2)$$

ρ being the density of the material, assumed to be homogeneous, of which the plate is made: ρ is therefore a constant.

The free vibrations of the uniform reference plate made of the same material and of constant thickness h_0 are governed by the equation

$$D_0 \nabla^4 w = -\rho h_0 \frac{\partial^2 w}{\partial t^2}$$

where

$$D_0 = \frac{E h_0^3}{12(1-\nu^2)}$$

For a simply-supported rectangular plate extending over a domain D defined by $0 \le x \le a$ and $0 \le y \le b$, the natural frequencies of the system are found to be

$$\omega_{mn} = \pi^2 \left[(\frac{m}{a})^2 + (\frac{n}{a})^2 \right] \sqrt{\frac{E h_0^2}{12\rho(1-\nu^2)}} \qquad (1.3)$$

The corresponding natural modes are

$$W_{mn}(x,y) = A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \qquad (1.4)$$

The fundamental frequency of vibration is

$$\omega_f = \omega_{11} = \pi^2 \left(\frac{1}{a^2} + \frac{1}{b^2} \right) \sqrt{\frac{E h_0^2}{12\rho(1-\nu^2)}} \qquad (1.5)$$

The variable thickness h of a plate occupying a domain D and having the same fundamental frequency of vibration ω_f than the uniform one made of the same material and occupying the same plane domain D   under the same boundary conditions, together with the corresponding mode w, have to satisfy the partial differential equation:

$$\nabla^2(D\nabla^2 w)+(1-\nu)\left(2\,\frac{\partial^2 D}{\partial x\partial y}\,\frac{\partial^2 w}{\partial x\partial y}-\frac{\partial^2 D}{\partial x^2}\,\frac{\partial^2 w}{\partial y^2}-\frac{\partial^2 D}{\partial y^2}\,\frac{\partial^2 w}{\partial x^2}\right)-\rho h\,\omega_f^2 w=0$$

(1.6)

For a simply-supported rectangular plate, the value of ω_f is given by (1.5).

The constraint equation (1.6) may be rewritten in terms of h and w only, as

$$h^2\nabla^4 w+6h\left[\frac{\partial h}{\partial x}\,\frac{\partial}{\partial x}(\nabla^2 w)+\frac{\partial h}{\partial y}\,\frac{\partial}{\partial y}(\nabla^2 w)\right]$$

$$+3\left\{\left[2\left(\frac{\partial h}{\partial x}\right)^2+h\frac{\partial^2 h}{\partial x^2}\right]\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)+2(1+\nu)\left[2\frac{\partial h}{\partial x}\frac{\partial h}{\partial y}+h\frac{\partial^2 h}{\partial x\partial y}\right]\frac{\partial^2 w}{\partial x\partial y}\right.$$

$$\left.+\left[2\left(\frac{\partial h}{\partial y}\right)^2+h\frac{\partial^2 h}{\partial y^2}\right]\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right)\right\}-\frac{12(1-\nu^2)}{E}\,\rho\omega_f^2 w=0$$

(1.7)

or, as an alternative way, in terms of D and w only as

$$\nabla^2(D\nabla^2 w)+(1-\nu)\left(2\,\frac{\partial^2 D}{\partial x\partial y}\,\frac{\partial^2 w}{\partial x\partial y}-\frac{\partial^2 D}{\partial x^2}\,\frac{\partial^2 w}{\partial y^2}-\frac{\partial^2 D}{\partial y^2}\,\frac{\partial^2 D}{\partial x^2}\right)-\beta^2 D^{1/3} w=0$$

(1.8)

where the positive constant β^2 is defined as:

$$\beta^2=\left[\frac{12(1-\nu^2)}{E}\right]^{1/3}\rho\,\omega_f^2$$

(1.9)

Our goal is to minimize the total mass of the plate, given by the surface intergral:

$$M = \iint_D \rho \, h \, dx \, dy \qquad (1.10)$$

and the optimization problem may be stated as follows:

Minimize the functional

$$J = \iint_D h \, dx \, dy \qquad (1.11)$$

subject to the constraint (1.7), or, alternatively,

Minimize the functional

$$K = \iint_D D^{1/3} \, dx \, dy \qquad (1.12)$$

subject to the constraint (1.8).

The boundary conditions can be expressed for the more general domain D in terms of the coordinates n and s in the directions normal and tangential to the boundary respectively:[32]

At a simply-supported edge,

$$w = 0$$

$$-D\nabla^2 w + (1-\nu)D\left(\frac{1}{R} \frac{\partial w}{\partial n} + \frac{\partial^2 w}{\partial s^2} \right) = 0$$

For a clamped edge,

$$w = 0$$

$$\frac{\partial w}{\partial n} = 0$$

In the case of a free edge,

$$-D\nabla^2 w + (1-\nu)D\left(\frac{1}{R}\frac{\partial w}{\partial n} + \frac{\partial^2 w}{\partial s^2}\right) = 0$$

$$D\frac{\partial}{\partial n}(\nabla^2 w) + (1-\nu)\frac{\partial}{\partial s}\left[D\left(\frac{\partial^2 w}{\partial n\partial s} - \frac{1}{R}\frac{\partial w}{\partial s}\right)\right] = 0$$

where the Laplacian has the form

$$\nabla^2 = \frac{\partial^2}{\partial n^2} + \frac{1}{R}\frac{\partial}{\partial n} + \frac{\partial^2}{\partial s^2}$$

and R denotes the radius of curvature of the boundary curve ∂D.

For a simply-supported rectangular plate, the boundary conditions read:

$$w = 0 \quad \text{and} \quad D\frac{\partial^2 w}{\partial x^2} = 0 \qquad \text{along} \qquad x = 0, a$$

$$w = 0 \quad \text{and} \quad D\frac{\partial^2 w}{\partial y^2} = 0 \qquad \text{along} \qquad y = 0, b \qquad (1.8a)$$

For clamped edges, they read:

$$w = 0 \quad \text{and} \quad \frac{\partial w}{\partial x} = 0 \qquad \text{along} \qquad x = 0, a$$

$$w = 0 \quad \text{and} \quad \frac{\partial w}{\partial y} = 0 \qquad \text{along} \qquad y = 0, b \qquad (1.8b)$$

2.   The Necessary Conditions

We will concentrate on the optimization problem under the second formulation, where we wish to minimize the surface integral

$$K = \iint_D D^{1/3}\, dxdy \qquad (2.1)$$

112

subject to the constraint (1.8), rewritten as:

$$D\nabla^4 w + 2 \frac{\partial D}{\partial x} \frac{\partial}{\partial x}(\nabla^2 w) + 2 \frac{\partial D}{\partial y} \frac{\partial}{\partial y}(\nabla^2 w)$$

$$+ \frac{\partial^2 D}{\partial x^2}\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right) + 2(1-\nu) \frac{\partial^2 D}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 D}{\partial y^2}\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right)$$

$$-\beta^2 D^{1/3} w = 0 \qquad\qquad (2.2)$$

In a similar fashion to that which was done in Part C, we break (2.2) into a system of first order partial differential equations as follows:

$$z_1 = w$$

$$\frac{\partial z_1}{\partial x} = z_2 \qquad\qquad (= \frac{\partial w}{\partial x})$$

$$\frac{\partial z_1}{\partial y} = z_3 \qquad\qquad (= \frac{\partial w}{\partial y})$$

$$\frac{\partial z_2}{\partial x} = z_4 \qquad\qquad (= \frac{\partial^2 w}{\partial x^2})$$

$$\frac{\partial z_2}{\partial y} = z_5 \qquad\qquad (= \frac{\partial^2 w}{\partial x \partial y})$$

$$\frac{\partial z_3}{\partial x} = z_5 \qquad\qquad (= \frac{\partial^2 w}{\partial x \partial y})$$

$$\frac{\partial z_3}{\partial y} = z_6 \qquad\qquad (= \frac{\partial^2 w}{\partial y^2})$$

$$\frac{\partial z_4}{\partial x} = z_7 \qquad\qquad (= \frac{\partial^3 w}{\partial x^3})$$

(continued)

113

$$\frac{\partial z_4}{\partial y} = z_8 \qquad (= \frac{\partial^3 w}{\partial x^2 \partial y})$$

$$\frac{\partial z_5}{\partial x} = z_8 \qquad (= \frac{\partial^3 w}{\partial x^2 \partial y})$$

$$\frac{\partial z_5}{\partial y} = z_9 \qquad (= \frac{\partial^3 w}{\partial x \partial y^2})$$

$$\frac{\partial z_6}{\partial x} = z_9 \qquad (= \frac{\partial^3 w}{\partial x \partial y^2})$$

$$\frac{\partial z_6}{\partial y} = z_{10} \qquad (= \frac{\partial^3 w}{\partial y^3})$$

$$\frac{\partial z_7}{\partial x} = u_1 \qquad (= \frac{\partial^4 w}{\partial x^4})$$

$$\frac{\partial z_7}{\partial y} = u_2 \qquad (= \frac{\partial^4 w}{\partial x^3 \partial y})$$

$$\frac{\partial z_8}{\partial x} = u_2 \qquad (= \frac{\partial^4 w}{\partial x^3 \partial y})$$

$$\frac{\partial z_8}{\partial y} = u_3 \qquad (= \frac{\partial^4 w}{\partial x^2 \partial y^2})$$

$$\frac{\partial z_9}{\partial x} = u_3 \qquad (= \frac{\partial^4 w}{\partial x^2 \partial y^2})$$

$$\frac{\partial z_9}{\partial y} = u_4 \qquad (= \frac{\partial^4 w}{\partial x \partial y^3})$$

(continued)

$$\frac{\partial z_{10}}{\partial x} = u_4 \qquad\qquad (= \frac{\partial^4 w}{\partial x \partial y^3})$$

$$\frac{\partial z_{10}}{\partial y} = u_5 \qquad\qquad (= \frac{\partial^4 w}{\partial y^4})$$

$$\frac{\partial D}{\partial x} = D_2$$

$$\frac{\partial D}{\partial y} = D_3$$

$$\frac{\partial D_2}{\partial x} = v_1 \qquad\qquad (= \frac{\partial^2 D}{\partial x^2})$$

$$\frac{\partial D_2}{\partial y} = [\, \beta^2 D^{1/3} z_1 - D(u_1 + 2u_3 + u_5) - 2D_2(z_7 + z_9) - 2D_3(z_8 + z_{10})$$

$$-v_1(z_4 + \nu z_6) - v_3(z_6 + \nu z_4)]/2(1-\nu)\,]\, z_5 \qquad (= \frac{\partial^2 D}{\partial x \partial y})$$

$$\frac{\partial D_3}{\partial x} = \frac{\partial D_2}{\partial y}$$

$$\frac{\partial D_3}{\partial y} = v_3 \qquad\qquad (= \frac{\partial^2 D}{\partial y^2}) \qquad (2.3)$$

In the case of a rectangular boundary with simply-supported edges, the boundary conditions read

$$z_1 = 0, \qquad z_4 = 0 \qquad\qquad \text{along} \qquad x = 0, a$$

$$z_1 = 0, \qquad z_6 = 0 \qquad\qquad \text{along} \qquad y = 0, b \qquad (2.3a)$$

We form the Hamiltonian:

$$H \triangleq D^{1/3} + \lambda_1 z_2 + \lambda_2 z_4 + \lambda_3 z_5 + \lambda_4 z_7 + \lambda_5 z_8 + \lambda_6 z_9 + \lambda_7 u_1 + \lambda_8 u_2 + \lambda_9 u_3 + \lambda_{10} u_4$$

$$+ \lambda_{11} D_2 + \lambda_{12} v_1 + \mu_1 z_3 + \mu_2 z_5 + \mu_3 z_6 + \mu_4 z_8 + \mu_5 z_9 + \mu_6 z_{10} + \mu_7 u_2 + \mu_8 u_3 + \mu_9 u_4$$

$$+ \mu_{10} u_5 + \mu_{11} D_3 + \mu_{13} v_3$$

$$+ \frac{\lambda_{13} + \mu_{12}}{2(1-\nu)z_5} \left\{ \beta^2 D^{1/3} z_1 - D(u_1 + 2u_3 + u_5) - 2D_2(z_7 + z_9) - 2D_3(z_8 + z_{10}) \right.$$

$$\left. -v_1(z_4 + \nu z_6) - v_3(z_6 + \nu z_4) \right\} \qquad (2.4)$$

The necessary conditions are:

$$\frac{\partial \lambda_1}{\partial x} + \frac{\partial \mu_1}{\partial y} = -\frac{\partial H}{\partial z_1} = -\frac{\lambda_{13} + \mu_{12}}{2(1-\nu)z_5} \beta^2 D^{1/3}$$

$$\frac{\partial \lambda_2}{\partial x} + \frac{\partial \mu_2}{\partial y} = -\frac{\partial H}{\partial z_2} = -\lambda_1$$

$$\frac{\partial \lambda_3}{\partial x} + \frac{\partial \mu_3}{\partial y} = -\frac{\partial H}{\partial z_3} = -\mu_1$$

$$\frac{\partial \lambda_4}{\partial x} + \frac{\partial \mu_4}{\partial y} = -\frac{\partial H}{\partial z_4} = -\lambda_2 + \frac{\lambda_{13} + \mu_{12}}{2(1-\nu)z_5}(v_1 + \nu w_3)$$

$$\frac{\partial \lambda_5}{\partial x} + \frac{\partial \mu_5}{\partial y} = -\frac{\partial H}{\partial z_5} = -\lambda_3 - \mu_2 + \frac{\lambda_{13} + \mu_{12}}{2(1-\nu)z_5^2} \left\{ \beta^2 D^{1/3} z_1 - D(u_1 + 2u_3 + u_5) \right.$$

$$\left. -2D_2(z_7 + z_9) - 2D_3(z_8 + z_{10}) - v_1(z_4 + \nu z_6) - v_3(z_6 + \nu z_4) \right\}$$

$$\frac{\partial \lambda_6}{\partial x} + \frac{\partial \mu_6}{\partial y} = -\frac{\partial H}{\partial z_6} = -\mu_3 + \frac{\lambda_{13} + \mu_{12}}{2(1-\nu)z_5}(v_3 + \nu v_1)$$

$$\frac{\partial \lambda_7}{\partial x} + \frac{\partial \mu_7}{\partial y} = -\frac{\partial H}{\partial z_7} = -\lambda_4 + \frac{\lambda_{13} + \mu_{12}}{(1-\nu)z_5} D_2$$

(continued)

116

$$\frac{\partial \lambda_8}{\partial x} + \frac{\partial \mu_8}{\partial y} = - \frac{\partial H}{\partial z_8} = -\lambda_5 - \mu_4 + \frac{\lambda_{13} + \mu_{12}}{(1-\nu)z_5} D_3$$

$$\frac{\partial \lambda_9}{\partial x} + \frac{\partial \mu_9}{\partial y} = - \frac{\partial H}{\partial z_9} = -\lambda_6 - \mu_5 + \frac{\lambda_{13} + \mu_{12}}{(1-\nu)z_5} D_2$$

$$\frac{\partial \lambda_{10}}{\partial x} + \frac{\partial \mu_{10}}{\partial y} = - \frac{\partial H}{\partial z_{10}} = - \mu_6 + \frac{\lambda_{13} + \mu_{12}}{(1-\nu)z_5} D_3$$

$$\frac{\partial \lambda_{11}}{\partial x} + \frac{\partial \mu_{11}}{\partial y} = - \frac{\partial H}{\partial D} = - \frac{1}{3} D^{-2/3} - \frac{\lambda_{13} + \mu_{12}}{2(1-\nu)z_5} \left\{ \frac{1}{3} \beta^2 D^{-2/3} z_1 - u_1 - 2u_3 - u_5 \right\}$$

$$\frac{\partial \lambda_{12}}{\partial x} + \frac{\partial \mu_{12}}{\partial y} = - \frac{\partial H}{\partial D_2} = -\lambda_{11} + \frac{\lambda_{13} + \mu_{12}}{(1-\nu)z_5} (z_7 + z_9)$$

$$\frac{\partial \lambda_{13}}{\partial x} + \frac{\partial \mu_{13}}{\partial y} = - \frac{\partial H}{\partial D_3} = -\mu_{11} + \frac{\lambda_{13} + \mu_{12}}{(1-\nu)z_5} (z_8 + z_{10})$$

$$\frac{\partial H}{\partial u_1} \equiv \lambda_7 - \frac{\lambda_{13} + \mu_{12}}{2(1-\nu)z_5} D = 0$$

$$\frac{\partial H}{\partial u_2} \equiv \lambda_8 + \mu_7 = 0$$

$$\frac{\partial H}{\partial u_3} \equiv \lambda_9 + \mu_8 - \frac{\lambda_{13} + \mu_{12}}{(1-\nu)z_5} D = 0$$

$$\frac{\partial H}{\partial u_4} \equiv \lambda_{10} + \mu_9 = 0$$

$$\frac{\partial H}{\partial u_5} \equiv \mu_{10} - \frac{\lambda_{13} + \mu_{12}}{2(1-\nu)z_5} D$$

$$\frac{\partial H}{\partial v_1} \equiv \lambda_{12} - \frac{\lambda_{13} + \mu_{12}}{2(1-\nu)z_5} (z_4 + \nu z_6)$$

$$\frac{\partial H}{\partial v_3} \equiv \mu_{13} - \frac{\lambda_{13} + \mu_{12}}{2(1-\nu)z_5} (z_6 + \nu z_4) \tag{2.5}$$

117

For a rectangular simply-supported plate, the boundary conditions read:

$$z_1 = \lambda_2 = \lambda_3 = z_4 = \lambda_5 = \lambda_6 = \lambda_7 = \lambda_8 = \lambda_9 = \lambda_{10} = \lambda_{11} = \lambda_{12} = \lambda_{13} = 0$$

along $x = 0, a$

$$z_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = z_6 = \mu_7 = \mu_8 = \mu_9 = \mu_{10} = \mu_{11} = \mu_{12} = \mu_{13} = 0$$

along $y = 0, b$ (2.5a)

As in the sandwich plate problem of Part C, we note that the first necessary condition

$$\frac{\partial \lambda_1}{\partial x} + \frac{\partial \mu_1}{\partial y} = - \frac{\lambda_{13} + \mu_{12}}{2(1-\nu)z_5} \beta^2 D^{1/3}$$

is very similar in appearance to equation (2.2), rewritten as:

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = \beta^2 D^{1/3} z_1 \tag{2.6}$$

where

$$P = \frac{\partial}{\partial x}\left[D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right)\right] + (1-\nu)\frac{\partial}{\partial y}\left[D \frac{\partial^2 w}{\partial x \partial y}\right]$$

$$Q = (1-\nu)\frac{\partial}{\partial x}\left[D \frac{\partial^2 w}{\partial x \partial y}\right] + \frac{\partial}{\partial y}\left[D\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right)\right]$$

and therefore leads us to assume the proportionality relations:

$$\lambda_1 = \frac{P}{\alpha} = -\frac{1}{\alpha}[D(z_7 + z_9) + D_2(z_4 + \nu z_6) + (1-\nu)D_3 z_5]$$

$$\mu_1 = \frac{Q}{\alpha} = -\frac{1}{\alpha}[D(z_8 + z_{10}) + (1-\nu)D_2 z_5 + D_3(z_6 + \nu z_4)]$$

$$\frac{\lambda_{13} + \mu_{12}}{2(1-\nu)z_5} = +\frac{z_1}{\alpha} , \tag{2.7}$$

118

α being a proportionality constant. This assumption is compatible with the boundary conditions (2.5a). We now proceed in the same manner as we did in Part C, computing λ_{12} and μ_{13} from the control equations:

$$\lambda_{12} = +\frac{z_1(z_4 + \nu z_6)}{\alpha}$$

$$\mu_{13} = +\frac{z_1(z_6 + \nu z_4)}{\alpha}$$

and rewriting the last three equations before the control equations as:

$$\frac{\partial \lambda_{11}}{\partial x} + \frac{\partial \mu_{11}}{\partial y} = -\frac{1}{3} D^{-2/3} - \frac{z_1}{\alpha} \left\{ \frac{1}{3} \beta^2 D^{-2/3} z_1 - u_1 - 2u_3 - u_5 \right\}$$

$$\frac{\partial \lambda_{12}}{\partial x} + \frac{\partial \mu_{12}}{\partial y} = -\lambda_{11} + \frac{2z_1(z_7 + z_9)}{\alpha}$$

$$\frac{\partial \lambda_{13}}{\partial x} + \frac{\partial \mu_{13}}{\partial y} = -\mu_{11} + \frac{2z_1(z_8 + z_{10})}{\alpha} \tag{2.8}$$

We differentiate the second equation above with respect to x, the third one with respect to y and add the results, yielding, in view of the top equation (2.8),

$$\frac{\partial^2 \lambda_{12}}{\partial x^2} + \frac{\partial^2 (\lambda_{13} + \mu_{12})}{\partial x \partial y} + \frac{\partial^2 \mu_{13}}{\partial y^2} = +\frac{1}{3} D^{-2/3} + \frac{z_1}{\alpha} \left\{ \frac{1}{3} \beta^2 D^{-2/3} z_1 - u_1 - 2u_3 - u_5 \right\}$$

$$+ \frac{2}{\alpha} \left\{ z_1(u_1 + 2u_3 + u_5) + z_2(z_7 + z_9) + z_3(z_8 + z_{10}) \right\} \tag{2.9}$$

The left hand side above is computed from the expressions for λ_{12}, $\lambda_{13} + \mu_{12}$ and μ_{13} derived earlier: (2.9) is rewritten as, after simplification:

$$(z_4 + z_6)^2 + 2(1-\nu)(z_5^2 - z_4 z_6) = \frac{1}{3} D^{-2/3} (\alpha + \beta^2 z_1^2)$$

or, in terms of the original w and D variables:

$$(\nabla^2 w)^2 + 2(1-\nu)\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right] = \frac{1}{3}D^{-2/3}(\alpha + \beta^2 w^2) \qquad (2.10)$$

Equation (2.10) is a second order nonlinear partial differential equation in the two dependent variables D, optimal flexural rigidity, and w, optimal displacement mode. Together with equation (1.8) it forms a system of simultaneous partial differential equations leading to the solution of the optimization problem.

(2.10) can also be rewritten in terms of the optimal thickness distribution h and w, as:

$$(\nabla^2 w)^2 + 2(1-\nu)\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right] = \frac{4(1-\nu^2)}{E}\rho\omega_f^2\frac{\alpha^* + w^2}{h^2} \qquad (2.11)$$

the constant α^* being defined as

$$\alpha^* = \frac{\alpha}{\beta^2},$$

and, together with (1.7), forms a system of two simultaneous partial differential equations for the two unknowns h, optimal thickness distribution, and w, corresponding displacement mode, the latter being known up to a constant multiplying factor only, as usual. α^* can be shown to be positive, exactly as for the sandwich plate equation. Let

$$\frac{4(1-\nu^2)}{E}\rho\omega_f^2\alpha^* = c^2$$

The solution w will be of the form cw_0, where w_0 satisfies:

$$(\nabla^2 w)^2 + 2(1-\nu)\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right] = 1 + \frac{4(1-\nu^2)}{E}\rho\omega_f^2 w^2/h^2 \qquad (2.12)$$

For a simply-supported rectangular plate, the solution of the optimization problem reduces to that of the two simultaneous partial differential equations:

$$(\nabla^2 w)^2 + 2(1-\nu)\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right] = \frac{\pi^4}{3}\left(\frac{1}{a^2} + \frac{1}{b^2}\right)^2 \frac{\alpha^* + w^2}{\tau^2} \quad (2.13)$$

$$\tau^2 \nabla^4 w + 6\tau\left[\frac{\partial \tau}{\partial x}\frac{\partial}{\partial x}(\nabla^2 w) + \frac{\partial \tau}{\partial y}\frac{\partial}{\partial y}(\nabla^2 w)\right]$$

$$+3\left[2\left(\frac{\partial \tau}{\partial x}\right)^2 + \tau\frac{\partial^2 \tau}{\partial x^2}\right]\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) + 2(1+\nu)\left[2\frac{\partial \tau}{\partial x}\frac{\partial \tau}{\partial y} + \tau\frac{\partial^2 \tau}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y}\right]$$

$$+\left[2\left(\frac{\partial \tau}{\partial y}\right)^2 + \tau\frac{\partial^2 \tau}{\partial y^2}\right]\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) - \pi^4\left(\frac{1}{a^2} + \frac{1}{b^2}\right)^2 w = 0 \quad (2.14)$$

$\tau = h/h_0$ being the non-dimensional optimal thickness distribution ($\tau = 1$ for the uniform reference structure).

The boundary conditions for simply-supported edges are:

$$w = 0 \quad \text{and} \quad \tau\frac{\partial^2 w}{\partial x^2} = 0 \qquad \text{along} \qquad x = 0, a$$

$$w = 0 \quad \text{and} \quad \tau\frac{\partial^2 w}{\partial y^2} = 0 \qquad \text{along} \qquad y = 0, b \qquad (2.14a)$$

For clamped edges:

$$w = 0 \quad \text{and} \quad \frac{\partial w}{\partial x} = 0 \qquad \text{along} \qquad x = 0, a$$

$$w = 0 \quad \text{and} \quad \frac{\partial w}{\partial y} = 0 \qquad \text{along} \qquad y = 0, b \qquad (2.14b)$$

A computational approach at this stage of the optimization process seems now inevitable, and the above system seems suitable to such an approach.

Equation (2.12) is very similar to the optimality condition derived by Suhubi[45] for the optimal plastic design of a plate using an entirely different approach.

THE OPTIMALITY CRITERION: A SUFFICIENT CONDITION

FOR AN EXTREMAL

The nonlinear partial differential equation to which the entire system of necessary conditions reduces in each of the three cases considered so far under the assumption of prescribed vanishing displacement at the edge is the expression, in each case, of an optimality criterion expressing the constance of the energy distribution of the system, which is a sufficient condition for an extremum and was first derived by Prager for a broad class of structural optimization problems. The system of two partial differential equations to be satisfied simultaneously or not by the optimal displacement mode and the optimal thickness distribution, which is formed by this optimality criterion and the original constraint equation, and to which each of the aforementioned cases reduces is therefore necessary and sufficient. For the shear plate, uniqueness of the optimal solution follows.

When the displacement is not required to vanish along the edge, Prager's criterion still applies, but has to be written in a three-dimensional space. The above fact agrees with the validity of the proportionality assumptions made in the course of the derivation of the optimality condition.

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Equations (B. 2. 8), (C. 2. 10) and (D. 2. 11) are nothing but the expression of a very general optimality criterion stating the uniformity of the energy distribution throughout the structure under consideration, valid for three dimensional structures to be optimally designed under the most general assumptions. It was first derived by Prager[46,47] and is an alternate form of stating the so-called "conservation of the Lagrangian density" which was pointed out by Ashley and McIntosh.[†]

[†]The term Lagrangian density which was introduced by Ashley and McIntosh[13] in optimal frequency design has its origin in the similarity between the Lagrangian of Classical Mechanics, difference between the kinetic and potential energy, and the expression $G - \varphi H$, difference between the potential energy density and the quantity $\rho \omega_f^2 w^2$, which can be viewed as a kinetic energy density.

Using Prager's notation,[46] the structure is to be designed to satisfy a single behavioral constraint prescribing the value of a scalar Φ itself characterized by a global minimum principle involving a field φ associated with the structure, and having one of the following forms:

$$\Phi = \min \int_V F[\varphi]dV \tag{1a}$$

$$\Phi = \min \frac{\int_V G[\varphi]dV}{\int_V H[\varphi]dV} \tag{1b}$$

V being the volume occupied by the structure. When the constraint is on the fundamental frequency of vibration, Φ is the square of the prescribed value ω_f of the frequency and, from Rayleigh's principle follows that the constraint is of the form (1b) above, where $G[\varphi]$ is twice the strain energy density of the field φ, and $H[\varphi]$ is $\rho|\varphi|^2$, where ρ is the density of the material.

Prager has proven that when the behavioral constraint is expressed by a minimum principle of the form (1a), a sufficient condition for an optimum is:

$$F = \text{Constant} \tag{2a}$$

on the portion of the surface S where vanishing surface tractions are prescribed.

When the constraint is in the form (1b), a sufficient condition is:

$$G - \Phi H = \text{Constant} \tag{2b}$$

on that same portion of the external surface S.

The above conditions can also be shown to be necessary. *

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*The general formulation of the sufficient conditions above is actually slightly more complex, and the best way to express them is, as did Prager, to use notations from set theory. The reader is referred to references 46, 47 for more precision.

A proof of the sufficiency of the condition of uniform energy distribution, which we believe to be original, will be given below for the case of interest to us, the minimum-mass design of a structure (one, two or three-dimensional) for a given fundamental frequency of free vibration. This proof makes use of Rayleigh's principle.

Let $\mathfrak{D}$ be the set of all structures with the same constant density ρ similar to the uniform reference structure D_0 and all having the same fundamental frequency of free vibration ω_f under the same support conditions. Among those, at least one, which we shall call $\overline{D}$, is assumed to be such that its energy distribution is constant throughout the volume $\overline{V}$ it encloses. If $\overline{\varphi}$ is the displacement field associated with it, $G[\overline{\varphi}]$ twice the strain energy density of the field, we assume that:

$$G[\overline{\varphi}] - \omega_f^2 \rho \overline{\varphi}^2 = c^2 \tag{3}$$

and, from Rayleigh's principle, ω_f is exactly given as:

$$\omega_f^2 = \frac{\displaystyle\int_{\overline{V}} G[\overline{\varphi}]dV}{\displaystyle\int_{\overline{V}} \rho \overline{\varphi}^2 dV} \tag{4}$$

rewritten as:

$$\int_{\overline{V}} \left\{ G[\overline{\varphi}] - \omega_f^2 \rho \overline{\varphi}^2 \right\} dV = 0 \tag{4a}$$

Now consider any structure D member of the above set and occupying a volume V, with the corresponding field φ. From Rayleigh's principle,

$$\omega_f^2 = \frac{\displaystyle\int_V G[\varphi]\,dV}{\displaystyle\int_V \rho\varphi^2\,dV} \tag{5}$$

and, $\overline{\varphi}$ being the field corresponding to the structure $\overline{D}$, we also have the inequality:

$$\omega_f^2 \leq \frac{\displaystyle\int_V G[\overline{\varphi}]\,dV}{\displaystyle\int_V \rho\overline{\varphi}^2\,dV} \tag{6}$$

rewritten as:

$$\int_V \left\{ G[\overline{\varphi}] - \omega_f^2\,\rho\overline{\varphi}^2 \right\} dV \geq 0 \tag{6a}$$

Subtracting (4a) from (6a),

$$\int_{V-\overline{V}} \left\{ G[\overline{\varphi}] - \omega_f^2\,\rho\overline{\varphi}^2 \right\} dV \geq 0 \tag{7}$$

and, by the use of (3), the inequality reduces to:

$$\int_{V-\overline{V}} dV \geq 0$$

or

$$\overline{V} \leq V \tag{7a}$$

The volume $\overline{V}$ occupied by the structure $\overline{D}$ satisfying the condition (3) is therefore smaller or equal to the volume V occupied by D. As the structure

D was arbitrarily chosen among the set $\mathfrak{D}$, we are led to the conclusion that a structure satisfying (3) has the smallest possible weight* among the structures belonging to the class $\mathfrak{D}$ under consideration. (3) is therefore a sufficient condition for an extremal of the weight under the conditions already stated.

The above sufficient condition is intuitively not at all surprising: this uniformity of the energy distribution is a very general and logical property encountered whenever one wants to make the best out of a given system, and should be considered as a law of nature. In the case of a vibrating plate with uniform thickness, the efforts are not uniformly distributed throughout the plate, and some parts undergo more stress than others. We might say that the uniform plate is not ideally designed for vibration purposes, and it reacts to vibrating conditions in an unorderly and anarchistic behavior, which is however the best it can do if it cannot modify its total shape. This is also a law of nature. Suppose now that the plate could modify its shape and mass: it would still comply to the laws of nature, but with more freedom now, and tend towards an optimal shape. The non-uniform plate would now be perfectly adapted to the vibrating conditions imposed on it, and the efforts would now be equally distributed throughout. This is the best of all plates under the given conditions, such that the energy distribution is uniform and the same role is assigned to any point of the structure. Understandably, such a structure will be optimally designed for the purpose at hand, here the first vibration frequency being held constant, and this optimal plate will have the lesser possible weight of all structures having the same fundamental frequency, its matter being the more properly distributed of all of them.

This uniformity of energy distribution attained by Nature in the best of all configurations of a system is also a very general principle going beyond structural optimization: Max Munk showed in 1918[48] that the minimum induced drag of a wing is obtained if the distribution of lift over the span is elliptic, corresponding to a uniform energy distribution over the wing area.

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*The material is assumed to be isotropic with uniform density ρ.

However, this "secret of successful engineering construction ... to make each part of a structure just as strong as the rest" has been known for many years, as illustrated by a funny little poem found in the ever interesting Recreations in Mathematics by H. E. Licks (D. Van Nostrand Company, Inc., New York, N.Y., 1917). This poem had been written "many years ago" by Oliver Wendell Holmes, and tells of the misadventures of a "one-hoss shay" which was optimally designed by a Massachusetts Deacon to last a hundred years. It begins:

> Have you heard of the wonderful one-hoss shay
> > That was built in such a logical way
> It ran a hundred years to a day?

The Deacon was resolved that the "shay" should be properly constructed:

> But the Deacon swore (as deacons do)
> > It should be so built that it couldn't break down,
> "Fur," said the Deacon, "'tis mightly plain
> > That the weakest spot must stan' the strain,
> And the way to fix it, as I maintain, is only jest
> > To make that place as strong as the rest."

> The wheels were just as strong as the thills,
> > And the floor was just as strong as the sills,
> And the panels just as strong as the floor,
> > And the whipple-tree neither less or more.
> And the back cross bar as strong as the fore
> > And spring and axle and hub encore.

The chaise was designed to run exactly a hundred years, and so it did. When that time arrived a parson was riding in it and the catastrophe came.

> All at once the horse stood still,
> > Close by the meetin'-house on the hill,
> First a shiver, and then a thrill,
> > Then something decidedly like a spill,
> And the parson was sitting on a rock,
> > At half-past nine by the meetin'-house clock.

You see of course, if you're not a dunce
    How it went to pieces all at once,
All at once and nothing first,
    Just like bubbles when they burst.

End of the wonderful one-hoss shay,
    Logic is logic! That's all I say.

And the lesson from this fable might be that the mathematical beauty of optimal design also brings with it rigorous consequences!

Let us now verify that we indeed obtained the optimality condition for the three structures we considered.

For the shear plate, the strain energy density is:

$$V = \frac{G}{2} \left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2 \right] \tag{8}$$

and H is simply:

$$H = \rho w^2 \tag{9}$$

so that (2a) is expressed under the form:

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 - \frac{\rho}{G} \omega_f^2 w^2 = \text{constant} \tag{10}$$

which is nothing but equation (B.2.8) which is therefore a necessary and sufficient condition for an extremum. This equation was derived for simply-supported edges, and was found to have a unique solution, given analytically by (B.2.9). It follows that it is _the_ solution to the optimization problem, and the shear plate problem possesses a unique solution as described in Part B.

The strain energy density of a sandwich plate is, in terms of the lateral deflection $w(x, y)$:[46]

$$V = \frac{E_s h_0^2}{4(1-\nu^2)} \left\{ \left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 + 2\nu \left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right] \right\} \tag{11}$$

expressing the optimality condition (2a) as:

$$\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 + 2\nu\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right] - \beta^2 w^2 = \text{constant} \tag{12}$$

where β^2 is a constant whose value is the same as the β^2 defined by (C.1.12).

This is nothing but another way of writing (C.2.10). Our proportionality assumption (C.2.5) was thus perfectly valid, and (C.2.10) is not only a necessary, but also a sufficient condition for optimality.

The fact that for sandwich structures the optimality condition does not involve the design parameter τ was pointed out by Prager and Taylor. [11]

For the classical plate with varying thickness, the strain energy density in bending is given by: [43]

$$V = \frac{Eh^2}{8(1-\nu^2)}\left\{(\nabla^2 w)^2 + 2(1-\nu)\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right]\right\} \tag{13}$$

and Prager's optimality criterion leads to the expression:

$$h^2\left\{(\nabla^2 w)^2 + 2(1-\nu)\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right]\right\} - \frac{4(1-\nu^2)}{E}\rho\omega_f^2 w^2 = \text{constant} \tag{14}$$

which is exactly (D.2.11).

The precision and beauty of Prager's theorem is even more evident when one considers the following important fact, which will also serve the purpose of clarifying a delicate point, which is the assumption of proportionality in the course of the derivation of the necessary conditions. This remark led to the conservation property; however, it was compatible with the boundary conditions only in the case of vanishing displacements at the edge: this was the case for the simply-supported shear plate, for the clamped or simply-supported sandwich plate and also for the elastic plate. On the other hand, Prager's theorem seems to lead to the same optimality condition, whatever the boundary conditions

are. A careful review of the conditions under which it is valid is therefore necessary at this point.

Prager's theorem applies to optimal structures occupying the volume V with the surface S. Each element of S is supposed to belong to one and only one of the sets S', S", S''', where nonvanishing surface tractions are prescribed on the elements of S' (for problems that we consider here S' = 0), vanishing surface tractions on the elements of S" (here the surface covered by the plate), and vanishing displacements on the elements of S'''. Moreover, the surfaces S' and S''' and the boundary conditions thereon are regarded as fixed "ingredients" of the design problem. S''' is the boundary of the plate when the displacement is prescribed to vanish along the edge. However, the portions of the boundary which are not supported belong to S", the plate being viewed in the broad sense of a three-dimensional structure, and cannot be regarded any more as fixed ingredients of the problem: in such cases, the optimization problem needs to be viewed in three dimensions, and will not yield an optimality condition as simple as (B. 2. 8), (C. 2. 10) or (D. 2. 11).

Prager's theorem therefore still applies, but with an added dimension. As we recall, when portions of the edge are free, the proportionality assumption expressed by (B. 2. 7), (C. 2. 5) or (D. 2. 7) is no longer compatible with the boundary conditions, rendering equations (B. 2. 8), (C. 2. 10) or (D. 2. 11) no longer valid.

This is one more evidence of the complete agreement between Prager's optimality criterion and our results, obtained by an entirely different method based on the calculus of variations in two dimensions.

CONCLUSIONS

In our approach to optimization problems in two dimensions, outlined in the theory presented in the first part of this work and applied to the subsequent structural optimization problems, we are confronted with a system of partial differential equations to be solved inside a domain, together with boundary conditions, both expressing a set of necessary and sufficient conditions for an optimum. A computational appraoch undertaken at this stage, although being in theory a classical problem of Numerical Analysis[49] would prove very awkward and especially time-consuming: for a simple structure such as a sandwich plate or an elastic plate and under a very simple constraint such as the fundamental frequency of vibration being fixed, the necessary conditions yield a system of 46 equations in 46 unknowns, mostly first-order partial differential equations! However, and this is one of the purposes of the present study, a careful study of these necessary conditions in each particular case permits, under assumptions later verified, to reduce them to only two partial differential equations in the important unknowns. In some cases we will be lucky enough to find an exact analytical solution, and in the majority we will have reduced the original problem to a reasonable amount of numerical computation, following the well-established pattern to solve numerically a single partial differential equation or a system of a limited number of the same.

This property leads us to make the remark that, in the case of optimization problems in one dimension, the necessary conditions for an extremum lead to two-point boundary value problems, requiring extreme caution: no universal numerical method has been found up to date to solve any kind of such problems (for example, the powerful transition-matrix procedure which gave excellent results in number of cases in structural optimization failed for the panel-flutter case where the non-linearity was too strong). However, for two-dimensional optimization procedures, following the method presented here, we are in the end confronted with a problem of a classical type and there is, in theory at least, a method permitting to solve it.

This is a very important feature differentiating optimization problems in one-dimensional space from those in higher dimensional spaces. This is why we will be less pessimistic than R. Chattopadhyay[50] about the existence of practical solutions for problems of optimal control of systems with distributed parameters, always keeping in mind that asking for the services of a computer should be the last thing to do, and should be done only after we have made sure that every possible method or trick failed. It is not our goal to discredit the computer: its help is tremendous in the majority of cases, and it brought the solution to inextricable practical problems, but some reflexion makes it more efficient and ... cheaper!

However, we will at this point raise several questions, as formulated in Ref. 50.

It is better to discretize the system itself in the first place, as did Haug,[19] than to discretize the equations expressing the necessary conditions for an extremal, obtained for the exact system? Are the approximate equations consistent with the original system equations? Is the discretization of the system valid? Do the approximate solutions converge to the actual one in the limit and do the errors made remain bounded? These questions are unfortunately very difficult to answer and are beyond the scope of the present investigation, but on their solution rests the possibility of obtaining results of practical interest.

It is our sincere hope that the present study will lay the theoretical foundations necessary to future investigations in this entirely new field of two-dimensional structural optimization, reducing any such problem to the search of the numerical solution of a simple system of partial differential equations: this is by itself a classical but unfortunately nontrivial problem, and attaining the solution will rely mainly on the skill of the researcher as well as the special nature of the structure investigated. The choice of two-dimensional systems to be investigated is of course very wide, and more general constraints can be considered, still following the pattern presented before. Of particular practical

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APPENDICES

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APPENDIX 1

INDEPENDENCE OF THE FORM OF THE SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS ON THE OPTIMIZATION PROBLEM

In Part A, we showed that the constraint defining the optimal problem for the shear plate, expressed in the form of the single partial differential equation of order 2 in w and 1 in h:

$$\frac{\partial}{\partial x}(h \frac{\partial w}{\partial x}) + \frac{\partial}{\partial y}(h \frac{\partial w}{\partial y}) + k^2 hw = 0 \tag{1}$$

could be put under the form of two systems of very different appearance, part of the control variables being in one of them the thickness h, in the other one its first derivatives.

Derivation of the necessary conditions for an optimum was made for the first case in section B. 2. 1 when the functional to be minimized is the integral of h over a plane domain D under the given constraint. Let us show now that the same answer is found if we start from the other system, obtained by the means described in (A. 1), and derive from it the necessary conditions.

With this system as a constraint, the Hamiltonian takes the form:

$$H = h + \lambda_1 z_2 + \lambda_2 u_1 + \lambda_3 u_2 + \lambda_4 v_1 + \mu_1 z_3 + \mu_2 u_2$$

$$-\mu_3 (u_1 + k^2 z_1 + \frac{v_1 z_2 + v_2 z_3}{h}) + \mu_4 v_2 \tag{2}$$

The necessary conditions for an extremal are thus found to be:

$$\frac{\partial \lambda_1}{\partial x} + \frac{\partial \mu_1}{\partial y} = k^2 \mu_3 \tag{3}$$

$$\frac{\partial \lambda_2}{\partial x} + \frac{\partial \mu_2}{\partial y} = -\lambda_1 + \frac{\mu_3 v_1}{h} \tag{4}$$

$$\frac{\partial \lambda_3}{\partial x} + \frac{\partial \mu_3}{\partial y} = -\mu_1 + \frac{\mu_3 v_2}{h} \tag{5}$$

$$\frac{\partial \lambda_4}{\partial x} + \frac{\partial \mu_4}{\partial y} = -1 - \mu_3 \frac{v_1 z_2 + v_2 z_3}{h^2} \tag{6}$$

$$\lambda_2 - \mu_3 = 0 \tag{7}$$

$$\lambda_3 + \mu_2 = 0 \tag{8}$$

$$\lambda_4 - \frac{\mu_3 z_2}{h} = 0 \tag{9}$$

$$\mu_4 - \frac{\mu_3 z_3}{h} = 0 \tag{10}$$

The form of the first equation suggests to try the substitution:

$$\lambda_1 = \frac{h z_2}{\alpha}$$

$$\lambda_2 = \frac{h z_3}{\alpha}$$

$$\mu_3 = -\frac{h z_1}{\alpha} \tag{11}$$

by analogy with the original constraint equation. Now,

$$\lambda_4 = -\frac{z_1 z_2}{\alpha} \tag{9}$$

$$\mu_4 = -\frac{z_1 z_3}{\alpha} \tag{10}$$

and (6) is rewritten as:

$$\frac{\partial}{\partial x}(z_1 z_2) + \frac{\partial}{\partial y}(z_1 z_3) = \alpha - \frac{z_1}{h}(v_1 z_2 + v_2 z_3) \tag{6}$$

or, in terms of w and h:

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 + w \nabla^2 w = \alpha - \frac{w}{h} \left(\frac{\partial h}{\partial x} \frac{\partial w}{\partial x} + \frac{\partial h}{\partial y} \frac{\partial w}{\partial y}\right) \tag{6}$$

But, from the original constraint,

$$\frac{\partial h}{\partial x} \frac{\partial w}{\partial x} + \frac{\partial h}{\partial y} \frac{\partial w}{\partial y} = - k^2 hw - h \nabla^2 w$$

so that (6) is put under the form:

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = \alpha + k^2 w^2 , \tag{6}$$

which is nothing but equation (B.2.8), the solution of which led to the optimal thickness distribution of the shear plate. It is easy to check that the hypotheses (11) are consistent with the system formed by the original constraints, the necessary conditions and the boundary conditions in the simply-supported case.

APPENDIX 2

ON ANOTHER METHOD TO SOLVE THE PARTIAL DIFFERENTIAL

EQUATION ARISING IN THE SHEAR PLATE OPTIMAL PROBLEM

A very interesting alternative approach to solve the equation:

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = c^2 + k^2 w^2 \tag{1}$$

and also applicable to more general boundary conditions than the one we considered, i.e., $w = 0$ along ∂D, has been suggested by Dr. E. O. A. Naumann.

Let us make the change of dependent variable, introducing the new function z defined by:

$$w = \frac{c}{k} \sinh(kz) \tag{2}$$

(1) reduces to:

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 1 \tag{3}$$

or, using standard notations,

$$p^2 + q^2 = 1 \tag{4}$$

This partial differential equation is a classical one, encountered in particular in the theory of plastic torsion of cylindrical beams[51]. It also arises in geometrical optics, and is investigated under that name in Ref. 35.

The system of five ordinary differential equations for the characteristics assumes the form:

$$\frac{dx}{ds} = 2p$$

$$\frac{dy}{ds} = 2q$$

(continued)

144

$$\frac{dz}{ds} = 2(p^2 + q^2)$$

$$\frac{dp}{ds} = 0$$

$$\frac{dq}{ds} = 0 \tag{5}$$

It follows that p and q are constants along any characteristic, and that:

$$x - x_0 = 2ps$$

$$y - y_0 = 2ps$$

$$z - z_0 = 2s \tag{6}$$

Hence the characteristics are straight lines. In view of the restriction (4) on p and q, they are precisely those lines which make an angle of 45° with the z-axis. A geometrical interpretation of those is given in Ref. 35, pp. 41-43.

Elimination of s in (6) results in:

$$p = \frac{x-x_0}{z-z_0}, \qquad\qquad q = \frac{y-y_0}{z-z_0}$$

and a solution of special interest is the cone of equation:

$$(z-z_0)^2 = (x-x_0)^2 + (y-y_0)^2 \tag{7}$$

It can be used to form a complete integral of equation (1). To satisfy the boundary conditions, the following conditions need to be met:

$$z = \zeta \quad \text{for} \quad x = \xi \quad \text{and} \quad y = \eta$$

and $d\zeta = 0$ along the given boundary curve $\eta = \eta(\xi)$.

Thus:

$$(z-\zeta)^2 = (x-\xi)^2 + (y-\eta)^2 \tag{8}$$

and, by differentiation:

$$(z-\zeta)d\zeta = (x-\xi)d\xi + (y-\eta)d\eta = 0$$

or

$$(x-\xi) + (y-\eta)\eta' = 0, \qquad \eta' = \frac{d\eta}{d\xi} \tag{9}$$

Equations (8) and (9) together with the relation between ξ and η expressed as:

$$\eta = \eta(\xi) \tag{10}$$

and its derivative with respect to ξ:

$$\eta' = \eta'(\xi) \tag{11}$$

are sufficient through elimination of ξ, η, η' to establish the desired function $z = z(x,y)$ which satisfies the partial differential equation (3) and the boundary condition. For complex functions $\eta(\xi)$, it may not be possible to find a single equation and ξ and/or η may remain as parameters of a set of equations. However, z is still uniquely defined.

In the simple case where the boundary ∂D is circular, the relation between ξ and η takes the form:

$$\xi^2 + \eta^2 = R^2 \tag{10}$$

together with the derivative:

$$\xi + \eta\eta' = 0 \tag{11}$$

Elimination of η' from (9) and (11) gives:

$$\eta x = \xi y$$

Substituting for ξ into (8) and (10) gives:

$$(z-\zeta)^2 = (x + \frac{x\eta}{y})^2 + (y-\eta)^2 = (x^2 + y^2)(1 - \frac{\eta}{y})^2 \tag{8}$$

$$R^2 = \frac{x^2\eta^2}{y^2} + \eta^2 = (x^2 + y^2)(\frac{\eta}{y})^2 \tag{10}$$

The parameter η/y can easily be eliminated, yielding,

$$(z-\zeta)^2 = R - \sqrt{x^2 + y^2}$$

or

$$z = \zeta \pm (R - \sqrt{x^2 + y^2})$$

Another simple example is the boundary defined by:

$$\eta = \xi \tag{10}$$

where

$$\eta' = 1 \tag{11}$$

(9) reads:

$$(x-\xi) + (y-\eta) = 0 \tag{9}$$

Eliminating ξ:

$$x + y - 2\eta = 0$$

or

$$\eta = \frac{x+y}{2}$$

(8) becomes:

$$(z-\zeta)^2 = (x - \frac{x+y}{2})^2 + (y - \frac{x+y}{2})^2$$

$$= \frac{1}{2}(x-y)^2 \tag{8}$$

or

$$z = \zeta \pm \frac{1}{\sqrt{2}}(x-y)$$

When the boundary is defined by:

$$\eta = b \tag{10}$$

147

i. e. :

$$\eta' = 0 \qquad (11)$$

one immediately obtains from (9):

$$x-\xi = 0 \qquad (9)$$

i. e. :

$$(z-\zeta)^2 = (y-b)^2$$

or

$$z = \zeta \pm (y-b)$$

Similarly for a boundary defined as:

$$\xi = a \qquad (10)$$

the solution is:

$$z = \zeta \pm (x-a)$$

The transformation of z into w is:

$$z - \zeta = \frac{1}{k} \left[\sinh^{-1} (\frac{kw}{c}) - \sinh^{-1} (\frac{kw_0}{c}) \right]$$

and, for $w_0 = 0$,

$$w = \frac{c}{k} \sinh[k(z-\zeta)]$$

For the case of a rectangular plate, Dr. Naumann's results presented above thus agree completely with the ones found in Chapter (B.2): for instance, in region (4) of the rectangular domain (figure 3), for the boundary $\eta = b$, we get:

$$z-\zeta = \pm (y-b)$$

148

i. e. :

$$w = \frac{c}{k} \, \sinh[\pm k(y-b)] = \frac{\epsilon c}{k} \, \sinh[k(y-b)], \qquad \epsilon = \pm 1$$

which is identical with the result we found.

Another interesting remark concerns the subject of which boundary controls what part of the domain. The z-functions are planes which intersect the xy plane at an angle of 45°. Thus, the intersecting boundaries within the domain are simply the bissectors of the angles of the boundaries, which are the dotted lines represented in figure 3. The solution z can be visualized as being a roof over the boundary base with a pitch angle of 45° all the way around (as well as its exact complement, i.e., a pitch angle of -45°).

One can also visualize the function z to be like a sand hill on top of any shaped boundary with the sand of such a consistency that it can only support a pitch angle of 45°.

To generalize the above results, it is obvious that the function $z-\zeta$ is $\pm \rho$, if ρ is the shortest distance of the particular point in the xy plane to the given boundary ∂D, which leads to the general solution for (1) given by:

$$w = \frac{\epsilon c}{k} \, \sinh(k\rho)$$

if the boundary condition is $w = 0$ along ∂D.

In case that w varies along ∂D, i.e., $d\zeta \neq 0$, equation (9) is still applicable and reads:

$$(z-\zeta) \frac{d\zeta}{d\xi} = (x-\xi) + (y-\eta)\eta' \tag{9a}$$

Again, equations (8), (9a), (10), (11), together with:

$$\frac{d\zeta}{d\xi} = f(\xi, \eta) \tag{12}$$

uniquely define the function $z = z(x,y)$ solution of (1), and the above procedure

can thus be used even if w varies in a known manner along the boundary ∂D. Our alternate approach of Chapter (B. 2) could also have been extended to this more general case, which is however of a limited practical value.

APPENDIX 3

A COMPUTER PROGRAM TO FIND THE OPTIMAL THICKNESS DISTRIBUTION

OF A SANDWICH PLATE

```
C     ************************************************
C     *           FUNCTION W                        *
C     ************************************************
C
C
C     CLASSICAL OPERATORS FOR THE PARTIAL DERIVATIVES
C     STARTING FUNCTION IS W = SIN(PI*X/10)*SIN(PI*Y/10)
C     TEST PERFORMED ON THE DOMAIN (0,10)*(0,10)
C     STEP .2 (49*49 MESH POINTS)
C
C     VALUE AT EACH POINT GOT BY SOLVING AW2+BW+C=0
C     ONLY POSITIVE ROOT KEPT AS RESULT , STOP OTHERWISE
C
      REAL *8 W(51,51),T(51,51),
     1WXX,WXY,WYY,
     2WXXX,hXXY,hXYY,WYYY,
     3WXXXX,WXXYY,WYYYY,
     4W2XX(51,51),W2XY(51,51),W2YY(51,51),
     5W3XXX(51,51),W3XXY(51,51),W3XYY(51,51),W3YYY(51,51),
     6W4XXXX(51,51),W4XXYY(51,51),W4YYYY(51,51),
     7H,H2,H3,H4,A,B,C,DELTA,DIF2,BETA2,HX,HY,TP,VIT,
     8AB1,AB2
C
C     SETTING UP A FEW CONSTANTS
C
      BETA2=0.0239
      H=0.2
      H2=H*H
      H3=H2*H
      H4=H3*H
      N=49
      N1=51
      A=-BETA2+10.64/H4
      K=1
C
C     INITIALISATION
C
      DO 3 I=1,N1
      DO 3 J=1,N1
      W(I,J)=DSIN((I-1)*H*3.1416/10.)*DSIN((J-1)*H*3.1416/10.)
    3 CONTINUE
C
   50 DO 1 IY=1,N
      DO 1 IX=1,N
      B=-5.32/H4*(W(IY+1,IX+2)+W(IY+1,IX)+h(IY+2,IX+1)+W(IY,IX+1))
C
      C=1.34/16.0/H4*(W(IY+2,IX+2)+W(IY,IX)-W(IY+2,IX)-W(IY,IX+2))**2
     1+(W(IY+1,IX)+W(IY+1,IX+2))**2/H4
```

```
2+(W(IY+2,IX+1)+W(IY,IX+1))**2/H4
3+0.66*(W(IY+1,IX)+W(IY+1,IX+2))*(W(IY+2,IX+1)+W(IY,IX+1))/H4-1.0
      DELTA=B*B-4.0*A*C
      IF (DELTA.LT.0.0) GOTO 30
C
  10  W(IY+1,IX+1)=(-B+DSQRT(DELTA))/(2.0*A)
  1   CONTINUE
C
      DIF2=0.0
      WRITE(6,199) K
      K=K+1
      IF (K.EQ.51) GOTO 30
      GOTO 50
  30  WRITE(6,100)
      CALL FLEVEL(W,-100.0,2.0,15,0.015)
      DO 4 J=1,26
      AB1=(J-1)*H
      WRITE(6,109) AB1
      DO 4 I=1,26
      AB2=(I-1)*H
      WRITE(6,108) AB2,W(J,I)
  4   CONTINUE
C
C
C     **********************************
C     ***FUNCTION T ********************
C     **********************************
C
C     INITIALISATION OF T
C
C
      DO 5 J=1,N1
      DO 5 I=1,N1
      T(I,J)=DABS(1.5*DSIN((J-1)*H*3.1416/5.)*DSIN((I-1)*H*3.1416/5.))
  5   CONTINUE
C
C
      DO 77 J=1,N
      DO 77 I=1,N
      WXX=(W(J+1,I+2)+W(J+1,I)-2.0*W(J+1,I+1))/H2
      WYY=(W(J+2,I+1)+W(J,I+1)-2.0*W(J+1,I+1))/H2
      WXY=(W(J+2,I+2)+W(J,I)-W(J+2,I)-W(J,I+2))/4.0/H2
      WXXY=(W(J+2,I)+W(J+2,I+2)-W(J,I)-W(J,I+2)
  1   -2.0*W(J+2,I+1)+2.0*W(J,I+1))/2.0/H3
      WXYY=(W(J+2,I+2)-W(J+2,I)+W(J,I+2)-W(J,I)
  1   -2.0*W(J+1,I+2)+2.0*W(J+1,I))/2.0/H3
      WXXYY=(W(J+2,I)+W(J+2,I+2)+W(J,I)+W(J,I+2)
  1   -2.0*(W(J+1,I)+W(J+1,I+2)+W(J,I+1)+W(J+2,I+1))+4.0*W(J+1,I+1))/H4
```

152

```
      IF ((J .EQ.1) .AND. (I .EQ. 1)) GOTO 60
      IF ((J .EQ. N) .AND. (I .EQ. 1)) GOTO 62
      IF ((J.EQ.N).AND.(I.EQ.N)) GOTO 64
      IF((J.EQ.1).AND.(I.EQ.N)) GOTO 66
      IF (I.EQ.1) GOTO 68
      IF (I.EQ.N) GOTO 70
      IF (J.EQ.1) GOTO 72
      IF (J.EQ.N) GOTO 74
C
C     CASE OF AN INTERIOR POINT
C
      WXXXX=(W(J+1,I+3)-4.0*W(J+1,I+2)+W(J+1,I-1)
     1-4.0*W(J+1,I)+6.0*W(J+1,I+1))/H4
      WYYYY=(W(J+3,I+1)-4.0*W(J+2,I+1)+W(J-1,I+1)
     1-4.0*W(J,I+1)+6.0*W(J+1,I+1))/H4
      WXXX=(W(J+1,I+3)-W(J+1,I-1)-2.0*W(J+1,I+2)+2.0*W(J+1,I))/2.0/H3
      WYYY=(W(J+3,I+1)-W(J-1,I+1)-2.0*W(J+2,I+1)+2.0*W(J,I+1))/2.0/H3
      GOTO 76
C
C     CASE OF AN EDGE POINT
C
   60 WXXXX=(W(J+1,I+3)+5.0*W(J+1,I+1)-4.0*W(J+1,I+2)-2.0*W(J+1,I))/H4
     1+(W(J+1,I+3)-3.0*W(J+1,I+1)+2.0*W(J+1,I))/3.0/H4
      WYYYY=(W(J+3,I+1)+5.0*W(J+1,I+1)-4.0*W(J+2,I+1)-2.0*W(J,I+1))/H4
     1+(W(J+3,I+1)-3.0*W(J+1,I+1)+2.0*W(J,I+1))/3.0/H4
      WXXX=(W(J+1,I+3)+W(J+1,I+1)-2.0*W(J+1,I+2))/2.0/H3
     1-(W(J+1,I+3)-3.0*W(J+1,I+1)+2.0*W(J+1,I))/6.0/H3
      WYYY=(W(J+3,I+1)+W(J+1,I+1)-2.0*W(J+2,I+1))/2.0/H3
     1-(W(J+3,I+1)-3.0*W(J+1,I+1)+2.0*W(J,I+1))/6.0/H3
      GOTO 76
   62 WXXXX=(W(J+1,I+3)+5.0*W(J+1,I+1)-4.0*W(J+1,I+2)-2.0*W(J+1,I))/H4
     1+(W(J+1,I+3)-3.0*W(J+1,I+1)+2.0*W(J+1,I))/3.0/H4
      WYYYY=(W(J-1,I+1)+5.0*W(J+1,I+1)-4.0*W(J,I+1)-2.0*W(J+2,I+1))/H4
     1+(W(J-1,I+1)-3.0*W(J+1,I+1)+2.0*W(J+2,I+1))/3.0/H4
      WXXX=(W(J+1,I+3)+W(J+1,I+1)-2.0*W(J+1,I+2))/2.0/H3
     1-(W(J+1,I+3)-3.0*W(J+1,I+1)+2.0*W(J+1,I))/6.0/H3
      WYYY=-(W(J-1,I+1)+W(J+1,I+1)-2.0*W(J,I+1))/2.0/H3
     1+(W(J-1,I+1)-3.0*W(J+1,I+1)+2.0*W(J+2,I+1))/6.0/H3
      GOTO 76
C
   64 WXXXX=(W(J+1,I-1)+5.0*W(J+1,I+1)-4.0*W(J+1,I)-2.0*W(J+1,I+2))/H4
     1+(W(J+1,I-1)-3.0*W(J+1,I+1)+2.0*W(J+1,I+2))/3.0/H4
      WYYYY=(W(J-1,I+1)+5.0*W(J+1,I+1)-4.0*W(J,I+1)-2.0*W(J+2,I+1))/H4
     1+(W(J-1,I+1)-3.0*W(J+1,I+1)+2.0*W(J+2,I+1))/3.0/H4
      WXXX=-(W(J+1,I-1)+W(J+1,I+1)-2.0*W(J+1,I))/2.0/H3
     1+(W(J+1,I-1)-3.0*W(J+1,I+1)+2.0*W(J+1,I+2))/6.0/H3
      WYYY=-(W(J-1,I+1)+W(J+1,I+1)-2.0*W(J,I+1))/2.0/H3
     1+(W(J-1,I+1)-3.0*W(J+1,I+1)+2.0*W(J+2,I+1))/6.0/H3
```

```
      GOTO 76
C
66    WXXXX=(W(J+1,I-1)+5.0*W(J+1,I+1)-4.0*W(J+1,I)-2.0*W(J+1,I+2))/H4
     1+(W(J+1,I-1)-3.0*W(J+1,I+1)+2.0*W(J+1,I+2))/3.0/H4
      WYYYY=(W(J+3,I+1)+5.0*W(J+1,I+1)-4.0*W(J+2,I+1)-2.0*W(J,I+1))/H4
     1+(W(J+3,I+1)-3.0*W(J+1,I+1)+2.0*W(J,I+1))/3.0/H4
      WXXX=-(W(J+1,I-1)+W(J+1,I+1)-2.0*W(J+1,I))/2.0/H3
     1+(W(J+1,I-1)-3.0*W(J+1,I+1)+2.0*W(J+1,I+2))/6.0/H3
      WYYY=(W(J+3,I+1)+W(J+1,I+1)-2.0*W(J+2,I+1))/2.0/H3
     1-(W(J+3,I+1)-3.0*W(J+1,I+1)+2.0*W(J,I+1))/6.0/H3
      GOTO 76
C
68    WXXXX=(W(J+1,I+3)+5.0*W(J+1,I+1)-4.0*W(J+1,I+2)-2.0*W(J+1,I))/H4
     1+(W(J+1,I+3)-3.0*W(J+1,I+1)+2.0*W(J+1,I))/3.0/H4
      WYYYY=(W(J+3,I+1)-4.0*W(J+2,I+1)+W(J-1,I+1)
     1-4.0*W(J,I+1)+6.0*W(J+1,I+1))/H4
      WXXX=(W(J+1,I+3)+W(J+1,I+1)-2.0*W(J+1,I+2))/2.0/H3
     1-(W(J+1,I+3)-3.0*W(J+1,I+1)+2.0*W(J+1,I))/6.0/H3
      WYYY=(W(J+3,I+1)-W(J-1,I+1)-2.0*W(J+2,I+1)+2.0*W(J,I+1))/2.0/H3
      GOTO 76
C
70    WXXXX=(W(J+1,I-1)+5.0*W(J+1,I+1)-4.0*W(J+1,I)-2.0*W(J+1,I+2))/H4
     1+(W(J+1,I-1)-3.0*W(J+1,I+1)+2.0*W(J+1,I+2))/3.0/H4
      WYYYY=(W(J+3,I+1)-4.0*W(J+2,I+1)+W(J-1,I+1)
     1-4.0*W(J,I+1)+6.0*W(J+1,I+1))/H4
      WXXX=-(W(J+1,I-1)+W(J+1,I+1)-2.0*W(J+1,I))/2.0/H3
     1+(W(J+1,I-1)-3.0*W(J+1,I+1)+2.0*W(J+1,I+2))/6.0/H3
      WYYY=(W(J+3,I+1)-W(J-1,I+1)-2.0*W(J+2,I+1)+2.0*W(J,I+1))/2.0/H3
      GOTO 76
C
72    WXXXX=(W(J+1,I+3)-4.0*W(J+1,I+2)+W(J+1,I-1)
     1-4.0*W(J+1,I)+6.0*W(J+1,I+1))/H4
      WYYYY=(W(J+3,I+1)+5.0*W(J+1,I+1)-4.0*W(J+2,I+1)-2.0*W(J,I+1))/H4
     1+(W(J+3,I+1)-3.0*W(J+1,I+1)+2.0*W(J,I+1))/3.0/H4
      WXXX=(W(J+1,I+3)-W(J+1,I-1)-2.0*W(J+1,I+2)+2.0*W(J+1,I))/2.0/H3
      WYYY=(W(J+3,I+1)+W(J+1,I+1)-2.0*W(J+2,I+1))/2.0/H3
     1-(W(J+3,I+1)-3.0*W(J+1,I+1)+2.0*W(J,I+1))/6.0/H3
      GOTO 76
C
74    WXXXX=(W(J+1,I+3)-4.0*W(J+1,I+2)+W(J+1,I-1)
     1-4.0*W(J+1,I)+6.0*W(J+1,I+1))/H4
      WYYYY=(W(J-1,I+1)+5.0*W(J+1,I+1)-4.0*W(J,I+1)-2.0*W(J+2,I+1))/H4
     1+(W(J-1,I+1)-3.0*W(J+1,I+1)+2.0*W(J+2,I+1))/3.0/H4
      WXXX=(W(J+1,I+3)-W(J+1,I-1)-2.0*W(J+1,I+2)+2.0*W(J+1,I))/2.0/H3
      WYYY=-(W(J-1,I+1)+W(J+1,I+1)-2.0*W(J,I+1))/2.0/H3
     1+(W(J-1,I+1)-3.0*W(J+1,I+1)+2.0*W(J+2,I+1))/6.0/H3
76    W2XX(J+1,I+1)=WXX
      W2XY(J+1,I+1)=WXY
```

```
      W2YY(J+1,I+1)=WYY
      W3XXX(J+1,I+1)=WXXX
      W3XXY(J+1,I+1)=WXXY
      W3XYY(J+1,I+1)=WXYY
      W3YYY(J+1,I+1)=WYYY
      W4XXXX(J+1,I+1)=WXXXX
      W4XXYY(J+1,I+1)=WXXYY
      W4YYYY(J+1,I+1)=WYYYY
   77 CONTINUE
C
      K=1
   55 DIF2=0.0
C
      DO 6 J=1,N
      DO 6 I=1,N
C
      A=-2.66/H2*(W2XX(J+1,L+1)+W2YY(J+1,I+1))
     1+W4XXXX(J+1,I+1)+2.0*W4XXYY(J+1,I+1)+W4YYYY(J+1,I+1)
     2-BETA2*W(J+1,I+1)
      B=(W2XX(J+1,I+1)+0.33*W2YY(J+1,I+1))*(T(J+1,I+2)+T(J+1,I))/H2
     1+1.34*(T(J+2,I+2)+T(J,I)-T(J+2,I)-T(J,I+2))*W2XY(J+1,I+1)/4.0/H2
     2+(0.33*W2XX(J+1,I+1)+W2YY(J+1,I+1))*(T(J+2,I+1)+T(J,I+1))/H2
     3+(W3XXX(J+1,I+1)+W3XYY(J+1,I+1))*(T(J+1,I+2)-T(J+1,I))/H
     4+(W3XXY(J+1,I+1)+W3YYY(J+1,I+1))*(T(J+2,I+1)-T(J,I+1))/H
     5-0.632*BETA2*W(J+1,I+1)
C
      TP=-B/A
      DIF2=DIF2+(T(I+1,J+1)-TP)**2
      T(J+1,I+1)=TP
    6 CONTINUE
C
      WRITE(6,200) K,DIF2
      IF((K.EQ.150) .OR. (DIF2.LT.0.001)) GOTO 80
      K=K+1
      GOTO 55
C
C     FINAL RESULTS FOR FUNCTION T
C
   80 WRITE(6,107)
      DO 7 J=1,26
      AB1=(J-1)*H
      WRITE(6,110) AB1
      DO 7 I=1,26
      AB2=(I-1)*H
      WRITE(6,111) AB2,T(J,I)
    7 CONTINUE
C
      HX=10./49.
```

```
      HY=HX
      VIT=0.0
      DO 2 I=2,49
      DO 2 J=2,49
      VIT=VIT+T(J-1,I-1)+T(J+2,I-1)+T(J-1,I+2)+
     1     T(J+2,I+2)-13.0*(T(J,I-1)+T(J+1,I-1)+
     2                      T(J-1,I)+T(J+2,I)+T(J-1,I+1)+T(J+2,I+1)
     3                      +T(J,I+2)+T(J+1,I+2))+169.0*(T(J,I)+T(J+1,I)
     4                      +T(J,I+1)+T(J+1,I+1))
    2 CONTINUE
C
      VIT=VIT*HX*HY/24.0/24.0
C
      WRITE(6,600) VIT
C
      CALL ELEVEL(T,-100.0,2.0,15,0.015)
  100 FORMAT(1H1,10X,'FUNCTION W ',////)
  107 FORMAT(1H1,10X,'FUNCTION T ',//)
  108 FORMAT(1H ,15X,E16.7,15X,E16.7,/)
  109 FORMAT(1H1, 'LINE Y=',E16.7,//,25X,'X',15X,'W(P)',/)
  110 FORMAT(1H1, 'LINE Y=',E16.7,//,25X,'X',15X,'T(P)',/)
  111 FORMAT(1H ,15X,E16.7,15X,E16.7,/)
  199 FORMAT(1H ,'ITERATION N. :',I3,/)
  200 FORMAT(1H ,'ITERATION N. :',I3,5X,'DELTA2 :',E16.7,/)
  600 FORMAT(1H1, 'VALUE OF THE INTEGRAL IS :',E16.7,//)
      RETURN
      END
```

```fortran
      SUBROUTINE FLEVEL(Z,WL,UL,NBL,TOL)
      REAL *8 Z,WL,UL,TOL
      DIMENSION Z(51,51)
C
C TRACES THE LEVEL-LINES FOR A TABULATED FUNCTION ( VALUES IN Z(J,I))
C WL IS THE LEVEL UNDER WHICH '?' IS PRINTED
C UL SAME FUNCTION FOR THE UPPER LEVEL
C NBL IS THE NUMBER OF LEVELS (STEP .1 ASSUMED)
C TOL IS THE HALF OF THE WIDTH OF THE BOUND
C
C PROGRAM PRINTS OUT THE LEVELS IDENTIFIED BY LETTERS ON ONE PAGE
C
      INTEGER LIT *2(15)/'A ','B ','C ','D ','E ','F ','G ','H ',
     1                   'I ','J ','K ','L ','M ','N ','O '/
      INTEGER PAGE *2(51,51)
      INTEGER *2 QUEST/'? '/,BLANK/'  '/
C
      DO 1 I=1,51
      DO 1 J=1,51
      PAGE(J,I)=BLANK
  1   CONTINUE
C
      DO 2 J=2,50
      DO 2 I=2,50
      IF ((Z(J,I).LT.WL).OR.(Z(J,I).GT.UL)) PAGE(J,I)=QUEST
  2   CONTINUE
C
      DO 3 K=1,NBL
      VAL=0.1*K
C
      DO 4 J=1,51
      DO 4 I=1,51
      IF ((Z(J,I).GT.(VAL-TOL)).AND.(Z(J,I).LT.(VAL+TOL)))
     1PAGE(J,I)=LIT(K)
  4   CONTINUE
  3   CONTINUE
C
      DO 5 J=1,51
      WRITE(6,500)(PAGE(52-J,I),I=1,51)
  5   CONTINUE
C
 500  FORMAT(1H ,20X,51A2)
      WRITE(6,501)
 501  FORMAT(1H1)
      RETURN
      END
```