Probabilistic inference for data science 2, Exam 27.3.2020

Answer to all questions. Calculators are allowed. Computer is allowed to do computations and coding, but no Googling! The second page includes a collection of potentially useful equations and R-scripts.

1. Let $X_1, \ldots, X_n$ be an iid sample from pdf

$$f(x|\theta) = \theta x^{(\theta-1)}, \quad 0 \leq x \leq 1, \quad 0 \leq \theta < \infty \tag{1}$$

   population. Find the MLE of θ.

2. Let $X_i = X(u_i)$ be random variable X at spatial locations u_1, u_2 and u_3. Let X_i be normally distributed with common mean μ, common variance σ^2 and correlation $\rho(X_i, X_j) = e^{-\|u_i - u_j\|/2}$. Consider locations $u_1 = (1, 0)$, $u_2 = (0, 0)$ and $u_3 = (0, 3)$.

   (a) Present the distribution of $\boldsymbol{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$

   (b) Present the likelihood of sample $\boldsymbol{X} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$.

3. Present the Central Limit Theorem for an iid population with known variance. Explain how does it generalize to populations with unknown variance.

4. Are the following statements true or false? Explain your answer, writing **just** true or false, will give you 0 points.

   (a) Let $\mathbf{x}$ be a bivariate Gaussian random variable, i.e. $\mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. The $\boldsymbol{\Sigma}$ is same in cases a), b) and c) in the Figure below?

   (b) Sampling from the infinite population is the same as sampling from a finite population.

   (c) Given a set of scalars $\{x_1, x_2, \ldots, x_N\}$, if you want to represent these N scalars by just one scalar, then the sample mean is the best approximator in terms of squared error.

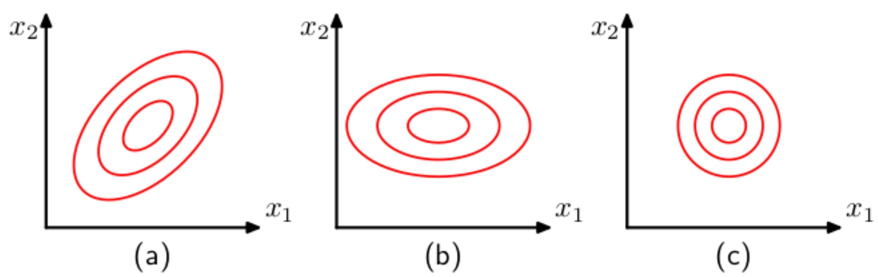
   (d) You will always want your estimator to be unbiased.

   (e) If $\mathsf{Var}X = \sigma^2$, then $P(|X - \mathsf{E}[X]| \geq 2\sigma) \leq \frac{1}{4}$. Additional question: what does the result mean?

5. Let $\boldsymbol{X} = \begin{pmatrix} X(u_1) \\ X(u_2) \\ X(u_3) \end{pmatrix}$ include the random variable X at spatial locations $u_1 = (1, 0)$, $u_2 = (0, 0)$, and $u_3 = (0, 1)$. Let $\boldsymbol{X}$ be normally distributed so that all three components have a common mean μ which is unknown, and $\sigma_i^2 = 2^2$ for all locations u_i is known. We assume that $\rho_{ij} = e^{-s_{ij}/2}$, where $s_{ij} = 2||u_i - u_j||$.

   (a) Find the maximum likelihood estimate of μ when assuming that $\boldsymbol{x} = \begin{pmatrix} 2 \\ 3 \\ 0.25 \end{pmatrix}$ has been observed. A numerical approximation is sufficient.

   (b) Change the assumption of the variance to $\sigma_i^2 = 1$ for all i. Find the MLE of μ and compare to (a).



Exam equation sheet, Introduction to statistical inference 2

$\mathsf{E}(\boldsymbol{X}_1\|\boldsymbol{x}_2) = \boldsymbol{\mu}_1 + \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_2^{-1}(\boldsymbol{x}_2 - \boldsymbol{\mu}_2)$	$\mathrm{var}(\boldsymbol{X}_1\|\boldsymbol{x}_2) = \boldsymbol{\Sigma}_1 - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_2^{-1}\boldsymbol{\Sigma}_{21}$
$f(x) = \frac{1}{\beta}e^{-x/\beta}$, $\mathsf{E}(X) = \beta$, $\mathrm{var}(X) = \beta^2$	$f(\boldsymbol{x}) = \frac{1}{\sqrt{(2\pi)^n\|\boldsymbol{\Sigma}\|}}e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})}$
$f_Z(z) = \int_{-\infty}^{\infty} f_X(w)f_Y(z-w)dw$	$(n-1)S^2/\sigma^2 \sim \chi^2(n-1)$
$f(x) = \frac{1}{\Gamma(p/2)2^{p/2}}x^{p/2-1}e^{-x/2}$	$f(t) = \frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2})}\frac{1}{(p\pi)^{1/2}}\frac{1}{(1+t^2/p)^{(p+1)/2}}$; $\mathsf{E}(T) = 0$; $\mathrm{var}(T) = \frac{p}{p-2}$
$\frac{S_X^2/\sigma_X^2}{S_Y^2/\sigma_Y^2} \sim F(n-1, m-1)$	$f(x) = \frac{\Gamma(\frac{p+q}{2})}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})}\left(\frac{p}{q}\right)^{p/2}\frac{x^{p/2-1}}{(1+\frac{p}{q}x)^{(p+q)/2}}$
$P(X \geq t) \leq \frac{\mathsf{E}(X)}{t}$	$P(\|X - \mathsf{E}(X)\| \geq t) \leq \frac{\mathrm{var}(X)}{t^2}$
$\lim_{n\to\infty} P(\|X_n - X\| < \epsilon) = 1$	$\lim_{n\to\infty} P(\|X_n - \mu\| < \epsilon) = 1$
If $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{p} a$ then $Y_nX_n \xrightarrow{d} aX$ and $X_n + Y_n \xrightarrow{d} X + a$.	$P(X = x\|r,p) = \binom{r+x-1}{x}p^r(1-p)^x$ $\mathsf{E}(X) = \frac{r(1-p)}{p}$; $\mathrm{var}(X) = \frac{r(1-p)}{p^2}$
$P(X = x\|n,p) = \binom{n}{x}p^x(1-p)^{n-x}$ $\mathsf{E}(X) = p$; $\mathrm{var}(X) = np(1-p)$	$\widehat{\boldsymbol{\beta}} = \left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}\boldsymbol{X}'\boldsymbol{y}$ $\widehat{\boldsymbol{\beta}} = \left(\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{X}\right)^{-1}\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{y}$
$f(x) = \frac{a}{b}\left(\frac{x}{b}\right)^{a-1}e^{-(x/b)^a}$; $E(X^k) = b^k\Gamma(1+k/a)$	$\Gamma(z) = \int_0^\infty z^{u-1}e^{-z}du$
$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}p^{\alpha-1}(1-p)^{\beta-1}$; $\mathsf{E}(X) = \frac{\alpha}{\alpha+\beta}$, $\mathrm{var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	$\pi(\theta\|\boldsymbol{x}) = \frac{f(\boldsymbol{x}\|\theta)\pi(\theta)}{m(\boldsymbol{x})}$
$f(x\|\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}$; $\mathsf{E}(X) = \mathrm{var}(X) = \lambda$	$\mathrm{var}_\theta\, W(\boldsymbol{X}) \geq \frac{\left(\frac{\partial}{\partial\theta}\mathsf{E}_\theta W(\boldsymbol{X})\right)^2}{\mathsf{E}_\theta\left(\left[\frac{\partial}{\partial\theta}\ln f(\boldsymbol{X}\|\theta)\right]^2\right)}$
$\mathsf{E}_\theta\left(\left[\frac{\partial}{\partial\theta}\ln f(\boldsymbol{X}\|\theta)\right]^2\right) = n\mathsf{E}_\theta\left(\left[\frac{\partial}{\partial\theta}\ln f(X\|\theta)\right]^2\right)$	$\mathsf{E}_\theta\left(\left[\frac{\partial}{\partial\theta}\ln f(X\|\theta)\right]^2\right) = -\mathsf{E}_\theta\left(\frac{\partial^2}{\partial\theta^2}\ln f(X\|\theta)\right)$
$\sqrt{n}\left(\tau(\widehat{\theta}) - \tau(\theta)\right) \xrightarrow{d} N(0, v(\theta))$	$\widehat{\mathrm{var}}\tau(\widehat{\theta}) \approx \frac{\left[\tau'(\theta)\right]^2\|_{\theta=\widehat{\theta}}}{-\frac{\partial^2}{\partial\theta^2}\ln L(\theta\|\boldsymbol{x})\|_{\theta=\widehat{\theta}}}$
$[I(\boldsymbol{\theta})]_{i,j} = \mathsf{E}_{\boldsymbol{\theta}}\left[\left(\frac{\partial}{\partial\theta_i}\ln f(\boldsymbol{X}\|\boldsymbol{\theta})\right)\left(\frac{\partial}{\partial\theta_j}\ln f(\boldsymbol{X}\|\boldsymbol{\theta})\right)\right]$	$e^{-t} = \lim_{n\to\infty}\left(1 - \frac{t}{n}\right)^n$
$f(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$	$f(x) = 1/(b-a)$; $\mathsf{E}(X) = \frac{b-a}{2}$; $\mathrm{var}\, X = \frac{(b-a)^2}{12}$

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> qt(0.95,df=13)
[1] 1.770933
> qt(0.975,df=13)
[1] 2.160369
> qt(0.95,df=14)
[1] 1.76131
> qt(0.975,df=14)
[1] 2.144787
> qt(0.95,df=15)
[1] 1.75305
> qt(0.975,df=15)
[1] 2.13145
> qnorm(0.95)
[1] 1.644854
> qnorm(0.975)
[1] 1.959964
> qnorm(0.99)
[1] 2.326348
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