STAT 600 - HW 3

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All Rcpp/RcppArmadillo can be found in my GitHub.

Derivation

First, I will derive the log-likelihood function given the completed data, $l\left(p, \lambda, \mu | \boldsymbol{Y}, \boldsymbol{\delta}\right)$.

$$\begin{aligned}
l\left(p, \lambda, \mu | \boldsymbol{Y}, \boldsymbol{\delta}\right) &= \log\left(L\left(p, \lambda, \mu | \boldsymbol{Y}, \boldsymbol{\delta}\right)\right) \\
&= \log\left(\prod_{i=1}^{n} p\left(Y_i, \delta_i | p, \lambda, \mu\right)\right) \\
&= \log\left(\prod_{i=1}^{n} \left(p\lambda \exp\left(-\lambda y_i\right)\right)^{\delta_i} \left(\left(1-p\right)\mu \exp\left(-\mu y_i\right)\right)^{1-\delta_i}\right) \\
&= \sum_{i=1}^{n} \log\left\{\left(p\lambda \exp\left(-\lambda y_i\right)\right)^{\delta_i} \left(\left(1-p\right)\mu \exp\left(-\mu y_i\right)\right)^{1-\delta_i}\right\} \\
&= \sum_{i=1}^{n} \left[\delta_i \left\{-\lambda y_i + \log\left(p\lambda\right)\right\} + \left(1-\delta_i\right)\left\{-\mu y_i + \log\left(\left(1-p\right)\mu\right)\right\}\right]
\end{aligned}$$

Then, the E-step in the EM algorithm is shown below.

$$\begin{aligned}
Q\left(p, \lambda, \mu | p^{(t)}, \lambda^{(t)}, \mu^{(t)}\right) &= E\left[l\left(p, \lambda, \mu | \boldsymbol{Y}, \boldsymbol{\delta}\right) | \boldsymbol{Y}, p^{(t)}, \lambda^{(t)}, \mu^{(t)}\right] \\
&= E\left[\sum_{i=1}^{n} \left[\delta_i \left\{-\lambda y_i + \log\left(p\lambda\right)\right\} + \left(1-\delta_i\right)\left\{-\mu y_i + \log\left(\left(1-p\right)\mu\right)\right\}\right] \middle| \boldsymbol{Y}, p^{(t)}, \lambda^{(t)}, \mu^{(t)}\right] \\
&= \sum_{i=1}^{n} E\left[\delta_i \left\{-\lambda y_i + \log\left(p\lambda\right)\right\} + \left(1-\delta_i\right)\left\{-\mu y_i + \log\left(\left(1-p\right)\mu\right)\right\} \middle| y_i, p^{(t)}, \lambda^{(t)}, \mu^{(t)}\right] \\
&= \sum_{i=1}^{n} \left\{\hat{\delta}_i^{(t)} \left[-\lambda y_i + \log\left(p\lambda\right)\right] + \left(1-\hat{\delta}_i^{(t)}\right)\left[-\mu y_i + \log\left(\left(1-p\right)\mu\right)\right]\right\}
\end{aligned}$$

where $\hat{\delta}_i^{(t)} = E\left[\delta_i | \, y_i, p^{(t)}, \lambda^{(t)}, \mu^{(t)}\right] = \frac{p^{(t)}\lambda^{(t)} \exp\left(-\lambda^{(t)} y_i\right)}{p^{(t)}\lambda^{(t)} \exp\left(-\lambda^{(t)} y_i\right) + \left(1-p^{(t)}\right)\mu^{(t)} \exp\left(-\mu^{(t)} y_i\right)}$

Then, I will derive the M-step by starting the finding the $p^{(t+1)}$.

$$\begin{aligned}
\frac{d}{dp} Q\left(p, \lambda, \mu | p^{(t)}, \lambda^{(t)}, \mu^{(t)}\right) &\overset{\text{set}}{=} 0 \\
\sum_{i=1}^{n} \left[\frac{\hat{\delta}_i^{(t)}}{p} - \frac{1-\hat{\delta}_i^{(t)}}{1-p} \right] &= 0 \\
\sum_{i=1}^{n} \left[(1-p)\hat{\delta}_i^{(t)} - \left(1-\hat{\delta}_i^{(t)}\right) p \right] &= 0 \\
\sum_{i=1}^{n} \hat{\delta}_i^{(t)} - np &= 0 \\
p^{(t+1)} &= \frac{1}{n} \sum_{i=1}^{n} \hat{\delta}_i^{(t)}
\end{aligned}$$

Below is the derivation for $\lambda^{(t+1)}$.

$$\begin{aligned}
\frac{d}{d\lambda} Q\left(p, \lambda, \mu | p^{(t)}, \lambda^{(t)}, \mu^{(t)}\right) &\overset{\text{set}}{=} 0 \\
\sum_{i=1}^{n} \left[-\hat{\delta}_i^{(t)} y_i + \frac{\hat{\delta}_i^{(t)}}{\lambda} \right] &= 0 \\
-\lambda \sum_{i=1}^{n} \hat{\delta}_i^{(t)} y_i + \sum_{i=1}^{n} \hat{\delta}_i^{(t)} &= 0 \\
\lambda^{(t+1)} &= \frac{\sum_{i=1}^{n} \hat{\delta}_i^{(t)}}{\sum_{i=1}^{n} \hat{\delta}_i^{(t)} y_i}
\end{aligned}$$

Lastly, this is the derivation for $\mu^{(t+1)}$

$$\begin{aligned}
\frac{d}{d\mu} Q\left(p, \lambda, \mu | p^{(t)}, \lambda^{(t)}, \mu^{(t)}\right) &\overset{\text{set}}{=} 0 \\
\sum_{i=1}^{n} \left[\left(1-\hat{\delta}_i^{(t)}\right) y_i + \frac{1-\hat{\delta}_i^{(t)}}{\mu} \right] &= 0 \\
-\mu \sum_{i=1}^{n} \left(1-\hat{\delta}_i^{(t)}\right) y_i + \sum_{i=1}^{n} \left(1-\hat{\delta}_i^{(t)}\right) &= 0 \\
\mu^{(t+1)} &= \frac{\sum_{i=1}^{n} \left(1-\hat{\delta}_i^{(t)}\right)}{\sum_{i=1}^{n} \left(1-\hat{\delta}_i^{(t)}\right) y_i}
\end{aligned}$$

Appendix

```r
knitr::opts_chunk$set(echo = FALSE)

library(tidyverse)
library(knitr)
library(Rcpp)
library(RcppArmadillo)
library(foreach)
library(doParallel)
library(ggplot2)
library(latex2exp)


path <- "/Users/kevin-imac/Desktop/Github - Repo/"
if(! file.exists(path)){
  path <- "/Users/kevinkvp/Desktop/Github Repo/"
}

# sourceCpp(paste0(path, "HW3EM/src/main.cpp"))

### User-defined functions ----------------------------------------------------
meanSD <- function(x, dplace = 5){
  mm <- round(mean(x), digits = dplace)
  ss <- round(sd(x), digits = dplace)
  paste0(mm, " (SD = ", ss, ")")
}
```