MAT320 Quiz #3

9/19/2023

Please answer the following questions, and write your name on top of the quiz.

Question 1. Is there a bijection from the set of rational numbers to the set of real numbers defined by decimal expansions

$$S = \{0.a_1a_2a_3\ldots \mid a_i \in \{1,3,5,7,9\} \text{ for all } i = 1,2,\ldots\}?$$

Question 2. A metric space is a set X with a function $X \times X \to \mathbb{R}$ satisfying certain axioms.

a) Write down the axioms that d should satisfy for (X, d) to be a metric space.

b) Let

$$X = \{f : [0,1] \to \mathbb{R} \mid f(x) > 0 \text{ for all } x \in [0,1], \text{ and } f \text{ continuous, bounded.}\}$$

Every such function has a Riemann integral $\int_0^1 f(x)dx$, which is a real number. Consider the function

$$d : X \times X \to \mathbb{R},$$
$$d(f,g) = \int_0^1 \frac{(f(x) - g(x))^2}{f(x)}\, dx.$$

For each axiom of a metric space, write whether the above function satisfies this axiom, and explain your reasoning.

Question 3.

a) Is the set $(0, 3]$ closed or open or both or neither in $\mathbb{R}$?

b) Is the empty set closed or open or both or neither in $\mathbb{R}$?

c) Give an example of a metric space X that is not $\mathbb{R}$, and an example of a closed set E in X and also an open set U in X for which $E \neq \emptyset, E \neq X$, and $U \neq \emptyset, U \neq X$.