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The Soda Can Optimization Problem: Getting Close to the Real Thing

Kirthi Premadasa, Paul Martin, Bryce Sprecher, Lai Yang, and Noah-Helen Dodge

Abstract: Optimizing the dimensions of a soda can is a classic problem that is frequently posed to freshman calculus students. However, if we only minimize the surface area subject to a fixed volume, the result is a can with a square edge-on profile, and this differs significantly from actual cans. By considering a more realistic model for the can that consists of six components, including varying wall thickness, and minimizing the total metal volume, we arrive at an optimal can that has dimensions more in-line with actual manufactured cans. This model indicates an optimal radius of 2.83 cm and an overall height of 15.2 cm, which is closer to the dimensions of real cans today than what is obtained from assuming the can is a right circular cylinder. The calculations involved would serve as a useful undergraduate modelling project.

Keywords: optimization, calculus, modelling.

1. INTRODUCTION

Minimizing the cost of a 12 ounce soda can is a classic optimization problem that is posed to many first-semester calculus students. The rationale for using this problem is that students can easily relate to a soda can and the large number of cans produced makes even small improvements significant. The basic model minimizes the surface area of a cylindrical can subject to the volume constraint of 12 fluid ounces (355 cm^3) and involves the following steps.

1. Obtain the surface area of the cylindrical soda can as a function of its height h and radius r, $A = 2\pi rh + 2\pi r^2$.

Address correspondence to Dr. Kirthi Premadasa, Department of Mathematics, University of Wisconsin - Baraboo/Sauk County, Mathematics, 1006 Connie Road, Baraboo, WI 53913, USA. E-mail: kirthi.premadasa@uwc.edu

2. Use the volume constraint ($355 = \pi r^2 h$) to eliminate one of the variables in the surface area function as $A = (710/r) + 2\pi r^2$.
3. Use calculus to show that the minimum area occurs at $r = 3.84$ cm and $h = 2r = 7.67$ cm

This differs significantly from an actual soda can where $r = 3.26$ cm and $h = 12.2$ cm.

We were able to locate two attempts to explain this discrepancy. One approach (posed in Stewart [2]), assumes that the circular tops and bottoms are cut from either square or hexagonal pieces and the excess metal is treated as waste. This makes the optimal height-to-diameter ratio slightly larger than one. A second model (Schroeder [1]) incorporates thicker material for the top and bottom, but the basic cylindrical shape is retained. This produced an optimal radius of 2.82 cm and height of 14.17 cm for a height-to-diameter ratio of 2.51. Here we present a model that includes variable metal thicknesses for each component of the can, but also takes into account the non-cylindrical shape of an actual soda can (Figure 1).

We start with a circular lid, conical frustums at the top and bottom of the cylindrical part, and an annular strip and inverted spherical cap at the bottom (Figure 2).

This treatment, as we will soon see, gives an optimal radius of 2.83 cm (in the cylindrical part) and a total height of 15.23 cm.

Figure 1. The non-cylindrical shape of a soda can.

Figure 2. The shape of the can showing the area components.

2. THE METHOD

Our basic approach is to consider a more realistic model of an actual 12 ounce soda can and find the dimensions that minimize cost. This is equivalent to minimizing the volume of aluminum metal in a can. A detailed diagram of our model can (Figure 3) shows the six components with parameters for each component described next.

We shall now describe some of the variables used in the method (Table 1).

We make assumptions about the shape parameters based on measurements from actual production cans to keep the number of variables down to just two, the height h and radius r of the cylindrical part of the can. Starting with the two frustum components, we assume the smaller radii of each (r_1 and r_2) are both 0.66 cm less than r. In other words $r_1 = r_2 = (r - r_1)$ cm. We also set the angles that the top and bottom conical frustums make with the horizontal (θ_1 and θ_2) to be constant in agreement with measured values from actual cans. The small annular ring on the bottom has outer radius equal to $r_2 = (r - 0.66)$ cm and we set the inside radius to $r_3 = (r - 0.98)$ cm. (Note that r_3 is also the base radius of the spherical cap.) One final assumption is assume that the angle of elevation α from the bottom edge of the spherical cap to its highest point be held constant.

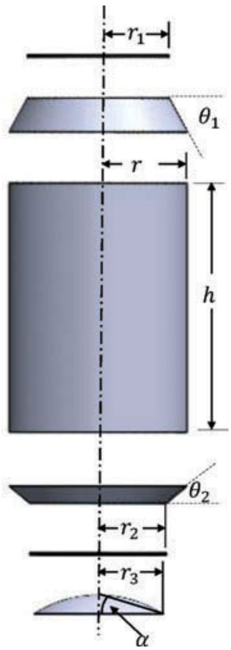

Figure 3. The model with parameters for each component.

3. THE CALCULATION

3.1. Build the function *MV*(*r*,*h*) for the volume of metal in a can

The surface area of the top conical frustum is given by $SA_{tf} = \pi\,(r + r_1)\,s_{tf}$. Similarly, the surface area of the bottom frustum is $SA_{bf} = \pi\,(r + r_2)\,s_{bf}$.

The surface area of a spherical cap is $SA_{sc} = \pi\left(r_3^2 + H^2\right)$. Let d_1, d_2, and d_3 be the thickness of the top, sides and frustums, and bottom, respectively. We now compute the volume of metal in each component with $MV_t, MV_{tf}, MV_c, MV_{bf}, MV_{as}$, and MV_{sc} denoting the metal volumes of the circular top, top frustum, cylindrical part, bottom frustum, annular strip, and spherical cap, respectively.

The top radius is $(r - 0.66)$ cm with thickness d_1, thus $MV_t = \pi\,(r - 0.66)^2 d_1$.

Next consider the two frustums with large and small radii at r cm and $(r - 0.66)$ cm, respectively. The slant lengths satisfy

$$\frac{s_{tf}}{0.66} = \sec\theta_1 \text{ and } \frac{s_{bf}}{0.66} = \sec\theta_2$$

Table 1. Description of the variables used in the metal volume calculation

Variable	Description
d_1	Thickness of the top
d_2	Thickness of the sides and the frustums
d_3	Thickness of the bottom
SA_{tf}	Surface area of the top frustum
SA_{bf}	Surface area of the bottom frustum
SA_{sc}	Surface area of the spherical cap
MV_t	Metal volume of the top
MV_{tf}	Metal volume of the top frustum
MV_c	Metal volume of the cylindrical part
MV_{bf}	Metal volume of the bottom frustum
MV_{as}	Metal volume of the annular strip
MV_{sc}	Metal volume of the spherical cap
MV	Total metal volume of the can
r	Radius of the cylindrical part of the can
r_1	Smaller radius of the top frustum
r_2	Smaller radius of the bottom frustum
r_3	Radius of the base of the spherical cap
h	Height of the cylindrical part of the can
H	Height of the spherical cap
s_{tf}	Slanted height of the top frustum
s_{bf}	Slanted height of the bottom frustum
θ_1	Angle that the top conical frustum makes with the horizontal
θ_2	Angle that the top bottom conical frustum makes with the horizontal
α	Angle that the chord joining the bottom edge of the spherical cap to its highest point makes with the horizontal

so that $MV_{tf} = \pi\,(2r - 0.66)\,0.66\sec\theta_1 d_2$ and $MV_{bf} = \pi\,(2r - 0.66)\,0.66\sec\theta_2 d_2$.

The cylindrical part has metal volume $MV_c = 2\pi rhd_2$.

The bottom annular strip has volume $MV_{as} = \pi\left((r - 0.66)^2 - (r - 0.98)^2\right)d_3 = 0.64\pi\,(r - 0.82)\,d_3$.

The spherical cap has base radius $(r - 0.98)$ cm, and the height H relates to the angle α as

$$\tan\alpha = \frac{H}{r_3} = \frac{H}{r - 0.98}.$$

Thus, the metal in the spherical cap is given by $V_{sc} = \pi[(r - 0.98)^2 + (r - 0.98)^2\tan^2\alpha]d_3$, which simplifies to $MV_{sc} = \pi(r - 0.98)^2\sec^2\alpha d_3$.

Summing the metal volume of the six components gives us MV as

$$MV = \pi (r - 0.66)^2 d_1 + \pi \ (2r - 0.66)\, 0.66 \sec \theta_1 d_2$$
$$+ \pi \ (2r - 0.66)\, 0.66 \sec \theta_2 d_2 + 2\pi\, rhd_2 + 0.64\pi \ (r - 0.82) d_3\, .$$
$$+ \pi (r - 0.98)^2 \sec^2\alpha \ d_3$$

Expanding and collecting terms this can be written as

$$MV = Qr^2 + Pr + 2\pi\, rhd_2 + N \tag{1}$$

where $Q = \pi(d_1 + \sec^2\alpha d_3)$, and

$$P = \pi [-\,(2)\,(0.66)\, d_1 + (2)\,(0.66) \sec \theta_1 d_2$$
$$+ (2)\,(0.66) \sec \theta_2 d_2 + 0.64 d_3 - 2\,(0.98) \sec^2\alpha \ d_3]$$
$$N = \pi [(0.66)^2 d_1 - (0.66)^2 \sec \theta_1 d_2 - (0.66)^2 \sec \theta_2 d_2$$
$$-\, 0.64\,(0.82)\, d_3 + (0.98)^2 \sec^2\alpha \ d_3]$$

3.2. Deriving the equation for the fluid volume constraint

We describe the additional variables used in this calculation in Table 2.

The volume of the top conical frustum is

$$V_{tf} = \frac{\pi\, h_{tf}}{3} \left(r_1^2 + rr_1 + r^2\right)$$

Similarly the volume of the bottom conical frustum is

$$V_{bf} = \frac{\pi\, h_{bf}}{3} \left(r_2^2 + rr_2 + r_2^2\right)$$

Table 2. Description of the additional variables used in the fluid volume constraint

Variable	Description
V_{tf}	Volume of the top frustum
V_{bf}	Volume of the bottom frustum
V_c	Volume of the cylinder
V_{sc}	Volume of the spherical cap
h_{tf}	Height of the top frustum
h_{bf}	Height of the bottom frustum

where h_{bf} is the height of the bottom frustum. Also, the volume of the spherical cap is

$$V_{sc} = \frac{\pi H}{6}\left(3r_3^2 + H^2\right)$$

We assume the liquid fills half of the top frustum.

The 355 cm^3 volume must be equal to 0.5 (Volume of top frustum) + volume of cylinder + volume of bottom frustum − volume of spherical cap. Using V_c to denote the volume of the cylinder, we get

$$355\,\mathrm{cm}^3 = \frac{1}{2}V_{tf} + V_c + V_{bf} - V_{sc}$$

Now let us calculate these volumes.

By using the angle θ_1 that the top frustum makes with the horizontal, we can write $h_{tf}/0.66 = \tan\theta_1$. Similarly, we get $h_{bf}/0.66 = \tan\theta_2$. Also, recall that both r_1 and r_2 equal $r - 0.66$ cm. Substituting these into the volume formula, we get the volume of the top frustum as

$$V_{tf} = \frac{\pi}{3}(0.66)\tan\theta_1\left((r-0.66)^2 + (r-0.66)\,r + r^2\right)$$

Likewise, the lower frustum volume can be written as

$$V_{bf} = \frac{\pi}{3}(0.66)\tan\theta_2\left((r-0.66)^2 + (r-0.66)\,r + r^2\right)$$

The volume of the cylindrical part is given by $V_c = \pi r^2 h$.

Recall that for the spherical cap that $H/r_3 = \tan\alpha$ and $r_3 = r - 0.98$. Thus, $H = (r - 0.98)\tan\alpha$ and

$$V_{sc} = \frac{\pi(r-0.98)^3\tan\alpha}{6}\left(3 + \tan^2\alpha\right)$$

Combining these expressions and setting to 355 cm^3 we get

$$355\mathrm{cm}^3 = \frac{0.66\pi}{3}\left(\frac{1}{2}\tan\theta_1 + \tan\theta_2\right)\left[r^2 + (r-0.66)^2 + (r-0.66)\,r\right]$$

$$+\,\pi r^2 h - \frac{\pi(r-0.98)^3\tan\alpha}{6}\left(3 + \tan^2\alpha\right)$$

Expanding and collecting terms, we get

$$355 = R_1 r^3 + Q_1 r^2 + P_1 r + \pi r^2 h + N_1 \tag{2}$$

where:

$$R_1 = -\frac{\pi}{6}\tan\alpha(3+\tan^2\alpha)$$

$$Q_1 = 0.66\pi\left(\frac{\tan\theta_1}{2}+\tan\theta_2\right)+\frac{\pi}{2}(3+\tan^2\alpha)\tan\alpha\,(0.98)$$

$$P_1 = -\pi(0.66)^2\left(\frac{\tan\theta_1}{2}+\tan\theta_2\right)-\frac{\pi}{2}(3+\tan^2\alpha)\tan\alpha(0.98)^2$$

$$N_1 = \frac{\pi}{3}(0.66)^3\left(\frac{\tan\theta_1}{2}+\tan\theta_2\right)+\frac{\pi}{6}(3+\tan^2\alpha)\tan\alpha(0.98)^3$$

3.3. Writing the surface area as a function of a single variable

Just as we do in the traditional treatment of the problem, we solve (2) for πrh and substitute this into the MV equation (1) to eliminate h

$$\pi rh = (355-N_1)\left(\frac{1}{r}\right)-P_1-Q_1r-R_1r^2 \tag{3}$$

Substituting (3) into (1), we get the total metal volume in terms of

$$\begin{aligned} MV &= Qr^2+Pr+2\left((355-N_1)\left(\frac{1}{r}\right)-P_1-Q_1r-R_1r^2\right)d_2+N \\ MV &= (Q-2d_2R_1)\,r^2+(P-2d_2Q_1)\,r+2d_2\,(355-N_1)\left(\frac{1}{r}\right)+(N-2d_2P_1) \end{aligned} \tag{4}$$

Next, we differentiate with respect to r and find the critical values to locate the minimum of MV and so the dimensions of the lowest-cost can. Before we do that, we list the parameter values as obtained from an actual 12 ounce soda can.

3.3. Measurements

The three metal thicknesses were measured from an actual can using a micrometer. The measurements are shown in Table 3.

4. THE OPTIMIZATION PROCESS

Substituting the measurements from Table 3 for the coefficients in (1), (2), and then (4), we get:, $Q \approx 0.177538$, $P \approx -0.103632$, $N \approx 0.040625$,

Table 3. Measurements of the metal thicknesses and the angles

Dimension	Measured value
d_1	0.020 cm
d_2	0.010 cm
d_3	0.03 cm
θ_1	$65° = 1.13$ radians
θ_2	$25° = 0.436$ radians
α	$25° = 0.436$ radians

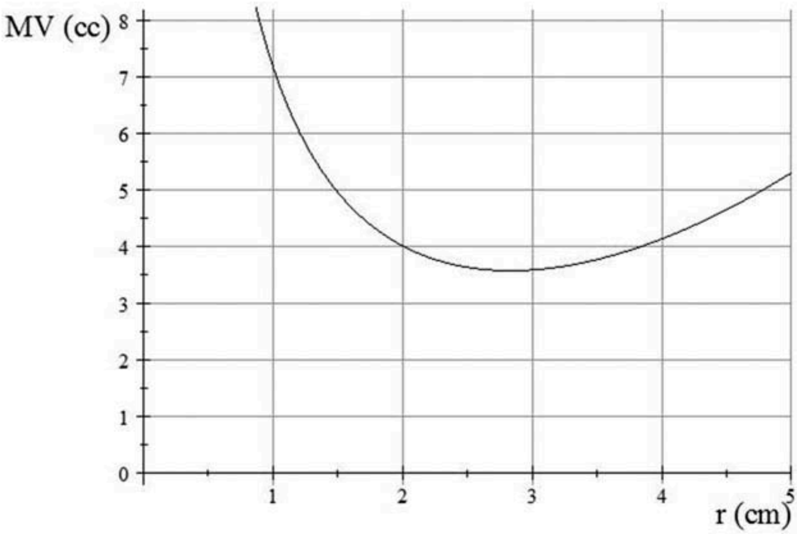

Figure 4. The graph of the total metal volume MV versus the radius r

$R_1 \approx -0.785565, Q_1 \approx 5.499692, P_1 \approx -4.368856,$ and $N_1 \approx 1.202574$. Substituting these values into (4), we obtain the total metal volume MV as

$$MV = 0.193249r^2 - 0.213626r + 0.128002 + \frac{7.075949}{r} \tag{5}$$

The graph of this rational function of r (Figure 4) shows a local minimum between $r = 2.5$ and $r = 3$.

Differentiating, we obtain

$$\frac{dMV}{dr} = -0.213626 - \frac{7.075949}{r^2} + 0.386498r$$

We obtain the real zero of this function to be $r \approx 2.83$ cm by using a computer algebra system.

This is a local minimum as MV'' is positive at $r = 2.83$ cm, but this is also the absolute minimum as there is only one critical point. This is also clearly seen from the graph of MV. Substituting $r = 2.83$ into (2) we get the height of the cylindrical part of the can to be $h = 13.51$cm. Adding the height of the two frusta, the total height of the optimal can comes to $13.51 + 0.66\,(\tan 65° + \tan 25°) = 15.23$ cm. These mathematically predicted dimensions are getting closer to an actual 12 ounce coke can, for which $r = 3.26$ cm and the overall height is 12.2 cm. The metal volume for our optimal 12 ounce soda can is $\approx 3.57\text{cm}^3$. With the density of aluminum at 2.7g/cm^3, our optimal model can mass would be $3.57\text{cm}^3 \times 2.7\text{g/cm}^3 \approx 9.6$g. This is significantly less than the actual ~12.9 g mass of a real soda can.

5. CONCLUSIONS

This project demonstrates that the refined six-component model of a soda can with varying wall thickness predicts an optimal can that is a bit closer to the can shape in current use. Further analysis of our model of a can shows that using the current can radius of 3.26 cm predicts a total height of 11.81 cm (very close to an actual can height of 12.1 cm). This indicates that the volume part of our model is pretty accurate. However, using $r = 3.26$ cm in our model predicts a total can mass of only 9.9 g. This is roughly 25% less than an actual can. This indicates that the model underestimates the mass in one or more parts of the can. This could be due to non-uniformity in the thicknesses across some of the six component areas and potentially the seam that connects the top to the can. An alternate vision is that there may be an opportunity to make cans more cheaply. With a potential of 25% reduction in aluminum usage, it seems a worthwhile endeavor to try to move real cans closer to the model can we studied here.

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BIOGRAPHICAL SKETCHES

Kirthi Premadasa obtained his MS and Ph.D. from Purdue University and has over 22 years of college teaching experience in Sri Lanka and the US. He is an Associate Professor of Mathematics at the University of Wisconsin-Baraboo/Sauk County. He is a Wisconsin Teaching fellow and was awarded the UW Colleges Chancellor award for teaching excellence in 2012 and the UW System's Alliant Energy Underkofler Excellence in Teaching Award in 2013. He is also the vice-chair of the UW-Colleges Math Department

Paul Martin obtained his Ph.D. in Mathematics at University of Wisconsin in 1994. Since then he has been teaching at the UW-Colleges. He is a full Professor of Mathematics and Associate Chair of the UW-MC mathematics department. He has won several teaching awards and he strives to make mathematics real to his students through applications and history.

Bryce Sprecher is a student at the University of Wisconsin-Baraboo/Sauk County. He plans to do a double major in computer science and biology with a neuroscience focus. He also plans to go to graduate school and research /work in artificial intelligence, nanotechnology, nano robotics, cognitive modeling, and neural prosthetics.

Lai Yang was born in Shanghai, China in 1993. She came to Wisconsin after her freshman year at East China Normal University. Then she decided to study at University of Wisconsin Baraboo/Sauk County. She plans to study Actuarial Science.

Noah-Helen Dodge is a Mathematics major at UW-Baraboo/Sauk Co. She hopes to someday earn her Ph.D. in mathematics and become a professor.