MAT320 Final Practice Problems

12/5/2023

Below are some practice problems. Problems will be added to this document throughout the week. You may hand in up to three problems for extra credit by Sunday Dec 15.

Problem 1. Consider the functions

$$f_n(x) = \frac{1}{1 + |x|^{\frac{n^3}{n^2+2024}}}$$

for $n = 100, 101, 102, \ldots$ Find

$$\lim_{n\to\infty} \int_0^\infty f_n(x)dx$$

where we take the limit over all $n > 100$. You may use the fact that

$$\frac{1}{1 + |x|^N}$$

is integrable over $\mathbb{R}$ for all $N \geq 2$.

Problem 2. Given a closed interval $[a, b] \subset \mathbb{R}$ with $b \geq a$ we write ℓ for its length:

$$\ell([a, b]) = (b - a).$$

If $b < a$ we set $\ell([a, b]) = \ell(\emptyset) = 0$.

For any $\alpha > 0$, and any subset $A \subset \mathbb{R}$, define

$$\mu_\alpha^*(A) = \liminf_{\delta \to 0^+} \left\{ \sum_{i=1}^\infty \ell([a_i, b_i])^\alpha \mid [a_i, b_i] \subset \mathbb{R} \text{ for each } i = 1, \ldots \infty, A \subset \bigcup_{i=1}^\infty [a_i, b_i], \ell([a_i, b_i]) < \delta \text{ for all } i = 1, \ldots, \infty \right\}.$$

Here the lim inf is taken as δ approaches zero from above.

a) Show that if $A \subset B \subset \mathbb{R}$ then $\mu_\alpha^*(A) \leq \mu_\alpha^*(B)$.

b) Show that when $\alpha = 1$ then $\mu_\alpha^*(A)$ agrees with the usual outer measure of A.

c) Show that if $\alpha > 1$ then $\mu_\alpha^*(A) = 0$ for any $A \subset \mathbb{R}$.

d) (This part is harder.) Show that for any collection of sets $B_i \subset \mathbb{R}$, $i = 1, \ldots, \infty$, we have

$$\mu_\alpha^* \left(\bigcup_{i=1}^\infty B_i \right) \leq \sum_{i=1}^\infty \mu_\alpha^*(B_i).$$

Thus, μ_α^* generalizes the outer measure.

Problem 3. Consider the function on the reals that computes the shortest distance from x to any integer:

$$D : \mathbb{R} \to \mathbb{R}, D(x) = d(x, \mathbb{Z}) = \min_{n \in \mathbb{Z}} |x - n|.$$

Given a pair of sequences of nonnegative real numbers

$$\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty},$$

we can consider the partial sums

$$f_N(x) = \sum_{n=1}^{N} a_n D(b_n x)$$

as functions $f_N : [0, 1] \to \mathbb{R}$ and see if there is a well-defined limit

$$f(x) = \sum_{n=1}^{\infty} a_n D(b_n x)$$

in some sense.

a) Show that if we set $a_n = (1/3)^n$ then for any sequence b_n the functions $f_N(x) \in C^0([0, 1])$ are a Cauchy sequence in the C^0 norm. As such, in this case the limit function f is a well defined element of $C^0([0, 1])$.

b) Show that if we set $a_n = (1/3)^n$ and $b_n = 2^n$ then the limit function $f : [0, 1] \to \mathbb{R}$ is Lipshitz.

For either of these questions it may be useful to recall that for any $0 < x < 1$ we have that the following sum converges:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1 - x}.$$

Problem 4. Recall the Banach space

$$\ell^2 = \{(a_n)_{n=1}^{\infty} \mid \sum_{n=1}^{\infty} |a_n|^2 < \infty\},$$

$$\|(a_n)\|_{\ell^2} = \sqrt{\sum_{n=1}^{\infty} |a_n|^2}.$$

Consider the subset

$$\mathcal{H} \subset \ell^2, \mathcal{H} = \{((a_n)_{n=1}^{\infty} \,|a_n \in [0, 1/n]\}.$$

a) Show that $\mathcal{H}$ does *not* contain any open ℓ^2 ball of positive radius, i.e. that $\mathcal{H}$ has empty interior.

b) Show that every sequence in $\mathcal{H}$ has a subsequence that is Cauchy in ℓ^2. You may use without proof the fact that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

Problem 5. Let Σ be a σ-algebra on a set X, and let $\mu : \Sigma \to \mathbb{R}$ be a countably additive measure such that $\mu(X) = 1$. (In other words, μ is a *probability measure* on X.)

Suppose that $f_n : X \to \mathbb{R}$ is a measurable function for each $n = 1, 2, \ldots$, i.e. such that $f_n^{-1}([a,b]) \in \Sigma$ for all $n \in \mathbb{N}$ and all $a, b \in \mathbb{R}$. Say that there is another measurable function $f : X \to \mathbb{R}$ as well.

Suppose that for every $\epsilon > 0$,

$$\mu(\{x \in X || f_n(x) - f(x)| > \epsilon\}) \to 0 \text{ as } n \to \infty.$$

(You do not have to prove that the sets $\{x \in X || f_n(x) - f(x)| > \epsilon\}$ are in Σ.)

a) Show that for every $\delta_1, \delta_2 > 0$, there exists an n such that for all $n > N$,

$$\mu(f_n^{-1}([a,b])) \leq \mu(f^{-1}([a - \delta_1, b + \delta_1])) + \delta_2.$$

b) Show that for every $[a,b] \in \mathbb{R}$,

$$\lim_{n\to\infty} \mu(f_n^{-1}([a,b])) = \mu(f^{-1}([a,b])).$$

You may use without proof the continuity of measure, i.e. that for any sequence of sets $A_i \in \Sigma$, we have

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} \mu\left(\bigcup_{i=1}^{n} A_i\right),$$

$$\mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} \mu\left(\bigcap_{i=1}^{n} A_i\right),$$

Problem 6. Suppose that we have a sequence of measurable functions $K_n : \mathbb{R} \to \mathbb{R}$ such that for all $n = 1, 2, \dots,$

- $K_n(x) \geq 0$ for all x,
- $$\int_{-\infty}^{\infty} K_n(x)dx = 1,$$

and also such that for all $\epsilon > 0$,

$$\lim_{n\to\infty} \int_{\{x\in\mathbb{R}||x|>\epsilon\}} K_n(x) = 0.$$

Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with bounded range (i.e. there exists a constant M such that $|f(x)| < M$ for all x.)

a) Show that the product function $x \mapsto f(x)K_n(x)$ is integrable over $\mathbb{R}$ for each n. (You will get partial credit if you show this under the assumption that $\{x \in \mathbb{R}|f(x) \neq 0\} \subset [a, b]$ for some finite $a, b \in \mathbb{R}$.)

b) Show that for any $x \in \mathbb{R}$

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} (f(x-y) - f(x))K_n(y)dy = 0.$$

(Hint: you should break up the integral above into several parts and bound each part separately. Remember that the continuity of f implies that for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $y \in [-\delta, +\delta]$ then $|f(x-y) - f(x)| < \epsilon$.)