Module 02: Small World Networks

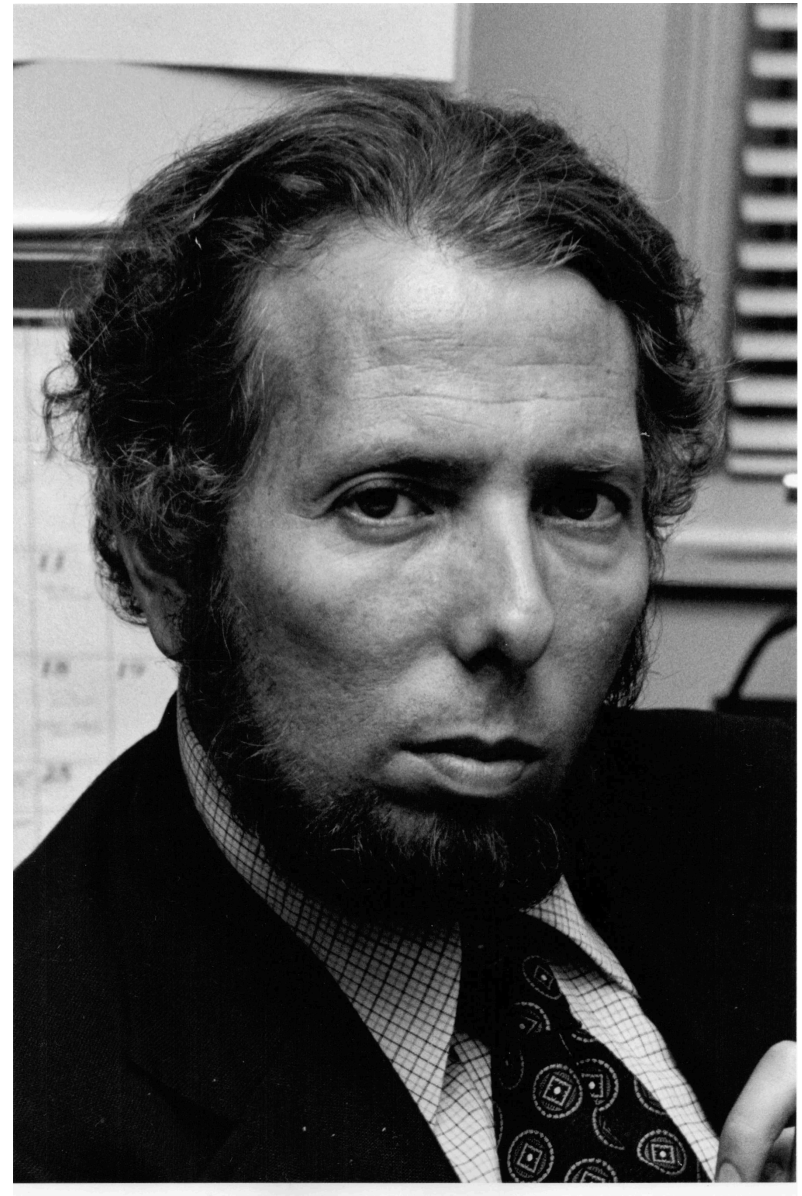
What You'll Learn in this Module

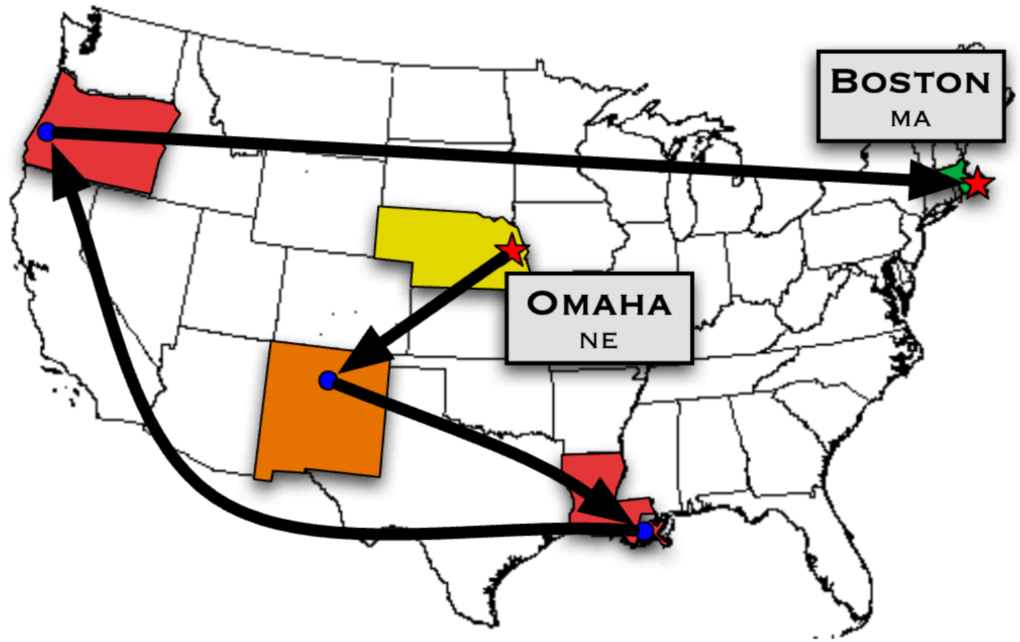
- How to measure **distance** between two nodes
- Clustering coefficient
- Small-world properties
- A mechanistic model for small-world networks: Watts-Strogatz model
- Libraries for network analysis

The Small-World Experiment

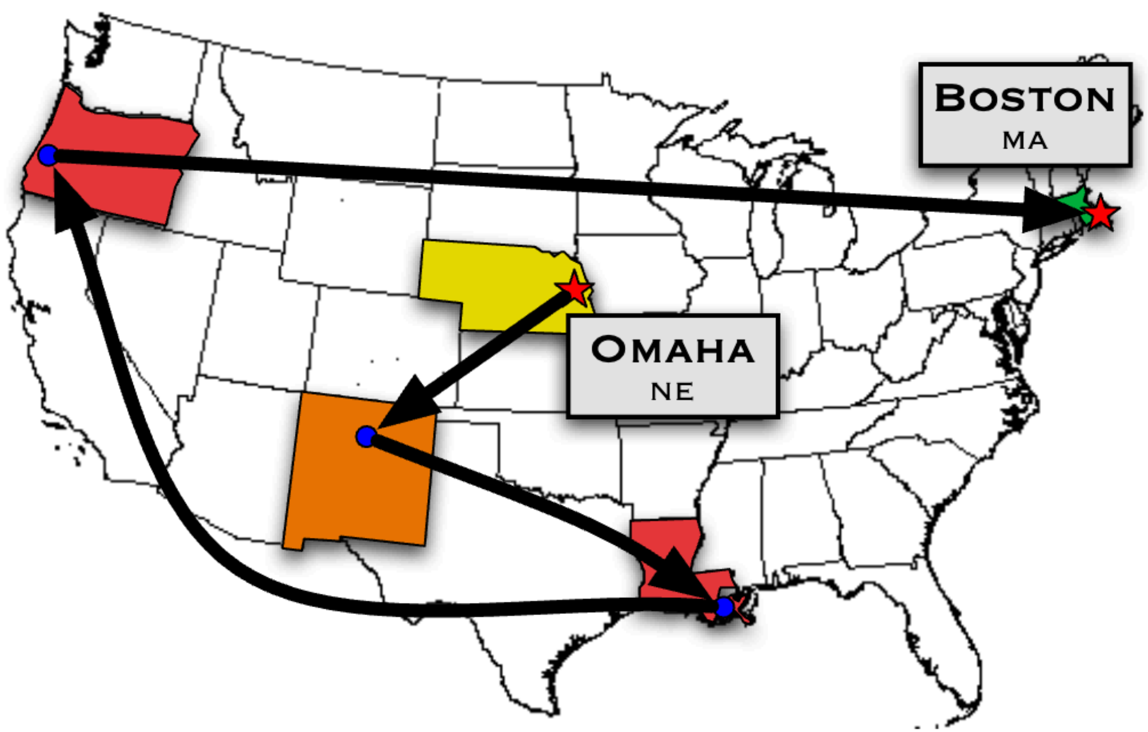
Stanley Milgram (1933-1984)

- American social psychologist
- Famous for obedience experiments
- Conducted groundbreaking research on social networks
- Revealed surprisingly short chains connecting people



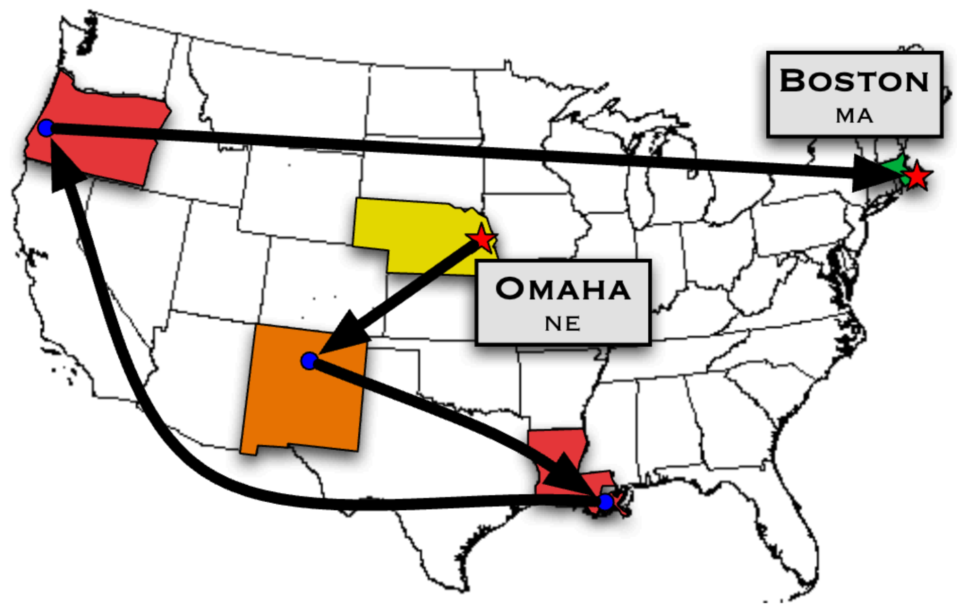

BOSTON
MA
OMAHA
NE

2. Recipients in Omaha, Nebraska, and Wichita, Kansas asked to forward a package to a target person in Boston if they knew them
3. If not, forward to someone they knew who might know the target
4. Chain continued until reaching the target



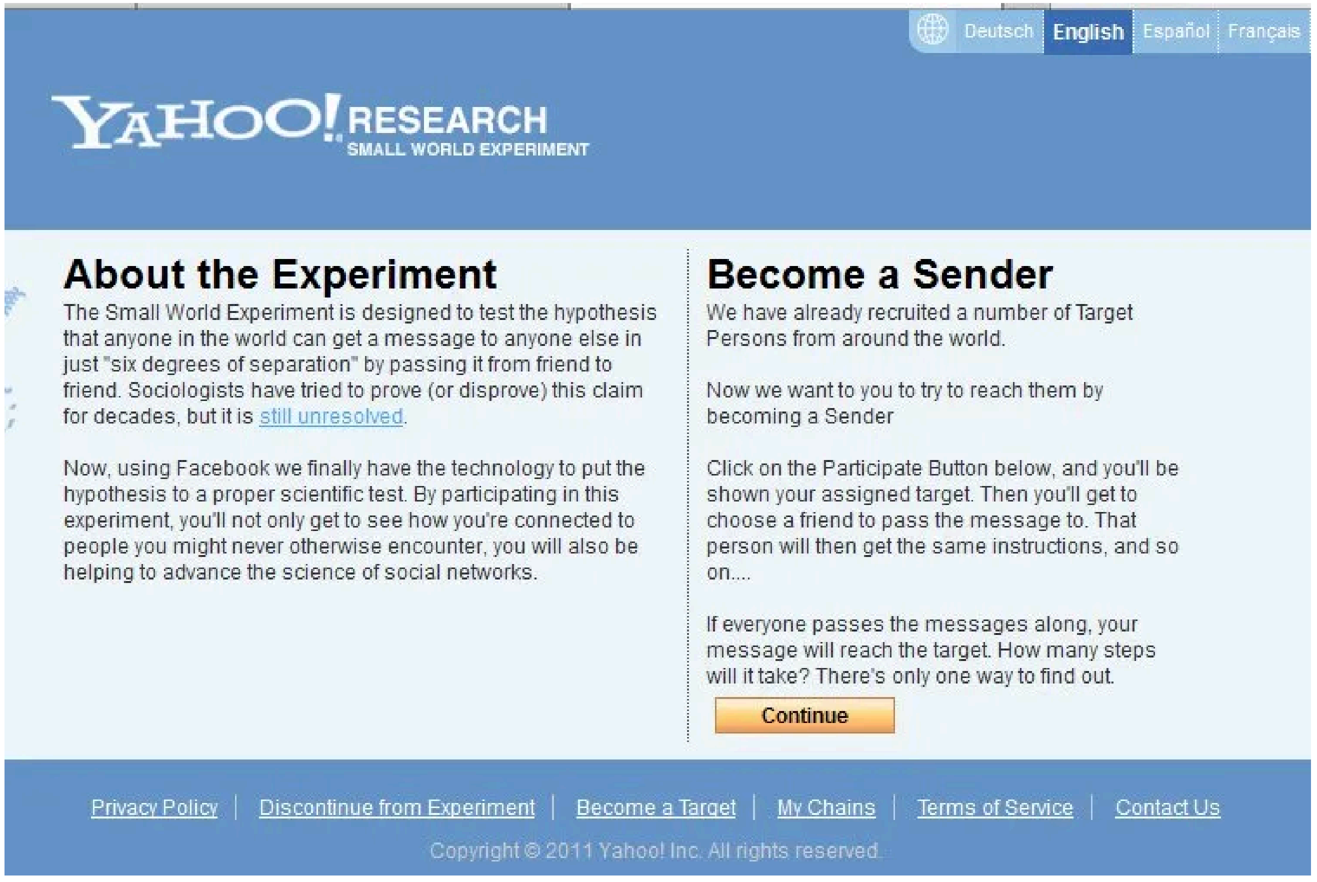
- Out of 160 letters sent, **64 successfully reached** the target
- Average chain length: **nearly 6 people**
- Later called **"six degrees of separation"**

Despite hundreds of millions of people in the US, their social network was remarkably compact!



Modern Confirmations:

- **Yahoo Research (2009)**: Email chains, average length ~4-7
- **Facebook Study (2012)**: 721M users, average path length 4.74



Experiencing Small-World: Wikirace

Play the game: WikiRace

- Start from one Wikipedia page
- Navigate to another page using only links
- Experience how few clicks separate any two topics

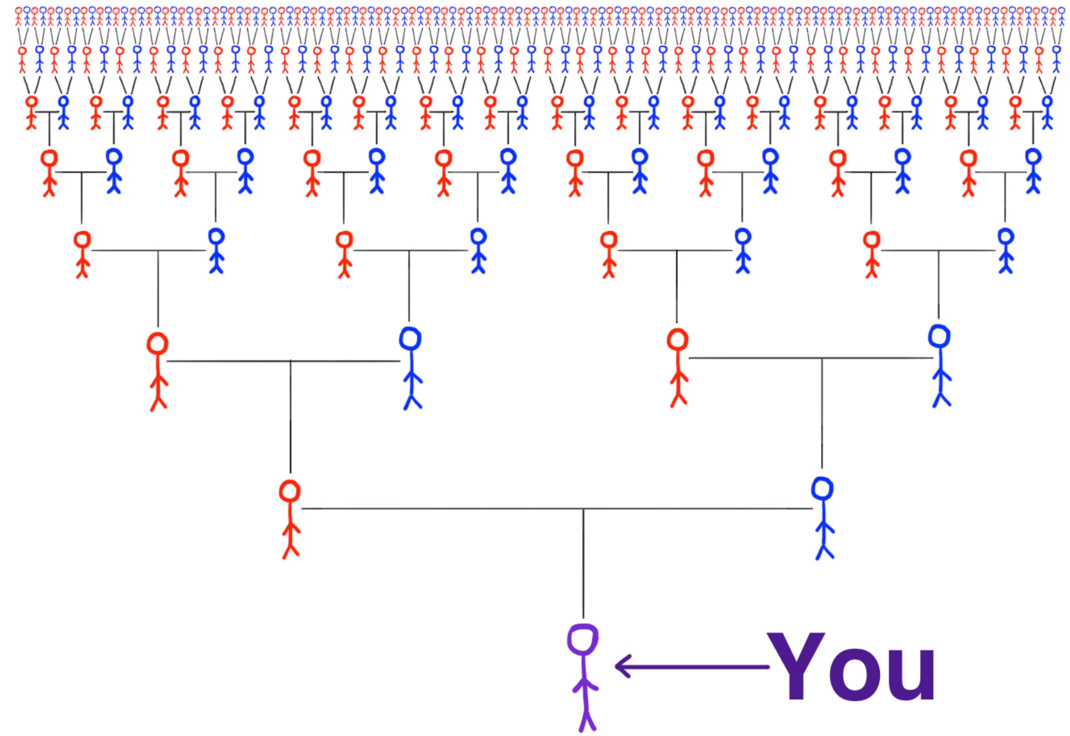
Question 🤔:

Why are people in the world connected by a small number of steps?

Think about your *family tree*:

How many ancestors do you have in each generation?

- 1 generation back (parents)
- 5 generations back?
- 10 generations back?



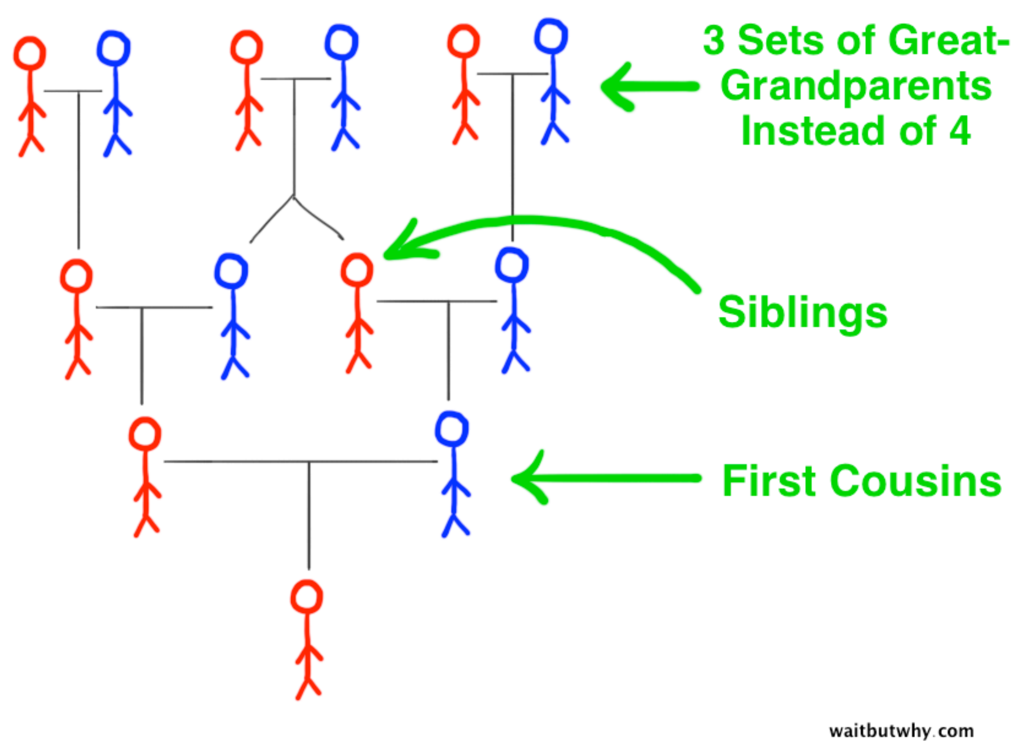
- Ancestors double each generation → **exponential growth**.
- In social networks, having more than 2 friends means you can reach billions in just a few steps. ← does this explain the small-world property 🤔?

Wait, think about 100 generations back

- 100 generation $\simeq$ 2000 years
- The number of ancestors is $2^{100} \simeq 10^{30}$
- But, population in 2000 years ago was only 200 million.

Then, what's wrong with the estimate 🤔?

- The family tree isn't a true tree—many ancestors overlap (due to incest).
- Local connections are more common in social networks—*your friends are also friends with each other*.
- Exponential growth alone doesn't explain short social distances!



Key question

- If people are connected locally, then our social networks are NOT small-world.
- But observations show that it is small-world.
- So, how can a network have lots of local connections and still remain globally compact 🤔?

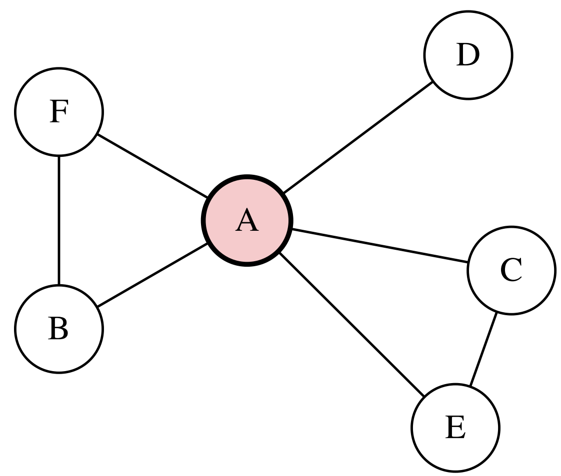
- Let's make it clear what we mean by local and global connections.

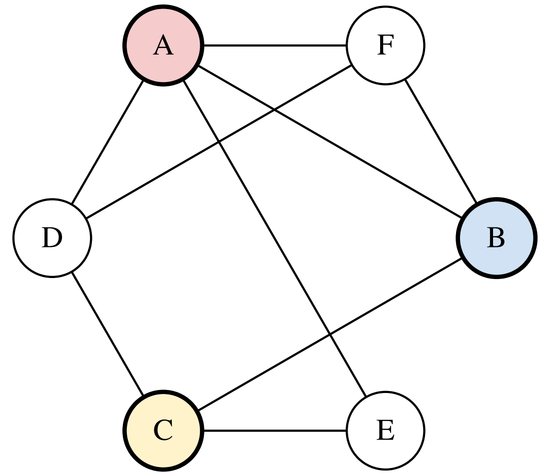
Clustering Coefficient (1)

Local clustering asks: given all your friends, how many of triangles you and your friends form, relative to the maximum possible number of triangles?

$$C_i = \frac{\text{\# of triangles involving } i \text{ and its neighbors}}{\text{\# of edges possibly exist in the neighborhood of } i}$$

- Node A has 5 neighbors
- Triangles with A: 2
- Possible triangles: $\binom{5}{2} = 10$
- $C_A = 2/10 = 0.2$





What are the local clustering coefficients of A, B and C?

$$C_i = \frac{\text{\# of triangles involving } i \text{ and its neighbors}}{\text{\# of edges possibly exist in the neighborhood of } i}$$

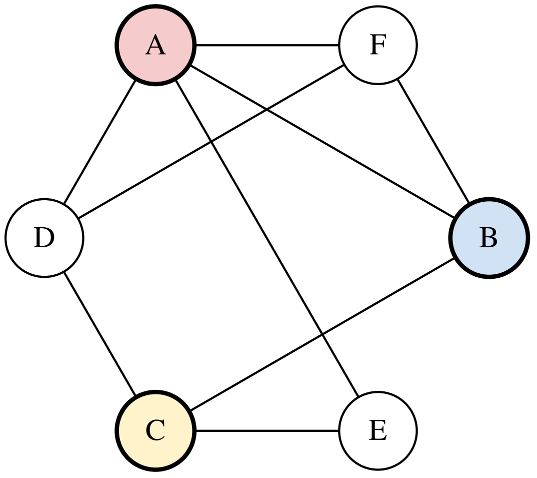
- A: 2/6 = 1/3
- B: 1/3 = 1/3
- C: 0

Clustering Coefficient (2)

Average clustering coefficient is the average of the local clustering coefficients of all nodes.

$$\overline{C} = \frac{1}{N} \sum_{i} C_i$$

$$\overline{C} = \frac{1}{6}\left(\underbrace{\frac{1}{3}}_{A} + \underbrace{\frac{1}{3}}_{B} + \underbrace{0}_{C} + \underbrace{\frac{1}{3}}_{D} + \underbrace{0}_{E} + \underbrace{\frac{2}{3}}_{F}\right) = \frac{5}{18}$$



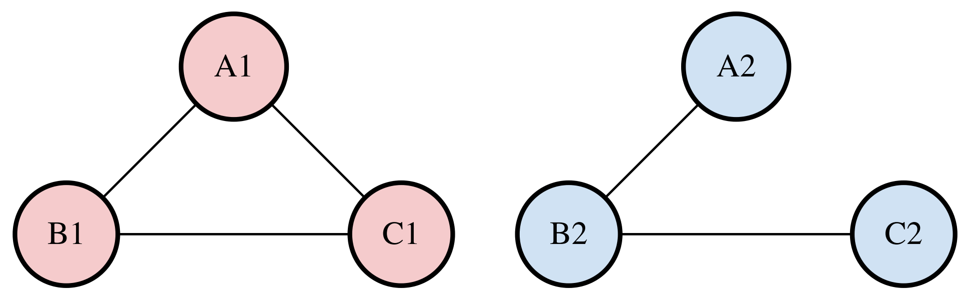
Clustering Coefficient (3)

Global clustering coefficient focuses on the total number of triangles in the network.

Definition

$$C = \frac{3 \times \text{number of triangles}}{\text{number of connected triplets}}$$

Connected triplets = A *sequence of three nodes* connected by at least two edges, forming either a closed triplet (triangle) or an open triplet (wedge).



Closed triplet (left) and open triplet (right)

Three types of clustering coefficients:

1. Local clustering coefficient → Density of triangles in a node's neighborhood
2. Average clustering coefficient → Average of the local clustering
3. Global clustering coefficient → Density of triangles in the entire network

Question:

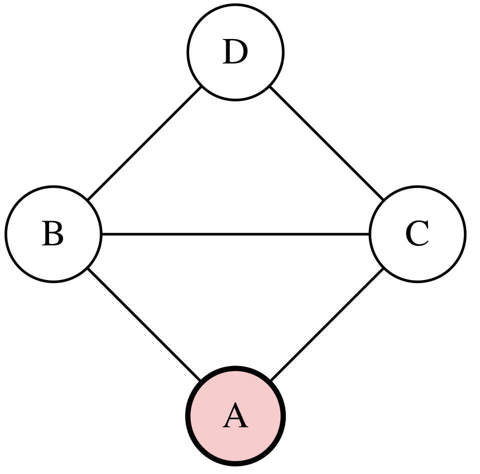
1. If a network has a high global clustering coefficient, does it necessarily have a high average local clustering coefficient?
2. If not, can you draw a network with high global clustering but low average local clustering coefficient?

Average Path Length (1)

Now, let's quantify the global connectivity via the average path length.

Distance between two nodes i and j is the minimum number of edges you need to traverse to get from one node to the other

Let's find the distance between A and D:



- Path 1: A → B → D (2 edges)
- Path 2: A → C → D (2 edges)
- Path 3: A → C → B → D (3 edges)

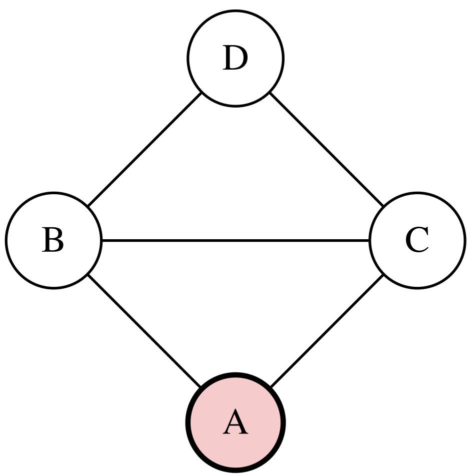
Even though there are multiple paths, the distance from A to D is **2 edges**.

Average Path Length (2)

Average path length L is the average distance between any two nodes:

$$\overline{L} = \frac{2}{N(N-1)} \sum_{i<j} d_{ij}$$

where d_{ij} is the shortest path length between node i and node j, and N is the number of nodes in the network.



$$L = \frac{1}{6}\left(\underbrace{1}_{A-B} + \underbrace{1}_{A-C} + \underbrace{2}_{A-D} + \underbrace{1}_{B-C} + \underbrace{1}_{B-D} + \underbrace{1}_{C-D}\right)$$

$$= \frac{7}{6} \simeq 1.16$$

Small-world networks are networks that have both *high clustering coefficient* and *short average path length*.

And we can quantify the "small-worldness" of a network by, for example,

$$\sigma_{\text{naive}} = \frac{\text{average clustering coefficient}}{\text{average path length}}$$

- But there is a problem 🤔
- Both clustering coefficient and average path length are correlated with the number of nodes N and edges M.
- Example: A small network has short average path length. Dense network has high clustering.

- Let's control for the effect of the number of nodes N and edges M.
- Think about rewiring the edges of the network randomly—this is called Erdős-Rényi random graph.
- This random network has the same number of nodes and edges but would have a different σ_{naive} value.
- Denoted by σ_{random} the average of σ_{naive} over many random networks.
- We normalize σ_{naive} by σ_{random}:

$$\sigma = \frac{\sigma_{\text{naive}}}{\sigma_{\text{random}}}$$

- If $\sigma > 1$, the network is small-world more than random networks.

Small-world Coefficient

$$\sigma = \frac{\sigma_{\text{naive}}}{\sigma_{\text{random}}}$$

- $\sigma > 1$: Strong small-world property
- $\sigma \approx 1$: Comparable to random network
- $\sigma < 1$: Anti-small-world

For Erdős-Rényi Random Graphs, we have:

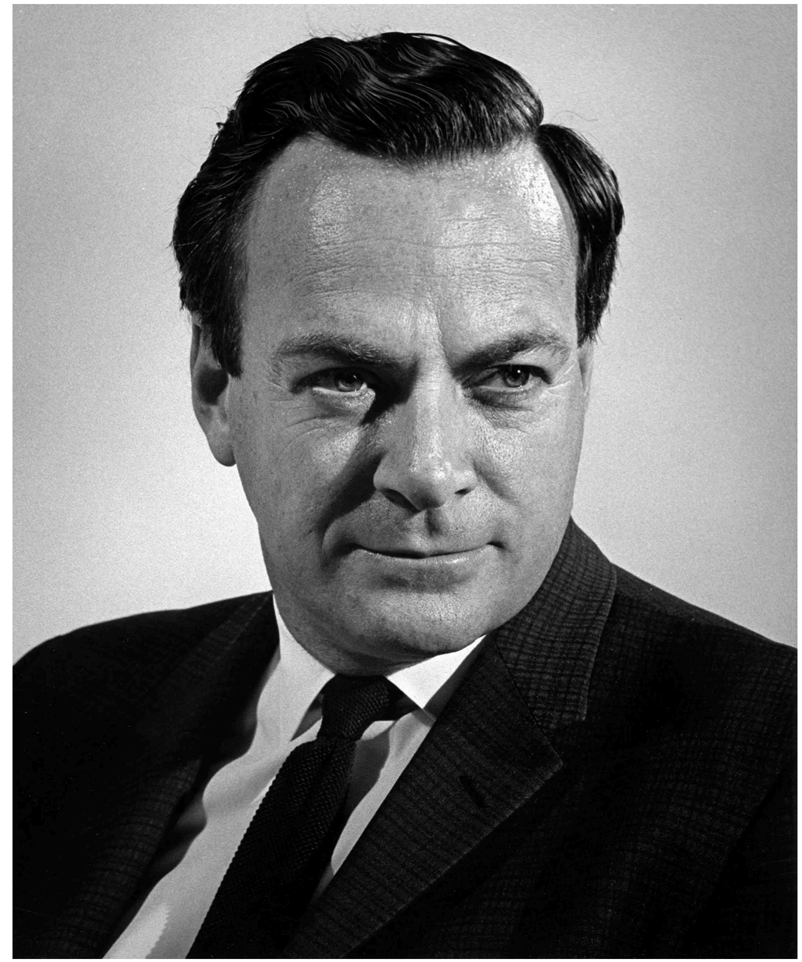
$$\sigma_{\text{random}} = \frac{\overline{C}_{\text{random}}}{\overline{L}_{\text{random}}}, \quad \overline{C}_{\text{random}} \approx \frac{\langle k \rangle}{N-1}, \quad \overline{L}_{\text{random}} \approx \frac{\ln N}{\ln \langle k \rangle}$$

where $\langle k \rangle$ is the average degree. See the lecture note for the derivation.

Now, we have a way to quantify the small-worldness of a network.

But we still don't know *why small-world networks emerge*.

> *"What I cannot create, I do not understand"*
>
> *—Richard Feynman*



What are the mechanism behind the small-world phenomenon 🤔?

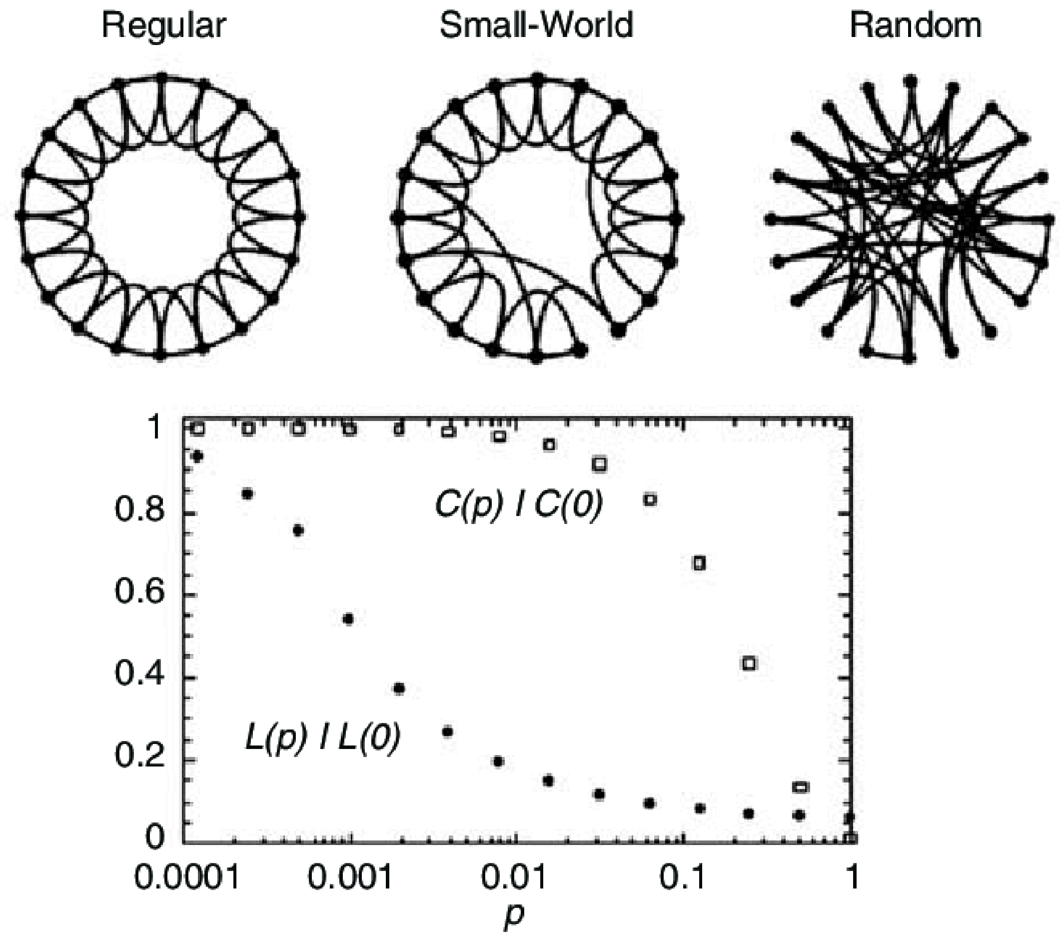
The Watts-Strogatz Model

Step 1: Create ring of N nodes connected to k nearest neighbors

- High clustering, long paths

Step 2: Randomly rewire each edge with probability p

- $p = 0$: regular lattice
- $p = 1$: random graph
- $0 < p < 1$: small-world



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Why Small-World Emerges

The Mechanism:

- Start with local clustering (ring lattice)
- Add a few long-range connections (rewiring)
- These "shortcuts" dramatically reduce path lengths
- Maintains high clustering while creating short paths

Examples:

- Biological networks: Neurons with local + long-range connections
- Technological networks: Internet with regional + continental links
- Social networks: Local friends + distant acquaintances

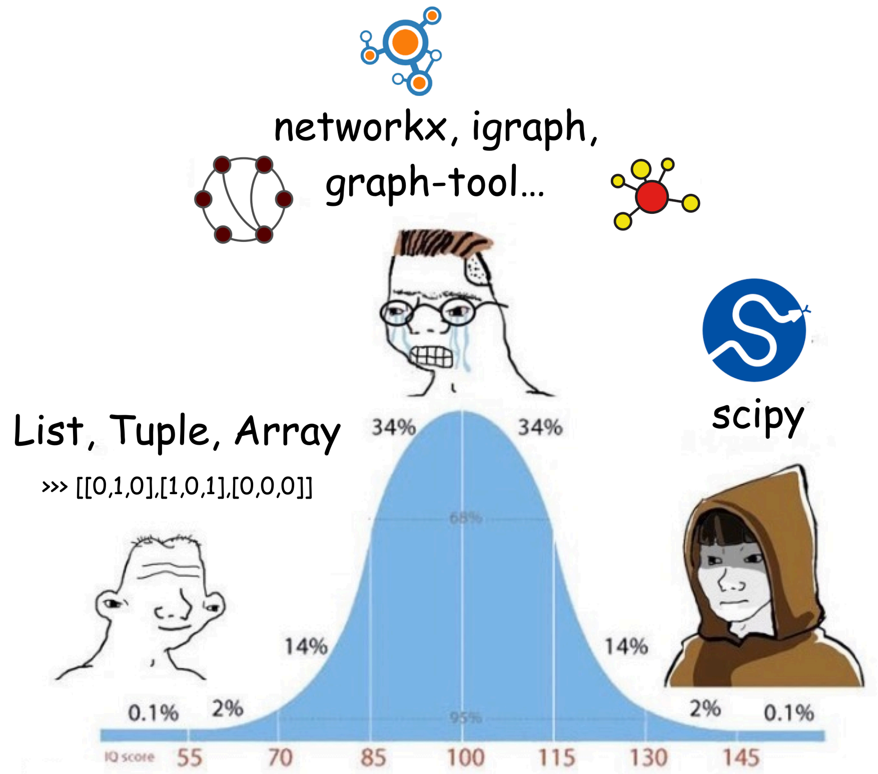
Key Takeaways

1. **Small-world networks** combine high clustering with short path lengths
2. **Milgram's experiment** revealed "six degrees of separation"
3. **Watts-Strogatz model** explains the mechanism through edge rewiring
4. **Quantification** possible through clustering coefficients and path lengths
5. **Long-range connections** are key to the small-world phenomenon

Next: We'll learn to compute shortest paths and connected components using igraph

Convenient libraries for network analysis

- networkx - a beginner-friendly library for network analysis
- igraph - a mature library with a wide range of algorithms
- graph-tool - specialized for stochastic block models
- scipy - efficient tools for analyzing large networks



Throughout this course, we'll primarily use `igraph`, a mature and robust library originally developed for R and later ported to Python.

First Step: Choose a notebook to work on

Marimo:

https://github.com/skojaku/adv-net-sci/notebooks/m02-small-world/starter.py

Open the terminal and run:

```
marimo edit --sandbox starter.py
```

or via `uv`

```
uvx marimo edit starter.py
```

Jupyter Notebook:

https://github.com/skojaku/adv-net-sci/notebooks/m02-small-world/starter.ipynb

Toy network

```python
import igraph

edge_list = [(0, 1), (1, 2), (0, 2), (0, 3)]

g = igraph.Graph() # Create an empty graph
g.add_vertices(4) # Add 4 vertices
g.add_edges(edge_list) # Add edges to the graph
```

Plot the graph.

```python
import matplotlib.pyplot as plt

fig, ax = plt.subplots(figsize=(5, 5))

# Draw the graph on the matplotlib axes using igraph
igraph.plot(
    g,
    bbox=(50, 50),
    vertex_label=list(range(4)),
    target=ax,
)
```

Path

Simple paths:

```
g.get_all_simple_paths(2, to=3)
```

Shortest path:

```
# Shortest path
g.get_shortest_paths(2, to=3)
```

Distance:

```
# Distance
g.distances(2, 3)
```

Connected Components

Find connected components:

```
components = g.connected_components()
```

Membership:

```
components.membership
```

Size:

```
components.size
```

The largest connected component:

```
components.giant()
```

Clustering coefficient

Local clustering coefficient:

```
g_cluster.transitivity_local_undirected()
```

Average clustering coefficient:

```
g_cluster.transitivity_avglocal_undirected()
```

Global clustering coefficient:

```
g_cluster.transitivity_undirected()
```

Watts-Strogatz Model

```
n_ws = 30 # Number of nodes
k_ws = 6 # Number of nearest neighbors in the ring lattice
p_rewire = 0.1 # Probability of rewiring each edge

g_smallworld = igraph.Graph.Watts_Strogatz(
    dim=1,
    size=n_ws,
    nei=k_ws // 2,
    p=p_rewire,
)
```

Compute the small-worldness σ using the formula below:

$$\sigma = \frac{\sigma_{\text{naive}}}{\sigma_{\text{random}}}, \text{ where } \sigma_{\text{naive}} = \frac{\text{average clustering coefficient}}{\text{average path length}}$$

$$\sigma_{\text{random}} = \frac{\overline{C}_{\text{random}}}{\overline{L}_{\text{random}}}, \quad \overline{C}_{\text{random}} \approx \frac{\langle k \rangle}{N-1}, \quad \overline{L}_{\text{random}} \approx \frac{\ln N}{\ln \langle k \rangle}$$

What's Next?

Module 03: Network Robustness

Does a network remain connected when one or more nodes fail?