MAT320 Problem Set 2

Due Sept 21, 2023

Please write your homework on paper neatly or type it up in LaTeX, and hand it in at the beginning of class next Thursday.

Royden $X.Y.Z$ refers Problem Z in Royden-Fitzpatrick, found in the collection of problems at the end of section $X.Y$.

Problem 1. (Royden 1.3.22) Prove that the set of subsets of $\mathbb{Q}$, namely $2^{\mathbb{Q}}$, is not countable. (Hint: One – of many – possible approaches is to use the fact that $\mathbb{Q}$ is countable in order construct an embedding of the set of all functions $\mathbb{N} \to \{0, 1, \ldots, 9\}$ into the set of subsets of $\mathbb{Q}$. Cantor's diagonal argument presented in class then shows that the set of all functions $\mathbb{N} \to \{0, 1, \ldots, 9\}$, is uncountable.)

Problem 2. (Royden 1.3.26) Show that $\mathbb{R} \times \mathbb{R}$ has the same cardinality as $\mathbb{R}$. (Hint: use the decimal expansions of real numbers.)

Remark. It is a tricky theorem to show that there is no bijection $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$ which is continuous and has a continuous inverse.

Problem 3. (Royden 1.4.27) Is the set of rational numbers closed or open or both or neither, as a subset of the real numbers with the usual distance metric on the reals?

Problem 4. Let $(X, d : X \times X \to \mathbb{R})$ be a metric space.

a) Show that for any triple of nonnegative real numbers a, b, c, if $a \leq b + c$ then

$$\frac{a}{1+a} \leq \frac{b}{1+b} + \frac{c}{1+c}.$$

b) Define

$$d' : X \times X \to \mathbb{R}, d'(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

Show that (X, d') is a metric space.

c) Show that

$$S_{d'} = \{d'(x,y) \mid x \in X, y \in X\}$$

is bounded. We call $\sup S_{d'}$ the *diameter of X with respect to the metric d'.*

Thus, with respect to d', X has a finite diameter, while with respect to d, X may or may not have an infinite diameter.

d) Show that if U is open in (X,d) then U is open in (X,d') and vice-versa.

This gives an example of a pair of metrics on any space which define the same topology.

Problem 5. (Royden 9.2.20)

Problem 6. (Royden 9.1.32). Hint: for (i), use the characterization of continuity that for a metric space X, a function $f : X \to \mathbb{R}$ is continuous if and only if for every interval $(a,b) \subset \mathbb{R}$, $f^{-1}(a,b) \subset X$ is open. Your other friend is the triangle inequality!

Extra credit.

Consider the set of functions $C[0,1]$ consisting of all continuous bounded functions $f : [0,1] \to \mathbb{R}$. Every such function has a Riemann integral. You can use properties of Riemann integrals from calculus.

a) Show that

$$d_1(f,g) = \int_0^1 |f(x) - g(x)| dx$$

defines a metric on $C[0,1]$.

b) Show that

$$d_2(f,g) = \sup\{|f(x) - g(x)| : x \in X\}$$

defines a metric on $C[0,1]$.

c) Give an example of a Cauchy sequence f_i with respect to d_1 such that $\lim_{i\to\infty} f_i$ does not exist in $C[0,1]$ with respect to the d_1 metric. Can you find such a sequence f_i which is *also* not Cauchy with respect to d_2?

d) We say that two metrics d_1, d_2 on X are equivalent when there is a positive number c such that

$$\frac{1}{c} d_1(x,y) \leq d_2(x,y) \leq c d_1(x,y)$$

for all pairs $(x,y) \in X \times X$. Here the constant c is *not* allowed to depend on the pair (x,y).

Show that the metrics d_1, d_2 defined above are *not* equivalent.

This fact is fundamentally an infinite-dimensional fact. The metrics above are induced from *norms* (we will discuss this in class on Sept 19); it turns out that any two metrics on a finite-dimensional vector space V coming from norms on V are equivalent.