MAT320 Midterm Answers

10/12/2023

Problem 1. For every $\epsilon > 0$, given an example of an open set $U_\epsilon \subset \mathbb{R}$ such that the Lebesgue measure of U_ϵ is less than ϵ, and such that U_ϵ is dense in $\mathbb{R}$. Please justify your answer.

Proof. Let $i \mapsto x_i$ be an enumeration of the rational numbers $\mathbb{Q}$, and let

$$U_\epsilon = \bigcup_{i=1}^{\infty} (x_i - \epsilon/2^{i+3}, x_i + \epsilon/2^{i+3}).$$

Then U_ϵ is open since it is a union of open sets, and it is dense since it contains the dense set $\mathbb{Q} = \{x_i \,|\, i = 1, 2, \ldots\}$. Its measure is less than ϵ since by the subadditivity of measure,

$$\mu(U_\epsilon) \leq \sum_{i=1}^{\infty} \mu((x_i - \epsilon/2^{i+3}, x_i + \epsilon/2^{i+3})) = \sum_{i=1}^{\infty} \frac{\epsilon}{2^{i+1}} = \epsilon/2 < \epsilon.$$

$\square$

Common errors:

- $\mathbb{Q}$ is not open.
- Any single interval that is dense will not have small measure (it will have to be all of $\mathbb{R}$); in particular intervals like $(0, \epsilon)$ are not dense by themselves.

Problem 2. Let $X = \mathbb{R}$ and

$$d : X \times X \to \mathbb{R},\ d(x, y) = \begin{cases} 0 \text{ if } x = y. \\ 1 \text{ else.} \end{cases}$$

a) Verify that (X, d) is a metric space.

b) Show that every subset of X is closed.

c) Describe (with proof) all compact subsets of X. Is the set of all compact subsets of X countable?

Proof. a) Clearly $d(x, y) \geq 0$, and $d(x, y) = 0$ exactly when $x = y$ by definition. Moreover $d(x, y) = d(y, x)$. The only nontrivial check is the triangle inequality: given $x, y, z \in X$, we have several cases. The first case is $x = y = z$ when all distances are zero. The second case is that $x = y, y \neq z$; in this case $x \neq z$, so $d(x, z) = 1 \leq 1 + 0 = d(x, y) + d(y, z)$. The case $x \neq y, y = z$ is similar. Then there is the case where $x \neq y, y \neq z$, in which case $d(x, y) + d(y, z) = 2 \geq 1 \geq d(x, z)$.

b) This is equivalent to the statement that every subset of X is open by taking complements. Unions of open balls are open and $B(x, \frac{1}{2}) = \{x\}$ for any $x \in X$; so any subset $S \subset X$ satisfies

$$S = \bigcup_{s \in S} B(s, \frac{1}{2}),$$

i.e. S is open.

c) Suppose $S \subset X$ has a finite number of elements. We claim that it is manifestly compact. Indeed, given an open cover $\{U_\alpha\}_{\alpha \in I}$ of S, we write $S = \{s_1, \ldots, s_k\}$ for some k, and then $s_1 \in U_{\alpha_i}$ for some $\alpha_i \in I$ for each of $i = 1, \ldots, k$. So then $\{U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_k}\}$ is a finte subcover of the cover $\{U_\alpha\}_{\alpha \in I}$ of S.

Suppose that $S \subset X$ is not finite. Then consider the open cover $\{B(s, 1/2)\}_{s \in S} = \{\{s\}\}_{s \in S}$ with index set $I = S$. The union of all the open sets of any finite subcover of this cover will only contain a finite number of elements in S, and so cannot contain all of S. So infinite subsets of X are not compact.

Thus the compact subsets of X are the finite subsets. In particular the number of compact subsets is uncountable since $\mathbb{R}$ is uncountable and every element $r \in \mathbb{R}$ defines a distinct compact set $\{r\}$.

$\square$

Common errors:

- For (c) many of you tried to use Heine-Borel. But Heine-Borel does not apply to metric spaces in general (what is a bounded subset of a general metric space, even?). Please keep in mind that there is not an an analog of Heine-Borel that applies to *general metric spaces.*
- In particular $[0, 1]$ is *not* compact in the above metric space, as the argument above shows.
- Just because we have a distance metric on the set $\mathbb{R}$ doesnt mean that the properties of the usual distance metric carry over to this new distance metric. *The same set can have many, many, completely unrelated metrics!* I could have found a bijection of $\mathbb{R}$ with the space of

continuous functions on $[0, 1]$ (yes, these two sets have the same cardinality!), and used the metric above to define a metric on the space of continuous functions on $[0, 1]$. The 'name' of a set does not mean much by itself!

- It is not clear what a 'bounded' subset of the above metric space is without clarification. A natural definition would be that the subset is contained in some closed ball. If one uses it this way then the above example shows that a 'closed and bounded' subset of a metric space may not be compact. But one should clarify this kind of usage.

Problem 3. Let Σ be a σ-algebra on a set X, and let $\mu : \Sigma \to \mathbb{R}$ be a countably additive measure on this σ-algebra. Given sets S, T, define $S\Delta T = (S \setminus T) \cup (T \setminus S)$.

a) Show (from the axioms of a σ-algebra) that if $S \in \Sigma$ and $T \in \Sigma$ then $S\Delta T \in \Sigma$.

b) Show that if $\mu(S\Delta T) = 0$ then $\mu(S) = \mu(T)$.

c) Show that if we define the relation $E \subset \Sigma \times \Sigma$ by

$$E = \{(S,T) \in \Sigma \times \Sigma \mid \mu(S\Delta T) = 0\}$$

then this is an equivalence relation. Feel free to use the notation $S \sim T$ or $S \sim_E T$ to denote the statement that $(S,T) \in E$.

d) Let Σ/E be the set of equivalence classes of Σ with respect to E. Show that the function

$$d^\mu : \Sigma/E \times \Sigma/E \to \mathbb{R},$$

which is defined by

$$d^\mu([S],[T]) = \mu(S\Delta T)$$

where $[S]$ and $[T]$ are the equivalence classes containing S and T respectively, makes $(\Sigma/E, d^\mu)$ into a metric space. (Part (c) of this problem shows that this function is well defined; you do not have to prove this!)

Proof. a) By monotonicity of measure, if $\mu(S\Delta T) = 0$ then $\mu(T \setminus S) = 0$. But $S \cup T = S \cup (T \setminus S)$, and S is disjoint from $T \setminus S$; thus by additivity of measure $\mu(S) = \mu(S) + \mu(T \setminus S) = \mu(S \cup T)$. Flipping the roles of S and T in this argument shows that $\mu(S \cup T) = \mu(T)$. So $\mu(S) = \mu(T)$.

b) This equivalence relation is reflexive since $\mu(S\Delta S) = \mu(\emptyset) = 0$. It is symmetric since $S\Delta T = T\Delta S$. The nontrivial part is to verify transitivity. Suppose we are given three subsets A, B, C such that $\mu(A\Delta B) = \mu(B\Delta C) = 0$. We need to verify that $\mu(A\Delta C) = 0$. But this follows from monotonicity and subadditivity of measure, together with the fact that

$$(A\Delta C) \subset (A\Delta B) \cup (B\Delta C). \tag{1}$$

I would have happily accepted a 'Venn Diagram proof' of this last fact above. Here is an actual proof: if $x \in A \setminus C$, then either $x \in B$ or $x \notin B$. If $x \in B$ then $x \in B \setminus C$, so $x \in B\Delta C$. If $x \notin B$ then $x \in A \setminus B$ so $x \in A\Delta B$. On the other hand, if $x \in C \setminus A$ then either $x \in B$ or $x \notin B$. In the first case $x \in B\Delta A$ and in the second case $x \in C\Delta B$.

c) This is almost the same as the previous arugment. If $d([S],[T]) = 0$ then $S\Delta T = 0$ so the equivalence classes $[S]$ and $[T]$ agree. Measure is nonnegative and $S\Delta T = T\Delta S$. Thus we have verified that $d(x,y) = d(y,x)$ and $d(x,y) \geq 0$ with $d(x,y) = 0$ iff $x = y$. It remains to verify the triangle inequality. But this follows from (1), which implies by monotonicity and subadditivity of measure that $d([A],[C]) \leq d([A],[B]) + d([B],[C])$ for any $A, B, C \in \Sigma$.

$\square$

Common problems: In (b), many of you used (a) to show that if $\mu(A\Delta B) = \mu(B\Delta C) = 0$ then $\mu(A) = \mu(B) = \mu(C)$, and then tried to conclude from that that $\mu(A\Delta C) = 0$. But this latter implication is entirely false in general. Indeed if you just know that $\mu(A) = \mu(B) = \mu(C)$ you might have that $A = B = C$. You have to use the something about the measures of the sets $A\Delta B$ and $B\Delta C$ being zero. Note: everyone who drew a Venn diagram for this problem actually happened to get the whole problem right.

Problem 4. Suppose that $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are Cauchy sequences in a metric space (X, d). Show that the sequence of real numbers $d(x_n, y_n)$ converges as $n \to \infty$. (Note: the metric space X is not assumed to be complete, and one should not use the existence of the completion of the metric space in this argument.)

Proof. By completeness of the real numbers it suffices to show that $c_n = d(x_n, y_n)$ is Cauchy. Given $\epsilon > 0$, there is an N_x such that $d(x_n, x_m) < \epsilon/2$ for $n, m > N_x$; and similarly there is an N_y such that $d(y_n, y_m) < \epsilon/2$ for $n, m > N_y$. So then for $n, m > N := \max(N_x, N_y)$, we have by the triangle inequality that

$$d(x_m, y_m) \leq d(x_n, y_n) + d(x_n, x_m) + d(y_n, y_m),$$

and the same argument shows that same equation holds with n and m switched. In other words we have that

$$|c_n - c_m| = |d(x_m, y_m) - d(x_n, y_n)| \leq d(x_n, x_m) + d(y_n, y_m) < \frac{2\epsilon}{2} = \epsilon.$$

Thus the sequence $\{c_n\}$ is Cauchy. $\square$

Common mistakes: The problem explicitly says that you can't assume X to be complete. Thus the Cauchy sequences $\{x_n\}$ and $\{y_n\}$ may not converge! Convergent sequences are always Cauchy, but Cauchy sqeuences may not converge – if they always converge then the ambient metric space is said to be *complete*.

Problem 5. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *increasing* if $x < y$ implies $f(x) \leq f(y)$, and is said to be *strictly increasing* if $x < y$ implies $f(x) < f(y)$.

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *measurable* if $f^{-1}((a, \infty))$ is measurable for all $a \in \mathbb{R}$. Equivalently, f is measurable if $f^{-1}([a, \infty))$ is measurable for all $a \in \mathbb{R}$, or equivalently if the preimage of every interval is measurable.

Prove that an increasing function is measurable. (Chapter 3 of the textbook states this as a theorem without proof; you cannot just invoke this statement. Prove that it is true!) Partial credit will be given if you only prove this under the assumption that f is strictly increasing.

Proof. If $x \in f^{-1}([a, \infty))$ then $f(x) \geq a$; so if $y > x$ then $f(y) \geq f(x) \geq a$; so $y \in f^{-1}([a, \infty))$. So

$$f^{-1}([a, \infty)) = \bigcup_{x \in f^{-1}([a,\infty))} [x, \infty).$$

Thus either $f^{-1}([a, \infty)) = (c, \infty)$ or $f^{-1}([a, \infty)) = [c, \infty)$ for $c = \inf f^{-1}([a, \infty))$. Both of these options make $f^{-1}([a, \infty))$ a measurable subset. $\square$

Common mistakes. Everyone who made a mistake made a different one. I tried to give partial credit for correct partial ideas even under strong assumptions on f.