

MAT320 Problem Set 8

Due Nov 30, 2023

Please write your homework on paper neatly or type it up in LaTeX, and hand it in at the beginning of class next Thursday. For us, *integrable* always means *Lebesgue integrable* unless otherwise specified.

Problem 1. Recall that for $f, g \in L^2([0, 1])$ we set

$$\langle f, g \rangle = \int f(x)g(x)dx.$$

Thus $\langle f, f \rangle = \|f\|_{L^2}^2$. Suppose that we have vectors $e_i \in L^2([0, 1])$, $i = 1, 2, \dots$, such that

$$\langle e_i, e_j \rangle = \delta_{ij}$$

where $\delta_{ij} = 1$ if $i = j$ and otherwise $\delta_{ij} = 0$.

a) Set

$$e_0 = 1 \in L^2([0, 1]),$$

$$\tilde{e}_n = \cos(2\pi nx) \in L^2([0, 1]), n = 1, 2, \dots$$

$$\tilde{f}_n = \sin(2\pi nx) \in L^2([0, 1]), n = 1, 2, \dots$$

What are positive constants c_n, d_n such that if we set $e_n = c_n \tilde{e}_n$, $f_n = d_n \tilde{f}_n$ then $\|e_n\|_{L^2} = \|f_n\|_{L^2} = 1$? You can use tables of antiderivatives from calculus or from the internet. It may be interesting for you to check for yourself that

$$\langle e_n, f_m \rangle = 0, \langle e_n, e_m \rangle = \langle f_n, f_m \rangle = \delta_{nm}$$

where $\delta_{nm} = 1$ if $n = m$ and otherwise $\delta_{nm} = 0$. (δ_{nm} is called the *Kronecker symbol*.)

Thus we can think of $\{e_0, e_1, f_1, e_2, f_2, \dots\}$ as a collection of vectors in $L^2([0, 1])$ which are pairwise orthogonal and each have length 1.

b) Suppose that

$$f(x) = \sum_{i=1}^{\infty} a_i e_i(x) \tag{1}$$

where *all but finitely many of the a_i are **zero***, so that the above sum is actually a finite sum. Show that we can recover a_n

$$a_n = \int_0^1 f(x) e_n(x). \quad (2)$$

- c) We prove in class that $L^2([0, 1]) \subset L^1([0, 1])$. Show that if a sequence of functions $f_i \in L^2([0, 1])$ converge to $f \in L^2([0, 1])$ in L^2 , then they converge in $L^1([0, 1])$. Using the “General Lebesgue Dominated Convergence Theorem” in Chapter 4, show that if (1) holds in the L^2 sense, that is, in the sense that for some sequence of numbers $a_i \in \mathbb{R}$, the partial sums

$$f_N(x) = \sum_{i=1}^N a_i e_i(x). \quad (3)$$

converge in L^2 to f , then

$$\int_0^1 f_N e_n \rightarrow \int_0^1 f e_n.$$

Conclude that (2) continues to hold whenever (1) holds in the sense described above.

- d) Show that if (1) holds in the sense described above then

$$\sum_{i=1}^{\infty} |a_i|^2 < \infty.$$

Conversely, show that if the above holds then the partial sums f_n (3) converge to some function in L^2 . (Restrict the summation to a big ‘box’ of indices, and think about the monotone convergence theorem.)

- e) Suppose that we try to differentiate the series (1) term by term with respect to x . Does that imply that the series

$$g(x) = \sum_{i=1}^{\infty} a_i \frac{d}{dx} e_i(x) \quad (4)$$

converges? The previous problem should come in handy here.

- f) Now, let us not assume that (1) holds in the L^2 -sense. Instead, let us simply define a_n via (2). Then we can define

$$T_n f(x) = \sum_{i=1}^n a_i e_i(x).$$

Show that

$$\langle T_n f, f - T_n f \rangle_{L^2} = 0.$$

Using Cauchy-Schwartz, show that

$$\sum_{i=1}^{\infty} |a_i|^2 \leq \|f\|_{L^2}^2 < \infty.$$

(Hint: think of $T_n f(x)$ as the projection of f_n to the linear subspace of $L^2([0, 1])$ spanned by $\{e_0, \dots, e_n\}$. This is just the statement that the length of a projection of a vector is bounded by the length of the original vector. Feel free to take a look at linear algebra textbooks remember how this proof goes!)

- g) **Extra credit.** Show that if $f(x)$ as in (1) is well defined in the sense that the partial sums converge in L_2 to some function, then the sum of the term-by-term integrals converges pointwise to a continuous function, and we have

$$\int_0^y f(x) dx = \sum_{i=1}^{\infty} a_i \int_0^y e_i(x) dx$$

for all y .

Problem 2. In this problem we will redo a variant of Dirichlet's proof of the L^2 convergence of Fourier series. Interestingly, this result is rarely proven outside of relatively advanced math classes, even though the rigorous proof is not really harder than the non-rigorous proofs that are often presented.

This proof will use the complex numbers. Hopefully, you are familiar with Euler's identity

$$e^{ix} = \cos(x) + i \sin(x);$$

if not we will take it on faith.

Our first step will be to massage the formulae of the previous problem.

We will set

$$a_n = \int_0^1 f(x) \cos(2\pi n x) dx$$

$$b_n = \int_0^1 f(x) \sin(2\pi n x) dx$$

and the analog of the Fourier series with this parameterization is

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n 2 \cos(2\pi n x) + b_n 2 \sin(2\pi n x). \quad (5)$$

a) If

$$f(x) = \tilde{a}_0 + \sum_{i=1}^{\infty} \tilde{a}_n e_n + \tilde{b}_n f_n,$$

holds in the L^2 sense, as does (5), where e_n and f_n are as in the previous problem, give expressions for \tilde{a}_i and \tilde{b}_i in terms of a_i and b_i . (Note that $e_n(x) \neq$

$2 \cos(2\pi nx)$ – instead, these two functions are only proportional. What is the proportionality constant?)

We will write

$$S_N f(x) = a_0 + \sum_{n=1}^N a_n 2 \cos(2\pi nx) + b_n 2 \sin(2\pi nx). \quad (6)$$

Our second step is to slightly massage this expression.

b) Show that

$$S_N f(x) = \int_0^1 \tilde{D}_N(x, y) f(y) dy$$

where

$$\tilde{D}_N(x, y) = 1 + 2 \sum_{n=1}^N (\cos(2\pi nx) \cos(2\pi ny) + \sin(2\pi nx) \sin(2\pi ny))$$

Moreover, using a trigonometric identity, show that

$$\tilde{D}_N(x, y) = 1 + 2 \sum_{n=1}^N \cos(2\pi n(x - y)) =: D_N(x - y). \quad (7)$$

The function $D_N(x)$ is called the *Dirichlet kernel*; we see that the partial sum of the fourier series is computed by taking the *convolution* with the Dirichlet kernel $D_n(x)$.

c) Now we will do some algebra. The complex numbers come in handy here.

c.1) Show that

$$\sum_{k=0}^n ar^k = a \frac{1 - r^{n+1}}{1 - r}.$$

c.2) Show that

$$\sum_{k=-n}^n r^k = r^{-n} \frac{1 - r^{2n+1}}{1 - r} = \frac{r^{-n-1/2} - r^{n+1/2}}{r^{-1/2} - r^{1/2}}.$$

c.3) Plugging in $r = e^{2\pi ix}$ show that

$$\sum_{k=-n}^n e^{2\pi i k x} = \frac{\sin((2n+1)\pi x)}{\sin(\pi x)}.$$

c.4) Show that

$$D_N(x) = \frac{\sin((2N+1)\pi x)}{\sin(\pi x)}.$$

We have proven two statements about two real-numbered quantities by using complex numbers!

d) By a change of variables, show that

$$S_N f(x) - f(x) = \int_0^1 D_N(y)(f(x+y) - f(x))dy.$$

(The fact that $D_N(y)$ is *even* may come in handy here, as may the expression (7)!)

e) We can write the expression above as

$$\int_0^1 D_N(y)(f(x+y) - f(x))dy = \int_0^1 \sin((2N+1)\pi y) \frac{f(x+y) - f(x)}{\sin \pi y} dy.$$

By looking at extra credit problems from earlier problem sets, show that if for each x ,

$$\frac{f(x+y) - f(x)}{\sin \pi y} \tag{8}$$

is integrable as a function on $[0, 1] \ni y$, then

$$\lim_{N \rightarrow \infty} S_N f(x) = f(x)$$

for each x ; that is, the functions $S_N f(x)$ converge pointwise to $f(x)$ everywhere.

Here we interpret $f(z)$ for $z \notin [0, 1]$ by making f 1-periodic, i.e. we set

$$f(z) = f(z - 1)$$

for all z .

f) Show that if $f(x)$ is differentiable with continuous derivative after being made 1-periodic then for every $x \in [0, 1]$, the functions (8) are integrable over $[0, 1] \ni y$. (Hint: The only confusing part is about the zeros of the denominator. Compute

$$\lim_{y \rightarrow 0} \frac{f(x+y) - f(x)}{\sin \pi y}$$

using L-Hospital's rule. Is the function (8) actually even *better* than integrable, as a function of y ?)

We have proven that the Fourier series for $f(x)$ pointwise everywhere to f for functions f with a continuous derivative. This is already very good. But we want to work with discontinuous functions f , too!

g) Using the previous problem show that $S_N f \rightarrow f$ in L^2 for functions like in part (f) of this problem.

h) You may take on faith (or look in the textbook, or think of a proof for yourself) that functions as in part (f) of this problem are dense in $L^2([0, 1])$.

Show, either using the previous problem or one of the homework problems from the last two problem sets, that the set of functions in $L^2([0, 1])$ whose fourier series converges to the original function in $L^2([0, 1])$ is a *closed* subset of $L^2([0, 1])$. This proves that Fourier series converge in L^2 for all elements of $L^2([0, 1])$, which includes discontinuous functions!