Lecture 10. Orthogonal Complements
November 2022

Orthogonal Complements and Projections

DEFINITION 10.1 If U is a subset of V, then the **orthogonal complement** of U, denoted by $U^\perp$, is the set of all vectors in V that are orthogonal to every vector in U:

$$U^\perp = \{\mathbf{x} \in V | \ \langle \mathbf{x}, \mathbf{y} \rangle = 0 \quad \forall \mathbf{y} \in U\}.$$

EXAMPLE 10.2 If U is a line in $\mathbb{R}^3$, then $U^\perp$ is the plane containing the origin that is perpendicular to U. If U is a plane in $\mathbb{R}^3$, then $U^\perp$ is the line containing the origin that is perpendicular to U.

The following lemma collects some useful properties of the orthogonal complement; the proof of parts (1)–(4) of this lemma is left as an exercise.

LEMMA 10.3 (Basic Properties of Orthogonal Complement) Let U be a subspace of an inner product space V. Then

(1) If U is a subset of V, then $U^\perp$ is a subspace of V.
(2) $\{\mathbf{0}\}^\perp = V$ and $V^\perp = \{\mathbf{0}\}$.
(3) If U is a subset of V, then $U \cap U^\perp \subset \{\mathbf{0}\}$.
(4) If U and W are subsets of V and $U \subset W$, then $W^\perp \subset U^\perp$.
(5) If $U = \operatorname{span}\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$, then $U^\perp = \{\mathbf{x} \in V | \ \langle \mathbf{x}, \mathbf{x}_i \rangle = 0 \text{ for } i = 1, 2, \ldots, k\}$.

Proof (5) Let $U = \operatorname{span}\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$; we must show that $U^\perp = \{\mathbf{x} \in V | \ \langle \mathbf{x}, \mathbf{x}_i \rangle = 0 \text{ for } i = 1, 2, \ldots, k\}$. If $\mathbf{x} \in U^\perp$, then $\langle \mathbf{x}, \mathbf{x}_i \rangle = 0$ for all i because each $\mathbf{x}_i$ is in U. Conversely, suppose that $\langle \mathbf{x}, \mathbf{x}_i \rangle = 0$ for all i; we must show that $\mathbf{x}$ is in $U^\perp$, that is, $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ for each $\mathbf{y}$ in U. Write $\mathbf{y} = r_1\mathbf{x}_1 + r_2\mathbf{x}_2 + \cdots + r_k\mathbf{x}_k$, where $r_i \in \mathbb{R}$, $1 \leqslant i \leqslant k$. Then

$$\langle \mathbf{x}, \mathbf{y} \rangle = r_1\langle \mathbf{x}, \mathbf{x}_1 \rangle + r_2\langle \mathbf{x}, \mathbf{x}_2 \rangle + \cdots + r_k\langle \mathbf{x}, \mathbf{x}_k \rangle = 0,$$

as required. $\square$

EXAMPLE 10.4 Find $U^\perp$ if $U = \operatorname{span}\{(1, -1, 2, 0), (3, 0, -2, 1)\}$ in $\mathbb{R}^4$.

Solution. By Lemma 10.3, $\mathbf{x} = (x, y, z, w)$ is in $U^\perp$ if and only if it is orthogonal to both $(1, -1, 2, 0)$ and $(1, 0, -2, 3)$; that is,

$$x - y + 2z = 0, \qquad 3x - 2z + w = 0$$

Gaussian elimination gives $U^\perp = \operatorname{span}\{(1, 1, 0, -3), (0, 2, 1, 2)\}$.

Recall that if U, W are subspaces of V, then V is the direct sum of U and W (written $V = U \oplus W$) if each element of V can be written in exactly one way as a vector in U plus a vector in W. The next result shows that every finite-dimensional subspace of V leads to a natural direct sum decomposition of V.

THEOREM 10.5 Suppose U is a finite-dimensional subspace of V. Then $V = U \oplus U^{\perp}$.

Proof. First we will show that $V = U + U^{\perp}$. To do this, suppose $\mathbf{x} \in V$. Let $\{\mathbf{e}_1, \dots \mathbf{e}_m\}$ be an orthonormal basis of U. Let $\mathbf{u} = \langle \mathbf{x}, \mathbf{e}_1 \rangle \mathbf{e}_1 + \cdots + \langle \mathbf{x}, \mathbf{e}_m \rangle \mathbf{e}_m$, $\mathbf{w} = \mathbf{x} - \mathbf{u}$.

Clearly $\mathbf{u} \in U$. For each $j = 1, \dots, m$ we have

$$\langle \mathbf{w}, \mathbf{e}_j \rangle = \langle \mathbf{x}, \mathbf{e}_j \rangle - \langle \mathbf{x}, \mathbf{e}_j \rangle = 0.$$

Thus $\mathbf{w} \in U^{\perp}$. This proves that $V = U + U^{\perp}$. By Lemma 10.3 (3), $U \cap U^{\perp} = \{\mathbf{0}\}$. So theorem follows. $\square$

COROLLARY 10.6 Suppose V is finite-dimensional and U is a subspace of V. Then

$$\dim U + \dim U^{\perp} = \dim V.$$

THEOREM 10.7 Suppose U is a finite-dimensional subspace of V. Then

$$U = (U^{\perp})^{\perp}.$$

Proof. First we will show that

$$U \subset (U^{\perp})^{\perp}.$$

To do this, suppose $\mathbf{u} \in U$. Then $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ for every $\mathbf{v} \in U^{\perp}$ (by the definition of $U^{\perp}$). Because $\mathbf{u}$ is orthogonal to every vector in $U^{\perp}$, we have $\mathbf{u} \in (U^{\perp})^{\perp}$.

To prove the inclusion in the other direction, suppose $\mathbf{v} \in (U^{\perp})^{\perp}$. By Theorem 10.5, we can write $\mathbf{v} = \mathbf{u} + \mathbf{w}$, where $\mathbf{u} \in U$ and $\mathbf{w} \in U^{\perp}$. We have

$$0 = \langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{w}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{w} \rangle,$$

whence $\mathbf{w} = \mathbf{0}$ and $\mathbf{v} = \mathbf{u} \in U$. $\square$

Hermitian Inner Product

Now we consider **complex vector spaces**.

DEFINITION 10.8 A **Hermitian inner product** on a complex vector space V is a function $\langle , \rangle : V \times V \to \mathbb{C}$ that assigns a number $\langle \mathbf{x}, \mathbf{y} \rangle$ to every pair $\mathbf{x}$, $\mathbf{y}$ of vectors in V in such a way that the following axioms are satisfied.

P1. $\langle \mathbf{x}, \mathbf{x} \rangle > 0$ for all $\mathbf{x} \neq \mathbf{0}$ (**positive definiteness**)
P2. $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$ for all $\mathbf{x}, \mathbf{y} \in V$ (**conjugate symmetry**)
P3. $\langle r\mathbf{x} + s\mathbf{y}, \mathbf{z} \rangle = r\langle \mathbf{x}, \mathbf{z} \rangle + s\langle \mathbf{y}, \mathbf{z} \rangle$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ and $r, s \in \mathbb{R}$ (**linearity in the first coordinate**)

A function satisfying P2–P3 is usually called **sesquilinear**.

A complex vector space V with a Hermitian inner product $\langle , \rangle$ will be called a **Hermitian inner product space**.

DEFINITION 10.9 The classic example of a Hermitian inner product is the standard one on $\mathbb{C}^n$. Given $\mathbf{z} = (z_1, z_2, \dots, z_n)$ and $\mathbf{w} = (w_1, w_2, \dots, w_n)$ in $\mathbb{C}^n$, define their **standard inner product** $\langle \mathbf{z}, \mathbf{w} \rangle$ by

$$\langle \mathbf{z}, \mathbf{w} \rangle = z_1\bar{w}_1 + z_2\bar{w}_2 + \cdots + z_1\bar{w}_n.$$

Clearly, if $\mathbf{z}$ and $\mathbf{w}$ actually lie in $\mathbb{R}^n$, then $\langle \mathbf{z}, \mathbf{w} \rangle = \mathbf{z} \cdot \mathbf{w}$ is the usual dot product.

LEMMA 10.10 Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{C}^n$, and let r, s denote any complex numbers. Then

$$\langle \mathbf{z}, r\mathbf{x} + s\mathbf{y} \rangle = \bar{r}\langle \mathbf{z}, \mathbf{x} \rangle + \bar{s}\langle \mathbf{z}, \mathbf{y} \rangle.$$

DEFINITION 10.11 If $\langle , \rangle$ is a Hermitian inner product on a space V, the **norm** $\|\mathbf{x}\|$ of a vector in V is defined by

$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}.$$

We define the **distance** between vectors $\mathbf{v}$ and $\mathbf{w}$ in an inner product space V to be

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|.$$

The proof of the following theorem is trivial.

THEOREM 10.12 If $\mathbf{z}$ is any vector in V, then

1. $\|\mathbf{z}\| \geqslant 0$ and $\|\mathbf{z}\| = 0$ if and only if $\mathbf{z} = \mathbf{0}$.
2. $\|\lambda\mathbf{z}\| = |\lambda| \cdot \|\mathbf{z}\|$ for all complex numbers λ.

DEFINITION 10.13 A vector $\mathbf{u}$ in $\mathbb{C}^n$ is called a **unit vector** if $\|\mathbf{u}\| = 1$. If $\mathbf{z} \neq \mathbf{0}$ is any nonzero vector in V, then $\mathbf{u} = \frac{1}{\|\mathbf{z}\|}\mathbf{z}$ is a unit vector.

EXAMPLE 10.14 In $\mathbb{C}^4$, find a unit vector $\mathbf{u}$ that is a positive real multiple of $\mathbf{z} = (1 + 2i, -i, 3 - i, 4)$.

Solution. $\|\mathbf{z}\| = \sqrt{5 + 1 + 10 + 16} = \sqrt{32} = 4\sqrt{2}$, so take $\mathbf{u} = \frac{1}{4\sqrt{2}}\mathbf{z}$.

DEFINITION 10.15 Two vectors $\mathbf{z}$ and $\mathbf{w}$ in $\mathbb{C}^n$ are **orthogonal** if $\langle \mathbf{z}, \mathbf{w} \rangle = 0$.

DEFINITION 10.16 As in the real case, a set of nonzero vectors $\{\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_m\}$ in $\mathbb{C}^n$ is called **orthogonal** if $\langle \mathbf{z}_i, \mathbf{z}_j \rangle = 0$ whenever $i \neq j$ and $\|\mathbf{z}_i\| \neq 0$ for each i. It is **orthonormal** if, in addition, $\|\mathbf{z}_i\| = 1$ for each i.