The past sections have taught techniques of integration for a few special cases; however, what do you do to integrate the product of two functions? This section will show *the substitution rule*, a method of simplifying such integrals for solution. In addition, it will show how the substitution rule can simplify definite integrals, even with special cases for symmetric functions.

THE SUBSITUTION RULE

The Substitution Rule, Integration by Substitution *(method)* – integrating by first substituting expressions to simplify the integrand, then solving with the evaluation theorem.

WHAT DOES INTEGRATION BY SUBSTITUTION MEAN?

Often, an integral will have no entry in the table of integrals *(section 5.3)*. Integration by substitution makes it possible to solve these kinds of integrals by first replacing expressions that can't be immediately integrated with those that can, then integrating the result.

The substitution rule tells us that if $u = g(x)$:

$$\int \mathrm{f}(\mathrm{g}(\mathrm{x}))\mathrm{g}'(\mathrm{x})\mathrm{dx} = \int \mathrm{f}(\mathrm{u})\mathrm{du}$$

Of course, the domain of f must contain interval I, g must have I in its domain and range, and $\mathrm{g}'(\mathrm{x})$ must have interval I as its domain.

WHERE WOULD YOU USE THE SUBSTITUTION RULE?

Because the substitution rule can integrate the product of functions, you might use it whenever you need to integrate two functions that are multiplied by another. For example:

1. **CLEANING**: Professor Chojnacki is re-organizing his room. The speed he organizes at (in $\mathrm{items/minute}$) when t minutes into the work is governed by $y' = \frac{\mathrm{x}+1}{\mathrm{x}+2} + \ln(x+2)$. How many items has he organized when 40 minutes into cleaning?
2. **PURE MATHEMATICS**: If $f(\mathrm{x}) = (\mathrm{x}+1)^3$, find $\int [\mathrm{f}^2(\mathrm{x}) \cdot 3(\mathrm{x}+1)^2]\mathrm{dx}$.

HOW WOULD YOU DO USE THE SUBSTITUTION RULE?

To use the substitution rule to simplify an integral:

1. Look at the integrand. See if the integrand can be split in half (i.e., by drawing a line through its factors) in a way that the antiderivative of one half can be the argument for the other half.
2. Call the antiderivative of that half u.
3. Replace dx with du at the end of the integral.
4. Remove u as a factor from the integrand.

$$\boxed{\begin{gathered}\int \mathrm{f}(\mathrm{g}(\mathrm{x}))\mathrm{g}'(\mathrm{x})\mathrm{dx} \\ u = g(x)\end{gathered}}$$

To use the substitution rule, find some part of the factors and call its antiderivative u.

LOOK IN THE BOOK: for examples of using the substitution rule, see examples 1-5 *(section 5.5)*.

SUBSTITUTION RULE AT WORK!

1. **CLEANING**: The speed he organizes at (in $\mathrm{items/minute}$) when t minutes into the work is governed by $y' = 9(\ln x + 2)^2 \left(\frac{1}{x+2}\right)$. How many items has he organized when $e - 2$ minutes into cleaning?

   $$y = \int \left[9(\ln x + 2)^2 \left(\frac{1}{x+2}\right)\right] \mathrm{dx}$$

   One expression that could be u is $\ln(\mathrm{x}+2)$.

   Applying the substitution rule yields

$$y = \int 9\mathrm{u}^2\mathrm{du}$$
$$y = 3\mathrm{u}^3$$
$$y = 3(\ln[\mathrm{x}+2])^3$$

At $e-2$ minutes into cleaning, then, Professor Chojnacki has cleaned $3(\ln[\mathrm{e}-2+2])^3 = 3$ items.

2. **PURE MATHEMATICS**: If $f(\mathrm{x}) = (\mathrm{x}+1)^3$ and $y = \int[\mathrm{f}^2(\mathrm{x}) \cdot 3(\mathrm{x}+1)^2]\mathrm{dx}$, find y.
One expression for u is $u = f(\mathrm{x}) = (\mathrm{x}+1)^3$.

$$y = \int [\mathrm{f}^2(\mathrm{x}) \cdot 3(\mathrm{x}+1)^2]\mathrm{dx}$$
$$y = \int \mathrm{u}^2\mathrm{du}$$
$$y = \frac{1}{3}\mathrm{u}^3$$
$$y = \frac{1}{3}([\mathrm{x}+1]^3)^3$$
$$y = \frac{1}{3}(x+1)^9$$

WHY IS THE SUBSTITUTION RULE TRUE?

The substation rule is built from the chain rule. If the chain rule can be converted to the substitution rule, then the substitution rule must be true. Here are the steps I used to reach this conclusion:

1. **START WITH TRUTH – CHAIN RULE**: *(section 2.5)*: $\frac{d}{dx}[\mathrm{f}(g(x))] = f'(g(x))g'(x)$.
2. **MODIFY VARIABLES**: *(step 1)*: $\frac{d}{dx}[\mathrm{F}(g(x))] = \mathrm{F}'(g(x))g'(x) = f(g(x))g'(x)$.
3. **MAKE INDEFINITE INTEGRAL**: *(section 5.3, step 2)*: $\int\left[\frac{d}{dx}[\mathrm{F}(g(x))]\mathrm{dx}\right] = \int[\mathrm{F}'(g(x))g'(x)dx]$.
4. **SIMPLIFY BY DEFINITION**: *(section 5.3, step 3)*: $F(g(x)) = \int[F'(g(x))g'(x)dx]$.
5. **SUBSTITUTE**: *(step 2)*: $u = g(x)$.
6. **SUBSTITUTE**: *(steps 4, 5)*: $F(g(x)) = F(u)$.
7. **EXPOUND UPON WHAT IS KNOWN**: *(step 6)*: $F(u) = \int\left[\frac{d}{du}[F(u)]du\right] = \int[F'(u)du]$.
8. **SUBSTITUE**: *(steps 4, 6, 7)*: $\int[F'(g(x))g'(x)dx] = F(g(x)) = F(u) = \int[F'(u)du]$.
9. **STATE CONCLUSION**: *(steps 5, 8)*: If $u = g(x)$, then $\int[f(g(x))g'(x)dx] = \int[f(u)du]$.

■ The substitution rule is true.

THE SUBSTITUTION RULE FOR DEFINITE INTEGRALS

The Substitution Rule for Definite Integrals *(method)* – simplifying definite integrals that have a common, innermost function in their integrand by moving it into the limits of integration.

WHAT DOES THE SUBSTITUTION RULE FOR DEFINITE INTEGRALS TELL US?

Sometimes a definite integral will have an integrand that repeatedly uses some common function $g(x)$ instead of standalone x. In such an integral, x will never occur alone; it will always be inside $g(x)$. These kinds of integrals can be simplified this way, substituting $u = g(x)$:

$$\int_a^b f(g(x))dx = \int_{g(a)}^{g(b)} f(u)du$$

There are only two requirements:

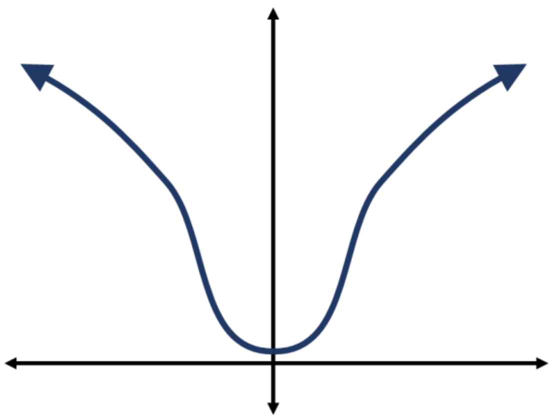
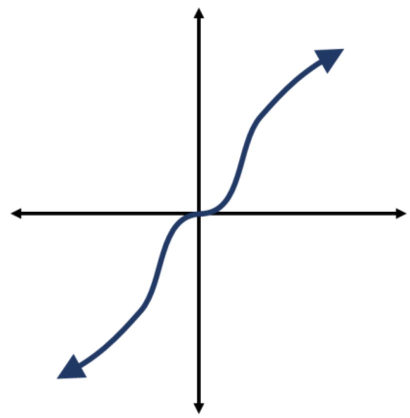
- g' must be continuous on $[a, b]$.
- f must be continuous on the range of $y = g'$.

SYMMETRY

Symmetry *(property)* – whether a function mirrors in some way across the x- and possibly y-axis.

WHAT KINDS OF SYMMETRY ARE THERE?

There are two kinds of symmetry, each with its own shortcut for calculating certain definite integrals:

Even	Odd
	
$f(-x) = f(x)$	$f(-x) = -f(x)$
$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$	$\int_{-a}^{a} f(x)dx = 0$
Area under the curve is combined (constructive interference), so area of whole is double of half.	Area under the curve is cancelled (destructive interference), so area of whole is zero.

WHAT DID YOU LEARN?

- What is the substitution rule? How do you use it to simplify an integral? Where would you?
- Why is the substitution rule true? How does this establish the substitution rule for definite integrals?
- What is the symmetry of a function? What kinds are there? How does it describe area under the curve?