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Preliminaries

Introduction to dynamical systems #1

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1 Notations and definitions

- **Vector**
  - An m-dimensional vector is denoted as

$$\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}, \quad \text{where} \quad v_i \in \mathbb{R}$$

  - The set of all m-dimensional vectors is denoted by $\mathbb{R}^m$
  - For $\boldsymbol{v}, \boldsymbol{u} \in \mathbb{R}^m$ and $\alpha \in \mathbb{R}$, we define scalar multiplication $\alpha\boldsymbol{v}$ and addition $\boldsymbol{v} + \boldsymbol{u}$ as

$$\alpha\boldsymbol{v} := \begin{bmatrix} \alpha v_1 \\ \alpha v_2 \\ \vdots \\ \alpha v_m \end{bmatrix}, \quad \boldsymbol{v} + \boldsymbol{u} := \begin{bmatrix} v_1 + u_1 \\ v_2 + u_2 \\ \vdots \\ v_m + u_m \end{bmatrix}$$

- **Matrix**
  - A real matrix of dimension $m \times n$ is denoted as

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \dots & \boldsymbol{a}_n \end{bmatrix}, \quad \text{where} \quad \boldsymbol{a}_j := \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} \in \mathbb{R}^m$$

  - The set of all $m \times n$ real matrices is denoted as $\mathbb{R}^{m \times n}$
  - $\boldsymbol{A}^\top$ denotes the transpose of $\boldsymbol{A}$
  - If $\boldsymbol{A}$ has an inverse, we denote it by $\boldsymbol{A}^{-1}$
  - The determinant of $\boldsymbol{A}$ is denoted by $|\boldsymbol{A}|$
  - We use $\boldsymbol{I}$ for an identity matrix and $\boldsymbol{O}$ for a null matrix

$$\boldsymbol{I} := \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}, \quad \boldsymbol{O} := \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

- **Some definitions**
  - For $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ and $\boldsymbol{v} \in \mathbb{R}^n$, we define $\boldsymbol{A}\boldsymbol{v} \in \mathbb{R}^m$ as

$$\boldsymbol{A}\boldsymbol{v} := \sum_{j=1}^{n} v_j \boldsymbol{a}_j = v_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + v_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + v_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

  - For $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ and $\alpha \in \mathbb{R}$, we define $\alpha\boldsymbol{A} \in \mathbb{R}^{m \times n}$ as

$$\alpha\boldsymbol{A} := \begin{bmatrix} \alpha\boldsymbol{a}_1 & \alpha\boldsymbol{a}_2 & \dots & \alpha\boldsymbol{a}_n \end{bmatrix}$$

  - For $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ and $\boldsymbol{B} \in \mathbb{R}^{m \times n}$, we define $\boldsymbol{A} + \boldsymbol{B} \in \mathbb{R}^{m \times n}$ as

$$\boldsymbol{A} + \boldsymbol{B} := \begin{bmatrix} \boldsymbol{a}_1 + \boldsymbol{b}_1 & \boldsymbol{a}_2 + \boldsymbol{b}_2 & \dots & \boldsymbol{a}_n + \boldsymbol{b}_n \end{bmatrix}$$

  - For $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ and $\boldsymbol{B} \in \mathbb{R}^{l \times m}$ we define $\boldsymbol{B}\boldsymbol{A} \in \mathbb{R}^{l \times n}$ as

$$\boldsymbol{B}\boldsymbol{A} := \begin{bmatrix} \boldsymbol{B}\boldsymbol{a}_1 & \boldsymbol{B}\boldsymbol{a}_2 & \dots & \boldsymbol{B}\boldsymbol{a}_n \end{bmatrix}$$

- **Some facts**
  - $\boldsymbol{C}(\boldsymbol{B}\boldsymbol{A}) = (\boldsymbol{C}\boldsymbol{B})\boldsymbol{A}$
  - $\boldsymbol{C}(\boldsymbol{B} + \boldsymbol{A}) = \boldsymbol{C}\boldsymbol{B} + \boldsymbol{C}\boldsymbol{A}$
  - $(\boldsymbol{C} + \boldsymbol{B})\boldsymbol{A} = \boldsymbol{C}\boldsymbol{A} + \boldsymbol{B}\boldsymbol{A}$
  - $(\boldsymbol{A}\boldsymbol{B})^\top = \boldsymbol{B}^\top \boldsymbol{A}^\top$
  - $(\boldsymbol{A}\boldsymbol{B})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$
  - $|\boldsymbol{A}\boldsymbol{B}| = |\boldsymbol{A}||\boldsymbol{B}|$
  - $|\boldsymbol{A}^{-1}| = |\boldsymbol{A}|^{-1}$
  - $|\boldsymbol{A}| \neq 0$ if and only if $\boldsymbol{A}$ is non-singular (i.e., $\boldsymbol{A}^{-1}$ exists)
  - For any $\boldsymbol{A} \in \mathbb{R}^{n \times n}$,

$$|\boldsymbol{A}| = \sum_{i=1}^{n} a_{ij}\tilde{a}_{ij} \quad \forall j, \qquad |\boldsymbol{A}| = \sum_{j=1}^{n} a_{ij}\tilde{a}_{ij} \quad \forall i,$$

    where $\tilde{a}_{ij}$ is the cofactor of a_{ij}, defined by

$$\tilde{a}_{ij} := (-1)^{i+j} \begin{vmatrix} a_{11} & \cdots & a_{1j-1} & a_{1j+1} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{i-11} & \cdots & a_{i-1j-1} & a_{i-1j+1} & \cdots & a_{i-1n} \\ a_{i+11} & \cdots & a_{i+1j-1} & a_{i+1j+1} & \cdots & a_{i+1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mj-1} & a_{mj+1} & \cdots & a_{mn} \end{vmatrix}$$

  - If $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ is nonsingular,

$$\boldsymbol{A}^{-1} = \frac{1}{|\boldsymbol{A}|}\tilde{\boldsymbol{A}}, \quad \text{where} \quad \tilde{\boldsymbol{A}} := \underbrace{\begin{bmatrix} \tilde{a}_{11} & \tilde{a}_{21} & \cdots & \tilde{a}_{n1} \\ \tilde{a}_{12} & \tilde{a}_{22} & \cdots & \tilde{a}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{1n} & \tilde{a}_{2n} & \cdots & \tilde{a}_{nn} \end{bmatrix}}_{\text{cofactor matrix}}$$

- **Partitioned matrices**
  - Consider a matrix of the form
$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$
  - Then, the determinant of A is
$$|A| = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = |A_{11}||A_{2|1}| = |A_{22}||A_{1|2}|$$

    and the inverse of A is
$$\begin{aligned} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1} &= \begin{bmatrix} A_{1|2}^{-1} & -A_{1|2}^{-1}A_{12}A_{22}^{-1} \\ -A_{22}^{-1}A_{21}A_{1|2}^{-1} & A_{22}^{-1} + A_{22}^{-1}A_{21}A_{1|2}^{-1}A_{12}A_{22}^{-1} \end{bmatrix} \\ &= \begin{bmatrix} A_{11}^{-1} + A_{11}^{-1}A_{12}A_{2|1}^{-1}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{2|1}^{-1} \\ -A_{2|1}^{-1}A_{21}A_{11}^{-1} & A_{2|1}^{-1} \end{bmatrix} \end{aligned}$$

    where
$$A_{1|2} := A_{11} - A_{12}A_{22}^{-1}A_{21}, \quad A_{2|1} := A_{22} - A_{21}A_{11}^{-1}A_{12}$$
- **Linear independence and determinant**
  - We say that $v_1, \dots, v_k \in \mathbb{R}^n$ are linearly independent if
$$c_1 v_1 + \cdots + c_k v_k = 0 \implies c_1 = \cdots = c_k = 0$$
  - For a function $f(x) := Ax$ with a matrix $A \in \mathbb{R}^{n \times k}$,
    - the following are equivalent:
      - · f is an injective function ($f(x) = f(x')$ implies $x = x'$)
      - · the column vectors of A are linearly independent
    - the following are equivalent:
      - · f is a surjective function ($f(\mathbb{R}^k) = \mathbb{R}^n$)
      - · the column vectors of A spans $\mathbb{R}^n$
  - For a function $f(x) := Ax$ with a square matrix $A \in \mathbb{R}^{n \times n}$, the following are equivalent:
    - f is an injective function
    - f is a surjective function
    - f is a bijective function (one-to-one)
  - For a square matrix $A \in \mathbb{R}^{n \times n}$, the following are equivalent:
    - column vectors of A are linearly independent
    - A is non-singular
    - $|A| \neq 0$
- **Sign of matrices**
  - For a square matrix $A \in \mathbb{R}^{n \times n}$,
    - A is positive definite if $x^\top A x > 0$ for any $x \neq \mathbf{0}$
    - A is positive semidefinite if $x^\top A x \geq 0$ for any x
    - A is negative definite if $x^\top A x < 0$ for any $x \neq \mathbf{0}$
    - A is negative semidefinite if $x^\top A x \leq 0$ for any x

- **Symmetric matrices**
  - A matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is called a symmetric matrix if $\boldsymbol{A}^\top = \boldsymbol{A}$
  - A matrix $\boldsymbol{P} \in \mathbb{R}^{n\times n}$ is called an orthogonal matrix if $\boldsymbol{P}^\top = \boldsymbol{P}^{-1}$
  - For a matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, the following are equivalent:
    - $\boldsymbol{A}$ is a symmetric matrix
    - $\boldsymbol{A}$ is decomposed as $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^{-1}$ where $\boldsymbol{\Lambda}$ is a diagonal matrix and $\boldsymbol{P}$ is an orthogonal matrix
  - For a symmetric matrix $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^{-1}$,
    - $\boldsymbol{A}$ is positive definite if and only if every diagonal element of $\boldsymbol{\Lambda}$ is positive
    - $\boldsymbol{A}$ is positive semidefinite if and only if every diagonal element of $\boldsymbol{\Lambda}$ is nonnegative
    - $\boldsymbol{A}$ is negative definite if and only if every diagonal element of $\boldsymbol{\Lambda}$ is negative
    - $\boldsymbol{A}$ is negative semidefinite if and only if every diagonal element of $\boldsymbol{\Lambda}$ is nonpositive

2 Matrix differentiation and integration

- **Differentiation**
  - For a function $f : \mathbb{R}^n \to \mathbb{R}$, we define

$$\frac{df(\boldsymbol{x})}{d\boldsymbol{x}} := \begin{bmatrix} \frac{\partial f(\boldsymbol{x})}{\partial x_1} & \frac{\partial f(\boldsymbol{x})}{\partial x_2} & \cdots & \frac{\partial f(\boldsymbol{x})}{\partial x_n} \end{bmatrix} \in \mathbb{R}^{1\times n}$$

  - For a function $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^m$, we define

$$\boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} f_1(\boldsymbol{x}) \\ f_2(\boldsymbol{x}) \\ \vdots \\ f_m(\boldsymbol{x}) \end{bmatrix} \implies \frac{d\boldsymbol{f}(\boldsymbol{x})}{d\boldsymbol{x}} := \begin{bmatrix} \frac{\partial f_1(\boldsymbol{x})}{\partial x_1} & \frac{\partial f_1(\boldsymbol{x})}{\partial x_2} & \cdots & \frac{\partial f_1(\boldsymbol{x})}{\partial x_n} \\ \frac{\partial f_2(\boldsymbol{x})}{\partial x_1} & \frac{\partial f_2(\boldsymbol{x})}{\partial x_2} & \cdots & \frac{\partial f_2(\boldsymbol{x})}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m(\boldsymbol{x})}{\partial x_1} & \frac{\partial f_m(\boldsymbol{x})}{\partial x_2} & \cdots & \frac{\partial f_m(\boldsymbol{x})}{\partial x_n} \end{bmatrix} \in \mathbb{R}^{m\times n},$$

    which is often called the Jacobian matrix of $\boldsymbol{f}$
  - For a function $\boldsymbol{F} : \mathbb{R} \to \mathbb{R}^{m\times n}$, we define

$$\boldsymbol{F}(x) = \begin{bmatrix} f_{11}(x) & f_{12}(x) & \cdots & f_{1n}(x) \\ f_{21}(x) & f_{22}(x) & \cdots & f_{2n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_{m1}(x) & f_{m2}(x) & \cdots & f_{mn}(x) \end{bmatrix} \implies \frac{d\boldsymbol{F}(x)}{dx} := \begin{bmatrix} \frac{df_{11}(x)}{dx} & \frac{df_{12}(x)}{dx} & \cdots & \frac{df_{1n}(x)}{dx} \\ \frac{df_{21}(x)}{dx} & \frac{df_{22}(x)}{dx} & \cdots & \frac{df_{2n}(x)}{dx} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{df_{m1}(x)}{dx} & \frac{df_{m2}(x)}{dx} & \cdots & \frac{df_{mn}(x)}{dx} \end{bmatrix}$$

- **Some facts**
  - For $\boldsymbol{A} \in \mathbb{R}^{m\times n}$,

$$\frac{d\boldsymbol{A}\boldsymbol{x}}{d\boldsymbol{x}} = \boldsymbol{A}$$

  - For $\boldsymbol{A} \in \mathbb{R}^{m\times l}$ and $\boldsymbol{g} : \mathbb{R}^n \to \mathbb{R}^l$,

$$\frac{d\boldsymbol{A}\boldsymbol{g}(\boldsymbol{x})}{d\boldsymbol{x}} = \boldsymbol{A}\frac{d\boldsymbol{g}(\boldsymbol{x})}{d\boldsymbol{x}}$$

  - For $\boldsymbol{F} : \mathbb{R} \to \mathbb{R}^{m\times l}$ and $\boldsymbol{G} : \mathbb{R} \to \mathbb{R}^{l\times n}$,

$$\frac{d\boldsymbol{F}(x)\boldsymbol{G}(x)}{dx} = \frac{d\boldsymbol{F}(x)}{dx}\boldsymbol{G}(x) + \boldsymbol{F}(x)\frac{d\boldsymbol{G}(x)}{dx}$$

- **Integration**
  - For a function $\boldsymbol{F} : \mathbb{R} \to \mathbb{R}^{m\times n}$, we define

$$\boldsymbol{F}(x) = \begin{bmatrix} f_{11}(x) & f_{12}(x) & \cdots & f_{1n}(x) \\ f_{21}(x) & f_{22}(x) & \cdots & f_{2n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_{m1}(x) & f_{m2}(x) & \cdots & f_{mn}(x) \end{bmatrix}$$
$$\implies \int \boldsymbol{F}(x)dx := \begin{bmatrix} \int f_{11}(x)dx & \int f_{12}(x)dx & \cdots & \int f_{1n}(x)dx \\ \int f_{21}(x)dx & \int f_{22}(x)dx & \cdots & \int f_{2n}(x)dx \\ \vdots & \vdots & \ddots & \vdots \\ \int f_{m1}(x)dx & \int f_{m2}(x)dx & \cdots & \int f_{mn}(x)dx \end{bmatrix}$$

3 Kronecker product and vectorization

- **Kronecker product**
  - For any pair of matrices $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ and $\boldsymbol{B} \in \mathbb{R}^{p\times q}$, we define their *Kronecker product* $\boldsymbol{A} \otimes \boldsymbol{B} \in \mathbb{R}^{mp\times nq}$ by

$$\boldsymbol{A} \otimes \boldsymbol{B} := \begin{bmatrix} a_{11}\boldsymbol{B} & a_{12}\boldsymbol{B} & \dots & a_{1n}\boldsymbol{B} \\ a_{21}\boldsymbol{B} & a_{22}\boldsymbol{B} & \dots & a_{2n}\boldsymbol{B} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}\boldsymbol{B} & a_{m2}\boldsymbol{B} & \dots & a_{mn}\boldsymbol{B} \end{bmatrix}$$

  - Example:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \otimes \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{11}b_{13} & a_{12}b_{11} & a_{12}b_{12} & a_{12}b_{13} \\ a_{11}b_{21} & a_{11}b_{22} & a_{11}b_{23} & a_{12}b_{21} & a_{12}b_{22} & a_{12}b_{23} \\ a_{11}b_{31} & a_{11}b_{32} & a_{11}b_{33} & a_{12}b_{31} & a_{12}b_{32} & a_{12}b_{33} \\ a_{21}b_{11} & a_{21}b_{12} & a_{21}b_{13} & a_{22}b_{11} & a_{22}b_{12} & a_{22}b_{13} \\ a_{21}b_{21} & a_{21}b_{22} & a_{21}b_{23} & a_{22}b_{21} & a_{22}b_{22} & a_{22}b_{23} \\ a_{21}b_{31} & a_{21}b_{32} & a_{21}b_{33} & a_{22}b_{31} & a_{22}b_{32} & a_{22}b_{33} \end{bmatrix}$$

  - Properties
    - $(\boldsymbol{A} \otimes \boldsymbol{B})(\boldsymbol{C} \otimes \boldsymbol{D}) = (\boldsymbol{AC}) \otimes (\boldsymbol{BD})$
    - $\boldsymbol{A}^k \otimes \boldsymbol{B}^k = (\boldsymbol{A} \otimes \boldsymbol{B})^k$ for any $k \in \mathbb{N}$
    - $(\alpha\boldsymbol{A} \otimes \beta\boldsymbol{B}) = \alpha\beta(\boldsymbol{A} \otimes \boldsymbol{B})$ for any $\alpha, \beta \in \mathbb{R}$
- **Vectorization**
  - We define $\text{vec}(\boldsymbol{A})$ as the column vector created by stacking the column vectors of $\boldsymbol{A}$, namely,

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_n \end{bmatrix} \implies \text{vec}(\boldsymbol{A}) := \begin{bmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \vdots \\ \boldsymbol{a}_n \end{bmatrix} \in \mathbb{R}^{mn}$$

- Example:

$$\operatorname{vec}\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}\right) = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{41} \\ a_{12} \\ a_{22} \\ a_{32} \\ a_{42} \\ a_{13} \\ a_{23} \\ a_{33} \\ a_{43} \end{bmatrix}$$

- Properties:
  - $\operatorname{vec}(\boldsymbol{A} + \boldsymbol{B}) = \operatorname{vec}(\boldsymbol{A}) + \operatorname{vec}(\boldsymbol{B})$ (by definition)
  - $\operatorname{vec}(\boldsymbol{ABC}) = (\boldsymbol{C}^\top \otimes \boldsymbol{A}) \operatorname{vec}(\boldsymbol{B})$
  - $\operatorname{vec}(\sum_{k=0}^{t} \boldsymbol{ABA}^\top) = \sum_{k=0}^{t} (\boldsymbol{A} \otimes \boldsymbol{A})^k \operatorname{vec}(\boldsymbol{B})$
- Example:

$$\operatorname{vec}\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}\right) = \begin{bmatrix} a_{11}b_{11}c_{11} + a_{12}b_{21}c_{11} + a_{11}b_{12}c_{21} + a_{12}b_{22}c_{21} \\ a_{21}b_{11}c_{11} + a_{22}b_{21}c_{11} + a_{21}b_{12}c_{21} + a_{22}b_{22}c_{21} \\ a_{31}b_{11}c_{11} + a_{32}b_{21}c_{11} + a_{31}b_{12}c_{21} + a_{32}b_{22}c_{21} \\ a_{11}b_{11}c_{12} + a_{12}b_{21}c_{12} + a_{11}b_{12}c_{22} + a_{12}b_{22}c_{22} \\ a_{21}b_{11}c_{12} + a_{22}b_{21}c_{12} + a_{21}b_{12}c_{22} + a_{22}b_{22}c_{22} \\ a_{31}b_{11}c_{12} + a_{32}b_{21}c_{12} + a_{31}b_{12}c_{22} + a_{32}b_{22}c_{22} \\ a_{11}b_{11}c_{13} + a_{12}b_{21}c_{13} + a_{11}b_{12}c_{23} + a_{12}b_{22}c_{23} \\ a_{21}b_{11}c_{13} + a_{22}b_{21}c_{13} + a_{21}b_{12}c_{23} + a_{22}b_{22}c_{23} \\ a_{31}b_{11}c_{13} + a_{32}b_{21}c_{13} + a_{31}b_{12}c_{23} + a_{32}b_{22}c_{23} \end{bmatrix}$$

$$= \begin{bmatrix} c_{11}a_{11} & c_{11}a_{12} & c_{21}a_{11} & c_{21}a_{12} \\ c_{11}a_{21} & c_{11}a_{22} & c_{21}a_{21} & c_{21}a_{22} \\ c_{11}a_{31} & c_{11}a_{32} & c_{21}a_{31} & c_{21}a_{32} \\ c_{12}a_{11} & c_{12}a_{12} & c_{22}a_{11} & c_{22}a_{12} \\ c_{12}a_{21} & c_{12}a_{22} & c_{22}a_{21} & c_{22}a_{22} \\ c_{12}a_{31} & c_{12}a_{32} & c_{22}a_{31} & c_{22}a_{32} \\ c_{13}a_{11} & c_{13}a_{12} & c_{23}a_{11} & c_{23}a_{12} \\ c_{13}a_{21} & c_{13}a_{22} & c_{23}a_{21} & c_{23}a_{22} \\ c_{13}a_{31} & c_{13}a_{32} & c_{23}a_{31} & c_{23}a_{32} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{12} \\ b_{22} \end{bmatrix} = \left(\begin{bmatrix} c_{11} & c_{21} \\ c_{12} & c_{22} \\ c_{13} & c_{23} \end{bmatrix} \otimes \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}\right) \begin{bmatrix} b_{11} \\ b_{21} \\ b_{12} \\ b_{22} \end{bmatrix}$$