1.1.

$$e\sigma(x) = 0 \leq 7 \Rightarrow 1$$

1) $$\frac{e_\sigma(x \leq 7) = 1 \qquad (z := x, \sigma) \Downarrow \sigma'}{(\underbrace{\text{if } x \leq 7 \text{ then } z := x \text{ else } z := y}_{A}, \sigma) \to \sigma'}$$

$$(A, \sigma) \Downarrow \sigma[z := x] \equiv \begin{array}{l} x \to 0 \\ y \to 1 \\ z \to 0 \end{array} \Bigg| \Rightarrow$$

$\sigma'(x) = 0$

$\sigma'(y) = 1$

$\sigma'(z) = 0$

2) $\sigma(x) = 6$, $\sigma(y) = 3$

$e(4 \leq x)$

$e(4) \leq e\sigma(x)$

$4 \leq 6 \to 1$

$$\frac{e_\sigma(4 \leq x) = 1 \qquad (x, \sigma) \Downarrow \sigma^3 \quad (\text{while } 4 \leq x \text{ do } x := x-1, \sigma_3)}{(\text{while } 4 \leq x \text{ do } x := x-1, \sigma) \Downarrow \sigma_2 \left(\begin{array}{l} \sigma(x) = 3 \\ \sigma(y) = 3 \end{array} \right)}$$

$e\sigma_2(4 \leq x) \Leftrightarrow e_{\sigma_2}(4) \leq e_{\sigma_2}(x)$

$4 \leq 3 \Rightarrow 0 \Rightarrow$ $$\frac{e\sigma_2(4 \leq x) = 0}{(\text{while } 4 \leq x \text{ do } x := x-y, \sigma_2) \Downarrow \sigma_2} \Rightarrow$$

$\Rightarrow \sigma_2 = \{\sigma_2(x) = 3, \sigma_2(y) = 3\}$

3. $e_\sigma(\neg(\overbrace{y = 0}^{b})) = \neg(e_\sigma(y) = e_\sigma(0)) =$

$= \neg(3 = 0) = 1$

$$\frac{e_\sigma(b) = 1 \quad (\overbrace{y := y-1 \; ; \; x := 2 * x}^{c}, \sigma) \Downarrow \sigma_3 \quad (\text{while } b \text{ do } c, \sigma_3) \Downarrow \sigma}{(\text{while } b \text{ do } c, \sigma) \Downarrow \sigma_2}$$

$\sigma_2 \equiv \{ y \to 2, \; x \to 2 \}$ $\qquad \sigma_3 \equiv y \to 1, \; x \to 4$

$e_{\sigma_2}(\neg(y=0)) = \neg(e_{\sigma_2}(y) = e_{\sigma_2}(0)) =$

$= \neg(2 = 0) = 1$

$$\frac{e_{\sigma_2}(b) = 1 \quad (c, \sigma_2) \Downarrow \sigma_4 \quad (\text{while } b \text{ do } c, \sigma_4) \Downarrow \sigma_3}{(\text{while } b \text{ do } c, \sigma_2) \Downarrow \sigma_2}$$

$e_{\sigma_3}(\neg(y = 0)) = \neg(1 = 0) = 1$

$\sigma_4 \equiv y \to 0, \; x \to 8$

$$\frac{e_{\sigma_3}(b) = 1 \quad (c, \sigma_3) \Downarrow \sigma_5 \quad (\text{while } b \text{ do } c, \sigma_5) \Downarrow \sigma_4}{(\text{while } b \text{ do } c, \sigma_3) \Downarrow \sigma_3}$$

$$\frac{e_{\sigma_4}(b) = 0}{(\text{while } b \text{ do } c, \sigma_4) \Downarrow \sigma_4}$$ $\Rightarrow \sigma_4 \equiv \{ \sigma_4(x) = 8, \; \sigma_4(y) = 0 \}$

2.1.

1) $f(x,y)$, $f(h(x),x)$, $f(x,h)$

$x = h(x) \Rightarrow$ eșec

$y = x$

2) $f(x, f(x, g(y)))$, $f(u,z)$, $f(g(y), y)$

$x = g(y)$

$f(x, g(y)) = y \Leftrightarrow y = f(x, g(y)) \Rightarrow$ eșec

3) $f(f(x,y), x)$, $f(g(y), z)$, $f(u, h(z))$

$g(y) = u$

$z = h(z) \Rightarrow$ eșec

4) $f(f(x,y), x)$, $f(v,u)$, $f(u, h(z))$

$v = u$

$u = h(z)$

$v = f(x,y)$

$u = x$

$v = u \Rightarrow v = x \Rightarrow x = f(x,y) \rightarrow$ eșec

5) $f(f(x,y), x)$, $f(v,u)$, $f(u,z)$

$v = f(x,y)$
$u = x$

$u = v \Rightarrow u = f(x,y)$ | $\Rightarrow f(x,y) = x \Leftrightarrow x = f(x,y) \Rightarrow$
$z = u$

$\Rightarrow$ eșec

6) $f(f(g(x), h(y)), h(z))$, $f(f(u, h(h(x))), h(y))$, $f(v,w)$

$f(g(x), h(y)) = f(u, h(h(x))) \Rightarrow$
$h(z) = h(y) \Rightarrow z = y$

$\Rightarrow u = g(x)$
$h(y) = h(h(x)) \Rightarrow y = h(x)$ ~~...~~

$v = f(u, h(h(x)))$
$w = h(y)$

$z = y$
$u = g(x)$
$w = h(y)$
~~...~~ $= y = h(x)$
$v = f(u, h(h(x))$

7) $p(x, x, z)$, $p(f(a,a), y, y)$, $p(f(x,a), e, z)$

$f(a,a) = f(x,a)$
$y = b$
$y = z$

$x = f(x,a)$ → eş ec
$x = b$

$x = z$

8) $p(x, x, z)$, $p(f(a,a), y, y)$, $p(x, b, z)$

$x = f(a,a)$
$b = y$
$z = y$

$\cancel{x = f(a,a)}$
$x = y$
$\cancel{z = y}$

$\Rightarrow$
$x = f(a,a)$
$b = y$
$z = y$
$x = y$

$\Rightarrow$ $y = b = z \neq x$

9) $p(x, x, z)$, $p(f(a,a), y, y)$, $p(x, f(a,a), z)$

$x = f(a,a)$
$x = y$
$z = y$

$\cancel{x = x}$

$\cancel{x = f(a,a)}$
$\cancel{z = z}$

$\Rightarrow$
$x = f(a,a)$
$x = y = z$

10) $p(f(x,a), g(y), z)$, $p(f(a,a), z, u)$, $p(v, a, z)$

$f(x,a) = f(a,a) \Rightarrow x = a$

$z = g(y)$

$z = u$

$v = f(x,a)$

$u = g(y)$

$\cancel{z = z}$

$x = a$

$z = u = g(y)$

$\Rightarrow v = f(x,a) = f(a,a)$

3.1.

1)

$((\lambda z.z)(\lambda q.(qq)))(\lambda s.(sa))$

$((\lambda z.z)[z := \lambda q.(qq)])(\lambda s.(sa))$

$(\lambda q.(qq))(\lambda s.(sa))$

$(\lambda q.(qq))[q := \lambda s.(sa)]$

$(\lambda s.(sa))(\lambda s.(sa))$

$(\lambda s.(sa))[s := \cancel{(sa)} \lambda s.(sa)]$

$(\lambda s.(sa))a$

$(\lambda s.(sa))[s := a]$

aa

2) $((\lambda z.z)(\lambda z.(zz)))(\lambda z.(zq))$

$((\lambda z.z)(\lambda a.(aa)))(\lambda z.(zq))$

$((\lambda z.z)[z := \lambda a.(aa)])(\lambda z.(zq))$

$(\lambda a.(aa))(\lambda z.(zq))$

$(\lambda a.(aa))[a := \lambda z.(zq)]$

$(\lambda z.(zq))(\lambda z.(zq))$

$(\lambda z.(zq))[z := \lambda z.(zq)]$

$(\lambda z.(zq))q$

$(\lambda z.(zq))[z := q]$

qq

3) $((\lambda s . \lambda q . (s q q))(\lambda a . a)) e$

$((\lambda s . \lambda q . (s q q))[s := \lambda a . a]) e$

$(\lambda q . (\lambda a . a . (q q))) e$

$(\lambda q . (\lambda a . a)[a := q q])) e$

$(\lambda q . (q q)) e$

$(\lambda q . (q q))[q := e]$

$e e$

4) $((\lambda s . \lambda q . (s q q))(\lambda q . q)) q$

$((\lambda s . \lambda q . (s q q))(\lambda a . a)) q$

$((\lambda s . \lambda q . (s q q))[s := \lambda a . a]) q$

$(\lambda q . (\lambda a . a (q q))) q$

$(\lambda q . (\lambda a . a [a := q q])) q$

$(\lambda q . (q q)) q$

$(\lambda q . (q q)) e$

$(\lambda q . (q q))[q := e]$

$e e$

5. $((\lambda n.(nn))(\lambda q.q))(\lambda q.q)$

$((\lambda n.(nn))[n := \lambda q.q])(\lambda q.q)$

$((\lambda q.q)(\lambda q.q))(\lambda q.q)$

$((\lambda q.q)(\lambda a.a))(\lambda e.e)$

$((\lambda q.q)[q := \lambda a.a])(\lambda e.e)$

$(\lambda a.a)(\lambda e.e)$

$(\lambda a.a)[a := \lambda e.e]$

$\lambda e.e$

3.2

1) $\lambda xyz.(x(yz))$

$C(\lambda x:X.\lambda y:Y.\lambda z:Z.(x(yz)), \emptyset, A) :=$

$C(\lambda y:Y.\lambda z:Z.(x(yz)), \{x:X\}, B) \cup \{A = X \to B\} :=$

$C(\lambda z:Z.(x(yz)), \{x:X, y:Y\}, C) \cup \{B = Y \to C\} :=$

$C(x(yz), \underbrace{\{x:X, y:Y, z:Z\}}_{\Gamma}, D) \cup \{C = Z \to D\} :=$

$C(x, \Gamma, E) \cup C(yz, \Gamma, F) \cup \{E = F \to D\}$

$C(y, \Gamma, G) \cup C(z, \Gamma, H) \cup \{G = H \to F\}$

$A = X \to B$

$B = Y \to C$

$C = Z \to D$

$E = F \to D$

$G = H \to F$

$Y = G$

$Z = H \quad X = E$

$\Rightarrow A = E \to (G \to (Z \to D)) = (F \to D) \to ((H \to F) \to (H \to D))$

2) $\lambda xy.(xy(\lambda z.y))$

$C(\lambda x:X.\lambda y:Y.(xy(\lambda z.y)), \emptyset, A) =$

$C(\lambda y:Y.(xy(\lambda z.y)), \{x:X\}, B) \cup \{A = X \to B\} =$

$C(xy(\lambda z.y), \underbrace{\{x:X, y:Y\}}_{\Gamma}, C) \cup \{B = Y \to C\} =$

$C(xy, \Gamma, D) \cup C(\lambda z.y, \Gamma, E) \cup \{D = E \to C\}$

$C(xy, \Gamma, D) = C(x, \Gamma, F) \cup C(y, \Gamma, G) \cup \{F = G \to D\}$

$C(\lambda z.y, \Gamma, E) = C(\lambda z:Z.y, \Gamma, E) =$

$= C(y, \Gamma \cup (z:Z), H) \cup \{E = Z \to H\}$

$A = X \to B$	$\Rightarrow A = F \to B$	$\Rightarrow A = F \to (G \to C)$
$B = Y \to C$	$B = G \to C$	$= F \to G \to C$
$D = E \to C$	$D = E \to C$	
$F = G \to D$	$F = G \to D$	
$Y = H$		
$X = F$		
$Y = G$		

SAU:

3.2.

2) $\lambda xy.(xy(\lambda z.y))$

$x : X$

$y : Y$

$z : Z$

$y : Y \Rightarrow (\lambda z.y) : Z \to Y$

$y : Y \qquad X = Y \to A$

$(x\,y) : A$

$\Rightarrow A = (Z \to Y) \to B$

deci $\underbrace{((xy)(\lambda z.y))}_{\alpha} : B$

$\lambda y.\alpha : Y \to B$

$\lambda x.xy.\alpha : X \to Y \to B$

$X = Y \to A = Y \to ((Z \to Y) \to B)$

Notăm $X = Y \to ((Z \to Y) \to B)$

Fie Gamma $= \{x : Y,\ y : Y,\ z : Z\}$

1. Din regula pt. variabile $\Rightarrow$ Gamma $\vdash y : Y$

"deduce"

2. $x : X,\ y : Y \vdash \lambda z.y : Z \to Y$ (regula pt. lambda)

3. $x : X,\ y : Y \vdash y : Y$ (reg. var)

4. $x : X,\ y : Y \vdash x : Y \to ((Z \to Y) \to B)$

(reg. pt. variabile)

3, 4 $\Rightarrow$ 5. $x : X,\ y : Y \vdash x\,y : (Z \to Y) \to B$

(reg. aplicației)

6. $x : X,\ y : Y \vdash (x\,y)(\lambda z . y) : B$

(din 5. și 2., reg. aplicației)

7. $x : X \vdash \lambda y . (x\,y)(\lambda z . y) : Y \to B$

(din 6., reg. lambda)

8. $\vdash \lambda x . \lambda y . (x\,y)(\lambda z . y) : X \to Y \to B$

3) $(\lambda xyz.\ zxy)(\lambda xyz.\ y)(\lambda xy.\ y) =$

$= (\lambda xyz.\ zxy)(\lambda abc.\ b)(\lambda de.\ e) =$

$= ((\lambda x.\lambda yz.\ zxy)[x := \lambda abc.\ b])(\lambda de.\ e) =$

$= (\lambda yz.\ z(\lambda abc.\ b)y)(\lambda de.\ e) =$

$= (\lambda yz.\ z(\lambda abc.\ b)y)[y := \lambda de.\ e] =$

$= \lambda z.\ z(\lambda abc.\ b)(\lambda de.\ e) =$

$= \lambda z.\ z(\lambda abc.\ b)[a := \lambda de.\ e] =$

$= \lambda z.\ z(\lambda bc.\ b) =$

$= \lambda z.\ z[z := \lambda bc.\ b] = \lambda bc.\ b$

$C(\lambda bc.\ b, \emptyset, A) = C(\lambda b:B.\ \lambda c:C.\ b, \emptyset, A) =$

$= C(\lambda c:C.\ b, \{b:B\}, D) \cup \{A = B \to D\}$

$= C(b, \{b:B, c:C\}, E) \cup \{D = C \to E\}$

$A = B \to D$
$D = C \to E$
$B = E$
$\Rightarrow$ $A = E \to D$
$D = C \to E$
$\Rightarrow$ $A = E \to (C \to E)$

1.2

1) $\{x = m \wedge y = n\} (x := x + y) \{x = m + n \wedge y = n\}$

$\{x = m + n \wedge y = n\} (y := x - y) \{x = m + n \wedge y = m\}$

$\{x = m + n \wedge y = m\}\ x := x - y\ \{x = n \wedge y = m\}$

(din regula de recuențiere => enunțul este corect)

2) Codul:

```
P := 0;
C := 1;
while C ≤ N do
    P := P + M;
    C := C + 1;
```

$P := 0;\ C := 1;$ => $\{P = 0 \wedge C = 1\}$

while ... => avem nevoie de invariant

invariant $= \{P = M * (C-1) \wedge 1 \le C \le N+1\}$

Verificare pt. $P = 0$ și $C = 1$:

$P = M * (C-1) \Leftrightarrow$

$0 = M * (1-1) \Leftrightarrow$

$0 = 0$ (adv)

$1 \le C \le N+1 \Leftrightarrow$

$1 \le 1 \le N+1$ (adv)

=> corect

Menținere: (cam ca la inducție)

După executarea comenzii $p := p + n$

$p' = p + n \Rightarrow p' = n * (c-1) + n \Rightarrow$

$\Rightarrow p' = n * c$

După executarea comenzii $c := c + 1$

$p' = n * c$

$c' = c + 1$

Verificăm invariantul pt. p' și c':

$p' = n * (c' - 1) \Leftrightarrow n * c = n * ((c+1) - 1) \Leftrightarrow$

$\Leftrightarrow n * c = n * c$ (adv)

$1 \leq c' \leq N + 1 \Leftrightarrow 1 \leq c + 1 \leq N + 1$ (adv)

La terminarea while-ului $c = N + 1$:

$p = n * (c - 1) \Leftrightarrow p = n * ((N+1) - 1) \Leftrightarrow$

$\Leftrightarrow p = n * N$ (adevărat)

2.2

1) 1) $r := p, q$?- w

2) $s := p, q$

3) $v := t, u$

4) $w := v, s$

5) t

6) q

7) u

8) p

$G0 = \neg w$

$C0 = w \vee \neg v \vee \neg s$

$G1 = \neg v \vee \neg s$

$C1 = v \vee \neg t \vee \neg u$

$G_2 = \neg t \vee \neg u \vee \neg s$

$C_2 = s \vee \neg p \vee \neg q$

$G_3 = \neg t \vee \neg u \vee \neg p \vee \neg q \Rightarrow G_3 = \perp$

2) 1) $q(x,y) := q(y,x), q(y, f(f(y)))$

2) $q(a, f(f(x)))$

$? = q(f(z), a)$

$G_0 = \neg q(f(z), a)$

$C_0 = q(x,y) \vee \neg q(y,x) \vee \neg q(y, f(f(y)))$

$f(z) = x$

$a = y$

$G_1 = \neg q(y, x) \vee \neg q(y, f(f(y)))$

$C_1 = q(a, f(f(x)))$

$y = a$

$x = f(f(x))$

$G_2 = \neg q(y, f(f(y)))$

$C_2 = q(a, f(f(x)))$

$y = a$

$f(f(y)) = f(f(x))$

$G_3 = \perp$

$x = f(z)$

$y = a$

$x = f(f(x))$

$f(f(y)) = f(f(x))$

$\Rightarrow x = y = a = f(z)$

$z = f(x)$

3) 1) $p(x) := q(x, f(y)), r(a)$

2) $p(x) := r(x)$

3) $q(x, y) := p(y)$

4) $r(x) := q(x, y)$

5) $r(f(b))$

$?- p(x), q(y, z)$

$G0 = \neg p(x) \vee \neg q(y, z)$

$C0 = p(T) \vee \neg r(T)$

$x = T$

$G1 = \neg r(T) \vee \neg q(y, z)$

$C1 = r(f(b))$

$T = f(b)$

$G_2 = \neg q(y, z)$

$C_2 = q(u, v) \vee \neg p(v)$

$y = u$

$z = u$

$G_3 = \neg p(v)$

$C_3 = p(w) \vee \neg r(w)$

$v = w$

$G_4 = \neg r(w)$

$C_4 = r(f(h))$

$w = f(h)$

$\Rightarrow G_5 = \perp$

$x = T$

$T = f(h)$

$x = f(h)$

$Y = u$

$z = w$

$w = f(h)$

$\Rightarrow x = f(h)$

$Y = u$

$z = f(h)$

$\Rightarrow x = z = f(h)$

$Y = u$