Matrix series

Introduction to dynamical systems #4

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1 Numerical series and convergence

1.1 Sequences and series

- **Cauchy sequences**
  - A sequence $\{a_k\}$ in $\mathbb{R}$ is said to be a *Cauchy sequence* if for every $\varepsilon > 0$, there is an integer $n \in \mathbb{N}$ such that
  $$l \geq m \geq n \implies |a_l - a_m| < \varepsilon$$
  - Cauchy criterion: for a sequence $\{a_k\}$ in $\mathbb{R}$, the following are euqivalent:
    - it converges to some $a \in \mathbb{R}$: for every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $|a_k - a| < \varepsilon$ for all $k \geq n$
    - it is a Cauchy sequence
- **Series**
  - Given a sequence $\{a_k\}$ in $\mathbb{R}$, we define another sequence $\{s_t\}$ in $\mathbb{R}$ by
  $$s_t := \textstyle\sum_{k=0}^{t} a_k = a_0 + a_1 + \ldots + a_t, \quad \forall t = 0, 1, 2, \ldots,$$
  which is called a *series* and denoted as $\sum_k a_k$
  - We say that a series $\sum_k a_k$ converges if $\{s_t\}$ converges
  - By the Cauchy criterion, the following are equivalent:
    - a series $\sum_k a_k$ converges
    - for any $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that
    $$l \geq m \geq n \implies \left|\textstyle\sum_{k=m}^{l} a_k\right| < \varepsilon \tag{1}$$
  - It immediately follows that $\lim_{k\to\infty} a_k = 0$ whenever $\sum_k a_k$ converges:
  $$\sum_k a_k \text{ is convergent} \implies (1) \text{ with } l = m \implies \lim_{k\to\infty} a_k = 0$$
- **Example**
  - If $a_k := x$ for all k and $|x| < 1$, then the series $\sum_k a_k$ converges:
  $$\textstyle\sum_{k=0}^{\infty} a_k = x^0 + x^1 + x^2 + \ldots = (1-x)^{-1} \tag{2}$$

1.2 Tests for convergence

- **Comparison test**
  - Consider a sequence $\{a_k\}$ in $\mathbb{R}$
  - If there exists another sequence $\{b_k\}$ in $\mathbb{R}$ such that
    - $\sum_k b_k$ converges, and
    - $|a_k| \leq b_k$ for all $k \geq n_0$ for some fixed $n_0 \in \mathbb{N}$,

    then $\sum_k a_k$ converges
  - Proof:
    - Fix $\varepsilon > 0$
    - Since $\sum_k b_k$ converges, there exists $n_1 \in \mathbb{N}$ such that

$$l \geq m \geq n_1 \implies \left| \sum_{k=m}^{l} b_k \right| < \varepsilon$$

    - Letting $n := \max\{n_0, n_1\}$, we then have $b_k \geq |a_k| \geq 0$ for all $k \geq n$ and thus

$$l \geq m \geq n \implies \left| \sum_{k=m}^{l} a_k \right| \leq \sum_{k=m}^{l} |a_k| \leq \sum_{k=m}^{l} b_k = \left| \sum_{k=m}^{l} b_k \right| < \varepsilon,$$

      which implies $\sum_k a_k$ converges

- **Root test**
  - Consider a sequence $\{a_k\}$ such that the limit

$$r := \lim_{k \to \infty} |a_k|^{1/k}$$

    exists in $\mathbb{R} \cup \{\infty\}$
  - If $r < 1$, then $\sum_k a_k$ converges because:
    - Since $r < 1$, one can choose $\beta \in \mathbb{R}$ such that $r < \beta < 1$
    - Since $\lim_{k \to \infty} |a_k|^{1/k} = r$, there exists $n_0 \in \mathbb{N}$ such that

$$k \geq n_0 \implies |a_k|^{1/k} < \beta \implies |a_k| < \beta^k$$

    - Since $\sum_k \beta^k$ is convergent, so is $\sum_k a_k$ by the comparison test
  - If $r > 1$, then $\sum_k a_k$ does not converge because:
    - Since $r > 1$ and since $\lim_{k \to \infty} |a_k|^{1/k} = r$, there exists $n \in \mathbb{N}$ such that

$$k \geq n \implies |a_k|^{1/k} > 1 \implies |a_k| > 1,$$

      which means $\lim_{k \to \infty} a_k \neq 0$, violating the necessary condition for series convergence

- **Ratio test**
  - Consider a sequence $\{a_k\}$ such that the limit

$$r := \lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right|$$

    exists in $\mathbb{R} \cup \{\infty\}$

- If $r < 1$, then $\sum_k a_k$ converges because:
  - Since $r < 1$, one can choose $\beta \in \mathbb{R}$ such that $r < \beta < 1$
  - Since $\lim_{k\to\infty} |a_{k+1}/a_k| = r$, there exists $n \in \mathbb{N}$ such that

$$k \geq n \implies |a_{k+1}/a_k| < \beta \implies |a_{k+1}| < \beta |a_k|,$$

    which implies $|a_k| < |a_n|\beta^{k-n}$ for all $k \geq n+1$
  - Since $\sum_k |a_n|\beta^{k-n}$ is convergent, so is $\sum_k a_k$ by the comparison test
- If $r > 1$, then $\sum_k a_k$ does not converge because:
  - Since $r > 1$ and since $\lim_{k\to\infty} |a_{k+1}/a_k| = r$, there exists $n \in \mathbb{N}$ such that

$$k \geq n \implies |a_{k+1}/a_k| > 1 \implies |a_{k+1}| > |a_k|,$$

    which means $\lim_{k\to\infty} a_k \neq 0$, violating the necessary condition for series convergence

- **Power series and radius of convergence**
  - A series $\sum_k a_k$ of the form

$$a_k := \alpha_k x^k \quad \forall k = 0, 1, 2, \ldots,$$

    is called a *power series*
  - **Root test**: consider a power series $\sum_k \alpha_k x^k$ such that

$$\alpha := \lim_{k\to\infty} |\alpha_k|^{1/k}$$

    exists in $\mathbb{R} \cup \{\infty\}$
    - Define $R \in \mathbb{R} \cup \{\infty\}$ by

$$R := \begin{cases} 0 & \text{if } \alpha \in \{-\infty, \infty\} \\ \infty & \text{if } \alpha = 0 \\ 1/\alpha & \text{otherwise} \end{cases},$$

      which is called the *radius of convergence* of the power series $\sum_k \alpha_k x^k$
    - If $|x| < R$, then $\sum_k \alpha_k x^k$ converges because $\lim_{k\to\infty} |\alpha_k x^k|^{1/k} = \lim_{k\to\infty} |\alpha_k|^{1/k}|x| = \alpha|x|$
    - Similarly, if $|x| > R$, then $\sum_k \alpha_k x^k$ does not converge
  - **Ratio test**: consider a power series $\sum_k \alpha_k x^k$ such that

$$\alpha := \lim_{k\to\infty} |\alpha_{k+1}/\alpha_k|$$

    exists in $\mathbb{R} \cup \{\infty\}$
    - The radius of convergence is given by $R := 1/\alpha$ because

$$\lim_{k\to\infty} \left| \frac{\alpha_{k+1} x^{k+1}}{\alpha_k x^k} \right| = \lim_{k\to\infty} \left| \frac{\alpha_{k+1}}{\alpha_k} \right| |x| = \alpha|x|$$

- **Examples**
  - What is the radius of convergence of the series defined in (2)?
  - Does the following series converge? What is the radius of convergence?

$$\sum_{k=0}^{\infty} \frac{1}{k!} x^k := 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots \tag{3}$$

2 Matrix series and convergence

2.1 Powers of matrices

- **Powers of Jordan blocks**
  - Let $\boldsymbol{J}_m(\lambda)$ be a Jordan block of size m:

$$\boldsymbol{J}_m(\lambda) := \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{bmatrix} = \lambda \boldsymbol{I} + \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}}_{=:\boldsymbol{Z}}$$

  - It follows form the binomial theorem[1] that, for any $k \in \mathbb{N}$,

$$(\boldsymbol{J}_m(\lambda))^k = (\lambda \boldsymbol{I} + \boldsymbol{Z})^k = \textstyle\sum_{l=0}^{k} \frac{k!}{l!(k-l)!} \lambda^{k-l} \boldsymbol{Z}^l,$$

where

$$\begin{aligned} \boldsymbol{Z}^0 &= \begin{bmatrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & \cdots & e_m \end{bmatrix} = \boldsymbol{I} \\ \boldsymbol{Z}^1 &= \begin{bmatrix} \mathbf{0} & e_1 & e_2 & e_3 & e_4 & e_5 & \cdots & e_{m-1} \end{bmatrix} \\ \boldsymbol{Z}^2 &= \begin{bmatrix} \mathbf{0} & \mathbf{0} & e_1 & e_2 & e_3 & e_4 & \cdots & e_{m-2} \end{bmatrix} \\ &\vdots \\ \boldsymbol{Z}^{m-1} &= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & e_1 \end{bmatrix} \\ \boldsymbol{Z}^l &= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix} = \boldsymbol{O} \quad \forall l \geq m \end{aligned}$$

  - Consider the case with $m = 4$, for example:

$$\boldsymbol{Z} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{Z}^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{Z}^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{Z}^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and thus

$$\begin{aligned} (\boldsymbol{J}_4(\lambda))^k &= \frac{k!}{0!(k-0)!} \lambda^{k-0} \boldsymbol{Z}^0 + \frac{k!}{1!(k-1)!} \lambda^{k-1} \boldsymbol{Z}^1 \\ &\quad + \frac{k!}{2!(k-2)!} \lambda^{k-2} \boldsymbol{Z}^2 + \frac{k!}{3!(k-1)!} \lambda^{k-3} \boldsymbol{Z}^3 + \boldsymbol{O} \\ &= \frac{k!}{0!(k-0)!} \lambda^{k-0} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \frac{k!}{1!(k-1)!} \lambda^{k-1} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &\quad + \frac{k!}{2!(k-2)!} \lambda^{k-2} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{k!}{3!(k-3)!} \lambda^{k-3} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

[1] For any $a, b \in \mathbb{R}$ and $k \in \mathbb{N}$, we have $(a+b)^k = a^k + ka^{k-1}b + \frac{k(k-1)}{2}a^{k-2}b^2 + \cdots + \frac{k(k-1)}{2}a^2b^{k-2} + kab^{k-1} + b^k = \sum_{l=0}^{k} \frac{k!}{l!(k-l)!} a^{k-l} b^l$. The binomial theorem is also true for square matrices $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{n \times n}$ provided that $\boldsymbol{AB} = \boldsymbol{BA}$.

- In general,

$$(\boldsymbol{J}_m(\lambda))^k = \begin{bmatrix} c_0(k) & c_1(k) & c_2(k) & \cdots & c_{m-1}(k) \\ 0 & c_0(k) & c_1(k) & \cdots & c_{m-2}(k) \\ 0 & 0 & c_0(k) & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & c_0(k) \end{bmatrix}, \quad c_l(k) := \begin{cases} \frac{k!}{l!(k-l)!}\lambda^{k-l} & k \geq l \\ 0 & k < l \end{cases}$$

- Notice that for each $l = 1, \ldots, m-1$

$$c_l(k) = \frac{k!}{l!(k-l)!}\lambda^{k-l} = \frac{k(k-1)\cdots(k-l+1)}{l(l-1)\cdots(l-l+1)}\lambda^{k-l} \leq k^l \lambda^{k-l} \quad \forall k \geq l,$$

  which, since $\lim_{k\to\infty} k^l \lambda^k = 0$ iff $|\lambda| < 1$, implies

$$\lim_{k\to\infty} c_l(k) = 0 \iff |\lambda| < 1$$

- Therefore

$$\lim_{k\to\infty} (\boldsymbol{J}_m(\lambda))^k = \boldsymbol{O} \iff |\lambda| < 1$$

- **Powers of Jordan matrices**
  - Let $\boldsymbol{J} \in \mathbb{R}^{n\times n}$ be a Jordan matrix of the following form:

$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{J}_{n_1}(\lambda_1) & \boldsymbol{O} & \cdots & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{J}_{n_2}(\lambda_2) & \cdots & \boldsymbol{O} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{O} & \boldsymbol{O} & \cdots & \boldsymbol{J}_{n_d}(\lambda_d) \end{bmatrix},$$

    where $n_1 + n_2 + \ldots + n_d = n$
  - The kth power of $\boldsymbol{J}$ is

$$\boldsymbol{J}^k = \begin{bmatrix} (\boldsymbol{J}_{n_1}(\lambda_1))^k & \boldsymbol{O} & \ldots & \boldsymbol{O} \\ \boldsymbol{O} & (\boldsymbol{J}_{n_2}(\lambda_2))^k & \ldots & \boldsymbol{O} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{O} & \boldsymbol{O} & \ldots & (\boldsymbol{J}_{n_d}(\lambda_d))^k \end{bmatrix}$$

    and therefore

$$\lim_{k\to\infty} \boldsymbol{J}^k = \boldsymbol{O} \iff \max\{|\lambda_1|, |\lambda_2|, \ldots, |\lambda_d|\} < 1$$

- **Powers of general matrices**
  - Consider a square matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$
  - Denote by $\rho(\boldsymbol{A}) \in \mathbb{R}_+$ the largest (in absolute terms) eigenvalue of $\boldsymbol{A}$, i.e.,

$$\rho(\boldsymbol{A}) := \max\{|\lambda| \in \mathbb{R}_+ \mid \lambda \text{ is an eigenvalue of } \boldsymbol{A}\},$$

    which is called the *spectral radius* of $\boldsymbol{A}$
  - The kth power of $\boldsymbol{A}$ is

$$\boldsymbol{A}^k = (\boldsymbol{V}\boldsymbol{J}\boldsymbol{V}^{-1})^k = \boldsymbol{V}\boldsymbol{J}^k\boldsymbol{V}^{-1}$$

    and therefore

$$\lim_{k\to\infty} \boldsymbol{A}^k = \boldsymbol{O} \iff \boldsymbol{V}\left(\lim_{k\to\infty} \boldsymbol{J}^k\right)\boldsymbol{V}^{-1} = \boldsymbol{O} \iff \rho(\boldsymbol{A}) < 1$$

  - Note: if $\rho(\boldsymbol{A}) < 1$ and $\rho(\boldsymbol{B}) < 1$, then $\rho(\boldsymbol{A} \otimes \boldsymbol{B}) < 1$

2.2 Matrix series and its convergence

- **Geometric series and Neumann series lemma**
  - Let $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ be an arbitrary square matrix
  - We define the *geometric series* generated by $\boldsymbol{A}$ as

$$\sum_{k=0}^{t} \boldsymbol{A}^k := \boldsymbol{A}^0 + \boldsymbol{A}^1 + \boldsymbol{A}^2 + \ldots + \boldsymbol{A}^t, \quad t = 0, 1, 2, 3, \ldots$$

  - The following result is called the *Neumann series lemma*:

$$\sum_{k=0}^{t} \boldsymbol{A}^k \text{ converges} \iff \rho(\boldsymbol{A}) < 1$$

  - In particular, if $\rho(\boldsymbol{A}) < 1$, then $\sum_k \boldsymbol{A}^k$ converges and

$$\lim_{t \to \infty} \sum_{k=0}^{t} \boldsymbol{A}^k = (\boldsymbol{I} - \boldsymbol{A})^{-1}$$

- **Sufficiency ($\Longleftarrow$)**
  - Notice that

$$\sum_{k=0}^{t} \boldsymbol{A}^k (\boldsymbol{I} - \boldsymbol{A}) = \left(\boldsymbol{A}^0 + \boldsymbol{A}^1 + \boldsymbol{A}^2 + \ldots + \boldsymbol{A}^t\right)(\boldsymbol{I} - \boldsymbol{A}) = \boldsymbol{I} - \boldsymbol{A}^{t+1}$$

  - Hence

$$\rho(\boldsymbol{A}) < 0 \implies \lim_{t \to \infty} \boldsymbol{A}^{t+1} = \boldsymbol{O} \implies \lim_{t \to \infty} \sum_{k=0}^{t} \boldsymbol{A}^k (\boldsymbol{I} - \boldsymbol{A}) = \boldsymbol{I} \implies \lim_{t \to \infty} \sum_{k=0}^{t} \boldsymbol{A}^k = (\boldsymbol{I} - \boldsymbol{A})^{-1}$$

- **Necessity ($\Longrightarrow$)**
  - If $(\lambda, \boldsymbol{v})$ is an eigenpair of $\boldsymbol{A}$, we have $\boldsymbol{A}^k \boldsymbol{v} = \lambda^k \boldsymbol{v}$ and thus

$$\left(\sum_{k=0}^{t} \boldsymbol{A}^k\right) \boldsymbol{v} = \sum_{k=0}^{t} \left(\boldsymbol{A}^k \boldsymbol{v}\right) = \sum_{k=0}^{t} \left(\lambda^k \boldsymbol{v}\right) = \left(\sum_{k=0}^{t} \lambda^k\right) \boldsymbol{v},$$

  which means that (because $\boldsymbol{v} \neq \boldsymbol{0}$)

$$\sum_{k=0}^{t} \boldsymbol{A}^k \text{ converges} \implies \sum_{k=0}^{t} \lambda^k \text{ converges} \implies |\lambda| < 1$$

  - Since this must be true for any eigenpair of $\boldsymbol{A}$, we conclude that $\rho(\boldsymbol{A}) < 1$

- **Examples**
  - Consider a square matrix

$$\boldsymbol{A} := \begin{bmatrix} 5/8 & -1/4 \\ -1/16 & 5/8 \end{bmatrix}$$

  - Does the series $\sum_k \boldsymbol{A}^k$ converge? If so, what is the limit?

- The characteristic polynomial of $\boldsymbol{A}$ is

$$\phi_{\boldsymbol{A}}(t) = \begin{vmatrix} 5/8-t & -1/4 \\ -1/16 & 5/8-t \end{vmatrix} = (3/4-t)(1/2-t),$$

  which means that the eigenvalues of $\boldsymbol{A}$ are $\lambda_1 := 3/4$ and $\lambda_2 := 1/2$
- Since $\rho(\boldsymbol{A}) = \max\{|\lambda_1|, |\lambda_2|\} = 3/4 < 1$, we know that $\sum_k \boldsymbol{A}^k$ must converge to

$$(\boldsymbol{I}-\boldsymbol{A})^{-1} = \begin{bmatrix} 1-5/8 & 1/4 \\ 1/16 & 1-5/8 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -2 \\ -1/2 & 3 \end{bmatrix}$$

- To verify this, we decompose $\boldsymbol{A}$ through eigenvectors:

$$(\boldsymbol{A}-\lambda_1\boldsymbol{I})\boldsymbol{v} = \boldsymbol{0} \iff \begin{bmatrix} 5/8-3/4 & -1/4 \\ -1/16 & 5/8-3/4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \boldsymbol{v} = \alpha\begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix} \quad \forall\alpha$$

  and

$$(\boldsymbol{A}-\lambda_2\boldsymbol{I})\boldsymbol{v} = \boldsymbol{0} \iff \begin{bmatrix} 5/8-1/2 & -1/4 \\ -1/16 & 5/8-1/2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \boldsymbol{v} = \alpha\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} \quad \forall\alpha$$

  so we choose

$$\boldsymbol{\Lambda} := \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad \boldsymbol{v}_1 := \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix}, \quad \boldsymbol{v}_2 := \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$$

  and

$$\boldsymbol{V} := \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1/2 & 1/2 \end{bmatrix} \implies \boldsymbol{V}^{-1} = \begin{bmatrix} 1/2 & -1 \\ 1/2 & 1 \end{bmatrix}$$

- It follows that

$$\boldsymbol{A}^k = \left(\boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1}\right)^k = \boldsymbol{V}\boldsymbol{\Lambda}^k\boldsymbol{V}^{-1} = \begin{bmatrix} 1 & 1 \\ -1/2 & 1/2 \end{bmatrix}\begin{bmatrix} (3/4)^k & 0 \\ 0 & (1/2)^k \end{bmatrix}\begin{bmatrix} 1/2 & -1 \\ 1/2 & 1 \end{bmatrix}$$

  and therefore

$$\begin{aligned} \sum_{k=0}^{t} \boldsymbol{A}^k &= \sum_{k=0}^{t} \left(\boldsymbol{V}\boldsymbol{\Lambda}^k\boldsymbol{V}^{-1}\right) \\ &= \boldsymbol{V}\left(\sum_{k=0}^{t} \boldsymbol{\Lambda}^k\right)\boldsymbol{V}^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ -1/2 & 1/2 \end{bmatrix}\begin{bmatrix} \sum_{k=0}^{t}(3/4)^k & 0 \\ 0 & \sum_{k=0}^{t}(1/2)^k \end{bmatrix}\begin{bmatrix} 1/2 & -1 \\ 1/2 & 1 \end{bmatrix} \\ &\to \begin{bmatrix} 1 & 1 \\ -1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1/2 & -1 \\ 1/2 & 1 \end{bmatrix} \quad (t \to \infty) \\ &= \begin{bmatrix} 3 & -2 \\ -1/2 & 3 \end{bmatrix} \end{aligned}$$

- What about the following matrix?

$$\boldsymbol{A} := \begin{bmatrix} 3/4 & -1/2 \\ -1/8 & 3/4 \end{bmatrix}$$

- **Convergence of general power series**
  - Consider a more general power series

$$\sum_{k=0}^{t} \alpha_k \boldsymbol{A}^k := \alpha_0 \boldsymbol{A}^0 + \alpha_1 \boldsymbol{A}^1 + \alpha_2 \boldsymbol{A}^2 + \ldots + \alpha_t \boldsymbol{A}^t, \quad t = 0, 1, 2, 3, \ldots,$$

  where α_k is not necessarily 1
  - We claim that

$$\sum_{k=0}^{t} |\alpha_k (\rho(\boldsymbol{A}))^k| \text{ converges } \implies \sum_{k=0}^{t} \alpha_k \boldsymbol{A}^k \text{ converges}$$

  - To prove this, let R be the radius of convergence of the series $\sum_k \alpha_k \rho^k$ so that the function

$$f(\rho) := \lim_{t\to\infty} \sum_{k=0}^{t} \alpha_k \rho^k \quad \forall \rho \in (-R, R)$$

  is well-defined, differentiable on $(-R, R)$, and

$$\frac{d^l f(\rho)}{d\rho^l} = \lim_{t\to\infty} \sum_{k=0}^{t} \frac{k!}{(k-l)!} \alpha_k \rho^{k-l} \quad \forall \rho \in (-R, R), \quad \forall l = 1, 2, \ldots$$

  meaning that the limit on the right-hand side exists for any $\rho \in (-R, R)$
  - Hence,

$$\sum_{k=0}^{t} |\alpha_k (\rho(\boldsymbol{A}))^k| \text{ converges } \implies \rho(\boldsymbol{A}) < R \implies \sum_{k=0}^{t} \frac{k!}{(k-l)!} |\alpha_k| (\rho(\boldsymbol{A}))^{k-l} \text{ converges}$$

  - Then, the above claim follows form the observation that

$$\begin{aligned}
\sum_{k=0}^{t} \alpha_k \boldsymbol{A}^k &= \boldsymbol{V} \left(\sum_{k=0}^{t} \alpha_k \boldsymbol{J}^k \right) \boldsymbol{V}^{-1} \\
&= \boldsymbol{V} \left(\begin{bmatrix} \sum_{k=0}^{t} \alpha_k (\boldsymbol{J}_{n_1}(\lambda_1))^k & \boldsymbol{O} & \cdots & \boldsymbol{O} \\ \boldsymbol{O} & \sum_{k=0}^{t} \alpha_k (\boldsymbol{J}_{n_2}(\lambda_2))^k & \cdots & \boldsymbol{O} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{O} & \boldsymbol{O} & \cdots & \sum_{k=0}^{t} \alpha_k (\boldsymbol{J}_{n_d}(\lambda_d))^k \end{bmatrix} \right) \boldsymbol{V}^{-1},
\end{aligned}$$

  where a typical element of $\sum_{k=0}^{t} \alpha_k (\boldsymbol{J}_{n_i}(\lambda_i))^k$ satisfies

$$\left| \sum_{k=0}^{t} \frac{k!}{l!(k-l)!} \alpha_k \lambda_i^{k-l} \right| \leq \frac{1}{l!} \sum_{k=0}^{t} \frac{k!}{(k-l)!} |\alpha_k| |\lambda_i|^{k-l} \leq \frac{1}{l!} \sum_{k=0}^{t} \frac{k!}{(k-l)!} |\alpha_k| (\rho(\boldsymbol{A}))^{k-l},$$

  which means that each element of $\sum_{k=0}^{t} \alpha_k (\boldsymbol{J}_{n_i}(\lambda_i))^k$ converges due to the comparison test
- **Example**
  - Consider a matrix sequence of the form

$$\sum_{k=0}^{\infty} \frac{1}{k!} \boldsymbol{A}^k := \boldsymbol{I} + \boldsymbol{A} + \frac{1}{2!} \boldsymbol{A}^2 + \frac{1}{3!} \boldsymbol{A}^3 + \ldots$$

  - Does this series converge? For any matrix $\boldsymbol{A}$?