MTH 659 Computational Wave Propagation Assignment 4

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1 Eigenvalue Problem for Plane Wave

Given

$$\begin{aligned} u_p &= \alpha_1 e^{i(pkh-\omega_h t)} \\ u_{p,1} &= \alpha_2 e^{i((p+\frac{1}{2})kh-\omega_h t)} \end{aligned}$$

Using the Maxima computer algebra system we implemented the algebraic simplification as below:

$$u_p = \alpha_1\, e^{i\,(h\,k\,p-\omega\,t)}$$

$$\frac{d^2 u}{dt^2} = -\alpha_1\,\omega^2\,e^{i\,(h\,k\,p-\omega\,t)}$$

Using the mass-lumped expression:

$$\frac{d^2 u}{dt^2} = \alpha_1\,e^{i\,(h\,k\,(p+1)-\omega\,t)} - \frac{c^2\,\left(14\,\alpha_1\,e^{i\,(h\,k\,p-\omega\,t)} - 8\,\left(\alpha_2\,e^{i\,(h\,k\,(p+0.5)-\omega\,t)} + \alpha_2\,e^{i\,(h\,k\,(p-0.5)-\omega\,t)}\right)\right)}{h^2} + \alpha_1\,e^{i\,(h\,k\,(p-1)-\omega\,t)}$$

$$\frac{d^2 u_{p,1}}{dt^2} = -\alpha_2\,\omega^2\,e^{i\,(h\,k\,(p+0.5)-\omega\,t)}$$

And the corresponding mass-lumped expression:

$$\frac{d^2 u_{p,1}}{dt^2} = \frac{4\,c^2\,\left(\alpha_1\,e^{i\,(h\,k\,(p+1)-\omega\,t)} - 2\,b\,e^{i\,(h\,k\,(p+0.5)-\omega\,t)} + \alpha_1\,e^{i\,(h\,k\,p-\omega\,t)}\right)}{h^2}$$

Equating the expression for $\frac{d^2 u}{dt^2}$

$$-\frac{\left(\alpha_1\,h^2\,e^{\frac{3\,i\,h\,k}{2}}\,\omega^2 + \alpha_1\,h^2\,e^{\frac{5\,i\,h\,k}{2}} + e^{i\,h\,k}\,\left(8\,\alpha_2\,c^2\,e^{i\,h\,k} - 14\,\alpha_1\,c^2\,e^{\frac{i\,h\,k}{2}} + 8\,\alpha_2\,c^2\right) + \alpha_1\,h^2\,e^{\frac{i\,h\,k}{2}}\right)\,e^{-i\,\omega\,t+i\,h\,k\,p-\frac{3\,i\,h\,k}{2}}}{h^2} = 0$$

Rearranging the above we get:
Equating the expression for $\frac{d^2 u_{p,1}}{dt^2}$

$$-\frac{\left(\alpha_2\,h^2\,e^{\frac{i\,h\,k}{2}}\,\omega^2 + 4\,\alpha_1\,c^2\,e^{i\,h\,k} - 8\,\alpha_2\,c^2\,e^{\frac{i\,h\,k}{2}} + 4\,\alpha_1\,c^2\right)\,e^{i\,h\,k\,p-i\,\omega\,t}}{h^2} = 0$$

Simplifying the above:

$$\begin{aligned} -\alpha_1 e^{-i\omega t+ihkp+ihk} - \frac{8\alpha_2 c^2 e^{-i\omega t+ihkp+\frac{ihk}{2}}}{h^2} &- \frac{8\alpha_2 c^2 e^{-i\omega t+ihkp-\frac{ihk}{2}}}{h^2} \\ &- \alpha_1 e^{-i\omega t+ihkp-ihk} - \alpha_1 \omega^2 e^{ihkp-i\omega t} + \frac{14\alpha_1 c^2 e^{ihkp-i\omega t}}{h^2} \\ &= 0 \end{aligned} \tag{1}$$

Simplifying further we get:

$$\omega_h^2 \alpha_1 = \frac{c^2}{h^2}(14\alpha_1 - 8\alpha_2(e^{ikh/2} + e^{-ikh/2}) + \alpha_1(e^{ikh} + e^{-ikh})) \tag{2}$$

as required.
Similarly for the second equation, we simplify it further to get:

$$-\frac{4\alpha_1 c^2 e^{-i\omega t+ihkp+ihk}}{h^2} - \alpha_2 \omega^2 e^{-i\omega t+ihkp+\frac{ihk}{2}} + \frac{8\alpha_2 c^2 e^{-i\omega t+ihkp+\frac{ihk}{2}}}{h^2} - \frac{4\alpha_1 c^2 e^{ihkp-i\omega t}}{h^2} = 0$$

Simplifying further by dividing the equation by $e^{i(hpk+hk+\omega t)}$, we get:

$$\omega_h^2 \alpha_2 = -\frac{4c^2}{h^2}(\alpha_1 e^{-ikh/2} - 2\alpha_2 + \alpha_1 e^{ikh/2}) \tag{3}$$

The Maxima code for the simplification is given below:

```
/* File     : wave.mac
   Author   : Sandeep Koranne (C) 2017. All rights reserved.
   Purpose  : 1D Wave propagation Symbol analysis
   Notation: a=alpha1, b=alpha2
*/

UP : a*exp(%i*(p*k*h - omega*t));
UPM1 : a*exp(%i*((p-1)*k*h - omega*t));
UPP1 : a*exp(%i*((p+1)*k*h - omega*t));

UP1 : b*exp(%i*((p+0.5)*k*h - omega*t));
UP1M1 : b*exp(%i*((p-0.5)*k*h - omega*t));

DP_EXP : diff( UP, t, 2);
DP1_EXP: diff( UP1, t, 2);

DP : (-c^2/h^2)*(14*UP - 8*(UP1M1 + UP1)) + UPP1 + UPM1;
DP1: (4*c^2/h^2)*( UP - 2*UP1 + UPP1);


S1 : DP_EXP - DP=0;
S1SIMP : ratsimp(expand(S1));
```

```
print(tex(expand(S1SIMP)));

S2 : DP1_EXP - DP1=0;
S2SIMP : ratsimp(expand(S2));

M : matrix( [q*(14 + 2*cos(k*h)), q*(-16*cos(k*h/2))],
            [q*(-8*cos(k*h/2)), q*8] );
N : charpoly( M, lambda);
print(N);

E: h^4*omegahp^2 + 4*c^2*h^2*(omega^2-6)*omegahp + 96*c^4*omega^2=0;
ES:solve(E,omegahp);
ESSOLN : map(rhs, ES);
print(ESSOLN);

ES1 : subst(omega=sin(kh/2),ESSOLN);
print(ES1);

print(tex(ratsimp(expand(ES1[1]))));
print(tex(ratsimp(expand(ES1[2]))));

T1 : taylor( ES1[1],kh,0,8 );
T2 : taylor( ES1[2],kh,0,8 );

print(tex(T1));
print(tex(T2));


NHATORIG : matrix( [14 + 2*cos(k*h), -16*cos(k*h/2)],
            [-8*cos(k*h/2), 8] );

NHAT : subst(k=10,NHATORIG);
print(NHAT);
[EVAL,EVEC] : eigenvectors(NHAT);
print(tex(EVAL[1]));
print(tex(EVEC[1]));

print(tex(EVAL[2]));
print(tex(EVEC[2]));
```

The equations above in Equation 1 and Equation 3, can be written in matrix notation as:

$$\omega_h^2 \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \frac{c^2}{h^2} \begin{bmatrix} 14 + 2\cos kh & -16\cos\frac{kh}{2} \\ -8\cos\frac{kh}{2} & 8 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

The right hand side matrix coefficient can be factored as

$$\frac{c^2}{h^2}\begin{bmatrix} 14+2\cos kh & -16\cos\frac{kh}{2} \\ -8\cos\frac{kh}{2} & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\frac{2c^2}{h^2}\begin{bmatrix} 7+\cos kh & -8\cos\frac{kh}{2} \\ -8\cos\frac{kh}{2} & 8 \end{bmatrix}$$

Substituting above we get:

$$\omega_h^2\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \frac{2c^2}{h^2}\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 7+\cos kh & -8\cos\frac{kh}{2} \\ -8\cos\frac{kh}{2} & 8 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \tag{4}$$

The above equation is a generalized eigenvalue equation with eigenvalue ω_h^2, the characteristic polynomial can be written as below:

$$(8q-\omega_h^2)((2\cos(kh)+14)q--\omega_h^2)-128*\cos^2(\frac{kh}{2}q^2)$$

where $q=\frac{c^2}{h^2}$. Let $\omega=\sin(\frac{kh}{2})$, then $\cos(kh)=1-2\omega^2$, and $\cos^2(kh/2)=1-\omega^2$. Substituting above and simplifying:

$$(8q-\omega_h^2)(-\omega_h^2+2-4\omega^2)+14)-128+128\omega^2=0$$

Expanding $q=\frac{c^2}{h^2}$, we get

$$\omega_h^4h^4+4c^2h^2(\omega^2-6)\omega_h^2+96c^4\omega^2=0$$

1.1 Solution to the numerical dispersion

Solving the above equation (again using Maxima) we get:

$$\left[\omega_{h,1}^2=-\frac{2\,c^2\,\sqrt{\omega^4-36\,\omega^2+36}+2\,c^2\,\omega^2-12\,c^2}{h^2},\omega_{h,2}^2=\frac{2\,c^2\,\sqrt{\omega^4-36\,\omega^2+36}-2\,c^2\,\omega^2+12\,c^2}{h^2}\right]$$

1.2 Taylor Expansion

We now substitute $\omega=sin(kh/2)$, to get:

$$\omega_{h,1}^2=-\frac{2\,c^2\,\sqrt{\sin^4\left(\frac{h\,k}{2}\right)-36\,\sin^2\left(\frac{h\,k}{2}\right)+36}+2\,c^2\,\sin^2\left(\frac{h\,k}{2}\right)-12\,c^2}{h^2}$$

and

$$\omega_{h,2}^2=\frac{2\,c^2\,\sqrt{\sin^4\left(\frac{h\,k}{2}\right)-36\,\sin^2\left(\frac{h\,k}{2}\right)+36}-2\,c^2\,\sin^2\left(\frac{h\,k}{2}\right)+12\,c^2}{h^2}$$

Performing Taylor expansion (in kh) using Maxima, we get:

$$\omega_{h,1}^2=\frac{c^2\,(kh)^2}{h^2}-\frac{c^2\,(kh)^6}{1440\,h^2}-\frac{c^2\,(kh)^8}{48384\,h^2}+\cdots$$

and

$$\omega_{h,2}^2=\frac{24\,c^2}{h^2}-\frac{2\,c^2\,(kh)^2}{h^2}+\frac{c^2\,(kh)^4}{12\,h^2}-\frac{c^2\,(kh)^6}{480\,h^2}+\frac{17\,c^2\,(kh)^8}{241920\,h^2}+\cdots$$

as required.

2 Eigenvalues and Eigenvectors of $\hat{N}_{1,2}$

Assume $c = 1$. For different values of $k = (1, 5, 10)$, define $N_p := \frac{kh}{2\pi r}$, where $r = 2$ for P2 FEM.
Above we have calculated $\hat{N}_{1,2}$ as:

$$\hat{N}_{1,2} = \frac{c^2}{h^2}\begin{bmatrix} 14 + 2\cos kh & -16\cos\frac{kh}{2} \\ -8\cos\frac{kh}{2} & 8 \end{bmatrix}$$

For $k = 1$, we compute the eigenvalues and eigenvectors as below:

$$\left[-\sqrt{\cos^2 h + 6\,\cos h + 128\,\cos^2\left(\frac{h}{2}\right) + 9} + \cos h + 11, \sqrt{\cos^2 h + 6\,\cos h + 128\,\cos^2\left(\frac{h}{2}\right) + 9} + \cos h + 11\right]$$

$$\left[\left[1, \frac{\sqrt{\cos^2 h + 6\,\cos h + 128\,\cos^2\left(\frac{h}{2}\right) + 9} + \cos h + 3}{16\,\cos\left(\frac{h}{2}\right)}\right]\right]$$

$$[1, 1]$$

$$\left[\left[1, -\frac{\sqrt{\cos^2 h + 6\,\cos h + 128\,\cos^2\left(\frac{h}{2}\right) + 9} - \cos h - 3}{16\,\cos\left(\frac{h}{2}\right)}\right]\right]$$

For $k = 5$:

$$\left[-\sqrt{\cos^2(5\,h) + 6\,\cos(5\,h) + 128\,\cos^2\left(\frac{5\,h}{2}\right) + 9} + \cos(5\,h) + 11, \sqrt{\cos^2(5\,h) + 6\,\cos(5\,h) + 128\,\cos^2\left(\frac{5\,h}{2}\right) + 9} + \cos\right.$$

$$\left[\left[1, \frac{\sqrt{\cos^2(5\,h) + 6\,\cos(5\,h) + 128\,\cos^2\left(\frac{5\,h}{2}\right) + 9} + \cos(5\,h) + 3}{16\,\cos\left(\frac{5\,h}{2}\right)}\right]\right]$$

$$[1, 1]$$

$$\left[\left[1, -\frac{\sqrt{\cos^2(5\,h) + 6\,\cos(5\,h) + 128\,\cos^2\left(\frac{5\,h}{2}\right) + 9} - \cos(5\,h) - 3}{16\,\cos\left(\frac{5\,h}{2}\right)}\right]\right]$$

For $k = 10$:

$$\left[-\sqrt{\cos^2(10\,h) + 6\,\cos(10\,h) + 128\,\cos^2(5\,h) + 9} + \cos(10\,h) + 11, \sqrt{\cos^2(10\,h) + 6\,\cos(10\,h) + 128\,\cos^2(5\,h) + 9} + \cos\right.$$

$$\left[\left[1, \frac{\sqrt{\cos^2(10\,h) + 6\,\cos(10\,h) + 128\,\cos^2(5\,h) + 9} + \cos(10\,h) + 3}{16\,\cos(5\,h)}\right]\right]$$

$$[1, 1]$$

$$\left[\left[1, -\frac{\sqrt{\cos^2(10\,h) + 6\,\cos(10\,h) + 128\,\cos^2(5\,h) + 9} - \cos(10\,h) - 3}{16\,\cos(5\,h)}\right]\right]$$

2.1 Plot of frequency of physical wave

Plot the error in the physical frequency, i.e., $\frac{\omega_{h,1}^2}{k^2} - 1$ against N_p, and illustrate the fourth order approximation (using a loglog plot). Since $c = 1$, $kh/2 = 2\pi N_p$, and $kh = 4\pi N_p$, therefore the Taylor expansions can be written as:

$$\frac{\omega_{h,1}^2}{k^2} - 1 = -(\frac{(kh)^4}{1440} + \frac{(kh)^6}{48384}) + O((kh)^8)$$

This was implemented and the resulting plot is shown in Figure 2.1 and we can observe fourth order approximation.

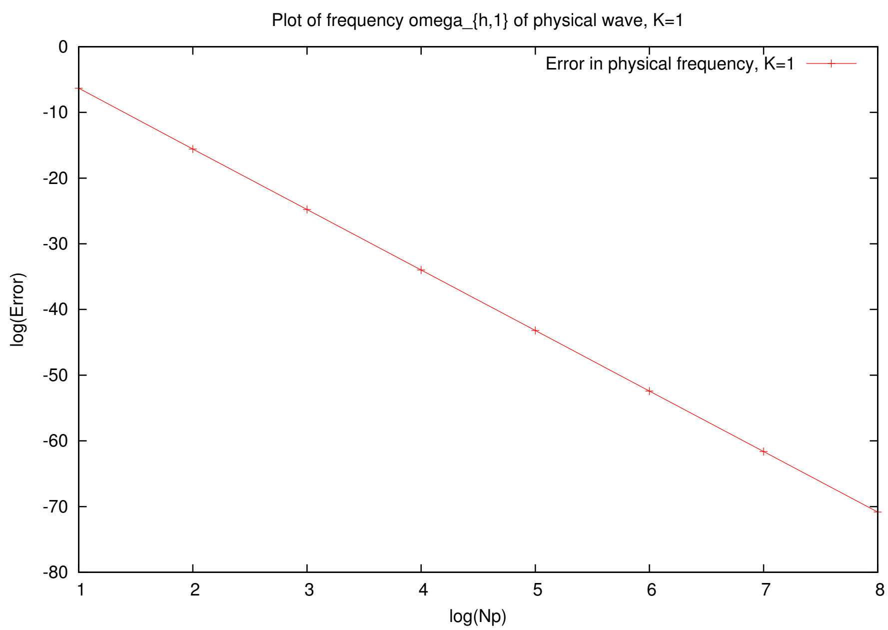

Figure 1: Plot of frequency $\omega_{h,1}$ of physical wave.

2.2 Plot of frequency of physical wave

Plot the error in the physical frequency, i.e., $\frac{\omega_{h,2}^2}{k^2}$ against N_p, and illustrate that this quantity goes to infinity as h goes to zero (using a loglog plot). Since $c = 1$, $kh/2 = 2\pi N_p$, and $kh = 4\pi N_p$, therefore the Taylor expansions can be written as:

$$\frac{\omega_{h,2}^2}{k^2} = (\frac{24}{k^2h^2} - 2 + \frac{(kh)^2}{12} - \frac{(kh)^4}{480} + O((kh)^6)$$

This was implemented and the resulting plot is shown in Figure 2.2 and we can observe the error going to infinity as $h \to 0$.

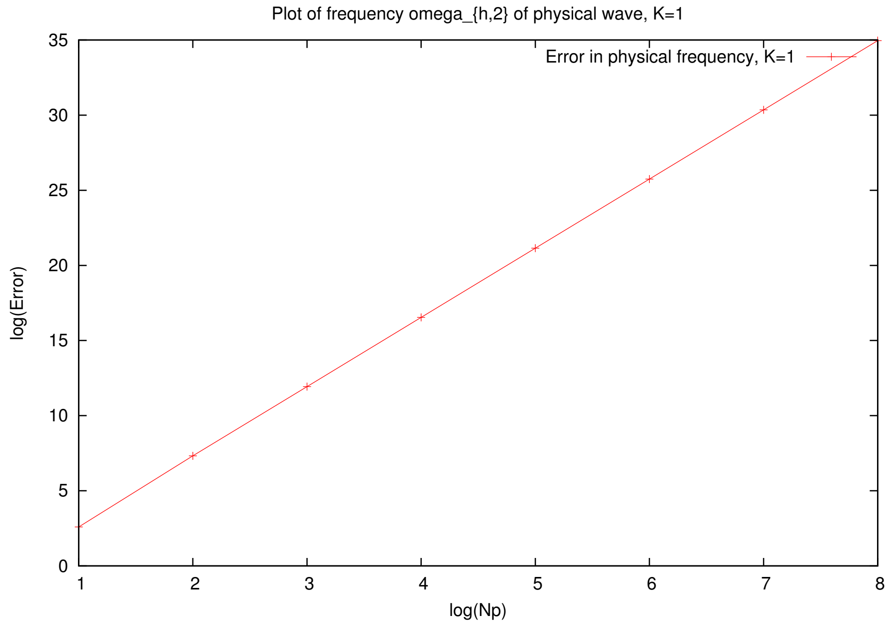

Figure 2: Plot of frequency $\omega_{h,2}$ of physical wave.

2.3 Plot of amplitude of parasitic wave

Given on pp. 198, we see:

$$\lambda_2 = k^2((\frac{24}{k^2h^2} - 2 + \frac{(kh)^2}{12} - \frac{(kh)^4}{480}$$

We have computed the eigenvalues and eigenvectors for $\omega_{h,2}$ and got the same values as shown in the text (see Maxima computation output above). Therefore $W_2 - Y_0$ can be computed as:

$$W_2 - Y_0 = -\frac{\sqrt{2}}{1152}(kh)^4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 2 * \frac{\sqrt{2}}{13824}(kh)^6 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

As recommended we use $Y_0 = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} \end{bmatrix}$. This was implemented and the resulting plot is shown in Figure 2.3 and we can observe the error showing fourth order decay.

2.4 Analysis of results

The amplitude of the parasitic wave is $O(h^4)$, which reduces the problem, but nevertheless the presence of this parasitic wave is a problem for the efficacy of the P2 FEM solution.
The implementation was completed in the Julia language and the code is shown in Listing 1.

```
# File   : taylor.jl
```

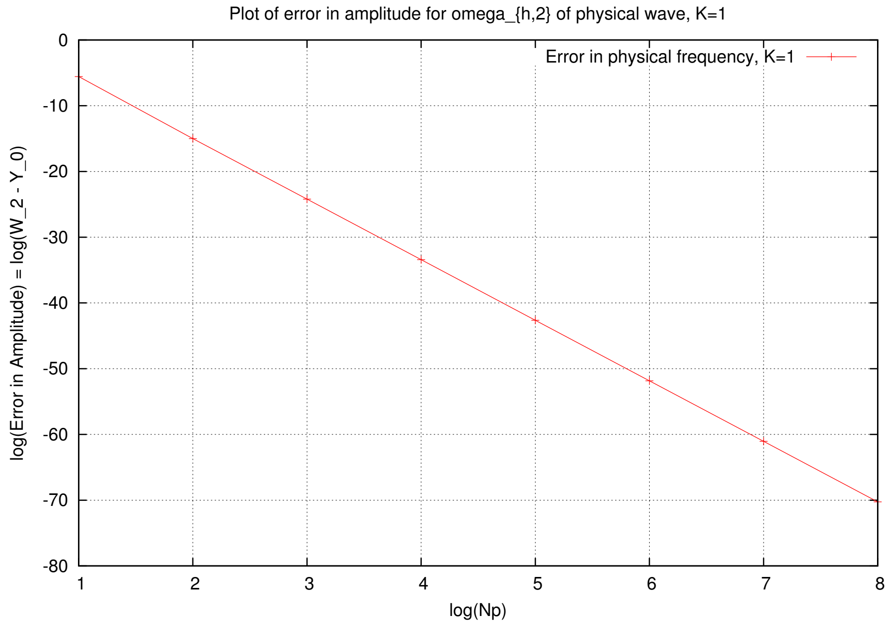

Figure 3: Plot of error in amplitude $\omega_{h,2}$ of physical wave.

```
# Author: Sandeep Koranne
#
K = 1;

#f(h) = ( (K^4*h^4)/1440 + (K^6*h^6)/48384);
#f(h) = ( 24/(K^2*h^2) - 2 + (K^2*h^2)/12 - (K^4*h^4)/480);
f(h) = sqrt(2)/1152*h^4 + 2*sqrt(2)/13824*h^6;
ANS=zeros(8,2);
for NP in collect(1:8)
    NPVALUE = 10^NP;
    h = 4*pi*K/NPVALUE;
     println(h);
    ANS[NP,1] = NP;
    ANS[NP,2] = log(f(h));
end
writedlm( "K3.txt", ANS, " " );
```

Listing 1: Implementation of P2 FEM for 1D wave propagation.