Lecture 12. Real Vector Spaces
May 2023

Vector Space Axioms

In this Lecture, we will extend the concept of a vector by using the basic properties of vectors in $\mathbb{R}^n$ as axioms, which if satisfied by a set of objects, guarantee that those objects behave like familiar vectors.

Let V be an arbitrary nonempty set of objects on which two operations are defined: addition, and multiplication by numbers called **scalars**. By **addition** we mean a rule for associating with each pair of objects $\mathbf{u}$ and $\mathbf{v}$ in V an object $\mathbf{u}+\mathbf{v}$ in V, called the sum of $\mathbf{u}$ and $\mathbf{v}$; by **scalar multiplication** we mean a rule for associating with each scalar k and each object $\mathbf{u}$ in V an object $k\mathbf{u}$ in V, called the scalar multiple of $\mathbf{u}$ by k.

DEFINITION 12.1 If the following axioms are satisfied by all objects $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in V and all scalars k and m, then we call V a **vector space** and we call the objects in V **vectors**.

1. $\mathbf{v}+\mathbf{w}=\mathbf{w}+\mathbf{v}$ [Commutative law for vector addition]
2. $\mathbf{v}+(\mathbf{w}+\mathbf{u})=(\mathbf{v}+\mathbf{w})+\mathbf{u}$ [Associative law for vector addition]
3. There is an object $\mathbf{0}$ in V, called a zero vector for V, such that $\mathbf{0}+\mathbf{v}=\mathbf{v}+\mathbf{0}=\mathbf{v}$ for all $\mathbf{v}\in\mathbf{V}$.
4. For each $\mathbf{v}\in\mathbf{V}$, there is $-\mathbf{v}\in\mathbf{V}$, called a *negative* of $\mathbf{v}$, such that $\mathbf{v}+(-\mathbf{v})=\mathbf{0}$.
5. $k(m\mathbf{v})=(km)\mathbf{v}$
6. $1\cdot\mathbf{v}=\mathbf{v}$
7. $k(\mathbf{v}+\mathbf{w})=k\mathbf{v}+k\mathbf{w}$ [Distributive law]
8. $(k+m)\mathbf{v}=k\mathbf{v}+m\mathbf{v}$ [Distributive law]

REMARK 12.2 In our course, scalars will be either real numbers or complex numbers. Vector spaces with real scalars will be called **real vector spaces** and those with complex scalars will be called **complex vector spaces**.

Observe that the definition of a vector space does not specify the nature of the vectors or the operations. Any kind of object can be a vector, and the operations of addition and scalar multiplication need not have any relationship to those on $\mathbb{R}^n$. The only requirement is that the eight vector space axioms be satisfied. In the examples that follow we will use four basic steps to show that a set with two operations is a vector space.

To Show That a Set with Two Operations Is a Vector Space

Step 1. Identify the set V of objects that will become vectors.
Step 2. Identify the addition and scalar multiplication operations on V.
Step 3. Verify that adding two vectors in V produces a vector in V, and multiplying a vector in V by a scalar also produces a vector in V. These are called **closure under addition** and **closure under scalar multiplication**.
Step 4. Confirm that Axioms 1–8 hold.

EXAMPLE 12.3 The Zero Vector Space.

Let V consist of a single object, which we denote by $\mathbf{0}$, and define

$$\mathbf{0} + \mathbf{0} = \mathbf{0} \qquad \text{and} \qquad k\mathbf{0} = \mathbf{0}$$

for all scalars k. It is easy to check that all the vector space axioms are satisfied. We call this **the zero vector space**.

Our second example is one of the most important of all vector spaces — the familiar space $\mathbb{R}^n$ defined as a set of n-tuples $(x_1, \ldots, x_n)$. It should not be surprising that the operations on $\mathbb{R}^n$ satisfy the vector space axioms because those axioms were based on known properties of operations on $\mathbb{R}^n$.

EXAMPLE 12.4 $\mathbb{R}^n$ Is a Vector Space.

In the next example our vectors will be matrices. This may be a little confusing at first because matrices are composed of rows and columns, which are themselves vectors (row vectors and column vectors). However, from the vector space viewpoint we are not concerned with the individual rows and columns but rather with the properties of the matrix operations as they relate to the matrix as a whole.

EXAMPLE 12.5 The Vector Space of $m \times n$ Matrices.

Let V be the set of $m \times n$ matrices with real entries, and take the vector space operations on V to be the usual operations of matrix addition and scalar multiplication. The set V is closed under addition and scalar multiplication because these operations produce $m \times n$ matrices as the end result. Axioms 1–8 are standard properties of matrix operations.

EXAMPLE 12.6 The Vector Space of Real-Valued Functions.

Let V be the set of real-valued functions that are defined at each x in the interval $(-\infty, \infty)$. If $\mathbf{f} = f(x)$ and $\mathbf{g} = g(x)$ are two functions in V and if k is any scalar, then define the operations of addition and scalar multiplication by

$$(\mathbf{f} + \mathbf{g})(x) = f(x) + g(x), \qquad (k\mathbf{f})(x) = kf(x)$$

The set V with these operations is denoted by the symbol $F(-\infty, \infty)$. We can prove that this is a vector space as follows.

If we add two functions that are defined at each x in the interval $(-\infty, \infty)$, then sums and scalar multiples of those functions must also be defined at each x in the interval $(-\infty, \infty)$. This proves that $F(-\infty, \infty)$ is closed under addition and scalar multiplication.

The function whose value at every point x in the interval $(-\infty, \infty)$ is zero when added to any other function $\mathbf{f}$ in $F(-\infty, \infty)$ produces $\mathbf{f}$ back again as the result.

Axiom 4 requires that for each function $\mathbf{f}$ in $F(-\infty, \infty)$ there exists a function $-\mathbf{f}$ in $F(-\infty, \infty)$, which when added to $\mathbf{f}$ produces the function $\mathbf{0}$. The function defined by $-\mathbf{f} = -f(x)$ has this property.

The validity of each of the other axioms follows from properties of real numbers.

Subspaces

It is often the case that some vector space of interest is contained within a larger vector space whose properties are known. Now we will show how to recognize when this is the case, we will explain how the properties of the larger vector space can be used to obtain properties of the smaller vector space, and we will give a variety of important examples.

We begin with some terminology.

DEFINITION 12.7 A subset W of a vector space V is called a **subspace** of V if W is itself a vector space under the addition and scalar multiplication defined on V.

In general, to show that a nonempty set W with two operations is a vector space one must verify the eight vector space axioms. However, if W is a subspace of a known vector space V, then certain axioms need not be verified because they are "inherited" from V. For example, it is not necessary to verify that $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ holds in W because it holds for all vectors in V including those in W. On the other hand, it is necessary to verify that W is closed under addition and scalar multiplication since it is possible that adding two vectors in W or multiplying a vector in W by a scalar produces a vector in V that is outside of W. The following theorem states that W is a subspace of V if and only if it is closed under addition and scalar multiplication.

THEOREM 12.8 If W is a set of one or more vectors in a vector space V, then W is a subspace of V if and only if the following conditions are satisfied.

(a) If $\mathbf{u}$ and $\mathbf{v}$ are vectors in W, then $\mathbf{u}+\mathbf{v}$ is in W.
(b) If k is a scalar and $\mathbf{u}$ is a vector in W, then $k\mathbf{u}$ is in W.

Proof. If W is a subspace of V, then all the vector space axioms hold in W together with conditions (a) and (b).

Conversely, assume that conditions (a) and (b) hold. Since Axioms 1, 2, 5, 6, 7, 8 are inherited from V, we only need to show that Axioms 3 and 4 hold in W. For this purpose, let $\mathbf{u}$ be any vector in W. It follows from condition (b) that $k\mathbf{u}$ is a vector in W for every scalar k. In particular, $0\mathbf{u}=\mathbf{0}$ and $(-1)\mathbf{u}=-\mathbf{u}$ are in W, which shows that Axioms 3 and 4 hold in W. □

REMARK 12.9 Note that every vector space has at least two subspaces, itself and its zero subspace.

EXAMPLE 12.10 The Zero Subspace.

If V is any vector space, and if $W=\{\mathbf{0}\}$ is the subset of V that consists of the zero vector only, then W is closed under addition and scalar multiplication since

$$\mathbf{0}+\mathbf{0}=\mathbf{0} \qquad \text{and} \qquad k\mathbf{0}=\mathbf{0}$$

for any scalar k. We call W the zero subspace of V.

EXAMPLE 12.11 Lines Through the Origin Are Subspaces of $\mathbb{R}^2$ and of $\mathbb{R}^3$

If W is a line through the origin of either $\mathbb{R}^2$ or $\mathbb{R}^3$, then adding two vectors on the line or multiplying a vector on the line by a scalar produces another vector on the line, so W is closed under addition and scalar multiplication.

EXAMPLE 12.12 Planes Through the Origin Are Subspaces of $\mathbb{R}^3$

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in a plane W through the origin of $\mathbb{R}^3$, then it is evident geometrically that $\mathbf{u}+\mathbf{v}$ and $k\mathbf{u}$ also lie in the same plane W for any scalar k. Thus W is closed under addition and scalar multiplication.

EXAMPLE 12.13 A Subset of $\mathbb{R}^2$ That Is Not a Subspace.

Let W be the set of all points (x, y) in $\mathbb{R}^2$ for which $x \geqslant 0$ and $y \geqslant 0$. This set is not a subspace of $\mathbb{R}^2$ because it is not closed under scalar multiplication. For example, $\mathbf{v} = (1, 1)$ is a vector in W, but $(-1)\mathbf{v} = (-1, -1)$ is not.

EXAMPLE 12.14 The Subspace $C(-\infty, \infty)$.

There is a theorem in calculus which states that a sum of continuous functions is continuous and that a constant times a continuous function is continuous. Rephrased in vector language, the set of continuous functions on $(-\infty, \infty)$ is a subspace of $F(-\infty, \infty)$. We will denote this subspace by $C(-\infty, \infty)$.

EXAMPLE 12.15 Functions with Continuous Derivatives.

A function with a continuous derivative is said to be continuously differentiable. There is a theorem in calculus which states that the sum of two continuously differentiable functions is continuously differentiable and that a constant times a continuously differentiable function is continuously differentiable. Thus, the functions that are continuously differentiable on $(-\infty, \infty)$ form a subspace of $F(-\infty, \infty)$. We will denote this subspace by $C^1(-\infty, \infty)$, where the superscript emphasizes that the first derivatives are continuous. To take this a step further, the set of functions with m continuous derivatives on $(-\infty, \infty)$ is a subspace of $F(-\infty, \infty)$ as is the set of functions with derivatives of all orders on $(-\infty, \infty)$. We will denote these subspaces by $C^m(-\infty, \infty)$ and $C^\infty(-\infty, \infty)$, respectively.

EXAMPLE 12.16 The Subspace of All Polynomials.

Recall that a polynomial is a function that can be expressed in the form

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n \tag{1}$$

where $a_0, a_1, \ldots, a_n$ are constants. It is evident that the sum of two polynomials is a polynomial and that a constant times a polynomial is a polynomial. Thus, the set W of all polynomials is closed under addition and scalar multiplication and hence is a subspace of $F(-\infty, \infty)$. We will denote this space by P_∞.

It is proved in calculus that polynomials are continuous functions and have continuous derivatives of all orders on $(-\infty, \infty)$. Thus, it follows that P_∞ is not only a subspace of $F(-\infty, \infty)$, as previously observed, but is also a subspace of $C^\infty(-\infty, \infty)$.

EXAMPLE 12.17 The Subspace of Polynomials of Degree $\leqslant n$.

Recall that the degree of a polynomial is the highest power of the variable that occurs with a nonzero coefficient. Thus, for example, if $a_n \neq 0$ in Formula (1), then that polynomial has degree n. We regard all constants to be polynomials of degree zero. For each nonnegative integer n the polynomials of degree n or less form a subspace of $F(-\infty, \infty)$. We will denote this space by P_n.

The following theorem provides a useful way of creating a new subspace from known subspaces.

THEOREM 12.18 If $W_1, W_2, \ldots, W_r$ are subspaces of a vector space V, then the intersection of these subspaces is also a subspace of V.

Proof. Let W be the intersection of the subspaces $W_1, W_2, \ldots, W_r$. This set is not empty because each of these subspaces contains the zero vector of V, and hence so does their intersection. Thus, it remains to show that W is closed under addition and scalar multiplication.

To prove closure under addition, let $\mathbf{u}$ and $\mathbf{v}$ be vectors in W. Since W is the intersection of $W_1, W_2, \ldots, W_r$, it follows that $\mathbf{u}$ and $\mathbf{v}$ also lie in each of these subspaces. Moreover, since these subspaces are closed under addition and scalar multiplication, they also all contain the vectors $\mathbf{u} + \mathbf{v}$ and $k\mathbf{u}$ for every scalar k, and hence so does their intersection W. This proves that W is closed under addition and scalar multiplication. □

Sometimes we will want to find the "smallest" subspace of a vector space V that contains all of the vectors in some set of interest. Recall that a vector $\mathbf{w}$ is called a **linear combination** of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, …, $\mathbf{v}_r$ in V if $\mathbf{w}$ can be expressed in the form

$$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r$$

where $k_1, k_2, \ldots, k_r$ are scalars.

THEOREM 12.19 If $S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}$ is a nonempty set of vectors in a vector space V, then:

(a) The set W of all possible linear combinations of the vectors in S is a subspace of V.
(b) The set W in part (a) is the "smallest" subspace of V that contains all of the vectors in S in the sense that any other subspace that contains those vectors contains W.

Proof (a). Let W be the set of all possible linear combinations of the vectors in S. We must show that W is closed under addition and scalar multiplication. To prove closure under addition, let

$$\mathbf{u} = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \cdots + c_r\mathbf{w}_r \qquad \text{and} \qquad \mathbf{v} = k_1\mathbf{w}_1 + k_2\mathbf{w}_2 + \cdots + k_r\mathbf{w}_r$$

be two vectors in W. It follows that their sum can be written as

$$\mathbf{u} + \mathbf{v} = (c_1 + k_1)\mathbf{w}_1 + (c_2 + k_2)\mathbf{w}_2 + \cdots + (c_r + k_r)\mathbf{w}_r$$

which is a linear combination of the vectors in S. Thus, W is closed under addition. We leave it for you to prove that W is also closed under scalar multiplication and hence is a subspace of V.

Proof (b). Let W' be any subspace of V that contains all of the vectors in S. Since W' is closed under addition and scalar multiplication, it contains all linear combinations of the vectors in S and hence contains W. $\square$

The following definition gives some important notation and terminology related to Theorem 12.19.

DEFINITION 12.20 If $S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}$ is a nonempty set of vectors in a vector space V, then the subspace W of V that consists of all possible linear combinations of the vectors in S is called the subspace of V generated by S, and we say that the vectors $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r$ **span** W. We denote this subspace as

$$W = \operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\} \qquad \text{or} \qquad W = \operatorname{span}(S).$$

In the case where S is the empty set, it will be convenient to agree that $\operatorname{span}(\emptyset) = \{\mathbf{0}\}$.

EXAMPLE 12.21 The Standard Unit Vectors Span $\mathbb{R}^n$.

Recall that the standard unit vectors in $\mathbb{R}^n$ are

$$\mathbf{e}_1 = (1, 0, 0, \ldots, 0), \quad \mathbf{e}_2 = (0, 1, 0, \ldots, 0), \quad \ldots, \quad \mathbf{e}_n = (0, 0, 0, \ldots, 1).$$

These vectors span $\mathbb{R}^n$ since every vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ in $\mathbb{R}^n$ can be expressed as

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n$$

which is a linear combination of $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$.

EXAMPLE 12.22 A Geometric View of Spanning in $\mathbb{R}^2$ and $\mathbb{R}^3$.

(a) If $\mathbf{v}$ is a nonzero vector in $\mathbb{R}^2$ or $\mathbb{R}^3$ that has its initial point at the origin, then $\operatorname{span}\{\mathbf{v}\}$, which is the set of all scalar multiples of $\mathbf{v}$, is the line through the origin determined by $\mathbf{v}$.

(b) If $\mathbf{v}_1$ and $\mathbf{v}_2$ are nonzero vectors in $\mathbb{R}^3$ that have their initial points at the origin, then $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$, which consists of all linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$, is the plane through the origin determined by these two vectors.