13: Quantum Dynamics

$$|\Psi, t\rangle = u(t, t_0)\, |\Psi, t_0\rangle$$

$$H(t) = i\hbar \frac{\partial}{\partial t} u(t, t_0)\, u^{+}(t, t_0)$$

$$i\hbar \frac{\partial}{\partial t} u(t, t_0) = H(t)\, u(t, t_0)$$

$$i\hbar \frac{d}{dt} u(t, t_0) = H(t)\, u(t, t_0)$$

H is time independent

$$H(t) = H$$

$$i\hbar \frac{du}{dt} = Hu$$

$$u = e^{-\frac{iHt}{\hbar}} u_0$$

$$i\hbar \frac{du}{dt} = i\hbar \left(\frac{-iH}{\hbar}\right) e^{-\frac{iHt}{\hbar}} u_0$$

$$= Hu$$

$$u(t, t_0) = e^{-\frac{iHt}{\hbar}} u_0$$

$$(t = t_0) = 1 = e^{-\frac{iHt_0}{\hbar}} u_0$$

$$u_0 = e^{\frac{iHt_0}{\hbar}}$$

$$u(t, t_0) = e^{-\frac{iH}{\hbar}(t - t_0)}$$ H time independent

B: Quantom Mechanics

$$e^{\alpha H}|\psi_n\rangle = e^{\alpha E_n}|\psi_n\rangle$$

if $H|\psi_n\rangle = E_n|\psi_n\rangle$

$H(t)$ is time dependent but $[\hat{H}(t_1), \hat{H}(t_2)] = 0 \ \forall t_1, t_2$

$$H = -\gamma \hat{B}(t) \cdot \hat{S}$$

$$= -\gamma \hat{B}_z(t) \cdot \hat{S}_z$$

$$u(t,t_0) = \exp\Big(\underbrace{-\frac{i}{\hbar}\int_{t_0}^{t} H(t')\,dt'}_{R(t)}\Big)$$

$$\dot{R}(t) = -\frac{i}{\hbar} H(t)$$

$$u = e^{R}$$

$$\frac{du}{dt} = \frac{d}{dt}\left(1 + R + \frac{1}{2}RR + \frac{1}{3!}RRR + \dots\right)$$

$$= \dot{R} + \frac{1}{2}(\dot{R}R + R\dot{R}) + \frac{1}{3!}(\dot{R}RR + R\dot{R}R + RR\dot{R}) + \dots$$

$$[\dot{R}, R] = 0$$

$$= \dot{R}e^{R}$$

$$= -\frac{i}{\hbar} H(t) u = \frac{du}{dt}$$

13: Quantum Dynamics

$H(t)$ general

$$u(t,t_0) = T\exp\left(-\frac{i}{\hbar}\int_{t_0}^{t} H(t')dt'\right)$$

$$= 1 + \left(\frac{-i}{\hbar}\right)\int_{t_0}^{t} H(t_1)dt_1 + \frac{1}{2}\left(-\frac{i}{\hbar}\right)^2 \int_{t_0}^{t} H(t')dt' \int_{t_0}^{t'} H(t'')dt''$$

$$\frac{d}{dx}\int_{x_0}^{x} g(x')dx' = g(x)$$

Heisenberg Picture

$\hat{A}_{\hat{S}}$

S for Schrödinger

$|\alpha,t\rangle, |\beta,t\rangle$

$\langle \alpha,t|\hat{A}_S|\beta,t\rangle \rightarrow$ matrix element

$= \langle \alpha,0|u^{+}(t,0)\,\hat{A}_S\, u(t,0)|\beta,0\rangle$

$\hat{A}_H(t) \equiv u^{+}(t,0)\,\hat{A}_S\, u(t,0)$

13: Quantum Dynamics

At $t=0 \Rightarrow A_H(t=0) = A_S(t=0)$

$\mathbb{1}_S \to \mathbb{1}_H = U^{\dagger}(t,0)\,\mathbb{1}_S U(t,0) = \mathbb{1}_S$

$\hat{C}_S = \hat{A}_S \hat{B}_S$

$\hat{C}_H = U^{\dagger}\hat{C}_S U = U^{\dagger}\hat{A}_S \hat{B}_S U = U^{\dagger}\hat{A}_S U U^{\dagger}\hat{B}_S U$

$\hat{C}_H = \hat{A}_H \hat{B}_H$

If $[A_S, B_S] = C_S$

$[A_H, B_H] = C_H$

$[x, p] = i\hbar$

$[x_H(t), p_H(t)] = i\hbar\mathbb{1}$

$H_H(t) = U^{\dagger}(t,0)\, H_S(t)\, U(t,0)$

$[H_S(t_1), H_S(t_2)] = 0, \forall t_1, t_2$

$H_H(t) = H_S(t)$ if $[H_S(t_1), H_S(t_2)] = 0$

$H_S(t) = H_S(\hat{x}, \hat{p}, t)$

$H_H(t) = U^{\dagger} H_S(\hat{x}, p, t)\, U$

$= H_S(x_H(t), p_H(t), t)$

$= H_S(\hat{x}, \hat{p}, t)$

13: Quantum Dynamics

$$\langle \psi, t | \hat{A}_S | \psi, t \rangle = \langle \psi, 0 | \hat{A}_H(t) | \psi, 0 \rangle$$

$$\langle \hat{A}_S \rangle = \langle \hat{A}_H(t) \rangle$$

$$i\hbar \frac{d}{dt} \hat{A}_H = i\hbar \frac{\partial u^+}{\partial t} \hat{A}_S u + i\hbar u^+ \hat{A}_S \frac{\partial u}{\partial t} + u^+ i\hbar \frac{\partial \hat{A}_S}{\partial t} u$$

$$i\hbar \frac{\partial u}{\partial t} = H_S u$$

$$i\hbar \frac{\partial u^+}{\partial t} = -u^+ H_S$$

$$i\hbar \frac{d}{dt} \hat{A}_H = -u^+ \hat{H}_S \hat{A}_S u + u^+ \hat{A}_S \hat{H}_S u + i\hbar \left(\frac{\partial \hat{A}_S}{\partial t}\right)_H$$

$$i\hbar \frac{d}{dt} \hat{A}_H(t) = [\hat{A}_H, \hat{H}_H] + i\hbar \left(\frac{\partial A_S}{\partial t}\right)_H$$

Suppose A_S has no explicit time dependence

$$i\hbar \frac{d\hat{A}_H}{dt} = [A_H, H_H(t)]$$

$$i\hbar \frac{d}{dt} \langle \psi, t | \hat{A}_S | \psi, t \rangle = i\hbar \frac{d}{dt} \langle \psi, 0 | \hat{A}_H(t) | \psi, 0 \rangle$$

$$= \langle \psi, 0 | i\hbar \frac{d}{dt} \hat{A}_H | \psi, 0 \rangle$$

$$= \langle \psi, 0 | [\hat{A}_H, \hat{H}_H] | \psi, 0 \rangle$$

$$i\hbar \frac{d}{dt} \langle \hat{A}_H(t) \rangle = \langle [\hat{A}_H, \hat{H}_H] \rangle$$

$$i\hbar \frac{d}{dt} \langle \hat{A}_S \rangle = \langle [\hat{A}_S, \hat{H}] \rangle$$

13: Quantum Dynamics

A time independent As is conserved if

$[\hat{A}_S, \hat{H}_S] = 0$

$\longrightarrow [\hat{A}_H, \hat{H}_H] = 0$

$\Longrightarrow \frac{d}{dt}\hat{A}_H = 0$

Harmonic Oscillator

$$H_S = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$

$$H_H = \frac{\hat{p}_H^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2_H(t)$$

$$i\hbar\frac{d}{dt}x_H = [x_H, H_H]$$
$$= [x_H, \frac{\hat{p}_H^2}{2m}]$$
$$= \frac{1}{2m}\hat{p}_H[i\hbar]\cdot 2$$

$$\frac{d\hat{x}_H}{dt} = \frac{1}{m}\hat{p}_H \iff \left.\frac{dx}{dt} = \frac{p}{m}\right\} CM$$

$$i\hbar\frac{d\hat{p}_H}{dt} = [\hat{p}_H, H_H]$$
$$= [\hat{p}_H, \frac{1}{2}m\omega^2 x_H^2]$$
$$= \frac{1}{2}m\omega^2\cdot 2\cdot x_H(-i\hbar)$$

$$\frac{dp_H}{dt} = -m\omega^2 x_H$$

13: Quantum Dynamics

$$\frac{d^2 x_H}{dt^2} = \frac{1}{M}\frac{dp_H}{dt}$$

$$= \frac{1}{m}\left(-m\omega^2 \hat{x}_H\right)$$

$$\frac{d^2 x_H}{dt^2} = -\omega^2 \hat{x}_H$$

$$\hat{x}_H(t) = \hat{A}\cos\omega t + \hat{B}\sin\omega t$$

$$\hat{p}_H(t) = m\frac{d\hat{x}}{dt}$$

$$= -m\omega\sin\omega t\, A + m\omega\cos\omega t\, B$$

$$t=0 \Rightarrow x_H(t) = \hat{A}$$
$$= \hat{x}$$

$$\Rightarrow p_H(t) = m\omega\hat{B}$$
$$= \hat{p}$$

$$\hat{B} = \frac{\hat{p}}{m\omega}$$

$$\hat{x}_H(t) = \hat{x}\cos\omega t + \frac{\hat{p}}{m\omega}\sin\omega t$$

$$\hat{p}_H(t) = \hat{p}\cos\omega t - m\omega\hat{x}\sin\omega t$$

$$H_H(t) = \frac{1}{2m}\left(\hat{p}\cos\omega t - m\omega\hat{x}\sin\omega t\right)^2 + \frac{1}{2}m\omega^2\left(\hat{x}\cos\omega t + \frac{\hat{p}}{m\omega}\sin\omega t\right)^2$$

$$= \frac{1}{2m}\cos^2\omega t\,\hat{p}^2 + \frac{1}{2m}m^2\omega^2\sin^2\omega t\,\hat{x}^2 - \frac{1}{2m}m\omega\sin\omega t\cos\omega t\left(\hat{p}\hat{x} + \hat{x}\hat{p}\right)$$

$$= \frac{1}{2}\frac{m\omega^2}{m^2\omega^2}\sin^2\omega t\, p^2 + \frac{1}{2}m\omega^2\cos^2\omega t\,\hat{x}^2$$

$$\frac{1}{2m} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$

14: Quantum Dynamics

$$\hat{x}_H(t) = \hat{x}\cos\omega t + \frac{\hat{p}}{m\omega}\sin\omega t$$

$$\hat{p}_H(t) = \hat{p}\cos\omega t - m\omega\hat{x}\sin\omega t$$

$$\hat{A}_H = u^{\dagger}(t,0)\hat{A}_S u(t,0) = e^{\frac{iHt}{\hbar}}\hat{A}_S e^{-\frac{iHt}{\hbar}}$$

$$\hat{a}(t) \equiv \hat{a}_H(t) = e^{-i\omega t}\hat{a}$$

$$\hat{a}^{\dagger}(t) \equiv \hat{a}_H^{\dagger}(t) = e^{i\omega t}\hat{a}^{\dagger}$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}^{\dagger}+\hat{a})$$

$$\hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a}^{\dagger}-\hat{a})$$

$T_{x_0} \equiv e^{\frac{-i\hat{p}x_0}{\hbar}}$ is unitary $x_0 \in \mathbb{R}$

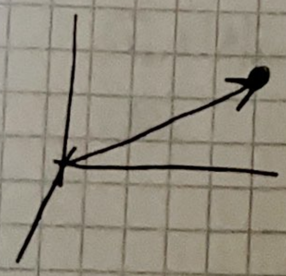
$$T_{x_0}T_{y_0} = T_{x_0+y_0}$$

$$\left.\begin{aligned} T_{x_0}^{\dagger} &= T_{-x_0} \\ T_{-x_0} &= (T_{x_0})^{\dagger} \end{aligned}\right\} \Rightarrow \text{unitary} \quad {}^{-1} = {}^{\dagger}$$

$$T_{x_0}^{\dagger}\hat{x}T_{x_0} = \hat{x} + x_0\mathbb{1}$$

$$T_{x_0}^{\dagger}\hat{p}T_{x_0} = \hat{p}$$

$|\psi\rangle$ ask $\langle\hat{x}\rangle_{|\psi\rangle}$



$\langle\hat{x}\rangle_{T_{x_0}|\psi\rangle}$

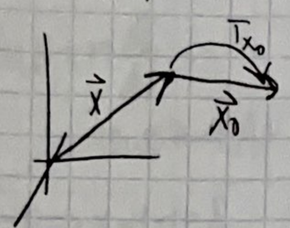
14: Quantum Dynamics

$$\langle \hat{x} \rangle_{T_{x_0}|\psi\rangle} = \langle \psi | T_{x_0}^\dagger \hat{x} T_{x_0} | \psi \rangle$$

$$= \langle \psi | \hat{x} + x_0 | \psi \rangle$$

$$= \langle \hat{x} \rangle_{|\psi\rangle} + x_0$$

$$\langle x \rangle_{T_{x_0}|\psi\rangle} = \langle \hat{x} \rangle_{|\psi\rangle} + x_0$$



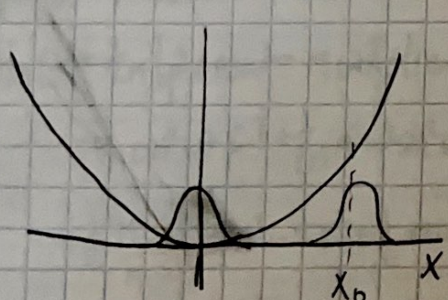
$$T_{x_0}|x\rangle = |x + x_0\rangle$$

$$|\psi\rangle \to \psi(x)$$

$$T_{x_0}|\psi\rangle \to \psi(x - x_0)$$

Coheurent states:

$$T_{x_0}|0\rangle = e^{-\frac{i\hat{p}x_0}{\hbar}}|0\rangle \equiv |\tilde{x}_0\rangle$$



$$\langle \tilde{x}_0 | \tilde{x}_0 \rangle = \langle 0 | 0 \rangle = 1$$

14: Quantum Dynamics

$$\psi_{x_0}(x) = \psi_0(x - x_0)$$

$$\uparrow$$

$$\langle x|0\rangle = \psi_0(x)$$

$$\langle \tilde{x}_0|A|\tilde{x}_0\rangle = \langle 0|T_{x_0}^{+} A T_{x_0}|0\rangle$$

$$\langle \tilde{x}_0|\hat{x}|\tilde{x}_0\rangle = \langle 0|(\hat{x}+x_0)|0\rangle$$

$$= x_0$$

$$\langle \tilde{x}_0|\hat{p}|\tilde{x}_0\rangle = \langle 0|\hat{p}|0\rangle$$

$$= 0$$

$$\langle \tilde{x}_0|H|\tilde{x}_0\rangle = \langle 0|\frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2(\hat{x}+x_0)^2|0\rangle$$

$$= \langle 0|H|0\rangle + \frac{1}{2}m\omega^2 x_0^2$$

$$= \frac{1}{2}\hbar\omega + \frac{1}{2}m\omega^2 x_0^2$$

$$\langle \tilde{x}_0|\hat{x}^2|\tilde{x}_0\rangle = x_0^2 + \frac{\hbar}{2m\omega}$$

$$\langle \tilde{x}_0|\hat{p}^2|\tilde{x}_0\rangle = \frac{m\hbar\omega}{2}$$

$$\langle \tilde{x}_0|\hat{x}\hat{p}+\hat{p}\hat{x}|\tilde{x}_0\rangle = 0$$

Time evolution

$$|\tilde{x}_0\rangle \rightarrow |\tilde{x}_0, t\rangle$$

$$\langle \tilde{x}_0, t|A|\tilde{x}_0, t\rangle = \langle \tilde{x}_0|A_H|\tilde{x}_0\rangle$$

$$\langle A\rangle_t \qquad (\equiv \langle 0|T_{x_0}^{+} A_H T_{x_0}|0\rangle)$$

14: Quantum Dynamics

$$\langle\hat{x}\rangle_{\tilde{x}_0}(t) = \langle\tilde{x}_0|\left(\hat{x}\cos\omega t + \frac{\hat{p}}{m\omega}\sin\omega t\right)|\tilde{x}_0\rangle$$

$$= x_0\cos\omega t$$

$$\langle\hat{p}\rangle_{\tilde{x}_0}(t) = \langle\tilde{x}_0|\left(\hat{p}\cos\omega t - m\omega\hat{x}\sin\omega t\right)|\tilde{x}_0\rangle$$

$$= -m\omega x_0\sin\omega t$$

$$= m\frac{d}{dt}\langle\hat{x}\rangle_t$$

$\Delta x(t), \Delta p(t)$

$$(\Delta x)^2(t) = \langle\tilde{x}_0,t|\hat{x}^2|\tilde{x}_0,t\rangle - \left(\langle\tilde{x}_0,t|\hat{x}|\tilde{x}_0,t\rangle\right)^2$$

$$= \langle\tilde{x}_0|x_H^2(t)|\tilde{x}_0\rangle - x_0^2\cos^2\omega t$$

$$= \langle\tilde{x}_0|\hat{x}^2\cos^2\omega t + \frac{\hat{p}^2}{m^2\omega^2}\sin^2\omega t + \frac{1}{m\omega}\cos\omega t\sin\omega t(\hat{x}\hat{p}+\hat{p}\hat{x})|\tilde{x}_0\rangle$$

$$= \left(x_0^2 + \frac{\hbar}{2m\omega}\right)\cos^2\omega t + \frac{m\hbar\omega}{2m^2\omega^2}\sin^2\omega t$$

$$\Rightarrow (\Delta x)^2(t) = \frac{\hbar}{2m\omega}$$

$$(\Delta p)^2(t) = \frac{m\hbar\omega}{2}$$

$$\Delta x(t)\Delta p(t) = \frac{\hbar}{2}$$

14: Quantum Dynamics

Energy Basis

$$|\tilde{x}_0\rangle = \exp\left(-\frac{i\hat{p}x_0}{\hbar}\right)$$

$$d_0^2 = \frac{\hbar}{m\omega}$$

$$|\tilde{x}_0\rangle = \exp\left(\frac{x_0}{\sqrt{2}d}(\hat{a}^+ - \hat{a})\right)|0\rangle$$

$$e^{X+Y} = e^X e^Y e^{-\frac{1}{2}[X,Y]}$$ if $[X,Y]$ commutes with X,Y

$$e^{\frac{x_0}{\sqrt{2}d_0}a^+ - \frac{x_0}{\sqrt{2}d_0}a}$$

$$= e^{\frac{x_0}{\sqrt{2}d_0}a^+} e^{-\frac{x_0}{\sqrt{2}d_0}a} e^{\frac{1}{2}(-)\left(\frac{x_0}{\sqrt{2}d_0}\right)^2(-1)}$$

$$|\tilde{x}_0\rangle = e^{\frac{x_0}{\sqrt{2}d_0}\hat{a}^+} e^{\frac{x_0}{\sqrt{2}d_0}\hat{a}} e^{-\frac{1}{4}\frac{x_0^2}{d_0^2}}|0\rangle$$

$$|\tilde{x}_0\rangle = e^{-\frac{1}{4}\frac{x_0^2}{d_0^2}} e^{\frac{x_0}{\sqrt{2}d_0}\hat{a}^+}|0\rangle$$

$$= e^{-\frac{1}{4}\frac{x_0^2}{d_0^2}} \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{x_0}{\sqrt{2}d}\right)^n (\hat{a}^+)^n |0\rangle$$

$$|n\rangle = \frac{(a^+)^n}{\sqrt{n!}}|0\rangle$$

$$= e^{\frac{1}{4}\frac{x_0^2}{d_0^2}} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}}\left(\frac{x_0}{\sqrt{2}d_0}\right)^n |n\rangle$$

$$= \sum_n c_n |n\rangle$$

c_n^2 = probability for $|\tilde{x}_0\rangle$ to be in $|n\rangle$

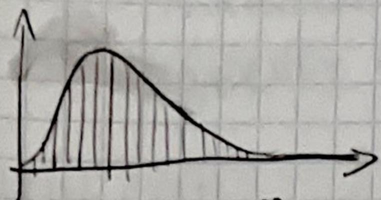
$$= \exp\left(-\frac{1}{2}\frac{x_0^2}{d_0^2}\right)\frac{1}{n!}\left(\frac{x_0^2}{2d_0^2}\right)^n$$

14: Quantum Dynamics

$$\lambda \equiv \frac{x_0^2}{2d_0^2}$$

$$c_n^2 \equiv e^{-\lambda} \frac{\lambda^n}{n!}$$

$$\sum_{n=0}^{\infty} c_n^2 = e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} = e^{-\lambda} e^{\lambda} = 1$$



$$\langle n \rangle = \sum_{n=0}^{\infty} n c_n^2 = \sum_{n=0}^{\infty} n e^{-\lambda} \frac{\lambda^n}{n!}$$

$$= e^{-\lambda} \lambda \frac{d}{d\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}$$

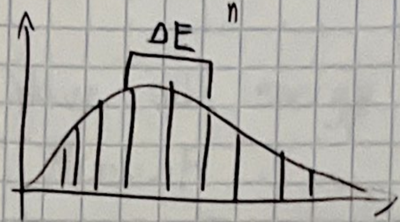
$$= e^{-\lambda} \lambda \frac{d}{d\lambda} e^{\lambda} = \lambda$$

14: Quantum Dynamics

$$\langle \tilde{x}_0 | \hat{N} | \tilde{x}_0 \rangle = \sum_{n,m} c_m^* \langle m | \hat{N} | n \rangle c_n$$

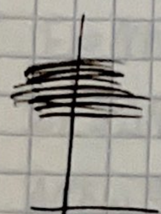
$$= \sum_{n,m} c_m c_n n \delta_{mn}$$

$$= \sum_n n c_n^2$$



$$(\Delta E)_{\tilde{x}_0} = \frac{\hbar \omega x_0}{\sqrt{2} d}$$

$$\frac{\Delta E}{\hbar \omega} = \frac{x_0}{d\sqrt{2}} \gg 1$$



$$\frac{\langle E \rangle}{\Delta E} \approx \frac{\frac{1}{2} m \omega^2 x_0^2}{\hbar \omega \frac{x}{\sqrt{2}}} = \frac{x_0}{\sqrt{2} d}$$

14: Quantum Dynamics

$$|\alpha\rangle \equiv \underbrace{D(\alpha)}_{\text{unitary}}|0\rangle \qquad \alpha \in \mathbb{C}$$

$$= \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})|0\rangle$$

$$\hat{a}|\alpha\rangle = [\hat{a}, e^{\alpha a^\dagger - \alpha^*\hat{a}}]|0\rangle$$

$$[A, e^B] = [A\ B]e^B \text{ is } [[A,B],B] = 0$$

$$= \alpha|\alpha\rangle$$

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

not hermitian

what?

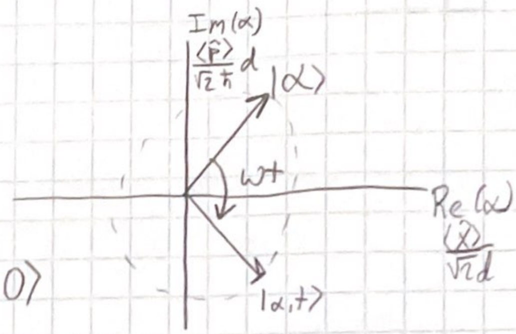
eigenvalues

$$\langle\alpha|\hat{x}|\alpha\rangle = \frac{d}{\sqrt{2}}\langle\alpha|\hat{a}+\hat{a}^\dagger|\alpha\rangle$$

$$= \frac{d}{\sqrt{2}}(\alpha+\alpha^*)$$

$$= \sqrt{2}d\,\mathrm{Re}(\alpha)$$

$$\langle\hat{p}\rangle = \langle\alpha|\hat{p}|\alpha\rangle \equiv \frac{\sqrt{2}\hbar}{d}\mathrm{Im}(\alpha)$$



$$|\alpha,t\rangle = e^{-\frac{iHt}{\hbar}} e^{\alpha(\hat{a}^\dagger - \alpha^*\hat{a})} e^{\frac{iHt}{\hbar}} e^{-\frac{iHt}{\hbar}}|0\rangle$$

$$= e^{\alpha(\hat{a}_H^\dagger(-t) - \alpha^* a_H(-t))}$$

$$= e^{\alpha e^{-i\omega t}\hat{a}^\dagger - \alpha^* e^{i\omega t}\hat{a}}\, e^{-\frac{i\omega t}{2}}|0\rangle$$

$$|\alpha,t\rangle = e^{-\frac{i\omega t}{2}}|e^{-i\omega t}\alpha\rangle$$

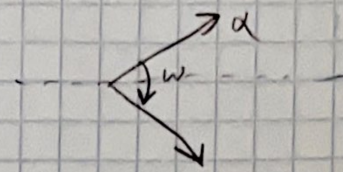
15: Quantum Dynamics

$|\alpha\rangle = D(\alpha)|0\rangle \Rightarrow D(\alpha) = e^{\alpha\hat{a}^{\dagger} - \alpha^{*}\hat{a}}$, $\alpha \in \mathbb{C}$

$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$

$|\alpha\rangle \longrightarrow |\alpha, t\rangle$

$= e^{-\frac{i\omega t}{2}}|e^{-i\omega t}\alpha\rangle$



Wave with energy E (light wave)

$A\cos\omega t$

$\Delta E \Delta t \sim \frac{\hbar}{2}$

phase $\phi = \omega t$

$\frac{\Delta\phi}{\omega} = \Delta t$

$E = N\hbar\omega$, N = # of photons

$\Delta E = \Delta N\hbar\omega$

$\Delta N\hbar\omega \frac{\Delta\phi}{\omega} \sim \frac{\hbar}{2}$

$\Delta N \Delta\phi \sim 1$

15: Quantum Dynamics

$$\langle \hat{N} \rangle_\alpha = \langle \alpha | a^\dagger a | \alpha \rangle$$
$$= \langle \alpha | \alpha^* \alpha | \alpha \rangle$$
$$= |\alpha|^2$$

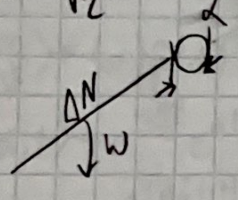
$$\langle \hat{N}^2 \rangle = \langle \alpha | a^\dagger a a^\dagger a | \alpha \rangle$$
$$= |\alpha|^2 \langle \alpha | a a^\dagger | \alpha \rangle$$
$$= |\alpha|^2 \langle \alpha | 1 + a^\dagger a | \alpha \rangle$$
$$= |\alpha|^2 (1 + |\alpha|^2)$$

$$\Delta N = \sqrt{|\alpha|^4 + |\alpha|^2 - |\alpha|^4} = |\alpha|$$

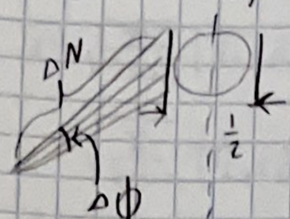
$$\Delta N = |\alpha|$$

$$\Delta p = \frac{\hbar}{\sqrt{2} d_0}$$

$$\Delta x = \frac{d_0}{\sqrt{2}}$$



15: Quantum Dynamics



$\Delta\phi \approx \frac{1}{|\alpha|}$

$\Delta N = |\alpha|$

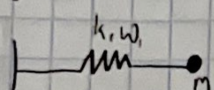
$\Delta\phi \Delta N \sim 1$ ✓

Expect the position if measured to be

$\langle \hat{x} \rangle \pm \Delta X$

$\frac{1}{\sqrt{2}\alpha_0}$

Squeeze States



$H_1 = \frac{p^2}{2m_1} + \frac{1}{2} m_1 \omega_1 x^2$

15: Quantum Dynamics

$$\left.\begin{aligned}\Delta x &= \sqrt{\frac{\hbar}{2m_1\omega_1}} \\ \Delta p &= \sqrt{\frac{\hbar m_1\omega_1}{2}}\end{aligned}\right\} \text{Ground State}$$

H_1 is valid $t<0$

particle in the ground state

$$H_2 \quad \frac{p^2}{2m_2} + \frac{1}{2}m_2\omega_2^2x^2$$

$t>0$ H_2 is valid

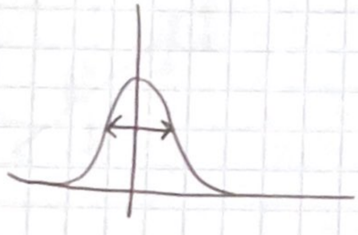
$$\Delta x = \sqrt{\frac{\hbar}{2m_2\omega_2}}\sqrt{\frac{m_2\omega_2}{m_1\omega_1}}$$

$$\Delta p = \sqrt{\frac{\hbar m_2\omega_2}{2}}\sqrt{\frac{m_1\omega_1}{m_2\omega_2}}$$

$$\Delta x \equiv e^{-\gamma}\sqrt{\frac{\hbar}{2m_2\omega_2}}$$

$$\Delta p \equiv e^{\gamma}\sqrt{\frac{\hbar m_2\omega_2}{2}}$$

$$e^{\gamma} \equiv \sqrt{\frac{m_1\omega_1}{m_2\omega_2}}$$



15: Quantum Dynamics

$$\hat{x} = \sqrt{\frac{\hbar}{2m_1\omega_1}}\left(\hat{a}_1 + \hat{a}_1^{+}\right)$$

$$= \sqrt{\frac{\hbar}{2m_2\omega_2}}\left(\hat{a}_2 + \hat{a}_2^{+}\right)$$

$$\hat{p} = -i\sqrt{\frac{m_1\omega_1\hbar}{2}}\left(\hat{a}_1 - \hat{a}_1^{+}\right)$$

$$= -i\sqrt{\frac{m_2\omega_2\hbar}{2}}\left(\hat{a}_2 - \hat{a}_2^{+}\right)$$

$$\hat{a}_1 + \hat{a}_1^{+} = e^{\gamma}\left(\hat{a}_2 + \hat{a}_2^{+}\right)$$

$$\hat{a}_1 - \hat{a}_1^{+} = e^{-\gamma}\left(\hat{a}_2 - \hat{a}_2^{+}\right)$$

$$\hat{a}_1 = \hat{a}_2 \cosh\gamma + \hat{a}_2^{+} \sinh\gamma$$

$$\hat{a}_1^{+} = \hat{a}_2^{+} \cos\gamma + \hat{a}_2 \sinh\gamma$$

$$|0\rangle_{(1)} = c_0|0\rangle_{(2)} + c\hat{a}_2^{+}a_2^{+}|0\rangle_{(2)}$$

$$\downarrow$$

$$\hat{a}_1|0\rangle_{(1)} = 0 \longrightarrow \hat{a}_2\cosh\gamma + \hat{a}_2^{+}\sinh\gamma$$

$$|0\rangle_{(1)} = N(\gamma) \; \text{mmm} \; |0\rangle_{(2)}$$

15: Quantum Dynamics

$|0\rangle_{(1)} = N(\gamma) e^{-\frac{1}{2} f(\gamma) \hat{a}_2^\dagger \hat{a}_2^\dagger} |0\rangle_{(2)}$

$(\hat{a}_2 \cosh\gamma + \hat{a}_2^\dagger \sinh\gamma) e^{-\frac{1}{2} f(\gamma) \hat{a}_2^\dagger \hat{a}_2^\dagger} |0\rangle_{(2)} = 0$

$\cosh\gamma [\hat{a}_2, e^{-\frac{1}{2} f \hat{a}_2^\dagger \hat{a}_2^\dagger}] |0\rangle_{(2)} + \sinh\gamma [\hat{a}_2^\dagger, e^{-\frac{1}{2} f a_2^\dagger}] |0\rangle = 0$

$[A, e^B] = [A, B] e^B$

if $[[A, B], B] = 0$

$\cosh\gamma [\hat{a}_2, -\frac{1}{2} f \hat{a}_2^\dagger \hat{a}_2^\dagger] e^{-\frac{1}{2} \cdots} |0\rangle + \sinh\gamma \, a_2^\dagger e^{\frac{1}{2} \cdots} |0\rangle$

$(-\cosh\gamma f a_2^\dagger e^{\cdots} + \sinh\gamma \, a_2^\dagger) e^{\cdots} |0\rangle = 0$

$f = \tanh\gamma$

$= (-\cos\gamma f + \sinh\gamma) a_2^\dagger e^{\cdots} |0\rangle = 0$

$|0\rangle_{(1)} = N(\gamma) \exp(-\frac{1}{2} \tanh\gamma \, \hat{a}_2^\dagger \hat{a}_2^\dagger) |0\rangle_{(2)}$

$1 = N^2 \langle 0| e^{-\frac{1}{2} \tan aa} e^{-\frac{1}{2} \tan a^\dagger a^\dagger} |0\rangle$

$${}_2\langle 0|0\rangle_1 = N(\gamma) \underbrace{{}_{(2)}\langle 0| e^{-\frac{1}{2} \tanh\gamma \, \hat{a}_2^\dagger \hat{a}_2^\dagger} |0\rangle_{(2)}}_{1}$$

15: Quantum Dynamics

$N(\gamma) = {}_2\langle 0|0\rangle_1$

$= \int {}_2\langle 0|x\rangle\langle x|0\rangle_1 \, dx$

$= \int (\psi_0^{(2)}(x))^* \psi_0^{(1)}(x) \, dx$

$= \frac{1}{\sqrt{\cosh\gamma}}$

$|0\rangle_1 = \frac{1}{\sqrt{\cosh\gamma}} \exp\left(-\frac{1}{2}\tanh\gamma \, \hat{a}_2^\dagger \hat{a}_2^\dagger\right)|0\rangle_2$

$H_1 \quad |0\rangle_1 \quad (m, \omega) \quad (\hat{a}, \hat{a}^\dagger)$

$|0\gamma\rangle = \frac{1}{\sqrt{\cosh\gamma}} \exp\left(-\frac{1}{2}\tanh\gamma \, \hat{a}^\dagger \hat{a}^\dagger\right)|0\rangle$

$\Delta x = e^{-\gamma}\sqrt{\frac{\hbar}{2m\omega}}$

$\Delta p = e^{\gamma}\sqrt{\frac{\hbar m\omega}{2}}$

$\langle 0_\gamma | 0_\gamma\rangle = 1$

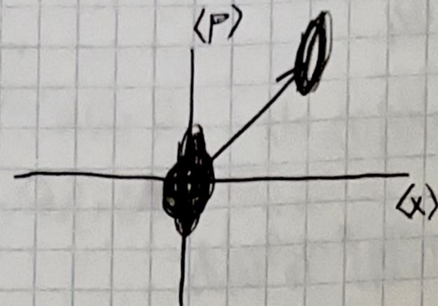
$\frac{1}{\sqrt{\cosh\gamma}} \exp\left(-\frac{1}{2}\tanh\gamma \, \hat{a}^\dagger \hat{a}^\dagger\right) = e^{\frac{\gamma}{2}(\hat{a}^\dagger\hat{a}^\dagger - \hat{a}\hat{a})}|0\rangle$

$|0_\gamma\rangle = S(\gamma)|0\rangle$

$S(\gamma) = e^{-\frac{\gamma}{2}(\hat{a}^\dagger\hat{a}^\dagger - \hat{a}\hat{a})}$

15: Quantum Dynamics

$S(\gamma)|0\rangle D(\alpha) \equiv |\alpha, \gamma\rangle$



$|0\rangle_{\infty} \sim \exp\left(-\frac{1}{2}\hat{a}^{+}\hat{a}^{+}\right)|0\rangle$

$\delta(x)_{|x=0\rangle}$

$\hat{x} = (\hat{a} + \hat{a}^{+}) e^{-\frac{1}{2}\hat{a}^{+}\hat{a}^{+}} |0\rangle = 0$

$|0\rangle_{-\infty} \sim \exp\left(\frac{1}{2}\hat{a}^{+}\hat{a}^{+}\right)|0\rangle \quad |p=0\rangle$

$|x\rangle = N\exp\left(\sqrt{\frac{2m\omega}{\hbar}}\, x\hat{a}^{+} - \frac{1}{2}\hat{a}^{+}\hat{a}^{+}\right)|0\rangle$

$|x\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega x^2}{2\hbar}} \exp\left(\sqrt{\frac{2m\omega}{\hbar}}\, x\hat{a}^{+} - \frac{1}{2}\hat{a}^{+}\hat{a}^{+}\right)|0\rangle$