MAT320 Quiz #10

11/28/2023

Please answer the following questions, and write your name on top of the quiz.

Question 1. Recall that

$$\ell^2 = \{(a_i)_{i=1}^{\infty} | \|(a_i)\|_{\ell^2} < \infty\}$$

$$\|(a_i)\|_{\ell^2} = \sum_{i=1}^{\infty} |a_i|^2.$$

Give a sequence of vectors in ℓ^2 of bounded ℓ^2 norm with no ℓ^2 convergent subsequence.

Question 2. Suppose that we have a pair of Banach spaces (B_1, B_2) and a linear map $A : B_1 \to B_2$ such that

$$\|Av\|_{B_2} \leq C\|v\|_{B_1}$$

for a constant C independent of v. Show that A is continuous.

Question 3.

a) Let s be any number greater than zero. Show that the set of sequences

$$\ell^{2,s} = \{(a_k)_{k\in\mathbb{Z}} \,:\, \sum_{i\in\mathbb{Z}}(1+k^2)^s|a_k|^2 < \infty\}$$

is closed under addition. Is it a Banach space under the norm

$$\|(a_k)\|_{\ell^{2,s}} = \sqrt{\sum_{i\in\mathbb{Z}}(1+k^2)^s|a_k|^2}?$$

b) Show, using the fact that there are positive constants C, D such that for all $k \in \mathbb{Z}$,

$$C(1+k^2)^s \geq 1^s + k^{2s} \geq D(1+k^2)^s \tag{1}$$

(you do not have to prove this fact!) that $\ell^{2,s}$ is the same vector space as the space of sequences

$$\mathcal{V}_{2,s} = \left\{(a_k)_{k\in\mathbb{Z}} \,:\, \sum_{i\in\mathbb{Z}}|a_k|^2 < \infty, \sum_{i\in\mathbb{Z}}k^{2s}|a_k|^2\right\}$$

c) **(Extra credit.)** Show that (1) holds for $0 < s < 1$. (You can use any theorems we have ever discussed in class or on the problem sets.) Show that if we give the space $\mathcal{V}_{2,s}$ the norm

$$\|(a_k)\|_{\mathcal{V}_{2,s}} = \|(a_k)\|_{\ell^2} + \|(k^s a_k)\|_{\ell^2}$$

Then the identification

$$\ell^{2,s} = \mathcal{V}_{2,s}$$

is a continuous bijection of Banach spaces with continuous inverse.