551 Lecture Notes
Spring 2022

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Contents

Chapter 1

Kinematics

1.1 Review of Calculus

1.1.1 Functions

It is a fact of life that nature can be modeled by mathematics, thus it is important to discuss the mathematical basis for physics. An important part of that is calculus. We must start with a notion of a function. A function is a mapping from an input to an output. We will be dealing with functions of real variables $f : \mathbb{R} \to \mathbb{R}$. If the input and the output of a function are real, then we have a real function. In 550, we only deal with real functions of real variables.

We can represent functions with formulas or expressions between dependent and independent variables.

Example

$$y = 3x + 2$$

$$y = \sin x$$

$$y = e^{\sin \frac{1}{\log x}}$$

We can have a domain on one axis and a range on another axis. This is the graphical interpretation of a function. The graphical interpretation is the same as the expression. These two interpretations can both be beneficial at different times. Often, we will be dealing with t as an independent variable where t is time.

1.1.2 Rates of Change and Derivatives

The rate of change is the change in the dependent variable with respect to the change in x. Rate of change= $\frac{\Delta y}{\Delta x}$. This means that you start at an x corresponding to y and move to x_1 which corresponds to y_1. Thus the rate of change becomes:

$$\frac{\Delta y}{\Delta x} = \frac{y_1 - y}{\Delta x} = \frac{f(x_1) - f(x)}{\Delta x}$$

This rate of change will correspond to the slope on the graph. More specifically, this will be the slope of the secant line between the two points. This rate of change will depend on the function, the starting x, and the ending x_1. This will also have a sign such that the rate of change can be either positive or negative. Just as a note, the slope will be zero when there is no change of y over a change in x. However, Δx is not infinitesimal meaning that the slope will only be an approximation.

To solve this we will introduce the derivative. Some of the notation is:

$$f'(x)$$

$$\frac{d}{dx}(f(x))$$

$$\frac{df}{dx}$$

$$\dot{x}$$

The definition of the derivative is the following:

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Graphically, the derivative corresponds to the slope of a line tangent to a function. Thus the slope of the tangent line= $\frac{dy}{dx}$. In physics though, we will use t instead of x for the independent variable.

The derivative depends on the function and x but not Δx. The derivative will also have a sign corresponding, again, to the slope.

1.1.3 Differentiation Methods

You can always use the graph to construct the derivative. You can additionally use the definition to find the derivative of a function given the formula.

Definition

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{1.1}$$

$$= \lim_{\Delta x \to 0} \frac{0}{\Delta x} = 0$$

$$\Rightarrow \frac{d}{dx}(5) = 0$$

This makes sense because the slope of a tangent to a horizontal line is zero.

$$f'(x) = \lim_{\Delta x \to 0} \frac{x + \Delta x - x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$$

$$\Rightarrow \frac{d}{dx}(x) = 1$$

This also makes sense because the slope of the line is 1. To generalize:

$$\frac{d}{dx}(ax) = a$$

In physics, we work with idealized functions so it is good to remember some of the derivatives in order to work through the math in the problem faster.

Definition

The Power Rule:

$$\frac{d}{dx}(x^n) = nx^{n-1} \tag{1.2}$$

For example:

$$\frac{d}{dx}(x^3) = 3x^2$$

Exponential Differentiation:

$$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$$

Logarithmic Differentiation:

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

Trigonometric Differentiation:

$$\frac{d}{dx}(\sin(ax)) = a\cos(ax)$$

$$\frac{d}{dx}(\cos(ax)) = -a\sin(ax)$$

We can exploit the properties of the derivative in order to generate some more rules for differentiation. One of the properties is linearity. The following is a linear combination of functions.

$$af(x) + bg(x)$$

To take the derivative of this:

$$\frac{d}{dx}(af(x) + bg(x)) = a\frac{df}{dx} + b\frac{dg}{dx}$$

From this we can arrive at the Product Rule which also exploits linearity:

Definition

The Product Rule

$$\frac{d}{dx}(f(x)g(x)) = \frac{df}{dx}g + f\frac{dg}{dx} \tag{1.3}$$

The Chain Rule deals with compositions of functions:

Definition

The Chain Rule

$$\frac{d}{dx}(f(g(x)) = f'(g(x)) \cdot g'(x) \tag{1.4}$$

Example

1)

$$f(x) = \sin(x^2)$$

$$f'(x) = 2x\cos(x^2)$$

2)

$$f(x) = ax^3 - bsin(e^x)$$

$$f'(x) = 3ax^2 - bcos(e^x)e^x$$

1.1.4 Antiderivatives and Integrals

We discussed earlier how you can differentiate a function. If a linear function has a slope of 3, for example, then the derivative is 3.

$$3x \to \frac{d}{dx} \to 3$$

We can additionally undo the differentiation by antidifferentiating the function or performing an indefinite integral:

$$3 \rightarrow \int dx \rightarrow 3x + C$$

Because the derivative reduces the order of polynomials, the constants vanish. Thus we must add an integration constant in order to indicate that the antiderivative can be a family of curves. This is why this integral is called indefinite. Thus the indefinite integral is the opposite of the derivative. Thus the rules for indefinite integration are the opposites of the rules for differentiation.

For definite integrals we will concern ourselves with finding the area under a curve on an integral between a and b. We can do this by approximate this by dividing the area under the curve into discrete rectangles. Taking the limit of the number of rectangles to infinity will give the exact, continuous area under the curve. The width of the rectangle will be a differential dx making the width a Δx in the limiting case. If we multiply $f(x)$ by dx we will get the differential area dA.

$$dA = f(x)dx$$

We will now sum the little areas

$$A = \sum_i f(x_i)\Delta x$$

$$A = \lim_{n\rightarrow\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$A = \int_a^b f(x)dx$$

To evaluate the definite integrals, we use the Fundamental Theorem of Calculus:

Definition

$$\int_a^b f(x)dx = F(b) - F(a) \tag{1.5}$$

Here the $F(x)$ is the anti derivative of $f(x)$.

Example

Find the area between $[0, 1]$ of $f(x) = x$

$$A = \int_0^1 xdx = \frac{1}{2}(1)^2 - \frac{1}{2}(0)^2 = \frac{1}{2}$$

For polynomials, this process is simple and we will only need to be dealing with polynomials in 551. In MTH590A, more complicated antidifferentiation techniques will be learned and applied in 552.

We can use this to solve equations like this:

$$\frac{dy}{dx} = f(x) \rightarrow dy = f(x)dx \rightarrow \int dy = \int f(x)dx \rightarrow y = \int f(x)dx + C$$

This is the method of separation of variables for solving separable differential equations.

1.2 1D Kinematics

1.2.1 Preliminaries

First we will start with some assumptions of classical mechanics. The first assumption is that space is 3D and is also Euclidean. This asserts that you can use the Pythagorean Theorem to measure distances making the metric tensor for the system a three by three identity matrix.

Definition

The Cartesian Metric Tensor

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.6}$$

The second assumption is that time can be measured with clocks. Clocks can additionally be synchronized. Once clocks are in sync, they will always agree on the time making time absolute. This is something that is not true in relativistic physics because that assumes a 4D spacetime.

Physics 551 will simplify objects in that we shall assume that all objects are points. In addition, we will only observe systems from far away. It should also be posed that, for now, the world that we live in is 1D (this will change for 2D kinematics).

In order to set up 1D kinematics, we need a line. On this line we will have a point $\boldsymbol{P}$ and an observer $\boldsymbol{O}$. Say that $\boldsymbol{P}$ is a distance x from $\boldsymbol{P}$. This implies that x is a coordinate. Once we have $x(t)$, that is all that we need to know about P. Now let's add another observer O' a distance x' from P. Here they will disagree how far away P is because $x \neq x'$. Now we move P to P_1, thus we will have some equations to describe the differences in x

$$\Delta x' = x_1' - x_0' = \Delta x$$

This means that both observers can agree on Δx and t so we can agree on the displacement and the time interval.

1.2.2 Quantities of Interest

$$\Delta x = x_f - x_i [=] m$$

$$\Delta t = t_f - t_i [=] s$$

$$\bar{v} = \frac{\Delta x}{\Delta t} [=] m/s$$

$$\bar{s} = |\bar{v}| [=] m/s$$

Example

Say we go from city A (x_A) to city B (x_b), in some time t_{AB} and then stop at x_B, and then return to A in time t_{BA}.

$$\bar{v} = \frac{\Delta x}{\Delta t} = 0 m/s$$

$$\bar{s} = \frac{d}{\Delta t} = \frac{2|x_B - x_a|}{t_{AB} + t_B + t_{BA}}$$

In order to create instantaneous velocity, we need to take the limit:

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

Definition

$$x(t) \to v(t) = \frac{dx(t)}{dt} \tag{1.7}$$

Acceleration will be similar to velocity because:

$$\bar{a} = \frac{\Delta v}{\Delta t} [=] m/s^2$$

We will take the same limit:

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$$

Definition

$$a(t) = \frac{dv(t)}{dt} = \frac{d^2x(t)}{dt^2} \tag{1.8}$$

1.2.3 Integration of the Equations of Motion

This means that from position, you can take two derivatives to get the acceleration. To get the other way around, we will integrate. Thus we must integrate to obtain the equations of motion:

$$a(t) = \frac{dv}{dt} \rightarrow \int_{v_0}^{v} dv = \int_{t_0}^{t} a dt$$

Definition

$$v(t) = v_0 + \int_{t_0}^{t} a(t) dt \tag{1.9}$$

$$x(t) = x_0 + \int_{t_0}^{t} v(t) dt \tag{1.10}$$

Now we can solve this:

$$x(t) = x_0 + \int_{t_0}^{t} \left(v_0 + \int_{t_0}^{t} a(t) dt \right)$$
$$= x_0 + \int_{t_0}^{t} v_0 dt + \int_{t_0}^{t} \int_{t_0}^{t} a(t) dt dt$$
$$= x_0 + v_0(t - t_0) + \int_{t_0}^{t} \int_{t_0}^{t} a(t) dt dt$$

Now we must solve for constant acceleration to get the Kinematic Equations for Constant acceleration:

$$a = \frac{dv}{dt} \rightarrow \int_{v_0}^{v} dv = \int_{t_0}^{t} a dt$$
$$v - v_0 = a \int_{t_0}^{t} dt = a(t - t_0)$$
$$v - v_0 = a(t - t_0)$$
$$v = \frac{dx}{dt} \rightarrow \int_{x_0}^{x} dx = \int_{t_0}^{t} v dt$$

$$x - x_0 = \int_{t_0}^{t} (v_0 + a(t - t_0))dt = \int_{t_0}^{t} v_0 dt + \int_{t_0}^{t} atdt - \int_{t_0}^{t} at_0 dt$$
$$= v_0(t - t_0) + \frac{1}{2}a(t - t_0)^2$$
$$x = x_0 + v_0(t - t_0) + \frac{1}{2}a(t - t_0)^2$$

Thus we now have the equations of motion:

Definition

$$v - v_0 = a(t - t_0) \tag{1.11}$$
$$x = x_0 + v_0(t - t_0) + \frac{1}{2}a(t - t_0)^2 \tag{1.12}$$

1.2.4 Homework: 1D Kinematics

Problems are from the 550 textbook "Physics for Scientists and Engineers".

24) $x = 100km$, $50km$ traveled at $60km/h$, and $50km$ traveled at $80km/h$.

$$t_A = \frac{50km}{60km/h} = 0.833h$$
$$t_B = \frac{50km}{80km/h} = 0.625h$$
$$t_{tot} = t_A + t_B = 1.458h$$
$$x_{tot} = 50km + 50km = 100km$$
$$\bar{v} = \frac{x_{tot}}{t_{tot}} = \frac{100km}{1.458h} = \boxed{68.6h}$$

Because you are traveling at a lower velocity for a longer time, the average velocity will end up being closer to the lower velocity than to the higher velocity.

40)

$$x(t) = 2.5t + 3.1t^2 - 4.5t^3$$

$$v(t) = \dot{x}(t) = 2.5 + 6.2t - 13.5t^2$$

From the velocity equation 0 and 2 can be plugged in such that $\boxed{v(0) = 2.5m/s}$ and $\boxed{v(2) = -39.1m/s}$.

$$a(t) = \dot{v}(t) = \ddot{x}(t) = 6.2 - 27t$$

The same goes for the acceleration equation such that for 0 and 2 $\boxed{a(0) = 6.2m/s^2}$ and $\boxed{a(2) = -47.8m/s^2}$.

$$\bar{v}_{0,2} = \frac{\Delta x}{\Delta t} = \frac{-18.6m - 0m}{2s} = \boxed{-9.3m/s}$$

This could alternatively be found using the average value formula:

$$\bar{v}_{0,2} = \frac{1}{2-0}\int_0^2 2.5 + 6.2t - 13.5t^2 dt = -9.3m/s$$

$$\bar{a}_{0,2} = \frac{\Delta v}{\Delta t} = \frac{-39.1m/s - 2.5m/s}{2s} = \boxed{-20.8m/s^2}$$

Here I needed to make sure that I was using the correct functions to find the average from.

42)

$$v(t) = Bt - Ct^2$$

Let $B = 6.0m/s^2$ and $C = 2.0m/s^3$. Then to find zeros, factor the expression and set equal to zero: Using the zero product property allows to solve for both cases: $0 = t(B - Ct) \Rightarrow \boxed{t = 0s}, t = \frac{B}{C} = \frac{6.0m/s^2}{2.0m/s^3} = \boxed{3.0s}$.

$$a(t) = \dot{v}(t) = B - 2Ct \Rightarrow 0 = B - 2Ct \Rightarrow t = \frac{B}{2C} \Rightarrow t = \frac{6.0m/s^2}{4.0m/s^3} = \boxed{1.5s}$$

To graph this, the maximum must be found which is equal to the velocity evaluated at $t = 1.5s$.

$$v(1.5) = 4.5m/s$$

The velocity at 4 must also be found:

$$v(4) = 6.0(4) - 3.0(4^2) = 24 - 32 = -8.0m/s$$

57) $50m$ to stop from $96km/h$.

$$96km/h\left(\frac{1000m}{1km}\right)\left(\frac{60min}{1hr}\right)\left(\frac{60s}{1min}\right) = 26.67m/s$$

$$a(x - x_0) = \frac{1}{2}(v^2 - v_0^2)$$

$$\Rightarrow a = \frac{\frac{1}{2}(v^2 - v_0^2)}{x - x_0} = \frac{\frac{1}{2}\left((-26.67m/s)^2\right)}{50m} = \boxed{-7.11m/s^2}$$

$$v = v_0 + at \Rightarrow t = \frac{v - v_0}{a} = \frac{-26.67m/s}{-7.112m/s^2} = \boxed{3.75s}$$

Dimensional analysis was important here so I will need to remember it from previous classes.
65) Acceleration of $a = 0.50m/s^2$ from v_0 across $x = 550m$ over $t = 15s$.

$$x = v_0 t + \frac{1}{2}at^2$$

$$\Rightarrow 550m = v_0(15s) + \frac{1}{2}(0.50m/s^2)(15s)^2$$

$$\Rightarrow 550m = v_0(15s) + 56.25m$$

$$\Rightarrow v_0 = \boxed{32.9m/s}$$

$$v = v_0 + at = (32.91m/s) + (0.50m/s^2)(15s) = \boxed{40.4m/s}$$

*Question: I know it would be tedious and unnecessary but would the Euler-Lagrange Equations would correctly describe the equations of motion if one finds the Lagrangian from kinetic and potential energies of the system?

1.3 Vectors

1.3.1 Vectors, Vector Spaces, and Basis Vectors

Properties are described often with magnitudes. These magnitudes can be described by scalars. However, other quantities need to be described by vectors which are rank one tensors. These vector quantities use both magnitude and direction in a multidimensional world.

A geometric vector can be described by an arrow going between two points. THhe direction of the vector is given by the direction of the arrow and the magnitude is given by the length of the vector. For a vector $\vec{A}$, its magnitude is given by A.

To describe these vectors, we need to describe the operations of these objects. We will cover the addition of two vectors in addition to the scalar multiplication.

$$\vec{B} = \alpha\vec{A}, \alpha \in \mathbb{R}$$

$$B = \alpha A$$

$$\vec{B}||\vec{A}$$

Given that $\alpha > 0$, then both vectors will point in the sane direction, in the case where $\alpha < 0$ then they point in opposite directions. To add two vectors, we add the vectors physically from tail to head. We can easily add these vectors by adding corresponding vector components.

Definition

Addition of vectors:

$$\vec{A} + \vec{B} = \vec{B} + \vec{A} \tag{1.13}$$

$$\vec{A} + \vec{B} = (A_i + B_i)\hat{e}_i \tag{1.14}$$

If $\vec{A}$ is a vector and $\vec{B}$ is a vector, then through linearity, with α, β as constants, the following can be defined to create a vector space.

Definition

Vector Spaces

$$\vec{C} = \alpha\vec{A} + \beta\vec{B} \tag{1.15}$$

If we have two vectors that are not parallel $\vec{u}, \vec{v}$, we can decompose a third vector $\vec{A}$ as a sum of the first two vectors. This is a concept from linear algebra.

$$A = \alpha\vec{u} + \beta\vec{v}$$

Each elements of the sum above is called a component of $\vec{A}$ and the scalars α, β are the scalar components of $\vec{A}$. In general, it is useful to call $\{\vec{u}, \vec{v}\}$ as being perpendicular to one another and being of unit length. This creates an orthonormal basis.

In the rectangular coordinate system, we use the x, y basis with unit vectors $\hat{e}_x, \hat{e}_y$. We can construct any vector as a linear combination of basis vectors and scalar multiples. Note that I like to use the $\hat{e}$ notation for basis vectors because it is most generalized to use indices. You may be more used to the $\hat{\imath}, \hat{\jmath}$ convention like is used in the figure below.

Definition

$$\vec{A} = A_x\hat{e}_x + A_y\hat{e}_y = A_i\hat{e}_i \tag{1.16}$$

From the components, we can find the magnitude of the vector:

$$A = \sqrt{A_iA_i}$$

We can also take the multiplication by a scalar:

$$\vec{B} = \alpha\vec{A}$$

$$\begin{aligned}\vec{B} = B_x\hat{e}_x + B_y\hat{e}_j &= \alpha(A_x\hat{e}_x + A_y\hat{e}_j) \\ &= \alpha A_x\hat{e}_x + \alpha A_y\hat{e}_y\end{aligned}$$

$$\begin{pmatrix} B_x \\ B_y \end{pmatrix} = \alpha\begin{pmatrix} A_x \\ A_y \end{pmatrix}$$

Example

$$\vec{A} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$\vec{A} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$$\vec{C} = \vec{A} + \vec{B} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

1.3.2 Vector Products and Vector Functions

In addition, we have the vector products: the dot product and the cross product. The dot product is the following:

Definition

The Dot Product

$$\vec{A}\cdot\vec{B} = AB\cos\theta \tag{1.17}$$

$$\vec{A}\cdot\vec{B} = A^iB^j\delta_{ij} \tag{1.18}$$

There is also the cross product which yields a vector:

Definition

The Cross Product

$$|\vec{A}\times\vec{B}| = AB\sin\theta \tag{1.19}$$

$$\vec{A}\times\vec{B} = \epsilon_{ijk}\hat{e}_i a^j b^k \tag{1.20}$$

We also want to introduce the notation of a vector function. A vector function is a vector of functions:

$$\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

We can take derivatives of these vector functions:

$$\frac{d}{dt}(\vec{r}(t)) = \frac{d}{dt}(x(t)\hat{e}_x + y(t)\hat{e}_y)$$

$$= \frac{d}{dt}(x(t)\hat{e}_x) + \frac{d}{dt}(y(t)\hat{e}_y)$$

$$= \frac{dx}{dt}\hat{e}_x + \frac{dy}{dt}\hat{e}_y$$

The general idea with these vector functions is that you can take derivatives and integrals of them by their parts.

1.4 2D Kinematics

1.4.1 Projectile Motion

We want to consider the observer situation that we had in one dimensional kinematics but in two dimensions. The position is modeled by a vector function $\vec{r}(t)$ from the observer to the point. In this course, for rectangular coordinates, we will use the basis vectors $\hat{e}_x$ and $\hat{e}_y$. We can decompose $\vec{r}(t)$ as the following:

Definition

$$\vec{r}(t) = x(t)\hat{e}_x + y(t)\hat{e}_y = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \tag{1.21}$$

$$r = \sqrt{x^2 + y^2} \tag{1.22}$$

$$\vec{r}(t) = \begin{pmatrix} r(t)\cos\theta(t) \\ r(t)\sin\theta(t) \end{pmatrix} \tag{1.23}$$

From this we can define the average velocity and the instantaneous velocity:

$$\bar{\vec{v}} = \frac{\vec{r}_f - \vec{r}_i}{t_f - t_i}$$

$$\vec{v} = \dot{\vec{r}}$$

The velocity vector is always tangent to the trajectory of the particle. Through a similar process to find average and instantaneous acceleration:

$$\bar{\vec{a}} = \frac{\vec{v}_f - \vec{v}_i}{t_f - t_i}$$

$$\vec{a} = \dot{\vec{v}} = \ddot{\vec{r}}$$

Basically, we recover the same relationships that we had in one dimension but we are dealing with vectors. It may be noticed that we have adopted a more compact system for derivative notation where we replace the Leibniz notation with dots such that $\frac{dx}{dt} = \dot{x}$.

We will now try using vector components. If we use a basis whose vectors do not change with time (ie: in General Relativity), then the derivative of a vector is given by its components such that:

$$\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$$\vec{v}(t) = \begin{pmatrix} x(\dot{t}) \\ y(\dot{t}) \end{pmatrix} = \begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix}$$

$$\vec{a}(t) = \begin{pmatrix} \dot{v}_x(t) \\ \dot{v}_y(t) \end{pmatrix} = \begin{pmatrix} a_x(t) \\ a_y(t) \end{pmatrix}$$

Example

$$\vec{r}(t) = \begin{pmatrix} 4 + 2t \\ 3 + 5t + 4t^2 \\ 2 - 2t - 3t^2 \end{pmatrix}$$

$$\vec{v}(t) = \begin{pmatrix} 2 \\ 5 + 8t \\ -2 - 6t \end{pmatrix}$$

$$\vec{a}(t) = \begin{pmatrix} 0 \\ 8 \\ -6 \end{pmatrix}$$

$$a = \sqrt{(8)^2 + (-6)^2} = 10$$

$$arg(\vec{a}) = \tan^{-1}\left(\frac{-6}{8}\right)$$

If you divide the vector by its magnitude, you can get the unit vector:

$$\frac{\vec{v}}{v} = \hat{e}$$

1.4.2 Equations of Motion for Multidimensional Kinematics

To start with constant acceleration, we start with the following acceleration:

$$\vec{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix}$$

$$\vec{a} = \dot{\vec{v}} \longrightarrow \int_{\vec{v_0}}^{\vec{v}} d\vec{v} = \int_{t_0}^{t} \vec{a} dt$$

$$\vec{v} - \vec{v_0} = \vec{a} \int_{t_0}^{t} dt = \vec{a}(t - t_0)$$

$$\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} v_{0_x} \\ v_{0_y} \end{pmatrix} + \begin{pmatrix} a_x \\ a_y \end{pmatrix} (t - t_0)$$

$$\vec{v} = \dot{\vec{r}} \longrightarrow \int_{\vec{r_0}}^{\vec{r}} d\vec{r} = \int_{t_0}^{t} \vec{v} dt$$

$$\vec{r}(t) = \vec{r_0} + \vec{v_0}(t - t_0) + \frac{1}{2}\vec{a}(t - t_0)^2$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} v_{0_x} \\ v_{0_y} \end{pmatrix} (t - t_0) + \frac{1}{2} \begin{pmatrix} a_x \\ a_y \end{pmatrix} (t - t_0)^2$$

From this we get the component based equations of motion:

Definition

$$v_x = v_{0_x} + a_x(t - t_0) \tag{1.24}$$

$$v_y = v_{0_y} + a_y(t - t_0) \tag{1.25}$$

$$x(t) = x_0 + v_{0_x}(t - t_0) + \frac{1}{2}a_x(t - t_0)^2 \tag{1.26}$$

$$y(t) = y_0 + v_{0_y}(t - t_0) + \frac{1}{2}a_y(t - t_0)^2 \tag{1.27}$$

For working in even more dimensions, we can use the same equations for each of our variables:

Definition

$$v_i = v_{0_i} + a_i(t - t_0) \tag{1.28}$$

$$x_i(t) = x_{i_0} + v_{0_{x_i}}(t - t_0) + \frac{1}{2}a_{x_i}(t - t_0)^2 \tag{1.29}$$

Now here is an example with projectile motion. We will assume that downward acceleration comes from gravity. We will use basis vectors $\hat{e}_x, \hat{e}_y$.

Example

Say a projectile is fired with initial velocity $\vec{v_0}$ and at an angle θ. The first thing to find is the acceleration vector and from that we can find everything:

$$\vec{a} = \begin{pmatrix} 0 \\ -g \end{pmatrix}$$

$$\begin{aligned} a_x &= 0 \\ v_x &= v_0 \cos\theta \\ x &= v_0 \cos\theta t \end{aligned}$$

$$\begin{aligned} a_y &= -g \\ v_y &= v_0 \sin\theta t - gt \\ y &= v_0 \sin\theta t - \tfrac{1}{2} g t^2 \end{aligned}$$

Now we want to find the trajectory as $y = f(x)$: In the x equation we will solve for t:

$$x = v_0 \cos\theta t \Rightarrow t = \frac{x}{v_0 \cos\theta}$$

Now we can plug this into the y equation:

$$y = v_0 \sin\theta \frac{x}{v_0 \cos\theta} - \frac{1}{2} g \left(\frac{x}{v_0 \cos\theta}\right)^2$$

$$= x \tan\theta - \frac{1}{2} \left(\frac{x}{v_0 \cos\theta}\right)^2$$

We will try to find the maximum of the parabola:

$$h_{max} = y_{max}$$

$$\dot{y}(t) = v_y = v_0 \sin\theta - gt = 0 \Rightarrow t_{h_{max}} = \frac{v_0 \sin\theta}{t}$$

The trajectory of the parabola can also be found: For this we can use symmetry so we have:

$$t_{max} = 2 t_{h_{max}} = \frac{2 v_0 \sin\theta}{g}$$

1.4.3 Homework: 2D Kinematics

8)

Equations of motion:

$$x(t) = 5t + 4t^2$$
$$y(t) = 3t^2 + 2t^3$$
$$z = 0$$

Taking the first derivative gives velocity (the power rule kills the z component because it is zero so I will not include it):

$$\dot{x}(t) = v_x = 5 + 8t$$
$$\dot{y}(t) = v_y = 6t + 6t^2$$

$$\vec{v} = \boxed{\begin{pmatrix} 5 + 8t \\ 6t + 6t^2 \end{pmatrix}}$$

Again for acceleration:

$$\dot{v}_x = a_x = 8$$
$$\dot{v}_y = a_y = 6 + 12t$$

$$\vec{a} = \boxed{\begin{pmatrix} 8 \\ 6 + 12t \end{pmatrix}}$$

$$\vec{v}(2) = \begin{pmatrix} 21 \\ 36 \end{pmatrix}$$

$$|\vec{v}(2)| = \sqrt{(21)^2 + (36)^2} = \boxed{41.7m/s}$$

18)

The initial position, velocity, and acceleration are given:

$$\vec{r} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\vec{v_0} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\vec{a} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

The equations of motion can now be derived from the acceleration:

$$a_x = 1 \longrightarrow v_x = \int dt + 3 = t + 3 \longrightarrow x = \int t + 3dt + 0 = \frac{1}{2}t^2 + 3t$$

$$a_y = 4 \longrightarrow v_y = \int -4dt + 2 = -4t + 2 \longrightarrow y = \int -4t + 2dt + 0 = -2t^2 + 2t$$

To find the extreme point of y, we set its derivative v_y to 0. Thus,

$$-4t + 2 = 0 \longrightarrow t = \boxed{\frac{1}{2}s}$$

Plugging back into x and y:

$$x\left(\frac{1}{2}\right) = 1.63m$$

$$y\left(\frac{1}{2}\right) = 0.5m$$

Then we have the position vector for $t = \frac{1}{2}s$

$$\vec{r} = \boxed{\begin{pmatrix} 1.63 \\ 0.5 \end{pmatrix}}$$

36)

To start, it is given that the angle of projection is $\theta = 30°$ and the initial velocity is $v_0 = 25m/s$. The projectile also starts $30m$ above ground level. To find the time where the projectile hits the ground, the equation for y can be set to $y = 0$.

$$0 = y_0 + v_{0y}t + \frac{1}{2}at^2$$

$$v_{0y} = v_0 \sin\theta = 23 \sin 30° m/s$$

$$a_y = -9.8m/s^2$$

$$0 = 30 + 25 \sin 30° t + \frac{1}{2}(-9.8)t^2$$

This can be solved using the quadratic equation:

$$t = \frac{-25 \sin 30° \pm \sqrt{(25 \sin 30°)^2 - 4(30)\left(\frac{1}{2}(-9.8)\right)}}{2\left(\frac{1}{2}(-9.8)\right)}$$

This provides two solutions, one is obviously incorrect:

$$t = -1.5s, 4.06s$$

Since the first is negative, the correct time of impact is $\boxed{4.06s}$.

41)

In this problem, the range r is to be twice the height h of a projectile's path. First we need the equations of motion.

$$a_x = 0$$
$$v_x = v_0 \cos\theta$$
$$x = v_0 \cos\theta t$$

$$a_y = -g$$
$$v_y = v_0 \sin\theta - gt$$
$$y = v_0 \sin\theta t - \tfrac{1}{2}gt^2$$

Next solve for the time of impact which will be at $x = r$ in the y equation:

$$0 = y = v_0 \sin\theta t - \frac{1}{2}gt^2 = t(v_0 \sin\theta - \frac{1}{2}gt)$$

This leaves two solutions: $t = 0$ and $t = \frac{2v_0 \sin\theta}{g}$

Plugging this t into the x equation will give us an expression for r.

$$r = v_0 \cos\theta = \frac{2v_0^2 \sin\theta \cos\theta}{g}$$

h is at the maximum of this parabola so it will be at the half point of the parabola which corresponds to the half of the range where we are at half of the time. So we can solve using the y equation:

$$h = y\left(\frac{t}{2}\right) = v_0 \sin\theta \frac{v_0 \sin\theta}{g} - \frac{v_0^2 \sin^2\theta}{2}$$

We now want to impose the condition that $r = 2h$ which we can do now that we have expressions for both h and r.

$$\frac{2v_0^2 \sin\theta \cos\theta}{g} = \frac{2v_0^2 \sin^2\theta}{g} - \frac{v_0^2 \sin^2\theta}{g}$$
$$\frac{2\cos\theta}{g} = \frac{2\sin\theta - \sin\theta}{g}$$
$$\frac{2\cos\theta}{g} = \frac{\sin\theta}{g}$$
$$2g\cos\theta = g\sin\theta$$
$$2 = \tan\theta \Rightarrow \theta = \tan^{-1}(2) = \boxed{63.4°}$$

53)

For this problem a gun is being shot at an angle θ from horizontal up a slope at an angle α from horizontal. The gun's bullet travels along parabolic motion a distance l on the slope. First, find the equations of motion:

$$a_x = 0$$
$$v_x = v_0 \cos\theta$$
$$x = v_0 \cos\theta t$$

$$a_y = -g$$
$$v_y = v_0 \sin\theta - gt$$
$$y = v_0 \sin\theta t - \tfrac{1}{2}gt^2$$

Next we note the relation that x and y have to l.

$$v_0 \cos\theta t = l\cos\alpha$$

$$v_0 \sin\theta t - \frac{1}{2}gt^2 = l\sin\alpha$$

We can exploit the relationship in the $x$equation to find an expression for t.

$$v_0 \cos\theta t = l\cos\alpha \Rightarrow t = \frac{l\cos\alpha}{v_0\cos\theta}$$

Plug the expression for t into the y equation:

$$y = v_0 \sin\theta - \frac{1}{2}g\frac{l^2\cos^2\alpha}{v_0^2\cos^2\theta} = l\sin\alpha$$

Now begin solving this equation for l:

$$\sin\theta\frac{l\cos\alpha}{\cos\theta} - \frac{1}{2}g\frac{l^2\cos^2\alpha}{v_0^2\cos\theta} = l\sin\alpha$$

$$\tan\theta\cos\alpha - \frac{1}{2}g\frac{l\cos^2\alpha}{v_0^2\cos^2\theta}$$

$$\tan\theta - \frac{gl\cos\alpha}{2v_0^2\cos^2\theta} = \tan\alpha$$

$$\frac{l}{v_0^2\cos^2\theta} = -\frac{2}{g\cos\alpha}(\tan\theta - \tan\alpha)$$

$$\boxed{l = -\frac{2v_0^2\cos^2\theta}{g\cos\alpha}(\tan\theta - \tan\alpha)}$$

Now we want to treat l as a function of θ and find its extreme values.

$$\frac{dl}{d\theta} = 0 = \frac{d}{d\theta}\left(\frac{-2v_0^2}{g\cos\alpha}\left(\cos^2\theta\frac{\sin\theta}{\cos\theta} - \cos^2\theta\tan\alpha\right)\right)$$

$$= -\frac{2v_0^2}{g\cos\alpha}(\cos^2\theta - \sin^2\theta - 2\cos\theta\sin\theta\tan\alpha)$$

We can now use some of the double angle rules from trigonometry to simplify:

$$0 = -\frac{2v_0^2}{g\cos\alpha}(\cos(2\theta) - \sin(2\theta)\tan\alpha)$$

$$\cos(2\theta) = \sin(2\theta)\tan\alpha$$

$$\cot(2\theta) = \tan\alpha = \cot\left(\alpha + \frac{\pi}{2}\right)$$

$$\boxed{2\theta = \alpha + \frac{\pi}{2} \Rightarrow \theta = \frac{\alpha}{2} + \frac{\pi}{4}}$$

1.5 Rotational Kinematics

1.5.1 Uniform Circular Motion

In rotational motion, we are dealing with one dimensional motion along a circle instead of a straight line. To locate the position, we can use the angle it is away from the horizontal θ. This situation is obviously in the situation where we have a fixed radius.

Similar to the way that we can look at an average translational velocity, we can also look at average angular velocity $\bar{\omega}$.

Definition

Average Angular Velocity:

$$\bar{\omega} = \frac{\Delta\theta}{\Delta t} \tag{1.30}$$

Angular Velocity:

$$\omega(t) = \dot{\theta}(t)[=]1/s \tag{1.31}$$

Average Angular Acceleration

$$\bar{\alpha} = \frac{\Delta\omega}{\Delta t} \tag{1.32}$$

Angular Acceleration:

$$\alpha(t) = \dot{\omega}(t) = \ddot{\theta}(t)[=]1/s^2 \tag{1.33}$$

Thus we can go from angular position to angular velocity to angular acceleration by differentiating $\theta(t)$.

Sometimes it is useful to use a different magnitude than ω and that is called the frequency which is:

Definition

Frequency:

$$f = \frac{\omega}{2\pi}[=]1/s[=]Hz \tag{1.34}$$

Period:

$$T = \frac{1}{f}[=]s \tag{1.35}$$

We integrate α to get the equations of motion:

$$\alpha = \dot{\omega} \to \omega(t) = \omega_0 + \int_{t_0}^{t} \alpha dt$$

$$\omega = \dot{\theta} \rightarrow \theta(t) = \theta_0 + \int_{t_0}^{t} \omega dt = \theta_0 + \omega(t - t_0) + \int_{t_0}^{t}\int_{t_0}^{t} \alpha dt dt$$

If α is constant, we arrive at the following equation:

Definition

$$\omega(t) = \omega_0 + \alpha(t - t_0) \tag{1.36}$$

$$\theta(t) = \theta_0 + \omega_0(t - t_0) + \frac{1}{2}\alpha(t - t_0)^2 \tag{1.37}$$

If we say $\alpha = 0$ then we have uniform circular motion so that $\omega(t) = \omega_0$and $\theta(t) = \theta_0 + \omega_0(t - t_0)$.

1.5.2 Connection to Equations of Linear Motion

We are going to be starting with a circle with a position vector $\vec{r}$ from the center O to the edge of the circle P. We could use the $\hat{e}_x$, $\hat{e}_y$ basis but we would rather use the polar basis $\hat{e}_r$ which points in the direction of the radius and $\hat{e}_\theta$ which points perpendicularly to the radius basis vector in the direction of motion. Note the following:

Definition

The Polar Metric Tensor:

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \tag{1.38}$$

$$\hat{e}_r = \cos\theta \hat{e}_x + \sin\theta \hat{e}_\theta$$

$$e_r = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

$$\hat{e}_\theta = -\sin\theta \hat{e}_x + \cos\theta \hat{e}_y$$

Now we need to take the derivative of these basis vectors:

$$\frac{d\hat{e}_r}{d\theta} = -\sin\theta \hat{e}_x + \cos\theta \hat{e}_y$$

$$\frac{d\hat{e}_\theta}{d\theta} = -\cos\theta \hat{e}_x - \sin\theta \hat{e}_y$$

Note the following relationship:

Definition

$$\frac{d\hat{e}_r}{d\theta} = \hat{e}_\theta \tag{1.39}$$

$$\frac{d\hat{e}_\theta}{d\theta} = -\hat{e}_r \tag{1.40}$$

To begin the relationship we need to find the position vector given a circle of radius $\boldsymbol{R}$:

$$\vec{r}(t) = R\hat{e}_r$$

$$\vec{v}(t) = \dot{\vec{r}} = \frac{d}{dt}(R\hat{e}_r) = R\frac{d}{dt}(\hat{e}_r)$$

We now need to change variables using the chain rule:

$$\vec{v}(t) = R\frac{d\hat{e}_r}{d\theta}\frac{d\theta}{dt}$$

$$= R\omega\hat{e}_\theta$$

This happens to prove that the angular velocity is perpendicular to the position vector.

Definition

Velocity:

$$\vec{v}(t) = R\omega\hat{e}_\theta \tag{1.41}$$

Now that we have the velocity, it is time to find the acceleration:

$$\vec{a}(t) = \dot{\vec{v}} = \frac{d}{dt}(R\omega\hat{e}_\theta)$$

$$\vec{a}(t) = \frac{dR}{dt}\omega\hat{e}_\theta + E\frac{d\omega}{dt}\hat{e}_\theta + R\omega\frac{d\hat{e}_\theta}{dt}$$

Again, use the chain rule:

$$= R\alpha\hat{e}_\theta + R\omega\frac{d\hat{e}_\theta}{d\theta}\frac{d\theta}{dt}$$

$$= R\alpha\hat{e}_\theta - R\omega^2\hat{e}_r$$

Thus we have the acceleration which points into the circle:

Definition

Acceleration:

$$\vec{a}(t) = R\alpha\hat{e}_\theta - R\omega^2\hat{e}_r \tag{1.42}$$

$$a_c = R\omega^2 = \frac{v^2}{r} \tag{1.43}$$

Example

At t_0, $\omega = 0$ and at t_1, we have $\omega = nrpm$ at which one waits for rotation to stop at t_2. What is the linear speed at the edge at t_1? What is the average angular acceleration between t_0 and t_1?

a)

$$v = R\omega = R\frac{2\pi n}{60} = \frac{n\pi R}{30}$$

b)

$$\bar{\alpha}_{0,1} = \frac{\Delta\omega}{\Delta t} = \frac{\pi n}{30t_1}$$

1.5.3 Homework: Rotational Mechanics

69 (Chapter 4))

Equations to start with:

$$a_c = \frac{v^2}{R}$$
$$v = R\omega$$
$$T = \frac{1}{f} = \frac{2\pi}{\omega} \Rightarrow \omega = \frac{2\pi}{T}$$

Now to find the a_c given the following values:

$$R = 6378100m$$

$$T = 1436min$$

$$\Rightarrow a_c = \frac{\left(R\frac{2\pi}{T}\right)^2}{R} = \frac{\left(6378100m\frac{2\pi}{(1436\cdot 60)}\right)^2}{6378100m} = \boxed{0.00339m/s^2}$$

$$a_{c,45^\circ} = \frac{(2\pi)^2 R\cos(45^\circ)}{T^2} = \frac{(2\pi)^2(6378100m)\cos(45^\circ)}{(1436\cdot 60)^2} = \boxed{0.024m/s^2}$$

70 (Chapter 4))

For the top:

$$a_{c,t} = \frac{v^2}{r} = \frac{\left(350kmh\left(\frac{1h}{3600s}\right)\left(\frac{1000m}{1km}\right)\right)^2}{500m} = \boxed{18.904m/s^2}$$

For the bottom

$$a_{c,b} = \frac{v^2}{r} = \frac{\left(620kmh\left(\frac{1h}{3600s}\right)\left(\frac{1000m}{1km}\right)\right)^2}{500m} = \boxed{59.32m/s^2}$$

In order to find relative impacts, add the normal acceleration due to gravity to a_c.

$$a_{r,t} = -9.81m/s^2 + 18.904m/s^2 = \boxed{9.904m/s^2}$$

$$a_{r,b} = -9.81m/s^2 - 59.32m/s^2 = \boxed{-69.13m/s^2}$$

12 (Chapter 12))

Initial conditions:

$$\omega_0 = 0, \omega_t = 500rad/s, t_{0,t} = 0.80s$$

$$\bar{\alpha}_{0,t} = \frac{500rad/s}{0.8s} = \boxed{625rad/s^2}$$

$$\bar{\alpha}_T = R\bar{\alpha}_{0,t} = (0.03m)(625rad/s^2) = \boxed{18.75m/s^2}$$

$$\bar{\alpha_c} = \omega^2 R = (50rad/s^2)(0.03m) = \boxed{75m/s^2}$$

$$\bar{\alpha} = \sqrt{al\bar{p}ha_T^2 + \bar{\alpha}_c^2} = \boxed{77.3m/s^2}$$

14 (Chapter 12))

$$d = 10000m$$

$$v = 900kmh = 250m/s$$

To find $\omega(0)$:

$$v = r\omega = d\omega \longrightarrow \omega = \frac{v}{d} = \frac{250m/s}{10000m} = \boxed{0.025rad/s}$$

To find $\omega(3)$, let β be the displacement angle:

If this is the case, then the component of $\vec{v}$ that is perpendicular to $\vec{r}$ is $v\cos\beta$ giving the following differential relationship between arclength and time:

$$ds = v\cos\beta dt$$

$$\Rightarrow \dot{\beta} = \frac{v\cos\beta}{r}$$

To find an expression for r:

$$r\cos\beta = d \Rightarrow r = \frac{d}{\cos\beta}$$

Substituting back into $\dot{\beta}$:

$$\dot{\beta} = \frac{v\cos\beta}{\frac{d}{\cos\beta}} = \frac{v\cos^2\beta}{d} = \frac{v\left(\frac{d}{r}\right)^2}{d}$$

Now the quantities in this expression need to be determined: In $3min = 180s$ the plane will travel $250m/s \cdot 180s = 4500m$. In addition: r can be found from the other two sides of the triangle:

$$r = \sqrt{(45000)^2 + (10000)^2} = 46100m$$

$$\omega(3) = \dot{\beta}(3) = \frac{250m/s\left(\frac{10000m}{46100m}\right)^2}{10000m} = \boxed{0.0012rad/s}$$

18 (Chapter 12))

$$r = 60cm$$

$$\bar{\alpha} = \frac{1.0cm/s(2\pi)}{20s} = \boxed{0.314rad/s^2}$$

$$\phi = \frac{1}{2}\alpha t^2 = \frac{1}{2}(0.314rad/s^2)(20s)^2 = \boxed{62.8rad}$$

$$s_{20} = r\phi = (0.6m)(62.8rad) = \boxed{37.68m}$$

For this problem set I would say that I feel comfortable with every problem except for 14. This problem took me a lot longer than I had thought because of all of the geometry needed to solve it. There were also a lot of moving pieces in this problem to deal with as well. Perhaps I will look at the extra problem set to get more practice with problems like 14.

1.6 Relative Motion

1.6.1 Addition of Velocities

In the past, we have assumed that physics can be modeled by masses as points moving along a trajectory. For us, we use a reference frame such that we have three axes to describe our 3D world. We can now add another observer that might be moving around. Thus we will need to consider relative motion and reference frames.

Say we want to think of a point P in a reference frame of x_0, y_0, z_0 and we have an observer O_1 with x_1, y_1, z_1. The vector connecting O_1 and P is the vector $\vec{r}_{1P}$. We can add another observer O_2 giving another vector to P $\vec{r}_{2P}$. Finally we have a vector connecting the two observes $\vec{r}_{12}$. Note that these vectors are connected by sums and differences.

Definition

$$\vec{r}_{0i} = \vec{r}_{0j} + \vec{r}_{ji} \tag{1.44}$$

$$\vec{v}_{0i} = \vec{v}_{0j} + \vec{v}_{ji} \tag{1.45}$$

$$\vec{a}_{0i} = \vec{a}_{0j} + \vec{a}_{ji} \tag{1.46}$$

Note that the equations above do not account for rotation. We can decompose as much as we want. It is also interesting to see what happens when we change the order of indices. This will create sign changes in the vectors creating the following relationship:

$$\vec{r}_{ji} = -\vec{r}_{ij}$$

$$\vec{v}_{ji} = -\vec{v}_{ij}$$

$$\vec{a}_{ji} = -\vec{a}_{ij}$$

Definition

The Galilean Transformation: If $\vec{a}_{21} = 0$:

$$\vec{v}_{01} = \vec{v}_{02} + \vec{v}_{21} \tag{1.47}$$

Example

As a train rolls by at 5.00 m/s, you see a cat on one of the flat- cars. The cat is walking toward the back of the train at a speed of 0.50 m/s relative to the car. On the cat is a flea which is walking from the cat's neck to its tail at a speed of 0.10 m/s relative to the cat. How fast is the flea moving relative to you?

$$\vec{v}_{FG} = \vec{v}_{FC} + \vec{v}_{CT} + \vec{v}_{TG}$$

1.6.2 Homework: Addition of Velocities

74)

$$\vec{v}_{RC} = \vec{v}_{RG} + \vec{v}_{CG}$$

$$\vec{v}_{RC} = 25\hat{e}_x - 10\hat{e}_y$$

$$v_{RC} = \sqrt{10^2 + 25^2} = \boxed{26.93 m/s}$$

$$arg(\vec{v}_{RC}) = \tan^{-1}\left(\frac{-10}{25}\right) = \boxed{21.8^\circ}$$

This problem was pretty easy as long as one can use the index formulas and use the basis vectors correctly. I had no problems here.

76)

$$\vec{v}_{SW} = \vec{v}_{WG} + \vec{v}_{SG}$$

$$\vec{v}_{SW} = -30\hat{e}_x + 330\hat{e}_y$$

$$v_{SW} = \sqrt{330^2 - 30^2} = \boxed{328.63m/s}$$

This problem felt similar to the last one and seemed to be fine for me to solve.

88)

$$\bar{v}_{RW} = \frac{1km}{2min} = \frac{1000m}{120s} = \boxed{8.3m/s}$$

$$\bar{v}_{RG} = \bar{v}_{RW} + \bar{v}_{WG} = 30km/h + 90km/h = \boxed{120km/h}$$

This problem was one dimensional making it easy for me to solve. I did need to draw a diagram because of all of the moving parts.

I found the problem set to be a lot easier for me to understand than the last two on kinematics. For this reason, I think I may want to look at some more challenging problems in the book. Perhaps some problems where one needs to calculate different equations of motion in order to determine the velocity additions. I also found the index notation here to be interesting because it is different that is used in GR. Instead of representing different components of a tensor, they describe from where a velocity is being observed.

Chapter 2

Dynamics

2.1 Laws and Basics

2.1.1 Reference Frames and Newton's First Law

To preface this, Newtonian Dynamics can alternatively done with the Euler Lagrange equations which simplify the calculations for the equations of motion. This deals with energies as opposed to the vectors and forces.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$$

These equations can directly find the equations of motion for any system. However, this is something that would be seen in an intermediate mechanics class such as PHY650.

When we see moving objects, we can think about our everyday experiences. Everything will stop at some point. Thus we can think of the natural state as being stationary. Thus we must perturb things in order to break their natural state like throwing a rock. However, we need to consider what our human intuition is and how it fails us.

An example of our intuition failing us is the Law of Inertia (Galileo). What happens when we get rid of resistive forces? Well then we would have no friction to stop anything so anything we throw. Now this means that the natural state is to be moving.

Definition

The Law of Inertia: An unperturbed object will continue on its state of motion unless acted upon.

This can be elaborated upon to say that there are frames called inertial frames in which unperturbed objects remain at constant velocity. Thus an inertial reference frame is a reference frame that moves are constant speed unless perturbed. The earth is an okay-ish inertial reference frame since we could have used it as one many years ago. This is no longer true because of our current measurements. We could also think of a Heliocentric frame which is also okay-ish. The best reference frame that would be inertial would be an observer moving through space at constant velocity in a straight line. It is impossible to have a completely unperturbed system. But the idea here is still effective in abstraction so we will usually consider our reference frames to be inertial.

2.1.2 Newton's Second Law

The basic idea here is the following. When you have two objects, they interact with each other, and, as a result, they change their state of motion. From this we can deduce the following:

$$\vec{F} \sim \frac{d\vec{v}}{dt} \sim \vec{a}$$

We conceptualize the interactions in vectors. Now different objects will react differently to the same force meaning that they have different inertia. To define the correct constant, so that inertia is additive, we put the constant on the righthand side of the equation giving Newton's Second Law:

Definition

$$\vec{F} = m_I \vec{a} \tag{2.1}$$

Newton identified the inertia to be proportional to the amount of matter in the object. Since the inertial mass is a number, the following holds:

$$\vec{F} || \vec{a}$$

Additionally, the inertial mass is not dependent on $\vec{v}$. Finally, note that implicitly, through the law of inertia, Newton's Second Law only works for inertial frames of reference. The relativistic affects will not be present in a realm less than ten percent of the speed of light. In many cases in 551, we will be working with frames that are very nearly inertial but this will not be the case later on.

If we are not in an inertial frame, the first thing to do is work as if we are in an inertial frame. Now in this frame we can solve using Newton's Second Law to solve the problem in an inertial frame. Then we need to go back to the non inertial frame. We do this by using relative motion from kinematics. An example of one of these problems is when someone is in a roller coaster and they need to measure a force. A roller coaster is not an inertial frame because it accelerates.

A final note is that we will only need to deal with the second derivative of position. The third derivative of the position is jerk but this will not matter for us and especially for forces. To compute where an object will move, we only need to consider the second derivative which is position.

Here we want to explain that Newton's Second Law is invariant to changes in the arrow of time.

$$t \longrightarrow t' = -t$$

$$\vec{a}' = \frac{d^2\vec{r}}{dt'^2}$$

$$= \frac{d^2\vec{r}}{dt^2}\left(\frac{dt}{dt'}\right)\left(\frac{dt}{dt'}\right) = \vec{a}$$

2.1.3 Newton's Third Law

The third law simply states the following: This is a result of space being symmetric.

Definition

$$\vec{F}_{ij} = -\vec{F}_{ji} \tag{2.2}$$

2.1.4 Homework: Introductory Dynamics

4)

$$\vec{F}_{BG} = m_B\vec{a}_B = 7.0m/s \cdot 50kg$$

$$\vec{F}_{GB} = m_G\vec{a}_G = 8.2m/s^2 \cdot m_G$$

$$\Rightarrow (7.0kg)(50kg) = (8.2m/s)(m_G) \Rightarrow m_g = \boxed{42.68kg}$$

This problem seemed to be a simple introduction to dynamics with all variables given. All that needed to be done was to apply Newton's Second Law.

20)

$$\vec{r} = \begin{pmatrix} 5 \times 10^4 t \\ 2.0 \times 10^4 t - 2.0 \times 10^5 t^2 \\ -4.0 \times 10^5 t^2 \end{pmatrix}$$

$$\frac{d\vec{r}}{dt} = \begin{pmatrix} 5 \times 10^4 \\ 2.0 \times 10^4 - 4.0 \times 10^5 t \\ -8.0 \times 10^5 t \end{pmatrix}$$

$$\frac{d^2\vec{r}}{dt^2} = \begin{pmatrix} 0 \\ -4.0 \times 10^5 \\ -8.0 \times 10^5 \end{pmatrix}$$

$$\vec{F} = m\vec{a} = 1.7 \times 10^{-27} \begin{pmatrix} 0 \\ -4.0 \times 10^5 \\ -8.0 \times 10^5 \end{pmatrix} = \begin{pmatrix} 0 \\ -6.8 \times 10^{-22} \\ -1.36 \times 10^{-21} \end{pmatrix}$$

$$F = \sqrt{(-6.8 \times 10^{-22})^2 + (-1.36 \times 10^{-21})^2} = \boxed{1.52 \times 10^{-21} N}$$

All that needed to be done was find the acceleration from position and use the second law to find force.

39)

$$T_2 = m_2 g = 3.0kg(9.81m/s^2) = \boxed{29.43N}$$

$$T_1 + T_2 = (m_1 + m_2)g = (3.0kg + 10.0kg)(9.81m/s^2) = \boxed{127.53N}$$

$$T = 126.53N - 29.43N = \boxed{98.1N}$$

This problem seemed fairly simple after applying knowledge from the free body diagrams. The addition of the masses is something I need to be careful of in the future. From the Lagrangian one could obtain (I only did for the first mass for now):

$$T_1 = \frac{1}{2} m \dot{x}_1^2$$

$$V_1 = m_1 g x_1$$

$$L_1 = T_1 - V_1 = \frac{1}{2} m \dot{x}_1^2 - m_1 g x_1$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = \frac{\partial L}{\partial x_1}$$

$$\frac{d}{dt}(m_1 \dot{x}_1) = m_1 g$$

$$m_1 \ddot{x}_1 = m_1 g$$

$$\ddot{x} = a = g$$

Thus the equations of motion are derived.

47)

$$F = \sum_i m_i a \Rightarrow a = \frac{F}{m_1 + m_2} = \frac{60N}{50kg} = \boxed{1.2m/s^2}$$

$$F_{21} = m_2 a = (30kg)(1.2m/s^2) = \boxed{36.0N}$$

The force in the opposite will be the same due to the symmetry with Newton's Third Law.

2.2 Forces

2.2.1 Contact Forces

We like to test what makes things fall and then stay on a surface. In order to take something off of a surface, we need to exert a force. This is due to gravity.

$$\vec{F}_g \sim m_g$$

We let m_g be the gravitational mass. But here the force is a vector and the mass is a scalar. This means that the constant of proportionality needs to be a vector so now we have:

$$\vec{F}_g = m_g \vec{g}$$

Now we want to learn about g which will give rise to something called the equivalence principle:

Definition

The Equivalence Principle: All objects fall to the ground with the same acceleration.

Well how can this be true? We have $\vec{F_{net}} = m_I \vec{a}$ but we also have $\vec{F}_g = m_g \vec{g}$. Thus we have the following:

$$\vec{a} = \frac{m_g}{m_I} \vec{g}$$

We have here that the inertial and gravitational masses must have the same ratio for all objects. Thus this constant in the fraction needs to be a constant but we will set it equal to one for the sake of simplicity:

Definition

$$m_g \equiv m_I \equiv 1 \tag{2.3}$$

$$\vec{a} = \vec{g} \tag{2.4}$$

On earth:

$$g \approx 9.81 m/s^2 \tag{2.5}$$

In PHY551:

$$g = 10 m/s^2 \tag{2.6}$$

In this class, we will consider gravity as a force. Newton also took the time to define mass with ρ as density:

Definition

Mass, density, and volume:

$$m = \rho V \tag{2.7}$$

Now we will take the time to discuss the contact forces. Imagine that we have two bodies and a plane tangent to both bodies. In this case, each will exert on the other thus giving us $\vec{F_{12}}$ and $\vec{F_{21}}$. We can decompose these forces into the normal line to the tangent plane and we shall call this the normal force $\vec{N}$. We can also project the force onto the tangent plane which will be called the frictional force. Thus given 2 objects in contact, we can always decompose their contact force into friction and normal force.

In physics 551, normal force is whatever is needed and no more which means that objects do not break.

Another note on weight: Scales measure the force that the scale applies to whatever is on top. This the scale measures the normal force. The scale assumes that it is in an inertial frame and that you are on top. It measures N and assumes some other force canceling it out mg. It says that, by Newton's Law, $\sum \vec{F} = m\vec{a}$ on the y axis. Thus the scale thinks $N = mg$ and takes $m = \frac{n}{g}$.

Imagine weighing yourself on an elevator: your friend, in an inertial reference frame, looks at you and sees your normal force N'. He applies Newton's Law $\sum \vec{F} = m\vec{a}$.

$$N' - mg = ma = m(g + a) > N$$

Thus we need to apply some transformations in order to get correct masses and weights.

2.2.2 Friction

The other component of the contact force is friction. Friction opposes the direction of motion. For example:

$$\frac{\vec{F}_F}{F_F} = -\frac{\vec{v}}{v}$$

Now that we have the direction, we need the magnitude:

$$F_F \sim N$$

Our constant of proportionality needs to be a scalar which we shall call μ. Because both of the forces are in Newtons, we do not need any dimensions on the coefficient of friction.

Definition

$$F_F = \mu N \tag{2.8}$$

The coefficient is determined empirically and is normally less than one and greater than zero.

We will decompose μ into static friction μ_s and dynamic friction μ_d. It is always the case that $\mu_s \geq \mu_d$.

In the static case, take $\vec{v}$ if $\mu = 0$.

For constant forces, we have the following such that acceleration is constant:

$$\vec{F}_{net} = const = m\vec{a}$$

2.2.3 Hooke's Law

Hooke's Law deals with springs and vibrations. Why do we deal with springs... because springs model oscillations which is something that is important because most objects are not perfectly rigid. In 552, this will be a main focus as you study vibrations.

Let's consider the spring with a length l_0 at the natural length, or equilibrium, of the spring. Now you compress the spring with a force F where the spring pushes back with F at a length l where the difference in length is Δl.

It turns out that from experiments:

$$F \sim \Delta l$$

Thus we get the Hooke's Law:

Definition

$$F = k\Delta l \tag{2.9}$$

We also get $k[=]N/m$. The direction of the forces goes against the direction of the displacement.

In we think about it, a higher k corresponds to a rigid spring while a lower k corresponds to a more elastic spring.

And now to discuss non forces. When you have one object, we will get two equations for two dimensions. So as we increase the number of objects, we doubly increase the amount of equations.

But for interactions, we deal with constraints for systems of several particles (4-6 particles). One constraint is a string and the other is a pulley (an evil device). The strings will fix the distance between objects or link the displacements. They do that by transmitting a "force" that we call tension $\vec{T}$. In 551, we deal with ideal strings which do not break and they are massless. This makes the magnitude of T constant across the whole string. This means that the tension is as strong as the normal force so that the string does not break thus making the tension a constraint.

Pulleys allow us to change the directions of tensions and link displacements through their radai. In practice, the pulley is a disk around which we place a string. The pulley is a disk which can turn and we will assume that we have an ideal pulley. This is a pulley that does not offer a resistance to spinning. This means that the pulley likely does not have any mass. This means that the tension on both sides of the pulley will be the same. These constraints will not be the case in 552 where the pulleys will have mass and they will also offer resistance.

2.2.4 Homework: Applications of Newton's Laws

51)

$$\frac{900N}{2\cos 85^\circ} = \boxed{5163.2N}$$

This problem was a simple application of Newton's Laws.

66)

$$F_x = mg\sin 35^\circ$$

$$F_y = F_N - mg\cos 35^\circ$$

$$a = g\sin 35^\circ = \boxed{5.63m/s^2}$$

For this problem, it was the components of the force that needed to be found and then they could be inserted into the laws to find the acceleration.

70)

$$m_1 a = T - m_1 g \Rightarrow T = m_1 a + m_1 g$$

$$m_2 a = m_2 g - T$$

$$m_2 a + m_1 a = m_2 g - m_1 g$$

$$\Rightarrow g = \frac{a(m_2 + m_1)}{m_2 - m_1}$$

$$x = \frac{1}{2}at^2 \Rightarrow a = \frac{2x}{t^2} = \frac{1}{40.95} = 0.02m/s^2$$

$$\Rightarrow g = \frac{0.02m/s^2(0.802kg)}{0.002kg} = \boxed{8.02m/s^2}$$

The problem here to determine the different accelerations and the acceleration to gravity. It is important here to include all decimal points because my answer is off by about one meter per second. I was also able to find the acceleration by using the Lagrangian Method.

74)

m_1:

$$T_1 - m_1 g = m_1 a_1 \Rightarrow a = \frac{T_1 - m_1 g}{m_1}$$

m_2:

$$m_2 g - T_2 = m_2 a_2 \Rightarrow a_2 = \frac{m_2 g - T_2}{m_2}$$

p_2:

$$2T_2 - T_1 = 0$$

$$\Rightarrow 2a_1 = a_2$$

$$\Rightarrow T_1 = m_1 g + m_1 a_1$$

$$\Rightarrow T_2 = \frac{1}{2}(m_1 g + m_1 a_1)$$

$$\Rightarrow m_2 g - \frac{1}{2} m_1 g - \frac{1}{2} m_1 a_1 = 2 m_2 a_1$$

$$\Rightarrow g\left(m_2 - \frac{1}{2} m_1\right) = a_1 \left(\frac{1}{2} m_1 + 2 m_2\right)$$

$$a_1 = \boxed{\frac{g\left(m_2 - \frac{1}{2} m_1\right)}{\frac{1}{2} m_1 + 2 m_2}}$$

$$a_2 = \boxed{\frac{2g\left(m_2 - \frac{1}{2} m_1\right)}{\frac{1}{2} m_1 + 2 m_2}}$$

$$a_{p_2} = \boxed{a_1}$$

This problem was a more challenging application of Newton's Laws to find the accelerations of the different parts of the system.

17 (Chapter 6))

$$F = ma$$

$$T \cos 30^\circ - \mu_D F_N = 0$$

$$F_N = mg - T \sin 30^\circ$$

$$T \cos 30^\circ - \mu_D mg + \mu_D T \sin 30^\circ = 0$$

$$T = \frac{\mu_D mg}{\cos 30^\circ + \mu_D \sin 30^\circ} = \frac{(0.6)(40kg)(9.81)}{\cos 30^\circ + 0.6 \sin 30^\circ} = \boxed{201.9N}$$

This problem looked into force at an angle including friction. I found the free body diagrams to be particularly important here.

2.2.5 The Inclined Plane

With this material covered, we are finished with new material for now. But we will need to practice some more complex systems before moving on to more complicated material.

Example

We want to find the maximum angle that the mass will stay and the acceleration when it does not. First we will use Newton's Second Law:

$$\vec{F}_{net} = \sum \vec{F} = m\vec{a}$$

Now consider all of the forces at play:

$$m\vec{g} + \vec{f_s} + \vec{F}_N = 0$$

Now we will decompose this using a coordinate system parallel to that of the incline:

$$\begin{pmatrix} mg\sin\theta \\ -mg\cos\theta \end{pmatrix} + \begin{pmatrix} -\mu_s F_N \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ F_N \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From this we have x and y equations. We will solve for the normal force in the y equation:

$$-mg\cos\theta + F_N = 0 \Rightarrow F_N = mg\cos\theta \Rightarrow mg\sin\theta - \mu_s mg\cos\theta = 0$$

$$\sin\theta = \mu_s mg\cos\theta \Rightarrow \theta = \tan^{-1}\mu_s$$

Now for acceleration:

$$m\vec{g} + \vec{f_k} + \vec{F}_N = ma$$

$$mg\sin\theta - \mu_k mg\cos\theta = ma \Rightarrow a = g(\sin\theta - \mu_k\cos\theta)$$

Example

We shall continue this example by putting our whole system into an elevator. To do this, we will need to edit our equation of forces to find the acceleration.

$$\begin{pmatrix} mg\sin\theta \\ -mg\cos\theta \end{pmatrix} + \begin{pmatrix} -\mu_s F_N \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ F_N \end{pmatrix} = \begin{pmatrix} -ma_E\sin\theta \\ ma_E\cos\theta \end{pmatrix}$$

Similarly to the last example, we will solve for normal force in the y equation and insert into the x equation:

$$mg\sin\theta - \mu_s mg\cos\theta - \mu_s ma_E\cos\theta = -ma_E\cos\theta$$

$$(g + a_E)\sin\theta - \mu_s(g + a_e)\cos\theta = 0$$

$$\theta = tan^{-1}\mu_s$$

Thus we have no change from the last problem.

Example

People in physics like elevators but they also like trains. To facilitate some horizontal movement, we will put the inclines plane in the train and see if there is any change to the angle:

$$\begin{pmatrix} mg\sin\theta \\ -mg\cos\theta \end{pmatrix} + \begin{pmatrix} -\mu_s F_N \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ F_N \end{pmatrix} = \begin{pmatrix} ma_T\cos\theta \\ ma_T\sin\theta \end{pmatrix}$$

Now solve for the normal force

$$F_N = ma_T\sin\theta + mg\cos\theta$$

$$mg\sin\theta - \mu_s ma_T\sin\theta + \mu_s mg\cos\theta = ma_T\cos\theta$$

$$(g - \mu_s a_T)\sin\theta - (a_T + g\mu_s)\cos\theta = 0$$

This is a different result from the one we had in the past two examples.

2.2.6 Homework: Springs, Pulleys, and Strings

25)

$$\vec{f}_v = -b\vec{v} \Rightarrow mg - b\vec{v} = 0$$

$$3.9\times 10^{-6}g\left(\frac{100g}{1kg}\right)(9.81m/s^2) - (2.8\times 10^{-5})\vec{v} = 0$$

$$\vec{v} = \frac{mg}{b} = \boxed{1336.39\times 10^{-3}m/s}$$

This problem was just a recognition of the second law from Newton.

29)

$\hat{e}_x$:

$$F_H\sin\theta - F_N = 0 \Rightarrow F_N = F_H\sin\theta$$

$\hat{e}_y$:

$$F_H\cos\theta + \mu_s F_N - mg = 0$$

$$F_H\cos\theta + \mu_s F_H\sin\theta - mg = 0$$

$$F_H = \frac{mg}{\cos\theta + \mu_s\sin\theta}$$

Differentiate with respect to θ

$$\frac{dF_H}{d\theta} = \frac{d}{d\theta}(\cos\theta + \mu_s\sin\theta) = -\sin\theta + \mu_s\cos\theta$$

Maximize by setting equal to zero:

$$-\sin\theta + \mu_s \cos\theta = 0 \Rightarrow \sin\theta = \mu_s \cos\theta$$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{\mu_s}{1}\right)$$

Given the value of the tangent, the values for the sine and cosine can be found using a trigonometric substitution with a right triangle:

$$\sin\theta = \frac{\mu_s}{\sqrt{1+\mu_s^2}}$$
$$\cos\theta = \frac{1}{\sqrt{1+\mu_s^2}}$$

Thus:

$$F_H = \frac{mg}{\frac{1}{\sqrt{1+\mu_s^2}} + \frac{\mu_s^2}{\sqrt{1+\mu_s^2}}} = \frac{mg}{\sqrt{1+\mu_s^2}}$$

Taking the force in the limiting case, we get the following:

$$\cos\theta + \mu_s \sin\theta = 0$$

$$\Rightarrow \tan\theta = \boxed{-\frac{1}{\mu_s}}$$

This problem was slightly more difficult because it involved a maximization problem. This was similar to the gun problem in kinematics.

32)

$$F_{net_1} = -\mu_1 F_{N_!} + F = m_1 a_1 \Rightarrow a_1 = \frac{-\mu_1 + F}{m_1}$$

$$F_{N_1} - m_1 g = 0 \rightarrow F_{N_1} = m_1 g$$

$$\Rightarrow -\mu_1 m_1 g + F = m_1 a_1$$

$$\Rightarrow a_1 = \boxed{\frac{-\mu_1 m_1 g + F}{m_1}}$$

$$F_{net_2} = \mu_1 F_{N_1} - \mu_2 F_{N_2} = m_2 a_2$$

$$F_{N_2} - F_{N_1} - m_2 g = 0 \Rightarrow F_{N_2} = F_{N_1} + m_2 g$$

$$F_{N_2} = m_1 g + m_2 g$$

$$m_1 \mu_1 a_1 + \mu_2 (m_1 g + m_2 g) = m_2 a_2$$

$$\Rightarrow a_2 = \boxed{\frac{\mu_1 m_1 - \mu_2(m_1 + m_2)}{m_2} g}$$

33)

$$m_2 g - T = m_2 a$$

$$F_N = m_1 g \cos\theta$$

$$f_k = \mu_k F_N = \mu_k m_1 g \cos\theta$$

$$\Rightarrow T - m_1 g \sin\theta - \mu_k m_1 g \cos\theta = m_1 a$$

$$m_2 g - m_1 g \sin\theta - \mu_k m_1 g \cos\theta = (m_1 + m_2) a$$

$$\Rightarrow a = \frac{m_2 g - m_1 g \sin\theta - \mu_k m_1 g \cos\theta}{m_1 + m_2}$$

$$\Rightarrow a = \boxed{3.67 m/s^2}$$

This problem was similar to the one studied in class. I would like to do these problems more systematically in the future with the column vectors like done in class.

47)

At the position with $x = 0$, the expression for force would be:

$$F = k_1 x + k_2 x = x(k_1 + k_2) = xk$$

This is true if we let $k_1 + k_2 = k$.

88)

$$m_2 g - T = 0 \Rightarrow m_2 g = -T$$

$$T - \mu_k F_N = m_1 a$$

$$F_{N_1} = m_1 g$$

$$\Rightarrow T - \mu_k m_1 g = m_1 a$$

$$-m_2 g - \mu_k m_1 g = m_1 a + m_2 a$$

$$\Rightarrow \boxed{a = \frac{-m_2 g - \mu_k m_1 g}{m_1 + m_2}}$$

This problem set was not too bad. I have been looking up how one can include conservative forces like friction into Lagrangians but it looks like that predicates the use of dampening factors. I suppose this is a scenario where it is more effective to use Newtonian Mechanics as opposed to Lagrangian or Hamiltonian.

2.3 Gravitation

2.3.1 Newton's Law of Gravity

So far we have looked at gravity on the surface of the earth which is a constant. Newton came and told us that any two point masses will attract each other with forces that is proportional to the masses and inversely proportional to the square of their distance. This is the first of many "inverse square laws" that will be learned in the future.

So if we have two particles, then:

$$F_G \sim \frac{m_1 m_2}{r^2}$$

We will need a constant of proportionality which is Newton's Gravitational Constant:

Definition

Newton's Law of Universal Gravitation:

$$F_G = \frac{G m_1 m_2}{r^2} \tag{2.10}$$

$$\vec{F}_G = \frac{G m_1 m_2}{r_{12}^2} \frac{\vec{r}_{12}}{r_{12}} \tag{2.11}$$

$$\vec{F}_G = \frac{G m_1 m_2}{|\vec{r}_2 - \vec{r}_1|^3} (\vec{r}_2 - \vec{r}_1)$$

Here are some observations: First, the force will go in the direction of the attracting particle. Second, we will always have positive attraction since mass is always positive, for now. Third, we are also at a force of a distance such that we are not looking at relativistic impacts such that it does not require a medium and the speed of propagation is infinite.

We will also look at the value of the constant $\boldsymbol{G}$.

Definition

$$G = 6.660 \times 10^{-11} \frac{m^3}{kgs^2} \tag{2.12}$$

This makes the gravity force very weak at small scales, and the strong, weak, and EM forces are so much more important at the quantum scales. But in the large scale, gravity is the most important.

Something that is important is that we have the inverse square law. This means that the strength falls off at large scales but it is never 0. This is something that is important to the gravitational and EM force but is not in the weak or strong forces because those forces are contained.

In the situation we have two masses where the first is much larger than the second. From this we have:

$$\vec{a}_{12} = \frac{\vec{F}_{12}}{m_1}$$

$$\vec{a}_{21} = \frac{\vec{F}_{21}}{m_2}$$

This will mean that the first mass will not move. We will call the big mass M and the small mass m. We will put M at the center and have $\vec{r}$ connecting the two. Thus:

$$\vec{F}_G = -G\frac{Mm}{r^2}\hat{e}_r = -G\frac{Mm}{r^3}\vec{r}$$

We can rewrite this with the gravitational field $\vec{g}(\vec{r})$

$$= \vec{g}(\vec{r})m$$

Definition

The Gravitational Field

$$\vec{g}(\vec{r}) = -G\frac{M}{r^2}\frac{\vec{r}}{r} \tag{2.13}$$

This is an example of a vector field. From this we can rederive the equivalence principle:

$$\vec{a} = \vec{g}(\vec{r})$$

This is all we need for a system of two particles, but if you have many masses, we can apply the superposition principle.

Definition

The force of attraction due to a mass is independent of the rest of the universe.

$$\vec{F}_1 = \vec{F}_{12} + \vec{F}_{13}$$
$$\vec{F}_i = \sum_{j=1}^{N} \vec{F}_{ij}$$
$$\vec{g}(\vec{r}) = \sum_{i=1}^{N} \vec{g}_i(\vec{r})$$

Example

Four values of equal mass on four corners of a square and the size of the square is l and we place an origin at the center. What is the gravitational field at the origin and what is the acceleration that a mass would experience.

Intuitively, the field at the origin will be 0.

$$\vec{g}(0,0) = 0$$

The second problem requires the use of the superposition principle:

$$\vec{g}_1 = \vec{g}(\vec{r}_1) = \vec{g}_2 + \vec{g}_3 + \vec{g}_4$$
$$= G\frac{m}{l^2}\begin{pmatrix} 0 \\ -1 \end{pmatrix} + G\frac{m}{l^2}\begin{pmatrix} -1 \\ 0 \end{pmatrix} + G\frac{m}{(\sqrt{2}l)^2}\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

2.3.2 Gravitational Field Continuously

Imagine that we have a cow and this cow has a certain density at each point if we put a coordinate system we have on the cow. We can define the density:

$$\rho(\vec{r}') = \frac{dm}{dv}$$

And we want to measure g in the direction from a point p some distance $\vec{r}$ away from the origin. We will cut the cow in very small cubes of cow which we can treat as point particles where we see a small amount of g to that cube. The vector going there is $\vec{r} - \vec{r}'$.

$$d\vec{g} = G\frac{dm}{|\vec{r} - \vec{r}'|^3}$$
$$\vec{g} = \int d\vec{g}$$
$$= \iiint_V \frac{G\rho(\vec{r}')}{|\vec{r}' - \vec{r}|^3}(\vec{r}' - \vec{r})d^3\vec{r}$$

If we have a spherical source, we can find the field a lot more easily, because this is the same as a point mass.

A sphere of uniform density is spherically symmetric. A sphere whose density increases as you get closer is also spherically symmetric. If someone made a hole in the middle of the sphere with constant density, then it is also spherically symmetric. If we have a thin shell that is a sphere with constant density then we have a spherically symmetric source. If we make a hole in the upper part of the sphere, then that is no longer spherically symmetric.

If we have holes in strange places, then we can still use the superposition principle to solve. A hole is the same as a hole with negative density.

2.3.3 Homework: Gravitation

3)

$$F_E = F_M$$

$$\Rightarrow \frac{Gm_E}{r^2} = \frac{Gm_M}{(l-r)^2}$$

$$l - r = r\sqrt{\frac{m_M}{m_E}}$$

$$\Rightarrow l = r + r\sqrt{\frac{m_M}{m_E}}$$

$$3.8 \times 10^8 m = r\left(1 + \sqrt{\frac{m_M}{M_e}}\right)$$

$$\rightarrow r = \boxed{3.42 \times 10^8 m}$$

Simple application of Newton's Law for Gravitation.

5)

$$F_S = \frac{Gm_P m_s}{r^2} = \boxed{0.415N}$$

$$F_M = \frac{Gm_P m_S}{r^2} = \boxed{0.023N}$$

This problem was a lot of looking up constants but it helped my understanding of perspective.

9)

$$ma = \frac{GMm}{r^2} \Rightarrow a = \frac{GM}{r^2}$$

Jupiter:

$$a_j = \frac{Gm_j}{r_j^2} = \boxed{25.87m/s^2}$$

Saturn:

$$a_s = \frac{Gm_s}{r_s^2} = \boxed{11.34m/s^2}$$

Uranus:

$$a_u = \frac{Gm_u}{r_u^2} = \boxed{8.68m/s^2}$$

This problem was also heavy use of the look up table for constants, but intuition can be used to at least guess the answers.

12)

$$\theta = \tan^{-1}\left(\frac{F_G}{mg}\right)$$

$$F_G = \frac{Gm_1m_2}{r^2} = 2.34 \times 10^8 N$$

$$mg = 14.715N$$

$$\tan\theta \approx \theta \Rightarrow \theta \approx \frac{F_G}{mg} = \boxed{1.59 \times 10^{-9\circ}}$$

This problem put into perspective how weak gravity is on the small scale and how much more important the strong force (gluons, quarks) and the EM force (electrons, antielectrons, and photons) are.

14)

$$F_G = \sum F_{Gi}$$

$$F_{Gi} = \frac{Gm^2}{a^2} \Rightarrow F_{net} = \frac{\cos 30^\circ Gm^2}{a^2} = \boxed{\frac{\sqrt{3}Gm^2}{2a^2}}$$

This made sense to me. I do have a question though. In a similar way that Coulombs Law has a corresponding Gauss Law, does the Universal Law of Gravitation have a corresponding Gauss Law?

2.4 Orbital Dynamics

2.4.1 Circular Dynamics

To deal with circular dynamics, one must revisit circular kinematics. Here we know that there are two components to angular acceleration. These are tangential acceleration and centripetal acceleration. From this we find:

$$\vec{a} = a_t \hat{e}_\theta + a_c \hat{e}_r$$

$$= R\alpha \hat{e}_\theta - R\omega^2 \hat{e}_r$$

Thus:

$$\vec{F}_{net} = m\vec{a}$$

There are multiple sources of a_c which could be gravity, tension, or normal force.

Example

Prove that the movement of a test particle (mass m) around a big mass M along a circular orbit is uniform. Give the value of the velocity. Also prove that the square of the period is proportional to the cube of the radius (Kepler's Third Law).

a)

$$\vec{F}_{net} = m\vec{a}$$

$$\begin{pmatrix} 0 \\ -\frac{GMm}{r^2} \end{pmatrix} = m \begin{pmatrix} 0 \\ -\frac{v^2}{r} \end{pmatrix}$$

$$\Rightarrow v = \sqrt{\frac{GM}{r}}$$

b)

$$T^2 = \frac{4\pi^2}{GM} r^3$$

Example

Imagine a racetrack with an inner radius of r_1 and a width of w. There are three particles. The first particle takes the inner radius, the second particle takes the outer radius, and the third particle which takes the outer radius to the inner radius. All of the curves traced out by these particles are circles. All particles have the same friction. What trajectory is optimal? This is with constant velocity.

The optimal trajectory is the one going on the inner radius because it has the smallest radius.

Example

Say you drop a mass into a fluid where $\vec{F}_f = -b\vec{v}$. Prove that $v(t) = \frac{mg}{b}\left(1 - e^{-\frac{bt}{m}}\right)$.

$$v(0) = \frac{mg}{b}(1 - e^0) = 0$$

$$v(t \to \infty) = \frac{m}{b}$$

$$F = ma \Rightarrow m\frac{dv}{dt} = -bv + mg$$

$$\frac{dv}{dt} = g - \frac{b}{m}v$$

$$\int \frac{dv}{g - \frac{b}{m}v} = \int dt$$

$$\Rightarrow \ln|1 - \frac{b}{mg}v| = -\frac{bt}{m}$$

$$\Rightarrow v(t) = \frac{mg}{b}\left(1 - e^{-\frac{bt}{m}}\right)$$

2.4.2 Homework: Circular Dynamics

51 (Chapter 6))

$$\vec{F} = m\vec{a}$$

$$\vec{F} = \begin{pmatrix} 0 \\ -mg \end{pmatrix} + \begin{pmatrix} 0 \\ -2T \end{pmatrix}$$

$$\Rightarrow -mg = 2T = ma = \frac{mv^2}{r}$$

$$\Rightarrow 2T = \frac{mv^2}{r} + mg$$

$$\Rightarrow T = \frac{\frac{mv^2}{r} + mg}{2}$$

$$= \frac{\frac{60kg(5m/s)^2}{5m} + 60kg(9.81m/s^2)}{2}$$

$$= \boxed{444.3N}$$

This problem was a great first application to the circular dynamics. One needs to use Newton's Laws to derive the equations of motion for the system.

64 (Chapter 6))

If we want to stop:

$$ma = -\mu_s mg \rightarrow a = -\mu_s g$$

$$2ad_1 = -v^2 \rightarrow -2\mu_s g d_1 = -v^2$$

$$\Rightarrow d_1 = -\frac{v^2}{2\mu_s g}$$

If we want to turn:

$$\mu_s mg = \frac{mv^2}{d_2} \rightarrow \mu_s g = \frac{v^2}{d_2}$$

$$\Rightarrow d_s = \frac{v^2}{\mu_s g}$$

In this case $\boxed{d_1 < d_2}$ so it is best to turn to avoid a car crash into the unforgiving brick wall.

It was interesting here to look at a problem with multiple scenarios.

65 (Chapter 6))

$$F = ma$$

$$\begin{pmatrix} 0 \\ -mg \end{pmatrix} + \begin{pmatrix} T\sin\phi \\ T\cos\phi \end{pmatrix} = \begin{pmatrix} \frac{mv^2}{r} \\ 0 \end{pmatrix}$$

$$\tan\phi = \frac{\frac{mv^2}{r}}{mg} = \frac{v^2}{rg} = \frac{v^2}{gl\sin\phi}$$

$$\Rightarrow \tan\phi = \frac{v^2}{gl\sin\theta}$$

$$\Rightarrow \boxed{v = \sqrt{\tan\phi g l \sin\phi}}$$

This problem was similar to the inclined plane in decomposition of vectors but different because of the rotation.

75 (Chapter 6))

$$v = \frac{r}{R} v_0$$

$$r = R + l \sin\alpha$$

$$F = ma$$

$$\begin{pmatrix} 0 \\ -mg \end{pmatrix} + \begin{pmatrix} T\sin\alpha \\ T\cos\alpha \end{pmatrix} = \begin{pmatrix} \frac{mv^2}{r} \\ 0 \end{pmatrix}$$

$$\tan\alpha = \frac{v^2}{rg} = \frac{v_0^2 r}{gR^2}$$

$$\Rightarrow v_0 = \sqrt{\frac{\tan\alpha g R^2}{r}}$$

$$= \sqrt{\frac{\tan\alpha g R^2}{R + l\sin\alpha}}$$

$$= \sqrt{\frac{(1)(9.81 m/s^2)(0.2m)^2}{(0.2m) + 0.3m\left(\frac{\sqrt{2}}{2}\right)}}$$

$$= \boxed{-0.97 m/s^2}$$

I think in this case, I like to work more in variable form. It is nice not to need to plug all of this into the calculator.

26 (Chapter 9))

$$T^2 = \frac{4\pi^2}{GM_s} r^3$$

$$\Rightarrow r^3 = T^2 \frac{GM_s}{4\pi^2}$$

$$r^3 = (4.32 \times 10^4 s) \frac{G(6 \times 10^{24} kg)}{4\pi^2}$$

$$\Rightarrow r = \boxed{2.66 \times 10^7 m}$$

$$v = \frac{2\pi r}{T} = \boxed{3.9 \times 10^3 m/s}$$

It is nice to see how the theories of the small translate to the theories of the large even in in the Newtonian Theories.

34 (Chapter 9))

$$F = \frac{Gm_1m_2}{(r_1+r_2)^2} = \frac{mv^2}{r} = \frac{4\pi^2mr}{T^2}$$

$$\Rightarrow \frac{Gm_1m_2}{(r_1+r_2)^2} = \frac{4\pi^2m_2r_2}{T^2} \to T^2 = \frac{4\pi^2}{G}\left(\frac{r_2}{m_1}\right)(r_1+r_2)^2$$

$$\Rightarrow \boxed{T^2 = \frac{4\pi^2}{G}\frac{(r_1+r_2)^3}{m_1+m_2}}$$

This problem makes intuitive sense to me and it was nice to do another proof.

I find these problems to be very interesting. It is going to be engaging to see what happens when we look at the energies in the gravitational fields.

2.5 Review

Example

Say we want to find the spring constant for parallel springs:

$$F = F_1 + F_2 = k_1\Delta x + k_2\Delta x = (k_1+k_2)\Delta x$$

$$k_p = \sum_i k_p$$

Now for springs in a series:

$$\Delta x = \Delta x_1 + \Delta x_2 = \frac{F}{k_1} + \frac{F}{k_2}$$

$$k_s = \frac{k_1k_2}{k_1+k_2}$$

$$\frac{1}{k_s} = \sum_i \frac{1}{k_i}$$

Example

Consider we have m_1 on a table spinning and m_2 is hanging off of the center of rotation spinning on its own circle. The string hanging to m_2 makes an angle θ with respect to the vertical. Find the radius of m_2's trajectory.

$$T = m_1 r_1 \omega^2$$

$$T \cos\theta = m_2 g$$

$$T \sin\theta = m_2 r_2 \omega^2$$

$$\Rightarrow m_2 g \frac{\sin\theta}{\cos\theta} = r_2 \omega^2$$

$$= m_2 g \frac{m_2 g}{\cos\theta} \frac{1}{m_1 r_1}$$

$$\Rightarrow r_2 = \frac{m_1}{m_2} \sin\theta r_1$$

Chapter 3

Energies

3.1 Work

3.1.1 The Dot Product

Before looking at work, we need to think about the dot product between two vectors. When we have two vectors $\vec{A}$ and $\vec{B}$, the scalar product is the following:

Definition

$$\vec{A} \cdot \vec{B} = AB\cos\theta_{AB} \tag{3.1}$$

We can break this into components as well if we do not know the magnitude

$$\vec{A} \cdot \vec{B} = A_i B_i \delta_{ij} \tag{3.2}$$

The scalar product commutes such that:

$$\begin{aligned}
\vec{A} \cdot \vec{B} &= AB\cos\theta_{AB} \\
&= BA\cos -\theta_{AB} \\
&= BA\cos\theta_{BA} \\
&= \vec{B} \cdot \vec{A}
\end{aligned}$$

The dot product will be larger than 0 when the cosine of the angle is positive which means that the angle must be between -90 and 90 degrees. Thus the vectors are roughly in the same direction. The dot product is 0 if either of the vectors are 0 or the vectors are perpendicular. The dot product is negative when the angle is between 90 and -90 degrees which means that the vectors are pointing roughly away from each other.

$$\vec{A} \cdot \vec{A} = AA\cos(0) = A^2$$

From this we can get a way to get the magnitude of a vector:

Definition

$$A = \sqrt{\vec{A} \cdot \vec{A}} \tag{3.3}$$

$$\begin{aligned}
\vec{A} \cdot \vec{B} &= AB\cos\theta = AB\cos\beta - \alpha \\
&= AB(\cos\beta\cos\alpha + \sin\beta\sin\alpha) \\
&= AB\left(\frac{B_x}{B}\frac{A_x}{A} + \frac{B_y}{B}\frac{A_y}{A}\right) \\
&= A_x B_x + A_y B_y \\
&= A_i B_i \delta_{ij}
\end{aligned}$$

Here are some properties of the scalar product:

$$\alpha\vec{A} \cdot \beta\vec{B} = \alpha\beta(\vec{A} \cdot \vec{B})$$

$$(\vec{A} + \vec{B}) \cdot \vec{C} = \vec{A} \cdot \vec{C} + \vec{B} \cdot \vec{C}$$

Now we would like to see what happens when we take derivatives of dot products. Now we have $\vec{A}(t)$ and $\vec{B}(t)$. We want to find:

$$\frac{d}{dt}(\vec{A}(t) \cdot \vec{B}(t))$$

$$\begin{aligned}
&= \frac{d}{dt}\left(\begin{pmatrix} A_x(t) \\ A_y(t) \end{pmatrix} \cdot \begin{pmatrix} B_x(t) \\ B_y(t) \end{pmatrix}\right) \\
&= \frac{d}{dt}(A_x(t)B_x(t) + A_y(t)B_y(t)) \\
&= \frac{d}{dt}(A_x(t)B_x(t)) + \frac{d}{dt}(A_y(t)B_y(t)) \\
&= \dot{A}_x(t)B_x(t) + A_x(t)\dot{B}_x(t) + \dot{A}_y(t)B_y(t) + A_y(t)\dot{B}_y(t) \\
&= \dot{\vec{A}}(t) + \vec{B}(t) + \vec{A}(t) \cdot \dot{\vec{B}}(t)
\end{aligned}$$

This means that the product rule applies to the dot product.

Definition

$$\frac{d}{dt}(\vec{A} \cdot \vec{B}) = \dot{\vec{A}}(t) + \vec{B}(t) + \vec{A}(t) \cdot \dot{\vec{B}}(t) \tag{3.4}$$

3.1.2 Element of Work

We know that the velocity is tangent to the trajectory and we know that $\vec{v} = \dot{\vec{r}}$ which means that the velocity is parallel to the vector $d\vec{r}$. Thus $d\vec{r}$ is also tangent to the trajectory. We can always decompose the acceleration into the tangent and normal accelerations. Adding these accelerations will give $\vec{a}$ such that:

Definition

$$\vec{a} = \vec{a}_T + \vec{a}_N \tag{3.5}$$

$$a_T = \frac{dv}{dt}$$

$$\vec{a}_T = \frac{dv}{dt}\frac{\vec{v}}{v}$$

From Newton's Laws, we know that mass will relate acceleration and force. So we can define the element of work:

Definition

Element of work:

$$dW = \vec{F} \cdot d\vec{r} \tag{3.6}$$

$$W[=]Nm = J \tag{3.7}$$

The element of the work is positive when the force points in the direction of movement. It is zero when there is no force or displacement. It is negative when the force is opposing the displacement.

3.1.3 Total Work

We discussed the element of work but we need a total element from A to B over a path C. In 551, the path will be a line. In general, the path can be anything which makes one introduce the line integral where one will need to parameterize the path to integrate it over multiple dimensions.

Definition

$$W = \int_C \vec{F} \cdot d\vec{r} \tag{3.8}$$

$$\oint_C \vec{F} \cdot d\vec{r} = 0 \tag{3.9}$$

There is a way to get rid of this integral for some cases.

$$W = \vec{F} \int_A^B dr = Fd\cos\theta$$

If we have 1D movement without constant force, we will have easy integrals. Our unit vector is $\hat{e}_x$.

$$\vec{F}(x) = F(x)\hat{e}_x$$
$$F\vec{r} = dx\hat{e}_x$$
$$\vec{F}\cdot d\vec{r} = F(x)dx$$
$$W = \int_{x_A}^{x_B} F(x)dx$$

In one dimension, the work is equal to the area below the force curve. This analogy is not so good for more dimensions where one must think of a nonuniformly dense rod moving through multiple dimensions.

In more dimensions, we will look at a very important theorem called the Work Energy Theorem.

$$\begin{aligned} W_{net} &= m\int_{\vec{r}_i}^{\vec{r}_f} (\vec{a}_T + \vec{a}_N)\cdot d\vec{r} \\ &= m\int_{\vec{r}_i}^{\vec{r}_f} \vec{a}_T\cdot d\vec{r} \\ &= m\int_{\vec{r}_i}^{\vec{r}_f} \frac{d\vec{v}}{dt}\frac{\vec{v}}{v}\cdot d\vec{r} \\ &= m\int_{\vec{v}_i}^{\vec{v}_f} \frac{\vec{v}}{v}\cdot \vec{v}dv \\ &= m\int_{\vec{v}_i}^{\vec{v}_f} vdv \\ &= m\left(\frac{1}{2}v_f^2 - \frac{1}{2}v_i^2\right) = T \end{aligned}$$

Here, T is the kinetic energy.

Definition

$$W_{net} = \Delta T = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 \tag{3.10}$$

The units of kinetic energy are also the same for work. Also, the theorem above is true only in inertial reference frames. Additionally, both work and kinetic energy depend on the frame.

3.1.4 Homework: Work

8)

$$F = 50N \quad \theta = 60^\circ \quad d = 1.6m$$

$$W = Fd\cos\theta = (50N)(1.6m(\cos(60^\circ)) = \boxed{40J}$$

This problem was just plugging in the numbers into the work equation.

11)

$$d = 2.5m \quad \mu = 0.45 \quad m = 60kg$$

$$\begin{pmatrix} -mg\sin\theta \\ -mg\cos\theta \end{pmatrix} + \begin{pmatrix} -\mu F_N \\ F_N \end{pmatrix} + \begin{pmatrix} F_H \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$F_N = mg\cos\theta$$

$$\rightarrow -mg\sin\theta - \mu mg\cos\theta + F_H = 0$$

$$\rightarrow F_H = mg\sin\theta + \mu mg\cos\theta$$

$$= (60kg)(9.81m/s^2)\sin(30^\circ) + (0.45)(60kg)(9.81m/s^2)\cos(30^\circ)$$

$$= 524N$$

$$\rightarrow W = (524N)(2.5m)\cos(0^\circ) = \boxed{2600J}$$

This problem was interesting since it involved using dynamics in order to solve this work problem. Goes to show how cumulative physics is.

19)

For the elevator:

$$T_1 - mg = ma \rightarrow T_1 = (1200kg)(9.81m/s^2) - (1200kg)(1.5m/s^2)$$

$$= 13567N$$

For the weight:

$$T_2 - mg = 1 \rightarrow T_2 = 8300N$$

$$x = x_0 + v_0t + \frac{1}{2}at^2$$

$$\rightarrow \Delta x = \frac{1}{2}(1.5m/s^2)(1s)^2 = 0.75m$$

$$W \int_C \vec{F} \cdot d\vec{r} = \int_0^{0.75} 13567dx - \int_0^{0.75} 8300dx$$
$$= 3900J$$

Now we just add the previous result to the zero acceleration work.

$$W_{tot} = \boxed{22408.5J}$$

This was slightly more involved but it was only one dimensional.

22)

$$k = 3.5 \cdot 10^4 N/m$$
$$dW = kxdx$$
$$W = k \int_0^{0.1} xdx = k \left[\frac{1}{2}x^2\right]_0^{0.1}$$
$$= \boxed{175J}$$
$$W = k \int_{0.1}^{0.2} xdx = \boxed{525J}$$

The problem here was a variable force so an integral was needed. But it was fairly simple given it was a general polynomial.

23)

$$W = \int_0^x F_x dx = 2J + 4J = \boxed{6J}$$

This was good practice with integrating based on a visual graph.

3.2 Potential

3.2.1 Potential Energy

In the last section, we covered work which we said was the following:

$$W = \int_C \vec{F} \cdot d\vec{r}$$

We could interpret this integral differently in order to make some of our calculations easier. We said, in one case, that:

$$W_{net} \int_A^B \vec{F}_{net} \cdot d\vec{r} = \Delta T$$

$$T = \frac{1}{2}mv^2$$

We defined the kinetic energy T as well here.

We will start some more calculations in the 1D world. Here:

$$W = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F} \cdot d\vec{r}$$

For 1D we only require a single unit vector. The position vector is thus $\vec{r} = x\hat{e}_x$. Over the course of motion, we will divide the trajectory into $d\vec{r} = dx\hat{e}_x$. Furthermore, we may have some force here which we can decompose into $\vec{F} = F_x\hat{e}_x$. Thus our work is:

$$W = \int_{x_i}^{x_f} F_x dx(\hat{e}_x \cdot \hat{e}_x)$$

$$= \int_{x_i}^{x_f} F_x dx$$

To make this integral easier, we will assume that force can only be a function of the position. Then we have:

$$W = \int_{x_i}^{x_f} F_x(x)dx$$

We can make the following statement by adding some V such that it is the anti derivative:

$$= -\int_{x_i}^{x_f} dV(x) = -\Delta V$$

Thus we get some relations with the kinetic energy theorem.

$$\Delta T + \Delta V = 0$$

$$\Delta E = 0$$

We label T the kinetic energy, V the potential energy, and E the total mechanical energy. These all have units of energy. Mechanical energy can also be a more generalized quantity called the Hamiltonian which would be covered in 650.

Definition

If

$$F_x = -\frac{dV}{dx} \tag{3.11}$$

then energy is conserved. Under the influence of a conservative force, the total mechanical energy is conserved.

We have established that the work is relative to the frame of reference, the kinetic energy is also relative to the frame reference, and the potential energy is absolute because it depends on position with respect to some point which is true in non relativistic physics. Because T is relative and V is not, then the total mechanical energy is relative.

In 3D, we can use the same work equation as before, but we will now need to use vector calculus.

$$W = \int_C \vec{F} \cdot d\vec{r} = \int -\nabla V \cdot d\vec{r}$$

$$\nabla V = \begin{pmatrix} \partial_x V \\ \partial_y V \\ \partial_z V \end{pmatrix}$$

From here we will still get the same results in terms of the conservation laws.

Now we shall assume that we have a potential V_1 which yields $F_1 = -\frac{dV_1}{dx}$ and V_2 such that $V_2 = V - 1 + c \longrightarrow F_2 = F_1$. This makes it important to examine the constant because it does not matter.

We should also note that the work done by a conservative force is path independent. If we start at A and end at B, it does not matter which way we go because the values at the beginning and the end are all that matter. Additionally, if we go in a circle such that A=B, then the work is zero because:

$$\oint_C \vec{F} \cdot d\vec{r} = 0$$

3.2.2 Constant Forces

The first force we will look at is the weight. The force is the following:

$$\vec{F} = -mg\hat{e}_y$$

$$F_y = -\frac{dV}{dy} = -mg \Rightarrow V_g(x) = mgy$$

Thus is an important result because this is how we will be working on the surface of the earth. At the surface, the potential energy is mgy so the work is

$$W_g = -mg\Delta y$$

3.2.3 Homework: Energy

Note: I use the convention where I call the kinetic energy T and the potential energy V.

48) We know the formalism that $E = T + V$ so we will look at the total energy at each point in the trajectory. At the bottom we have no potential energy so $E = T_i$ while at the top there is no kinetic energy so $E = V_f$. Thus at the middle of the trajectory we will have a point where $E = \frac{1}{2}T_i + \frac{1}{2}V_f$. From this it can be implied that this height will be one half of the max height.

50) We are given the formula for the force such that $F_x = -ax + bx^3$. We will thus evaluate this between the endpoints using the work formula.

$$W = \int_C \vec{F} \cdot d\vec{r}$$

$$= \int_{x_1}^{x_2} -ax + bx^3 dx$$

$$= \left[-\frac{a}{2}x^2 + \frac{b}{4}x^4 \right]_{x_1}^{x_2}$$

$$= \boxed{-\frac{a}{2}x_2^2 + \frac{b}{4}x_2^4 + \frac{a}{2}x_1^2 - \frac{b}{4}x_2^2}$$

The kinetic energy is the same result by using the net change theorem from calculus.

54) Here we need only use the potential energy and kinetic energy theorem.

$$T_i + V_i = T_f + V_f$$

$$T_i = V_f$$

$$T_i = (3200)(9.81)(6) = \boxed{1.9 \times 10^5 J}$$

These are interesting problems but I would like to see what happens when these integrals get a little bit messier.

3.3 Conservation

3.3.1 Elastic Energy

Another force that we have looked at earlier but not examined is the force done by a spring. This is called the elastic force which is a consequence of Hooke's Law. We shall now derive this. From a small displacement dx, we have:

$$dW = -kxdx$$

Now we can integrate to find the total work:

$$W = \int_0^{\Delta x} -kxdx$$

$$= -k\left[\frac{x}{2}\right]_0^{\Delta x}$$

$$= -\frac{1}{2}k\Delta x^2$$

Thus we have found an important relation:

Definition

Work done by a spring

$$W = -\frac{1}{2}kx\Delta x \tag{3.12}$$

Now we can also look at the potential energy. The general rule is that $\boldsymbol{F} = -\nabla V$ but we only need look at the x component, so:

$$F_x = -kx = -\frac{dV_e}{dx}$$

$$\Rightarrow \frac{dV_e}{dx} = kx$$

$$\Rightarrow V_e = \frac{1}{2}k\Delta x^2$$

We have found an important relation.

$$W = -\Delta V$$

This is an example of a conservative force since the force only depends on the position.

3.3.2 Gravitational Forces

We would now like to take the time to look at gravity again now that we have some more physical tools. With our new tools, we can use Newton's Law of Gravity to find the gravitational potential. We can now do this derivation:

$$\vec{F} = -\frac{GMm}{r^2}\hat{e}_r = F_r\hat{e}_r$$

We can identify this as the derivative of some potential.

$$-\frac{GMm}{r^2} = -\frac{dV_g}{dr}$$

We can solve this differential equation by separating variables. We use the following antiderivative:

$$\int \frac{1}{x^2}dx = -\frac{1}{x} + C$$

Thus:

Definition

Gravitational potential:

$$V_g = -\frac{GMm}{r} \tag{3.13}$$

This should make us a little bit angry because this is not similar to the result that we got for potential earlier. In order to reconcile this, we will set up a situation. Here, we will pull in some mass from $R + h$ to R. Thus we can find the work.

$$\begin{aligned} W &= -\Delta V \\ &= -\left(-\frac{GMm}{R+h} + \frac{GMm}{R}\right) \\ &= \frac{GMm}{R}\left(\frac{1}{1+\frac{h}{R}} - 1\right) \end{aligned}$$

We will Taylor expand the fraction by using a first order Taylor polynomial which says that. to first order, $\frac{1}{1+\epsilon} = 1 + \epsilon$.

$$\begin{aligned} &\approx \frac{GMm}{R}\left(-\frac{h}{R}\right) \\ &-\frac{GMm}{R^2}h \\ &= -mgh \\ &\Rightarrow U \approx mgh \end{aligned}$$

3.3.3 Non-Conservative Forces

Many forces that we have looked at do not have the same conservation properties as the ones we have looked at so far. One such example is friction. To show this example, say we have a block. One path we could take is to slide the block directly across a surface from A to B. Another path is that we move the block from A to C and then from C to B. We will compute the work for the first path:

$$W_{AB} = \int_A^B -\mu mg dx = -\mu mg \int_A^B dx = -\mu mg(A - B)$$

Now we will compute the work for the second.

$$W_{AB} = W_{AC} + W_{CB} = -\mu mg(C - A) + \mu mg(B - C)$$

Thus we get the following relation which makes no sense for a conservative force.

$$\Rightarrow W_{AB} - 2\mu mg(C - B) \neq W_{AB}$$

This means that, for friction, work is path independent. This means that the force is not conservative. This means that the path depends on velocity and not only the position.

For energy conservation, forces are wither conservative or non conservative so we can decompose work accordingly.

Definition

$$W_{net} = W_{non} + W_{con} \tag{3.14}$$

We can arrive at the following:

$$W_{non} = \Delta E$$

Now to end, we will say that at the microscopic level, all forces are conservative. However at the macroscopic level, we have some path dependent forces that need to be dealt with accordingly.

3.3.4 Homework: Potential and Conservation

2) We know the x force $F_x = 2x^3 + 1$ so we will compute the work on both paths.

$$W_1 = \int_{C_1} F_x dx = \int_{x_1}^{x_2} 2x^3 - 1dx$$

$$= \frac{1}{2}x_2^4 - x_2 - \frac{1}{2}x_1^4 + x_1$$

$$W_2 = \int_{C_2} F_x dx = \int_{x_2}^{x_1} 2x^3 - 1dx$$

$$= \frac{1}{2}x_1^4 - x_1 - \frac{1}{2}x_2^4 + x_2$$

$$W_{net} = \sum W$$

$$= \frac{1}{2}x_2^4 - x_2 - \frac{1}{2}x_1^4 + x_1 + \frac{1}{2}x_1^4 - x_1 - \frac{1}{2}x_2^4 + x_2$$

$$\boxed{= 0J}$$

This makes the generalized statement for conservation that any work done in a loop will be zero such that:

$$\oint_C \vec{F} \cdot d\vec{r} = 0$$

9) We know the potential $V = 2x^2 + x^4$ and we can find the force through this relation:

$$F = -\nabla V \Rightarrow F_x = -\partial_x V = -\frac{dV}{dx}$$

$$\Rightarrow F_x = -\frac{d}{dx}(2x^2 + x^4) = \boxed{-4x - 4x^3 N}$$

20)

$$F_{net} = ma$$

$$f_k = ma_x = -\mu_k mg$$

$$\Rightarrow a_x = \mu_k g$$

$$\Rightarrow v = \mu_k g(t - t_0)$$

$$\Rightarrow \Delta t = \frac{v}{\mu_k g}$$

$$v^2 = 2a\Delta x = 2\mu_k g \Delta x \Rightarrow \boxed{\Delta x = \frac{v^2}{2\mu_k g}}$$

This problem set was very helpful in establishing path independence of conservative forces.

3.4 Analysis of Potential energy

3.4.1 1D Movement

If we have one dimension of movement, then we will have one dimension, and the energy here will be conserved. Thus:

$$E = E_0 + T + V(x)$$

$$\frac{1}{2}mv^2 + V(x) = E_0 \rightarrow v = \sqrt{\frac{2}{m}(E_0 - V(x))}$$

Now we will assume some potential curve where we asymptote at zero and then achieve a negative minimum value, and then approach the x axis as we go to infinity. We will take one energy state E_1 which is a horizontal line across the curve which we will call an energy level. We will now look at $v^2(x)$ which needs to be positive or zero which means that $E_0 - V(x) \geq 0$. In the case of this particle, at E_1, the potential is larger than the energy thus the particle can never go to the left of where the particle is E_1. To the right, there is a point where the potential is higher than E_1 which creates another forbidden region. Thus: $x \in [x_1, x_2]$.

If the allowed region of space for a particle is bounded, we call this a bounded state. Now we shall consider $E_2 > 0$ which creates its own forbidden regions. But with E_2, to the right, it is the greatest point so it has no bound on the right side which makes E_2 and unbounded state.

At x_1 and x_2, the velocity is zero because these are called turning points. If we are at any other region, then the distance between the energy level to the potential curve will give us the kinetic energy such that:

$$v = \sqrt{\frac{2}{m}(E_1 - V(x))}$$

We also know that $F_x = -\frac{dV}{dx}$, so if we take the slope of a potential energy curve, we will get the negative force. So if the slope of the curve is positive, the force will be negative, and the same goes for the opposite. At the minimum of the potential curve, the slope will be zero which means that the force is also zero.

Definition

If

$$\frac{dV}{dx} = 0 \tag{3.15}$$

and it is a minimum, then

$$F = 0 \tag{3.16}$$

This is called an equilibrium point where the kinetic energy is at its maximum.

Note that particles will oscillate between the turning points. This is why this situation is often called the simple harmonic oscillator and is good for approximating potentials at specific places where it may look positive quadratic.

We can also have maxima on the potential curve, but it will not be stable such that any perturbation will remove the particle from the equilibrium solution.

The other possibility is an inflection point which is also unstable.

Definition

The only stable points are minima of the potential curve.

3.4.2 Springs

When we look at the potential of a spring, which has a parabolic energy, we will learn that all minima are essentially parabolas by Taylor expansions. Because of this, we will use the potential of the spring, the harmonic oscillator, to approximate many different stable points.

Example

Given $V(x) = 2x^4 - x^2$ find the equilibrium points and the turning points where $E = 1, 0.5J$.

To find the equilibrium we need to find the derivative $\dot{V} = 8x^3 - 2x$. We can set to zero to find the equilibrium.

3.4.3 Power

Sometimes it is important to know how much work is done at some period of time.

Definition

$$P = \frac{dW}{dt}[=]W \tag{3.17}$$

$$P = \frac{dW}{dt} = \frac{\vec{F} \cdot d\vec{v}}{dt} = \vec{F} \cdot \vec{V}$$

$$\int_0^W dW = \int_0^t P dt$$

Definition

$$W = \int_0^t P(t)dt = \int_C \vec{F} \cdot d\vec{r} \tag{3.18}$$

3.4.4 Homework: Potential and Power

25)

$$V(x) = mgx + \frac{1}{2}kx^2$$
$$F_x = -\partial_x V$$
$$= -mg - kx$$
$$F_x = 0 \Rightarrow -mg - kx = 0$$
$$\Rightarrow x = -\frac{mg}{k}$$
$$= \frac{(70)(-9.81)}{150} = -4.57m$$
$$T = E - V = 6180 - mg(-4.57) - \frac{1}{2}k(-4.57)^2 = \frac{1}{2}mv^2$$
$$\Rightarrow v = \boxed{6.5m/s}$$
$$F_x = -mg - kx = ma$$
$$\Rightarrow a = \frac{-mg - kx}{m} = 21.7m/s^2$$

28) According to the graph, the turning points are $0.1m$ for E_1, $0.25m$ and $3m$ for E_2, and $0.5m$ and 1.25m for E_3. The speed is maximized at $0.8m$ for all levels and minimized at the turning points. E_1 is an unbounded energy state while the others are bounded states.

65)

$$60W\left(\frac{1kW}{1000W}\right)\left(\frac{365d}{1y}\right)\left(\frac{24hr}{1d}\right)\left(\frac{1.5}{1kw \cdot hr}\right) = \boxed{79\$}$$

104)

$$E = T + V = \frac{1}{2}kx^2$$
$$\Delta T = W_f$$
$$0 - T_i = -\mu m f \Delta x \Rightarrow T_i = \boxed{47J}$$
$$47 = \frac{1}{2}kx^2 = \frac{1}{2}(120)x^2$$
$$\Rightarrow x = \boxed{0.89m}$$

This was a nice problem set that helped with the analysis of the potential curves and the energies of systems.

3.5 Orbits

3.5.1 Kepler's Laws

These problems refer to the two body problem which is really a disguised one body problem. In the one body problem, we have one mass moving which is under the influence of an external potential which does not change because of m. The two body problem assumes two particles with a big mass M and a small mass m. It is proven that we can reduce this to a one body problem (this proof is a 650 topic).

We will make the approximation such that if $M >> m$ then M is stationary making it a good place for us to place the origin. Additionally, the force will now be:

$$\vec{F}_g = -\frac{GMm}{r^2}\hat{e}_r$$

Another approximation is that M is a spherically symmetric point mass. We do this to avoid triple integrals.

We will need to look at the conic sections to get a better idea of the 2 body problem. The sections are the circle, ellipse, parabola, and hyperbola.

Definition

Circle:

$$x^2 + y^2 = r^2 \tag{3.19}$$

Ellipse:

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \tag{3.20}$$

Parabola:

$$y = ax^2 \tag{3.21}$$

Hyperbola:

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1 \tag{3.22}$$

All of these sections are curves in the plane. They are also second order which means that they have leading power of two. They are also called conic sections because they are all cut from the a cone. We can also write this in toe form of polar coordinates:

$$r = \frac{l}{1 + e\cos\theta}$$

Zeroth Law: The orbits of planets are inside of planes. From this we can determine that we are dealing with one dimensional motion but the dimension is curved in the shape of the orbit. If the velocity is not parallel to the acceleration, then they will define a plane. so $d\vec{v}$ is in the plane which means that velocity will never leave the plane.

First Law: Planets go around the sun following ellipses with the sun placed at one focus of the ellipse. More generally, trajectories are conic sections with M in one of the foci. To prove this, write the acceleration of m in polar coordinates. Then use Newton's Second Law and the Gravitational Law to get a second order differential equation for r as a function of θ. Finally, plug as a solution $r = \frac{1}{1+e\cos\theta}$.

Second Law: The line connecting the sun to the planet sweeps equal areas in equal times. We will solve this with the cross product: see the appendix for that since cross product will not be used until PHY552:

$$d\vec{A} = \frac{1}{2}(\vec{r} \times \dot{\vec{r}})$$

Third Law: The period of the orbit is proportional to the cube of the orbit. Derivation is in test 2 question 9.

3.5.2 Kepler 1 Proof

This is not rigorous per se but it will generate the necessary framework. Take a mass m in an elliptical orbit around M. The total energy is the following:

$$E = T + V = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

The velocity of m will have two components, a radial component $\dot{r}$ and an angular component $\dot{\theta}$. Thus we can now rewrite the energy keeping in mind the polar metric tensor:

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{GMm}{r}$$

We can also write the angular momentum, a topic from 552:

$$L = mr^2\dot{\theta}$$

We will now make a substitution such that $u = \frac{1}{r}$ which means that $\dot{\theta} = L\frac{u^2}{m}$. Now we can evaluate this as an integral:

$$\begin{aligned} \theta &= \int \frac{L}{m}u^2 dt \\ &= \int \frac{L}{m}u^2 \frac{dt}{du} du \end{aligned}$$

Now this integral is solvable which yields a conic section

3.5.3 Energy of Orbits

We know that gravity is a conservative force making it work path independent and it gives us a potential energy:

Definition

$$V = -\frac{GMm}{r} \tag{3.23}$$

If the motion is radial, then we have a one dimensional problem. Then we can analyze the movement by looking at the potential energy. Now how energy look? Well it is a variation on the basis rational function so it will fall off positively towards zero as r goes to infinity meaning that there are no minima or equilibrium points. The slope of the potential curve for positive r will always be positive so the force will be negative which pulls our mass close to zero for all $r > 0$.

Now let's say that I kick the particle away from the mass at the middle of the potential curve with negative energy. It will continue moving to the right until it reaches the turning point where the particle will again crash into the zero. If I give an energy of zero or more, then the particle is unbounded and it will thus escape. From here we arrive at the following relation for escape velocity:

Definition

$$v \geq \sqrt{2}\sqrt{\frac{GM}{r}} = v_e \tag{3.24}$$

In general, the movement of objects under the influence of gravity is not only radial, but there is also an angular component. We also have 2D because our conic sections are in the plane. Luckily, there is a workaround, which is to decouple the radial r and the angular θ part of the movement such that:

$$\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$

$$E = T + V = \frac{1}{2}mv^2 - \frac{GMM}{r}$$

$$= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m\frac{(r^2\dot{\theta})}{r^2} - \frac{GMm}{r}$$

It turns out that $r^2\dot{\theta} = L$ is constant where L is a concept called the angular momentum. We will keep going:

$$= \frac{1}{2}mv_r^2 + \frac{1}{2}m\frac{L^2}{r^2} - \frac{GMm}{r}$$

Thus we get an expression for the energy that is the same as a one dimensional effective potential.

We will now try to find the energy of a circular orbit:

$$E_c = T_c + V_c = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

$$a_c = \frac{GM}{r^2} \Rightarrow v_x = \sqrt{\frac{GM}{r}}$$

$$\Rightarrow E_c = -\frac{1}{2}\frac{GMm}{r} = -T_c$$

The energy for an ellipse, we will find that:

Definition

$$E_e = -\frac{1}{2}\frac{GMm}{a} \tag{3.25}$$

We are now ready to dive into practice which means that we are finished with theory for the term which also means it is time for me to edit these notes!

3.6 Review

Example

We will look at the conveyor problem again where we have a block on the belt with a mass m and a friction μ_k. We have an observer on the belt and an observer off of the belt so explain the work with respect to each of them.

From the stationary, the forces we have are normal and gravitational and some acceleration to the right as well as friction to the left.

$$W_{box} = W_{F_n} + W_W = W_f$$

$$= \int_{x_i}^{x_f} F_f dx = \frac{1}{2}mv^2$$

Now for the work done as seen from on the belt. Here we have some acceleration to the right and friction to the left. Thus we will apply the same argument such that we will have:

$$W_{box} = -\frac{1}{2}mv^2$$

Example

Imagine we have a satellite on a geosynchronous orbit which is an orbit which does not change with respect to the observer. The period of this orbit is one day. The angular motion is also uniform so we have uniform angular motion and the orbit is also a circle on the equator. The radius is roughly six times the radius of the earth. We will also have uniform speed:

$$v_c = \sqrt{\frac{HM}{R_e}}$$

Now a rocket misfires and we now have lower velocity. Now we need to determine how the orbit will change due to this lower amount of energy which leaves us in a bound orbit. Our orbit is also now an ellipse. We are at the appocenter and now we need to look at the energy of the orbit.

3.6.1 Homework: Orbits

38) We are given the following values:

$$d_p = 8.78 \times 10 -^{10}\ m$$

$$v_p = 5.45 \times 10^4 m/s$$
$$d_a = 5.28 \times 10^{12} m$$

We will use this relation as well:

$$v_p d_p = v_a d_a$$
$$\Rightarrow v_a = \boxed{906.0 m/s}$$

44) Here we are given the following and we need to find the ratio:

$$d_a = 1.52 \times 10^8 km$$
$$d_p = 1.47 \times 10^8 km$$
$$\frac{d_a}{d_p} = \boxed{1.034}$$

47) We want to look at two definitions for velocities and reconcile the factor:

$$v_e = \sqrt{\frac{2GM}{r}}$$
$$v_c = \sqrt{\frac{GM}{r}}$$

Both of these equations differ by a factor of $\boxed{\sqrt{2}}$.

60) For this problem we know three speeds so we want to see if any one of the speeds cater to a circular orbit:

$$r = 2.00 \times 10^7 m$$
$$v_c = \sqrt{\frac{GM}{r}} = \boxed{4.47 km/s}$$

From here we see that satellite II at $4.47 km/s$ will be circular while the other two at $3.47 km/s$ and $5.47 km/s$ will travel on elliptical orbits.

69)

$$E = T + V$$
$$= \frac{1}{2}mv^2 - \frac{GMm}{r}$$

Now we will solve for one kinetic term:

$$\frac{1}{2}mv_1^2 - \frac{GMm}{r_1} = \frac{1}{2}Mv^2 - \frac{GMm}{r_2}$$
$$\Rightarrow \frac{1}{2}mv_1^2 - \frac{1}{2}Mv_2^2 = GMm\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

$$\Rightarrow \frac{1}{2}mv_1^2\left(1-\left(\frac{r_1}{r_2}\right)^2\right) = GMm\left(\frac{1}{r_1}-\frac{1}{r_2}\right)$$

$$\Rightarrow \frac{1}{2}mv_1^2\left(\frac{r_2^2-r_1^2}{r_2^2}\right) = GMm\left(\frac{r_2-r_1}{r_1r_2}\right)$$

$$\Rightarrow \frac{1}{2}mv_1^2 = GMm\left(\frac{r_1}{r_2}\right)\frac{1}{r_1+r_2}$$

Now we can fit this into the energy:

$$E = GMm\left(\left(\frac{r_1}{r_2}\right)\left(\frac{1}{r_1+r_2}\right)-\frac{1}{r_1}\right)$$

$$= \boxed{-\frac{GMm}{r_1+r_2}}$$

71)

$$E = \frac{1}{2}mv_1^2 - \frac{GMm}{r}$$

$$= m(8.2\times 10^5)$$

$$8.2\times 10^5 = \frac{GM}{r_1+r_2} \Rightarrow r_1+r_2 = 6.0\times 10^6$$

$$\Rightarrow v = \frac{r_1v_1}{r_2} = \boxed{8.17m/s}$$

Chapter 4

Appendix

4.1 Significant Figure Rules

Rule 1: All digits that are nonzero are significant.
Rule 2: Any zeros between nonzero digits are significant.
Rule 3: Zeros on the left of the first nonzero digits are not significant. They instead indicate where the decimal point is.
Rule 4: Any zeros to the right of the decimal point are significant (this includes the zero before the decimal point as well).
Rule 5: Any numbers to the right of the last nonzero digit is not always significant. This rule can be avoided by using scientific notation.

For adding or subtracting, round to the amount of decimal places as the number with the fewest decimal places in the sum or difference. The same goes for multiplying or dividing.

4.2 Mathematics

4.2.1 Trigonometric Identities

In 550 trigonometry will be used to model some of the systems that we deal with. It will be additionally important to use the identities that will make it easier to evaluate some of the trigonometric expressions.

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \sin\phi\cos\theta$$

$$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$$

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

4.2.2 Common Derivatives

These are all of the common derivatives needed for the course. Note that certain trigonometric functions are not present as well as the inverse trigonometric functions. Differentiating those functions will be a part of MTH580 while integrating them will be left to MTH590A.

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx}$$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(f(x)g(x)) = \frac{df}{dx}g + f\frac{dg}{dx}$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{df}{dx}g - f\frac{dg}{dx}}{(g(x))^2}$$

$$\frac{d}{dx}(e^{ax}) = ae^{ax}$$

$$\frac{d}{dx}(\ln ax) = \frac{1}{x}$$

$$\frac{d}{dx}(\sin(ax)) = a\cos(ax)$$

$$\frac{d}{dx}(\cos(ax)) = -a\sin(ax)$$

4.2.3 Common Integrals

As should be noted, we add integration constants to all integrals. This appendix includes general integration techniques as well as elementary formulas. Note that partial fractions and trigonometric substitutions may be needed to evaluate certain integrals. More advanced techniques for line integrals and multivariable integrals may be needed for work and electricity and magnetism in the later half of 551 and the electromagnetism portion of the sequence.

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

$$\int x^n dx = \frac{1}{n+1}x^{n+1}$$

$$\int u dv = uv - \int v du$$

$$\int f(g(x))g'(x)dx = \int f(u)du$$

$$\int e^{ax}dx = \frac{1}{a}e^{ax}$$

$$\int \frac{dx}{x+a} = \ln|x+a|$$

$$\int \sin(ax)dx = -\frac{1}{a}\cos(ax)$$

$$\int \cos(ax)dx = \frac{1}{a}\sin(ax)$$

4.2.4 The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus allows us to evaluate definite integrals by finding the antiderivative of a function. It also makes it clear that the derivative is the opposite as the indefinite integral.

$$\int_a^b f(x)dx = F(b) - F(a)$$

4.2.5 Vector Products

This course will make use of vectors and their products. Shown below are the dot and cross products in trigonometric and normal notation. The normal notation makes use of the Einstein Summation Convention such that if an index is repeated twice, it is implied that it is to be summed over. Additionally, in the normal notation, the Kronecker Delta and the Levi Civita Symbol are used to restrict the values of the indices.

$$\vec{A}\cdot\vec{B} = AB\cos\theta$$

$$\vec{A}\cdot\vec{B} = a^i b^j \delta_{ij}$$

$$|\vec{A}\times\vec{B}| = AB\sin\theta$$

$$\vec{A}\times\vec{B} = \epsilon_{ijk}\hat{e}_i a^j b^k$$

4.2.6 Partial Differentiation

Partial differentiation will likely not be needed in 551 but it is still good to know. If we have a function of more than one variable then we can take derivatives of more than one variable. If we have $f(x, y)$ then we can differentiate with respect to x or y. The differentiation rules here are the same as in single variable calculus but we treat the variables that are not being differentiated as constants. The notation we use is the following:

$$\frac{\partial}{\partial x}(f(x, y))$$

Example:

$$f(x, y) = x^3y^2$$
$$\frac{\partial f}{\partial x} = f_x = 3x^2y^2$$
$$\frac{\partial f}{\partial y} = f_y = 2x^3y$$

Additionally, we define the gradient which is the sum of all partial derivatives of first order:

$$\nabla f(x_i) = \frac{\partial f}{\partial x_i}$$