Parallelization of QR decomposition

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Contents

What is a QR decomposition?

It is

- $A = Q \cdot R$
- A is SPD (symmetric positive definite)
- Q is orthogonal
- R is upper triangular
- it is one way to compute eigen numbers

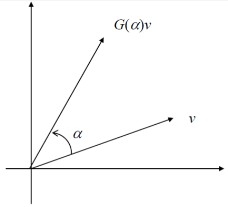
It is not

- the best way to compute solution of $Ax = B$
- the Gaussian elimination method is better

Basic facts

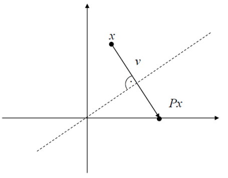
We make matrix A upper triangular by

Givens rotation



- numerically unstable
- requires one more cycle

Householder reflection



- numerically stable
- faster than Givens

Givens rotation

$$\begin{pmatrix} \star & \star & \star \\ \star & \star & \star \\ \star & \star & \star \end{pmatrix} \cdot G_1 \rightarrow \begin{pmatrix} \star & \star & \star \\ \star & \star & \star \\ 0 & \star & \star \end{pmatrix} \cdot G_2 \rightarrow \begin{pmatrix} \star & \star & \star \\ 0 & \star & \star \\ 0 & \star & \star \end{pmatrix} \cdot G_3 \rightarrow \begin{pmatrix} \star & \star & \star \\ 0 & \star & \star \\ 0 & 0 & \star \end{pmatrix}$$

$$G(i,k,\alpha) = \begin{array}{cc} \begin{pmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & c & \cdots & s & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & -s & \cdots & c & \cdots & 0 \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{pmatrix} & \begin{matrix} \\ \\ i \\ \\ k \\ \\ \\ \end{matrix} \end{array}$$

$$A \cdot (G_1 G_2 G_3) = R \rightarrow Q = G_3^T G_2^T G_1^T$$

Householder reflection

$$\begin{pmatrix} \star & \star & \star \\ \star & \star & \star \\ \star & \star & \star \end{pmatrix} \cdot P_1 \rightarrow \begin{pmatrix} \star & \star & \star \\ 0 & \star & \star \\ 0 & \star & \star \end{pmatrix} \cdot P_2 \rightarrow \begin{pmatrix} \star & \star & \star \\ 0 & \star & \star \\ 0 & 0 & \star \end{pmatrix}$$

$$P_i = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \cdots & \cdots & \vdots \\ \vdots & \cdots & \bigstar & \cdots & \bigstar \\ \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \bigstar & \cdots & \bigstar \end{pmatrix}$$

$$A \cdot (P_1 P_2) = Q \rightarrow R = P_2^T P_1^T$$

Which one should we choose?

Which one should we choose?

Which one should we choose?

Which one should we choose?

Householder

How to use the QR decomposition to find eigenvalues?

It's simple. We compute matrices Q, R and update matrix A

$$A_{new} = Q' \cdot A \cdot Q.$$

We repeat QR decomposition with new matrix A. After 6 iterations for matrices less than 100 (approximately 10 iterations for matrices until 1000) the matrix A is like diagonal.

The items on diagonal of A are approximations of eigenvalues.

Example in matlab

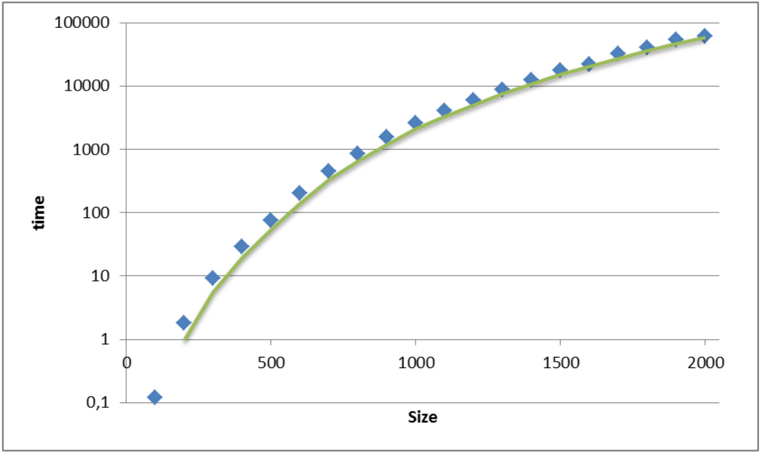
Givens rotation

```
[m,n] = size(A);
Q = eye(m,m); R = A;
for j=1:m
    for i = m:(-1):j+1
        x = R(:,j);
        if norm([x(i-1),x(i)]) > 0
            c = x(i-1)/norm([x(i-1),x(i)]);
            s=-x(i)/norm([x(i-1),x(i)]);
            G=eye(m);
            G([i-1,i],[i-1,i]) = [c,s;-s,c];
            R=G'*R;
            Q=Q*G;
        end
    end
end
```

Hauseholder reflection

```
[m,n] = size(A);
Q = eye(m,m); R = A;
for j=1:min(m,n)
    x = R(j:m,j);
    v=sign(x(1))*norm(x)*eye(m-j+1,1)-x;
    if norm(v) > 0
        v=v/norm(v);
        P=eye(m);
        P(j:m,j:m)=P(j:m,j:m)-2*v*v';
        R=P*R;
        Q=Q*P;
    end
end
```

Sequential approach



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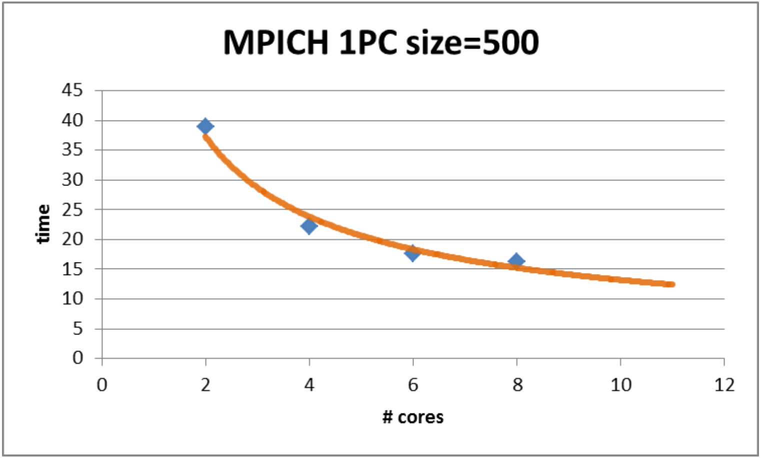
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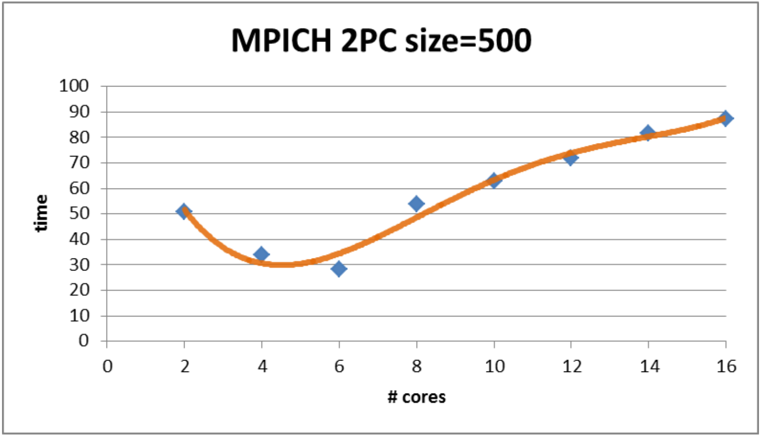
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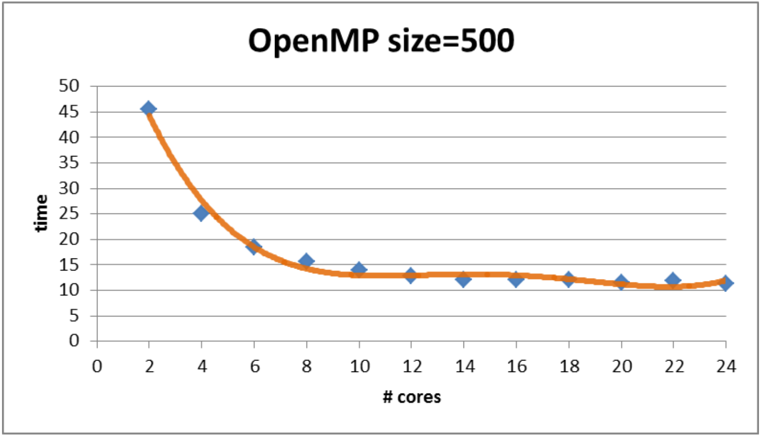
OpenMP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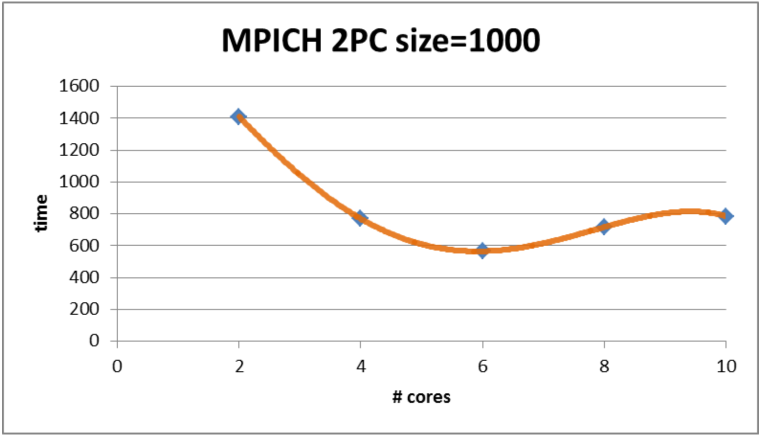
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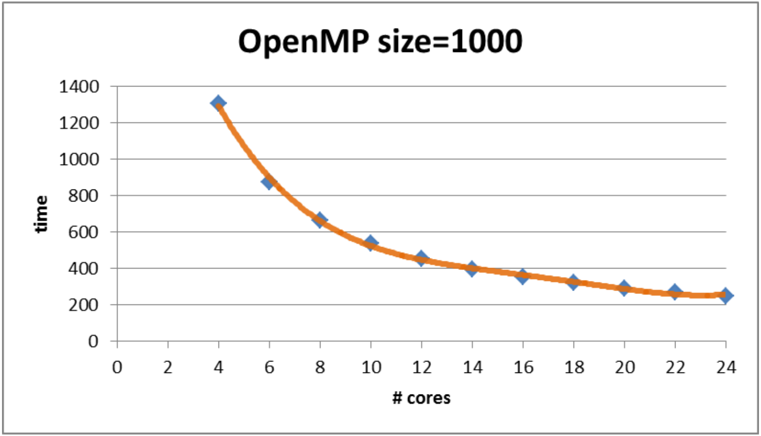
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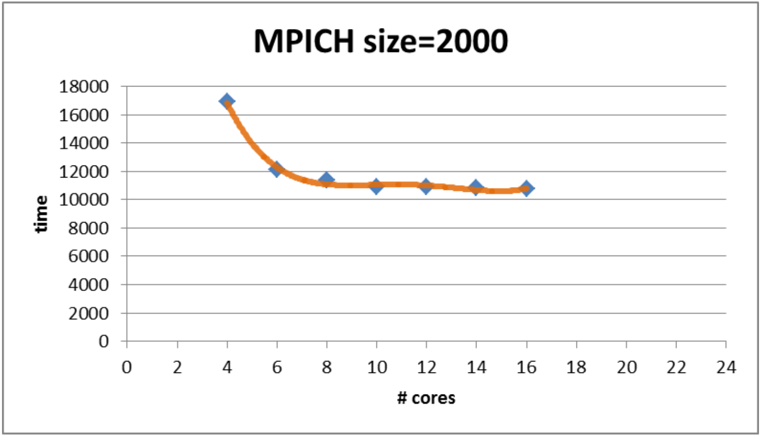
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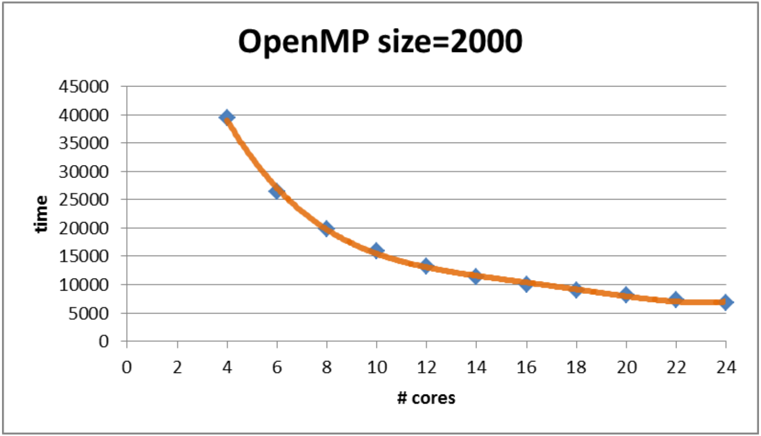
OpenMP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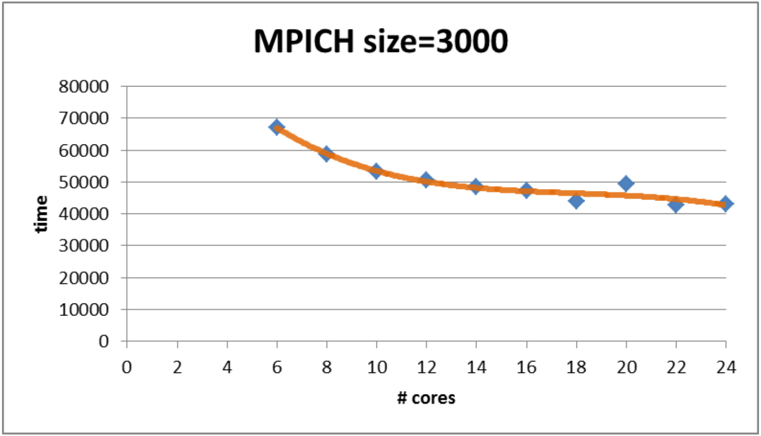
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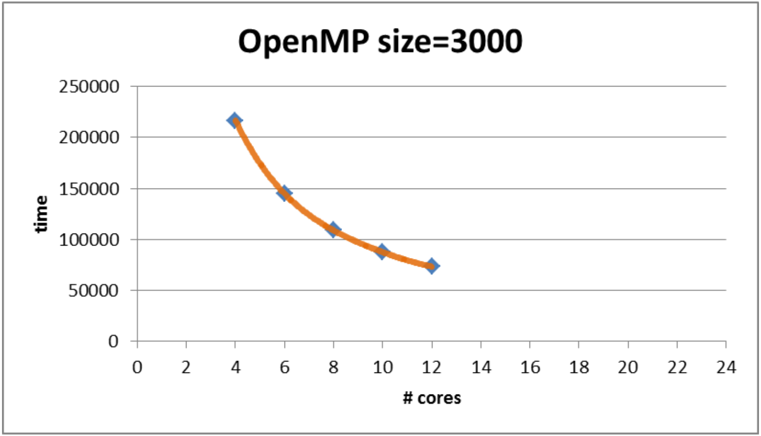
OpenMP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20
22
24
cores

Thanks

Thank you!