MAT320 Problem Set 9

Due Dec 7, 2023

Please write your homework on paper neatly or type it up in LaTeX, and hand it in at the beginning of class next Thursday. For us, *integrable* always means *Lebesgue integrable* unless otherwise specified.

Throughout this problem set, let (Ω, Σ, μ) *be a* ***probablility triple****: namely,* Ω *is a set,* Σ *is a* σ*-algebra on* Ω*, and* $\mu : \Sigma \to \mathbb{R}$ *is a probability measure.* We will sometimes refer to elements of Σ as *events*, and we may denote $\mu(A)$ by $\mathbb{P}(A)$, since this is the *probability that* A *occurs.*

Problem 1. (Converse Borel-Cantelli Theorem.)

We will prove the following statement: given events A_i which are *mutually independent* such that

$$\sum_i \mathbb{P}(A_i) = \infty,$$

then infinitely many of the A_i will occur with probability 1, i.e. writing

$$\text{“}\{A_i\}\text{i.o.”} := \bigcap_{n=1}^{\infty} \bigcup k = n^{\infty} A_k$$

then

$$\mathbb{P}\left(\{A_i\}\text{i.o.}\right) = 0.$$

a) Let's show, equivalently, that

$$1 - \mathbb{P}\left((\{A_i\}\text{i.o.})^c\right) = 0,$$

where the c denotes taking complement in Ω. Show using de-Morgan's laws and monotonicity of measure that

$$\mathbb{P}((\{A_i\}\text{i.o.})^c) \leq \sum_{n=1}^{\infty} \mathbb{P}\left(\bigcap_{k=n}^{\infty} (A_k^c)\right)$$

b) The next step is to show that each term in the sum on the right hand side above is zero. Show, using mutual independence of the A_i, that for any m,

$$\mathbb{P}\left(\bigcap_{k=n}^{\infty} (A_k^c)\right) \leq \prod_{i=1}^{n+m} \mathbb{P}(A_i^c). \tag{1}$$

c) Show that $1 - p \leq e^{-p}$ for $p \in [0, 1]$. Conclude that

$$\prod_{i=1}^{n+m} (1 - \mathbb{P}(A_i)) \leq \prod_{i=1}^{n+m} \exp(-\mathbb{P}(A_i))$$

goes to zero as $m \to \infty$ and thus that the quantity on the left hand side of (1) is zero. Finish the proof of the statement claimed.

Problem 2. (Application of Borel-Cantelli converse)

Let us imagine flipping an infinite number of independent coins such that the i-th coin flips heads with probability $p_i \in [0, 1]$. This can be made rigorous sense of as follows: the Caratheodory extension theorem lets you construct a probability measure on the the set

$$\Omega = \{0, 1\}^\infty$$

with σ-algebra Σ that is generated by the collection of subsets

$$\bigcup_n P(\{0, 1\})^n \times \{0, 1\}^\infty \subset \{0, 1\}^\infty$$

(where $P(S)$ is the powerset of S) such that, writing

$$X_i((a_1, a_2, a_3, \ldots,)) = a_i \in \{0, 1\}$$

we interpret the event $\{X_i = 1\}$ as the event that the i-th coin flips heads, and we have that

$$\mathbb{P}((X_{i_1} = 1) \cap (X_{i_2} = 1) \cap \ldots \cap (X_{i_n} = 1)) = p_{i_1} p_{i_2} \ldots p_{i_n}.$$

a) Suppose that we have a sequence of biased but independent coin flips where the n-th coin flip will be heads with probability $1/n$. Show that nevertheless, heads will be flipped infinitely often with probability 1.

b) Now suppose they are even more biased, i.e. the n-th coin flip will be heads with probability only $1/n^2$. Show that with probability 1, only finitely many of the coins will ever flip heads! This is true *even though there is still a positive probability of flipping heads* for each of the infinitely-many coin flips!

Problem 3. (Options Payoffs.) An *option* is a kind of bet, specified by a payoff function $f(x)$. Buying an option on X from someone (where X is something like the difference between the price of a bushel of wheat in 6 months and the price of a bushel of wheat now) entitles you to get they payoff $P_f(X) = \max(X, f(X))$. Usually, X is something "random"; in this previous example with bushels of wheat, you have to wait 6 months to find out the value of X, but you want to buy this option from someone (like Goldman Sachs) then you have to buy it *today*.

A standard example is a *call option*, which is defined by

$$f(x) = \max(x, c).$$

A call option never pays more than the underlying X if $X \geq c$, but protects you from downside if $X < c$. Here c is called the *strike price*; for simplicity we will study the case where $c = 0$.

Remember that a *random variable* is a Borel-measurable function $X : \Omega \to \mathbb{R}$, i.e. $X^{-1}([a, b]) \in \Sigma$ for any a, b. The *expectation value* of X is

$$\mathbb{E}[X] = \int_\Omega X d\mu.$$

a) Show that for a call option any strike price,

$$\mathbb{E}[P_f(X)] \geq \mathbb{E}[X],$$

no matter what the distribution of X is. Show in fact that this happens whenever the function $P_f(x)$ is *convex*. Show that this follows from f being convex. Thus, *convexity has value*, and you are usually charged for it.

b) The *strong law of large numbers* for independent and identically distributed random variables $\{Y_i\}_{i=1}^\infty$ with $\mathbb{E}[Y_i] < \infty$, says that with probability 1,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n Y_i = \mathbb{E}[Y_1].$$

Here a *random variable* is a Borel-measurable function $Y_i : \Omega \to \mathbb{R}$. You do not have to prove this in the problem set, although it is easy to show with the tools that we have. Show that the set

$$\left\{ x \in \Omega : \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n Y_i(x) = \mathbb{E}[Y_1] \right\}$$

is in Σ, the σ-algebra of events, so the statement of the strong law of large numbers (namely, that the probability of the above event is 1) is well defined.

c) Say $X_i \sim N(a, b)$ are a sequence of independent and identically distributed normal random variables with mean a and variance b. Show that it is possible to find $a > 0$, $b > 0$, and a positive price d for a call option with strike price zero, such that almost surely,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n (P_f(X_i) - d - X_i) = A > a \tag{2}$$

but, writing $g(y)$ for the function which is 1 if $y < 0$ and otherwise $g(0) = 0$, that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n g(P_f(X_i) - d) = B > 1/2. \tag{3}$$

Imagine that X_i is the profit you get by buying a bushel of wheat on month i and selling it on month $i+1$; since $a > 0$ this means that most of the time you make money if you just buy wheat and then sell it a month later. The above (2) shows that buying wheat *and* the call option for price d every time will cause you to make *more* money in the long run than just buying the wheat; however, by (3) if you do this, then on *most months* you will actually *lose money* rather than making money! Hedging the risk that $X_i < 0$ turns out to be a good idea, although it may take some psychological determination to keep doing it for many losing months.

This is a strange psychological aspect of making bets that have asymmetric, convex payoffs: you might look like you are doing badly most of the time, even though in the long run you will come out ahead, because occasionally your payoff will make up for your small intermediate losses.

Feel free to use a calculator to pick the numbers a, b, d!