Master's Thesis

The Fourier transform on $\ell^1(\mathbb{Z})_+$ and Spectrum preservers

Kaito Sato

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Graduate School of Science and Technology, Niigata University

Chapter 1

Abstract

In this paper, we characterize the form of spectrally additive surjective maps on $\ell^1(\mathbb{Z})_+$, as well as spectrally multiplicative surjective maps on the set of all characteristic functions in $\ell^1(\mathbb{Z})_+$. The image of $\ell^1(\mathbb{Z})$ under the Fourier transform is a Banach algebra known as the Wiener algebra $A(\mathbb{T})$. By Bochner's theorem, the set $\widehat{\ell^1(\mathbb{Z})_+} \subset A(\mathbb{T})$ coincides with the collection of all positive definite functions in $A(\mathbb{T})$.

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Chapter 2

Introduction

Let $C(X, E)$ be the linear space of all continuous functions on a compact Hausdorff space X with values in a locally convex space E. Theorem 1.1 in [1] asserts that a map $T : C(X, E) \to C(Y, E)$ that satisfies $\mathrm{Ran}(TF - TG) \subset \mathrm{Ran}(F - G)$ for all $F, G \in C(X, E)$ is of the form $TF = T(1 \otimes 0_E) + F \circ \varphi$ where $\varphi : Y \to X$ is a continuous map. The forms of spectrally additive surjective maps and spectrally multiplicative surjective maps on Banach algebras were characterized by O. Hatori, T. Miura, and H. Takagi in [2]. Theorem 3.1 in [2] asserts that a surjective map T from a unital Banach algebra A into a unital semi-simple commutative Banach algebra B that satisfies $\sigma(T(a) + T(b)) \subset \sigma(a + b)$ holds for all $a, b \in A$ is linear and multiplicative. Here, $\sigma(x) := \{\lambda \in \mathbb{C} \mid \lambda - x \text{ is invertible}\}$ denotes the spectrum of an element x in a unital Banach algebra A. Theorem 3.2 in [2] asserts that if a surjective map T from a semi-simple unital commutative Banach algebra A into a unital commutative Banach algebra B satisfies $\sigma(TfTg) = \sigma(fg)$ holds for every pair f and g in A, then B is also semi-simple and T is of the form $(Tf)(y) = \tau(y)f(\Phi(y))$ $(y \in M_B)$ for some $\tau \in B$ with $\sigma(\tau) \subset \{-1, 1\}$ and a homeomorphism $\Phi : M_B \to M_A$, where M_A and M_B denote the maximal ideal space of A and B, respectively.

Theorem 1.1 in [1] and Theorem 3.1 in [2] are shown by applying the theorem of Kowalski and Słodkowski [3]. Their result applies to maps defined on the entire algebra. Consequently, the corresponding problem for proper subsets of Banach algebras has remained open. In particular, for a Banach algebra where a positive cone can be defined, considering the problem on the cone is highly significant. In this setting, the algebraic structure is no longer preserved, which prevents the direct application of the theorem of Kowalski and Słodkowski. Motivated by this observation, we investigate the problem in such a restricted setting.

We describe the form of a surjective map T satisfying the range condition $\mathrm{Ran}(\widehat{T(f)} + \widehat{T(g)}) = \mathrm{Ran}(\hat{f} + \hat{g})$ for every $f, g \in \ell^1(\mathbb{Z})_+$. Such a map T is shown to be of the form $T(f) = f \circ \mathrm{id}_{\mathbb{Z}}$ or $T(f) = f \circ (-\mathrm{id}_{\mathbb{Z}})$. Here, the Fourier transform of f is defined by $\hat{f}(z) := \sum_{n\in\mathbb{Z}} f(n)z^{-n}$ for every $z \in \mathbb{T}$, and the range of $\hat{f}$ coincides with the spectrum of f. Hence, we examine the problem in the setting of $\ell^1(\mathbb{Z})$, where the positive cone can be naturally defined.

In addition, we describe the form of the surjective map T satisfying the range condition $\mathrm{Ran}(\widehat{Tf} \cdot \widehat{Tg}) = \mathrm{Ran}(\hat{f} \cdot \hat{g})$ for every $f, g \in \{\chi_n \in \ell^1(\mathbb{Z})_+ \mid n \in \mathbb{Z}\}$. Here, the characteristic function χ_n is defined by $\chi_n(m) = 1$ and $\chi_n(m) = 0$ for every $n \neq m$.

The first result concerning spectrally additive maps is joint work with my supervisor S. Oi, which has appeared on arXiv:2512.01173 and been accepted for publication in Archiv der Mathematik [4]. The second result is an original contribution of this thesis. As an important first step toward these results, we consider surjective maps T on $\ell^1(\mathbb{Z})_+$ satisfying $\mathrm{Ran}(\widehat{T(f)}) = \mathrm{Ran}(\hat{f})$ for all $f \in \ell^1(\mathbb{Z})_+$. We show that any such map T is of the form $T(f) = f \circ \varphi$ where $\varphi : \mathbb{Z} \to \mathbb{Z}$ is bijective. Although this bijectivity can be obtained as a consequence of known general results, we present a direct and self-contained proof in the present setting.

The proof proceeds by first determining the form of T on characteristic functions, and then extending this description to the whole space $\ell^1(\mathbb{Z})_+$.

Chapter 3

Preliminaries

In this chapter, we establish several lemmas that will be used in the subsequent chapters.

3.1 Basic notation

We denote by $\mathbb{Z}$ the set of all integers, and by $\mathbb{T}$ the unit circle in $\mathbb{C}$. The Banach algebra $\ell^1(\mathbb{Z})$ consists of all functions $f : \mathbb{Z} \to \mathbb{C}$ whose series of absolute values converges. We denote by $\ell^1(\mathbb{Z})_+$ the positive cone of $\ell^1(\mathbb{Z})$, that is,

$$\ell^1(\mathbb{Z})_+ := \{f \in \ell^1(\mathbb{Z}) \mid f(n) \geq 0 \text{ for all } n \in \mathbb{Z}\}.$$

For each $n \in \mathbb{Z}$, we denote by χ_n the characteristic function of n, defined by $\chi_n(n) = 1$ and $\chi_n(m) = 0$ for each $m \neq n$.

3.2 Wiener algebra

We write $A(\mathbb{T})$ for the image of $\ell^1(\mathbb{Z})$ under the Fourier transform. We equip this space with the norm defined by $\|\hat{f}\| := \|f\|_1$. With pointwise multiplication, addition, and the norm $\|\cdot\|$, $A(\mathbb{T})$ becomes a Banach algebra.

3.3 Bochner's Theorem

Theorem (Bochner's Theorem)**.** *Let G be a locally compact abelian group, and let φ be a continuous function on G. Then φ is positive definite if and only if there is a non-negative measure $\mu \in M(\Gamma)$ such that*

$$\varphi(x) = \int_\Gamma \gamma(x) d\mu(\gamma) \quad (x \in G).$$

We write $A(\mathbb{T})_+$ for the set of all positive definite functions in $A(\mathbb{T})$. By Bochner's theorem, this space coincides with the image of $\ell^1(\mathbb{Z})_+$ under the Fourier transform.

Chapter 4

Lemmas

In this chapter, we introduce the Lemmas that will be used in subsequent chapters.

4.1 Properties of $\{z^n + z^m \mid z \in \mathbb{T}\} \cap 2\,\mathbb{T}$

We denote by $M_{n,m}$ the set $\{z^n + z^m \mid z \in \mathbb{T}\} \cap 2\,\mathbb{T}$ for every $n, m \in \mathbb{Z}$ with $n \neq m$.

Note that $\mathrm{Ran}(\widehat{\chi_n} + \widehat{\chi_m}) = \{z^n + z^m \mid z \in \mathbb{T}\}$ holds for all $n, m \in \mathbb{Z}$ by the definition of the Fourier transform of $\ell^1(\mathbb{Z})$. We use the lemma concerning the set $M_{n,m}$, that is,

Lemma 1.

$$\begin{aligned} M_{n,m} &= \{2(e^{i\frac{2k\pi}{m-n}})^m \mid k = 0, 1, \ldots, |m-n| - 1\} \\ &= \{2(e^{i\frac{2k\pi}{m-n}})^n \mid k = 0, 1, \ldots, |m-n| - 1\}, \\ |M_{n,m}| &= \frac{|m-n|}{\gcd(m,n)}. \end{aligned}$$

Here, $|M_{n,m}|$ is the cardinality of $M_{n,m}$.

Proof. Fix any $n, m \in \mathbb{Z}$ with $n \neq m$. If $|z^n + z^m| = 2$ holds, we have $z^n + z^m = 2z^n = 2z^m$. This implies the first equality.

Let us show the second equality. We denote by ℓ the denominator of the reduced form of $\frac{m}{m-n}$. Since $M_{n,m}$ consists of all $|\ell|$-th roots of $2 \in 2\,\mathbb{T}$, we have $|M_{n,m}| = |\ell|$. Thus we conclude the second equality using the Euclidean algorithm. $\square$

4.2 Maps preserving the range of z^n

We also use the lemma concerning the range condition as follows.

Lemma 2. *Let $T : \ell^1(\mathbb{Z})_+ \to \ell^1(\mathbb{Z})_+$ be a surjection that satisfies*

$$\mathrm{Ran}(\widehat{T(K\chi_n)}) = \mathrm{Ran}(\widehat{K\chi_n})$$

for all $\chi_n \in \ell^1(\mathbb{Z})_+$ and $K \geq 0$. Then $S := T|_{\{K\chi_n \mid n \in \mathbb{Z}, K \geq 0\}}$ is described as $S(K\chi_n) = K\chi_{\varphi(n)}$ by a bijection $\varphi : \mathbb{Z} \to \mathbb{Z}$.

Proof. Fix $n \in \mathbb{Z}$ and $K > 0$ arbitrarily. In the case $n = 0$, we have

$$\mathrm{Ran}(\widehat{K\chi_0}) = \mathrm{Ran}(Kz^0) = \{K\}.$$

Since the Fourier transform is injective, we obtain $T(K\chi_0) = K\chi_0$.

Assume that $n \neq 0$, and write $T(K\chi_n) = \{c_j \geq 0\}_{j \in \mathbb{Z}}$. Set $Z := \{j \in \mathbb{Z} \mid c_j > 0\}$. It is easy to see that $Z \neq \{0\}$ and $Z \neq \emptyset$. Suppose that $|Z| \geq 2$. We assume that $k \neq \ell$ for some $k, \ell \in Z$. We have

$$\begin{aligned} |\widehat{T(K\chi_n)}(e^{-i\pi/4k\ell})| &\leq \sum_{j \neq k,\ell} |c_j (e^{-i\pi/4k\ell})^j| + |c_k(e^{-i\pi/4\ell}) + c_\ell(e^{-i\pi/4k})| \\ &< \sum_{j \neq k,\ell} |c_j| + |c_k + c_\ell| = K. \end{aligned}$$

Note that the second inequality follows from the fact that the angles between $e^{-i\pi/4\ell}$ and $e^{-i\pi/4k}$ fall within the range $(0, \pi/2)$. However, we also have

$$K\,\mathbb{T} = \mathrm{Ran}(\widehat{K\chi_n}) = \mathrm{Ran}(\widehat{T(K\chi_n)})$$

by the assumption. Thus we obtain $|\widehat{T(K\chi_n)}(z)| = K$ for all $z \in \mathbb{T}$, which yields a contradiction. Hence $|Z| = 1$. As a result, we obtain $T(\chi_n) = \chi_{m_n}$ where $Z = \{m_n\}$. It is clear from the preceding discussion that m_n does not depend on the choice of $K > 0$. Then we obtain the map $\varphi : \mathbb{Z} \ni n \mapsto m_n \in \mathbb{Z}$ satisfies $T(K\chi_n) = K\chi_{\varphi(n)}$.

By the same argument as that used to show $|Z| = 1$, the injectivity of φ is obtained. Thus, φ is a bijection. $\square$

Chapter 5

Main Result

5.1 Spectrally additive surjective maps on the $\ell^1(\mathbb{Z})_+$

Our first result is as follows.

Theorem 1. *Let $T : \ell^1(\mathbb{Z})_+ \to \ell^1(\mathbb{Z})_+$ be a surjection that satisfies*

$$\operatorname{Ran}(\widehat{T(f)} + \widehat{T(g)}) = \operatorname{Ran}(\hat{f} + \hat{g})$$

for all $f, g \in \ell^1(\mathbb{Z})_+$. Then the map T is of the form $T(f) = f \circ \varphi$ where $\varphi = \mathrm{id}_{\mathbb{Z}}$ or $\varphi = -\mathrm{id}_{\mathbb{Z}}$.

In order to prove this, first we characterize the form of the map T restricted to the set of all characteristic functions. Then we expand the result to the whole space $\ell^1(\mathbb{Z})_+$. This is crucial since characteristic functions form a natural generating set of $\ell^1(\mathbb{Z})_+$.

Lemma 3. *Let $T : \ell^1(\mathbb{Z})_+ \to \ell^1(\mathbb{Z})_+$ be a surjective map that satisfies*

$$\operatorname{Ran}(\widehat{T(f)} + \widehat{T(g)}) = \operatorname{Ran}(\hat{f} + \hat{g})$$

for all $f, g \in \ell^1(\mathbb{Z})_+$. Then the map restricted to the set of all characteristic functions, that is, $S := T|_{\{\chi_n | n \in \mathbb{Z}\}}$ is of the form $S(\chi_n) = \chi_{\varphi(n)}$ where $\varphi = \mathrm{id}_{\mathbb{Z}}$ or $\varphi = -\mathrm{id}_{\mathbb{Z}}$.

Remark 1. *Since T maps characteristic functions to characteristic functions, there exists a map $\varphi : \mathbb{Z} \to \mathbb{Z}$ such that $T(\chi_n) = \chi_{\varphi(n)}$ for all $n \in \mathbb{Z}$.*

Proof. Fix $m \in \mathbb{Z} \setminus \{0, 1\}$ arbitrarily. Since

$$\{z + z^m; z \in \mathbb{T}\} = \{z^{\varphi(1)} + z^{\varphi(m)}; z \in \mathbb{T}\} \tag{5.1}$$

holds, thus $M_{1,m} = M_{\varphi(1),\varphi(m)}$. Hence

$$\varphi(m) - \varphi(1) = G(m - 1) \tag{5.2}$$

holds where G is either $\gcd(\varphi(m), \varphi(1))$ or its negative. Let $r \in \mathbb{R} \setminus \mathbb{Q}$ be an arbitrary irrational number. Then we have

$$e^{ir\pi} + e^{imr\pi} = e^{i\varphi(1)\theta_0} + e^{i\varphi(m)\theta_0}$$

for some $\theta_0 \in [0, 2\pi)$. A direct computation shows that

$$\left|\cos \frac{m-1}{2} r\pi\right| = \left|\cos \frac{\varphi(m) - \varphi(1)}{2} \theta_0\right| \tag{5.3}$$

and

$$e^{i(\frac{\varphi(m)+\varphi(1)}{2}\theta_0 - \frac{m+1}{2} r\pi)} \in \mathbb{R}\,. \tag{5.4}$$

There are two possibilities according to (5.3). For the first case, $\cos \frac{m-1}{2} r\pi = \cos \frac{\varphi(m)-\varphi(1)}{2}\theta_0$, we have $\frac{m-1}{2} r\pi = \pm \frac{\varphi(m)-\varphi(1)}{2}\theta_0 + 2n\pi$ for some $n \in \mathbb{Z}$. For the second case, $\cos \frac{m-1}{2} r\pi =$

$-\cos\frac{\varphi(m)-\varphi(1)}{2}\theta_0$, that is, $\cos\frac{m-1}{2}r\pi = \cos\frac{\varphi(m)-\varphi(1)}{2}\theta_0 + \pi$, we have $\frac{m-1}{2}r\pi = \pm(\frac{\varphi(m)-\varphi(1)}{2}\theta_0 + \pi) + 2n\pi$ for some $n \in \mathbb{Z}$. In either case, we obtain

$$(m-1)r\pi = \pm(\varphi(m) - \varphi(1))\theta_0 + 2n\pi \tag{5.5}$$

for some $n \in \mathbb{Z}$ by (5.3). By (5.2) and (5.5), we obtain

$$(m-1)r\pi = \pm G(m-1)\theta_0 + 2n\pi. \tag{5.6}$$

Multiplying both sides by $\frac{1}{m-1}$ and writing $\frac{2n}{m-1}$ as $q \in \mathbb{Q}$, we have

$$r\pi = \pm G\theta_0 + q\pi. \tag{5.7}$$

Thus we have

$$G\theta_0 = \pm(r\pi - q\pi) \tag{5.8}$$

for some $q \in \mathbb{Q}$. By (5.4), we have

$$\frac{\varphi(m)+\varphi(1)}{2}\theta_0 - \frac{m+1}{2}r\pi = n\pi$$

for some $n \in \mathbb{Z}$. Multiplying both sides by $2G$ and substituting (5.8), we obtain

$$\begin{aligned}(m+1)Gr\pi &= (\varphi(m)+\varphi(1))G\theta_0 - 2nG\pi \\ &= \pm(\varphi(m)+\varphi(1))(r\pi - q\pi) - 2nG\pi.\end{aligned}$$

Thus we obtain

$$((m+1)G \mp (\varphi(m) - \varphi(1)))r = \pm(\varphi(m)+\varphi(1))q - 2nG.$$

Since only r is irrational and all the others are rational numbers, it should hold that

$$(m+1)G \mp (\varphi(m) - \varphi(1)) = 0.$$

Thus we have either $-G(m+1) = \varphi(m) + \varphi(1)$ or $G(m+1) = \varphi(m) + \varphi(1)$. In the first case, combining the equation and (5.2), we obtain $\varphi(m) = 0$. This contradicts (5.1). In the second case, combining the equation and (5.2), we have

$$\varphi(m) = Gm \text{ and } \varphi(1) = G.$$

Hence we conclude that $\varphi = \mathrm{id}_{\mathbb{Z}}$ or $\varphi = -\mathrm{id}_{\mathbb{Z}}$. $\square$

Lemma 4. *Fix $f, g \in \ell^1(\mathbb{Z})_+$ arbitrarily. If*

$$\begin{cases}\mathrm{Ran}(\hat{f} + nz) &= \mathrm{Ran}(\hat{g} + nz) \\ \mathrm{Ran}(\hat{f} + nz^{-1}) &= \mathrm{Ran}(\hat{g} + nz^{-1})\end{cases}$$

holds for every $n \in \mathbb{Z}$, then $f = g$ holds.

Remark 2. *Here $\hat{f} + nz$ denotes the function $z \mapsto \hat{f}(z) + nz$ on $\mathbb{T}$.*

Proof. Fix $\theta \in [0, 2\pi)$ arbitrarily. For any $n \in \mathbb{N}$, there exists $\theta_n \in [0, 2\pi)$ such that

$$\hat{f}(e^{i\theta}) + ne^{i\theta} = \hat{g}(e^{i\theta_n}) + ne^{i\theta_n}$$

holds. Since the linear term dominates as $n \to \infty$, it follows that $e^{i\theta_n} \to e^{i\theta}$ as $n \to \infty$. Define $I_n := \hat{f}(e^{i\theta}) - \hat{g}(e^{i\theta_n})$. By rewriting the equation and multiplying both sides by its complex conjugate, we have

$$ne^{i\theta_n}(\overline{ne^{i\theta_n}}) = (I_n + ne^{i\theta})(\overline{I_n + ne^{i\theta}}).$$

Hence

$$\mathrm{Re}(I_n e^{-i\theta}) = -\frac{1}{2n}|I_n|^2.$$

There exists $\theta'_n \in [0, 2\pi)$ such that

$$\hat{f}(e^{i\theta}) + ne^{-i\theta} = \hat{g}(e^{i\theta'_n}) + ne^{-i\theta'_n}$$

by our assumption. By the same argument as above, we define I'_n analogously. Then we have

$$\mathrm{Re}(I'_n e^{i\theta}) = -\frac{1}{2n}|I'_n|^2.$$

Combining the previous two equations as $n \to \infty$, we obtain

$$\hat{f}(e^{i\theta}) - \hat{g}(e^{i\theta}) = 0$$

holds for any $\theta \in [0, 2\pi)$ with $\theta \neq 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$. Hence we conclude that $f = g$ by the continuity of $\hat{f}, \hat{g}$ and injectivity of the Fourier transform. $\square$

Proof of Theorem 1. Suppose that a map $T : \ell^1(\mathbb{Z})_+ \to \ell^1(\mathbb{Z})_+$ is surjective and satisfies

$$\mathrm{Ran}(\widehat{T(f)} + \widehat{T(g)}) = \mathrm{Ran}(\hat{f} + \hat{g})$$

for every pair $f, g \in \ell^1(\mathbb{Z})_+$. Then

$$\begin{aligned}\mathrm{Ran}(\hat{f} + \widehat{\chi_1}) &= \mathrm{Ran}(\widehat{T(f)} + \widehat{T(n\chi_1)}) \\ &= \begin{cases}\mathrm{Ran}(\widehat{T(f)} + \widehat{n\chi_1}) & (\varphi = \mathrm{id}_{\mathbb{Z}}) \\ \mathrm{Ran}(\widehat{T(f)} + \widehat{n\chi_{-1}}) & (\varphi = -\mathrm{id}_{\mathbb{Z}})\end{cases}\end{aligned}$$

holds. In the case of $\varphi = \mathrm{id}_{\mathbb{Z}}$, $T(f) = f$ can be deduced from the previous Lemma. Since $\hat{f}$ is a positive definite function, $\hat{f}(z) = \overline{\hat{f}(z^{-1})}$ holds for every $z \in \mathbb{T}$. In the case of $\varphi = -\mathrm{id}_{\mathbb{Z}}$, we have

$$\begin{aligned}\mathrm{Ran}(\hat{f} + \widehat{n\chi_1}) &= \mathrm{Ran}(\widehat{T(f)} + \widehat{n\chi_{-1}}) \\ &= \mathrm{Ran}(\overline{\widehat{T(f)}} + \widehat{n\chi_1}) \\ &= \mathrm{Ran}(\widehat{T(f) \circ -\mathrm{id}_{\mathbb{Z}}} + \widehat{n\chi_1})\end{aligned}$$

holds. Therefore, $T(f) \circ -\mathrm{id}_{\mathbb{Z}} = f$ i.e., $T(f) = f \circ -\mathrm{id}_{\mathbb{Z}}$ can be deduced from the previous Lemma again. $\square$

5.2 Spectrally multiplicative surjective maps on the $\ell^1(\mathbb{Z})_+$

Our second result is as follows.

Theorem 2. *Let $T : \ell^1(\mathbb{Z})_+ \to \ell^1(\mathbb{Z})_+$ be a surjective map that satisfies*

$$\mathrm{Ran}(\widehat{T(f)} \cdot \widehat{T(g)}) = \mathrm{Ran}(\hat{f} \cdot \hat{g})$$

for all $f, g \in \ell^1(\mathbb{Z})_+$. Then the map restricted to the set of all characteristic functions, that is, $S := T|_{\{\chi_n \in \ell^1(\mathbb{Z})_+ | n \in \mathbb{Z}\}}$ is of the form $S(\chi_n) = \chi_{\varphi(n)}$ where $\varphi = \mathrm{id}_{\mathbb{Z}}$ or $\varphi = -\mathrm{id}_{\mathbb{Z}}$.

In order to prove this result, we first establish a proposition.

Lemma 5. *Let $T : \ell^1(\mathbb{Z})_+ \to \ell^1(\mathbb{Z})_+$ be a surjective map that satisfies*

$$\mathrm{Ran}(\widehat{T(f)} \cdot \widehat{T(g)}) = \mathrm{Ran}(\hat{f} \cdot \hat{g})$$

for all $f, g \in \ell^1(\mathbb{Z})_+$. Then the equation

$$\mathrm{Ran}(\widehat{T(f)}) = \mathrm{Ran}(\hat{f}) \text{ for all } f \in \ell^1(\mathbb{Z})_+$$

holds.

Proof. The fact that

$$\begin{aligned}\{1\} &= \mathrm{Ran}(1)\\ &= \mathrm{Ran}(\widehat{\chi_0}\cdot\widehat{\chi_0})\\ &= \mathrm{Ran}(\widehat{T(\chi_0)}\cdot\widehat{T(\chi_0)})\end{aligned}$$

ensures that $\widehat{T(\chi_0)} = 1$ or $\widehat{T(\chi_0)} = -1$.

Fix $f \in \ell^1(\mathbb{Z})_+$ arbitrarily. Thus,

$$\begin{aligned}\mathrm{Ran}(\hat{f}\cdot 1) &= \mathrm{Ran}(\hat{f}\cdot\widehat{\chi_0})\\ &= \mathrm{Ran}(\widehat{T(f)}\cdot\widehat{T(\chi_0)})\\ &= \begin{cases}\mathrm{Ran}(\widehat{T(f)}\cdot 1) & \cdots(\widehat{T(\chi_0)} = 1)\\ \mathrm{Ran}(\widehat{T(f)}\cdot(-1)) & \cdots(\widehat{T(\chi_0)} = -1)\end{cases}\\ &= \mathrm{Ran}(\widehat{T(f)})\end{aligned}$$

ensures that

$$\mathrm{Ran}(\widehat{T(f)}) = \mathrm{Ran}(\hat{f}) \text{ for all } f \in \ell^1(\mathbb{Z})_+.$$

□

In what follows, we use this fact without further mention.

Then we establish two lemmas to prove our second result.

Lemma 6. *Let $T : \ell^1(\mathbb{Z})_+ \to \ell^1(\mathbb{Z})_+$ be a surjective map satisfying*

$$\mathrm{Ran}(\widehat{T(f)}\cdot\widehat{T(g)}) = \mathrm{Ran}(\hat{f}\cdot\hat{g})$$

for all $f, g \in \ell^1(\mathbb{Z})_+$. Then it follows that

$$\varphi(m) = -\varphi(-m) \text{ for any } m \in \mathbb{Z},$$

where φ is the bijection given in Lemma 2.

Proof. Fix $n \in \mathbb{Z}$ arbitrarily. We observe that

$$\begin{aligned}\{1\} &= \mathrm{Ran}(z^0)\\ &= \mathrm{Ran}(z^n\cdot z^{-n})\\ &= \mathrm{Ran}(\widehat{\chi_n}\cdot\widehat{\chi_{-n}})\\ &= \mathrm{Ran}(\widehat{T(\chi_n)}\cdot\widehat{T(\chi_{-n})})\\ &= \mathrm{Ran}(\widehat{\chi_{\varphi(n)}}\cdot\widehat{\chi_{\varphi(-n)}}),\end{aligned}$$

where the last equality follows from the Lemma 2. This ensures that $\varphi(n) = -\varphi(-n)$.

□

Lemma 7. *Let $T : \ell^1(\mathbb{Z})_+ \to \ell^1(\mathbb{Z})_+$ be a surjective map satisfying*

$$\mathrm{Ran}(\widehat{T(f)}\cdot\widehat{T(g)}) = \mathrm{Ran}(\hat{f}\cdot\hat{g})$$

for all $f, g \in \ell^1(\mathbb{Z})_+$. Then it follows that

$$T(\chi_0 + \chi_n) = \chi_0 + \chi_{\varphi(n)},$$

where φ is the bijection given in Lemma 2.

Proof. Fix $n \in \mathbb{Z}$ arbitrarily. Routine calculations show that

$$\begin{aligned} 1 + \mathbb{T} &= \operatorname{Ran}(\widehat{\chi_0 + \chi_n}) \\ &= \operatorname{Ran}(1 + z^n) \\ &= \operatorname{Ran}((z^{-n} + 1)z^n) \\ &= \operatorname{Ran}(\widehat{\chi_{-n} + \chi_0}\widehat{\chi_n}) \\ &= \operatorname{Ran}(\widehat{\chi_0 + \chi_{-n}}\widehat{\chi_n}), \end{aligned}$$

and

$$1 + \mathbb{T} = \operatorname{Ran}(\widehat{\chi_0 + \chi_{-n}}).$$

Combining them and by using the assumption, we have

$$\begin{cases} \operatorname{Ran}(\widehat{T(\chi_0 + \chi_{-n})}) & = 1 + \mathbb{T}, \\ \operatorname{Ran}(\widehat{T(\chi_0 + \chi_{-n})}\widehat{\chi_{\varphi(n)}}) & = 1 + \mathbb{T}\,. \end{cases} \tag{5.9}$$

For any $z \in \mathbb{T}$, put $\omega(z) = \widehat{T(\chi_0 + \chi_{-n})}(z) - 1 - z^{-\varphi(n)}$. Then it is enough to prove $\omega(z) = 0$.

Fix $z \in \mathbb{T}$ with $z^{\varphi(n)} \neq -1, 1$. Substituting z into (5.9), we obtain

$$\begin{cases} |1 + z^{-\varphi(n)} + \omega(z) - 1| & = 1, \\ |(1 + z^{-\varphi(n)} + \omega(z))z^{\varphi(n)} - 1| & = 1. \end{cases}$$

By routine calculations, we have

$$\begin{cases} |z^{-\varphi(n)} + \omega(z)| = 1, \\ |1 + \omega(z)| = 1. \end{cases}$$

That is,

$$\begin{cases} \omega(z) \in -z^{-\varphi(n)} + \mathbb{T}, \\ \omega(z) \in -1 + \mathbb{T}\,. \end{cases}$$

Since $z^{-\varphi(n)} \neq 1, -1$, it follows that

$$\omega(z) = -(1 + z^{-\varphi(n)}) \ \text{ or } \ \omega(z) = 0.$$

In the case of $\omega(z) = -(1 + z^{-\varphi(n)})$, it follows that

$$\begin{aligned} \widehat{T(\chi_0 + \chi_{-n})}(z) &= 1 + z^{-\varphi(n)} + \omega(z) \\ &= 1 + z^{-\varphi(n)} - (1 + z^{-\varphi(n)}) \\ &= 0. \end{aligned}$$

Here, we consider

$$\begin{aligned} \Lambda &:= \{\lambda \in \mathbb{T} \,|\, 1 + \lambda^{\varphi(n)} + \omega(\lambda) \neq 0\} \\ &= \{\lambda \in \mathbb{T} \,|\, 1 + \lambda^{\varphi(n)} \neq 0\}. \end{aligned}$$

Here, the second equality follows from the contraposition of the previous statement.

Since $z^{-\varphi(n)} \neq -1$, it follows that $\omega(z) = 0$. By using the continuity of ω, we conclude $\omega(z') = 0$ for any $z' \in \mathbb{T}$. □

In fact, we conclude Theorem 2 directly by using the previous lemma.

Proof of Theorem 2. Fix $n \in \mathbb{Z}$ $(n > 0)$ arbitrarily. By using the previous lemma, we have

$$\begin{aligned} \operatorname{Ran}(\widehat{\chi_0 + \chi_n} \cdot \widehat{\chi_1}) &= \operatorname{Ran}(\widehat{T(\chi_0 + \chi_n)} \cdot \widehat{T(\chi_1)}) \\ &= \operatorname{Ran}(\widehat{\chi_0 + \chi_{\varphi(n)}} \cdot \widehat{\chi_{\varphi(1)}}). \end{aligned}$$

Thus we obtain

$$\operatorname{Ran}(z+z^{n+1}) = \operatorname{Ran}(z^{\varphi(1)} + z^{\varphi(n)+\varphi(1)}). \tag{5.10}$$

By using Lemma 1, we have

$$\frac{|(n+1)-1|}{\gcd(n+1,1)} = \frac{|(\varphi(n)+\varphi(1))-\varphi(1)|}{\gcd(\varphi(n)+\varphi(1),\varphi(1))}.$$

Thus we obtain

$$n = \frac{|\varphi(n)|}{\gcd(\varphi(n),\varphi(1))}.$$

Since the statement $\gcd(\varphi(n),\varphi(1)) \geq 1$ holds, the conclusion

$$|\varphi(n)| = n$$

follows.

Our next goal is to derive the desired result by removing the absolute value signs and using the symmetry $\varphi(n') = -\varphi(-n')$.

Assume that $\varphi(n) = -n\varphi(1)$. Substituting this into (5.10), we obtain either

$$\operatorname{Re}(z+z^{n+1}) = \operatorname{Re}(z^{-1}+z^{n-1}) \tag{5.11}$$

or

$$\operatorname{Re}(z+z^{n+1}) = \operatorname{Re}(z+z^{-(n-1)}). \tag{5.12}$$

Without loss of generality, we may restrict ourselves to the second case.

Fix $r \in \mathbb{R}\setminus\mathbb{Q}$ arbitrarily. By (5.11), we have

$$e^{ir\pi} + e^{i(n+1)r\pi} = e^{i\theta} + e^{-i(n-1)\theta}$$

for some $\theta \in [0,2\pi)$.

The fact that

$$e^{ik\theta'} + e^{i\ell\theta'} = 2\cos(\frac{k-\ell}{2}\theta')e^{i\theta'\frac{k+\ell}{2}} \text{ for all } k,\ell \in \mathbb{Z}, \theta' \in [0,2\pi)$$

ensures that

$$2\cos\left(\frac{n}{2}r\pi\right)e^{\frac{i(n+2)r\pi}{2}} = 2\cos\left(\frac{-n}{2}\theta\right)e^{-\frac{i(n+2)\theta}{2}}.$$

There are two possible equations as follows.

$$\begin{cases} \cos(\frac{n}{2}r\pi) = \cos(\frac{-n}{2}\theta) & \text{and} \quad e^{\frac{i(n+2)r\pi}{2}} = e^{-\frac{i(n+2)\theta}{2}} \\ \cos(\frac{n}{2}r\pi) = -\cos(\frac{-n}{2}\theta) & \text{and} \quad e^{\frac{i(n+2)r\pi}{2}} = -e^{-\frac{i(n+2)\theta}{2}} \end{cases}$$

In both cases, we have

$$\begin{cases} nr\pi = n\theta + 2k\pi \\ (n+2)r\pi = (-n+2)\theta + 2\ell\pi \end{cases}$$

for some $k,\ell \in \mathbb{Z}$. We obtain

$$\begin{cases} r\pi = (-n+1)\theta + (\ell-k)\pi \\ (n+1)r\pi = \theta + (\ell+k)\pi. \end{cases}$$

Combining them, it follows that

$$(1-(n-1)(n+1))r\pi = -(n+1)(\ell+k)\pi + (\ell-k)\pi.$$

By a routine calculation, we have

$$n^2 r = -(n+1)(\ell+k) + (\ell-k).$$

Since $n > 0$, the right hand side is nonzero. It follows that $n^2 = 0$, since r is irrational. However, this contradicts the assumption that $n > 0$. Therefore, $\varphi(n) = n\varphi(1)$. Based on $\varphi(n') = -\varphi(-n')$ holds for any $n' \in \mathbb{Z}$, the result extends to all $n \in \mathbb{Z}$. This is the desired result. $\square$

Chapter 6

Future Work and Open Problems

First, we aim to extend the result Theorem 2 to the whole space.

Moreover, our results are formulated only the specific space $\ell^1(\mathbb{Z})_+$, and it is natural to ask whether they extend more general settings. For example, it may be natural to investigate the problem for locally compact groups.

Moreover, in attempting such a generalization, it is necessary to refine the proofs of Theorem 1 and 2, as well as the auxiliary lemmas on which they depend. The current arguments rely essentially on the structure of $\ell^1(\mathbb{Z})_+$.

Chapter 7

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References

[1] Y. Enami, *Range preserving maps between the spaces of continuous functions with values in a locally convex space,* Proceedings of the American Mathematical Society, 2021.

[2] O. Hatori, T. Miura and H. Takagi, *Unital and multiplicatively spectrum-preserving surjections between semi-simple commutative Banach algebras are linear and multiplicative,* Journal of Mathematical Analysis and Applications, 2007.

[3] S. Kowalski and Z. Słodkowski, *A characterization of multiplicative linear functionals in Banach spaces,* Studia Math, 1980.

[4] S. Oi and K. Sato, *Spectrally additive maps on the positive cones of the Wiener algebra,* Archiv der Mathematik, to appear.