Serii Fourier

Concepte de bază:

Def: O funcție f se numește periodică de perioadă T dc, $\exists\, T > 0$ a.î. $f(x+T) = f(x)$ $\forall\, x$ din dom. fct f.

Exemple:

- $\sin x, \cos x \to T = 2\pi$ (perioada principala)
- $\sin 4x \to T = \frac{\pi}{2}$
- $\{x\} \to T = 1$

Def: $f: \mathbb{R} \to \mathbb{R}$

a) f pară dacă $f(-x) = f(x)$, $\forall\, x \in \mathbb{R}$

- $\cos x, |x|, x^2, x^{2k}$

b) f impară dacă $f(-x) = -f(x)$

- $\sin x, \operatorname{tg} x, x, x^{2k+1}$

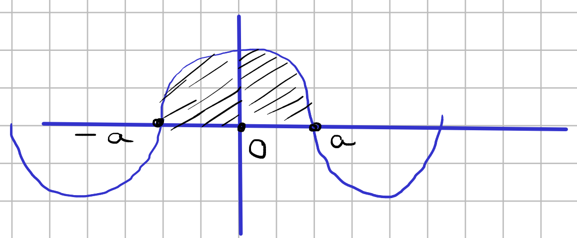
Proprietăți: $f: \mathbb{R} \to \mathbb{R}$ și $a > 0$

a) f pară

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

Proof:
$$\int_{-a}^{a} f(x)\,dx = \underbrace{\int_{-a}^{0} f(x)\,dx}_{\substack{x=-y \\ 1\,dx = -1\,dy}} + \int_{0}^{a} f(x)\,dx = \int_{0}^{a} \underbrace{f(-y)}_{f(y)}\,dy + \int_{0}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

$\cos x \to$ pară



b) f impar

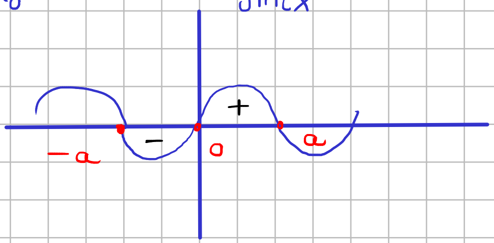
$$\int_{-a}^{a} f(x)\,dx = 0$$

Proof: $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx = \int_{0}^{a} \underbrace{f(-y)}_{-f(y)}\,dy + \int_{0}^{a} f(x)\,dx = 0$

$x=-y$

$1\,dx = -1\,dy$

$\sin x \rightarrow$ impar



Def: Două funcții continue $f, g : [a, b] \rightarrow \mathbb{R}$ se numesc ortogonale pe intervalul $[a,b]$ dacă $\int_{a}^{b} f(x)\cdot g(x)\,dx = 0$

OBS: O mulțime de fct. se numește ortogonală dacă oricare 2 sunt ortogonale.

Teoremă:

Mulțimea $\underbrace{\{1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos nx, \sin nx\}}_{\text{SISTEM TRIGONOMETRIC}}$ este ortogonală pe $[-\pi, \pi]$

Mai precis dacă $m, n \in \mathbb{N}^*$ au loc

i) $\int_{-\pi}^{\pi} \cos mx \cdot \cos nx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}$

ii) $\int_{-\pi}^{\pi} \cos mx \cdot \sin nx = 0$

iii) $\int_{-\pi}^{\pi} \sin mx \cdot \sin nx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}$

iv) $\int_{-\pi}^{\pi} \cos mx\,dx = \int_{-\pi}^{\pi} \sin mx\,dx = 0$

Obs: Funcțiile din sistemul trigon. sunt liniar ind.

Teoremă: Fie V, un spațiu vectorial peste corpul $\mathbb{K}$, înzestrat cu produs scalar și S o mulțime ortogonală formată cu vectori nenuli din V. Atunci S liniar independentă:

$$S = \{v_1, \dots, v_n\}$$

$\boxed{R}$ $v_1, \dots, v_n \in V$ lin. indep dacă singura alegere a scalarilor $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{K}$ a.î. $\alpha_1 v_1 + \dots + \alpha_n v_n = 0$ este $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$

P.p. abs că vectorii din S sunt l. dependenți $\Rightarrow \exists\, \alpha_1, \dots, \alpha_n$ NU TOȚI NULI a.î. $\alpha_1 v_1 + \dots + \alpha_n v_n = 0$

$$< \alpha_1 v_1 + \dots + \alpha_n v_n, v_1 > = < 0, v_1 >$$

$$\alpha_1 \underbrace{< v_1, v_1 >}_{> 0} + \alpha_2 \underbrace{< v_2, v_1 >}_{0} + \dots + \alpha_n \underbrace{< v_n, v_1 >}_{0} = 0$$

$$\Rightarrow \alpha_1 = 0 \quad \Rightarrow \quad \alpha_2 v_2 + \dots + \alpha_n v_n = 0$$

$$< \alpha_2 v_2 + \dots + \alpha_n v_n, v_2 > = < 0, v_2 > = 0$$

$$\Rightarrow \alpha_2 = 0$$

$$\vdots$$

$$\alpha_n = 0$$

Fie $m \in \mathbb{N}^*$. Vom arăta că $\{1, \cos kx, \sin kx : k \in \mathbb{N}\}$ e lin. indep

P.p. abs că sunt liniar dependente $\Rightarrow \exists\, b_0, b_k, a_k \in \mathbb{R}$ nu toți nuli a.î.

$$b_0 + \sum_{k=1}^{m} b_k \cos kx + a_k \sin kx = 0$$

$$b_0 + b_1 \cos x + a_1 \sin x + b_2 \cos 2x + a_2 \sin 2x + \dots + b_m \cos mx + a_m \sin mx = 0 \,\Big|\, \int_{-\pi}^{\pi}$$

$$b_0 \cdot \int_{-\pi}^{\pi} 1\, dx + \sum_{k=1}^{m} b_k \cdot \underbrace{\int_{-\pi}^{\pi} \cos kx\, dx}_{0} + a_k \cdot \underbrace{\int_{-\pi}^{\pi} \sin kx\, dx}_{0} = 0$$

$$b_0 \cdot x \Big|_{-\pi}^{\pi} = 0 \iff 2\pi\, b_0 = 0 \Rightarrow b_0 = 0$$

$b_1 \cos x + a_1 \sin x + \ldots + b_m \cos mx + a_m \sin mx = 0 \quad | \cdot \cos x$ și integrăm de la $-\pi$ la π

$$b_1 \int_{-\pi}^{\pi} \underbrace{\cos^2 x}_{\substack{\| \\ \frac{1+\cos 2x}{2}}} dx + a_1 \underbrace{\sin x \cos x}_{0} + \sum_{k=2}^{m} b_k \cdot \int_{-\pi}^{\pi} \underbrace{\cos kx \cdot \cos x}_{0} dx + a_k \cdot \int_{-\pi}^{\pi} \underbrace{\sin kx \cdot \sin x}_{0} dx = 0$$

$\pi \cdot b_1 = 0 \Rightarrow b_1$

$\vdots$

inductiv $b_0 = b_1 = \ldots = b_m = a_1 = a_2 = \ldots = a_m = 0$ (dd)

$\Rightarrow$ liniar indep.

Cu ajutorul funcțiilor anterioare formăm seria:

$$b_0 + b_1 \cos x + a_1 \sin x + \ldots + b_n \cos nx + a_n \sin nx + \ldots$$

Căutăm coeficienții $b_0, b_1, \ldots, b_n, a_1, a_2, \ldots, a_n, \ldots$ astfel încât

$$f(x) = b_0 + b_1 \cos x + a_1 \sin x + \ldots + b_n \cos nx + a_n \sin nx + \ldots$$

$$b_0 + b_1 \cos x + a_1 \sin x + \ldots + b_n \cos nx + a_n \sin nx + \ldots = f(x) \quad \Big| \int_{-\pi}^{\pi}$$

$$2\pi b_0 = \int_{-\pi}^{\pi} f(x)\, dx \Rightarrow \boxed{b_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx}$$

$$b_0 + b_1 \cos x + a_1 \sin x + \ldots + b_n \cos nx + a_n \sin nx + \ldots = f(x) \quad \Big| \sin kx, \text{ apoi } \int_{-\pi}^{\pi}$$

$$a_k \cdot \int_{-\pi}^{\pi} \sin^2 kx\, dx = \int_{-\pi}^{\pi} f(x) \cdot \sin kx\, dx$$

$$\int_{-\pi}^{\pi} \sin^2 kx\, dx = \int_{-\pi}^{\pi} \frac{1-\cos 2x}{2} dx = \frac{1}{2} \cdot 2\pi + \frac{1}{2} \frac{\sin 2x}{2} \Big|_{-\pi}^{\pi}$$

$$= \pi + \frac{1}{4} (\underbrace{\sin 2\pi + \sin 2\pi}_{0}) = \pi$$

$$\Rightarrow \boxed{a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \sin kx\, dx}$$

$b_0 + b_1\cos x + a_1 \sin x + \ldots + b_n \cos nx + a_n \sin nx + \ldots = f(x) \mid \cos kx$ și integrăm

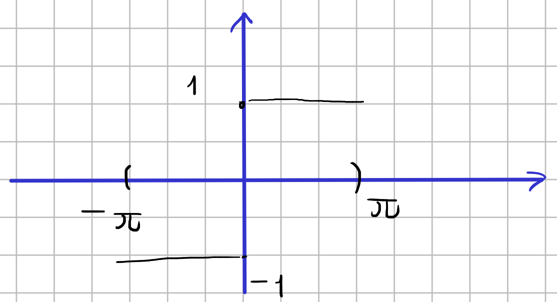
$$b_k \cdot \int_{-\pi}^{\pi} \underbrace{\cos^2 kx}_{\frac{1+\cos 2x}{2}} \, dx = \int_{-\pi}^{\pi} f(x) \cos kx \, dx$$

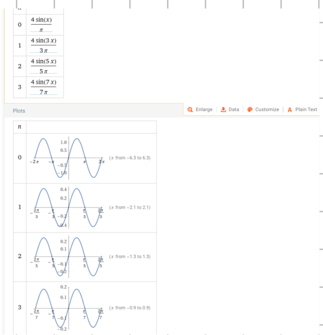
$$b_k \cdot \pi = \int_{-\pi}^{\pi} f(x) \cdot \cos kx \, dx \Rightarrow \boxed{b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \cos kx \, dx}$$

Exemplu:

$$f(x) = \begin{cases} -1, & -\pi \le x < 0 \\ 1, & 0 \le x < \pi \end{cases}$$

Dezv f în serie fourier





$$b_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left(\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx \right)$$

$\cos(k\pi) = (-1)^k$

$$= \frac{1}{2\pi}\left(-x \Big|_{-\pi}^{0} + x \Big|_{0}^{\pi} \right) = \frac{1}{2\pi}\left(-\pi + \pi \right) = 0$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \sin kx \, dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} -\sin kx \, dx + \int_{0}^{\pi} \sin kx \, dx \right)$$

$$= \frac{1}{\pi}\left(\frac{\cos kx}{k} \Big|_{-\pi}^{0} - \frac{\cos kx}{k} \Big|_{0}^{\pi} \right) = \frac{1}{k\pi}\left(1 - (-1)^k - (-1)^k + 1 \right)$$

$$= \frac{2}{k\pi}\left(1 - (-1)^k \right) = \begin{cases} 0, & k \text{ par } = 2n \\ \frac{4}{k\pi}, & k \text{ impar} \to 2n+1 \end{cases}$$

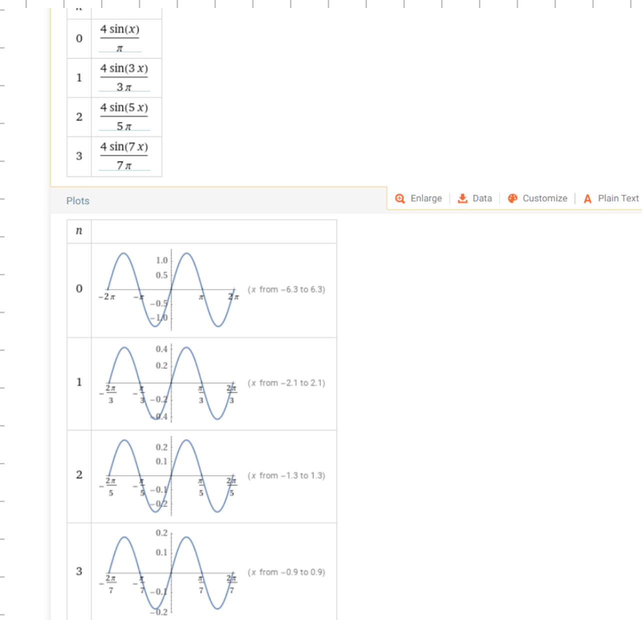
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot \cos nx\, dx = \frac{1}{\pi}\left(-\int_{-\pi}^{0}\cos nx\, dx + \int_{0}^{\pi}\cos nx\, dx\right)$$

$$= \frac{1}{\pi}\left(-\frac{\sin nx}{n}\Big|_{-\pi}^{0} + \frac{\sin nx}{n}\Big|_{0}^{\pi}\right) = 0$$

$b_n = 0, \forall\ n \in \mathbb{N}$

$a_1 = \frac{4}{\pi},\ a_2 = 0,\ a_3 = \frac{4}{3\pi},\ a_4 = 0,\ a_5 = \frac{4}{5\pi}, \ldots$

$$\Rightarrow f(x) = \sum_{n=0}^{\infty} \frac{4}{(2n+1)\pi}\cdot \sin[(2n+1)x] = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin[(2n+1)\cdot x]}{2n+1}$$



Serii Fourier Complexe:

FORMULA LUI EULER: $e^{it} = \cos t + i\sin t$ (1)

$t \mapsto -t \quad e^{-it} = \cos(-t) + i\sin(-t) = \cos t - i\sin t$ (2)

din (1)+(2) $\Rightarrow 2\cos t = e^{it} + e^{-it} \Rightarrow \cos t = \frac{e^{it}+e^{-it}}{2}$

din (1)−(2) $\Rightarrow 2i\sin t = e^{it} - e^{-it} \Rightarrow \sin t = \frac{e^{it}-e^{-it}}{2i}$

$$\cos nx = \frac{e^{inx}+e^{-inx}}{2}$$

$$\sin nx = \frac{e^{inx}-e^{-inx}}{2i} = -i\left(\frac{e^{inx}-e^{-inx}}{2}\right)$$

$$f(x) \sim b_0 + \sum_{n=1}^{\infty} b_n \cos nx + a_n \sin nx = b_0 + \sum_{n=1}^{\infty} b_n \cdot \frac{e^{inx}+e^{-inx}}{2} - i a_n \cdot \frac{e^{inx}-e^{-inx}}{2}$$

$$= b_0 + \sum_{n=1}^{\infty} \frac{(b_n - i a_n)}{2} e^{inx} + \frac{(b_n + i a_n)}{2} e^{-inx}$$

Folosim notațiile: $c_0 = b_0$

$$c_n = \frac{b_n - i a_n}{2}$$

$$c_{-n} = \frac{b_n + i a_n}{2}$$

$z = a + ib$

$\bar{z} = a - ib$

! OBS: $c_{-n} = \overline{c_n}$

$$f(x) \sim c_0 + \sum_{n=}^{\infty} c_n e^{inx} + c_{-n} \cdot e^{-inx} = c_0 + \sum_{n=1}^{\infty} c_n \cdot e^{inx} + \underbrace{\sum_{n=1}^{\infty} c_{-n} \cdot e^{-inx}}$$

$n \to -n$

$$\sum_{n=-\infty}^{-1} c_n \cdot e^{inx}$$

$$\Rightarrow f(x) = \sum_{n=-\infty}^{\infty} c_n \cdot e^{inx}$$

$$\boxed{c_n = \frac{b_n - i a_n}{2} = \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} f(x)\underbrace{(\cos nx - i \sin nx)}_{e^{-inx}}\right) dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cdot e^{-inx}\, dx}$$

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n \cdot e^{inx} \quad \text{unde } c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cdot e^{-inx}\, dx$$

Definiție: Fie $g : \mathbb{R} \to \mathbb{R}$. Spunem că g e continuă pe porțiuni dacă pentru orice interval compact $[a, b] \subset \mathbb{R}$ există o diviziune $a = x_0 < x_1 < \ldots < x_n = b$ a.î. g continuă pe fiecare interval deschis (x_{k-1}, x_k) și g are limitele laterale finite în fiecare pct x_k

Definiție: Fie $g: \mathbb{R} \to \mathbb{R}$. Spunem că g e de clasă pe porțiuni dacă pentru orice interval compact $[a,b] \subset \mathbb{R}$ există o diviziune $a = x_0 < x_1 < \dots < x_n = b$ a.î. g de clasă C^1 (→ derivabilă, cu derivata continuă) pe fiecare interval deschis (x_{k-1}, x_k) și g' are limitele laterale finite în fiecare pct x_k

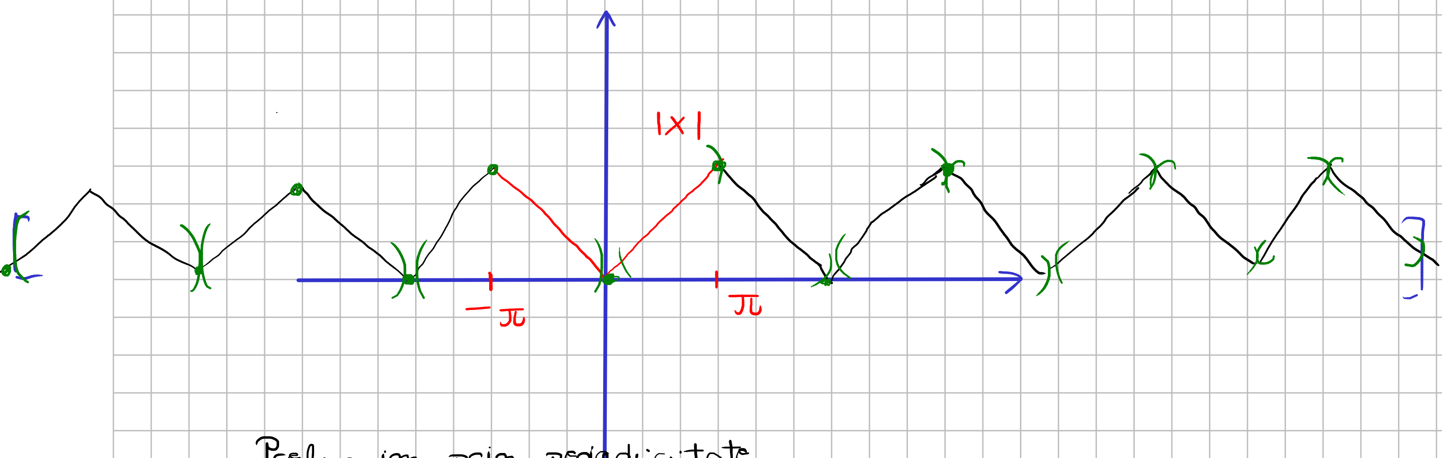
Teoremă (Dirichlet) Fie $g: \mathbb{R} \to \mathbb{R}$ 2π periodică de clasă C^1 pe porțiuni. Atunci seria Fourier a lui g converge punctual în fiecare $x \in \mathbb{R}$:

- dacă x = pct de continuitate, limita e $g(x)$
- dacă x = pct de disc, limita $\frac{g(x^-) + g(x^+)}{2}$

Ex 2: Fie funcția $f(x) = |x|$, $x \in [-\pi, \pi]$

a) Calc coef Fourier și seria Fourier pe $\mathbb{R}$

b) Calc. suma $\frac{1}{1^2} + \frac{1}{3^2} + \dots + \frac{1}{5^2} + \dots$



Prelungim prin periodicitate

$f \to$ continuă pe porțiuni

$$b_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |x|\, dx = \frac{1}{2\pi}\left(\int_{-\pi}^{0} -x\, dx + \int_0^{\pi} x\, dx \right)$$

$$= \frac{1}{2\pi}\left(-\frac{x^2}{2}\Big|_{-\pi}^{0} + \frac{x^2}{2}\Big|_0^{\pi} \right) = \frac{1}{2\pi}\left(\frac{\pi^2}{2} + \frac{\pi^2}{2} \right) = \frac{\pi}{2}$$

f pară $\Rightarrow a_n = 0, \forall n \in \mathbb{N}$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos nx \, dx = \frac{2}{\pi}\int_0^{\pi} x \cos nx \, dx = \frac{2}{\pi}\int_0^{\pi} x \cdot \left(\frac{\sin nx}{n}\right)' dx$$

$$= \frac{2}{\pi}\left(x \cdot \frac{\sin nx}{n} \Big|_0^{\pi} - \int_0^{\pi} \frac{\sin nx}{n} \, dx\right)$$

$$= \frac{2}{\pi}\left(0 + \frac{\cos nx}{n^2} \Big|_0^{\pi}\right) = \frac{2((-1)^n - 1)}{\pi n^2} = \begin{cases} 0, & n \text{ par} \\ -\frac{4}{n^2\pi}, & n \text{ impar} \end{cases}$$

$$f(x) = \frac{\pi}{2} + \sum_{n=0}^{\infty} \frac{-4}{\pi(2n+1)^2} \cdot \cos(2n+1)x$$

$x = 0$ Seria Fourier converge $\frac{\pi}{2} + \sum_{n=0}^{\infty} \frac{-2}{\pi(2n+1)^2} \xrightarrow{\infty} f(0) = 0$

$$-\frac{4}{\pi}\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \rightarrow -\frac{\pi}{2}$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \rightarrow \frac{\pi^2}{8}$$

Ex 2. Să se dezvolte în serie Fourier complexă, funcția periodică de perioadă 2π

$$f(x) = \begin{cases} -1, & -\pi \le x < 0 \\ 2, & 0 \le x < \pi \end{cases}$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) \cdot e^{-inx} dx = \frac{1}{2\pi}\left(-\int_{-\pi}^{0} e^{-inx} dx + 2\int_0^{\pi} e^{-inx} dx\right)$$

$$= \frac{1}{2\pi} \cdot \left(\frac{e^{-inx}}{in} \Big|_{-\pi}^{0} - 2\frac{e^{-inx}}{in} \Big|_0^{\pi}\right) =$$

$$= \frac{1}{2\pi}\left(\frac{1 - e^{in\pi} - 2e^{-in\pi} + 2}{ni}\right) = \frac{3 - e^{in\pi} - 2e^{-in\pi}}{2n\pi i}$$

$$e^{in\pi} = \cos n\pi + i \sin n\pi = (-1)^n$$

$$e^{-in\pi} = \cos n\pi - i\sin n\pi = (-1)^n$$

$$\Rightarrow \quad c_n = \frac{3\left(1-(-1)^n\right)}{2n\pi i} = \begin{cases} \frac{3}{2(2k+1)\pi i}, & n \text{ impar} \\ 0, & n \text{ par} \end{cases}$$

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2}$$

$$f(x) \sim \frac{1}{2} + \frac{3}{2\pi i} \cdot \sum_{k=-\infty}^{\infty} \frac{1}{2k+1} \cdot e^{(2k+1)ix}$$