27: 3D Vector Fields

At every point $\vec{F} = \langle P, Q, R \rangle$

Functions of (x, y, z)

Force fields

$\vec{F} = \frac{-c\langle x, y, z\rangle}{\rho^3}$

Velocity fields $\vec{v}$

Gradient fields

$u = u(x, y, z)$

$\nabla u = \langle u_x, u_y, u_z \rangle$

Flux

$\int_C \vec{F} \cdot \hat{n}\, ds$

Flux measured through a surface in 3D

$\vec{F}$ vector field, S a surface in space

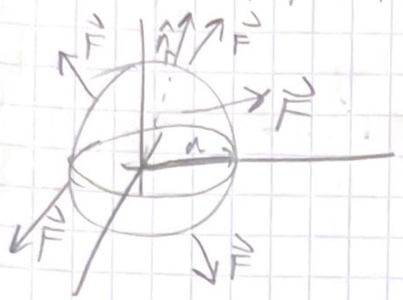
$\hat{n}$ = unit normal to S

27: 3D Vector Fields

$\text{Flux} = \iint_S \vec{F} \cdot \hat{n}\, dS$

$d\vec{S} = \hat{n}\, dS$

Flux of $\vec{F} = \langle x, y, z \rangle$ through sphere $r = a$ $(0,0,0)$



$\hat{n} \parallel \vec{F}$

$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \hat{n}\, dS$

$\hat{n} = \frac{1}{a} \langle x, y, z \rangle$

$\vec{F} \cdot \hat{n} = |\vec{F}||\hat{n}| = |\vec{F}|$

$\iint_S \vec{F} \cdot \hat{n}\, dS = \iint_S a\, dS = a \iint_S dS = 4\pi a^3$

Same sphere $\vec{F} = z\hat{k}$, $\vec{F} \cdot \hat{n} = \frac{z^2}{a}$

$\iint_S \vec{F} \cdot \hat{n}\, dS = \iint_S \frac{z^2}{a}\, dS$ $\quad dS = \ldots\, d\phi\, d\theta$

$\Delta S = (a \sin\phi\, \Delta\theta)(a\, \Delta\phi)$

$dS = a^2 \sin^2 \phi\, d\phi\, d\theta$

$\int_0^{2\pi} \int_0^{\pi} \frac{a^2 \cos^2 \phi}{a} a^2 \sin\phi\, d\phi\, d\theta = 2\pi a^3 \left[-\frac{1}{3} \cos^3 \phi \right]_0^{\pi} = \frac{4}{3}\pi a^3$

27: 3D Vector Fields

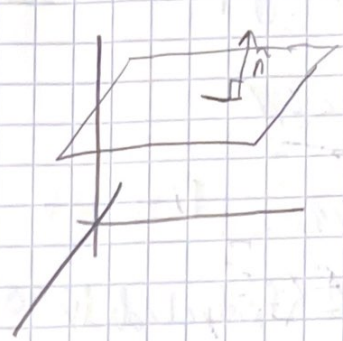
Use geometry

Set up $\iint_S \vec{F} \cdot \hat{n}\, dS$

S = horizontal plane ($z = a$)

$\hat{n} = \pm \hat{k}$

$dS = dx\,dy \Leftrightarrow dy\,dx$ (vertical)



$z = f(x,y)$

$\hat{n}\,dS = \pm\langle -f_x, -f_y, 1\rangle\, dx\,dy$

28: The Divergence Theorem

Flux of $\vec{F}$ through S is $\iint_S \vec{F}\cdot\hat{n}\,dS$

If S is the graph of a function $z=f(x,y)$

$\hat{n}\,dS = \pm\langle -f_x, -f_y, 1\rangle\,dx\,dy$

$\vec{u}\times\vec{v} = \Delta S\cdot\hat{n}$

$\hat{n}\,\Delta S = \vec{u}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & k \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix}\Delta x\Delta y = \langle -f_x, -f_y, 1\rangle\Delta x\Delta y$

$\vec{F} = z\hat{k}$ through $z=x^2+y^2$ above unit disk

$\iint_S \vec{F}\cdot\hat{n}\,dS \Rightarrow \iint_S \langle 0,0,z\rangle\cdot\langle -2x,-2y,1\rangle\,dx\,dy$

$= \iint_S (x^2+y^2)\,dx\,dy = \iint_{\mathbb{D}} (x^2+y^2)\,dx\,dy = \int_0^{2\pi}\int_0^1 r^2\,r\,dr\,d\theta$

$= \int_0^{2\pi} \left[\frac{1}{4}r^4\right]_0^1 d\theta = \int_0^{2\pi} \frac{1}{4}\,d\theta = \frac{1}{4}\theta\Big|_0^{2\pi} = \frac{1}{4}2\pi = \frac{\pi}{2}$

Given

$S: \begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}$

$\langle x,y,z\rangle = \vec{r} = \vec{r}(u,v)$

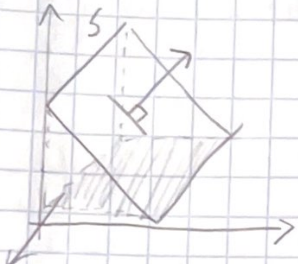
28: The Divergence Theorem

$\hat{n}\,dS$

$\frac{\partial \vec{r}}{\partial u}\Delta u = \langle \frac{\partial x}{\partial u}\Delta u \ldots \rangle$

$\hat{n}\Delta S = \left(\frac{\partial \vec{r}}{\partial u}\Delta u\right) \times \left(\frac{\partial \vec{r}}{\partial v}\Delta v\right) = \left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right)\Delta u \Delta v$

If we know normal $\vec{N}$



$\cos\alpha = \frac{\vec{N}\cdot\hat{k}}{|\vec{N}||k|}$

$\hat{n}\Delta S = \frac{|\vec{N}|\hat{n}}{\vec{N}\cdot\hat{k}}\Delta A = \pm\frac{\vec{N}}{\vec{N}\cdot\hat{k}}\Delta A$

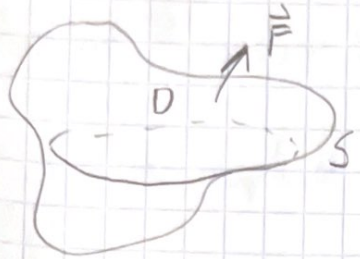
$\hat{n}\,dS = \pm\frac{\vec{N}}{\vec{N}\cdot\hat{k}}\,dx\,dy$

If S is a closed surface enclosing D $\hat{n}$ out, $\vec{F}$ is differentiable in D

$$\oiint_S \vec{F}\cdot d\vec{S} = \iiint_D \operatorname{div}(\vec{F})\,dV$$

$\operatorname{div}(P\hat{i} + Q\hat{j} + R\hat{k}) = P_x + Q_y + R_z$

29: The Divergence Theorem



$$\oiint_S \vec{F}\cdot d\vec{S} = \iiint_D \mathrm{Div}(\vec{F})\,dV$$

$$\oiint_S \langle P,Q,R\rangle\cdot\hat{n}\,dS = \iiint_D (P_x+Q_y+R_z)\,dV$$

∇ Notation

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

$$\nabla\cdot\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$\mathrm{div}(\vec{F})$ = source rate

$$\iiint_D \mathrm{div}(\vec{F})\,dV = \oiint_S \vec{F}\cdot\hat{n}\,dS$$

29: The Divergence Theorem

$$\oiint_S \langle 0,0,R\rangle \cdot \hat{n}\, dS = \iiint_D R_z\, dV$$

R:

$$\iiint_D R_z\, dV = \iint_u \int_{z(x,y)}^{z_2(x,y)} R_z\, dz\, dx\, dy$$

$$\iiint_D R_z\, dV = \iint_u [R(x,y,z_2(x,y)) - R(x,y,z_1(x,y))]\, dx\, dy$$

$$\oiint_S \langle 0,0,R\rangle \cdot \hat{n}\, dS = \iint_{top} + \iint_{Bottom} + \iint_{Sides}$$

$$\iint_B \langle 0,0,R\rangle \cdot \hat{n}\, dS = \iint_B -R\, dx\, dy = \iint -R(x,y,z_1(x,y))\, dx\, dy$$

$$\iiint_D R_z\, dV = \oiint_{top+bottom+sides} \langle 0,0,R\rangle \cdot \hat{n}\, dS$$

$u(x,y,z,t)$ $\quad \frac{\partial u}{\partial t} = k\nabla^2 u = k\nabla \cdot \nabla u \quad div(\nabla)$

$$\oiint_S \vec{F} \cdot \hat{n}\, dS = \frac{d}{dt}\left(\iiint_D u\, dV\right) = -\iiint_D \frac{\partial u}{\partial t}\, dV$$

$$div(\vec{F}) = -\frac{\partial u}{\partial t}$$

30: Line Integrals in Space

$\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$

Curve C in space

$W = \int_C \vec{F} \cdot d\vec{r} \quad dr = \langle dx, dy, dz \rangle$

$\int_C P\,dx + Q\,dy + R\,dz$

$\vec{F} = \langle yz, xz, xy \rangle$

$C: x = t^3, y = t^2, z = t \quad 0 \le t \le 1$

$dx = 3t^2 dt \quad dy = 2t\,dt \quad dz = 1\,dt$

$\int_C \vec{F} \cdot d\vec{r} = \int_C yz\,dx + xz\,dy + xy\,dz = \int_C t^3 \cdot 3t^2 dt + t^4 \cdot 2t\,dt + t^5 dt$

$= \int_0^1 6t^5 dt = [t^6]_0^1 = 1$

$\vec{F} = \nabla(x,y,z) = \langle yz, xz, xz \rangle$

$\int_C \nabla f \cdot d\vec{r} = f(P_1) - f(P_0)$

$f(1,1,1) - f(0,0,0) = 1$

$P\,dx + Q\,dy + R\,dz = \text{exact } (= df) \Leftrightarrow \ldots$

$f_x = 2xy \underset{\int dx}{\Rightarrow} f = x^2y + g(y,z)$

$\downarrow \frac{\partial}{\partial y}$

$f_y = x^2 + z^3 = x^2 + g_y \Rightarrow g_y = z^3 \underset{\int dy}{\Rightarrow} g = yz^3 + h(z)$

$f_z = 3yz^2 - 4z^3 \Rightarrow x^2y + yz^3 + h(z) \Rightarrow h = -z^4 + C \Rightarrow f = x^2y + yz^3 - z^4 + C$

30: Line Integrals in Space

Curl in 3D: if $\vec{F} = P\hat{\imath} + Q\hat{\jmath} + R\hat{k} \Rightarrow \text{curl}(\vec{F}) = (R_y - Q_z)\hat{\imath} + (P_z - R_x)\hat{\jmath} + (Q_x - P_y)\hat{k}$

If $\vec{F}$ defined

$\vec{F}$ conservative $\Leftrightarrow \text{curl}(\vec{F}) = 0$

$\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$

$\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$

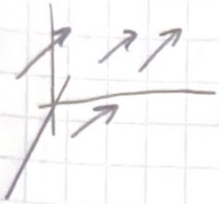
$\nabla \cdot \langle P, Q, R \rangle = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \text{div}(\vec{F})$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{\imath} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\hat{\jmath} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k} = \text{curl}(\vec{F})$$

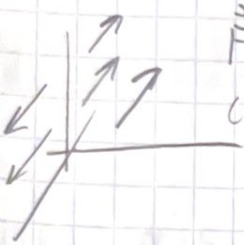
$\vec{v} = \langle -\omega y, \omega x, 0 \rangle \quad \text{curl}(\vec{v}) = 2\omega\hat{k}$

31: Stoke's Theorem

$curl(\vec{F}) = \nabla \times \vec{F}$



$\vec{F} = \langle a,b,c \rangle$
$curl(\vec{F}) = 0$



$\vec{F} = \langle x,0,0 \rangle$
$curl(\vec{F}) = 0$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n} \, dS$$

S=any surface bounded by C

C, S compatible

RHR: thumb along C positively the index finger tangent to S

middle finger points $\hat{n}$

S= portion of xy bounded by C

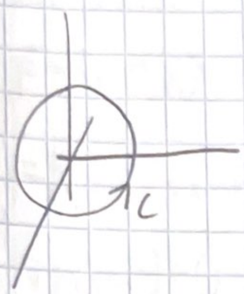
$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C P dx + Q dy = \iint_S (\nabla \times \vec{F}) \cdot \hat{k} \, dS$$

$$= \iint_S (Q_x - P_y) \, dx dy$$

31: Stokes Theorem

Greens = special case (xy) of stoke's

$\vec{F} = z\hat{i} + x\hat{j} + y\hat{k}$ around unit circle in xy CCW



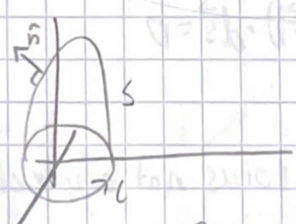
$$\oint_C z\,dx + x\,dy + y\,dz$$

$x = \cos t$
$y = \sin t$
$z = 0$

$$= \int_0^{2\pi} 0 + \cos t(\cos t\,dt) + 0$$

$$= \int_0^{2\pi} \cos^2 t\,dt = \pi$$

$z = 1 - x^2 - y^2$



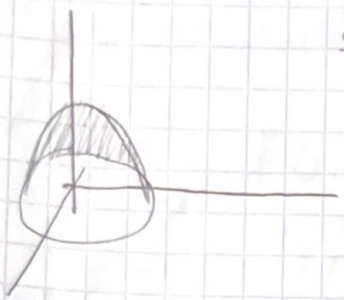
$$\iint_S (\nabla \times F) \cdot \hat{n}\,dS \rightarrow \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = (1)\hat{i} - (-1)\hat{j} + (1)\hat{k} = \langle 1,1,1 \rangle$$

$$\hat{n}\,dS = \langle -f_x, -f_y, 1 \rangle dx\,dy = \langle 2x, 2y, 1 \rangle dx\,dy$$

$$\iint_S \langle 1,1,1 \rangle \cdot \langle 2x, 2y, 1 \rangle dx\,dy = \iint_S 2x + 2y + 1\,dx\,dy = \pi$$

32: Stoke's Theorem

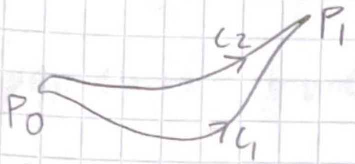
$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla \times \vec{F})\cdot \hat{n}\, dS$$



space with origin removed is simply connected

if $\vec{F} = \nabla f \Rightarrow \text{curl}(\vec{F}) = 0$

Assume $\text{curl}(\vec{F}) = 0$



$$\int_{C_1} \vec{F}\cdot d\vec{r} - \int_{C_2} \vec{F}\cdot d\vec{r} = \oint_C \vec{F}\cdot d\vec{r}$$

$$= \iint_S (\text{curl}(\vec{F}))\cdot d\vec{S} = 0$$

Surface of torus not simply connected

Two independent loops which bound nothing

$$\iint_{S_1} (\nabla\times\vec{F})\cdot\hat{n}\,dS - \iint_{S_2} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\,dS$$

32: Stokes Theorem

$$= \iiint_D \operatorname{div}(\nabla \times \vec{F})\, dV$$

$$\operatorname{div}(\nabla \times \vec{F}) \equiv 0$$

$$\vec{F} = \langle P, Q, R \rangle \Rightarrow \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz}$$

$$\nabla \cdot (\nabla \times \vec{F}) \equiv 0$$

$$\iiint_R f\, dV$$

- rectangular: $dV = dz\,dx\,dy$
- cylindrical: $dV = dz\,r\,dr\,d\theta$
- spherical: $dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$
  - mass
  - average value
  - moment of inertia
  - gravitational attraction at $(0,0,0)$

$$\iint_S \vec{F} \cdot n\, dS$$

$$\hat{n}\, dS = \ldots dx\,dy \Rightarrow \iint \ldots dx\,dy$$

$$\int_C \vec{F}\, d\vec{r} = \int_C P\,dx + Q\,dy + R\,dz$$

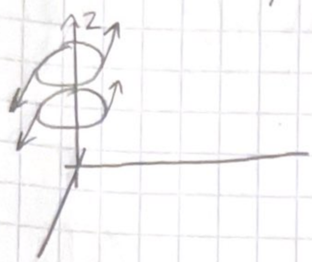
paramaterize $C \Rightarrow$ express in terms of single variable

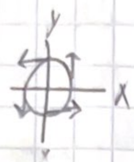
$$\iint_S \vec{F} \cdot \hat{n}\, dS = \iiint_D (\operatorname{div}(\vec{F}))\, dV$$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, dS$$

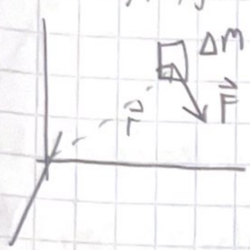
33: Maxwell's Equations

Curl: $\vec{v}$ velocity field $\Rightarrow$ curl($\vec{v}$) measures 2·angular velocity vector





$\vec{v} = \omega(-y\hat{i} + x\hat{j}) \Rightarrow \nabla \times \vec{v} = 2\omega\hat{k}$



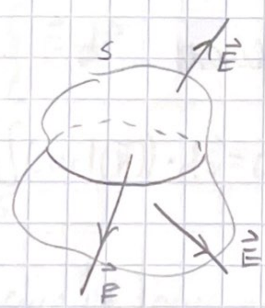
$\vec{F}\Delta m \Rightarrow \vec{\tau} = \vec{r} \times \vec{F}\Delta m$

$\vec{E}$, $\vec{B}$

$\vec{F} = q\vec{E}$, $\vec{F} = q\vec{v} \times \vec{B}$

Gauss Coulomb Law

$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$

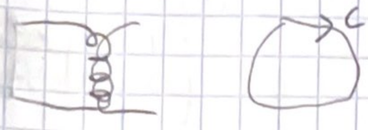


$$\oiint_S \vec{E} \cdot d\vec{S} = \iiint_D \mathrm{div}(\vec{E})\, dV = \frac{1}{\varepsilon_0} \iiint_D \rho\, dV = \frac{Q}{\varepsilon_0}$$

3B: Maxwell's Equations

Faraday's Law

$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$



$$\oint_C \vec{E} \cdot d\vec{r} = \iint_S (\nabla \times \vec{E}) \cdot d\vec{S} = \iint_S \left(-\frac{\partial \vec{B}}{\partial t}\right) \cdot d\vec{S}$$

$\nabla \cdot \vec{B} = 0$ (Gauss Magnitism)

$\nabla \times \vec{B} = \mu_0 \vec{J} + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$ (Maxwell-Ampere)