MAT320 Quiz #4

9/26/2023

Please answer the following questions, and write your name on top of the quiz.

Question 1.

a) Is the set of integers $\mathbb{Z} \subset \mathbb{R}$ compact?

b) Is the set
$$\{0\} \cup \{\frac{1}{n} \,|\, n = 1, 2, 3, 4, 5, \ldots\}$$
compact in $\mathbb{R}$?

Question 2.

a) Write down the definition of a Cauchy sequence in a metric space X.

b) What does it mean for a metric space to be complete?

c) Give an example of a metric space which is *not* complete.

Question 3.

Give an example of a sequence $a_n \in \mathbb{R}$, $n = 1, 2, \ldots$, such that the associated sequence of sums

$$s_n = \sum_{k=1}^{n} a_k$$

is Cauchy, but the sequence

$$s'_n = \sum_{k=1}^{n} |a_k|$$

is not.

Question 4.

Is the sequence

$$f_k : [0, 1] \to \mathbb{R},$$
$$f_k(x) = \min(kx^2, 1)$$

where $k = 1, 2, \ldots$, Cauchy in the metric space

$$X = \left\{ g : [0, 1] \to \mathbb{R} \mid g \text{ continuous }, \int_0^1 |g(x)|\, dx < \infty \right\}$$

where

$$d : X \times X \to \mathbb{R}$$

is

$$d(f, g) = \int_0^1 |f(x) - g(x)|\, dx?$$

If it is Cauchy, does it have a limit, and if it does, what is the limit? You do not have to explain why the above metric space is indeed a metric space, etc.