MAT320 Problem Set 2

Due Sept 19, 2024

Please write your homework on paper neatly or type it up in LaTeX, and hand it in at the beginning of class next Thursday.

Royden $X.Y.Z$ refers Problem Z in Royden-Fitzpatrick, found in the collection of problems at the end of section $X.Y$.

Problem 1. (Royden 1.3.22) Prove that the set of subsets of $\mathbb{Q}$, namely $2^{\mathbb{Q}}$, is not countable. (Hint: One – of many – possible approaches is to use the fact that $\mathbb{Q}$ is countable in order construct an embedding of the set of all functions $\mathbb{N} \to \{0, 1, \ldots, 9\}$ into the set of subsets of $\mathbb{Q}$. Cantor's diagonal argument presented in class then shows that the set of all functions $\mathbb{N} \to \{0, 1, \ldots, 9\}$, is uncountable.)

Problem 2. (Royden 1.3.26) Show that $\mathbb{R} \times \mathbb{R}$ has the same cardinality as $\mathbb{R}$. (Hint: use the decimal expansions of real numbers.)

Remark. It is a tricky theorem of topology to show that there is no bijection $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$ which is continuous and has a continuous inverse. You are not expected to prove or investigate this in the class!

Problem 3. (Royden 1.4.27) Is the set of rational numbers closed or open or both or neither, as a subset of the real numbers with the usual distance metric on the reals?

Problem 4. Let $(X, d : X \times X \to \mathbb{R})$ be a metric space.

a) Show that for any triple of nonnegative real numbers a, b, c, if $a \leq b + c$ then

$$\frac{a}{1+a} \leq \frac{b}{1+b} + \frac{c}{1+c}.$$

b) Define

$$d' : X \times X \to \mathbb{R}, d'(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

Show that (X, d') is a metric space.

c) Show that
$$S_{d'} = \{d'(x,y) \,|\, x \in X, y \in X\}$$
is bounded. We call $\sup S_{d'}$ the *diameter of X with respect to the metric d'*. Thus, with respect to d', X has a finite diameter, while with respect to d, X may or may not have an infinite diameter.

d) Show that if U is open in (X, d) then U is open in (X, d') and vice-versa.

This gives an example of a pair of metrics on any space which define the same topology.

Problem 5. (Royden 9.2.20)

Extra credit (1/2 a problem) (Royden 9.3.32). Hint: for (i), use the characterization of continuity that for a metric space X, a function $f : X \to \mathbb{R}$ is continuous if and only if for every interval $(a, b) \subset \mathbb{R}$, $f^{-1}(a, b) \subset X$ is open. Your other friend is the triangle inequality!

Extra credit. (1 full problem)

Consider the set of functions $C[0, 1]$ consisting of all continuous bounded functions $f : [0, 1] \to \mathbb{R}$. Every such function has a Riemann integral. You can use properties of Riemann integrals from calculus.

a) Show that
$$d_1(f, g) = \int_0^1 |f(x) - g(x)|dx$$
defines a metric on $C[0, 1]$.

b) Show that
$$d_2(f, g) = \sup\{|f(x) - g(x)| : x \in X\}$$
defines a metric on $C[0, 1]$.

c) Give an example of a Cauchy sequence f_i with respect to d_1 such that $\lim_{i\to\infty} f_i$ does not exist in $C[0, 1]$ with respect to the d_1 metric. Can you find such a sequence f_i which is *also* not Cauchy with respect to d_2?

d) We say that two metrics d_1, d_2 on X are equivalent when there is a positive number c such that
$$\frac{1}{c} d_1(x, y) \leq d_2(x, y) \leq c d_1(x, y)$$
for all pairs $(x, y) \in X \times X$. Here the constant c is *not* allowed to depend on the pair (x, y).

Show that the metrics d_1, d_2 defined above are *not* equivalent.

This fact is fundamentally an infinite-dimensional fact. The metrics above are induced from *norms* (we will discuss this later in the class); it turns out that any two metrics on a finite-dimensional vector space V coming from norms on V are equivalent.