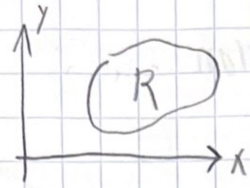
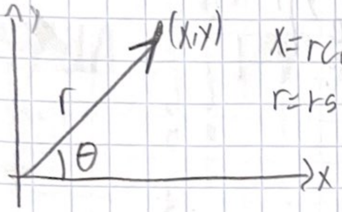
17: Polar Coordinates

$$\iint_R f(x,y)\,dA$$



$$= \iint f(x,y)\,dy\,dx$$

$$\iint_R 1-x^2-y^2\,dA$$



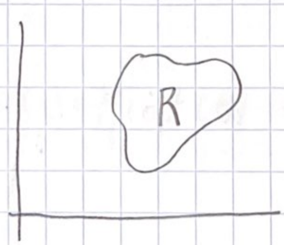
$x = r\cos\theta$

$r = r\sin\theta$

$$\int_0^{\frac{\pi}{2}}\int_0^1 (1-r^2)\,r\,dr\,d\theta = \int_0^{\frac{\pi}{2}} \left[\tfrac{1}{2}r^2 - \tfrac{1}{4}r^4\right]_0^1 d\theta = \int_0^{\frac{\pi}{2}} \tfrac{1}{4}\,d\theta = \frac{\pi}{8}$$

1) Find Area of R

$$\text{Area}(R) = \iint_R 1\,dA$$



Mass of a flat object with density δ = mass per unit

$$\Delta m = \delta\Delta A \Rightarrow m = \iint_R \delta \cdot dA$$

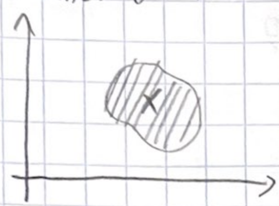
17: Polar Coordinates

2) Average value of F on a region

$$\bar{f} = \frac{1}{A}\iint_R f\,dA$$

$$w\bar{f} = \frac{1}{Mass(R)}\iint_R f\delta\,dA$$

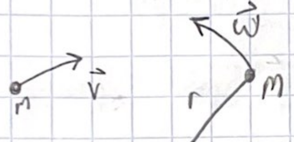
3) Center of Mass



$(\bar{x}, \bar{y})$, $\bar{x} = \frac{1}{m}\iint_R x\delta\,dA$

$\bar{y} = \frac{1}{m}\iint_R y\,\delta\,dA$

4) Moment of Inertia (how hard it is to rotate)



$ke = \frac{1}{2}mv^2 = \frac{1}{2}mr^2\omega^2$

$\Delta m \approx \delta\,\Delta A$

about $(0,0) = \iint_R r^2\delta\,dA = I_0$

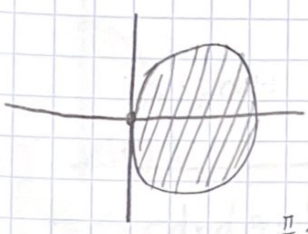
$$I_x = \iint_R y^2\,\delta\,dA$$

17: Polar Coordinates

disk of $r=a$ $(\delta=1)$

$$I_0=\iint r^2\,dA=\int_0^{2\pi}\int_0^{a} r^2\,r\,dr\,d\theta=\int_0^{2\pi}\left[\tfrac{1}{4}r^4\right]_0^{a}d\theta$$

$$=\int_0^{2\pi}\tfrac{1}{4}a^4\,d\theta=\frac{\pi a^4}{2}$$



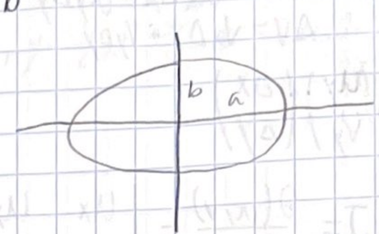
$$I_0=\iint_R r^2\,dA=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2a\cos\theta} r^2\,r\,dr\,d\theta=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\tfrac{1}{4}r^4\right]_0^{2a\cos\theta}d\theta$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}4a^4\cos^4\theta\,d\theta=\tfrac{3}{2}\pi a^4$$

18: Change of Variables

Area of ellipse with axes, a, b

$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$



$$\iint_R dx\,dy$$

$R: \{(x,y) \mid \left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2<1\}$

$$\begin{cases} \frac{x}{a}=u \rightarrow du=\frac{1}{a}dx \\ \frac{y}{b}=v \rightarrow dv=\frac{1}{b}dy \end{cases}$$

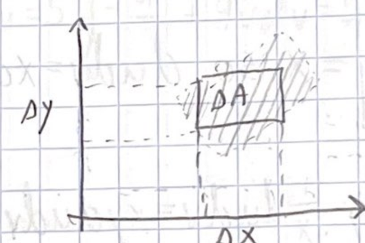
$$\iint_R dx\,dy=\iint_{u^2+v^2<1} du\,dv=\pi ab$$

In general: find scaling factor ($dxdy$ vs $dudv$)

$u=3x-2y,\ v=x+y$

$dA=dxdy$

$dA'=dudv$



$A'=\left|\begin{matrix}3 & 1\\ -2 & 1\end{matrix}\right|=5 \Rightarrow dA'=5dA$

$$\iint dx\,dy=\iint \frac{1}{5}du\,dv$$

18: Change of Variables

$u = u(x,y)$ $\quad \Delta u = u_x \Delta x + u_y \Delta y$
$v = v(x,y)$ $\quad \Delta v = v_x \Delta x + v_y \Delta y$

$$\begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix} \approx \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

Jacobian: $J = \dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$

$$du\,dv = |J|\,dx\,dy = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| dx\,dy$$

$x = r\cos\theta$
$y = r\sin\theta$, $\quad \dfrac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$

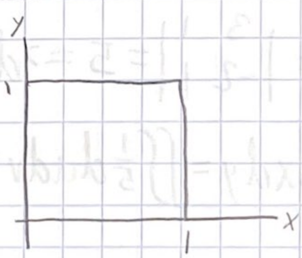
$$dx\,dy = r\,dr\,d\theta$$

$\displaystyle\int_0^1\!\!\int_0^1 x^2 y\,dx\,dy$ by change of variables: $u = x$, $v = xy$

$\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 0 \\ y & x \end{vmatrix} = x$ so $du\,dv = x\,dx\,dy$

$$x^2 y\,dx\,dy = x^2 y \frac{1}{x} du\,dv = xy\,du\,dv = v\,du\,dv$$

$\displaystyle\iint_R v\,du\,dv = \int_0^1\!\!\int_v^1 v\,du\,dv$



18: Change of Variables

$\iint_R x+y\,dA$ R: bounded: $y=-2x$, $y=\frac{1}{2}x-\frac{15}{2}$, $y=-2x+10$, $y=\frac{1}{2x}$

$x=u+2v$, $y=v-2u$

$y=2x: v-2u=2u-4v \rightarrow v=0$

$y=\frac{1}{2}x-\frac{15}{2}: 2v-4u=u+2v-15 \rightarrow u=3$

$y=-2x+10: v-2u=-2u-4v+10 \rightarrow v=2$

$y=\frac{1}{2}x: 2v-4u=u+2v \rightarrow u=0$

$J=\begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix}=5$

$\int_0^2\int_0^3 (u+2v)+(v-2u)5\,du\,dv = 5\int_0^2\int_0^3 -u+3v\,du\,dv$

$=5\int_0^2 \left[-\frac{1}{2}u^2+3vu\right|_0^3 dv = 5\int_0^2 -\frac{9}{2}+9v\,dv = 5\left[-\frac{9}{2}v+\frac{9}{2}v^2\right|_0^2$

$=45$

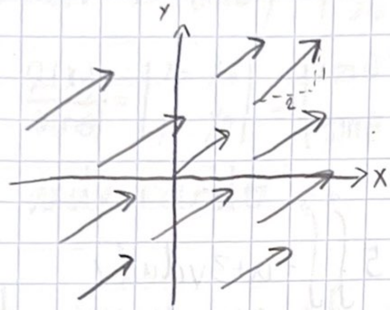
14: Vector Fields

$\vec{F} = M\hat{i} + N\hat{j}$ M and Y are functions of x and y

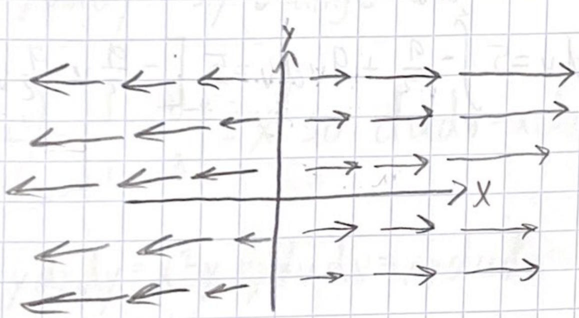
At each point in the plane, there is a vector $\vec{F}$ that depends on x any y

Examples
- velocity $\vec{v}$
- force field $\vec{F}$

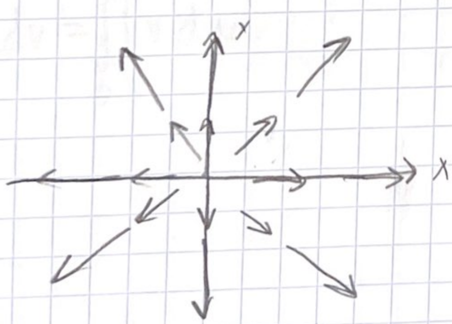
$\vec{F} = 2\hat{i} + \hat{j}$



$\vec{F} = x\hat{i}$



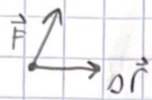
$\vec{F} = x\hat{i} + y\hat{j}$



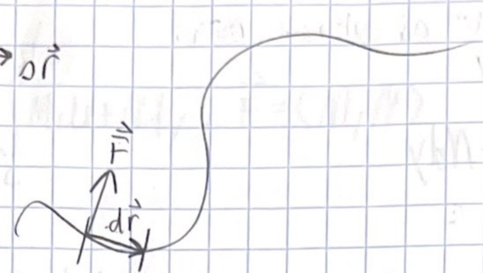
14: Vector Fields

Work and line integral

$W = \vec{F} \cdot \Delta \vec{r}$



$W = \int_C \vec{F} \cdot d\vec{r}$



$\left(\lim_{\Delta r_i \to 0} \sum_i \vec{F} \cdot \Delta r_i \right)$

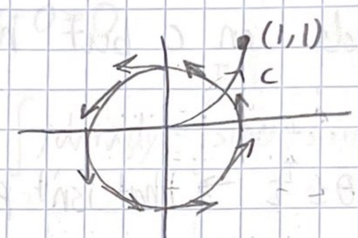
$= \sum_i \vec{F} \cdot \left(\frac{\Delta \vec{r}}{\Delta t} \Delta t \right)$

$\Downarrow \vec{v} = \frac{d\vec{r}}{dt}$

$\int_{t_i}^{t_f} \vec{F} \cdot \frac{d\vec{r}}{dt} dt$

$\vec{F} = -y\hat{i} + x\hat{j}$

$C: \{x = t,\ y = t^2,\ 0 \le t \le 1\}$



$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F} \cdot \frac{dr}{dt}$

$= \int_0^1 \langle -t^2, t \rangle \cdot \langle 1, 2t \rangle dt = \int_0^1 -t^2 + 2t^2 \, dt$

$= \int_0^1 t^2 dt = \frac{1}{3}$

14: Vector Fields

$\vec{F} = \langle M, N \rangle$

$d\vec{r} = \langle dx, dy \rangle$

$\vec{F} \cdot d\vec{r} = M\,dx + N\,dy$

$\int_C \vec{F} \cdot d\vec{r} = \int_C M\,dx + N\,dy$

$= \int_C -y\,dx + x\,dy$ in terms of

$x = t \quad \rightarrow \quad dx = dt$

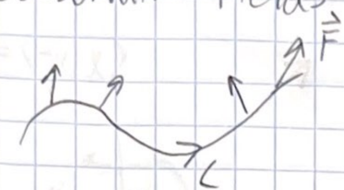
$y = t^2 \qquad\quad dy = 2t\,dt$

$\int_C -t^2\,dt + t \cdot 2t\,dt = \int_0^1 t^2\,dt = \frac{1}{3}$

$\int_C \vec{F} \cdot d\vec{r}$ depends on C but not puramaterization

$\begin{cases} x = \sin\theta \\ y = \sin^2\theta \end{cases} \quad 0 \leq \theta \leq \frac{\pi}{2} \rightarrow$ that isn't practical

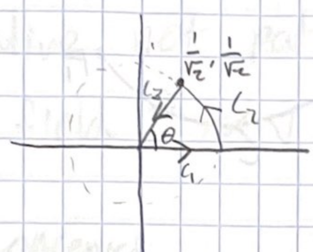
10: Conservative Fields



$$\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot\hat{T}\,ds = \int_C M\,dx + N\,dy, \quad \vec{F} = \langle M, N\rangle$$

$y\hat{i} + x\hat{j}$

$$\int_C \vec{F}\cdot d\vec{r}$$



$C = C_1 + C_2 + C_3$

$0 \le \theta \le \frac{\pi}{4}$

C_1: $y = 0 \Rightarrow \int_{C_1} y\,dx + x\,dy = \int_{C_1} 0\,dx + 0 = 0$

C_2: $x = \cos\theta$, $y = \sin\theta$, $0 \le \theta \le \frac{\pi}{4}$; $dx = -\sin\theta\,d\theta$, $dy = \cos\theta\,d\theta$

$$\Rightarrow \int_{C_2} y\,dx + x\,dy = \int_0^{\frac{\pi}{4}} \sin\theta(-\sin\theta)\,d\theta + \cos\theta\cdot\cos\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \cos^2\theta - \sin^2\theta\,d\theta = \left[\frac{1}{2}\sin 2\theta\right|_0^{\frac{\pi}{4}} = \frac{1}{2}$$

C_3: $x = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}t$, $y = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}t$, $0 \le t \le 1$

$$\Rightarrow \int_{-C_3} = -\int_{C_3} \Rightarrow \int_0^{\frac{1}{\sqrt{2}}} t\,dt + t\,dt = \left[t^2\right|_0^{\frac{1}{\sqrt{2}}} = -\frac{1}{2}$$

$$\int_{C_1} + \int_{C_2} + \int_{C_3} = 0 + \frac{1}{2} - \frac{1}{2} = 0$$

20: Conservative Fields

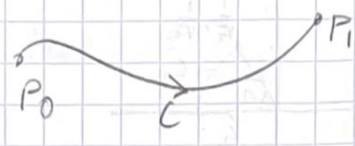
Special case: $\vec{F} = \nabla f$ = gradient field

($f(x,y)$) is called potential of $\vec{F}$

Then we can simplify the $\int_C \vec{F}\cdot d\vec{r}$

Fundimental Theorem of Line Integrals

$$\int_C \nabla f \cdot d\vec{r} = f(P_1) - f(P_0)$$



$$\int_C f_x\,dx + f_y\,dy = \int_C df = f(P_1) - f(P_0)$$

$$\int_C \nabla f \cdot d\vec{r} \Rightarrow C: x = x(t),\ y = y(t)$$

$$dx = x'(t)dt,\ dy = y'(t)dt$$

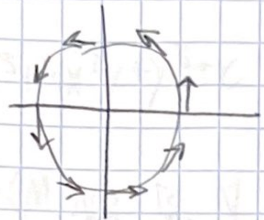
$$\int_C \left(f_x \frac{dx}{dt} + f_y \frac{dy}{dt}\right) dt = \int_C \frac{df}{dt}\,dt = \int_{t_0}^{t_1} \frac{df}{dt}\,dt$$

$\vec{F} = \nabla f \Rightarrow$ conservative

$$\int_C \vec{F}\cdot d\vec{r} = 0$$

20: Conservative Fields

$\vec{F} = \langle -y, x \rangle$



$\vec{F}$ is tangent

$\vec{F} \cdot \hat{T} = |\vec{F}| = 1$

Not conservative, not path independent

If a force field $\vec{F}$ is ∇ of potential

$\vec{F} = \nabla f$

Energy is conserved