Parallelization of QR decomposition

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Contents

What is a QR decomposition?

It is

- $A = Q \cdot R$
- A is SPD (symmetric positive definite)
- Q is orthogonal
- R is upper triangular
- it is one way to compute eigen numbers – and not bad

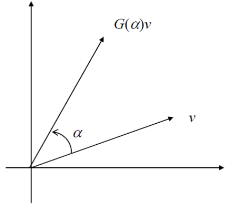
It is not

- the best way to compute solution of $Ax = B$
- even the Gaussian elimination method is better

Basic facts

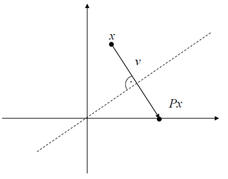
We make matrix A upper triangular by

Givens rotation



- numerically unstable
- requires one more cycle

Householder reflection



- numerically stable
- faster than Givens

Givens rotation

$$\begin{pmatrix} \star & \star & \star \\ \star & \star & \star \\ \star & \star & \star \end{pmatrix} \cdot G_1 \rightarrow \begin{pmatrix} \star & \star & \star \\ \star & \star & \star \\ 0 & \star & \star \end{pmatrix} \cdot G_2 \rightarrow \begin{pmatrix} \star & \star & \star \\ 0 & \star & \star \\ 0 & \star & \star \end{pmatrix} \cdot G_3 \rightarrow \begin{pmatrix} \star & \star & \star \\ 0 & \star & \star \\ 0 & 0 & \star \end{pmatrix}$$

$$G(i,k,\alpha) = \begin{array}{c} \begin{pmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & c & \cdots & s & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & -s & \cdots & c & \cdots & 0 \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{pmatrix} \end{array} \begin{array}{l} \\ \\ i \\ \\ k \\ \\ \\ \end{array}$$

$$A \cdot (G_1 G_2 G_3) = R \rightarrow Q = G_3^T G_2^T G_1^T$$

Householder reflection

$$\begin{pmatrix} \star & \star & \star \\ \star & \star & \star \\ \star & \star & \star \end{pmatrix} \cdot P_1 \rightarrow \begin{pmatrix} \star & \star & \star \\ 0 & \star & \star \\ 0 & \star & \star \end{pmatrix} \cdot P_2 \rightarrow \begin{pmatrix} \star & \star & \star \\ 0 & \star & \star \\ 0 & 0 & \star \end{pmatrix}$$

$$P_i = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \cdots & \cdots & \vdots \\ \vdots & \cdots & \star & \cdots & \star \\ \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \star & \cdots & \star \end{pmatrix}$$

$$A \cdot (P_1 P_2) = Q \rightarrow R = P_2^T P_1^T$$

Which one to choose?

Which one to choose?

Which one to choose?

Which one to choose?

Householder

How to use the QR decomposition to find eigenvalues?

It's simple. We compute matrices Q, R and update matrix A

$$A_{new} = Q' \cdot A \cdot Q.$$

We repeat QR decomposition with new matrix A. After 6 iterations for matrixes less than 100 (approximately 10 iterations for matrices till 1000) the matrix A is like diagonal.

The items on diagonal of A are approximations of eigenvalues.

Example in matlab

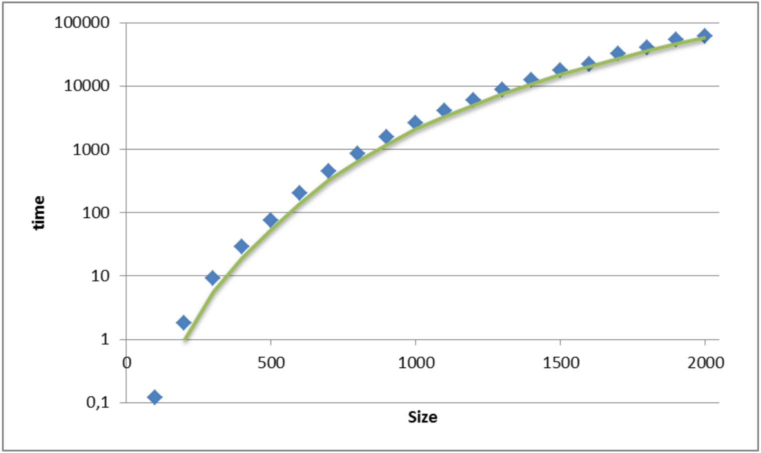
Givens rotation

```
[m,n] = size(A);
Q = eye(m,m); R = A;
for j=1:m
    for i = m:(-1):j+1
        x = R(:,j);
        if norm([x(i-1),x(i)]) > 0
            c = x(i-1)/norm([x(i-1),x(i)]);
            s=-x(i)/norm([x(i-1),x(i)]);
            G=eye(m);
            G([i-1,i],[i-1,i]) = [c,s;-s,c];
            R=G'*R;
            Q=Q*G;
        end
    end
end
```

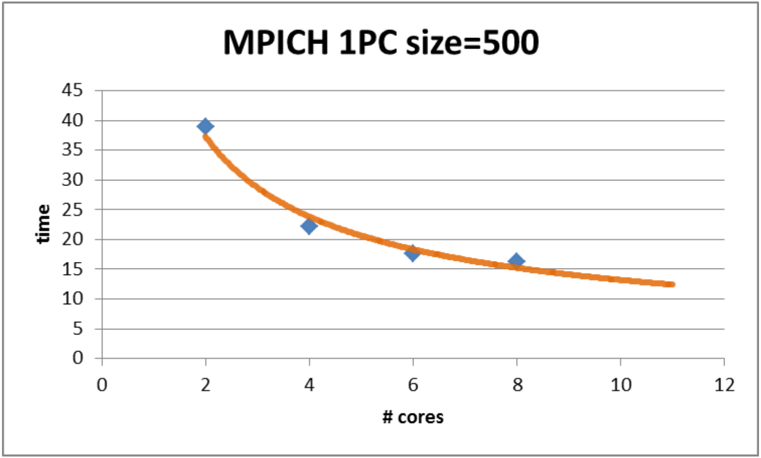
Householder reflection

```
[m,n] = size(A);
Q = eye(m,m); R = A;
for j=1:min(m,n)
    x = R(j:m,j);
    v=sign(x(1))*norm(x)*eye(m-j+1,1)-x;
    if norm(v) > 0
        v=v/norm(v);
        P=eye(m);
        P(j:m,j:m)=P(j:m,j:m)-2*v*v';
        R=P*R;
        Q=Q*P;
    end
end
```

Sequential approach



MPICH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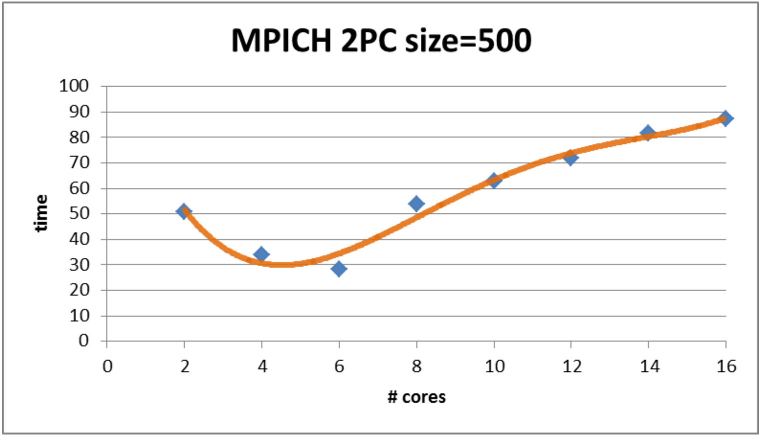
MPICH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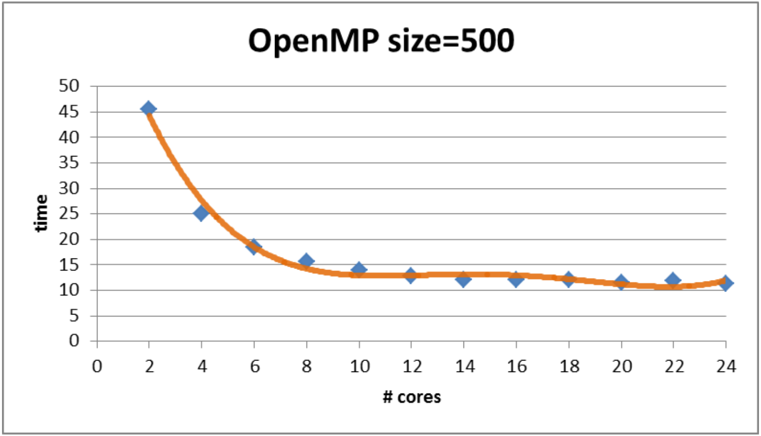
OpenMP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22
24

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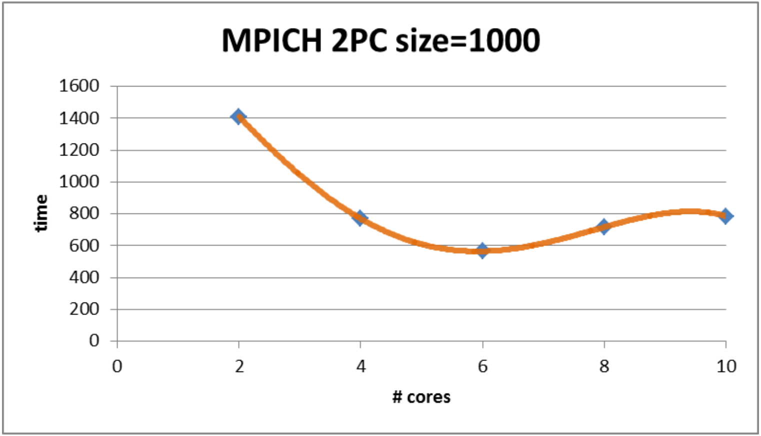
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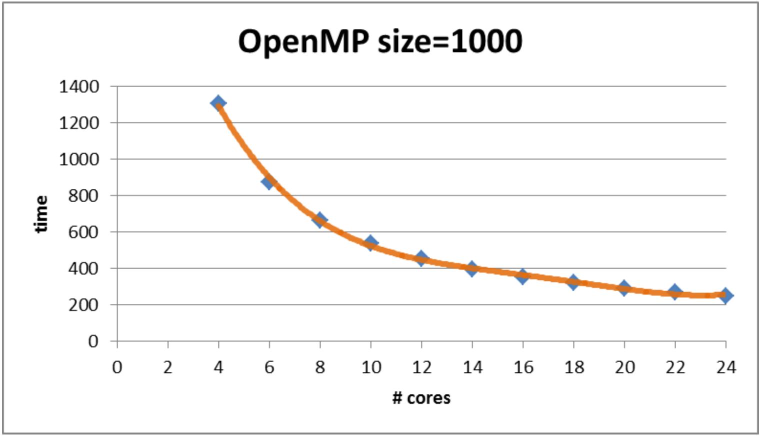
OpenMP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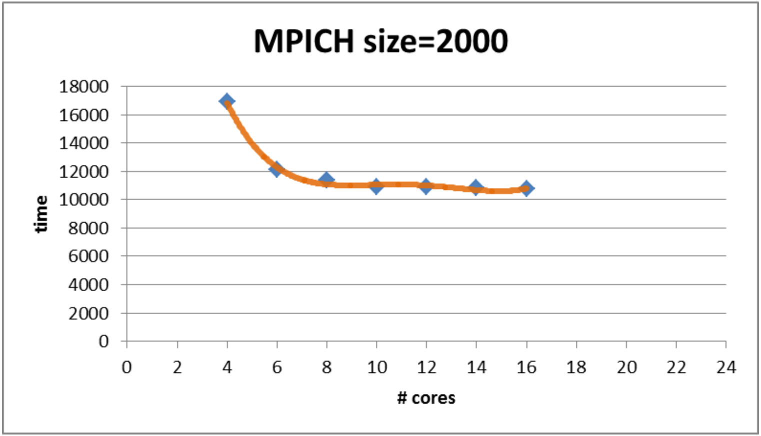
20
22
24

MPICH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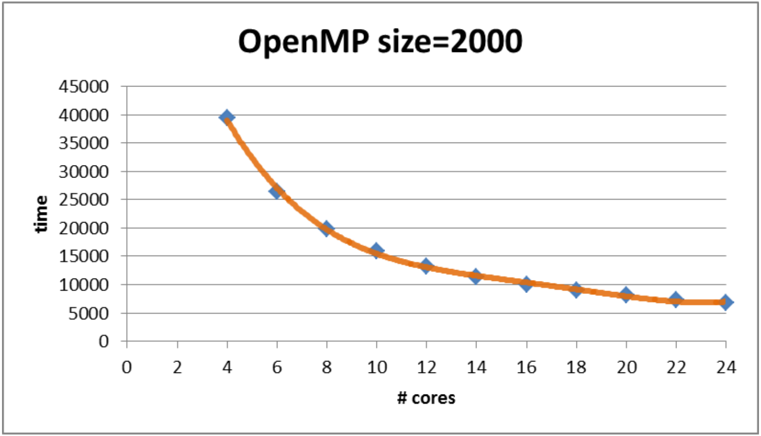
cores

OpenMP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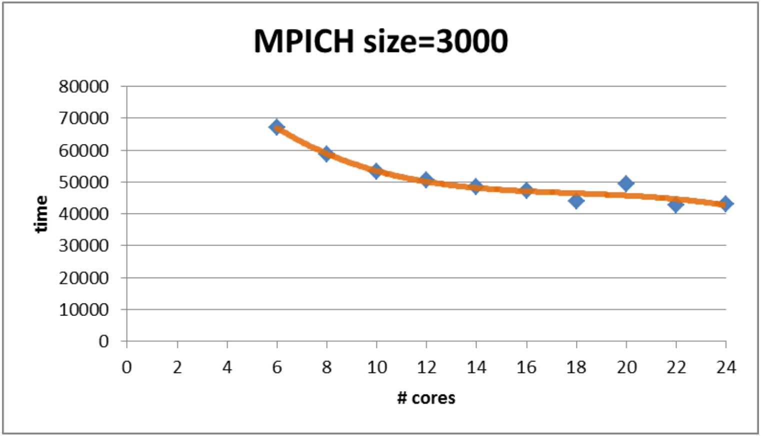
cores

MPICH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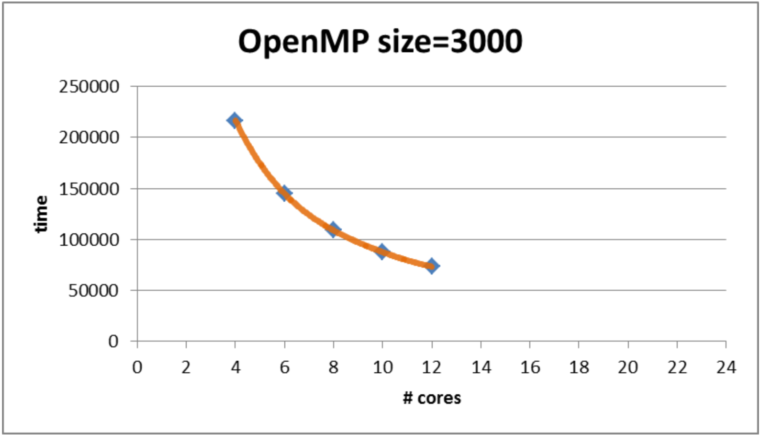
cores

OpenMP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10
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14
16
18
20
22
24
cores

Thanks

Thank you for your attention