Lecture 13. Self-Adjoint Operators
November 2022

Linear Functionals on Inner Product Spaces

EXAMPLE 13.1 Consider $\mathbb{R}^3$ with a standard inner product. Then the functional $f : \mathbb{R}^3 \to \mathbb{R}$ given by

$$f(\mathbf{x}) = 4x_1 - x_2 + 5x_3$$

can be represented as

$$f(\mathbf{x}) = \langle \mathbf{x}, \mathbf{v} \rangle, \qquad \mathbf{v} = (4, -1, 5)^T.$$

If $\mathbf{v} \in V$, then the map that sends $\mathbf{x}$ to $\langle \mathbf{x}, \mathbf{v} \rangle$ is a linear functional on V. The next result shows that every linear functional on V is of this form.

THEOREM 13.2 (Riesz Representation Theorem) Suppose V is finite-dimensional inner product space and $f : V \to \mathbb{R}$ is a linear functional on V. Then there is a unique vector $\mathbf{v} \in V$ such that

$$f(\mathbf{x}) = \langle \mathbf{x}, \mathbf{v} \rangle \qquad \forall \mathbf{x} \in V.$$

Proof. First we show there exists a vector $\mathbf{v} \in V$ such that $f(\mathbf{x}) = \langle \mathbf{x}, \mathbf{v} \rangle$ for every $\mathbf{x} \in V$. Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be an orthonormal basis of V. Then, by linearity,

$$\begin{aligned} f(\mathbf{x}) = f(\langle \mathbf{x}, \mathbf{e}_1 \rangle \mathbf{e}_1 + \cdots + \langle \mathbf{x}, \mathbf{e}_n \rangle \mathbf{e}_n) = \langle \mathbf{x}, \mathbf{e}_1 \rangle f(\mathbf{e}_1) + \cdots + \langle \mathbf{x}, \mathbf{e}_n \rangle f(\mathbf{e}_n) \\ = \langle \mathbf{x}, f(\mathbf{e}_1)\mathbf{e}_1 + \cdots + f(\mathbf{e}_n)\mathbf{e}_n \rangle \end{aligned}$$

for every $\mathbf{x} \in V$. Thus setting

$$\mathbf{v} = f(\mathbf{e}_1)\mathbf{e}_1 + \cdots + f(\mathbf{e}_n)\mathbf{e}_n$$

we have $f(\mathbf{x}) = \langle \mathbf{x}, \mathbf{v} \rangle$ for every $\mathbf{x} \in V$, as desired.

Now we prove that only one vector $\mathbf{v} \in V$ has the desired behavior. Suppose $\mathbf{v}_1, \mathbf{v}_2 \in V$ are such that

$$f(\mathbf{x}) = \langle \mathbf{x}, \mathbf{v}_1 \rangle = \langle \mathbf{x}, \mathbf{v}_2 \rangle \qquad \forall \mathbf{x} \in V.$$

Then

$$0 = \langle \mathbf{x}, \mathbf{v}_1 \rangle - \langle \mathbf{x}, \mathbf{v}_2 \rangle = \langle \mathbf{x}, \mathbf{v}_1 - \mathbf{v}_2 \rangle \qquad \forall \mathbf{x} \in V.$$

Taking $\mathbf{x} = \mathbf{v}_1 - \mathbf{v}_2$ shows that $\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{0}$, completing the proof of the uniqueness part of the result. □

Adjoints

Let V and W be two finite-dimensional inner product spaces. Suppose $f : V \to W$ is a linear map. To simplify reasoning, through the lecture, we will use notation $f\mathbf{x}$ rather then $f(\mathbf{x})$.

DEFINITION 13.3 Let V and W be two inner product spaces. Suppose $f : V \to W$ is linear. The **adjoint** of f is the map $f^* : W \to V$ such that

$$\langle f\mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, f^*\mathbf{y} \rangle$$

for every $\mathbf{x} \in V$ and every $\mathbf{y} \in W$.

To see why the definition above makes sense, fix $\mathbf{y} \in W$ and consider the linear functional on V that maps $\mathbf{x} \in V$ to $\langle f\mathbf{x}, \mathbf{y}\rangle$. This linear functional depends on f and $\mathbf{y}$. By the Riesz Representation Theorem, there exists a unique vector in V such that this linear functional is given by taking the inner product with it. We call this unique vector $f^*\mathbf{y}$.

EXAMPLE 13.4 Consider $\mathbb{R}^2$ and $\mathbb{R}^3$ with a standard inner product. Define $f : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$f(x_1, x_2, x_3) = (x_1 - x_2, 2x_1 - 3x_3).$$

Find a formula for f^*.

Solution. Here f^* will be a function from $\mathbb{R}^2$ to $\mathbb{R}^3$. To compute f^*, fix a point $(y_1, y_2) \in \mathbb{R}^2$. Then for every $(x_1, x_2, x_3) \in \mathbb{R}^3$ we have

$$\begin{aligned} \langle (x_1, x_2, x_3), f^*(y_1, y_2)\rangle = \langle f(x_1, x_2, x_3), (y_1, y_2)\rangle &= x_1y_1 - x_2y_1 + 2x_1y_2 - 3x_3y_2 \\ &= \langle (x_1, x_2, x_3), (y_1 + 2y_2, -y_1, -3y_2)\rangle. \end{aligned}$$

Thus $f^*(y_1, y_2) = (y_1 + 2y_2, -y_1, -3y_2)$.

In the example above, f^* turned out to be not just a function but a linear map. This is true in general, as shown by the next theorem. The proofs of the next two results use a common technique: flip f^* from one side of an inner product to become f on the other side.

THEOREM 13.5 The adjoint is a linear map, i.e., if $f \in L(V, W)$, then $f^* \in L(W, V)$.

Proof. Suppose $f \in L(V, W)$. Fix $\mathbf{y}_1, \mathbf{y}_2 \in W$. If $\mathbf{x} \in V$, then

$$\begin{aligned} \langle \mathbf{x}, f^*(\mathbf{y}_1 + \mathbf{y}_2)\rangle = \langle f\mathbf{x}, \mathbf{y}_1 + \mathbf{y}_2\rangle &= \langle f\mathbf{x}, \mathbf{y}_1\rangle + \langle f\mathbf{x}, \mathbf{y}_2\rangle \\ &= \langle \mathbf{x}, f^*\mathbf{y}_1\rangle + \langle \mathbf{x}, f^*\mathbf{y}_2\rangle = \langle \mathbf{x}, f^*\mathbf{y}_1 + f^*\mathbf{y}_2\rangle, \end{aligned}$$

which shows that $f^*(\mathbf{y}_1 + \mathbf{y}_2) = f^*\mathbf{y}_1 + f^*\mathbf{y}_2$.

Fix $\mathbf{y} \in W$ and $\lambda \in \mathbb{R}$. If $\mathbf{x} \in V$, then

$$\langle \mathbf{x}, f^*(\lambda\mathbf{y})\rangle = \langle f\mathbf{x}, \lambda\mathbf{y}\rangle = \lambda\langle f\mathbf{x}, \mathbf{y}\rangle = \lambda\langle \mathbf{x}, f^*\mathbf{y}\rangle = \langle \mathbf{x}, \lambda f^*\mathbf{y}\rangle,$$

which shows that $f^*(\lambda\mathbf{y}) = \lambda f^*(\mathbf{y})$. Thus f^* is a linear map, as desired. □

THEOREM 13.6 (Properties of the Adjoint)

(a) $(f + g)^* = f^* + g^*$ for all $f, g \in L(V, W)$;
(b) $(\lambda f)^* = \lambda f^*$ for all $f \in L(V, W)$ and $\lambda \in \mathbb{R}$;
(c) $(f^*)^* = f$ for all $f \in L(V, W)$;
(d) $I^* = I$, where I is the identity operator on V;
(e) $(g \circ f)^* = f^* \circ g^*$ for all $f \in L(V, W)$ and $g \in L(W, U)$.

Proof is straightforward.

The next result shows the relationship between the kernel and the image of a linear map and its adjoint.

THEOREM 13.7 (Kernel and Image of f^*) Suppose $f \in L(V, W)$. Then

(a) $\operatorname{Ker} f^* = (\operatorname{Im} f)^\perp$;
(b) $\operatorname{Im} f^* = (\operatorname{Ker} f)^\perp$;
(c) $\operatorname{Ker} f = (\operatorname{Im} f^*)^\perp$;
(d) $\operatorname{Im} f = (\operatorname{Ker} f^*)^\perp$.

Proof. We begin by proving (a). Let $\mathbf{y} \in W$. Then

$$\mathbf{y} \in \operatorname{Ker} f^* \;\Leftrightarrow\; f^*\mathbf{y} = \mathbf{0} \;\Leftrightarrow\; \langle \mathbf{x}, f^*\mathbf{y} \rangle = 0 \;\forall \mathbf{x} \in V \;\Leftrightarrow\; \langle f\mathbf{x}, \mathbf{y} \rangle = 0 \;\forall \mathbf{x} \in V$$

The last implies that $\mathbf{y} \in (\operatorname{Im} f)^\perp$. Thus $\operatorname{Ker} f^* = (\operatorname{Im} f)^\perp$, proving (a).

If we take the orthogonal complement of both sides of (a), we get (d). Replacing f with f^* in (a) gives (c), where we have used Theorem 13.6(c). Finally, replacing f with f^* in (d) gives (b). $\square$

The next result shows how to compute the matrix of f^* from the matrix of f.

THEOREM 13.8 Let $f \in L(V, W)$, where V and W are inner product spaces. Suppose the inner products in V and W are described by Gram matrices G and G' with respect to bases B in V and B' in W, respectively. Then the matrices of f and f^* with respect to these bases are related by

$$A^* = G^{-1} A^T G'.$$

Proof. By definition of f^*, for every $\mathbf{x} \in V$ and every $\mathbf{y} \in W$ we have

$$\langle f\mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, f^*\mathbf{y} \rangle.$$

In matrix representation, this equality takes the form

$$\mathbf{x}^T A^T G' \mathbf{y} = (A\mathbf{x})^T G' \mathbf{y} = \mathbf{x}^T G A^* \mathbf{y}.$$

Since this is true for every $\mathbf{x} \in V$ and every $\mathbf{y} \in W$, then

$$A^T G' = G A^* \qquad \Longleftrightarrow \qquad A^* = G^{-1} A^T G'. \quad \square$$

COROLLARY 13.9 Let $f \in L(V, W)$ and suppose that the bases in V and W are orthonormal. Then $A^* = A^T$.

DEFINITION 13.10 An operator $f : V \to V$ is called **self-adjoint** if $f = f^*$. In other words, $f \in L(V)$ is self-adjoint if and only if

$$\langle f\mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, f\mathbf{y} \rangle$$

for all $\mathbf{x}, \mathbf{y} \in V$.

PROPOSITION 13.11 Given an inner product space with an orthonormal basis, an operator $f : V \to V$ is self-adjoint if and only if the matrix of f with respect to this basis is symmetric, i.e. satisfies $A = A^T$.