The derivative of a function tells much about its shape. Is it increasing or decreasing? Is it concave upward or downward? Where are its minima and maxima? The signs of functions tell much about them, as signs do elsewhere in life. Section 4.3 shows what the signs of derivatives tell about the shapes of graphs and their extrema, organized by different tests of derivatives.

INCREASING/DECREASING TEST

Increasing/Decreasing Test, I/D Test *(test)* – a test to determine if a function is increasing or decreasing across an interval.

WHAT THE I/D TEST TELLS

For a function f and an interval I in the function's domain, the function may fall into one of these cases:

If $f'(x) > 0$ for all $x \in I$	➔	The graph of f is increasing in I.
If $f'(x) < 0$ for all $x \in I$	➔	The graph of f is decreasing in I.

Table 1

NOTE: the I/D Test only will tell if the function is increasing across the *entire* interval or decreasing across the *entire* interval, but not both or a mix of the two.

WHY THE I/D TEST IS TRUE

- Suppose a function f is increasing from x_1 to x_2.
- Thus, $f'(x) > 0$ whhen $x_1 \leq x \leq x_2$.
- Thus, f is differentiable on $[x_1, x_2]$.
- According to the mean value theorem *(section 4.2)*, there is a number c in $[x_1, x_2]$ such that $\mathrm{f}(x_2) - \mathrm{f}(x_1) = f'^{(c)(x_2 - x_1)}$.
- Because f is assumed to be increasing all the way from x_1 to x_2, $f'(c)$ must be positive.
- Because $x_2 > x_1$, $(x_2 - x_1)$ must be positive also.
- Therefore, $\mathrm{f}'^{(c)(x_2 - x_1)}$ must be positive.
- Therefore, $f(x_2) - f(x_1)$ must be positive.
- Therefore, $f(x_2) > f(x_1)$.

HOW TO USE THE I/D TEST

The I/D test can tell if a function is increasing or decreasing on an interval. To do, simply identify which case the function fits into for its interval in the table above (Table 1).

The I/D test can also be used to find *where* a function is increasing or decreasing. To do so,

1. Take the derivative of the function.
2. Factor the derivative.
3. Take the critical numbers *(section 4.1)* of the factors.
4. Divide the derivative's domain into intervals separated by the critical numbers.
5. In each interval, determine for each factor whether it is positive or negative.
6. Cancel out each negative factor with another negative one.
7. If there's one negative factor left over, the function is decreasing in that interval. If not, it's increasing.

THE FIRST DERIVATIVE TEST

The First Derivative Test *(test)* – a test that tells if a critical number in a function is a local minimum, maximum, or neither.

WHAT THE FIRST DERIVATIVE TEST TELLS

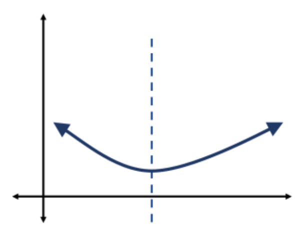
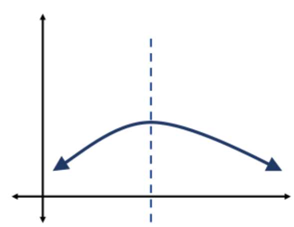
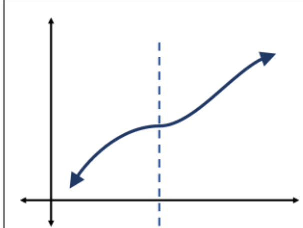
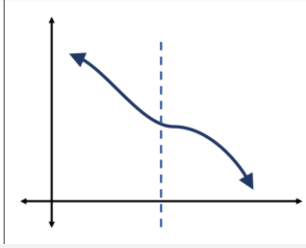
For a function f and critical number c in f,

If f' goes from negative to positive at c	➔	f has a local minimum at c.
If f' goes from positive to negative at c	➔	f has a local maximum at c.
If f' stays positive or negative at c	➔	f has no extremum at c.

Table 2

WHY THE FIRST DERIVATIVE TEST IS TRUE

Consider that, at a critical number c (represented by a dashed line), a function f resembles one of the graphs below, possibly giving it an extremum. (See table below)

Negative to Positive	Positive to Negative	Positive to positive	Negative to negative
			
Minimum	Maximum	(no extremum)	

HOW TO USE THE FIRST DERIVATIVE TEST

The first derivative test can classify a critical number in a function as a maximum, minimum, or neither. To do so, identify which case the function's critical number fits into in Table 2.

CONCAVITY TEST

Concave up/downward *(description)* – the direction a function "opens". (See Table 3)

Inflection Point *(value)* – where a function changes its concave direction. (See Figure 1)

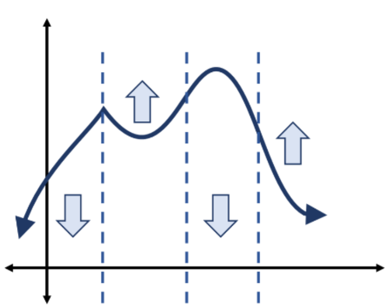


Figure 1

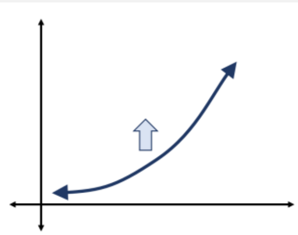
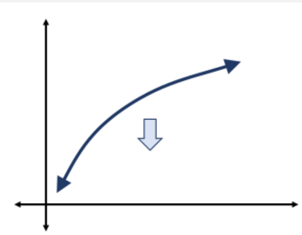
Concave Upward	Concave Downward
	
The function "opens" upward, over itself.	The function "opens" downward, under itself.

Table 3

Concavity Test *(test)* – a test used to find whether a function f is concave upward or downward in an interval I.

WHAT THE CONCAVITY TEST TELLS

For a function f and interval I in f,

If $f''(x) > 0$ for all $x \in I$	➔	The graph of f is concave upward in I.
If $f''(x) < 0$ for all $x \in I$	➔	The graph of f is concave downward in I.

Table 4

WHY THE CONCAVITY TEST IS TRUE

- Let's assume that, in an interval, "concave" is the direction that a function is increasing toward.
- Then the function's first derivative should be increasing that direction (positive or negative).
- Then the function's second derivative should be positive or negative across the interval.

[for actual, full proof, see Appendix D of textbook]

HOW TO USE THE CONCAVITY TEST

The concavity test tells if an interval of a function is concave upward or downward (or neither). To find out, simply prove that one of the two conditions in Table 4 is true, then accept the consequences.

THE SECOND DERIVATIVE TEST

The Second Derivative Test *(test)* – a test that tells mathematically if a critical number in a function is a maximum, minimum, or neither.

WHAT THE SECOND DERIVATIVE TEST TELLS

For a function f and critical number c in f,

If $f'(c) = 0$ and $f''(c) > 0$	➔	f has a local minimum at c.
If $f'(c) = 0$ and $f''(c) < 0$	➔	f has a local maximum at c.
If $f'(c) = 0$ and $f''(c) = 0$	➔	f has no extremum at c.

Table 5

WHY THE SECOND DERIVATIVE TEST IS TRUE

- From the first derivative test, if f' changes from negative to positive, there is a minimum, and vice versa for a maximum.
- The second derivative, f'' tells if f' changes from negative to positive or vice versa.
- Therefore, the second derivative test is an accurate mathematical representation of the first derivative test. (This assume that the first derivative test is true.)

HOW TO USE THE SECOND DERIVATIVE TEST

The second derivative test tells where maxima and minima are in a function f. To find them,

1. Find the critical numbers in f.
2. For each critical number c, identify which case of Table 5 it fits into, then accept the consequences.

HOW WOULD YOU ANSWER?

- What do derivatives tell about the direction a function goes? How do you know?
- What does it mean for a function to be “concave upward”? How can you find out if, and where, it is?
- How do derivatives tell where minima and maxima are? Why is this true?