

MAT320 Problem Set 3

Due Sept 26, 2024

Please write your homework on paper neatly or type it up in LaTeX, and hand it in at the beginning of class next Thursday. Please remember that if you don't do every homework problem, that does not mean that you are going to do badly in this class.

Royden $X.Y.Z$ refers Problem Z in Royden-Fitzpatrick, found in the collection of problems at the end of section $X.Y$.

Problem 1. Royden 1.5.42.

Problem 2. Royden 1.6.48. You may use without proof that

$$|\sin x - x| \leq x^2/2.$$

Problem 3. Royden 9.5.72.

Problem 4. We say that a metric space X is *separable* if there is countable dense subset of X . Show that every compact metric space is separable.

Problem 5. We proved in class that if $E \subset X$ is compact (where X is a metric space), then every sequence $\{x_i\}_{i=1}^\infty \subset E$ has a convergent subsequence. It turns out (and is proven in Royden-Fitzpatrick, Section 9.5, Theorem 16) that the converse holds – if all sequences in $E \subset X$ have convergent subsequences then E is compact. We will use this fact as a black box in this problem.

Suppose we are given a countable collection of metric spaces (X_i, d_i) , $i = 1, 2, \dots$. Write

$$\mathcal{X} = X_1 \times X_2 \times \dots$$

for the infinite cartesian product, and define

$$d : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$$

by

$$d((x_1, x_2, \dots), (y_1, y_2, \dots)) = \sum_{i=1}^{\infty} \frac{d_i(x_i, y_i)}{2^i(1 + d_i(x_i, y_i))}.$$

- a) Show that (X, d) is a metric space.
- b) Given compact subsets $K_i \subset X_i$ for each $i = 1, 2, \dots$, show that

$$K_1 \times K_2 \times \dots \subset \mathcal{X}$$

is compact.

Extra Credit. Let X be a metric space. For every pair of compact subsets $K_1 \subset X$ and $K_2 \subset X$, define

$$d(K_1, K_2) = \max\left\{\sup_{x \in K_1} \inf_{y \in K_2} d(x, y), \sup_{y \in K_2} \inf_{x \in K_1} d(y, x)\right\}.$$

This is a function

$$\mathcal{S} \times \mathcal{S} \rightarrow \mathbb{R}$$

where

$$\mathcal{S} = \{K \subset X \mid K \text{ compact}\}.$$

- a) For $i = 1, 2$, where

$$K_i^\epsilon = \cup_{x \in K_i} B(x, \epsilon).$$

Show that

$$d(K_1, K_2) = \inf\{\epsilon > 0 : K_1 \subset K_2^\epsilon \text{ and } K_2 \subset K_1^\epsilon\}.$$

- b) Show that this makes \mathcal{S} into a metric space.
- c) Show that if X is compact then \mathcal{S} is compact.