

Answers to Study Questions - Lecture 10

1. $C = \rho c_p$, where ρ is the density of the material. For water we know that $\rho = 1 \times 10^3 \text{ kg m}^{-3}$ (i.e. $1 \text{ kg} = 1\ell = 10 \times 10 \times 10 \text{ cm}$), therefore $4.180 \times 10^3 \text{ J kg}^{-1} \text{ K}^{-1} \times 10^3 \text{ kg m}^{-3} = \underline{4.180 \times 10^6 \text{ J m}^{-3} \text{ K}^{-1}}$.
2. $P = 55\%$ for a dry soil means $\theta_a = 0.55$ and $\theta_m = 1 - \theta_a = 0.45$:

$$\begin{aligned} C &= \theta_m C_m + \theta_a C_a \\ &\approx \theta_m C_m \\ &= 0.45 \times 2.1 \text{ MJ m}^{-3} \text{ K}^{-1} \\ &= \underline{0.945 \text{ MJ m}^{-3} \text{ K}^{-1}} \end{aligned}$$

Subscripts a and m refer to air and mineral matter, respectively. Note, the second term is small compared to the first one ($\theta_a C_a = 0.45 \times 0.0012 \text{ MJ m}^{-3} \text{ K}^{-1} = 0.00066 \text{ MJ m}^{-3} \text{ K}^{-1}$) and can be neglected.

3. $P = 55\%$ for the saturated case means $\theta_w = 0.55$ and $\theta_m = 1 - \theta_w = 0.45$:

$$\begin{aligned} C &= \theta_m C_m + \theta_w C_w \\ &= 0.45 \times 2.1 \text{ MJ m}^{-3} \text{ K}^{-1} + 0.55 \times 4.18 \text{ MJ m}^{-3} \text{ K}^{-1} \\ &= 0.945 \text{ MJ m}^{-3} \text{ K}^{-1} + 2.299 \text{ MJ m}^{-3} \text{ K}^{-1} \\ &= \underline{3.24 \text{ MJ m}^{-3} \text{ K}^{-1}} \end{aligned}$$

Subscripts a and m refer to air and mineral matter, respectively. Note, the second term is small compared to the first one ($\theta_a C_a = 0.45 \times 0.0012 \text{ MJ m}^{-3} \text{ K}^{-1} = 0.00066 \text{ MJ m}^{-3} \text{ K}^{-1}$) and can be neglected.

4. $P = 50\%$ and $\theta_a = 0.30$ means $\theta_w = P - \theta_a = 0.20$. An organic to mineral ratio of 1.5 (3/2) means $(1 - P) = 0.5$ is made up of 0.3 θ_o and 0.2 θ_m :

$$\begin{aligned} C &= \theta_m C_m + \theta_o C_o + \theta_w C_w \\ &= 0.3 \times 2.5 \text{ MJ m}^{-3} \text{ K}^{-1} + 0.2 \times 2.1 \text{ MJ m}^{-3} \text{ K}^{-1} \\ &\quad + 0.2 \times 4.18 \text{ MJ m}^{-3} \text{ K}^{-1} \\ &= \underline{2.0 \text{ MJ m}^{-3} \text{ K}^{-1}} \end{aligned}$$

5. $\Delta\theta_w = 0.1$:

$$\begin{aligned}\Delta C &= \Delta\theta_w C_w \\ &= 0.1 \times 4.18 \text{ MJ m}^{-3} \text{ K}^{-1} \\ &= \underline{0.418 \text{ MJ m}^{-3} \text{ K}^{-1}}\end{aligned}$$

C of the soil will increase by $0.418 \text{ MJ m}^{-3} \text{ K}^{-1}$.

6. The warming rate of a material is defined by:

$$\frac{\Delta T}{\Delta t} = \frac{1}{C} \frac{\Delta Q_G}{\Delta z}$$

Q_G at 0 cm depth (surface) is $+100 \text{ W m}^{-2}$, Q_G at 10 cm must be zero because there is no energy distributed to lower layers, i.e. only topmost 10 cm experience heating, hence $\Delta Q_G = 100 \text{ W m}^{-2} - 0 \text{ W m}^{-2} = 100 \text{ W m}^{-2}$ (same as $\text{J s}^{-1} \text{ m}^{-2}$):

$$\frac{\Delta T}{\Delta t} = \frac{100 \text{ J s}^{-1} \text{ m}^{-2}}{2 \text{ MJ m}^{-3} \text{ K}^{-1} \times 0.1 \text{ m}} = 0.0005 \text{ K s}^{-1} = \underline{1.8 \text{ K h}^{-1}}.$$

7. Fourier's Law: $Q_G = -k \Delta T / \Delta z = -k(T_2 - T_1) / (z_2 - z_1)$. It states that the flow rate of heat conducted through a solid material (or still fluid) is proportional to the temperature gradient.

8. Use Fourier's Law:

$$Q_G = -k \frac{\Delta T}{\Delta z}$$

You insert the thermal conductivity of $k = 0.27 \text{ W m}^{-1} \text{ K}^{-1}$:

$$\begin{aligned}Q_G &= -0.27 \text{ W m}^{-1} \text{ K}^{-1} \frac{20^\circ\text{C} - 18.5^\circ\text{C}}{0.02 \text{ m} - 0.06 \text{ m}} \\ &= -0.27 \text{ W m}^{-1} \text{ K}^{-1} \frac{1.5 \text{ K}}{-0.04 \text{ m}} \\ &= \underline{10.1 \text{ W m}^{-2}}\end{aligned}$$

9. Again we use Fourier's Law, but rearranged:

$$-Q_G \frac{\Delta z}{\Delta T} = k$$

We can directly plug-in the inverse of the gradient ($\Delta T / \Delta z = -0.5 \text{ K cm}^{-1}$) into $\Delta z / \Delta T$:

$$\begin{aligned}k &= -20 \text{ W m}^{-2} \times -0.02 \text{ K m}^{-1} \\ &= \underline{+0.4 \text{ W m}^{-1} \text{ K}^{-1}}\end{aligned}$$

10. The thermal diffusivity κ tells us how quickly temperature waves propagate down into the soil, and κ is defined by:

$$\begin{aligned}\kappa &= \frac{k}{C} = \frac{k}{\rho c_p} \\ &= \frac{0.4 \text{ J s}^{-1} \text{ m}^{-1} \text{ K}^{-1}}{1.4 \times 10^3 \text{ kg m}^{-3} 1.8 \times 10^3 \text{ J kg}^{-1} \text{ K}^{-1}} \\ &= \underline{0.16 \times 10^{-6} \text{ m}^2 \text{ s}^{-1}}\end{aligned}$$

Note the fine - but important - difference between the symbol k for thermal conductivity (Latin k) and the symbol κ for thermal diffusivity (Greek ‘kappa’).

11. The thermal admittance μ is strictly speaking a surface property. It defines how well a surface can accept or release heat.

$$\begin{aligned}\mu &= \sqrt{k C} = \sqrt{k \rho c_p} \\ &= \sqrt{0.4 \text{ J s}^{-1} \text{ m}^{-1} \text{ K}^{-1} \times 1.4 \times 10^3 \text{ kg m}^{-3} \times 1.8 \times 10^3 \text{ J kg}^{-1} \text{ K}^{-1}} \\ &= \sqrt{1.008 \times 10^6 \text{ J}^2 \text{ m}^{-4} \text{ K}^{-2} \text{ s}^{-1}} \\ &= \underline{1004 \text{ J m}^{-2} \text{ K}^{-1} \text{ s}^{-1/2}}\end{aligned}$$

12. M_m and M_o is the mass of mineral and organic material, respectively, in one m^3 of soil.

The total mass of the dry soil M in one cubic-metre is given by the bulk density ($\rho_s = 1.4 \text{ Mg m}^{-3}$):

$$M = M_m + M_o = \rho_s \times 1 \text{ m}^3 = 1.4 \text{ Mg m}^{-3} \times 1 \text{ m}^3 = 1.4 \text{ Mg}$$

f_o is the organic mass fraction (given: $f_o = 0.25$) which is the mass of organic material to the total mass

$$f_o = \frac{M_o}{M} = \frac{M_o}{M_o + M_m} = 0.25$$

solving for M_m and M_o :

$$M_m = M \times (1 - f_o) = 1.4 \text{ Mg} \times (1 - 0.25) = 1.4 \text{ Mg} \times 0.75 = 1.05 \text{ Mg}$$

$$M_o = M \times f_o = 1.4 \text{ Mg} \times 0.25 = 0.35 \text{ Mg}$$

13. Using the mass of organic and mineral material contained in one cubic metre (determined in Question 12), we can formulate the densities of organic (ρ_o) and mineral material (ρ_m) in the same soil:

$$\rho_o = \frac{M_o}{1\text{m}^3 \times \theta_o}$$

$$\rho_m = \frac{M_m}{1\text{m}^3 \times \theta_m}$$

Where ($1\text{m}^3 \times \theta_o$) is the volume of organic material in one cubic metre, and ($1\text{m}^3 \theta_m$) is the volume of mineral material in one cubic metre.

Lecture 10, slide 5 provides $c_m = 0.8\text{J kg}^{-1} \text{K}^{-1}$ and $c_o = 1.9\text{J kg}^{-1} \text{K}^{-1}$. Using $C = \rho c$ allows us to then determine the heat capacity of organic (C_o) and mineral material (C_m) in this soil:

$$C_o = \rho_o c_o$$

$$C_m = \rho_m c_m$$

The composite heat capacity of the dry soil (C_s) is the sum of the compound heat capacities weighted by the respective volume fractions:

$$C_s = \theta_o C_o + \theta_m C_m$$

replacing C_o by $\rho_o c_o$ (and same for C_m) then gives:

$$C_s = \theta_o \rho_o c_o + \theta_m \rho_m c_m$$

replacing ρ_o by $\frac{M_o}{1\text{m}^3 \theta_o}$ (and same for ρ_m) then gives:

$$C_s = \theta_o \frac{M_o}{1\text{m}^3 \theta_o} c_o + \theta_m \frac{M_m}{1\text{m}^3 \theta_m} c_m$$

Note that then θ_o and θ_m cancel out:

$$C_s = \frac{M_o}{1\text{m}^3} c_o + \frac{M_m}{1\text{m}^3} c_m$$

Inserting the values:

$$C_s = \frac{0.35 \text{ Mg}}{1 \text{ m}^3} 1.9 \text{ kJ kg}^{-1} \text{ K}^{-1} + \frac{1.05 \text{ Mg}}{1 \text{ m}^3} 0.8 \text{ kJ kg}^{-1} \text{ K}^{-1} =$$

$$0.665 \text{ MJ m}^{-3} \text{ K}^{-1} + 0.84 \text{ MJ m}^{-3} \text{ K}^{-1} = 1.505 \text{ MJ m}^{-3} \text{ K}^{-1}$$

We learn from this exercise that generally we can rewrite the composite heat capacity of a soil (C_s) from using heat capacities of compound substances and volume fractions:

$$C_s = \theta_o C_o + \theta_m C_m + \dots$$

to using specific heat and known mass (M_o , M_m , etc.) of the compound substances in a soil volume V_s :

$$C_s = \frac{M_o}{V_s} c_o + \frac{M_m}{V_s} c_m + \dots$$

where V_s is the volume of the soil, and M_o and M_m is the mass of organic and mineral material in the same volume V_s . This has the advantage of avoiding assuming any specific density of organic and mineral material, which is difficult to determine practically.