Lecture 11. Bilinear and Quadratic Forms
November 2022

Bilinear Forms

DEFINITION 11.1 Let V be a real vector space. A function $b : V \times V \to \mathbb{R}$ that assigns to every pair of vectors $\mathbf{x}$, $\mathbf{y}$ a real number $b(\mathbf{x}, \mathbf{y})$ is called a **bilinear form** if it is linear with respect to each vector:

- $b(r\mathbf{x} + s\mathbf{y}, \mathbf{z}) = r \cdot b(\mathbf{x}, \mathbf{z}) + s \cdot b(\mathbf{y}, \mathbf{z})$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ and $r, s \in \mathbb{R}$ (**linearity in the first coordinate**)
- $b(\mathbf{z}, r\mathbf{x} + s\mathbf{y}) = r \cdot b(\mathbf{z}, \mathbf{x}) + s \cdot b(\mathbf{z}, \mathbf{y})$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ and $r, s \in \mathbb{R}$ (**linearity in the second coordinate**)

A bilinear form is called **symmetric** if $b(\mathbf{x}, \mathbf{y}) = b(\mathbf{y}, \mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in V$.

EXAMPLE 11.2 An inner product on a real vector space V is a symmetric bilinear form.

EXAMPLE 11.3 A form $b(\mathbf{x}, \mathbf{y}) = x_1y_2 - x_2y_1$ on $\mathbb{R}^2$ is bilinear, but not symmetric.

EXAMPLE 11.4 A form $b(\mathbf{x}, \mathbf{y}) = x_1y_1 - x_2y_2$ on $\mathbb{R}^2$ is bilinear and symmetric, but not positive definite, so it is not an inner product.

Given a basis $B = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ in the vector space V we can represent a bilinear form by a matrix as follows. Expand

$$\mathbf{x} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n, \qquad \mathbf{y} = y_1\mathbf{e}_1 + \cdots + y_n\mathbf{e}_n.$$

Then by definition of a bilinear form we have

$$b(\mathbf{x}, \mathbf{y}) = b\left(\sum_{j=1}^{n} x_j\mathbf{e}_j, \sum_{j=1}^{n} y_j\mathbf{e}_j\right) = \sum_{i,j} x_i b(\mathbf{e}_i, \mathbf{e}_j) y_j = (\mathbf{x}_B)^T A_B \mathbf{y}_B,$$

where

$$A_B = \begin{pmatrix} b(\mathbf{e}_1, \mathbf{e}_1) & b(\mathbf{e}_1, \mathbf{e}_2) & \ldots & b(\mathbf{e}_1, \mathbf{e}_n) \\ b(\mathbf{e}_2, \mathbf{e}_1) & b(\mathbf{e}_2, \mathbf{e}_2) & \ldots & b(\mathbf{e}_2, \mathbf{e}_n) \\ \vdots & \vdots & \ddots & \vdots \\ b(\mathbf{e}_n, \mathbf{e}_1) & b(\mathbf{e}_n, \mathbf{e}_2) & \ldots & b(\mathbf{e}_n, \mathbf{e}_n) \end{pmatrix}$$

is a matrix of b with respect to basis B and by $\mathbf{x}_B$ and $\mathbf{y}_B$ we mean the coordinate vectors of $\mathbf{x}$ and $\mathbf{y}$ written as columns.

EXAMPLE 11.5 A bilinear form $b(\mathbf{x}, \mathbf{y}) = 7x_1y_1 + 3x_1y_2 - x_2y_1 + 5x_2y_2$ on $\mathbb{R}^2$ has a matrix $A = \begin{pmatrix} 7 & 3 \\ -1 & 5 \end{pmatrix}$ with respect to the standard basis.

REMARK 11.6 A bilinear form b is symmetric if and only if the matrix A is symmetric.

THEOREM 11.7 Let $B = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ and $B' = \{\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_n\}$ be two bases in the vector space V and let P be a transition matrix from B to B'. If b is a bilinear form having the matrix A_B with respect to the basis B, then the matrix $A_{B'}$ of this form with respect to the new basis is computed by the formula

$$A_{B'} = P^T A_B P.$$

Proof. Consider two arbitrary vectors $\mathbf{x}, \mathbf{y} \in V$ and suppose that they have coordinates $\mathbf{x}_B$, $\mathbf{y}_B$ with respect to B and coordinates $\mathbf{x}_{B'}$, $\mathbf{y}_{B'}$ with respect to B'. Then we have

$$\mathbf{x}_B = P\mathbf{x}_{B'}, \qquad \mathbf{y}_B = P\mathbf{y}_{B'}$$

and

$$b(\mathbf{x}, \mathbf{y}) = (\mathbf{x}_B)^T A_B \mathbf{y}_B = (P\mathbf{x}_{B'})^T A_B (P\mathbf{y}_{B'}) = (\mathbf{x}_{B'})^T (P^T A_B P)\mathbf{y}_{B'} = (\mathbf{x}_{B'})^T A_{B'} \mathbf{y}_{B'}.$$

Since this equality holds for any two vectors, Theorem follows. □

DEFINITION 11.8 Two $n \times n$ matrices A and B are called **congruent**, written $A \overset{c}{\sim} B$, if $B = P^T AP$ for some invertible matrix P.

Here are some properties of congruence:

1. $A \overset{c}{\sim} A$ for all A.
2. If $A \overset{c}{\sim} B$, then $B \overset{c}{\sim} A$.
3. If $A \overset{c}{\sim} B$ and $B \overset{c}{\sim} C$, then $A \overset{c}{\sim} C$.
4. If $A \overset{c}{\sim} B$, then A is symmetric if and only if B is symmetric.

LEMMA 11.9 Two symmetric matrices represent the same bilinear quadratic form with respect to different bases if and only if they are congruent.

DEFINITION 11.10 Given a bilinear form b, the rank of a matrix A_B of b with respect to some basis B is called a **rank of the bilinear form** q.

LEMMA 11.11 The rank of a bilinear form is well-defined, it does not depend on the choice of a basis of V.

Proof. Lemma follows from the formula $A_{B'} = P^T A_B P$ and the fact that $\operatorname{rank}(AB) \leqslant \operatorname{rank} A$ and $\operatorname{rank}(AB) \leqslant \operatorname{rank} B$. □

LEMMA 11.12 The determinants of two congruent matrices are either both zero or have the same sign.

Proof. If $B = P^T AP$, then $|B| = |P^T AP| = |P|^2 \cdot |A|$. □

Quadratic Forms

DEFINITION 11.13 A function $q : V \to \mathbb{R}$ is called a **quadratic form** if there exists a bilinear form $b : V \times V \to \mathbb{R}$ such that $q(\mathbf{x}) = b(\mathbf{x}, \mathbf{x})$ for every $\mathbf{x} \in V$.

EXAMPLE 11.14 In $\mathbb{R}^2$, $q(\mathbf{x}) = 7x_1^2 + 2x_1x_2 + 5x_2^2$ is a quadratic form, since $q(\mathbf{x}) = b(\mathbf{x}, \mathbf{x})$ for the bilinear form $b(\mathbf{x}, \mathbf{y}) = 7x_1y_1 + 3x_1y_2 - x_2y_1 + 5x_2y_2$.

Also, $q(\mathbf{x}) = b_1(\mathbf{x}, \mathbf{x})$ for the symmetric bilinear form $b_1(\mathbf{x}, \mathbf{y}) = 7x_1y_1 + x_1y_2 + x_2y_1 + 5x_2y_2$.

LEMMA 11.15 Let V be a real vector space. For every quadratic form q, there exists a unique symmetric bilinear form b such that $q(\mathbf{x}) = b(\mathbf{x}, \mathbf{x})$ for every $\mathbf{x} \in V$.

Proof. Since q is quadratic, there exists a bilinear form b_0 such that $q(\mathbf{x}) = b_0(\mathbf{x}, \mathbf{x})$ for every $\mathbf{x} \in V$. Let

$$b(\mathbf{x}, \mathbf{y}) = \frac{b_0(\mathbf{x}, \mathbf{y}) + b_0(\mathbf{y}, \mathbf{x})}{2}.$$

Then b is a symmetric bilinear form and $b(\mathbf{x}, \mathbf{x}) = b_0(\mathbf{x}, \mathbf{x})$ for every $\mathbf{x} \in V$. Thus, q is also induced by b.

To show that b is unique, it suffices to note that

$$b(\mathbf{x}, \mathbf{y}) = \frac{b(\mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}) - b(\mathbf{x}, \mathbf{x}) - b(\mathbf{y}, \mathbf{y})}{2} = \frac{q(\mathbf{x}+\mathbf{y}) - q(\mathbf{x}) - q(\mathbf{y})}{2}.$$

So, a symmetric bilinear form is completely defined by q. □

THEOREM 11.16 If q is a quadratic form on V and B is a basis of V, then

$$q(\mathbf{x}) = (\mathbf{x}_B)^T A_B \mathbf{x}_B$$

for a unique symmetric matrix A_B. Also, if B' is another basis of V and P is a transition matrix, then

$$A_{B'} = P^T A_B P.$$

Proof follows from Theorem 11.7. □

THEOREM 11.17 (Diagonalization Theorem) For any quadratic form q, there is a basis such that the matrix of q is diagonal:

$$q(\mathbf{x}) = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \cdots + \lambda_n x_n^2.$$

Proof proceeds by induction on n, where $n = \dim V$.

For $n = 1$ the statement is evident.

If $b(\mathbf{x}, \mathbf{y}) = 0$ for all $\mathbf{x}, \mathbf{y} \in V$, then the matrix of q is zero, so theorem holds.

If $b \neq 0$, then there exists a vector $\mathbf{e}_1 \neq \mathbf{0}$ such that $q(\mathbf{e}_1) = b(\mathbf{e}_1, \mathbf{e}_1) \neq 0$. Consider a linear function $f : V \to \mathbb{R}$ given by $f(\mathbf{x}) = b(\mathbf{x}, \mathbf{e}_1)$. This function is not trivial because $f(\mathbf{e}_1) \neq 0$. Therefore, $\dim \operatorname{Im} f = 1$ and, by Dimension theorem, $\dim \operatorname{Ker} f = n - 1$. Thus, there is a basis $\{\mathbf{e}_2, \mathbf{e}_3, \ldots, \mathbf{e}_n\}$ in $\operatorname{Ker} f$ such that $b|_{\operatorname{Ker} f}$ is diagonal, i.e.,

$$b(\mathbf{e}_i, \mathbf{e}_j) = 0 \qquad i \neq j, \quad i, j = 2, \ldots n.$$

We also have $b(\mathbf{e}_1, \mathbf{e}_j) = 0$ for $j = 2, \ldots, n$.

We want to prove that the vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ form the basis. Since $\dim V = n$ and so we have the right number of vectors, it suffices to show that these vectors are linearly independent. Assume to the contrary that

$$k_1 \mathbf{e}_1 + k_2 \mathbf{e}_2 + \cdots + k_n \mathbf{e}_n = 0,$$

where not all coefficients are zero. It must be $k_1 \neq 0$, because the vectors $\mathbf{e}_2, \ldots, \mathbf{e}_n$ are linearly independent. Then it is possible to express $\mathbf{e}_1$ in terms of $\mathbf{e}_2, \ldots, \mathbf{e}_n$ as follows:

$$\mathbf{e}_1 = \alpha_2 \mathbf{e}_2 + \cdots + \alpha_n \mathbf{e}_n,$$

whence

$$0 \neq f(\mathbf{e}_1) = f(\alpha_2\mathbf{e}_2 + \cdots + \alpha_n\mathbf{e}_n) = \alpha_2 f(\mathbf{e}_2) + \cdots + \alpha_n f(\mathbf{e}_n) = 0,$$

because $\{\mathbf{e}_2, \mathbf{e}_3, \ldots, \mathbf{e}_n\} \in \operatorname{Ker} f$. This leads us to a contradiction.

It was shown that with respect to the basis $B = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$, the quadratic form q has a diagonal matrix. $\square$

COROLLARY 11.18 Any symmetric matrix in congruent to a diagonal one.

DEFINITION 11.19 Let $q = q(\mathbf{x})$ be any quadratic form in n variables. We say that q is **completely diagonalized** if in some variables $\mathbf{y}$ it has the form

$$q(\mathbf{y}) = y_1^2 + \cdots + y_k^2 - y_{k+1}^2 - \cdots - y_r^2.$$

COROLLARY 11.20 Any quadratic form can be completely diagonalized.

Proof. Let $q = q(\mathbf{x})$ be any quadratic form in n variables. By Diagonalization Theorem, there is basis such that

$$q = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \cdots + \lambda_n x_n^2.$$

By reordering the basis vectors, if necessary, we may assume that $\lambda_1, \ldots, \lambda_k$ are positive and $\lambda_{k+1}, \ldots, \lambda_r$ are negative. Then

$$q = \left(\sqrt{\lambda_1}\, x_1\right)^2 + \left(\sqrt{\lambda_k}\, x_k\right)^2 - \left(\sqrt{-\lambda_{k+1}}\, x_{k+1}\right)^2 - \cdots - \left(\sqrt{-\lambda_r}\, x_r\right)^2.$$

Now if we set

$$y_1 = \sqrt{\lambda_1}\, x_1, \ldots, y_k = \sqrt{\lambda_k}\, x_k,$$
$$y_{k+1} = \sqrt{-\lambda_{k+1}}\, x_{k+1}, \ldots, y_r = \sqrt{-\lambda_k}\, x_r,$$
$$y_{r+1} = x_{r+1}, \ldots, y_n = x_n.$$

we obtain the required result. $\square$