Eigenvalues and eigenvectors

Introduction to dynamical systems #2

Hiroaki Sakamoto

§ Contents

1 Eigenvalues and eigenvectors

1.1 Definition

- **Definition of eigenvalues and eigenvectors**
  - Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ be a square matrix.
  - If a real value $\lambda \in \mathbb{R}$ and a non-zero vector $\boldsymbol{v} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}$ jointly satisfy

$$\boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{v},$$

    we say that λ is an *eigenvalue* of $\boldsymbol{A}$ and $\boldsymbol{v}$ is an *eigenvector* of $\boldsymbol{A}$ associated with λ (and we also call $(\lambda, \boldsymbol{v})$ an *eigenpair* of $\boldsymbol{A}$)
  - Notice:
    - If $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are both eigenvectors associated with the same eigenvalue λ of $\boldsymbol{A}$, then so is their linear combination $\alpha_1 \boldsymbol{v}_2 + \alpha_2 \boldsymbol{v}_2$ because

$$\boldsymbol{A}\left(\alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2\right) = \left(\alpha_1 \boldsymbol{A}\boldsymbol{v}_1 + \alpha_2 \boldsymbol{A}\boldsymbol{v}_2\right) = \left(\alpha_1 \lambda \boldsymbol{v}_1 + \alpha_2 \lambda \boldsymbol{v}_2\right) = \lambda\left(\alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2\right)$$

    - If $(\lambda, \boldsymbol{v})$ is an eigenpair of $\boldsymbol{A}$, then $(\lambda^k, \boldsymbol{v})$ is an eigenpair of $\boldsymbol{A}^k$ for any $k \in \mathbb{N}$ because $\boldsymbol{A}^k \boldsymbol{v} = \boldsymbol{A}^{k-1}\boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{A}^{k-1}\boldsymbol{v} = \lambda^2 \boldsymbol{A}^{k-2}\boldsymbol{v} = \cdots = \lambda^k \boldsymbol{A}^{k-k}\boldsymbol{v} = \lambda^k \boldsymbol{v}$
    - Provided that $\boldsymbol{A}$ is non-singular (which is the case if and only if 0 is not an eigenvalue of $\boldsymbol{A}$; See below),

$$(\lambda, \boldsymbol{v}) \text{ is an eigenpair of } \boldsymbol{A} \iff (\lambda^{-1}, \boldsymbol{v}) \text{ is an eigenpair of } \boldsymbol{A}^{-1}$$

    - If $\boldsymbol{A} \in \mathbb{R}^{m\times m}$ has eigenpairs $(\lambda_1, \boldsymbol{v}_1), (\lambda_2, \boldsymbol{v}_2), \ldots, (\lambda_m, \boldsymbol{v}_m)$ and $\boldsymbol{B} \in \mathbb{R}^{p\times p}$ has eigenpairs $(\mu_1, \boldsymbol{u}_1), (\mu_2, \boldsymbol{u}_2), \ldots, (\mu_p, \boldsymbol{u}_p)$, then, for any $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, p$,

$$(\boldsymbol{A} \otimes \boldsymbol{B})(\boldsymbol{v}_i \otimes \boldsymbol{u}_j) = (\boldsymbol{A}\boldsymbol{v}_i \otimes \boldsymbol{B}\boldsymbol{u}_j) = (\lambda_i \boldsymbol{v}_i \otimes \mu_j \boldsymbol{u}_j) = \lambda_i \mu_j (\boldsymbol{v}_i \otimes \boldsymbol{u}_j),$$

      meaning that $(\lambda_i \mu_j, \boldsymbol{v}_i \otimes \boldsymbol{u}_j)$ is an eigenpair of $\boldsymbol{A} \otimes \boldsymbol{B}$, which implies that $\boldsymbol{A} \otimes \boldsymbol{B} \in \mathbb{R}^{mp\times mp}$ has the following mp eigenvalues

$$\lambda_i \mu_j \quad \forall i = 1, 2, \ldots, m, \quad \forall j = 1, 2, \ldots, p$$

- **Example**
  - Consider a square matrix
$$A := \begin{bmatrix} 5/2 & -1 \\ 1 & 0 \end{bmatrix}$$
  - Then $\lambda_1 := 2$ is an eigenvalue of A and $v_1 := (2, 1)^\top$ is an eigenvector of A associated with λ_1 because
$$Av_1 = \begin{bmatrix} 5/2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \lambda_1 v_1$$
  - Also, $\lambda_2 := 1/2$ is another eigenvalue of A and $v_2 := (1, 2)^\top$ is an eigenvector of A associated with λ_2 because
$$Av_2 = \begin{bmatrix} 5/2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \lambda_2 v_2$$

1.2 Characteristic polynomials

- **Definition of characteristic polynomials**
  - Let $A \in \mathbb{R}^{n \times n}$ be a square matrix.
  - Define a function $\phi_A : \mathbb{R} \to \mathbb{R}$ by
$$\phi_A(t) := |A - tI| \quad \forall t \in \mathbb{R},$$
    where $I \in \mathbb{R}^{n \times n}$ is the identity matrix. We call ϕ_A the *characteristic polynomial* of A
- **Useful results**
  - Note that λ is an eigenvalue of A if and only if $\phi_A(\lambda) = 0$ because:
    - $|A - \lambda I| = 0$ iff $(A - \lambda I)$ is singular (i.e., not invertible)
    - $(A - \lambda I)$ is singular iff column vectors of $(A - \lambda I)$ are linearly dependent
    - column vectors of $(A - \lambda I)$ are linearly dependent iff $(A - \lambda I)v = \mathbf{0}$ for some $v \neq \mathbf{0}$
  - In general, $A \in \mathbb{R}^{n \times n}$ has $m \leq n$ distinct eigenvalues $\lambda_1, \ldots, \lambda_m$ if and only if
$$\phi_A(t) = (\lambda_1 - t)^{k_1} (\lambda_2 - t)^{k_2} \cdots (\lambda_m - t)^{k_m},$$
    where $k_i \in \mathbb{N}$ is called the *algebraic multiplicity* of λ_i, satisfying $k_1 + k_2 + \ldots + k_m = n$
  - An immediate consequence:
$$A \text{ is singular} \iff |A| = 0 \iff \phi_A(0) = 0 \iff 0 \text{ is an eigenvalue of } A$$
  - If A is a triangle matrix, the eigenvalues of A are the diagonal elements of A because
$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \implies \phi_A(t) = (a_{11} - t)(a_{22} - t) \cdots (a_{nn} - t)$$
  - The determinant of A equals the product of all eigenvalues of A (including duplicates):
$$|A| = |A - 0I| = \phi_A(0) = (\lambda_1 - 0)^{k_1} (\lambda_2 - 0)^{k_2} \cdots (\lambda_m - 0)^{k_m} = \lambda_1^{k_1} \lambda_2^{k_2} \cdots \lambda_m^{k_m}$$

- **Examples**
  - Consider a square matrix
$$A := \begin{bmatrix} 5/2 & -1 \\ 1 & 0 \end{bmatrix},$$
whose characteristic polynomial is
$$\phi_A(t) := \left| \begin{bmatrix} 5/2 & -1 \\ 1 & 0 \end{bmatrix} - t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = \begin{vmatrix} 5/2 - t & -1 \\ 1 & -t \end{vmatrix} = -\left(\frac{5}{2} - t\right) t + 1 = (t-2)\left(t - \frac{1}{2}\right),$$
meaning that $\lambda_1 := 2$ and $\lambda_1 := 1/2$ are two eigenvalues of A and their algebraic multiplicity is 1
  - Consider another square matrix
$$A := \begin{bmatrix} 5 & 4 \\ -1 & 1 \end{bmatrix},$$
whose characteristic polynomial is
$$\phi_A(t) = \begin{vmatrix} 5 - t & 4 \\ -1 & 1 - t \end{vmatrix} = (t-3)^2,$$
which means that $\lambda_1 := 3$ is the unique eigenvalue of A and its algebraic multiplicity is 2

1.3 Solving for eigenvectors

- **Procedure**

1. Given a square matrix $A \in \mathbb{R}^{n \times n}$, find eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of A by solving $\phi_A(\lambda) = 0$ for λ
2. For each λ_i, solve the linear system of equations $Av = \lambda_i v$ for $v \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, i.e.,
$$(A - \lambda_i I)\, v = \mathbf{0} \iff v = \dots,$$
which is an eigenvector of A associated with λ_i

- **Example**
  - Consider a square matrix
$$A := \begin{bmatrix} 5/2 & -1 \\ 1 & 0 \end{bmatrix}$$
  - We know that
$$\phi_A(\lambda) = 0 \iff \lambda = 2, \frac{1}{2}$$
so let $\lambda_1 := 2$, $\lambda_2 := 1/2$
  - Solve $(A - \lambda_1 I)v = \mathbf{0}$ for v, i.e.,
$$\begin{bmatrix} 5/2 - 2 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff v_1 = 2v_2 \iff \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \alpha \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \forall \alpha \in \mathbb{R},$$
meaning that $v_1 := (2,1)^\top$ is an eigenvector of A associated with $\lambda_1 = 2$
  - Similarly, solve $(A - \lambda_2 I)v = \mathbf{0}$ for v, i.e.,
$$\begin{bmatrix} 5/2 - 1/2 & -1 \\ 1 & -1/2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff v_1 = \frac{1}{2} v_2 \iff \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \forall \alpha \in \mathbb{R},$$
meaning that $v_2 := (1,2)^\top$ is an eigenvector of A associated with $\lambda_2 = 1/2$

2 Diagonalization

2.1 Definition

- **Definition of diagonalizable matrices**
  - A square matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is said to be *diagonalizable* if there exists a nonsingular (i.e., invertible) matrix $\boldsymbol{V} \in \mathbb{R}^{n\times n}$ and a diagonal matrix $\boldsymbol{\Lambda} \in \mathbb{R}^{n\times n}$ such that

$$\boldsymbol{A} = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1}$$

  - $\boldsymbol{\Lambda}$ is the 'simplest' matrix that is *similar* to $\boldsymbol{A}$

- **Remark**
  - Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be (arbitrarily chosen) n eigenvalues of $\boldsymbol{A}$ and let $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ be the associated eigenvectors
  - Let $\boldsymbol{\Lambda} := \mathrm{diag}(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^{n\times n}$ and $\boldsymbol{V} := [\boldsymbol{v}_1 \ldots \boldsymbol{v}_n] \in \mathbb{R}^{n\times n}$
  - Since $\boldsymbol{A}\boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i$ for $i = 1, 2, \ldots, n$, we have

$$\boldsymbol{A}\begin{bmatrix}\boldsymbol{v}_1 & \boldsymbol{v}_2 & \ldots & \boldsymbol{v}_n\end{bmatrix} = \begin{bmatrix}\boldsymbol{A}\boldsymbol{v}_1 & \boldsymbol{A}\boldsymbol{v}_2 & \ldots & \boldsymbol{A}\boldsymbol{v}_n\end{bmatrix} = \begin{bmatrix}\boldsymbol{v}_1 & \boldsymbol{v}_2 & \ldots & \boldsymbol{v}_n\end{bmatrix}\begin{bmatrix}\lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n\end{bmatrix}$$

    or $\boldsymbol{A}\boldsymbol{V} = \boldsymbol{V}\boldsymbol{\Lambda}$
  - Thus, $\boldsymbol{A}$ is diagonalizable whenever $\boldsymbol{V}$ is non-singular
  - $\boldsymbol{V}$ is non-singular if and only if its column vectors are linearly independent
  - $\boldsymbol{A}$ is diagonalizable whenever **one can find n linearly independent eigenvectors of $\boldsymbol{A}$**

- **A sufficient condition for diagonalization**
  - A square matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is diagonalizable if it has n distinct eigenvalues[1]
  - Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the n distinct eigenvalues of $\boldsymbol{A}$ and let $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ be the associated eigenvectors so that

$$\boldsymbol{A}\boldsymbol{V} = \boldsymbol{V}\boldsymbol{\Lambda}$$

  - Since $\lambda_1, \ldots, \lambda_n$ are distinct from each other, $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are linearly independent:
    - If $\boldsymbol{v}_i$ and $\boldsymbol{v}_j$ are linearly dependent, there exists $c \neq 0$ such that $c\boldsymbol{v}_i = \boldsymbol{v}_j$ and thus

$$c(\lambda_i \boldsymbol{v}_i) = c(\boldsymbol{A}\boldsymbol{v}_i) = \boldsymbol{A}(c\boldsymbol{v}_i) = \boldsymbol{A}\boldsymbol{v}_j = \lambda_j \boldsymbol{v}_j = \lambda_j(c\boldsymbol{v}_i) = c(\lambda_j \boldsymbol{v}_i),$$

      which, because $c \neq 0$ and $\boldsymbol{v}_i \neq \boldsymbol{0}$, implies $\lambda_i = \lambda_j$, a contradiction
    - By induction, one can conclude $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are linearly independent
  - Since $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are linearly independent, the matrix $\boldsymbol{V} = [\boldsymbol{v}_1 \ldots \boldsymbol{v}_n]$ is nonsingular
  - Then there exists the inverse $\boldsymbol{V}^{-1}$ and therefore

$$\boldsymbol{A} = \boldsymbol{A}\boldsymbol{V}\boldsymbol{V}^{-1} = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1},$$

    meaning that $\boldsymbol{A}$ is diagonalizable

[1] This is a sufficient, but not necessary, condition for a matrix to be diagonalizable.

- **Similar matrices**
  - We say that two matrices, $\boldsymbol{A}$ and $\boldsymbol{B}$, are *similar* if there exits a non-singular $\boldsymbol{C}$ such that

$$\boldsymbol{A} = \boldsymbol{C}\boldsymbol{B}\boldsymbol{C}^{-1}$$

  - If $\boldsymbol{A}$ and $\boldsymbol{B}$ are similar, then
    - $|\boldsymbol{A}| = |\boldsymbol{B}|$ because

$$|\boldsymbol{A}| = |\boldsymbol{C}\boldsymbol{B}\boldsymbol{C}^{-1}| = |\boldsymbol{C}||\boldsymbol{B}||\boldsymbol{C}^{-1}| = |\boldsymbol{C}||\boldsymbol{B}||\boldsymbol{C}|^{-1} = |\boldsymbol{B}|$$

    - $\boldsymbol{A}$ and $\boldsymbol{B}$ have the same characteristic polynomial because

$$\phi_{\boldsymbol{A}}(t) = |\boldsymbol{A} - t\boldsymbol{I}| = |\boldsymbol{C}\boldsymbol{B}\boldsymbol{C}^{-1} - t\boldsymbol{I}| = |\boldsymbol{C}||\boldsymbol{B} - t\boldsymbol{I}||\boldsymbol{C}^{-1}| = |\boldsymbol{B} - t\boldsymbol{I}| = \phi_{\boldsymbol{B}}(t) \quad \forall t \in \mathbb{R}$$

    - $\boldsymbol{A}$ and $\boldsymbol{B}$ have the same set of eigenvalues
    - $\boldsymbol{A}$ is diagonalizable if and only if $\boldsymbol{B}$ is diagonalizable

2.2 Examples

- **Example 1**
  - Consider a square matrix

$$\boldsymbol{A} := \begin{bmatrix} 5/2 & -1 \\ 1 & 0 \end{bmatrix},$$

which we know has $\lambda_1 = 2, \lambda_2 = 1/2$ as eigenvalues and $\boldsymbol{v}_1 = (2,1)^\top, \boldsymbol{v}_2 = (1,2)^\top$ as associated eigenvectors

  - Define

$$\boldsymbol{V} := \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad \boldsymbol{\Lambda} := \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

  - Then

$$|\boldsymbol{V}| = 3, \quad \boldsymbol{V}^{-1} = \frac{1}{|\boldsymbol{V}|} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix}$$

and therefore

$$\boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix} = \begin{bmatrix} 5/2 & -1 \\ 1 & 0 \end{bmatrix} = \boldsymbol{A}$$

- **Example 2**
  - Consider a square matrix

$$\boldsymbol{A} := \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$$

  - The characteristic polynomial is

$$\phi_{\boldsymbol{A}}(t) = \begin{vmatrix} 1-t & 0 & -1 \\ 1 & 2-t & 1 \\ 2 & 2 & 3-t \end{vmatrix} = (1-t)(2-t)(3-t),$$

which means that $\lambda_1 := 1$, $\lambda_2 := 2$, $\lambda_3 := 3$ are the eigenvalues of $\boldsymbol{A}$

- Solving $Av = \lambda_1 v$ for v yields

$$\begin{bmatrix} 1-1 & 0 & -1 \\ 1 & 2-1 & 1 \\ 2 & 2 & 3-1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \mathbf{0} \iff \begin{matrix} v_2 = -v_1 \\ v_3 = 0 \end{matrix} \iff \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \forall \alpha \in \mathbb{R}$$

so we choose the following v_1 as an eigenvector associated with λ_1:

$$v_1 := \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

- Solving $Av = \lambda_2 v$ for v yields

$$\begin{bmatrix} 1-2 & 0 & -1 \\ 1 & 2-2 & 1 \\ 2 & 2 & 3-2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \mathbf{0} \iff \begin{matrix} v_1 = -2v_2 \\ v_3 = -v_1 \end{matrix} \iff \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \alpha \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}, \quad \forall \alpha \in \mathbb{R}$$

so we choose the following v_2 as an eigenvector associated with λ_2:

$$v_2 := \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$

- Solving $Av = \lambda_3 v$ for v yields

$$\begin{bmatrix} 1-3 & 0 & -1 \\ 1 & 2-3 & 1 \\ 2 & 2 & 3-3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \mathbf{0} \iff \begin{matrix} v_2 = -v_1 \\ v_3 = -2v_1 \end{matrix} \iff \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \quad \forall \alpha \in \mathbb{R}$$

so we choose the following v_3 as an eigenvector associated with λ_3:

$$v_3 := \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$$

- Define

$$V := \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & -1 \\ 0 & 2 & -2 \end{bmatrix}, \quad \boldsymbol{\Lambda} := \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

- Then

$$V^{-1} = \begin{bmatrix} 0 & -1 & 1/2 \\ -1 & -1 & 0 \\ -1 & -1 & -1/2 \end{bmatrix}$$

and therefore

$$V\boldsymbol{\Lambda}V^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & -1 \\ 0 & 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1/2 \\ -1 & -1 & 0 \\ -1 & -1 & -1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix} = A$$

- **Example 3**
  - Consider a square matrix
$$A := \begin{bmatrix} 5 & 4 \\ -1 & 1 \end{bmatrix}$$
  - The characteristic polynomial is
$$\phi_A(t) = \begin{vmatrix} 5-t & 4 \\ -1 & 1-t \end{vmatrix} = (3-t)^2$$
    which means that $\lambda_1 := 3$ is the unique eigenvalue of A
  - Solving $Av = \lambda_1 v$ for v yields
$$\begin{bmatrix} 5-3 & 4 \\ -1 & 1-3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \mathbf{0} \iff v_1 = -2v_2 \iff \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \alpha \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \forall \alpha \in \mathbb{R},$$
    meaning that we can only choose one linearly independent eigenvector
  - Matrix A is not diagonalizable
- **Example 4**
  - Consider a square matrix
$$A := \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 2 & 1 & 5 \end{bmatrix}$$
  - The characteristic polynomial is
$$\phi_A(t) = \begin{vmatrix} 1-t & 3 & 0 \\ 0 & 1-t & 0 \\ 2 & 1 & 5-t \end{vmatrix} = (1-t)^2(5-t)$$
    which means that $\lambda_1 := 1$, $\lambda_2 := 5$ are the eigenvalues of A
  - Solving $Av = \lambda_1 v$ for v yields
$$\begin{bmatrix} 1-1 & 3 & 0 \\ 0 & 1-1 & 0 \\ 2 & 1 & 5-1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \mathbf{0} \iff \begin{array}{c} v_2 = 0 \\ v_1 = -2v_3 \end{array} \iff \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \alpha \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \quad \forall \alpha \in \mathbb{R}$$
    so we choose the following v_1 as an eigenvector associated with λ_1:
$$v_1 := \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$
  - Solving $Av = \lambda_2 v$ for v yields
$$\begin{bmatrix} 1-5 & 3 & 0 \\ 0 & 1-5 & 0 \\ 2 & 1 & 5-5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \mathbf{0} \iff \begin{array}{c} 4v_1 = 3v_2 \\ v_2 = 0 \end{array} \iff \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \alpha \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \forall \alpha \in \mathbb{R}$$
    so we choose the following v_2 as an eigenvector associated with λ_1:
$$v_2 := \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
  - Matrix A is not diagonalizable

- **Example 5**
  - Consider a square matrix
$$A := \begin{bmatrix} 1 & 0 & 0 \\ 6 & -2 & -6 \\ -2 & 1 & 3 \end{bmatrix}$$
  - The characteristic polynomial is
$$\phi_A(t) = \begin{vmatrix} 1-t & 0 & 0 \\ 6 & -2-t & -6 \\ -2 & 1 & 3-t \end{vmatrix} = -t(1-t)^2$$
which means that $\lambda_1 := 0$, $\lambda_2 := 1$ are the eigenvalues of $\boldsymbol{A}$
  - Solving $\boldsymbol{A}\boldsymbol{v} = \lambda_1 \boldsymbol{v}$ for $\boldsymbol{v}$ yields
$$\begin{bmatrix} 1-0 & 0 & 0 \\ 6 & -2-0 & -6 \\ -2 & 1 & 3-0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \mathbf{0} \iff \begin{matrix} v_1 = 0 \\ v_2 = -3v_3 \end{matrix} \iff \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \alpha \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}, \quad \forall \alpha \in \mathbb{R}$$
so we choose the following $\boldsymbol{v}_1$ as an eigenvector associated with λ_1:
$$\boldsymbol{v}_1 := \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}$$
  - Solving $\boldsymbol{A}\boldsymbol{v} = \lambda_2 \boldsymbol{v}$ for $\boldsymbol{v}$ yields
$$\begin{bmatrix} 1-1 & 0 & 0 \\ 6 & -2-1 & -6 \\ -2 & 1 & 3-1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \mathbf{0} \iff v_1 = \tfrac{1}{2}v_2 + v_3 \iff \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \forall \alpha, \beta$$
so we choose the following as two linearly independent eigenvectors associated with λ_2:
$$\boldsymbol{v}_2 := \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \boldsymbol{v}_3 := \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$
  - Note that
    - $\boldsymbol{A}$ does not have 3 distinct eigenvalues
    - yet $\boldsymbol{A}$ has 3 linearly independent eigenvectors (one for λ_1, two for λ_2)
    - we say that λ_1 and λ_2 has *geometric multiplicity* of 1 and 2, respectively
  - In fact, $\boldsymbol{A}$ is diagonalizable by defining
$$\boldsymbol{V} := \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ -3 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{\Lambda} := \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
  - Then
$$\boldsymbol{V}^{-1} = \begin{bmatrix} 2 & -1 & -2 \\ 3 & -1 & -3 \\ -2 & 1 & 3 \end{bmatrix}$$
and therefore
$$\boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ -3 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & -2 \\ 3 & -1 & -3 \\ -2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 6 & -2 & -6 \\ -2 & 1 & 3 \end{bmatrix} = \boldsymbol{A}$$