1) Algoritmul $G(A, b) \simeq O\left(\frac{2n^3}{3}\right)$

pentru $k = 1 : n-1$

pentru $i = k+1 : n$

$a_{ik} \leftarrow \mu_{ik} = \frac{a_{ik}}{a_{kk}}$

pentru $i = k+1 : n$

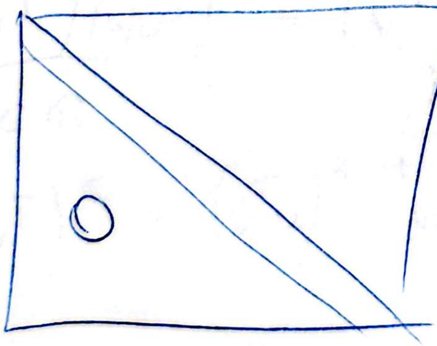
pentru $j = k+1 : n$

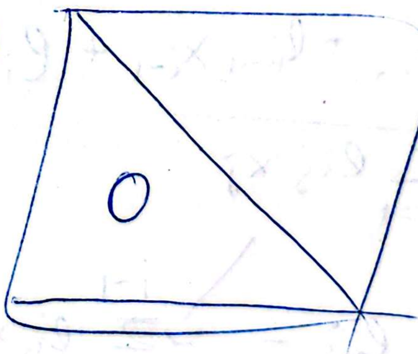
$a_{ij} \leftarrow a_{ij} - \mu_{ik} a_{kj}$

2). Fie matricile structurate A:

a) Superior Hessenberg

b) Sup Δ + linia n nenulă

a) 

b) 

Adaptați eficient alg G.

a) pentru $k = 1 : n-1$

$$a_{k+1\,k} = \frac{a_{k+1}}{a_{kk}} \quad (= -\mu_{k+1\,k})$$

pentru $i = k+1 : n$

$$a_{k+1\,i} = \cancel{\frac{a}{kk} - a_{ik}}$$

$$a_{k+1\,i} + a_{ki} \cdot \frac{a_{k+1\,k}}{a_{kk}}$$

$\Rightarrow O(n^2)$

b) ~~$k = n-1$~~

~~pentru $i = k+1 : n-1$~~

~~$a_{ik} = \frac{a_{ik}}{a_{kk}}$~~

~~$a_{ki} = \frac{a_{ki}}{a_{ii}}$~~

(ord) ✓

pentru $k = 1 : n-1$

$$a_{nk} = \frac{a_{nk}}{a_{kk}}$$

pentru $i = k+1 : n$

$$a_{ni} = a_{ni} + a_{ki} \cdot \frac{a_{nk}}{a_{kk}}$$

c) pentru $k = 1 : n-1$

$$a_{k+1\,k} = \frac{a_{k+1\,k}}{a_{kk}}$$

$$a_{nk} = \frac{a_{nk}}{a_{kk}}$$

$\Rightarrow O(n^2)$

pentru $i = k+1 : n$

$$a_{k+1\,i} = a_{k+1\,i} + a_{ki} \cdot \frac{a_{k+1\,k}}{a_{kk}}$$

$$a_{ni} = a_{ni} + a_{nk}/a_{kk} \cdot a_{ki}$$

CamScanner

SEMINAR

1) Fie datele $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 2 & 1 \end{bmatrix}$, $b = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. Rezolvați $\min_x \|Ax - b\|_2^2$

a) prin E.N. → ecuații normale

b) prin factorizarea QR

a) 1. Calculăm $C = A^T A$, $d = A^T b$

2. Cholesky $C = GG^T$

3. Rezolvăm sistemul

1) $C = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix}$

$d = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

2) $g_{11} = \sqrt{c_{11}} = \sqrt{5}$

$\cancel{g_{22} = \sqrt{\quad}}$

$\begin{bmatrix} g_{11} & 0 \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} g_{11} & g_{21} \\ 0 & g_{22} \end{bmatrix} = \begin{bmatrix} g_{11}^2 & g_{11}g_{21} \\ g_{11}g_{21} & g_{21}^2 + g_{22}^2 \end{bmatrix}$

$g_{11} g_{21} = 3$

$\sqrt{5}\, g_{21} = 3$

$g_{21} = \cancel{\frac{3\sqrt{5}}{5}}\ \frac{3}{\sqrt{5}} \Rightarrow G = \begin{bmatrix} \sqrt{5} & 0 \\ \frac{3}{\sqrt{5}} & \frac{\sqrt{6}}{\sqrt{5}} \end{bmatrix}$

$g_{21}^2 + g_{22}^2 = 3$

$\frac{9}{5} + g_{22}^2 = 3$

$g_{22} = \sqrt{3 - \frac{9}{5}}$

$= \sqrt{\frac{6}{5}}$

3)

$G\underbrace{G^T_x}_{y} = d$

$$\begin{cases} Gy = d \\ G^T x = y \end{cases}$$

I $Gy = d$

$$\begin{bmatrix} \sqrt{5} & 0 \\ \frac{3}{\sqrt{5}} & \frac{\sqrt{6}}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$\sqrt{5} y_1 = 3$

$y_1 = \frac{3}{\sqrt{5}}$

$\frac{3}{\sqrt{5}} y_1 + \frac{\sqrt{6} y_2}{\sqrt{5}} = 2$

$\frac{9}{5} + \frac{\sqrt{6} y_2}{\sqrt{5}} = 2$

$\frac{9}{5} = \sqrt{5} y_2 \sqrt{6} = 10$

$\sqrt{5} y_2 \sqrt{6} = 1$

$y_2 = \frac{1}{\sqrt{5} \cdot \sqrt{6}} = \frac{1}{\sqrt{30}}$

II $G^T x = \begin{bmatrix} \frac{3}{\sqrt{5}} \\ \frac{1}{\sqrt{30}} \end{bmatrix}$

$$\begin{bmatrix} \sqrt{5} & \frac{3}{\sqrt{5}} \\ 0 & \frac{\sqrt{6}}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{5}} \\ \frac{1}{\sqrt{30}} \end{bmatrix}$$

$\cancel{\sqrt{5}}\sqrt{5} x_1 + \frac{3 x_2}{\sqrt{5}} = \frac{3}{\sqrt{5}}$

$x_1 = \frac{1}{2}$

1) $5x_1 + 3x_2 = 3 \iff 5x_1 + \frac{1}{2} = 3 \to x_1$

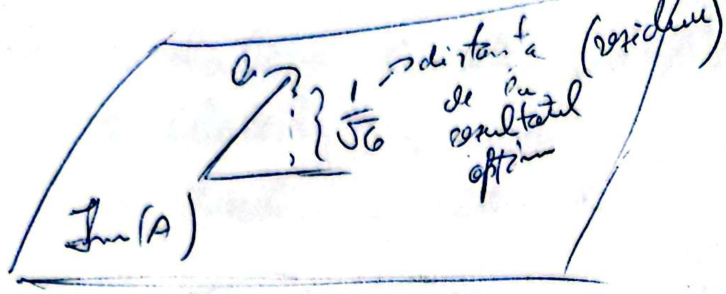
2) $\frac{\sqrt{6}}{\sqrt{5}} x_2 = \frac{1}{\sqrt{30}}$

$x_2 = \frac{\sqrt{5}}{\sqrt{6} \cdot \sqrt{30}} = \frac{\cancel{\sqrt{5}}}{6\cancel{\sqrt{5}}} = \frac{1}{6}$

Rezidui:

$$\left\| \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{6} \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\| = \left\| \begin{matrix} \frac{1}{2} + \frac{1}{6} - 1 \\ \frac{1}{6} \\ 1 + \frac{1}{6} - 1 \end{matrix} \right\| = \left\| \begin{matrix} -\frac{1}{3} \\ \frac{1}{6} \\ \frac{1}{6} \end{matrix} \right\| =$$

$$= \sqrt{\frac{2}{6^2} + \frac{1}{9}} = \sqrt{\frac{1}{18} + \frac{2}{9}} = \sqrt{\frac{5}{18}} = \sqrt{\frac{1}{6}} = \frac{1}{\sqrt{6}}$$



a) 1) Factorizăm $G_2 R$: $A = GR$

2) Calculăm $d = G^T a$

3) Rezolvăm $R'x = d'$)

$U = I - \frac{uu^T}{\beta}$, $\beta = \frac{\|u\|_2^2}{2}$

$$u = \begin{bmatrix} 0 \\ \dots \\ 0 \leftarrow k-1 \\ x_k + \sigma \\ x_{k+1} \\ \dots \\ x_m \end{bmatrix} \qquad \sigma = \sqrt{\sum_{j=k}^{m} x_j^2}$$

↳ norma de la k la m

1)

$a = \begin{bmatrix} 0 \\ \sigma \\ 2 \end{bmatrix}$ ~~$U = I$~~

~~$\beta = \frac{\sigma^2 + 4}{2}$~~ $\|u\| = \sqrt{10 + 2\sqrt{5}}$

$\sigma_1 = \sqrt{1+4} = \sqrt{5}$ $\beta = \frac{10 + 2\sqrt{5}}{2} = 5 + 5\sqrt{5}$

(specific primei coloane)

$$a = \begin{bmatrix} 1 + \sqrt{5} \\ 0 \\ 2 \end{bmatrix}$$

$$U_1 = I - \frac{\begin{bmatrix} 1 + \sqrt{5} \\ 0 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 + \sqrt{5} & 0 & 2 \end{bmatrix}}{5 + \sqrt{5}} =$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{\begin{pmatrix} 6+2\sqrt{5} & 0 & 2+2\sqrt{5} \\ 0 & 0 & 0 \\ 2+2\sqrt{5} & 0 & 4 \end{pmatrix}}{5+\sqrt{5}} =$$

$$= \begin{bmatrix} 1 - \frac{6+2\sqrt{5}}{5+\sqrt{5}} & 0 & -\left(\frac{2+2\sqrt{5}}{5+\sqrt{5}}\right) \\ 0 & 1 & 0 \\ -\left(\frac{2+2\sqrt{5}}{5+\sqrt{5}}\right) & 0 & 1 - \frac{4}{5+\sqrt{5}} \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} -1 & 0 & -2 \\ 0 & \sqrt{5} & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

Aplicăm U_1 la A și b

$A^+ = U_1 A$

$b^+ = U_1 b$

primul pas a fost să triangularizăm prima coloană

$$A^+ = \frac{1}{\sqrt{5}}\begin{bmatrix} -1 & 0 & -2 \\ 0 & \sqrt{5} & 0 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 2 & 1 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} -5 & -3 \\ 0 & \sqrt{5} \\ 0 & -1 \end{bmatrix}$$

$\rightarrow$ acum vrem să-l anulăm pe asta

$$b^+ = \frac{1}{\sqrt{5}}\begin{bmatrix} -1 & 0 & -2 \\ 0 & \sqrt{5} & 0 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} -3 \\ 0 \\ -1 \end{bmatrix}$$

$$\sigma_2 = \sqrt{\frac{1}{5}(5+1)} = \sqrt{\frac{6}{5}}$$

$$u_2 = \begin{bmatrix} 0 \\ 1+\sigma_2 \\ \frac{-1}{\sqrt{5}} \end{bmatrix} \rightarrow$$ adăugăm σ la indicele pe care vrem să-l Δ (o e mai el 0)

$$U_2 = I - \frac{u_2 u_2^T}{\frac{\|u\|^2}{2}}$$

$$\frac{\|u\|^2}{2} = \frac{0 + \left(1+\sqrt{\frac{6}{5}}\right)^2 + \frac{1}{5}}{2} = \frac{\frac{1}{5} + 1 + 2\sqrt{\frac{6}{5}} + \frac{6}{5}}{5}$$

$$\frac{\frac{12}{5} + 2\sqrt{\frac{6}{5}}}{5}$$

$$U_2 = I - \frac{\begin{pmatrix} 0 \\ 1+\frac{\sqrt{6}}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1+\frac{\sqrt{6}}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix}}{\beta}$$

$$U_2 = I - \begin{pmatrix} 0 & 0 & 0 \\ 0 & \left(\frac{1+\sqrt{6}}{\sqrt{5}}\right)^2 & -\frac{1}{\sqrt{5}}\left(1+\frac{\sqrt{6}}{\sqrt{5}}\right) \\ 0 & -\frac{1}{\sqrt{5}}\left(1+\frac{\sqrt{6}}{\sqrt{5}}\right) & \frac{1}{5} \end{pmatrix} \cdot \frac{5}{\sqrt{30}\left(1+\sqrt{30}\right)}$$

analog. U_1

$A^{++} = U_2 \cdot A^{+}$

$b^{++} = U_2 \cdot b^{+}$

$\Rightarrow A^{++} = \begin{pmatrix} x & x \\ x & x \\ 0 & 0 \end{pmatrix}$

calculăm $d = G^T b = \begin{pmatrix} x \\ x \\ 0 \end{pmatrix}$.

$\Rightarrow$ rezolvăm ecuațiile x

SEMINAR

1)

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 7 \end{bmatrix}$$

$$\det(A - x i) = 0$$

$$\begin{bmatrix} 4-x & 0 \\ 0 & 7-x \end{bmatrix} = 0$$

$\Rightarrow x_1 = 4$

$x_2 = 7$ } valorile proprii

$Av_1 = \lambda_1 v_1$

$A[v_1 \; v_2] = [\lambda_1 v_1 \quad \lambda_2 v_2]$

$$\begin{bmatrix} 4 & 0 \\ 0 & 7 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 4 v_1 \\ 4 v_2 \end{bmatrix}$$

$$\left.\begin{aligned} 4v_1 &= 4v_1 \\ 7v_2 &= 4v_2 \end{aligned}\right| \Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} v_1$$

$Av_2 = \lambda_2 v_2$

$$\begin{bmatrix} 4 & 0 \\ 0 & 7 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 7 v_1 \\ 7 v_2 \end{bmatrix}$$

$$\begin{aligned} 4v_1 &= 7v_1 \\ 7v_2 &= 7v_2 \end{aligned} \Rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

2) $\begin{bmatrix} 3 & 8 \\ 0 & 9 \end{bmatrix}$

$$\begin{bmatrix} 3-\lambda & 8 \\ 0 & 9-\lambda \end{bmatrix} = 0$$

$(3-\lambda)(9-\lambda) = 0$

$\lambda_1 = 3$

$\lambda_2 = 9$

$$\begin{bmatrix} 3 & 8 \\ 0 & 9 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 3v_1 \\ 3v_2 \end{bmatrix}$$

$3v_1 + 8v_2 = 3v_1$

$9v_2 = 3v_2$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 3 & 8 \\ 0 & 9 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 9v_1 \\ 9v_2 \end{bmatrix}$$

$3v_1 + 8v_2 = 9v_1 \Leftrightarrow$

$9v_2 = 9v_2$

$6v_1 = 8v_2$

$3v_1 = 4v_2$

~~$\begin{bmatrix} \frac{4}{3} \\ 1 \end{bmatrix}$~~

$\begin{bmatrix} 4 \\ 3 \end{bmatrix}$

3) $A = \begin{bmatrix} 0 & m \\ f & 0 \end{bmatrix}$ $\cancel{\begin{bmatrix} 0 & m \\ f & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \sqrt{mf}\, v_1 \\ \sqrt{mf}\, v_2 \end{bmatrix}}$

$\det(A - \lambda i) = 0$

$\cancel{mf = 0}$

$\det\left(\begin{bmatrix} -\lambda & m \\ f & -\lambda \end{bmatrix}\right) = 0$

$\lambda^2 - mf = 0$

$\lambda = \pm\sqrt{mf}$

$v_1 = e_2$
$v_2 = e_1$
$\left(\begin{matrix} \text{e permutare} \\ \text{de} \quad 1 \quad 0 \\ \quad\ \ 0 \quad 1 \end{matrix}\right)$

$A = \begin{pmatrix} 0 & m \\ f & 0 \end{pmatrix}$

$PA = \begin{pmatrix} f & 0 \\ 0 & m \end{pmatrix}$

$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

4) $A = U \cdot D \cdot U^T$

a) $A^2 = A \cdot A = U \cdot D \cdot U^T \cdot U \cdot D \cdot U^T = U D^2 U^T$

e) $A^{-1} = (U \cdot D \cdot U^T)^{-1}\ \cancel{(U \cdot D^{-1} \cdot U^T)} = U D^{-1} U^T$

c) $C = A + 4i = UDU^T + 4i = UDU^T + U4iU^T =$
$= U(D + 4i)U^T$

5) $T = \begin{bmatrix} 0,6 & 0,2 \\ 0,4 & 0,8 \end{bmatrix}$

$P_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$P_1 = TP_0 = \begin{bmatrix} 0,6 & 0,2 \\ 0,4 & 0,8 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \cancel{\begin{bmatrix} 0,6 \\ 0,4 \end{bmatrix}} \begin{bmatrix} 0,6 \\ 0,4 \end{bmatrix}$

CS CamScanner

$$P_n = T^n P_0$$

$$\det|T - \lambda i| = 0$$

$$\det\begin{bmatrix} 0{,}6-\lambda & 0{,}2 \\ 0{,}4 & 0{,}8-\lambda \end{bmatrix} = 0$$

$$(0{,}6-\lambda)(0{,}8-\lambda) - 0{,}08 = 0$$

$$0{,}48 - 0{,}6\lambda - 0{,}8\lambda + \lambda^2 - 0{,}08 = 0$$

$$\lambda^2 - 0{,}14\lambda + 0{,}4 = 0$$

sau

$$\lambda_1 = 1$$

$$\lambda_2 = 0{,}4$$

$$v_1 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$T^n = \underbrace{\begin{bmatrix} 1/2 & 1 \\ 1 & -1 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0{,}4 \end{bmatrix}^n}_{S} \underbrace{\begin{bmatrix} 1/2 & 1 \\ 1 & -1 \end{bmatrix}^{-1}}_{U^T}$$

$$= \begin{bmatrix} 1/2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0{,}4^n \end{bmatrix} \begin{bmatrix} 1/2 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$$

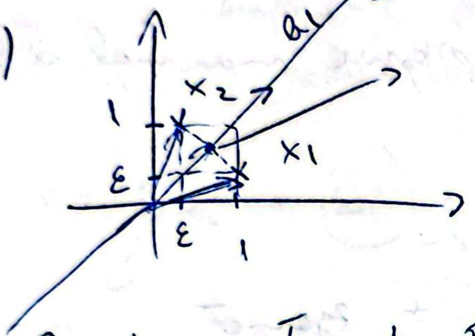
$$\Downarrow \quad 0$$

nu e simetrică
deci ne trebuie
inversa

1) Calculați prima componentă principală pentru:

a) $X = \begin{bmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{bmatrix}$

b) $X = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

a) 

x_1 reconstruit ~~(proiecția pe v_1)~~

x_2 reconstruit ~~(proiecția pe v_1)~~

Caimic

$\Rightarrow$ doar $K = 1$ nu e destul pt. nu putem distinge între x_1, x_2

$$S = \frac{1}{2} X \cdot X^T = \frac{1}{2} \begin{bmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{bmatrix} \begin{bmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1+\varepsilon^2 & 2\varepsilon \\ 2\varepsilon & 1+\varepsilon^2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1+\varepsilon^2}{2} & \varepsilon \\ \varepsilon & \frac{1+\varepsilon^2}{2} \end{bmatrix}$$

Obs: $A \in \mathbb{R}^{n \times n}$

$\Lambda(A) = \{\lambda_1, \dots, \lambda_n\}$

$\Lambda(\alpha I + A) = \{\lambda_1 + \alpha, \dots, \lambda_n + \alpha\}$

(PVD)

primary value decomposition

Obs: $A = uv^T$, $u, v \in \mathbb{R}^n$

$\Lambda(A) = \{u^T v, 0, \dots, 0\}$

$\Rightarrow A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \underset{e}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} [1 \; 1 \; 1]$

$\Lambda(A) = \{3, 0, 0\}$

$$\Rightarrow S = \begin{bmatrix} \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \end{bmatrix} + \begin{bmatrix} \frac{(1-\varepsilon)^2}{2} & 0 \\ 0 & \frac{(1-\varepsilon)^2}{2} \end{bmatrix} = \varepsilon e e^T + \frac{(1-\varepsilon)^2}{2} I_2$$

$$\Lambda(e e^T) = \Lambda\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \{2, 0\}$$

$$\Lambda(S) = \left\{ 2\varepsilon + \frac{(1-\varepsilon)^2}{2}, \ 0 + \frac{(1-\varepsilon)^2}{2} \right\}$$

$\varepsilon > 0$

$$S u_1 = \left(\frac{(1-\varepsilon)^2}{2} + 2\varepsilon \right) u_1$$

$$\left[\cancel{\frac{(1-\varepsilon)^2}{2}} I_2 + \varepsilon e e^T \right] u_1 = \left(\cancel{\frac{(1-\varepsilon)^2}{2}} + 2\varepsilon \right) u_1$$

$$\cancel{\varepsilon} e e^T u_1 = 2 \cancel{\varepsilon} u_1$$

$$\begin{cases} \cancel{[u_1]_1} + [u_1]_2 = \cancel{2}[u_1]_1 \\ [u_1]_1 + \cancel{[u_1]_2} = \cancel{2}[u_1]_2 \end{cases} \Rightarrow \left\{ u = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \Rightarrow$$

$$\Rightarrow u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$z_1 = u_1^T x_1 = \frac{1+\varepsilon}{\sqrt{2}}$$

$$z_2 = u_1^T x_2 = \frac{1+\varepsilon}{\sqrt{2}}$$

$$\Lambda(S) = \left\{ 2\varepsilon + \frac{(1-\varepsilon)^2}{2}, \frac{(1-\varepsilon)^2}{2} \right\}$$

$$J_1 = \frac{(1-\varepsilon)^2}{2} \ \Big| \ \varepsilon \simeq 1, \Rightarrow J_1 \simeq 0$$

epsilon converge către 1 ⇒ J_1 converge 0

2) Calculați SVD pt:

a) $A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$

b) $B = \begin{bmatrix} 4 & 4 \\ 3 & 3 \end{bmatrix}$

a) $A = U \Sigma V^T$

$A^T A = \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix}$

$A A^T = \begin{bmatrix} 32 & 0 \\ 0 & 18 \end{bmatrix}$

$\Rightarrow \Sigma = \begin{bmatrix} \underset{\sigma_1}{4\sqrt{2}} & 0 \\ 0 & \overset{\sigma_2}{3\sqrt{2}} \end{bmatrix}$

$\Downarrow$

$U = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix} v_1 = 32 v_1$

$\begin{cases} 25[v_1]_1 + 7[v_1]_2 = 32[v_1]_1 \\ 7[v_1]_1 + 25[v_1]_2 = 32[v_1]_2 \end{cases} \Rightarrow v_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$v_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$V = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

deci $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} 4\sqrt{2} & 0 \\ 0 & 3\sqrt{2} \end{pmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}$

b) $B^T B = \begin{pmatrix} 25 & 25 \\ 25 & 25 \end{pmatrix} \Rightarrow \Lambda(B^T B) = \{50, 0\}$

$$\Sigma = \begin{bmatrix} 5\sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}$$

$B^T B v_1 = 50 v_1$

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} v_1 = 2 v_1$

$\begin{cases} [v_1]_2 = [v_1]_1 \\ [v_1]_1 = [v_1]_2 \end{cases} \Rightarrow v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\left. \begin{array}{l} v_2 \perp v_1 \\ \|v_2\| = 1 \end{array} \right| \Rightarrow v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$V = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$B B^T = \begin{pmatrix} 32 & 24 \\ 24 & 18 \end{pmatrix}$

$\begin{pmatrix} 32 & 24 \\ 24 & 18 \end{pmatrix} u_1 = 50 u_1$

$\left. \begin{array}{l} 32 [u_1]_1 + 24 [u_1]_2 = 50 [u_1]_1 \\ 24 [u_1]_1 + 18 [u_1]_2 = 50 [u_1]_2 \end{array} \right| \Rightarrow u_1 = \frac{1}{5} \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

$18 u_{11} - 24 u_{12} = 0$

$\cancel{24 u_{11} - 32 u_{12} = 0}$

$\left. \begin{array}{l} u_2 \perp u_1 \\ \|u_2\| = 1 \end{array} \right| \Rightarrow u_2 = \cancel{\frac{1}{5} \begin{bmatrix} +4 \\ -3 \end{bmatrix}}$

$\frac{1}{5} \begin{bmatrix} 3 \\ -4 \end{bmatrix}$

$$U = \frac{1}{5}\begin{pmatrix} 4 & 3 \\ 3 & -4 \end{pmatrix}$$

$$B = \frac{1}{5}\begin{bmatrix} 4 & 3 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} 5\sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\frac{1}{\sqrt{2}}$$

(SVD completă)

$$B = \underbrace{5\sqrt{2}}_{\sigma_1} \underbrace{\begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}}_{u_1} \overbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}^{v_1} \quad \text{(SVD redusă)}$$

3) Rezolvați pe baza SVD:

a) $Ax = b$, b) $Bx = b$ $\quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

a) $U \Sigma V^T x = b$

$$\Sigma \underbrace{V^T x}_{y} = U^T b$$

$$\begin{cases} \Sigma y = U^T b \Rightarrow y_i = \dfrac{u_i^T b}{\sigma_i} \\ V^T x = y \Rightarrow x = V y \end{cases}$$

$$\begin{bmatrix} 4\sqrt{2} & 0 \\ 0 & 3\sqrt{2} \end{bmatrix} y = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow y = \begin{bmatrix} 1/8\sqrt{2} \\ 1/3\sqrt{2} \end{bmatrix}$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} x = y \Rightarrow x = \begin{bmatrix} -1/24 \\ 7/24 \end{bmatrix}$$

b) $\min_{x} \|Bx - b\| =$ Una schimbări norma => îl dăm factor comun

$= \min_{x} \|\Sigma V^T x - U^T b\|_2^2$

$= \min_{y = V^T x} \|\Sigma y - U^T b\|_2^2$

$$\begin{cases} y^* = \arg\min_{y} \|\Sigma y - U^T b\|_2^2 = \\ V^T x = y^* \end{cases}$$

$= \arg\min_{y_1, y_2} \left\| \begin{pmatrix} 5\sqrt{2} & 0 \\ 0 & 0 \end{pmatrix} y - \begin{pmatrix} 7/5 \\ -1/5 \end{pmatrix} \right\|_2^2$

$= \arg\min_{y_1} \|5\sqrt{2}\, y_1 - 7/5\|_2^2$

$\Rightarrow y_1^* = \frac{1}{5\sqrt{2}} \cdot \frac{7}{5} = \frac{7}{25\sqrt{2}}$

$y_2^* = 0$ (nu influențează eroarea)

$x = Vy$

$x = V \begin{bmatrix} 7/25\sqrt{2} \\ 0 \end{bmatrix} = \frac{7}{50} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (x^* unic)

SEMINAR

1) $\left(\begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \cdot \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} \right)^2 = (6 + 12 + 20)^2 = (38)^2 = 1444$

$k(x,y) = (x^T y)^2 = 1444$

$k_2(x,y) = \varphi(x)^T \cdot \varphi(y)$

$\varphi(x) = [\boxed{x_1^2} \quad x_1x_2 \quad x_1x_3 \quad x_2^2 \quad x_2x_3 \quad x_{31} \quad x_3 \quad \dots]$

slide

$\varphi(x)^T \varphi(y) = x_1^2 y_1^2 + x_1x_2y_1y_2 + \dots$

$= [4 \quad 6 \quad 8 \quad 6 \quad 9 \quad 12 \quad 8 \quad 12 \quad 16]^T \cdot [9 \quad 12 \quad 15 \quad 12$

$16 \quad 20 \quad 15 \quad 20 \quad 24] = 36 + 72 + 120 + 72 + 144 + 240 + 120 + 240 +$

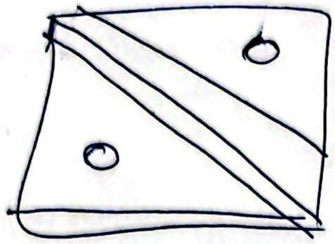
$+ 400 = 1444$

1) Adaptați alg. G. pt. un sist. de forma:

$$Ax = b, \quad A \in \mathbb{R}^{n \times n}, \quad b \in \mathbb{R}^n$$

unde $a_{ij} = 0$, pt. $|i-j| \geq 2$, $i \geq n-1$

se modifică doar elementul $k+1$, restul sunt 0 !!

$\Rightarrow$ 

de eliminat

$$\begin{array}{c|c} & k \\ & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

μ_{21}, μ_{61}

$j \in \{k+1\}$

$\Rightarrow i \in \{k+1, n\}$ doar 2 modificări

$a_{indici} = \{$

Alg. G.:

pentru $k = 1 : n-1$
- pentru $i = k+1 : n$
  - $a_{in} = \mu_{in} = \dfrac{a_{in}}{a_{ik}}$
- pentru $j = k+1 : n$
  - pentru $i = k+1 : n$
    - $a_{ij} = a_{ij} - \mu_{ik} a_{kj}$

$O(n^3)$

pentru $k = 1 : n-1$

$O(n)$

- $a_{k+1\, n} = \mu_{k+1\, n} = \dfrac{a_{k+1\, n}}{a_{k+1\, k}}$
- $a_{n\, n} = \mu_{n\, n} = \dfrac{a_{\cancel{n+1}}}{a_{n\, k}}$
- $a_{k+1\, k+1} = a_{k+1\, k+1} - \mu_{k+1\, k} \cdot a_{k\, k+1}$
- $a_{n\, k+1} = a_{n\, k+1} - \mu_{n\, k} \, a_{k\, k+1}$

↳ alg. ~~inferior~~ superior Δ

era mai ușor inferior Δ că trebuie să scăpăm doar de el. de deasupra diag.

2) $\min\limits_{x} \|Ax - b\|_2^2$

$$U = I_m - \frac{2}{\|u\|^2} u \cdot u^T$$

$$A = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$u_1 = \begin{bmatrix} 1 + \sqrt{3} \\ 1 \\ 1 \end{bmatrix}$$

$$U = I_3 - \frac{2}{3+\sqrt{3}} \begin{bmatrix} (1+\sqrt{3})^2 & 1+\sqrt{3} & 1+\sqrt{3} \\ 1+\sqrt{3} & 1 & 1 \\ 1+\sqrt{3} & 1 & 1 \end{bmatrix}$$

$$UA = \begin{bmatrix} 1 + \frac{6+4\sqrt{3}}{3+\sqrt{3}} \\ 0 \\ 0 \end{bmatrix}$$

$$\|Ax - b\| = \|UAx - Ub\|$$

$$Ub = \begin{bmatrix} 1 - \frac{4+2\sqrt{3}}{3+2\sqrt{3}} \\ \frac{-(1+\sqrt{3})}{3+\sqrt{3}} \\ \frac{-(1+\sqrt{3})}{2+\sqrt{3}} \end{bmatrix}$$

$$\|Ax - b\| = \left\| \begin{bmatrix} \\ \end{bmatrix} \right\| = \begin{bmatrix} \frac{6+4\sqrt{3}}{3+\sqrt{3}} \cancel{x} - \left(1 - \frac{4+2\sqrt{3}}{3+2\sqrt{3}}\right) \\ (\;) \\ (\;) \end{bmatrix}$$

$$\Rightarrow x^* = \frac{1}{3}$$

A doua metodă:

$$\left\| \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} x - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\|_2^2 = 3x^2 - 2x + c$$

$$\Rightarrow \text{vrem } \min_x 3x^2 - 2x$$