MAT320 Uniform Continuity Lecture Notes

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Earlier, we discussed how to characterize compact subsets of $\mathbb{R}^n$. Heine-Borel explains that these are exactly the closed subsets which are bounded (i.e. they fit in a big ball of finite radius).

We can also consider infinite dimensional real vector spaces, e.g.

$$\mathbb{R}^\infty = \{(x_i)_{i=1}^\infty : x_i = 0 \text{ for } i >> 0.\}$$

This space has a natural metric

$$d((x_i), (y_i)) = \sqrt{\sum_i (x_i - y_i)^2}.$$

In fact, it has *many* natural metrics; we will go into this later.

Similarly, there are function spaces like the space of all continuous functions

$$C^0([0,1]) = \{f : [0,1] \to \mathbb{R} | f \text{ continuous}\}$$

and these tend to have natural metrics. In this latter space we will take the *sup*-norm metric

$$d(f,g) = \sup_{x \in [0,1]} |f(x) - g(x)|.$$

(This makes sense because continuous functions achieve their minima and maxima on compact sets.) Actually this latter space makes sense for any compact metric space X, i.e there is $C^0(X)$ where we replace $[0,1]$ with X in the definitions above everywhere. We can drop the condition that X is compact by just requiring this to be the set of such continous functions f such that $d(f,0) < \infty$.

We can ask: what are the compact subsets of vector spaces like *these*? This is very useful in numerical analysis – if we optimize a continuous function over a compact subset of a function space we will know that the optimum exists.

In this lecture we will answer this question for the metric space $C^0([0,1])$ above.

Let (X, d_X) and (Y, d_Y) be metric spaces. One way of saying that a function $f : X \to Y$ is continuous is to state that there exists a monotonically nondecreasing function $\omega_f : X \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that

$$d_Y(f(x) - g(y)) \leq \omega_f(x, d_X(x,y)),$$

$$\lim_{r\to 0} \omega_f(x,r) = 0 \text{ for all } x \in X.$$

Indeed we can pick $\omega_f(x,\delta)$ to be the ϵ in the ϵ-δ definition.

Definition 1. *A function $f : X \to Y$ is* uniformly continuous *when we can fund a function ω_f as above which is* independent of x*, i.e. $\omega_f(x,r) = \omega_f(r)$. The function ω_f is called a modulus of continuity of f.*

For example, if $X = Y = \mathbb{R}$ and f is differentiable with continuous derivative $|f'(x)| < C$ then we have that

$$|f(x) - f(y)| \leq C|x - y|.$$

Any function satisfying the inequality above is called *Lipshitz*; these functions are always continuous, but don't have to be differentiable (consider $f(x) = |x|$.) Here $\omega_f(r) = Cr$.

Example of a continuous function which is not uniformly continuous: $f : (0,1) \to \mathbb{R}$, $f(x) = 1/x$.

Relatedly, a sequence of functions $f_i : X \to \mathbb{R}$ converges *pointwise* to a function $f : X \to \mathbb{R}$ if $\lim_{i\to\infty} f_i(x) = f(x)$ for each $x \in X$, while it converges *uniformly* if for every ϵ there is an N (independent of $x \in X$) such that for all $n > N$, $|f_n(x) - f(x)| < \epsilon$ for all $x \in X$ at once. It is an exercise that sequences of functions $f_i \in C^0(X)$ converge in this metric space when they converge uniformly to some continuous function.

However,

Proposition 1. *Any continuous $\mathbb{R}$-valued function on a compact metric space is uniformly continuous.*

Proof. We are given a function $\omega_f(x,r)$ as above and we want to produce a new function $\omega'_f(r)$ which is independent of x. Well we know that that for $\epsilon > 0$ and any $x \in X$ we can find r_x such that $\omega_f(x, r_x) < \epsilon$. Thus consider the open cover $\{B(x, r_x/2)\}_{x\in X}$ of X; this has a finite subcover with indices $x_1, \ldots, x_k$. So for $y \in X$, $y \in B(x_i, r_{x_i}/2)$, and $|f(y) - f(x_i)| < \epsilon$. Set $r = \min_i r_{x_i}/2$; then for any $z \in B(y, r/2) \subset B(x_i, r)$

$$|f(z) - f(y)| \leq |f(x_i) - f(y)| + |f(z) - f(x_i)| \leq 2\epsilon.$$

But y was arbitrary. So we can set

$$\omega'_f(r/2) = 2\epsilon.$$

$\square$

Intuitively, compactness tells us lets us reduce the infinite number of conditions at each scale ϵ defining continuity to a finite number of conditions at each scale ϵ, and take the weakest condition in each of those finite collections.

Back to compactness. Both $\mathbb{R}^\infty$ and $C^0([0,1])$ are *normed spaces*: they are vector spaces V with metrics which come from *norms*, i.e. functions $\|\cdot\| : X \to \mathbb{R}$ such that

- $\|x\| \geq 0$,
- $\|cx\| = |c|\|x\|$ for $x \in V$, $c \in \mathbb{R}$, and
- $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality).

Then the metric is $d(x, y) = \|x - y\|$.

Proposition 2. *The closed unit ball in an infinite dimensional real normed vector space is not compact.*

This is in the book but I'll just prove these two cases. On one hand, for $\mathbb{R}^\infty$, we can choose the sequence

$$e_i = (0, \ldots, 0, 1, 0 \ldots) \in B(0, 1)$$

and we have that $|e_i - e_j| = \sqrt{2}$ if $i \neq j$, so this is a sequence in the closed unit ball which has no subsequence which is Cauchy. Thus in particular this sequence has no convergent subsequences.

Similarly, for $C^0([0, 1])$ we can take

$$f_n(x) = \sin(2\pi nx) \in B(0, 1).$$

The reason that no subsequence is Cauchy is because if we chose a subsequence f_{n_k}, then if we look at a fixed $f_{n_{k_0}}$ then for $k_1 >> k_0$ the function $f_{n_{k_1}}$ will oscillate a million times in the time it takes $f_{n_{k_0}}$ to oscillate once, so the difference in sup norms $d(f_{n_{k_0}}, f_{n_{k_1}})$ will converge to 2 as $k_1 \to \infty$ for every k_1.

So nothing like Heine-Borel works anymore. What do we do instead? Well, the idea is that the problem with the f_n is that they 'oscillate more and more as $n \to \infty$'. Let's solve this by controlling the rate of oscillation; one way to measure this is in terms of the function ω_f described earlier.

Definition 2. *Let X be a metric space. A set of functions $E \subset \{f : X \to \mathbb{R}; f$ cont. $\}$ is* equicontinous *of for each $f \in E$ we can pick $\omega_f(r)$ to be a function $\omega(r)$ that is* independent of f.

A good example is the set of functions in $C^0([0, 1])$ of fixed lipshitz constant. Note that the sequence f_n above is not equicontinous! But we could chose $\omega_f(x) = C|x - y|^\alpha$ instead for fractional $\alpha > 1$; this would constitute a bound on a "Hölder norm" of f. We will dicuss these later when we talk more about Banach spaces.

Theorem 1 (Arzela-Ascoli). *Let X be a compact metric space. A set $E \subset C^0(X, \mathbb{R})$ is compact if and only if it is closed, bounded (i.e. $E \subset B_{C^0(X,\mathbb{R})}(0, R)$ for some R), and equicontinuous.*

Proof. We will only prove one implication, the one implying continuity. The idea is just like the proof of the proposition that continuous functions are uniformly continous, but now we have control on small balls on how much the functions in E can oscillate.

Let $f_i \in E$ be a sequence. In a homework problem we proved that E has a contable dense subset $D = \{x_j\}$. We will first find a subsequence $g_i = f_{k_i}$ of f_i such that $\lim_{i\to\infty} g_i(x_j) =: g(x_j)$ exists for each $x_j \in D$. This just requires boundedness: for each each x_j, the sets $\{f_i(x_j)\}_{i=1}^{\infty}$ are sequences in $[-R, R]$ (by the definition of the sup norm and the boundedness of E) so subsequences of these sequences converge. We then construct g_i via a diagonal argument from these subsequences.

Now we want to show that (a) the function $D \to \mathbb{R}$, $x_j \to g(x_j)$ defined this way is continuous in the subspace metric on D, and thus extends automatically to a continous function $g : X \to \mathbb{R}$; and subsequently that a subsequence of g_i (and thus of f_i) converges to this g everywhere.

Let's check that $g : D \to \mathbb{R}$ is continuous. For $x_i, x_j \in D$, we must have that $|g(x_i) - g(x_j)| < \omega(d_K(x_i, x_j))$ since absolute value is taking differences is continous, $\lim_k g_k(x_i) = g(x_i)$, the corresponding inequality is satisfied with $g \to g_k$ for all k, and the right hand side is independent of k.

So g extends to a continuous function on X with the same modulus of continity ω. Well, for every $\delta > 0$ we can find finitely many $x_1, \ldots, x_{k_\delta} \in X$ such that X is covered by $\{B(x_j, \delta)\}_{j=1}^{k_\delta}$. Given $\epsilon > 0$, we choose δ such that $\omega(\delta) < \epsilon/3$. So then for $y \in X$, $y \in B(x_i, \delta)$ for some x_i, and

$$|g(y)-g_j(y)| \leq |g(y)-g(x_i)|+|g(x_i)-g_j(x_i)|+|g_j(x_i)-g_j(y)| \leq 2\epsilon/3+|g(x_i)-g_j(x_i)|.$$

But we know that there is an N such that for $j > N$, $|g(x_i)-g_j(x_i)| < \epsilon/3$ for any one of the finitely many x_i (take the maximum of the Ns for the corresponding sequences). So we have that g_j converves uniformly to g, i.e. $g_j \to g$ in $C^0(X)$. □

We won't show this in class, but an easy application of the mean value theorem proves that the set

$$\{f : [0,1] \to \mathbb{R} : |f(x)-f(y)| < C \text{ for any pair} x, y \in [0,1], f(0) < D\} \subset C^0([0,1])$$

is closed, bounded, and equicontinous; so this is an 'infinite dimensional' compact metric space.