Lecture 1. Linear Maps

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Examples and Elementary Properties

In $\mathbb{R}^2$, rotations about the origin and reflections in a line through the origin studied in the course of analytic geometry are examples of linear map or a linear operator.

In the first two lectures, we will describe linear maps in general, introduce the kernel and image of a linear map, and prove a useful result (called the dimension theorem) that relates the dimensions of the kernel and image.

DEFINITION 1.1 If V and W are two vector spaces, a function $f : V \to W$ is called a **linear map** if it satisfies the following axioms.

A1 $f(\mathbf{v} + \mathbf{u}) = f(\mathbf{v}) + f(\mathbf{u})$ for all $\mathbf{v}$ and $\mathbf{u}$ in V.
A2 $f(k\mathbf{v}) = kf(\mathbf{v})$ for all $\mathbf{v}$ in V and k in $\mathbb{R}$.

In the special case where $V = W$, the linear map is called a **linear operator** on the vector space V.

REMARK 1.2

1. $\mathbf{v} + \mathbf{u}$ and $k\mathbf{v}$ here are computed in V, while $f(\mathbf{v}) + f(\mathbf{u})$ and $kf(\mathbf{v})$ are in W.
2. We say that f preserves addition if A1 holds, and that f preserves scalar multiplication if A2 holds.

The following example lists three important linear maps that will be referred to later.

EXAMPLE 1.3 If V and W are vector spaces, the following are linear maps:

- **Identity operator** $V \to V$ denoted by $1_V : V \to V$ where $1_V(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v}$ in V.
- **Zero map** $V \to W$ denoted by $0 : V \to W$ where $0(\mathbf{v}) = \mathbf{0}$ for all $\mathbf{v} \in V$.
- **Scalar operator** $V \to V$ denoted by $k : V \to V$ where $k(\mathbf{v}) = k\mathbf{v}$ for all $\mathbf{v}$ in V. (Here k is any real number.)

The next example gives two important maps of matrices. Recall that the **trace** $\operatorname{tr} A$ of an $n \times n$ matrix A is the sum of the entries on the main diagonal.

EXAMPLE 1.4 Show that the transposition and trace defined by

- $R : M_{mn} \to M_{nm}$, where $R(A) = A^T$ for all A in M_{mn}.
- $S : M_{nn} \to R$ where $S(A) = \operatorname{tr} A$ for all A in M_{nn}

are both linear maps.

The next theorem collects three useful properties of all linear maps. They can be described by saying that, in addition to preserving addition and scalar multiplication (these are the axioms), linear maps preserve the zero vector, negatives, and linear combinations.

THEOREM 1.5 Let $f : V \to W$ be a linear map, then

1. $f(\mathbf{0}) = \mathbf{0}$ (f preserves the zero vector).

2. $f(-\mathbf{v}) = -f(\mathbf{v})$ for all $\mathbf{v}$ in V (f preserves the negative of a vector).
3. $f(k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r) = k_1 f(\mathbf{v}_1) + k_2 f(\mathbf{v}_2) + \cdots + k_r f(\mathbf{v}_r)$ for all scalars k_i and all vectors $\mathbf{v}_i$ in V.

Proof.

1. $f(\mathbf{0}) = f(0\mathbf{v}) = 0f(\mathbf{v}) = \mathbf{0}$ for any $\mathbf{v}$ in V.
2. $f(-\mathbf{v}) = f((-1)\mathbf{v}) = (-1)f(\mathbf{v}) = -f(\mathbf{v})$ for any $\mathbf{v}$ in V.
3. This can be proved by the method of mathematical induction. □

DEFINITION 1.6 Two linear maps $f : V \to W$ and $g : V \to W$ are called **equal** (written $f = g$) if they have the same action; that is, if $f(\mathbf{v}) = g(\mathbf{v})$ for all $\mathbf{v}$ in V.

PROPOSITION 1.7 Let $f : V \to W$ and $g : V \to W$ be two linear maps. Suppose that $V = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. If $f(\mathbf{v}_i) = g(\mathbf{v}_i)$ for each i, then $f = g$.

COROLLARY 1.8 Let $V = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. Let $f : V \to W$ be a linear map. If $f(\mathbf{v}_1) = \cdots = f(\mathbf{v}_n) = 0$, then $f = 0$.

Proposition 1.7 can be expressed as follows: If we know what a linear map $f : V \to W$ does to each vector in a spanning set for V, then we know what f does to every vector in V. If the spanning set is a basis, we can say much more.

THEOREM 1.9 (Uniqueness Theorem) Let V and W be vector spaces and let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis of V. Given any vectors $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n$ in W (they need not be linearly independent), there exists a unique linear map $f : V \to W$ satisfying $f(\mathbf{v}_i) = \mathbf{w}_i$ for each $i = 1, 2, \ldots, n$.

Proof. 1) Show that there really is such a linear map. Given $\mathbf{x}$ in V, we must specify $f(\mathbf{x})$ in W. Because $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis of V, we have $\mathbf{x} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n$, where $x_1, \ldots, x_n$ are uniquely determined by $\mathbf{x}$. Hence we may define $f : V \to W$ by

$$f(\mathbf{x}) = f(x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n) = x_1\mathbf{w}_1 + x_2\mathbf{w}_2 + \cdots + x_n\mathbf{w}_n.$$

This satisfies $f(\mathbf{v}_i) = \mathbf{w}_i$ for each i; the verification that f is linear is straightforward.

2) If g is any other such map, then $f(\mathbf{v}_i) = \mathbf{w}_i = g(\mathbf{v}_i)$ holds for each i, so $g = f$ by Proposition 1.7. Hence f is unique. □

This theorem shows that linear maps can be defined almost at will: Simply specify where the basis vectors go, and the rest of the action is dictated by the linearity. Moreover, Proposition 1.7 shows that deciding whether two linear maps are equal comes down to determining whether they have the same effect on the basis vectors.

EXAMPLE 1.10 Find a linear map $f : P_2 \to M_{22}$ such that

$$f(1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad f(x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad f(x^2) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Solution. The set $\{1, x, x^2\}$ is a basis of P_2, so for every vector $p = a + bx + cx^2$ in P_2 we have

$$f[p(x)] = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

Matrices of Linear Maps

DEFINITION 1.11 If A is $m\times n$ matrix, the map $f_A:\ \mathbb{R}^n \to \mathbb{R}^m$ given by $f_A(\mathbf{v}) = A\mathbf{v}$ is called the **matrix map induced by** A.

LEMMA 1.12 Any matrix map $f_A:\ \mathbb{R}^n \to \mathbb{R}^m$ given by $f_A(\mathbf{v}) = A\mathbf{v}$ is a linear map.

Proof. We have

$$f_A(\mathbf{v}+\mathbf{u}) = A(\mathbf{v}+\mathbf{u}) = A\mathbf{v} + A\mathbf{u} = f_A(\mathbf{v}) + f_A(\mathbf{u})$$

and

$$f_A(k\mathbf{v}) = A(k\mathbf{v}) = k(A\mathbf{v}) = kf_A(\mathbf{v})$$

hold for all $\mathbf{v}$ and $\mathbf{u}$ in $\mathbb{R}^n$ and all scalars k. Hence f_A satisfies A1 and A2, and so is linear. □

Recall that if we fix a basis $E = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ in a vector space V, then any vector $\mathbf{x} \in V$ can be uniquely represented as $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$. In that case by **coordinate vector** of $\mathbf{x}$ we mean the vector $\mathbf{x}_E = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.

THEOREM 1.13 Let V and W be finite-dimensional vector spaces, and $f : V \to W$ be a linear map. Suppose that $E = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ and $E' = \{\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_m\}$ are bases of V and W. Then there exists a unique $m \times n$ matrix A such that $f(\mathbf{x}) = \mathbf{y}$ if and only if $\mathbf{y}_{E'} = A\mathbf{x}_E$, where $\mathbf{x}_E$ and $\mathbf{y}_{E'}$ are coordinate vectors of $\mathbf{x} \in V$ and $\mathbf{y} \in W$ relative to the bases E and E'.

Proof. From Lemma 1.12 it follows that the map defined as multiplication by a matrix is linear.

Given the basis $E = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$, where

$$(\mathbf{e}_1)_E = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad (\mathbf{e}_2)_E = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad (\mathbf{e}_n)_E = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix},$$

and a linear map f, define the matrix

$$A = [f(\mathbf{e}_1)_{E'} \quad f(\mathbf{e}_2)_{E'} \quad \ldots \quad f(\mathbf{e}_n)_{E'}],$$

whose columns are the coordinates of the vectors $f(\mathbf{e}_i)$ relative to the basis E'.

Consider a linear map given by $\mathbf{y}_{E'} = A\mathbf{x}_E$, where $\mathbf{x}_E$ and $\mathbf{y}_{E'}$ are coordinate vectors of $\mathbf{x} \in V$ and $\mathbf{y} \in W$ relative to the bases E and E'. It is clear that

$$f(\mathbf{e}_1)_{E'} = A \cdot (\mathbf{e}_1)_E, \quad \ldots, \quad f(\mathbf{e}_n)_{E'} = A \cdot (\mathbf{e}_n)_E.$$

Now statement follows from Uniqueness Theorem 1.9. □

REMARK 1.14

1. In what follows we will say that f is **induced by** a matrix A with respect to bases V and W.
2. Since this matrix is unique, we can speak of **the** matrix of a linear map.
3. Every linear map f is actually a matrix map with respect to some bases.

EXAMPLE 1.15 Define $f : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$f\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + 3x_2 - x_3 \\ -2x_1 + 5x_3 \end{pmatrix}.$$

Show that f is a linear map and find its matrix.

Solution. Since

$$f\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad f\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \quad f\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \end{pmatrix},$$

by Theorem 1.13, the matrix of f is

$$A = [f(\mathbf{e}_1) \quad f(\mathbf{e}_2) \quad f(\mathbf{e}_3)] = \begin{pmatrix} 1 & 3 & -1 \\ -2 & 0 & 5 \end{pmatrix}.$$

Let $f : V \to W$ be a linear map. Suppose $E = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ and $E' = \{\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_m\}$ are bases of V and W. From now on, to simplify reasoning we will identify the vectors to their coordinate vectors relative to these bases. So we will omit subscripts by writing $\mathbf{x}$ instead of $\mathbf{x}_E$. From Theorem 1.13 we know that there is a matrix A such that $f(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$. This matrix is given by

$$A = [f(\mathbf{e}_1) \quad f(\mathbf{e}_2) \quad \ldots \quad f(\mathbf{e}_n)].$$

THEOREM 1.16 Let $f : V \to W$ and $g : W \to U$ be linear maps.

1. The composition $g \circ f$ is again a linear map.
2. Suppose f and g are represented by matrices A_f and A_g relative to bases E, E', and E'' of V, W, and U, respectively. Then the matrix corresponding to the composition $g \circ f$ is equal to the product of matrices corresponding to f and g in the appropriate order:

$$A_{g \circ f} = A_g \cdot A_f.$$

Proof. Take an arbitrary vector $\mathbf{v} \in V$. It is mapped to the vector $\mathbf{w} \in W$ such that $\mathbf{w} = A_f\mathbf{v}$. The vector $\mathbf{w}$, in turn, is mapped to the vector $\mathbf{u} \in U$ such that $\mathbf{u} = A_g\mathbf{w}$. Therefore, eventually, the vector $\mathbf{v}$ is transformed to the vector $\mathbf{u}$ by the formula

$$\mathbf{u} = A_g\mathbf{w} = A_g(A_f\mathbf{v}) = (A_gA_f)\mathbf{v}. \quad \square$$

Change of Basis and Linear Maps

Let $f : V \to W$ be a linear map of finite-dimensional vector spaces. Suppose f is represented by a matrix A relative to bases E and E' (call them "old" bases). Now suppose that B is another basis of V and B' is another basis of W (call them "new" bases). We want to know the coordinates of $f(\mathbf{x})$ with respect to B', given the coordinates $\mathbf{x}_B$ of $\mathbf{x}$ with respect to B.

THEOREM 1.17 Suppose that $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $B' = \{\mathbf{b}'_1, \ldots, \mathbf{b}'_m\}$ are bases of V and W and that $f : V \to W$ is a linear map. If A is a matrix representing f in old coordinates, then the matrix $A_{B,B'}$ which represents f relative to the bases B and B' can be computed by the formula

$$A_{B,B'} = P_{B'}^{-1} A P_B,$$

where P_B and $P_{B'}$ are the transition matrices to the basis B in V and to the basis B' in W, respectively. (Recall that a the change-of-basis matrix (also called transition matrix) is the matrix whose columns are the coordinate vectors of the new basis vectors on the old basis.)

Proof. We know that for any $\mathbf{x} \in V$, $\mathbf{x} = P_B \mathbf{x}_B$; and, similarly, for any $\mathbf{y} \in W$, $\mathbf{y} = P_{B'} \mathbf{y}_{B'}$. Suppose $\mathbf{y} = f(\mathbf{x})$. Assume that in old coordinates, $\mathbf{y} = A\mathbf{x}$. We want to find a matrix $A_{B,B'}$ such that $\mathbf{y}_{B'} = A_{B,B'} \mathbf{x}_B$.

If we start with a vector $\mathbf{x}$ in B coordinates, then $\mathbf{x} = P_B \mathbf{x}_B$ will give us the old coordinates. We can then perform the linear map on $\mathbf{x}$ using the matrix A,

$$\mathbf{y} = A\mathbf{x} = A P_B \mathbf{x}_B,$$

giving us the vector $\mathbf{y}$ in old coordinates in W. Substituting $\mathbf{y} = P_{B'} \mathbf{y}_{B'}$ we get

$$P_{B'} \mathbf{y}_{B'} = A P_B \mathbf{x}_B \qquad \Longleftrightarrow \qquad \mathbf{y}_{B'} = P_{B'}^{-1} A P_B \mathbf{x}_B.$$

Since this is true for any $\mathbf{x} \in V$, we conclude that

$$A_{B,B'} = P_{B'}^{-1} A P_B$$

is the matrix of the linear map in the new bases. □