

# Diffg

seb

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## Contents

<b>1 Categories</b>	<b>2</b>
1.1 Abstracting . . . . .	3
1.2 Abelian Categories . . . . .	6
1.3 Exact Sequences . . . . .	8
1.4 Additive and Exact Functors . . . . .	10
1.5 Projective Objects . . . . .	11
<b>2 Derived Functors</b>	<b>13</b>
<b>3 Extensions</b>	<b>14</b>

## About

# 1 Categories

**Kernel, Image, Cokernel in  $\mathbf{AB}$ .** Let  $\mathcal{C} = \mathbf{AB}$  be the category of abelian groups and  $f : X \rightarrow Y$  a (group) (homo)morphism.

- $\text{kern}(f) \subseteq X$ , the kernel of  $f$  is the subgroup (object) of  $X$  containing those elements  $x \in X$  such that  $f(x) = 0$ .
- $\text{im}(f) \subseteq Y$ , the image of  $f$ , is the subgroup (object) of  $Y$  containing those elements  $y \in Y$  such that  $y = f(x)$  for some  $x \in X$ .
- $\text{cokern}(f) = Y/\text{im}(f)$  is the quotient of  $Y$  by the image of  $f$ .

**Faithful Functor.** A functor  $F : (\mathcal{C}) \rightarrow (\mathcal{D})$  is called faithful if for fixed objects  $X, Y \in \mathcal{C}$  it maps morphisms  $X \rightarrow Y$  injectively to morphisms  $F(X) \rightarrow F(Y)$ .

**Note.** A faithful functor must not be injective on objects are the set of all morphisms.

**Concrete Category.** A concrete category is a pair  $(\mathcal{C}, U)$  where  $\mathcal{C}$  is a category and  $U : \mathcal{C} \rightarrow \mathbf{SET}$  is a faithful functor.

## 1.1 Abstracting

The following notions lift the properties “injective” and “surjective” as well as “(co)kernel” and “image” to categories. In particular to categories where the objects do not form sets (or have elements).

**Monic Morphism.** Let  $f : X \rightarrow Y$  be a morphism in a category  $\mathcal{C}$ . It is called monic (mono) if for any  $g, h : A \rightarrow X$  we have left cancellation

$$f \circ g = f \circ h \implies g = h \quad (A \rightrightarrows X \rightarrow Y).$$

**Note.** In SET a morphism is mono if and only if it is injective.

**Epic Morphism.** Let  $f : X \rightarrow Y$  be a morphism in a category  $\mathcal{C}$ . It is called epic (epi) if for any  $g, h : Y \rightarrow A$  we have right cancellation

$$g \circ f = h \circ f \implies g = h \quad (X \rightarrow Y \rightrightarrows A).$$

**Note.** Again, in SET epic morphisms are precisely the surjective maps.

**Note.** In RING the embedding  $f : \mathbb{Z} \hookrightarrow \mathbb{Q}$  is epic (but not surjective). To see this, we need to show that any (ring) homomorphisms  $f, g : \mathbb{Q} \rightarrow A$  agree on  $\mathbb{Z}$ .

One way to show this is by showing that if  $f, g$  agree on  $\mathbb{Z}$  then they must be equal: Any (ring) homomorphism  $\mathbb{Q} \rightarrow A$  is uniquely determined by its values in  $\mathbb{Z}$ .

In fact for any ring  $A$  there exists at most one ring homomorphism  $\mathbb{Q} \rightarrow A$ .

**Initial, Terminal and Zero Object** Let  $\mathcal{C}$  be a category. An object  $I$  in  $\mathcal{C}$  is called initial if for any object  $X$  there exists precisely one morphism  $I \rightarrow X$ .

An object  $M$  is called terminal if for any object  $X$  there exists precisely one morphism  $X \rightarrow M$ .

If an object  $0$  is (isomorphic to) both initial and terminal object, it is called zero object.

**Note.** Initial, terminal and zero objects are unique up to unique isomorphism.

**Note.** Let  $0$  be the zero object and  $f : X \rightarrow Y$ . Then the compositions  $0 \rightarrow X \rightarrow Y$  and  $X \rightarrow Y \rightarrow 0$  are the unique morphisms  $0 \rightarrow Y$  and  $X \rightarrow 0$  respectively.

**Proposition. Zero Morphism.** Let  $0$  be a zero object in  $\mathcal{C}$ . Then for any objects  $X$  and  $Y$  there exists a distinguished morphism

$$X \rightarrow 0 \rightarrow Y, \quad X \xrightarrow{0} Y,$$

the zero morphism.

**Note.** Let  $f : X \rightarrow Y$ . Then the composites  $X \xrightarrow{f} Y \xrightarrow{0} Z$  and  $W \xrightarrow{0} X \xrightarrow{f} Y$  are given by  $X \xrightarrow{0} Z$  and  $W \xrightarrow{0} Y$  respectively.

**Note.** Let  $f : X \rightarrow Y$ . If for all  $g : Y \rightarrow Z$  we have  $g \circ f = 0$ , then  $f = 0$  since we may choose  $g = \text{id}_Y$  thus  $0 = \text{id}_Y \circ f = f$ .

**Products and Coproducts.** Let  $\mathcal{C}$  be a category. Let  $X, Y$  and  $X \times Y, X \sqcup Y$  be objects with morphisms  $p_x, p_y$  and  $i_x, i_y$  respectively. Then  $X \times Y$  and  $X \sqcup Y$  are called product and coproduct respectively if for any  $T, t_x, t_y$  there exists a unique morphism  $\varphi$  such that the following diagrams commute

**Kernel and Cokernel.** Let  $f : X \rightarrow Y$  be a morphism in a category  $\mathcal{C}$  with 0. The kernel and cokernel of  $f$  are objects together with morphisms  $i : \ker(f) \rightarrow X$ ,  $p : Y \rightarrow \text{coker}(f)$  such that the following for any  $T, j$  or  $T, q$  there exists a unique morphism  $\varphi$  such that the following diagrams commute if the outer squares commute

**Note.** Kernels and cokernels are unique up to unique isomorphism.

**Note.** Consider the zero morphism  $X \xrightarrow{0} Y$ . Then  $\ker(0) = \text{id}_X$  and  $\text{coker}(0) = \text{id}_Y$ .

**Note.** Let  $X$  be an object. Then  $\ker(\text{id}_X) = 0 = \text{coker}(\text{id}_X)$ . This is an instance of the following general fact.

**Proposition.** Let  $f : X \rightarrow Y$  in a category with kernels and cokernels. Then

1. If  $f$  is mono then  $\ker(f) = 0$ ,
2. If  $f$  is epi then  $\text{coker}(f) = 0$ .

**Proof.** (1) Assume  $f$  is mono. We show that 0 satisfies the universal property of the kernel. If we take any  $T \rightarrow 0$  such that  $f \circ t = 0$  (outer square commutes) then since  $f$  is mono we get  $f \circ t = 0 = f \circ 0$  and  $t = 0$  is forced which makes the whole diagram commute as desired.

(2) Omitted since it is dual.

**Note.** The converse does not necessarily hold (if  $\mathcal{C}$  is not Abelian, see below).

**Proposition.** Any kernel is mono and any cokernel is epic.

**Proof.**

**Image.** Let  $\mathcal{C}$  be a category and  $f : X \rightarrow Y$  be a morphism. Then we define the image of  $f$  to be an object  $\text{im}(f)$  together with morphisms  $m : \text{im}(f) \rightarrow Y$  and  $e : X \rightarrow \text{im}(f)$  such that  $m$  is mono and for any  $T$  and  $e', m'$  there exists a unique morphism  $\varphi$  such that the following diagram commutes if the outer square commutes

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 e \searrow & \downarrow \varphi & \nearrow m \\
 & \text{im}(f) & \\
 e' \swarrow & & \nearrow m' \\
 T & &
 \end{array}$$

## 1.2 Abelian Categories

**Ab-Category** An Ab-Category is a category  $\mathcal{C}$  that can be equipped with the following structure. For any objects  $X$  and  $Y$  the set  $\text{Hom}(A, B)$  has an abelian group structure

$$+ : \text{Hom}(A, B) \times \text{Hom}(A, B) \rightarrow \text{Hom}(A, B)$$

that is compatible with composition, i.e.

$$f \circ (g + h) = (f \circ g) + (f \circ h), \quad (f + g) \circ h = (f \circ h) + (g \circ h).$$

**Note.** The neutral element of  $\text{Hom}(A, B)$  is the zero morphism  $A \rightarrow 0 \rightarrow B$ .

**Additive Category.** An Ab-category is called additive if it has a zero object and admits products and coproducts for all objects.

**Characterization of Additive Categories.** A category  $\mathcal{C}$  admits an additive (abelian group) structure on  $\text{Hom}(A, B)$  for all objects  $A, B$  if and only if

1.  $\mathcal{C}$  has a zero object,
2.  $\mathcal{C}$  has products and coproducts for all (pairs of) objects
3. For each object  $A$  the maps

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : A \sqcup A \rightarrow A \times A$$

are isomorphisms (where the 0 map denotes the composite  $A \rightarrow 0 \rightarrow A$ ).

**Biproduct.** Let  $\mathcal{C}$  be an additive category. Then for any objects  $X$  and  $Y$  we have  $X \times Y \cong X \sqcup Y$ . Thus instead of writing the (co)product we write  $X \oplus Y$ .

**Abelian Category.** An additive category is called abelian if

1. Every morphism has a kernel and cokernel,
2. Every mono is the kernel of its cokernel,
3. Every epi is the cokernel of its kernel.

**Characterization of Abelian Categories.** A category  $\mathcal{C}$  is an abelian category if and only if

1.  $\mathcal{C}$  has a zero object,
2.  $\mathcal{C}$  has products and coproducts for all objects,
3.  $\mathcal{C}$  has kernel and cokernel for all morphisms,
4. Every mono is a kernel and every epi is a cokernel.

**Image in an Abelian Category.** Let  $\mathcal{C}$  be an Abelian category and  $f : X \rightarrow Y$  be a morphism. Then

$$\text{im}(f) = \ker(Y \xrightarrow{p} \text{coker}(f))$$

### 1.3 Exact Sequences

In this subsection we work with abelian categories.

**Homology and Exactness.** The notion of exactness applies to a pair of maps

$$A \xrightarrow{f} B \xrightarrow{g} C$$

such that  $g \circ f = 0$ , i.e. the composite factors through 0. Consider the diagrams:

In the right column we have our pair of maps  $f$  and  $g$ .  $f$  factors through its image and includes via the monomorphism  $t$ .

Here, the composite  $g \circ t : \text{im}(f) \rightarrow C$  factors through zero. Thus, by the universal property of the kernel  $\ker(g)$  there exists a unique dashed morphism  $\varphi : \text{im}(f) \rightarrow \ker(g)$ . The cokernel of that map  $\text{coker}(\varphi)$  is called the homology object at  $B$ . If  $\text{coker}(\varphi) = 0$  and thus  $\varphi$  an isomorphism, then we say  $A \xrightarrow{f} B \xrightarrow{g} C$  is exact at  $B$ .

**Short Exact Sequence, Extension.** A short exact sequence is a sequence of morphisms

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

such that it is exact at  $A, B$  and  $C$ . We call  $B$  an extension of  $C$  by  $A$ .

**Split Short Exact Sequence.** Consider the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C \longrightarrow 0 \\ & & \parallel & & \downarrow \varphi & & \parallel \\ 0 & \longrightarrow & A & \xrightarrow{(1,0)^T} & A \oplus C & \xrightarrow{(0,1)} & C \longrightarrow 0 \end{array}$$

We call the short exact sequence  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  split exact if there exists an isomorphism  $\varphi$  (dashed) such that the diagram commutes.

When can give two equivalent criteria for the above sequence to be split short exact. The following are equivalent

1. An isomorphism  $\varphi$  exists such that the above diagram commutes,
2. There exists  $s : C \rightarrow B$  such that  $g \circ s = \text{id}_C$ ,

3. There exists  $r : B \rightarrow A$  such that  $r \circ f = \text{id}_A$ .

**Short Exact Sequence, Kernels and Cokernels.** Let

$$A \xrightarrow{f} B \xrightarrow{g} C$$

be a sequence such that  $g \circ f = 0$ . Then it is a short exact sequence if and only if  $A \cong \ker(g)$  and  $C \cong \text{cokern}(f)$ .

## 1.4 Additive and Exact Functors

**Additive Functor.** Let  $\mathcal{A}$  and  $\mathcal{B}$  be additive categories (Ab-categories is sufficient). A functor  $F : \mathcal{A} \rightarrow \mathcal{B}$  is called an additive (or Ab-functor) if for any objects  $A, A' \in \mathcal{A}$  the map

$$F : \text{Hom}(A, A') \rightarrow \text{Hom}(F(A), F(A')), \quad f \mapsto F(f)$$

is a group homomorphism, i.e.  $F(f + g) = F(f) + F(g)$  and  $F(0) = 0$ .

**Proposition. Additive I.** Let  $F : \mathcal{A} \rightarrow \mathcal{B}$  be an Ab-Functor between Ab-categories. Then

1.  $F(0) = 0$  (preserves initial, terminal and zero objects),
2.  $F(A \times B) = F(A) \times F(B)$  (preserves products),
3.  $F(A \sqcup B) = F(A) \sqcup F(B)$  (preserves coproducts).

**Proposition. Additive II.** Let  $F : \mathcal{A} \rightarrow \mathcal{B}$  be a functor between additive categories that preserves 0, products and coproducts. Then it is additive.

**Left Exact Functor.** Let  $\mathcal{A}$  and  $\mathcal{B}$  be abelian categories and  $F : \mathcal{A} \rightarrow \mathcal{B}$  be an additive functor.  $F$  is called left exact if it preserves kernels: Given a morphism  $f : A \rightarrow B$  we have that

$$F(\ker(f)) \cong \ker(F(f)).$$

i.e. applying  $F$  to  $\ker(f)$  is a kernel of  $F(f)$ .

Equivalently, a left exact functor preserves the exactness of sequences

$$0 \longrightarrow A \longrightarrow B \longrightarrow C.$$

**Right Exact Functor.** Dually,  $F : \mathcal{A} \rightarrow \mathcal{B}$  is called right exact if it preserves cokernels or equivalently exactness of sequences of the form

$$A \longrightarrow B \longrightarrow C \longrightarrow 0.$$

**Exact Functor.**  $F : \mathcal{A} \rightarrow \mathcal{B}$  is called exact if it preserves both kernels and cokernels or equivalently the exactness of short exact sequences

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0.$$

## 1.5 Projective Objects

In this subsection we work with Abelian categories.

**Projective Object.** Let  $P \in \mathcal{C}$  be an object.  $P$  is called projective if any short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow P \longrightarrow 0$$

splits.

**Characterizing Projective Objects.** The following are equivalent

1.  $P \in \mathcal{C}$  is projective,
2. Any epi  $p : B \rightarrow P$  splits, i.e. there exists (a section)  $s : P \rightarrow B$  such that  $p \circ s = \text{id}_P$ ,
3. Let  $f : P \rightarrow Y$  be any morphism. Then we can factor  $f$  along any epi  $g : X \rightarrow Y$ , i.e. there exists a dashed arrow such that

$$\begin{array}{ccc} & P & \\ & \swarrow \exists! & \downarrow f \\ X & \xrightarrow{g} & Y \end{array}$$

commutes,

4. For any  $X \in \mathcal{C}$  the Hom-functor  $\text{Hom}(X, -) : \mathcal{C} \rightarrow \text{AB}$  is exact.

**Proof.**

**Properties of Projective Objects.** The coproduct of projective objects is projective. If  $X \oplus Y$  is projective iff  $X$  and  $Y$  are projective.

**Enough Projectives.** An abelian category  $\mathcal{C}$  is said to have enough projectives if for each object  $X \in \mathcal{C}$  there exists a projective object  $P \in \mathcal{C}$  and an epimorphism  $P \twoheadrightarrow X$  onto  $X$ .

**Projective Resolution.** A projective resolution of an object  $A \in \mathcal{C}$  is an exact sequence

$$\begin{array}{ccccccc} \dots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 \\ & & & & & \downarrow & \\ & & & & & & A \\ & & & & & & \longrightarrow 0 \end{array}$$

of projective objects  $P_i$ . If  $\mathcal{C}$  has enough projective a projective resolution exists for any object  $A \in \mathcal{C}$ .

**Proof.** We construct  $P_\bullet$  inductively starting with an projective object  $P =: P_0$  for

$\ker(A \xrightarrow{e_{-1}} 0)$  such that  $P_0 \rightarrow \ker(e_{-1})$  yielding the diagram

$$\begin{array}{ccccc} P_0 & \xrightarrow{\quad} & A & & \\ e_0 \searrow & & \parallel & & \downarrow e_{-1} \\ & & \ker(e_{-1}) & & 0 \end{array}$$

and continuing this process by finding projective objects for the objects  $\ker(e_i)$

$$\begin{array}{ccccccc} \dots & \xrightarrow{\quad} & P_1 & \xrightarrow{\quad} & P_0 & \xrightarrow{\quad} & A \\ & \searrow e_2 & \swarrow & \searrow e_1 & \swarrow & \searrow e_0 & \searrow \\ & \ker(e_1) & & \ker(e_0) & & \ker(e_{-1}) & 0 \end{array}$$

we obtain the dashed arrows as the composition of maps factoring in and out of the kernels.

## 2 Derived Functors

In this section we work with abelian categories.

We have seen that  $\text{Hom}(-, -) : \mathcal{A}^{\text{op}} \times \mathcal{A} \rightarrow \text{AB}$  is not a exact functor but only a left exact functor, failing to preserve short exact sequences at the right end. In this section we develop a tool to quantify by how much it fails to preserve exactness at the right when applied to a short exact sequence.

Generally, from a left exact functor  $F : \mathcal{A} \rightarrow \mathcal{B}$  we derive a sequence of new functors  $R^i(F) : \mathcal{A} \rightarrow \mathcal{B}$  such that  $R^0(F) = F$  and for any short exact sequence in  $\mathcal{A}$

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

we not only obtain a exact sequence

$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C)$$

but also a long exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & F(A) & \longrightarrow & F(B) & \longrightarrow & F(C) \\ & & & & & & \downarrow \\ & & \hookrightarrow R^1(F)(A) & \longrightarrow & R^1(F)(B) & \longrightarrow & R^1(F)(C) \\ & & & & & & \downarrow \delta \\ & & & \hookrightarrow R^2(F)(A) & \longrightarrow & R^2(F)(B) & \longrightarrow \dots \end{array}$$

with connecting morphisms  $\delta$  where  $R^1(F)(A)$  (and the tail of the sequence) is a measure for the extent that  $F$  fails to be (right) exact.

The sequence  $R^i(F)$  for  $i \geq 0$  together with the connecting map  $\delta$  is called a cohomological delta functor.

We then show that the way we constructed this cohomological delta functor, the derived functor of  $F$ , is not arbitrary but really ‘the’ derived functor. It is a universal cohomological delta functor.

Universal cohomological delta functors are those that determine natural transformations for delta functors completely by the 0th degree. But any derived functor of  $F$ , in its 0th degree, is naturally isomorphic to  $F$  by assumption.

### 3 Extensions

**Definition. Extensions.** Fix two objects  $A, C$  and let  $B$  and  $B'$  be extensions of  $C$  by  $A$ . Then we say these two extensions are equivalent if there exists an isomorphism  $\varphi$  such that

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow 0 \\ & & \parallel & & \downarrow \varphi & & \parallel \\ 0 & \longrightarrow & A & \longrightarrow & B' & \longrightarrow & C & \longrightarrow 0 \end{array}$$

commutes. This is an equivalence relation on the set of extensions of  $C$  by  $A$ . The set  $\text{Ext}(C, A)$  denotes the set those extensions modulo equivalence.

The map

$$\text{Ext}(-, -) : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \text{SET}$$

yields functors  $\text{Ext}(-, C)$  and  $\text{Ext}(A, -)$ .

**Note.** A priori, these functors map into SET, the category of large sets (and not into small sets, SET).

**Note.** For any  $A, C$  there exists the trivial extension  $A \oplus C$ .

**Proposition.** Let  $f : A \rightarrow B$  be a morphism. Then

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker(f) & \hookrightarrow & A & \xrightarrow{f} & B & \twoheadrightarrow & \text{coker}(f) & \longrightarrow 0 \\ & & & & \searrow & \nearrow & & & & \\ & & & & \text{im } f & & & & & \\ & & & & \swarrow & \searrow & & & & \\ 0 & & & & & & 0 & & & \end{array}$$

is exact everywhere.