№ 3.2.

а) $f(x) = x\ln x + (1-x)\ln(1-x)$, $x \in (0,1)$

$$f'(x) = \ln x + 1 - 1 - \ln(1-x) = \ln\left(\frac{x}{1-x}\right)$$

$$f''(x) = \frac{1-x}{x} \cdot \left(\frac{1}{1-x} + \frac{x}{(1-x)^2}\right) =$$

$$= \frac{1-x}{x}\,\frac{1}{(1-x)^2} = \frac{1}{x(1-x)}$$

Т.к. $x \in (0,1) \Rightarrow d^2 f > 0 \Rightarrow$ ф-ия выпукла.

№ 3.3.

а) $f(x) = e^x$

$$f^*(x^*) = \sup_{x \in \mathbb{R}} (\langle x^*, x\rangle - e^x) = \sup_{x \in \mathbb{R}} (x^* x - e^x)$$

1) $g(x) = x^* x - e^x$

$g'(x) = x^* - e^x = 0 \Rightarrow x = \ln x^*$ $(x^* > 0)$

$g''(x) = -e^x \Rightarrow x = \ln x^*$ - max

2) Если $x^* = 0 \Rightarrow f^*(x^*) = \sup(-e^x) = 0$

$x \in \mathbb{R}$

3) Если $x^* < 0$, то $f^*(x^*) = +\infty$

Ответ: $f^*(x^*) = \begin{cases} +\infty, & x^* < 0 \\ 0, & x^* = 0 \\ x^*(\ln x^* - 1), & x^* > 0 \end{cases}$

2) $\delta\{0\}$ - индикаторная ф-я мн-ва $A = \{0\}$

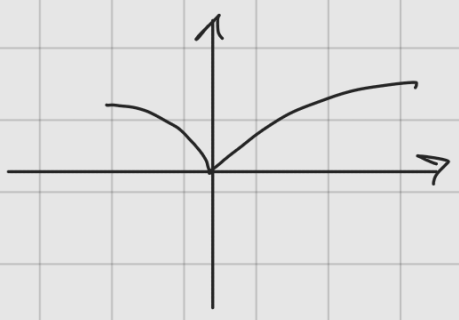
$$\delta = \begin{cases} 0, & x = 0 \\ +\infty, & x \neq 0 \end{cases}$$

$$\delta^*(x^*) = \sup_{x \in \mathbb{R}} (x^* x - \delta(x)) = (x^* x - \delta(x))\big|_{x=0} =$$

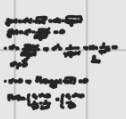
$$= 0.$$

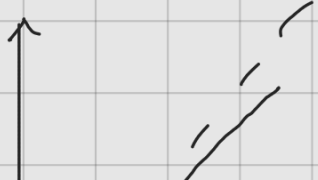
№3.5

а) $f(x) = \sqrt{|x|}$



1) $f^*(x^*) = \sup_{x \in \mathbb{R}} (x^* x - \sqrt{|x|})$





$x^* > 0$ — Аналогично при $x^* < 0$

При $x^* = 0 \to 0 \Rightarrow f^*(x^*) = \begin{cases} +\infty, & x^* \neq 0 \\ 0, & x = 0 \end{cases}$

2) $f^{**}(x) = \sup\limits_{x^* \in \mathbb{R}^n} (x^* x - f^*(x^*)) \equiv 0$

б) $f(x) = \sin x$

1) $f^*(x^*) = \sup\limits_{x \in \mathbb{R}} (x^* x - \sin x) =$

$$= \begin{cases} +\infty, & x^* < 0 \\ 1, & x^* = 0 \\ +\infty, & x^* > 0 \end{cases}$$

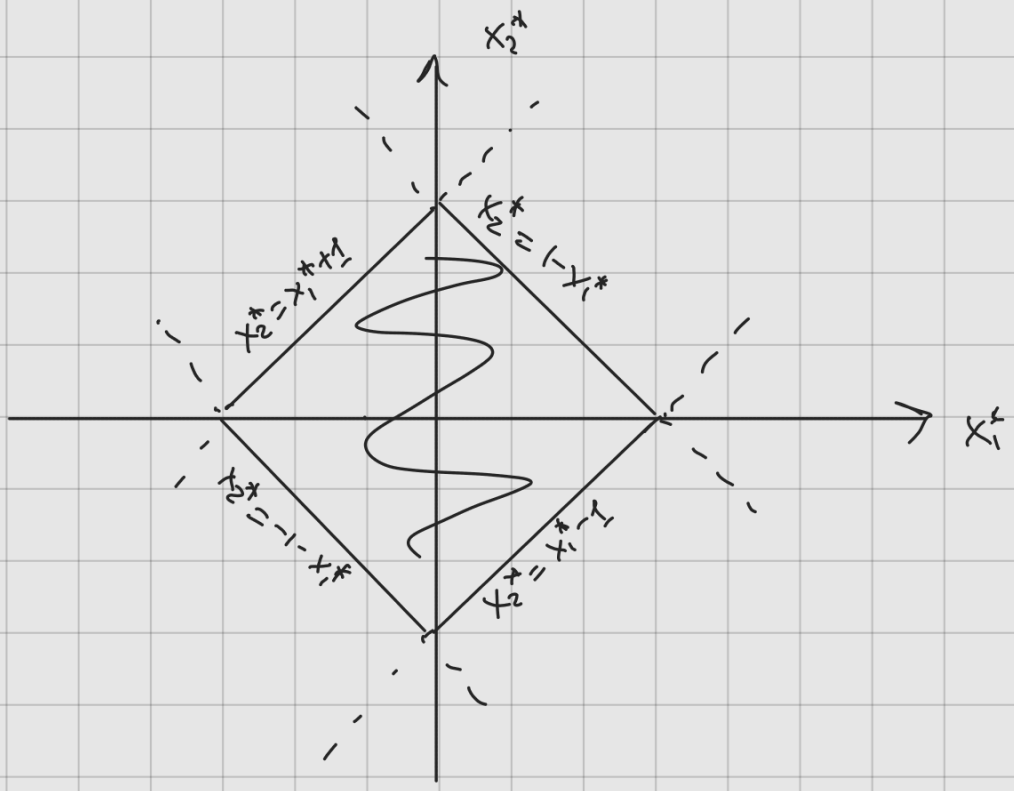
2) $f^{**}(x) = \sup\limits_{x^* \in \mathbb{R}} (x^* x - f^*(x^*)) = -1$

N 3.7

а) $A = \{(-1,-1), (-1,1), (1,-1), (1,1)\}$

$A^\circ = \{ x^* \in X^* \mid \langle x^*, x \rangle \le 1, \forall x \in X \}$

$$\begin{cases} -x_1^* - x_2^* \le 1 \\ -x_1^* + x_2^* \le 1 \\ x_1^* - x_2^* \le 1 \\ x_1^* + x_2^* \le 1 \end{cases}$$



$A^\circ = \{ (x_1^*, x_2^*) \in \mathbb{R}^2 \mid |x_1^*| + |x_2^*| \le 1 \}$

б) $A = \{ (x_1, x_2) \mid x_1^2 + x_2^2 \le 1 \}$

$A^\circ = \{ (x_1^*, x_2^*) \mid x_1^{*2} + x_2^{*2} \le 1 \}$

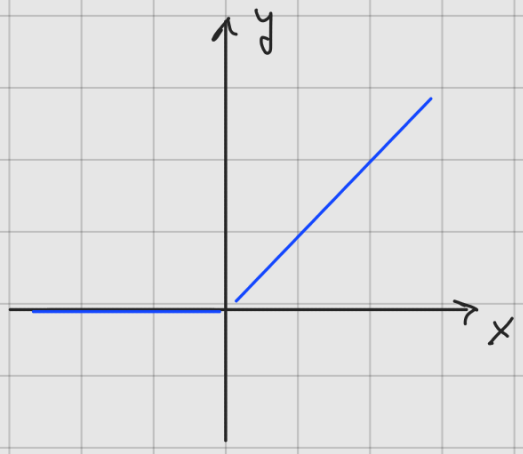
A и A° – круги, все вектора единичные, поэтому любое скалярное произведение таких векторов ≤ 1

№ 3.8

а) $f(x) = |x|$

$$\partial f(x) = \begin{cases} -1, & x < 0 \\ [-1, 1], & x = 0 \\ 1, & x > 0 \end{cases}$$

б) $\max\{x, 0\} = f(x)$



$$\partial f(x) = \begin{cases} 0, & x < 0 \\ [0, 1], & x = 0 \\ 1, & x > 0 \end{cases}$$

в) $f(x) = \max\{-x, 0\}$

$$\partial f(x) = \begin{cases} -1, & x < 0 \\ [-1, 0], & x = 0 \\ 0, & x > 0 \end{cases}$$