MAT320 Problem Set 2

Due Oct 5, 2023

Please write your homework on paper neatly or type it up in LaTeX, and hand it in at the beginning of class next Thursday.

Royden $X.Y.Z$ refers Problem Z in Royden-Fitzpatrick, found in the collection of problems at the end of section $X.Y$.

Problem 1. Royden 2.3.11.

Problem 2. Royden 2.3.14.

Problem 3. We say that $f : [a, b] \to \mathbb{R}$ is *Lipshitz* if there is a constant $c \geq 0$ such that for all $u, v \in [a, b]$,

$$|f(u) - f(v)| \leq c|u - v|.$$

Show that the image of a set of measure zero under a Lipshitz function has measure zero.

(We will see on October 3 that there is a continuous function $f : [0, 1] \to \mathbb{R}$ and a set $C \subset [0, 1]$ of measure zero such that $f(E)$ is a measurable set of measure 1.)

Problem 4. Let $0 < \alpha < 1$. We define a subset $F_\alpha \subset [0,1]$, by defining

$$F_\alpha = \cap_{n=1}^{\infty} F_\alpha^n$$

where F_α^n is a union of intervals each of equal length, and $F_\alpha^0 = [0,1]$ and F_α^n is produced from F_α^{n-1} by removing an open interval of length $\alpha/(3^n)$ from the middle each of the intervals comprising F_α^{n-1}. Thus, if

$$F_\alpha^{n-1} = \bigcup_{i=1}^{n_k} [x_i - a_i, x_i + a_i],$$

for some real numbers x_i and real numbers $a_i > 0$, then

$$F_\alpha^n = \bigcup_{i=1}^{n_k} ([x_i - a_i, x_i - \alpha/(2 * 3^n)] \cup [x_i + \alpha/(2 * 3^n), x_i + a_i].$$

Show that F_α is closed and uncountable, and compute the measure of F.

Problem 5. Let f be a continuous function and let B be a Borel set. Show that $f^{-1}(B)$ is a Borel set.

Extra credit. Given a subset E of a metric space X, we say that a *boundary point* of E is a point $x \in X$ such that for all $\epsilon > 0$, $B(x, \epsilon)$ contains some point in E and also some other point in $X \setminus E$. Let ∂E be the set of boundary points of E. (Note that ∂E may or may not contain points of E.)

- Show that if E is closed then $\partial E \subset E$.
- Show that ∂E is always closed.
- Show that $\partial([0,1] \setminus F_\alpha)$ has measure greater than zero, where F_α is defined as above, and we take some α such that $0 < \alpha < 1$.