MAT320 Problem Set 2

Due Sept 28, 2023

Please write your homework on paper neatly or type it up in LaTeX, and hand it in at the beginning of class next Thursday.

Royden $X.Y.Z$ refers Problem Z in Royden-Fitzpatrick, found in the collection of problems at the end of section $X.Y$.

Problem 1. a) Royden 2.2.1, b) Royden 2.2.3.

Problem 2. Royden 9.5.72.

Problem 3. We say that a metric space X is *separable* if there is countable dense subset of X. Show that every compact metric space is separable.

Problem 4. We proved in class that if $E \subset X$ is compact (where X is a metric space), then every sequence $\{x_i\}_{i=1}^{\infty} \subset E$ has a convergent subsequence. It turns out (and is proven in Royden-Fitzpatrick, Section 9.5, Theorem 16) that the converse holds – if all sequences in $E \subset X$ have convergent subsequences then E is compact. We will use this fact as a black box in this problem.

Suppose we are given a countable collection of metric spaces (X_i, d_i), $i = 1, 2, \ldots$. Write

$$\mathcal{X} = X_1 \times X_2 \times \ldots$$

for the infinite cartesian product, and define

$$d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$$

by

$$d((x_1, x_2, \ldots), (y_1, y_2, \ldots)) = \sum_{i=1}^{\infty} \frac{d_i(x_i, y_i)}{2^i(1 + d_i(x_i, y_i))}.$$

a) Show that (X, d) is a metric space.

b) Given compact subsets $K_i \subset X_i$ for each $i = 1, 2, \ldots$, show that

$$K_1 \times K_2 \times \ldots \subset \mathcal{X}$$

is compact.

Problem 5. Show if U is an open subset of a metric space X such that $X \subset X$ is compact, then U is a countable union of closed sets. (If you do it for just $[0, 1]$, that is partial credit.)

Extra Credit. Let X be a metric space. For every pair of compact subsets $K_1 \subset X$ and $K_2 \subset X$, define

$$d(K_1, K_2) = \max\{ \sup_{x \in K_1} \inf_{y \in K_2} d(x, y), \sup_{y \in K_2} \inf_{x \in K_1} d(y, x)\}.$$

This is a function

$$\mathcal{S} \times \mathcal{S} \to \mathbb{R}$$

where

$$\mathcal{S} = \{K \subset X \,|\, K \text{ compact } \}.$$

a) For $i = 1, 2$, where

$$K_i^\epsilon = \cup_{x \in K_i} B(x, \epsilon).$$

Show that

$$d(K_1, K_2) = \inf\{\epsilon > 0 : K_1 \subset K_2^\epsilon \text{ and } K_2 \subset K_1^\epsilon\}.$$

b) Show that this makes $\mathcal{S}$ into a metric space.

c) (Really, very extra credit.) Show that if X is compact then $\mathcal{S}$ is compact.