Discrete-time models

Introduction to dynamical systems #6

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1 Discrete-time linear dynamical system

1.1 Definitions

- **Linear dynamical system**
  - A discrete-time *linear dynamical system* is a system of difference equations of the form

    $$\boldsymbol{x}_{t+1} = \boldsymbol{A}\boldsymbol{x}_t + \boldsymbol{B}\boldsymbol{u}_t \quad t = 0, 1, \dots,$$

    where
    - $\boldsymbol{x}_t \in \mathbb{R}^m$: state vector at t
    - $\boldsymbol{A} \in \mathbb{R}^{m \times m}$: system matrix
    - $\boldsymbol{u}_t \in \mathbb{R}^n$: control (input) vector at t
    - $\boldsymbol{B} \in \mathbb{R}^{m \times n}$: diffusion matrix
  - Starting with some *initial state* $\boldsymbol{x}_0$, the state at any t is given by

    $$\boldsymbol{x}_t = \boldsymbol{A}\boldsymbol{x}_{t-1} + \boldsymbol{B}\boldsymbol{u}_{t-1} = \boldsymbol{A}(\boldsymbol{A}\boldsymbol{x}_{t-2} + \boldsymbol{B}\boldsymbol{u}_{t-2}) + \boldsymbol{B}\boldsymbol{u}_{t-1} = \dots = \boldsymbol{A}^t\boldsymbol{x}_0 + \sum_{k=0}^{t-1} \boldsymbol{A}^{t-1-k}\boldsymbol{B}\boldsymbol{u}_k$$
  - We want to know how $\boldsymbol{x}_t$ evolves over time depending on $\boldsymbol{A}$
  - In particular, a linear dynamical system is said to be *homogeneous* if $\boldsymbol{B} = \boldsymbol{O}$, i.e.,

    $$\boldsymbol{x}_{t+1} = \boldsymbol{A}\boldsymbol{x}_t \quad t = 0, 1, \dots,$$

    where the behavior of $\boldsymbol{x}_t$ is completely characterized by $\boldsymbol{A}$ and $\boldsymbol{x}_0$
- **Equilibrium**
  - Consider the case where the control is constant at $\boldsymbol{u}$:

    $$\boldsymbol{x}_{t+1} = \boldsymbol{A}\boldsymbol{x}_t + \boldsymbol{b}, \quad t = 0, 1, \dots, \quad \text{where} \quad \boldsymbol{b} := \boldsymbol{B}\boldsymbol{u} \tag{1}$$
  - The homogeneous system is a special case of this with $\boldsymbol{u} = \boldsymbol{0}$
  - We define an *equilibrium point* of (1) as $\bar{\boldsymbol{x}}$ that solves

    $$\bar{\boldsymbol{x}} = \boldsymbol{A}\bar{\boldsymbol{x}} + \boldsymbol{b},$$

    which obviously depends on both $\boldsymbol{A}$ and $\boldsymbol{b}$

- **Stability**
  - An equilibrium point $\bar{\boldsymbol{x}}$ of (1) is said to be *asymptotically stable* if, starting from **any initial state** $\boldsymbol{x}_0$, the state trajectory satisfies

$$\lim_{t\to\infty} \boldsymbol{x}_t = \bar{\boldsymbol{x}}$$

  - Notice that, for any initial state $\boldsymbol{x}_0$ and for any constant input $\boldsymbol{b}$, the state trajectory $\{\boldsymbol{x}_t\}$ satisfies

$$(\boldsymbol{x}_{t+1} - \bar{\boldsymbol{x}}) = \boldsymbol{A}(\boldsymbol{x}_t - \bar{\boldsymbol{x}}) \quad \forall t = 0, 1, \ldots,$$

    implying the stability of an equilibrium point is determined by $\boldsymbol{A}$ alone (irrespective of $\boldsymbol{b}$)
  - More generally, the state trajectory $\{\boldsymbol{x}_t\}$ may or may not converge to the equilibrium point, depending on the initial state $\boldsymbol{x}_0$
  - We define the *stable manifold* of an equilibrium point $\bar{\boldsymbol{x}} \in \mathbb{R}^m$ as the set of initial state from which the state trajectory converges to the equilibrium point:

$$W(\bar{\boldsymbol{x}}) := \left\{ \boldsymbol{x}_0 \in \mathbb{R}^m \;\middle|\; \lim_{t\to\infty} \boldsymbol{x}_t = \bar{\boldsymbol{x}} \right\}$$

  - An equilibrium point $\bar{\boldsymbol{x}}$ is asymptotically stable if and only if $W(\bar{\boldsymbol{x}}) = \mathbb{R}^m$

1.2 Examples

- **Example 1**
  - Consider the following homogeneous one-dimensional linear dynamical system:

$$x_{t+1} = a x_t$$

  - Observe:
    - the unique equilibrium point of the system is $\bar{x} = 0$ unless $a = 1$
    - the state trajectory is given by $x_t = a^t x_0$
    - $\bar{x} = 0$ is asymptotically stable if and only if $|a| < 1$
  - The stable manifold of $\bar{x} = 0$ is

$$W(\bar{x}) = \begin{cases} \mathbb{R} \quad \text{(i.e., asymptotically stable)} & \text{if } |a| < 1 \\ \{0\} & \text{if } |a| \geq 1 \end{cases}$$

- **Example 2**
  - Consider the following one-dimensional linear dynamical system:

$$x_{t+1} = a x_t + b, \quad a \neq 1$$

  - Observe:
    - the unique equilibrium point of the system is $\bar{x} = \frac{b}{1-a}$
    - the state trajectory is given by $x_t = a^t x_0 + \frac{1-a^t}{1-a} b$
    - the equilibrium point is asymptotically stable if and only if $|a| < 1$
  - The stable manifold is

$$W(\bar{x}) = \begin{cases} \mathbb{R} & \text{if } |a| < 1 \\ \{\frac{1}{1-a} b\} & \text{if } |a| \geq 1 \end{cases}$$

- **Example 3**
  - Consider the following two-dimensional linear dynamical system:

$$\begin{bmatrix} x_{1,t+1} \\ x_{2,t+1} \end{bmatrix} = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix}$$

  - Notice that, in this case, the evolution of $x_{1,t}$ and $x_{2,t}$ are independent:

$$x_{i,t+1} = a_i x_{i,t} \quad i = 1, 2$$

  - Observe:
    - if $a_1 \neq 1$ and $a_2 \neq 1$, the unique equilibrium point of the system is $\bar{\boldsymbol{x}} = \boldsymbol{0}$
    - the state trajectory is given by

$$\begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix} = \begin{bmatrix} a_1^t & 0 \\ 0 & a_2^t \end{bmatrix} \begin{bmatrix} x_{1,0} \\ x_{2,0} \end{bmatrix} = \begin{bmatrix} a_1^t x_{1,0} \\ a_2^t x_{2,0} \end{bmatrix}$$

    - the equilibrium point is asymptotically stable if and only if $\max\{|a_1|, |a_2|\} < 1$
  - The stable manifold is

$$W(\bar{\boldsymbol{x}}) = \begin{cases} \mathbb{R}^2 & \text{if } |a_1| < 1 \text{ and } |a_2| < 1 \\ \{0\} \times \mathbb{R} & \text{if } |a_1| \geq 1 \text{ and } |a_2| < 1 \\ \mathbb{R} \times \{0\} & \text{if } |a_1| < 1 \text{ and } |a_2| \geq 1 \\ \{0\} \times \{0\} & \text{if } |a_1| \geq 1 \text{ and } |a_2| \geq 1 \end{cases}$$

- **Example 4**
  - Consider the following m-dimensional linear dynamical system:

$$\boldsymbol{x}_{t+1} = \boldsymbol{A}\boldsymbol{x}_t, \quad \boldsymbol{A} := \begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_m \end{bmatrix}, \quad a_i \neq 1 \ \forall i = 1, \dots, m$$

  - Observe:
    - the unique equilibrium point of the system is $\bar{\boldsymbol{x}} = \boldsymbol{0}$
    - the state trajectory is given by

$$\boldsymbol{x}_t = \boldsymbol{A}^t \boldsymbol{x}_0 = \begin{bmatrix} a_1^t & 0 & \dots & 0 \\ 0 & a_2^t & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_m^t \end{bmatrix} \begin{bmatrix} x_{1,0} \\ x_{2,0} \\ \vdots \\ x_{m,0} \end{bmatrix} = \begin{bmatrix} a_1^t x_{1,0} \\ a_2^t x_{2,0} \\ \vdots \\ a_m^t x_{m,0} \end{bmatrix}$$

    - the equilibrium point is asymptotically stable if and only if

$$\max\{|a_1|, |a_2|, \dots, |a_m|\} < 1$$

  - The stable manifold is

$$W(\bar{\boldsymbol{x}}) = \left\{ \boldsymbol{x}_0 \in \mathbb{R}^m \,\middle|\, x_{i,0} = 0 \text{ for } i \text{ such that } |a_i| \geq 1 \right\}$$

- **Example 5**
  - Consider the following m-dimensional linear dynamical system:

$$\boldsymbol{x}_{t+1} = \boldsymbol{A}\boldsymbol{x}_t + \boldsymbol{b}, \quad \boldsymbol{A} := \begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_m \end{bmatrix}$$

  - Assume $a_i \neq 1$ for all $i = 1, \dots, m$ (otherwise there would be no stable equilibrium point)
  - Observe:
    - the unique equilibrium point of the system is

$$\bar{\boldsymbol{x}} = (\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b} = \begin{bmatrix} \frac{1}{1-a_1} & 0 & \dots & 0 \\ 0 & \frac{1}{1-a_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{1-a_2} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = \begin{bmatrix} \frac{1}{1-a_1} b_1 \\ \frac{1}{1-a_2} b_2 \\ \vdots \\ \frac{1}{1-a_m} b_m \end{bmatrix}$$

    - the state trajectory is given by

$$\boldsymbol{x}_t - \bar{\boldsymbol{x}} = \boldsymbol{A}^t(\boldsymbol{x}_0 - \bar{\boldsymbol{x}})$$

    or

$$\boldsymbol{x}_t = \bar{\boldsymbol{x}} + \boldsymbol{A}^t(\boldsymbol{x}_0 - \bar{\boldsymbol{x}}) = \begin{bmatrix} a_1^t x_{1,0} + \frac{1-a_1^t}{1-a_1} b \\ a_2^t x_{2,0} + \frac{1-a_2^t}{1-a_2} b \\ \vdots \\ a_m^t x_{m,0} + \frac{1-a_m^t}{1-a_m} b \end{bmatrix}$$

    - the equilibrium point is asymptotically stable if and only if

$$\max\{|a_1|, |a_2|, \dots, |a_m|\} < 1$$

  - The stable manifold is

$$W(\bar{\boldsymbol{x}}) = \left\{ \boldsymbol{x}_0 \in \mathbb{R}^m \;\middle|\; x_{i,0} = \frac{1}{1-a_i} b_i \text{ for } i \text{ such that } |a_i| \geq 1 \right\}$$

- **Example 6**
  - Consider the following two-dimensional linear dynamical system:

$$\begin{bmatrix} x_{1,t+1} \\ x_{2,t+1} \end{bmatrix} = \begin{bmatrix} 5/4 & -3/2 \\ -3/8 & 5/4 \end{bmatrix} \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix}$$

  - The unique equilibrium point of the system is

$$\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} 5/4 & -3/2 \\ -3/8 & 5/4 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} \iff \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

  - Is this equilibrium point asymptotically stable?
  - How does $\boldsymbol{x}_t$ evolve over time?

2 Characterization

2.1 General method

- **Trajectory and stability**
  - Consider the linear dynamical system
$$x_{t+1} = Ax_t + b \quad t = 0, 1, \ldots,$$
  - Assume A does not have 1 as its eigenvalue (otherwise no stable equilibrium point)
  - Since $\lambda = 1$ is not an eigenvalue of A, we know that $(I - A)$ is non-singular (why?), and the unique equilibrium point is
$$\bar{x} := (I - A)^{-1}b$$
    and we may write
$$x_t = \bar{x} + A^t(x_0 - \bar{x}), \quad t = 0, 1, \ldots \tag{2}$$
  - Hence,
$$\bar{x} \text{ is asymptotically stable} \iff \lim_{t\to\infty} A^t = O \iff \rho(A) < 1$$
  - Using the Jordan decomposition $A = VJV^{-1}$, one can rewrite (2) as
$$x_t = \bar{x} + VJ^tV^{-1}(x_0 - \bar{x}), \quad t = 0, 1, \ldots$$
  - In particular, if A is diagonalizable, we have $J = \Lambda := \mathrm{diag}(\lambda_1, \ldots, \lambda_m)$ and
$$x_{t+1} = \bar{x} + V\Lambda^tV^{-1}(x_0 - \bar{x}) \quad t = 0, 1, \ldots,$$
    in which case the stable manifold can be expressed as
$$W(\bar{x}) = \left\{ x_0 \in \mathbb{R}^m \;\middle|\; e_i^\top V^{-1}(x_0 - \bar{x}) = 0 \text{ for } i \text{ such that } |\lambda_i| \geq 1 \right\}$$

- **Eigenvectors of the system matrix**
  - Let $A \in \mathbb{R}^{m\times m}$ be a square matrix and consider the linear system of difference equations
$$x_{t+1} = Ax_t \quad \forall t = 0, 1, 2, 3, \ldots \tag{3}$$
  - If (λ, v) is an eigenpair of A, then
$$x_t := \lambda^t v \quad \forall t$$
    solves (3) with $x_0 = v$ because
$$x_t := \lambda^t v \implies x_{t+1} = \lambda^{t+1}v = \lambda^t\lambda v = \lambda^t Av = A\lambda^t v = Ax_t$$
  - If $(\lambda_1, v_1), (\lambda_2, v_2)$ are eigenpairs of A, then for any $z_1, z_2 \in \mathbb{R}$,
$$x_t := z_1\lambda_1^t v_1 + z_2\lambda_2^t v_2 \quad \forall t$$
    solves (3) with $x_0 = z_1v_1 + z_2v_2$
  - If $(\lambda_1, v_1), (\lambda_2, v_2), \ldots, (\lambda_m, v_m)$ are eigenpairs of A, then for any $z_1, z_2, \ldots, z_m \in \mathbb{R}$,
$$x_t := z_1\lambda_1^t v_1 + z_2\lambda_2^t v_2 + \ldots + z_m\lambda_m^t v_m \quad \forall t$$
    solves (3) with $x_0 = z_1v_1 + z_2v_2 + \ldots + z_mv_m$

- Hence, if $(\lambda_1, \boldsymbol{v}_1), (\lambda_2, \boldsymbol{v}_2), \ldots, (\lambda_m, \boldsymbol{v}_m)$ are **linearly independent** eigenpairs of $\boldsymbol{A}$, then for any arbitrary initial state $\boldsymbol{x}_0 \in \mathbb{R}^m$, one can find $z_1, z_2, \ldots, z_m \in \mathbb{R}$ such that

$$\boldsymbol{x}_0 = z_1\boldsymbol{v}_1 + z_2\boldsymbol{v}_2 + \ldots + z_m\boldsymbol{v}_m$$

  and the state trajectory from $\boldsymbol{x}_0$ can be written as

$$\boldsymbol{x}_t = z_1\lambda_1^t\boldsymbol{v}_1 + z_2\lambda_2^t\boldsymbol{v}_2 + \ldots + z_m\lambda_m^t\boldsymbol{v}_m = \begin{bmatrix}\boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_m\end{bmatrix}\begin{bmatrix}\lambda_1^t & 0 & \ldots & 0\\ 0 & \lambda_2^t & \ldots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \ldots & \lambda_m^t\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ \vdots\\ z_m\end{bmatrix} = \boldsymbol{V}\Lambda^t\boldsymbol{z} \quad \forall t$$

- **Long-run behavior and dominant mode**
  - Consider the following m-dimensional linear dynamical system:

$$\boldsymbol{x}_{t+1} = \boldsymbol{A}\boldsymbol{x}_t + \boldsymbol{b} \quad \forall t \quad \text{with some initial state } \boldsymbol{x}_0 \in \mathbb{R}^m$$

  - Suppose that $\boldsymbol{A} \in \mathbb{R}^{m\times m}$ is diagonalizable and let $(\lambda_1, \boldsymbol{v}_1), (\lambda_2, \boldsymbol{v}_2), \ldots, (\lambda_m, \boldsymbol{v}_m)$ be linearly independent eigenpairs of $\boldsymbol{A}$
  - Then, there exist $z_1, z_2, \ldots, z_m \in \mathbb{R}$ such that

$$\boldsymbol{x}_0 - \bar{\boldsymbol{x}} = z_1\boldsymbol{v}_1 + z_2\boldsymbol{v}_2 + \ldots z_m\boldsymbol{v}_m = \underbrace{\begin{bmatrix}\boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_m\end{bmatrix}}_{\boldsymbol{V}}\underbrace{\begin{bmatrix}z_1\\ z_2\\ \vdots\\ z_m\end{bmatrix}}_{\boldsymbol{z}}$$

  and thus

$$\boldsymbol{x}_t - \bar{\boldsymbol{x}} = \boldsymbol{A}^t(\boldsymbol{x}_0 - \bar{\boldsymbol{x}}) = z_1\lambda_1^t\boldsymbol{v}_1 + z_2\lambda_2^t\boldsymbol{v}_2 + \cdots + z_m\lambda_m^t\boldsymbol{v}_m \quad \forall t$$

  - Let λ_i be the *dominant eigenvalue* of $\boldsymbol{A}$, i.e., $|\lambda_i| > |\lambda_j|$ for all $j \neq i$
  - Then, for sufficiently large t,

$$\frac{1}{\lambda_i^t}(\boldsymbol{x}_t - \bar{\boldsymbol{x}}) = z_1\left(\frac{\lambda_1}{\lambda_i}\right)^t\boldsymbol{v}_1 + \cdots + z_i\left(\frac{\lambda_i}{\lambda_i}\right)^t\boldsymbol{v}_i + \cdots + z_m\left(\frac{\lambda_m}{\lambda_i}\right)^t\boldsymbol{v}_m \approx z_i\boldsymbol{v}_i,$$

  or

$$\boldsymbol{x}_t \approx \bar{\boldsymbol{x}} + \lambda_i^t z_i\boldsymbol{v}_i, \quad \text{where} \quad z_i = \boldsymbol{e}_i^\top\boldsymbol{V}^{-1}(\boldsymbol{x}_0 - \bar{\boldsymbol{x}})$$

  - Observe:
    - the long-term state is essentially determined by the eigenvector associated with the dominant eigenvalue of $\boldsymbol{A}$
    - if $\rho(\boldsymbol{A}) < 1$, the rate at which the state converges to the equilibrium point is ultimately governed by the dominant eigenvalue
    - if $\rho(\boldsymbol{A}) = 1$ (i.e, the dominant eigenvalue has magnitude of 1),

$$\lim_{t\to\infty}\boldsymbol{x}_t = \bar{\boldsymbol{x}} + z_i\boldsymbol{v}_i,$$

      in which case $\boldsymbol{x}_t$ neither diverges to infinity nor converges to $\bar{\boldsymbol{x}}$ (we call such situation as *marginally stable*) and the limit depends on the initial state (through z_i)

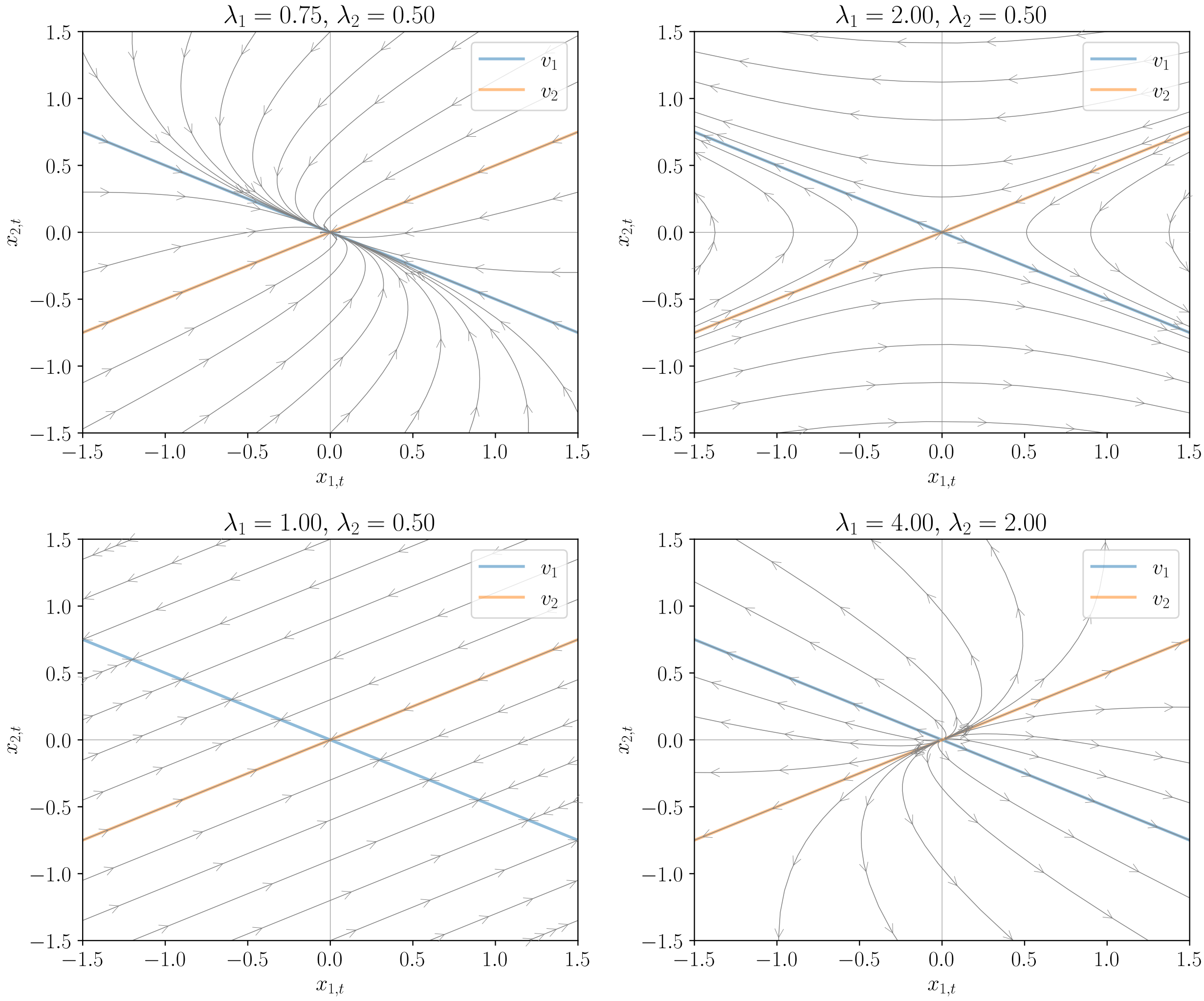

Figure 1: Phase diagrams for example 1 (top left), example 2 (top right), example 4 (bottom left), and example 5 (bottom right)

2.2 Examples

- **Example 1**
  - Consider the following two-dimensional linear dynamical system:

$$\begin{bmatrix} x_{1,t+1} \\ x_{2,t+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 5/8 & -1/4 \\ -1/16 & 5/8 \end{bmatrix}}_{\boldsymbol{A}} \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix}$$

  - The state trajectory $\{\boldsymbol{x}_t\}$ from arbitrary $\boldsymbol{x}_0 \in \mathbb{R}^2$ is

$$\boldsymbol{x}_t = \boldsymbol{A}^t \boldsymbol{x}_0 = \begin{bmatrix} 5/8 & -1/4 \\ -1/16 & 5/8 \end{bmatrix}^t \boldsymbol{x}_0,$$

    which is not easy to characterize
  - So we look at the characteristic polynomial

$$\phi_{\boldsymbol{A}}(t) = \begin{vmatrix} 5/8 - t & -1/4 \\ -1/16 & 5/8 - t \end{vmatrix} = (3/4 - t)(1/2 - t),$$

    implying that the eigenvalues of $\boldsymbol{A}$ are $\lambda_1 := 3/4$ and $\lambda_2 := 1/2$

- We already see that the unique equilibrium point $\bar{\boldsymbol{x}} := \mathbf{0}$ is asymptotically stable
- To characterize the behavior of $\{\boldsymbol{x}_t\}$ more explicitly, we derive eigenvectors:

$$(\boldsymbol{A} - \lambda_1 \boldsymbol{I})\boldsymbol{v} = \mathbf{0} \iff \begin{bmatrix} 5/8 - 3/4 & -1/4 \\ -1/16 & 5/8 - 3/4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \boldsymbol{v} = \alpha \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix} \quad \forall \alpha$$

and

$$(\boldsymbol{A} - \lambda_2 \boldsymbol{I})\boldsymbol{v} = \mathbf{0} \iff \begin{bmatrix} 5/8 - 1/2 & -1/4 \\ -1/16 & 5/8 - 1/2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \boldsymbol{v} = \alpha \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} \quad \forall \alpha$$

so we choose

$$\boldsymbol{\Lambda} := \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad \boldsymbol{v}_1 := \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix}, \quad \boldsymbol{v}_2 := \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$$

and

$$\boldsymbol{V} := \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1/2 & 1/2 \end{bmatrix} \implies \boldsymbol{V}^{-1} = \begin{bmatrix} 1/2 & -1 \\ 1/2 & 1 \end{bmatrix}$$

- Now we can express the state trajectory $\{\boldsymbol{x}_t\}$ from arbitrary $\boldsymbol{x}_0 \in \mathbb{R}^2$ as

$$\boldsymbol{x}_t = \boldsymbol{A}^t \boldsymbol{x}_0 = \boldsymbol{V}\boldsymbol{\Lambda}^t \boldsymbol{V}^{-1}\boldsymbol{x}_0 = \begin{bmatrix} 1 & 1 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} (3/4)^t & 0 \\ 0 & (1/2)^t \end{bmatrix} \begin{bmatrix} 1/2 & -1 \\ 1/2 & 1 \end{bmatrix} \boldsymbol{x}_0$$

- The trajectory converges to $\bar{\boldsymbol{x}} = \mathbf{0}$ regardless of $\boldsymbol{x}_0$
- Given an initial state $\boldsymbol{x}_0$, defining $\boldsymbol{z}$ as

$$\boldsymbol{z} := \boldsymbol{V}^{-1}\boldsymbol{x}_0, \quad \text{or} \quad z_i := \boldsymbol{e}_i^\top \boldsymbol{V}^{-1}\boldsymbol{x}_0, \quad i = 1, 2$$

allows us to write

$$\boldsymbol{x}_t = \lambda_1^t z_1 \boldsymbol{v}_1 + \lambda_2^t z_2 \boldsymbol{v}_2 \quad \forall t$$

- Since λ_1 is the dominant eigenvalue, the long-term behavior is characterized by

$$\boldsymbol{x}_t \approx \lambda_1^t z_1 \boldsymbol{v}_1 = \left(\frac{3}{4}\right)^t z_1 \begin{bmatrix} 1 \\ -1/2 \end{bmatrix}, \quad \text{where} \quad z_1 := \boldsymbol{e}_1^\top \boldsymbol{V}^{-1}\boldsymbol{x}_0 = \frac{1}{2}x_{1,0} - x_{2,0}$$

for sufficiently large t

- **Example 2**
  - Consider another two-dimensional linear dynamical system:

$$\begin{bmatrix} x_{1,t+1} \\ x_{2,t+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 5/4 & -3/2 \\ -3/8 & 5/4 \end{bmatrix}}_{\boldsymbol{A}} \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix}$$

  - The state trajectory $\{\boldsymbol{x}_t\}$ from arbitrary $\boldsymbol{x}_0 \in \mathbb{R}^2$ is

$$\boldsymbol{x}_t = \boldsymbol{A}^t \boldsymbol{x}_0 = \begin{bmatrix} 5/4 & -3/2 \\ -3/8 & 5/4 \end{bmatrix}^t \boldsymbol{x}_0,$$

which is not easy to characterize

- So we look at the characteristic polynomial

$$\phi_A(t) = \begin{vmatrix} 5/4 - t & -3/2 \\ -3/8 & 5/4 - t \end{vmatrix} = (2-t)(1/2-t),$$

  implying that the eigenvalues of $\boldsymbol{A}$ are $\lambda_1 := 2$ and $\lambda_2 := 1/2$
- We already see that the unique equilibrium point $\bar{\boldsymbol{x}} := \boldsymbol{0}$ is NOT asymptotically stable
- To characterize the behavior of $\{\boldsymbol{x}_t\}$ more explicitly, we derive eigenvectors:

$$(\boldsymbol{A} - \lambda_1 \boldsymbol{I})v = \boldsymbol{0} \iff \begin{bmatrix} 5/4 - 2 & -3/2 \\ -3/8 & 5/4 - 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff v = \alpha \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix} \quad \forall \alpha$$

  and

$$(\boldsymbol{A} - \lambda_2 \boldsymbol{I})v = \boldsymbol{0} \iff \begin{bmatrix} 5/4 - 1/2 & -3/2 \\ -3/8 & 5/4 - 1/2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff v = \alpha \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} \quad \forall \alpha$$

  so we choose

$$\boldsymbol{\Lambda} := \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad v_1 := \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix}, \quad v_2 := \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$$

  and

$$\boldsymbol{V} := \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1/2 & 1/2 \end{bmatrix} \implies \boldsymbol{V}^{-1} = \begin{bmatrix} 1/2 & -1 \\ 1/2 & 1 \end{bmatrix}$$

- Now we can express the state trajectory $\{\boldsymbol{x}_t\}$ from arbitrary $\boldsymbol{x}_0 \in \mathbb{R}^2$ as

$$\boldsymbol{x}_t = \boldsymbol{A}^t \boldsymbol{x}_0 = \boldsymbol{V}\boldsymbol{\Lambda}^t \boldsymbol{V}^{-1} \boldsymbol{x}_0 = \begin{bmatrix} 1 & 1 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 2^t & 0 \\ 0 & (1/2)^t \end{bmatrix} \begin{bmatrix} 1/2 & -1 \\ 1/2 & 1 \end{bmatrix} \boldsymbol{x}_0$$

- The trajectory converges to $\bar{\boldsymbol{x}} = \boldsymbol{0}$ only if we choose $\boldsymbol{x}_0$ to nullify the impact of 2^t
- More specifically, we need to choose $\boldsymbol{x}_0$ in such a way that

$$\begin{bmatrix} 1/2 & -1 \\ 1/2 & 1 \end{bmatrix} \boldsymbol{x}_0 = \begin{bmatrix} 0 \\ \dots \end{bmatrix} \iff \boldsymbol{e}_1^\top \underbrace{\begin{bmatrix} 1/2 & -1 \\ 1/2 & 1 \end{bmatrix}}_{\boldsymbol{V}^{-1}} \boldsymbol{x}_0 = 0 \iff \frac{1}{2} x_{1,0} - x_{2,0} = 0$$

- Hence, the stable manifold is

$$W(\bar{\boldsymbol{x}}) = \left\{ \boldsymbol{x}_0 = (x_{1,0}, x_{2,0}) \in \mathbb{R}^2 \,\middle|\, \frac{1}{2} x_{1,0} - x_{2,0} = 0 \right\}$$

- Given an initial state $\boldsymbol{x}_0$, defining $\boldsymbol{z}$ as

$$\boldsymbol{z} := \boldsymbol{V}^{-1} \boldsymbol{x}_0, \quad \text{or} \quad z_i := \boldsymbol{e}_i^\top \boldsymbol{V}^{-1} \boldsymbol{x}_0, \quad i = 1, 2$$

  allows us to write

$$\boldsymbol{x}_t = \lambda_1^t z_1 \boldsymbol{v}_1 + \lambda_2^t z_2 \boldsymbol{v}_2 \quad \forall t$$

- Since λ_1 is the dominant eigenvalue, the long-term behavior is characterized by

$$\boldsymbol{x}_t \approx \lambda_1^t z_1 \boldsymbol{v}_1 = 2^t z_1 \begin{bmatrix} 1 \\ -1/2 \end{bmatrix}, \quad \text{where} \quad z_1 := \boldsymbol{e}_1^\top \boldsymbol{V}^{-1} \boldsymbol{x}_0 = \frac{1}{2} x_{1,0} - x_{2,0}$$

  for sufficiently large t

- **Example 3**
  - Consider a slightly modified version of the example above:

$$\begin{bmatrix} x_{1,t+1} \\ x_{2,t+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 5/4 & -3/2 \\ -3/8 & 5/4 \end{bmatrix}}_{\boldsymbol{A}} \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix} + \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{\boldsymbol{b}}$$

  - Since 1 is not an eigenvalue of $\boldsymbol{A}$, we know that $\boldsymbol{I} - \boldsymbol{A}$ is non-singular and

$$(\boldsymbol{I} - \boldsymbol{A})^{-1} = \begin{bmatrix} 1 - 5/4 & 3/2 \\ 3/8 & 1 - 5/4 \end{bmatrix}^{-1} = \begin{bmatrix} 1/2 & 3 \\ 3/4 & 1/2 \end{bmatrix}$$

  - The unique equilibrium point is

$$\bar{\boldsymbol{x}} = (\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b} = \begin{bmatrix} 1/2 & 3 \\ 3/4 & 1/2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

  - The state trajectory $\{\boldsymbol{x}_t\}$ from arbitrary $\boldsymbol{x}_0 \in \mathbb{R}^2$ is

$$\begin{aligned} \boldsymbol{x}_t &= \bar{\boldsymbol{x}} + \boldsymbol{A}^t(\boldsymbol{x}_0 - \bar{\boldsymbol{x}}) = \bar{\boldsymbol{x}} + \begin{bmatrix} 5/4 & -3/2 \\ -3/8 & 5/4 \end{bmatrix}^t (\boldsymbol{x}_0 - \bar{\boldsymbol{x}}), \\ &= \bar{\boldsymbol{x}} + \boldsymbol{V}\boldsymbol{\Lambda}^t\boldsymbol{V}^{-1}(\boldsymbol{x}_0 - \bar{\boldsymbol{x}}) = \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 2^t & 0 \\ 0 & (1/2)^t \end{bmatrix} \begin{bmatrix} 1/2 & -1 \\ 1/2 & 1 \end{bmatrix} \left(\boldsymbol{x}_0 - \begin{bmatrix} 4 \\ 2 \end{bmatrix} \right) \end{aligned}$$

  - The trajectory converges to $\bar{\boldsymbol{x}} = (4, 2)^\top$ only if we choose $\boldsymbol{x}_0$ to nullify the impact of 2^t
  - More specifically, we need to choose $\boldsymbol{x}_0$ in such a way that

$$\begin{bmatrix} 1/2 & -1 \\ 1/2 & 1 \end{bmatrix} (\boldsymbol{x}_0 - \bar{\boldsymbol{x}}) = \begin{bmatrix} 0 \\ \dots \end{bmatrix} \iff \boldsymbol{e}_1^\top \underbrace{\begin{bmatrix} 1/2 & -1 \\ 1/2 & 1 \end{bmatrix}}_{\boldsymbol{V}^{-1}} (\boldsymbol{x}_0 - \bar{\boldsymbol{x}}) = 0 \iff \frac{1}{2}(x_{1,0} - 4) - (x_{2,0} - 2) = 0$$

  - Hence, the stable manifold is

$$W(\bar{\boldsymbol{x}}) = \left\{ \boldsymbol{x}_0 = (x_{1,0}, x_{2,0}) \in \mathbb{R}^2 \,\middle|\, \frac{1}{2}(x_{1,0} - 4) - (x_{2,0} - 2) = 0 \right\}$$

  - Given an initial state $\boldsymbol{x}_0$, defining $\boldsymbol{z}$ as

$$\boldsymbol{z} := \boldsymbol{V}^{-1}(\boldsymbol{x}_0 - \bar{\boldsymbol{x}}), \quad \text{or} \quad z_i := \boldsymbol{e}_i^\top \boldsymbol{V}^{-1}(\boldsymbol{x}_0 - \bar{\boldsymbol{x}}), \quad i = 1, 2$$

    allows us to write

$$\boldsymbol{x}_t = \bar{\boldsymbol{x}} + \lambda_1^t z_1 \boldsymbol{v}_1 + \lambda_2^t z_2 \boldsymbol{v}_2 \quad \forall t$$

  - Since λ_1 is the dominant eigenvalue, the long-term behavior is characterized by

$$\boldsymbol{x}_t \approx \bar{\boldsymbol{x}} + \lambda_1^t z_1 \boldsymbol{v}_1 = 2^t z_1 \begin{bmatrix} 1 \\ -1/2 \end{bmatrix}, \quad \text{where} \quad z_1 := \boldsymbol{e}_1^\top \boldsymbol{V}^{-1}(\boldsymbol{x}_0 - \bar{\boldsymbol{x}}) = \frac{1}{2}(x_{1,0} - 4) - (x_{2,0} - 2)$$

    for sufficiently large t
  - The phase diagram looks exactly the same as in example 2, except that now the center of the graph is replaced by $\bar{\boldsymbol{x}} = (4, 2)^\top$

- **Example 4**
  - Consider the following two-dimensional linear dynamical system:
$$\begin{bmatrix} x_{1,t+1} \\ x_{2,t+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 3/4 & -1/2 \\ -1/8 & 3/4 \end{bmatrix}}_{A} \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix}$$
  - The characteristic polynomial of $\boldsymbol{A}$ is
$$\phi_{\boldsymbol{A}}(t) = \begin{vmatrix} 3/4 - t & -1/2 \\ -1/8 & 3/4 - t \end{vmatrix} = (1-t)(1/2-t),$$
    implying that the eigenvalues of $\boldsymbol{A}$ are $\lambda_1 := 1$ and $\lambda_2 := 1/2$
  - Note that $\boldsymbol{I} - \boldsymbol{A}$ is singular because $\boldsymbol{A}$'s eigenvalues include 1:
    - obviously, $\boldsymbol{0}$ is *an* equilibrium point of the system
    - there are many other equilibrium points, and in fact, any eigenvector associated with the unit eigenvalue is an equilibrium point because
$$\boldsymbol{A}(\alpha \boldsymbol{v}_1) = \lambda_1(\alpha \boldsymbol{v}_1) = \alpha \boldsymbol{v}_1 \quad \forall \alpha \in \mathbb{R}$$
    - none of these equilibrium points is asymptotically stable
  - To characterize the behavior of $\{\boldsymbol{x}_t\}$ more explicitly, we derive eigenvectors:
$$(\boldsymbol{A} - \lambda_1 \boldsymbol{I})\boldsymbol{v} = \boldsymbol{0} \iff \begin{bmatrix} 3/4 - 1 & -1/2 \\ -1/8 & 3/4 - 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \boldsymbol{v} = \alpha \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix} \quad \forall \alpha$$
    and
$$(\boldsymbol{A} - \lambda_2 \boldsymbol{I})\boldsymbol{v} = \boldsymbol{0} \iff \begin{bmatrix} 3/4 - 1/2 & -1/2 \\ -1/8 & 3/4 - 1/2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \boldsymbol{v} = \alpha \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} \quad \forall \alpha$$
    so we choose
$$\boldsymbol{\Lambda} := \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad \boldsymbol{v}_1 := \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix}, \quad \boldsymbol{v}_2 := \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$$
    and
$$\boldsymbol{V} := \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1/2 & 1/2 \end{bmatrix} \implies \boldsymbol{V}^{-1} = \begin{bmatrix} 1/2 & -1 \\ 1/2 & 1 \end{bmatrix}$$
  - Now we can express the state trajectory $\{\boldsymbol{x}_t\}$ from arbitrary $\boldsymbol{x}_0 \in \mathbb{R}^2$ as
$$\boldsymbol{x}_t = \boldsymbol{A}^t \boldsymbol{x}_0 = \boldsymbol{V}\boldsymbol{\Lambda}^t \boldsymbol{V}^{-1} \boldsymbol{x}_0 = \begin{bmatrix} 1 & 1 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (1/2)^t \end{bmatrix} \begin{bmatrix} 1/2 & -1 \\ 1/2 & 1 \end{bmatrix} \boldsymbol{x}_0$$
  - Given an initial state $\boldsymbol{x}_0$, defining $\boldsymbol{z}$ as
$$\boldsymbol{z} := \boldsymbol{V}^{-1}\boldsymbol{x}_0, \quad \text{or} \quad z_i := \boldsymbol{e}_i^\top \boldsymbol{V}^{-1} \boldsymbol{x}_0, \quad i = 1, 2$$
    allows us to write
$$\boldsymbol{x}_t = \lambda_1^t z_1 \boldsymbol{v}_1 + \lambda_2^t z_2 \boldsymbol{v}_2 = z_1 \boldsymbol{v}_1 + \lambda_2^t z_2 \boldsymbol{v}_2 \to z_1 \boldsymbol{v}_1 = \left(\frac{1}{2} x_{1,0} - x_{2,0} \right) \boldsymbol{v}_1 \quad t \to \infty$$
    meaning that the state trajectory
    - moves in parallel with $\boldsymbol{v}_2$
    - converges to a particular point on the set $\{\boldsymbol{x} \in \mathbb{R}^2 \mid \boldsymbol{x} = \alpha \boldsymbol{v}_1, \alpha \in \mathbb{R}\}$

- **Example 5**
  - Consider the following two-dimensional linear dynamical system:

$$\begin{bmatrix} x_{1,t+1} \\ x_{2,t+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 3 & -2 \\ -1/2 & 3 \end{bmatrix}}_{\boldsymbol{A}} \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix}$$

  - The characteristic polynomial of $\boldsymbol{A}$ is

$$\phi_{\boldsymbol{A}}(t) = \begin{vmatrix} 3-t & -2 \\ -1/2 & 3-t \end{vmatrix} = (4-t)(2-t),$$

    implying that the eigenvalues of $\boldsymbol{A}$ are $\lambda_1 := 4$ and $\lambda_2 := 2$
  - The unique equilibrium point of the system is $\bar{\boldsymbol{x}} = \boldsymbol{0}$, which is not asymptotically stable because $\rho(\boldsymbol{A}) \geq 1$
  - We can express the state trajectory $\{\boldsymbol{x}_t\}$ from arbitrary $\boldsymbol{x}_0 \in \mathbb{R}^2$ as

$$\boldsymbol{x}_t = \boldsymbol{A}^t \boldsymbol{x}_0 = \boldsymbol{V}\boldsymbol{\Lambda}^t \boldsymbol{V}^{-1}\boldsymbol{x}_0 = \begin{bmatrix} 1 & 1 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 4^t & 0 \\ 0 & 2^t \end{bmatrix} \begin{bmatrix} 1/2 & -1 \\ 1/2 & 1 \end{bmatrix} \boldsymbol{x}_0$$

  - Obviously,

$$\lim_{t\to\infty} \boldsymbol{x}_t = \begin{bmatrix} 1 & 1 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} \lim_{t\to\infty} 4^t & 0 \\ 0 & \lim_{t\to\infty} 2^t \end{bmatrix} \begin{bmatrix} 1/2 & -1 \\ 1/2 & 1 \end{bmatrix} \boldsymbol{x}_0 = \begin{bmatrix} \infty \\ \infty \end{bmatrix}$$

    unless $\boldsymbol{x}_0 = \bar{\boldsymbol{x}} = \boldsymbol{0}$
  - Given an initial state $\boldsymbol{x}_0$, defining $\boldsymbol{z}$ as

$$\boldsymbol{z} := \boldsymbol{V}^{-1}\boldsymbol{x}_0, \quad \text{or} \quad z_i := \boldsymbol{e}_i^\top \boldsymbol{V}^{-1}\boldsymbol{x}_0, \quad i = 1, 2$$

    allows us to write

$$\boldsymbol{x}_t = \lambda_1^t z_1 \boldsymbol{v}_1 + \lambda_2^t z_2 \boldsymbol{v}_2 \quad \forall t$$

  - Since $\lambda_1 = 4$ is the dominant eigenvalue, the long-term behavior is characterized by

$$\boldsymbol{x}_t \approx \lambda_1^t z_1 \boldsymbol{v}_1 = 4^t z_1 \begin{bmatrix} 1 \\ -1/2 \end{bmatrix}, \quad \text{where} \quad z_1 := \boldsymbol{e}_1^\top \boldsymbol{V}^{-1}\boldsymbol{x}_0 = \frac{1}{2}x_{1,0} - x_{2,0}$$

    for sufficiently large t