What the audience knows. What they know they don't know. What they don't know they don't know. How we're going to know it.

FUNDEMENTAL THEOREM OF CALCULUS (PART 1)

Fundemental Theorem of Calculus (Part 1), FTC1 *(theorem)* – a theorem that says the derivative of the integral of a function is that very same original function.

CALCULUS KEY: the fundamental theorem of calculus (part 1) connects integrals to derivatives.

WHAT DOES FTC1 TELL?

For any function f that is continuous[1] on $[\mathrm{a}, \mathrm{b}]$, a corresponding function g exists (where $a \leq x \leq b$):

$$g(\mathrm{x}) = \int_{\mathrm{a}}^{\mathrm{x}} \mathrm{f(t)dt}$$

FTC1 says that the derivative of g is f. That is, $\mathrm{g}'(\mathrm{x}) = f(\mathrm{x})$, whenever $a < x < b$.

LOOK IN THE BOOK: for examples showing this, see examples 1 and 2 *(section 5.4)*.

WHERE WOULD YOU USE FTC1?

Anywhere you want an integral's derivative, you might use FTC1. For example:

1. **HOMEWORK**: Keith gets homework every day. The amount of homework he gets on day x is $h(\mathrm{x})$. The total amount of homework he ever got in the year, then, is $w(\mathrm{x}) = \int_0^{\mathrm{x}} \mathrm{h(t)dt}$. At what rate will the total amount of work, $w(\mathrm{x})$, increase as the days go on?
2. **EXERCISE**: Jonny runs around the block every day. He runs at a varying speed. At m minutes into his exercise, the number of steps Jonny has stepped is $s(\mathrm{m}) = \int_0^{\mathrm{m}} [50 \sin \pi \mathrm{t} + 150]\mathrm{dt}$. How many steps per minute is he running at when 3.5 minutes into exercise?
3. **TILING**: Bill is remodeling his house. The room he's working on is a square with a side length of x feet. Each tile is 1 square foot; that means the floor needs x^2 tiles. Bill gets better and faster at installing the tiles the more he does them. Specifically, after installing t tiles, the next tile will take only $\frac{1}{\mathrm{t}}$ minutes to install. The formula for how many minutes it will take Bill to tile the whole floor, then, is $f(\mathrm{x}) = \int_0^{\mathrm{x}^2} \frac{1}{\mathrm{t}} \mathrm{dt}$. How will $f(\mathrm{x})$ increase as x increases?

Where to use FTC1

- Upper limit is changing.
- Integrand is not dependent on x.
- You want to find the integral's derivative.

HOW WOULD YOU USE FTC1?

FTC1 lets you take the derivative of an integral, with respect to the integral's upper limit[2]. Combined with the chain rule *[section 2.5]*, it tells that if you have any integral where both

1. the integrand[3] does not depend on x
2. the upper limit depends on x (either it is x or a function of x)

$$\frac{d}{dx} \int_a^{h(x)} \mathrm{f(t)dt}$$

FTC1 solves these kinds of problems – the derivative of an integral as its upper limit changes

---

[1] **Continuous**: existing as part of the curve of a function; not a single, isolated point *(section 1.5)*.

[2] **Upper limit**: what goes on the top of the "∫" symbol *(section 5.2)*. For example, 5 is the upper limit in $\int_1^5 \mathrm{xdx}$.

[3] **Integrand**: what goes inside the integral *(section 5.2)*. For example, in "$\int [\mathrm{f(x)dx}]$", the integrand is "$f(\mathrm{x})$".

To find the integral's derivative over x, just take the integrand at the right and multiply it by the upper limit's derivative over x. Here's an illustration:

If it looks like this…	…then it can be this.
$\frac{d}{dx}\int_a^{h(x)} f(t)dt$	$\mathrm{h}'(\mathrm{x})f\big(h(x)\big)$

LOOK IN THE BOOK: for examples of applying FTC1, see examples 3-5 *(section 5.4)*.

FTC1 AT WORK!

1. **HOMEWORK**: *Keith's total homework is $w(x) = \int_0^x h(t)dt$. At what rate will the total amount of work, $w(x)$, increase as the days go on?* Keith is asking what the derivative of $w(\mathrm{x})$ is, that is, he wants to know what is $\frac{\mathrm{d}}{\mathrm{dx}}\int_0^{\mathrm{x}} \mathrm{h}(\mathrm{t})\mathrm{dt}$. FTC1 says to take the derivative of the upper limit ($\mathrm{x}' = 1$) and multiply it by the integrand at the right ($h(\mathrm{x})$). Combined, this makes $h(\mathrm{x})$. Keith's homework increases by $h(\mathrm{x})$ as the days go on.
2. **EXERCISE**: *At m minutes into his exercise, the number of steps Jonny has stepped is $s(\mathrm{m}) = \int_0^{\mathrm{m}}[50\sin \pi\mathrm{t} + 150]\mathrm{dt}$. How many steps per minute is he running at when 3.5 minutes into exercise?* Jonny wants to know what the derivative of $s(\mathrm{m})$ is, that is, $\frac{\mathrm{d}}{\mathrm{dm}}\int_0^{\mathrm{m}}[50\sin \pi\mathrm{t} + 150]\mathrm{dt}$. FTC1 says to take the derivative of the upper limit ($\mathrm{m}' = 1$) and multiply it by the integrand at the 3.5 (which is $50\sin 3.5\pi + 150 = 100$) This gives $1 \cdot 100 = 100$. Jonny is running at a speed of 100 steps/sec.
3. **TILING**: *The formula for how many minutes it will take Bill to tile the whole floor, then, is $f(\mathrm{x}) = \int_0^{\mathrm{x}^2}\frac{1}{\mathrm{t}}\mathrm{dt}$. How will $f(\mathrm{x})$ increase as x increases?* Bill wants to know the derivative of f by x; that is, $\mathrm{f}'(\mathrm{x})$. Mathematically, $\mathrm{f}'(\mathrm{x}) = \frac{\mathrm{d}}{\mathrm{dx}}\int_0^{\mathrm{x}^2}\frac{1}{\mathrm{t}}\mathrm{dt}$. FTC1 tells Bill to differentiate the upper limit (x^2), which gives $2x$. Bill also takes the integrand ($\frac{1}{\mathrm{t}}$) and substitutes it with the upper limit, giving $\frac{1}{\mathrm{x}^2} = \mathrm{x}^{-2}$. Multiplied together, this is $(2\mathrm{x})(\mathrm{x}^{-2}) = 2\mathrm{x}^{-1} = \frac{2}{\mathrm{x}}$. To answer the original question: $\mathrm{f}'(\mathrm{x}) = \frac{2}{\mathrm{x}}$.

WHY IS FTC1 TRUE?

FTC1 says that $\frac{\mathrm{d}}{\mathrm{dx}}\int_{\mathrm{a}}^{\mathrm{x}} \mathrm{f}(\mathrm{t})\mathrm{dt} = f(\mathrm{x})$. We can test this by simplifying the first side of the equation and seeing if it can equal $f(\mathrm{x})$. Here are the steps and reasons I used to reach this conclusion:

1. **DEFINE NOTATION**: $g(x) = \int_a^x f(t)dt$.
2. **SUBSTITUTE**: *(step 1)*: $\frac{\mathrm{d}}{\mathrm{dx}}\int_a^x f(t)dt$ now equals $\frac{\mathrm{d}}{\mathrm{dt}}g(\mathrm{x}) = \mathrm{g}'(\mathrm{x})$.
3. **CONVERT TO LIMIT FORM OF DERIVATIVE**: *(section 2.1; step 2)*: $g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$.
4. **SUBSTITUTE**: *(steps 3, 1)*: $g(\mathrm{x}+\mathrm{h}) - g(\mathrm{x}) = \int_{\mathrm{a}}^{\mathrm{x}+\mathrm{h}} \mathrm{f}(\mathrm{t})\mathrm{dt} - \int_{\mathrm{a}}^{\mathrm{x}} \mathrm{f}(\mathrm{t})\mathrm{dt}$.
5. **APPLY INTEGRAL LAWS**: *(section 5.2)*: $\int_a^{x+h} f(t)dt - \int_a^x f(t)dt = \int_x^{x+h} f(t)dt$.
6. **APPLY EXTREME VALUE THEOREM**: *(section 4.1; step 5)*: Suppose $y = f(\mathrm{t})$ where t is in $[\mathrm{x}, \mathrm{x}+\mathrm{h}]$. There are variables u and v that give minimum and maximum values of y. The minimum $m = f(\mathrm{u})$; the maximum $M = f(\mathrm{v})$.
7. **SETUP AN INEQUALITY**: *(step 5, 6)*: Since m is minimum and M is maximum, $mh \leq \int_x^{x+h} f(t)dt \leq Mh$.
8. **SUBSTITUTE**: *(steps 6, 7)*: $f(\mathrm{u})h \leq \int_x^{x+h} f(t)dt \leq f(v)h$.
9. **SIMLIFY**: *(step 8)*: $f(\mathrm{u}) \leq \frac{1}{\mathrm{h}}\int_x^{x+h} f(t)dt \leq f(v)$.

10. **SUBSTITUTE**: *(steps 9, 5)*: $f(\mathrm{u}) \le \frac{1}{\mathrm{h}}\left[\int_a^{x+h} f(t)dt - \int_a^x f(t)dt\right] \le f(v)$.
11. **SUBSTITUTE**: *(steps 10, 1)*: $f(\mathrm{u}) \le \frac{1}{\mathrm{h}}[g(x+h) - \mathrm{g}(\mathrm{x})] \le f(v)$.
12. **SUBSTITUTE AND ADD LIMIT**: *(steps 11, 3)*: $\lim_{h\to 0} f(\mathrm{u}) \le g'(x) \le \lim_{h\to 0} f(v)$.
13. **EVALUATE EXTREME VALUES**: *(steps 12, 6)*: If h approaches 0, then within the range $[\mathrm{x}, \mathrm{x}+\mathrm{h}]$ there is only one value in the range, so the extreme values equal; $[\mathrm{f}(\mathrm{u}) = \mathrm{m}] = [\mathrm{f}(\mathrm{v}) = \mathrm{M}]$, thus $u = v = x$, thus $\lim_{\mathrm{h}\to 0} \mathrm{f}(\mathrm{u}) = \lim_{\mathrm{h}\to 0} \mathrm{f}(\mathrm{v}) = f(\mathrm{x})$.
14. **APPLY SQUEEZE THEOREM**: *(section 1.4; steps 12, 13)*: $g'(x) = f(\mathrm{x})$.
15. **SUBSTITUTE**: *(steps 1, 14)*: $\frac{\mathrm{d}}{\mathrm{dx}}\int_{\mathrm{a}}^{\mathrm{x}} \mathrm{f}(\mathrm{t})\mathrm{dt} = f(\mathrm{x})$.

■ FTC1 is true!

FUNDAMENTAL THEOREM OF CALCULUS (PART 2)

Fundamental Theorem of Calculus (Part 2), FTC2 *(theorem)* – a theorem that says the integral of a function is the difference of any antiderivative[4] of that same function at the limits of integration[5].

CALCULUS KEY: the fundamental theorem of calculus (part 2) connects integrals to antiderivatives.

WHAT DOES FTC2 TELL?

For any function f continuous on $[\mathrm{a}, \mathrm{b}]$, and for any antiderivative of f, called F, this is true:

$$\int_{\mathrm{a}}^{\mathrm{b}} \mathrm{f}(\mathrm{x})\mathrm{dx} = FF(\mathrm{b}) - FF(\mathrm{a})$$

NOTE: this works for *any* antiderivative, if $\mathrm{F}' = f$ *(section 4.7)*.

CONNECT THE DOTS: FTC2 looks just like the net change theorem *(section 5.3)*, except it integrates $f(\mathrm{x})$ instead of $\mathrm{F}'(\mathrm{x})$. FTC2 is the same as the evaluation theorem *(section 5.3)*.

WHERE WOULD YOU USE FTC2?

FTC2 finds the integral of a function. Anywhere you want to find the integral of a function, then, you might use FTC2.

HOW WOULD YOU USE FTC2?

To integrate any function f from a to b; that is, to find $\int_{\mathrm{a}}^{\mathrm{b}} \mathrm{f}(\mathrm{x})\mathrm{dx}$, FTC2 says to:

1. Find F, the antiderivative of f.
2. Evaluate F at the upper (b) and lower limits (a).
3. Subtract the value at the lower limit from the one at the upper limit; that is, $F(\mathrm{b}) - F(\mathrm{a})$.
4. $\int_{\mathrm{a}}^{\mathrm{b}} \mathrm{f}(\mathrm{x})\mathrm{dx} = FF(\mathrm{b}) - FF(\mathrm{a})$.

FTC2 AND TWO OTHER THEOREMS

The only difference between FTC2 and the net change theorem (section 5.3) is that the net change theorem integrates $F'(x)$ but FTC2 integrates $f(x)$. FTC1 just showed that $F'(x) = f(x)$. Therefore, wherever you would use the net change theorem, you could use FTC2.

---

[4] **Antiderivative**: the inverse (opposite) of the derivative. If f is the derivative of F, then F is an antiderivative of f.

[5] **Limits of integration**: the region where the integral is being found. For example, in $\int_{\mathrm{a}}^{\mathrm{b}} \mathrm{f}(\mathrm{x})\mathrm{dx}$, the limits of integration are a and b. Specifically, a is called the **lower limit** and b is the **upper limit**.

FTC2 is the same as the evaluation theorem *(section 5.3)*. It says exactly what FTC2 says.

WHY IS FTC2 TRUE?

FTC2 says that $\int_a^b f(x)dx = F(b) - F(a)$. We can see if this is true by simplifying the left side to make it equal to the right side. Here are the steps I used to reach this conclusion:

1. **START WITH TRUTH – NET CHANGE THEOREM**: *(section 5.3)*, net change theorem says $\int_a^b F'(x)dx = F(b) - F(a)$.
2. **SIMPLIFY**: *(section 4.7)*: since $F'(x) = f(x)$, then $\int_a^b F'(x)dx = \int_a^b f(x)dx = F(b) - F(a)$.

■ FTC2 is true.

MEAN VALUE THEOREM FOR INTEGRALS

Average Value *(property)* – the mean value of a function within a certain interval.

NOTATION FOR AVERAGE VALUE: To express the average value of a function f, write f_{ave}.

Mean Value Theorem for Integrals *(theorem)* – a theorem that says, for a function f continuous in an interval, there is a number c where $f(c)$ equals the function's average in that interval.

HOW WOULD YOU CALCULATE THE AVERAGE VALUE OF A FUNCTION?

To find the average value of f on the interval $[a, b]$, use this equation:

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx$$

LOOK IN THE BOOK: for proof of why this is true and example of calculating the average of a function, see subheading "Average Value of a Function" *(section 5.4)*.

WHAT DOES THE MEAN VALUE THEOREM FOR INTEGRALS TELL?

The mean value theorem for integrals tells us that if a function f is continuous on interval $[a, b]$, there is some number c in $[a, b]$ such that $f(c) = f_{ave}$.

Using algebra, the mean value theorem can be used to find just where in the function's domain[6] does the function equal its average. It guarantees that such a value *does* exist.

WHERE WOULD YOU USE THE MEAN VALUE THEOREM FOR INTEGRALS?

Whenever you want to find where a function equals its average in a certain interval, you might use the mean value theorem for integrals. For example:

1. **TYPING**: Matthew is writing a report in statistics class. He doesn't have a lot of practice, so he feels like he is always doing worse than normal. He wants to know if that's true.
2. **ROLLER COASTERS**: Josh and his friends visited an amusement park. One ride was called the *Reaxelerator*. The *Reaxelerator's* speed at t fractions of the one-minute-ride in is $v(t) = t^2 - t^3$. Afterward, when they were driving home, Josh wondered whether the *Reaxelerator* was ever at an average speed, and if so, where.

[6] **Domain**: the x-values that a function can work with; what can go in to it where something will come out. For example, in $f(x) = 1/x$, the domain would be all real numbers except 0, because $1/0$ is undefined.

HOW WOULD YOU USE THE MEAN VALUE THEOREM FOR INTEGRALS?

The mean value theorem for integrals says that if a function f is continuous on the interval $[\mathrm{a}, \mathrm{b}]$, then there is some number c in $[\mathrm{a}, \mathrm{b}]$ such that $f(\mathrm{c}) = \mathrm{f}_{\mathrm{ave}}$. Thus, you might use the mean value theorem for integrals to do either:

1. Be assured that there is an instance of the average value in such a function's range.
2. Find where such a function equals its average value in an interval.

LOOK IN THE BOOK: for examples of the mean value theorem for integrals, see example 7 *(section 5.4)*.

MEAN VALUE THEOREM FOR INTEGRALS AT WORK!

1. **TYPING**: *Matthew … feels like he is always doing worse than normal [at typing]. He wants to know if that's true.* Suppose we could make a function that measures Matthew's performance at typing. The mean value theorem for integrals says that there is some point of time where that function's average (or, "normal") is a real point. So, Matthew is not always typing worse than normal.
2. **ROLLER COASTERS**: *The Reaxelerator's speed at t fractions of the one-minute-ride in is $v(t) = t^2 - t^3$ … Josh wondered whether the Reaxelerator was ever at an average speed, and if so, where.* The mean value theorem assures us that *yes*, there was *some* moment in time where the *Reaxelerator* was going at average speed. What was it? Josh calculates the average speed: $v_{\mathrm{ave}} = \frac{1}{1-0}\int_0^1 \mathrm{v(t)dt} = \frac{1}{3}\mathrm{t}^3 - \frac{1}{4}\mathrm{t}^4\Big]_0^1 = \frac{1}{12}$. He then solves for t in the equation $v(\mathrm{t}) = \frac{1}{12}$. Using an online graphing calculator, he finds that $t \approx 0.3612$ and 0.8963. Multiplying by 60 to convert to seconds, Josh finds that the *Reaxelerator* was at average speed at 22 seconds and at 54 seconds into the ride.

WHY IS THE MEAN VALUE THEOREM FOR INTEGRALS TRUE?

The mean value theorem for integrals says that $f(\mathrm{c}) = \mathrm{f}_{\mathrm{ave}} = \frac{1}{\mathrm{b-a}}\int_{\mathrm{a}}^{\mathrm{b}} \mathrm{f(t)dt}$. If this equation can be derived from another equation, one established to be true, then the mean value theorem for integrals is true. I found that these steps show it to be true:

1. **START WITH TRUTH – MEAN VALUE THEOREM FOR DERIVATIVES**: *(section 4.2)*: For a function f continuous on $[\mathrm{a}, \mathrm{b}]$, there is some number c such that $f(\mathrm{b}) - f(\mathrm{a}) = \mathrm{f}'(\mathrm{c})(\mathrm{b} - \mathrm{a})$.
2. **TAKE THE ANTIDERIVATIVES**: *(section 4.7)*: $F(\mathrm{b}) - F(\mathrm{a}) = f(\mathrm{c})(\mathrm{b} - \mathrm{a})$.
3. **SUBSTITUTE – FTC1**: *(section 5.4)*: $\int_{\mathrm{a}}^{\mathrm{b}} \mathrm{f(x)dx} = f(\mathrm{c})(\mathrm{b} - \mathrm{a})$.
4. **USE ALGEBRA**: $f(\mathrm{c}) = \frac{1}{\mathrm{b-a}}\int_{\mathrm{a}}^{\mathrm{b}} \mathrm{f(x)dx}$.

■ The mean value theorem for integrals is true.

WHAT DO YOU THINK?

- What is the fundamental theorem of calculus? What does part 1 do? part 2? Why are they true?
- Where would you use the fundamental theorem of calculus? Why is it so important to calculus?
- What is the average of a function? How do you find it?
- What does the mean value theorem for integrals tell? How does it do this? Why is it true?