Lecture 15. Change of Basis, Null Space, and Linear Systems
May 2023

Isomorphism of Vector Spaces

In mathematics, an **isomorphism** is a structure-preserving bijection between two structures of the same type. Two mathematical structures are **isomorphic** if an isomorphism exists between them.

DEFINITION 15.1 Two vector spaces V and W are **isomorphic** if there is a bijection $T : V \to W$ that preserves addition and scalar multiplication, i.e., for all vectors $\mathbf{u}$ and $\mathbf{v}$ in V, and all scalars $k \in \mathbb{R}$ we have

$$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}), \qquad T(k\mathbf{v}) = kT(\mathbf{v}).$$

The correspondence T is called an **isomorphism of vector spaces**. When V and W are isomorphic, but the specific isomorphism is not named, we'll just write $V \cong W$.

The following two lemmata are trivial and we leave them without proof.

LEMMA 15.2 If $T : V \to W$ is an isomorphism of vector spaces, then its inverse $T^{-1} : W \to V$ is also an isomorphism.

LEMMA 15.3 If $T : V \to W$ and $S : W \to U$ are both isomorphisms of vector spaces, then so is their composition $(S \circ T) : V \to U$.

THEOREM 15.4 If $T : V \to W$ is an isomorphism, then T carries linearly independent sets to linearly independent sets, spanning sets to spanning sets, and bases to bases.

Proof. For the first statement, let S be a set of linearly independent vectors in V. We'll show that its image $T(S)$ is a set of linearly independent vectors in W. If $\mathbf{0}$ were a nontrivial linear combination of vectors in $T(S)$, then an application of T^{-1} would yield a nontrivial linear combination of vectors in S, but there is none since S is independent. Therefore, $T(S)$ is linearly independent.

For the second statement, let $\mathbf{w}$ be any vector in W, then $T^{-1}(\mathbf{w})$ is a linear combination of vectors in V. Apply T to that linear combination to see that $\mathbf{w}$ is a linear combination of vectors in W. Since T carries both independent and spanning sets from V to W, it carries bases to bases. □

THEOREM 15.5 (Isomorphism Theorem) If V and W are finite dimensional vector spaces, then $V \cong W$ if and only if $\dim V = \dim W$.

Proof. If $T : V \to W$ is an isomorphism, then it carries a basis to a basis, so $\dim V = \dim W$.

Conversely, let $\dim V = \dim W = n$. Suppose that $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ and $\{\mathbf{f}_1, \dots, \mathbf{f}_n\}$ are bases of V and W, respectively. Define T on the basis vectors by $T(\mathbf{e}_j) = \mathbf{f}_j$, $j = 1, \dots, n$. The action of T on any vector $\mathbf{v} \in V$ is prescribed by

$$\mathbf{v} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n \quad \mapsto \quad T(\mathbf{v}) = x_1\mathbf{f}_1 + \cdots + x_n\mathbf{f}_n.$$

Then T is well-defined and bijective by uniqueness of basis representation. Clearly, it preserves addition and scalar multiplication. $\square$

If $B = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for a vector space V, and if $[\mathbf{x}]_B = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ is the coordinate vector of $\mathbf{x}$ relative to B, then the mapping

$$\mathbf{x} \mapsto [\mathbf{x}]_B \tag{1}$$

creates a bijection between vectors in the general vector space V and vectors in $\mathbb{R}^n$. We call (1) the **coordinate mapping** relative to B from V to $\mathbb{R}^n$.

There is a natural isomorphism between any n-dimensional vector space V and $\mathbb{R}^n$.

To find the coordinates of a vector $\mathbf{x}$ with respect to a basis B, we need to solve the system of linear equations

$$\mathbf{x} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n,$$

which, in matrix form, is

$$\mathbf{x} = (\mathbf{v}_1 \quad \mathbf{v}_2 \quad \dots \quad \mathbf{v}_n) \cdot [\mathbf{x}]_B.$$

DEFINITION 15.6 If $B = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ and $B' = \{\mathbf{v}_1', \mathbf{v}_2', \dots, \mathbf{v}_n'\}$ are two bases of V, the matrix

$$P = (\mathbf{v}_1' \quad \mathbf{v}_2' \quad \dots \quad \mathbf{v}_n'),$$

whose columns are coordinates of the basis vectors of B' in B, is called the **transition matrix from B to B'**. In matrix form, the relationship between two bases can be written

$$(\mathbf{v}_1' \quad \mathbf{v}_2' \quad \dots \quad \mathbf{v}_n') = (\mathbf{v}_1 \quad \mathbf{v}_2 \quad \dots \quad \mathbf{v}_n) \cdot \begin{pmatrix} p_{11} & \dots & p_{1n} \\ \vdots & \ddots & \vdots \\ p_{n1} & \dots & p_{nn} \end{pmatrix} \tag{2}$$

It is clear that the matrix P^{-1} is the transition matrix from B' to B.

In order to emphasise the connection of a transition matrix P with the corresponding bases B and B', we will sometimes denote the matrix by $P_{B\to B'}$ or $P_{BB'}$.

EXAMPLE 15.7 Let B be the following set of vectors of $\mathbb{R}^n$:

$$B = \left\{(1,\ 2,\ -1)^T,\ (2,\ -1,\ 4)^T,\ (3,\ 2,\ 1)^T\right\}$$

1. Show that B is a basis.
2. Given the B coordinates of a vector $\mathbf{v}$, find its standard coordinates, where $[\mathbf{v}]_B = (4,\ 1,\ -5)^T$.
3. Find the B coordinates of a vector $\mathbf{u} = (5,\ 7,\ -3)^T$.

Solution. 1) Form the matrix with the vectors of B as columns and find its determinant:

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & 2 \\ -1 & 4 & 1 \end{vmatrix} = 4 \neq 0.$$

As the determinant is nonzero, the vectors are linearly independent. Since we have exactly 3 linearly independent vectors in $\mathbb{R}^3$, then B is a basis.

2) $\mathbf{v} = 4\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} - 5\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -9 \\ -3 \\ -5 \end{pmatrix}.$

3) To find $[\mathbf{u}]_B$ we must solve the vector equation

$$x_1\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + x_2\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + x_3\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ -3 \end{pmatrix}.$$

By Cramer's rule or by the Gaussian elimination method, we get the answer: $[\mathbf{u}]_B = (1, -1, 2)^T$. $\square$

EXAMPLE 15.8 Check whether the following polynomials form a basis for P_2:

$$p_1 = x^2 + 1, \qquad p_2 = x^2 - 1, \qquad p_3 = 2x - 1.$$

If so, represent the polynomial $p_4 = 5x^2 + 2x + 2$ relative to this basis.

Solution. 1) By the coordinate isomorphism, we can write the polynomials as vectors:

$$p_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad p_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad p_3 = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}.$$

Now construct a matrix A that has that vectors as columns and find its determinant: $|A| = 4 \neq 0$. Since we have 3 linearly independent vectors in three-dimensional vector space, the vectors are a basis.

2) We need to solve the vector equation

$$x_1\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + x_2\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + x_3\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 2 \end{pmatrix}.$$

The solution is as follows: $x_1 = 4$, $x_2 = 1$, $x_3 = 1$, so $p_4 = 4p_1 + p_2 + p_3$. $\square$

Change of Basis

A basis that is suitable for one problem may not be suitable for another, so it is a common process in the study of vector spaces to change from one basis to another. Now we will study problems related to changing bases.

Given two bases B and B' of V with transition matrix $P_{BB'}$, how do we change from coordinates in the basis B' to coordinates in the basis B? The answer is quite simple and given in the following theorem.

THEOREM 15.9 If B and B' are two bases of V, with $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $B' = \{\mathbf{v}'_1, \mathbf{v}'_2, \ldots, \mathbf{v}'_n\}$, and P is the transition matrix from B to B', then for each vector $\mathbf{x}$ in V, the coordinate vector $[\mathbf{x}]_B$ is related to the coordinate vector $[\mathbf{x}]_{B'}$ by the equation

$$[\mathbf{x}]_B = P \cdot [\mathbf{x}]_{B'}. \tag{3}$$

Proof. For any vector $\mathbf{x} \in V$ we have

$$\mathbf{x} = (\mathbf{v}_1 \quad \mathbf{v}_2 \quad \ldots \quad \mathbf{v}_n) \cdot [\mathbf{x}]_B = (\mathbf{v}'_1 \quad \mathbf{v}'_2 \quad \ldots \quad \mathbf{v}'_n) \cdot [\mathbf{x}]_{B'}.$$

By (2), this can be written

$$(\mathbf{v}_1 \quad \mathbf{v}_2 \quad \ldots \quad \mathbf{v}_n) \cdot [\mathbf{x}]_B = (\mathbf{v}_1 \quad \mathbf{v}_2 \quad \ldots \quad \mathbf{v}_n)P \cdot [\mathbf{x}]_{B'}.$$

Therefore, eliminating $(\mathbf{v}_1 \quad \mathbf{v}_2 \quad \ldots \quad \mathbf{v}_n)$ from both sides, we get (3). □

EXAMPLE 15.10 Check that each of the sets of vectors

$$B = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}, \qquad S = \left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \end{pmatrix} \right\}$$

is a basis of $\mathbb{R}^2$. Given a vector $\mathbf{x} \in \mathbb{R}^2$ with $[\mathbf{x}]_B = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$ find the coordinates of $\mathbf{x}$ in the basis S.

Solution. Introduce the matrices

$$A_1 = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$$

formed by vectors from B and S as columns. Since $|A_1| = 3 \neq 0$ and $|A_2| = 1 \neq 0$, the given sets of vectors are bases of $\mathbb{R}^2$.

Find the transition matrix from S to B. By (2), we have

$$A_1 = A_2 P \quad \Longrightarrow \quad P = A_2^{-1} A_1 = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} -8 & -7 \\ 5 & 4 \end{pmatrix}.$$

Therefore,

$$[\mathbf{x}]_S = P \cdot [\mathbf{x}]_B = \begin{pmatrix} -8 & -7 \\ 5 & 4 \end{pmatrix} \begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} -25 \\ 16 \end{pmatrix}. \quad \square$$

Null Space of a Matrix

DEFINITION 15.11 If A is an $m \times n$ matrix, then the **null space** of A, denoted $\operatorname{null} A$, consists of all solutions $\mathbf{x} \in \mathbb{R}^n$ of the homogeneous system $A\mathbf{x} = \mathbf{0}$, that is,

$$\operatorname{null} A = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}.$$

THEOREM 15.12 If A is an $m \times n$ matrix, then $\operatorname{null} A$ is a subspace of $\mathbb{R}^n$.

Proof. The zero vector $\mathbf{0}$ in $\mathbb{R}^n$ lies in $\operatorname{null} A$ because $A\mathbf{0} = \mathbf{0}$. If $\mathbf{x}_1$ and $\mathbf{x}_2$ are in $\operatorname{null} A$, then $\mathbf{x}_1 + \mathbf{x}_2$ is in $\operatorname{null} A$ because $A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = \mathbf{0} + \mathbf{0} = \mathbf{0}$. In addition, if $\mathbf{x} \in A$ and $k \in \mathbb{R}$, then $k\mathbf{x} \in A$ as $A(k\mathbf{x}) = k(A\mathbf{x}) = k\mathbf{0} = \mathbf{0}$. Hence $\operatorname{null} A$ is a subspace of $\mathbb{R}^n$. $\square$

The dimension of the null space of A is called the **nullity** of A.

DEFINITION 15.13 A matrix constructed from basis vectors of $\operatorname{null} A$ as columns is called a **fundamental matrix** of the system $A\mathbf{x} = \mathbf{0}$.

EXAMPLE 15.14 Given a matrix $F = \begin{pmatrix} 1 & 3 \\ 0 & 1 \\ 2 & -2 \end{pmatrix}$ find a homogeneous system, for which matrix F is fundamental.

Solution. We can restate the task as follows: find a matrix A such that $AF = 0$, where 0 is a zero matrix of an appropriate size, and $\operatorname{rank} A = 3 - \operatorname{rank} F = 1$.

If we transpose $AF = 0$, we get $F^T A^T = 0$, where F is a given matrix and A^T is an unknown matrix which can be considered as a fundamental matrix of linear system $F^T\mathbf{x} = \mathbf{0}$. To solve this system, we use Gaussian elimination method:

$$\begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -8 \end{pmatrix}.$$

The solution is $(x_1, x_2, x_3)^T = h(-2, 8, 1)^T$, $h \in \mathbb{R}$. So $A = (-2 \quad 8 \quad 1)$. It is easy to check that $AF = 0$ and F is a fundamental matrix of $A\mathbf{x} = \mathbf{0}$. $\square$

THEOREM 15.15 (Kronecker–Capelli Consistency Theorem) A system of linear equations $A\mathbf{x} = \mathbf{b}$ is consistent if and only if the rank of A is equal to the rank of $(A|\mathbf{b})$:

$$\operatorname{rank} A = \operatorname{rank}(A|\mathbf{b}).$$

Proof. A system of linear equations $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the column space of A. Indeed, the product $A\mathbf{x}$ can be expressed as a linear combination of column vectors $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n$ of A with coefficients from $\mathbf{x}$; that is,

$$A\mathbf{x} = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_n\mathbf{c}_n. \tag{4}$$

Thus, a linear system, $A\mathbf{x} = \mathbf{b}$, of m equations in n unknowns can be written as

$$x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_n\mathbf{c}_n = \mathbf{b} \tag{5}$$

from which we conclude that $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is expressible as a linear combination of the column vectors of A. By Rank Theorem, the rank of A is equal to the dimension of $\operatorname{col} A = \mathbf{span}\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$. As $\mathbf{b} \in \mathbf{span}\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$, then

$$\operatorname{col}(A|\mathbf{b}) = \mathbf{span}\{\mathbf{b}, \mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\} = \mathbf{span}\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\} = \operatorname{col} A.$$

Thus the dimensions of those vector spaces are the same, so $\operatorname{rank} A = \operatorname{rank}(A|\mathbf{b})$. $\square$

Now we will establish the relationship between the solutions of a homogeneous linear system $A\mathbf{x} = \mathbf{0}$ and the solutions (if any) of a nonhomogeneous linear system $A\mathbf{x} = \mathbf{b}$ that has the same coefficient matrix. These are called *corresponding linear systems.*

THEOREM 15.16 (Representation of a General Solution of a Nonhomogeneous Linear System) The general solution of a consistent linear system $A\mathbf{x} = \mathbf{b}$ can be obtained by adding any specific solution of $A\mathbf{x} = \mathbf{b}$ to the general solution of $A\mathbf{x} = \mathbf{0}$.

Proof. Let $\mathbf{x}_0$ be any specific solution of $A\mathbf{x} = \mathbf{b}$, let W denote the solution set of $A\mathbf{x} = \mathbf{0}$, and let $\mathbf{x}_0 + W$ denote the set of all vectors that result by adding $\mathbf{x}_0$ to each vector in W. We must show that if $\mathbf{x}$ is a vector in $\mathbf{x}_0 + W$, then $\mathbf{x}$ is a solution of $A\mathbf{x} = \mathbf{b}$, and conversely that every solution of $A\mathbf{x} = \mathbf{b}$ is in the set $\mathbf{x}_0 + W$.

Assume first that $\mathbf{x}$ is a vector in $\mathbf{x}_0 + W$. This implies that $\mathbf{x}$ is expressible in the form $\mathbf{x} = \mathbf{x}_0 + \mathbf{w}$, where $A\mathbf{x}_0 = \mathbf{b}$ and $A\mathbf{w} = \mathbf{0}$. Thus,

$$A\mathbf{x} = A(\mathbf{x}_0 + \mathbf{w}) = A\mathbf{x}_0 + A\mathbf{w} = \mathbf{b} + \mathbf{0} = \mathbf{b}$$

which shows that $\mathbf{x}$ is a solution of $A\mathbf{x} = \mathbf{b}$.

Conversely, let $\mathbf{x}$ be any solution of $A\mathbf{x} = \mathbf{b}$. To show that $\mathbf{x}$ is in the set $\mathbf{x}_0 + W$ we must show that $\mathbf{x}$ is expressible in the form

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{w} \tag{6}$$

where $\mathbf{w}$ is in W (i.e., $A\mathbf{w} = \mathbf{0}$). We can do this by taking $\mathbf{w} = \mathbf{x} - \mathbf{x}_0$. This vector obviously satisfies (6), and it is in W since

$$A\mathbf{w} = A(\mathbf{x} - \mathbf{x}_0) = A\mathbf{x} - A\mathbf{x}_0 = \mathbf{b} - \mathbf{b} = \mathbf{0}. \quad \square$$

REMARK 15.17 If we interpret vector addition as translation, then the theorem states that if $\mathbf{x}_0$ is any specific solution of $A\mathbf{x} = \mathbf{b}$, then the entire solution set of $A\mathbf{x} = \mathbf{b}$ can be obtained by translating the solution space of $A\mathbf{x} = \mathbf{0}$ by the vector $\mathbf{x}_0$. A set obtained by translating (away from the origin) the vector subspace by the translation vector (the vector added to all the elements of the vector space) is called an **affine subspace** of the vector space.

Keeping in mind that the null space of A is the same as the solution space of $A\mathbf{x} = \mathbf{0}$, we can rephrase that theorem in the following vector form.

THEOREM 15.18 If $\mathbf{x}_0$ is any solution of a consistent linear system $A\mathbf{x} = \mathbf{b}$, and if $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a basis for the null space of A, then every solution of $A\mathbf{x} = \mathbf{b}$ can be expressed in the form

$$\mathbf{x} = \mathbf{x}_0 + k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r. \tag{7}$$

Conversely, for all choices of scalars $k_1, k_2, \ldots, k_r$, the vector $\mathbf{x}$ in this formula is a solution of $A\mathbf{x} = \mathbf{b}$.

The vector $\mathbf{x}_0$ in Formula (7) is called a **particular solution** of $A\mathbf{x} = \mathbf{b}$, and the remaining part of the formula is called the **general solution** of $A\mathbf{x} = \mathbf{0}$. With this terminology Theorem 15.18 can be rephrased as:

The general solution of a consistent linear system can be expressed as the sum of a particular solution of that system and the general solution of the corresponding homogeneous system.

EXAMPLE 15.19 Find the general solution of a linear system

$$\begin{cases} x_1 - 5x_3 + 4x_4 &= 7, \\ x_2 + 11x_3 - 2x_4 &= -3 \end{cases}$$

and represent it as the sum of a particular solution of that system and the general solution of the corresponding homogeneous system.

Solution. Taking x_1, x_2 as leading variables and x_3, x_4 as parameters, we can write the solution in the form

$$\begin{cases} x_1 &= 7 + 5h_1 - 4h_2, \\ x_2 &= -3 - 11h_1 + 2h_2, \\ x_3 &= h_1, \\ x_4 &= h_2, \end{cases}$$

which can be written in the vector form as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 7 \\ -3 \\ 0 \\ 0 \end{pmatrix} + h_1 \begin{pmatrix} 5 \\ -1 \\ 1 \\ 0 \end{pmatrix} + h_2 \begin{pmatrix} -4 \\ 2 \\ 0 \\ 1 \end{pmatrix}.$$

Here h_1, h_2 are any real numbers, $\mathbf{x}_0 = (7, -3, 0, 0)^T$ is a particular solution of the linear system, and $\mathbf{v}_1 = (5, -1, 1, 0)^T$, $\mathbf{v}_2 = (-4, 2, 0, 1)^T$ span the solution space and form the basis of null A.

The solution can also be represented in the form $\mathbf{x} = \mathbf{x}_0 + F \cdot \mathbf{h}$, where F is a fundamental matrix of the associated homogeneous linear system as follows:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 7 \\ -3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 5 & -4 \\ -1 & 2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}. \quad \square$$

Exercises

1. Find the general solution of the following homogeneous systems

(a) $$\begin{cases} x_1 - x_2 - 6x_3 + x_4 + 2x_5 = 0, \\ x_1 + x_2 + x_3 + x_4 + 2x_5 = 0, \\ 4x_1 + x_2 - 5x_3 + 3x_4 + 6x_5 = 0, \\ 2x_1 + 3x_2 + 4x_3 + 3x_4 + 6x_5 = 0, \end{cases}$$

(b) $$\begin{cases} 2x_1 + 2x_2 - x_3 + x_5 = 0, \\ -x_1 - x_2 + 2x_3 - 3x_4 + 5x_5 = 0, \\ x_1 + x_2 + x_3 - 3x_4 + 6x_5 = 0. \end{cases}$$

and represent it in the form $\mathbf{x} = F \cdot \mathbf{h}$, where F is a fundamental matrix of the corresponding homogeneous system and $\mathbf{h}$ is an arbitrary vector of the appropriate size.

Answer.

$$\text{(a)}\ \begin{pmatrix} 2 & 0 \\ -7 & 0 \\ 2 & 0 \\ 0 & 1 \\ 3 & -2 \end{pmatrix} \cdot \begin{pmatrix} h_1 \\ h_2 \end{pmatrix},\ h_1, h_2 \in \mathbb{R}; \qquad \text{(b)}\ \begin{pmatrix} -1 & 1 & -7 \\ 1 & 0 & 0 \\ 0 & 2 & -11 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix},\ h_1, h_2, h_3 \in \mathbb{R}.$$

2. Find the general solution of each of the following systems

$$\begin{cases} 3x_1 - 2x_2 - x_3 - x_4 = 1, \\ 3x_1 - 2x_2 + 5x_3 + 4x_4 = 2, \\ 6x_1 - 4x_2 + 4x_3 + 3x_4 = 3; \end{cases}$$

and represent it in the form $\mathbf{x} = \mathbf{x}_0 + F \cdot \mathbf{h}$, where $\mathbf{x}_0$ is a particular solution of that system and $F \cdot \mathbf{h}$ is the general solution of the associated homogeneous system.

Answer. $\begin{pmatrix} 0 \\ 0 \\ 6 \\ -7 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 10 & -15 \\ -12 & 18 \end{pmatrix} \cdot \begin{pmatrix} h_1 \\ h_2 \end{pmatrix},\ h_1, h_2 \in \mathbb{R}.$

3. Check whether the following matrices form a basis for the space of 2×2 matrices:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & -1 \\ 3 & 7 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} -4 & 1 \\ 1 & 0 \end{pmatrix}.$$

If so, find the coordinates of $\begin{pmatrix} 8 & -5 \\ 9 & 9 \end{pmatrix}$ relative to this basis.

Answer. Yes, $(2, 1, 5, -1)$.