20.1: Translatiin Operator

$$\hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x}$$

$$e^{\frac{i\hat{p}a}{\hbar}}$$

$$e^{\frac{i\hat{p}a}{\hbar}}\Psi(x) = e^{a\frac{\partial}{\partial x}}\Psi(x)$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\left(a\frac{d}{dx}\right)^n\Psi(x)$$

$$= \Psi(x) + a\frac{d\Psi}{dx} + \frac{1}{2}a^2\frac{d^2\Psi}{dx^2} + \dots$$

$$= \Psi(x+a)$$

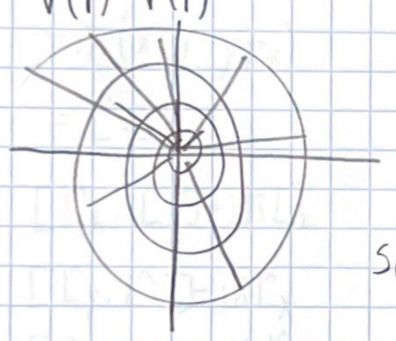
Quantum Mechanics in 3D central potentials and Angular momentum

$$\vec{p} = \frac{\hbar}{i}\vec{\nabla}$$

$$p_x = \frac{\hbar}{x}\frac{\partial}{\partial x},\ p_y = \frac{\hbar}{y}\frac{\partial}{\partial y},\ p_z = \frac{\hbar}{z}\frac{\partial}{\partial z}$$

$$-\frac{\hbar^2}{2m}\nabla^2\Psi(\vec{r}) + V(\vec{r})\Psi(\vec{r}) = E\Psi(\vec{r})$$

$$V(\vec{r}) = V(r)$$



spherically symmetric

20.1: Translation Operator (continued)

$$\nabla^2 \psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)\psi$$

$$-\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right) \equiv (\vec{L})^2$$

Time independent is relevant where there exists a two body problem with the potential function $V(\vec{x}_1, \vec{x}_2)$ is a function of $V(|\vec{x}_1 - \vec{x}_2|)$

20.2: Angular Momentum Operators

$\vec{L} = \vec{r} \times \vec{p}$ classically

$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y$

$\hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z$

$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$

Hermitian: $(\hat{L}_x)^\dagger = (\hat{p}_z)^\dagger(\hat{y})^\dagger - (\hat{p}_y)^\dagger(\hat{z})^\dagger = \hat{p}_z\hat{y} - \hat{p}_y\hat{z} = \hat{L}_x$

$\downarrow$

correct

$\hat{L}_i$'s are observable

$[\hat{L}_x, \hat{L}_y] = [yp_z - zp_y, zp_x - xp_z]$

$[yp_z, zp_x] + [zp_y, xp_z]$

$= [yp_z, z]p_x + x[zp_y, p_z]$

$= y[p_z, z]p_x + x[z, p_z]p_y$

$= i\hbar(xp_y - yp_x)$

$= \hat{L}_z$

$[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z$

$[\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x$

$[\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y$

$[\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z$

$\vdots \qquad \vdots$

20.3: Commuting Observables

Can have simultanious eigenstates of $\hat{L}_x, \hat{L}_y, \hat{L}_z$, NO!

$$\hat{L}_x \phi_0 = \lambda_x \phi_0$$

$$\hat{L}_y \phi_0 = \lambda_y \phi_0$$

$$\hat{L}_z \phi_0 = \lambda_z \phi_0$$

$$[\hat{L}_x, \hat{L}_y]\phi_0 \to i\hbar \hat{L}_z \phi_0$$

$$\to i\hbar \lambda_z \phi_0$$

$$L_x L_y \phi_0 - L_y L_x \phi_0 = \lambda_x \lambda_y \phi_0 - \lambda_y \lambda_x \phi_0$$

$$0 = \lambda_z \phi_0 \Rightarrow \lambda_z = 0$$

$$\boxed{\lambda_x \lambda_y \lambda_z = 0} \leftarrow \text{No!}$$

$$\vec{L}^2 = \hat{L}_x \hat{L}_x + \hat{L}_y \hat{L}_y + \hat{L}_z \hat{L}_z$$

$$[\hat{L}_x, \vec{L}^2] = [\hat{L}_x, \hat{L}_x \hat{L}_x + \hat{L}_y \hat{L}_y + \hat{L}_z \hat{L}_z]$$

$$= [\hat{L}_x, \hat{L}_y]\hat{L}_y + [\hat{L}_y, \hat{L}_x]\hat{L}_y + [\hat{L}_x, \hat{L}_z]\hat{L}_z + [\hat{L}_z, \hat{L}_x]\hat{L}_z$$

$$= i\hbar L_z L_y + i\hbar L_y L_z - i\hbar L_y L_z - i\hbar L_z L_y$$

$$= 0$$

20.3: Commuting Observables (continued)

$$[\hat{L}_x, \vec{L}^2] \Rightarrow \begin{cases} [\hat{L}_x, \vec{L}^2] = 0 \\ [\hat{L}_y, \vec{L}^2] = 0 \\ [\hat{L}_z, \vec{L}^2] = 0 \end{cases}$$

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$$

$$r = \sqrt{x^2+y^2+z^2} \quad \theta = \cos^{-1}\left(\frac{z}{r}\right) \quad \phi = \tan^{-1}\left(\frac{y}{z}\right)$$

$$\frac{\partial}{\partial\phi} = \frac{\partial y}{\partial\phi}\frac{\partial}{\partial y} + \frac{\partial x}{\partial\phi}\frac{\partial}{\partial x} + \frac{\partial z}{\partial\phi}\frac{\partial}{\partial z}$$

$\because z$ does not depend

$$= x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

$$L_z = \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

$$L_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}$$

$$\vec{L}^2 = \hat{L}_x\hat{L}_x + \hat{L}_y\hat{L}_y + \hat{L}_z\hat{L}_z = -\hbar\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

20.4: Simultaneous Eigenstates

$(\vec{L}^2), L_z$

$\psi_{lm}(\theta,\phi)$

$\hat{L}_z \psi_{lm} = \hbar m \psi_{lm} \quad m \in \mathbb{R}$

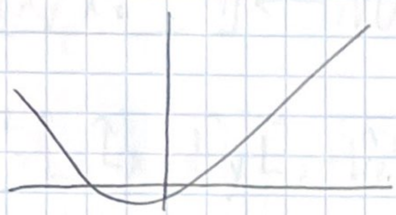
$\vec{L}^2 \psi_{lm} = \hbar^2 \lambda \psi_{lm}$

$(\psi_{lm}, \vec{L}^2 \psi_{lm}) = \hbar^2 \lambda \underbrace{(\psi_{lm}, \psi_{lm})}_{1}$

$(L_x \psi_{lm}, L_x \psi_{lm}) + \dots y, z$

$\vec{L}^2 \psi_{lm} = \hbar^2 \lambda \psi_{lm} = \hbar^2 l(l+1) \psi_{lm}$

$l \in \mathbb{R}, \lambda > 0$



$\frac{\hbar}{i} \frac{\partial}{\partial \phi} \psi_{lm} = \hbar m \psi_{lm}$

$\frac{\partial}{\partial \phi} \psi_{lm} = i m \psi_{lm}$

$\psi_{lm} = e^{im\phi} P_l^m(\theta)$

10.4: Simultaneous Eigenstates (continued)

$$\psi_{\ell m}(\theta, \phi + 2\pi) = \psi_{\ell m}(\theta, \phi)$$

$$m \in \mathbb{Z}$$

$$-\hbar^2 \left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)\psi_{\ell m} = \hbar^2 \ell(\ell+1)\psi_{\ell m}$$

$$\frac{\partial^2}{\partial\phi^2} \rightarrow (im)^2 = -m^2$$

$$\sin\theta \frac{d}{d\theta}\left(\sin\theta \frac{dP_\ell^m}{d\theta}\right) + \left(\ell(\ell+1)\sin^2\theta - m^2\right)P_\ell^m = 0$$

$$x = \cos\theta$$

$$\frac{d}{dx} = -\frac{1}{\sin\theta}\frac{d}{d\theta}$$

$$\sin\theta \frac{d}{d\theta} = -(1-x^2)\frac{d}{dx}$$

$$\frac{d}{dx}\left((1-x^2)\frac{dP_\ell^m}{dx}\right) + \left(\ell(\ell+1) - \frac{m^2}{1-x^2}\right)P_\ell^m(x) = 0$$

$$\psi_{\ell m} = N_{\ell m}\, e^{im\phi} P_\ell^m(\cos\theta)$$

20.4: Simultaneous Eigenstates (continued)

$$\frac{d}{dx}\left((1-x^2)\frac{dP_\ell}{dx}\right)+\ell(\ell+1)P_\ell=0$$

$$P_\ell(x)=\sum_k a_k x^k$$

Coefficient x^k

$$(k+1)(k+2)a_{k+2}+(\ell(\ell+1)-k(k+1))a_k=0$$

$$\frac{a_{k+2}}{a_k}=\frac{-(\ell(\ell+1)-k(k+1))}{(k+1)(k+2)}$$

$$P_\ell(x)=\#x^\ell+\ldots+\#$$

$\ell=0,1,2,3,\ldots$

$P_\ell(x)$ = Legendre Polynomials

21.1: Associated Legendre Functions

$\hat{L}_z \psi_{lm} = \hbar m \psi_{lm}$

$\hat{L}^2 \psi_{lm} = \hbar^2 l(l+1) \psi_{lm}$

$\nabla^2 \psi = \frac{1}{r} \frac{d}{dr^2}(r\psi) + \frac{1}{r^2}\left(-\frac{\hat{L}^2}{\hbar^2}\right)\psi$

$$\frac{d}{dx}\left((1-x^2)\frac{dP_l^m}{dx}\right) + \left(l(l+1) - \frac{2m^2}{1-x^2}\right)P_l^m(x) = 0$$

$$P_l(x) = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l$$

$$L^2 = L_x^2 + L_y^2 + L_z^2$$

$$P_l^m(x) \equiv (1-x^2)^{\frac{|m|}{2}}\left(\frac{d}{dx}\right)^{|m|} P_l(x)$$

↳ solves Differential equation

$-l \leq m \leq l$

↳ no more solutions

$l = 0 \rightarrow m = 0$

$l = 1 \rightarrow m = -1, 0, 1$

$$\begin{cases} Y_{lm}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}(-1)^m e^{im\phi} P_l^m(\cos\theta), & 0 \leq m \leq l \\ Y_{lm}(\theta, \phi) = (-1)^m (Y_{l-m}(\theta, \phi))^*, & m < 0 \end{cases}$$

$$Y_{00} = \frac{1}{\sqrt{4\pi}}$$

$$Y_{1\pm1} = \mp\sqrt{\frac{3}{8\pi}}\, e^{\pm i\phi} \sin\theta$$

21.2: Orthonormality of Spherical Harmonics

$$\int d\Omega = \int_{\theta=0}^{\theta=\pi} \underbrace{\sin\theta \, d\theta}_{-d(\cos\theta)} \int_0^{2\pi} d\phi$$

$$= \int_{-1}^{1} d(\cos\theta) \int_0^{2\pi} d\phi$$

$$\int d\Omega \, (Y^*_{\ell' m'}(\theta,\phi))(Y_{\ell m}(\theta,\phi))$$
$$= \delta_{\ell'\ell}\,\delta_{m'm}$$

$$\frac{-\hbar^2}{2m}\nabla^2\Psi + V(r)\Psi = E\Psi$$

$$\frac{-\hbar^2}{2m}\left[\frac{1}{r}\frac{d^2}{dr^2} r\Psi - \frac{1}{\hbar^2 r^2} L^2\Psi\right] + V(r)\Psi = E\Psi$$

$$\Psi(\vec{r}) = R_E(r) Y_{\ell m}(\theta,\phi)$$

21.2: Orthonormality of Spherical Harmonics

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r}\frac{d^2}{dr^2} rR_E - \frac{\ell(\ell+1)}{r^2}R_E\right] + V(r)R_E = ER_E(r)$$

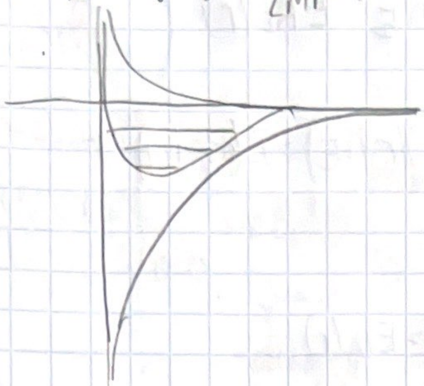
$$-\frac{\hbar^2}{2m}\frac{d^2}{dr}(rR_E) + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2} rR_E + V(r)(rR_E) = E(rR_E(r))$$

$$u(r) \equiv rR_E(r)$$

$$-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}u(r)\left[V(r) + \frac{\hbar^2}{2mr^2}\ell(\ell+1)\right]u(r) = Eu(r)$$

21.3: Effective Potential

$$V_{eff}(r) = V(r) + \frac{\hbar^2}{2mr^2}(\ell+1)$$



$$\psi(\vec{r}) = \frac{u(r)}{r} Y_{\ell,m}(\theta,\phi)$$

$$\int |\psi|^2 d^3x = 1$$

$$= \int_0^\infty r^2 dr \int Y_{\ell m}^*(\theta,\phi) Y_{\ell m}(\theta,\phi) \frac{|u|^2}{r^2} d\Omega$$

$$= \int_0^\infty |u(r)|^2 dr$$

$$= 1$$

as $r \to 0$:

$$DE \Rightarrow \frac{-\hbar^2}{2m}\frac{d^2u}{dr^2} + \frac{\hbar^2 \ell(\ell+1)}{2mr^2} u = 0$$

$$\frac{d^2u}{dr^2} = \frac{\ell(\ell+1)}{r^2} u \quad \text{for } r \to 0$$

$$u = \begin{cases} r^{\ell+1} \Leftarrow \\ r^{-\ell} \times \to \text{too divergent} \end{cases}$$

21.4: Hydrogen Atom Body

proton $\vec{X}_p, \vec{P}_p$

electron $\vec{x}_e, \vec{p}_e$

$[(\vec{x}_p)_i, (\vec{p}_p)_j] = i\hbar\delta_{ij}$

$[(\vec{x}_e)_i, (\vec{p}_e)_j] = i\hbar\delta_{ij}$

$\hat{x}\,\hat{y}\,\hat{z} \quad \hat{x}_i \quad i=1,2,3$

$\hat{p}_x\,\hat{p}_y\,\hat{p}_z \quad \hat{p}_i \quad i=1,2,3$

$[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$

$|\Psi(\vec{x}_e, \vec{x}_p)|^2 d^3\vec{x}_e\, d^3x_p = dP$

$$\frac{(\vec{p}_p)^2}{2m_p} + \frac{(\vec{p}_e)^2}{2m_e} + V(|\vec{x}_e - \vec{x}_p|)$$

$$\left.\begin{aligned} \vec{P} &\equiv \vec{p}_p + \vec{p}_e \\ &= \underbrace{m_p\vec{x}_p + m_e\vec{x}_p}_{m_p + m_e} \end{aligned}\right\} \text{CM quantum variables}$$

$[(\vec{x})_i, (\vec{p})_j] \overset{?}{=} i\hbar\delta_{ij}?$

$$\left[\frac{m_p(x_p)_i + m_e(x_e)_i}{m_p m_e}, (\vec{p}_p)_i + (\vec{p}_e)_i\right] = i\hbar\left(\frac{m_p}{m_p+m_e} + \frac{m_e}{m_p+m_e}\right)$$

21.4: Hydrogen Atom Body (continued)

$\vec{x} \equiv \vec{x}_e - \vec{x}_p$

$\vec{p} \equiv \alpha \vec{p}_e - \beta \vec{p}_p \quad \alpha, \beta ?$

$[(\vec{x})_i, (\vec{p})_j] = i\hbar \delta_{ij}$

$\alpha + \beta = 1$

$[\vec{p}, \vec{x}] \rightarrow \alpha m_e - \beta m_p = 0$

$\alpha = \frac{m_p}{m_e + m_p}$

$\beta = \frac{m_e}{m_e + m_p}$

$\mu \equiv \frac{m_e m_p}{m_e + m_p}$

$M = m_e + m_p$

$\alpha = \frac{\mu}{m_e}, \beta = \frac{\mu}{m_p}$

$\vec{x} = x_e - x_p, \quad \vec{p} \equiv \mu \left(\frac{\vec{p}_e}{m_e} - \frac{\vec{p}_p}{m_p} \right)$

$$\frac{(\vec{p}_p)^2}{2m_p} + \frac{(\vec{p}_e)^2}{2m_e} = \frac{1}{2m_p} \left(\frac{m_p}{M} \vec{P} - \vec{p} \right)^2 + \frac{1}{2m_e} \left(\frac{m_e}{M} \vec{P} + \vec{p} \right)^2$$

$$K_{in_{op}} = \frac{1}{2M} \vec{P}^2 + \frac{1}{2\mu} \vec{p}^2$$

22.1: Center of Mass

$$H = \frac{(\vec{p}_p)^2}{2m_p} + \frac{(\vec{p}_e)^2}{2m_e} + V(|\vec{x}_e - \vec{x}_p|)$$

$(\vec{P}, \vec{X})$ CM

$(\vec{p}, \vec{x})$ R-Motion

$$H = \frac{(\vec{P})^2}{2M} + \frac{(\vec{p})^2}{2\mu} + V(|\vec{x}|)$$

$$\vec{P} = \frac{\hbar}{i} \nabla_X$$

$$\vec{p} = \frac{\hbar}{i} \nabla_x$$

$$\Psi(\vec{X}, \vec{x}) = \Psi_{cm}(\vec{X}) \Psi(\vec{x})$$

$$H\Psi = E\Psi$$

$$\left(\frac{\vec{P}^2}{2M}\right) \Psi_{cm}(\vec{X}) \Psi_{rel}(\vec{x}) + \left(\frac{(\vec{p})^2}{2\mu} \Psi_{rel}(\vec{x}) + V(|\vec{x}|) \Psi_{rel}(\vec{x})\right) \Psi_{cM}(\vec{X})$$

$$= E \Psi_{cm}(\vec{X}) \Psi_{rel}(\vec{x})$$

$$\frac{1}{\Psi_{cm}(\vec{X})} \left(\frac{P^2}{2M} \Psi_{cM}(\vec{X})\right) + \frac{1}{\Psi_{rel}(\vec{x})} \left(\frac{\vec{p}^2}{2\mu} \Psi_{rel} + V(|x|) \Psi_{rel}\right) = E$$

$$\frac{\vec{P}^2}{2M} \Psi_{cm}(X) = E_{cm} \Psi_{cm}(X)$$

22.1 Center of Mass (continued)

$$\frac{\vec{P}^2}{2\mu}\psi_{rel}(\vec{x}) + V(r)\psi_{rel}(\vec{x}) = E_{rel}\psi_{rel}(\vec{x}), \quad r = |\vec{x}|$$

$$E = E_{CM} + E_{rel}$$

Important combination

22.2: Hydrogen Atom

$V(r) = -\frac{e^2}{r}$

$V(r) = \frac{-Ze^2}{r}$　　Z = protons

Bohr Radius: $\frac{\hbar^2}{2ma_0^2} = \frac{e^2}{a_0}$

$a_0 = \frac{\hbar^2}{me^2}$

$= \frac{\hbar^2 c^2}{e^2 mc^2}$

$= \frac{\hbar c}{\frac{e^2}{\hbar c} mc^2}$

$= \frac{197\,\text{meVfm}}{\frac{1}{137} 0.5\,\text{MeV}}$

$= 0.529\,\text{Å}$

$= 53\,\text{pm}$

$\frac{e^2}{a_0} = \left(\frac{e^2}{\hbar c}\right)^2 (mc^2)$

$= \left(\frac{1}{137}\right)^2 511{,}000\,\text{eV} \approx 27.2\,\text{eV}$

$\alpha a_0 = \bar{\lambda}_c = \frac{\hbar}{mc} = 400\,\text{fm}$

$\alpha^2 a_0 \approx 2.8\,\text{fm}$

22.3: Hydrogen Schrödinger

$E<0$

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}+\frac{\hbar\ell(\ell+1)}{2mr^2}-\frac{Ze^2}{r}\right)u=Eu$$

$$\psi(r,\theta,\phi)=\frac{u(r)}{r}Y_{\ell m}(\theta,\phi)$$

$r=\frac{a_0}{2Z}x$ ← unitless quantity

$$\frac{2Z^2e^2}{a_0}\left(-\frac{d^2}{dx^2}+\frac{\ell(\ell+1)}{x^2}-\frac{1}{x}\right)u=Eu$$

$$\left(-\frac{d^2}{dx^2}+\frac{\ell(\ell+1)}{x^2}-\frac{1}{x}\right)u=-\kappa^2u$$

$$\kappa^2=-\frac{E}{\left(\frac{2Z^2e^2}{a_0}\right)}$$

$x\to\infty$

$$\frac{d^2u}{dx^2}=\kappa^2u$$

$\Rightarrow u=e^{\pm\kappa x}$

$$\rho\equiv\kappa x=\frac{2\kappa Z}{a_0}r$$

$$\left(-\frac{d^2}{d\rho^2}+\frac{\ell(\ell+1)}{\rho^2}-\frac{1}{\kappa\rho}\right)u=-u$$

22.3: Hydrogen Schrödinger (continued)

$\rho \to \infty$

$$\frac{d^2 u}{d\rho^2} = u$$

$$u = e^{\pm \rho}$$

$r \to 0$

$$u \sim \rho^{l+1} \quad \rho \to 0$$

$$u(\rho) = \rho^{l+1} w(\rho) e^{-\rho}$$

$$\rho \frac{d^2 w}{d\rho^2} + 2(l+1-\rho)\frac{dw}{d\rho} + \left(\frac{1}{\kappa} - 2(l+1)\right) w = 0$$

22.4: Series solution

$$\sum_{k=0}^{\infty} a_k \rho^k = W$$

$$\frac{a_{k+1}}{a_k} = \frac{2(k+l+1)-\frac{1}{\kappa}}{(k+1)(k+2l+2)}$$

$$= \frac{2}{k} \text{ as } k \to \infty$$

$$= \frac{2}{k+1}$$

$$a_{k+1} = \frac{2}{k+1} a_k$$

$$a_k = \frac{2^k a_0}{k!}$$

$$\sum_k a_k \rho^k = a_0 \sum \frac{2^k}{k!} \rho^k$$

$$= a_0 e^{2\rho}$$

Polynomial of degree N

$a_N \neq 0$

$a_{N+1} = 0$

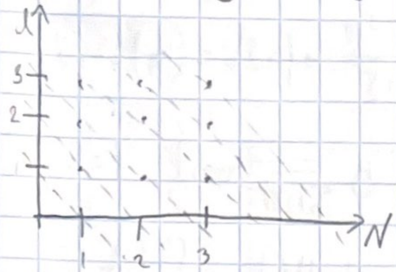
$$\frac{1}{\kappa} = 2(N+l+1)$$

$$\frac{1}{2\kappa} = N+l+1 \equiv n, \begin{cases} n \in \mathbb{Z} \\ n \geq 1 \end{cases}$$

$$E = -\frac{2Z^2e^2}{a_0}\kappa^2$$

$$E_n = \frac{-Z^2e^2}{2a_0}\frac{1}{n^2}$$

22.5: Hydrogen Eigenstates



for each n: $N + l = n-1$

$$0 \le l \le n-1$$
$$0 \le N \le n-1$$

l	0	1	. . .	n-1
N	n-1	n-2	. . .	0

$$\psi_{nlm} = \rho^l W(\rho) e^{-\rho} Y_{lm}(\theta, \phi)$$
$$= A \left(\frac{r}{a_0}\right)^l (P(x)) e^{\frac{-Zr}{na_0}} Y_{lm}(\theta, \phi)$$

$$\psi_{100} = \frac{1}{\sqrt{\pi a^3}} e^{\frac{-r}{a_0}}, Z=1$$