MAT320 Problem Set 8

Due Dec 5, 2024

Please write your homework on paper neatly or type it up in LaTeX, and hand it in at the beginning of class next Thursday. For us, *integrable* always means *Lebesgue integrable* and measurable unless otherwise specified. This homework may seem rather long, but many of the problems have extremely short answers!

The total point value of this Problem Set for your grade is 5 problems, divided evenly among the 15 non-extra-credit pieces 1.a, 1.b, .., 2.a, etc; the pieces like 2.c.1 each weigh as much as pieces like 1.c.

Problem 1. Recall that for $f, g \in L^2([0,1])$ we set

$$\langle f, g\rangle = \int f(x)g(x)dx.$$

Thus $\langle f, f\rangle = \|f\|_{L^2}^2$. Suppose that we have vectors $e_i \in L^2([0,1])$, $i = 1, 2, \ldots$, such that

$$\langle e_i, e_j\rangle = \delta_{ij}$$

where $\delta_{ij} = 1$ if $i = j$ and otherwise $\delta_{ij} = 0$.

a) Set

$$e_0 = 1 \in L^2([0,1]),$$
$$\tilde{e}_n = \cos(2\pi nx) \in L^2([0,1]), n = 1, 2, \ldots$$
$$\tilde{f}_n = \sin(2\pi nx) \in L^2([0,1]), n = 1, 2, \ldots$$

What are positive constants c_n, d_n such that if we set $e_n = c_n\tilde{e}_n$, $f_n = d_n\tilde{d}_n$ then $\|e_n\|_{L^2} = \|f_n\|_{L^2} = 1$? You can use tables of antiderivatives from calculus or from the internet. It may be interesting for you to check for yourself that

$$\langle e_n, f_m\rangle = 0, \langle e_n, e_m\rangle = \langle f_n, f_m\rangle = \delta_{nm}$$

where $\delta_{nm} = 1$ if $n = m$ and otherwise $\delta_{nm} = 0$. (δ_{nm} is called the *Kronecker symbol.*)

Thus we can think of $\{e_0, e_1, f_1, e_2, f_2, \ldots\}$ as a collection of vectors in $L^2([0,1])$ which are pairwise orthogonal and each have length 1.

b) Suppose that

$$f(x) = \sum_{i=1}^{\infty} a_i e_i(x) \tag{1}$$

where *all but finitely many of the* a_i *are* ***zero***, so that the above sum is actually a finite sum. Show that we can recover a_n

$$a_n = \int_0^1 f(x) e_n(x). \tag{2}$$

c) We proved in an earlier homework $L^2([0,1]) \subset L^1([0,1])$. Show that if a sequence of functions $f_i \in L^2([0,1])$ converge to $f \in L^2([0,1])$ in L^2, then they converge in $L^1([0,1])$. Using the "General Lebesgue Dominated Convergence Theorem" in Chapter 4 (or some other method!) show that if (1) holds in the L^2 sense, that is, in the sense that for some sequence of numbers $a_i \in \mathbb{R}$, the partial sums

$$f_N(x) = \sum_{i=1}^{N} a_i e_i(x). \tag{3}$$

converge in L^2 to f, then

$$\int_0^1 f_N e_n \to \int_0^1 f e_n.$$

Conclude that (2) continues to hold whenever (1) holds in the sense described above.

d) Show that if (1) holds in the sense described above then

$$\sum_{i=1}^{\infty} |a_i|^2 < \infty. \tag{4}$$

Hint – the e_i behave like an orthonormal basis! Conversely, show that if the above holds then the partial sums f_n (3) converge to some function in L^2. This is one of the harder problems, so I will give the following hint: knowing that (4) holds can be shown to imply directly that the f_N form a Cauchy sequence in $L^2([0,1])$. Remember that (4) implies that

$$\lim_{k\to\infty} \sum_{i=k}^{\infty} |a_i|^2 = 0.$$

e) Extra credit: 1/2 problem. Suppose that we try to differentiate the series (1) term by term with respect to x. Does does that imply that the series

$$g(x) = \sum_{i=1}^{\infty} a_i \frac{d}{dx} e_i(x) \tag{5}$$

converges? The previous problem should come in handy here.

f) Now, let us not assume that (1) holds in the L^2-sense. Instead, let us simply define a_n via (2). Then we can define

$$T_n f(x) = \sum_{i=1}^{n} a_i e_i(x).$$

Show that

$$\langle T_n f, f - T_n f\rangle_{L^2} = 0.$$

Using Cauchy-Schwartz, show that

$$\sum_{i=1}^{\infty} |a_i|^2 \leq \|f\|_{L^2}^2 < \infty.$$

(Hint: think of $T_n f(x)$ as the projection of f_n to the linear subspace of $L^2([0,1])$ spanned by $\{e_0, \dots, e_n\}$. This is just the statement that the length of a projection of a vector is bounded by the length of the original vector. Feel free to take a look at linear algebra textbooks remember how this proof goes!)

g) **Extra credit. 1/2 problem.** Show that if $f(x)$ as in (1) is well defined in the sense that the partial sums converge in $L^2([0,1])$ to some function, then the sum of the term-by-term integrals converges pointwise to a continuous function, and we have

$$\int_0^y f(x)dx = \sum_{i=1}^{\infty} a_i \int_0^y e_i(x)dx$$

for all y.

Problem 2. In this problem we will redo a variant of Dirichlet's proof of the L^2 convergence of Fourier series. Interestingly, this result is rarely proven outside of relatively advanced math classes, even though the rigorous proof is not really harder than the non-rigorous proofs that are often presented.

This proof will use the complex numbers. Hopefully, you are familiar with Euler's identity

$$e^{ix} = \cos(x) + i\sin(x);$$

if not we will take it on faith.

Our first step will be to massage the formulae of the previous problem.

We will set

$$a_n = \int_0^1 f(x)\cos(2\pi nx)\,dx$$

$$b_n = \int_0^1 f(x)\sin(2\pi nx)\,dx$$

and the analog of the Fourier series with this parameterization is

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n 2\cos(2\pi nx) + b_n 2\sin(2\pi nx). \tag{6}$$

a) If

$$f(x) = \tilde{a}_0 + \sum_{i=1}^{\infty} \tilde{a}_n e_n + \tilde{b}_n f_n,$$

holds in the L^2 sense, as does (6), where e_n and f_n are as in the previous problem, give expressions for $\tilde{a}_i$ and $\tilde{b_i}$ in terms of a_i and b_i. (Note that $e_n(x) \neq 2\cos(2\pi nx)$ – instead, these two functions are only proportional. What is the proportionality constant?)

We will write

$$S_N f(x) = a_0 + \sum_{n=1}^{N} a_n 2\cos(2\pi nx) + b_n 2\sin(2\pi nx). \tag{7}$$

Our second step is to slightly massage this expression.

b) Show that

$$S_N f(x) = \int_0^1 \tilde{D}_N(x,y) f(y) dy$$

where

$$\tilde{D}_N(x,y) = 1 + 2\sum_{n=1}^{N} (\cos(2\pi nx)\cos(2\pi ny) + \sin(2\pi nx)\sin(2\pi ny))$$

Moreover, using a trigonometric identity, show that

$$\tilde{D}_N(x,y) = 1 + 2\sum_{n=1}^{N} \cos(2\pi n(x-y)) =: D_N(x-y). \tag{8}$$

The function $D_N(x)$ is called the *Dirichlet kernel*; we see that the partial sum of the fourier series is computed by taking the *convolution* with the Dirichlet kernel $D_n(x)$.

c) Now we will do some algebra. The complex numbers come in handy here.

c.1) Show that

$$\sum_{k=0}^{n} ar^k = a\frac{1-r^{n+1}}{1-r}.$$

c.2) Show that

$$\sum_{k=-n}^{n} r^k = r^{-n}\frac{1-r^{2n+1}}{1-r} = \frac{r^{-n-1/2} - r^{n+1/2}}{r^{-1/2} - r^{1/2}}.$$

c.3) Plugging in $r = e^{2\pi i x}$ show that

$$\sum_{k=-n}^{n} e^{2\pi i k x} = \frac{\sin((2n+1)\pi x)}{\sin(\pi x)}.$$

c.4) Show that

$$D_N(x) = \frac{\sin((2N+1)\pi x)}{\sin(\pi x)}.$$

We have proven two statements about two real-numbered quantities by using complex numbers!

d) By a change of variables, show that

$$S_N f(x) - f(x) = \int_0^1 D_N(y)(f(x+y) - f(x))dy.$$

(The fact that $D_n(y)$ is *even* may come in handy here, as may the expression (8)!)

e) We can write the expression above as

$$\int_0^1 D_N(y)(f(x+y) - f(x))dy = \int_0^1 \sin((2N+1)\pi y)\frac{f(x+y) - f(x)}{\sin \pi y} dy.$$

By looking at extra credit problems from earlier problem sets, show that if for each x,

$$\frac{f(x+y) - f(x)}{\sin \pi y} \tag{9}$$

is integrable as a function on $[0,1] \ni y$, then

$$\lim_{N\to\infty} S_N f(x) = f(x)$$

for each x; that is, the functions $S_N f(x)$ converge pointwise to $f(x)$ everywhere.

Here we interpret $f(z)$ for $z \notin [0,1]$ by making f 1-periodic, i.e. we set

$$f(z) = f(z-1)$$

for all z.

f) Show that if $f(x)$ is differentiable with continuous derivative after being made 1-periodic then for every $x \in [0,1]$, the functions (9) are integrable over $[0,1] \ni y$. (Hint: The only confusing part is about the zeros of the denominator. Compute

$$\lim_{y\to 0} \frac{f(x+y) - f(x)}{\sin \pi y}$$

using L-Hospital's rule. Is the function (9) actually even *better* than integrable, as a function of y?)

We have proven that the Fourier series for $f(x)$ pointwise everywhere to f for functions f with a continuous derivative. This is already very good. But we want to work with discontinuous functions f, too!

g) Using the previous problem show that $S_N f \to f$ in L^2 for functions like in part (f) of this problem.

Extra credit. 1/2 problem) You may take on faith (or look in the textbook, or think of a proof for youself) that functions as in part (f) of this problem are dense in $L^2([0,1])$.

Show that the set of functions f in $L^2([0,1])$ such that $S_N f \to f$ in $L^2([0,1])$, is necessarily is a *closed* subset of $L^2([0,1])$. Conclude that that Fourier series converge in L^2 for all elements of $L^2([0,1])$, which includes discontinuous functions!