MAT320 Problem Set 4 Answers

October 11, 2023

Royden $X.Y.Z$ refers Problem Z in Royden-Fitzpatrick, found in the collection of problems at the end of section $X.Y$.

Problem 1. Royden 2.3.11.

Proof. Suppose the σ-algebra A contains (a, ∞) for all a. Then since

$$[a, \infty) = \cap_{n=1}^{\infty}(a - 1/n, \infty)$$

it contains $[a, \infty)$ for all a. It also contains the complements of each of these, namely $(-\infty, a)$ and $(-\infty, a]$ for every a. But then it contains $[a, b] = [a, \infty) \cap (-\infty, b]$, $(a, b) = (a, \infty) \cap (-\infty, b)$, $(a, b] = (a, \infty) \cap (-\infty, b]$ and so forth. $\square$

Problem 2. Royden 2.3.14.

Proof. Suppose E has positive outer measure. By countable subadditivity of outer measure,

$$\mu^*(E) \leq \sum_{j=-\infty}^{\infty} \mu^*(E \cap [j, j+1]).$$

Since $\mu^*(E) > 0$, at least one of the $\mu^*(E \cap [j, j+1])$ must be greater than zero. $\square$

Problem 3. We say that $f : [a, b] \to \mathbb{R}$ is *Lipshitz* if there is a constant $c \geq 0$ such that for all $u, v \in [a, b]$,

$$|f(u) - f(v)| \leq c|u - v|.$$

Show that the image of a set of measure zero under a Lipshitz function has measure zero.

(We will see on October 3 that there is a continuous function $f : [0, 1] \to \mathbb{R}$ and a set $C \subset [0, 1]$ of measure zero such that $f(E)$ is a measurable set of measure 1.)

Proof. Say $E \subset [a, b]$ is of measure zero. Then for any $\epsilon > 0$ there is a countable collection I_i of open intervals such that $E \subset \cup_{i=1}^{\infty} I_i$ and $\sum_i \ell(I_i) < \epsilon$. I claim that $f(I_i)$ is in some open interval of size $c\ell(I_i)$. If this holds then $f(E)$ is covered by a countable collection of open intervals $I'_j \supset f(I_i)$ such that $\sum_{j=1}^{\infty} \ell(I'_j) = \sum_{j=1}^{\infty} c\ell(I_i) < c\epsilon$. Since ϵ was arbitrarily small and c doesn't depend on ϵ this proves the desired claim.

To produce I'_j from I_j, we choose the midpoint $x_j \in I_j$. Then any $x'_j \in I_j$ satisfies $|x'_j - x_j| < \ell(I_j)/2$, so $|f(x'_j) - f(x_j)| < c\ell(I_j)/2$. In other words, $f(I_j)$ is contained in $[f(x_j) - c\ell(I_j)/2, f(x_j) + c\ell(I_j)/2]$, which is the desired claim. $\square$

Problem 4. Let $0 < \alpha < 1$. We define a subset $F_\alpha \subset [0,1]$, by defining

$$F_\alpha = \cap_{n=1}^{\infty} F_\alpha^n$$

where F_α^n is a union of intervals each of equal length, and $F_\alpha^0 = [0,1]$ and F_α^n is produced from F_α^{n-1} by removing an open interval of length $\alpha/(3^n)$ from the middle each of the intervals comprising F_α^{n-1}. Thus, if

$$F_\alpha^{n-1} = \bigcup_{i=1}^{n_k} [x_i - a_i, x_i + a_i],$$

for some real numbers x_i and real numbers $a_i > 0$, then

$$F_\alpha^n = \bigcup_{i=1}^{n_k} \left([x_i - a_i, x_i - \alpha/(2*3^n)] \cup [x_i + \alpha/(2*3^n), x_i + a_i] \right).$$

Show that F_α is closed and uncountable, and compute the measure of F.

Proof. F_α^n contains F_1^n, for all n, and so F_α contains F_1, which is the Cantor set. Since the Cantor set is uncountable, so is F_α. It is an infinite intersection of closed sets, so its closed. By induction we show that F_α^n has 2^n intervals of equal length, so we remove 2^n intervals from F_α^n to get F_α^{n+1}. Thus the measure of F_α is

$$1 - \left(\frac{\alpha}{3} + \frac{2\alpha}{3^2} + \dots \right) = 1 - \frac{\alpha}{3} \sum_{n=0}^{\infty} \left(\frac{2}{3} \right)^n = 1 - \frac{\alpha}{3(1-(2/3))} = 1 - \alpha.$$

□

Problem 5. Let f be a continuous function and let B be a Borel set. Show that $f^{-1}(B)$ is a Borel set.

Proof. Say f is a continuous function $X \to Y$ for some metric spaces X, Y.

Let $Borel_X$ be the σ-alegebra of Borel sets of X, and similarly for $Borel_Y$. Let $\mathbb{A}$ be the set of σ algebras A on X such that A contains all open subsets of X. Then

$$Borel_X = \cap_{A \in \mathbb{A}} A.$$

Given a σ-algebra A on X, we have an associated σ algebra $f(A)$ on X defined by

$$f(A) = \{E \subset Y | f^{-1}(E) \in A\}.$$

One can verify the axioms of a σ-algebra using the fact that f^{-1} commutes with all set-theoretic operations like union and intersection. We claim that if $A \in \mathbb{A}$ then $f(A)$ contains the open sets of Y: if U is open in Y, then $f^{-1}(U)$ is open in X by continuity, so $f^{-1}(U) \in A$ by the definition of $\mathbb{A}$, so $U \in f(A)$. This is true for all $A \in \mathbb{A}$; one sees immediately that

$$f(Borel_X) = \cap_{A \in \mathbb{A}} f(A).$$

Writing $\mathbb{B}$ for the set of σ algebras on Y containing the open sets of Y, we have that $f(A) \in \mathbb{B}$ for every $A \in \mathbb{A}$; so

$$f(Borel_X) = \cap_{A \in \mathbb{A}} f(A) \supset \cap_{B \in \mathbb{B}} B = Borel_Y.$$

In other words, if $E \in Borel_Y$ then $E \in f(Borel_X)$, i.e. $f^{-1}(E) \in Borel_X$ – which is what we wanted to show. $\square$

Extra credit. Given a subset E of a metric space X, we say that a *boundary point* of E is a point $x \in X$ such that for all $\epsilon > 0$, $B(x, \epsilon)$ contains some point in E and also some other point in $X \setminus E$. Let ∂E be the set of boundary points of E. (Note that ∂E may or may not contain points of E.)

- Show that if E is closed then $\partial E \subset E$.
- Show that ∂E is always closed.
- Show that $\partial([0, 1] \setminus F_\alpha)$ has measure greater than zero, where F_α is defined as above, and we take some α such that $0 < \alpha < 1$.

Proof.

- A boundary point x of E is clearly a point of closure of E, as the latter is just the condition that $B(x, \epsilon)$ contains points of E for any ϵ. Since closed sets contain all their points of closure this proves the claim.
- Suppose x is a point of closure of ∂E. Then for any $\epsilon > 0$ there is a point y of ∂E in $B(x, \epsilon/2)$. But then there are also points z_0 of E and z_1 of $X \setminus E$ in $B(y, \epsilon/2)$. By the triangle inequality, z_0 and z_1 are both in $B(x, \epsilon)$. This was true for all $\epsilon > 0$. So $x \in \partial E$.
- Clearly $\partial E = \partial(X \setminus E)$ for any $E \subset X$. So $\partial[0, 1] \setminus F_\alpha$ is ∂F_α; we show that $\partial F_\alpha = F_\alpha$ (which is interesting)! Now F_α is closed so $\partial F_\alpha \subset F_\alpha$; so we only need to show that for any $x \in \partial F_\alpha$ and any $\epsilon > 0$, there is a point of $[0, 1] \setminus F_\alpha$ which is in $B(x, \epsilon)$.

  To see this we estimate the sizes of each of the 2^k intervals of F_α^k. Each time we at least cut the intervals in two, so they are of size strictly less than $1/2^k$ by induction. So if we choose k such that $1/2^k < \epsilon/2$ then $B(x, \epsilon)$ contains the entire interval of F_α^k containing x; and when producing F_α^{k+1} from F_α^k we remove something from this interval, i.e. $B(x, \alpha)$ intersects $[0, 1] \setminus F_\alpha^{k+1} \subset [0, 1] \subset F_\alpha$.

$\square$