Diffg

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1 Abelian Groups and Modules

Note. R-Module. Let R be a ring with 1. We define the category of left R-modules R-Mod to be abelian groups (written additively) with an additional left action of ring elements in R, written $r \cdot m$ such that

1. $r \cdot (x + y) = r \cdot x + r \cdot y$,
2. $(r + s) \cdot x = r \cdot x + s \cdot y$,
3. $(rs) \cdot x = r \cdot (sx)$,
4. $1 \cdot x = x$.

Similarly we define Mod-R of right R-modules. If R is commutative then there is not difference between left and right modules.

Note. Abelian Groups as Modules. Any $\mathbb{Z}$-module merely carries the structure of an abelian group since the above axioms force $r \cdot m = m + m + \cdots + m$.

Note. R as a R-module. Any ring R can be made into a module by setting the action of R on the abelian group of R to be the ring multiplication, i.e. $r \cdot s = rs$.

Note. Free Module. An R-module M is called free if it admits a basis, i.e. any m can be written as a sum $m = r_1 b_1 + \ldots + r_n b_n$ for elements $r_i \in R$ and $b_i \in M$ (may not be a finite sum).
Equivalently, free modules are isomorphic to the direct product of copies of R as R-modules.

Proposition. Hom-Set as R-Module. Let R be commutative and M, N be R-modules. Then the set $\mathrm{Hom}(N, M)$ carries the structure of a R-module by

$$(r \cdot f)(m) = f(r \cdot m)$$

for all $r \in R$, $m \in M$ and $f : M \to N$. In particular the functor $\mathrm{Hom}(-, -) : R\text{-Mod} \to \mathrm{Ab}$ now lands in R-Mod. We write $\mathrm{Hom}_R(M, N)$ to emphasize the module structure.

2 Categories

Kernel, Image, Cokernel in $\text{A}\textsc{b}$. Let $\mathcal{C} = \text{A}\textsc{b}$ be the category of abelian groups and $f : X \to Y$ a (group) (homo)morphism.

- $\text{kern}(f) \subseteq X$, the kernel of f is the subgroup (object) of X containing those elements $x \in X$ such that $f(x) = 0$.
- $\text{im}(f) \subseteq Y$, the image of f, is the subgroup (object) of Y containing those elements $y \in Y$ such that $y = f(x)$ for some $x \in X$.
- $\text{cokern}(f) = Y/\text{im}(f)$ is the quotient of Y by the image of f.

Faithful Functor. A functor $F : (C) \to (D)$ is called faithful if for fixed objects $X, Y \in \mathcal{C}$ it maps morphisms $X \to Y$ injectively to morphisms $F(X) \to F(Y)$.

Note. A faithful functor must not be injective on objects are the set of all morphisms.

Concrete Category. A concrete category is a pair $(\mathcal{C}, U)$ where $\mathcal{C}$ is a category and $U : \mathcal{C} \to \text{S}\textsc{et}$ is a faithful functor.

2.1 Abstracting

The following notions lift the properties "injective" and "surjective" as well as "(co)kernel" and "image" to categories. In particular to categories where the objects do not form sets (or have elements).

Monic Morphism. Let $f : X \to Y$ be a morphism in a category $\mathcal{C}$. It is called monic (mono) if for any $g, h : A \to X$ we have left cancellation

$$f \circ g \;=\; f \circ h \implies g = h \qquad (A \rightrightarrows X \to Y).$$

Note. In $\text{S}\textsc{et}$ a morphism is mono if and only if it is injective.

Epic Morphism. Let $f : X \to Y$ be a morphism in a category $\mathcal{C}$. It is called epic (epi) if for any $g, h : Y \to A$ we have right cancellation

$$g \circ f \;=\; h \circ f \implies g = h \qquad (X \to Y \rightrightarrows A).$$

Note. Again, in $\text{S}\textsc{et}$ epic morphisms are precisely the surjective maps.
Note. In $\text{R}\textsc{ing}$ the embedding $f : \mathbb{Z} \hookrightarrow \mathbb{Q}$ is epic (but not surjective). To see this, we need to show that any (ring) homomorphisms $f, g : \mathbb{Q} \to A$ agree on $\mathbb{Z}$.
One way to show this is by showing that if f, g agreen on $\mathbb{Z}$ then they must be equal: Any (ring) homomorphism $\mathbb{Q} \to A$ is uniquely determined by its values in $\mathbb{Z}$.
In fact for any ring A there exists at most one ring homomorphism $\mathbb{Q} \to A$.

Initial, Terminal and Zero Object Let $\mathcal{C}$ be a category. An object I in $\mathcal{C}$ is called initial if for any object X there exists precisely one morphism $I \to X$.
An object M is called terminal if for any object X there exists precisely one morphism $X \to T$.
If an object 0 is (isomorphic to) both initial and terminal object, it is called zero object.

Note. Initial, terminal and zero objects are unique up to unique isomorphism.
Note. Let 0 be the zero object and $f : X \to Y$. Then the compositions $0 \to X \to Y$ and $X \to Y \to 0$ are the unique morphisms $0 \to Y$ and $X \to 0$ respectively.

Proposition. Zero Morphism. Let 0 be a zero object in $\mathcal{C}$. Then for any objects X and Y there exists a distinguished morphism

$$X \to 0 \to Y, \qquad X \xrightarrow{0} Y,$$

the zero morphism.

Note. Let $f : X \to Y$. Then the composities $X \xrightarrow{f} Y \xrightarrow{0} Z$ and $W \xrightarrow{0} X \xrightarrow{f} Y$ are given by $X \xrightarrow{0} Z$ and $W \xrightarrow{0} Y$ respectively.
Note. Let $f : X \to Y$. If for all $g : Y \to Z$ we have $g \circ f = 0$, then $f = 0$ since we may choose $g = \mathrm{id}_Y$ thus $0 = \mathrm{id}_Y \circ f = f$.

Products and Coproducts. Let $\mathcal{C}$ be a category. Let X, Y and $X \times Y, X \sqcup Y$ be objects with morphisms p_x, p_y and i_x, i_y respectively. Then $X \times Y$ and $X \sqcup Y$ are called product and coproduct respectively if for any T, t_x, t_y there exists a unique morphism φ such that the following diagrams commute

$$\begin{tikzcd}
& T \arrow[d, dashed, "\varphi"'] \arrow[ddl, bend right, "t_x"'] \arrow[ddr, bend left, "t_y"] & \\
& X \times Y \arrow[dl, "p_x"'] \arrow[dr, "p_y"] & \\
X & & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[dr, "i_x"] \arrow[ddr, bend right, "t_x"'] & & Y \arrow[dl, "p_y"'] \arrow[ddl, bend left, "t_y"] \\
& X \sqcup Y \arrow[d, dashed, "\varphi"'] & \\
& T &
\end{tikzcd}$$

Kernel and Cokernel. Let $f : X \to Y$ be a morphism in a category $\mathcal{C}$ with 0. The kernel and cokernel of f are objects together with morphisms $i : \ker(f) \to X$, $p : Y \to \operatorname{coker}(f)$ such that the following for any T, j or T, q there exists a unique morphism φ such that the following diagrams commute if the outer squares commute

$$
\begin{tikzcd}
T \arrow[dr, dashed, "\varphi"] \arrow[drr, bend left, "j"] \arrow[ddr, bend right, "0"'] & & \\
& \ker(f) \arrow[r, "i"] \arrow[d, "0"'] & X \arrow[d, "f"] \\
& 0 \arrow[r, "0"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "0"'] & Y \arrow[d, "p"] \arrow[ddr, bend left, "q"] & \\
0 \arrow[r, "0"] \arrow[drr, bend right, "0"'] & \operatorname{coker}(f) \arrow[dr, dashed, "\varphi"] & \\
& & T
\end{tikzcd}
$$

Note. Kernels and cokernels are unique up to unique isomorphism.
Note. Consider the zero morphism $X \overset{0}{\to} Y$. Then $\ker(0) = \mathrm{id}_X$ and $\operatorname{coker}(0) = \mathrm{id}_Y$.
Note. Let X be an object. Then $\ker(\mathrm{id}_X) = 0 = \operatorname{coker}(\mathrm{id}_X)$. This is an instance of the following general fact.

Proposition. Let $f : X \to Y$ in a category with kernels and cokernels. Then

1. If f is mono then $\ker(f) = 0$,
2. If f is epi then $\operatorname{coker}(f) = 0$.

Proof. (1) Assume f is mono. We show that 0 satisfies the universal property of the kernel. If we take any $T \to 0$ such that $f \circ t = 0$ (outer square commutes) then since f is mono we get $f \circ t = 0 = f \circ 0$ and $t = 0$ is forced which makes the whole diagram commute as desired.
(2) Omitted since it is dual.

Note. The converse does not necessarily hold (if $\mathcal{C}$ is not Abelian, see below).

Proposition. Any kernel is mono and any cokernel is epic.

Proof.

Image. Let $\mathcal{C}$ be a category and $f : X \to Y$ be a morphism. Then we define the image of f to be an object $\operatorname{im}(f)$ together with morphisms $m : \operatorname{im}(f) \to Y$ and $e : X \to \operatorname{im}(f)$ such that m is mono and for any T and e', m' there exists a unique morphism φ such that the following diagram commutes if the outer square commutes

$$
\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "e"] \arrow[ddr, bend right, "e'"'] & & Y \\
& \operatorname{im}(f) \arrow[ur, hook, "m"'] \arrow[d, dashed, "\varphi"'] & \\
& T \arrow[uur, bend right, "m'"'] &
\end{tikzcd}
$$

2.2 Abelian Categories

Ab-Category An Ab-Category is a category $\mathcal{C}$ that can be equipped with the folloing structure. For any objects X and Y the set $\mathrm{Hom}(A, B)$ has an abelian group structure

$$+ : \mathrm{Hom}(A, B) \times \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, B)$$

that is compatbile with compoisiton, i.e

$$f \circ (g + h) \;=\; (f \circ g) + (f \circ h), \qquad (f + g) \circ h \;=\; (f \circ h) + (g \circ h).$$

Note. The neutral element of $\mathrm{Hom}(A, B)$ is the zero morphism $A \to 0 \to B$.

Additive Category. An Ab-category is called additive if it has a zero object and admits products and coproducts for all objects.

Characterization of Additive Categories. A category $\mathcal{C}$ admits an additive (abelian group) structure on $\mathrm{Hom}(A, B)$ for all objects A, B if and only if

1. $\mathcal{C}$ has a zero object,
2. $\mathcal{C}$ has products and coproducts for all (pairs of) objects
3. For each object A the maps

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : \quad A \sqcup A \to A \times A$$

are isomorphisms (where the 0 map denotes the composite $A \to 0 \to A$).

Biproduct. Let $\mathcal{C}$ be an additive category. Then for any objects X and Y we have $X \times Y \cong X \sqcup Y$. Thus instead of writing the (co)product we write $X \oplus Y$.

Abelian Category. An additive category is called abelian if

1. Every morphism has a kernel and cokernel,
2. Every mono is the kernel of its cokernel,
3. Every epi is the cokernel of its kernel.

Characterization of Abelian Categories. A category $\mathcal{C}$ is an abelian category if and only if

1. $\mathcal{C}$ has a zero object,
2. $\mathcal{C}$ has products and coproducts for all objects,
3. $\mathcal{C}$ has kernel and cokernel for all morphisms,
4. Every mono is a kernel and every epi is a cokernel.

Image in an Abelian Category. Let $\mathcal{C}$ be an Abelian category and $f : X \to Y$ be a morphism. Then

$$\mathrm{im}(f) \;=\; \ker\bigl(Y \xrightarrow{p} \mathrm{coker}(f)\bigr)$$

2.3 Exact Sequences

In this subsection we work with abelian categories.

Homology and Exactness. The notion of exactness applies to a pair of maps

$$A \xrightarrow{f} B \xrightarrow{g} C$$

such that $g \circ f = 0$, i.e. the composite factors through 0. Consider the diagrams:

$$\begin{tikzcd}
\mathrm{im}(f) \arrow[rrd, bend left, "t"] \arrow[rdd, bend right] & & A \arrow[ll] \arrow[d, "f"] \\
 & & B \arrow[d, "g"] \\
 & 0 \arrow[r] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{im}(f) \arrow[rrd, bend left, "t"] \arrow[rd, dashed, "\varphi"] \arrow[rdd, bend right] & & A \arrow[ll] \arrow[d, "f"] \\
 & \ker(g) \arrow[r] \arrow[d] & B \arrow[d, "g"] \\
 & 0 \arrow[r] & C
\end{tikzcd}$$

In the right column we have our pair of maps f and g. f factors through its image and includes via the monomorphism t.
Here, the composite $g \circ t : \mathrm{im}(f) \to C$ factors through zero. Thus, by the universal property of the kernel $\ker(g)$ there exists a unique dashed morphism $\varphi : \mathrm{im} f \to \ker(g)$. The cokernel of that map $\mathrm{coker}(\varphi)$ is called the homology object at B. If $\mathrm{coker}(\varphi) = 0$ and thus φ and isomorphism, then we say $A \xrightarrow{f} B \xrightarrow{g} C$ is exact at B.

Short Exact Sequence, Extension. A short exact sequence is a sequence of morphisms

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

such that it is exact at A, B and C. We call B an extension of C by A.

Split Short Exact Sequence. Consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "g"] \arrow[d, dashed, "\varphi"] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r, "(1{,}0)^T"'] & A \oplus C \arrow[r, "(0{,}1)"'] & C \arrow[r] & 0
\end{tikzcd}$$

We call the short exact sequence $0 \to A \to B \to C \to 0$ split exact if there exists an isomorphism φ (dashed) such that the diagram commutes.
When can give two equivalent criteria for the above sequence to be split short exact. The following are equivalent

1. An isomorphism φ exists such that the above diagram commutes,
2. There exists $s : C \to B$ such that $g \circ s = \mathrm{id}_C$,
3. There exists $r : B \to A$ such that $r \circ f = \mathrm{id}_A$.

Short Exact Sequence, Kernels and Cokernels. Let

$$A \xrightarrow{f} B \xrightarrow{g} C$$

be a sequence such that $g \circ f = 0$. Then it is a short exact sequence if and only if $A \cong \ker(g)$ and $C \cong \operatorname{cokern}(f)$.

2.4 Additive and Exact Functors

Additive Functor. Let $\mathcal{A}$ and $\mathcal{B}$ be additive categories (Ab-categories is sufficient). A functor $F : \mathcal{A} \to \mathcal{B}$ is called an additive (or Ab-functor) if for any objects $A, A' \in \mathcal{A}$ the map

$$F : \mathrm{Hom}(A, A') \to \mathrm{Hom}\big(F(A), F(A')\big), \quad f \mapsto F(f)$$

is a group homomorphism, i.e. $F(f + g) = F(f) + F(g)$ and $F(0) = 0$.

Proposition. Additive I. Let $F : \mathcal{A} \to \mathcal{B}$ be an Ab-Functor between Ab-categories. Then

1. $F(0) = 0$ (preserves initial, terminal and zero objects),
2. $F(A \times B) = F(A) \times F(B)$ (preserves products),
3. $F(A \sqcup B) = F(A) \sqcup F(B)$ (preserves coproducts).

Proposition. Additive II. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between additive categories that preserves 0, products and coproducts. Then it is additive.

Left Exact Functor. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and $F : \mathcal{A} \to \mathcal{B}$ be an additive functor. F is called left exact if it preserves kernels: Given a morphism $f : A \to B$ we have that

$$F\big(\ker(f)\big) \;\cong\; \ker\big(F(f)\big).$$

i.e. applying F to $\ker(f)$ is a kernel of $F(f)$.
Equivalently, a left exact functor preserves the exactness of sequences

$$0 \longrightarrow A \longrightarrow B \longrightarrow C.$$

Right Exact Functor. Dually, $F : \mathcal{A} \to \mathcal{B}$ is called right exact if it preserves cokernels or equivalently exactness of sequences of the form

$$A \longrightarrow B \longrightarrow C \longrightarrow 0.$$

Exact Functor. $F : \mathcal{A} \to \mathcal{B}$ is called exact if it preserves both kernels and cokernels or equivalently the exactness of short exact sequences

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0.$$

2.5 Projective Objects

In this subsection we work with Abelian categories.

Projective Object. Let $P \in \mathcal{C}$ be an object. P is called projective if any short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow P \longrightarrow 0$$

splits.

Characterizing Projective Objects. The following are equivalent

1. $P \in \mathcal{C}$ is projective,
2. Any epi $p : B \to P$ splits, i.e. there exists (a section) $s : P \to B$ such that $p \circ s = \mathrm{id}_P$,
3. Let $f : P \to Y$ be any morphism. Then we can factor f along any epi $g : X \to Y$, i.e. there exists a dashed arrow such that

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists!"'] \arrow[d, "f"] \\ X \arrow[r, two heads, "g"] & Y \end{tikzcd}$$

   commutes,
4. For any $X \in \mathcal{C}$ the Hom-functor $\mathrm{Hom}(X, -) : \mathcal{C} \to \textsc{Ab}$ is exact.

Proof.

Properties of Projective Objects. The coproduct of projective objects is projective. If $X \oplus Y$ is projective iff X and Y are projective.

Enough Projectives. An abelian category $\mathcal{C}$ is said to have enough projectives if for each object $X \in \mathcal{C}$ there exists a projective object $P \in \mathcal{C}$ and an epimorphism $P \twoheadrightarrow X$ onto X.

Projective Resolution. A projective resolution of an object $A \in \mathcal{C}$ is an exact sequence $P_\bullet$

$$\begin{tikzcd} \cdots \arrow[r] & P_2 \arrow[r] & P_1 \arrow[r] & P_0 \arrow[d, two heads] \\ & & & A \arrow[r] & 0 \end{tikzcd}$$

of projective objects P_i. If $\mathcal{C}$ has enough projective a projective resolution exists for any object $A \in \mathcal{C}$.

Proof. We construct $P_\bullet$ inductively starting with an projective object $P =: P_0$ for $\ker(A \xrightarrow{e_{-1}} 0)$ such that $P_0 \to \ker(e_{-1})$ yielding the diagram

$$\begin{tikzcd}
P_0 \arrow[rr, two heads] \arrow[dr, two heads, "e_0"'] & & A \arrow[dr, "e_{-1}"] \arrow[dl, equal] & \\
& \ker(e_{-1}) & & 0
\end{tikzcd}$$

and continuing this process by finding projective objects for the objects $\ker(e_i)$

$$\begin{tikzcd}
\cdots \arrow[rr, dashed] \arrow[dr, two heads, "e_2"'] & & P_1 \arrow[rr, dashed] \arrow[dr, two heads, "e_1"'] & & P_0 \arrow[rr, dashed] \arrow[dr, two heads, "e_0"'] & & A \arrow[dr] \arrow[dl, equal] & \\
& \ker(e_1) \arrow[ur, hook] & & \ker(e_0) \arrow[ur, hook] & & \ker(e_{-1}) & & 0
\end{tikzcd}$$

we obtain the dashed arrows as the composition of maps factoring in and out of the kernels.

2.6 Adjoints

Definition. Adjoint. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors. An adjunction $F \dashv G$ is a natural isomorphism of functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \textsc{Set}$

$$\mathrm{Hom}_{\mathcal{D}}(F(-), -) \;\cong\; \mathrm{Hom}_{\mathcal{C}}(-, G(-)).$$

For $f \in \mathrm{Hom}_{\mathcal{D}}(F(X), Y)$ we often write $\overline{f} \in \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$ to be the image under this isomorphism and vice versa.

Example. Free and Forgetful. In many categories such as Grp, Ab and R-Mod we have an adjunction

$$\mathrm{free}(-) : \textsc{Set} \to \textsc{Grp} \qquad \dashv \qquad \mathrm{forgetful}(-) : \textsc{Grp} \to \textsc{Set}.$$

To see this, note that specifying a map on a free group is the same as specifying a map (of sets) on the underlying set.

Example. Discrete and Forgetful and Coarse. In Top this previous example works by replacing 'free' with 'discrete topology', a functor from Set to Top. Continuous maps of spaces with the disrecte topology are all possible (set) maps.
TODO

Example. Diagonal and Product. TODO

Example. Currying. Let $S \in \textsc{Set}$ be fixed. Then the functors $\textsc{Set} \to \textsc{Set}$

$$- \times S \quad \vdash \quad \mathrm{Hom}(S, -)$$

are adjoint. To see this we show that for all $X, Y \in \textsc{Set}$ we have a natural isomorphism

$$\mathrm{Hom}(X \times S, Y) \;\cong\; \mathrm{Hom}(X, \mathrm{Hom}(S, Y)).$$

But it is clear, that specifying a map $f : X \times S \to Y$ is the same as specifying a map $g : X \to (S \to Y)$. In particular we map

$$f \mapsto \big(x \mapsto f(x, -)\big), \qquad g \mapsto \big((x, s) \mapsto g(x)(s)\big).$$

Definition. Unit. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $F \dashv G$. Then for each $X \in \mathcal{C}$ we have

$$1_{F(X)} \;\in\; \mathrm{Hom}_{\mathcal{D}}(F(X), F(X)) \;\cong\; \mathrm{Hom}_{\mathcal{C}}\big(X, (G \circ F)(X)\big)$$

by the natural isomorphism of the adjunction and we the image of $1_{F(X)}$ under this isomoprhism is given by $\eta_X : X \to (G \circ F)(X)$ produces a natural transformation

$$\eta : 1_{\mathcal{C}} \to G \circ F$$

that we call unit of the adjunction. We can write a morphism $\alpha : F(X) \to Y$ under the adjunction as

$$\overline{\alpha} \;=\; \overline{\alpha \circ 1_{F(X)}} \;=\; G \circ \alpha \circ \overline{1_{F(X)}} \;=\; G \circ \alpha \circ \eta_X.$$

Proposition. Propoerties of Functors in Adjunction. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $F \dashv G$. Then

1. G preserves products, terminal objects and pullbacks,
2. F preserves coproducts, initial objects and pushouts,
3. if $\mathcal{C}$ and $\mathcal{D}$ are additive, then F and G are additive functors,
4. if $\mathcal{C}$ and $\mathcal{D}$ are abelian, then F is right exact and G is left exact.

3 Derived Functors

In this section we work with abelian categories.

We have seen that $\mathrm{Hom}(-,-) : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathrm{AB}$ is not a exact functor but only a left exact functor, failing to preserve short exact sequences at the right end. In this section we develop a tool to quanitfy by how much it fails to preserve exactness at the right when applied to a short exact sequence.

Generally, from a left exact functor $F : \mathcal{A} \to \mathcal{B}$ we derive a sequence of new functors $R^i(F) : \mathcal{A} \to \mathcal{B}$ such that $R^0(F) = F$ and for any short exact sequence in $\mathcal{A}$

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

we not only obtain a exact sequence

$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C)$$

but also a long exact sequence

$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C) \xrightarrow{\delta} R^1(F)(A) \longrightarrow R^1(F)(B) \longrightarrow R^1(F)(C) \xrightarrow{\delta} R^2(F)(A) \longrightarrow R^2(F)(B) \longrightarrow \cdots$$

with connecting morphisms δ where $R^1(F)(A)$ (and the tail of the sequence) is a measure for the extent that F fails to be (right) exact.

The sequence $R^i(F)$ for $i \geq 0$ together with the connecting map δ is called a cohomological delta functor.
We then show that the way we constructed this cohomological delta functor, the derived functor of F, is not arbitrary but really 'the' derived functor. It is a universal cohomological delta functor.
Universal cohomological delta functors are those that determine natural transformations for delta functors completely by the 0th degree. But any derived functor of F, in its 0th degree, is naturally isomorphic to F by assumption. Thus the derived functor, regardless of it's construction, is uniquely determined up to natural isomorphism of functors.

3.1 Functorial Resolutions

Let $\mathcal{A}$ be an abelian category with enough projectives. Given an object $A \in \mathcal{A}$ we a guaranteed to find a projective resolution $P_\bullet \twoheadrightarrow A$. Only from this however it is not clear how to construct a map $\mathcal{A} \to \mathrm{Ch}(\mathcal{A})_{\geq 0}$ and much less so how to make it into a functor.

It is in fact not possible to to so with $\mathrm{Ch}(\mathcal{A})_{\geq 0}$. Instead we need to move to the category of chain complexes up to chain homotopies $K(\mathcal{A})_{\geq 0}$.

Proposition. Let $\mathcal{A}$ be an abelian category with enough projectives and $f : A \to B$ a morphism in $\mathcal{A}$. Let $P_\bullet \to A$ and $Q_\bullet \to B$ be projective resolutions of A and B respectively. Then there exists a chain map $f_\bullet : P_\bullet \to Q_\bullet$ that extends f such that the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & P_1 \arrow[r] \arrow[d, dashed, "f_1"'] & P_0 \arrow[r] \arrow[d, dashed, "f_0"'] & A \arrow[r] \arrow[d, "f"'] & 0 \\
\cdots \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

The chain map $f_\bullet$ is unique up to chain homotopy.

Definition. Projective Resolutions as Category. We define the full additive subcategories

1. $\mathrm{Proj}(\mathcal{A}) \subset \mathcal{A}$ of projective objects,
2. $K_{\geq 0}(\mathrm{Proj}(\mathcal{A}))$ of chain complexes of projective objects up to homotopy in non-negative degree,
3. $K^0_{\geq 0}(\mathrm{Proj}(\mathcal{A}))$ of chain complexes of projective objects up to homotopy in non-negative degree such that $H_n(P_\bullet) = 0$ for all $n \geq 0$ and $P_\bullet \in K_{\geq 0}(\mathrm{Proj}(\mathcal{A}))$.

Note. Every object in (3) is a projective resolution

$$\cdots \longrightarrow P_1 \xrightarrow{p_1} P_0 \longrightarrow \mathrm{coker}(p_1) \longrightarrow 0$$

of the cokernel of the last differential p_1.

Proposition. Let $\mathcal{A}$ be an abelian category with enough projectives. Then the functor

$$H_0 : K^0_{\geq 0}(\mathrm{Proj}(\mathcal{A})) \to \mathcal{A}$$

is an equivalence of categories. Thus there exists an inverse functor denote

$$P_\bullet(-) : \mathcal{A} \to K^0_{\geq 0}(\mathrm{Proj}(\mathcal{A})) \subset K_{\geq 0}(\mathcal{A})$$

that is unique up to natural isomorphism.

3.2 Derived Functors

Definition. Left Derived Functor. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories such that $\mathcal{A}$ has enough projectives. Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor. We define functors

$F_i : \mathcal{A} \to \mathcal{B}$ for all $i \geq 0$ by

$$\begin{tikzcd}
A \arrow[rr, dashed, "F_i"] \arrow[d, shift right, "P_\bullet(-)"'] & & B \\
K^0_{\geq 0}(\mathrm{Proj}(\mathcal{A})) \arrow[u, shift right, "H_0"'] \arrow[rr, "K(F)"'] & & K_{\geq 0}(B) \arrow[u, "H_i"']
\end{tikzcd}$$

We also write L_iF for F_i.

Definition. Right Derived Functor. Similarly, we define the right derived functor of a left exact functor $F : \mathcal{A} \to \mathcal{B}$ whenever $\mathcal{A}$ has enough injectives. Instead of a projective resolution we use an injective resolution to obtain a cochain complex, apply cohomology and obtain the right derived functor denote by R^iF.

3.3 (Universal) Delta Functors

In the previous subsection we saw how to construct the derived functor and showed that is actually a functor, independent of the choice of projective / injective resolution chosen. Now we introduce additional structure that naturally arises in this process, a long exact sequence, and show that derived functors are universal delta functors.

Definition. Homological Delta Functor. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and $T_n : \mathcal{A} \to \mathcal{B}$ additive functors. If for each short exact sequence

$$s \ := \ \big(0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \big)$$

in $\mathcal{A}$ and each $n \in \mathbb{N}_{\geq 0}$ there exists a morphism

$$\delta^s_n : T_n(C) \to T_{n-1}(A)$$

(depending on s) such that the sequence

$$\begin{tikzcd}
\cdots \arrow[r] & T_1(B) \arrow[r] & T_1(C) \arrow[dll, rounded corners, "\delta_1"'] & \\
T_0(A) \arrow[r] & T_0(B) \arrow[r] & T_0(C) \arrow[r] & 0
\end{tikzcd}$$

is exact and $\delta^s_\bullet$ is natural in s, i.e. for any morphism of short exact sequences $s \mapsto t$ given as a chain map

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 & s \arrow[d] \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 & t
\end{tikzcd}$$

the diagram

$$\begin{tikzcd}
T_n(C) \arrow[r, "\delta^s"] \arrow[d, "h"'] & T_{n-1}(A) \arrow[d, "f"] \\
T_n(C') \arrow[r, "\delta^t"'] & T_{n-1}(A)
\end{tikzcd}$$

commutes for all $n \in \mathbb{N}_{\geq 1}$.

We now show that homology itself is a delta functor (dropping the non-negative degree constraint). The derived functors will then inherit the structure from homology.

Proposition. Homology is a Homological Delta Functor. Let $\mathcal{A}$ be an abelian category and $H_n : \mathrm{Ch}(\mathcal{A}) \to \mathcal{A}$ be the homology functor for $n \in \mathbb{Z}$. Then the functors H_n for $n \in \mathbb{Z}$ can be assembled into a long exact sequence with a connecting morphism δ that is natural with respect to the short exact sequence. Thus homology is a homological delta functor.

Proof. We get δ and its naturality from the snake lemma.

Proposition. Left Derived Functor as Homological Delta Functor. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories such that $\mathcal{A}$ has enough projectives and $F : \mathcal{A} \to \mathcal{B}$ be right exact. Let

$$\begin{tikzcd}
0 \arrow[r] & P'_\bullet \arrow[r] \arrow[d] & P_\bullet \arrow[r] \arrow[d] & P''_\bullet \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r] & A \arrow[r] & A'' \arrow[r] & 0
\end{tikzcd}$$

be a short exact sequence in A, that for each term has a projective resolution that together assemble into a degree-wise short exact sequence of projective resolutions.
We apply the induced functor F to this short exact sequence to obtain

$$0 \longrightarrow F(P'_\bullet) \longrightarrow F(P_\bullet) \longrightarrow F(P''_\bullet) \longrightarrow 0$$

that is again short exact, because F preserves the degree-wise exactness of sequences of projective objects (this is due to the following: Any short exact sequence of projective objects splits, thus by additivity F preserves it). Applying homology H_n yields a long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & F_1(A) \arrow[r] & F_1(A'') \arrow[dll, rounded corners, "\delta_1"] & \\
F_0(A') \arrow[r] & F_0(A) \arrow[r] & F_0(A'') \arrow[r] & 0
\end{tikzcd}$$

assembling the derived functors F_i for F into a long exact sequence as desired.

This construction has two issues. First, it is not clear that we can find projective resolutions of A', A and A'' such that they fit together into a short exact sequence themselves.

Additionally, a morphism of short exact sequences needs to induce a morphism of the projective resolutions such that this chain map commutes with the surjection onto the original sequence. A priori we only get this up to chain homotopy. The next two propisitions given an answer to these two isses.

Proposition. Horseshoe Lemma. Given a short exact sequence and projective resolutions

$$\begin{tikzcd} & P'_\bullet \arrow[d] & & P''_\bullet \arrow[d] & \\ 0 \arrow[r] & A' \arrow[r] & A \arrow[r] & A'' \arrow[r] & 0 \end{tikzcd}$$

we can find a projective resolution $P_\bullet = P'_\bullet \oplus P''_\bullet$ (degree-wise direct sum) of A such that $P'_\bullet \to P_\bullet \to P''_\bullet$ is degree-wise short exact.

Proposition. Given a morphism of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & B' \arrow[r] & B \arrow[r] & B'' \arrow[r] & 0 \end{tikzcd}$$

we can find projective resolutions of the upper and lower short exact sequence respectively and a morphism

$$\begin{tikzcd} 0 \arrow[r] & P'_\bullet \arrow[r] \arrow[d] & P_\bullet \arrow[r] \arrow[d] & P''_\bullet \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & Q'_\bullet \arrow[r] & Q_\bullet \arrow[r] & Q''_\bullet \arrow[r] & 0 \end{tikzcd}$$

such that the the epimorphisms $P'_\bullet \twoheadrightarrow A'$, $Q'_\bullet \twoheadrightarrow B'$, $P_\bullet \twoheadrightarrow A$, and so forth commoute with the chain maps.

Proof. Arrow category construction.

Definition. Map of Delta Functors. S_n, δ and T_n, δ be homological δ-functors $\mathcal{A} \to \mathcal{B}$. A map of homological δ-functors is given by a natural transformation $\alpha_n : S_n \to T_n$ for each $n \in \mathbb{N}_0$ such that for any short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

the diagram

$$\begin{tikzcd} S_n(C) \arrow[r, "\delta"] \arrow[d, "\alpha_n(C)"'] & S_{n-1}(A) \arrow[d, "\alpha_{n-1}(A)"] \\ T_n(C) \arrow[r, "\delta"'] & T_{n-1}(A) \end{tikzcd}$$

commutes for all $n \geq 1$.

Definition. Universal Delta Functor. A homological δ-functor $T_n : \mathcal{A} \to \mathcal{B}$ is called universal if for all homological δ-functors $S_n : \mathcal{A} \to \mathcal{B}$ and all natural transformations $\alpha_0 : S_0 \to T_0$ there exists a unique map of δ-functors $\alpha : S \to T$ that extends α_0.

Proposition. Unique Universal Delta Functor. Let S and T both be universal δ-functors $\mathcal{A} \to \mathcal{B}$ such that $S_0 \cong T_0$ is a natural isomorphism of functors. Then $S_n \cong T_n$ for all $n \in \mathbb{N}_0$ and we say that S and T are isomorphic as δ-functors.

We now want to show that any derived functor is a universal δ-functor. From the above proposition it then follows that the δ-functor structure of the derived functor is unique up to isomorphism of δ-functors, since any derived functors of F agree in degree 0 by assumption.
To show this, we show that (a) coeffacable δ-functors are universal and (b) that derived functors are coeffaceable.

Definition. Coeffaceable Delta Functor. Let $T : \mathcal{A} \to \mathcal{B}$ be a δ-functor. It is called coeffacable if for any object $A \in \mathcal{A}$ there exists $P \in \mathcal{A}$ and a epimorphism $u : P \twoheadrightarrow A$ such that $T_n(u) = 0$ for all $n \geq 1$ (but not necessarily for $n = 0$).

Proposition. Coeffacable Delta Functors are Universal. Let $T : \mathcal{A} \to \mathcal{B}$ be a coeffaceable δ-functor. Then it is universal, i.e. for any δ-functor $S : \mathcal{A} \to \mathcal{B}$ and natural transformation $\alpha_0 : S_0 \to T_0$ we can extend α_0 uniquely to a map of δ-functors.

Proof.

Proposition. Left Derived Functor is Coeffaceable. The left derived functor L_nF of any right exact functor $F : \mathcal{A} \to \mathcal{B}$ is coeffacable.

Proof. Given any $A \in \mathcal{A}$ we find a projective object $P \in \mathcal{A}$ such that $P \twoheadrightarrow A$. Applying L_iF to P we find the projective resolution of P by

$$0 \longrightarrow P \xrightarrow{\quad\cong\quad} P \longrightarrow 0$$

that is concentrated in degree 0 and thus has zero homology in all degress $n \geq 1$.

4 Ext and Extensions

Definition. Ext. Let $\mathcal{A}$ be an abelian category. The functor $\mathrm{Hom}(-,-) : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to$ $\textsc{Ab}$ is left exact in both arguments. Let $B \in \mathcal{A}$ be fixed. We define

$$\mathrm{Ext}^i(-,B) \;:=\; R^i\mathrm{Hom}(-,B) \;=\; H^i(-) \circ K(-) \circ P_\bullet(-).$$

Note that we use a projective resolution for the first argument of $\mathrm{Hom}(-,-)$ since a projective resolution in $\mathcal{A}$ is an injective resolution in $\mathcal{A}^{\mathrm{op}}$ as required.
As a derived functor of $\mathrm{Hom}(-,B)$ is $\mathrm{Ext}(-,B)$ a universal cohomological delta functor.

Definition. Extensions. Fix two objects A, C and let B and B' be extensions of C by A. Then we say these two extensions are equivalent if there exists an isomorphism φ such that

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d, dashed, "\varphi"] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r] & B' \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

commutes. This is an equivalence relation on the set of extensions of C by A. The set $\mathrm{Ext}(C,A)$ denotes the set those extensions modulo equivalence.
The map

$$\mathrm{Ext}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{SET}$$

yields functors $\mathrm{Ext}(-,C)$ and $\mathrm{Ext}(A,-)$.

Note. A priori, these functors map into SET, the category of large sets (and not into small sets, $\textsc{Set}$).
Note. For any A, C there exists the trivial extension $A \oplus C$.

Proposition. Let $f : A \to B$ be a morphism. Then

$$\begin{tikzcd}
0 \arrow[r] & \ker(f) \arrow[r, hook] & A \arrow[rr, "f"] \arrow[dr, two heads] & & B \arrow[r, two heads] & \mathrm{coker}(f) \arrow[r] & 0 \\
& & & \mathrm{im} f \arrow[ur, hook] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

is exact.

5 Tensor and Tor

Definition. R-Balanced Map. Let $M \in$ Mod-R and $N \in$ R-Mod and $A \in$ Ab. A map $\beta : M \times N \to A$ is called R-balanced if

1. $\beta(a+b, x) = \beta(a, x) + \beta(b, x)$,
2. $\beta(a, x+y) = \beta(a, x) + \beta(a, y)$,
3. $\beta(ar, x) = \beta(a, rx)$.

Definition. Tensor Product. Let $M \in$ Mod-R and $N \in$ R-Mod. The tensor product of M and N over R is an abelian group denoted $M \otimes_R N$ together with an R-balanced map denote $- \otimes - : M \times N \to M \otimes_R N$ such that for any R-balanced map $\beta : M \times N \to A$ to some abelian group A there exists a unique group homomorphism $\varphi : M \otimes_R N \to A$ such that the following diagram commutes:

$$\begin{tikzcd}
M \times N \arrow[rr, "- \otimes_R -"] \arrow[dr, "\beta"'] & & M \otimes_R N \arrow[dl, dashed, "\varphi"] \\
& A &
\end{tikzcd}$$

Proposition. The Tensor Product Exists. For any right R-module M and left R-module N the tensor product exists.

Proof. We write $\mathbb{Z}[M \times N]$ for the free abelian group that is generated by all elements in $M \times N$. Let $U \subset \mathbb{Z}[M \times N]$ be the subgroup generated by elements of the form

1. $(m + m', n) - (m, n) - (m', n)$,
2. $(m, n + n') - (m, n) - (m, n')$,
3. $(mr, n) - (m, rn)$

for all $m, m' \in M$, $n, n' \in N$ and $r \in R$. Then we define $M \otimes_R N := \mathbb{Z}[M \times N]/U$ and the composition

$$- \otimes_R - : \quad M \times N \xrightarrow{i} \mathbb{Z}[M \times N] \xrightarrow{p} \mathbb{Z}[M \times N]/U$$

is R-balanced. Furthermore, let A be an abelian group. Then

$$\{\text{arbitrary maps } M \times N \to A\} \quad \overset{1:1}{\longleftrightarrow} \quad \{\mathbb{Z} - \text{linear maps } \mathbb{Z}[M \times N]\}$$

and specifying a R-balanced map $\beta : M \times N \to A$ is equivalent to specifying a $\mathbb{Z}$-linear map $\varphi : \mathbb{Z}[M \times N] \to A$ such that $\varphi|_U = 0$. But specifying such a map φ is the same as specifying the map on the quotient $M \otimes_R N$.

Note. We can specify an element of the tensor product on the equivlance classes (with respect to the quotient by U) of basis elements $M \times N$ of $\mathbb{Z}[M \times N]$.

Note. Let $M \in R$-Mod. Then $R \otimes_R M \cong M$. Similarly for right R-modules.
Note. The tensor product is associative.
Note. If R is commutative, then the tensor product is commutative.

Proposition. The Tensor Product is a Bifunctor. Let $M \in$ Mod-R and $N \in R -$ Mod be module. Then the tensor product

$$- \otimes_R - : \text{Mod-}R \ \times \ R\text{-Mod} \ \to \ \text{Ab}$$

is a bifunctor.

Note. The tensor product is thus a functor in both variables. For a fixed M we have a functor $N \to M \times N$ given by $(M, -)$. We then define

$$(M \otimes_R -) \ := \ (- \otimes_R -) \circ (M, -) : \ R\text{-Mod} \to \text{Ab}.$$

Similar for the left argument.

Proposition. Tensor as R-Module. Let R be commutative and M, N be R-modules. Then the tensor product (as a functor) lands in R-Mod.

Proposition. The Tensor-Hom Adjunction. Let R be a commutative ring and M be a R-module. Then

$$- \otimes_R M : \ \dashv \ \ \mathrm{Hom}_R(M, -)$$

is an adjunction R-Mod $\longleftrightarrow$ R-Mod.

Proof. For fixed M we need to find a natural isomorphism

$$\mathrm{Hom}_R(L \otimes_R N, M) \ \cong \ \mathrm{Hom}_R\big(L, \mathrm{Hom}_R(M, N)\big).$$

But this is a special case of the currying adjunction. Note that specifying a map on the left hand side is the same as specifying an R-balanced and R-linear map $L \times N \to M$ by the universal property of the tensor product.
On the other hand, the right-hand side are R-linear maps $L \to (M \to N)$ where $(M \to N)$ must be R-linear as well.
This is in 1:1 correspondence, thus a bijection of sets. Naturality still needs checking.

Proposition. The Tensor Product is Right Exact. Let R be a commutative ring. The tensor product

$$(- \otimes_R -) : R\text{-Mod} \times R\text{-Mod} \to R\text{-Mod}$$

is right exact in both variables.

Proof. Right exactness of the first variable follows from the tensor-hom-adjunction. Since the tensor product 'commutes' for commutative rings, i.e. $M \otimes_R N = N \otimes_R M$

right exactness of the second variable follows.
For non-commutative bimodules right exactness of the second variable can be shown by passing to the opposite ring.

Definition. Tor. Let R be a ring and $N \in R$-Mod be fixed. We define

$$\mathrm{Tor}_n^R(-,N) : \text{Mod-}R \to \text{Ab}, \qquad \mathrm{Tor}_n^R(-,N) = L_n^R(- \otimes_R N)$$

to be the n-th left derived functor of $(- \otimes_R N)$. If R is commutative both the tensor product and thus also Tor_N^R land in R-Mod.

Definition. Flat Module. A right R-module is called flat if $(M \times_R -)$ is exact. Similarly for left R-modules.
Since $(M \otimes_R -)$ is right exact M is flat if and only if it preserves kernels (or equivalently monos (?)).

Proposition. Projectives are Flat. Projective left and right R-modules are flat.

Proposition. Tor is Symmetric. Let $M \in$ Mod-R and $N \in R$-Mod be fixed. We have a natural isomorphism

$$L_n^R(- \otimes_R N) \;\cong\; L_n^R(M \otimes_R -).$$

Definition. Acyclic. Let $F : \mathcal{A} \to \mathcal{B}$ be left (right) exact. An object $A \in \mathcal{A}$ is called F-acyclic if $R^n F(A) = 0$ ($L_n F(A) = 0$) for all $n \geq 0$.

Note. Flat R-modules are $(- \otimes -)$-acyclic, projective objects are $\mathrm{Hom}(-, B)$ acyclic and injective objects $\mathrm{Hom}(A, -)$-acyclic.

Proposition. Derived Functor via Acylic Resolution. Let $\mathcal{A}$ be an abelian category with enough projectives and $F : \mathcal{A} \to \mathcal{B}$ a right exact functor. Let

$$X_\bullet \twoheadrightarrow A$$

be a acyclic resolution of any $A \in \mathcal{A}$. Then

$$L_n F(A) \;\cong\; H_n\big(F(X_\bullet)\big).$$

We obtain the derived functor by this acyclic resolution.

6 The Universal Coefficient Theorem

In this section we work exclusively with $\mathbb{Z}$-modules, i.e. abelian groups, and drop the ring sup/subscripts.

Definition. (Co)Homology with Coefficients. Let A be an abelian group and $C_\bullet$ a chain complex of free abelian groups, i.e. free $\mathbb{Z}$-modules. We define

$$H_n(C_\bullet, A) \;:=\; H_n(C_\bullet \otimes A) \quad \text{and} \quad H^n(C_\bullet, A) \;:=\; H^n\big(\mathrm{Hom}(C_\bullet, A)\big)$$

to be the homology and cohomology with coefficents in A respectively.

Theorem. Universal Coefficient Theorem. For general right exact functors. Let $\mathcal{A}$ be an abelian category that has enough projectives and let $F : \mathcal{A} \to \mathcal{B}$ be right exact. Further, let $(C_\bullet, d_\bullet)$ be a chain complex in $\mathrm{Ch}(\mathcal{A})$ such that C_n and $\mathrm{im}(d_n)$ are F-acyclic for all $n \in \mathbb{N}_0$ (or more generally $n \in \mathbb{Z}$).
Then we have a natural short exact sequence

$$0 \longrightarrow F\big(H_n(C_\bullet)\big) \longrightarrow H_n\big(F(C_\bullet)\big) \longrightarrow L_1\big(F(H_{n-1}(C_\bullet))\big) \longrightarrow 0$$

for all $n \geq 1$. If all $\mathrm{im}(d_n)$ are projective, then this short exact sequence splits, although not canonically.

Theorem. Universal Coefficient Theorem. Let $A \in \textsc{Ab}$ and $C_\bullet$ be a chain complex of free abelian groups. Then we have natural short exact sequences

$$0 \longrightarrow H_n(C_\bullet) \otimes A \longrightarrow H_n((C_\bullet \otimes A) \longrightarrow \mathrm{Tor}_1\big(H_{n-1}(C_\bullet), A\big) \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{Ext}^1\big(H_{n-1}(C_\bullet, A)\big) \longrightarrow H^n\big(\mathrm{Hom}(C_\bullet, A)\big) \longrightarrow \mathrm{Hom}\big(H_n(C_\bullet), A\big) \longrightarrow 0$$

that split non-canonically. In particular we have isomorphisms

$$H_n(C_\bullet, A) \;\cong\; \big(H_n(C_\bullet) \otimes A\big) \,\oplus\, \mathrm{Tor}_1\big(H_{n-1}(C_\bullet), A\big)$$

and

$$H^n(C_\bullet, A) \;\cong\; \mathrm{Hom}\big(H_n(C_\bullet), A\big) \,\oplus\, \mathrm{Ext}^1\big(H_{n-1}(C_\bullet), A\big).$$

Proof. This follows from the general version of the theorem with $F = (- \otimes A) : \textsc{Ab} \to \textsc{Ab}$ and $F = \mathrm{Hom}(-, A) : \textsc{Ab} \to \textsc{Ab}^{\mathrm{op}}$. In particular we assume $C_\bullet$ to be free and thus projective and thus F-acyclic. Also $\mathrm{im}(d_n)$ is free again as a subgroup of a free group.

7 Group Homology

Note. G-Module. Let G be a group. We define the category of left G-modules G-Mod to be abelian groups (written additively) with an additional left action of group elements in G, written $g \cdot m$ such that

1. $g \cdot (x + y) = g \cdot x + g \cdot y$,
2. $(gh) \cdot x = g \cdot (hx)$,
3. $1 \cdot x = x$.

G-modules are similar to R-modules, but lack the multiplication of ring elements. A G-module homomorphism $f : M \to N$ is a group homomorphism of abelian groups that satisfy $f(g \cdot a) = g \cdot f(a)$ (one could say it is G-linear).

Note. G-Mod is an abelian category.

Note. Fix a group G. Denote by BG the category with one element and morphisms $* \to *$ equal to elements in G. Then

$$\mathrm{Fun}(BG, \mathrm{Ab}) \;\cong\; \text{G-Mod} \quad \text{by} \quad F \mapsto F(*)$$

where g facts on $F(*)$ by the group homomorphism $F(g) : F(*) \to F(*)$, i.e. $g \cdot m = F(g)(m) \in F(*)$ for all $m \in F(*)$ and $g \in G$.

Note. Trivial G-Module. Let G be a fixed group and M be an abelian group. Then there is a distinguished G-module structure on M by $g \cdot m = m$ for all $g \in G$. This is called the trivial G-module of M, denoted by $\mathrm{triv}(M)$ and $\underline{M}$.
In fact for a fixed group G the map $\mathrm{triv}(-) : \mathrm{Ab} \to \text{G-Mod}$ is a exact and fully faithful functor. If $f : M \to N$ is a morphism of abelian groups then clearly $\mathrm{triv}(f) = f$ respects the (trivial) scalar multiplication with elements from G.

Definition. Integral Group Ring. Let G be a group. We can form a free $\mathbb{Z}$-module by

$$\mathbb{Z}G \;:=\; \bigoplus_{g \in G} \mathbb{Z}g.$$

This module also carries the structure of a ring with multiplication on a basis of $\mathbb{Z}G$ given by the group law of G, i.e. $g_i g_j = g_i \cdot g_j$ that extends $\mathbb{Z}$-linearly.

Example. Let $G = \{1, a, a^2\}$ be the cyclic group of three elements, multiplication of the ring $\mathbb{Z}G$ of elements $r = z_0 1 + z_1 a + z_2 a^2$ and $s = w_0 1 + w_1 a + w_2 a^2$ is given by

$$rs \;=\; z_0 w_0 1 + z_0 w_1 a + z_0 w_2 a^2 + \cdots$$

where $z_i w_i$ denotes multiplication in $\mathbb{Z}$.

Definition. Augmentation Map. Let G be a group. The canonical ring homomorphism

$$\varepsilon : \mathbb{Z}G \to \mathbb{Z}, \qquad g \mapsto 1$$

is called augmentation map and sum all coefficients of any element $m \in \mathbb{Z}G$.

Example. Let $G = C_n$ be the cyclic groups on n elements. Then $\mathbb{Z}G \cong \mathbb{Z}[t]/(t^n - 1)$, the ring of polynomials of finite degree at most $n - 1$.

Example. Let $G = C_\infty$ be the infinite cycle group. Then $\mathbb{Z}G \cong \mathbb{Z}[t^{-1}, t]$, the ring of Laurent polynomials.

Proposition. G-Modules as Ring Modules. Let G be a fixed group and M a G-module. Then we can view M as a R-module (ring module) for $R = \mathbb{Z}G$ by

$$g \cdot m \;\mapsto\; (1g) \cdot m, \qquad \left(\sum_{i=1}^{|G|} z_i g_i \right) \cdot m \;\mapsto\; \sum z_i (g_i \cdot m).$$

This is an isomorphism of categories G-Mod and $\mathbb{Z}G$-Mod.

Definition. Group of (Co)Invariants. Let G be a fixed group and M a G-module. We define abelian groups

$$M^G = \{ m \in M \mid g \cdot m = m \quad \forall g \in G \}$$

and

$$M_G = M / (g \cdot m - m \mid g \in G, \; m \in M)$$

called group of invariants and group of coinvariants respectively. M^G is a subgroup of M and M_G is a quotient of M, both discarding the G-module structure.
Note that a G-module morphism $f : M \to N$ descends to morphisms $f^G : M^G \to N^G$ and $f_G : M_G \to N_G$. The former is well defined since for any $a \in M^G$ and $g \in G$ we have

$$g \cdot f(a) = f(g \cdot a) = f(a) \in N^G.$$

Thus $(-)^G$ and $(-)_G$ are functors G-Mod to Ab.

Proposition. Invariants and Adjoints. The functors triv$(-)$, $(-)^G$ and $(-)_G$ are adjoint as follows

$$\begin{tikzcd} G - \text{Mod} \arrow[d, bend right=60, "(-)_G"'] \arrow[d, bend left=60, "(-)^G"] \\ \text{Ab} \arrow[u, "\text{triv}(-)"] \end{tikzcd}$$

such that $(-)_G \dashv \text{triv}(-)$ and $\text{triv}(-) \dashv (-)^G$, i.e. coinvariants are left-adjoint to the triv and invariants are right adjoint to triv.

Proposition. The invariant functor $(-)^G$ is additive and left exact, the coinvariant functor $(-)_G$ is additive and right exact.

Note. For a fixed group G the category G-Mod has enough projectives and injectives since it is isomorphic to $\mathbb{Z}G$-Mod.

Definition. Group (Co)Homology. Let G be a fixed group. We define the functors

$$H^n(G,-) \;:=\; R^n(-)^G, \qquad H_n(G,-) \;:=\; L_n(-)_G.$$

Proposition. Reinterpreting G-Modules. We have natural isomorphisms

$$(-)_G \;\cong\; \mathbb{Z} \otimes_{\mathbb{Z}G} -, \qquad (-)^G \;\cong\; \mathrm{Hom}_{\mathbb{Z}G-\textsc{Mod}}(\mathbb{Z},-)$$

where $\mathbb{Z}$ appears as the (trivial) $\mathbb{Z}G$-module.