MAT320 Problem Set 1

Due Sept 14, 2023

Please write your homework on paper neatly or type it up in LaTeX, and hand it in at the beginning of class next Thursday.

Royden $X.Y.Z$ refers Problem Z in Royden-Fitzpatrick, found in the collection of problems at the end of section $X.Y$.

Problem 1. Show that a composition of surjective maps is surjective, a composition of injective maps is injective, and a composition of bijective maps is bijective. Give an example of maps f and g such that

- f is surjective, g is injective, and $f \circ g$ is neither,
- f is injective, g is surjective, and $f \circ g$ is neither.

Problem 2. Let S and S' be subsets of the real number which are both bounded above. Show that if $S \subset S'$ then $\sup(S) \leq \sup(S')$.

Problem 3. Royden 1.1.3.

Problem 4. From the axioms of the real numbers, show that between every pair of real numbers there is a rational number, and also between every pair of real numbers there is an irrational number. (Feel free to use the fact that there *is* an irrational number, like $\sqrt{2}$). Hint: use the Archimedean property of the reals!

Problem 5. Royden 1.2.13. Feel free to use any theorem through section 1.3 of of Royden.
(Hint: Problems 2 and 4 may be helpful!)

Problem 6. *Equivalence relations.* Verify, from the axioms, that the following are equivalence relations:

- The equivalence relation E_1 on $\mathbb{Z}$ given by $x \sim_{E_1} y$ if x has the same parity as y (i.e. $x \equiv y \pmod 2$), and $x \equiv y \pmod 3$.

- Let X be some set, and let 2^X be the set of subsets of X. The equivalence relation E_2 on 2^X given by $S \sim_{E_2} T$ if there is a bijection from S to T. (Hint: Problem 1!)
- Let $f : X \to Y$ be a function. The relation E_3 on X given by $x \sim_{E_3} y$ if $f(x) = f(y)$.

Show that the following is *not* an equivalence relation:

- The relation $<$ on $\mathbb{Q}$ given by $x < y$ if x is less than y.
- The relation E_4 that on $\mathbb{Q}$ that $x \sim_{E_4} y$ if $x \geq 5$.

Extra credit, for those that want to work on their proof-writing skills. Given an equivalence relation E_5 on some set X, we define a new set $X/{\sim_{E_5}}$ by

$$X/{\sim_{E_5}} = \left\{ S \in 2^X \;\middle|\; \begin{array}{c} S \neq \emptyset, \text{ and} \\ (\text{for all } x \in S, y \in X, , \text{ if } x \sim_{E_5} y \text{ then } y \in S), \\ \text{and (for all } x \in S, y \in S, \text{ we have } x \sim_{E_5} y). \end{array} \right\}.$$

It takes a bit of work to show (although you will have essentially shown this in the previous problem) that the elements of $X/{\sim_{E_5}}$ are *mutually disjoint*: namely, if $S_1, S_2 \in X/{\sim_{E_5}}$, then

$$\text{either } S_1 = S_2 \text{ or } S_1 \cap S_2 = \emptyset.$$

The elements of $X/{\sim_{E_5}}$ are called the *equivalence classes* with respect to E_5. Moreover,

$$\cup_{S \in X/\sim_{E_5}} S = X.$$

Therefore, there is a well defined function $f_{E_5} : X \to X/{\sim_{E_5}}$ for which $f_{E_5}(x)$ is the *unique* element $S \in X/{\sim_{E_5}}$ such that $x \in S$.

The set $X/{\sim_{E_5}}$ is called the *quotient* of X by the equivalence relation E_5, and the function f_{E_5} is called the *quotient map.*

Prove the above claims about elements of $X/{\sim_{E_5}}$. Prove that the above function f_{E_5} is well-defined. Show that for any other function

$$g : X \to Y$$

such that if $x \sim_{E_5} y$ then $g(x) = g(y)$, there is a *unique* function $\bar{g} : X/{\sim_{E_5}} \to Y$ such that

$$\bar{g} \circ f_{E_5} = g.$$