MAT320 Homework Answers For Final Prep

12/11/2023

1 Some Homework Answers and Some Practice Problems

Pset 3: Q5

There were lots of answers for this. The fact that $X \subset X$ is compact doesn't matter; when I assigned this problem I thought it might be a simplifying assumption for some of you.

Given a set $S \subset X$ write $S_\epsilon = \cup_{x \in S} B(x, \epsilon)$. Then for any $\epsilon > 0$, $S_\epsilon \supset S$ and S_ϵ is open. Moreover, if S is closed then $S = \cap_{n=1}^{\infty} S_{1/n}$. Indeed if $x \in S_{1/n}$ for all n then for every n, there is a point $x_n \in S$ such that $d(x, x_n) < 1/n$. But this exactly means that x is a limit point of S, so by the closure of S we have $x \in S$.

Set $S = U^c$; then S is closed. So by de Morgan's laws,

$$U = S^c = \left(\bigcap_{n=1}^{\infty} S_{1/n} \right)^c = \bigcup_{n=1}^{\infty} S_{1/n}^c.$$

Each $S_{1/n}^c$ is the complement of a closed set, so it's open; moreover $S_{1/n}^c \subset U$ since $U^c \subset S_{1/n}$.

HW6 Q2 (Royden 4.3.21) part b)

From a) of this problem, we know that is suffices to assume that f is a simple function. Lets call the interval $[a, b]$. Let's write

$$f = \sum_{i=1}^{k} a_i \mathbf{1}_{E_i}$$

where $E_i \subset [a, b]$ is some measurable set. Given E_i there is a bounded open set $U_i \supset E_i$ with $|\mu(U_i) - \mu(E_i)| < \epsilon'$. Every open subset of $\mathbb{R}$ is a countable union of open intervals, so we can write

$$\mu(U_i) = \sup_j \mu(V_i^j)$$

where $V_i^j \subset V_i$ is a union of the first j of these disjoint intervals comprising U_i, under some enumeration of them. Thus for sufficiently large j we have

$$|\mu(V_i^j) - \mu(U_i)| \leq \epsilon'.$$

By choosing j large enough this inequality will be satisfied for all i simultaneously since $|\mu(V_i^j) - \mu(U_i)|$ is monotonically decreasing to zero as $j \to \infty$.

Write

$$h = \sum_{i=1}^{k} a_i \mathbf{1}_{V_i^j}.$$

This is a step function since it is a finite sum of step functions. It has bounded support. By linearity of integration we have

$$\int |f - h| \leq \sum_{i=1}^{k} \int |a_i \mathbf{1}_{E_i} - a_i \mathbf{1}_{V_i^j}| = \sum_{i=1}^{k} |a_i| \int |\mathbf{1}_{E_i} - \mathbf{1}_{V_i^j}|$$

$$\leq \sum_{i=1}^{k} |a_i| \int |\mathbf{1}_{E_i} - \mathbf{1}_{U_i}| + |a_i| \int |\mathbf{1}_{U_i} - \mathbf{1}_{V_i^j}| \leq \sum_{i=1}^{k} |a_i| 2\epsilon'.$$

Since $E_i \subset U_i$ and $V_i^j \subset V_i^j$, the quantities being integrated are just $|\mu(V_i^j) - \mu(U_i)|$ and $|\mu(U_i) - \mu(E_i)|$ respectively (we are using the measurability of E here). We conclude by setting

$$\epsilon' = \frac{\epsilon}{2} \left(\sum_{i=1}^{k} |a_i| \right)^{-1}.$$

HW6 Q4

Consider a Cauchy sequence $f_i \in C^1([0,1])$. We wish to show that it converges. Since $\|g\|_{C^1} \geq \|g'\|_{C^0}$ and also $\|g\|_{C^1} \geq \|g\|_{C^0}$ for any $g \in C^1([0,1])$, this implies that the sequences $f_i \in C^0([0,1])$ and $f_i' \in C^0([0,1])$ are Cauchy, and so

$$f_i \to f, f_i' \to g \text{ in } C^0([0,1]).$$

We would like to show that $f' = g$ – that the derivative of the limit is the limit of the derivatives. One way to do this is to show that

$$f(x) = \int_0^x g(y) dy.$$

But this follows for the corresponding statements

$$f_i(x) = \int_0^x f_i'(y) dy.$$

because we can exchange Riemann integrals and uniform limits (this theorem is stated in the book; I sketched it in class.)

But now we are done: given $\epsilon > 0$, there are N_1, N_2 such that for all $i > N_1$ or $j > N_2$ we have

$$\|f_i - f\|_{C^0} < \epsilon/2, \|f_i' - f'\|_{C^0} < \epsilon/2$$

respectively. So then for $i > N = \max(N_1, N_2)$ we have $\|f_i - f\|_{C^1} < \epsilon$.

HW6 Q5 (Royden 13.1.6) part iii

We wish to show that if X is a normed linear space then addition and scalar multiplication are continuous. For addition this follows from part i immediately once we show that if $x_i \to x$ and $y_i \to y$ in X, then $(x_i, y_i) \to (x, y)$ in $X \times X$. The norm on $X \times X$ is the sum of the norms on the two coordinates. So we have

$$\|(x_i, y_i) - (x, y)\| = \|x_i - x\| + \|y_i - y\|;$$

we have $\|(x_i, y_i) - (x, y)\| \to 0$ if $\|x_i - x\| \to 0$ and $\|y_i - y\| \to 0$ as $i \to \infty$. This proves the desired statement for addition.

For multiplication we need to show that if $(a_i, x_i) \to (a, x)$ in $\mathbb{R} \times X$ then $a_i x_i \to ax$ in X. The former means that $a_i \to a$ in $\mathbb{R}$ and $x_i \to X$ in X. So then

$$\|a_i x_i - ax\| = \|a_i x_i - ax_i + ax_i - ax\| \leq \|a_i x_i - ax_i\| + \|ax_i - ax\| = |a_i - a|\|x_i\| + |a|\|x_i - x\|.$$

This then follows by properties we know about limits of sequences of real numbers, specifically that if $f_i \to f$ in $\mathbb{R}$ and $g_i \to g$ in $\mathbb{R}$ then $f_i g_i \to fg$ in $\mathbb{R}$.

HW7 Q1 part 4 and part 5 I believe the TA had comments for this.

HW7 Q2 The initial part can be found in `https://math.stackexchange.com/questions/589791/proving-holders-inequality-using-jensens-inequality` in the 3rd answer. (The other answers to this question are also pedagogically interesting!) The rest of it is proven in the book in Chapter 7.

HW6 Q4 Compute: for $a > 0$, $a \neq 1$,

$$\int x^{-a} = x^{a+1}/a, \text{ so } \int_1^\infty x^{-a} < \infty \text{ if } a > 1.$$

(For larger a, the value of the function decreases faster as $x \to \infty$, making the integral "more convergent".)

The case $a = 1$ has $\log x$ as an antiderivative and the corresponding integral $\int_1^\infty 1/x$ is infinite.

From this we can immediately produce all of our examples: if $1 < p < q < \infty$ then $\mathbf{1}_{[1,\infty)} x^{-\epsilon}$ is in L^p if and only if $p\epsilon > 1$. So we need to find ϵ so that $q\epsilon > 1$ but $p\epsilon < 1$; choosing $\epsilon = (1 + (q - p)/2)/q$ does the trick.

HW7 extra credit (in particular, how to show that $L^1([0,1]; g)$ is a Banach space)

There is a bijection

$$\phi : L^1([0,1]; g) \to L^1([0,1])$$

$$f \mapsto \phi(f) = fg.$$

This bijection has the property that

$$\|f\|_g = \|\phi(f)\|_{L^1([0,1])},$$

i.e. it preserves norms. Thus is f_i is a Cauchy sequence in $L^1([0,1]; g)$ then $\phi(f_i)$ is a Cauchy sequence in $L^1([0,1])$, and so converges to some $\phi(f)$ for some

$f \in \phi : L^1([0,1]; g)$; but since ϕ preserves norms this means that $f_i \to f$ in $L^1([0,1]; g)$. So $L^1([0,1]; g)$ is Banach.

HW8 Q1 part d and part g

d) Forwards direction: if the partial sums f_N converge to f in L^2, then their norms converge, so

$$\lim_N \|f_N\|^2 = \|f\| < \infty.$$

(Here we are using the L^2 norms.) By orthogonality of the functions e_i we have that

$$\|f_N\|^2 = \left\langle \sum_{i=1}^N a_i e_i, \sum_{j=1}^N a_j e_j \right\rangle = \sum_{i=1}^N \sum_{j=1}^N a_i a_i \langle e_i, e_j \rangle = \sum_{i=1}^N a_i^2.$$

Thus the partial sums of the squares of the a_i converge to some finite number.

Backwards direction: There are various arguments. Here's a straightforward one: if the sum of the squares of the a_i is finite, then it converges, so the sequence of partial sums the $|a_i|^2$ is a Caucy sequence; thus for any ϵ there exists an N such that for all $n > m > N$, we have that

$$\sum_{i=1}^n |a_i|^2 - \sum_{i=1}^m |a_i|^2 = \sum_{i=m+1}^N |a_i|^2 < \epsilon.$$

But we compute again using orthonormality that

$$\|f_n - f_m\|^2 = \left\| \sum_{i=m+1}^n a_i e_i \right\| = \sum_{i=m+1}^N |a_i|^2.$$

Since ϵ was arbitrarily small this implies that the partial sums f_N are a Cauchy sequence in L^2, and thus must converge.

g) The fact that if f is integrable then

$$\int_0^y f(x)dx$$

is continuous was proven in class (it follows from an easy application of dominated convergence; the book shows in fact that such a function is absolutely continuous.) So we just need to show that the partial sums of the term-wise integrals converge to this function (i.e. the formula in the problem set.) So we just need to exchange a limit and an integral, for each fixed y. We know that the partial sums converge $\sim_{i=1}^n a_i e_i$ converge in L^2 to something, so they converge in L^1 to something; so by general Lebesgue dominated convergence we can exchange the limit and the integral.

HW8 Q2 part g and part h There are many answers possible for either of these.

part g) We have that $S_N f \to f$ pointwise everywhere for continuous functions f with a continuous derivative. Note that $f \in L^2([0,1])$. We wish to show that they converge in L^2 as well. We know by Problem 1f that there is a bound $\|f\|^2 \geq \|S_N f\|^2$, so by Problem $1f$ we have that $S_N f \to g$ in L^2 for some $g \in L^2([0,1])$. But then g is the pointwise limit of $S_N f$ almost everywhere, so $g = f$ almost everywhere; so $g = f$ as elements of $L^2([0,1])$.

part h) Consider the sequence space

$$\ell^2 = \{(a_i)_{i\in\mathbb{Z}} \,|\, \sum_{i\in\mathbb{Z}} a_i^2 < \infty\}.$$

I said that you can use that this is a Banach space under the norm

$$\|(a_i)\|_{\ell^2} = \sqrt{\sum_{i\in\mathbb{Z}} |a_i|^2}$$

(proving that this is true is a very good exercise!) Problem 1 shows that there is a map

$$\phi : \ell^2 \to L^2([0,1]), (a_i) \to \phi((a_i)) = a_0 + \sum_{i=1}^{\infty} a_i e_i + a_{-i} f_i$$

which *preserves the norms*: a direct computation shows that

$$\|(a_i)\|_{\ell^2} = \|\phi((a_i))\|_{L^2([0,1])}$$

if all but finitely many of the a_i are zero, and thus (since the norms are continuous, and the subset

$$\ell^2_f \subset \ell^2, \ell^2_f = ((a_i) \in \ell^2, \text{ all but finitely many of the } a_i \text{ are zero}\}$$

is dense; so by continuity of both norms we have that ϕ preserves the norms for any element $(a_i) \in \ell^2$. Now, an element is in the image of ϕ exactly when its partial Fourier sums converge in L^2 to itself. (The image of ϕ is the set of functions expressible as a Fourier series, and if a function is expressible as a Fourier series, then its Fourier coefficients are the ones you expect to get by Problem 1 c.) Suppose you have a sequence of functions $f^i = \phi((a_i^j)) \in Im(\phi)$ which has a limit in $L^2([0,1])$. We need to show that this limit is in $Im(\phi)$. But since f^i has a limit in $L^2([0,1])$ then it is a Cauchy sequence, i.e. for all $\epsilon >$ there is an N such that for all $n, m > N$ we have

$$\|f^n - f^m\|_{L^2([0,1])} = \|(a_i^n) - (a_i^m)\|_{\ell^2} < \epsilon;$$

so the vectors $a^n = (a_i^n)$ in ℓ^2 are Cauchy, so they have a limit $a = (a_i)$. Setting $f = \phi(a)$ we have that $a^n \to a$ in ℓ^2, so $\phi(a^n) = f^n \to \phi(a) = f$ by the fact that ϕ preserves norms.

Another possible argument is to show using a triangle inequality that if $f^i \to f$ in L^2 with f^i continuously differentiable, then since the partial Fourier sums of f_i converge to f_i then the same holds for the partial fourier sums of f. E.g.: We have by the triangle inequality that

$$\|S_N f - f\| \leq \|S_N f - S_N f^i\| + \|S_N f^i - f^i\| + \|f^i - f\|.$$

Now problem $1f$ implies that

$$\|S_N f - S_N f^i\| = \|S_N(f - f^i)\| \le \|f - f^i\|.$$

So

$$\|S_N f - f\| \le 2\|f^i - f\| + \|S_N f^i - f^i\|$$

For any $\epsilon > 0$ we can find an i such that $\|f^i - f\| < \epsilon/4$, and then find an N such that $\|S_N f^i - f^i\| < \epsilon/2$; so we are done.

HW9 Q3 part b and part c I will post an answer to b. (Nothing like c will be tested on the exam.)

Concretely, we need to show that, given a sequence of measurable functions $Y_i : \Omega \to \mathbb{R}$ that are measurable with respect to a σ-algebra Σ on Ω (so $Y_i^{-1}(A) \in \Sigma$ for all i and all Borel subsets $A \subset \mathbb{R}$), then the set

$$S = \{x \in \Omega : \lim_{n\to\infty} \sum_{i=1}^{n} Y_i(x) = c\} \subset \Omega$$

is an element of Σ. This argument is extremely similar to the argument in the book that a pointwise limit of measurable functions is measurable. Indeed, if the pointwise limit $\lim_{n\to\infty} \sum_{i=1}^{n} Y_i(x)$ existed for all x, then defining Y to be this pointwise limit, we would just be looking at $Y^{-1}(\{c\})$, which is is in Σ if Y is measurable with respect to Σ.

However this pointwise limit may not exist for many $x \in \Omega$; we will thus give a direct argument. What does it mean that

$$\lim_{n\to\infty} \sum_{i=1}^{n} Y_i(x) = c?$$

It means that for all $\epsilon > 0$ there exists an N such that for all $n > N$, the

$$\sum_{i=1}^{n} Y_i(x) \in [c - \epsilon, c + \epsilon].$$

Now the set $S_{\epsilon,n}$ of x such that the above condition holds is an element of Σ, since a finite sum of measurable functions is measurable (the argument from the book immediately generalizes to this case) and the set $[c - \epsilon, c + \epsilon]$ is Borel. So we need to translate the "for all ϵ there exists an N" into measurable language terms. Given N, the set of points which are in *all* $S_{\epsilon,n}$ for all $n > N$ is

$$\cap_{n>N} S_{\epsilon,n}.$$

We have a a condition for all $\epsilon > 0$, so we will have an intersection some sets with a parameter $1/z$, $z = 1, 2, 3....$ How do we articulate that there *exists* an N? Well, thats a *union.* Consider

$$\bigcap_{z=1}^{\infty} \left(\bigcup_{N=1}^{\infty} \left(\cap_{n>N} S_{1/z,n}\right) \right)$$

If x is in the above set that means that for all z, there exists an N such that for all $n > N$, $x \in S_{1/z,n}$. But that's exactly what we want! So the above quantity is exactly the set S, and it is clearly in Σ since each $S_{1/z,n}$ is in Σ.

2 Some Practice Problems

Chapter 1 1.3.26, 1.4.33, 1.5.42, 1.6.48, 1.6.55

Chapter 2 2.4.18, 2.4.19, 2.5.27, 2.7.35, 2.7.45

Chapter 3 3.1.1, 3.2.14, 3.2.22

Chapter 4 4.3.22, 4.3.26, 4.5.37, 4.6.47, (challenge) 4.6.48.

Chapter 6 6.2.15, 6.3.29, 6.6.62, 6.6.68,

Chapter 7 7.2.21 is *very* good for building some intuition but is a little tricky. 7.3.34, 7.4.41, 4.4.48 (also a bit hard but very insightful).

Chapter 9 Most of the basic problems on open/closed sets are good review. A nice exercise to make sure you really undertand the definitions is thinking through 9.4.49. Another nice one is 9.5.68.

Chapter 10 This is mostly to make sure you understand something about $C^0([a, b])$. 10.1.5, 10.1.12,

The first few problems of Chapter 13.2 may also be of interest.