MAT320 Problem Set 4

Due Oct 3, 2023

Please write your homework on paper neatly or type it up in LaTeX, and hand it in at the beginning of class next Thursday.

Royden $X.Y.Z$ refers Problem Z in Royden-Fitzpatrick, found in the collection of problems at the end of section $X.Y$.

Problem 1 Royden 2.1.1, 2.1.3.

Problem 2. Please solve one of the following two problems. You can pick which one! For this problem, you can't use results proven after section 2.2 in the book! Royden 2.2.6, 2.2.7.

Problem 3. Royden 2.2.9.

Problem 4. Royden 2.3.11.

Problem 5. Let $0 < \alpha < 1$. We define a subset $F_\alpha \subset [0,1]$, by defining

$$F_\alpha = \cap_{n=1}^{\infty} F_\alpha^n$$

where F_α^n is a union of intervals each of equal length, and $F_\alpha^0 = [0,1]$ and F_α^n is produced from F_α^{n-1} by removing an open interval of length $\alpha/(3^n)$ from the middle each of the intervals comprising F_α^{n-1}. Thus, if

$$F_\alpha^{n-1} = \bigcup_{i=1}^{n_k} [x_i - a_i, x_i + a_i],$$

for some real numbers x_i and real numbers $a_i > 0$, then

$$F_\alpha^n = \bigcup_{i=1}^{n_k} \left([x_i - a_i, x_i - \alpha/(2 * 3^n)] \cup [x_i + \alpha/(2 * 3^n), x_i + a_i] \right).$$

Show that F_α is closed and uncountable. (We will prove something very helpful for this in class on Tuesday!) Using the axioms of the Lebesgue measure, compute the measure of F.

Extra credit (1/2 problem): Royden 2.3.14.

Extra credit. Let f be a continuous function and let B be a Borel set. Show that $f^{-1}(B)$ is a Borel set.