16: Two State Systems

$|\alpha\rangle = D(\alpha)|0\rangle$

$D = e^{\alpha \hat{a}^{+} - \alpha^{*}\hat{a}}$

$a|\alpha\rangle = \alpha|\alpha\rangle$

$$E = \frac{1}{2}\iiint_V \epsilon_0\left[\vec{E}^2(\vec{r},t) + c^2\vec{B}^2(\vec{r},t)\right] d^3x$$

$$E_x(z,t) = \sqrt{\frac{2}{\varepsilon_0 V}}\,\omega q(t)\sin kz$$

$$cB_y(z,t) = \sqrt{\frac{2}{\varepsilon_0 V}}\,p(t)\cos kz$$

$$E = \frac{1}{2}\left(p^2(t) + \omega^2 q^2(t)\right)$$

$[p] = [\sqrt{E}]$

$[q] = [t\sqrt{E}]$

$[pq] = [tE]$

$= [\hbar]$

$$H = \frac{1}{2}\left(\hat{p}^2 + \omega^2\hat{q}^2\right)$$

16: Two State Systems

$$i\hbar \frac{d}{dt} P_H(t) = [H, \hat{p}_H]$$

$$H = \frac{1}{2}(\hat{p}^2 + \omega^2 \hat{q}^2) \quad m \to 1$$

$$\hat{q} = \sqrt{\frac{\hbar}{2\omega}}(\hat{a} + \hat{a}^\dagger)$$

$$\hat{p} = \frac{1}{i}\sqrt{\frac{\omega\hbar}{2}}(\hat{a} - \hat{a}^\dagger)$$

$$H = \hbar\omega(\hat{a}^\dagger \hat{a} + \frac{1}{2})$$

$$H = \hbar\omega(\hat{N} + \frac{1}{2})$$

$$\hat{q}(t) = \sqrt{\frac{\hbar}{2\omega}}(e^{-i\omega t}\hat{a} + e^{i\omega t}\hat{a}^\dagger$$

$$\hat{E}_x(z,t) = \mathcal{E}_0(e^{-i\omega t}\hat{a} + e^{i\omega t}\hat{a}^\dagger)\sin kz$$

$$\mathcal{E}_0 = \sqrt{\frac{\hbar\omega}{\varepsilon_0 V}}$$

$|n\rangle$

16: Two State Systems

$|n\rangle$

$$\langle E_x\rangle_{|n\rangle} = \mathcal{E}_0\left(e^{-i\omega t}\langle n|\hat{a}|n\rangle + e^{i\omega t}\langle n|\hat{a}^{+}|n\rangle\right)\sin kz$$

$$= 0$$

$|\alpha\rangle$

$$\langle E_x\rangle_{|\alpha\rangle} = \mathcal{E}_0\left(e^{-i\omega t}\langle \alpha|\hat{a}|\alpha\rangle + e^{i\omega t}\langle \alpha|\hat{a}^{+}|\alpha\rangle\right)\sin kz$$

$$= \mathcal{E}_0\left(\alpha e^{-i\omega t} + \alpha^* e^{i\omega t}\right)\sin kz$$

$$= 2\mathcal{E}_0 \mathrm{Re}\left(\alpha e^{-i\omega t}\right)\sin kz$$

$$\alpha = |\alpha| e^{i\theta}$$

$$= 2\mathcal{E}_0 |\alpha| \cos(\omega t - \theta)\sin kz$$

$$\langle H\rangle = \hbar\omega\left(\langle\hat{N}\rangle + \tfrac{1}{2}\right)$$

$$= \hbar\omega\left(|\alpha|^2 + \tfrac{1}{2}\right)$$

Spin precession

$$\vec{\mu} = \frac{q}{2m}\vec{S}$$

$$\rightarrow \vec{\mu} = g\frac{q}{2m}\vec{S}$$

16: Two state systems

$\hat{H} = -\vec{\mu}\cdot\vec{B}$

$= -\gamma\vec{S}\cdot\vec{B}$

$= -\gamma\vec{B}\cdot\vec{S}$

$= -\gamma[B_x\hat{S}_x + B_y\hat{S}_y + B_z\hat{S}_z]$

If $\vec{B} = B\hat{z}$

$\Rightarrow H = -\gamma B\hat{S}_z$

$u(t,0) = \exp\left(-\frac{iH_s t}{\hbar}\right) =$

$= \exp\left(\frac{-i(-\gamma Bt)\hat{S}_z}{\hbar}\right)$

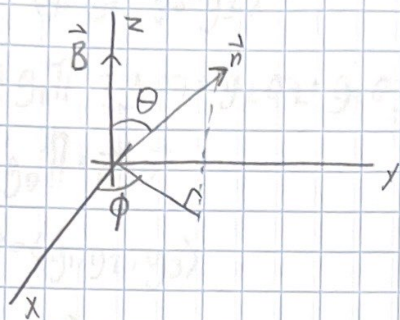
$R_{\vec{m}}(\alpha) = \exp\left(\frac{-i\alpha\hat{S}_{\vec{n}}}{\hbar}\right)$

$\hat{S}_{\vec{n}} = \vec{n}\cdot\hat{S}$

spin states

16: Two State systems

Rotation by angle α with respect to $\vec{n}$



$$|\Psi,0\rangle = \cos\frac{\theta_0}{2}|+\rangle + \sin\frac{\theta_0}{2}e^{i\phi_0}|-\rangle$$

$$H_s|+\rangle = -\gamma B S_z|+\rangle$$

$$= -\frac{\gamma B\hbar}{2}|+\rangle$$

$$H_s|-\rangle = -\gamma B S_z|-\rangle$$

$$= \frac{\gamma B\hbar}{2}|-\rangle$$

$$|\Psi,t\rangle = e^{\frac{-iH_s t}{\hbar}}|\Psi,0\rangle$$

$$= \cos\frac{\theta_0}{2}e^{\frac{-i\left(-\frac{\gamma B\hbar}{2}\right)t}{\hbar}}|+\rangle + \sin\frac{\theta_0}{2}e^{\frac{i\left(\frac{\gamma B\hbar}{2}\right)t}{\hbar}}e^{i\phi_0}|-\rangle$$

$$|\Psi,t\rangle = \cos\frac{\theta_0}{2}e^{\frac{i\gamma Bt}{2}}|+\rangle + \sin\frac{\theta_0}{2}e^{-\frac{i\gamma Bt}{2}}e^{i\phi_0}|-\rangle$$

$$= e^{\frac{i\gamma Bt}{2}}\left(\cos\frac{\theta_0}{2} + \sin\frac{\theta_0}{2}e^{i(\phi_0-\gamma Bt)}\right)|-\rangle$$

$$\theta(t) = \theta_0$$

$$\phi(t) = \phi_0 - \gamma Bt$$

16: Two State systems

$\vec{\mu}$ in a $\vec{B}$ field $\Rightarrow$ torque

$\vec{\tau} = \vec{\mu} \times \vec{B}$

$\frac{d\vec{S}}{dt} = \vec{\tau}$

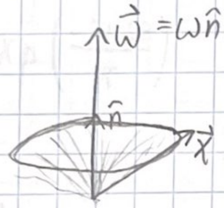
$= \vec{\mu} \times \vec{B}$

$= \gamma \vec{S} \times \vec{B}$

$= -(\gamma \vec{B}) \times \vec{S}$

$\frac{d\vec{S}}{dt} = -(\gamma \vec{B}) \times \vec{S}$

$\frac{d\vec{x}}{dt} = \vec{\omega} \times \vec{x}$



$\vec{\omega}_L = -\gamma \vec{B}$ - Larmour frequency

$H = \vec{\omega}_L \cdot \vec{S}$

16: Two State Systems

$$H = \begin{pmatrix} g_0+g_3 & g_1-ig_2 \\ g_1+ig_2 & g_0-g_3 \end{pmatrix}$$

$$= g_0\mathbb{1} + g_1\sigma_1 + g_2\sigma_2 + g_3\sigma_3$$

$$= g_0\mathbb{1} + \vec{g}\cdot\vec{\sigma}$$

$$\vec{g} = \langle g_1, g_2, g_3 \rangle$$

$$\vec{g} = g\vec{n}$$

$$H = g_0\mathbb{1} + gn\cdot\vec{\sigma}$$

$$n\cdot\vec{\sigma}\,|n;\pm\rangle = \pm|n;\pm\rangle$$

$$\vec{S} = \frac{\hbar}{2}\vec{\sigma}$$

$$\vec{n}\cdot\vec{S} = |n;\pm\rangle = \pm\frac{\hbar}{2}|n;\pm\rangle$$

$$\vec{\omega}_L = \frac{2\vec{g}}{\hbar}$$

17: Two State Systems

$H = g_0 \mathbb{1} + \vec{g}\cdot\vec{\sigma} = g\mathbb{1}$

$= g_0 \mathbb{1} + \vec{\omega}_L \cdot \vec{S}$

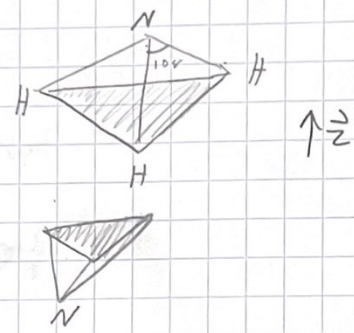
$E = g_0 \pm g \begin{matrix} \nearrow |\vec{n};+\rangle \\ \searrow |\vec{n};-\rangle \end{matrix}$

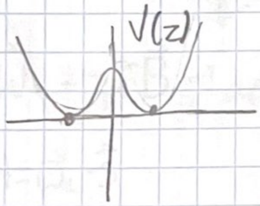
$\vec{\omega}_L = \dfrac{2\vec{g}}{\hbar}$

$|+\rangle = |1\rangle$

$|-\rangle = |2\rangle$

Ammonia NH_3





$|1\rangle = |\uparrow\rangle$ N-up

$|2\rangle = |\downarrow\rangle$ N down

17: Two State Systems

$H = \begin{pmatrix} E_0 & -\Delta \\ -\Delta & E_0 \end{pmatrix} \quad \Delta > 0$

$\langle 1|H|2\rangle$

$\langle 2|H|1\rangle$

$= E_0 \mathbb{1} - \Delta \sigma_1$

$\vec{g} = \Delta \hat{e}_x$

$g = \Delta$

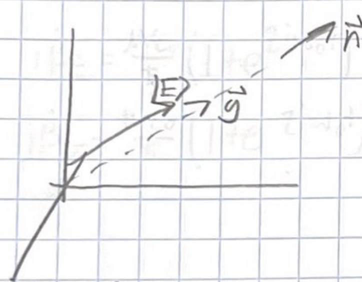
$E_E = E_0 + \Delta$

$E_0 = E_0 - \Delta$

$G_{ap} = 2\Delta$

$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle)$

$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$



$\vec{g} = -\Delta \hat{x}$

17: Two State Systems

$$|\Psi,0\rangle = |\uparrow\rangle = \frac{1}{\sqrt{2}}(|E\rangle + |G\rangle)$$

$$|\Psi,t\rangle = e^{\frac{-iEt}{\hbar}}\left(\cos\left(\frac{t\Delta}{\hbar}\right)|\uparrow\rangle + i\sin t\left(\frac{t\Delta}{\hbar}\right)|\downarrow\rangle\right)$$

$$P_{|\uparrow\rangle} = \cos^2\left(\frac{t\Delta}{\hbar}\right)$$

$$P_{|\downarrow\rangle} = \sin^2\left(\frac{t\Delta}{\hbar}\right)$$

$$\omega_L = \frac{2\Delta}{\hbar}$$

$$P \to \mu$$

$$E = -\vec{\mu}\cdot\vec{\mathcal{E}}$$

$$\vec{\mathcal{E}} = \mathcal{E}\hat{z}$$

$$\vec{\mu} = -\mu\hat{z}$$

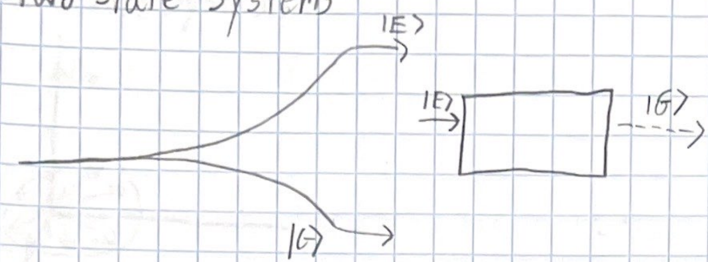
$$H = \begin{pmatrix} E_0 - \mu\mathcal{E} & -\Delta \\ -\Delta & E_0 - \mu\mathcal{E} \end{pmatrix}$$

$$= E_0 \mathbb{1} - \Delta\sigma_1 + \mu\mathcal{E}\sigma_3$$

$$E_E = E_0 + \sqrt{\Delta^2 + (\mu\mathcal{E})^2} = E_0 + \Delta + \frac{1}{2}\frac{(\mu\mathcal{E})^2}{\Delta} + \ldots$$

$$E_G = E_0 - \sqrt{\Delta^2 - (\mu\mathcal{E})^2} = E_0 - \Delta - \frac{1}{2}\frac{(\mu\mathcal{E})^2}{\Delta} - \ldots$$

17: Two State Systems



$$H'_{\substack{|E\rangle=|1'\rangle \\ |G\rangle=|2'\rangle}} = \begin{pmatrix} E_0+\Delta & \mu\mathcal{E} \\ \mu_\mathcal{E} & E_0-\Delta \end{pmatrix}$$

$$\Psi(t) = \begin{pmatrix} C_E(t) \\ C_G(t) \end{pmatrix}$$

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix} C_E \\ C_G \end{pmatrix} = \begin{pmatrix} \Delta & \mu\mathcal{E} \\ \mu\mathcal{E} & -\Delta \end{pmatrix}\begin{pmatrix} C_E \\ C_G \end{pmatrix}$$

$$\begin{pmatrix} C_E(t) \\ C_G(t) \end{pmatrix} = \begin{pmatrix} e^{-\frac{i\Delta t}{\hbar}}\beta_E(t) \\ e^{\frac{i\Delta t}{\hbar}}\beta_G(t) \end{pmatrix}$$

$$i\hbar\frac{d}{dt}\begin{pmatrix} \beta_E \\ \beta_G \end{pmatrix} = \begin{pmatrix} 0 & e^{i\omega_0 t}\mu\mathcal{E} \\ e^{-i\omega_0 t}\mu_\mathcal{E} & 0 \end{pmatrix}\begin{pmatrix} \beta_E \\ \beta_G \end{pmatrix}$$

$$\mathcal{E}(t) = 2\mathcal{E}_0\cos\omega_0 t$$
$$= \mathcal{E}_0(e^{i\omega_0 t} + e^{-i\omega_0 t})$$

$$i\dot{\beta}_E = \frac{\mu\mathcal{E}_0}{\hbar}(1+e^{2i\omega_0 t})\beta_G(t)$$

$$i\dot{\beta}_G = \frac{\mu\mathcal{E}_0}{\hbar}(1+e^{-2i\omega_0 t})\beta_E(t)$$

17; Two State Systems

$$i\dot{\beta}_E = \frac{\mu\varepsilon_0}{\hbar}\beta_G$$

$$i\dot{\beta}_G = \frac{\mu\varepsilon_0}{\hbar}\beta_E$$

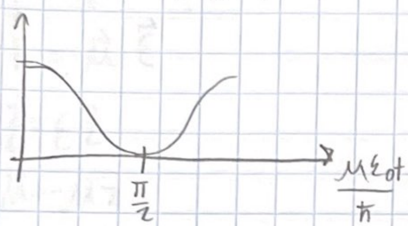
$$ii\ddot{\beta}_E = \frac{\mu\varepsilon_0}{\hbar} i\dot{\beta}_G$$

$$= \left(\frac{\mu\varepsilon_0}{\hbar}\right)^2 \beta_E$$

$$\ddot{\beta}_E = \left(-\frac{\mu\varepsilon_0}{\hbar}\right)\beta_E$$

$$\beta_E(t) = \cos\left(\frac{\mu\varepsilon_0 t}{\hbar}\right)$$

$$P_E(t) = \cos^2\left(\frac{\mu\varepsilon_0 t}{\hbar}\right)$$



$$\frac{\mu\varepsilon_0 T}{\hbar} = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$$

$$\Delta N \Delta\phi \geq \frac{1}{2}$$

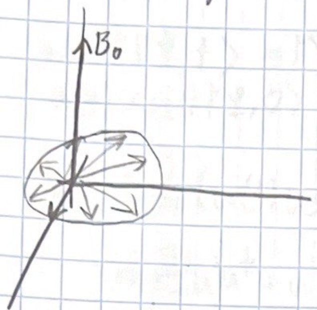
$$\langle\hat{N}\rangle_\alpha = |\alpha|^2 \equiv n$$

$$\Delta N = |\alpha| = \sqrt{n}$$

$$\Delta N \Delta\phi \simeq \frac{1}{2}$$

$$\Delta\phi \equiv \frac{1}{2\sqrt{n}}$$

17: Two State Systems



$\vec{B} = B_0\hat{z} + B_1(\cos\omega t\,\hat{x} - \sin\omega t\,\hat{y})$

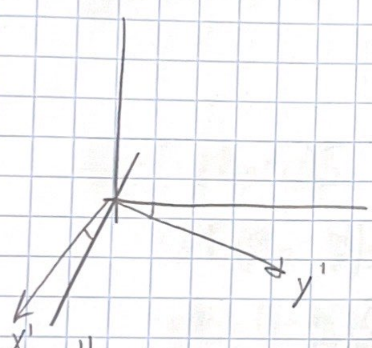
$H_S(t) = -\gamma\vec{B}(t)\cdot\vec{S}$

$= -\gamma[B_0\vec{S}_z + B_1(\hat{S}_x\cos\omega t - \hat{S}_y\sin\omega t)]$

$i\hbar\frac{\partial}{\partial t}|\Psi\rangle = H|\Psi\rangle$

$U|\Psi\rangle = |\Psi'\rangle$

$U^\dagger H_S U = H$



$H_S = 0$

$U = e^{\frac{-i\omega t\hat{S}_z}{\hbar}}$

$H_R = \omega S\hat{z}$

17: Two State Systems

$$|\psi_R\rangle = U|\psi\rangle$$

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = H_S|\psi\rangle$$

$$i\hbar\frac{\partial}{\partial t}|\psi_R\rangle = i\hbar\frac{\partial}{\partial t}U|\psi\rangle$$

$$= i\hbar\left(\frac{\partial}{\partial t}U\right)U^{+}|\psi_R\rangle + UH_SU^{+}|\psi\rangle$$

$$i\hbar\frac{\partial}{\partial t}|\psi_R\rangle = \left(UH_SU^{+} + i\hbar\frac{\partial}{\partial t}(U)U^{+}\right)|\psi_R\rangle$$

$$i\hbar\frac{\partial}{\partial t}(U)U^{+} = \omega\hat{S}_z$$

$$U = e^{-\frac{i\omega t S_z}{\hbar}}$$

$$H_R = UH_SU^{+} + i\hbar\frac{\partial}{\partial t}(U)U^{+}$$

$$U = e^{\frac{i\omega t S_z}{\hbar}}$$

$$|\psi\rangle = e^{\frac{i\omega t\hat{S}_z}{\hbar}}|\psi_R(t)\rangle$$

$$\psi|R\rangle = U\psi$$

18: Multiparticle States

$$u(t)|\Psi,t\rangle \equiv |\Psi_R,t\rangle$$
$$u(t)\,u_s(t)|\Psi,0\rangle$$

$$H_R = i\hbar\frac{\partial}{\partial t}(u(t)u_s(t))u_s^\dagger(t)u^\dagger(t)$$
$$= i\hbar\frac{\partial}{\partial t}(u)u^\dagger + u(t)\,i\hbar\frac{\partial}{\partial t}(u_s(t))u_s^\dagger\,u^\dagger(t)$$

$$H_R = Hu^\dagger u(t)\,H_s(t)u^\dagger(t)$$

$$H_R = \omega\hat{S}_z + e^{-\frac{i\omega t\hat{S}_z}{\hbar}}\left(-\gamma\left(B_0\hat{S}_z + B_1(\cos\omega t\,\hat{S}_x - \sin\omega t\,\hat{S}_y)\right)\right)e^{\frac{i\omega t\hat{S}_z}{\hbar}}$$

$$i\hbar\frac{\partial}{\partial t}|\Psi_R,t\rangle = H_R|\Psi_R,t\rangle$$

$$H_R = (\omega - \gamma B_0)\hat{S}_z - \gamma B_1 e^{-\frac{i\omega t\hat{S}_z}{\hbar}}(\cos\omega t\,\hat{S}_x - \sin\omega t\,\hat{S}_y)e^{\frac{i\omega t\hat{S}_z}{\hbar}}$$

$$e^{A}Be^{-A}$$

$$\frac{d}{dt}M(t) = e^{-\frac{i\omega t\hat{S}_z}{\hbar}}\left(\frac{-i\omega}{\hbar}[\hat{S}_z, \cos\omega t\,\hat{S}_x - \sin\omega t\,\hat{S}_y] - \omega\sin\omega t\,\hat{S}_x - \omega\cos\omega t\,\hat{S}_y\right)e^{\cdots}$$

$$\left(-\frac{i\omega}{\hbar}(i\hbar\hat{S}_y\cos\omega t + \sin\omega t\, i\hbar\hat{S}_x)\right) = 0$$

$$H_R = (-\gamma B_0 + \omega)\hat{S}_z - \gamma B_1\hat{S}_x$$
$$= -\gamma\left[\left(B_0 - \frac{\omega}{\gamma}\right)\hat{S}_z + B_1\hat{S}_x\right]$$
$$= -\gamma\left[B_0\left(1 - \frac{\omega}{\omega_0}\right)\hat{S}_z + B_1\hat{S}_x\right]$$

18: Multiparticle systems

$H_R = -\gamma \vec{B}_R \cdot \vec{S}$

$\vec{B}_R = B_0\left(1 - \frac{\omega}{\omega_0}\right)\hat{z} + B_1\hat{x}$

$|\Psi, t\rangle = U^{+}|\Psi_R, t\rangle$

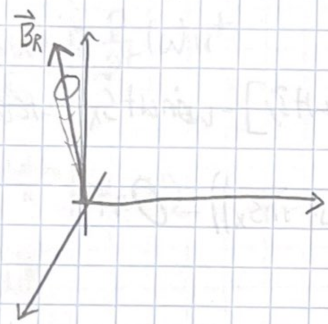
$= e^{\frac{i\omega t S_z}{\hbar}} e^{\frac{i\gamma \vec{B}_R \cdot \vec{S} t}{\hbar}} |\Psi, 0\rangle$

$\frac{d}{dt} e^{tA} B e^{-tA} = e^{tA}[A, B]e^{-tA}$

$B_1 << B_0$

$\omega << \omega_0$

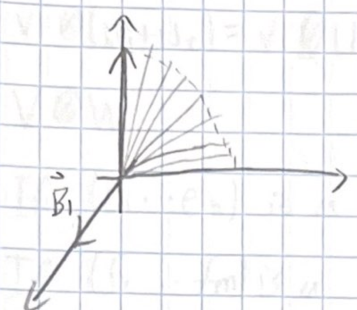
$\vec{B}_R \cong B_0\hat{z} + B_1\hat{x}$

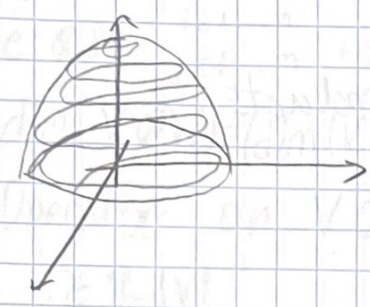


$|B_R| \cong B_0$

18: Multiparticle Systems

$B_R = B_1 \hat{x}$





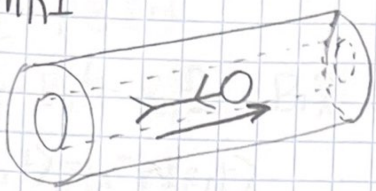
$\omega_1 T = \frac{\pi}{2}$

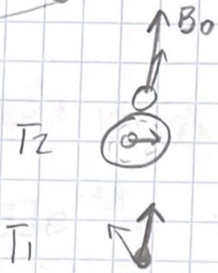
$\gamma B_1 T = \frac{\pi}{2}$

$T = \frac{\pi}{2\gamma B_1}$

18: Multiparticle Systems

MRI





Multi particle states and tensor products

Particle 1: - complex vector space V
- operators $T_1, T_2, \ldots$

Particle 2: - complex vector space W
- operators $S_1, S_2, \ldots$

$v \in V$
$w \in W$

(v, w) information about each particle

$v \otimes w$ an element of a new vector space $\in V \otimes W$

$$\underset{V}{\underset{\cap}{v}} \otimes \underset{W}{\underset{\cap}{w}} \in V \otimes W$$

$$(\alpha v) \otimes w = \alpha\, v \otimes w = v \otimes (\alpha w)$$

$v_1 \otimes w_1 \in V \otimes W$
$v_2 \otimes w_2 \in V \otimes W$
$\alpha(v_1 \otimes w_1) + \beta(v_2 \otimes w_2) \in V \otimes W$

16: Multiparticle systems

$(v_1+v_2)\otimes w_1 = v_1\otimes w_1 + v_2\otimes w_2$

$v_1\otimes(w_1+w_2) = v_1\otimes w_1 + v_1\otimes w_2$

$V\otimes W$

If $(e_1 \cdots e_n)$ is a basis for V

If $(f_1 \ldots f_m)$ is a basis for W

$e_i\otimes f_j \quad \begin{matrix} i=1\ldots n \\ j=1\ldots m \end{matrix}$ forms basis of $V\otimes W$

$\dim(V\otimes W) = (\dim(V))(\dim(W))$

Operators on $V\otimes W$

$T\in\mathcal{L}(V)$

$S\in\mathcal{L}(W)$

$T\otimes S\in\mathcal{L}(V\otimes W)$

$T\otimes S(v\otimes w)$

$T\otimes S(v\otimes w) = (Tv)\otimes(Sw)$

$T_1\in\mathcal{L}(V)\longrightarrow T_1\otimes 1\!\!1$
$\in\mathcal{L}(V\otimes W)$

$S_1\in\mathcal{L}(W)\longrightarrow 1\!\!1\otimes S$

$(T_1\otimes 1\!\!1)(1\!\!1\otimes S_1)(v\otimes w) = (T_1\otimes 1\!\!1)(v\otimes S_1 w) = (T_1 v)\otimes S_1 w$

$(1\!\!1\otimes S_1)(T_1\otimes 1\!\!1)(v\otimes w) = (T_1 v)\otimes(S_1 w)$

16: Multiparticle Systems

$H_T = H_1 \otimes \mathbb{1} + \mathbb{1} \otimes H_2$

Two spin $\frac{1}{2}$

$|+\rangle_1, |-\rangle_1$

$|+\rangle_2, |-\rangle_2$

$|+\rangle_1 \otimes |+\rangle_2$

$|+\rangle_1 \otimes |-\rangle_2$

$|-\rangle_1 \otimes |+\rangle_2$

$|-\rangle_1 \otimes |-\rangle_2$

$|\psi\rangle = \alpha_1 |+\rangle_1 \otimes |+\rangle_2 + \alpha_2 |+\rangle \otimes |-\rangle + \alpha_3 |-\rangle \otimes |+\rangle + \alpha_4 |-\rangle \otimes |-\rangle$

$S_z^{TOT} |\psi\rangle = \hbar(\alpha_1 |+\rangle \otimes |+\rangle - \alpha_4 |-\rangle \otimes |-\rangle)$