Systems Modelling and Control: Tutorial 2

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Problem 1

Fourier Transforms

A periodic signal, $x(t)$, with period T has Fourier coefficients c_k^x such that:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k^x e^{jk\omega_0 t}, \quad \omega_0 = \frac{2\pi}{T}, \quad -\infty < t < \infty$$

Question a

If $v(t) = x(t-1)$, then:

$$\begin{aligned} v(t) &= \sum_{k=-\infty}^{\infty} c_k^x e^{jk\omega_0(t-1)} \\ &= \sum_{k=-\infty}^{\infty} c_k^x e^{jk\omega_0 t - jk\omega_0} \\ &= \sum_{k=-\infty}^{\infty} c_k^x e^{jk\omega_0 t} e^{-jk\omega_0} \\ &= \sum_{k=-\infty}^{\infty} e^{-jk\omega_0} c_k^x e^{jk\omega_0 t} \end{aligned}$$

Hence, we find that $c_k^v = e^{-jk\omega_0} c_k^x$.

Question b

If $v(t) = \frac{dx(t)}{dt}$, then:

$$v(t) = \frac{d}{dt}\left(\sum_{k=-\infty}^{\infty} c_k^x e^{jk\omega_0 t} \right)$$

We note at this point that generally it is not appropriate to simply move a differential operator inside an infinite series, however, in this instance we will assume that the necessary conditions for performing this operation are satisfied. Hence, we get:

$$\begin{aligned} v(t) &= \sum_{k=-\infty}^{\infty} c_k^x \frac{d}{dt}(e^{jk\omega_0 t}) \\ &= \sum_{k=-\infty}^{\infty} c_k^x jk\omega_0 (e^{jk\omega_0 t}) \\ &= \sum_{k=-\infty}^{\infty} jk\omega_0 c_k^x e^{jk\omega_0 t} \end{aligned}$$

Problem 1 continued on next page. . .

Hence, we find that $c_k^v = jk\omega_0 c_k^x$.

Question c

If $v(t) = x(t)e^{j\omega_0 t}$, then:

$$\begin{aligned} v(t) &= e^{j\omega_0 t} \sum_{k=-\infty}^{\infty} c_k^x e^{jk\omega_0 t} \\ &= \sum_{k=-\infty}^{\infty} c_k^x e^{j\omega_0 t} e^{jk\omega_0 t} \\ &= \sum_{k=-\infty}^{\infty} c_k^x e^{j\omega_0 t(k+1)} \end{aligned}$$

Since the sum is over an infinite series, we can change c_x^k to c_x^{k-1} and change $(k+1)$ to simply k and we get the following:

$$v(t) = \sum_{k=-\infty}^{\infty} c_{k-1}^x e^{j\omega_0 tk}$$

Hence, we find that $c_k^v = c_{k-1}^x$.

Question d

If $v(t) = x(t)\cos(\frac{2\pi}{T}t)$, then:

$$v(t) = \cos(\frac{2\pi}{T}t) \sum_{k=-\infty}^{\infty} c_k^x e^{jk\omega_0 t}$$

Since $\omega_0 = \frac{2\pi}{T}$, we can re write this as:

$$
\begin{aligned}
v(t) &= cos(\omega_0 t) \sum_{k=-\infty}^{\infty} c_k^x e^{jk\omega_0 t} \\
&= \sum_{k=-\infty}^{\infty} c_k^x e^{jk\omega_0 t} cos(\omega_0 t) \\
&= \sum_{k=-\infty}^{\infty} c_k^x e^{jk\omega_0 t} \frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t}) \\
&= \frac{1}{2} \sum_{k=-\infty}^{\infty} c_k^x e^{jk\omega_0 t}(e^{j\omega_0 t} + e^{-j\omega_0 t}) \\
&= \frac{1}{2} \sum_{k=-\infty}^{\infty} c_k^x e^{jk\omega_0 t} e^{j\omega_0 t} + c_k^x e^{jk\omega_0 t} e^{-j\omega_0 t} \\
&= \frac{1}{2} \sum_{k=-\infty}^{\infty} c_k^x e^{j\omega_0 t(k+1)} + c_k^x e^{j\omega_0 t(k-1)}
\end{aligned}
$$

Using the same idea that we used in question (c), we can re-write the series as follows:

$$
\begin{aligned}
v(t) &= \frac{1}{2} \sum_{k=-\infty}^{\infty} c_{k-1}^x e^{j\omega_0 tk} + c_{k+1}^x e^{j\omega_0 tk} \\
&= \sum_{k=-\infty}^{\infty} \frac{1}{2}(c_{k-1}^x + c_{k+1}^x) e^{jk\omega_0 t}
\end{aligned}
$$

Hence, we find that $c_k^v = \frac{1}{2}(c_{k-1}^x + c_{k+1}^x)$.

Problem 2

Fourier Transforms

A continuous-time signal $x(t)$ has the Fourier transform:

$$\mathscr{F}\{x(t)\} = X(\omega) = \frac{1}{j\omega + b}$$

The original signal, $x(t)$, which has the Fourier transform shown above is:

$$x(t) = e^{-bt} \cdot u(t)$$

Question a

If $v(t) = x(5t-4)$, then rewritten, we can say that $v(t) = x(5(t - {}^{4}\!/\!_{5}))$. Using standard Fourier transforms, we can see that:

$$\begin{aligned}\mathscr{F}\{v(t)\} &= \frac{1}{5}X(\omega)e^{-j\omega\frac{4}{5}} \\ &= \frac{1}{5(j\omega + b)}e^{-j\omega\frac{4}{5}}\end{aligned}$$

Question b

If $v(t) = t^2 x(t)$, then using standard Fourier transforms, we can see that:

$$\mathscr{F}\{v(t)\} = j^2 \frac{d^2}{d\omega^2}\left(\frac{1}{j\omega + b}\right)$$

Now, we see that:

$$\frac{d}{d\omega}(j\omega + b)^{-1} = -j(j\omega + b)^{-2}$$

And that:

$$\frac{d}{d\omega}(-j(j\omega + b)^{-2}) = -2(j\omega + b)^{-3}$$

Hence, we find that:

$$\begin{aligned}\mathscr{F}\{v(t)\} &= j^2 \cdot (-\frac{2}{(j\omega + b)^3}) \\ &= \frac{2}{(j\omega + b)^3}\end{aligned}$$

Question c

If $v(t) = x(t)e^{j2t}$, then using standard Fourier transforms, we can see that:

$$\begin{aligned}\mathscr{F}\{v(t)\} &= X(\omega - 2) \\ &= \frac{1}{j(\omega - 2) + b}\end{aligned}$$

Question d

If $v(t) = x(t)cos(4t)$, then using standard Fourier transforms, we can see that:

$$\begin{aligned}\mathscr{F}\{v(t)\} &= \frac{1}{2}\Bigg[X(\omega + 4) + X(\omega - 4)\Bigg] \\ &= \frac{1}{2}\left[\frac{1}{j(\omega + 4) + b} + \frac{1}{j(\omega - 4) + b}\right]\end{aligned}$$

Question e

If $v(t) = \frac{d^2x(t)}{dt^2}$, then using standard Fourier transforms, we can see that:

$$\begin{aligned}\mathscr{F}\{v(t)\} &= (j\omega)^2 X(\omega) \\ &= -1 \cdot \omega^2 \cdot \frac{1}{j\omega + b} \\ &= -\frac{\omega^2}{j\omega + b}\end{aligned}$$

Question f

If $v(t) = x(t) * x(t)$, then using standard Fourier transforms, we can see that:

$$\begin{aligned}\mathscr{F}\{v(t)\} &= X(\omega) \cdot X(\omega) \\ &= \frac{1}{j\omega + b} \cdot \frac{1}{j\omega + b} \\ &= \frac{1}{(j\omega + b)^2}\end{aligned}$$

Question g

If $v(t) = [x(t)]^2$, then $v(t) = [e^{-bt} \cdot u(t)]^2$. Hence, rewriting $v(t)$ we get:

$$v(t) = e^{-2bt}, \quad \forall t > 0$$

Hence, taking the Fourier transform, we get:

Problem 2 continued on next page...

$$\begin{aligned}
\mathscr{F}\{v(t)\} &= \int_0^\infty e^{-2bt}e^{-j\omega t}dt \\
&= \int_0^\infty e^{-(2b+j\omega)t}dt \\
&= -\frac{1}{j\omega+2b}\left[e^{-(2b+j\omega)t}\right]_0^\infty \\
&= -\frac{1}{j\omega+2b}\left[\lim_{t\to\infty} e^{-(2b+j\omega)t} - 1\right] \\
&= -\frac{1}{j\omega+2b}\cdot(-1) \\
&= \frac{1}{j\omega+2b}
\end{aligned}$$

Question h

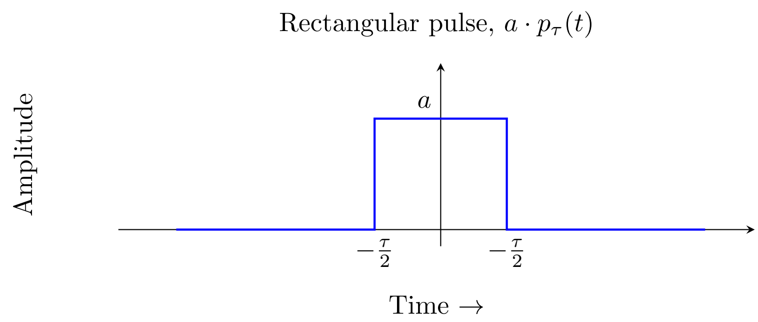
If $v(t) = \frac{1}{jt-b}$, then:

$\boxed{\text{Was unable to answer this question successfully.}}$

Problem 3

Fourier Transforms

Consider the general rectangular pulse, $p_\tau(t)$, multiplied by some amplitude a shown below.



We know that the Fourier transform of some signal, $x(t)$, is given by:

$$\mathscr{F}\{x(t)\} = X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Hence, for the rectangular pulse, we get that:

$$\begin{aligned}
\mathscr{F}\{a \cdot p_\tau(t)\} &= \int_{-\infty}^{\infty} a \cdot p_\tau(t)e^{-j\omega t}dt \\
&= a\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} p_\tau(t)e^{-j\omega t}dt \\
&= a\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} e^{-j\omega t}dt \\
&= a\left[-\frac{1}{j\omega}e^{-j\omega t}\right]_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \\
&= -\frac{a}{j\omega}\left(e^{-j\omega\frac{\tau}{2}} - e^{j\omega\frac{\tau}{2}}\right) \\
&= -\frac{a}{j\omega}\left(\cos(\frac{\omega\tau}{2}) - j\sin(\frac{\omega\tau}{2}) - \cos(\frac{\omega\tau}{2}) - j\sin(\frac{\omega\tau}{2})\right) \\
&= \frac{2a}{\omega}\sin(\frac{\omega\tau}{2})
\end{aligned}$$

Now, given that:

$$\text{sinc}(A\omega) = \frac{\sin(A\pi\omega)}{A\pi\omega}$$

Hence, we can see that:

$$\sin(A\omega) = \sin(\frac{\omega\tau}{2})$$
$$\therefore A = \frac{\tau}{2\pi}$$

Problem 3 continued on next page. . .

Hence, we can write our transform in terms of the sinc function:

$$\begin{aligned}\mathscr{F}\{a \cdot p_\tau(t)\} &= 2a \cdot \frac{\sin(A\pi\omega)}{A\pi\omega} \cdot A\pi \\ &= 2a \cdot A\pi \cdot \operatorname{sinc}(A\omega) \\ &= 2a \cdot \frac{\tau}{2\pi}\pi \operatorname{sinc}(\frac{\tau\omega}{2\pi}) \\ &= a\tau \operatorname{sinc}(\frac{\tau\omega}{2\pi})\end{aligned}$$

Finally, suppose that we had a pulse rectangle $p_2(t)$ with an amplitude of 1, then the Fourier transform would be:

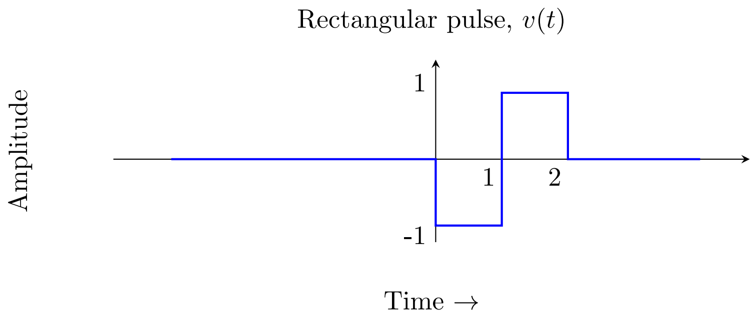
$$\begin{aligned}\mathscr{F}\{p_2(t)\} &= 1 \cdot 2 \cdot \operatorname{sinc}(\frac{2\omega}{2\pi}) \\ &= 2 \cdot \operatorname{sinc}(\frac{\omega}{\pi})\end{aligned}$$

Problem 4

Fourier Transforms

Question a

Consider the following signal $v(t)$:



This pulse can be written as follows:

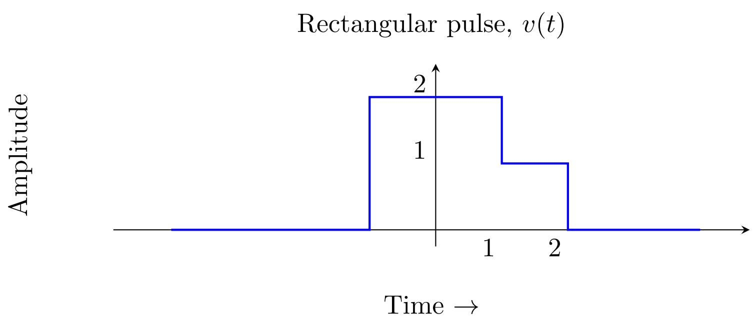
$$v(t) = -p_1(t - 1/2) + p_1(t - 3/2)$$

The Fourier transform of $v(t)$, using standard transforms we get:

$$\mathscr{F}\{v(t)\} = -1 \cdot \text{sinc}(\frac{\omega}{2\pi})e^{-j\omega 1/2} + \text{sinc}(\frac{\omega}{2\pi})e^{-j\omega 3/2}$$

Question b

Consider the following signal $v(t)$:



This pulse can be written as follows:

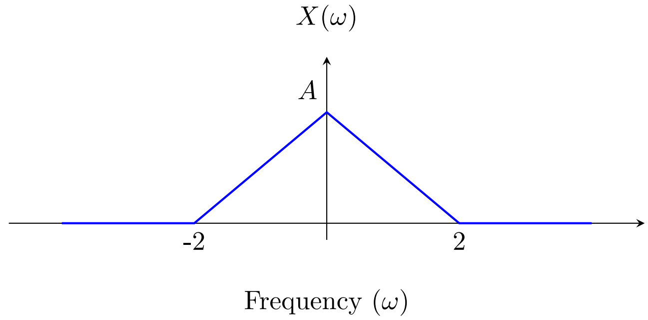
$$v(t) = p_3(t - 1/2) + p_2(t)$$

The Fourier transform of $v(t)$, using standard transforms we get:

$$\mathscr{F}\{v(t)\} = -1 \cdot \text{sinc}(\frac{\omega}{2\pi})e^{-j\omega 1/2} + \text{sinc}(\frac{\omega}{2\pi})e^{-j\omega 3/2}$$

Problem 5

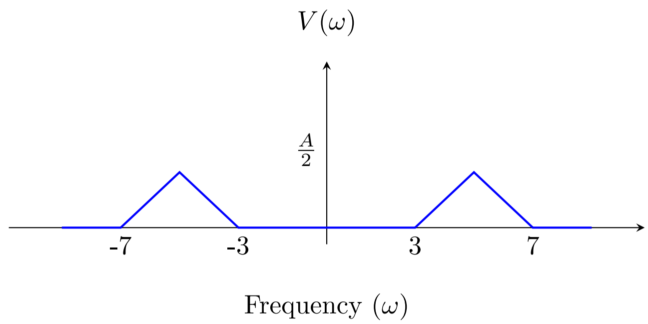
Consider the signal $x(t) \longleftrightarrow X(\omega)$. The plot of $X(\omega)$ is shown below.



Suppose a new signal, $v(t)$, was composed by multiplying $x(t)$ by $\cos(5t)$. The Fourier transform would be:

$$x(t) \cdot \cos(5t) \longleftrightarrow \frac{1}{2} \cdot \Big[X(\omega + 5) + X(\omega - 5)\Big].$$

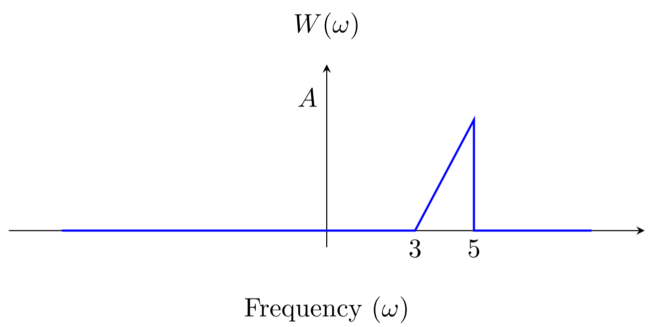
The plot of the Fourier transform of $v(t)$, $V(\omega)$, is shown below.



Suppose now that the signal $v(t)$ was passed through some filter $H_1(\omega)$, such that:

$$H_1(\omega) = \begin{cases} 2 & \text{if } 3 \leq |\omega| \leq 5 \\ 0 & \text{all other } \omega \end{cases}$$

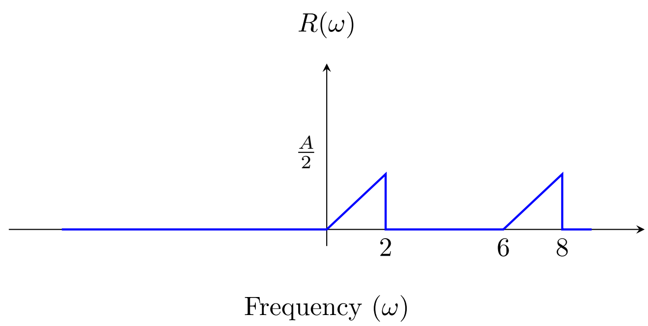
If $W(\omega) = H_1(\omega)V(\omega)$, then a plot of $W(\omega)$ is shown below:



Now, if $w(t) \longleftrightarrow W(\omega)$, then we can create some new signal, $r(t)$, such that $r(t) = w(t) \cdot \cos(3t)$. The new signal has the following transform:

$$w(t) \cdot \cos(3t) \longleftrightarrow \frac{1}{2}\Big[W(\omega + 3) + W(\omega - 3)\Big]$$

The plot of the Fourier transform of $r(t)$, $R(\omega)$, is shown below:



Finally, suppose that $r(t)$ was passed through some filter $H_2(\omega)$, such that:

$$H_2(\omega) = \begin{cases} 2 & \text{if } |\omega| \leq 3 \\ 0 & \text{all other } \omega \end{cases}$$

If $Y(\omega) = H_2(\omega)R(\omega)$, then a plot of $Y(\omega)$ is shown below: