№2(3)

$$\lim_{\alpha \to 1} \int_2^4 \frac{x\,dx}{1+x^2+\alpha^6} =$$

Т.к подынтегральная фун-ция непр по x и α, то

$$= \int_2^4 \lim_{\alpha \to 1} \frac{x}{1+x^2+\alpha^6}\,dx = \int_2^4 \frac{x\,dx}{2+x^2} =$$

$$= \frac{1}{2} \int_2^4 \frac{x\,dx}{1+\frac{x^2}{2}} = \frac{1}{2} \int_2^4 \frac{d\left(\frac{x^2}{2}\right)}{1+\frac{x^2}{2}} =$$

$$= \frac{1}{2} \ln\left(1+\frac{x^2}{2}\right)\Big|_2^4 = \frac{1}{2}\left(\ln 9 - \ln 3\right) = \ln\sqrt{3}$$

№14(2)

$$J(\alpha) = \int_\alpha^{2\alpha} \frac{\sin \alpha x}{x}\,dx$$

$$J'(\alpha) = \int_\alpha^{2\alpha} \cos \alpha x\,dx + \frac{\sin 2\alpha^2}{\alpha} - \frac{\sin \alpha^2}{\alpha} =$$

$$= \frac{\sin \alpha x}{\alpha}\Big|_\alpha^{2\alpha} + \frac{1}{\alpha}\left(\sin 2\alpha^2 - \sin \alpha^2\right) =$$

$$= \frac{2}{\alpha}\left(\sin 2\alpha^2 - \sin \alpha^2\right)$$

N17.

• $$J(\alpha) = \int_0^{b} \frac{dx}{x^2+\alpha^2} = \frac{1}{\alpha^2}\int_0^{b} \frac{dx}{1+\frac{x^2}{\alpha^2}} = \frac{1}{\alpha}\int_0^{b} \frac{d\left(\frac{x}{\alpha}\right)}{1+\frac{x^2}{\alpha^2}} =$$

$$= \frac{1}{\alpha}\operatorname{arctg}\left(\frac{x}{\alpha}\right)\Big|_0^{b} = \frac{1}{\alpha}\operatorname{arctg}\left(\frac{b}{\alpha}\right)$$

• $$J'(\alpha) = \int_0^{b} \frac{-2\alpha\, dx}{(x^2+\alpha^2)^2} = -\frac{1}{\alpha^2}\operatorname{arctg}\left(\frac{b}{\alpha}\right) - \frac{1}{\alpha}\cdot\frac{\frac{b}{\alpha^2}}{1+\frac{b^2}{\alpha^2}} =$$

$$= -\frac{1}{\alpha^2}\operatorname{arctg}\left(\frac{b}{\alpha}\right) - \frac{b}{\alpha(\alpha^2+b^2)} \Rightarrow$$

$$\Rightarrow \int_0^{b} \frac{dx}{(x^2+\alpha^2)^2} = \frac{1}{2\alpha^3}\operatorname{arctg}\left(\frac{b}{\alpha}\right) + \frac{b}{2\alpha^2(\alpha^2+b^2)}$$

N18

2) $$J(\alpha) = \int_0^{\pi} \ln(1-2\alpha\cos x+\alpha^2)\,dx \;; \quad |\alpha|<1$$

$$J'(\alpha) = \int_0^{\pi} \frac{-2\cos x+2\alpha}{1-2\alpha\cos x+\alpha^2}\,dx =$$

$$t = \alpha - \cos x \Rightarrow \cos x = \alpha - t$$

$$x = \arccos(\alpha - t)$$

$$dx = \frac{1}{\sqrt{1-(\alpha-t)^2}}\, dt$$

$$= \int_{\alpha-1}^{\alpha} \frac{-2t}{1-\alpha^2+\alpha t+\alpha t}$$

№4.

$f(x)$ принимает положит. значения на $[0,1]$ и непр. Показать, что у $I(\alpha) = \int_0^1 \frac{\alpha}{x^2+\alpha^2} f(x)\,dx$ разрыв при $\alpha = 0$

$$I(\alpha) = \alpha \int_0^1 \frac{f(x)\,dx}{x^2+\alpha^2}$$

Рассмотрим при $\alpha > 0$:

$f(x)$ и $\frac{1}{x^2+\alpha^2}$ – положит. и непр $\Rightarrow$

$\Rightarrow \frac{f(x)}{x^2+\alpha^2}$ – положит. и непр. на $[0,1] \Rightarrow$

$$\Rightarrow \int_0^1 \frac{f(x)\,dx}{x^2+\alpha^2} > 0$$

$$\int_0^1 \frac{f(x)\,dx}{x^2+\alpha^2} = \frac{1}{\alpha}\operatorname{arcctg}\frac{1}{\alpha} f(1) - \frac{1}{\alpha}\int_0^1 \operatorname{arcctg}\frac{x}{\alpha} f'(x)\,dx$$

$$= \frac{1}{\alpha}\left(\operatorname{arcctg}\frac{1}{\alpha} f(1) - \int_0^1 \operatorname{arcctg}\frac{x}{\alpha}\cdot f'(x)\,dx\right) \geq$$

$$\geq \frac{1}{\alpha}\operatorname{arcctg}\frac{1}{\alpha}\left(f(1) - \int_0^1 f'(x)\,dx\right) =$$

$$= \frac{1}{\alpha}\operatorname{arcctg}\frac{1}{\alpha}\cdot f(0)$$ — как видно

При $\alpha \to 0+0$ данное значение увеличивается $\Rightarrow$

$\Rightarrow I(\alpha) = \alpha \int_0^1 \frac{f(x)\,dx}{x^2+\alpha^2}$ - возрастает при $\alpha \to 0+0$

Аналогично $I(\alpha)$ уменьшается при $\alpha \to 0-0$

Таким образом $\lim\limits_{\alpha\to 0+0} I(\alpha) \neq \lim\limits_{\alpha \to 0-0} I(\alpha) \Rightarrow$

$\Rightarrow$ разрыв в $\alpha = 0$.

№ 18

2) $J(\alpha) = \int\limits_0^{\pi} \ln(1 - 2\alpha\cos x + \alpha^2)\,dx$; $|\alpha| < 1$

$$J'(\alpha) = \int\limits_0^{\pi} \frac{-2\cos x + 2\alpha}{1 - 2\alpha\cos x + \alpha^2}\,dx =$$

Сделаем универсальную триг. замену:

$t = \operatorname{tg}\frac{x}{2}$; $\cos x = \frac{1-t^2}{1+t^2}$, $dx = \frac{2dt}{1+t^2}$

$$J'(\alpha) = \int\limits_0^{+\infty} \frac{2t^2 - 2 + 2\alpha + 2\alpha t^2}{1 + t^2 - 2\alpha + 2\alpha t^2 + \alpha^2 + \alpha^2 t^2} \frac{2}{1+t^2}\,dt =$$

$$= 4\int\limits_0^{+\infty} \frac{t^2(\alpha+1) + (\alpha-1)}{t^2(\alpha+1)^2 + (\alpha-1)^2} \frac{1}{1+t^2}\,dt =$$

$$= 4\int\limits_0^{+\infty} \left(\frac{\frac{\alpha^2-1}{2\alpha}}{t^2(\alpha+1)^2 + (\alpha-1)^2} + \frac{\frac{1}{2\alpha}}{1+t^2} \right) dt =$$

$$= \frac{4(\alpha-1)(\alpha+1)}{2\alpha} \int\limits_0^{+\infty} \frac{dt}{(\alpha-1)^2 + t^2(\alpha+1)^2} + \frac{4}{2\alpha}\int\limits_0^{+\infty} \frac{dt}{1+t^2} =$$

$$= \frac{2}{\alpha} \operatorname{arctg}\left(t\,\frac{\alpha+1}{\alpha-1}\right)\Big|_0^{+\infty} + \frac{2}{\alpha}\operatorname{arctg}(t)\Big|_0^{+\infty} =$$

$$= -\frac{\pi}{\alpha} + \frac{\pi}{\alpha} = 0 \Rightarrow \left.\begin{array}{l} I(\alpha) = const \\ I(0) = 0 \end{array}\right| \Rightarrow I(\alpha) = 0$$