MAT320 Problem Set 7

Due Nov 21, 2024

Royden $X.Y.Z$ refers Problem Z in Royden-Fitzpatrick, found in the collection of problems at the end of section $X.Y$.

Problem 1. (Jensen's inequality) A function $\phi : \mathbb{R} \to \mathbb{R}$ is *convex* if

$$t\phi(x) + (1-t)\phi(y) \geq \phi(t(x) + (1-t)y) \text{ for any } t \in [0,1], \text{ any } x, y \in \mathbb{R}.$$

1. Show that if ϕ is twice differentiable with piecewise continuous second derivative then if $\phi''(x) \geq 0$ everywhere then ϕ is convex.

2. Show, by induction from the case $n = 2$, that given nonnegative numbers $w_i > 0$, $i = 1, \ldots, n$, such that $\sum_{i=1}^n w_i = 1$, then for any numbers c_i,

$$\phi\left(\sum_{i=1}^n w_i c_i\right) \leq \sum_{i=1}^n w_i \phi(c_i)).$$

   (Hint: Think of the w_i as 'weights'. One of the w_i, say w_n, must be strictly less than one; so we can divide and multiply the other w_i by $(1 - w_n)$ and try to use the convexity of ϕ.)

3. Show that for any simple function $f : [0,1] \to \mathbb{R}$ that

$$\phi\left(\int_0^1 f(x)dx\right) \leq \int_0^1 \phi(f(x))dx.$$

4. Show that if ϕ is continuous, $f : [0,1] \to \mathbb{R}$ is integrable, and $\phi \circ f : [0,1] \to \mathbb{R}$ is integrable as well, then the above inequality continues to hold.

It turns out that convex functions on $\mathbb{R}$ are continuous, so the assumption is actually unnecessary.

Problem 2. Show that if $1 \leq p < q < \infty$ then $L^p([0,1]) \subset L^q([0,1])$. (Hint: the $p = 2$ case was discussed in class, and follows from Cauchy-Schwarz. The Hölder inequality is the generalization of Cauchy-Schwarz.) Show that this inclusion is not an equality via an explicit example of a function in $L^q([0,1])$ that is not in $L^p([0,1])$.

Problem 3. One does not only have to consider L^p spaces on $[0,1]$ or on bounded intervals; for example, one can consider

$$L^p(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \text{ measurable } ||f|^p \text{ integrable}\}/\sim$$

where the equivalence relation identifies functions which agree almost everywhere, and

$$\|f\|_{L^p} = \left(\int |f|^p\right)^{1/p}.$$

Show that now if $1 < p < q < \infty$ then $L^p(\mathbb{R})$ is *not* a subset of $L^q(\mathbb{R})$ by an explicit example. (Hint: flip the example from Problem 3 'diagonally'!)

Problem 4. (Riemann-Lebesgue Lemma.)

Let $f : [a,b] \to \mathbb{R}$ be an integrable function. Show that the functions $f_n : [a,b] \to \mathbb{R}$, $f_n(x) = f(x)\sin(nx)$ are Lebesgue integrable over $[a,b]$. Show that if f is a step function then

$$\lim_{n\to\infty} \int_a^b f(x)\, \sin(nx)\, dx = 0.$$

Conclude, using the fact that continuous functions on an interval are uniformly approximable by step functions, that the same holds whenever f is just continuous.

Extra credit. (1 problem). Show that for any measurable function $f : [a,b] \to \mathbb{R}$, there is a sequence of step functions $f_i : [a,b] \to \mathbb{R}$ which converge pointwise almost everywhere to f. (Hint: you can reduce to the case where f is a simple function.) Conclude that the conclusion of problem 4 holds under the weaker assumption that f is integrable. (Remark: This is the key fact that lets you prove that Fourier series converge to the original function, in an appropriate sense!)