An improved batch processing CP approach

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3.2.1 Lower bound on $L_{\max}$

A lower bound on $L_{\max}$ can be found using the lower bound on the completion date of each bucket q, where a bucket is defined as the set of batches with due date δ_q:

$$L_{\max} \geq C_{\max,q} - \delta_q \quad \forall q \tag{3.1}$$

This works because the buckets up to bucket q are guaranteed to contain all jobs with due dates $d \leq \delta_q$, since the batches within the buckets are ordered by earliest due date (EDD) in the optimal solution. The buckets up to bucket q will likely also contain some later ($d > \delta_q$) jobs in the optimal solution but this does not affect the validity of the lower bound.

Now we need to find $C_{\max,q}$, or at least a lower bound on it, in polynomial time. The simplest approach simply considers the "total elastic area", i.e. the sum of all $s_j p_j$ products:

$$C_{\max,q} \geq \lceil \frac{1}{b} \sum_{j:d_j \leq \delta_q} s_j p_j \rceil \quad \forall q \tag{3.2}$$

A better lower bound on $C_{\max,q}$ would be given by a preemptive-cumulative schedule. Unfortunately, minimizing $C_{\max}$ for such problems is equivalent to solving a standard bin-packing problem, which requires exponential time.

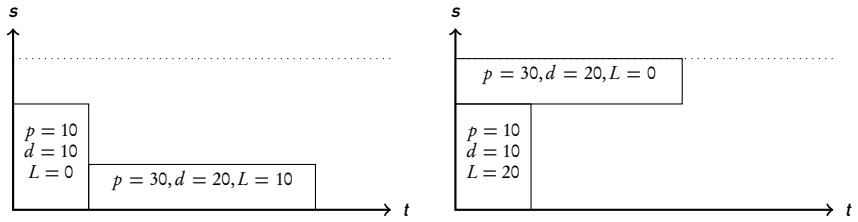


Figure 3.1: Overzealous batch elimination can increase $L_{\max}$

3.2.2 Upper bound on $L_{\max}$

An upper bound on $L_{\max}$ can be found by using a dispatch rule to find a feasible, if not optimal, schedule. A good approach could be the "best-fit" heuristic proposed in the original paper. This has not been implemented yet.

3.2.3 Bounding the number of batches n^k

Initially, the number of batches needed is assumed to be equal to the number of jobs: $n_k = n_j$. Reducing n_k by pre-computing the maximum number of batches needed shrinks the x_{jk} matrix.

Unfortunately, we cannot make a general statement that optimal solutions never have more batches than other feasible solutions – a simple counterexample is shown in figure 3.1.[1]

. . .

. . .

I have a long list of ideas here, none of which I've been able to prove right or wrong, so I've commented them out and made this page end here.

[1] To be more precise, we cannot state that at least one optimal solution is in the subset of feasible solutions that uses the fewest number of batches – a dominance situation that could be exploited, were it true.

3.3 MIP model used

3.3.1 Original approach

This is Malapert's original approach:

$$\sum_{k \in K} x_{jk} = 1 \qquad \forall j \in J \tag{3.3}$$

$$\sum_{j \in J} s_j x_{jk} \leq b \qquad \forall k \in K \tag{3.4}$$

$$p_j x_{jk} \leq P_k \qquad \forall j \in J, \forall k \in K \tag{3.5}$$

$$C_{k-1} + P_k = C_k \qquad \forall k \in K \tag{3.6}$$

$$(d_{max} - d_j)(1 - x_{jk}) + d_j \geq D_k \qquad \forall j \in J, \forall k \in K \tag{3.7}$$

$$D_{k-1} \leq D_k \qquad \forall k \in K \tag{3.8}$$

$$C_k - D_k \leq L_{\max} \qquad \forall k \in K \tag{3.9}$$

$$C_k \geq 0, P_k \geq 0 \text{ and } D_k \geq 0 \qquad \forall k \in K \tag{3.10}$$

The constraint 3.8 is the EDD constraint.

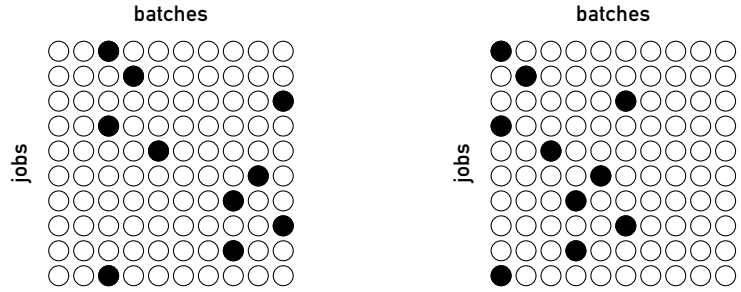
3.3.2 Grouping empty batches

The given formulation lacks a rule that ensures that no empty batch is followed by a non-empty batch. Empty batches have no processing time and a due date only bounded by $d_{\max}$, so they can be sequenced between non-empty batches without negatively affecting $L_{\max}$. Since, however, desirable schedules have no empty batches scattered throughout, we can easily reduce the search space by disallowing such arrangements. The idea is illustrated in Figure 3.2.

A mathematical formulation is

$$\sum_{j \in J} x_{j,k-1} = 0 \rightarrow \sum_{j \in J} x_{jk} = 0 \quad \forall k \in K. \tag{3.11}$$

To implement this, we can write constraints in terms of an additional set of binary variables, e_k, indicating whether a batch k is empty or not:



(a) Without dominance rule (b) With dominance rule

Figure 3.2: Dominance rule to eliminate empty batches followed by non-empty batches

$$e_k + \sum_{j \in J} x_{jk} \geq 1 \qquad \forall k \in K, \tag{3.12}$$

$$n_j(e_k - 1) + \sum_{j \in J} x_{jk} \leq 0 \quad \forall k \in K. \tag{3.13}$$

Constraint 3.12 enforces $e_k = 1$ when the batch k is empty. Constraint 3.13 enforces $e_k = 0$ otherwise, since the sum term will never exceed n_j. The rule 3.11 can now be expressed as $e_{k-1} = 1 \rightarrow e_k = 1$, and implemented as follows:

$$e_k - e_{k-1} \geq 0 \quad \forall k \in K. \tag{3.14}$$

We can also prune any attempts to leave the first batch empty by adding a constraint $e_0 = 0$.

This, while it works just fine, actually takes longer than without it.

3.3.3 No postponing of jobs to later batches

Since the jobs are already sorted by non-decreasing due dates, it makes sense to explicitly instruct the solver never to attempt to push jobs into batches with a greater index than their own: even if every job had its own batch, it would be unreasonable to ever postpone a job to a later batch.

$$x_{jk} = 0 \quad \forall j, k : j > k \tag{3.15}$$

3.4 CP model

[Some introduction ...]

3.4.1 Bin-packing and cumulative constraints

This makes sure the jobs are distributed into the batches such that no batch exceeds the capacity b:

$$\texttt{pack}(J, K, b) \tag{3.16}$$

The cumulative constraint functions similarly, but instead of packing discrete bins, it enforces non-overlapping constraint on the temporal (`IntervalVariable`) J variables. For this to work directly with the `assignments` array used by `pack` (i.e. without a x_{jk} matrix), it seems I need to define my own search goal class – I still need to do this).

$$\texttt{cumul}(J, b) \tag{3.17}$$

3.4.2 Temporal constraints

These constraints are implemented using `IntervalVar` objects, which offer properties such as `lengthOf` or `endOf`.

$$P_k \geq \max_{j:B_j=k} p_j \qquad \forall k \in K \tag{3.18}$$

$$D_k \leq \min_{j:B_j=k} d_j \qquad \forall k \in K \tag{3.19}$$

$$C_k + P_{k+1} = C_{k+1} \qquad \forall k \in K \tag{3.20}$$

$$L_{\max} \geq \max_k (C_k - D_k) \tag{3.21}$$

The first constraint ensures that each batch is as long as its longest job. The second constraint ensures that the earliest-due job sets the due date. The third constraint defines batch completion dates, and the fourth constraint defines $L_{\max}$.

The first two temporal constraints were (rather, will be) implemented in custom constraint propagator classes using the job assignments propagated by the bin-packing constraint. They exploit the temporal features of `IntervalVars`.

3.4.3 Constraint on number of batches with length $P_k > p$

Since batches take on the processing time of their longest job, there is at least one batch with $P = \max_j p_j$:

$$\texttt{globalCardinality}(|P_k = \max_j p_j| = 1) \tag{3.22}$$

We can proceed to fill batches with jobs, ordered by non-increasing processing time, based on algorithm 1.

Algorithm 1 Generating lower bounds on batch lengths

```
J* ← J                                          ▷ initialize all jobs as unassigned jobs
n_k ← 1; S_k ← {0}; P_{k,min} ← {0}             ▷ Create one empty batch of size and length zero
sort J* by processing time, non-increasing
repeat
    j ← J*.pop()                                ▷ select job for assignment, longest job first
    loop through all n_k existing batches k, first batch first
        k_p ← ∅                                 ▷ no feasible batch
        c_min = b                               ▷ currently known minimum remaining capacity
        if s_j < b − S_k and b − S_k < c_min then
            k_p ← k_p; c_min ← b − S_k
        end if
    end loop
    if |k_p| = 1 then
        S_{k_p} ← S_{k_p} + s_j                 ▷ assign job j to batch k_p
    else
        if n_k < LB(n_k) then
            n_k ← n_k + 1                       ▷ open new batch
            S_{n_k} ← s_j; P_{n_k,min} ← p_j    ▷ assign s_j and p_j to the new batch
        else
            leave the loop now and end.
        end if
    end if
until J* is empty
```

At the end of this algorithm, we can state:

$$\texttt{globalCardinality}(|P_{k-1,\min} > P_k \geq P_{k,\min}| = 1) \quad \forall k \in \{k_0, \ldots, k_{LB(n_k)}\} \tag{3.23}$$

The algorithm sorts jobs by non-increasing p, and then fills batches job by job. If a job fits into a previous batch, it is assigned there. If a job fits into multiple previous batches, it is assigned to the batch with the smallest remaining capacity. As I found out today, this is called *best-fit decreasing rule*. I can expand on why this works, if necessary.

Unfortunately, the optimal solution may perform better in terms of "vertical" (s_j) bin packing, and may thus require fewer batches. We therefore need to find a lower bound $LB(n_k)$ on the number of batches, and we can only guarantee the first $LB(n_k)$ of the above constraints to hold in the optimal solution. Finding a true lower bound is a two-dimensional bin packing problem, which runs in exponential time … so we have to come up with an even lower bound – right now I can only think of j_0, the number of jobs ordered by decreasing s_j that can never fit into a batch together.

This whole method can surely be improved somehow by realizing a priori that some jobs must go before others?

Furthermore, if all jobs have different processing times, all batches will have different processing times as well: `alldifferent`(P_k). If m out of n_j jobs have different processing times, we can still enforce `k_alldifferent`(P_k, m). Propagation rules for this constraint were given in [?]. I can't find anything on this w.r.t. CP Optimizer, so I may have to implement this myself … time permitting.

3.4.4 Temporal constraints on a job's start date

Given any partial assignment of jobs and an open job j, we can reason that

- if the first batch with a due date later than the job is k, then the job cannot be part of a batch after k – this would result in a non-EDD sequence of batches.
- if the first batches up to $k-1$ offer not enough capacity for j due to the given partial assignment, then the job cannot be part of a batch before k.

Since batches are *not* dynamically created like in Malapert's solution but fixed from the start, any partial assignment that fails due to these constraints cannot be part of an optimal solution.

This constraint is redundant with both the ($C_{k+1} \geq C_k$) and `packing` constraints, but may help accelerate the propagation in some cases.

3.4.5 Restricting the search to *before* $L_{\max}$

Once a feasible (but possibly non-optimal) solution is found that has $L_{\max}$ in batch $k_{L_{\max}}$, then this solution dominates all solutions with other job configurations *after* $k_{L_{\max}}$ – improving L in those batches has no impact on $L_{\max}$, and worsening it never leads to a better solution.

Therefore, once all jobs are assigned to a feasible solution, the search should be limited to the following jobs:

- those jobs in the first batch that are due with the second batch k_2 or later,
- all jobs in batches $\{k_2, \ldots, k_{L_{\max}-1}\}$,
- all jobs in $k_{L_{\max}}$ for which $p_j > j_{L_{\max}}$,
- $j_{L_{\max}}$ itself, but it can only be scheduled earlier.

Once another, better feasible solution is found, this constraint on "searchable" variables must be updated. If no better solution is found using those variables, the solution is optimal. I think this could maybe speed up the search, but I have no idea how to implement this dominance rule as a constraint. The documentation offers `SearchPhase` functions that can guide the search, but I haven't experimented with that yet.

3.4.6 Grouping empty batches

Just like in the MIP model, we can force empty batches to the back and thus establish dominance of certain solutions. The implementation is much easier than in the MIP model:

$$\texttt{IfThen}(C_{k+2} > C_k, C_{k+1} > C_k) \quad \forall k \in \{k_1, \ldots, k_{n_k-2}\} \tag{3.24}$$

3.4.7 No postponing of jobs to later batches

Just like in the MIP model, jobs should never go into a batch with an index greater than their own:

$$x_{jk} = 0 \quad \forall j, k : j > k \tag{3.25}$$

This is implemented as $\texttt{assignments}_j \leq k$.

This constraint is analogous to the concept of finding an upper bound on n_k – essentially, it would be very helpful to find an upper bound on the latest batch for *every* job.

3.5 Test data for evaluation

Malapert uses benchmark data by Daste. For design purposes, I created my own set of randomized job lists, with $s_j, p_j \in [1,20]$ and $d_j \in [1,10n]$ where n is the number of jobs. Here is the Python code used:

```
import csv
import random

random.seed(None)

for i in range(10,100,20):
    with open( "data_" + str(i), "wb") as csvfile:
        csvwriter = csv.writer(csvfile, delimiter=",")
        csvwriter.writerow(['s_j', 'p_j', 'd_j'])
        for job in range(i):
            csvwriter.writerow([random.randrange(1,20),
               random.randrange(1,20),
               random.randrange(1,5*i)])
    csvfile.close()
```

CSV files can be read in with `IloCsvReader` and `IloCsvLine`.

My solution

4.1 MIP formulation improvements

4.2 CP formulation improvements

Discussion

5.1 Introduction

Books to read: Model Building in Mathematical Programming by HP Williams

Mathematical programming models all involve optimization. They want to minimize or maximize some objective function.

There are linear programming models, non-linear programming models and integer programming models.

Constraint programming and Integer Programming are twins and can usually be translated into one another. In CP each variable has a finite domain of possible values. Constraints connect/restrict the possible combinations of values which the variables cantake. These constraints are of a richer variety then the linear constraints of IP and are usually expressed in the form of predicates, such as "all_different(x1, x2, x3)". The same condition could be imposed via IP, but it's more troublesome to formulate. Once one of the variables has been set to one of the values in its domain (either temporarily or permanently), this predicate would imply that this value must be taken out of the domain of all other variables ("propagation"). In this way constraint propagation is used until a feasible set of values is found (or not).

CP is useful where a problem function has no objective function and we are just looking for a feasible solution. We can't prove optimality, although we could using IP.

Comparisons and connectionsbetween IP and CLP are discussed by Barth (1995), Bockmayr and Kasper (1998), Brailsford et al. (1996), Prolland Smith (1998), Darby-Dowman and Little (1998), Hooker (1998) and Wilson and Williams (1998). Theformulation of the all_different predicate using IP is discussed by Williams and Yan (1999).

The first approach to solving a multi-objective problem is to solve the model a number of times with each objective function in turn. The result may give an idea of what to do next.

5.1.1 Chapter 8: Integer Programming

The real power of IP as a method of modelling is that of binary constraints, where variables can take on values of 0 or 1. Maybe it should be called "discrete programming." Precise definitions of those problems that can be formulated by IP models are given by Meyer (1975) and Jeroslow (1987).

Problems with discrete inputs and outputs are the most obvious candidates for IP modelling ("lumpy inputs and outputs"). Sometimes solving an LP and rounding to the nearest integer works well, but sometimes it doesn't, as demonstrated in Williams, first example in 8.2. The smaller the variables (say, <5), the greater significance those rounding problems will have.

When input variables have small domains, say, machine capacities, then again this may rule out LP relaxations. A good example of an IP problem is the knapsack assignment problem. A single constraint is that the capacity of the knapsack cannot be exceeded. Any LP formulation would always fill the knapsack 100%, ignoring the discrete size of the objects. Two other well-known types of problems include *set partitioning problems* and *aircrew scheduling problems*.

Integer programming models cannot be solved directly, but need to be brute-forced in a tree search manner. The search space, however, can be greatly reduced by a number of bounding techniques often depending on the nature of the problem. That is why the general method of IP solving is called *branch-and-bound*.

So-called *cutting planes methods* usually start by solving an IP problem as LP. If the resulting solution is integral, we're happy. Otherwise extra constraints (cutting planes) are added to the problem, further constraining it until an integer solution is found (or, if none can be found, we're out of luck). Cutting planes make for nice illustrations they are not very efficient with large problems. Cutting planes were invented by Gomory (1958).

Enumerative methods are generally applied to pure binary problems, where the search tree is pruned. The best known of these methods is Balas's additive algorithm described by Balas (1965). Other methods are givenby Geoffrion (1969). A good overall exposition is given in Chapter 4 of Garfinkel and Nemhauser (1972).

There are so-called *pseudo-boolean methods* to solve pure binary problems that take boolean constraints as inputs. That may be comfortable for the user sometimes, but is rarely used in practice.

Generally, branch-and-bound methods first solve the LP relaxation to check whether we're lucky enough to find an integer solution. If not, a tree search is performed.

5.1.2 Chapter 9: Building IP models I.

Binary variables are often called 01-variables. Decision variables could also have a domain like $\{0,1,2\}$ or $\{4,12,102\}$. Decision variables, especially the 01 kind, can be linked to the state of continuous variables like this: $x - M\delta \leq 0 \leftrightarrow x > 0 \rightarrow \delta = 1$, where we know that $x < M$ is always true.

To use a 01 variable to indicate whether the following is satisfied:

$$2x_1 + 3x_2 \leq 1 \tag{5.1}$$

$$x_1 \leq 1 \tag{5.2}$$

$$x_2 \leq 1, \tag{5.3}$$

so, mathematically speaking,

$$\delta = 1 \rightarrow 2x_1 + 3x_2 \leq 1 \tag{5.4}$$

$$2x_1 + 3x_2 \leq 1 \rightarrow \delta = 1, \tag{5.5}$$

Then we can argue that at most, $2x_1 + 3x_2 = 5$, so

$$2x_1 + 3x_2 + 4\delta \leq 5$$

will ensure that $\delta = 1$ forces the equation to be true: use $M = 2 + 3 - 1$ to find 4. In order to enforce $\delta = 1$ if the equation is true, use $m = 0 + 0 - 1$ and write

$$2x_1 + 3x_2 + \delta \geq 1$$

All kinds of logical conditions can be modelled using 01 variables, although it's not always obvious how to capture them in that format.

Logical conditions are sometimes expressed within a Constraint Logic Programming language as discussed inSection 2.4. The tightest way of expressing certain examples using linear programming constraints is described byHooker and Yan (1999) and Williams and Yan (1999). There is a close relationship between logic and 01 integerprogramming, which is explained in Williams (1995), Williams and Brailsford (1999) and Chandru and Hooker(1999).

Special Ordered Sets Two very common types of restriction arise in mathematical programming, so two concepts (SOS1 and SOS2) have been developed. An SOS1 is a set of variables within which exactly one variable must be non-zero and the rest zero. An SOS2 is a set where at most two can be non-zero, and the two variables must be adjacent in the input ordering. Using a branch-and-bound algorithm specialized for SOS1 or SOS2 sets can speed things up greatly.

Disjunctive constraints It is possible to define a set of constraints and postulate that at least one of them be satisfied.

5.1.3 Special kinds of IP models

Most practical IP models do not all into any of these categories but arise as MIP models often extending an existing LP model. Here are some examples.

Set covering problems We have a set $S = \{1, 2, 3, \ldots, m\}$. We have a bunch of subsets $\mathcal{S}$ of subsets, each associated with a cost. Now cover all members of S using the least-cost members of $\mathcal{S}$.

Knapsack problem These are the really simple ones with just one constraint, namely the constraint of not being able to take more items with you than the knapsack can carry while maximizing the value of the taken objects.

Quadratic assignment problem Two sets of objects S and T of the same size require the objects to be matched pairwise. There are costs associated with pairs of pairs, that is the cost $c_{ij,kl}$ is the cost of assigning i to j while also assigning k to l. This cost will be incurred if both δ_{ij} and δ_{kl} are `true`, i.e. $\delta_{ij}\delta_{kl} = 1$. The objective function is a quadratic expression in 0/1 variables:

$$\text{Minimize} \sum_{\substack{i,j,k,l=1 \\ k>l}}^{n} c_{ij,kl}\delta_{ij}\delta_{kl} \tag{5.6}$$

The quadratic version is practically unsolveable, so to be able to enumerate the possible assignments the above objective function has to be split up into separate constraints, which obviously means there will be an explosion in problem size as the number of variables grow.

5.1.4 How to formulate a good model

According to Williams, it is easy to build IP models that, while correct, are really inefficient. Fortunately, with some knowledge of what happens behind the scenes (and some practice), sucky models can often be improved. One good method to know (although it doesn't help by itself) is to turn a general integer variable into a bunch of 01 variables. Say γ is a general, non-negative integer variable with a known upper bound of u, then we can replace it with $\delta_0 + 2\delta_1 + 4\delta_2 + 8\delta_3 + \cdots + 2^n\delta_n$.

Example problem http://www.scribd.com/doc/49547850/Model-Building-in-Mathematical-Programming

5.2 How this could be solved

We're trying to create an mixed integer linear programming model (MILP). In the original paper, they had one non-linear constraint:

$$(d_{max} - d_j)(1 - x_{jk}) \geq D_k - d_j \quad \forall j \in J, \forall k \in K \tag{5.7}$$

This can easily be turned into the simpler

$$D_k - (d_j - d_{max})x_{jk} \leq d_{max} \quad \forall j \in J, \forall k \in K \tag{5.8}$$

5.2.1 Possible improvements

We cannot modify the solution process itself – that's the whole point of the exercise, after all. Maybe we can preprocess to limit the domains of some decision variables. This is where "preordering" may come into play.

Limiting batch due date Since we start with as many batches as jobs, the solver is free to put every job into a batch of the same number. To make things more efficient, it can also put jobs into batches before, but it would never make sense to put a job into a batch with a higher number, so we have a limit that

$$x_{jk} = 0 \quad \forall j, k : j > k \tag{5.9}$$