MAT320 Quiz #8

11/7/2023

Please answer the following questions, and write your name on top of the quiz.

Question 1. Let $f_k : [0,1] \to \mathbb{R}$ and $f : [0,1] \to \mathbb{R}$ be measurable functions. If $f_k \to f$ almost everywhere, does it follow that $\int_0^1 f_k \to \int_0^1 f$ as $k \to \infty$? State yes or no, and if no, give a counterexample.

Question 2.

a) State the (Lebesgue) dominated convergence theorem.

b) Let $f : [0,1] \to \mathbb{R}$ be an integrable function. Show that $f_k(x) = x^k f(x)$ defines an integrable function $f_k : [0,1] \to \mathbb{R}$.

c) Show that $\lim_{k\to\infty} \int_0^1 x^k f(x) = 0$.

Extra credit. **(1/2 Q)** With the notation above, compute $\lim_{k\to\infty} \int_0^1 kx^k f(x)$ in terms of values of f, assuming that f is differentiable with continuous derivative.

Question 3.

a) State the monotone convergence theorem.

b) Suppose $f : \mathbb{R} \to \mathbb{R}$ is measurable and nonnegative. Suppose that

$$\int_{-\infty}^{\infty} \frac{n^2}{n^2 + x^2} f(x)dx \leq 1 \text{ for all } n = 1, 2, \ldots.$$

Show that f is integrable.