Lecture 3. Isomorphism and Dual Spaces
September 2022

The Dimension Theorem

The dimension theorem is one of the most useful results in all of linear algebra.

THEOREM 3.1 (Dimension Theorem) Let $f : V \to W$ be any linear map and assume that $\operatorname{Ker} f$ and $\operatorname{Im} f$ are both finite dimensional. Then V is also finite dimensional and

$$\dim V = \dim(\operatorname{Ker} f) + \dim(\operatorname{Im} f)$$

Proof. Every vector in $\operatorname{Im} f = f(V)$ has the form $f(\mathbf{v})$ for some $\mathbf{v}$ in V. Hence let $\{f(\mathbf{b}_1), f(\mathbf{b}_2), \dots, f(\mathbf{b}_k)\}$ be a basis of $\operatorname{Im} f$, where the $\mathbf{b}_i$ lie in V. Let $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_r\}$ be any basis of $\operatorname{Ker} f$. Then $\dim(\operatorname{Im} f) = k$ and $\dim(\operatorname{Ker} f) = r$, so it suffices to show that $B = \{\mathbf{e}_1, \dots, \mathbf{e}_r, \mathbf{b}_1, \dots, \mathbf{b}_k\}$ is a basis of V.

1. B spans V. If $\mathbf{v}$ lies in V, then $f(\mathbf{v})$ lies in $\operatorname{Im} f$, so

$$f(\mathbf{v}) = t_1 f(\mathbf{b}_1) + t_2 f(\mathbf{b}_2) + \cdots + t_k f(\mathbf{b}_k), \quad t_i \in \mathbb{R}.$$

This implies that $\mathbf{v} - t_1\mathbf{b}_1 - t_2\mathbf{b}_2 - \cdots - t_k\mathbf{b}_k$ lies in $\operatorname{Ker} f$ and so is a linear combination of $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_r$. Hence $\mathbf{v}$ is a linear combination of the vectors in B.

2. B is linearly independent. Suppose that t_i and s_j in $\mathbb{R}$ satisfy

$$t_1\mathbf{e}_1 + \cdots + t_r\mathbf{e}_r + s_1\mathbf{b}_1 + \cdots + s_k\mathbf{b}_k = \mathbf{0}. \tag{1}$$

Applying f gives $s_1 f(\mathbf{b}_1) + s_2 f(\mathbf{b}_2) + \cdots + s_k f(\mathbf{b}_k) = \mathbf{0}$ (because $f(\mathbf{e}_i) = \mathbf{0}$ for each i). Hence the independence of $\{f(\mathbf{b}_1), f(\mathbf{b}_2), \dots, f(\mathbf{b}_k)\}$ yields $s_1 = \cdots = s_k = 0$. But then (1) becomes

$$t_1\mathbf{e}_1 + \cdots + t_r\mathbf{e}_r = \mathbf{0}$$

so $t_1 = \cdots = t_r = 0$ by the independence of $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_r\}$. This proves that B is linearly independent. $\square$

Isomorphisms

Often two vector spaces can consist of quite different types of vectors but, on closer examination, turn out to be the same underlying space displayed in different symbols.

DEFINITION 3.2 A linear map $f : V \to W$ is called an **isomorphism** if it is both onto and one-to-one. The vector spaces V and W are said to be **isomorphic** if there exists an isomorphism $f : V \to W$, and we write $V \cong W$ when this is the case.

EXAMPLE 3.3 Isomorphic spaces can "look" quite different. For example, $M_{2\times 2} \cong P_3$ because the map $f : M_{2\times 2} \to P_3$ given by $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + bx + cx^2 + dx^3$ is an isomorphism.

An isomorphism $f : V \to W$ induces a pairing

$$\mathbf{v} \leftrightarrow f(\mathbf{v})$$

between vectors $\mathbf{v}$ in V and vectors $f(\mathbf{v})$ in W that preserves vector addition and scalar multiplication. Hence, as far as their vector space properties are concerned, the spaces V and W are identical except for notation.

The following theorem gives a very useful characterization of isomorphisms: They are the linear maps that preserve bases.

THEOREM 3.4 If V and W are finite dimensional spaces, the following conditions are equivalent for a linear map $f : V \to W$.

1. f is an isomorphism.
2. If $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is any basis of V, then $\{f(\mathbf{e}_1), f(\mathbf{e}_2), \ldots, f(\mathbf{e}_n)\}$ is a basis of W.
3. There exists a basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ of V such that $\{f(\mathbf{e}_1), f(\mathbf{e}_2), \ldots, f(\mathbf{e}_n)\}$ is a basis of W.

Proof. (1) $\Rightarrow$ (2). Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be a basis of V. If

$$t_1 f(\mathbf{e}_1) + \cdots + t_n f(\mathbf{e}_n) = \mathbf{0}$$

with t_i in $\mathbb{R}$, then $f(t_1\mathbf{e}_1 + \cdots + t_n\mathbf{e}_n) = \mathbf{0}$, so $t_1\mathbf{e}_1 + \cdots + t_n\mathbf{e}_n = \mathbf{0}$ (because $\operatorname{Ker} f = \{\mathbf{0}\}$). But then each $t_i = 0$ by the independence of the $\mathbf{e}_i$, so $\{f(\mathbf{e}_1), \ldots, f(\mathbf{e}_n)\}$ is independent. To show that it spans W, choose $\mathbf{w}$ in W. Because f is onto, $\mathbf{w} = f(\mathbf{v})$ for some $\mathbf{v}$ in V, so write $\mathbf{v} = t_1\mathbf{e}_1 + \cdots + t_n\mathbf{e}_n$. Then $\mathbf{w} = f(\mathbf{v}) = t_1 f(\mathbf{e}_1) + \cdots + t_n f(\mathbf{e}_n)$, proving that $\{f(\mathbf{e}_1), \ldots, f(\mathbf{e}_n)\}$ spans W.

(2) $\Rightarrow$ (3). This is because V has a basis.

(3) $\Rightarrow$ (1). If $f(\mathbf{v}) = \mathbf{0}$, write $\mathbf{v} = v_1\mathbf{e}_1 + \cdots + v_n\mathbf{e}_n$ where each v_i is in $\mathbb{R}$. Then

$$\mathbf{0} = f(\mathbf{v}) = v_1 f(\mathbf{e}_1) + \cdots + v_n f(\mathbf{e}_n),$$

so $v_1 = \cdots = v_n = 0$ by (3). Hence $\mathbf{v} = \mathbf{0}$, so $\ker f = \{\mathbf{0}\}$ and f is one-to-one. To show that f is onto, let $\mathbf{w}$ be any vector in W. By (3) there exist $\mathbf{w}_1, \ldots, \mathbf{w}_n$ in $\mathbb{R}$ such that

$$\mathbf{w} = w_1 f(\mathbf{e}_1) + \cdots + w_n f(\mathbf{e}_n) = f(w_1\mathbf{e}_1 + \cdots + w_n\mathbf{e}_n).$$

Thus f is onto. $\square$

The following theorem shows that two vector spaces V and W have the same dimension if and only if they are isomorphic.

THEOREM 3.5 (Isomorphism Theorem) If V and W are finite dimensional vector spaces, then $V \cong W$ if and only if $\dim V = \dim W$.

Proof. $\Rightarrow$ If $V \cong W$, then there exists an isomorphism $f : V \to W$. Since V is finite dimensional, let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be a basis of V. Then $\{f(\mathbf{e}_1), \ldots, f(\mathbf{e}_n)\}$ is a basis of W by Theorem 3.4, so $\dim W = n = \dim V$.

$\Leftarrow$ Let V and W be vector spaces of dimension n, and suppose that $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ and $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ are bases of V and W, respectively. Theorem 1.9 asserts that there exists a linear map $f : V \to W$ such that $f(\mathbf{e}_i) = \mathbf{b}_i$ for each $i = 1, 2, \ldots, n$. Then

$\{f(\mathbf{e}_1), \dots, f(\mathbf{e}_n)\}$ is evidently a basis of W, so f is an isomorphism by Theorem 3.4. Furthermore, the action of f is prescribed by

$$f(r_1\mathbf{e}_1 + \cdots + r_n\mathbf{e}_n) = r_1\mathbf{b}_1 + \cdots + r_n\mathbf{b}_n,$$

so isomorphisms between spaces of equal dimension can be easily defined as soon as bases are known. □

COROLLARY 3.6 If V is a vector space and $\dim V = n$, then V is isomorphic to $\mathbb{R}^n$.

If V is a vector space of dimension n, note that there are important explicit isomorphisms $V \to \mathbb{R}^n$. Fix a basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ of V and write $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ for the standard basis of $\mathbb{R}^n$. Since there is a unique linear map $f : V \to \mathbb{R}^n$ given by

$$f(x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n) = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

where each x_i is in $\mathbb{R}$. Moreover, $f(\mathbf{v}_i) = \mathbf{e}_i$ for each i so f is an isomorphism by Theorem 3.4, called the **coordinate isomorphism** corresponding to the basis B.

Inverse Linear Maps

THEOREM 3.7 Let V and W be finite-dimensional vector spaces. The following conditions are equivalent for a linear map $f : V \to W$.

1. f is an isomorphism.
2. There exists a linear map $g : W \to V$ such that $gf = 1_V$ and $fg = 1_W$.

Moreover, in this case g is also an isomorphism and is uniquely determined by f: If $\mathbf{w} \in W$ is written as $\mathbf{w} = f(\mathbf{v})$, then $g(\mathbf{w}) = \mathbf{v}$.

Proof. (1) $\Rightarrow$ (2). If $B = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is a basis of V, then $D = \{f(\mathbf{e}_1), \dots, f(\mathbf{e}_n)\}$ is a basis of W by Theorem 2.16. Hence, define a linear map $g : W \to V$ by

$$g[f(\mathbf{e}_i)] = \mathbf{e}_i \qquad \text{for each } i. \tag{2}$$

Since $\mathbf{e}_i = 1_V(\mathbf{e}_i)$, this gives $gf = 1_V$ by Corollary 1.7. But applying f gives $f[g[f(\mathbf{e}_i)]] = f(\mathbf{e}_i)$ for each i, so $fg = 1_W$ (again by Corollary 1.7).

(2) $\Rightarrow$ (1). If $f(\mathbf{v}) = f(\mathbf{v}_1)$, then $g[f(\mathbf{v})] = g[f(\mathbf{v}_1)]$. Because $gf = 1_V$ by (2), this reads $\mathbf{v} = \mathbf{v}_1$; that is, f is one-to-one. Given $\mathbf{w}$ in W, the fact that $fg = 1_W$ means that $\mathbf{w} = f[g(\mathbf{w})]$, so f is onto.

Finally, g is uniquely determined by the condition $gf = 1_V$ because this condition implies (2) and g is an isomorphism because it carries the basis D to B. As to the last assertion, given $\mathbf{w}$ in W, write $\mathbf{w} = r_1f(\mathbf{e}_1) + \cdots + r_nf(\mathbf{e}_n)$. Then $\mathbf{w} = f(\mathbf{v})$, where $\mathbf{v} = r_1\mathbf{e}_1 + \cdots + r_n\mathbf{e}_n$. Then $g(\mathbf{w}) = \mathbf{v}$ by (2). □

DEFINITION 3.8 Given an isomorphism $f : V \to W$, the unique isomorphism $g : W \to V$ satisfying $gf = 1_V$ and $fg = 1_W$ is called the **inverse** of f and is denoted by f^{-1}.

Hence $f : V \to W$ and $f^{-1} : W \to V$ are related by the **fundamental identities**:

$$f^{-1}[f(\mathbf{v})] = \mathbf{v} \text{ for all } \mathbf{v} \in V \qquad \text{and} \qquad f[f^{-1}(\mathbf{w})] = \mathbf{w} \text{ for all } \mathbf{w} \in W.$$

In other words, each of f and f^{-1} reverses the action of the other. In particular, equation (2) in the proof of Theorem 3.1 shows how to define f^{-1} using the image of a basis under the isomorphism f.

THEOREM 3.9 Let $f : V \to W$ be an invertible linear map (isomorphism) represented by a matrix A relative to bases E, E' of V, W, respectively. Then the matrix corresponding to the inverse of f is A^{-1}.

Proof. Since $f^{-1}f = 1_V$, by Theorem 1.16 we have $A_{f^{-1}}A_f = I$. Thus $A_{f^{-1}} = A_f^{-1}$. □

Vector Space of Linear Maps

DEFINITION 3.10 If V and W are vector spaces, the set of all linear maps from V to W will be denoted by

$$L(V,W) = \{f \mid f : V \to W \text{ is a linear map}\}.$$

Given f and g in $L(V,W)$ and $k \in \mathbb{R}$, define $f + g : V \to W$ and $k \cdot f : V \to W$ by

1. $(f + g)(\mathbf{v}) = f(\mathbf{v}) + g(\mathbf{v})$ for all $\mathbf{v} \in V$;
2. $(k \cdot f)(\mathbf{v}) = k \cdot f(\mathbf{v})$ for all $\mathbf{v} \in V$.

LEMMA 3.11 $L(V,W)$ is a vector space.

Proof. The proof that kf and $f + g$ are linear and that all axioms hold are routine verifications. The zero vector in $L(V,W)$ is the zero map, the negative of f is $(-1)f$. □

DEFINITION 3.12 If V is a vector space, the space $V^* = L(V,\mathbb{R})$ of all **linear functionals** on V is called the **dual vector space** (or just **dual space** for short) of V. Elements of the algebraic dual space V^* are sometimes called **covectors** or **one-forms**.

LEMMA 3.13 If $\dim V = n$, $\dim W = m$, then $\dim L(V,W) = mn$.

Proof. Since any linear map is uniquely defined by its $m \times n$ matrix with respect to some fixed bases in V and W, the vector space $L(V,W)$ is isomorphic to the space of $m \times n$ matrices, which is mn-dimensional. □

DEFINITION 3.14 Let V be a finite-dimensional vector space. Given a basis $B = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ of V, let $\mathbf{e}^i : V \to \mathbb{R}$[1] for each $i = 1, 2, \ldots, n$ be the linear map that assigns to each vector $\mathbf{v}$ its i-th coordinate:

$$\mathbf{e}^i(\mathbf{v}) = \mathbf{e}^i(v_1\mathbf{e}_1 + \cdots + v_n\mathbf{e}_n) = v_i, \qquad i = 1, 2, \ldots, n.$$

It is clear that maps $\mathbf{e}^i$ are linear and satisfy the property

$$\mathbf{e}^i(\mathbf{e}_j) = \delta^i_j = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

[1]The superscript here is the index, not an exponent.

Symbol δ^i_j is called the **Kronecker delta** symbol. This property is referred to as **bi-orthogonality property**.

THEOREM 3.15 The following statements about the dual space hold.

(a) $\mathbf{v} = \mathbf{e}^1(\mathbf{v})\mathbf{e}_1 + \mathbf{e}^2(\mathbf{v})\mathbf{e}_2 + \cdots + \mathbf{e}^n(\mathbf{v})\mathbf{e}_n$ for all $\mathbf{v} \in V$.
(b) $f = f(\mathbf{e}_1)\mathbf{e}^1 + f(\mathbf{e}_2)\mathbf{e}^2 + \cdots + f(\mathbf{e}_n)\mathbf{e}^n$ for all $f \in V^*$.
(c) $\{\mathbf{e}^1, \mathbf{e}^2, \ldots, \mathbf{e}^n\}$ is a basis of V^* (called the **dual basis** to B).

Proof. (a) Write $\mathbf{v} = v_1\mathbf{e}_1 + \cdots + v_n\mathbf{e}_n$. By definition, $v_i = \mathbf{e}^i(\mathbf{v})$ and the statement follows.

(b) Given $f : V \to \mathbb{R}$ and $\mathbf{v} \in V$, using (a) and linearity of f, we have:

$$f(\mathbf{v}) = f[\mathbf{e}^1(\mathbf{v})\mathbf{e}_1 + \cdots + \mathbf{e}^n(\mathbf{v})\mathbf{e}_n] = \mathbf{e}^1(\mathbf{v})f(\mathbf{e}_1) + \cdots + \mathbf{e}^n(\mathbf{v})f(\mathbf{e}_n).$$

(c) It spans V^* by (b). If $r_1\mathbf{e}^1 + \cdots + r_n\mathbf{e}^n = \mathbf{0}$, where $r_i \in \mathbb{R}$, then apply this to $\mathbf{e}_j$:

$$0 = \mathbf{0}(\mathbf{e}_j) = (r_1\mathbf{e}^1 + \cdots + r_n\mathbf{e}^n)(\mathbf{e}_j) = r_1\mathbf{e}^1(\mathbf{e}_j) + \cdots + r_n\mathbf{e}^n(\mathbf{e}_j) = r_j.$$

Hence $\{\mathbf{e}^1, \mathbf{e}^2, \ldots, \mathbf{e}^n\}$ is linearly independent. $\square$

COROLLARY 3.16 Vector spaces V and V^* are isomorphic.