MAT320 Practice Problems

10/3/2023

Problem. A *rectangle* is a subset $(a,b) \times (c,d) \subset \mathbb{R}^2$ where $b > a$ and $d > c$. We write
$$l\,(\,(a,b) \times (c,d)\,) = (b-a)(d-c).$$
For a subset $S \subset \mathbb{R}^2$, define
$$\mu^*(S) = \left\{ \sum_{i=1}^{\infty} \ell(R_i) \,|\, R_i \text{ a rectangle for } i = 1, \ldots; S \subset \bigcup_{i=1}^{\infty} R_i \right\}.$$
For $a = (a_1, a_2) \in \mathbb{R}^2$ and $r > 0$, write
$$B(a,r) = \{(x_1,x_2) \in \mathbb{R}^2 | \sqrt{(x_1-a_1)^2 + (x_2-a_2)^2}\}.$$
Prove that $\mu^*(B(a,r)) = r^2 \mu^*(B((0,0),1))$.

Problem.
Find the measure of the set of all numbers $x \in [0,1]$ for which there are no 4s in any decimal expansion of x.

Problem. Let A_i be subsets of a metric space X for $i = 1, 2, \ldots,$. Prove that if $B = \cup_{i=1}^{\infty} A_i$ then $\overline{B} \supset \cup_{i=1}^{\infty} A_i$. Show that the latter inclusion does not have to be an equality.

Problem. An F_δ set is a set that is a countable union of closed sets. Show that continuous functions $f : \mathbb{R} \to \mathbb{R}$ map F_δ sets to F_δ sets.
(Hint: show first that the image of a compact set under a continuous function is compact.)

Problem. Problem 2.4.19.

Problem. Show that a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous except at a finite number of points is measurable. (On the quiz, I would remind you of the definition of a measurable function.)

Problem. Is the subset
$$\bigcup_{n=2}^{\infty} [1/n, 1-1/n] \times \{1/n\} \subset \mathbb{R}^2$$
compact? Here we use the standard distance metric on $\mathbb{R}^2$ given by
$$d((v_1,v_2),(w_1,w_2)) = \sqrt{(v_1-w_1)^2 + (v_2-w_2)^2}.$$