APC 523 Final Project

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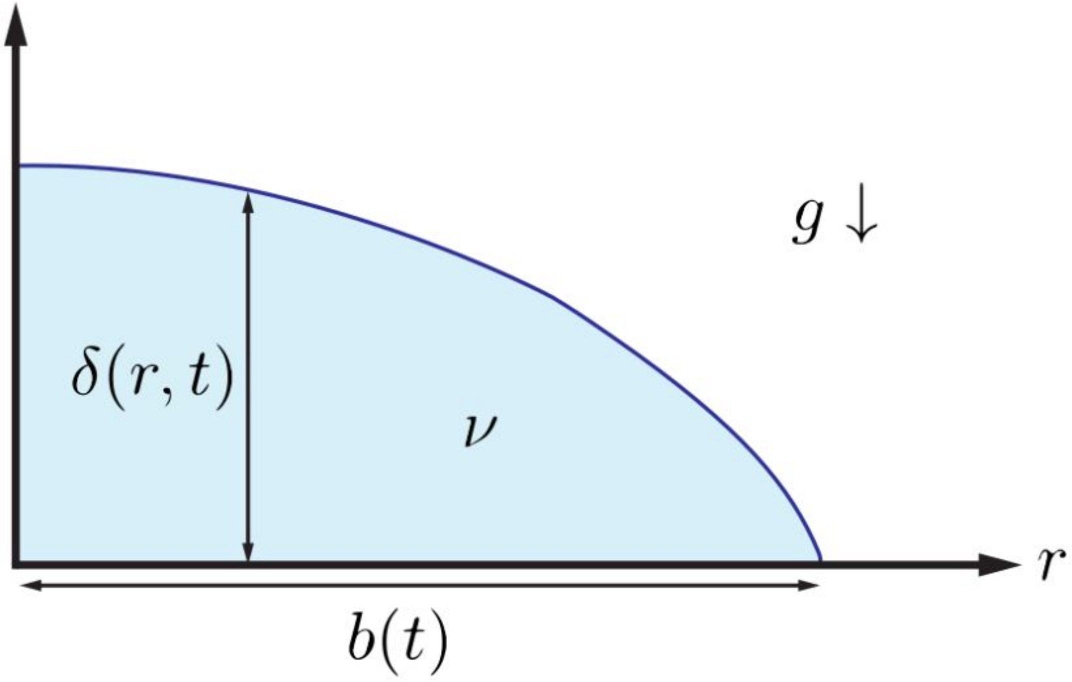
Selected Problem

- We model numerically a spreading viscous drop under gravity *g* and viscosity ν
- $\delta(r,t)$ is the thickness profile
- *g* and ν are gravity and viscosity
- *b(t)* is the radius



Problem Definition

- This problem can be modeled as a nonlinear, parabolic boundary value problem defined by:

$$\delta^3 \left[\frac{\partial^2 \delta}{\partial r^2} + \frac{1}{r}\frac{\partial \delta}{\partial r}\right] + 3\delta^2 \left[\frac{\partial \delta}{\partial r}\right]^2 - 3\frac{\nu}{g}\frac{\partial \delta}{\partial t} = 0$$

$$\delta(r = b(t), t) = 0$$

$$\frac{\partial \delta}{\partial r}(r = 0, t) = 0$$

$$2\pi \int_0^{b(t)} r\delta(r,t)dr = constant$$

- The boundary conditions define the drop edge and ensure smooth solutions at the axis of symmetry
- The integral condition ensures the volume of the drop is conserved

Problem Reformulation

- The problem is easier to solve by making a change of variables: $\xi = r/b$
  It now reads:

$$\frac{\delta^3}{b^2}\left[\frac{\partial^2\delta}{\partial\xi^2} + \frac{1}{\xi}\frac{\partial\delta}{\partial\xi}\right] + \frac{3\delta^2}{b^2}\left[\frac{\partial\delta}{\partial\xi}\right]^2 - 3\frac{\nu}{g}\frac{\partial\delta}{\partial t} = 0$$

$$\delta(\xi = 1, t) = 0$$

$$\frac{\partial\delta}{\partial t}(\xi = 0, t) = 0$$

$$2\pi\int_0^{b(t)} r\delta(r,t)dr = constant$$

- We then solve for $\delta(\xi, t)$ on $\xi^n \in [0, 1]$, and use the integral condition to find b(t) at each time step

Numerical Methods

We solve the PDE using both explicit and implicit methods

Explicit

$$\frac{3\nu}{g}\frac{\delta_i^{n+1}-\delta_i^n}{\Delta t} = \frac{(\delta_i^n)^3}{(b^n)^2}\left[\frac{\delta_{i+1}^n - 2\delta_i^n + \delta_{i-1}^n}{\Delta\xi^2} + \frac{1}{\xi_i}\frac{\delta_{i+1}^n-\delta_{i-1}^n}{2\Delta\xi}\right] + \frac{3(\delta_i^n)^2}{(b^n)^2}\left[\frac{\delta_{i+1}^n-\delta_{i-1}^n}{2\Delta\xi}\right]^2$$

$$\delta_0^{n+1}=\delta_1^{n+1} \text{ and } \delta_N^{n+1}=0, \qquad \frac{\pi}{2}\sum_{i=1}^{N} (b^{n+1})^2(\xi_i^2-\xi_{i-1}^2)(\delta_i^n+\delta_{i-1}^n) = constant$$

- Simple explicit formulation with 2nd order finite difference approximation
- Trapezoidal approximation for integral condition
- Adaptive time-stepping used for stability:
    - For each step, form problem as:

      $$\delta^{n+1} = \mathbb{A}^n\delta^n$$
    - For stability, $\rho(\mathbb{A}) \leq 1$
    - Shrink time step until stable for each step
- Method is slow due to poor stability, requiring very small time step

Implicit

$$\frac{3\nu}{g}\frac{\delta_i^{n+1}-\delta_i^n}{\Delta t} = \frac{(\delta_i^{n+1})^3}{(b^n)^2}\left[\frac{\delta_{i+1}^{n+1} - 2\delta_i^{n+1} + \delta_{i-1}^{n+1}}{\Delta\xi^2} + \frac{1}{\xi_i}\frac{\delta_{i+1}^{n+1}-\delta_{i-1}^{n+1}}{2\Delta\xi}\right] + \frac{3(\delta_i^{n+1})^2}{(b^n)^2}\left[\frac{\delta_{i+1}^{n+1}-\delta_{i-1}^{n+1}}{2\Delta\xi}\right]^2$$

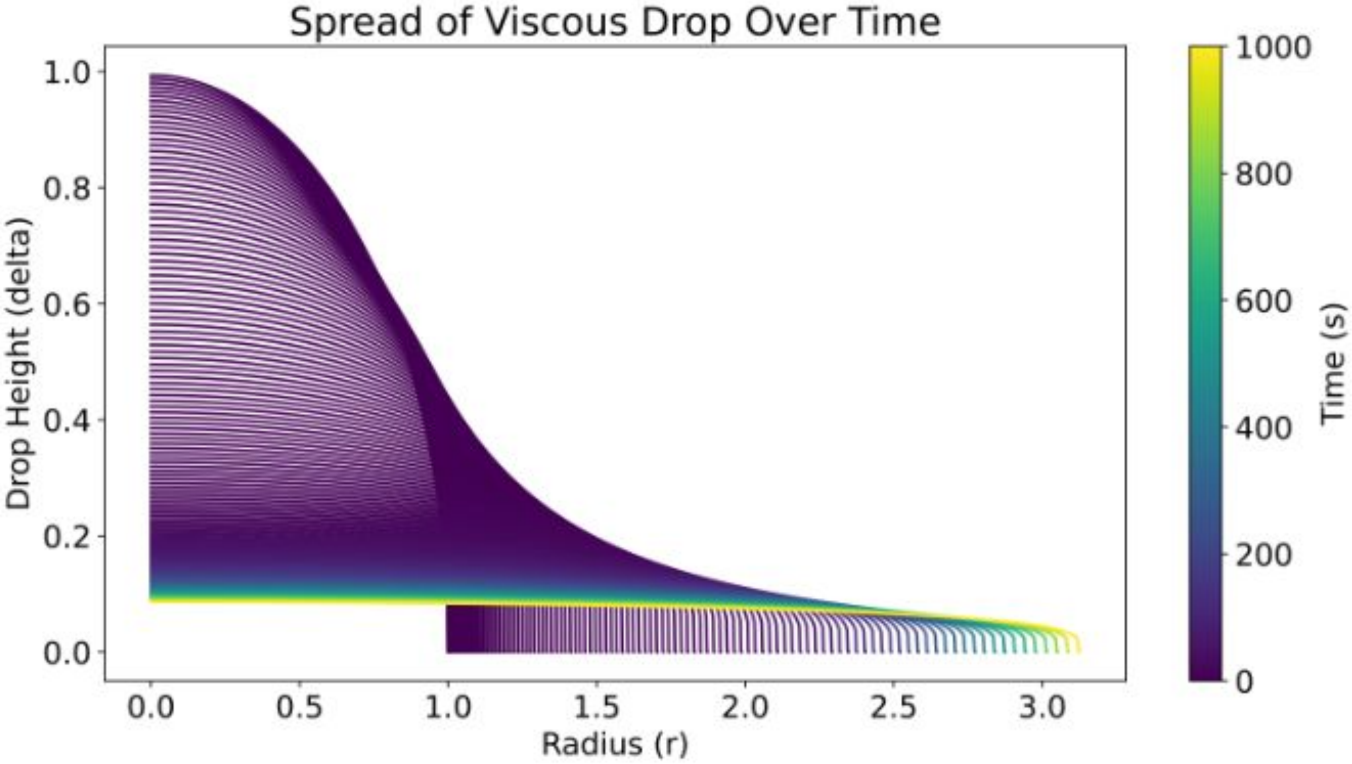
$$\delta_0^{n+1}=\delta_1^{n+1} \text{ and } \delta_N^{n+1}=0, \qquad \sum_{i=1}^{k} w_i f(r_i) = constant$$

- Implicit formulation with 2nd order finite difference approximation
- Gauss-Legendre integration
- Adaptive time-stepping used for speed:
    - Since problem is parabolic, we expect change to slow with time
    - Use 10% increase in time step with each step
- Method is far more stable
    - Code to be run to long times
    - Higher N can be used in reasonable amount of computation time

Results: Drop Profile vs Time

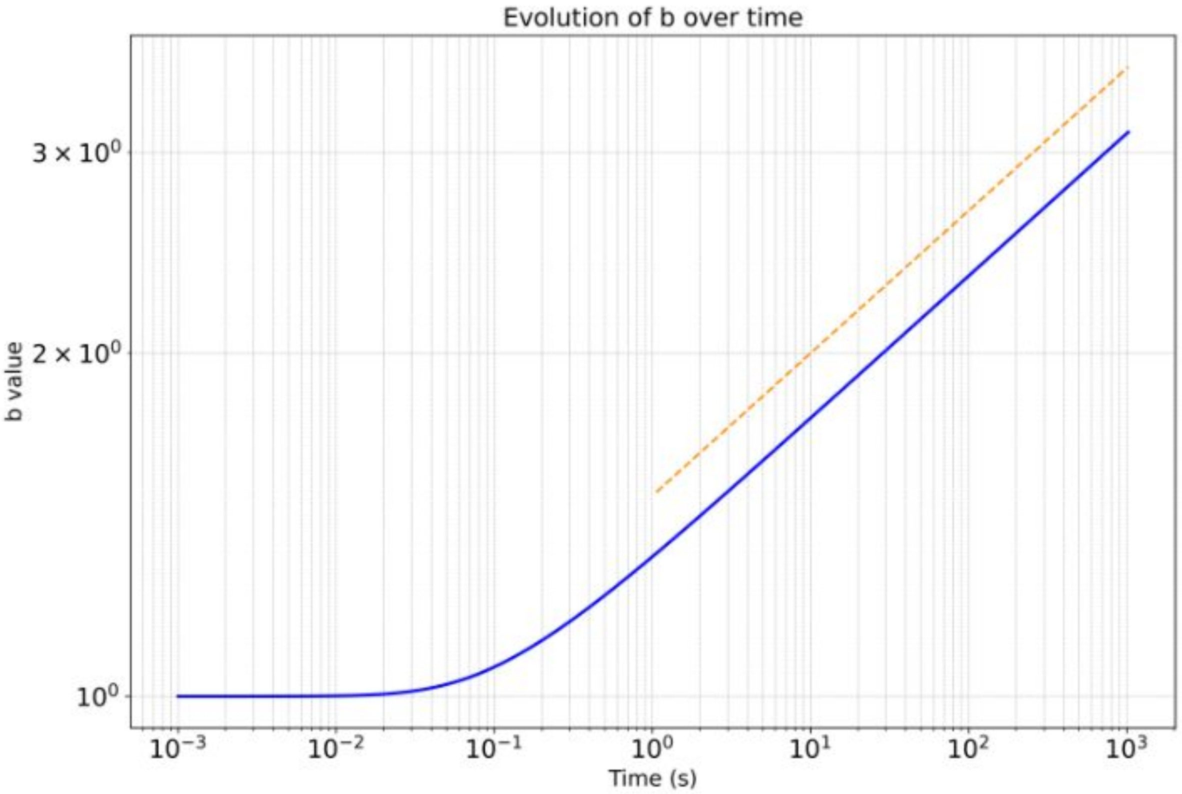
- ν = 1, g = 9.81
- Initialized as sphere with b(0) = 1

- Drop spreads, first quickly then more slowly as expected



Results: Long Time Behavior

- $b(t) \sim t^{1/8}$ at long times
- This is consistent with the analytical similarity solution [2]



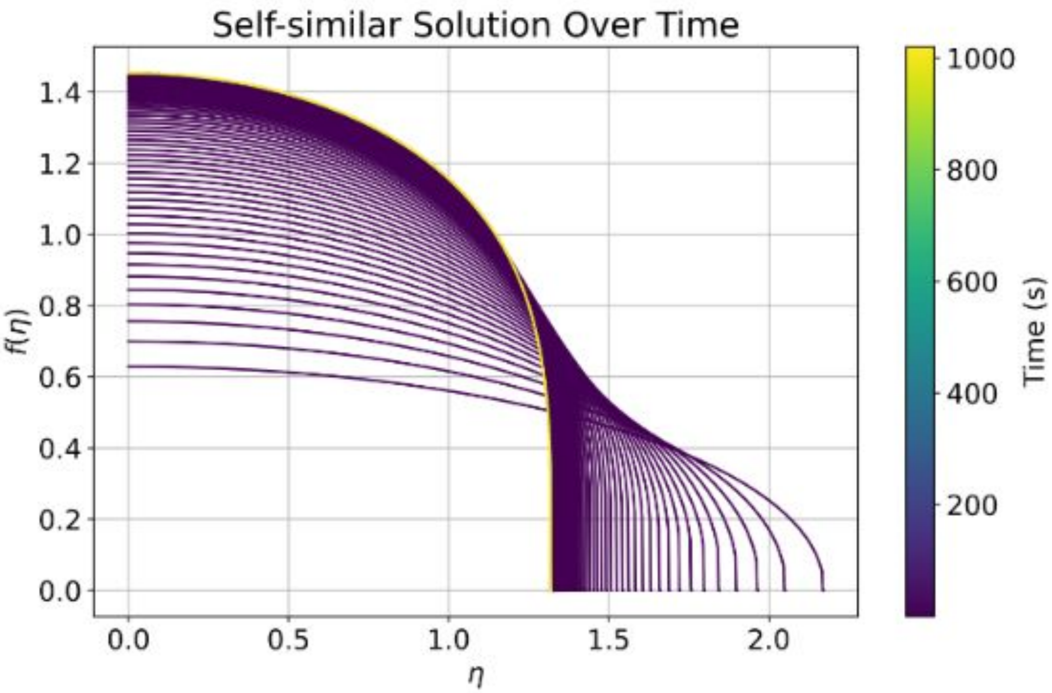
[2] L. Deike, “MAE501 problem set,” 2023.

Results: Similarity Solution Convergence

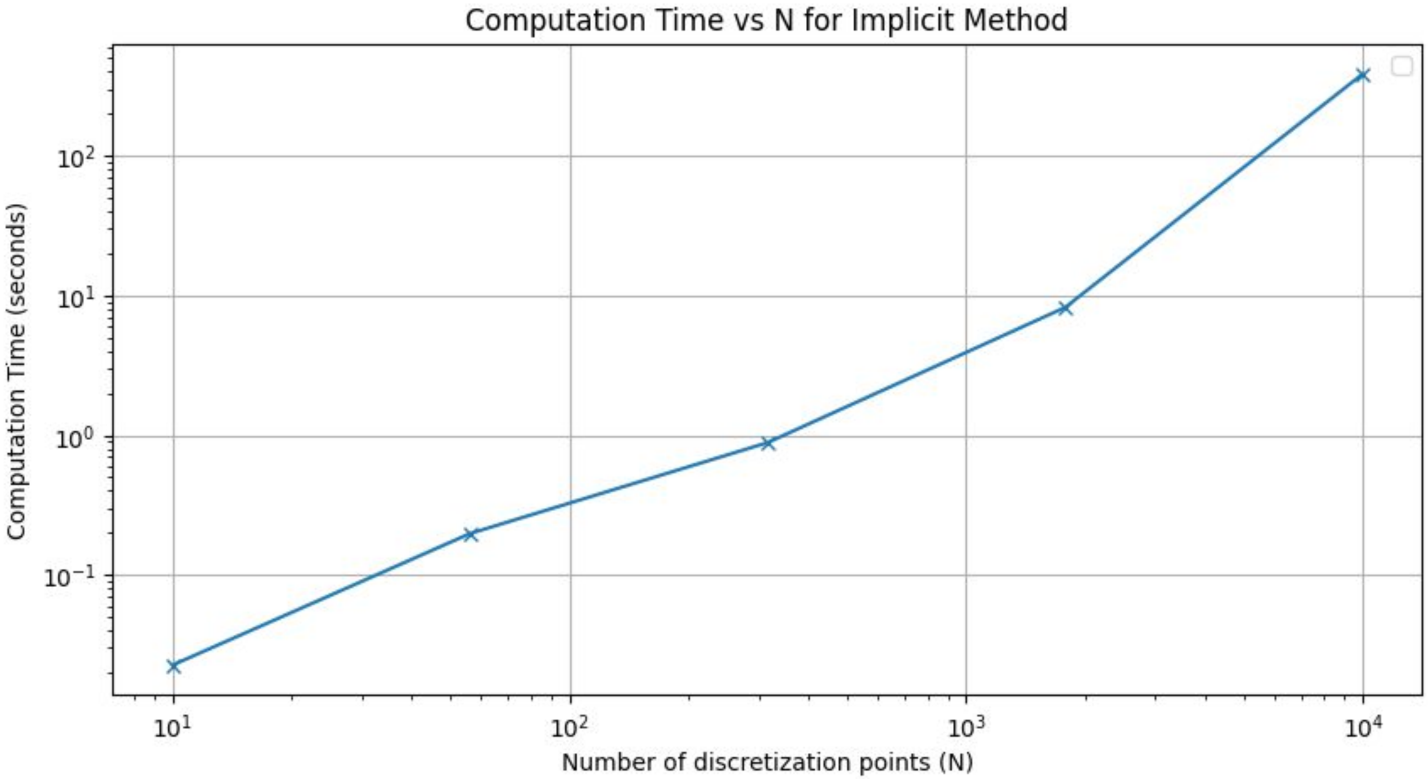
- For long times, the solution converges to one line (yellow).
- The similarity solution is given by:

$$\delta(r,t) = \left(\frac{3a\nu}{16\pi g t}\right)^{1/4} f\left(r\left(\frac{3\nu\pi^3}{a^3 g t}\right)^{1/8}\right)$$

- Where $\eta = r\left((3\nu\pi^3)/(a^3 g t)\right)^{1/8}$



Results: Time Complexity



Summary

- Viscous drop spreading modeled and solved numerically using a second order, nonlinear, parabolic PDE with boundary conditions and an integral conservation constraint
- Problem reformulated and solved numerically using explicit and implicit finite difference equations with numerical integration
- Implicit method is far more stable for large time steps, and can be used with adaptive time stepping to solve for long times accurately
- Results are verified against theory (similarity solutions and long-term behavior)