Continuous-time models

Introduction to dynamical systems #7

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§ Contents

1 Continuous-time linear dynamical system

1.1 Definitions

- **Linear dynamical system**
  - A continuous-time *linear dynamical system* is a system of differential equations of the form

$$\frac{d}{dt}\boldsymbol{x}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \quad t \in \mathbb{R}_+,$$

    where
    - $\boldsymbol{x}(t) \in \mathbb{R}^m$: state vector at t
    - $\boldsymbol{A} \in \mathbb{R}^{m \times m}$: system matrix
    - $\boldsymbol{u}(t) \in \mathbb{R}^n$: control (input) vector at t
    - $\boldsymbol{B} \in \mathbb{R}^{m \times n}$: diffusion matrix
  - Starting with some *initial state* $\boldsymbol{x}(0)$, we want to know how $\boldsymbol{x}(t)$ evolves over time depending on $\boldsymbol{A}$
  - In particular, a linear dynamical system is said to be *homogeneous* if $\boldsymbol{B} = \boldsymbol{O}$, i.e.,

$$\frac{d}{dt}\boldsymbol{x}(t) = \boldsymbol{A}\boldsymbol{x}(t) \quad t \in \mathbb{R}_+,$$

    where the behavior of $\boldsymbol{x}(t)$ is completely characterized by $\boldsymbol{A}$ and $\boldsymbol{x}(0)$

- **Equilibrium**
  - Consider the case where the control is constant at $\boldsymbol{u}$:

$$\frac{d}{dt}\boldsymbol{x}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \underbrace{\boldsymbol{B}\boldsymbol{u}}_{\boldsymbol{b}}, \quad t \in \mathbb{R}_+, \tag{1}$$

  - The homogeneous system is a special case of this with $\boldsymbol{u} = \boldsymbol{0}$
  - We define an *equilibrium point* of (1) as $\bar{\boldsymbol{x}}$ that solves

$$\boldsymbol{0} = \boldsymbol{A}\bar{\boldsymbol{x}} + \boldsymbol{b},$$

    which obviously depends on both $\boldsymbol{A}$ and $\boldsymbol{b}$

- **Stability**
  - An equilibrium point $\bar{\boldsymbol{x}}$ of (1) is said to be *asymptotically stable* if, starting from **any initial state** $\boldsymbol{x}(0)$, the state trajectory satisfies
$$\lim_{t\to\infty} \boldsymbol{x}(t) = \bar{\boldsymbol{x}}$$
  - Notice that, for any initial state $\boldsymbol{x}(0)$ and for any constant input $\boldsymbol{b}$, the state trajectory $\{\boldsymbol{x}(t)\}$ satisfies
$$\frac{d}{dt}(\boldsymbol{x}(t) - \bar{\boldsymbol{x}}) = \boldsymbol{A}(\boldsymbol{x}(t) - \bar{\boldsymbol{x}}) \quad \forall t \in \mathbb{R}_+,$$
which implies that the stability of an equilibrium point is determined by $\boldsymbol{A}$ alone
  - More generally, the state trajectory $\{\boldsymbol{x}(t)\}$ may or may not converge to the equilibrium point, depending on the initial state $\boldsymbol{x}(0)$
  - We define the *stable manifold* of an equilibrium point $\bar{\boldsymbol{x}} \in \mathbb{R}^m$ as the set of initial state from which the state trajectory converges to the equilibrium point:
$$W(\bar{\boldsymbol{x}}) := \left\{ \boldsymbol{x}(0) \in \mathbb{R}^m \,\middle|\, \lim_{t\to\infty} \boldsymbol{x}(t) = \bar{\boldsymbol{x}} \right\}$$
  - An equilibrium point $\bar{\boldsymbol{x}}$ is asymptotically stable if and only if $W(\bar{\boldsymbol{x}}) = \mathbb{R}^m$

1.2 Examples

- **Example 1**
  - Consider the following homogeneous one-dimensional linear dynamical system:
$$\dot{x}(t) = ax(t) \quad \text{where} \quad \dot{x}(t) := \frac{d}{dt}x(t)$$
  - Observe:
    - the equilibrium point of the system is $\bar{x} = 0$ unless $a = 0$
    - the state trajectory is given by $x(t) = e^{at}x(0)$ because
$$\dot{x}(t) = ax(t) \implies \frac{d}{dt}\ln(x(t)) = a \implies \ln(x(t)/x(0)) = \int_0^t a d\tau = at$$
    - $\bar{x} = 0$ is asymptotically stable if and only if $a < 0$
- **Example 2**
  - Consider the following one-dimensional linear dynamical system:
$$\dot{x}(t) = ax(t) + b, \quad a \neq 0$$
  - Observe:
    - the equilibrium point of the system is $\bar{x} = -\frac{b}{a}$
    - the state trajectory is given by
$$x(t) = \bar{x} + e^{at}(x(0) - \bar{x}) = e^{at}x(0) - \left(1 - a^{at}\right)\frac{b}{a}$$
    because
$$\dot{x}(t) = ax(t) + b \implies \frac{d}{dt}\left(x(t) - \bar{x}\right) = a(x(t) - \bar{x}) \implies x(t) - \bar{x} = e^{at}(x(0) - \bar{x})$$
    - the equilibrium point is asymptotically stable if and only if $a < 0$

- **Example 3**
  - Consider the following two-dimensional linear dynamical system:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

  - Notice that, in this case, the evolution of $x_1(t)$ and $x_2(t)$ are independent:

$$\dot{x}_i(t) = a_i x_i(t) \quad i = 1, 2$$

  - Observe:
    - if $a_1 \neq 0$ and $a_2 \neq 0$, the unique equilibrium point of the system is

$$\bar{\boldsymbol{x}} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

    - the state trajectory is given by

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{a_1 t} x_1(0) \\ e^{a_2 t} x_2(0) \end{bmatrix} = \begin{bmatrix} e^{a_1 t} & 0 \\ 0 & e^{a_2 t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \quad \text{or} \quad \boldsymbol{x}(t) = e^{\boldsymbol{A}t} \boldsymbol{x}(0)$$

    - the equilibrium point is asymptotically stable if and only if $\max\{a_1, a_2\} < 0$

- **Example 4**
  - Consider the following m-dimensional linear dynamical system:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t), \quad \boldsymbol{A} := \begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_m \end{bmatrix}, \quad a_i \neq 0 \ \forall i = 1, \dots, m$$

  - Observe:
    - the unique equilibrium point of the system is

$$\bar{\boldsymbol{x}} = \boldsymbol{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

    - the state trajectory is given by

$$\boldsymbol{x}(t) = \begin{bmatrix} e^{a_1 t} x_1(0) \\ e^{a_2 t} x_2(0) \\ \vdots \\ e^{a_m t} x_m(0) \end{bmatrix} = \begin{bmatrix} e^{a_1 t} & 0 & \dots & 0 \\ 0 & e^{a_2 t} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{a_m t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_m(0) \end{bmatrix} \quad \text{or} \quad \boldsymbol{x}(t) = e^{\boldsymbol{A}t} \boldsymbol{x}(0)$$

    - the equilibrium point is asymptotically stable if and only if

$$\max\{a_1, a_2, \dots, a_m\} < 0$$

  - The stable manifold is

$$W(\bar{\boldsymbol{x}}) = \left\{ \boldsymbol{x}(0) \in \mathbb{R}^m \,\middle|\, x_i(0) = 0 \text{ for } i \text{ such that } a_i \geq 0 \right\}$$

- **Example 5**
  - Consider the following m-dimensional linear dynamical system:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}, \quad \boldsymbol{A} := \begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_m \end{bmatrix}$$

  - Assume $a_i \neq 0$ for all $i = 1, \dots, m$ (otherwise no stable equilibrium point)
  - Observe:
    - the unique equilibrium point of the system is

$$\bar{\boldsymbol{x}} = -\boldsymbol{A}^{-1}\boldsymbol{b} = \begin{bmatrix} -\frac{b_1}{a_1} \\ \vdots \\ -\frac{b_m}{a_m} \end{bmatrix}$$

    - the state trajectory is given by

$$\begin{bmatrix} x_1(t) + \frac{b_1}{a_1} \\ x_2(t) + \frac{b_1}{a_2} \\ \vdots \\ x_m(t) + \frac{b_1}{a_m} \end{bmatrix} = \begin{bmatrix} e^{a_1 t}\left(x_1(0) + \frac{b_1}{a_1}\right) \\ e^{a_2 t}\left(x_2(0) + \frac{b_2}{a_m}\right) \\ \vdots \\ e^{a_m t}\left(x_m(0) + \frac{b_m}{a_m}\right) \end{bmatrix} = \begin{bmatrix} e^{a_1 t} & 0 & \dots & 0 \\ 0 & e^{a_2 t} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{a_m t} \end{bmatrix} \begin{bmatrix} x_1(0) + \frac{b_1}{a_1} \\ x_2(0) + \frac{b_1}{a_2} \\ \vdots \\ x_m(0) + \frac{b_1}{a_m} \end{bmatrix}$$

    or

$$\boldsymbol{x}(t) - \bar{\boldsymbol{x}} = e^{\boldsymbol{A}t}(\boldsymbol{x}(0) - \bar{\boldsymbol{x}}) \quad \text{or} \quad \boldsymbol{x}(t) = \bar{\boldsymbol{x}} + e^{\boldsymbol{A}t}(\boldsymbol{x}(0) - \bar{\boldsymbol{x}})$$

    - the equilibrium point is asymptotically stable if and only if

$$\max\{a_1, a_2, \dots, a_m\} < 0$$

  - The stable manifold is

$$W(\bar{\boldsymbol{x}}) = \left\{ \boldsymbol{x}(0) \in \mathbb{R}^m \,\middle|\, x_i(0) = -\frac{b_i}{a_i} \text{ for } i \text{ such that } a_i \geq 0 \right\}$$

- **Example 6**
  - Consider the following two-dimensional linear dynamical system

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{\boldsymbol{A}} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{2}$$

  - Suppose $\boldsymbol{A}$ is diagonalizable

$$\boldsymbol{A} = \underbrace{\begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{bmatrix}}_{\boldsymbol{V}} \underbrace{\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}}_{\boldsymbol{\Lambda}} \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{bmatrix}^{-1}$$

- For each t, define $z(t) \in \mathbb{R}^2$ as

$$\boldsymbol{z}(t) := \boldsymbol{V}^{-1}\boldsymbol{x}(t) \quad \text{or} \quad \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} := \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{bmatrix}^{-1} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \tag{3}$$

  which means

$$\dot{\boldsymbol{z}}(t) = \boldsymbol{V}^{-1}\dot{\boldsymbol{x}}(t) = \boldsymbol{V}^{-1}\left(\boldsymbol{A}\boldsymbol{x}(t)\right) = \boldsymbol{V}^{-1}\left(\boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1}\boldsymbol{V}\boldsymbol{z}(t)\right) = \boldsymbol{\Lambda}\boldsymbol{z}(t)$$

  or

$$\begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 z_1(t) \\ \lambda_2 z_2(t) \end{bmatrix}$$

- Solving this system is straightforward:

$$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} z_1(0) \\ e^{\lambda_2 t} z_2(0) \end{bmatrix} = \underbrace{\begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}}_{e^{\boldsymbol{\Lambda} t}} \begin{bmatrix} z_1(0) \\ z_2(0) \end{bmatrix}$$

- Since $\boldsymbol{x}(t) = \boldsymbol{V}\boldsymbol{z}(t)$, it follows that

$$\boldsymbol{x}(t) = \boldsymbol{V}\boldsymbol{z}(t) = \boldsymbol{V}e^{\boldsymbol{\Lambda} t}\boldsymbol{V}^{-1}\boldsymbol{V}\boldsymbol{z}(0) = e^{\boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1}t}\boldsymbol{x}(0) = e^{\boldsymbol{A}t}\boldsymbol{x}(0)$$

- **Example 7**
  - Consider the same dynamical system as (2)
  - This time, $\boldsymbol{A}$ is NOT diagonalizable and the Jordan decomposition yields

$$\boldsymbol{A} = \underbrace{\begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{bmatrix}}_{\boldsymbol{V}} \underbrace{\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}}_{\boldsymbol{J}} \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{bmatrix}^{-1}$$

  - For each t, define $z(t) \in \mathbb{R}^2$ as (3), which means

$$\dot{\boldsymbol{z}}(t) = \boldsymbol{V}^{-1}\dot{\boldsymbol{x}}(t) = \boldsymbol{V}^{-1}\left(\boldsymbol{A}\boldsymbol{x}(t)\right) = \boldsymbol{V}^{-1}\left(\boldsymbol{V}\boldsymbol{J}\boldsymbol{V}^{-1}\boldsymbol{V}\boldsymbol{z}(t)\right) = \boldsymbol{J}\boldsymbol{z}(t)$$

  or

$$\begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} \lambda z_1(t) + z_2(t) \\ \lambda z_2(t) \end{bmatrix}$$

  - First focusing on the second equation, we have

$$e^{-\lambda t}\dot{z}_2(t) - e^{-\lambda t}\lambda z_2(t) = 0 \implies z_2(t) = e^{\lambda t} z_2(0),$$

  which in turn allows us to solve the first equation as

$$e^{-\lambda t}\dot{z}_1(t) - e^{-\lambda t}\lambda z_1(t) = z_2(0) \implies z_1(t) = e^{\lambda t} z_1(0) + t e^{\lambda t} z_2(0),$$

  and therefore

$$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} e^{\lambda t} z_1(0) + t e^{\lambda t} z_2(0) \\ e^{\lambda t} z_2(0) \end{bmatrix} = \underbrace{\begin{bmatrix} e^{\lambda t} & t e^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}}_{e^{\boldsymbol{J} t}} \begin{bmatrix} z_1(0) \\ z_2(0) \end{bmatrix}$$

  - Since $\boldsymbol{x}(t) = \boldsymbol{V}\boldsymbol{z}(t)$, it follows that

$$\boldsymbol{x}(t) = \boldsymbol{V}\boldsymbol{z}(t) = \boldsymbol{V}e^{\boldsymbol{J} t}\boldsymbol{V}^{-1}\boldsymbol{V}\boldsymbol{z}(0) = e^{\boldsymbol{V}\boldsymbol{J}\boldsymbol{V}^{-1}t}\boldsymbol{x}(0) = e^{\boldsymbol{A}t}\boldsymbol{x}(0)$$

2 Characterization

2.1 General method

- **Trajectory and stability**
  - Consider the linear dynamical system
$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b} \quad t \in \mathbb{R}_+ \tag{4}$$
  - Assume $\boldsymbol{A}$ does not have 0 as its eigenvalue (otherwise no stable equilibrium point)
  - Since $\lambda = 0$ is not an eigenvalue of $\boldsymbol{A}$, we know that $\boldsymbol{A}$ is non-singular (why?), and the unique equilibrium point is
$$\bar{\boldsymbol{x}} := -\boldsymbol{A}^{-1}\boldsymbol{b}$$
and we may write
$$\boldsymbol{x}(t) = \bar{\boldsymbol{x}} + e^{\boldsymbol{A}t}(\boldsymbol{x}(0) - \bar{\boldsymbol{x}}), \quad t \in \mathbb{R}_+ \tag{5}$$
because
$$(4) \implies \frac{d}{dt}(\boldsymbol{x}(t) - \bar{\boldsymbol{x}}) = \boldsymbol{A}(\boldsymbol{x}(t) - \bar{\boldsymbol{x}}) \implies \frac{d}{dt}\left(e^{-\boldsymbol{A}t}(\boldsymbol{x}(t) - \bar{\boldsymbol{x}})\right) = \boldsymbol{0}$$
  - Hence,
$$\bar{\boldsymbol{x}} \text{ is asymptotically stable} \iff \lim_{t\to\infty} e^{\boldsymbol{A}t} = \boldsymbol{O} \iff \rho(e^{\boldsymbol{A}}) < 1$$
  - Using the Jordan decomposition $\boldsymbol{A} = \boldsymbol{V}\boldsymbol{J}\boldsymbol{V}^{-1}$, one can rewrite (5) as
$$\boldsymbol{x}(t) = \bar{\boldsymbol{x}} + \boldsymbol{V}e^{\boldsymbol{J}t}\boldsymbol{V}^{-1}(\boldsymbol{x}(0) - \bar{\boldsymbol{x}}), \quad t \in \mathbb{R}_+$$
  - In particular, if $\boldsymbol{A}$ is diagonalizable, we have $\boldsymbol{J} = \boldsymbol{\Lambda} := \mathrm{diag}(\lambda_1, \ldots, \lambda_m)$ and
$$\boldsymbol{x}(t) = \bar{\boldsymbol{x}} + \boldsymbol{V}e^{\boldsymbol{\Lambda}t}\boldsymbol{V}^{-1}(\boldsymbol{x}(t) - \bar{\boldsymbol{x}}) \quad t \in \mathbb{R}_+,$$
in which case the stable manifold can be expressed as
$$W(\bar{\boldsymbol{x}}) = \left\{\boldsymbol{x}(0) \in \mathbb{R}^m \,\middle|\, \boldsymbol{e}_i^\top \boldsymbol{V}^{-1}(\boldsymbol{x}(0) - \bar{\boldsymbol{x}}) = 0 \text{ for } i \text{ such that } \lambda_i \geq 0\right\}$$

- **Using eigenvectors as a basis**
  - Let $\boldsymbol{A} \in \mathbb{R}^{m\times m}$ be a square matrix and consider the linear system of differential equations
$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) \quad \forall t \in \mathbb{R}_+ \ldots \tag{6}$$
  - If $(\lambda, \boldsymbol{v})$ is an eigenpair of $\boldsymbol{A}$, then
$$\boldsymbol{x}(t) := e^{\lambda t}\boldsymbol{v} \quad \forall t$$
solves (6) with $\boldsymbol{x}(0) = \boldsymbol{v}$ because
$$\boldsymbol{x}(t) = e^{\lambda t}\boldsymbol{v} \implies \dot{\boldsymbol{x}}(t) = \lambda e^{\lambda t}\boldsymbol{v} = e^{\lambda t}\lambda\boldsymbol{v} = e^{\lambda t}\boldsymbol{A}\boldsymbol{v} = \boldsymbol{A}e^{\lambda t}\boldsymbol{v} = \boldsymbol{A}\boldsymbol{x}(t)$$
  - If $(\lambda_1, \boldsymbol{v}_1), (\lambda_2, \boldsymbol{v}_2)$ are eigenpairs of $\boldsymbol{A}$, then for any $z_1, z_2 \in \mathbb{R}$,
$$\boldsymbol{x}(t) := z_1 e^{\lambda_1 t}\boldsymbol{v}_1 + z_2 e^{\lambda_2 t}\boldsymbol{v}_2 \quad \forall t$$
solves (6) with $\boldsymbol{x}(0) = z_1\boldsymbol{v}_1 + z_2\boldsymbol{v}_2$

- If $(\lambda_1, \boldsymbol{v}_1), (\lambda_2, \boldsymbol{v}_2), \ldots, (\lambda_m, \boldsymbol{v}_m)$ are eigenpairs of $\boldsymbol{A}$, then for any $z_1, z_2, \ldots, z_m \in \mathbb{R}$,

$$\boldsymbol{x}(t) := z_1 e^{\lambda_1 t}\boldsymbol{v}_1 + z_2 e^{\lambda_2 t}\boldsymbol{v}_2 + \ldots + z_m e^{\lambda_m t}\boldsymbol{v}_m \quad \forall t$$

  solves (6) with $\boldsymbol{x}(0) = z_1\boldsymbol{v}_1 + z_2\boldsymbol{v}_2 + \ldots + z_m\boldsymbol{v}_m$

- Hence, if $(\lambda_1, \boldsymbol{v}_1), (\lambda_2, \boldsymbol{v}_2), \ldots, (\lambda_m, \boldsymbol{v}_m)$ are **linearly independent** eigenpairs of $\boldsymbol{A}$, then for any arbitrary initial state $\boldsymbol{x}(0) \in \mathbb{R}^m$, one can find $z_1, z_2, \ldots, z_m \in \mathbb{R}$ such that

$$\boldsymbol{x}(0) = z_1\boldsymbol{v}_1 + z_2\boldsymbol{v}_2 + \ldots + z_m\boldsymbol{v}_m$$

  and the state trajectory from $\boldsymbol{x}(0)$ can be written as

$$\boldsymbol{x}(t) = z_1 e^{\lambda_1 t}\boldsymbol{v}_1 + z_2 e^{\lambda_2 t}\boldsymbol{v}_2 + \ldots + z_m e^{\lambda_m t}\boldsymbol{v}_m = \boldsymbol{V}e^{\boldsymbol{\Lambda}t}\boldsymbol{z} \quad \forall t$$

- **Long-run behavior and dominant mode**
  - Consider the following m-dimensional linear dynamical system:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b} \quad \forall t \quad \text{with some initial state } \boldsymbol{x}(0) \in \mathbb{R}^m$$

  - Suppose that $\boldsymbol{A} \in \mathbb{R}^{m\times m}$ is diagonalizable and let $(\lambda_1, \boldsymbol{v}_1), (\lambda_2, \boldsymbol{v}_2), \ldots, (\lambda_m, \boldsymbol{v}_m)$ be linearly independent eigenpairs of $\boldsymbol{A}$
  - Then, there exist $z_1, z_2, \ldots, z_m \in \mathbb{R}$ such that

$$\boldsymbol{x}_0 - \bar{\boldsymbol{x}} = z_1\boldsymbol{v}_1 + z_2\boldsymbol{v}_2 + \ldots z_m\boldsymbol{v}_m = \underbrace{\begin{bmatrix}\boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_m\end{bmatrix}}_{\boldsymbol{V}} \underbrace{\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \end{bmatrix}}_{\boldsymbol{z}}$$

  and thus

$$\boldsymbol{x}(t) - \bar{\boldsymbol{x}} = e^{\boldsymbol{A}t}(\boldsymbol{x}(0) - \bar{\boldsymbol{x}}) = z_1 e^{\lambda_1 t}\boldsymbol{v}_1 + z_2 e^{\lambda_2 t}\boldsymbol{v}_2 + \ldots + z_m e^{\lambda_m t}\boldsymbol{v}_m \quad \forall t$$

  - Let λ_i be the *dominant eigenvalue* of $\boldsymbol{A}$, i.e., $\lambda_i > \lambda_j$ for all $j \neq i$
  - Then, for sufficiently large t,

$$e^{-\lambda_i t}(\boldsymbol{x}(t) - \bar{\boldsymbol{x}}) = z_1 e^{-(\lambda_i - \lambda_1)}\boldsymbol{v}_1 + \cdots + z_i\boldsymbol{v}_i + \cdots + z_m e^{-(\lambda_i - \lambda_m)}\boldsymbol{v}_m \approx z_i\boldsymbol{v}_i,$$

  or

$$\boldsymbol{x}(t) \approx \bar{\boldsymbol{x}} + e^{\lambda_i t} z_i\boldsymbol{v}_i, \quad \text{where} \quad z_i = \boldsymbol{e}_i^\top \boldsymbol{V}^{-1}(\boldsymbol{x}(0) - \bar{\boldsymbol{x}})$$

  - Observe:
    - the long-term state is essentially determined by the eigenvector associated with the dominant eigenvalue of $\boldsymbol{A}$
    - if $\rho(e^{\boldsymbol{A}}) < 1$, the rate at which the state converges to the equilibrium point is ultimately governed by the dominant eigenvalue
    - if $\rho(e^{\boldsymbol{A}}) = 1$ (i.e, the dominant eigenvalue is zero),

$$\lim_{t\to\infty} \boldsymbol{x}(t) = \bar{\boldsymbol{x}} + z_i\boldsymbol{v}_i,$$

      in which case $\boldsymbol{x}(t)$ neither diverges to infinity nor converges to $\bar{\boldsymbol{x}}$ (we call such situation as *marginally stable*) and the limit depends on the initial state (through z_i)

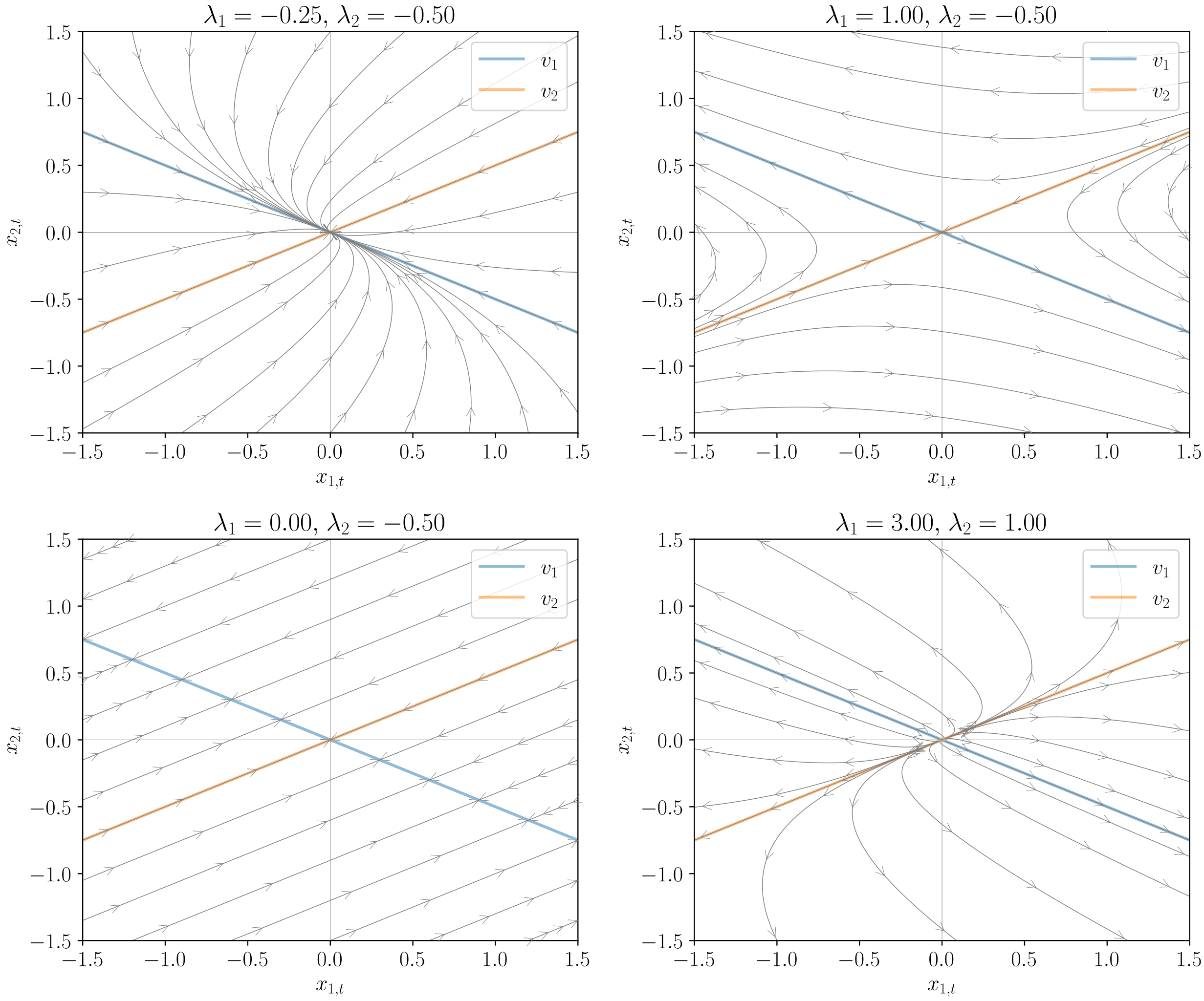

Figure 1: Phase diagrams for example 1 (top left), example 2 (top right), example 4 (bottom left), and example 5 (bottom right)

2.2 Examples

- **Example 1**
  - Consider the following two-dimensional linear dynamical system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -3/8 & -1/4 \\ -1/16 & -3/8 \end{bmatrix}}_{\boldsymbol{A}} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

  - The state trajectory $\{\boldsymbol{x}(t)\}$ from arbitrary $\boldsymbol{x}_0 \in \mathbb{R}^2$ is

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}_0,$$

    which is not easy to characterize
  - So we look at the characteristic polynomial

$$\phi_{\boldsymbol{A}}(t) = \begin{vmatrix} -3/8 - t & -1/4 \\ -1/16 & -3/8 - t \end{vmatrix} = (-1/4 - t)(-1/2 - t),$$

    implying that the eigenvalues of $\boldsymbol{A}$ are $\lambda_1 := -1/4$ and $\lambda_2 := -1/2$

- We already see that the unique equilibrium point $\bar{\boldsymbol{x}} := \boldsymbol{0}$ is asymptotically stable
- To characterize the behavior of $\{\boldsymbol{x}(t)\}$ more explicitly, we derive eigenvectors:

$$(\boldsymbol{A} - \lambda_1 \boldsymbol{I})\boldsymbol{v} = \boldsymbol{0} \iff \begin{bmatrix} -3/8 + 1/4 & -1/4 \\ -1/16 & -3/8 + 1/4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \boldsymbol{v} = \alpha \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix} \quad \forall \alpha$$

  and

$$(\boldsymbol{A} - \lambda_2 \boldsymbol{I})\boldsymbol{v} = \boldsymbol{0} \iff \begin{bmatrix} -3/8 + 1/2 & -1/4 \\ -1/16 & -3/8 + 1/2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \boldsymbol{v} = \alpha \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} \quad \forall \alpha$$

  so we choose

$$\boldsymbol{\Lambda} := \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad \boldsymbol{v}_1 := \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix}, \quad \boldsymbol{v}_2 := \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$$

  and

$$\boldsymbol{V} := \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1/2 & 1/2 \end{bmatrix} \implies \boldsymbol{V}^{-1} = \begin{bmatrix} 1/2 & -1 \\ 1/2 & 1 \end{bmatrix}$$

- Now we can express the state trajectory $\{\boldsymbol{x}(t)\}$ from arbitrary $\boldsymbol{x}_0 \in \mathbb{R}^2$ as

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0) = \boldsymbol{V}e^{\boldsymbol{\Lambda}t}\boldsymbol{V}^{-1}\boldsymbol{x}(0) = \begin{bmatrix} 1 & 1 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} e^{-\frac{1}{4}t} & 0 \\ 0 & e^{-\frac{1}{2}t} \end{bmatrix} \begin{bmatrix} 1/2 & -1 \\ 1/2 & 1 \end{bmatrix} \boldsymbol{x}(0)$$

- The trajectory converges to $\bar{\boldsymbol{x}} = \boldsymbol{0}$ regardless of $\boldsymbol{x}(0)$
- Given an initial state $\boldsymbol{x}(0)$, defining $\boldsymbol{z}$ as

$$\boldsymbol{z} := \boldsymbol{V}^{-1}\boldsymbol{x}(0), \quad \text{or} \quad z_i := \boldsymbol{e}_i^\top \boldsymbol{V}^{-1}\boldsymbol{x}(0), \quad i = 1, 2$$

  allows us to write

$$\boldsymbol{x}(t) = e^{\lambda_1 t} z_1 \boldsymbol{v}_1 + e^{\lambda_2 t} z_2 \boldsymbol{v}_2 \quad \forall t$$

- Since $\lambda_1 = -1/4$ is the dominant eigenvalue, the long-term behavior is characterized by

$$\boldsymbol{x}(t) \approx e^{\lambda_1 t} z_1 \boldsymbol{v}_1 = e^{-\frac{1}{4}t} z_1 \begin{bmatrix} 1 \\ -1/2 \end{bmatrix}, \quad \text{where} \quad z_1 := \boldsymbol{e}_1^\top \boldsymbol{V}^{-1}\boldsymbol{x}(0) = \frac{1}{2}x_1(0) - x_2(0)$$

  for sufficiently large t

- **Example 2**
- Consider another two-dimensional linear dynamical system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1/4 & -3/2 \\ -3/8 & 1/4 \end{bmatrix}}_{\boldsymbol{A}} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- The state trajectory $\{\boldsymbol{x}(t)\}$ from arbitrary $\boldsymbol{x}(0) \in \mathbb{R}^2$ is

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0),$$

  which is not easy to characterize

- So we look at the characteristic polynomial

$$\phi_A(t) = \begin{vmatrix} 1/4 - t & -3/2 \\ -3/8 & 1/4 - t \end{vmatrix} = (1-t)(-1/2-t),$$

  implying that the eigenvalues of $\boldsymbol{A}$ are $\lambda_1 := 1$ and $\lambda_2 := -1/2$
- Since $\lambda_1 > 0$, the unique equilibrium point $\bar{\boldsymbol{x}} := \boldsymbol{0}$ is NOT asymptotically stable
- To characterize the behavior of $\{\boldsymbol{x}(t)\}$ more explicitly, we derive eigenvectors:

$$(\boldsymbol{A} - \lambda_1 \boldsymbol{I})v = \boldsymbol{0} \iff \begin{bmatrix} 1/4 - 1 & -3/2 \\ -3/8 & 1/4 - 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \boldsymbol{v} = \alpha \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix} \quad \forall \alpha$$

  and

$$(\boldsymbol{A} - \lambda_2 \boldsymbol{I})v = \boldsymbol{0} \iff \begin{bmatrix} 1/4 + 1/2 & -3/2 \\ -3/8 & 1/4 + 1/2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \boldsymbol{v} = \alpha \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} \quad \forall \alpha$$

  so we choose

$$\boldsymbol{\Lambda} := \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad \boldsymbol{v}_1 := \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix}, \quad \boldsymbol{v}_2 := \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$$

  and

$$\boldsymbol{V} := \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1/2 & 1/2 \end{bmatrix} \implies \boldsymbol{V}^{-1} = \begin{bmatrix} 1/2 & -1 \\ 1/2 & 1 \end{bmatrix}$$

- Now we can express the state trajectory $\{\boldsymbol{x}(t)\}$ from arbitrary $\boldsymbol{x}(0) \in \mathbb{R}^2$ as

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0) = \boldsymbol{V}e^{\boldsymbol{\Lambda}t}\boldsymbol{V}^{-1}\boldsymbol{x}(0) = \begin{bmatrix} 1 & 1 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{-\frac{1}{2}t} \end{bmatrix} \begin{bmatrix} 1/2 & -1 \\ 1/2 & 1 \end{bmatrix} \boldsymbol{x}(0)$$

- The trajectory converges to $\bar{\boldsymbol{x}} = \boldsymbol{0}$ only if we choose $\boldsymbol{x}(0)$ to nullify the impact of e^t
- More specifically, we need to choose $\boldsymbol{x}(0)$ in such a way that

$$\begin{bmatrix} 1/2 & -1 \\ 1/2 & 1 \end{bmatrix} \boldsymbol{x}(0) = \begin{bmatrix} 0 \\ \dots \end{bmatrix} \iff \boldsymbol{e}_1^\top \underbrace{\begin{bmatrix} 1/2 & -1 \\ 1/2 & 1 \end{bmatrix}}_{\boldsymbol{V}^{-1}} \boldsymbol{x}(0) = 0 \iff \frac{1}{2}x_1(0) - x_2(0) = 0$$

- Hence, the stable manifold is

$$W(\bar{\boldsymbol{x}}) = \left\{ \boldsymbol{x}(0) = (x_1(0), x_2(0)) \in \mathbb{R}^2 \;\middle|\; \frac{1}{2}x_1(0) - x_2(0) = 0 \right\}$$

- Given an initial state $\boldsymbol{x}(0)$, defining $\boldsymbol{z}$ as

$$\boldsymbol{z} := \boldsymbol{V}^{-1}\boldsymbol{x}(0), \quad \text{or} \quad z_i := \boldsymbol{e}_i^\top \boldsymbol{V}^{-1}\boldsymbol{x}(0), \quad i = 1, 2$$

  allows us to write

$$\boldsymbol{x}(t) = e^{\lambda_1 t} z_1 \boldsymbol{v}_1 + e^{\lambda_2 t} z_2 \boldsymbol{v}_2 \quad \forall t$$

- Since λ_1 is the dominant eigenvalue, the long-term behavior is characterized by

$$\boldsymbol{x}(t) \approx e^{\lambda_1 t} z_1 \boldsymbol{v}_1 = e^t z_1 \begin{bmatrix} 1 \\ -1/2 \end{bmatrix}, \quad \text{where} \quad z_1 := \boldsymbol{e}_1^\top \boldsymbol{V}^{-1}\boldsymbol{x}(0) = \frac{1}{2}x_1(0) - x_2(0)$$

  for sufficiently large t

- **Example 3**
  - Consider a slightly modified version of the example above:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1/4 & -3/2 \\ -3/8 & 1/4 \end{bmatrix}}_{\boldsymbol{A}} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{\boldsymbol{b}}$$

  - Since 0 is not an eigenvalue of $\boldsymbol{A}$, we know that $\boldsymbol{A}$ is non-singular and

$$\boldsymbol{A}^{-1} = \begin{bmatrix} -1/2 & -3 \\ -3/4 & -1/2 \end{bmatrix}$$

  - The unique equilibrium point is

$$\bar{\boldsymbol{x}} = -\boldsymbol{A}^{-1}\boldsymbol{b} = -\begin{bmatrix} -1/2 & -3 \\ -3/4 & -1/2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

  - The state trajectory $\{\boldsymbol{x}(t)\}$ from arbitrary $\boldsymbol{x}(0) \in \mathbb{R}^2$ is

$$\begin{aligned} \boldsymbol{x}(t) &= \bar{\boldsymbol{x}} + e^{\boldsymbol{A}t}(\boldsymbol{x}(0) - \bar{\boldsymbol{x}}) = \bar{\boldsymbol{x}} + \boldsymbol{V}e^{\boldsymbol{\Lambda}t}\boldsymbol{V}^{-1}(\boldsymbol{x}(0) - \bar{\boldsymbol{x}}) \\ &= \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{-\frac{1}{2}t} \end{bmatrix} \begin{bmatrix} 1/2 & -1 \\ 1/2 & 1 \end{bmatrix} \left(\boldsymbol{x}(0) - \begin{bmatrix} 4 \\ 2 \end{bmatrix} \right) \end{aligned}$$

  - The trajectory converges to $\bar{\boldsymbol{x}} = (4, 2)^\top$ only if we choose $\boldsymbol{x}(0)$ to nullify the impact of e^t
  - More specifically, we need to choose $\boldsymbol{x}(0)$ in such a way that

$$\begin{bmatrix} 1/2 & -1 \\ 1/2 & 1 \end{bmatrix} (\boldsymbol{x}(0) - \bar{\boldsymbol{x}}) = \begin{bmatrix} 0 \\ \cdots \end{bmatrix} \iff \frac{1}{2}(x_1(0) - 4) - (x_2(0) - 2) = 0$$

  - Hence, the stable manifold is

$$W(\bar{\boldsymbol{x}}) = \left\{ \boldsymbol{x}(0) = (x_1(0), x_2(0)) \in \mathbb{R}^2 \,\middle|\, \frac{1}{2}(x_1(0) - 4) - (x_2(0) - 2) = 0 \right\}$$

  - Given an initial state $\boldsymbol{x}(0)$, defining $\boldsymbol{z}$ as

$$\boldsymbol{z} := \boldsymbol{V}^{-1}(\boldsymbol{x}(0) - \bar{\boldsymbol{x}}), \quad \text{or} \quad z_i := \boldsymbol{e}_i^\top \boldsymbol{V}^{-1}(\boldsymbol{x}(0) - \bar{\boldsymbol{x}}), \quad i = 1, 2$$

    allows us to write

$$\boldsymbol{x}(t) = \bar{\boldsymbol{x}} + e^{\lambda_1 t} z_1 \boldsymbol{v}_1 + e^{\lambda_2 t} z_2 \boldsymbol{v}_2 \quad \forall t$$

  - Since $\lambda_1 = 1$ is the dominant eigenvalue, the long-term behavior is characterized by

$$\boldsymbol{x}(t) \approx \bar{\boldsymbol{x}} + e^{\lambda_1 t} z_1 \boldsymbol{v}_1 = e^t z_1 \begin{bmatrix} 1 \\ -1/2 \end{bmatrix}, \quad \text{where} \quad z_1 := \boldsymbol{e}_1^\top \boldsymbol{V}^{-1}(\boldsymbol{x}(0) - \bar{\boldsymbol{x}}) = \frac{1}{2}(x_1(0) - 4) - (x_2(0) - 2)$$

    for sufficiently large t
  - The phase diagram looks exactly the same as in example 2, except that now the center of the graph is replaced by $\bar{\boldsymbol{x}} = (4, 2)^\top$

- **Example 4**
  - Consider the following two-dimensional linear dynamical system:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} -1/4 & -1/2 \\ -1/8 & -1/4 \end{bmatrix}}_{A} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

  - The characteristic polynomial of A is

$$\phi_A(t) = \begin{vmatrix} -1/4 - t & -1/2 \\ -1/8 & -1/4 - t \end{vmatrix} = -t(-1/2 - t),$$

    implying that the eigenvalues of A are $\lambda_1 := 0$ and $\lambda_2 := -1/2$
  - Note that A is singular because its eigenvalues include 0:
    - obviously, $\mathbf{0}$ is *an* equilibrium point of the system
    - there are many other equilibrium points, and in fact, any eigenvector associated with the zero eigenvalue is an equilibrium point because

$$A(\alpha v_1) = \lambda_1(\alpha v_1) = \mathbf{0} \quad \forall \alpha \in \mathbb{R}$$

    - none of these equilibrium points is asymptotically stable
  - To characterize the behavior of $\{x(t)\}$ more explicitly, we derive eigenvectors:

$$(A - \lambda_1 I)v = \mathbf{0} \iff \begin{bmatrix} -1/4 - 0 & -1/2 \\ -1/8 & -1/4 - 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff v = \alpha \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix} \quad \forall \alpha$$

    and

$$(A - \lambda_2 I)v = \mathbf{0} \iff \begin{bmatrix} -1/4 + 1/2 & -1/2 \\ -1/8 & -1/4 + 1/2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff v = \alpha \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} \quad \forall \alpha$$

    so we choose

$$\Lambda := \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad v_1 := \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix}, \quad v_2 := \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$$

    and

$$V := \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1/2 & 1/2 \end{bmatrix} \implies V^{-1} = \begin{bmatrix} 1/2 & -1 \\ 1/2 & 1 \end{bmatrix}$$

  - Now we can express the state trajectory $\{x(t)\}$ from arbitrary $x(0) \in \mathbb{R}^2$ as

$$x(t) = e^{At}x(0) = Ve^{\Lambda t}V^{-1}x(0) = \begin{bmatrix} 1 & 1 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{-\frac{1}{2}t} \end{bmatrix} \begin{bmatrix} 1/2 & -1 \\ 1/2 & 1 \end{bmatrix} x(0)$$

  - Given an initial state $x(0)$, defining z as

$$z := V^{-1}x(0), \quad \text{or} \quad z_i := e_i^\top V^{-1}x(0), \quad i = 1, 2$$

    allows us to write

$$x(t) = e^{\lambda_1 t}z_1 v_1 + e^{\lambda_2 t}z_2 v_2 = z_1 v_1 + e^{-\frac{1}{2}t}z_2 v_2 \to z_1 v_1 = \left(\frac{1}{2}x_1(0) - x_2(0)\right) v_1 \quad t \to \infty$$

    meaning that the state trajectory
    - moves in parallel with v_2
    - converges to a particular point on the set $\{x \in \mathbb{R}^2 \mid x = \alpha v_1, \alpha \in \mathbb{R}\}$

- **Example 5**
  - Consider the following two-dimensional linear dynamical system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & -2 \\ -1/2 & 2 \end{bmatrix}}_{A} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

  - The characteristic polynomial of $\boldsymbol{A}$ is

$$\phi_A(t) = \begin{vmatrix} 2-t & -2 \\ -1/2 & 2-t \end{vmatrix} = (3-t)(1-t),$$

    implying that the eigenvalues of $\boldsymbol{A}$ are $\lambda_1 := 3$ and $\lambda_2 := 1$
  - The unique equilibrium point of the system is $\bar{\boldsymbol{x}} = \boldsymbol{0}$, which is not asymptotically stable because $\rho(e^{\boldsymbol{A}}) \geq 1$
  - We can express the state trajectory $\{\boldsymbol{x}(t)\}$ from arbitrary $\boldsymbol{x}(0) \in \mathbb{R}^2$ as

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0) = \boldsymbol{V}e^{\boldsymbol{\Lambda}t}\boldsymbol{V}^{-1}\boldsymbol{x}(0) = \begin{bmatrix} 1 & 1 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 \\ 0 & e^t \end{bmatrix} \begin{bmatrix} 1/2 & -1 \\ 1/2 & 1 \end{bmatrix} \boldsymbol{x}(0)$$

  - Obviously,

$$\lim_{t\to\infty} \boldsymbol{x}(t) = \begin{bmatrix} 1 & 1 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} \lim_{t\to\infty} e^{3t} & 0 \\ 0 & \lim_{t\to\infty} e^t \end{bmatrix} \begin{bmatrix} 1/2 & -1 \\ 1/2 & 1 \end{bmatrix} \boldsymbol{x}(0) = \begin{bmatrix} \infty \\ \infty \end{bmatrix}$$

    unless $\boldsymbol{x}(0) = \bar{\boldsymbol{x}} = \boldsymbol{0}$
  - Given an initial state $\boldsymbol{x}(0)$, defining $\boldsymbol{z}$ as

$$\boldsymbol{z} := \boldsymbol{V}^{-1}\boldsymbol{x}(0), \quad \text{or} \quad z_i := \boldsymbol{e}_i^\top \boldsymbol{V}^{-1}\boldsymbol{x}(0), \quad i = 1,2$$

    allows us to write

$$\boldsymbol{x}(t) = e^{\lambda_1 t} z_1 \boldsymbol{v}_1 + e^{\lambda_2 t} z_2 \boldsymbol{v}_2 \quad \forall t$$

  - Since $\lambda_1 = 3$ is the dominant eigenvalue, the long-term behavior is characterized by

$$\boldsymbol{x}(t) \approx e^{\lambda_1 t} z_1 \boldsymbol{v}_1 = e^{3t} z_1 \begin{bmatrix} 1 \\ -1/2 \end{bmatrix}, \quad \text{where} \quad z_1 := \boldsymbol{e}_1^\top \boldsymbol{V}^{-1}\boldsymbol{x}(0) = \frac{1}{2}x_1(0) - x_2(0)$$

    for sufficiently large t

3 Discretization of continuous-time systems

- **General solution**
  - Consider a general linear dynamical system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \quad t \in \mathbb{R}_+ \tag{7}$$

  - Note that

$$\begin{aligned} (7) \implies \frac{d}{dt}\left(e^{-\boldsymbol{A}t}\boldsymbol{x}(t)\right) &= e^{-\boldsymbol{A}t}\dot{\boldsymbol{x}}(t) - \boldsymbol{A}e^{-\boldsymbol{A}t}\boldsymbol{x}(t) = e^{-\boldsymbol{A}t}\boldsymbol{B}\boldsymbol{u}(t) \quad t \in \mathbb{R}_+ \qquad (8) \\ \implies e^{-\boldsymbol{A}t}\boldsymbol{x}(t) - e^{-\boldsymbol{A}0}\boldsymbol{x}(0) &= \int_0^t e^{-\boldsymbol{A}\tau}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau \quad t \in \mathbb{R}_+ \end{aligned}$$

    and therefore, the state trajectory can be written as

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0) + \int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau \quad t \in \mathbb{R} \tag{9}$$

- **Discretization**
  - In some cases (like when observations can only be made at discrete points in time), one might want to convert the continuous-time model into the corresponding discrete-time model
  - Note that (8) implies that for all $t, \Delta t \in \mathbb{R}$

$$e^{-\boldsymbol{A}(t+\Delta t)}\boldsymbol{x}(t+\Delta t) - e^{-\boldsymbol{A}t}\boldsymbol{x}(t) = \int_t^{t+\Delta t} e^{-\boldsymbol{A}\tau}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau$$

  and thus

$$\boldsymbol{x}(t+\Delta t) = e^{\boldsymbol{A}\Delta t}\boldsymbol{x}(t) + \int_t^{t+\Delta t} e^{\boldsymbol{A}(t+\Delta t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau \tag{10}$$

  - In particular, choosing $\Delta t = 1$ yields

$$\boldsymbol{x}(t+1) = e^{\boldsymbol{A}}\boldsymbol{x}(t) + \int_t^{t+1} e^{\boldsymbol{A}(t+1-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau \quad \forall t = 0,1,2,\ldots, \tag{11}$$

  - Let us consider the situation where one can only change the value of $u(t)$ at discrete points in time $t = 0,1,2,\ldots$:

$$u(t) = \begin{cases} u_0 & \forall t \in [0,1) \\ u_1 & \forall t \in [1,2) \\ u_2 & \forall t \in [2,3) \\ \vdots \end{cases}$$

  then

$$\int_t^{t+1} e^{\boldsymbol{A}(t+1-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau = \int_t^{t+1} e^{\boldsymbol{A}(t+1-\tau)}d\tau\boldsymbol{B}\boldsymbol{u}_t = \int_0^1 e^{\boldsymbol{A}\tau}d\tau\boldsymbol{B}\boldsymbol{u}_t = \boldsymbol{A}^{-1}(e^{\boldsymbol{A}} - \boldsymbol{I})\boldsymbol{B}\boldsymbol{u}_t,$$

  and therefore, the state trajectory satisfies the following difference equations

$$\boldsymbol{x}(t+1) = \boldsymbol{A}_d\boldsymbol{x}(t) + \boldsymbol{B}_d\boldsymbol{u}_t \quad \forall t = 0,1,2,\ldots, \tag{12}$$

  where

$$\boldsymbol{A}_d := e^{\boldsymbol{A}}, \quad \boldsymbol{B}_d := \boldsymbol{A}^{-1}(e^{\boldsymbol{A}} - \boldsymbol{I})\boldsymbol{B}$$

  - (12) is a discretized version of the continuous model (7)
  - Conversely, if a discrete-time dynamical system of the form

$$\boldsymbol{x}_{t+1} = \boldsymbol{A}\boldsymbol{x}_t + \boldsymbol{B}\boldsymbol{u}_t \quad \forall t = 0,1,2,\ldots$$

  is given, one can transform it into a corresponding continuous-time model as

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}_c\boldsymbol{x}(t) + \boldsymbol{B}_c\boldsymbol{u}(t) \quad \forall t \in \mathbb{R}_+,$$

  where

$$\boldsymbol{A}_c := \ln(\boldsymbol{A}), \quad \boldsymbol{B}_c = (\boldsymbol{A} - \boldsymbol{I})^{-1}\ln(\boldsymbol{A})\boldsymbol{B}$$