# 12.1: Nondegeneracy of Bound States

For 1D Potentials over $x \in (-\infty, \infty)$ there are no degenerate bound states. (normalizable energy eigenstates ($\Psi \to 0$ $x \to \pm\infty$))

With $V(x)$ real the energy eigenstates can be chosen to be real for 1-D Probabilities, the bound states are real up to a phase.

Given a complex $\Psi(x)$ then:

$$\Psi_{re}(x) \equiv \frac{\Psi(x) + \Psi^*(x)}{2}$$

$$\Psi_{im}(x) \equiv \frac{1}{2i}(\Psi(x) - \Psi^*(x))$$

real solutions with the same energy

Since there are no degenerate bound states, then $\Psi_{im}(x) \sim \Psi_{re}(x)$

$$\Psi_{im}(x) = c\Psi_{re}(x)$$
$$c \in \mathbb{R}$$

$$\begin{aligned}\Psi &= \Psi_{re} + i\Psi_{im} \\ &= \Psi_{re} + ic\Psi_{re} \\ &= (1+ic)\Psi_{re}\end{aligned}$$

$$\Psi = e^{i\operatorname{Arg}(1+ic)}\sqrt{1+c^2}\,\Psi_{re}$$

## 12.2: Potentials $V(-x)=V(x)$

If $V(-x)=V(x)$, the energy eigenstates can be chosen to be even or odd under $x \to -x$

For 1D potentials, the bound states are either even or odd.

$$\psi''(x) + \frac{2m}{\hbar}(E-V(x))\psi(x)=0$$

$\psi''$ = second derivative of $\psi$ evaluated at $x$

$$\varphi(x) \equiv \psi(-x)$$

$$\frac{d\varphi(x)}{dx} = \psi'(-x)(-1)$$

$$\frac{d^2\varphi(x)}{dx^2} = \psi''(-x)(-1)(-1)$$
$$= \psi''(-x)$$

$$\psi''(-x) + \frac{2m}{\hbar^2}(E-V(x))\psi(-x)=0$$

$$\frac{d^2\varphi(x)}{dx^2} + \frac{2m}{\hbar^2}(E-V(x))\varphi(x)=0$$

Both $\psi(x)$ and $\varphi(x)=\psi(-x)$ are solutions to the Schrödinger Equation with the same energy.

$$\psi_s(x) \equiv \frac{1}{2}(\psi(x)+\psi(-x))$$
$$\psi_a(x) \equiv \frac{1}{2}(\psi(x)-\psi(-x))$$

12.2: Potentials $V(x) = V(-x)$ (continued)

$\psi_s(-x) = \psi_s(x)$

$\psi_a(-x) = -\psi_a(x)$

even/odd

1D bound states:

$\psi(-x) = c\,\psi(x)$

$c \in \mathbb{R}$

$\underbrace{\psi(x) = c\,\psi(-x) \Rightarrow c^2 = 1}_{\text{two solutions}}$

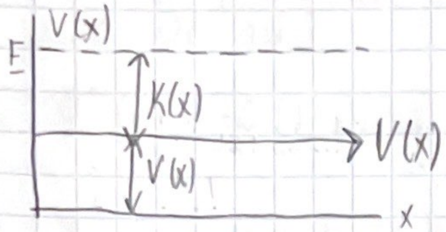
$c = 1 \Rightarrow \psi$ is even

$c = -1 \Rightarrow \psi$ is odd

# 12.3: Local de Broglie Wavelength

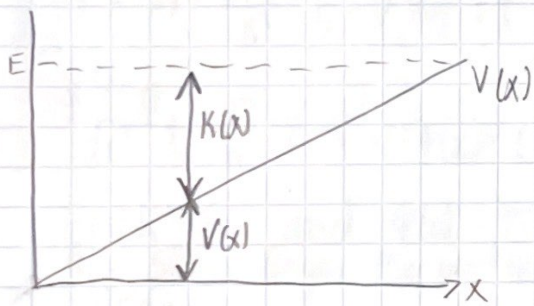
Qualitative insights

$E = K(x) + V(x)$




$K = \frac{p^2}{2m}$

$\lambda = \frac{h}{p}$

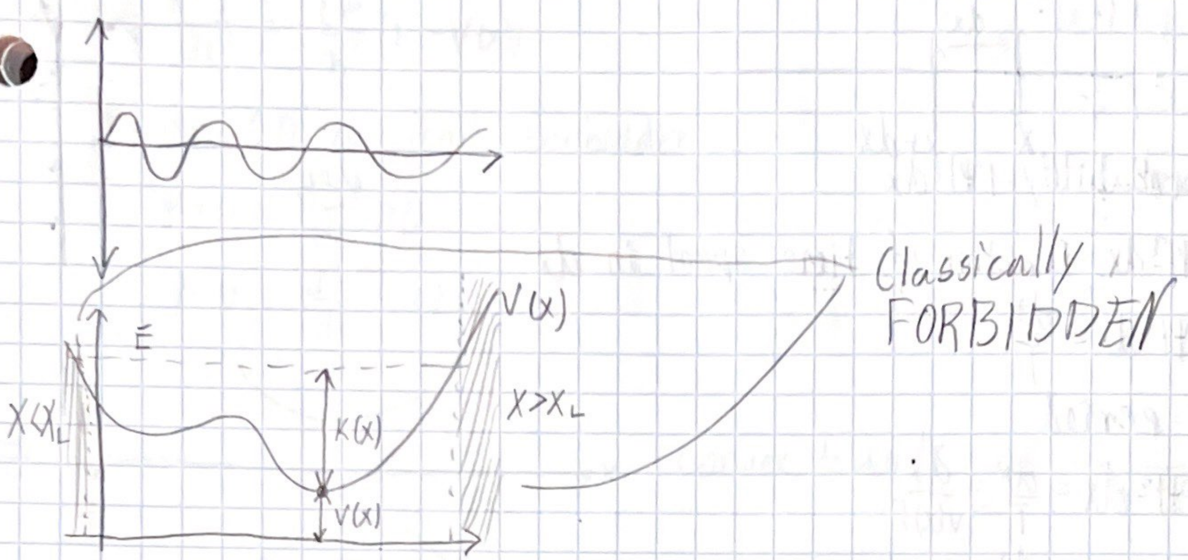




$K(x) \longrightarrow$ decreasing

$P(x) \longrightarrow$ decreasing

$\lambda(x) \rightarrow$ increasing

12.3: Local deBroglie Wavelength (continued)

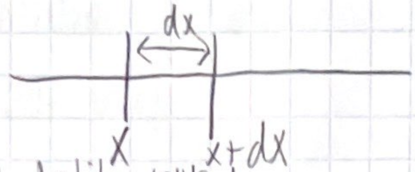



$x_R$ and $x_L$ are turning points as they classically loose all kinetic energy

$$\lambda \frac{dV}{dx} \ll V(x)$$

$$\lambda(x) = \frac{h}{p(x)} = \frac{h}{\sqrt{2mK(x)}}$$

# 12.4: Correspondence Principle




probability $= |\psi|^2 dx$

$|\psi|^2 dx$ = fraction of time spent in dx

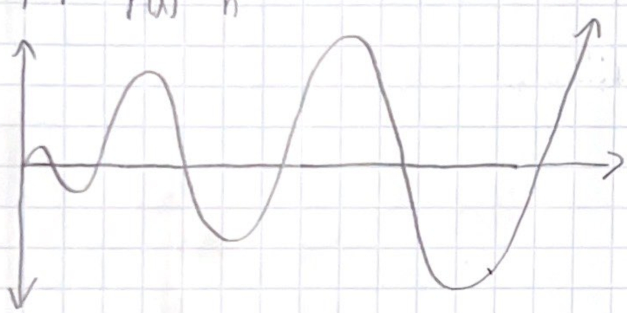
$|\psi|^2 dx = \frac{dt}{T}$

T = period

$|\psi|^2 dx = \frac{dt}{T} = \frac{dx}{v(x)T}$

$= \frac{dx}{p(x)} \frac{m}{T}$

$|\psi|^2 \sim \frac{1}{p(x)} = \frac{\lambda(x)}{h}$



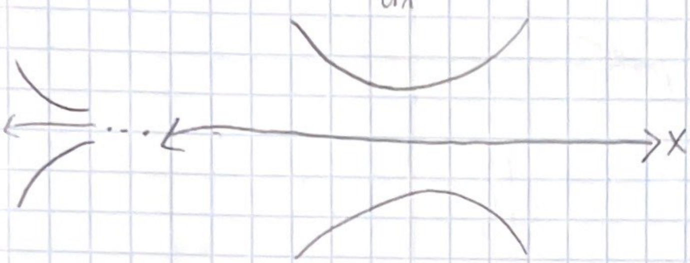
# 12.5: The Local Picture

$$\frac{1}{\psi}\frac{d^2\psi}{dx^2} = -\frac{2m}{\hbar^2}(E-V(x))$$

$E-V(x)<0$ classically forbidden

$\psi>0 \Rightarrow \frac{d^2\psi}{dx^2}>0$

$\psi<0 \Rightarrow \frac{d^2\psi}{dx^2}<0$

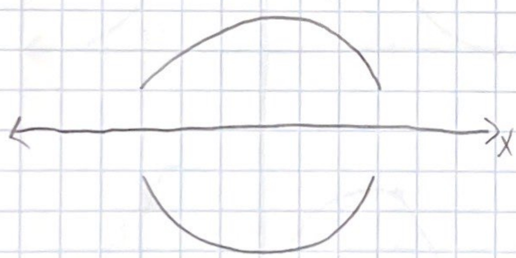



Convex towards axis ...

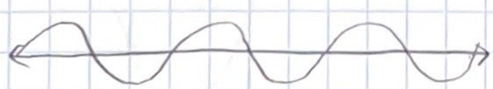
$E-V(x)>0$ classically allowed

$\psi>0 \Rightarrow \frac{d^2\psi}{dx^2}<0$

$\psi<0 \Rightarrow \frac{d^2\psi}{dx^2}>0$




Concave toward axis



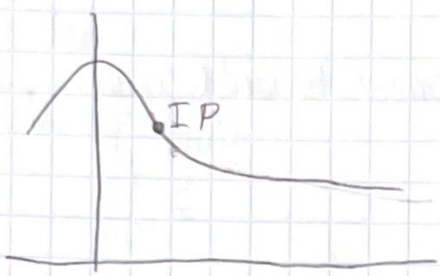
12.5: The Local Picture (continued)

$E - V(x_0) = 0$

$x_0$ is a turning point

$\frac{1}{\psi}\frac{d^2\psi}{dx^2} = 0$

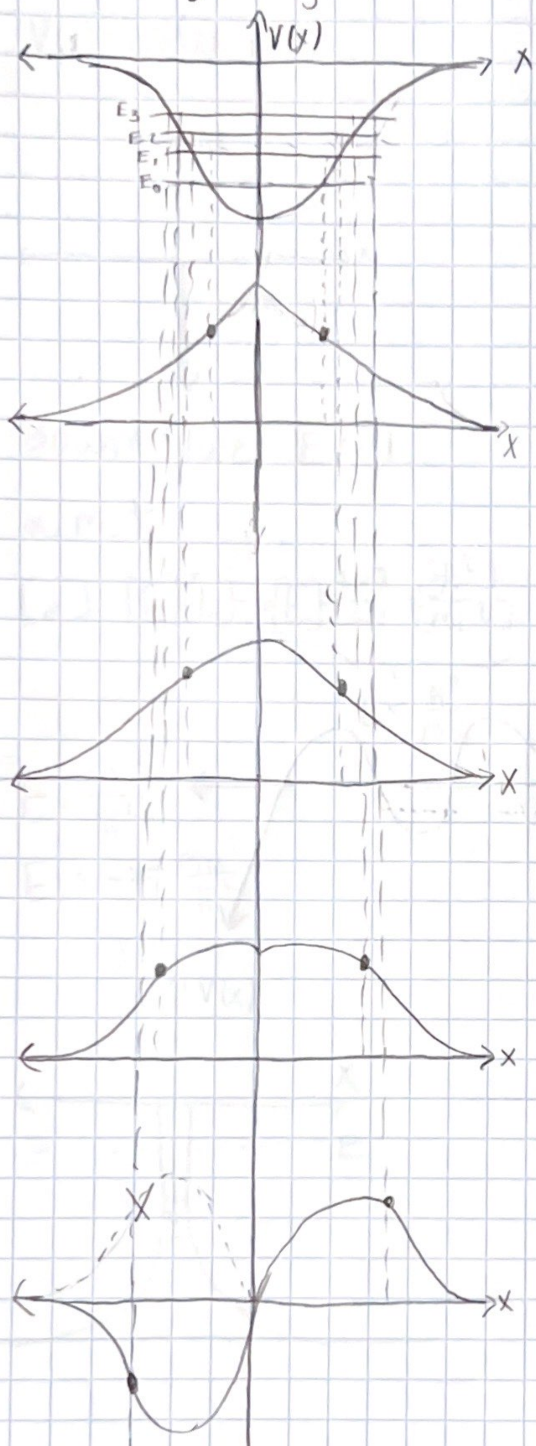
If $\psi \neq 0$, then $\left.\frac{d^2\psi}{dx^2}\right|_{x_0} = 0 \Rightarrow$ inflection point




$\frac{d^2\psi}{dx^2} = (E-V)\psi$

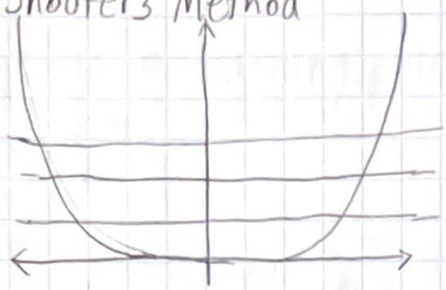
Inflection points also at the nodes

# 12.6: Energy Eigenstates Symmetric Potential



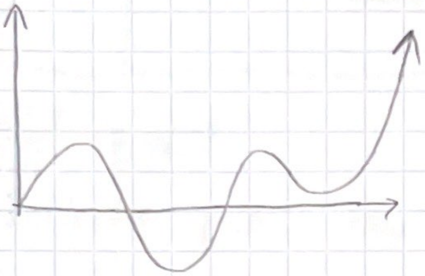
12.6: Energy Eigenstates Symmetric potential (continued)

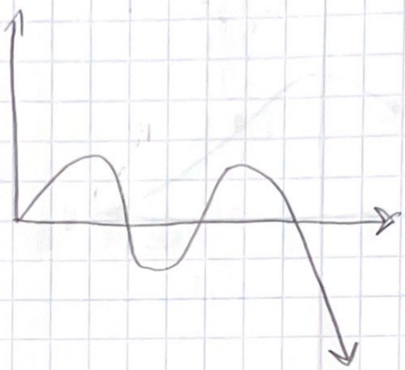
Shooter's Method

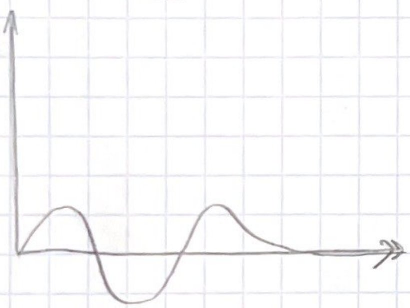


Pick energy $E_0$

$\left.\begin{array}{l}\Psi(x=0)=1 \\ \Psi'(x=0)=0\end{array}\right\}$ Integrate

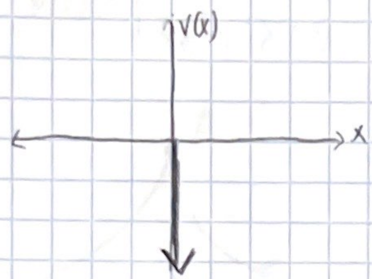






B.1: Delta function Potential Preliminaries:

$V(x) = -\alpha\delta(x),\ \alpha > 0$



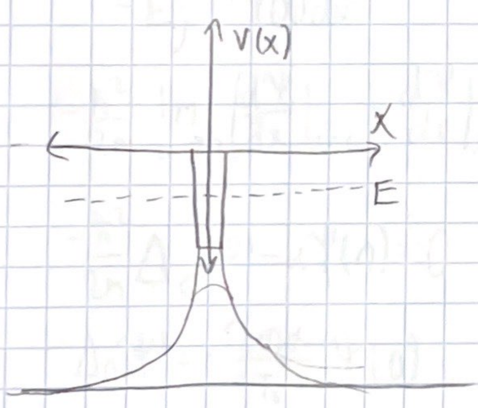

Bound States ($E < 0$)

$\alpha, m, \hbar$

$[\alpha] = [E]\cdot[L] \rightarrow [E] = \underbrace{\frac{[\alpha]}{[L]} = \frac{[\hbar^2]}{[mL^2]}}_{L = \frac{\hbar^2}{m\alpha}}$

$E = \frac{m\alpha^2}{\hbar^2}$

$E_b = -\# \frac{m\alpha^2}{\hbar^2}$



## 13.1: Delta Function Potential Preliminaries (continued)

$-\frac{\hbar^2}{2m}\psi'' = E\psi \quad x \neq 0$

$\hookrightarrow \psi'' = -\frac{2mE}{\hbar}\psi$

$= \kappa^2\psi, \ \kappa^2 = -\frac{2mE}{\hbar^2} > 0$

$= \psi'' = \kappa^2\psi(x)$

$e^{-\kappa x}, e^{\kappa x}$

$\cosh(\kappa x), \sinh(\kappa x)$

For an odd bound state (first exited state)

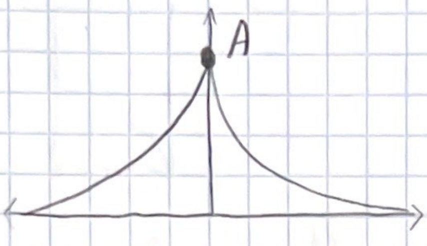
$\psi = 0, x = 0$

$\psi(x) = \sinh \kappa x$?? no

13.2 Delta Function Potential Bound State

$Ae^{-\kappa|x|} = \psi(x \neq 0)$

$$\psi(x) = \begin{cases} Ae^{-\kappa x}, & x > 0 \\ Ae^{\kappa x}, & x < 0 \end{cases}$$




$\int |\psi|^2 dx = 1$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi(x) = E\psi(x)$$

Integrate

$$\int_{-\epsilon}^{\epsilon} dx \longrightarrow -\frac{\hbar^2}{2m}\left(\frac{d\psi}{dx}\Big|_{x=\epsilon} - \frac{d\psi}{dx}\Big|_{x=-\epsilon}\right) + \int_{-\epsilon}^{\epsilon} -\alpha\delta(x)\psi(x)dx$$

$$= E\int_{-\epsilon}^{\epsilon}\psi(x)dx$$

$$-\frac{\hbar^2}{2m}\lim_{\epsilon\to 0}\left(\frac{d\psi}{dx}\Big|_{x=\epsilon} - \frac{d\psi}{dx}\Big|_{x=-\epsilon}\right) + \alpha\psi(0) \equiv 0$$

$$-\frac{\hbar^2}{2m}\Delta_0\psi' - \alpha\psi(0) = 0$$

$$\Delta_0\psi' = -\frac{2m\alpha}{\hbar^2}\psi(0)$$

13.2: Delta Function Potential Bound State (continued)

$V(x) = -\alpha\delta(x)$

$-\kappa A e^{-\kappa\epsilon} - \left(\kappa A e^{-\kappa\epsilon}\right) = -\frac{2m\alpha}{\hbar^2}A$

$\lim_{\epsilon \to 0} \Rightarrow -2\kappa A = -\frac{2m\alpha}{\hbar^2}A$

$\kappa = \frac{m\alpha}{\hbar^2}$

$E = -\frac{\hbar^2\kappa^2}{2m}$

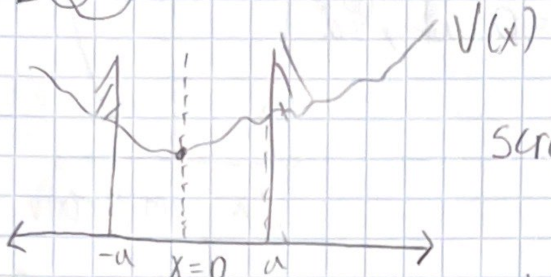
$= -\frac{\hbar^2 m^2 \alpha^2}{2m\hbar^4}$

$E = -\frac{m\alpha^2}{2\hbar^2}$

# 13.3: Node Theorem

$\psi_1, \psi_2, \psi_3 \ldots$ energy eigenstates of 1D $V(x)$
$E_1 < E_2 < E_3 < \ldots$

$\psi_n$ has $(n-1)$ nodes



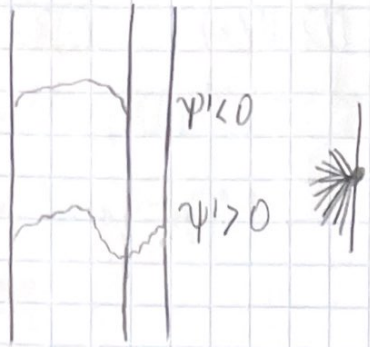

Screened Potential $V_a(x)$

$\psi(x)=0, \psi'(x)=0$ is not possible

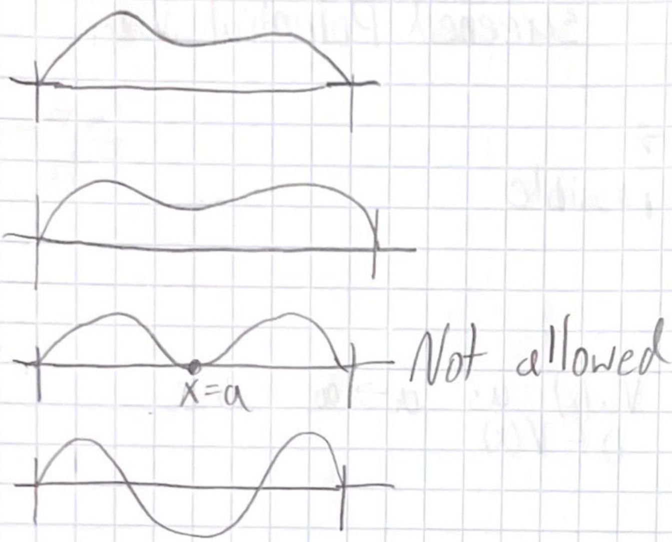
$$V_a(x) = \begin{cases} V(x), & |x|<a \\ \infty, & |x|>a \end{cases}$$

Bound states of $V_a(x)$ as $a \to \infty$ are the bound states of $V(x)$

## 13.3: Node Theorem (continued)




some point at which $\psi$ and $\psi' = 0$ at $x = a$



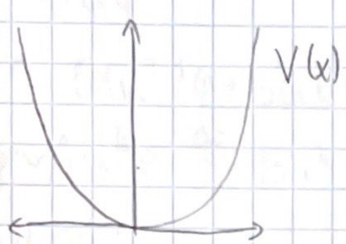
# 13.4: Simple Harmonic Oscillator Differential Equation

$E = KE + PE = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$

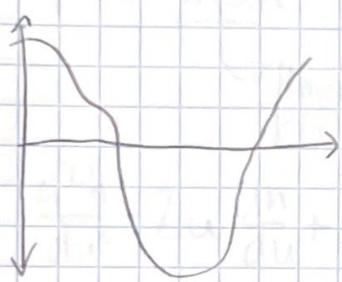
$\omega = \sqrt{\frac{k}{m}}$

$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$

$[\hat{x}, \hat{p}] = i\hbar$




$V(x) = \frac{1}{2}m\omega^2 x^2$



$\hat{H}\varphi_n(x) = E\varphi_n(x)$

$-\frac{\hbar^2}{2m}\frac{d^2\varphi}{dx^2} + \frac{1}{2}m\omega^2 x^2 \varphi(x) = E\varphi(x)$

$x = au$

$[a] = [L]$

$E = \frac{\hbar^2}{ma^2} = m\omega^2 a^2$

$a^2 = \frac{\hbar}{m\omega}$

## 13.4: Simple Harmonic Oscillator Differential Equation (continued)

$$\frac{-\hbar^2}{2ma^2}\frac{d^2\varphi}{du^2}+\frac{1}{2}m\omega^2a^2u^2\varphi=E\varphi$$

$$-\frac{1}{2}\hbar\omega\frac{d^2\varphi}{du^2}+\frac{1}{2}\hbar\omega u^2\varphi=E\varphi$$

$$\frac{2}{\hbar\omega}\rightarrow\quad-\frac{d^2\varphi}{du^2}+u^2\varphi=\frac{2E}{\hbar\omega}\varphi$$

$$\varepsilon=\frac{2E}{\hbar\omega}$$

$\uparrow$

unit free energy

$$E=\frac{\hbar\omega}{2}\varepsilon$$

$$\frac{d^2\varphi}{du^2}+u^2\varphi=E\varphi$$

$$\frac{d^2\varphi}{du^2}=(u^2-E)\varphi$$

## 13.5: Behavior of the Differential Equation as $u \to \pm\infty$?

$$\frac{d^2\varphi}{du^2} \simeq u^2\varphi$$

$$\varphi \overset{?}{=} u^k e^{\frac{\alpha u^2}{2}}$$

$$\varphi' \simeq \alpha u \varphi$$

$$\varphi'' \cong (\alpha u^2)\varphi + \text{subleading} \quad (u \to \infty)$$

$$\varphi \sim A u^k e^{-\frac{u^2}{2}} + B u^k e^{\frac{u^2}{2}} \quad u \to \pm\infty$$

WLOG:

$$\varphi(u) = h(u) e^{-\frac{u^2}{2}}$$

Hope that this is a polynomial $P(x)$

$$\frac{d^2h}{du^2} - 2u\frac{dh}{du} + (\varepsilon - 1)h = 0$$

Differential equation for $h$