Lecture 13. Linear Dependence and Basis
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Linear Dependence and Independence of Vectors

DEFINITION 13.1 If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a nonempty set of vectors in a vector space V, then S is said to be **linearly dependent** if the vector equation

$$\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \cdots + \lambda_k \mathbf{v}_k = \mathbf{0} \tag{1}$$

can be satisfied with coefficients λ_1, λ_2, $\ldots$, λ_k that are not all zero.

DEFINITION 13.2 If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a nonempty set of vectors in a vector space V, then S is said to be a **linearly independent** set if the only coefficients satisfying the vector equation (1) are $\lambda_1 = 0$, $\lambda_2 = 0$, $\ldots$, $\lambda_k = 0$.

LEMMA 13.3 (Dependent Lemma) If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of two or more vectors in a vector space V, then S is linearly dependent if and only if there is a vector in S that can be expressed as a linear combination of the others.

Proof. Assume first that S is linearly dependent. We will show that if the equation

$$\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \cdots + \lambda_k \mathbf{v}_k = \mathbf{0}$$

can be satisfied with coefficients that are not all zero, then at least one of the vectors in S must be expressible as a linear combination of the others. To be specific, suppose that $\lambda_1 \neq 0$. Then we can rewrite the last equation as

$$\mathbf{v}_1 = \left(-\frac{\lambda_2}{\lambda_1}\right)\mathbf{v}_2 + \cdots + \left(-\frac{\lambda_k}{\lambda_1}\right)\mathbf{v}_k$$

which expresses $\mathbf{v}_1$ as a linear combination of the other vectors in S.

Conversely, suppose that at least one of the vectors is expressible as a linear combination of the others, say

$$\mathbf{v}_1 = x_2 \mathbf{v}_2 + \cdots + x_r \mathbf{v}_k$$

which we can rewrite as

$$\mathbf{v}_1 + (-x_2)\mathbf{v}_2 + \cdots + (-x_r)\mathbf{v}_k = \mathbf{0}$$

But this means that the vector equation is satisfied by $\lambda_1 = 1$, $\lambda_2 = -x_2$, $\ldots$, $\lambda_k = -x_k$, the first coefficient being nonzero. Thus, the vectors in S must be linearly dependent. $\square$

Basis

DEFINITION 13.4 If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an ordered set of vectors in a vector space V, then S is called a **basis** for V if:

(a) Any vector in V is expressible as a linear combination of the vectors in S (that is, S **spans** V).

(b) S is linearly independent.

If we think of a basis as describing a coordinate system for a vector space V, then part (a) of this definition guarantees that there are enough basis vectors to provide coordinates for all vectors in V, and part (b) guarantees that there is no interrelationship between the basis vectors.

EXAMPLE 13.5 (The Standard Basis for $\mathbb{R}^n$) It is evident that the standard unit vectors $\mathbf{e}_1 = (1, 0, 0, \dots, 0)$, $\mathbf{e}_2 = (0, 1, 0, \dots, 0)$, $\dots$, $\mathbf{e}_n = (0, 0, 0, \dots, 1)$ span $\mathbb{R}^n$ and that they are linearly independent.

THEOREM 13.6 (Uniqueness of Basis Representation) If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for a vector space V, then every vector $\mathbf{v}$ in V can be expressed in the form $\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n$ in exactly one way.

Proof. Since S spans V, it follows from the definition of a spanning set that every vector in V is expressible as a linear combination of the vectors in S. To see that there is only one way to express a vector as a linear combination of the vectors in S, suppose that some vector $\mathbf{v}$ can be written as

$$\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n$$

and also as

$$\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n$$

Subtracting the second equation from the first one gives

$$\mathbf{0} = (x_1 - k_1)\mathbf{v}_1 + (x_2 - k_2)\mathbf{v}_2 + \cdots + (x_n - k_n)\mathbf{v}_n$$

Since the right side of this equation is a linear combination of vectors in S, the linear independence of S implies that

$$x_1 - k_1 = 0, \quad x_2 - k_2 = 0, \quad \dots, x_n - k_n = 0$$

that is, $x_1 = k_1$, $x_2 = k_2$, $\dots$, $x_n = k_n$. Thus, the two expressions for $\mathbf{v}$ are the same. $\square$

The following theorem is one of the most useful results in linear algebra.

THEOREM 13.7 (Fundamental Theorem) Suppose a vector space V can be spanned by n vectors. If a set of m vectors in V is linearly independent, then $m \leqslant n$.

Proof. Let $V = \mathbf{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$, and suppose that $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\}$ is an independent set in V. Then

$$\mathbf{u}_1 = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n, \qquad a_i \in \mathbb{R}.$$

As $\mathbf{u}_1 \neq \mathbf{0}$, not all of the a_i are zero, say $a_1 \neq 0$ (after relabelling the $\mathbf{v}_i$). It easy to verify that $V = \mathbf{span}\{\mathbf{u}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$. Hence, write

$$\mathbf{u}_2 = b_1\mathbf{u}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \cdots + c_n\mathbf{v}_n.$$

Then some $c_i \neq 0$ because $\{\mathbf{u}_1, \mathbf{u}_2\}$ is independent; so, as before, $V = \mathbf{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{v}_n\}$, again after possible relabelling of the $\mathbf{v}_i$. If $m > n$, this procedure continues until all the vectors $\mathbf{v}_i$ are replaced by the vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$. In particular,

$$V = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}.$$

But then $\mathbf{u}_{n+1}$ is a linear combination of $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ contrary to the independence of the $\mathbf{u}_i$. Hence, the assumption $m > n$ cannot be valid, so $m \leqslant n$ and the theorem is proved. □

COROLLARY 13.8 (Steinitz Exchange Lemma) If $V = \mathbf{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, and if $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ is an independent set in V, then m of the (spanning) vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ can be replaced by the (independent) vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ so that the resulting set will still span V.

THEOREM 13.9 (Invariance Theorem) Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ and $\{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_m\}$ be two bases of a vector space V. Then $n = m$.

Proof. Because $V = \mathbf{span}\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ and the set $\{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_m\}$ is independent, it follows from Fundamental theorem that $m \leqslant n$. Similarly $n \leqslant m$, so $n = m$, as asserted. □

Theorem 13.9 guarantees that no matter which basis of V is chosen it contains the same number of vectors as any other basis. Hence there is no ambiguity about the following definition.

DEFINITION 13.10 If $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is a basis of the nonzero vector space V, the number n of vectors in the basis is called the **dimension** of V, and we write $\dim V = n$. The zero vector space $\{\mathbf{0}\}$ is defined to have dimension 0: $\dim\{\mathbf{0}\} = 0$.

EXAMPLE 13.11 Show that the set V of all symmetric 2×2 matrices is a vector space, and find the dimension of V.

Solution. A matrix A is symmetric if $A^T = A$. If A and B lie in V, then

$$(A+B)^T = A^T + B^T = A + B \qquad \text{and} \qquad (kA)^T = kA^T = kA.$$

Hence $A + B$ and kA are also symmetric. As the 2×2 zero matrix is also in V, this shows that V is a vector space (being a subspace of M_{22}). Now a matrix A is symmetric when entries directly across the main diagonal are equal, so each 2×2 symmetric matrix has the form

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + c\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Hence the set $B = \left\{\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right), \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right), \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)\right\}$ spans V, and it can be easily verified that B is linearly independent. Thus B is a basis of V, so $\dim V = 3$. □

Subspaces and Bases

Up to this point, we have had no guarantee that an arbitrary vector space has a basis – and hence no guarantee that one can speak at all of the dimension of V. However, we will show that any space that is spanned by a finite set of vectors has a (finite) basis: The proof requires the following basic lemma, of interest in itself, that gives a way to enlarge a given independent set of vectors.

LEMMA 13.12 (Independent Lemma) Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be an independent set of vectors in a vector space V. If $\mathbf{u} \in V$ but $\mathbf{u} \notin \mathbf{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, then $\{\mathbf{u}, \mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is also independent.

Proof. Let $t\mathbf{u} + k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + t_k\mathbf{v}_k = \mathbf{0}$; we must show that all the coefficients are zero. If $t \neq 0$, then $\mathbf{u}$ is in $\mathbf{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, contrary to our assumption. If $t = 0$, then $t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_k\mathbf{v}_k = \mathbf{0}$ so the rest of the t_i are zero by the independence of $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$. This is what we wanted. $\square$

Note that the converse of Lemma 13.12 is also true: if $\{\mathbf{u}, \mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is independent, then $\mathbf{u}$ is not in $\mathbf{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$.

DEFINITION 13.13 A vector space V is called **finite dimensional** if it can be spanned by a finite set of vectors. Otherwise, V is called **infinite dimensional**.

LEMMA 13.14 If V is a finite dimensional vector space, then any independent subset of V can be enlarged to a finite basis of V.

Proof. Suppose that I is an independent subset of V. If $\mathbf{span}\, I = V$ then I is already a basis of V. If $\mathbf{span}\, I \neq V$, choose $\mathbf{u}_1 \in V$ such that $\mathbf{u}_1 \notin \mathbf{span}\, I$. Hence the set $I \cup \{\mathbf{u}_1\}$ is independent by Lemma 13.12. If $\mathbf{span}(I \cup \{\mathbf{u}_1\}) = V$ we are done; otherwise choose $\mathbf{u}_2 \in V$ such that $\mathbf{u}_2 \notin \mathbf{span}(I \cup \{\mathbf{u}_1\})$. Hence $I \cup \{\mathbf{u}_1, \mathbf{u}_2\}$ is independent, and the process continues. We claim that a basis of V will be reached eventually. Indeed, if no basis of V is ever reached, the process creates arbitrarily large independent sets in V. But this is impossible by the fundamental theorem because V is finite dimensional and so is spanned by a finite set of vectors. $\square$

COROLLARY 13.15 Let V be a finite dimensional vector space spanned by m vectors.

1. V has a finite basis, and $\dim V \leqslant m$.
2. Every independent set of vectors in V can be enlarged to a basis of V by adding vectors from any fixed basis of V.

THEOREM 13.16 If U is a subspace of a finite dimensional vector space V, then

1. U is finite dimensional and $\dim U \leqslant \dim V$.
2. Every basis of U is a part of a basis of V.
3. If $\dim U = \dim V$, then $U = V$.

Proof 1. This is clear if $U = \{\mathbf{0}\}$. Otherwise, let $\mathbf{u} \neq \mathbf{0}$ in U. Then $\{\mathbf{u}\}$ can be enlarged to a finite basis B of U by Lemma 13.14, proving that U is finite dimensional. But B is independent in V, so $\dim U \leqslant \dim V$ by the fundamental theorem.

2. This is clear if $U = \{\mathbf{0}\}$ because V has a basis; otherwise, it follows from Corollary 13.15.

3. Now assume $\dim U = \dim V = n$, and let B be a basis of U. Then B is an independent set in V. If $U \neq V$, then $\mathbf{span}\, B \neq V$, so B can be extended to an independent set of $n+1$ vectors in V by Lemma 13.14. This contradicts the fundamental theorem because V is spanned by $\dim V = n$ vectors. Hence $U = V$. $\square$

EXAMPLE 13.17 Find a basis of P_3 containing the independent set $\{1+x,\ 1+x^2\}$.

Solution. The standard basis of P_3 is $\{1,\ x,\ x^2,\ x^3\}$, so including two of these vectors will do. If we use 1 and x^3, the result is $\{1,\ 1+x,\ 1+x^2,\ x^3\}$. This is independent because the polynomials have distinct degrees, and so is a basis. Of course, including $\{1,\ x\}$ or $\{1,\ x^2\}$ would not work! $\square$

EXAMPLE 13.18 Show that the space P of all polynomials is infinite dimensional.

Solution. For each $n \geqslant 1$, P has a subspace P_n of dimension $n+1$. Suppose P is finite dimensional, say $\dim P = m$. Then $\dim P_n \leqslant \dim P$ by Theorem 13.16, that is $n+1 \leqslant m$. This is impossible since n is arbitrary, so P must be infinite dimensional. $\square$

EXAMPLE 13.19 If $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_k$ are independent columns in $\mathbb{R}^n$ and $k < n$, show that they are the first k columns in some invertible $n \times n$ matrix.

Solution. By Theorem 13.16, expand $\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_k\}$ to a basis $\{\mathbf{c}_1,\ \mathbf{c}_2, \ldots, \mathbf{c}_k, \mathbf{c}_{k+1}, \ldots, \mathbf{c}_n\}$ of $\mathbb{R}^n$. Then the matrix

$$A = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \ldots & \mathbf{c}_k & \mathbf{c}_{k+1} & \ldots & \mathbf{c}_n \end{bmatrix}$$

with this basis as its columns is an $n \times n$ matrix and it is invertible. $\square$

EXAMPLE 13.20 If a is a number, let W denote the subspace of all polynomials in P_n that have a as a root:

$$W = \{p(x) \mid p(x) \in P_n \quad \text{and} \quad p(a) = 0\}.$$

Show that $\{(x-a), (x-a)^2, \ldots, (x-a)^n\}$ is a basis of W.

Solution. Observe first that $(x-a), (x-a)^2, \ldots, (x-a)^n$ are members of W, and that they are independent because they have distinct degrees. Write

$$U = \mathbf{span}\{(x-a), (x-a)^2, \ldots, (x-a)^n\}.$$

Then we have $U \subset W \subset P_n$, $\dim U = n$, and $\dim P_n = n+1$. Hence $n \leqslant \dim W \leqslant n+1$ by Theorem 13.16. Since $\dim W$ is an integer, we must have $\dim W = n$ or $\dim W = n+1$. But then $W = U$ or $W = P_n$, again by Theorem 13.16. Because $W \neq P_n$, it follows that $W = U$, as required. $\square$

Lemma 13.14 gives a way to enlarge independent sets to a basis; by contrast, Lemma 13.21 shows that spanning sets can be cut down to a basis.

LEMMA 13.21 Let V be a finite dimensional vector space. Any spanning set for V can be cut down (by deleting vectors) to a basis of V.

Proof. Since V is finite dimensional, it has a finite spanning set S. Among all spanning sets contained in S, choose S_0 containing the smallest number of vectors. It suffices to show that S_0 is independent (then S_0 is a basis, proving the lemma). Suppose, on the contrary, that S_0 is not independent. Then some vector $\mathbf{u} \in S_0$ is a linear combination of the set $S_1 = S_0 \setminus \{\mathbf{u}\}$ of vectors in S_0 other than $\mathbf{u}$. It follows that span $S_0 = \mathbf{span}\, S_1$, that is, $V = \mathbf{span}\, S_1$. But S_1 has fewer elements than S_0 so this contradicts the choice of S_0. Hence S_0 is independent after all. $\square$

EXAMPLE 13.22 Find a basis of P_3 in the spanning set $S = \{1, x+x^2, 2x-3x^2, 1+3x-2x^2, x^3\}$.

Solution. Since $\dim P_3 = 4$, we must eliminate one polynomial from S. It cannot be x^3 because the span of the rest of S is contained in P_2. But eliminating $1+3x-2x^2$ does leave a basis. Note that $1+3x-2x^2$ is the sum of the first three polynomials in S. $\square$

In general, to show that a set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, we must show that the vectors are linearly independent and $\mathbf{span}\, V$. However, if we know that V has dimension n (so that S contains the right number of vectors for a basis), then it suffices to check either linear independence or spanning — the remaining condition will hold automatically. This is the content of the following theorem.

THEOREM 13.23 Let V be an n-dimensional vector space V, and suppose S is a set of exactly n vectors in V. Then S is a basis for V if either S spans V or S is linearly independent.

Proof. Assume first that S is independent. By Theorem 13.16, S is contained in a basis B of V. Since S has the same number of vectors as B, it follows that $S = B$.

Conversely, assume that S spans V, so by Lemma 13.21, S contains a basis B. Again S and B have the same number of vectors, thus $S = B$. $\square$

THEOREM 13.24 Let A be an $n \times n$ matrix. Columns of A are a basis of $\mathbb{R}^n$ if and only if $\det A \neq 0$.

Proof. 1) Let $\mathbf{c}_1, \ldots, \mathbf{c}_n$ denote columns of A and $\det A \neq 0$. Consider the dependency equation $x_1\mathbf{c}_1 + \cdots + x_n\mathbf{c}_n = \mathbf{0}$. In matrix notation, it takes the form $A\mathbf{x} = \mathbf{0}$, where $\det A \neq 0$. By Cramer's rule, it has a unique solution that is trivial. Therefore, the vectors are linearly independent. By Theorem 13.23, they are a basis of $\mathbb{R}^n$.

2) If columns of A are linearly dependent, then by Corollary 9.19, $\det A = 0$. This completes the proof. $\square$

EXAMPLE 13.25 Check whether the vectors $(1,2,3)^T$, $(-3,0,5)^T$, $(7,-1,2)^T$ form a basis in $\mathbb{R}^3$.

Solution. Construct the matrix with these vectors as columns and compute the determinant:

$$\begin{vmatrix} 1 & -3 & 7 \\ 2 & 0 & -1 \\ 3 & 5 & 2 \end{vmatrix} = 96 \neq 0,$$

so these vectors are a basis. $\square$

Exercises

1. Which of the following vectors are linear combinations of $\mathbf{u} = (0, -2, 2)$ and $\mathbf{v} = (1, 3, -1)$:

$$\mathbf{a}_1 = (2, 2, 2)^T, \quad \mathbf{a}_2 = (0, 4, 5)^T, \quad \mathbf{a}_3 = (0, 0, 0)^T?$$

Answer. $\mathbf{a}_1$ and $\mathbf{a}_3$.

2. Which of the following matrices are linear combinations of

$$A = \begin{pmatrix} 4 & 0 \\ -2 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 2 \\ 1 & 4 \end{pmatrix}:$$

$$\text{(a)} \begin{pmatrix} 6 & -8 \\ -1 & -8 \end{pmatrix}, \quad \text{(b)} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{(c)} \begin{pmatrix} -1 & 5 \\ 7 & 1 \end{pmatrix}?$$

Answer. (a) and (b).

3. Show that $\mathbf{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \mathbf{span}\{\mathbf{w}_1, \mathbf{w}_2\}$ if

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 6 \\ 4 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}, \quad \mathbf{w}_1 = \begin{pmatrix} 1 \\ -2 \\ -5 \end{pmatrix}, \quad \mathbf{w}_2 = \begin{pmatrix} 0 \\ 8 \\ 9 \end{pmatrix}.$$

4. Determine whether the following polynomials span P_2:

$$p_1 = 1 - x + 2x^2, \quad p_2 = 3 + x,$$
$$p_3 = 5 - x + 4x^2, \quad p_4 = -2 - 2x + 2x^2.$$

Answer. No.