Subspații vectoriale

Def: Fie $V/_K$ sp. vect. și $V' \subseteq V$, $V' \neq \emptyset$

V' s.n. subsp. vect. al lui V dacă: e închisă (stabilă) la adunarea vect. și la înmulțirea cu scalari

i.e. $\left[\begin{array}{l} (\forall)\ v_1, v_2 \in V' \Rightarrow v_1 + v_2 \in V' \\ (\forall)\ \alpha \in K, v \in V' \Rightarrow \alpha v \in V' \end{array}\right]$

P $V' \subseteq V$
ssp. vect. $\Leftrightarrow \left[(\forall)\ v_1, v_2 \in V',\ \alpha_1, \alpha_2 \in K \Rightarrow \alpha_1 v_1 + \alpha_2 v_2 \in V'\right]$

Exemple: ① $\{0_V\}$, $V \subseteq V$
ssp. vect. improprii

② $\mathbb{R}_n[x] \subset \mathbb{R}[x]$
ssp. vect.

③ Fie $V/_K$ sp. vect.

$W \subseteq V$

Submulț. $\not\ni 0_V \Rightarrow W$ nu este subsp. vect.

$\boxed{\text{Apl}}$ Fie $\mathbb{R}^n/_{\mathbb{R}}$ sp. vect. real

$$S(A) = \{(x_1, \dots, x_n) \in \mathbb{R}^n \,/\, \sum_{j=1}^{n} a_{ij} x_j = 0, (\forall)\, i = \overline{1,m}\}$$

$A = (a_{ij})_{\substack{i=\overline{1,m} \\ j=\overline{1,n}}}$, $m \leq n$, $rg\,A = m$

Dem. că: $S(A) \subset \mathbb{R}^n$ ssp. vect. $\left(\dim_{\mathbb{R}} S(A) = n - m\right)$

Sol: Fie $x, y \in S(A)$, $\alpha, \beta \in \mathbb{R}$ $\Rightarrow \sum_{j=1}^{n} a_{ij} x_j = 0$, $\sum_{j=1}^{n} a_{ij} y_j = 0$, $(\forall)\, i = \overline{1,m}$ $\Rightarrow$

$$\Rightarrow \sum_{j=1}^{n} a_{ij}(\alpha x_j + \beta y_j) = \alpha \sum_{j=1}^{n} a_{ij} x_j + \beta \sum_{j=1}^{n} a_{ij} y_j = 0$$

$\Rightarrow \alpha x + \beta y \in S(A) \Rightarrow S(A) \subset \mathbb{R}^n$ ssp. vect.

$\dim_{\mathbb{R}} S(A) = n - m = n - rg\,A$ (conf. th.)

Consecințe:

1) $\mathbb{R}^2/_{\mathbb{R}}$

- $\{0_{\mathbb{R}^2}\}$, $\mathbb{R}^2$ ssp. vect. triviale (improprii) → 0, resp. 2-dim.
- $d = \{(x_1, x_2) \in \mathbb{R}^2 \,/\, a_1 x_1 + a_2 x_2 = 0\} \subset \mathbb{R}^2$ ssp. vect. 1-dim (dr. vect $\ni 0_{\mathbb{R}^2}$)

$rg\,(a_1\ a_2) = 1$ (i.e. $a_1^2 + a_2^2 > 0$)

2) $\mathbb{R}^3/\mathbb{R}$

· $\{0_{\mathbb{R}^3}\}$, $\mathbb{R}^3$ ssp. vect. improprii (de dim. 0, resp. 3)

· $d = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \;/\; \begin{cases} a_1 x_1 + a_2 x_2 + a_3 x_3 = 0 \\ b_1 x_1 + b_2 x_2 + b_3 x_3 = 0 \end{cases}, \; rg \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} = 2 \right\} \subset \mathbb{R}^3$ ssp. vect. 1-dim

(dr. vect. $\ni 0_{\mathbb{R}^3}$)

· $\mathcal{P} = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 \;/\; a_1 x_1 + a_2 x_2 + a_3 x_3 = 0, \; rg(a_1\ a_2\ a_3) = 1 \} \subset \mathbb{R}^3$ ssp. vect. 2-dim

(plan vect. $\ni 0_{\mathbb{R}^3}$)

$\boxed{\text{Ap1}}$ Fie sp. vect. $\mathbb{R}^3/\mathbb{R}$ și $U = \{(x, y, z) \in \mathbb{R}^3 / x + 2y + 3z = 0\} \subset \mathbb{R}^3$

a) Stabiliți dacă $\underset{\text{ssp. vect.}}{U \subset \mathbb{R}^3}$

b) Determinați $\dim_{\mathbb{R}} U = ?$

<u>Rez</u> a) Fie $v_1, v_2 \in U \Rightarrow v_1 = (x_1, y_1, z_1)$, $x_1 + 2y_1 + 3z_1 = 0$

$\alpha_1, \alpha_2 \in \mathbb{R}$ $\quad v_2 = (x_2, y_2, z_2)$, $x_2 + 2y_2 + 3z_2 = 0$

Ar. că: $\alpha_1 v_1 + \alpha_2 v_2 \in U$

$$\alpha_1 v_1 + \alpha_2 v_2 = (\underbrace{\alpha_1 x_1 + \alpha_2 x_2}_{x}, \underbrace{\alpha_1 y_1 + \alpha_2 y_2}_{y}, \underbrace{\alpha_1 z_1 + \alpha_2 z_2}_{z})$$

$$x + 2y + 3z = \alpha_1 x_1 + \alpha_2 x_2 + 2(\alpha_1 y_1 + \alpha_2 y_2) + 3(\alpha_1 z_1 + \alpha_2 z_2) =$$

$$= \alpha_1 \underbrace{(x_1 + 2y_1 + 3z_1)}_{0} + \alpha_2 \underbrace{(x_2 + 2y_2 + 3z_2)}_{0} = 0 \Rightarrow$$

$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 \in U$

Deci: $\underset{\text{ssp. vect.}}{U \subset \mathbb{R}^3}$

b) (V1) Folosim aplicația (teoretica) anterioară:

În cazul nostru $A = (1\ 2\ 3) \in \mathcal{M}_{(1,3)}(\mathbb{R})$

$\underset{\text{ssp. vect}}{U = S(A) \subset \mathbb{R}^3}$ și $\dim_{\mathbb{R}} U = 3 - rg A = 3 - 1 = 2$

$\Rightarrow U \subset \mathbb{R}^3$
subsp. vect. ($\ni 0_{\mathbb{R}^3}$)

(V_2) Determinăm, în mod explicit, o bază pt. U

Fie $v \in U$
arbitrar

$v = (x, y, z)$, $x + 2y + 3z = 0$
$\Rightarrow x = -2y - 3z$

$U \ni v = (-2y-3z, y, z) = (-2y, y, 0) + (-3z, 0, z)$
$= y\underbrace{(-2, 1, 0)}_{u_1} + z\underbrace{(-3, 0, 1)}_{u_2} = y u_1 + z u_2 \Rightarrow \underset{\text{sist. de generatori}}{S' = \{u_1, u_2\} \subset U}$

În plus, se poate arăta că $S' \subset U$
s.v. lin. indep. + sist. de gen.

$\Rightarrow \underset{\text{bază}}{S' \subset U} \Rightarrow \dim_{\mathbb{R}} = 2$

$\boxed{T}$ Fie $U = \{(x, y, z) \in \mathbb{R}^3 / -x + 3y + z = 0\} \subset \mathbb{R}^3/_{\mathbb{R}}$

a) Stabiliți dacă $U \subset \mathbb{R}^3$
subsp. vect.

b) Determinați $\dim_{\mathbb{R}} U$.

T. dimensiunii (Grassmann)

Fie $V/_K$ sp. vect. (finit dimensional) și $V_1, V_2 \subseteq V$
subsp. vect.

Atunci $\boxed{\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)}$

$\boxed{\text{Apl}}$ Fie $V_1 = \{(x, y, 0) \,/\, x, y \in \mathbb{R}\} \subset \mathbb{R}^3$

$V_2 = \{(t, 0, t) \,/\, t \in \mathbb{R}\}$

a) Ar. că $\underset{\text{ssp vect.}}{V_1, V_2 \subset \mathbb{R}^3}$ și precizați dimensiunile lor

b) Determinați $V_1 + V_2 = ?$

Rez: a) Fie $u_1, u_2 \in V_1 \Rightarrow u_1 = (x_1, y_1, 0)$, $x_1, y_1 \in \mathbb{R}$

$\alpha_1, \alpha_2 \in \mathbb{R}$ $\quad u_2 = (x_2, y_2, 0)$, $x_2, y_2 \in \mathbb{R}$

$\alpha_1 u_1 + \alpha_2 u_2 = (\alpha_1 x_1 + \alpha_2 x_2, \alpha_1 y_1 + \alpha_2 y_2, 0)$, unde $\alpha_1 x_1 + \alpha_2 x_2,\ \alpha_1 y_1 + \alpha_2 y_2 \in \mathbb{R}$

$\Rightarrow \alpha_1 u_1 + \alpha_2 u_2 \in V_1 \Rightarrow \underset{\text{ssp. vect.}}{V_1 \subset \mathbb{R}^3}$

$V_1 \ni (x, y, 0) = x e_1 + y e_2 \Rightarrow \underset{\text{bază}}{B_1 = \{e_1, e_2\} \subset V_1} \Rightarrow \dim B_1 = 2$

$x, y \in \mathbb{R}$ (plan vectorial)

Analog se arată că: $\underset{\text{ssp. vect}}{V_2 \subset \mathbb{R}^3}$ și $\dim V_2 = 1$ (dr. vect.)

$\underset{\text{bază}}{B_2 = \{(1, 0, 1)\} \subset V_2}$

$V_1 + V_2 = \langle V_1 \cup V_2 \rangle$

Aplicăm t. dimensiunii (Grassmann):

$\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$

Obs: $V_1 \cap V_2 \ni v \Rightarrow x = y = t = 0 \Rightarrow V_1 \cap V_2 = \{0_{\mathbb{R}^3}\}$

$\Rightarrow \dim V_1 + V_2 = 2 + 1 - 0 = 3$

Dar: $\underset{\text{ssp. vect.}}{V_1 + V_2 \subseteq \mathbb{R}^3}$ $\Rightarrow \boxed{V_1 \oplus V_2 = \mathbb{R}^3}$ suma directă

(deoarece $V_1 \cap V_2 = \{0_{\mathbb{R}^3}\}$)

$\boxed{\text{T}}$ Fie $V_1 = \{(x, y, 0) \,/\, x, y \in \mathbb{R}\}$

$V_2 = \{(u, 0, v) \,/\, u, v \in \mathbb{R}\}$

a) Ar. că: $\underset{\text{ssp. vect}}{V_2 \subset \mathbb{R}^3}$ și precizați $\dim V_2$

b) Dem. că: $V_1 + V_2 = \mathbb{R}^3$ (E adev. și rel. $V_1 \oplus V_2 = \mathbb{R}^3$?)

$\boxed{\text{Apl}}$ Fie $V_1 = \{(x, y, 0) / x, y \in \mathbb{R}\}$

(T) $V_2 = \{(u, 0, v) / u, v \in \mathbb{R}\}$

a) Ar. că: $V_2 \subset \mathbb{R}^3$ ssp. vect. și precizați $\dim V_2$.

b) Dem. că: $V_1 + V_2 = \mathbb{R}^3$ (E adev. și rel. $V_1 \oplus V_2 = \mathbb{R}^3$?)

Rez.

a) Fie $w_1, w_2 \in V_2 \Rightarrow w_1 = (u_1, 0, v_1)$, $w_2 = (u_2, 0, v_2)$, $u_1, v_1, u_2, v_2 \in \mathbb{R}$

$\alpha_1, \alpha_2 \in \mathbb{R}$

$\alpha_1 w_1 + \alpha_2 w_2 = (\alpha_1 u_1 + \alpha_2 u_2, 0, \alpha_1 v_1 + \alpha_2 v_2)$, unde $\alpha_1 u_1 + \alpha_2 u_2, \alpha_1 v_1 + \alpha_2 v_2 \in \mathbb{R}$

$\Rightarrow \alpha_1 w_1 + \alpha_2 w_2 \in V_2 \Rightarrow V_2 \subset \mathbb{R}^3$ ssp. vect.

$V_2 \ni (u, 0, v) = u e_1 + v e_3 \Rightarrow B_2 = \{e_1, e_3\}$ bază $\Rightarrow \dim V_2 = 2$ (plan vectorial)

$u, v \in \mathbb{R}$

b) T. dimensiunii (Grassmann)

$$\boxed{\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)}$$

$V_1 \cap V_2 \ni v \Rightarrow \begin{cases} x = u \ (\stackrel{\text{not}}{=} t) \\ y = 0 \\ v = 0 \end{cases} \Rightarrow V_1 \cap V_2 = \{(t, 0, 0) / t \in \mathbb{R}\}$

$= \{t e_1 / t \in \mathbb{R}\} = \langle e_1 \rangle$

$\Rightarrow \dim V_1 \cap V_2 = 1$

Avem: $\dim V_1 = \dim V_2 = 2$

Așadar: $\dim(V_1 + V_2) = 2 + 2 - 1 = 3$.

$$\left.\begin{array}{l} \dim(V_1+V_2)=3 \\ \text{Dar}: \underset{\text{ssp. vect.}}{V_1+V_2} \subseteq \mathbb{R}^3 \end{array}\right| \Rightarrow \boxed{V_1+V_2=\mathbb{R}^3}$$

! Relația $V_1 \oplus V_2 = \mathbb{R}^3$ nu este adevărată deoarece:

$V_1 \cap V_2 \neq \{0_{\mathbb{R}^3}\}$ (mai exact, $V_1 \cap V_2 = \langle e_1 \rangle$).

(T)* Fie $V_1 = \{A \in \mathcal{M}_n(\mathbb{R}) \,/\, \underset{\text{urma}}{Tr}\, A = 0\}$

$$V_2 = \{A \in \mathcal{M}_n(\mathbb{R}) \,/\, A = \lambda I_n, \lambda \in \mathbb{R}\}$$

a) Ar. că: $\underset{\text{ssp. vect.}}{V_1, V_2} \subset \mathcal{M}_n(\mathbb{R})$

b) Dem. că $V_1 \oplus V_2 = \mathcal{M}_n(\mathbb{R})$

c) Verificați teorema dimensiunii în acest caz.