MAT320 Practice Problems Answer Key

10/8/2024

Problem. For every $\epsilon > 0$, given an example of an open set $U_\epsilon \subset \mathbb{R}$ such that the Lebesgue measure of U_ϵ is less than ϵ, and such that U_ϵ is dense in $\mathbb{R}$. Please justify your answer.

Proof. Let $i \mapsto x_i$ be an enumeration of the rational numbers $\mathbb{Q}$, and let

$$U_\epsilon = \bigcup_{i=1}^{\infty}(x_i - \epsilon/2^{i+3}, x_i + \epsilon/2^{i+3}).$$

Then U_ϵ is open since it is a union of open sets, and it is dense since it contains the dense set $\mathbb{Q} = \{x_i \,|\, i = 1, 2, \ldots\}$. Its measure is less than ϵ since by the subadditivity of measure,

$$\mu(U_\epsilon) \leq \sum_{i=1}^{\infty} \mu((x_i - \epsilon/2^{i+3}, x_i + \epsilon/2^{i+3})) = \sum_{i=1}^{\infty} \frac{\epsilon}{2^{i+1}} = \epsilon/2 < \epsilon.$$

□

Problem. Cantor set problem on problem set 4.

Proof. (Sketch.) Closedness: We have that F_α^0 is closed and F_α^{i+1} is F_α^i minus an open set, thus all F_α^i are closed by induction. So their intersection, namely F_α, is closed. There is a bijection

$$f : 2^{\mathbb{N}} \to F_\alpha.$$

defined as follows: we take the infinite binary string $S = x_1x_2x_3\ldots$ to the unique element of the set $\cap_{n=1}^{\infty} F_\alpha^{i,S_i}$, with notation as below. We write $S_i = x_1 \ldots x_i$ for the first i symbols of S. We write F_α^{i,S_i} for the $(2^{i-1}x_1 + 2^{i-2}x_2 + \ldots + x_i + 1)$-th interval in the decomposition of F_α^i into intervals, when ordered from the left (we make sense of this for $i = 1$ by dropping all the terms except the last one). Thus for any infinite binary string S, the the sequence of intervals F_α^{i,S_i} is a nested sequence of intervals of radius converging to zero, so their intersection is a single point (by the Nested Set Theorem). Thus the map f described above is well defined. It is a bijection because every $x \in F^\alpha$ must for every i be contained in a unique one of the intervals $F_\alpha^{i,S}$ ranging over strings S of length i, which lets us reconstruct the total infinite string S from x.

Thus we know that F_α is uncountable. It's closed, so it's measurable; in fact, by continuity of measure we have that

$$m(F_\alpha) = \lim_{n\to\infty} F_\alpha^n = \lim_{n\to\infty} 1 - \sum_{i=1}^{n} \frac{\alpha 2^{n-1}}{3^n} = 1 - \alpha(1) = 1 - \alpha.$$

□

Problem. Show that the intersection of any collection of σ-algebras on X is a σ-algebra. Do the extra credit problem on problem set 4!

Proof. Let X be a set and let $\{\Sigma_i\}_{i\in I}$ be a collection of σ-algebras. We claim that $\cap_{i\in I}\Sigma_i =: \Sigma$ is a σ-algebra. Indeed, the empty set is an element of Σ_i for all $i \in I$, so it is an element of their intersection. If A is an element of Σ_i for all $i \in I$, then A^c is an element of Σ_i for every $i \in I$, and is thus an element of Σ. Finally, if $\{A_j\}_{j=1}^{\infty}$ is a collection of sets such that $A_j \in \Sigma_i$ for every $j \in \mathbb{N}$ and $i \in I$, then for every $i \in I$, $\cup_{j=1}^{\infty} A_j \in \Sigma_i$, thus $\cup_{j=1}^{\infty} A_j \in \Sigma$.

Extra credit problem:

Suppose that $E \subset \mathbb{R}$ is Borel. We need to show that $f^{-1}(E)$ is Borel, i.e. that E is contained in every σ-algebra Σ such that $(a, b) \subset \Sigma$ for all $a, b \in \mathbb{R}$ (in other words, that Σ *contains every interval*). Write

$$f(X) = \{f(x)|x \in X\}$$

for any set X that is a subset of the domain of f; thus $f(X)$ is the image of X under f. We verify that for a collection of subsets $F_i \subset \mathbb{R}$, $i = 1, 2, \ldots$, that,

$$\cup_{i=1}^{\infty} f(F_i) = f\left(\cap_{i=1}^{\infty} F_i\right), f(\emptyset) = \emptyset, f(F_i^c) = f(F_i)^c.$$

Thus

$$f(\Sigma) := \{f(F)|F \in \Sigma\}$$

is a σ-algebra on $\mathbb{R}$. Clearly, if $f^{-1}(E) \notin \Sigma$ then $f(f^{-1}(E)) = E \notin f(\Sigma)$. But we claim that $f(\Sigma)$ contains all intervals. Indeed, $f^{-1}((a, b))$ is an open subset of $\mathbb{R}$, thus a countable union of intervals; thus in particular it is contained in Σ since Σ contains all intervals and is a σ-algebra. But then $f(f^{-1}((a, b))) = (a, b) \in f(\Sigma)$, so $f(\Sigma)$ contains all intervals, thus all Borel sets. This is a contradiction; we have shown that $f^{-1}(E) \in \Sigma$. $\square$

Problem. An F_δ set is a set that is a countable union of closed sets. Show that continuous functions $f : \mathbb{R} \to \mathbb{R}$ map F_δ sets to F_δ sets.

(Hint: show first that the image of a compact set under a continuous function is compact.)

Proof. Let $f : X \to Y$ be continuous. Given a countable collection of closed sets $F_i \subset X$, $i = 1, 2, \ldots,$, we first note that if F_i is bounded, then it is compact; and then $f(F_i)$ is also compact and thus closed. Indeed, given any cover of $f(F_i)$, if there is an open cover $\{U_\alpha\}_{\alpha\in I}$ of $f(F_i)$, then $\{f^{-1}(U_\alpha)\}_{\alpha\in I}$ is an open cover of $f(F_i)$ by the continuity of f. As such the latter has a finite subcover, say $\{f^{-1}(U_1), \ldots, f^{-1}(U_r)\}$. But $f(f^{-1}(T)) = T$ for any $T \subset \mathbb{R}$; so if

$$F_i \subset f^{-1}(U_1) \cup \ldots \cup f^{-1}(U_r) \text{ then } f(F_i) \subset U_1 \cup \ldots \cup U_r.$$

In other words, there was a finite subcover of the cover $\{U_\alpha\}_{\alpha\in I}$ of $f(F_i)$. So we have show that continuous functions take compact sets to compact sets.

Now for every F_i, writing $F_i^j = F_i \cap [j, j+1]$ for $j \in \mathbb{Z}$, we see that $F_i = \cup_{j\in\mathbb{Z}} F_i^j$. Thus

$$F = \bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} \bigcup_{j=-\infty}^{\infty} F_{ij};$$

so

$$f(F) = \bigcup_{i=1}^{\infty} f(F_i) = \bigcup_{i=1}^{\infty} \bigcup_{j=-\infty}^{\infty} f(F_{ij}).$$

The quantity on the right above is a countable union of closed sets. $\square$

Problem. Suppose that $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are Cauchy sequences in a metric space (X, d). Show that the sequence of real numbers $d(x_n, y_n)$ converges as $n \to \infty$. (Note: the metric space X is not assumed to be complete, and one should not use the existence of the completion of the metric space in this argument.)

Proof. By completeness of the real numbers, it suffices to show that the sequence $c_n = d(x_n, y_n)$ is a Cauchy sequence. Given $\epsilon > 0$, choose N_1 such that for $n, m > N_1$, $d(x_n, x_m) < \epsilon/3$; similarly, choose N_2 such that for $n, m > N_2$, $d(y_n, y_m) < \epsilon/4$. Then for $n, m \geq max(N_1, N_2) + 1 =: N$ we have

$$d(x_n, y_n) \leq d(x_n, x_N) + d(x_N, y_N) + d(y_N, y_n);$$

so

$$|d(x_n, y_n) - d(x_N, y_N)| \leq \epsilon/2.$$

The same of course holds for n replaced by m. Thus

$$|d(x_n, y_n) - d(x_m, y_m)| \leq |d(x_n, y_n) - d(x_N, y_N)| + |d(x_N, y_N) - d(x_m, y_m)| \leq 2\epsilon/2 = \epsilon.$$

Every inequality above is simply an application of the triangle inequality, together with the fact that distances are nonnegative. We have proven that the sequence $\{c_n\}$ is Cauchy. □

Problem. Royden Problem 2.4.19.

Proof. By theorem 2.4.11 of the textbook, a set E is measurable if and only if for any $\epsilon > 0$, there is an open set $O \supset E$ such that $m^*(O \backslash E) < \epsilon$ (here m^* is the outer measure as defined in the textbook, which we denoted by μ^* in the lectures). Thus, if E is *not* measurable then there is an $\epsilon > 0$ such that for *any* open set $O \supset E$, we have $m^*(O \setminus E) \geq \epsilon$. However, if E has finite outer measure $m^*(E)$, then for any $\epsilon_2 > 0$ then there is an open set $V \supset E$ such that $m^*(V) < m^*(E) + \epsilon_2$. In particular, letting ϵ_2 range over $1/n$, $n = 1, 2, \ldots$, and taking the intersection of the corresponding open sets, we see that there is a Borel set $\widetilde{V} \supset E$ such that $m^*(V) = m^*(E)$. Choose an open set $U \supset \widetilde{V}$ such that $m^*(U) < m^*(\widetilde{V}) + \epsilon = m^*(E) + \epsilon$. Then

$$m^*(U \setminus E) \geq m^*(U \setminus \widetilde{V}) = m^*(U) - m^*(\widetilde{V}) < \epsilon$$

where the first inequality is by monotonicity of the outer measure. But $U \supset \widetilde{V} \supset E$; so this contradicts the non-measurability of E. □

Problem. Give examples of subsets of $\mathbb{R}^2$ that are

- Countable, compact: $\{(0, 1/i) | i = 1, 2 \ldots\} \cup \{(0, 0)\}$
- Uncountable, compact, $[0, 1] \times \{0\}$, $[0, 1] \times [0, 1]$, anything closed and bounded (i.e. contained in a ball of finite radius)
- Uncountable, compact, have countably many pairwise distinct subsets which are simulatenously closed and open (with respect to the subspace metric):

$$(\{1/i | i = 1, 2 \ldots\} \cup \{0\}) \times [0, 1].$$