STAT 600 - HW 5

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All Rcpp/RcppArmadillo can be found in my GitHub.

(a)

Below is the derivation of the likelihood function, $p\left(\boldsymbol{X}|\theta, \lambda_1, \lambda_2, \alpha\right)$.

$$
\begin{aligned}
p\left(\boldsymbol{X}|\theta, \lambda_1, \lambda_2, \alpha\right) &= \prod_{j=1}^{112} P\left(X_j|\lambda_1, \lambda_2, \alpha\right) \\
&= \left[\prod_{j=1}^{\theta} \frac{e^{-\lambda_1}\lambda_1^{X_j}}{X_j!}\right]\left[\prod_{j=\theta+1}^{112} \frac{e^{-\lambda_2}\lambda_2^{X_j}}{X_j!}\right] \\
&= \frac{\lambda_1^{\sum_{j=1}^{\theta} X_j} e^{-\lambda_1\theta} \lambda_2^{\sum_{j=\theta+1}^{112} X_j} e^{-\lambda_2(112-\theta)}}{\prod_{j=1}^{112} X_j!} \\
&\propto \lambda_1^{\sum_{j=1}^{\theta} X_j} \lambda_2^{\sum_{j=\theta+1}^{112} X_j} e^{-\lambda_1\theta} e^{-\lambda_2(112-\theta)}
\end{aligned}
$$

In order to perform Gibbs sampler, we need four conditional probabilities derived below.

$$
\begin{aligned}
p\left(\theta|\lambda_1, \lambda_2, \alpha, \boldsymbol{X}\right) &\propto p\left(\boldsymbol{X}|\theta, \lambda_1, \lambda_2, \alpha\right) p\left(\theta\right) p\left(\lambda_1|\alpha\right) p\left(\lambda_2|\alpha\right) p\left(\alpha\right) \\
&\propto p\left(\boldsymbol{X}|\theta, \lambda_1, \lambda_2, \alpha\right) p\left(\theta\right) \\
&= \lambda_1^{\sum_{j=1}^{\theta} X_j} \lambda_2^{\sum_{j=\theta+1}^{112} X_j} e^{-\lambda_1\theta} e^{-\lambda_2(112-\theta)} \frac{1}{111} \mathbb{I}_{\theta \in \{1,2,\cdots,111\}} \\
&= \lambda_1^{\sum_{j=1}^{\theta} X_j} \lambda_2^{\sum_{j=\theta+1}^{112} X_j} e^{-\theta(\lambda_1-\lambda_2)} \mathbb{I}_{\theta \in \{1,2,\cdots,111\}}
\end{aligned}
$$

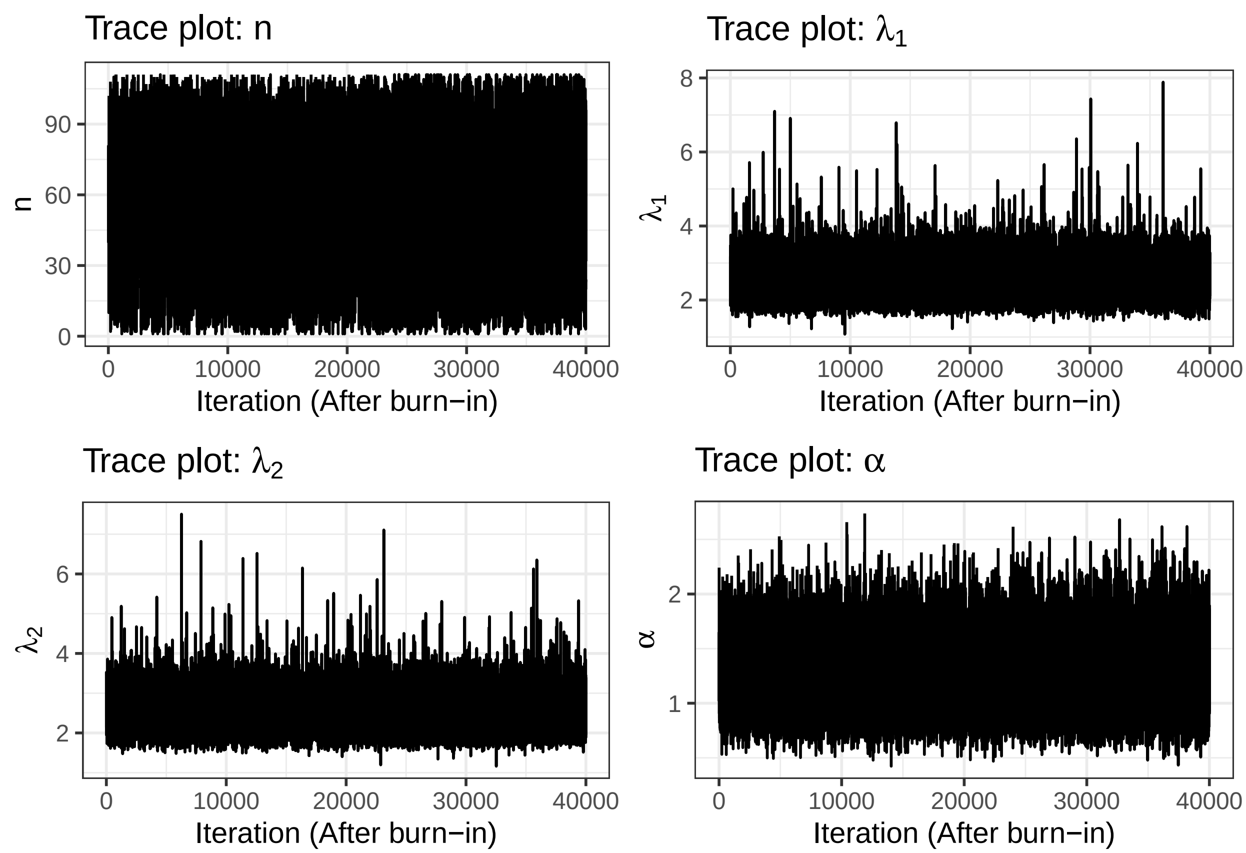
$$
\begin{aligned}
p\left(\lambda_1|\theta, \lambda_2, \alpha, \boldsymbol{X}\right) &\propto p\left(\boldsymbol{X}|\theta, \lambda_1, \lambda_2, \alpha\right) p\left(\theta\right) p\left(\lambda_1|\alpha\right) p\left(\lambda_2|\alpha\right) p\left(\alpha\right) \\
&\propto p\left(\boldsymbol{X}|\theta, \lambda_1, \lambda_2, \alpha\right) p\left(\lambda_1|\alpha\right) \\
&= \lambda_1^{\sum_{j=1}^{\theta} X_j} \lambda_2^{\sum_{j=\theta+1}^{112} X_j} e^{-\lambda_1\theta} e^{-\lambda_2(112-\theta)} \frac{\alpha^3}{\Gamma\left(3\right)} \lambda_1^2 e^{-\alpha\lambda_1} \\
&\propto \lambda_1^{\sum_{j=1}^{\theta} X_j} e^{-\lambda_1\theta} \lambda_1^2 e^{-\alpha\lambda_1} \\
&\propto \lambda_1^{2+\sum_{j=1}^{\theta} X_j} e^{-\lambda_1(\theta+\alpha)} \\
&\equiv \text{Gamma}\left(3 + \sum_{j=1}^{\theta} X_j, \theta + \alpha\right)
\end{aligned}
$$

$$
\begin{aligned}
p\left(\lambda_2|\theta,\lambda_1,\alpha,\boldsymbol{X}\right) &\propto p\left(\boldsymbol{X}|\theta,\lambda_1,\lambda_2,\alpha\right)p\left(\theta\right)p\left(\lambda_1|\alpha\right)p\left(\lambda_2|\alpha\right)p\left(\alpha\right)\\
&\propto p\left(\boldsymbol{X}|\theta,\lambda_1,\lambda_2,\alpha\right)p\left(\lambda_2|\alpha\right)\\
&= \lambda_1^{\sum_{j=1}^{\theta}X_j}\lambda_2^{\sum_{j=\theta+1}^{112}X_j}e^{-\lambda_1\theta}e^{-\lambda_2(112-\theta)}\frac{\alpha^3}{\Gamma\left(3\right)}\lambda_2^2e^{-\alpha\lambda_2}\\
&\propto \lambda_2^{\sum_{j=\theta+1}^{112}X_j}e^{-\lambda_2(112-\theta)}\lambda_2^2e^{-\alpha\lambda_2}\\
&\propto \lambda_2^{2+\sum_{j=\theta+1}^{112}X_j}e^{-\lambda_2(112-\theta+\alpha)}\\
&\equiv \text{Gamma}\left(3+\sum_{j=\theta+1}^{112}X_j, 112-\theta+\alpha\right)
\end{aligned}
$$

$$
\begin{aligned}
p\left(\alpha|\theta,\lambda_1,\lambda_2,\boldsymbol{X}\right) &\propto p\left(\boldsymbol{X}|\theta,\lambda_1,\lambda_2,\alpha\right)p\left(\theta\right)p\left(\lambda_1|\alpha\right)p\left(\lambda_2|\alpha\right)p\left(\alpha\right)\\
&\propto p\left(\lambda_1|\alpha\right)p\left(\lambda_2|\alpha\right)p\left(\alpha\right)\\
&= \frac{\alpha^3}{\Gamma\left(3\right)}\lambda_1^2e^{-\alpha\lambda_1}\frac{\alpha^3}{\Gamma\left(3\right)}\lambda_2^2e^{-\alpha\lambda_2}\frac{10}{\Gamma\left(10\right)}\alpha^9e^{-10\alpha}\\
&\propto \alpha^{18}e^{-\alpha(10+\lambda_1+\lambda_2)}\\
&\equiv \text{Gamma}\left(19,10+\lambda_1+\lambda_2\right)
\end{aligned}
$$

(b)

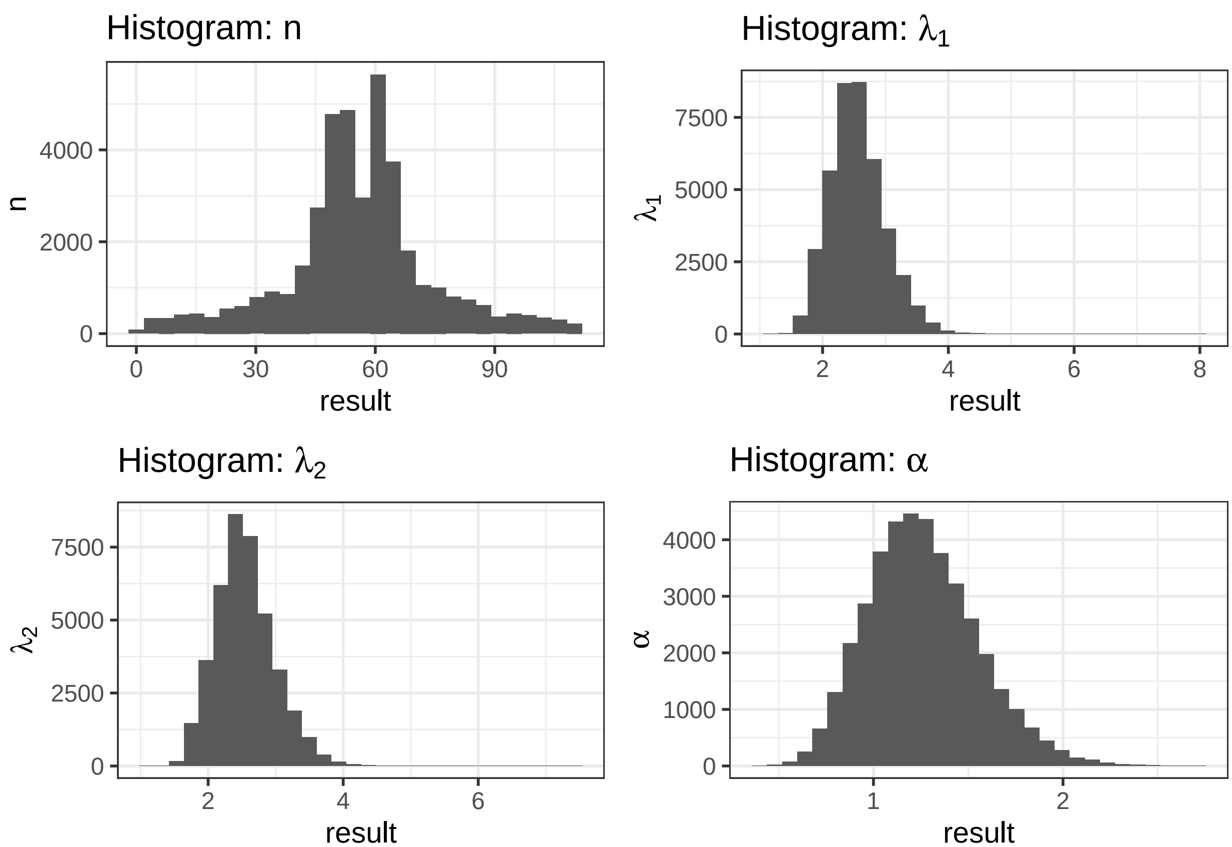
I will run the model for 50,000 iterations while letting the first 10,000 iterations as a burn-in.



(c)

	Mean (SD)
n	55.834 (SD = 18.763)
lambda1	2.558 (SD = 0.448)
lambda2	2.553 (SD = 0.444)
alpha	1.26 (SD = 0.291)

	95% Lower HPD	95% Upper HPD
n	11.0000000	97.000000
lambda1	1.7557302	3.448155
lambda2	1.7212786	3.406064
alpha	0.7288066	1.847579



(d)

(e)

For this question, I will let the prior for λ_i to be the half-Normal distribution. Specifically, $p\left(\lambda_i|\sigma_i^2\right) = \frac{\sqrt{2}}{\sqrt{\pi\sigma_i^2}}e^{-\frac{\lambda_i^2}{2\sigma_i^2}}\mathbb{I}_{\lambda_i>0}$ for i = 1, 2.

The conditional probability for θ still be the same as in part (a). Below are the derivation for the conditional probability for λ_i.

$$
\begin{aligned}
p\left(\lambda_1|\theta,\lambda_2,\boldsymbol{X}\right) &\propto p\left(\boldsymbol{X}|\theta,\lambda_1,\lambda_2\right)p\left(\theta\right)p\left(\lambda_1\right)p\left(\lambda_2\right)\\
&\propto p\left(\boldsymbol{X}|\theta,\lambda_1,\lambda_2\right)p\left(\lambda_1\right)\\
&= \lambda_1^{\sum_{j=1}^{\theta}X_j}\lambda_2^{\sum_{j=\theta+1}^{112}X_j}e^{-\lambda_1\theta}e^{-\lambda_2(112-\theta)}\frac{\sqrt{2}}{\sqrt{\pi\sigma_1^2}}e^{-\frac{\lambda_1^2}{2\sigma_1^2}}\\
&\propto \lambda_1^{\sum_{j=1}^{\theta}X_j}e^{-\lambda_1\theta-\frac{\lambda_1^2}{2\sigma_1^2}}
\end{aligned}
$$

$$
\begin{aligned}
p\left(\lambda_2|\theta,\lambda_1,\boldsymbol{X}\right) &\propto p\left(\boldsymbol{X}|\theta,\lambda_1,\lambda_2\right)p\left(\theta\right)p\left(\lambda_1\right)p\left(\lambda_2\right)\\
&\propto p\left(\boldsymbol{X}|\theta,\lambda_1,\lambda_2\right)p\left(\lambda_2\right)\\
&= \lambda_1^{\sum_{j=1}^{\theta}X_j}\lambda_2^{\sum_{j=\theta+1}^{112}X_j}e^{-\lambda_1\theta}e^{-\lambda_2(112-\theta)}\frac{\sqrt{2}}{\sqrt{\pi\sigma_2^2}}e^{-\frac{\lambda_2^2}{2\sigma_2^2}}\\
&\propto \lambda_2^{\sum_{j=\theta+1}^{112}X_j}e^{-\lambda_2(112-\theta)-\frac{\lambda_2^2}{2\sigma_2^2}}
\end{aligned}
$$

(f)

Proposal distribution: $q\left(\lambda_i^*|\lambda_i\right) = \text{Gamma}\left(\lambda_i, 1\right)$.

Appendix

```r
knitr::opts_chunk$set(echo = FALSE)

library(tidyverse)
library(knitr)
library(Rcpp)
library(RcppArmadillo)
library(foreach)
library(doParallel)
library(ggplot2)
library(latex2exp)
library(gridExtra)
library(HDInterval)

path <- "/Users/kevin-imac/Desktop/Github - Repo/"
if(! file.exists(path)){
  path <- "/Users/kevinkvp/Desktop/Github Repo/"
}

sourceCpp(paste0(path, "HW5MCMC/src/main.cpp"))

### Import the data
dat <- read.table(paste0(path, "HW5MCMC/coal.dat"), header = TRUE)

### Run the model
set.seed(31807)
result <- gibbsGamma(iter = 50000, dat = dat$disasters)

### Function: Trace plot
tpGGplot <- function(resultMat, colIndex, burnin, yLab, titleLab){
  totiter <- nrow(result)
  data.frame(iter = 1:(totiter - burnin), result = result[-(1:burnin), colIndex]) %>%
    ggplot(aes(x = iter, y = result)) +
    geom_line() +
    theme_bw() +
    labs(x = "Iteration (After burn-in)", y = yLab, title = titleLab)
}

p1 <- tpGGplot(result, 1, 10000, TeX("n"), "Trace plot: n")
p2 <- tpGGplot(result, 2, 10000, TeX("$\\lambda_{1}$"), TeX("Trace plot: $\\lambda_{1}$"))
p3 <- tpGGplot(result, 3, 10000, TeX("$\\lambda_{2}$"), TeX("Trace plot: $\\lambda_{2}$"))
p4 <- tpGGplot(result, 4, 10000, TeX("$\\alpha$"), TeX("Trace plot: $\\alpha$"))

grid.arrange(p1, p2, p3, p4)

meanSD <- function(x, dplace = 3){
  mm <- round(mean(x), digits = dplace)
  ss <- round(sd(x), digits = dplace)
  paste0(mm, " (SD = ", ss, ")")
}

### Mean and SD
```

```r
data.frame(sapply(1:4, function(x){meanSD(result[-(1:10000), x])})) %>%
  `rownames<-`(c("n", "lambda1", "lambda2", "alpha")) %>%
  kable(col.names = "Mean (SD)")

### HDI
sapply(1:4, function(x){as.numeric(hdi(result[-(1:10000), x]))}) %>%
  t() %>%
  `rownames<-`(c("n", "lambda1", "lambda2", "alpha")) %>%
  kable(col.names = c("95% Lower HPD", "95% Upper HPD"))

### Function: Histogram
htGGplot <- function(resultMat, colIndex, burnin, yLab, titleLab){
  totiter <- nrow(result)
  data.frame(iter = 1:(totiter - burnin), result = result[-(1:burnin), colIndex]) %>%
    ggplot(aes(x = result)) +
    geom_histogram() +
    theme_bw() +
    labs(y = yLab, title = titleLab)
}

h1 <- htGGplot(result, 1, 10000, TeX("n"), "Histogram: n")
h2 <- htGGplot(result, 2, 10000, TeX("$\\lambda_{1}$"), TeX("Histogram: $\\lambda_{1}$"))
h3 <- htGGplot(result, 3, 10000, TeX("$\\lambda_{2}$"), TeX("Histogram: $\\lambda_{2}$"))
h4 <- htGGplot(result, 4, 10000, TeX("$\\alpha$"), TeX("Histogram: $\\alpha$"))

grid.arrange(h1, h2, h3, h4)

### Half-Normal
resultHN <- gibbsHalfN(iter = 50000, s2_1 = 1, s2_2 = 1, dat = dat$disasters)
```