Nonlinear models

Introduction to dynamical systems #8

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1 Nonlinear dynamical system

1.1 Continuous-time model

- **General Dynamical System**
  - A (time-invariant) continuous-time *dynamical system* is a system of differential equations of the form
$$\frac{d}{dt}\boldsymbol{x}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t)) \quad t \in \mathbb{R}_+,$$
    where
    - $\boldsymbol{x}(t) \in \mathbb{R}^m$: state vector at time t
    - $\boldsymbol{u}(t) \in \mathbb{R}^n$: control (input) vector at time t
    - $\boldsymbol{f} \in \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$: vector-valued function
  - A linear dynamical system is a special case where
$$\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$
  - Starting with some *initial state* $\boldsymbol{x}(0)$, we want to know how $\boldsymbol{x}(t)$ evolves over time depending on $\boldsymbol{f}$ and $\boldsymbol{u}$.
- **Equilibrium**
  - Consider the case where the control input remains constant at $\bar{\boldsymbol{u}}$:
$$\frac{d}{dt}\boldsymbol{x}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \bar{\boldsymbol{u}}) \quad t \in \mathbb{R}_+ \tag{1}$$
  - An *equilibrium point* $\bar{\boldsymbol{x}}$ of (1) is defined as a solution to
$$\boldsymbol{0} = \boldsymbol{f}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}}),$$
    which depends on both the system dynamics $\boldsymbol{f}$ and the control input $\bar{\boldsymbol{u}}$
- **Stability**
  - An equilibrium point $\bar{\boldsymbol{x}}$ of (1) is said to be *stable* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that
$$\|\boldsymbol{x}(0) - \bar{\boldsymbol{x}}\| < \delta \implies \|\boldsymbol{x}(t) - \bar{\boldsymbol{x}}\| < \varepsilon \quad \forall t > 0,$$

namely, the state trajectory stays arbitrarily close to the equilibrium point as long as the initial state is close enough to the equilibrium point

- An equilibrium point $\bar{\boldsymbol{x}}$ of (1) is said to be *asymptotically stable* if a) it is stable (as define above), and b) there exists $\bar{\delta} > 0$ such that

$$\|\boldsymbol{x}(0) - \bar{\boldsymbol{x}}\| < \bar{\delta} \implies \lim_{t\to\infty} \boldsymbol{x}(t) = \bar{\boldsymbol{x}},$$

namely, the state trajectory actually converges to the equilibrium point as long as the initial state is close enough to the equilibrium point

- **Example 1**
  - Consider the following one-dimensional dynamical system:

$$\dot{x}(t) = ax(t) + b(x(t))^{\gamma}, \quad ab < 0, \gamma \in \mathbb{R} \setminus \{1\}, \tag{2}$$

which generalizes the linear dynamical system $\dot{x}(t) = ax(t) + b$ where $\gamma = 0$

  - Observe:
    - For $\gamma = 0$ (linear case), the system has a unique equilibrium point:

$$a\bar{x} + b = 0 \iff \bar{x} = -\frac{b}{a}$$

    - For $\gamma \neq 0$, the system has two (trivial and non-trivial) equilibrium points:

$$a\bar{x} + b\bar{x}^{\gamma} = 0 \iff \bar{x} = 0 \text{ or } \bar{x} = \left(-\frac{b}{a}\right)^{\frac{1}{1-\gamma}}$$

    - Since (2) is a Bernoulli differential equation,[1] the exact solution can be derived as

$$x(t) = \left(\left((x(0))^{1-\gamma} + \frac{b}{a}\right) e^{(1-\gamma)at} - \frac{b}{a}\right)^{\frac{1}{1-\gamma}} \tag{3}$$

    - Trivial equilibrium, $\bar{x} = 0$, is asymptotically stable if and only if $\gamma > 1$ and $a < 0$
    - Non-trivial equilibrium, $\bar{x} = (-b/a)^{\frac{1}{1-\gamma}}$, is asymptotically stable if and only if $(1 - \gamma)a < 0$

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[1] Notice that (2) may be rewritten as

$$(x(t))^{-\gamma}\dot{x}(t) = a(x(t))^{1-\gamma} + b \quad \implies \quad \frac{d}{dt}(x(t))^{1-\gamma} = (1-\gamma)a(x(t))^{1-\gamma} + (1-\gamma)b \quad \forall t,$$

which in turn implies

$$\frac{d}{dt}\left\{(x(t))^{1-\gamma} - (\bar{x})^{1-\gamma}\right\} = (1-\gamma)a\left((x(t))^{1-\gamma} - (\bar{x})^{1-\gamma}\right) \quad \forall t, \quad \bar{x} := \left(-\frac{b}{a}\right)^{\frac{1}{1-\gamma}}$$

or

$$\frac{d}{dt}\ln\left((x(t))^{1-\gamma} - (\bar{x})^{1-\gamma}\right) = (1-\gamma)a \quad \forall t.$$

Therefore, integrating both sides over $[0, t]$ yields

$$\ln\left(\frac{(x(t))^{1-\gamma} - (\bar{x})^{1-\gamma}}{(x(0))^{1-\gamma} - (\bar{x})^{1-\gamma}}\right) = (1-\gamma)at \quad \implies \quad (x(t))^{1-\gamma} - (\bar{x})^{1-\gamma} = \left((x(0))^{1-\gamma} - (\bar{x})^{1-\gamma}\right)e^{(1-\gamma)at},$$

which gives (3).

- **Example 2**
  - Consider the following two-dimensional dynamical system:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} a_1 x_1(t) - b_1(x_1(t) + x_2(t))x_1(t) \\ a_2 x_2(t) - b_2(x_1(t) + x_2(t))x_2(t) \end{bmatrix}, \quad a_i, b_i > 0, \quad \frac{a_1}{b_1} \neq \frac{a_2}{b_2} \tag{4}$$

  - Observe:
    - The system has a trivial equilibrium point, $\bar{x}_1 = \bar{x}_2 = 0$
    - There is no equilibrium point where both $\bar{x}_1$ and $\bar{x}_2$ are non-zero,[2] indicating that at least one of them must be zero
    - It follows that there are only two non-trivial equilibrium points:

$$\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} a_1/b_1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ a_2/b_2 \end{bmatrix}$$

    - Are these equilibrium points stable? If so, in what condition?

1.2 Discrete-time model

- **General dynamical system**
  - A (time-invariant) discrete-time *dynamical system* is a system of difference equations of the form

$$\boldsymbol{x}_{t+1} = \boldsymbol{f}(\boldsymbol{x}_t, \boldsymbol{u}_t) \quad t = 0, 1, 2, \ldots,$$

    where
    - $\boldsymbol{x}_t \in \mathbb{R}^m$: state vector at t
    - $\boldsymbol{u}_t \in \mathbb{R}^n$: control (input) vector at t
    - $\boldsymbol{f} \in \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$: vector-valued function
  - A linear dynamical system is a special case where

$$\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$

  - Starting with some *initial state* $\boldsymbol{x}_0$, we want to know how $\boldsymbol{x}_t$ evolves over time depending on $\boldsymbol{f}$ and $\boldsymbol{u}$

- **Equilibrium**
  - Consider the case where the control is constant at $\bar{\boldsymbol{u}}$:

$$\boldsymbol{x}_{t+1} = \boldsymbol{f}(\boldsymbol{x}_t, \bar{\boldsymbol{u}}) \quad t = 0, 1, 2, \ldots, \tag{5}$$

  - We define an *equilibrium point* of (5) as $\bar{\boldsymbol{x}}$ that solves

$$\bar{\boldsymbol{x}} = \boldsymbol{f}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})$$

    which obviously depends on both $\boldsymbol{f}$ and $\bar{\boldsymbol{u}}$

[2] If $(\bar{x}_1, \bar{x}_2)$ is an equilibrium point with $\bar{x}_1 \neq 0$ and $x_2 \neq 0$, then (4) implies

$$a_1 + b_1(\bar{x}_1 + \bar{x}_2) = \frac{\dot{\bar{x}}_1}{\bar{x}_1} = 0 = \frac{\dot{\bar{x}}_2}{\bar{x}_2} = a_2 + b_2(\bar{x}_1 + \bar{x}_2),$$

which is only possible when $\frac{a_1}{b_1} = \frac{a_2}{b_2}$, a contradiction.

- **Stability**
  - An equilibrium point $\bar{\boldsymbol{x}}$ of (5) is said to be *stable* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that
$$\|\boldsymbol{x}_0 - \bar{\boldsymbol{x}}\| < \delta \implies \|\boldsymbol{x}_t - \bar{\boldsymbol{x}}\| < \varepsilon \quad \forall t = 1, 2, \ldots$$
  - An equilibrium point $\bar{\boldsymbol{x}}$ of (5) is said to be *asymptotically stable* if a) it is stable, and b) there exists $\bar{\delta} > 0$ such that
$$\|\boldsymbol{x}_0 - \bar{\boldsymbol{x}}\| < \bar{\delta} \implies \lim_{t\to\infty} \boldsymbol{x}_t = \bar{\boldsymbol{x}},$$
- **Example 3**
  - Consider the following one-dimensional dynamical system:
$$x_{t+1} = \frac{ax_t + b}{cx_t + 1}, \quad a \neq 1, \tag{6}$$
which generalizes the linear dynamical system $x_{t+1} = ax_t + b$ where $c = 0$
  - Observe:
    - If $c = 0$, the system has a unique equilibrium point:
$$\bar{x} = a\bar{x} + b \iff \bar{x} = \frac{b}{1-a}$$
    - If $c \neq 0$, the system has two equilibrium points:
$$\bar{x} = \frac{a\bar{x} + b}{c\bar{x} + 1} \iff \bar{x} = \begin{cases} -\frac{1}{2}\frac{1-a}{c} + \sqrt{\left(\frac{1}{2}\frac{1-a}{c}\right)^2 + \frac{b}{c}} =: \bar{x}_+ > 0 \\ -\frac{1}{2}\frac{1-a}{c} - \sqrt{\left(\frac{1}{2}\frac{1-a}{c}\right)^2 + \frac{b}{c}} =: \bar{x}_- < 0 \end{cases}$$
  - One may rewrite (6) as
$$\frac{1}{x_{t+1} - \bar{x}} = \underbrace{\frac{1 + c\bar{x}}{a - c\bar{x}}}_{=:\alpha} \frac{1}{x_t - \bar{x}} + \frac{c}{a - c\bar{x}}, \quad \forall \bar{x} \in \{\bar{x}_+, \bar{x}_-\},$$
which implies that the state trajectory satisfies
$$\frac{1}{x_t - \bar{x}} = \alpha^t \frac{1}{x_0 - \bar{x}} + \frac{1 - \alpha^t}{1 - \alpha} \frac{c}{a - c\bar{x}},$$
or
$$x_t = \bar{x} + \left(\frac{1}{x_0 - \bar{x}}\alpha^t + \frac{1 - \alpha^t}{1 - \alpha}\frac{c}{a - c\bar{x}}\right)^{-1}, \quad \bar{x} \in \{\bar{x}_+, \bar{x}_-\} \tag{7}$$
where
$$\alpha = \frac{1 + c\bar{x}}{a - c\bar{x}} = \begin{cases} \frac{1+c\bar{x}_+}{a-c\bar{x}_+} = \frac{\frac{1}{2}\frac{1+a}{c} + \sqrt{\left(\frac{1}{2}\frac{1-a}{c}\right)^2 + \frac{b}{c}}}{\frac{1}{2}\frac{1+a}{c} - \sqrt{\left(\frac{1}{2}\frac{1-a}{c}\right)^2 + \frac{b}{c}}} & \text{if } \bar{x} = \bar{x}_+ \\ \frac{1+c\bar{x}_-}{a-c\bar{x}_-} = \frac{\frac{1}{2}\frac{1+a}{c} - \sqrt{\left(\frac{1}{2}\frac{1-a}{c}\right)^2 + \frac{b}{c}}}{\frac{1}{2}\frac{1+a}{c} + \sqrt{\left(\frac{1}{2}\frac{1-a}{c}\right)^2 + \frac{b}{c}}} & \text{if } \bar{x} = \bar{x}_- \end{cases}$$
  - It follows from (7) that an equilibrium point $\bar{x}$ is asymptotically stable if and only if
$$\lim_{t\to\infty}\left(\frac{1}{x_0 - \bar{x}}\alpha^t + \frac{1 - \alpha^t}{1 - \alpha}\frac{c}{a - c\bar{x}}\right) = \pm\infty \text{ if } x_0 \text{ is close enough to } \bar{x} \iff |\alpha| > 1,$$
from which we conclude that $\bar{x}_+$ is asymptotically stable whereas $\bar{x}_-$ is not

- **Example 4**
  - Consider the following two-dimensional dynamical system:

$$\begin{bmatrix} x_{1,t+1} \\ x_{2,t+1} \end{bmatrix} = \begin{bmatrix} a_1 x_{1,t} + b_1 x_{2,t}^2 \\ b_2 x_{1,t} + a_2 x_{2,t} \end{bmatrix}, \quad a_1, a_2 \neq 1, \quad b_1, b_2 \neq 0 \tag{8}$$

  - Observe:
    - There are two equilibrium points:

$$\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} a_1 \bar{x}_1 + b_1 \bar{x}_2^2 \\ b_2 \bar{x}_1 + a_2 \bar{x}_2 \end{bmatrix} \iff \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{1-a_1}{b_1} \left(\frac{1-a_2}{b_2} \right)^2 \\ \frac{1-a_1}{b_1} \frac{1-a_2}{b_2} \end{bmatrix}$$

    - Are these equilibrium points stable? If so, in what condition?

2 Linearization

2.1 Continuous-time model

- **Linearized dynamical system**
  - Consider the following dynamical system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t)) \quad \text{or} \quad \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_m(t) \end{bmatrix} = \begin{bmatrix} f_1(\boldsymbol{x}(t), \boldsymbol{u}(t)) \\ f_2(\boldsymbol{x}(t), \boldsymbol{u}(t)) \\ \vdots \\ f_m(\boldsymbol{x}(t), \boldsymbol{u}(t)) \end{bmatrix} \tag{9}$$

  - Let $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})$ be *an* equilibrium point of the system

$$\boldsymbol{f}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}}) = \boldsymbol{0}$$

  - If $(\boldsymbol{x}(t), \boldsymbol{u}(t))$ is close to $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})$,

$$\boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t)) \approx \underbrace{\boldsymbol{f}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})}_{=\boldsymbol{0}} + \frac{d\boldsymbol{f}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})}{d\boldsymbol{x}} (\boldsymbol{x}(t) - \bar{\boldsymbol{x}}) + \frac{d\boldsymbol{f}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})}{d\boldsymbol{u}} (\boldsymbol{u}(t) - \bar{\boldsymbol{u}})$$

    where

$$\frac{d\boldsymbol{f}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})}{d\boldsymbol{x}} = \begin{bmatrix} \frac{\partial f_1(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})}{\partial x_1} & \frac{\partial f_1(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})}{\partial x_2} & \cdots & \frac{\partial f_1(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})}{\partial x_m} \\ \frac{\partial f_2(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})}{\partial x_1} & \frac{\partial f_2(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})}{\partial x_2} & \cdots & \frac{\partial f_2(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})}{\partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})}{\partial x_1} & \frac{\partial f_m(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})}{\partial x_2} & \cdots & \frac{\partial f_m(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})}{\partial x_m} \end{bmatrix}, \quad \frac{d\boldsymbol{f}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})}{d\boldsymbol{u}} = \begin{bmatrix} \frac{\partial f_1(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})}{\partial u_1} & \frac{\partial f_1(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})}{\partial x_2} & \cdots & \frac{\partial f_1(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})}{\partial u_n} \\ \frac{\partial f_2(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})}{\partial u_1} & \frac{\partial f_2(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})}{\partial x_2} & \cdots & \frac{\partial f_2(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})}{\partial u_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})}{\partial u_1} & \frac{\partial f_m(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})}{\partial x_2} & \cdots & \frac{\partial f_m(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})}{\partial u_n} \end{bmatrix}$$

  - Hence, in a neighborhood of $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})$, dynamical system (9) can be approximated by the following *linear* system

$$\frac{d}{dt}(\boldsymbol{x}(t) - \bar{\boldsymbol{x}}) = \boldsymbol{A}(\boldsymbol{x}(t) - \bar{\boldsymbol{x}}) + \boldsymbol{B}(\boldsymbol{u}(t) - \bar{\boldsymbol{u}}), \quad \text{where} \quad \boldsymbol{A} := \frac{d\boldsymbol{f}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})}{d\boldsymbol{x}}, \quad \boldsymbol{B} := \frac{d\boldsymbol{f}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})}{d\boldsymbol{u}}$$

  - Stability of each equilibrium point and the state trajectory around it can be characterized based on the system matrix $\boldsymbol{A}$
  - Since $\boldsymbol{A}$ depends on $\bar{\boldsymbol{x}}$, we need to use different $\boldsymbol{A}$ for analyzing stability of different equilibrium points $\bar{\boldsymbol{x}}$

- **Example 1**
  - Let us revisit Example 1 and consider the following one-dimensional dynamical system:
$$\dot{x}(t) = \underbrace{ax(t) + b(x(t))^\gamma}_{f(x(t))}, \quad ab < 0, \gamma \in \mathbb{R} \setminus \{1\}$$
  - Linearly approximating the system around an equilibrium point $\bar{x}$ yields
$$\frac{d}{dt}(\bar{x}(t) - \bar{x}) = \underbrace{\left(a + \gamma b(\bar{x})^{\gamma-1}\right)}_{f'(\bar{x})}(x(t) - \bar{x}),$$
    which indicates that the equilibrium point is asymptotically stable if and only if
$$f'(\bar{x}) = a + \gamma b(\bar{x})^{\gamma-1} < 0$$
  - Assume $\gamma \neq 0$ so that the system has two equilibrium points:
$$\bar{x} = 0 \quad \text{and} \quad \bar{x} = \left(-\frac{b}{a}\right)^{\frac{1}{1-\gamma}}$$
  - Observe that
    - $\bar{x} = 0$ is asymptotically stable if and only if
$$a + \gamma b(\bar{x})^{\gamma-1}\big|_{\bar{x}=0} < 0 \iff a < 0 \text{ and } \gamma > 1$$
    - $\bar{x} = (-b/a)^{\frac{1}{1-\gamma}}$ is asymptotically stable if and only if
$$a + \gamma b\left(\bar{x}\right)^{\gamma-1}\Big|_{\bar{x}=(-b/a)^{\frac{1}{1-\gamma}}} < 0 \iff (1-\gamma)a < 0$$
  - The linearized model allows us to reach the same conclusion without explicitly solving for the state trajectory

- **Example 2**
  - Let us revisit Example 2 and consider the following two-dimensional dynamical system:
$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} a_1x_1(t) - b_1(x_1(t) + x_2(t))x_1(t) \\ a_2x_2(t) - b_2(x_1(t) + x_2(t))x_2(t) \end{bmatrix}}_{f(\boldsymbol{x}(t))}, \quad a_i, b_i > 0, \quad \frac{a_1}{b_1} \neq \frac{a_2}{b_2},$$
    which we know has three equilibrium points:
$$\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} a_1/b_1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ a_2/b_2 \end{bmatrix}$$
  - Linearly approximating the system around an equilibrium point $\bar{\boldsymbol{x}}$ yields
$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} a_1 - 2b_1\bar{x}_1 - b_1\bar{x}_2 & -b_1\bar{x}_1 \\ -b_2\bar{x}_2 & a_2 - 2b_2\bar{x}_2 - b_2\bar{x}_1 \end{bmatrix}}_{\frac{df(\bar{x})}{dx}} \begin{bmatrix} x_1(t) - \bar{x}_1 \\ x_2(t) - \bar{x}_2 \end{bmatrix}$$
    which indicates that the equilibrium point is asymptotically stable if and only if
$$\rho\left(e^{\frac{df(\bar{x})}{dx}}\right) < 1 \iff \text{eigenvalues of } \frac{d\boldsymbol{f}(\bar{\boldsymbol{x}})}{d\boldsymbol{x}} \text{ are all negative}$$

- For $\bar{\boldsymbol{x}} = (0,0)^\top$, we have
$$\left.\frac{d\boldsymbol{f}(\bar{\boldsymbol{x}})}{d\boldsymbol{x}}\right|_{\bar{\boldsymbol{x}}=\begin{bmatrix}0\\0\end{bmatrix}} = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$$
and therefore the equilibrium point is not stable because both of the eigenvalues are positive ($a_1 > 0$ and $a_2 > 0$)
- For $\bar{\boldsymbol{x}} = (a_1/b_1, 0)^\top$, we have
$$\left.\frac{d\boldsymbol{f}(\bar{\boldsymbol{x}})}{d\boldsymbol{x}}\right|_{\bar{\boldsymbol{x}}=\begin{bmatrix}a_1/b_1\\0\end{bmatrix}} = \begin{bmatrix} -a_1 & -a_1 \\ 0 & -\left(\frac{a_1}{b_1} - \frac{a_2}{b_2}\right) b_2 \end{bmatrix}$$
and therefore the equilibrium point is asymptotically stable if and only if $\frac{a_1}{b_1} > \frac{a_2}{b_2}$
- For $\bar{\boldsymbol{x}} = (0, a_2/b_2)^\top$, we have
$$\left.\frac{d\boldsymbol{f}(\bar{\boldsymbol{x}})}{d\boldsymbol{x}}\right|_{\bar{\boldsymbol{x}}=\begin{bmatrix}0\\a_2/b_2\end{bmatrix}} = \begin{bmatrix} -\left(\frac{a_2}{b_2} - \frac{a_1}{b_1}\right) b_1 & 0 \\ -a_2 & -a_2 \end{bmatrix}$$
and therefore the equilibrium point is asymptotically stable if and only if $\frac{a_2}{b_2} > \frac{a_1}{b_1}$

2.2 Discrete-time model

- **Linearized dynamical system**
  - Consider the following dynamical system
$$\boldsymbol{x}_{t+1} = \boldsymbol{f}(\boldsymbol{x}_t, \boldsymbol{u}_t) \quad \text{or} \quad \begin{bmatrix} x_{1,t+1} \\ x_{2,t+1} \\ \vdots \\ x_{m,t+1} \end{bmatrix} = \begin{bmatrix} f_1(\boldsymbol{x}_t, \boldsymbol{u}_t) \\ f_2(\boldsymbol{x}_t, \boldsymbol{u}_t) \\ \vdots \\ f_m(\boldsymbol{x}_t, \boldsymbol{u}_t) \end{bmatrix} \tag{10}$$
  - Let $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})$ be *an* equilibrium point of the system: $\boldsymbol{f}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}}) = \bar{\boldsymbol{x}}$
  - If $(\boldsymbol{x}_t, \boldsymbol{u}_t)$ is close to $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})$,
$$\boldsymbol{f}(\boldsymbol{x}_t, \boldsymbol{u}_t) \approx \underbrace{\boldsymbol{f}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})}_{=\bar{\boldsymbol{x}}} + \frac{d\boldsymbol{f}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})}{d\boldsymbol{x}}(\boldsymbol{x}_t - \bar{\boldsymbol{x}}) + \frac{d\boldsymbol{f}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})}{d\boldsymbol{u}}(\boldsymbol{u}_t - \bar{\boldsymbol{u}})$$
where
$$\frac{d\boldsymbol{f}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})}{d\boldsymbol{x}} = \begin{bmatrix} \frac{\partial f_1(\bar{\boldsymbol{x}},\bar{\boldsymbol{u}})}{\partial x_1} & \frac{\partial f_1(\bar{\boldsymbol{x}},\bar{\boldsymbol{u}})}{\partial x_2} & \cdots & \frac{\partial f_1(\bar{\boldsymbol{x}},\bar{\boldsymbol{u}})}{\partial x_m} \\ \frac{\partial f_2(\bar{\boldsymbol{x}},\bar{\boldsymbol{u}})}{\partial x_1} & \frac{\partial f_2(\bar{\boldsymbol{x}},\bar{\boldsymbol{u}})}{\partial x_2} & \cdots & \frac{\partial f_2(\bar{\boldsymbol{x}},\bar{\boldsymbol{u}})}{\partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m(\bar{\boldsymbol{x}},\bar{\boldsymbol{u}})}{\partial x_1} & \frac{\partial f_m(\bar{\boldsymbol{x}},\bar{\boldsymbol{u}})}{\partial x_2} & \cdots & \frac{\partial f_m(\bar{\boldsymbol{x}},\bar{\boldsymbol{u}})}{\partial x_m} \end{bmatrix}, \quad \frac{d\boldsymbol{f}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})}{d\boldsymbol{u}} = \begin{bmatrix} \frac{\partial f_1(\bar{\boldsymbol{x}},\bar{\boldsymbol{u}})}{\partial u_1} & \frac{\partial f_1(\bar{\boldsymbol{x}},\bar{\boldsymbol{u}})}{\partial x_2} & \cdots & \frac{\partial f_1(\bar{\boldsymbol{x}},\bar{\boldsymbol{u}})}{\partial u_n} \\ \frac{\partial f_2(\bar{\boldsymbol{x}},\bar{\boldsymbol{u}})}{\partial u_1} & \frac{\partial f_2(\bar{\boldsymbol{x}},\bar{\boldsymbol{u}})}{\partial x_2} & \cdots & \frac{\partial f_2(\bar{\boldsymbol{x}},\bar{\boldsymbol{u}})}{\partial u_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m(\bar{\boldsymbol{x}},\bar{\boldsymbol{u}})}{\partial u_1} & \frac{\partial f_m(\bar{\boldsymbol{x}},\bar{\boldsymbol{u}})}{\partial x_2} & \cdots & \frac{\partial f_m(\bar{\boldsymbol{x}},\bar{\boldsymbol{u}})}{\partial u_n} \end{bmatrix}$$
  - Hence, in a neighborhood of $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})$, dynamical system (10) can be approximated by the following linear system
$$\boldsymbol{x}_{t+1} - \bar{\boldsymbol{x}} = \boldsymbol{A}(\boldsymbol{x}_t - \bar{\boldsymbol{x}}) + \boldsymbol{B}(\boldsymbol{u}_t - \bar{\boldsymbol{u}}), \quad \text{where} \quad \boldsymbol{A} := \frac{d\boldsymbol{f}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})}{d\boldsymbol{x}}, \quad \boldsymbol{B} := \frac{d\boldsymbol{f}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})}{d\boldsymbol{u}}$$
  - Stability of each equilibrium point and the state trajectory around it can be characterized based on the system matrix $\boldsymbol{A}$
  - Note that we need to use different $\boldsymbol{A}$ for different equilibrium point

- **Example 3**
  - Let us revisit Example 3 and consider the following one-dimensional dynamical system:

$$x_{t+1} = \underbrace{\frac{ax_t + b}{cx_t + 1}}_{f(x_t)}, \quad a \neq 1, \quad c \neq 0,$$

  which we know has two equilibrium points:

$$\bar{x} = \frac{a\bar{x} + b}{c\bar{x} + 1} \iff \bar{x} = \begin{cases} -\frac{1}{2}\frac{1-a}{c} + \sqrt{\left(\frac{1}{2}\frac{1-a}{c}\right)^2 + \frac{b}{c}} =: \bar{x}_+ > 0 \\ -\frac{1}{2}\frac{1-a}{c} - \sqrt{\left(\frac{1}{2}\frac{1-a}{c}\right)^2 + \frac{b}{c}} =: \bar{x}_- < 0 \end{cases}$$

  - Linearly approximating the system around an equilibrium point $\bar{x}$ yields

$$x_{t+1} - \bar{x} = \underbrace{\frac{a - c\bar{x}}{c\bar{x} + 1}}_{f'(\bar{x})}(x_t - \bar{x}),$$

  which indicates that the equilibrium point is asymptotically stable if and only if

$$\left|f'(\bar{x})\right| < 1 \iff \left|\frac{a - c\bar{x}}{c\bar{x} + 1}\right| < 1 \iff \left|\frac{1 + c\bar{x}}{a - c\bar{x}}\right| > 1,$$

  - Since

$$\left|\frac{1 + c\bar{x}}{a - c\bar{x}}\right| = \begin{cases} \left|\frac{\frac{1}{2}\frac{1+a}{c} + \sqrt{\left(\frac{1}{2}\frac{1-a}{c}\right)^2 + \frac{b}{c}}}{\frac{1}{2}\frac{1+a}{c} - \sqrt{\left(\frac{1}{2}\frac{1-a}{c}\right)^2 + \frac{b}{c}}}\right| > 1 & \text{if } \bar{x} = \bar{x}_+ \\ \left|\frac{\frac{1}{2}\frac{1+a}{c} - \sqrt{\left(\frac{1}{2}\frac{1-a}{c}\right)^2 + \frac{b}{c}}}{\frac{1}{2}\frac{1+a}{c} + \sqrt{\left(\frac{1}{2}\frac{1-a}{c}\right)^2 + \frac{b}{c}}}\right| < 1 & \text{if } \bar{x} = \bar{x}_- \end{cases}$$

  we conclude that $\bar{x}_+$ is asymptotically stable whereas $\bar{x}_-$ is not stable
  - The linearized model allows us to reach the same conclusion without explicitly solving for the state trajectory

- **Example 4**
  - Let us revisit Example 4 and consider the following two-dimensional dynamical system:

$$\begin{bmatrix} x_{1,t+1} \\ x_{2,t+1} \end{bmatrix} = \underbrace{\begin{bmatrix} a_1 x_{1,t} + b_1 x_{2,t}^2 \\ b_2 x_{1,t} + a_2 x_{2,t} \end{bmatrix}}_{f(x_t)}, \quad a_1, a_2 \neq 1, \quad b_1, b_2 \neq 0,$$

  which we know has two equilibrium points

$$\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{1-a_1}{b_1}\left(\frac{1-a_2}{b_2}\right)^2 \\ \frac{1-a_1}{b_1}\frac{1-a_2}{b_2} \end{bmatrix}$$

  - We are not able to explicitly solve for the state trajectory here so we turn to linearization
  - To fix context, assume $|a_1| < 1$ and $|a_2| < 1$

- Linearly approximating the system around an equilibrium point $\bar{\boldsymbol{x}}$ yields

$$\begin{bmatrix} x_{1,t+1} - \bar{x}_1 \\ x_{2,t+1} - \bar{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} a_1 & 2b_1\bar{x}_2 \\ b_2 & a_2 \end{bmatrix}}_{\frac{df(\bar{x})}{dx}} \begin{bmatrix} x_{1,t} - \bar{x}_1 \\ x_{2,t} - \bar{x}_2 \end{bmatrix},$$

  which indicates that the equilibrium point is asymptotically stable if and only if

$$\rho\left(\frac{d\boldsymbol{f}(\bar{\boldsymbol{x}})}{d\boldsymbol{x}}\right) < 1 \iff \text{eigenvalues of } \frac{d\boldsymbol{f}(\bar{\boldsymbol{x}})}{d\boldsymbol{x}} \text{ are within unit circle}$$

- For $\bar{\boldsymbol{x}} = (0,0)^\top$, we have

$$\left.\frac{d\boldsymbol{f}(\bar{\boldsymbol{x}})}{d\boldsymbol{x}}\right|_{\bar{\boldsymbol{x}}=\begin{bmatrix}0\\0\end{bmatrix}} = \begin{bmatrix} a_1 & 0 \\ b_2 & a_2 \end{bmatrix}$$

  and therefore the equilibrium point is asymptotically stable (since $|a_1| < 1$ and $|a_2| < 1$)

- For $\bar{\boldsymbol{x}} = \left(\frac{1-a_1}{b_1}\left(\frac{1-a_2}{b_2}\right)^2, \frac{1-a_1}{b_1}\frac{1-a_2}{b_2}\right)^\top$, we have

$$\left.\frac{d\boldsymbol{f}(\bar{\boldsymbol{x}})}{d\boldsymbol{x}}\right|_{\bar{\boldsymbol{x}}=\begin{bmatrix}\frac{1-a_1}{b_1}\left(\frac{1-a_2}{b_2}\right)^2\\ \frac{1-a_1}{b_1}\frac{1-a_2}{b_2}\end{bmatrix}} = \begin{bmatrix} a_1 & 2\frac{(1-a_1)(1-a_2)}{b_2} \\ b_2 & a_2 \end{bmatrix}$$

  and observe

  - The characteristic polynomial is

$$\phi_{\frac{df(\bar{x})}{dx}}(t) = (a_1 - t)(a_2 - t) - 2(1-a_1)(1-a_2)$$

    and in particular

$$\phi_{\frac{df(\bar{x})}{dx}}(1) = -(1-a_1)(1-a_2) < 0$$

  - Hence, one of the eigenvalue is strictly greater than 1, implying that the equilibrium point is not stable