Free Algebras

1. Words and Free Monoid

Let $X = \{x_i \mid i \in I\}$ be a nonempty (possibly indexed) set.

- A **word** over X is a finite sequence of elements from X, written as

$$x_{i_1} x_{i_2} \cdots x_{i_m}.$$

  The empty word is denoted by 1 and has length 0.
  Define multiplication by juxtaposition:

$$(x_{i_1} \cdots x_{i_m}) \cdot (x_{j_1} \cdots x_{j_n}) := x_{i_1} \cdots x_{i_m} x_{j_1} \cdots x_{j_n}.$$

With this multiplication, the set of all words forms a monoid (with identity 1), denoted X^*, called the **free monoid** on X.

Each element $w \in X^*$ (i.e., a word) can be uniquely described as

$$w = x_{i_1}^{k_1} x_{i_2}^{k_2} \cdots x_{i_r}^{k_r}, \quad \text{with } x_{i_j} \in X,\ k_j \in \mathbb{N}, \text{ and } x_{i_j} \neq x_{i_{j+1}}.$$

Equality in X^*: Two words $w, w' \in X^*$ are equal if they contain the same sequence of symbols in the same order.

2. Free Algebra

Let $\mathbb{F}$ be a field. The **free algebra** on X over $\mathbb{F}$, denoted $\mathbb{F}\langle X \rangle$, is defined as the monoid algebra $\mathbb{F}[X^*]$.

Elements: An element $f \in \mathbb{F}\langle X \rangle$ is a finite linear combination of words:

$$f = \sum_{i=1}^{m} \lambda_i w_i, \quad \lambda_i \in \mathbb{F},\ w_i \in X^*.$$

Two such sums are equal if, after combining like terms (i.e., words that are equal in X^*), their coefficients agree.

Operations:

- **Addition:** Combine like monomials.
- **Scalar multiplication:** $\lambda f = \sum \lambda\lambda_i w_i$
- **Multiplication:** Extend bilinearly via

$$(\lambda w)(\mu v) = (\lambda\mu)(wv),$$

where wv is the concatenation in X^*.

Algebra Axioms: $\mathbb{F}\langle X\rangle$ is an associative unital algebra over $\mathbb{F}$, satisfying:

- Associativity of addition and multiplication
- Distributivity
- Scalar compatibility
- Identity element $1 \in \mathbb{F}\langle X\rangle$

3. Monomials, Degree, and Homogeneity

Monomial: A monomial is a nonzero scalar multiple of a word: λw where $\lambda \in \mathbb{F}$, $w \in X^*$.

Length and Degree:

- Length of a word $w = x_{i_1} \cdots x_{i_m}$: $\ell(w) := m$, and $\ell(1) := 0$.
- Degree of a nonzero polynomial $f = \sum \lambda_i w_i$:

$$\deg(f) := \max\{\ell(w_i) \mid \lambda_i \neq 0\}.$$

Homogeneous Polynomial: All monomials in f have the same length (degree).

Multilinear Polynomial: Each indeterminate appears *exactly once* in each monomial. Formally,

$$f(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} \lambda_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}, \quad \lambda_\sigma \in \mathbb{F}.$$

4. Universal Property

Let A be any unital $\mathbb{F}$-algebra and $f : X \to A$ a function. Then there exists a unique unital algebra homomorphism:

$$\tilde{f} : \mathbb{F}\langle X\rangle \to A$$

such that $\tilde{f}(x) = f(x)$ for all $x \in X$.

This is the **universal property** of $\mathbb{F}\langle X\rangle$, making it the free object in the category of unital $\mathbb{F}$-algebras.

5. Evaluation

Given $f(x_1, \ldots, x_n) \in \mathbb{F}\langle X \rangle$ and elements $a_1, \ldots, a_n \in A$, the evaluation:

$$f(a_1, \ldots, a_n)$$

is obtained by replacing each x_i by a_i, preserving order.

Example: If $f = x_1^2 x_2 x_1 - x_2 x_3 + 1$, then

$$f(a_1, a_2, a_3) = a_1^2 a_2 a_1 - a_2 a_3 + 1 \in A.$$

6. Domain Property

The free algebra $\mathbb{F}\langle X \rangle$ is a domain. That is, for any nonzero $f, g \in \mathbb{F}\langle X \rangle$,

$$\deg(fg) = \deg(f) + \deg(g).$$