Lecture 16. Sum of Subspaces
June 2023

Dimension Theorem for Matrices

The following theorem interprets nullity in the context of a homogeneous linear system and establishes a fundamental relationship between rank and nullity of a matrix.

THEOREM 16.1 (Dimension Theorem for Matrices) If A is a matrix with n columns, then

$$\operatorname{rank}(A) + \dim \operatorname{null}(A) = n. \tag{1}$$

Proof. Since A has n columns, the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ has n unknowns (variables). These fall into two distinct categories: the leading variables and the free variables. Thus,

$$(\text{number of leading variables}) \quad + \quad (\text{number of free variables}) = n$$

But the number of leading variables is the same as the number of leading 1's in any row echelon form of A, which is the same as the rank of A.

To complete the proof, we will show that the number of free variables in the general solution of $A\mathbf{x} = \mathbf{0}$ is the same as the nullity of A, i.e., the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$.

Let $\operatorname{rank} A = r$. Suppose the matrix A is put into a reduced row-echelon form. On relabelling variables, it can be represented in the following form:

$$\left(\begin{array}{c|c} I_r & C \\ \hline 0 & 0 \end{array}\right) = \left(\begin{array}{cccc|cccc} 1 & 0 & \dots & 0 & c_{1,r+1} & c_{1,r+2} & \dots & c_{1,n} \\ 0 & 1 & \dots & 0 & c_{2,r+1} & c_{2,r+2} & \dots & c_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & c_{r,r+1} & c_{r,r+2} & \dots & c_{r,n} \\ \hline 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{array}\right)$$

This means that the linear system $A\mathbf{x} = \mathbf{0}$ is equivalent to the following:

$$\begin{cases} x_1 \qquad\qquad + c_{1,r+1}x_{r+1} + c_{1,r+2}x_{r+2} + \dots + c_{1,n}x_n = 0, \\ \quad x_2 \qquad + c_{2,r+1}x_{r+1} + c_{2,r+2}x_{r+2} + \dots + c_{2,n}x_n = 0, \\ \qquad\qquad \dots \\ \qquad\quad x_r + c_{r,r+1}x_{r+1} + c_{r,r+2}x_{r+2} + \dots + c_{r,n}x_n = 0 \end{cases}$$

Here $x_1, \dots, x_r$ represent leading variables, whereas $x_{r+1}, \dots, x_n$ represent free variables, or parameters. This system can be solved for the leading variables in terms of the parameters as follows:

$$\begin{cases} x_1 = -c_{1,r+1}x_{r+1} - c_{1,r+2}x_{r+2} - \dots - c_{1,n}x_n, \\ x_2 = -c_{2,r+1}x_{r+1} - c_{2,r+2}x_{r+2} - \dots - c_{2,n}x_n, \\ \qquad \dots \\ x_r = -c_{r,r+1}x_{r+1} - c_{r,r+2}x_{r+2} - \dots - c_{r,n}x_n \end{cases}$$

or in the vector form as

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_r \\ x_{r+1} \\ \vdots \\ x_n \end{pmatrix} = x_{r+1} \begin{pmatrix} -c_{1,r+1} \\ \vdots \\ -c_{r,r+1} \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + x_n \begin{pmatrix} -c_{1,n} \\ \vdots \\ -c_{r,n} \\ 0 \\ \vdots \\ 1 \end{pmatrix} \tag{2}$$

We assert that the vectors

$$\mathbf{y}_{r+1} = \begin{pmatrix} -c_{1,r+1} \\ \vdots \\ -c_{r,r+1} \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \qquad , \ldots , \qquad \mathbf{y}_n = \begin{pmatrix} -c_{1,n} \\ \vdots \\ -c_{r,n} \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

form the basis of $\operatorname{null} A$. Indeed, from (2) it follows that the vectors $\mathbf{y}_{r+1}, \ldots, \mathbf{y}_n$ span $\operatorname{null} A$. We need to show that these vectors are linearly independent. Consider the vector equation

$$k_{r+1}\mathbf{y}_{r+1} + \cdots + k_n\mathbf{y}_n = \mathbf{0}.$$

Equating coordinates from $i = r+1$ to n implies that $k_{r+1} = \cdots = k_n = 0$.

Therefore, the vectors $\mathbf{y}_{r+1}, \ldots, \mathbf{y}_n$ form a basis for $\operatorname{null} A$, and so $\dim \operatorname{null} A = n - r$. Theorem follows. □

THEOREM 16.2 (Fredholm Alternative Theorem) A system of linear equations $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}^T\mathbf{y} = \mathbf{0}$ for every column vector $\mathbf{y}$ such that $A^T\mathbf{y} = \mathbf{0}$.

Proof. 1) Suppose the system $A\mathbf{x} = \mathbf{b}$ is consistent. This implies, by Consistency Theorem, that $\mathbf{b}$ is in the column space of A, that is, $\mathbf{b} = k_1\mathbf{c}_1 + \cdots + k_n\mathbf{c}_n$. If $A^T\mathbf{y} = \mathbf{0}$, then $\mathbf{c}_i^T\mathbf{y} = \mathbf{0}$ for every $i = 1, \ldots, n$. Therefore,

$$\mathbf{b}^T\mathbf{y} = (k_1\mathbf{c}_1 + \cdots + k_n\mathbf{c}_n)^T\mathbf{y} = k_1\mathbf{c}_1^T\mathbf{y} + \cdots + k_n\mathbf{c}_n^T\mathbf{y} = \mathbf{0}.$$

2) Conversely, suppose $\mathbf{b}^T\mathbf{y} = \mathbf{0}$ for every column vector $\mathbf{y}$ such that $A^T\mathbf{y} = \mathbf{0}$. This means that two linear systems

$$A^T\mathbf{y} = \mathbf{0} \qquad \text{and} \qquad (A|\mathbf{b})^T\mathbf{y} = \mathbf{0}$$

have the same null space N. If $\mathbf{y} \in \mathbb{R}^m$ and $\dim N = k$, then by Dimension Theorem, $\operatorname{rank}(A^T) + k = m$. At the same time, $\operatorname{rank}(A|\mathbf{b})^T + k = m$. This implies that $\operatorname{rank}(A^T) = \operatorname{rank}(A|\mathbf{b})^T$ and, by Consistency Theorem, the system $A\mathbf{x} = \mathbf{b}$ is consistent. □

Sum and Intersection of Subspaces

If U and W are subspaces of a vector space V, there are three related subsets that are of interest, their **sum** $U+W$, **union** $U \cup W$, and **intersection** $U \cap W$, defined by

$$U + W = \{\mathbf{u} + \mathbf{w} \mid \mathbf{u} \in U, \text{ and } \mathbf{w} \in W\}$$
$$U \cup W = \{\mathbf{v} \in V \mid \mathbf{v} \in U \text{ or } \mathbf{v} \in W\}$$
$$U \cap W = \{\mathbf{v} \in V \mid \mathbf{v} \text{ in both } U \text{ and } W\}$$

EXAMPLE 16.3 The set $U \cup W$ is not generally a subspace of V, since if, say, $\mathbf{u} \in U$, $\mathbf{u} \notin W$ and $\mathbf{w} \in W$, $\mathbf{w} \notin U$, then $\mathbf{u} + \mathbf{w} \notin U \cup W$; in which case it is not closed under addition. The subset $U + W$ contains both U and W, but is generally much larger.

THEOREM 16.4

1. The sets $U + W$ and $U \cap W$ are subspaces of V.
2. The set $U \cap W$ is contained in both U and W.
3. The set $U + W$ is the "smallest" subspace of V that contains both U and W.

Proof. The sets $U + W$ and $U \cap W$ are subspaces of V because they are closed under addition and scalar multiplication.

Prove that the set $U + W$ is the "smallest" subspace of V that contains both U and W. By this, we mean that if S is a subspace of V and we have both $U \subset S$ and $W \subset S$, then $U + W \subset S$. To see this, we can simply note that for any $\mathbf{u} \in U$ and any $\mathbf{w} \in W$, we will have $\mathbf{u} \in S$ and $\mathbf{w} \in S$ and so, because S is a subspace, $\mathbf{u} + \mathbf{w} \in S$. This shows that any vector of the form $\mathbf{u} + \mathbf{w}$ is in S, which means that $U + W \subset S$. $\square$

We will now prove a useful fact about the dimensions of these spaces. The proof is a good illustration of how the theorems in this lecture are used.

THEOREM 16.5 (Grassman Formula for Vector Space Dimensions) Suppose that U and W are finite dimensional subspaces of a vector space V. Then $U + W$ is finite dimensional and

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W).$$

Proof. Since $U \cap W \subset U$, it has a finite basis, say $\{\mathbf{v}_1, \ldots, \mathbf{v}_d\}$. Extend it to a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_d, \mathbf{u}_1, \ldots, \mathbf{u}_m\}$ of U. Similarly extend $\{\mathbf{v}_1, \ldots, \mathbf{v}_d\}$ to a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_d, \mathbf{w}_1, \ldots, \mathbf{w}_p\}$ of W. Then

$$U + W = \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_d, \mathbf{u}_1, \ldots, \mathbf{u}_m, \mathbf{w}_1, \ldots, \mathbf{w}_p\},$$

so $U+W$ is finite dimensional. For the rest, it suffices to show that $\{\mathbf{v}_1, \ldots, \mathbf{v}_d, \mathbf{u}_1, \ldots, \mathbf{u}_m, \mathbf{w}_1, \ldots, \mathbf{w}_p\}$ is independent. Suppose that

$$r_1\mathbf{v}_1 + \cdots + r_d\mathbf{v}_d + s_1\mathbf{u}_1 + \cdots + s_m\mathbf{u}_m + t_1\mathbf{w}_1 + \cdots + t_p\mathbf{w}_p = 0 \tag{3}$$

where the r_i, s_j, and t_k are scalars. Then

$$r_1\mathbf{v}_1 + \cdots + r_d\mathbf{v}_d + s_1\mathbf{u}_1 + \cdots + s_m\mathbf{u}_m = -(t_1\mathbf{w}_1 + \cdots + t_p\mathbf{w}_p)$$

is in U (left side) and also in W (right side), and so is in $U \cap W$. Hence $(t_1\mathbf{w}_1 + \cdots + t_p\mathbf{w}_p)$ is a linear combination of $\{\mathbf{v}_1, \ldots, \mathbf{v}_d\}$, so $t_1 = \cdots = t_p = 0$, because $\{\mathbf{v}_1, \ldots, \mathbf{v}_d, \mathbf{w}_1, \ldots, \mathbf{w}_p\}$ is independent. Similarly, $s_1 = \cdots = s_m = 0$, so (3) becomes $r_1\mathbf{v}_1 + \cdots + r_d\mathbf{v}_d = 0$. It follows that $r_1 = \cdots = r_d = 0$, as required. □

Direct Sum of Subspaces

A sum of two subspaces is sometimes a direct sum.

DEFINITION 16.6 A vector space V is said to be the **direct sum** of subspaces U and W if

$$U \cap W = \{\mathbf{0}\} \qquad \text{and} \qquad U + W = V.$$

In this case we write $V = U \oplus W$. Given a subspace U, any subspace W such that $V = U \oplus W$ is called a **complement** of U in V.

EXAMPLE 16.7 Let $U = \{(x, y, 0) \in \mathbb{R}^3 \mid x, y \in \mathbb{R}\}$, $W = \{(0, 0, z) \in \mathbb{R}^3 \mid z \in \mathbb{R}\}$. Then $\mathbb{R}^3 = U \oplus W$.

However, if instead $W = \{(0, z, w) \in \mathbb{R}^3 \mid z, w \in \mathbb{R}\}$, then $\mathbb{R}^3 = U + W$ but is not the direct sum of U and W.

EXAMPLE 16.8 Let

$$U = \{p \in P_{2m+1} \,|\, p(x) = a_0 + a_2x^2 + \cdots + a_{2m}x^{2m}\},$$
$$W = \{p \in P_{2m+1} \,|\, p(x) = a_1x + a_3x^3 + \cdots + a_{2m+1}x^{2m+1}\}.$$

Then $P_{2m+1} = U \oplus W$.

THEOREM 16.9 Let $U, W \subset V$ be subspaces and $V = U + W$. Then $V = U \oplus W$ if either of the conditions hold.

1. $U \cap W = \{\mathbf{0}\}$.
2. If $\mathbf{0} = \mathbf{u} + \mathbf{w}$ with $\mathbf{u} \in U$ and $\mathbf{w} \in W$, then $\mathbf{u} = \mathbf{w} = \mathbf{0}$.
3. Every $\mathbf{v} \in V$ can be uniquely written as $\mathbf{v} = \mathbf{u} + \mathbf{w}$ for $\mathbf{u} \in U$ and $\mathbf{w} \in W$.
4. $\dim(U + W) = \dim U + \dim W$.
5. If B_1 is a basis of U and B_2 is a basis of W, then $B_1 \cup B_2$ is a basis of V.

Proof. $1 \Rightarrow 2$. Suppose $U \cap W = \{\mathbf{0}\}$. Assume that $\mathbf{0} = \mathbf{u} + \mathbf{w}$, where $\mathbf{u} \in U$ and $\mathbf{w} \in W$. Then $\mathbf{u} = -\mathbf{w}$ is in both U and W. Hence $\mathbf{u} = \mathbf{w} = \mathbf{0}$.

$2 \Rightarrow 3$. Suppose condition 2 holds. Since $V = U + W$ we have that, for all $\mathbf{v} \in V$, there exist $\mathbf{u} \in U$ and $\mathbf{w} \in W$ such that $\mathbf{v} = \mathbf{u} + \mathbf{w}$. Suppose $\mathbf{v} = \mathbf{u}_1 + \mathbf{w}_1$ with $\mathbf{u}_1 \in U$ and $\mathbf{w}_1 \in W$. Subtracting the two equations, we obtain

$$\mathbf{0} = (\mathbf{u} - \mathbf{u}_1) + (\mathbf{w} - \mathbf{w}_1),$$

where $\mathbf{u} - \mathbf{u}_1 \in U$ and $\mathbf{w} - \mathbf{w}_1 \in W$. By condition 2, this implies $\mathbf{u} - \mathbf{u}_1 = \mathbf{0}$ and $\mathbf{w} - \mathbf{w}_1 = \mathbf{0}$, or equivalently $\mathbf{u} = \mathbf{u}_1$ and $\mathbf{w} = \mathbf{w}_1$, as desired.

$3 \Rightarrow 1$. Suppose condition 3 holds. If $\mathbf{u} \in U \cap W$, then $\mathbf{0} = \mathbf{u} + (-\mathbf{u})$ with $\mathbf{u} \in U$ and $-\mathbf{u} \in W$. Since $\mathbf{0}$ can be uniquely written as $\mathbf{0} = \mathbf{0} + \mathbf{0}$, we have $\mathbf{u} = \mathbf{0}$ and $-\mathbf{u} = \mathbf{0}$ so that $U \cap W = \{\mathbf{0}\}$.

$1 \Rightarrow 5.$ follows from Grassman formula. Actually, if $U \cap W = \{\mathbf{0}\}$, then there are no vectors $\mathbf{v}_i$ in the proof of Grassman formula, and the argument shows that if $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ and $\{\mathbf{w}_1, \ldots, \mathbf{w}_p\}$ are bases of U and W respectively, then $\{\mathbf{u}_1, \ldots, \mathbf{u}_m, \mathbf{w}_1, \ldots, \mathbf{w}_p\}$ is a basis of $U + W$.

$5 \Rightarrow 4.$ If $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ and $\{\mathbf{w}_1, \ldots, \mathbf{w}_p\}$ are bases of U and W respectively and $\{\mathbf{u}_1, \ldots, \mathbf{u}_m, \mathbf{w}_1, \ldots, \mathbf{w}_p\}$ is a basis of $U + W$, then $\dim(U + W) = \dim U + \dim W$.

$4 \Rightarrow 1.$ If $\dim(U+W) = \dim U + \dim W$, then, by Grassman formula, $\dim(U \cap W) = 0$, which means that $U \cap W = \{\mathbf{0}\}$. □

The definition of the direct sum of two subspaces can be generalized to m subspaces $U_1, U_2, \ldots, U_m$.

EXAMPLE 16.10 Let

$$U_1 = \{(x, y, 0) \in \mathbb{R}^3\}, \qquad U_2 = \{(0, 0, z) \in \mathbb{R}^3\}, \qquad U_3 = \{(0, y, y) \in \mathbb{R}^3\}.$$

Then certainly $\mathbb{R}^3 = U_1 + U_2 + U_3$, but $\mathbb{R}^3 \neq U_1 \oplus U_2 \oplus U_3$ since, for example,

$$(0, 0, 0) = (0, 1, 0) + (0, 0, 1) + (0, -1, -1).$$

But $U_1 \cap U_2 = U_1 \cap U_3 = U_2 \cap U_3 = \{\mathbf{0}\}$.

THEOREM 16.11 Let $U_1, \ldots, U_m$ be subspaces of V. Assume $V = U_1 + \cdots + U_m$; that is, every $\mathbf{v}$ in V can be written (in at least one way) in the form

$$\mathbf{v} = \mathbf{u}_1 + \cdots + \mathbf{u}_m, \qquad \mathbf{u}_i \in U_i.$$

Then the following conditions are equivalent.

1. $U_i \cap (U_1 + \cdots + U_{i-1} + U_{i+1} + \cdots + U_m) = \{\mathbf{0}\}$ for each $i = 1, 2, \ldots, m$.
2. If $\mathbf{u}_1 + \cdots + \mathbf{u}_m = \mathbf{0}$, where $\mathbf{u}_i$ in U_i, then $\mathbf{u}_i = \mathbf{0}$ for each i.
3. If $\mathbf{u}_1 + \cdots + \mathbf{u}_m = \mathbf{u}'_1 + \cdots + \mathbf{u}'_m$, $\mathbf{u}_i$ and $\mathbf{u}'_i$ in U_i, then $\mathbf{u}_i = \mathbf{u}'_i$ for each i.
4. $\dim(U_1 + U_2 + \cdots + U_m) = \dim U_1 + \dim U_2 + \cdots + \dim U_m$.
5. If B_1, B_2, $\ldots$, B_m are bases of U_1, U_2, $\ldots$, U_m, respectively, then $B_1 \cup B_2, \cup \cdots \cup B_m$ is a basis of V.

We will omit the proof of this theorem.

DEFINITION 16.12 Let $U_1, \ldots, U_m$ be subspaces of V and assume that

$$V = U_1 + \cdots + U_m.$$

Then V is said to be a **direct sum** of subspaces $U_1, \ldots, U_m$, if for every $\mathbf{v} \in V$ there exist unique vectors $\mathbf{u}_i \in U_i$ for $1 \leqslant i \leqslant m$ such that

$$\mathbf{v} = \mathbf{u}_1 + \cdots + \mathbf{u}_m.$$

We write $V = U_1 \oplus U_2 \oplus \cdots \oplus U_m$.

EXAMPLE 16.13 Find the projection of $\mathbf{x}$ onto $V_1 = \operatorname{span}\{\mathbf{a}_1, \mathbf{a}_2\}$ along $V_2 = \operatorname{span}\{\mathbf{b}\}$ if

$$\mathbf{x} = \begin{pmatrix} -6 \\ 5 \\ 0 \end{pmatrix}, \quad \mathbf{a}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 4 \\ 1 \\ 7 \end{pmatrix}.$$

Solution. The sum $V_1 \oplus V_2$ is direct, since the vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}$ are linearly independent. Solving vector equation

$$x_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} + x_3 \begin{pmatrix} 4 \\ 1 \\ 7 \end{pmatrix} = \begin{pmatrix} -6 \\ 5 \\ 0 \end{pmatrix}$$

yields $x_1 = 1$, $x_2 = 3$, $x_3 = -1$. Thus, $\mathbf{x} = \mathbf{a}_1 + 3\mathbf{a}_2 - \mathbf{b}$. The projection of $\mathbf{x}$ onto L_1 is V_1-component of this sum, so the answer is $\mathbf{a}_1 + 3\mathbf{a}_2 = (-2, 6, 7)$. □

EXAMPLE 16.14 Find the dimension and a basis of the sum and the intersection of $V_1 = \operatorname{span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $V_2 = \operatorname{span}\{\mathbf{b}_1, \mathbf{b}_2\}$ if

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 3 \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 3 \end{pmatrix}, \quad \mathbf{a}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{b}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{b}_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \end{pmatrix}.$$

Solution. Form the matrix by $\mathbf{a}_i$ and $\mathbf{b}_j$ as columns and reduce it by elementary row operations taking leading 1's from LHS:

$$\left(\begin{array}{ccc|cc} 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & 1 & 1 \\ 2 & 1 & -1 & 1 & 2 \\ 3 & 3 & 1 & 1 & 3 \end{array}\right) \to \left(\begin{array}{ccc|cc} 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 3 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

As it is impossible to take leading 1's of the left, we choose the pivot on the right in position 44 and continue the process. Eventually, we get the matrix

$$\left(\begin{array}{ccc|cc} 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

We see that $\dim V_1 = 3$, $\dim V_2 = 2$, $\dim(V_1 + V_2) = 4$, $\dim(V_1 \cap V_2) = 1$. If we denote the column vectors of the resulting matrix by $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$, $\mathbf{w}_4$, $\mathbf{w}_5$, we see that the vector $\mathbf{w}_5 - \mathbf{w}_4$ is in both $\operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ and $\operatorname{span}\{\mathbf{w}_4, \mathbf{w}_5\}$, so that $V_1 \cap V_2 = \operatorname{span}\{\mathbf{b}_2 - \mathbf{b}_1\} = \operatorname{span}\{(0, 0, 1, 2)^T\}$. □

Exercises

1. Find the projection of $\mathbf{x} = \begin{pmatrix} 1 \\ -7 \\ 5 \\ -2 \end{pmatrix}$ onto $\operatorname{span}\{\mathbf{a}_1, \mathbf{a}_2\}$ along $\operatorname{span}\{\mathbf{b}_1, \mathbf{b}_2\}$ if

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} 3 \\ -5 \\ 7 \\ 2 \end{pmatrix}, \quad \mathbf{b}_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix}, \quad \mathbf{b}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 3 \end{pmatrix}.$$

Answer. $\mathbf{a}_2 - \mathbf{a}_1 = (2, -6, 6, 1)^T$.

2. Find the dimension and a basis of the sum and the intersection of $V_1 = \operatorname{span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $V_2 = \operatorname{span}\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ if

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix}, \mathbf{a}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 3 \end{pmatrix}, \mathbf{a}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 3 \end{pmatrix}, \mathbf{b}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{b}_2 = \begin{pmatrix} 3 \\ 3 \\ 3 \\ 3 \end{pmatrix}, \mathbf{b}_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix}.$$

Answer. $\dim(V_1 + V_2) = 4$, $B_{V_1+V_2} = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}_1, \mathbf{b}_3\}$, $\dim(V_1 \cap V_2) = 0$.

3. Find the dimension and a basis of the sum and the intersection of $V_1 = \operatorname{span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$ and $V_2 = \operatorname{span}\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4\}$ if

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{a}_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 2 \end{pmatrix}, \quad \mathbf{a}_4 = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \end{pmatrix},$$

$$\mathbf{b}_1 = \begin{pmatrix} 4 \\ 6 \\ 1 \\ 4 \end{pmatrix}, \quad \mathbf{b}_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \quad \mathbf{b}_3 = \begin{pmatrix} 1 \\ 1 \\ -2 \\ 3 \end{pmatrix}, \quad \mathbf{b}_4 = \begin{pmatrix} 2 \\ 4 \\ 1 \\ 1 \end{pmatrix}.$$

Answer. $\dim(V_1 + V_2) = 4$, $B_{V_1+V_2} = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{b}_2\}$, $\dim(V_1 \cap V_2) = 2$, $B_{V_1 \cap V_2} = \{\mathbf{b}_2 + \mathbf{b}_3, \mathbf{b}_1 - \mathbf{b}_2\}$.