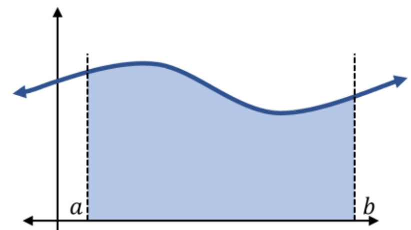
In the last section, the limit of the sum of infinity rectangles showed us the area under a curve. This is actually a method of evaluating the *definite integral*. This section will show what the definite integral is, where it's used, how to evaluate it, and properties known to be true about the definite integral.

THE DEFINITE INTEGRAL



The definite integral is the area under the curve.

The Definite Integral *(object)* – the combination of a definite[1] range $[a, b]$ of infinitesimal data under the curve of $f(x)$.

NOTATION FOR INTEGRATION: $\int_a^b f(x)dx$.

WHAT DOES THE DEFINITE INTEGRAL MEAN?

To find the area under the curve f from a to b, find the sum of infinity rectangles that make up the area *(section 5.1)*. Those tiny slices each have the height of $f(x)$ and the width of Δx. The definite integral also tells us the area under the curve, and it is written like this: $\int_a^b f(x)dx$

Each part has a purpose in the definite integral. Consider:

Symbol	What it's called	What it means
$\int$	Integration sign	This is the integral operation – the area.
a	Lower limit of integration	The integral starts at point a.
b	Upper limit of integration	The integral ends at point b.
$f(x)$	Integrand	The height of each slice.
dx	(no official name)	The width of the tiny slices that $f(x)$ is across.

CONNECT THE DOTS: $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)dx$. Replace Σ with ∫, Δx with dx, and f(x*) with f(x).

WHERE WOULD YOU USE THE DEFINITE INTEGRAL?

Because integrals mean the area under the curve, you might use them to find antiderivatives *(section 4.7)* in a range. For example:

- A painter is painting the equation $y = |\sin x|$ on a wall, where x and y are in feet and the x-axis is 10 feet across the floor. He fills in all the area under the curve. How much paint will he use?
- You know a function $v(t)$ that tells the speed of a rocket on the up-down axis. How far did the rocket move up or down over a certain duration of time?
- A 256-page book is written, starting at a rate of 1 page every day. Over the course of a week, the rate of writing gradually increases to one more page per day. How many days will it take for the book to be written?
- A factory line operates in direct proportion to the number of workers operating in it. If the number of workers is increased by 30% every hour, by what percent of average operations per hour will the factory have improved to over the next 4 hours?

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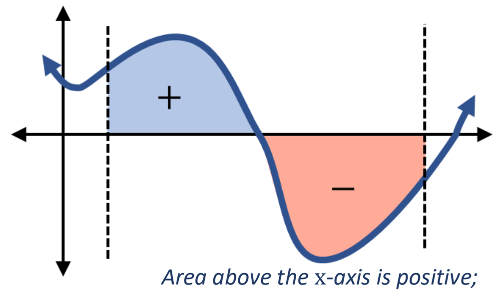
[1] **Definite**: clearly defined; explicitly precise.

HOW WOULD YOU DO EVALUATE THE DEFINITE INTEGRAL?



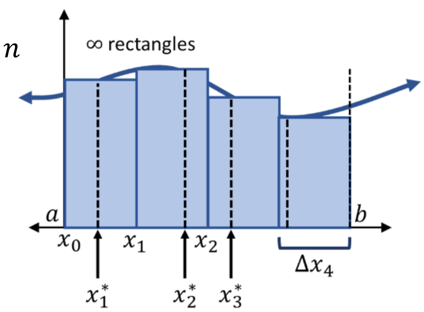
Area above the x-axis is positive; area under the x-axis is negative.

To find evaluate[2] the definite integral $\int_a^b f(x)dx$:

1. Convert the integral to a summation of infinite, regular rectangles, using these formulas:
   $x_i^* = a + i\Delta x$
   $$\Delta x = \frac{b-a}{n}$$
   $$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$
2. Apply algebra *(section 1.1)*, summations rules *(section 5.1)*, and limit laws *(sections 1.3-6; 3.7)*.

NOTE: Area above the x-axis is positive and adds to the integral's area; area below the x-axis is negative and subtracts from the integral's area.

A MORE TECHNICAL DEFINTIION OF THE DEFINITE INTEGRAL



The Reimann sum is the sum of infinity rectangles. Their heights are $f(x_i^*)$, where x_i^* is any value in subinterval i.

If a function $f(x)$ exists, then in the interval $[a, b]$ for any integer n, there are n number of **subintervals**. All them together are called a **partition** P of $[a, b]$.

The ith subinterval $[x_{i-1}, x_i]$ has the width $\Delta x_i = x_i - x_{i-1}$.

Samples points are chosen, one for each subinterval, written x_i^*. They can be anywhere in the subinterval; on the far-left, far-right, right in the middle, or somewhere in-between.

The **midpoint rule** lets the sample point be in the middle of its subinterval. $x_i^* = \frac{1}{2}(x_{i-1} + x_i)$.

The **Reimann sum** for a partition P is the sum of the areas of infinity rectangles defined from the partition described above. (This means that if a rectangle goes *under* the x-axis, it has a *negative* area, and would *subtract*, not add, its area from the sum)

Since the width of each subinterval (Δx_i) can be different, to find the *definite* integral, limit the maximum of Δx_i to 0. The **definite integral** then is this:

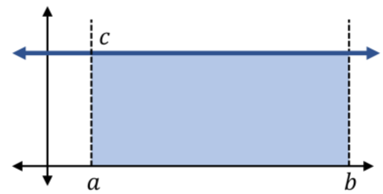
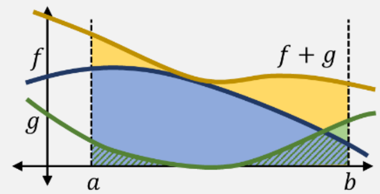
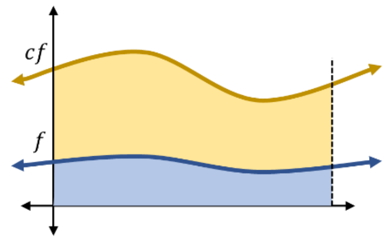
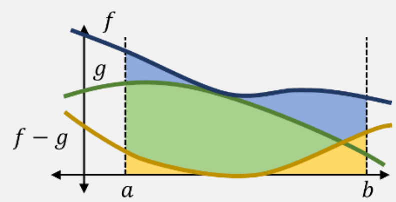
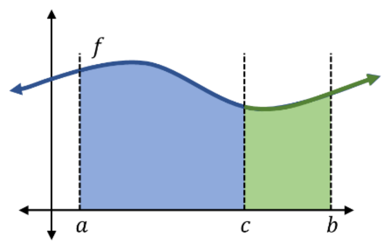
$$\int_a^b f(x)dx = \lim_{\max \Delta x_i = 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

PROPERTIES OF THE DEFINITE INTEGRAL

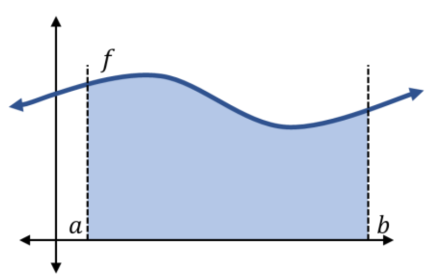
Like many mathematical objects, integrals can be converted between forms using rules. Here are some:

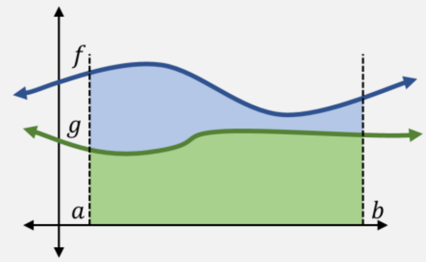
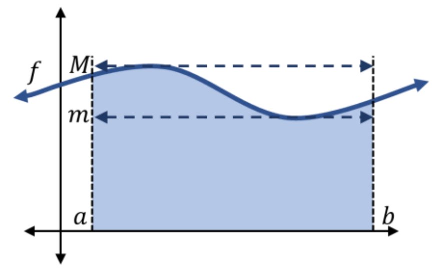
If it looks like this…	…then it can be this.	Here's why

[2] **Evaluate**: find the value of.

$\int_a^b c\,dx =$	$c(b-a)$	
$\int_a^b [f(x)+g(x)]dx =$	$\int_a^b f(x)dx + \int_a^b g(x)dx$	
$\int_a^b cf(x)dx =$	$c\int_a^b f(x)dx$	
$\int_a^b [f(x)-g(x)]dx =$	$\int_a^b f(x)dx - \int_a^b g(x)dx$	
$\int_a^c f(x)dx + \int_c^b f(x)dx =$	$\int_a^b f(x)dx$	

As well as properties that convert from one form to another, integrals have properties that tell how they *compare* with other expressions.

If this is true…	…then this is true.	Here's why
If $f(x) \geq 0$ for $a \leq x \leq b$,	then $\int_a^b f(x)dx \geq 0$	

If $f(\mathrm{x}) \geq g(\mathrm{x})$ for $a \leq x \leq b$,	then $\int_{\mathrm{a}}^{\mathrm{b}} \mathrm{f(x)dx} \geq \int_{\mathrm{a}}^{\mathrm{b}} \mathrm{g(x)dx}$	
If $m \leq f(\mathrm{x}) \leq M$ for $a \leq x \leq b$,	then $m(\mathrm{b}-\mathrm{a}) \leq \int_{\mathrm{a}}^{\mathrm{b}} \mathrm{f(x)dx} \leq M(\mathrm{b}-\mathrm{a})$	

LOOK IN THE BOOK: for proof that these rules are true, see pages 276-8 in the textbook.

WHAT DID YOU LEARN?

- What is the definite integral? How do you evaluate it? Where would you use it?
- What are some properties of the definite integral? What do they do? Why are they true?
- What is another way to evaluate the definite integral? Why is it true? How does it do this?