MAT320 Practice Problems Answer Key

10/9/2023

Problem. A *rectangle* is a subset $(a,b) \times (c,d) \subset \mathbb{R}^2$ where $b > a$ and $d > c$. We write

$$l\,(\,(a,b) \times (c,d)\,) = (b-a)(d-c).$$

For a subset $S \subset \mathbb{R}^2$, define

$$\mu^*(S) = \left\{ \sum_{i=1}^{\infty} \ell(R_i) \,|\, R_i \text{ a rectangle for } i = 1, \ldots; S \subset \bigcup_{i=1}^{\infty} R_i \right\}.$$

For $a = (a_1, a_2) \in \mathbb{R}^2$ and $r > 0$, write

$$B(a,r) = \{(x_1,x_2) \in \mathbb{R}^2 | \sqrt{(x_1-a_1)^2 + (x_2-a_2)^2}\}.$$

Prove that $\mu^*(B(a,r)) = r^2 \mu^*(B((0,0),1))$.

Proof. Given a set $S \subset \mathbb{R}^2$, write

$$bS = \{bs : s \in S\} \text{ for } b \in \mathbb{R}, \text{ and}$$

$$S + v = \{s + v : s \in S\} \text{ for } v \in \mathbb{R}^2.$$

The map

$$R \mapsto rR + a$$

(where r and a are as in the problem statement) defines a bijection $\phi : \mathcal{S} \to \mathcal{S}$, where $\mathcal{S}$ is the set of rectangles. If $T \subset S$ then $bT + v \subset bS + v$; and if $T_i \subset \mathbb{R}^2$ for $i = 1, 2, \ldots$, then

$$\bigcup_{i=1}^{\infty} (bT_i + v) = b\left(\bigcup_{i=1}^{\infty} T_i\right) + v;$$

as such, the bijection ϕ induces a bijection

$$\begin{aligned} \left\{ \{R_i\}_{i=1}^{\infty} | R_i \text{ a rectangle }, S \subset \bigcup_{i=1}^{\infty} R_i \right\} \mapsto & \quad \left\{ \{rR_i + a\}_{i=1}^{\infty} | R_i \text{ a rectangle }, S \subset \bigcup_{i=1}^{\infty} R_i \right\} \\ = & \quad \left\{ \{R'_i\}_{i=1}^{\infty} | R'_i \text{ a rectangle }, rS + a \subset \bigcup_{i=1}^{\infty} R'_i \right\}. \end{aligned}$$

Setting $S = B((0,0),1)$, we see that $rS + a = B(a,r)$. Moreover $\ell(rR + a) = r^2\ell(R)$ for any rectangle R. As such, we see that

$$\left\{\sum_{i=1}^{\infty} r^2\ell(R_i)|R_i \text{ a rectangle }, B((0,0),1) \subset \bigcup_{i=1}^{\infty} R_i\right\} = \left\{\sum_{i=1}^{\infty} \ell(R_i')|R_i' \text{ a rectangle }, B(a,r) \subset \bigcup_{i=1}^{\infty} R_i'\right\}$$

as sets. So the infima of the left hand side and the right hand side are equal; but this is what we were trying to show. □

Problem.

Find the measure of the set of all numbers $x \in [0,1]$ for which there are no 4s in any decimal expansion of x.

Proof. We describe the set in a way analogous to the Cantor set. The set of numbers $x \in [0,1]$ which have a 4 in the first digit of some decimal expansion is $[4/10, 5/10]$. The set of numbers which have a 4 in the second digit of some decimal expansion is

$$[0.04, 0.05] \cup [0.14, 0.15] \cup \ldots \cup [0.94, 0.95].$$

In general, the set of numbers which have a 4 in the k-th digit of some decimal expansion is of the form

$$[0.0\ldots04, 0.0\ldots05] \cup [0.0\ldots14, 0.0\ldots15] \cup \ldots \cup [0.9\ldots94, 0.9\ldots95]$$

where there are 10^{k-1} separate intervals in the above disjoint union.

Write $F_{10}^k \subset [0,1]$ to be the set of numbers which do *not* have a 4 in the first k digits of any of their decimal expansions. Then $F_{10}^0 = [0,1]$ since the condition is vacuous, and F_{10}^k comes from removing a finite collection of closed intervals from F_{10}^{k-1}, each of measure $1/(10)^k$. In particular, the set of numbers F described in the problem is a countable intersection of measurable sets, and is thus measurable.

We claim that the measure of F is

$$1 - \left(\frac{1}{10} + \frac{9}{100} + \frac{9*9}{1000} + \frac{9*9*9}{10000} + \ldots\right). \tag{1}$$

In other words, we first remove all the numbers with a 4 in its first decimal digit. The remaining numbers all have some digit other than 4 in their first digit (for any decimal expansion); so there are 9 remaining possible first digits, and we remove all numbers with a 4 in the second digit and any one of these first digits. The remaining numbers have, for their first two digits in their decimal expansions, some two digit number with no 4s; there are $9*9$ of these. So we remove all the numbers with any of these choices for their first two digits of their decimal expansions, and a 4 in the third digit. This process goes on: there are 9^{k-1} strings of $k-1$ digits containing no 4s, and to get F_{10}^k from F_{10}^{k-1} we remove 9^{k-1} intervals each of length 10^k from F_{10}^{k-1}.

It remains to sum (1). To do this we write this as

$$1 - \frac{1}{10}\sum_{k=0}^{\infty}\left(\frac{9}{10}\right)^k = 1 - \frac{1}{10}\left(\frac{1}{1-\frac{9}{10}}\right) = 1 - \frac{1}{10}(10) = 0.$$

This may seem surprising, but heuristically, the probability that the k-th digit is a 4 has probability 1/10, and these events are independent, so the probability that none of the digits is a 4 should be $\Pi_{k=1}^{\infty}(9/10)^k = 0$. The above computation is simply a more careful version of this argument. $\square$

Problem. Let A_i be subsets of a metric space X for $i = 1, 2, \ldots,$. Prove that if $B = \cup_{i=1}^{\infty} A_i$ then $\overline{B} \supset \cup_{i=1}^{\infty} A_i$. Show that the latter inclusion does not have to be an equality.

Proof. Let $x \in X$. If for every $\epsilon > 0$, $B(x, \epsilon)$ contains a point of A_j for some fixed j independent of ϵ, then $B(x, \epsilon)$ clearly contains a point of B as well. So if x is a point of closure of A_j then it is a point of closure of B. This proves the first claim in the problem.

For the second part of the problem, let $A_j = \{1/j\} \cup \{1/j + 1/k : k = 1, 2, \ldots\}$. Then each A_j is closed so $\overline{A_j} = A_j$. Set $B = \cup_{i=1}^{\infty} A_i$. Then B is not closed. Indeed, $0 \notin B$ since every element of A_j is strictly positive for any j. But $1/j \in B$ for $j = 1, 2, \ldots$. So 0 is a point of closure of B that is not in B. $\square$

Problem. An F_δ set is a set that is a countable union of closed sets. Show that continuous functions $f : \mathbb{R} \to \mathbb{R}$ map F_δ sets to F_δ sets.

(Hint: show first that the image of a compact set under a continuous function is compact.)

Proof. Let $f : X \to Y$ be continuous. Given a countable collection of closed sets $F_i \subset X$, $i = 1, 2, \ldots,$, we first note that if F_i is bounded, then it is compact; and then $f(F_i)$ is also compact and thus closed. Indeed, given any cover of $f(F_i)$, if there is an open cover $\{U_\alpha\}_{\alpha \in I}$ of $f(F_i)$, then $\{f^{-1}(U_\alpha)\}_{\alpha \in I}$ is an open cover of $f(F_i)$ by the continuity of f. As such the latter has a finite subcover, say $\{f^{-1}(U_1), \ldots, f^{-1}(U_r)\}$. But $f(f^{-1}(T)) = T$ for any $T \subset \mathbb{R}$; so if

$$F_i \subset f^{-1}(U_1) \cup \ldots \cup f^{-1}(U_r) \text{ then } f(F_i) \subset U_1 \cup \ldots \cup U_r.$$

In other words, there was a finite subcover of the cover $\{U_\alpha\}_{\alpha \in I}$ of $f(F_i)$. So we have show that continuous functions take compact sets to compact sets.

Now for every F_i, writing $F_i^j = F_i \cap [j, j+1]$ for $j \in \mathbb{Z}$, we see that $F_i = \cup_{j \in \mathbb{Z}} F_i^j$. Thus

$$F = \bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} \bigcup_{j=-\infty}^{\infty} F_{ij};$$

so

$$f(F) = \bigcup_{i=1}^{\infty} f(F_i) = \bigcup_{i=1}^{\infty} \bigcup_{j=-\infty}^{\infty} f(F_{ij}).$$

The quantity on the right above is a countable union of closed sets. $\square$

Problem. Royden Problem 2.4.19.

Proof. By theorem 2.4.11 of the textbook (also proven in class), a set E is measurable if and only if for any $\epsilon > 0$, there is an open set $O \supset E$ such that $m^*(O \setminus E) < \epsilon$ (here m^* is the outer measure as defined in the textbook, which we denoted by μ^* in the lectures). Thus, if E is *not* measurable then there is an $\epsilon > 0$ such that for *any* open set $O \supset E$, we have $m^*(O \setminus E) \geq \epsilon$. However, if E has finite outer measure $m^*(E)$, then for any $\epsilon_2 > 0$ then there is an open set $V \supset E$ such that $m^*(V) < m^*(E) + \epsilon_2$. In particular, letting ϵ_2 range over $1/n$, $n = 1, 2, \ldots$, and taking the intersection of the corresponding open sets, we see that there is a Borel set $\widetilde{V} \supset E$ such that $m^*(V) = m^*(E)$. Choose an open set $U \supset \widetilde{V}$ such that $m^*(U) < m^*(\widetilde{V}) + \epsilon = m^*(E) + \epsilon$. Then

$$m^*(U \setminus E) \geq m^*(U \setminus \widetilde{V}) = m^*(U) - m^*(\widetilde{V}) < \epsilon$$

where the first inequality is by monotonicity of the outer measure. But $U \supset \widetilde{V} \supset E$; so this contradicts the non-measurability of E. $\square$

Problem. Show that a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous except at a finite number of points is measurable. (On the quiz, I would remind you of the definition of a measurable function.)

Proof. A function is measurable if the preimage of any open set is measurable. Write the set of points of continuity of f as

$$(-\infty, a_0) \cup (a_0, a_1) \cup \ldots \cup (a_{r-1}, a_r) \cup (a_r, \infty),$$

and for shorthand write $I_0 = (-\infty, a_0)$, $I_\ell = (a_{\ell-1}, a_\ell)$ for $\ell = 1, 2, \ldots, r$, and $I_{r+1} = (a_r, \infty)$. Write f_ℓ for the restriction of f to I_ℓ. Then for any open set $U \subset \mathbb{R}$,

$$f^{-1}(U) = \left(\cup_{\ell=0}^{r+1} f^{-1}(U) \cap I_\ell\right) \cup (U \cap \{a_0, \ldots, a_r\}).$$

Now $(U \cap \{a_0, \ldots, a_r\})$ is finite and so is measurable. Thus, if we show that $f^{-1}(U) \cap I_\ell$ is measurable for each $\ell = 0, \ldots, r+1$, we are done since finite unions of measurable sets are measurable. But $f^{-1}(U) \cap I_\ell = f_\ell^{-1}(U)$. Each f_ℓ is a continuous funciton, and the preimage of an open set under a continuous function is continuous. So indeed $f^{-1}(U) \cap I_\ell$ is open for each $\ell = 1, \ldots, r+1$. $\square$

Problem. Is the subset

$$\bigcup_{n=2}^{\infty} [1/n, 1 - 1/n] \times \{1/n\} \subset \mathbb{R}^2$$

compact? Here we use the standard distance metric on $\mathbb{R}^2$ given by

$$d((v_1, v_2), (w_1, w_2)) = \sqrt{(v_1 - w_1)^2 + (v_2 - w_2)^2}).$$

Proof. No because it is not closed:

$$\lim_{n \to \infty} (1/2, 1/n) = (1/2, 0),$$

and the right hand side is not in the set while the terms in the limit in the left hand side are all in the set. Thus $(1/2, 0)$ is a point of closure of the set which is not in the set. $\square$