Deep Dive: Taylor Series Expansion of $\ln(1-x)$

Professor's Detailed Notes

Abstract

This document provides a rigorous derivation of the Taylor (Maclaurin) series for $f(x) = \ln(1-x)$ centered at $x = 0$. We explore two derivation methods, analyze the radius of convergence, and discuss the specific inequalities used in probabilistic proofs (e.g., Bloom Filters and the Birthday Paradox).

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1 Introduction

The Taylor series representation of a function $f(x)$ centered at a is given by:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{1}$$

When $a = 0$, this is called a **Maclaurin series**:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n \tag{2}$$

Our goal is to find this series for $f(x) = \ln(1-x)$.

2 Method 1: Direct Differentiation (The "Brute Force" Approach)

We compute the n-th derivatives of $f(x)$ and evaluate them at $x = 0$.

2.1 Step-by-Step Derivatives

Let $f(x) = \ln(1 - x)$.

1. **Zeroth Derivative ($n = 0$):**

$$f(0) = \ln(1 - 0) = \ln(1) = 0$$

2. **First Derivative ($n = 1$):** Using the chain rule ($\frac{d}{dx}\ln(u) = \frac{u'}{u}$):

$$f'(x) = \frac{1}{1-x} \cdot (-1) = -(1-x)^{-1}$$

At $x = 0$:

$$f'(0) = -1$$

3. **Second Derivative ($n = 2$):**

$$f''(x) = -(-1)(1-x)^{-2} \cdot (-1) = -(1-x)^{-2}$$

At $x = 0$:

$$f''(0) = -1$$

4. **Third Derivative ($n = 3$):**

$$f'''(x) = -(-2)(1-x)^{-3} \cdot (-1) = -2(1-x)^{-3}$$

At $x = 0$:

$$f'''(0) = -2$$

5. **Fourth Derivative ($n = 4$):**

$$f^{(4)}(x) = -2(-3)(1-x)^{-4} \cdot (-1) = -6(1-x)^{-4}$$

At $x = 0$:

$$f^{(4)}(0) = -6$$

2.2 Identifying the Pattern

Looking at the sequence of derivatives at $x = 0$: $0, -1, -1, -2, -6, \ldots$ The pattern for $n \geq 1$ is:

$$f^{(n)}(0) = -(n-1)! \tag{3}$$

2.3 Constructing the Series

Substituting this into the general Maclaurin formula:

$$\begin{aligned}
\ln(1-x) &= f(0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\
&= 0 + \sum_{n=1}^{\infty} \frac{-(n-1)!}{n!} x^n \\
&= -\sum_{n=1}^{\infty} \frac{(n-1)!}{n \cdot (n-1)!} x^n \quad \text{(Since } n! = n \cdot (n-1)!\text{)} \\
&= -\sum_{n=1}^{\infty} \frac{1}{n} x^n
\end{aligned}$$

Result:

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots \tag{4}$$

3 Method 2: Integration of Geometric Series (The Elegant Approach)

Your lecture notes mention that "useful tricks for deriving new Taylor series are differentiation and integration" (Bloom Filters 4.pdf, Page 5). This is that trick.

3.1 The Geometric Series

Recall the sum of a geometric series for $|t| < 1$:

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n = 1 + t + t^2 + t^3 + \dots$$

3.2 Relating to Natural Log

Notice that the derivative of our target function is related to this series:

$$\frac{d}{dx} \ln(1-x) = \frac{-1}{1-x}$$

Therefore, we can express $\ln(1-x)$ as an integral:

$$\ln(1-x) = \int_0^x \frac{-1}{1-t}\, dt$$

3.3 Term-by-Term Integration

Substitute the series expansion into the integral:

$$
\begin{aligned}
\ln(1-x) &= -\int_0^x \left(\sum_{n=0}^{\infty} t^n \right) dt \\
&= -\sum_{n=0}^{\infty} \int_0^x t^n \, dt \\
&= -\sum_{n=0}^{\infty} \left[\frac{t^{n+1}}{n+1} \right]_0^x \\
&= -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}
\end{aligned}
$$

Re-indexing the sum by letting $k = n+1$ (so when $n = 0, k = 1$), we get the same result:

$$\ln(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k} \tag{5}$$

4 Nuance: Convergence and Bounds

4.1 Radius of Convergence

Using the **Ratio Test**, the series converges when:

$$\lim_{n\to\infty} \left| \frac{x^{n+1}/(n+1)}{x^n/n} \right| = |x| \lim_{n\to\infty} \frac{n}{n+1} = |x| < 1$$

Thus, the series is valid for $-1 < x < 1$.

- **At $x = -1$:** The series becomes $-\sum \frac{(-1)^n}{n}$, which is the alternating harmonic series (Converges).
- **At $x = 1$:** The series becomes $-\sum \frac{1}{n}$, which is the harmonic series (Diverges toward $-\infty$).

Interval of Convergence: $[-1, 1)$.

4.2 Useful Inequalities (For Bloom Filters/Birthday Paradox)

In your course notes (Bloom Filters 4.pdf, Page 6), the Taylor series is used to approximate probabilities.

1. The Linear Upper Bound: Since all terms in the expansion $\ln(1-x) = -x - \frac{x^2}{2} - \dots$ are negative (for $0 < x < 1$):

$$\ln(1-x) \leq -x$$

This is used to show $1 - x \leq e^{-x}$.

2. The Quadratic Lower Bound: For small x, the notes derive a tighter bound by keeping the second term:

$$\ln(1-x) \geq -x - x^2$$

This "two-sided bound" is critical for proving that the approximations in the Birthday Paradox are accurate.

5 Converting Intractable Products into Solvable Sums

In probability, we often calculate the probability of *successive independent events*. This results in a product of probabilities, which is algebraically difficult to solve.

In the analysis of probabilistic algorithms (like Hashing and Bloom Filters), we frequently encounter the expression $\ln(1-x)$ where x is a very small number (close to 0). We use the Taylor expansion for three specific strategic reasons.

Example from Birthday Paradox: The probability that n people have unique birthdays is the product:

$$P(\text{unique}) = \prod_{k=0}^{n-1} \left(1 - \frac{k}{365}\right)$$

Solving for n directly from this product is nearly impossible

The Strategy:

1. Apply ln to turn the product into a sum:

$$\ln(P) = \sum_{k=0}^{n-1} \ln\left(1 - \frac{k}{365}\right)$$

2. **Apply Taylor Series:** Replace the complex log term with a simple polynomial. Since $x = \frac{k}{365}$ is small, $\ln(1-x) \approx -x$.

$$\ln(P) \approx \sum_{k=0}^{n-1} \left(-\frac{k}{365}\right)$$

3. Now we can pull out the constant and use the arithmetic sum formula $\sum k = \frac{n(n-1)}{2}$.

Without the Taylor expansion, we would be stuck with the logarithm inside the summation.

6 The "Exponential Approximation" Trick

In Computer Science, we frequently use the inequality $1 - x \leq e^{-x}$. This is derived directly from the Taylor series of $\ln(1-x)$.

Derivation:

$$\ln(1-x) \approx -x \implies 1 - x \approx e^{-x}$$

Application in Bloom Filters: When analyzing Bloom Filters, we determine the probability that a bit remains 0 after n insertions into m bits.

$$P(\text{bit is } 0) = \left(1 - \frac{1}{m}\right)^{kn}$$

Calculating limits with $\left(1 - \frac{1}{m}\right)$ is tedious. Using the Taylor approximation, we instantly simplify:

$$\left(1 - \frac{1}{m}\right)^{kn} \approx \left(e^{-1/m}\right)^{kn} = e^{-kn/m}$$

This form allows us to easily use calculus to find the optimal number of hash functions k.

7 Justification: Why is it safe?

The Taylor series for $\ln(1-x)$ is an alternating series (for $x > 0$):

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots$$

For very small x (like $1/365$ or $1/m$), the higher-order terms ($\frac{x^2}{2}, \frac{x^3}{3}$) vanish very quickly.

- **First Order Term ($-x$):** Dominates the value.
- **Error Term ($-\frac{x^2}{2}$):** Provides a tight error bound if we need more precision.

In the Birthday Paradox with $n \approx 23$, the quadratic correction is small (≈ 0.014), confirming that the linear approximation is highly accurate.

8 Summary

We use the expansion $\ln(1-x) \approx -x$ because it transforms **products** (hard to solve) into **sums** (easy to solve) and allows us to approximate **polynomials** as **exponentials** ($1 - x \approx e^{-x}$), which are much easier to manipulate in calculus.