The last sections showed how to find the derivative of a quantity – for example, the speed of a moving object from its position over time. What if you want to know the opposite – how far a moving object goes, just knowing its speed? And how does this relate to the area under a curve? A special kind of limit combined with *sigma notation* will tell us the answer.

SIGMA NOTATION

Sum, **Summation** *(operation)* – the addition of a sequence (collection) of numbers.

Sigma notation *(notation)* – a way to write the sum of a sequence f, from a to b, using the Σ sign.

WHAT DOES THE SIGMA MEAN?

Suppose we have integers a and b, and a sequence f_x or function $f(x)$ that is defined on $[a, b]$. The sigma sign expands the sequence to add all the values together. A summation might be written like this:

$$\sum_{x=a}^{b} f(x) = f(a) + f(a+1) + f(a+2) + \cdots + f(b)$$

It may also be written like this, using subscripts:

$$\sum_{x=a}^{b} f_x = f_a + f_{a+1} + f_{a+2} + \cdots + f_b$$

WHERE WOULD YOU USE SIGMA NOTATION?

Anywhere you want to find the sum (addition) of all the numbers (or other kinds of mathematical expressions) in a sequence, you may want to use sigma notation. For example:

- To find the mean (average) of a sequence of numbers, you must add them all, then divide the sum by the number of numbers. Sigma notation describes it this way: $\text{Avg.}(x) = \sum_{i=1}^{n} [x_i \div n]$.
- To find how many days a project will last, you must add the lengths of each component of the project (if the entire project is developed in serial – where only one part is worked on at a time). You can use sigma to do that - $\sum \text{days}$.
- To find the total number of people in a building, you must add the number of people in each floor level. You can use sigma notation to do that too.

SOME RULES ABOUT SUMMATIONS

Use the table below to convert summations from one form to another.

If it looks like this...	then it can be like this,	given these condition(s)
$\sum [cf(x)] =$	$c \sum [f(x)]$	c is a constant
$\sum [a_x + b_x] =$	$\sum a_x + \sum b_x$	
$\sum [a_x - b_x] =$	$\sum a_x - \sum b_x$	
$\sum_{x=a}^{b} f_x =$	$\sum_{x=1}^{b} f_x - \sum_{x=1}^{a} f_x$	a and b are constants.

$\sum_{x=1}^{n} 1 =$	n	
$\sum_{x=1}^{n} x =$	$\frac{1}{2}n(n+1)$	
$\sum_{x=1}^{n} x^2 =$	$\frac{1}{6}n(n+1)(2n+1)$	
$\sum_{x=1}^{n} x^3 =$	$\left[\frac{n(n+1)}{2}\right]^2$	

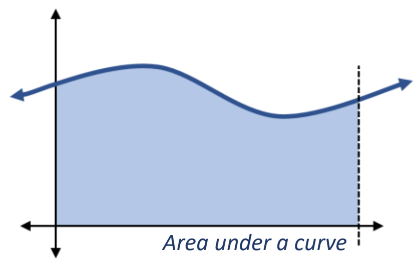
HOW TO KNOW THESE RULES ARE TRUE: you can prove these rules true with inductive reasoning, with a picture, or by looking in appendix B in the textbook (among some ways to see their truthfulness).

AREA UNDER A CURVE

Area *(value)* – the extent that a figure, such as a shape, covers the plane.

WHAT IS THE AREA UNDER A CURVE?

If you have a function and you fill in the space under its curve (between it and the x-axis), a function can have area. The figure to the right illustrates this.



WHERE WOULD YOU USE THE AREA UNDER A CURVE?

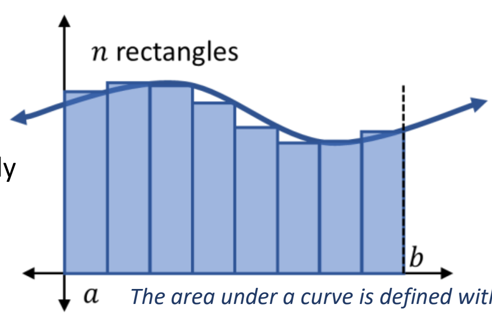
The area under the curve means the definite integrate *(section 5.2)*, which means the antiderivative *(section 4.7)*, so if you can find it then you can mathematically find values such as:

- If you know a function that tells how fast a car (or rocket) is going at a given time, can you find out how far it travels over the next twenty seconds?
- How far does a train go and thus how many revolutions do the wheels turn if it travels at a speed that varies according to a certain function over a certain duration of time?
- If water leaks from a bottle at a certain speed, and it started out with a certain amount of water and leaks faster by a certain speed every second, how long until the bottle is empty?

HOW DO YOU FIND THE AREA UNDER A CURVE?

To find the area under a curve:

1. Define *where* under the curve you want to find the area; an interval from a to b, that is, $[a, b]$.
2. Make little rectangles that touch the curve, consistently either on their top-left or top-right corner. (If they touch on their top-right corner, they're called *right-approximating*; on the top-left makes them *left-approximating*.)
3. Take the sum of all the rectangles' areas. For more accuracy, use more rectangles. For complete accuracy and precision, use infinity rectangles.



The area under a curve is defined with right-approximating rectangles

THE LIMITS OF MATH: infinite precision sounds impossible, but limits help us out *[sections 1.3-6; 3.7]*.

NOTATION FOR THE SUM OF RECTANGLES: for the area of n rectangles that touch their top-right corners to the curve, write R_n. For rectangles that touch on their top-left corner, use L_n.

A CLOSER LOOK AT STEP 3 – ADDING ALL THE AREAS

The area of one rectangle is length times width. The sum of all the areas is expressed by sigma notation:

$$R_n = \sum_{i=1}^{n} [(\text{width})(\text{length})]$$

What are the width and length of the rectangles?

Width: In these calculations for the right-approximating sums, the width of each rectangle is the same, and all the n rectangles are supposed to span from a to b, so the width of each rectangle should be $(b-a) \div n$. To find the x, or left, of the rectangle, just multiply the width by the rectangle's index. (The index will start at one for right-approximating sums, at zero for left-approximating sums.)

Length: The length (or height) of each rectangle should be enough to make its key corner touch the curve (top-right corner for R_n). The formula for the area of n rectangles whose top-right corners touch the curve of f is, then:

Plugging in these expressions gives:

$$R_n = \sum_{i=1}^{n} \left[\frac{b-a}{n} f\left(\mathrm{a} + i\left(\frac{b-a}{n}\right)\right)\right]$$

Using sigma rules to simplify the expression:

$$R_n = \frac{b-a}{n} \sum_{i=1}^{n} \left[f\left(a + i\left(\frac{b-a}{n}\right)\right)\right]$$

LOOK IN THE BOOK: for examples of making rectangles to approximate the area under a curve, see example 1 *(section 5.1)*.

USING INFINITY RECTANGLES

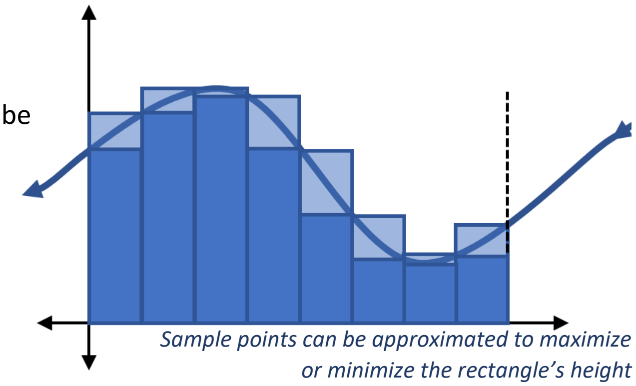
To use an infinite number of rectangles, just limit n to approach infinity.

$$A_{\text{curve}} = \lim_{n \to \infty} \mathrm{R_n}$$

Using limit laws and sigma laws, you *might* then be able to evaluate the area under the curve to a definite number, a precise value *(section 5.2)*.

LOOK IN THE BOOK: for examples of using infinity rectangles with limit laws and sigma laws to find the area under a curve, see examples 2 and 3 *(section 5.1)*.

A MORE TECHNICAL EXPLANATION



Sample points can be approximated to maximize or minimize the rectangle's height

Each rectangle (at index i) is defined on an interval $[x_{i-1}, x_i]$, where $x_i = \mathrm{i}(b-a) \div n + a$. As n approaches infinity, the height of the rectangle can be *any* $\mathrm{f}(\mathrm{x}_\mathrm{i}^*)$, where x_i^* is called a **sample point**, such that $x_i^* \in [x_{i-1}, x_i]$.

Using the previous method for describing the area with rectangles, if you were to use a finite number of rectangles, there exists at least two methods for finding samples points, such that one method maximizes the sum and one minimizes it. The larger measure is called the upper sums, and the smaller one the lower sums. The area A under the curve is the unique number that is smaller than all the upper sums and larger than all the lower sums.

Since the sample point is defined on an interval that is inclusive from both sides, the right-approximating sums and the left-approximating sums give the same answer.

DISTANCE OVER TIME

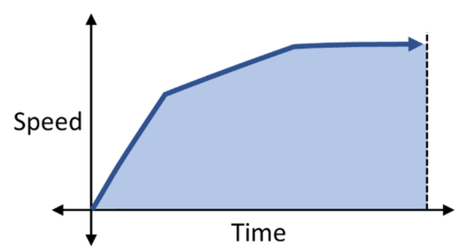
Distance *(value)* – a measure of length in space; the smallest amount of space between two points.

Time *(value)* – a measure of the degree to which materials may mutate. *[Debatable]*

Speed *(value)* – how much distance changes as time changes; the derivative of distance by time.

Velocity *(value)* – speed in a direction, a vector; more technically: the derivative of position by time; how distance changes as time changes.

HOW WOULD YOU FIND VELOCITY AND DISTANCE?



In a graph of speed over time, distance traveled is the area under the curve

Describing these variables with math can look this way:

- $t =$ time.
- $x(t) =$ distance traveled at t time.
- $\vec{x}(t) =$ position at t time.
- $v(t) =$ speed at t time.
- $\vec{v}(t) =$ velocity at t time.

If speed is distance traveled over time (it is), then speed is the derivative of distance traveled. The same with velocity and position. Mathematically:

- $\mathrm{x}'(t) = v(t)$.
- $\vec{x}'(t) = \vec{v}(t)$.

Therefore, if you want to find distance traveled or the position change over a definite interval of time, find the area under the curve of speed or velocity. The figure above illustrates this.

HOW WOULD YOU ANSWER?

- What is sigma notation? What does it mean? Where is it used? How can you simplify it?
- What does the "area under the curve" mean? How do you find it?
- What is velocity? How do you find it from distance traveled? How do you use velocity to find distance?