ENG252 Dynamics: Practical 3

Shane Reynolds

September 25, 2019

1 Introduction

1.1 Moment of Inertia

Consider a body with mass m undergoing rotational motion, with angular acceleration α, around a fixed axis OO'. Graphically this scenario is depicted in Figure 1. If we want to calculate the Moment of this body around the axis OO', then we need to consider the moments of each and every infinitesimally small mass particle that make up the body. The moment of a single particle mass is given by:

$$M = \boldsymbol{r} \times \boldsymbol{F} \tag{1}$$

We note $\boldsymbol{r}$ is the position vector of the particle from point where the moment is to calculated, and $\boldsymbol{F}$ is the force acting on the particle. If our motion is constrained to a 2D plane, equation (1) simplifies to the well known equation:

$$M = F \times d \tag{2}$$

The quantity F is the scalar magnitude of the force acting orthogonal to the shortest line connecting the particle mass to the moment point of calculation; and d is the distance between between these two points. Using (2) we can calculate the moment M_O about axis OO' for a small mass element, dm, of the body in Figure 1:

$$M_O = Fr = a_t \; dm \; r \tag{3}$$

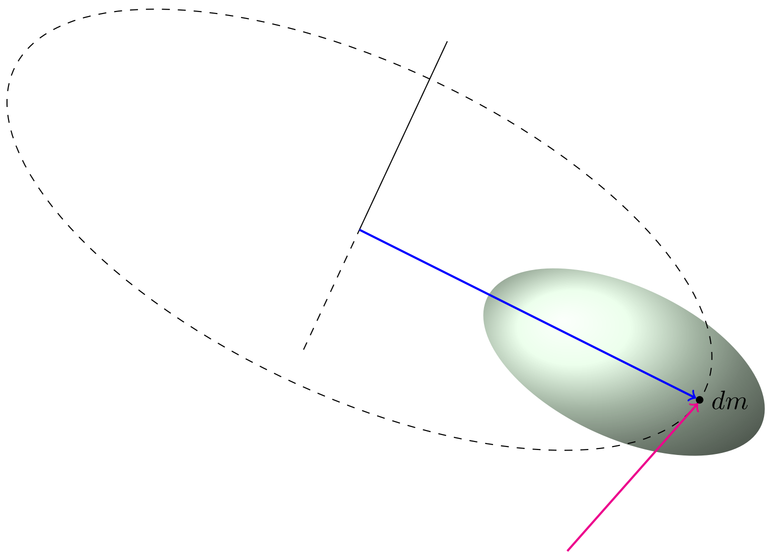


Figure 1: A body of mass m rotating about a fixed axis OO' with some angular acceleration α.

According to Giancoli a particle undergoing fixed axis rotation can re-express tangential acceleration a_t, as αr, where r is the distance from the centre of rotation to the particle. We can now rite (3) as:

$$M_O = \alpha r \; dm \; r \tag{4}$$

This is convenient since α remains constant for all infinitesimally small particles in the body meaning that the only variable that needs consideration is r. In fact, to calculate the sum of the moments of the body around axis OO' we need to integrate the right hand side of equation (4), which yields:

$$\sum M_O = \alpha \int r^2 dm \tag{5}$$

Equation (5) is often thought of as somewhat analogous to $\sum F = ma$, but for rotational motion. In fact since α is the angular acceleration, the integral in equation (5) is often referred to as the resistance of a body to change it's state of rotation. In the literature this quantity is denoted I_O and referred to as the Moment of Inertia and is defined as:

$$I = \int r^2 dm \tag{6}$$

For a body with a uniform mass density ρ, we note that $dm = \rho dV$, where dV is an infinitesimally small volume located a distance of r from the centre of rotation. Equation (6) can be written as:

$$I = \rho \int r^2 dV \tag{7}$$

Equation (7) is deceptively simple, however, the evaluation of I can be fiendishly difficult for axes of rotation which do not pass through the body's centre of mass. In practice, (7) is typically only used to determine the moment of inertia through the body's mass centre, I_G. To find a moment of inertia around an axis that does not pass through the mass centre, the parallel axis theorem is often applied. The theorem derivation is beyond the scope of this paper, however, the result can be seen in equation (8).

$$I_O = I_G + md^2 \tag{8}$$

Equation (8) tells us if the moment of inertia around an axis passing through the mass centre of a body I_G is known, then the moment of inertia around any parallel axis I_O is calculated with an additive translation of I_G by the body mass m multiplied by the square of orthogonal Euclidean distance between the two axes d.

1.2 Radius of Gyration

Meriam and Kraige define the radius of gyration k as the radial distance from some axis of rotation such that if the whole mass of the body were concentrated to a point, then the moment of inertia about the given axis would be identical in value to the moment of inertia about the axis using the actual distribution of the body's mass. Mathematically, for some mass m, if I_O is the moment of inertia about some axis O then the radius of gyration is defined as:

$$k = \sqrt{\frac{I_O}{m}} \tag{9}$$

The radius of gyration has a similar parallel axes theorem as that seen for moments of inertia in equation (8). If we know the radius of gyration for the axis passing through the mass centre, k_G, then a parallel axis O at distance d has radius of gyration given by:

$$k_O^2 = k_G^2 + d^2 \tag{10}$$

1.3 Determining Moments of Inertia with a Trifilar

A Trifilar is a simple apparatus used to determine an object's Mass Moment of Inertia. The device consists of a circular platform with some mass m, which is suspended by three wire filaments of length L. Filaments are placed at 120^o separation from each other, at a distance of r from the disk centre. An object is placed in the centre of the device to determine it's mass moment of interia. Figure 2 shows an unloaded Trifilar disk in a state of rotation.

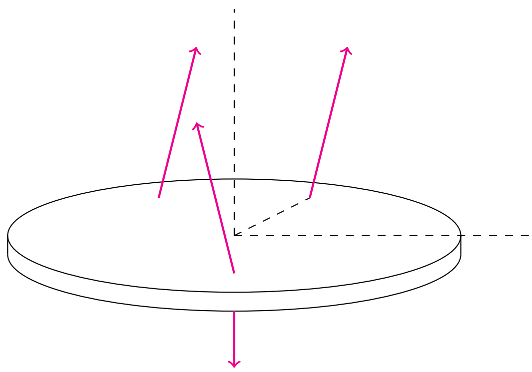


Figure 2: A Trifilar is a simple apparatus used to determine the moment of inertia for different objects.

If the disk is given some initial rotational perturbation, the interaction between the weight of the disk mg, and the forces from the filaments T cause a rotational motion. An expression for natural frequency of oscillation f_n of the disk and object can be derived by considering the sum of forces in the vertical direction and the moments about the disk centre. The derivation is beyond the scope of this paper, however, the result can be seen in equation (9) below:

$$f_n = \frac{r}{2\pi k}\sqrt{\frac{g}{L}} \tag{11}$$

An object's moment of inertia is found by placing it on the disk, and perturbing the disk such that it only undergoes rotational motion, oscillating about the disk centre. Natural oscillation frequency can be determined by observing period of oscillation T_n for the disk and object and applying equation (9) below:

$$f_n = \frac{1}{T_n} \tag{12}$$

Rearranging (11), allows for the radius of gyration to be expressed as a function of f_n, r, and L:

$$k = \frac{r}{2\pi f_n}\sqrt{\frac{g}{L}} \tag{13}$$

Equating equations (9) and (13) allows us to solve for the mass moment of inertia about the mass centre:

$$I = \frac{mg}{L}\left(\frac{r}{2\pi f_n}\right)^2 \tag{14}$$

1.4 Scope

This practical requires the Mass Moment of Inertia to be determined for various objects, using a trifilar. The first object is a wooden disk, which makes up part of the trifilar; and the second object is a small bolt like mass. The trifilar apparatus can be seen in Figure 3; and the bolt like mass is shown in Figure 4. To determine the Mass Moment of Inertia, the trifilar will be rotationally perturbed and natural periods of oscillation will be observed. Captured experimental data will be used to calculate the moment of inertia, which will then be compared to the theoretically derived results.

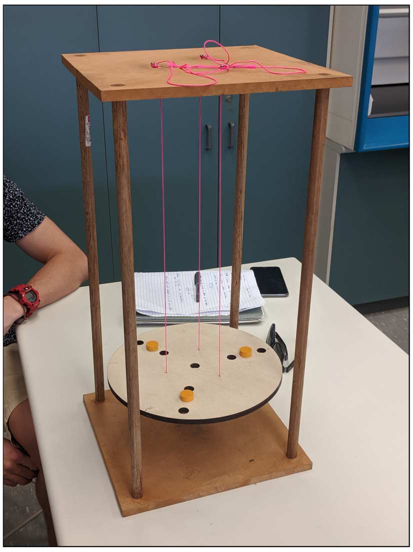


Figure 3: The trifilar rig consists of a wooden disk suspended by three filament wires. Masses can be placed in holes provided that are in the disk.

Figure 4: The small mass that is placed in the holes cut out of the trifilar disk.

2 Results

Dimensional measurements of the Trifilar were captured prior to undertaking the experiment. Measurements included disk weight m_d; disk radius r_d; filament length L; the radius from centre of disk to inner set of holes d_1 the radius from centre of disk to middle set of holes d_2; the radius from centre of disk to outer set of holes d_3; and the radius from the disk centre to the filament base r. These measurements are presented in Table 1 below. To better understand where the masses were mounted on the disk, the Trifilar disk geometries are shown in Figure 3.

Table 1: Trifilar parameter measurements

Description	Value	Units
Disk Mass (m_d)	0.200	kg
Disk Radius (r_d)	0.140	m
Filament Length (L)	0.450	m
Distance to Inner Holes (d_1)	0.090	m
Distance to Middle Holes (d_2)	0.090	m
Distance to Outer Holes (d_3)	0.090	m
Distance to Filament (r)	0.049	m

Table 2: Dimensions of the small masses

Description	Value	Units
Object Mass (m_{obj})	0.020	kg
Upper Cylinder Radius (r_1)	0.090	m
Upper Cylinder Height (h_1)	0.007	m
Lower Cylinder Radius (r_2)	0.090	m
Lower Cylinder Height (h_2)	0.007	m

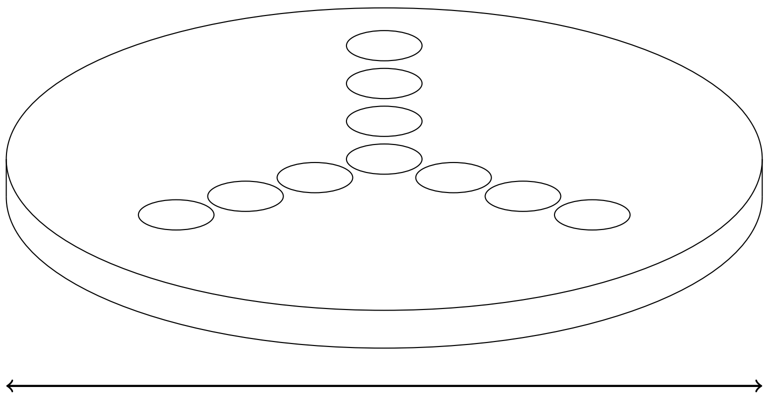


Figure 5: Geometries of the Trifilar disk and holes where small masses were mounted.

Measurement information was also taken about the small objects, including their mass m_{obj}; radius of upper cylinder r_1; height of the upper cylinder h_1; radius of the lower cylinder r_2; and the height of the lower cylinder h_2. These measurements are presented in Table 2 above. The object geometries can be seen in Figure 4.

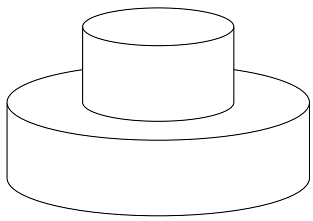


Figure 6: The small objects are essentially a small cylinder stacked on top of a larger cylinder

The trifilar was rotationally perturbed and allowed to oscillate. The time taken to complete 10 oscillations was recorded. This was repeated 5 times for both the disk, and the disk with mass objects placed in the middle holes. The tabulated results can be found in Table 3. Measurement error was reduced by taking an average of results for the disk, and for the disk plus objects. Average values can be found in the final column of Table 3.

Table 3: Five trials were undertaken to record the natural period of oscillation for 10 oscillations. This was done for both the disk, and the disk plus mass objects in the middle holes

Object	$(T_{10})_1$ [s]	$(T_{10})_2$ [s]	$(T_{10})_3$ [s]	$(T_{10})_4$ [s]	$(T_{10})_5$ [s]	$(T_{10})_{avg}$ [s]
Disk	31.91	31.50	30.97	30.82	30.78	31.21
Disk + 3 Objects (middle)	33.38	33.40	33.38	32.44	33.53	33.23

3 Calculations

3.1 Mass Moment of Inertia for Disk

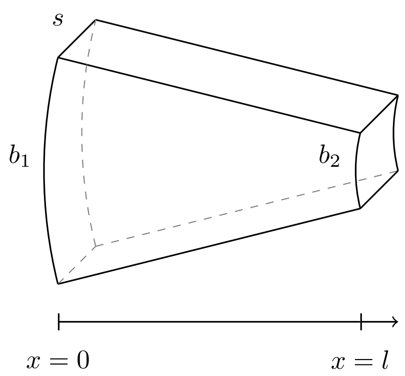
3.1.1 Disk experimental moment of inertia

3.1.2 Disk theoretical moment of inertia

3.2 Mass Moment of Inertia for Disk and Objects

3.2.1 Disk and objects experimental moment of inertia

3.2.2 Disk and objects theoretical moment of inertia



4 Discussion

5 Conclusion