

A New MIP Model for Parallel-Batch Scheduling with Non-Identical Job Sizes

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Abstract. Parallel batch machine problems arise in numerous manufacturing settings. They have recently been addressed by Malapert et al., who proposed the implementation of a new **sequenceEDD** global constraint. The latter, in combination with **pack**, forms the current state-of-the-art approach and performs significantly better than a basic MIP formulation of the problem. In this paper, we present three improvements to this basic MIP model and show that they boost its performance and solution quality to match and exceed that of the CP model.

↑ **CB: To be fixed later.**

1 Introduction

Despite the widespread application of mixed integer programming (MIP) technology to optimization problems in general and scheduling problems specifically,¹ there is a significant body of work that demonstrates the superiority of constraint programming (CP) and hybrid approaches for a number of classes of scheduling problems [?]. While the superiority is often a result of strong inference techniques embedded in global constraints [?], the strong performance is often the result of more detailed, problem-specific implementation in the form of specialized global constraints [1] or instantiations of decomposition techniques [?]. The flexibility of CP and hybrid techniques which facilitates such implementation is undoubtedly positive from the perspective of solving specific problems better. However, the flexibility is in some ways in opposition to the “holy grail” of CP [2]: to enable users to model and solve problems without implementing anything new at all.

↑ **CB: Some of the references above are still missing. Did you have any particular papers in mind for some_sched_papers?**

Our over-arching thesis is that, in fact, MIP technology is closer to this goal than CP, at least in the context of combinatorial optimization problems. In our investigation of this thesis, we are developing MIP models for scheduling problems where the current state of the art is customized CP or hybrid approaches. Heinz et al. [3] showed that on a class of resource allocation and scheduling

¹ For example, of the 58 papers published in the *Journal of Scheduling* in 2012, 19 use MIP, more than any other single approach.

problems, a MIP model could be designed that was competitive with the state-of-the-art logic-based Benders decomposition approach. This paper represents a similar contribution in different scheduling problem: a parallel batch processing problem which has previously been attacked by MIP, branch-and-price [11], and CP [1]. The latter represents the current state of the art.

In this paper, we propose a MIP model inspired by the idea of modifying a canonical feasible solution to arrive at an optimal solution. The definition of our objective function in this novel context is not intuitive until we reason algorithmically about how constraints and assignments interact – a strategy usually reserved for local search techniques. Indeed, we suggest that the analogy between branching on independent decision variables and making moves between neighbouring schedules should be explored in more detail for a range of combinatorial problems.

The balance of the paper is organized as follows. In next section we present the formal problem definition and discuss existing approaches. In Section 3 we present and prove a number of propositions that allows us to propose a novel MIP model for the problem before formally defining the model in Section 4. Section 5 presents our empirical results, demonstrating that the performance of the new model is superior to the existing CP model, both in terms of mean time to find optimal solutions (especially on smaller problems) and in terms of solution quality when optimal solutions could not be found within the time limit (especially on larger problems).

2 Background

Batch machines with limited capacity exist in many manufacturing settings: as ovens in semiconductor manufacturing [4], autoclaves in the production of carbon fiber parts [1], and processing tanks in the pharmaceutical industry [5]. In this paper, we tackle the problem of minimizing the maximum lateness, L_{\max} , in a single machine parallel batching problem where each job has an individual due date and size.

We use the following notation: a set \mathcal{J} of n jobs, is to be assigned to a set of batches $\mathcal{B} = \{B_1, \dots, B_n\}$. Every job j has three properties: its processing time p_j , its size s_j and its due date d_j . Jobs can be assigned to arbitrary batches, as long as the sum of the sizes of the jobs in any batch does not exceed the machine capacity, b .

The single machine processes one batch at a time. Each batch B_k has a *batch start date* S_k , a *batch processing time*, defined as the longest of processing times of all jobs assigned to the batch, $P_k = \max_{j \in B_k}(p_j)$, and a *batch completion date*, which must fall before the start time of the next batch, $C_k = S_k + P_k \leq S_{k+1}$.

The lateness of a job j , L_j , is the completion time of its batch C_k less its due date d_j . The objective function is to minimize the maximum lateness over all jobs, $L_{\max} = \max_{j \in \mathcal{J}}(L_j)$. Since we are interested in the maximum lateness, only the earliest-due job in each batch matters and we define it as the *batch due date* $D_k = \min_{j \in B_k}(d_j)$.

Previous authors have used the format established by Graham et al. [6] to summarize the problem as $1|p\text{-batch}; b < n; \text{non-identical}|L_{\max}$ [?, 1], where $p\text{-batch}; b < n$ represents the parallel-batch nature and the finite capacity of the resource. A simpler version with identical job sizes was shown to be strongly NP-hard by Brucker et al. [7]; this problem, therefore, is no less difficult.

Figure 1 shows a solution to a sample problem with eight jobs and a resource with capacity $b = 20$. The last batch has the maximum lateness $L_5 = C_5 - D_5 = 70 - 39 = 31$.

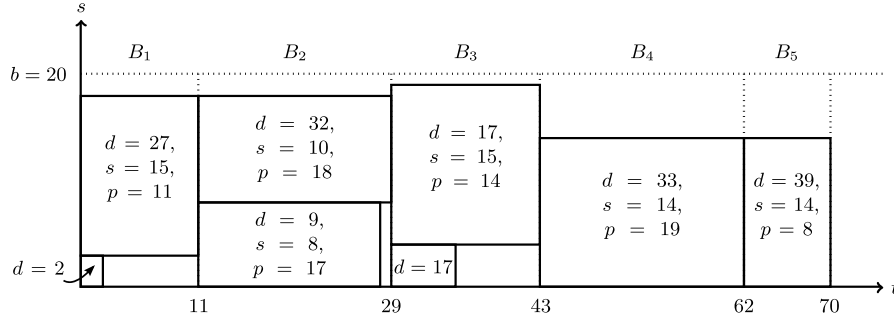


Fig. 1. An optimal solution to an example problem with eight jobs (values for s_j and p_j are not shown for the two small jobs in batches 1 and 3, respectively).

2.1 Reference MIP model

The problem is formally defined by MIP Model 2.1, used by Malapert et al. [1] for comparison with their CP model (see below).

The decision variable, x_{jk} is a binary variable that is one if and only if job j is assigned to batch k . Constraints (2) ensure that each job j is assigned to exactly one batch k . Constraints (3) ensure that no batch exceeds the machine capacity, b . Constraints (4) define each batch's processing time P_k as the maximum processing time of the jobs j assigned to it. Constraints (5) define each batch's completion time C_k as that of the previous batch, plus the batch's processing time. Constraints (7) define the objective value L_{\max} . Constraints (8) sort the batches by due date, based on a well-known dominance rule: there exists an optimal solution with batches scheduled in earliest-due-date-first order (EDD). This is due to the fact that if all jobs are already assigned, the problem reduces to a polynomial-time solvable single machine problem ($1|D_k|L_{\max}$) [8].

2.2 Previous Work

Malapert et al. [1] present a CP formulation of the problem (see Model 2.2) which relies on two global constraints: **pack** [9], which constrains the job-to-

$$\begin{aligned}
\text{Min. } & L_{\max} & (1) \\
\text{s.t. } & \sum_{k \in \mathcal{B}} x_{jk} = 1 & \forall j \in \mathcal{J} & (2) \\
& \sum_{j \in \mathcal{J}} s_j x_{jk} \leq b & \forall k \in \mathcal{B} & (3) \\
& p_j x_{jk} \leq P_k & \forall j \in \mathcal{J}, \forall k \in \mathcal{K} & (4) \\
& C_{k-1} + P_k = C_k & \forall k \in \mathcal{B} & (5) \\
& (d_{\max} - d_j)(1 - x_{jk}) + d_j \geq D_k & \forall j \in \mathcal{J}, \forall k \in \mathcal{B} & (6) \\
& C_k - D_k \leq L_{\max} & \forall k \in \mathcal{B} & (7) \\
& D_{k-1} \leq D_k & \forall k \in \mathcal{B} & (8) \\
& x_{jk} \in \{0, 1\}, C_k \geq 0, P_k \geq 0, D_k \geq 0 & \forall j \in \mathcal{J}, \forall k \in \mathcal{B} & (9)
\end{aligned}$$

Model 2.1. Reference MIP model

batch assignments such that no capacity limits are violated, and **sequenceEDD**, which enforces the EDD order over the batches. The implementation of the latter constraint is the main contribution of the paper and is primarily responsible for the strong performance. It includes a set of pruning rules that update the lower and upper bounds on L_{\max} and on the number of batches every time **sequenceEDD** is triggered by a job-batch assignment. Based on these bounds, other assignments are then eliminated from the set of feasible assignments.

$$\begin{aligned}
\text{Min. } & L_{\max} & (10) \\
\text{s.t. } & \text{maxOfASet}(P_k, B_k, [p_j]_k, 0) & \forall k \in \mathcal{B} & (11) \\
& \text{minOfASet}(D_k, B_k, [d_j]_k, d_{\max}) & \forall k \in \mathcal{B} & (12) \\
& \text{pack}(B_k, A_j, S_k, M, s_j) & & (13) \\
& \text{sequenceEDD}(B_k, D_k, P_k, M, L_{\max}) & & (14)
\end{aligned}$$

Model 2.2. CP model proposed by Malapert et al.

Constraints (11) define P_k as the maximum of the set of processing times $[p_j]_k$ belonging to the jobs assigned to batch B_k , with a minimum value of 0 (note that the notation is adapted from Malapert et al. to match that in this paper). Constraints (12) define D_k as the minimum of the due dates $[d_j]_k$ associated with the set of jobs assigned to the batch B_k , with a maximum of d_{\max} , the largest due date among all given jobs. Constraint (13) implements the limited batch capacity b . It uses propagation rules incorporating knapsack-based reasoning, as well as a dynamic lower bound on the number of non-empty batches M [1, 10]. Note that this constraint handles the channeling between the

set of jobs assigned to batch B_k , and the assigned batch index A_j for each job j . The limited capacity is enforced by setting the domain of the batch loads S_k to $[0, b]$. Constraint (14) ensures that the objective value L_{\max} is equal to the maximum lateness of the batches scheduled according to the EDD rule.

The problem has also been addressed with a detailed branch-and-price algorithm [11], which is described in [1] as follows: each batch is a column in the column generation master problem. A solution of the master problem is a feasible sequence of batches. The objective of the subproblem is to find a batch which improves the current solution of the master problem. Malapert et al. [1] showed that their CP model, was significantly faster than the branch-and-price algorithm which itself was more efficient than the reference MIP model.

Other authors have examined similar problems: Azizoglu & Webster [12] provide an exact method and a heuristic for the same problem, but minimize makespan (C_{\max}) instead of L_{\max} , similarly to the work by Dupont and Dhaenens-Flipo [13]. Exact methods have been proposed for multi-agent variants with different objective functions by Sabouni and Jolai [14], for makespan minimization on single batch machines by Kashan et al. [15], and for makespan minimization on parallel batch machines with different release dates [16]. An extensive review of MIP models in batch processing is given by Grossmann [5].

3 Exploiting the Structure of the Parallel-Batching Problem

In this section, we provide a series of observations about the parallel-batch scheduling problem that allow us to develop a novel MIP formulation.

3.1 The single-EDD schedule and assigning jobs to earlier batches

We can exploit the EDD rule to eliminate $\frac{1}{2}(n^2 - n)$ of the n^2 potential job-batch assignments a priori.

We first re-index all jobs in non-decreasing due date (and in non-decreasing processing time in case of a tie), which can be done in $\mathcal{O}(n \log n)$ time. For the remainder of this paper, consider all jobs to be indexed in this way. We then introduce the notion of the *single-EDD* schedule, in which each batch B_k contains the *single* job j matching its index (i.e., $x_{jk} = 1$ exactly when $j = k$), such that EDD is always satisfied. We refer to j as the *host job* of batch B_k , while other jobs assigned to B_k , if any, are *guest jobs*.

Proposition 1. *There exists an optimal solution in which job j is assigned to batch $B_k \forall j \leq k$.*

Before giving a proof, we point out the following:

Lemma 1. *Consider a schedule S in which B_k is the earliest-scheduled batch such that its host job j is assigned to a later-scheduled batch B_{k+m} . In this schedule, B_k is either empty or $D_k = d_j$ (although $j \notin B_k$).*

Proof. If B_k is non-empty, $D_k \geq d_j$: since B_k is defined as the earliest scheduled batch whose host job is scheduled later, B_k cannot host jobs due before d_j . But if $D_k > d_j$ then EDD is violated, as $D_{k+m} = d_j$. Thus, B_k is empty, or $D_k = d_j$ (due to other jobs with due date d_j assigned to B_k). \square

Schedule S clearly violates the constraint given in Proposition 1. We now prove the proposition by showing that S is not uniquely optimal:

Proof. Consider again schedule S .

Note first that $D_{k+m} = d_j$. To ensure EDD, we require that no batch $B_q, k \leq q < k+m$, is due after d_j . By Lemma 1, we only need to consider the following two cases:

- B_k is empty, so $P_k = 0$. Since EDD is not violated, we know that $D_q = d_j \forall B_q, k \leq q \leq k+m$. In this case, we can instead assign all jobs from B_{k+m} into B_k , such that $P_{k+m} = 0$. L_{\max} will stay constant, as the completion time of the last-scheduled of all batches due at d_j does not change.
- B_k is non-empty and due at $D_k = d_j$ (although $j \notin B_k$), due to at least one job g from a later-scheduled batch for which $d_g = d_j$, which is assigned to B_k . In this case, since $D_k = d_j = D_{k+m}$ and since EDD is not violated, *all* batches B_q where $k \leq q \leq k+m$ must be due at d_j . But then we can re-order these batches such that their respective earliest-due jobs are once again assigned to their single-EDD indices. The jobs in B_{k+m} (including j) will be assigned to B_k as a result. L_{\max} is not affected by this re-assignment, as the completion time of the last-scheduled batch due at d_j does not change.

\square

We thus introduce the following constraint to exclude solutions in which jobs are assigned to later batches than their single-EDD batches.

$$x_{jk} = 0 \quad \forall \{j \in \mathcal{J}, k \in \mathcal{B} | j < k\} \quad (15)$$

We can also show that in every non-empty batch B_k , the earliest-due job j must be the single-EDD job (such that $j = k$). This means that when batch B_k 's host job j is assigned to an earlier batch, no other jobs can be assigned to B_k ; a batch that is *host-less* must be empty. This requirement rests on the following proposition:

Proposition 2. *There exists an optimal solution that has no host-less, non-empty batches.*

Proof. Consider an optimal schedule, ordered by EDD, in which batch B_k is the last-scheduled batch which is host-less but non-empty: instead of its host job j (with $j = k$), only a set G of later-due guest jobs is assigned to B_k (where $j \notin G$).

The earliest-due job $g \in G$ must have the same due date as batch B_{k+1} : if it is due later, EDD is violated; if it is due earlier, G is not a set of later-due guest

jobs. Job g 's own host batch B_g (which is host-less) cannot itself be due later than $d_g = D_{k+1}$ – this would require B_g to have guest jobs from later batches, but we defined B_k as the last batch with this property – therefore, $D_q = D_{k+1}$ for all batches $B_q, k \leq q \leq g$.

Then we can re-assign the guest jobs G from B_k into B_g , such that g is again host job in its own single-EDD batch. This re-assignment has no impact on L_{\max} since it makes $P_k = 0$, resulting in the same completion time of the set of all batches with batch due date $D = D_{k+1}$. \square

The above proposition translates to the following constraint:

$$x_{kk} \geq x_{jk} \quad \forall \{j \in \mathcal{J}, k \in \mathcal{B} | j > k\} \quad (16)$$

This observation allows us to define the due date of *all* batches to be the due date of their respective host jobs: $D_k = d_j \forall \{j \in \mathcal{J}, k \in \mathcal{K} | j = k\}$. Note that this rule holds even for empty batches B_k : $P_k = 0$, so $C_k = C_{k-1}$; but $D_{k-1} \leq D_k$ due to this rule, so $L_{k-1} \geq L_k$ and thus L_k has no impact on L_{\max} .

3.2 Using changes to the single-EDD schedule to calculate the objective value

We can formulate each batch's lateness, L_k , as its pre-computed lateness in the single-EDD schedule, modified by the assignment of jobs into and out of batches $B_h, h \leq k$.

We first define $\mathcal{B}^* \subseteq \mathcal{B}$ as the set of batches B_k which, given any EDD schedule, are the last-scheduled among all batches with due date D_k , since we can make the following observation:

Proposition 3. *Given a set of batches with equal due date in a schedule, we only need consider the lateness of the one scheduled last.*

Proof. In an EDD ordering, the lateness of the batch scheduled last is greater than (or equal to, in the case of an empty batch) the lateness values of all other batches sharing its due date as it has the latest completion date. \square

This fact allows us to reduce the number of constraints defining L_{\max} , as we only need to consider batches \mathcal{B}^* as potential candidates for L_{\max} .

To simplify the following arguments, we also define the term *move* as the re-assignment of a job j from its single-EDD batch B_k to an earlier batch $B_h, h < k$, such that $x_{jk} = 0$ and $x_{jh} = 1$ and j is a guest job in B_h . Any schedule can thus be understood as a set M of such moves, executed in arbitrary order starting from the single-EDD schedule. To define the objective function, we consider below the change in L_{\max} associated with individual moves, going from one schedule to another, instead of reasoning about the L_{\max} value of an entire schedule at once.

Consider any EDD schedule, such as the one shown in Figure 2a. Moving a job j from its single-EDD batch $B_{k=j}$ into an earlier batch B_e has the following effect:

- the lateness of all batches $B_i, i \geq k$ is reduced by p_j , as in Figure 2(b),
- the lateness of all batches $B_h, h \geq e$ is increased by $\max(0, p_j - P_e)$, where P_e is the processing time of batch B_e before j is moved into it, as in Figure 2(c).

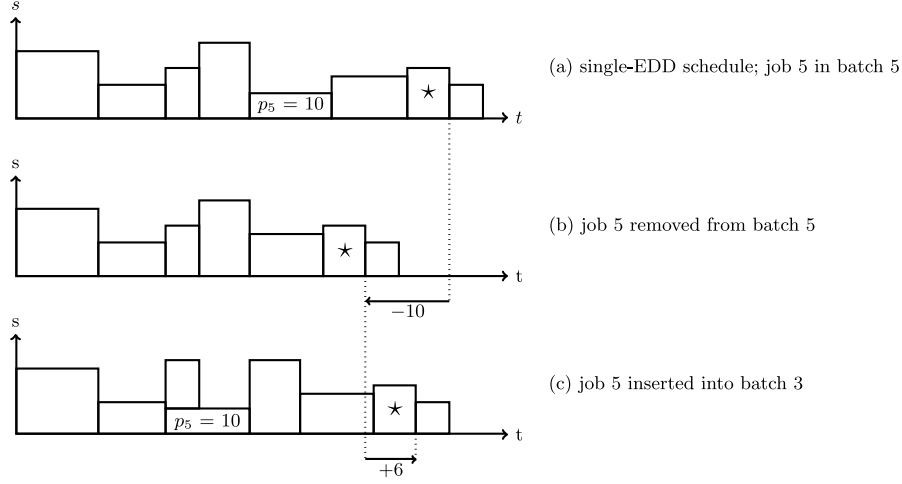


Fig. 2. Moving a job in a single-EDD schedule. Job 5 (marked “ $p_5 = 10$ ”) is moved from its single-EDD batch 5 into the earlier batch 3. This changes the lateness of job 7 (marked \star) from $L_{7,\text{single}}$ to $L_{7,\text{single}} - 10 + 6 = L_{7,\text{single}} - 4$.

In any batch, only the host job’s lateness is relevant to L_{\max} . In other words, the lateness of batch B_k equals the lateness of job $j = k$, unless the job was moved into an earlier batch (in which case $P_k = 0$ due to Proposition 2 and $L_k = L_{k-1}$). Therefore, we can understand the lateness of batch B_k as its lateness in the single-EDD schedule, written as $L_{k,\text{single}}$, modified by the summed effect that all moves of other jobs into and out of batches $h \leq k$ have on the completion time of B_k :

$$L_k = L_{k,\text{single}} + \sum_{h \leq k} \underbrace{P'_h - p_h(2 - x_{hh})}_{T_h} \quad \forall k \in \mathcal{B}^* \quad (17)$$

$$P'_k \geq p_j x_{jk} \quad \forall \{j \in \mathcal{J}, k \in \mathcal{B} | j \geq k\} \quad (18)$$

$$P'_k \geq p_j \quad \forall \{j \in \mathcal{J}, k \in \mathcal{B} | j = k\} \quad (19)$$

where $P'_k = \max(P_k, p_k) \forall k \in \mathcal{B}$ as defined in constraints (18) and (19).

For every batch $B_k \in \mathcal{B}^*$, consider the possible scenarios for all batches $B_h, h \leq k$:

- Batch B_h holds its host job. Then $x_{hh} = 1$ and the summand T_h evaluates to $P'_h - p_h$. If B_h has guest jobs, then $P'_h - p_h > 0$ if any of them are longer than the host job; if all guests are shorter, $P'_h = p_h$ and $T_h = 0$.
- Batch B_h is hostless and thus empty. We require $T_h = -p_h$ in accordance with Figure 2(b). To achieve this, we state in constraints (19) that P'_h never drop below the length of its host job, even when $P_h = 0$. With this in effect, the minimization objective enforces $P'_h = p_h$ and $T_h = P'_h - 2p_h = -p_h$.

Effectively thus, we add to L_k the increase in processing time due to guests, $\max(0, P'_h - p_h)$, for every non-empty batch B_h ; we subtract from L_k the host job processing time p_h for every empty batch B_h . This is congruent with Figure 2 above.

Note that the P'_k variables used here are not the same as the physically meaningful P_k due to constraints (19), which is why we distinguish them with a prime mark.

The net sum of these additions and subtractions to and from $L_{k,\text{single}}$ adjusts the lateness of batch k to its correct number given the values of x_{jk} .

Proposition 4. *Constraints (17)–(19) are sufficient to define L_{\max} .*

Proof. We first express the lateness of the first batch as its single-EDD lateness, plus $\max(P_1 - p_1, 0)$ since guest jobs may cause $P_1 > p_1$:

$$L_1 = L_{1,\text{single}} + \underbrace{P'_1 - p_1(2 - x_{11})}_{=\max(P_1 - p_1, 0)}. \quad (20)$$

We then hypothesize that the above is generally true for any batch L_k :

$$L_k = L_{k,\text{single}} + \sum_{h \leq k} P'_h - p_h(2 - x_{hh}) \quad (21)$$

To show how an expression for L_{k+1} follows from equation (21), we relate L_{k+1} to L_k :

$$L_{k+1} = L_k + \underbrace{P'_{k+1}}_{C_{k+1} - C_k} - \underbrace{(d_{k+1} - d_k)}_{D_{k+1} - D_k} \quad (22)$$

But the difference $L_{k+1} - L_k$ can also be written in terms of single-EDD lateness values and processing time adjustments due to guests or host-less-ness, all of which are expressed in known terms:

$$P_{k+1} - (d_{k+1} - d_k) = L_{k+1,\text{single}} - L_{k,\text{single}} + \begin{cases} \max(P_{k+1} - p_{k+1}, 0) & x_{k+1,k+1} = 1 \\ -p_{k+1} & x_{k+1,k+1} = 0 \end{cases}. \quad (23)$$

Note that the conditional expression is equivalent to $P'_{k+1} - p_{k+1}(2 - x_{k+1,k+1})$. We can now rewrite our (21) for L_{k+1} and arrive at

$$L_{k+1} = \left[L_{k,\text{single}} + \sum_{h \leq k} P'_h - p_h(2 - x_{hh}) \right] \quad (24)$$

$$+ L_{k+1,\text{single}} - L_{k,\text{single}} + P'_{k+1} - p_{k+1}(2 - x_{k+1,k+1}), \quad (25)$$

which, after cancelling out $L_{k,\text{single}}$ terms, becomes

$$L_{k+1} = L_{k+1,\text{single}} + \sum_{h \leq k+1} P'_h - p_h(2 - x_{hh}) \quad (26)$$

and agrees with (21). Since (26) follows from (21), and the latter is true for the base case of $k = 1$, (17) is true for all k .

Note also that if a batch $B_k \in \mathcal{B}^*$ is empty, then $d_k = d_{k-1}$ and $x_{kk} = 0$ if $B_{k-1} \notin \mathcal{B}^*$, so $L_k = L_{k-1}$; and $d_k > d_{k-1}$, so $L_k < L_{k-1}$, if $B_{k-1} \in \mathcal{B}^*$.

$$\begin{aligned} L_k &= [L_{k-1,\text{single}} + P_k - (d_k - d_{k-1})] \\ &\quad + \left[\sum_{h \leq k-1} P'_h - p_h(2 - x_{hh}) \right] + P'_k - p_k(2 - x_{kk}) = L_{k-1} \end{aligned}$$

as $d_k = d_{k-1}$ and x_{kk} if B_k is empty. □

↑ **CB:** Maybe there's a way to elegantly combine the informal explanation and the induction somehow?

3.3 Additional lazy constraints

Additionally, *lazy constraints* [17] are used in the model. Lazy constraints can be generated in the thousands prior to solving, but are not immediately used in the model. Instead, they are checked whenever an integral solution is found, and only those that are violated are added to the LP model. In practice, only few of the lazy constraints are used in the solution process. Nevertheless, they can noticeably improve solving time in some cases.

Symmetry-breaking rule This rule creates an explicit, arbitrary preference for certain solutions. Consider two schedules S_1 and S_2 . Both schedules contain batches B_h and B_k , both of which are holding their respective host jobs only. Two jobs j and i are now assigned as the only guests to the two batches; furthermore $\max(p_i, p_j) \leq \min(p_h, p_k)$, $\max(s_h, s_k) + \max(s_j, s_i) \leq b$ and $\min(d_j, d_i) \geq \max(d_h, d_k)$. If $j \rightarrow B_h$ and $i \rightarrow B_k$ in schedule S_1 and vice versa in S_2 , then the constraint renders S_2 infeasible.

$$\begin{aligned} & \forall \{j, i \in \mathcal{J}, \\ & \quad h, k \in K \\ & \quad | \ h < k < j < i \wedge \\ & \quad [p_i \leq k_h \wedge b - s_h \geq s_i \\ & \quad \quad \forall i \in \{j, i\}, \\ & \quad \quad \forall h \in \{h, k\}]\} \\ 2(4 - x_{hh} - x_{kk} - x_{jh} - x_{jk} - x_{ih} - x_{ik} \\ & \quad + \sum_{\substack{g \\ g \neq j \\ g \neq i}} (x_{gh} + x_{gk})) \geq x_{jk} + x_{ih} \end{aligned} \quad (27)$$

The left-hand side of the equation evaluates to zero exactly when the above conditions are met, which in turn disallows the assignment given on the right. For all other job/batch pairings, the left side evaluates to at least two, which places no constraint on the right hand side at all.

This kind of symmetry-breaking rule can theoretically be extended to situations with $m > 2$ batches, with the number of constraints growing combinatorially with m . Since it takes a constant but appreciable time to generate these constraints prior to solving, we have in our trials kept to the simplest variant shown here, and limited their use to problem instances with $n \geq 50$ jobs.

Dominance rule on required assignments A schedule is not uniquely optimal if a job j is left in its single-EDD batch although there is capacity for it in an earlier batch. This constraint can be expressed logically as: if a job j can be safely assigned to B_k without violating the capacity constraint, then j must be moved somewhere, or B_k must be empty (or both).

The left side of the above *if-then* statement is written as $(1.0 + b - s_j - \sum_{\substack{i=k \\ i \neq j}}^{n_j} s_i x_{ik})/b$, which evaluates to 1.0 or greater iff s_k plus the sizes of guest jobs in k sum to less than $b - s_j$. The constraint is written as follows:

$$2 - x_{jj} - x_{kk} \geq \left(1.0 + b - s_j - \sum_{\substack{i=k \\ i \neq j}}^{n_j} s_i x_{ik} \right) / b \quad \begin{array}{l} \forall \{k \in \mathcal{B}, j \in \mathcal{J} \\ |j > k \wedge p_k \geq p_j \\ \wedge s_k + s_j \leq b\} \end{array} \quad (28)$$

As with the rule above, we have found that only more difficult problems with $n \geq 50$ benefit from these constraints.

4 A New MIP Model

The full novel MIP model we are proposing is defined in Model 4.1.

Constraints (30) and (31) are uniqueness and capacity constraints: batches have to remain within capacity b , and every job can only occupy one batch.

Constraints (32) and (33) define the value of P_k for every batch k as the longest p of all jobs in k , but at least p_k . This is required in (35), which follows the explanation above.

Constraints (34) ensure that no job is moved into a host-less batch, i.e. in order to move job j into batch k ($x_{jk} = 1$), job k must still be in batch k ($x_{kk} = 1$).

Constraints (36) implement the requirement that jobs are only moved into earlier batches.

Constraints (37) and (38) implement the additional lazy constraints described above.

$$\text{Min. } L_{\max} \quad (29)$$

$$\text{s.t. } \sum_k x_{jk} = 1 \quad \forall j \in \mathcal{J} \quad (30)$$

$$\sum_j s_j x_{jk} \leq b \quad \forall k \in \mathcal{B} \quad (31)$$

$$P'_k \geq p_j x_{jk} \quad \forall \{j \in \mathcal{J}, k \in \mathcal{B} | j \geq k\} \quad (32)$$

$$P'_k \geq p_j \quad \forall \{j \in \mathcal{J}, k \in \mathcal{B} | j = k\} \quad (33)$$

$$x_{kk} \geq x_{jk} \quad \forall \{j \in \mathcal{J}, k \in \mathcal{B} | j > k\} \quad (34)$$

$$L_{\max} \geq L_{k,\text{single}} + \sum_{h \leq k} P'_h - p_h(2 - x_{hh}) \quad \forall k \in \mathcal{B}^* \quad (35)$$

$$x_{jk} = 0 \quad \forall \{j \in \mathcal{J}, k \in \mathcal{B} | j < k\} \quad (36)$$

$$\begin{aligned} (*) \quad & 2(4 - x_{k_1, k_1} - x_{k_2, k_2} - x_{j_1, k_1} - x_{j_1, k_2} - x_{j_2, k_1} - x_{j_2, k_2} \\ & + \sum_{\substack{j \\ j \neq j_1 \\ j \neq j_2}} (x_{j, k_1} + x_{j, k_2})) \geq x_{j_1, k_2} + x_{j_2, k_1} \\ & \forall \{j_1, j_2 \in \mathcal{J}, \\ & \quad k_1, k_2 \in \mathcal{K} \\ & \quad | k_1 < k_2 < j_1 < j_2 \wedge \\ & \quad [p_i \leq k_h \wedge b - s_h \geq s_i \\ & \quad \quad \forall i \in \{j_1, j_2\}, \\ & \quad \quad \forall h \in \{k_1, k_2\}]\} \end{aligned} \quad (37)$$

$$\begin{aligned} (*) \quad & 2 - x_{jj} - x_{kk} \geq \left(1.0 + b - s_j - \sum_{\substack{i=k \\ i \neq j}}^{n_j} s_i x_{ik} \right) / b \\ & \forall \{k \in \mathcal{B}, j \in \mathcal{J} \\ & \quad | j > k \wedge p_k \geq p_j \\ & \quad \wedge s_k + s_j \leq b\} \end{aligned} \quad (38)$$

Model 4.1. Move-based MIP model. Constraints marked (*) are lazy constraints.

5 Empirical comparison

We empirically compared the performance of the CP model by Malapert et al. and Model 4.1. Both models were run on 120 benchmark instances provided by Malapert et al. (i.e. 40 instances of each $n_j = \{20, 50, 75\}$). The benchmarks are generated as specified in [11], with a capacity of $b = 10$ and values for p_j , s_j and d_j distributed as follows:

$$p_j = U[1, 99] \quad (39)$$

$$s_j = U[1, 10] \quad (40)$$

$$d_j = U[0, 0.1] \cdot \tilde{C}_{\max} + U[1, 3] \cdot p_j \quad (41)$$

where $U[a, b]$ is a uniform distribution between a and b , and $\tilde{C}_{\max} = \frac{1}{bn} \cdot (\sum_{j=1}^{n_j} s_j \cdot \sum_{j=1}^{n_j} p_j)$ is an approximation of the time required to process all jobs.

The MIP benchmarks were run using CPLEX 12.5 [18] on an Intel i7 Q740 CPU (1.73 GHz) and 8 GB RAM in single-thread mode, with CPLEX parameters `Probe = Aggressive` and `MIPEmphasis = Optimality` (the latter for $n = 20$ only). The CP was implemented using the Choco solver library [19] and run on the same machine using the same problem instances.² Solving was aborted after a time of 3600 seconds (1 hour).

The reference MIP model solves fewer than a third of all instances to optimality within the time limit. The branch-and-price model proposed by Daste [11] also performs considerably worse than CP [1]. Neither of the two is included in these results.

5.1 Results

The overview in Table 1 below shows aggregated results that demonstrate the performance and robustness of our new model.

As shown in Figure 3, our MIP model performed better overall on instances with $n_j = 20$ and $n_j = 75$, while MIP and CP perform similarly well on intermediate problems ($n_j = 50$).

Wherever an optimal solution was not found within the 1-hour time limit, the improved MIP model achieved a significantly better solution quality: out of the 40 instances with $n_j = 75$, 22 were solved to optimality by both CP and MIP, 13 were solved to optimality by the MIP only, and 5 were solved by neither model within an hour.

6 Discussion

Our results show that the improved MIP model is an improvement over previous approaches, demonstrating that at least in this case, the performance of custom

² The authors would like to extend a warm thank-you to Arnaud Malapert for both providing his code and helping us run it.

n_j	optimal soln. found by	instances	solving time [s]		absolute gap	
			CP	MIP	CP	MIP
20	both models	40/40	0.42	0.04	0	0
50	both models	40/40	5.67	4.16	0	0
75	both models	22/40	49.30	52.88	0	0
	CP model only	0/40	—	—	—	—
	MIP model only	13/40	> 3600	139.86	323.46	0
	neither model	5/40	> 3600	> 3600	310.40	25.00

Table 1. Summary of empirical results. Values are geometric means for solving time and arithmetic means for absolute gaps. Note that no relative gaps are given because negative lower bounds result in misleading percentages; see Figure 4 for a more detailed gap comparison.

implementations can indeed be matched and exceeded by a comparatively simple mathematical formulation.

A comparison of solution quality where no optimal schedule was found confirms the robustness of the improved MIP model: as Figure 4 illustrates, the gap ($UB(L_{\max}) - LB(L_{\max})$) is consistently larger in the CP model. This means that even with very difficult problems, our model will often give near-optimal solutions more quickly than Malapert et al.’s CP model.

A likely reason for the new model’s performance is its reduced size: while the old model required $3n^2 + 8n$ constraints over $n^2 + 3n$ variables, our model uses fewer than $2.5n^2 + 2.5n$ constraints over $n^2 + n$ variables. In addition, CPLEX’ presolve methods are more effective on our model (see Table 2).

Mean over 120 instances	Reference MIP		Improved MIP	
	Variables	Constraints	Variables	Constraints
Before presolve	7415.14	3033.81	4291.48	2882.76
After presolve	2209.34	1513.06	754.57	687.13
Reduction	−70.2%	−50.1%	−82.4%	−76.1%

Table 2. Average numbers of variables and constraints (i.e. columns and rows) in reference and improved MIP models before and after processing by CPLEX’ presolve routines.

Figure 5 shows the evolution of bounds for the three models over the first few seconds. It is based on the solvers’ logs for the 40 instances with $n = 75$, which contain only irregular timestamps. Yet even such approximate data demonstrates that it is the upper bound on which both the CP and our improved MIP defeat the reference MIP.

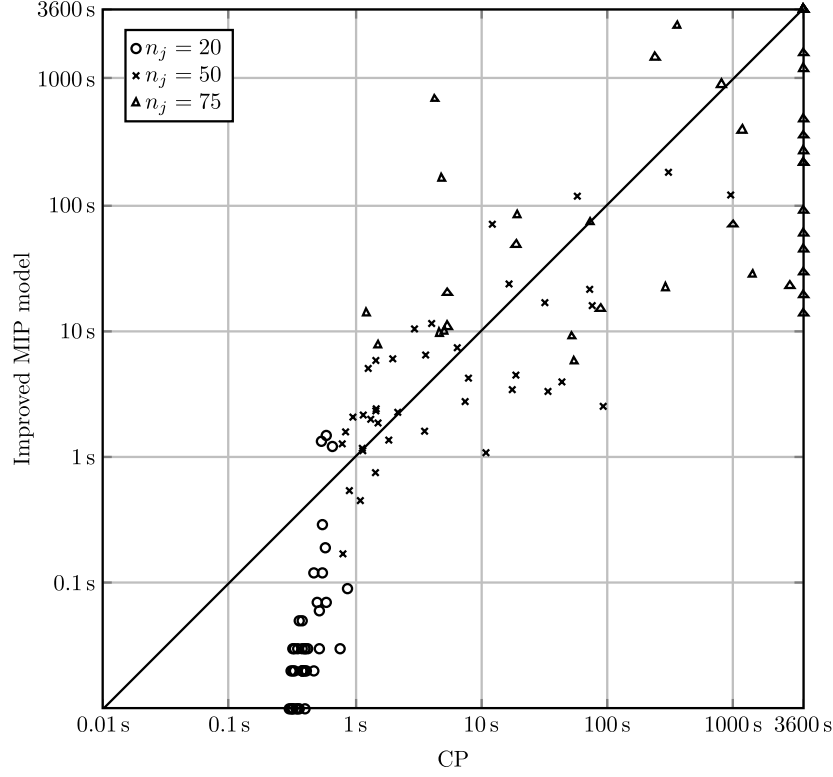


Fig. 3. Performance comparison over 120 instances, each represented by one data point. Horizontal/vertical coordinates correspond to solving time by CP model and improved MIP model, respectively. Note that 16 instances were not solved to optimality within an hour by either the CP model or both models.

Mathematical models also have the general benefit of being more readily understandable, straightforward to implement and reasonably easy to adapt to new, similar problems.

6.1 Correlation between problem configuration and solving time

The mean number of jobs per batch (n_j/n_k) in the optimal solution strongly correlates with the time needed to solve to optimality. This is unsurprising given that a problem with a great number of large jobs (i.e. large s_j) will be of relatively low difficulty in terms of bin packing.

An excellent predictor of solving time therefore appears to be what Baptiste & LePape [20] call the *disjunction ratio*: the ratio between the number of job pairs which cannot run in parallel (as $s_{j_1} + s_{j_2} > b$) and the total number of job pairs n_j^2 . For a sample of 40 random instances of $n_j = 30$, we found

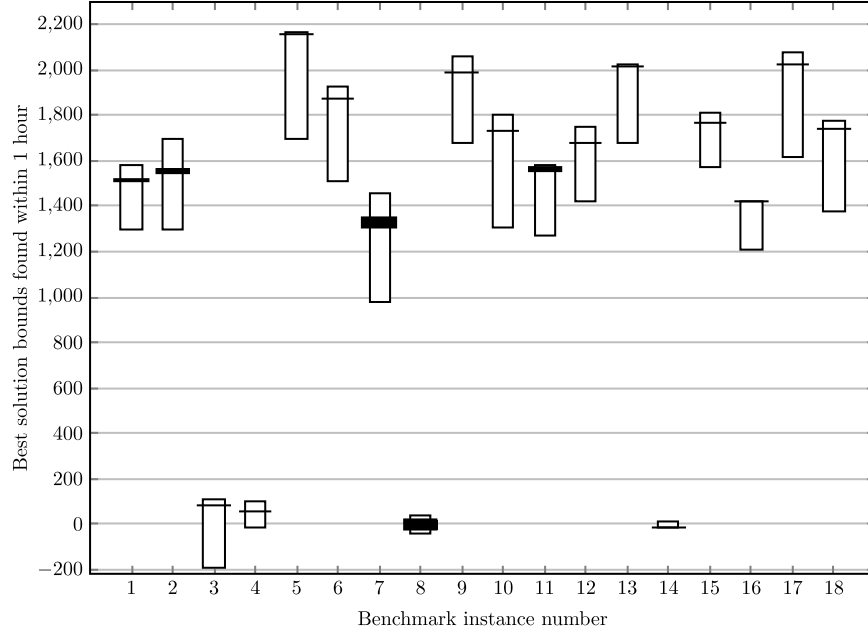


Fig. 4. Comparison of solution quality for the 18 instances that were not solved to optimality within an hour by either the CP model or both models. White bars represent the LB-UB gap achieved by the CP model, black bars the LB-UB gap achieved by the improved MIP model (straight line if solved to optimality).

a correlation between n_j/n_k and solution time of $r = 0.72$ and a correlation between disjunction ratio and solution time of $r = -0.64$.

As a corollary, the solving time is greatly dependent on the capacity of the machine. Larger capacities, relative to the average job size \bar{s}_j , will correlate with longer solving times as the disjunction ratio decreases.

↑ **CB: how important is this section? Since we're using n now instead of n_j and n_k , we'd have to redefine those ... wouldn't a simple sentence mentioning the disjunction ratio suffice?**

6.2 Generalizing the search technique used in the improved MIP model

The effects of binary decision variables on the objective function can often be considered discretely, which allows us to reason about them algorithmically even though they constitute an entirely declarative model. Indeed, the concept of *moves* typically appears in local search techniques, such as the *Large Neighbourhood Search* [21] where moves correspond to the removal and re-insertion of jobs from and into the schedule, similarly to what we propose in Section 3.2 above.

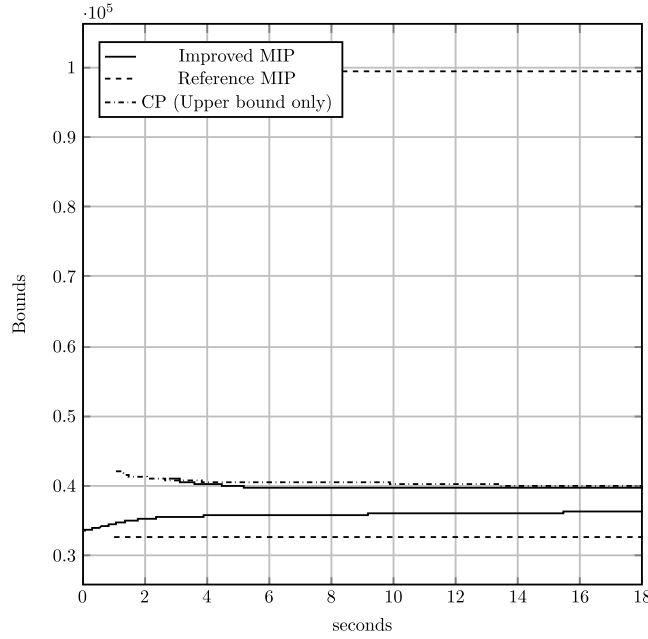


Fig. 5. Evolution of upper and lower bounds (i.e. best known feasible solutions and LP relaxation solutions at the nodes) over the first ≈ 18 seconds. The left cutoffs of each line indicate the approximate mean time at which the respective bound was first found.

Analogous to the notion of “neighbourhood” is the set of schedules than can be reached via feasible moves, starting from any other schedule. We can then easily represent the objective function as a sum of changes from some canonical solution. Such models, while non-standard, may provide substantial insight into problem structure that can be exploited in a MIP (or CP) formulation.

7 Conclusion

CB: Needs rewriting later.

By examining the batch scheduling problem carefully, we were able to identify redundancies in the original model. We proposed a new model and showed that its performance is comparable to, or better than that of the state-of-the-art constraint programming implementation.

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