Sensitivity Analysis

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1 Introduction

To determine sensitivity of the option value with respect to several parameters, we basically have to determine the (be it discrete) derivative of the option value with respect to the particular parameter. To determine this, we first need to calculate the option value. There are several ways to do this, of which three will be treated here. Section 2 will start off using the Black-Scholes partial differential equation, where an elaborate sensitivity analysis will be done. Section 3 will follow with the more straightforward way of running a Monte Carlo simulation as is done in the first assignment of this course. Section 4 contains a third way to calculate option value by making use of a binomial model. Both section 3 and 4 compare results with the Black-Scholes method to say something on the sensitivities of these methods. The report ends with a conclusion and some remarks on the Black-Scholes method.

2 Black-Scholes

A most prominent way to calculate option value, is by using the Black-Scholes partial differential equation, defined as follows:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{1}$$

where V is the option value, σ the implied volatility (we will use 'volatility' in this report), t time to expiry, r the risk-free interest rate (also called drift in this report) and S the current stock price. Determining the solution can be done in several ways. We use the `Matlab` function `blsprice`, which derives cumulative probability distributions analytically and uses the table-values to determine a solution. It uses σ, t, r, S, exercise price K and yield as input and gives V. Determining sensitivity, then, can be done in several ways. We choose to first determine the sensitivity of V with respect to each one of the asked variables (S, σ, t and r) keeping the rest constant at a reference value (given in Tab. 1). This reference (annual values of the first assignment) is for call options in-the-money and for put options out-of-the-money. After this 'one-dimensional' analysis, we will focus on the broader picture. There, as it is on the real market, all four factors may vary and this may alter our view of the sensitivity of the option value, leading to a 'multi-dimensional' analysis. As already mentioned, option sensitivity to a specific variable, is basically its change with respect to changes in the specific variable, i.e. its derivative. This will be used in the following to describe sensitivity.

Table 1: Reference values used in the sensitivity analysis.

Variable	Symbol	Value
Exercise Price	K_0	€12,-
Risk-free interest rate	r_0	0.02 per year
Volatility	σ_0	0.2 per year
Starting stock price	S_0	€11,-
Yield	-	0 (default)
Time to expiry	t_0	82/252 year

2.1 One-dimensional sensitivity

2.1.1 Stock price S

To investigate sensitivity to the starting stock price, we calculate the option price for starting stock prices ranging from €8 to €18. The results are shown in Fig. 1. The upper panels are as expected: higher starting stock prices mean higher call option value because the chance for you to exercise will be higher, as you already start with a better position with respect to the exercise price (vice versa for put options, hence the decrease with starting stock price). The derivatives of V with respect to S for call and put options are shown in the lower panels, where we see that the sensitivity of call options to the starting stock price is larger (positive) for larger S (up to 1 €/€). For put options this is larger (negative) for smaller S (down to -1 €/€). Concerning the stock price with respect to the exercise price, it is important to discuss three situations: we call a call option in-the-money if the strike price is lower than the starting stock price (i.e. $K < S$), at-the-money if they are equal ($K = S$) and out-of-the-money if the strike price is higher than the starting stock price ($K > S$). For put options, this is the other way around ($K > S$ is in-the-money and $K < S$ is out-of-the-money). There is no significant deviation from the continuous effect of increasing S when in either of the domains in-, at- or out-of-the-money. The second derivative of V with respect to S might be having a peak at-the-money, however, but this can also be coincidence.

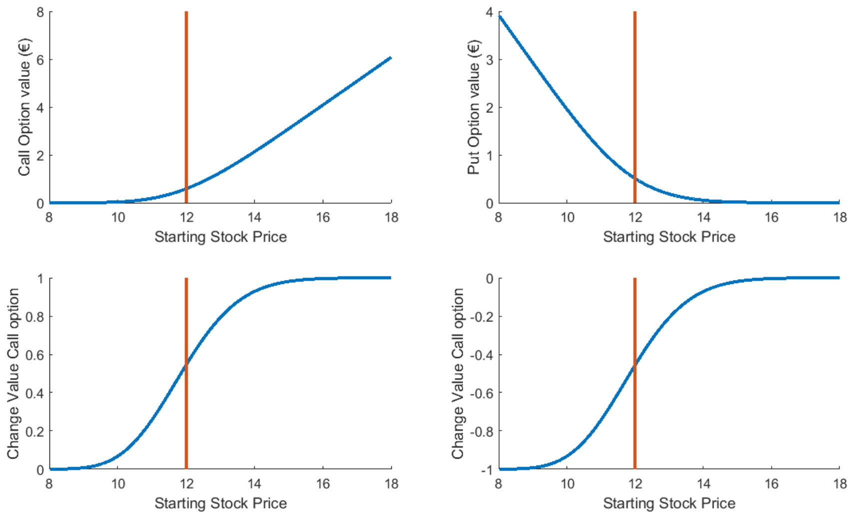


Figure 1: Sensitivity of option value to changes in starting stock price (in €). The results for call options are displayed on the left side, and for put options on the right side. The red vertical lines show the position of the exercise price.

2.1.2 Implied volatility σ

The upper panels in Fig. 2 are relatively straightforward. Both the value of call and put options increase with increased volatility. This is expected, as volatility induces strong variance of which options profit, and people do have not exercise if this variance turns the wrong way. In the lower panels, the sensitivity of the call and put options with respect to changes in σ are displayed (note the heavily exaggerated σ-axis, to show the development of the lower panels). At first, the sensitivity to σ rises very rapidly with increased σ. However, around $\sigma \approx 0.5$, it drops again but remains above zero. For other situations of in-, at- and out-of-the-money, the picture is qualitatively the same.

2.1.3 Duration (expiry time) t

For both call and put options, lengthening the expiry time (Fig. 3) will increase the value of the option. This makes sense as the longer the expiry time, the more time there is to let the market volatility change the course of the stock value, which in turn makes a call or put option valuable. The results of sensitivity to changes in expiry time are comparable with those of sensitivity to changes in volatility: rapid increased sensitivity for small expiry times, and sensitivity drops slowly (retaining positive values) after that. Note that for small t, the sensitivity of put option value w.r.t. t is negative. For other situations of in-, at- and out-of-the-money, the picture is qualitatively the same although the sharp peak in the lower panels is shifted or smoothened when increasing S relative to K. The negative feature seen in the lower right panel may also disappear.

2.1.4 Interest rate r

The results are displayed in Fig. 4. It is visible that call options increase in value with higher drift rates per year, while put options decrease in value. Both of these observations can be explained by the fact that with higher drifts, the market itself pushes the stock price up to even above exercise price, which increases call option value and decreases put option value. Sensitivities to changes in drift are very large (negative) at low drift levels for put options, and positive for call options, with a maximum around roughly 75% per year. Note that these figures contain unrealistic drift rate levels, but this nicely illustrates hypothetical consequences of

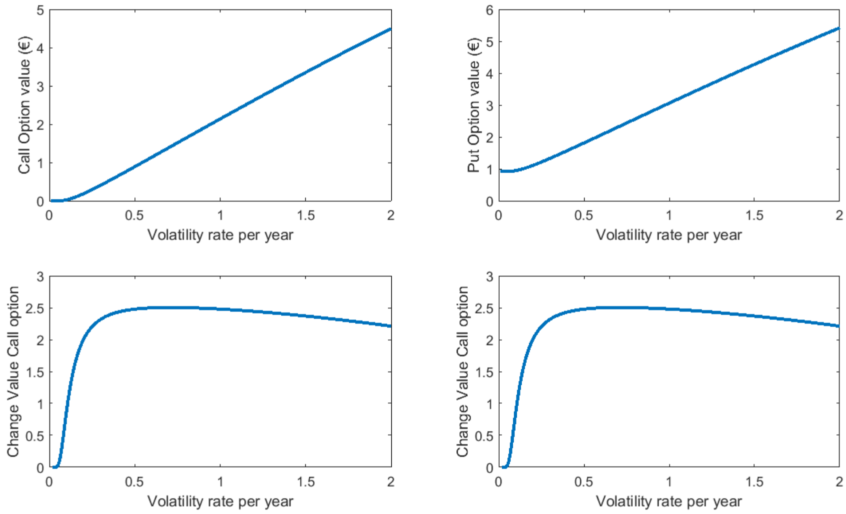


Figure 2: Sensitivity of option value to changes in implied volatility. The results for call options are displayed on the left side, and for put options on the right side.

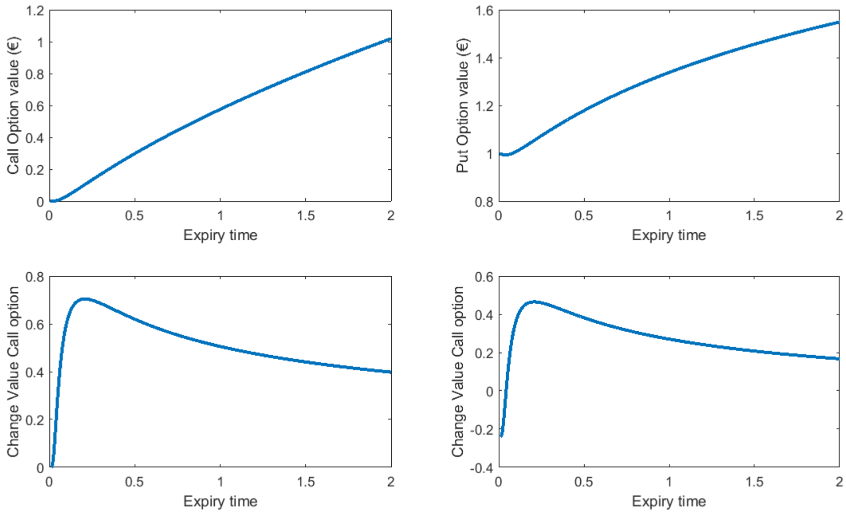


Figure 3: Sensitivity of option value to changes in expiry time (in years). The results for call options are displayed on the left side, and for the put options on the right side.

different drift rates to sensitivity of the option value. For other situations of in-, at- and out-of-the-money, the picture is qualitatively the same, although the peak in sensitivity of call options (lower left panel) may shift to the left or even disappear.

2.2 Multi-dimensional sensitivity

2.2.1 Simplification of the multi-dimensional problem

From the earlier paragraphs, a summary can be made which is visible in Tab. 2. As one can however imagine, these sensitivities are highly non-linear so that with different reference values, this table might look different. This is why we introduce a multi-dimensional sensitivity analysis. As we can at maximum plot 4D plots (3D axis plus coloring), we can do such an analysis only if we assume one of the parameters roughly constantly affecting V, that is, the changes in V due to changes in this parameter is invariant in sign if we change the

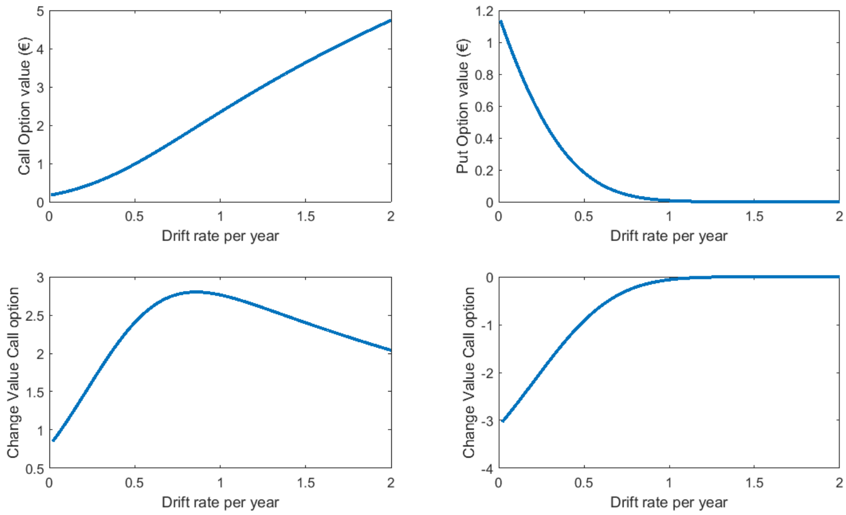


Figure 4: Sensitivity of option value to changes in interest rate. The results for call options are displayed on the left side, and for put options on the right side.

reference values. The choice is relatively obvious if we look at Tab. 2, namely the starting stock price S. Although the graphs of $V(S)$ and $\frac{\partial V}{\partial S}$ are quite curved, we signs are always the same. This is expected as higher stock prices will directly increase call option value due to more expected profit and decrease put option value due to less expected profit. So for now, keep $S =$ €11. That the behavior of V due to changes in S mainly depends on S itself, is of course partly an assumption. But the following nicely illustrates at least the effect of the other non-linearities.

Table 2: Summary of 1D sensitivity analyses, i.e. keeping the rest constant. A + sign indicates positive values, a - sign negative values and a +/- positive for small and negative for higher values of the parameter (-/+ vice versa).

Variable	Call	Put
$\partial V/\partial S$	+	-
$\partial^2 V/\partial S^2$	+	+
$\partial V/\partial \sigma$	+	+
$\partial^2 V/\partial \sigma^2$	+/-	+/-
$\partial V/\partial t$	+	-/+
$\partial^2 V/\partial t^2$	+/-	+/-
$\partial V/\partial r$	+	-
$\partial^2 V/\partial r^2$	+/-	+

2.2.2 Four-dimensional results

As mentioned earlier, using S constant, we can visualize V for various values of σ, r and t. Results for the sensitivities are visualized in Fig. 5. These figures can give us insight in points where the general (3D) sensitivity of the option value is very high or very low. This is interesting because that way, we can define (keeping S constant) extremes as points of large fragility or robustness of the option value with respect to changes in the variables. The figures contain 100x100 pixels (more would strongly increase computational time), which might explain some visible discontinuities. First on the reading of these figures. Colors indicate the option value gradient: call option gradients on the left and put option gradients to the right. The displayed gradient is calculated as the vectorial sum of the three gradients in each direction: per one unit drift in the x direction, per one unit volatility in the y direction and per year in the z direction. The figures are made to see three dimensional positions of the largest (or smallest) sensitivities of the option value. The top two panels (i.e. one for call options and one for put options) show for each r x σ coordinate a colored dot at a time (z) coordinate where the sensitivity is highest, whose color then shows the magnitude of this high sensitivity. Lower values

are skipped, so not the whole horizontal plane is filled. This means that to find the highest sensitivities, one should look at the yellowish colors in the top two panels. The lower two panels show, in contrast to the upper two, minima for every r x σ coordinate.
In the hypothetical scenario that the domains of r, σ and t are all $[0, 1]$ (which can of course be ridiculous), points of large fragility (high sensitivity; preferably yellowish colors) can be found in the upper panels. For call options these can be found when expiry time is low and drift and volatility are high. This is interesting, as for low drift and volatility, the maximum sensitivity can be found at high expiry time. Put options show high sensitivity in several corners. This clearly shows the high non-linearity of the BS equation. Dark blue colors in the lower panels show points of large robustness (low sensitivity) of the option value to changes in the parameters. Roughly independent there of σ, call option sensitivity is very low at low values of r and t. Put option sensitivity is very low in a small band crossing the r-σ plane diagonally with low values of t, and it is very low at high values of t combined with low values of r and high values of σ.

3 Monte Carlo method

In this section we show our findings of the Monte Carlo method and compare them with the Black-Scholes model. Monte Carlo methods are useful for simulating a large number of random walks of possible option prices. These prices are averaged out to produce a value which should be a realistic valuation over the possible outcomes. Monte Carlo methods can handle a lot of parameters, but their reliance on large numbers can make them hard to compute. To compare the Monte Carlo method with the Black-Scholes model, we use the script of assignment 1 and modify it to output both call and put prices. When using the same parameters as in section 2, the Monte Carlo method returns the same results over 10,000 runs. Since the pricing in large Monte Carlo simulations are the same as in the Black-Scholes model, we can state that their sensitivity will be the same as well. Although, of course, pricing as calculated by the Monte Carlo method is also dependent on the amount of runs.

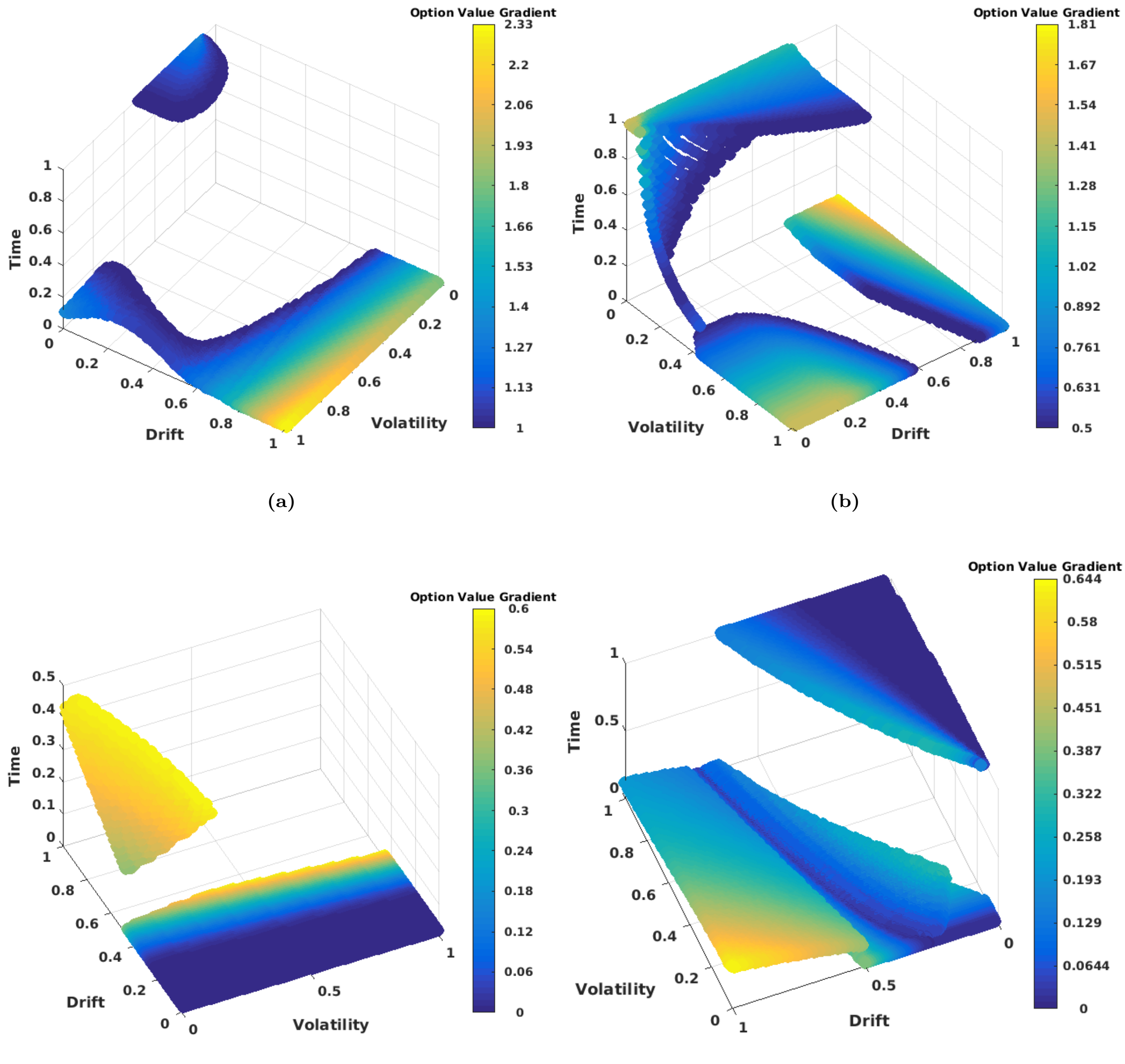
Table 3: Monte Carlo Simulation results with different sizes N, values V for call and put options and the standard deviations σ and computational time T. Results are averages over 30 trials.

		Call			Put		
N	T (s)	V_{call}	σ_{call}	margin of error call	V_{put}	σ_{put}	margin of error put
10	6.27	0.1638	0.1204	0,074625	1.0772	0.3766	0,233419
50	12.52	0.1818	0.0687	0,019043	1.1019	0.1176	0,032597
100	22.97	0.1833	0.0406	0,007958	1.1201	0.1014	0,019874
200	41.08	0.1869	0.0414	0,005738	1.0966	0.0695	0,009632
500	94.765	0.1834	0.0215	0,001885	1.1051	0.0365	0,003199
1,000	164.38	0.1861	0.0137	0,000849	1.1145	0.0246	0,001525
2,000	359.74	0.1854	0.0125	0,000548	1.1108	0.0210	0,00092
5,000	801.14	0.1857	0.0066	0,000183	1.1178	0.0131	0,000363
10,000	1859.19	0.1849	0.0045		1.1144	0.0100	

Table 3 shows the results for varying N between 10 and 10,000. Each run creates a call and a put vector of 30 trials. The values μ_{call}, σ_{call}, μ_{put}, and σ_{put} are the averages and standard deviations of these vectors. To assert whether the result from our samples are close to the large simulations a margin of error is calculated. The formula used for this is $z \cdot \sqrt{\frac{\rho(1-\rho)}{N}}$, where z is the z-value, ρ is the sample proportion and N is the sample size. This shows that the smallest sample size the gives the correct price in two decimals with a 95% confidence interval is at N=200, meaning that increasing N will not effectively change the resulting option value. So for the sensitivity analysis, we can safely use N=200.

3.1 Complexity

While the outcomes of the Black-Scholes model and the Monte Carlo method of assignment 1 yield similar results, the Monte Carlo simulation has a serious computational disadvantage. This difference lies in the fact that the Black-Scholes model can be calculated analytically, while for the Monte Carlo simulation a number of trials is necessary to be able to derive conclusions. Doing just one or several trials would lead to unreliable



(a) (b)

(c) (d)

Figure 5: Sensitivity of option value to changes in interest rate (drift), volatility and expiry time. The results for call options are displayed on the left side, and for the put options on the right side. Surfaces at maximum option value gradients per σ and r are shown on the top panels and surfaces at minima are displayed in the bottom.

conclusions, since the return values would be influenced too much by the randomness in the function. These differences in the amount of calculations are thus substantial: the computation time for a `blsprice` is around 0.09 s, while a Monte Carlo simulation of N=10,000 is around 60.77 s. While in assignment 1 this did not present any problems, doing this when calculating for example the sensitivity in a vector `[0:0.01:10]` will result in a program that takes an estimated 1012 min to calculate. This is too slow for performing analysis in real time. On the other hand a sample size too small will lead to less confident conclusions, so a trade off between time and precision has to be made.

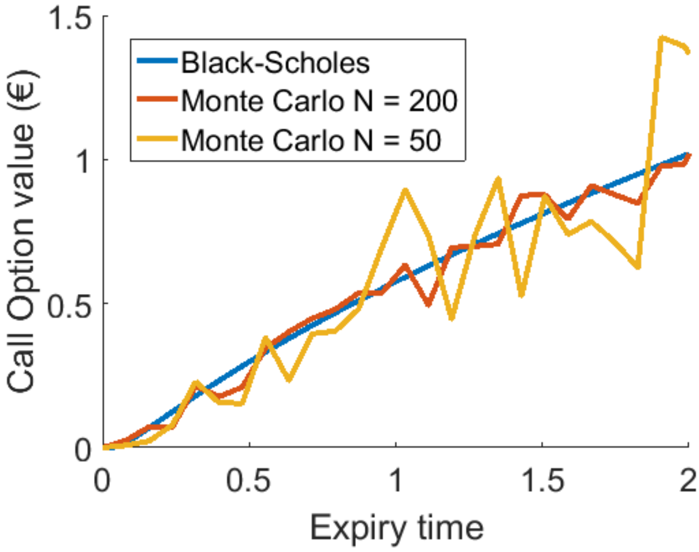


Figure 6: Sensitivity of call option value to changes in expiry time (in years). Colors as indicated in the legend.

3.2 Parameters

As mentioned before, a correct application of the Monte Carlo method will yield the same results as a Black-Scholes model. However, the amount of trials is an extra parameter which can influence the sensitivity. As showed in 3 a lower sample size will lead to less reliable option price valuations. Therefore we assume that plotting a Monte Carlo method with a smaller sample size *ceteris paribus* will result in a more fluctuating path than a bigger sample size. 6 shows that this is correct. The reason for this is that a smaller sample size has less prices to average out, so it will be more exposed to outliers in a normal distribution.

4 Binomial Option Pricing Model Simulation

The Binomial Option Pricing Model (BOPM) is a model first proposed by Cox, Ross, and Rubinstein in 1979 [1]. There are lots of deviations on the original model. We, however, will use the original and denote it as the CRR model. Essentially, the model uses a discrete-time model of the varying price over time of the underlying financial instrument. It is a very simple model that uses an iterative procedure to price options, allowing for the specification of nodes, or points in time, during the time span between the valuation date and the option's expiration date.

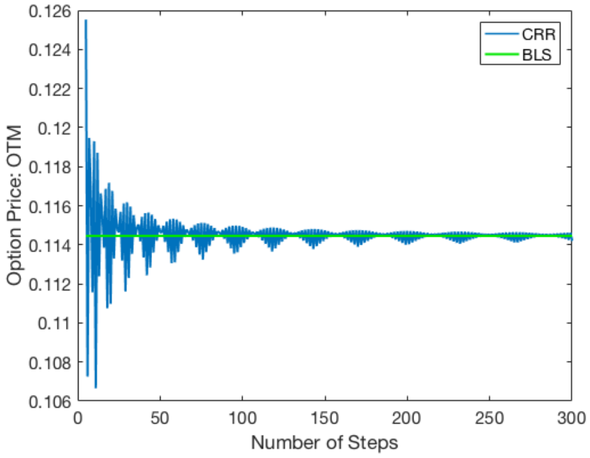
4.1 Comparing the Binomial and Black-Scholes option pricing models

For this reason the CRR is often used to evaluate American options. Therein it obviously differs from the Black-Scholes method, that assumes a fixed exercise date -and can therefore only be used for European options. Although computationally slower than the Black–Scholes formula, binomial models in general can be used for a broader range of applications, and are also easier to use.
We have compared the pricing of a call option by using the BS method with using the CRR method. Again the reference values are used (Tab. 1). The CRR model valuates the price of an option on different time-steps. Often this is visualized as a tree structure, where larger tree structures mean more steps. In order to compare the BS method to the CRR method, we tried to investigate how both option prices will converge to each other as the number of steps increase. Note that the BS option price obviously remains as the number of steps increase. This is due the fact that the BS method does not valuate at discrete-time events. We will make a distinction between two scenarios. In Fig. 7 three scenarios are presented: one where the call option is out-of-the-money (a), one where the call option is in-the-money (b) and one where the call option is at-the-money (c).

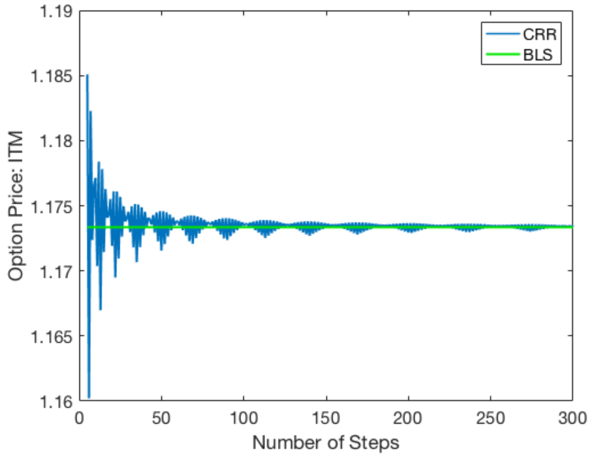
[1] Cox, J.C., Ross, S. A., Rubinstein, M. (1979). Option Pricing: A Simplified Approach. *J. of Fin. Econ. (3)*, 229

4.2 Parameters

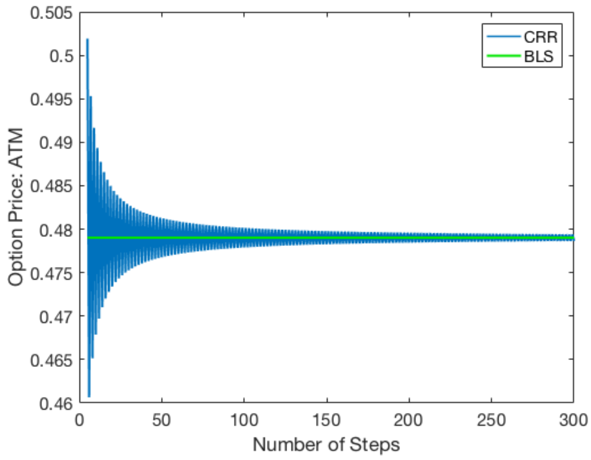
In both figures we see that the CRR option price converges to the BS option price, as the number of steps increases. That Binomial Option Pricing models actually fully converge to the BS if steps near the infinite, is proven by Cox, Ross, and Rubinstein. Furthermore, the number of steps is an important parameter with respect to the sensitivity. As you can see, when the number of steps is chosen in the right way, the CRR option valuation converges towards the option valuation of the BS. And at that point they are almost the same, and therefore it is not unimaginable that both models have a similar sensitivity.



(a)



(b)



(c)

Figure 7: This figure shows the convergence of the CRR model's option price to the BS model's option price, for an out-of-the-money situation (a), an in-the-money situation (b), and an at-the-money situation (c).

5 Conclusive Remarks

As mentioned before, the Black-Scholes model proved to be much faster than Monte Carlo simulations. However, the Black-Scholes model is not without issues. First of all, it assumes that there is a risk-free interest rate to compare to. Unfortunately, even the least risky stocks or bonds still are not without risk. Second, it applies only to European options. Third, it implies the volatility remains unchanged during the period between buying the option and the expiring date. This is a huge assumption, since volatility can change a lot[2]. Fourth,

[2]Schwert, G.W. (1989). Why Does Stock Market Volatility Change Over time. *The Journal of Finance, Vol XLIV*, p. 1145

it assumes that the option follows a lognormal distribution (i.e. that the the returns on the underlying have a normal distribution). This assumption does not hold for every stock either, since an analysis of the Dow Jones showed that big price changes are far more common than Gaussian models allow[3]. Finally, it assumes that no dividend is applicable. This is also not a realistic assumption, since dividends can influence whether a trade is profitable or not.

The Monte Carlo simulation is less vulnerable for these disadvantages, since we can modify the function more easily. For example, instead of using random numbers from a normal distribution, we can use other distributions in instances where a different distribution is assumed.

Concluding, three methods of estimating option value have been analyzed: analytically using Black-Scholes, numerically using Monte Carlo and by a binomial model. The main conclusion of this report is that, if the amount of simulations in the Monte Carlo and the number of steps in the binomial model are chosen well, the three methods give approximately the same results. Concerning sensitivity, the different variables have different effects on call and put options (as summarized by Tab. 2). The non-linearity of the system is clearly shown when analyzing the option value behavior in a multi-dimensional way (Fig. 5).

[3]Mandelbrôt (2004), the Misbehavior of Markets. p. 138-163.