MAT320 Problem Set 7

Due Nov 16, 2023

Please write your homework on paper neatly or type it up in LaTeX, and hand it in at the beginning of class next Thursday. For us, *integrable* always means *Lebesgue integrable* unless otherwise specified.

This problem set may seem long, but several of the problems have rather short answers.

Problem 1. (Jensen's inequality) A function $\phi : \mathbb{R} \to \mathbb{R}$ is *convex* if

$$t\phi(x) + (1-t)\phi(y) \geq \phi(t(x) + (1-t)y) \text{ for any } t \in [0,1], \text{ any } x, y \in \mathbb{R}.$$

1. Show that if ϕ is twice differentiable with piecwise continuous second derivative then if $\phi''(x) \geq 0$ everywhere then ϕ is convex.
2. Show, by induction from the case $n = 2$, that given nonnegative numbers $w_i \geq 0$, $i = 1, \ldots, n$, such that $\sum_{i=1}^n w_i = 1$, then for any numbers c_i,

$$\phi\left(\sum_{i=1}^n w_i c_i\right) \leq \sum_{i=1}^n w_i \phi(c_i)).$$

   (Hint: Think of the w_i as 'weights'. One of the w_i, say w_n, must be strictly less than one; so we can divide and multiply the other w_i by $(1 - w_n)$ and try to use the convexity of ϕ.)
3. Show that for any step function $f : [0,1] \to \mathbb{R}$ that

$$\phi\left(\int_0^1 f(x)dx\right) \leq \int_0^1 \phi(f(x))dx.$$

4. Show that if ϕ is continuous, $f : [0,1] \to \mathbb{R}$ is integrable, and $\phi(f(x)) \geq 0$ for all $x \in [0,1]$, then the above inequality continues to hold.

One can drop the assumptions about the continuity of ϕ and the nonnegativity of $\phi(f(x))$, but it slightly technical to do so.

Problem 2. (Hölder's inequality.) The notion of a convex function makes sense for $\phi : (0, \infty) \to \mathbb{R}$.

- Show that $-\log x$ is convex as a function on $(0, \infty)$.
- We say that a pair of numbers $1 < p < q < \infty$ are *conjugate exponents* if

$$\frac{1}{p} + \frac{1}{q} = 1.$$

  By choosing $t = 1/p$, $x = u^p$, $y = v^q$, use the convexity of $-\log x$ to derive Young's inequality: for $u, v \geq 0$, we have

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q}$$

  whenever p and q are conjugate exponents.

- Now let $u : [0, 1] \to \mathbb{R}$ and $v : \mathbb{R}$ be functions in $L^p([0, 1])$ and $L^q([0, 1])$ for a pair of conjugate exponents. Show that uv is integrable and that

$$\int uv \leq \|u\|_{L^p} \|v\|_{L^q}.$$

  Hint: first prove that the above holds if $\|u\|_{L^p} = \|v\|_{L^q} = 1$ and then finish by rescaling u and v.

Problem 3. Show that if $1 \leq p < q < \infty$ then $L^q([0, 1]) \subset L^p([0, 1])$. (Hint: the $p = 2$ case was discussed in class, and follows from Cauchy-Schwarz. Above, we proved a generalization of Cauchy-Schwarz.) Show that this inclusion is not an equality via an explicit example of a function in $L^q([0, 1])$ that is not in $L^p([0, 1])$.

Problem 4. One does not only have to consider L^p spaces on $[0, 1]$ or on bounded intervals; for example, one can consider

$$L^p(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \text{ measurable } | |f|^p \text{ integrable}\} / \sim$$

where the equivalence relation identifies functions which agree almost everywhere, and

$$\|f\|_{L^p} = \left(\int |f|^p \right)^{1/p}.$$

Show that now if $1 < p < q < \infty$ then $L^q(\mathbb{R})$ is *not* a subset of $L^p(\mathbb{R})$ by an explicit example. (Hint: flip the example from Problem 3 'diagonally'!)

Extra credit. Let $g : [0,1] \to \mathbb{R}$ be an arbitrary measurable function such that $g(x) > 0$ for all $x \in \mathbb{R}$. Consider the space of functions

$$L^1([0,1]; g) = \{f : [0,1] \to \mathbb{R} \text{ measurable } | |fg| \text{ integrable}\}/\sim$$

with the norm

$$\|f\|_g = \int_0^1 |fg|.$$

Show that this space is a Banach space. (Hint: one can give a much easier answer than just running the whole proof again!). Show that for any measurable function $h : [0,1] \to \mathbb{R}$, there is a function g as above such that $h \in L^1([0,1]; g)$.