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Matrix exponential

Introduction to dynamical systems #5

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1 Exponential function

1.1 Real exponential function

- **Definition**
  - Recall that the real exponential function e^a can be defined as

$$e^a := \sum_{k=0}^{\infty} \frac{1}{k!} a^k = 1 + a + \frac{1}{2!} a^2 + \frac{1}{3!} a^3 + \dots,$$

which is well defined for every $a \in \mathbb{R}$ (why?)

- **Properties**
  - $e^0 = 1$
  - $e^a > 0$
  - $e^{a+b} = e^a e^b$
  - $(e^a)^k = e^{ka}$
  - $\frac{d}{dt} e^{at} = a e^{at}$
  - $\lim_{t \to \infty} e^{at} = 0$ if and only if $a < 0$

1.2 Matrix exponential function

- **Definition**
  - Similar to the real exponential function, we define the matrix exponential function as

$$e^{\boldsymbol{A}} := \sum_{k=0}^{\infty} \frac{1}{k!} \boldsymbol{A}^k = \boldsymbol{I} + \boldsymbol{A} + \frac{1}{2!} \boldsymbol{A}^2 + \frac{1}{3!} \boldsymbol{A}^3 + \dots,$$

which is also well defined for every square matrix $\boldsymbol{A}$ (why?)

  - Given $\boldsymbol{A}$, if there exists a matrix $\boldsymbol{B}$ such that

$$e^{\boldsymbol{B}} = \boldsymbol{A},$$

we call it the logarithm of $\boldsymbol{A}$ and write $\ln(\boldsymbol{A}) := \boldsymbol{B}$

- **Properties**

1. $e^{\boldsymbol{O}} = \boldsymbol{I}$ where $\boldsymbol{O}$ is the zero matrix
2. $e^{\boldsymbol{A}^\top} = \left(e^{\boldsymbol{A}}\right)^\top$
3. If $\boldsymbol{A}$ and $\boldsymbol{B}$ commute[1], then $\boldsymbol{B}e^{\boldsymbol{A}t} = e^{\boldsymbol{A}t}\boldsymbol{B}$ for any $t \in \mathbb{R}$
4. If $\boldsymbol{A}$ is non-singular, then $e^{\boldsymbol{A}\boldsymbol{B}\boldsymbol{A}^{-1}} = \boldsymbol{A}e^{\boldsymbol{B}}\boldsymbol{A}^{-1}$
5. For any $s, t \in \mathbb{R}$, we have $e^{\boldsymbol{A}s}e^{\boldsymbol{A}t} = e^{\boldsymbol{A}(s+t)}$
6. For any $\boldsymbol{A}$, $e^{\boldsymbol{A}}$ is non-singular and $(e^{\boldsymbol{A}})^{-1} = e^{-\boldsymbol{A}}$
7. $\frac{d}{dt}e^{\boldsymbol{A}t} = \boldsymbol{A}e^{\boldsymbol{A}t}$
8. If $\boldsymbol{A}$ and $\boldsymbol{B}$ commute, then $e^{\boldsymbol{A}}e^{\boldsymbol{B}} = e^{\boldsymbol{A}+\boldsymbol{B}}$
9. For any $k \in \mathbb{N}$, $(e^{\boldsymbol{A}})^k = e^{\boldsymbol{A}k}$
10. If $\boldsymbol{A}$ and $\boldsymbol{B}$ commute, so do $e^{\boldsymbol{A}}$ and $e^{\boldsymbol{B}}$
11. If $(\lambda, \boldsymbol{v})$ is an eigenpair of $\boldsymbol{A}$, then $(e^{\lambda t}, \boldsymbol{v})$ is an eigenpair of $e^{\boldsymbol{A}t}$ for all $t \in \mathbb{R}$
12. $\lim_{t\to\infty} e^{\boldsymbol{A}t} = \boldsymbol{O}$ if and only if the eigenvalues of $\boldsymbol{A}$ are all negative
13. If $\boldsymbol{A}$ is a diagonal matrix, so is $e^{\boldsymbol{A}}$ and

$$\boldsymbol{A} = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_m \end{bmatrix} \implies e^{\boldsymbol{A}} = \begin{bmatrix} e^{a_1} & 0 & \cdots & 0 \\ 0 & e^{a_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{a_m} \end{bmatrix}$$

14. If $\boldsymbol{J}_m(\lambda)$ is a Jordan block of size m, then

$$e^{\boldsymbol{J}_m(\lambda)t} = \begin{bmatrix} e^{\lambda t} & \frac{t}{1!}e^{\lambda t} & \frac{t^2}{2!}e^{\lambda t} & \cdots & \frac{t^{m-1}}{(m-1)!}e^{\lambda t} \\ 0 & e^{\lambda t} & \frac{t}{1!}e^{\lambda t} & \cdots & \frac{t^{m-2}}{(m-2)!}e^{\lambda t} \\ 0 & 0 & e^{\lambda t} & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & e^{\lambda t} \end{bmatrix} \quad \forall t \in \mathbb{R}$$

15. For a Jordan matrix $\boldsymbol{J} \in \mathbb{R}^{n\times n}$,

$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{J}_{n_1}(\lambda_1) & \boldsymbol{O} & \cdots & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{J}_{n_2}(\lambda_2) & \cdots & \boldsymbol{O} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{O} & \boldsymbol{O} & \cdots & \boldsymbol{J}_{n_d}(\lambda_d) \end{bmatrix} \implies e^{\boldsymbol{J}t} = \begin{bmatrix} e^{\boldsymbol{J}_{n_1}(\lambda_1)t} & \boldsymbol{O} & \cdots & \boldsymbol{O} \\ \boldsymbol{O} & e^{\boldsymbol{J}_{n_2}(\lambda_2)t} & \cdots & \boldsymbol{O} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{O} & \boldsymbol{O} & \cdots & e^{\boldsymbol{J}_{n_d}(\lambda_d)t} \end{bmatrix} \quad \forall t \in \mathbb{R}$$

16. If μ is an eigenvalue of $e^{\boldsymbol{A}}$, then $\mu > 0$ and $\lambda := \ln(\mu)$ is an eigenvalue of $\boldsymbol{A}$
17. The eigenvalues of $\boldsymbol{A}$ are all negative if and only if $\rho(e^{\boldsymbol{A}}) < 1$

[1] We say that two matrices $\boldsymbol{A}, \boldsymbol{B}$ commute if $\boldsymbol{A}\boldsymbol{B} = \boldsymbol{B}\boldsymbol{A}$. Obviously, $\boldsymbol{A}^k$ and $\boldsymbol{A}^l$ commute for any $k, l \in \mathbb{N}$. Also, if $\boldsymbol{A}$ and $\boldsymbol{B}$ commute, so do $(\boldsymbol{A}+\boldsymbol{B})$ and $\boldsymbol{A}$. Moreover, if $\boldsymbol{A}$ and $\boldsymbol{B}$ commute, so do $\boldsymbol{A}$ and $\boldsymbol{B}^k$ for any k.

- **Proof**

1. $e^{\boldsymbol{O}} = \boldsymbol{I}$ where $\boldsymbol{O}$ is the zero matrix (by definition)
2. $e^{\boldsymbol{A}^\top} = \left(e^{\boldsymbol{A}}\right)^\top$ because

$$\begin{aligned} e^{\boldsymbol{A}^\top} &:= I + \boldsymbol{A}^\top + \frac{1}{2!}(\boldsymbol{A}^\top)^2 + \frac{1}{3!}(\boldsymbol{A}^\top)^3 + \cdots \\ &= I^\top + \boldsymbol{A}^\top + \frac{1}{2!}(\boldsymbol{A}^2)^\top + \frac{1}{3!}(\boldsymbol{A}^3)^\top + \cdots \\ &= \left(I + \boldsymbol{A} + \frac{1}{2!}\boldsymbol{A}^2 + \frac{1}{3!}\boldsymbol{A}^3 + \cdots\right)^\top = (e^{\boldsymbol{A}})^\top \end{aligned}$$

3. If $\boldsymbol{A}$ and $\boldsymbol{B}$ commute, then $\boldsymbol{B}e^{\boldsymbol{A}t} = e^{\boldsymbol{A}t}\boldsymbol{B}$ for any $t \in \mathbb{R}$ because

$$\begin{aligned} \boldsymbol{B}e^{\boldsymbol{A}t} &= \boldsymbol{B} + \boldsymbol{B}\boldsymbol{A}t + \frac{t^2}{2!}\boldsymbol{B}\boldsymbol{A}^2 + \frac{t^3}{3!}\boldsymbol{B}\boldsymbol{A}^3 + \cdots \\ &= \boldsymbol{B} + \boldsymbol{A}\boldsymbol{B}t + \frac{t^2}{2!}\boldsymbol{A}^2\boldsymbol{B} + \frac{t^3}{3!}\boldsymbol{A}^3\boldsymbol{B} + \cdots = e^{\boldsymbol{A}t}\boldsymbol{B} \end{aligned}$$

4. If $\boldsymbol{A}$ is non-singular, then $e^{\boldsymbol{A}\boldsymbol{B}\boldsymbol{A}^{-1}} = \boldsymbol{A}e^{\boldsymbol{B}}\boldsymbol{A}^{-1}$ because

$$\begin{aligned} e^{\boldsymbol{A}\boldsymbol{B}\boldsymbol{A}^{-1}} &:= \boldsymbol{I} + \boldsymbol{A}\boldsymbol{B}\boldsymbol{A}^{-1} + \frac{1}{2!}(\boldsymbol{A}\boldsymbol{B}\boldsymbol{A}^{-1})^2 + \frac{1}{3!}(\boldsymbol{A}\boldsymbol{B}\boldsymbol{A}^{-1})^3 + \cdots \\ &= \boldsymbol{A}\boldsymbol{A}^{-1} + \boldsymbol{A}\boldsymbol{B}\boldsymbol{A}^{-1} + \frac{1}{2!}\boldsymbol{A}\boldsymbol{B}^2\boldsymbol{A}^{-1} + \frac{1}{3!}\boldsymbol{A}\boldsymbol{B}^3\boldsymbol{A}^{-1} + \cdots \\ &= \boldsymbol{A}\left(\boldsymbol{I} + \boldsymbol{B} + \frac{1}{2!}\boldsymbol{B}^2 + \frac{1}{3!}\boldsymbol{B}^3 + \cdots\right)\boldsymbol{A}^{-1} = \boldsymbol{A}e^{\boldsymbol{B}}\boldsymbol{A}^{-1} \end{aligned}$$

5. For any $s, t \in \mathbb{R}$, we have $e^{\boldsymbol{A}s}e^{\boldsymbol{A}t} = e^{\boldsymbol{A}(s+t)}$ because

$$\begin{aligned} e^{\boldsymbol{A}s}e^{\boldsymbol{A}t} &= \left(\sum_{j=0}^{\infty}\frac{\boldsymbol{A}^j s^j}{j!}\right)\left(\sum_{k=0}^{\infty}\frac{\boldsymbol{A}^k t^k}{k!}\right) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\boldsymbol{A}^j s^j}{j!}\frac{\boldsymbol{A}^k t^k}{k!} \\ &= \sum_{j=0}^{\infty}\sum_{l=j}^{\infty}\frac{\boldsymbol{A}^j s^j}{j!}\frac{\boldsymbol{A}^{l-j}t^{l-j}}{(l-j)!} = \sum_{l=0}^{\infty}\sum_{j=0}^{l}\frac{\boldsymbol{A}^j s^j}{j!}\frac{\boldsymbol{A}^{l-j}t^{l-j}}{(l-j)!} \\ &= \sum_{l=0}^{\infty}\frac{\boldsymbol{A}^l}{l!}\underbrace{\sum_{j=0}^{l}\frac{l!}{j!(l-j)!}s^j t^{l-j}}_{=(s+t)^l \text{ binomial theorem}} = \sum_{j=0}^{\infty}\frac{(\boldsymbol{A}(s+t))^l}{l!} = e^{\boldsymbol{A}(s+t)} \end{aligned}$$

where the fourth equality uses the fact that

$$\begin{aligned} \sum_{j=0}^{\infty}\sum_{l=j}^{\infty}x_{j,l} &= x_{0,0} + x_{0,1} + x_{0,2} + x_{0,3} + \cdots \\ &\qquad + x_{1,1} + x_{1,2} + x_{1,3} + x_{1,4} + \cdots \\ &\qquad\qquad + x_{2,2} + x_{2,3} + x_{2,4} + x_{2,5} + \cdots \\ &= \sum_{j=0}^{0}x_{j,0} + \sum_{j=0}^{1}x_{j,1} + \sum_{j=0}^{2}x_{j,2} + \sum_{j=0}^{3}x_{j,3} + \cdots = \sum_{l=0}^{\infty}\sum_{j=0}^{l}x_{j,l} \end{aligned}$$

6. $e^{\boldsymbol{A}}$ is invertible for any $\boldsymbol{A}$ and $(e^{\boldsymbol{A}})^{-1} = e^{-\boldsymbol{A}}$ because

$$e^{\boldsymbol{A}}e^{-\boldsymbol{A}} = e^{\boldsymbol{A}\times 1}e^{\boldsymbol{A}\times -1} = e^{\boldsymbol{A}\times(1-1)} = e^{\boldsymbol{A}\times 0} = e^{\boldsymbol{O}} = \boldsymbol{I}$$

7. $\frac{d}{dt}e^{\boldsymbol{A}t} = \boldsymbol{A}e^{\boldsymbol{A}t}$ because defining $\boldsymbol{F} : \mathbb{R} \to \mathbb{R}^{n\times n}$ by $\boldsymbol{F}(t) := e^{\boldsymbol{A}t}$ gives

$$\begin{aligned}
\frac{d\boldsymbol{F}(t)}{dt} &:= \lim_{h\to 0} \frac{\boldsymbol{F}(t+h) - \boldsymbol{F}(t)}{h} = \lim_{h\to 0} \frac{e^{\boldsymbol{A}(t+h)} - e^{\boldsymbol{A}t}}{h} = \lim_{h\to 0} \frac{e^{\boldsymbol{A}t}e^{\boldsymbol{A}h} - e^{\boldsymbol{A}t}}{h} = e^{\boldsymbol{A}t} \lim_{h\to 0} \frac{e^{\boldsymbol{A}h} - I}{h} \\
&= e^{\boldsymbol{A}t} \lim_{h\to 0} \frac{\frac{1}{1!}\boldsymbol{A}h + \frac{1}{2!}\boldsymbol{A}^2h^2 + \frac{1}{3!}\boldsymbol{A}^3h^3 + \cdots}{h} \\
&= e^{\boldsymbol{A}t} \lim_{h\to 0} \left(\frac{1}{1!}\boldsymbol{A} + \frac{1}{2!}\boldsymbol{A}^2h + \frac{1}{3!}\boldsymbol{A}^3h^2 + \cdots \right) = e^{\boldsymbol{A}t}\boldsymbol{A} = \boldsymbol{A}e^{\boldsymbol{A}t}
\end{aligned}$$

8. If $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{B} \in \mathbb{R}^{n\times n}$ commute, then $e^{\boldsymbol{A}}e^{\boldsymbol{B}} = e^{\boldsymbol{A}+\boldsymbol{B}}$ because defining $\boldsymbol{F} : \mathbb{R} \to \mathbb{R}^{n\times n}$ by

$$\boldsymbol{F}(t) := e^{(\boldsymbol{A}+\boldsymbol{B})t}e^{-\boldsymbol{B}t}e^{-\boldsymbol{A}t} \quad \forall t \in \mathbb{R}$$

yields

$$\begin{aligned}
\frac{d\boldsymbol{F}(t)}{dt} &= (\boldsymbol{A}+\boldsymbol{B})e^{(\boldsymbol{A}+\boldsymbol{B})t}e^{-\boldsymbol{A}t}e^{-\boldsymbol{B}t} + e^{(\boldsymbol{A}+\boldsymbol{B})t}e^{-\boldsymbol{A}t}(-\boldsymbol{B})e^{-\boldsymbol{B}t} + e^{(\boldsymbol{A}+\boldsymbol{B})t}(-\boldsymbol{A})e^{-\boldsymbol{A}t}e^{-\boldsymbol{B}t} \\
&= (\boldsymbol{A}+\boldsymbol{B})e^{(\boldsymbol{A}+\boldsymbol{B})t}e^{-\boldsymbol{A}t}e^{-\boldsymbol{B}t} - \boldsymbol{B}e^{(\boldsymbol{A}+\boldsymbol{B})t}e^{-\boldsymbol{A}t}e^{-\boldsymbol{B}t} - \boldsymbol{A}e^{(\boldsymbol{A}+\boldsymbol{B})t}e^{-\boldsymbol{A}t}e^{-\boldsymbol{B}t} \\
&= (\boldsymbol{A}+\boldsymbol{B}-\boldsymbol{B}-\boldsymbol{A})e^{(\boldsymbol{A}+\boldsymbol{B})t}e^{-\boldsymbol{A}t}e^{-\boldsymbol{B}t} = \boldsymbol{O},
\end{aligned}$$

which means $\boldsymbol{F}(t)$ is independent of t and in particular

$$e^{\boldsymbol{A}+\boldsymbol{B}}e^{-\boldsymbol{B}}e^{-\boldsymbol{A}} = \boldsymbol{F}(1) = \boldsymbol{F}(0) = e^{(\boldsymbol{A}+\boldsymbol{B})0}e^{-\boldsymbol{B}0}e^{-\boldsymbol{A}0} = \boldsymbol{I}.$$

9. For any $k \in \mathbb{N}$, $(e^{\boldsymbol{A}})^k = e^{\boldsymbol{A}k}$ because $\boldsymbol{A}$ and $\boldsymbol{A}$ commute and thus

$$(e^{\boldsymbol{A}})^2 = e^{\boldsymbol{A}}e^{\boldsymbol{A}} = e^{\boldsymbol{A}+\boldsymbol{A}} = e^{\boldsymbol{A}2}$$

10. If $\boldsymbol{A}$ and $\boldsymbol{B}$ commute, so do $e^{\boldsymbol{A}}$ and $e^{\boldsymbol{B}}$ because

$$e^{\boldsymbol{A}}e^{\boldsymbol{B}} = e^{\boldsymbol{A}+\boldsymbol{B}} = e^{\boldsymbol{B}+\boldsymbol{A}} = e^{\boldsymbol{B}}e^{\boldsymbol{A}}$$

11. If $(\lambda, \boldsymbol{v})$ is an eigenpair of $\boldsymbol{A}$, then $e^{\boldsymbol{A}t}\boldsymbol{v} = e^{\lambda t}\boldsymbol{v}$ because

$$\boldsymbol{A}^k\boldsymbol{v} = \lambda\boldsymbol{A}^{k-1}\boldsymbol{v} = \lambda^2\boldsymbol{A}^{k-2}\boldsymbol{v} = \lambda^k\boldsymbol{v} \quad \forall k \in \mathbb{N}$$

and therefore

$$\begin{aligned}
e^{\boldsymbol{A}t}\boldsymbol{v} &= \left(I + \frac{1}{1!}\boldsymbol{A}t + \frac{1}{2!}\boldsymbol{A}^2t^2 + \frac{1}{3!}\boldsymbol{A}^3t^3 + \cdots \right)\boldsymbol{v} \\
&= \left(\boldsymbol{v} + \frac{1}{1!}\boldsymbol{A}\boldsymbol{v}t + \frac{1}{2!}\boldsymbol{A}^2\boldsymbol{v}t^2 + \frac{1}{3!}\boldsymbol{A}^3\boldsymbol{v}t^3 + \cdots \right) \\
&= \left(\boldsymbol{v} + \frac{1}{1!}\lambda\boldsymbol{v}t + \frac{1}{2!}\lambda^2\boldsymbol{v}t^2 + \frac{1}{3!}\lambda^3\boldsymbol{v}t^3 + \cdots \right) \\
&= \left(1 + \frac{1}{1!}(\lambda t) + \frac{1}{2!}(\lambda t)^2 + \frac{1}{3!}(\lambda t)^3 + \cdots \right)\boldsymbol{v} = e^{\lambda t}\boldsymbol{v}
\end{aligned}$$

12. $\lim_{t\to\infty} e^{\boldsymbol{A}t} = \boldsymbol{O}$ if and only if the eigenvalues of $\boldsymbol{A}$ are all negative because

$$\lim_{k\to\infty} (e^{\boldsymbol{A}})^k = \boldsymbol{O} \iff \rho(e^{\boldsymbol{A}}) < 1 \iff \max\{e^{\lambda_1}, \ldots, e^{\lambda_n}\} < 1 \iff \max\{\lambda_1, \ldots, \lambda_n\} < 0$$

13. If $\boldsymbol{A}$ is a diagonal matrix, so is $e^{\boldsymbol{A}}$ and

$$\boldsymbol{A} = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_m \end{bmatrix} \implies e^{\boldsymbol{A}} = \begin{bmatrix} e^{a_1} & 0 & \cdots & 0 \\ 0 & e^{a_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{a_m} \end{bmatrix}$$

because

$$e^{\boldsymbol{A}} = \sum_{k=0}^{\infty} \frac{1}{k!}\boldsymbol{A}^k = \begin{bmatrix} \sum_{k=0}^{\infty} \frac{1}{k!} a_1^k & 0 & \cdots & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{1}{k!} a_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sum_{k=0}^{\infty} \frac{1}{k!} a_m^k \end{bmatrix} = \begin{bmatrix} e^{a_1} & 0 & \cdots & 0 \\ 0 & e^{a_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{a_m} \end{bmatrix}$$

14. If $\boldsymbol{J}_m(\lambda)$ is a Jordan block of size m, then

$$e^{\boldsymbol{J}_m(\lambda)t} = \begin{bmatrix} e^{\lambda t} & \frac{t}{1!}e^{\lambda t} & \frac{t^2}{2!}e^{\lambda t} & \cdots & \frac{t^{m-1}}{(m-1)!}e^{\lambda t} \\ 0 & e^{\lambda t} & \frac{t}{1!}e^{\lambda t} & \cdots & \frac{t^{m-2}}{(m-2)!}e^{\lambda t} \\ 0 & 0 & e^{\lambda t} & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & e^{\lambda t} \end{bmatrix} \quad \forall t \in \mathbb{R}$$

because

$$(\boldsymbol{J}_m(\lambda)t)^k = t^k \begin{bmatrix} c_0(k) & c_1(k) & c_2(k) & \cdots & c_{m-1}(k) \\ 0 & c_0(k) & c_1(k) & \cdots & c_{m-2}(k) \\ 0 & 0 & c_0(k) & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & c_0(k) \end{bmatrix}, \quad c_l(k) := \begin{cases} \frac{k!}{l!(k-l)!}\lambda^{k-l} & k \geq l \\ 0 & k < l \end{cases}$$

and

$$e^{\boldsymbol{J}_m(\lambda)t} := \sum_{k=0}^{\infty} \frac{1}{k!}(\boldsymbol{J}_m(\lambda)t)^k = \begin{bmatrix} \sum_{k=0}^{\infty} \frac{t^k}{k!}c_0(k) & \sum_{k=0}^{\infty} \frac{t^k}{k!}c_1(k) & \sum_{k=0}^{\infty} \frac{t^k}{k!}c_2(k) & \cdots & \sum_{k=0}^{\infty} \frac{t^k}{k!}c_{m-1}(k) \\ 0 & \sum_{k=0}^{\infty} \frac{t^k}{k!}c_0(k) & \sum_{k=0}^{\infty} \frac{t^k}{k!}c_1(k) & \cdots & \sum_{k=0}^{\infty} \frac{t^k}{k!}c_{m-2}(k) \\ 0 & 0 & \sum_{k=0}^{\infty} \frac{t^k}{k!}c_0(k) & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \sum_{k=0}^{\infty} \frac{t^k}{k!}c_0(k) \end{bmatrix}$$

where

$$\sum_{k=0}^{\infty} \frac{t^k}{k!}c_l(k) = \sum_{k=l}^{\infty} \frac{t^k}{k!}c_l(k) = \sum_{k=l}^{\infty} \frac{t^l t^{k-l}}{k!} \frac{k!}{l!(t-l)!}\lambda_i^{k-l} = \frac{t^l}{l!} \sum_{k=l}^{\infty} \frac{1}{(k-l)!}(\lambda_i t)^{k-l} = \frac{t^l}{l!}e^{\lambda_i t}$$

15. For a Jordan matrix $\boldsymbol{J} \in \mathbb{R}^{n \times n}$,

$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{J}_{n_1}(\lambda_1) & \boldsymbol{O} & \cdots & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{J}_{n_2}(\lambda_2) & \cdots & \boldsymbol{O} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{O} & \boldsymbol{O} & \cdots & \boldsymbol{J}_{n_d}(\lambda_d) \end{bmatrix} \implies e^{\boldsymbol{J}t} = \begin{bmatrix} e^{\boldsymbol{J}_{n_1}(\lambda_1)t} & \boldsymbol{O} & \cdots & \boldsymbol{O} \\ \boldsymbol{O} & e^{\boldsymbol{J}_{n_2}(\lambda_2)t} & \cdots & \boldsymbol{O} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{O} & \boldsymbol{O} & \cdots & e^{\boldsymbol{J}_{n_d}(\lambda_d)t} \end{bmatrix}$$

because

$$e^{Jt} := \sum_{k=0}^{\infty} \frac{t^k}{k!} J^k = \begin{bmatrix} \sum_{k=0}^{\infty} \frac{1}{k!}(J_{n_1}(\lambda_1)t)^k & O & \cdots & O \\ O & \sum_{k=0}^{\infty} \frac{1}{k!}(J_{n_2}(\lambda_2)t)^k & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & \sum_{k=0}^{\infty} \frac{1}{k!}(J_{n_d}(\lambda_d)t)^k \end{bmatrix}$$

16. Decompose A as VJV^{-1} where J is the Jordan normal form of A. Since $e^{A} = Ve^{J}V^{-1}$, we know that e^{A} and e^{J} are similar. Because e^{J} is a triangle matrix, it follows that eigenvalues of e^{A} are diagonal elements of e^{J}, which is $e^{\lambda_1}, e^{\lambda_2}, \ldots, e^{\lambda_n}$ where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of A. Therefore, if μ is an eigenvalue of e^{A}, there must exist an eigenvalue λ of A such that $\mu = e^{\lambda} > 0$.
17. Immediate from the results (11 and 16) above

1.3 Examples

- **Example 1**
  - Consider a square matrix

$$A := \begin{bmatrix} 1 & 0 & 0 \\ 6 & -2 & -6 \\ -2 & 1 & 3 \end{bmatrix}$$

  - Let us say we want to compute the exponential of A:

$$e^{A} := I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \ldots,$$

  the right-hand side of which is not easy to deal with
  - Notice that A is decomposed as

$$\begin{bmatrix} 1 & 0 & 0 \\ 6 & -2 & -6 \\ -2 & 1 & 3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 1 \\ -3 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} 2 & -1 & -2 \\ 3 & -1 & -3 \\ -2 & 1 & 3 \end{bmatrix}}_{V^{-1}}$$

  so

$$e^{A} = e^{V\Lambda V^{-1}} = Ve^{\Lambda}V^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ -3 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e \end{bmatrix} \begin{bmatrix} 2 & -1 & -2 \\ 3 & -1 & -3 \\ -2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} e & 0 & 0 \\ -6+6e & 3-2e & 6-6e \\ 2-2e & -1+e & -2+3e \end{bmatrix}$$

- **Example 2**
  - Consider a square matrix

$$A := \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$$

  - Let us say we want to compute e^{At} for each real $t \in \mathbb{R}$
  - Notice that A is decomposed as

$$\begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}}_{J} \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}}_{V^{-1}}$$

  so

$$e^{At} = Ve^{Jt}V^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} e^{3t} & te^{3t} \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} te^{3t} + e^{3t} & te^{3t} \\ -te^{3t} & e^{3t} - te^{3t} \end{bmatrix}$$