MAT320 Practice Problems

10/3/2023

Problem. A *rectangle* is a subset $(a,b)\times(c,d)\subset\mathbb{R}^2$ where $b>a$ and $d>c$. We write

$$l\,(\ (a,b)\times(c,d)\ ) = (b-a)\times(c-d).$$

For a subset $S\subset\mathbb{R}^2$, define

$$\mu^*(S) = \left\{\sum_{i=1}^{\infty}\ell(R_i)\,|\,R_i \text{ a rectangle for } i=1,\ldots; S\subset\bigcup_{i=1}^{\infty}R_i\right\}.$$

For $a=(a_1,a_2)\in\mathbb{R}^2$ and $r>0$, write

$$B(a,r)=\{(x_1,x_2)\in\mathbb{R}^2|\sqrt{(x_1-a_1)^2+(x_2-a_2)^2}\}.$$

Prove that $\mu^*(B(a,r)) = r^2\mu^*(B((0,0),1))$.

Problem.
Find the measure of the set of all numbers $x\in[0,1]$ for which there are no 4s in any decimal expansion of x.

Problem. Let A_i be subsets of a metric space X for $i=1,2,\ldots,$. Prove that if $B=\cup_{i=1}^{\infty}A_i$ then $\overline{B}\supset\cup i=1^{\infty}A_i$. Show that the latter inclusion does not have to be an equality.

Problem. An F_δ set is a set that is a countable union of closed sets. Show that continuous functions map F_δ sets to F_δ sets.

Problem. Problem 2.4.19.

Problem. Show that a function $f:\mathbb{R}\to\mathbb{R}$ that is continuous except at a finite number of points is measurable. (On the quiz, I would remind you of the definition of a measurable function.)

Problem. Is the subset

$$\bigcup_{n=2}^{\infty}[1/n,1-1/n]\times\{1/n\}\subset\mathbb{R}^2$$

compact?