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Jordan normal form

Introduction to dynamical systems #3

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1 Jordan normal form

1.1 Jordan block and Jordan matrix

- **Jordan block**
  - A square matrix of the following form is called a *Jordan block* of size m:

$$J_m(\lambda) := \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{bmatrix} \in \mathbb{R}^{m\times m}$$

  - For example,

$$J_1(\lambda) = \lambda, \quad J_2(\lambda) = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \quad J_3(\lambda) = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

- **Jordan matrix**
  - A block diagonal matrix of the form

$$J = \begin{bmatrix} J_{n_1}(\lambda_1) & 0 & \cdots & 0 \\ 0 & J_{n_2}(\lambda_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_{n_d}(\lambda_d) \end{bmatrix}$$

  is called a *Jordan matrix*, where $\lambda_1, \lambda_2, \ldots, \lambda_d$ can be the same value or different values
  - Examples of a Jordan matrix:

$$J = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \quad J = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}, \quad J = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad J = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

1.2 Generalized eigenvectors

- **Definition**
  - Let λ be an eigenvalue of $\boldsymbol{A} \in \mathbb{R}^{n\times n}$
  - We call $\boldsymbol{v} \in \mathbb{R}^n$ a *generalized eigenvector of rank* $m \in \mathbb{N}$ associated with λ if

$$(\boldsymbol{A} - \lambda \boldsymbol{I})^m \boldsymbol{v} = \boldsymbol{0} \quad \text{and} \quad (\boldsymbol{A} - \lambda \boldsymbol{I})^{m-1} \boldsymbol{v} \neq \boldsymbol{0}$$

  - Note that an eigenvector is nothing but a generalized eigenvector of rank $m = 1$

$$(\boldsymbol{A} - \lambda \boldsymbol{I})^1 \boldsymbol{v} = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{v} = (\boldsymbol{A} - \lambda \boldsymbol{I})^{1-1} \boldsymbol{v} \neq \boldsymbol{0}$$

  - Important fact:
    - An eigenvalue does **not always** have as many linearly independent **eigenvectors** as its algebraic multiplicity
    - Any eigenvalue **always** has as many linearly independent **generalized eigenvectors** as its algebraic multiplicity
- **Example 1**
  - Consider a 3×3 square matrix

$$\boldsymbol{A} := \begin{bmatrix} 3 & -1 & 1 \\ 2 & 0 & 2 \\ 1 & -1 & 3 \end{bmatrix},$$

whose characteristic polynomial is

$$\phi_{\boldsymbol{A}}(t) = \begin{vmatrix} 3-t & -1 & 1 \\ 2 & 0-t & 2 \\ 1 & -1 & 3-t \end{vmatrix} = (2-t)^3,$$

meaning that $\lambda := 1$ is the unique eigenvalue of $\boldsymbol{A}$ (with algebraic multiplicity of 3)

  - Then

$$\boldsymbol{v} := \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

is a generalized eigenvector of rank 2 associated with $\lambda = 2$, because

$$(\boldsymbol{A} - \lambda \boldsymbol{I})^2 \boldsymbol{v} = \begin{bmatrix} 3-2 & -1 & 1 \\ 2 & 0-2 & 2 \\ 1 & -1 & 3-2 \end{bmatrix}^2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \boldsymbol{0}$$

and

$$(\boldsymbol{A} - \lambda \boldsymbol{I}) \boldsymbol{v} = \begin{bmatrix} 3-2 & -1 & 1 \\ 2 & 0-2 & 2 \\ 1 & -1 & 3-2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \neq \boldsymbol{0}$$

  - In fact, since $(\boldsymbol{A} - \lambda \boldsymbol{I})^2 = \boldsymbol{O}$, any vector $\boldsymbol{v} = (v_1, v_2, v_3)^\top$ is a generalized eigenvector of rank 2 associated with $\lambda = 2$ as long as $(\boldsymbol{A} - \lambda \boldsymbol{I}) \boldsymbol{v} \neq \boldsymbol{0}$ (i.e., as long as $\boldsymbol{v}$ is not an eigenvector itself)

1.3 Jordan decomposition

- **Jordan decomposition theorem**
  - Consider a square matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$
  - There always exists a Jordan matrix $\boldsymbol{J} \in \mathbb{R}^{n\times n}$ and a nonsingular matrix $\boldsymbol{V} \in \mathbb{R}^{n\times n}$ (consisting of n generalized eigenvectors) such that

$$\boldsymbol{A} = \boldsymbol{V}\boldsymbol{J}\boldsymbol{V}^{-1}, \quad \text{where} \quad \boldsymbol{J} = \begin{bmatrix} \boldsymbol{J}_{n_1}(\lambda_1) & 0 & \cdots & 0 \\ 0 & \boldsymbol{J}_{n_2}(\lambda_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{J}_{n_d}(\lambda_d) \end{bmatrix}$$

    and $\lambda_1, \lambda_2, \ldots, \lambda_d$ are eigenvalues (with possibility of duplicates) of $\boldsymbol{A}$
  - The matrix $\boldsymbol{J}$ above is called the *Jordan normal form* of $\boldsymbol{A}$

- **More details**
  - There exist $\lambda_1, \lambda_2, \ldots, \lambda_d \in \mathbb{R}$ and $n_1, n_2, \ldots, n_d \in \mathbb{N}$ such that
    1. $\lambda_1, \lambda_2, \ldots, \lambda_d$ are eigenvalues of $\boldsymbol{A}$ (with possibility of duplicates)
    2. $n_1 + n_2 + \ldots + n_d = n$
    3. for each $i = 1, \ldots, d$, there exist n_i generalized eigenvectors $\boldsymbol{v}_{i,1}, \boldsymbol{v}_{i,2}, \ldots, \boldsymbol{v}_{i,n_i}$ associated with λ_i such that
       - $\boldsymbol{v}_{i,1}$ is a generalized eigenvector of rank 1 satisfying $(\boldsymbol{A} - \lambda_i \boldsymbol{I})\boldsymbol{v}_{i,1} = \boldsymbol{0}$ and $\boldsymbol{v}_{i,1} \neq \boldsymbol{0}$
       - $\boldsymbol{v}_{i,2}$ is a generalized eigenvector of rank 2 satisfying $(\boldsymbol{A} - \lambda_i \boldsymbol{I})\boldsymbol{v}_{i,2} = \boldsymbol{v}_{i,1}$
       - $\boldsymbol{v}_{i,3}$ is a generalized eigenvector of rank 3 satisfying $(\boldsymbol{A} - \lambda_i \boldsymbol{I})\boldsymbol{v}_{i,3} = \boldsymbol{v}_{i,2}$

         $\vdots$
       - $\boldsymbol{v}_{i,n_i}$ is a generalized eigenvector of rank n_i satisfying $(\boldsymbol{A} - \lambda_i \boldsymbol{I})\boldsymbol{v}_{i,n_i} = \boldsymbol{v}_{i,n_i-1}$
  - Then $\boldsymbol{v}_{1,1}, \boldsymbol{v}_{1,2}, \ldots, \boldsymbol{v}_{1,n_1}, \boldsymbol{v}_{2,1}, \ldots, \boldsymbol{v}_{2,n_2}, \ldots, \boldsymbol{v}_{d,n_d}$ are n linearly independent vectors
  - Note that for each $i = 1, 2, \ldots, d$,

$$(\boldsymbol{A} - \lambda_i \boldsymbol{I}) \underbrace{\begin{bmatrix} \boldsymbol{v}_{i,1} & \boldsymbol{v}_{i,2} & \ldots & \boldsymbol{v}_{i,n_i} \end{bmatrix}}_{=:\boldsymbol{V}_i \in \mathbb{R}^{n\times n_i}} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{v}_{i,1} & \ldots & \boldsymbol{v}_{i,n_i-1} \end{bmatrix},$$

    which implies

$$\boldsymbol{V}_i \boldsymbol{J}_{n_i}(\lambda_i) = \boldsymbol{V}_i \left(\lambda_i \boldsymbol{I} + \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 0 & 0 \end{bmatrix} \right) = \lambda_i \boldsymbol{V}_i + \begin{bmatrix} \boldsymbol{0} & \boldsymbol{v}_{i,1} & \ldots & \boldsymbol{v}_{i,n_i-1} \end{bmatrix} = \boldsymbol{A}\boldsymbol{V}_i$$

  - Hence, defining $\boldsymbol{V} := [\boldsymbol{V}_1 \ \boldsymbol{V}_2 \ \ldots \ \boldsymbol{V}_d]$ gives

$$\boldsymbol{V} \begin{bmatrix} \boldsymbol{J}_{n_1}(\lambda_1) & 0 & \cdots & 0 \\ 0 & \boldsymbol{J}_{n_2}(\lambda_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{J}_{n_d}(\lambda_d) \end{bmatrix} = \begin{bmatrix} \underbrace{\boldsymbol{V}_1 \boldsymbol{J}_{n_1}(\lambda_1)}_{\boldsymbol{A}\boldsymbol{V}_1} & \underbrace{\boldsymbol{V}_2 \boldsymbol{J}_{n_2}(\lambda_2)}_{\boldsymbol{A}\boldsymbol{V}_2} & \cdots & \underbrace{\boldsymbol{V}_d \boldsymbol{J}_{n_d}(\lambda_d)}_{\boldsymbol{A}\boldsymbol{V}_d} \end{bmatrix} = \boldsymbol{A}\boldsymbol{V}$$

  - Since $\boldsymbol{V}$ is nonsingular, we therefore obtain

$$\boldsymbol{A} = \boldsymbol{A}\boldsymbol{V}\boldsymbol{V}^{-1} = \boldsymbol{V}\boldsymbol{J}\boldsymbol{V}^{-1}$$

2 Examples

2.1 2×2 matrix

- **Example 2**
  - Consider a square matrix

$$\boldsymbol{A} := \begin{bmatrix} 5 & 4 \\ -1 & 1 \end{bmatrix}$$

  - We know that
    - the characteristic polynomial of $\boldsymbol{A}$ is

$$\phi_A(t) = \begin{vmatrix} 5-t & 4 \\ -1 & 1-t \end{vmatrix} = (3-t)^2$$

    - $\boldsymbol{A}$ only has one eigenvalue $\lambda_1 := 3$ with $\boldsymbol{v}_{1,1} := (2, -1)^\top$ being an associated eigenvector
    - $\boldsymbol{A}$ is not diagonalizable (no other linearly independent eigenvector)
  - But the Jordan decomposition theorem shows that $\boldsymbol{A}$ is "semi-diagonalizable" in the sense that there exists a non-zero vector $\boldsymbol{v}_{1,2} \in \mathbb{R}^2$ such that $\boldsymbol{v}_{1,1}, \boldsymbol{v}_{1,2}$ are linearly independent and

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{v}_{1,1} & \boldsymbol{v}_{1,2} \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}}_{\boldsymbol{J}} \begin{bmatrix} \boldsymbol{v}_{1,1} & \boldsymbol{v}_{1,2} \end{bmatrix}^{-1},$$

    where $\boldsymbol{J}$ is the Jordan normal form of $\boldsymbol{A}$
  - Procedure to find such $\boldsymbol{v}_{1,2}$
    1. The discussion above indicates that $\boldsymbol{v}_{1,2}$ is a generalized eigenvector of rank 2 associated with λ_1, satisfying

$$(\boldsymbol{A} - \lambda_1 \boldsymbol{I})\boldsymbol{v}_{1,2} = \boldsymbol{v}_{1,1}$$

    2. Hence, we solve the following system of equations for $\boldsymbol{v}$:

$$(\boldsymbol{A} - \lambda_1 \boldsymbol{I})\boldsymbol{v} = \boldsymbol{v}_{1,1} \implies \boldsymbol{v} = \ldots$$

    3. For the current example,

$$\begin{bmatrix} 5-3 & 4 \\ -1 & 1-3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \iff v_1 + 2v_2 = 1 \iff \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -2 \\ 1 \end{bmatrix} \quad \forall \alpha \in \mathbb{R}$$

    4. So we choose the following $\boldsymbol{v}_{1,2}$ as a generalized eigenvector associated with λ_1:

$$\boldsymbol{v}_{1,2} := \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

  - Defining

$$\boldsymbol{V} := \begin{bmatrix} \boldsymbol{v}_{1,1} & \boldsymbol{v}_{1,2} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \implies \boldsymbol{V}^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix},$$

    one can verify that

$$\boldsymbol{V}\boldsymbol{J}\boldsymbol{V}^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ -1 & 1 \end{bmatrix} = \boldsymbol{A}$$

- **Example 3**
  - Consider a square matrix

$$A := \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$$

  - The characteristic polynomial is

$$\phi_A(t) = \begin{vmatrix} 4-t & 1 \\ -1 & 2-t \end{vmatrix} = (4-t)(2-t)+1 = (3-t)^2,$$

    which means A has only one eigenvalue $\lambda_1 := 3$
  - Solving $(A - \lambda_1 I)v = \mathbf{0}$ for v yields

$$\begin{bmatrix} 4-3 & 1 \\ -1 & 2-3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff v_1 = -v_2 \iff \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \forall \alpha$$

    so we choose the following $v_{1,1}$ as an eigenvector associated with λ_1:

$$v_{1,1} := \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

  - Since there is no other linearly independent eigenvector, we seek to find a rank-2 generalized eigenvector $v_{1,2}$ associated with λ_1 such that

$$(A - \lambda_1 I)v_{1,2} = v_{1,1},$$

    which, once we found such $v_{1,2}$, would imply

$$Av_{1,1} = \lambda_1 v_{1,1} = \begin{bmatrix} v_{1,1} & v_{1,2} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ 0 \end{bmatrix} \quad \text{and} \quad Av_{1,2} = v_{1,1} + \lambda_1 v_{1,2} = \begin{bmatrix} v_{1,1} & v_{1,2} \end{bmatrix} \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}$$

    and thus

$$A \begin{bmatrix} v_{1,1} & v_{1,2} \end{bmatrix} = \begin{bmatrix} v_{1,1} & v_{1,2} \end{bmatrix} \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix} \implies A = \begin{bmatrix} v_{1,1} & v_{1,2} \end{bmatrix} \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix} \begin{bmatrix} v_{1,1} & v_{1,2} \end{bmatrix}^{-1}$$

  - Solving $(A - \lambda_1 I)v = v_{1,1}$ for v yields

$$\begin{bmatrix} 4-3 & 1 \\ -1 & 2-3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \iff v_1 = 1 - v_2 \iff \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \forall \alpha$$

    so we choose the following $v_{1,2}$ as a generalized eigenvector associated with λ_1:

$$v_{1,2} := \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

  - Defining

$$V := \begin{bmatrix} v_{1,1} & v_{1,2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \implies V^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix},$$

    one can verify that

$$VJV^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}}_{=J} \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix} = A$$

2.2 3×3 matrix

- **Example 4**
  - Consider a square matrix

$$A := \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 2 & 1 & 5 \end{bmatrix}$$

  - We know that
    - $\phi_A(t) = -(t-1)^2(t-5)$ so eigenvalues are $\lambda_1 := 1$ (twice) and $\lambda_2 := 5$
    - Only one linearly independent eigenvector associated with λ_1, such as $v_{1,1} := (2, 0, -1)^\top$
    - Only one linearly independent eigenvector associated with λ_2, such as $v_{2,1} := (0, 0, 1)^\top$
    - A is not diagonalizable
  - We need another generalized eigenvector which, together with $v_{1,1}$ and $v_{2,1}$, gives us 3 linearly independent vectors in total
  - Since λ_1 has the algebraic multiplicity of 2, it must have two generalized eigenvectors
    1. Use $v_{1,1}$ as the first generalized eigenvector (of rank 1)
    2. Find the second generalized eigenvector, $v_{1,2}$, such that

$$(A - \lambda_1 I)v_{1,2} = v_{1,1},$$

so that A would be decomposed as

$$A = \begin{bmatrix} v_{1,1} & v_{1,2} & v_{2,1} \end{bmatrix} \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} v_{1,1} & v_{1,2} & v_{2,1} \end{bmatrix}^{-1}$$

  - Solving $(A - \lambda_1 I)v = v_{1,1}$ for v yields

$$\begin{bmatrix} 1-1 & 3 & 0 \\ 0 & 1-1 & 0 \\ 2 & 1 & 5-1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \iff \begin{matrix} v_2 = \frac{2}{3} \\ v_1 + 2v_3 = -\frac{5}{6} \end{matrix} \iff \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -\frac{5}{6} \\ \frac{2}{3} \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \quad \forall \alpha$$

  - So we choose the following $v_{1,2}$ as a generalized eigenvector of rank 2 associated with λ_1:

$$v_{1,2} := \begin{bmatrix} -\frac{5}{6} \\ \frac{2}{3} \\ 0 \end{bmatrix}$$

  - Defining

$$V := \begin{bmatrix} v_{1,1} & v_{1,2} & v_{2,1} \end{bmatrix} = \begin{bmatrix} 2 & -5/6 & 0 \\ 0 & 2/3 & 0 \\ -1 & 0 & 1 \end{bmatrix} \implies V^{-1} = \begin{bmatrix} 1/2 & 5/8 & 0 \\ 0 & 3/2 & 0 \\ 1/2 & 5/8 & 1 \end{bmatrix},$$

one can verify that

$$VJV^{-1} = \begin{bmatrix} 2 & -5/6 & 0 \\ 0 & 2/3 & 0 \\ -1 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}}_{=J} \begin{bmatrix} 1/2 & 5/8 & 0 \\ 0 & 3/2 & 0 \\ 1/2 & 5/8 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 2 & 1 & 5 \end{bmatrix} = A$$

- **Example 5**
  - Consider a square matrix
$$A := \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ 0 & -1 & 1 \end{bmatrix}$$
  - The characteristic polynomial is
$$\phi_A(t) = \begin{vmatrix} 2-t & 1 & 1 \\ 1 & 3-t & 2 \\ 0 & -1 & 1-t \end{vmatrix} = -(t-2)^3$$
which means A has only one eigenvalue $\lambda_1 := 2$ (three times)
  - Solving $(A - \lambda_1 I)v = \mathbf{0}$ for v yields
$$\begin{bmatrix} 2-2 & 1 & 1 \\ 1 & 3-2 & 2 \\ 0 & -1 & 1-2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff \begin{matrix} v_1 = v_2 \\ v_3 = -v_2 \end{matrix} \iff \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad \forall \alpha$$
so we choose the following $v_{1,1}$ as an eigenvector associated with λ_1:
$$v_{1,1} := \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$
  - No other linearly independent eigenvector, thus we seek to find two more generalized eigenvectors $v_{1,2}, v_{1,3}$ associated with λ_1 so that
$$\begin{bmatrix} v_{1,1} & v_{1,2} & v_{1,3} \end{bmatrix} \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix} \begin{bmatrix} v_{1,1} & v_{1,2} & v_{1,3} \end{bmatrix}^{-1} = A$$
  - Such generalized eigenvectors must satisfy
$$(A - \lambda_1 I)v_{1,2} = v_{1,1}, \quad \text{and} \quad (A - \lambda_1 I)v_{1,3} = v_{1,2}$$
  - First, solving $(A - \lambda_1 I)v = v_{1,1}$ for v yields
$$\begin{bmatrix} 2-2 & 1 & 1 \\ 1 & 3-2 & 2 \\ 0 & -1 & 1-2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \iff \begin{matrix} v_2 + v_3 = 1 \\ v_1 + v_3 = 0 \end{matrix} \iff \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \quad \forall \alpha$$
so we choose the following $v_{1,2}$ as another generalized eigenvector associated with λ_1:
$$v_{1,2} := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$
  - Given this $v_{1,2}$, we solve $(A - \lambda_1 I)v = v_{1,2}$ for v to obtain
$$\begin{bmatrix} 2-2 & 1 & 1 \\ 1 & 3-2 & 2 \\ 0 & -1 & 1-2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \iff \begin{matrix} v_2 + v_3 = 0 \\ v_1 + v_3 = 1 \end{matrix} \iff \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \quad \forall \alpha$$

so we choose the following $v_{1,3}$ as yet another generalized eigenvector associated with λ_1:

$$v_{1,3} := \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Defining

$$\boldsymbol{V} := \begin{bmatrix} v_{1,1} & v_{1,2} & v_{1,3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \implies \boldsymbol{V}^{-1} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix},$$

one can verify that

$$\boldsymbol{V}\boldsymbol{J}\boldsymbol{V}^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}}_{=\boldsymbol{J}} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ 0 & -1 & 1 \end{bmatrix} = \boldsymbol{A}$$

- **Example 1**
  - Let us revisit Example 1 and consider a square matrix

$$\boldsymbol{A} := \begin{bmatrix} 3 & -1 & 1 \\ 2 & 0 & 2 \\ 1 & -1 & 3 \end{bmatrix},$$

for which we already know that $\lambda := 1$ is the unique eigenvalue

  - Solving $(\boldsymbol{A} - \lambda \boldsymbol{I})v = \boldsymbol{0}$ for v yields

$$(\boldsymbol{A} - \lambda \boldsymbol{I})v = \boldsymbol{0} \iff \begin{bmatrix} 3-2 & -1 & 1 \\ 2 & 0-2 & 2 \\ 1 & -1 & 3-2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff v_2 = v_1 + v_3,$$

implying that any non-zero vector of the form

$$v = \begin{bmatrix} \alpha \\ \alpha + \beta \\ \beta \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \forall \alpha, \beta$$

is an eigenvector associated with λ

  - Observe that
    - one can choose two linearly independent eigenvectors associated with λ, such as

$$v\big|_{(\alpha,\beta)=(1,0)} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad v\big|_{(\alpha,\beta)=(0,1)} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

    - there are many other ways of choosing two linearly independent eigenvectors, like

$$v\big|_{(\alpha,\beta)=(1,1)} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \text{and} \quad v\big|_{(\alpha,\beta)=(1,-1)} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

- When the eigenspace has multiple degrees of freedom (here we have two free parameters, α and β), you need to carefully choose eigenvectors; otherwise you would not be able to find generalized eigenvectors of higher ranks
  - To demonstrate the point, let us say we choose

$$\boldsymbol{v}_{1,1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{v}_{2,1} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \tag{1}$$

    as two linearly independent eigenvectors associated with $\lambda = 2$
  - Then, there does not exist $\boldsymbol{v}_{1,2}$ such that $(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{v}_{1,2} = \boldsymbol{v}_{1,1}$ because

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{v} = \boldsymbol{v}_{1,1} \iff \begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \iff \begin{aligned} v_1 - v_2 + v_3 &= 1 \\ v_1 - v_2 + v_3 &= 1/2 \\ v_1 - v_2 + v_3 &= 0 \end{aligned}$$

  - There is no $\boldsymbol{v}_{2,2}$ to satisfy $(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{v}_{2,2} = \boldsymbol{v}_{2,1}$, either, because

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{v} = \boldsymbol{v}_{2,1} \iff \begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \iff \begin{aligned} v_1 - v_2 + v_3 &= 0 \\ v_1 - v_2 + v_3 &= 1/2 \\ v_1 - v_2 + v_3 &= 1 \end{aligned}$$

  - Hence, if we use (1) as eigenvectors, we cannot find a generalized eigenvector that allows us to complete the Jordan decomposition
- Here is a procedure you can use when the eigenspace of an eigenvalue λ has multiple degrees of freedom:
  1. First, find a generalized eigenvector of highest rank (rank 2, in this example) associated with λ, namely, find $\boldsymbol{v}_{1,2}$ such that

$$(\boldsymbol{A} - \lambda \boldsymbol{I})^2 \boldsymbol{v}_{1,2} = \boldsymbol{0} \quad \text{and} \quad (\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{v}_{1,2} \neq \boldsymbol{0} \tag{2}$$

  2. Then find a generalized eigenvector of lower rank (rank 1, in this example) by defining $\boldsymbol{v}_{1,1}$ as

$$\boldsymbol{v}_{1,1} := (\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{v}_{1,2}, \tag{3}$$

     which is, by construction, an eigenvector associated with λ
  3. Finally, find another eigenvector $\boldsymbol{v}_{2,1}$ that is linearly independent of $\boldsymbol{v}_{1,1}$
- Note that in this procedure, we find (generalized) eivenvectors in the reverse order
  - previously, we first find an eigenvector $\boldsymbol{v}_{i,1}$ and then compute a generalized eigenvector $\boldsymbol{v}_{i,2}$ by solving $(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{v}_{i,2} = \boldsymbol{v}_{i,1}$ for $\boldsymbol{v}_{i,2}$
  - here, we first find a generalized eigenvector $\boldsymbol{v}_{i,2}$ and then compute an eigenvector $\boldsymbol{v}_{i,1}$ by solving $(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{v}_{i,2} = \boldsymbol{v}_{i,1}$ for $\boldsymbol{v}_{i,1}$
- For Example 1, first find $\boldsymbol{v}_{1,2}$ that satisfies (2):
  - we have

$$(\boldsymbol{A} - \lambda \boldsymbol{I})^2 = \begin{bmatrix} 3-2 & -1 & 1 \\ 2 & 0-2 & 2 \\ 1 & -1 & 3-2 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

    meaning that $(\boldsymbol{A} - \lambda \boldsymbol{I})^2 \boldsymbol{v} = \boldsymbol{0}$ for any vector $\boldsymbol{v}$

- thus, $\boldsymbol{v}$ is a generalized eigenvector of rank 2 if and only if it is not an eigenvector itself, i.e.,

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{v} \neq \boldsymbol{0} \iff v_1 - v_2 + v_3 \neq 0$$

- for example, we can choose the following

$$\boldsymbol{v}_{1,2} := \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

◦ Based on this $\boldsymbol{v}_{1,2}$, we choose an eigenvector by (3):

$$\boldsymbol{v}_{1,1} := (\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{v}_{1,2} = \begin{bmatrix} 3-2 & -1 & 1 \\ 2 & 0-2 & 2 \\ 1 & -1 & 3-2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

◦ Finally, choose another eigenvector $\boldsymbol{v}_{2,1}$ that is linearly independent of $\boldsymbol{v}_{1,1}$

- recall that the eigenvector associated with $\lambda = 2$ must be of the form

$$\boldsymbol{v} = \begin{bmatrix} \alpha \\ \alpha + \beta \\ \beta \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \forall \alpha, \beta$$

- $\boldsymbol{v}_{1,1}$ above is the one where $(\alpha, \beta) = (1, 1)$
- any (α, β) with $\alpha \neq \beta$ would give us an eigenvector that is linearly independent of $\boldsymbol{v}_{1,1}$
- for example, setting $(\alpha, \beta) = (1, 0)$ yields

$$\boldsymbol{v}_{2,1} := \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

◦ Then $\boldsymbol{A}$ should be decomposed as:

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{v}_{1,1} & \boldsymbol{v}_{1,2} & \boldsymbol{v}_{2,1} \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}}_{=:\boldsymbol{J}} \begin{bmatrix} \boldsymbol{v}_{1,1} & \boldsymbol{v}_{1,2} & \boldsymbol{v}_{2,1} \end{bmatrix}^{-1}$$

where $\lambda_1 = \lambda_2 = \lambda = 2$

◦ In fact,

$$\boldsymbol{V} := \begin{bmatrix} \boldsymbol{v}_{1,1} & \boldsymbol{v}_{1,2} & \boldsymbol{v}_{2,1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \implies \boldsymbol{V}^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

and

$$\boldsymbol{V}\boldsymbol{J}\boldsymbol{V}^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ 2 & 0 & 2 \\ 1 & -1 & 3 \end{bmatrix} = \boldsymbol{A}$$