Speculator Best Response Game

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1 Background

In this section, we describe the model from [1]. A speculator deposits Ethereum (N_t – more generally could be any basket of collateral) with the protocol. Upon doing so they are now liable to return these stablecoins. Furthermore, the amount of collateral provided must be $> \beta L_t$. Often, β is chosen to be greater than 1 in a process of *overcollateralization* In return, the protocol mints stablecoins. The total supply is S_t. In a one-speculator model $L_t = S_t$. This quantity shows up on the liabilities side of the speculator's balance sheet. The reverse process will occur if the speculator chooses to remove some of their collateral. In this case, they remove stablecoins from supply which we assume reduces their leverage constraint, however this comes at the cost of selling some of their current ETH holdings (N_t).

Furthermore, we take a step forward with this formulation in considering the protocol's decision over the so-called "stability fee." This fee can be viewed as a one-time interest payment paid by the speculator upon settling their position with Maker Dao. So, if $\alpha = 1.1$ rather than paying the original quantity of DAI borrowed, the speculator will pay back this original value, plus 10%. Unlike interest, this value does not accrue (for that reason may be confusing to use that analogy). However, we can note that this fee can be dynamically updated. This is how Reflexer Labs updates their redemption price for RAI (which however is not intended to target $1 peg). The adjustment of redemption price should indirectly affect market price and allow us to drive it to some target. Again, this is done by looking a the speculator (primary participant's) decision process.

What is a good utility function for the speculator then? We follow the argument laid out in [1]. In this case, the speculator hopes to maximize their long term equity. This is given as $A_t - E[p_t^S]L_t = A_t - L_t$ (assets - liabilities) where the expected stablecoin price $E[p_t^S]$ is $1 over the long term. The speculator's assets at time t are given by $A_t = N_t p_t^E + \Delta_t p(L_t)$. Notice that when the speculator chooses to mint, we assume they immediately repurchase ETH to add to their vault. Thus their value (in ETH) increases while stablecoin supply does as well (adding leverage) burn supply, their assets are reduced by the corresponding amount ($ paid to repurchase supply).

We also refer to the speculator as "player 2" as they are the follower in a leader-follower stackelberg game. Thus, we can encode the speculator's decision as solving the following optimization problem (U indexed by 2)

$$U^2 = \left(N_t p_t^E + \Delta_t p(L_t)\right) - \alpha L_t$$

$$s.t. \;\; A_t - \beta L_t > 0 \text{ and } L_t = L_{t-1} + \Delta_t$$

where we have added *alpha* to represent the "stability fee," a concept we return to later.
A_t is shorthand for "Assets held at time t" Since the speculator maximizes

utility, we could also formulate this as minimizing cost:

$$J^2 = -U^2 = -\left(N_t p_t^E + \Delta_t p(L_t)\right) + (\alpha + \delta\alpha_t)L_t$$

1.1 Vault Dynamics Example

This example illustrates some intuition as to the decision faced by the speculator over a single time step. Suppose $N_t = 2$, $p^E = \$5$, $L_t = 5$ and let's suppose the speculator expects ETH to go down $r = .90$ in the next time step. Then, their vault constraint becomes: $\frac{N_t}{L_t} = \frac{10}{5} = 2 \to \frac{9}{5} = 1.8$. On the other hand, they could choose to sell \$1 value of ETH at its current price to repurchase stablecoins $\to \frac{10-1}{5-1} \to \frac{9}{4} = 2.25$. In fact this trade has increased their collateralization ratio. It can be noted that it is due to this dynamic, the authors of [1] argue, that a deleveraging spiral occurs.

On the other hand, if the speculator expects ETH to go up (say, $r = 1.1$) then their vault constraint can only improve. In such a case, the speculator might as well consider increasing their position (to better maximize utility) as they face little to no penalty in terms of their leverage constraint. For example, using numbers from the last time but with the new return $\frac{10}{5} \to \frac{(11+1)}{(5+1)} = 2$. Here $\Delta = 1$ and the leverage constraint has not changed over a single time step due to appreciation in ETH holdings ($10 \to 11$) but the speculator has increased their utility.

2 Protocol

The protocol is concerned with peg stability. In our case this will mean setting supply as near to demand as possible through changes on stability fees. However, we will make these changes gradually (smoothly) to allow market to react and plan accordingly

$$J^1 = \sum_{t=1}^{T} ||D_t - S_t||^2 + ||\delta\alpha_t||^2$$

$$J^1 = \sum_{t=1}^{T} ||D_t - (S_{t-1} + \Delta_t)||^2 + ||\delta\alpha_t||^2$$

3 Stackelberg Game

In the formulation, we should consider what are "shared" constraints versus private constraints to each player. I propose that the supply, collateral and vault constraints belong *privately* to the speculator. My reasoning for this is that it is their decisions that control supply, not the protocol's. The protocol can only attempt to influence it. The rare constraint belongs to the protocol and is updated at every time step according to their decision. The idea is now to solve a Stackelberg game between the speculator and the protocol out across a finite (discrete) time horizon T. Conceretely, that means we solve an optimization problem involving variables indexed from time $t \in \{0, 1, ..., T\}$ where $t = 0$ represents the initial conditions of the system.

Now, I attempt to formalize the problem and notation. We model the decentralized stablecoin as a shared dynamical system in which both the protocol and speculator participate. Market demand at time t (D_t) is treated as exogenous to the system but we assume we have access to a forecast of this demand which is a finite sequence of numbers denoted $D_{0:T}$. Note that we are also assuming both the speculator and the protocol have access to a forecast of Ethereum prices over the next T time steps which is denoted $p^E_{t\in\{0:T\}}$. The state variables are $S_t, N_t, \alpha_t \; \forall t$ which represent quantity of supply, quantity of collateral, and current stability fee, respectively. Note that the protocol only has direct control over this fee, while the first two quantities are affected directly by speculator decisions. The speculator's decision variables are $u^2 = [\Delta_0, ..., \Delta_{T-1}]$ which are burn/minting actions and the protocol's decisions are denoted $u^1 = [\delta\alpha_0, ..., \delta\alpha_{T-1}]$ which are changes to the stability fee.

3.1 State

We denote: $\mathbf{X} = \left[S_0, N_0, \alpha_0, S_1, \dots N_T \alpha T\right]$ which is the total concatenation of state variables across the time horizon T and $\mathbf{U} = \left[u^1_{0:T-1}, u^2_{0:T-1}\right]$ where $u^1 = \left[\delta\alpha_{0:T}\right]$ and $u^2 = \left[\Delta_{0:T}\right]$. We also define $X^1 = \left[\alpha_{0:T}, u^1_{0:T}\right]$ and $X^2 = \left[S_{0:T}, N_{0:T}, u^2_{0:T}\right]$ to represent the partitioning of all primal variables into the sets which affect of the respective optimization problems individually.

3.2 Constraints

Player 2 has the following *private* constraints. These are the vault, supply and ETH (collateral) bank dynamics which are the speculator's decision variables.

$$A_t - \beta S_t \geq 0 \;\; (c_V)$$

$$S_t = S_{t-1} + \Delta_t \;\; (c_S)$$

$$N_t = N_{t-1} + \frac{p^S}{p^E}\Delta_t \;\; (c_N)$$

Which must be satisfied for all t.

Player 1 has the private constraint:

$$\alpha_t = \alpha_{t-1} + u_t \;\; (c_\alpha)$$

which is the evolution of the rate over time and it is treated *exogenous* to the speculator. However, it appears in the speculator's cost function.

Then, we have a Stackelberg leader-follower game of the following form:

$$
\begin{aligned}
\min_{X_t^1, \delta\alpha_t} \quad & J^1(X_t^1, \delta\alpha_t) \\
& \text{s.t.} \quad \alpha_t = \alpha_{t-1} + \delta\alpha_t \\
& \min_{X_t^2, \Delta_t} \quad J^2(X_t^2, \Delta_t) \\
& \qquad \text{s.t.} \quad S_t = S_{t-1} + \Delta_t \\
& \qquad\qquad N_t = N_{t-1} + \frac{p^S}{p^E}\Delta_t \\
& \qquad\qquad A_t - \beta S_t \geq 0
\end{aligned}
$$

Following ideas given in [2], we will attempt to solve the nested problem by re-framing it as *open-loop* bilevel game and add the KKT conditions of player 2 (speculator) across $\{0, 1, \ldots, T\}$ to the protocol's constraints (player 1). For this, we need player 2's Lagrangian. We take the gradient with respect to *player 2's decision variables* (check this, because I'm unsure if this should also include the state variables) and set to 0 to satisfy first-order necessary conditions. We must also add the complementarity conditions for player 2's inequality constraints (which in our case are just the vault constraints at every time step)

Note, I think the answer to my question in red above is that we should be taking ∇ wrt the variables relevant to player 2's decision and constraints, that is $X^2 = \{S_t \cup N_t \cup u^2_{0:T}\} = \left[S_0, N_0, \ldots, N_T, \Delta_0, \ldots \Delta_T\right]$

Player 2's Lagrangian:

$$\mathcal{L}^2(X^2, u^2, \lambda) = \sum_{t=0}^{T} J_t^2 - \lambda_{\mathbf{V}}{}^\intercal c_V - \lambda_{\mathbf{S}}{}^\intercal c_S - \lambda_{\mathbf{N}}{}^\intercal c_N$$

Let the concatenation of player 2's multipliers be given by $\lambda = \left[\lambda_V \lambda_S \lambda_N\right]$ where each component is divided up further into T variables, one for each time step.

Then, player 2's KKT Conditions are given by:

$$
\begin{aligned}
\nabla_{X^2}\mathcal{L}^2(X^{2*}, \lambda^*) = 0 \\
c_s(X^*), c_B(X^*) = 0 \\
c_V(X^*) \geq 0 \\
0 \leq \lambda_V \perp c_v(X^*, \lambda^*) \geq 0
\end{aligned}
$$

The subset of player 2's dual and decision variables which satisfy their KKT conditions are concatenated with player 1's decision variables in the final formulation of the optimization problem. Thus, the primal variables of the final problem "y" are given by $y = \left[X^1_{0:T}, \lambda^*, X^{2*}_{0:T}\right]$. Recall that $X^{2*}_{0:T}$ is the concatenation of S^*_t, N^*_t, u^{2*}_t across all t. We let X^1 denote the concatenation of player 1's state and decision variables which are $[\alpha_t, \delta\alpha_t]$ across all t Also, we can use a different symbol to denote Player 1 (protocol's multipliers) μ although this does not appear below.

Finally, we get the bilevel optimization problem we can solve to extract the protocol's optimal decisions under this framework:

$$
\begin{aligned}
\min_{X^1_{0:T}, \lambda^*, X^{2*}_{0:T}} \quad & \sum_{t=1}^{T} ||D_t - (S_{t-1} + \Delta_t)||^2 + ||u^1_t||^2 \\
\text{s.t.} \quad & \alpha_{t+1} - \alpha_t - \delta\alpha_t = 0 \ \ \forall t_{1:T} \\
& \nabla \mathcal{L}^2(X^{2*}, \lambda^*) = 0 \\
& S_t - S_{t-1} - \Delta_t \ \ \ = 0 \ \ \forall t \\
& N_t - N_{t-1} - \frac{p^S}{p^E}\Delta_t = 0 \ \ \forall t \\
& c_V(X^*) \geq 0 \ \ \text{vault constraint} \\
& 0 \leq \lambda_V \perp c_v(X^*, \lambda^*) \geq 0 \ \ \text{complementarity} \\
& \lambda^* \geq 0 \\
& X_{lb} \leq \{X^{1*}, X^{2*}\} < X_{ub} \ \ \text{primal bounds}
\end{aligned}
$$

where p^S is given by the *market-clearing* condition from [1] at every time step:

$$
p^S = \frac{D_t}{S_{t-1} + \Delta_t}
$$

4 Other Considerations

- Based on the analysis [1], even this model may suffer when their is constant downward pressure on ETH
- Addition of uncorrelated assets to reserves, and as acceptable Collateral $\rightarrow$ *Multi-Collateral Dai.* In fact, maybe our framework and our paper is ripe for this analysis (capable of handling additional complexity like this)
- External forces on DAI price are not considered
- Addition of Lending Protocols or Protocol-Owned Curve Liquidity Pools will enrich the problem (distributed control)

- To-Do : still wanting to find how this can be applied to siegniorage models, custodial coins, my hunch is it can be easily extended to fractional-reserve models (Frax or Titan)

References

[1] Ariah Klages-Mundt and Andreea Minca. "(In)stability for the blockchain: Deleveraging spirals and stablecoin attacks". In: (2021).

[2] Oguzhan Akcin et al. "A control theoretic approach to infrastructure-centric blockchain tokenomics". In: *arXiv preprint arXiv:2210.12881* (2022).