5.1: Momentum and Energy Operator

$\Psi(x,t) = e^{ikx - i\omega t}$

$p = \hbar k$
$E = \hbar \omega$

Non Reletivistic particles:
$E = \frac{p^2}{2m}$

$\underbrace{\frac{\hbar}{i}\frac{\partial}{\partial x}}_{\text{operator}} \Psi(x,t) = \hbar k \, \Psi(x,t) = p \, \Psi(x,t)$

Momentum operator $\hat{p}$
$\hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x}$
$\hat{p}\Psi(x,t) = p\Psi(x,t)$

If this holds, then:
$\Psi(x,t)$ is an eigenstate of $\hat{p}$

(operator A, φ: $\hat{A}\varphi = \alpha\varphi$)
with eigenvalue p

$\Psi(x,t)$ is a state of definite momentum

$i\hbar \frac{\partial \Psi}{\partial t} = i\hbar(-i\omega)\Psi = \hbar\omega\Psi = E\Psi(x,t)$

$\downarrow$

$E = \frac{p^2}{2m}$

$E\Psi = O\Psi$
(invent an 'O')

$E\Psi = \frac{p^2}{2m}\Psi = \frac{p}{2m}(p \cdot \Psi) = \frac{p}{2m}\frac{\hbar}{i}\frac{\partial \Psi}{\partial x}$
$= \frac{1}{2m}\frac{\hbar}{i}\frac{\partial}{\partial x}\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\Psi\right)$

5.1: Momentum and Energy Operator (continued)

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}=E\psi$$

$\hat{E}=-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$ = Energy operator

The energy operator times wave function is energy times the wave function

$$\hat{E}\psi=E\psi$$

ψ is an energy eigenstate

$$\hat{E}=\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}=\frac{1}{2m}\hat{p}^2$$

$$\hat{p}\hat{p}=\frac{\hbar}{i}\frac{\partial}{\partial x}\frac{\hbar}{i}\frac{\partial}{\partial x}$$

Combines to Schrödinger Equation

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}$$

5.2: Free Schrödinger Equation

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}$$

try $\psi=e^{ikx-i\omega t}$

$$i\hbar(-i\omega)\psi=-\frac{\hbar^2}{2m}(ik)^2\psi$$

$$\hbar\omega=\frac{\hbar^2k^2}{2m}\rightarrow E=\frac{p^2}{2m}$$

$$\psi(x,t)=\int_{-\infty}^{\infty}\Phi(k)e^{ikx-i\omega(k)t}dk$$

$\downarrow$

This solves the Schrödinger Equation

$$v_g=\left.\frac{d\omega}{dk}\right|_{k_0}=\frac{dE}{dp}=\frac{d}{dp}\left(\frac{p^2}{2m}\right)=\frac{p}{m}=v$$

5.2: Free Schrödinger Equation (continued)

Remarks

Ψ can not be real ($\Psi \notin \mathbb{R}$)

not a usual wave equation

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 \phi}{\partial t^2} = 0 \Rightarrow \text{solutions} \in \mathbb{R}$$

5.3: General Schrödinger Equation

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{E}\Psi$$

$$\hat{E} = \frac{\hat{p}^2}{2m}$$

Potential: $V(x,t)$

E = Kinetic + Potential

$$\hat{E} \stackrel{?}{=} \frac{\hat{p}^2}{2m} + V(x,t)$$

$\uparrow$

Hamiltonian $\hat{H}$

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}\Psi$$

$$i\hbar \frac{\partial}{\partial t}\Psi(x,t) = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,t)\right)\Psi = \text{Full Equation}$$

$V(x,t)$ should be thought as an operator

Introduce an operator $\hat{x}$, which acting on functions of x multiplies them by x

$$\hat{x}f(x) \equiv xf(x)$$

Operators:

$\hat{p}$

$\hat{x}$

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}, t)$$

$$\hat{x}\hat{p}\phi - \hat{p}\hat{x}\phi = 0?$$

5.3: General Schrödinger Equation (continued)

$AB\phi \equiv A(B\phi)$

$\hat{x}\hat{p}\phi - \hat{p}\hat{x}\phi = \hat{x}(\hat{p}\,\phi(x,t)) - \hat{p}(\hat{x}\,\phi(x,t))$

$= \hat{x}\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\phi(x,t)\right) - p\,(x\,\phi(x,t))$

$= \frac{\hbar}{i} x \frac{\partial}{\partial x}\phi(x,t) - \frac{\hbar}{i}\frac{\partial}{\partial x}(x\phi) = -\frac{\hbar}{i}\phi = i\hbar\phi$

$(\hat{A}+\hat{B})\phi = \hat{A}\phi + \hat{B}\phi$

$(\hat{x}\hat{p} - \hat{p}\hat{x})\phi = i\hbar\phi$

$\hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar$ Equality between operators

$[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$ = commutator of $\hat{A}$ and $\hat{B}$

$[\hat{x},\hat{p}] = i\hbar$

5.4: Commutators, Matrices and 3-D Schrödinger

Operators	Matrices
Wave functions	Vectors
Eigenstates	Eigenvectors

$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\vec{S} = \frac{\hbar}{2}\sigma$

$\sigma_1\sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$

$\sigma_2\sigma_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$

$[\sigma_1,\sigma_2] = \sigma_1\sigma_2 - \sigma_2\sigma_1 = 2\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = 2i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma^{-3}$

$[\sigma_1,\sigma_2] = 2i\sigma_3$

5.4: Commutators, Matrices, and 3-D Schrodinger (continued)

$p_x = \frac{\hbar}{i}\frac{\partial}{\partial x} \rightarrow p_1 = \frac{\hbar}{i}\frac{\partial}{\partial x_1}$

$p_y = \frac{\hbar}{i}\frac{\partial}{\partial y} \rightarrow p_2 = \frac{\hbar}{i}\frac{\partial}{\partial x_2}$

$p_z = \frac{\hbar}{i}\frac{\partial}{\partial z} \rightarrow p_3 = \frac{\hbar}{i}\frac{\partial}{\partial x_3}$

$e^{i\vec{k}\cdot\vec{x} - i\omega t} \qquad \vec{p} = \hbar\vec{k}$

$p_k = \frac{\hbar}{i}\frac{\partial}{\partial x_k}$
$k = 1, 2, 3$

$\hat{\vec{p}} = \frac{\hbar}{i}\nabla$

$$\hat{\vec{p}}\, e^{i\vec{k}\cdot\vec{x} - i\omega t} = \frac{\hbar}{i}\nabla e^{i\vec{k}\cdot\vec{x} - i\omega t}$$
$$= \frac{\hbar}{i} i\vec{k} e^{\cdots}$$
$$= \hbar\vec{k}\Psi$$

$$\hat{H} = \frac{(\hat{\vec{p}})^2}{2m} + V(\vec{x},t)$$

$$\hat{\vec{p}}^2 = \hat{\vec{p}}\cdot\hat{\vec{p}} = \frac{\hbar}{i}\nabla\cdot\frac{\hbar}{i}\nabla = -\hbar^2\nabla^2$$

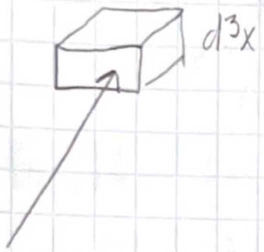
$$i\hbar\frac{\partial}{\partial t} = \left(\frac{\hbar^2}{2m}\nabla^2 + V(\vec{x},t)\right)\Psi(x,t)$$

$$[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$$

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

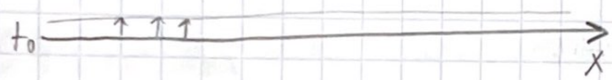
5.5: Interpritation of the Wavefunctron

$\Psi(x,t)$ does not tell how much of the particle is at x at time t but rather what is the probability to find it at x at time t.



$$dP(\vec{x},t) = |\Psi(\vec{x},t)|^2 d^3x$$

single particle: $\int_{all} |\Psi(\vec{x},t)|^2 d^3x = 1$



6.1: Normalizable Wavefunctions

$$i\hbar \frac{\partial \Psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,t)\right)\Psi(x,t)$$

$$|\Psi(x,t)|^2 dx$$

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = 1 \quad \text{Holds for } t = t_0$$

$$\int_{-\infty}^{\infty} \Psi^*(x,t)\,\Psi(x,t)\,dx = 1$$

In order to guarentee this will hold:

$$\lim_{x \to \pm\infty} \Psi(x,t) = 0$$

$\lim_{x \to \pm\infty} \frac{\partial \Psi}{\partial x}$ is bounded

6.1: Normalizable Wavefunctions (continued)

$\Psi(x,t)$

$$\int |\Psi|^2 dx = N \neq 1$$

Ψ is normalizable meaning it can be normalized
Use instead Ψ':

$$\Psi' = \frac{\Psi}{\sqrt{N}}$$

$$\int |\Psi'|^2 dx = \int \frac{|\Psi|^2}{N} dx = \frac{1}{N}\int |\Psi|^2 dx = 1$$

6.2: Hermiticity of the Hamiltonian

If $\int \Psi^*(x,t_0)\Psi(x,t_0)dx = 1$ at $t=t_0$
then it must hold for later times for $t > t_0$

Define: $\rho(x,t)$ = probability density $= \Psi^*(x,t)\Psi(x,t)$

$$N(t) \equiv \int \rho(x,t)\,dx$$

$N(t_0)=1$
Will Schrödinger Equation guarentee $\frac{dN}{dt}=0$?

$$\frac{dN}{dt} = \int \frac{\partial}{\partial t}\rho(x,t)\,dx$$

$$= \int \left(\left(\frac{\partial \Psi^*}{\partial t}\right)\Psi + \Psi^*\left(\frac{\partial \Psi}{\partial t}\right)\right) dx$$

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}\Psi$$

6.2: Hermiticity of the Hamiltonian (continued)

$$\frac{\partial \Psi}{\partial t} = -\frac{i}{\hbar}\hat{H}\Psi$$

$$i\hbar\left(\frac{\partial \Psi}{\partial t}\right)^* = (\hat{H}\Psi)^*$$

$$-i\hbar\frac{\partial \Psi^*}{\partial t} = (\hat{H}\Psi)^*$$

$$\frac{\partial \Psi^*}{\partial t} = \frac{i}{\hbar}(\hat{H}\Psi)^*$$

$$\frac{dN}{dt} = \int \frac{i}{\hbar}\left((\hat{H}\Psi)^*\Psi - \Psi^*(H\Psi)\right)dx$$

For this to be zero:

$$\int(\hat{H}\Psi)^*\Psi\,dx = \int \Psi^*(H\Psi)\,dx$$

True if $\hat{H}$ is a hermitian operator

The hermitian operator $\hat{H}$ satisfies that:

$$\int(\hat{H}\Psi_1)^*\Psi_2\,dx = \int \Psi_1^*\hat{H}\Psi_2\,dx$$

In general, given an operator T, one defines its hermitian conjugate $T^\dagger$ as follows:

$$\int \Psi_1^* T\,\Psi_2\,dx \rightarrow \int (T^\dagger\Psi_1)^*\Psi_2\,dx$$

T is hermitian if $T^\dagger = T$

6.3: Probability Current

$$\frac{dN}{dt} = \int \frac{i}{\hbar}(\hat{H}\psi)^*\psi - \psi^*\hat{H}\psi \, dx$$

$$\frac{i}{\hbar}(\hat{H}\psi)^*\psi - \psi^* H\psi = \frac{\partial\rho}{\partial t}$$

$$\frac{\partial\rho}{\partial t} = \frac{i}{\hbar}\left(-\frac{\hbar^2}{2m}\frac{\partial^2\psi^*}{\partial x^2}\psi + V(x,t)\psi^*\psi + \frac{\hbar^2}{2m}\psi^*\frac{\partial^2\psi}{\partial x^2} - \psi^* V(x,t)\psi\right)$$

$$\frac{\partial\rho}{\partial t} = \frac{-i\hbar}{2m}\left(\frac{\partial^2\psi^*}{\partial x^2}\psi - \psi^*\frac{\partial^2\psi}{\partial x^2}\right)$$

$$\frac{\partial\rho}{\partial t} = -\frac{i\hbar}{2m}\frac{\partial}{\partial x}\left(\frac{\partial\psi^*}{\partial x}\psi - \psi^*\frac{\partial\psi}{\partial x}\right) = -\frac{\partial}{\partial x}\left(\frac{\hbar}{2im}\left(\psi^*\frac{\partial\psi}{\partial x} - \psi\frac{\partial\psi^*}{\partial x}\right)\right)$$

$$z - z^* = 2i\,\mathrm{Im}(z)$$

$$\frac{\partial\rho}{\partial t} = -\frac{\partial}{\partial x}\underbrace{\left(\frac{\hbar}{m}\mathrm{Im}\left(\psi^*\frac{\partial\psi}{\partial x}\right)\right)}_{J(x,t)}$$

$$J(x,t) = \frac{\hbar}{m}\mathrm{Im}\left(\psi^*\frac{\partial\psi}{\partial x}\right)$$

$$\underbrace{\frac{\partial\rho}{\partial t} = -\frac{\partial J}{\partial x} \rightarrow \frac{\partial\rho}{\partial t} + \frac{\partial J}{\partial x} = 0}_{\text{Current Conservation}}$$

$$[\psi] = \frac{1}{\sqrt{L}} \qquad \int|\psi|^2 dx = 1$$

$$\left[\psi^*\frac{\partial\psi}{\partial x}\right] = \frac{1}{L^2}$$

$$[\hbar] = \frac{ML^2}{T}$$

$$\left[\frac{\hbar}{m}\right] = \frac{L^2}{T}$$

$$[J] = \frac{L^2}{T} = \frac{1}{T}$$

6.4: Three Dimensional Current

$\vec{J}(x,t) = \frac{\hbar}{m} \mathrm{Im}(\psi^* \nabla \psi)$

$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0$

$[J] = \frac{1}{L^2} \cdot \frac{1}{T}$ probability per unit area and unit time

$\frac{dN}{dt} = \int_{-\infty}^{\infty} \frac{\partial \rho}{\partial t} dx = -\int_{-\infty}^{\infty} \frac{\partial J}{\partial x} dx = -[J(x=\infty, t) - J(x=-\infty, t)]$

$J = \frac{\hbar}{2im} \underbrace{\left(\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}\right)}_{\to 0}$

$\underbrace{\frac{dN}{dt} = 0}_{\text{Consistency}}$

Two ideas
1) existence of a current for probability
2) $\hat{H}$ is a hermitian operator

	EM	QM
ρ	charge density	probability density
Q	charge in Volume	probability to find particle in volume
J	current density	probability current density

$Q_V(t) = \int_V \rho(x,t) d^3x$

$\frac{dQ}{dt}(t) = \int_V \frac{\partial \rho}{\partial t} d^3x = -\int_V \nabla \cdot \vec{J} d^3x$

6.4: Three Dimensional Current (continued)

$$\frac{dQ}{dt}(t) = -\int_S \vec{J}\cdot d\vec{a} \quad \left(\frac{\partial\rho}{\partial t} + \nabla\cdot\vec{J} = 0\right)$$

$$P_{ab} = \int_a^b \rho(x,t)\,dx$$

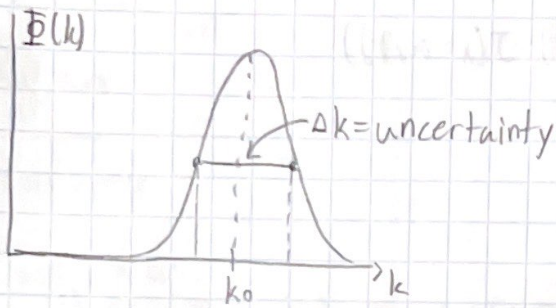
$$\frac{dP_{ab}}{dt} = \int_a^b \frac{\partial\rho}{dt}\,dx = -\int_a^b \frac{\partial J}{\partial x}\,dx = -(J(x=b,t) - J(x=a,t))$$

$$\frac{dP_{ab}}{dt} = -J(b,t) + J(a,t)$$

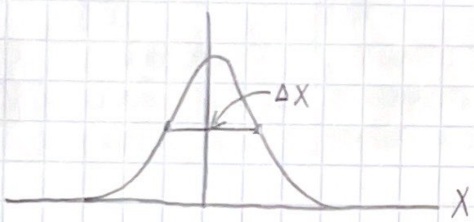
7.1: Wavepackets and Fourier

$$\underbrace{\Psi(x,0)=\frac{1}{\sqrt{2\pi}}\int \Phi(k)e^{ikx}dk}_{\text{Fourier}}$$

$$\underbrace{\Phi(k)=\frac{1}{\sqrt{2\pi}}\int \Psi(x,0)e^{-ikx}dx}_{\text{Reverse Fourier}}$$



$|\Psi(x,0)|$ peaks at $x=0$



Δx = width

7.2: Reality Condition

Is $\Psi(x,0)$ real?

$$(\Psi(x,0))^* = \frac{1}{\sqrt{2\pi}}\int (\Phi(k))^* e^{-ikx} dk$$

$\downarrow$

$k \to -k$

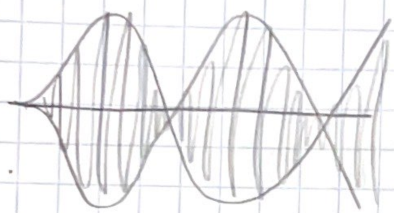
$\downarrow$

$$= \frac{1}{\sqrt{2\pi}}\int \Phi^*(k) e^{ikx} dk$$

$$\stackrel{?}{=} \frac{1}{\sqrt{2\pi}}\int \Phi(k) e^{ikx} dk = \Psi(x,0)$$

Reality requires $\frac{1}{\sqrt{2\pi}}\int (\Phi(k) - \Phi^*(-k)) e^{ikx} dx = 0$

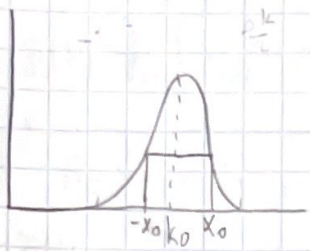
$(\Phi(-k))^* = \Phi(k)$ is condition for reality



2.3: Widths and Uncertainty

$k = k_0 + \tilde{k}$

$\Psi(x,0) = \frac{1}{\sqrt{2\pi}} e^{ik_0 x} \int \Phi(k_0+\tilde{k}) e^{i\tilde{k}x} d\tilde{k}$



Relevant region of integration for $\tilde{k}$ is from $[-\frac{\Delta k}{2}, \frac{\Delta k}{2}]$

For any x different from zero, the phase in the integral will range over $[-\frac{\Delta k}{2}x, \frac{\Delta k}{2}x]$

Total phase excersion is $\Delta k \cdot x$

If $\Delta k \cdot x \ll 1$, then get contribution

If $\Delta k \cdot x \gg 1$, then the contribution goes to 0

$\Psi(x,0)$ = sizable in an interval $x \in (-x_0, x_0)$ for $\Delta k \cdot x_0$ roughly value of 1

$\Delta k x_0 \approx 1$

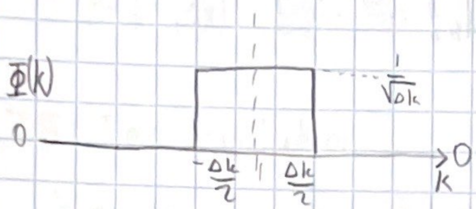
$\Delta x = 2x_0$

$\Delta k \Delta x \approx 1$

7.3: Widths and Uncertainty (continued)

Since $P = \hbar k$, $\Delta p = \hbar \Delta k \Rightarrow \Delta p \Delta x \approx \hbar$

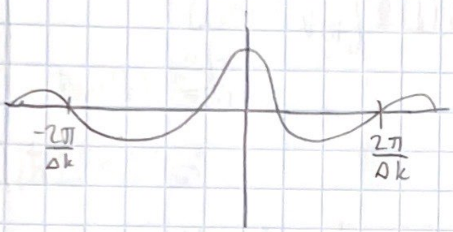
$\Delta x \Delta p \geqslant \frac{\hbar}{2}$



$$\Psi(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{\Delta k}{2}}^{\frac{\Delta k}{2}} \frac{1}{\sqrt{\Delta k}} e^{ikx} dx$$

$$= \frac{1}{\sqrt{2\pi \Delta k}} \left[\frac{e^{ikx}}{ix} \Big|_{-\frac{\Delta k}{2}}^{\frac{\Delta k}{2}} \right.$$

$$= \frac{\Delta k}{2\pi} \frac{\sin\left(\frac{\Delta k x}{2}\right)}{\left(\frac{\Delta k x}{2}\right)}$$



$\Delta x = \frac{2\pi}{\Delta k}$

$\Delta x \Delta k \approx 2\pi$

7.4: Shape Changes in a Wave

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int \Phi(k) e^{ikx} e^{-i\omega(k)t} dk$$

$$\omega(k) = \omega(k_0) + (k-k_0)\frac{d\omega}{dk}\Big|_{k_0} + \frac{1}{2}(k-k_0)^2 \frac{d^2\omega}{dk^2}\Big|_{k_0} + \dots$$

$$\frac{d\omega}{dk} = \frac{dE}{dt} = \frac{p}{m} = \frac{\hbar k}{m}$$

$$\frac{d^2\omega}{dk^2} = \frac{\hbar}{m}$$

$$\frac{d^3\omega}{dk^3} = 0$$

$$e^{-i\omega(k)t} = e^{\dots - i\frac{1}{2}(k-k_0)^2 \frac{\hbar}{m} t}$$

$$(k-k_0)^2 \frac{\hbar}{m}|t| \ll 1$$

$$(\Delta k)^2 \frac{\hbar |t|}{m} \ll 1$$

$$\frac{(\Delta p)^2 |t|}{\hbar m} \ll 1$$

$$|t| \ll \frac{\hbar m (\Delta x)^2}{\hbar^2}$$

$$\Delta x \Delta p \simeq \hbar$$

$$\frac{\Delta p |t|}{m} \leq \frac{\hbar}{\Delta p}$$

$$\frac{\Delta p}{m}|t| \ll \Delta x$$

7.4: Shape Changes in a Wave (continued)

$\Delta x = 10^{-10}$ m for an electron, how long does it remain localized

$$t \approx \frac{m}{\hbar}(\Delta x)^2 = \frac{mc^2}{\hbar c} \cdot \frac{(\Delta x)^2}{c} = 10^{-16} \text{sec}$$

7.5: Time Evolution

Suppose you know $\Psi(x,0)$:

1) calculate $\Phi(k)$

$$\Phi(k) = \frac{1}{\sqrt{2\pi}} \int \Psi(x,0) e^{-ikx} dx$$

2) With this rewrite

$$\Psi(x,0) = \frac{1}{\sqrt{2\pi}} \int \Phi(k) e^{ikx} dk$$

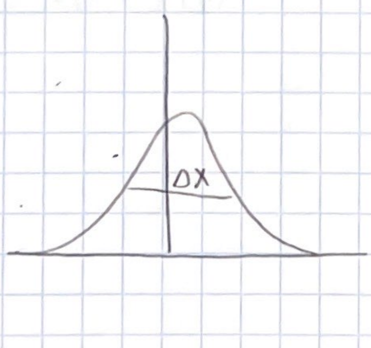
3)
$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int \Phi(k) e^{i(kx-\omega(k)t)} dk$$

$$\hbar\omega(k) = \frac{\hbar^2 k^2}{2m}$$

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2}$$

4) Do the k integral

$$\Psi_a(x,0) = \frac{e^{-\frac{x^2}{4a^2}}}{(2\pi)^{\frac{1}{4}} \sqrt{a}}$$



$\Delta x \simeq a$

$$\tau = \frac{2ma^2}{\hbar}$$

8.1: Fourier Transforms and Delta Functions

Uncovering Momentum Space

$$\Psi(x) = \frac{1}{\sqrt{2\pi}} \int \Phi(k) e^{ikx} dk$$

$$\Phi(k) = \frac{1}{\sqrt{2\pi}} \int \Psi(x) e^{-ikx} dx$$

$\Phi(k)$ has the same information as $\Psi(x)$

$\Phi(k)$ is the weight of the planewaves in the superposition

$$\Psi(x) = \frac{1}{\sqrt{2\pi}} \int e^{ikx} dk \frac{1}{\sqrt{2\pi}} \int \Psi(x') e^{-ikx'} dx'$$

$$\Psi(x) = \int_{-\infty}^{\infty} \Psi(x') dx' \underbrace{\frac{1}{2\pi} \int e^{ik(x-x')} dk}_{\delta(x'-x)}$$

Delta function

$k \to -k$

$$\frac{1}{2\pi} \int e^{ik(x'-x)} dk$$

$$\delta(x-x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x')} dk$$

prop
$$\begin{cases} \delta(-x) = \delta(x) \\ \delta(ax) = \frac{1}{|a|} \delta(x) \end{cases}$$

8.2: Parseval Identity

$$\int \underbrace{\Psi^*(x)}_{\downarrow\, k} \underbrace{\Psi(x)}_{\downarrow\, k'} dx$$

$$= \int dx \frac{1}{\sqrt{2\pi}} \int \Phi^*(k) e^{-ikx} dk \, \frac{1}{\sqrt{2\pi}} \int \Phi(k') e^{ik'x} dk'$$

$$= \int \Phi^*(k) dk \int \Phi(k') dk' \underbrace{\frac{1}{2\pi} \int e^{i(k'-k)x} dx}_{\delta(k'-k)}$$

$$= \int \Phi^*(k) \Phi(k) dk$$

Parseval's Theorem:

$$\int |\Psi(x)|^2 dx = \int |\Phi(k)|^2 dk$$

Go to momentum language:

$$p = \hbar k$$

$$\frac{dp}{\hbar} = dk$$

$$\Phi(k) = \tilde{\Phi}(p)$$

8.2: Parseval Identity (continued)

$$\Psi(x) = \frac{1}{\sqrt{2\pi}} \int \tilde{\Phi}(p) e^{\frac{ipx}{\hbar}} \frac{dp}{\hbar}$$

$$\tilde{\Phi}(p) = \frac{1}{\sqrt{2\pi}} \int \Psi(x) e^{-\frac{ipx}{\hbar}} dx$$

Let $\tilde{\Phi}(p) \to \Phi(p)\sqrt{\hbar}$

$$\left.\begin{aligned} \Psi(x) &= \frac{1}{\sqrt{2\pi\hbar}} \int \Phi(p) e^{\frac{ipx}{\hbar}} dp \\ \Phi(p) &= \frac{1}{\sqrt{2\pi\hbar}} \int \Psi(x) e^{\frac{-ipx}{\hbar}} dx \end{aligned}\right\} \text{Fourier in momentum}$$

Parseval:

$$\int |\Psi(x)|^2 dx = \int |\Phi(p)|^2 dp$$

8.3: Three Dimensional Fourier

$|\Phi(p)|^2 dp$ = probability to find particle with momentum in the range $[p, p+dp]$

$$\left.\begin{aligned} \Psi(\vec{x}) &= \frac{1}{(2\pi\hbar)^{\frac{3}{2}}} \int \Phi(\vec{p}) e^{\frac{i\vec{p}\cdot\vec{x}}{\hbar}} d^3\vec{p} \\ \Phi(\vec{p}) &= \frac{1}{(2\pi\hbar)^{\frac{3}{2}}} \int \Psi(\vec{x}) e^{\frac{-i\vec{p}\cdot\vec{x}}{\hbar}} d^3\vec{x} \end{aligned}\right\} \text{3-D case}$$

8.3: Three Dimensional Fourier

$$\delta^3(x-x') = \frac{1}{(2\pi)^3}\int e^{i\vec{k}(\vec{x}-\vec{x}')} d^3\vec{k} \quad \Big\} \text{ 3-D Delta}$$

$$\int |\psi(\vec{x})|^2 d^3\vec{x} = \int |\Phi(\vec{p})|^2 d^3\vec{p} \quad \Big\} \text{ 3-D Parseval}$$

8.4: Expectatron Values of Operators

Random variable (Q) which can take values in the set $\{Q_1 \ldots Q_n\}$ with probabilities $\{p_1 \ldots p_n\}$

$$\langle Q \rangle \equiv \sum_{i=1}^{n} Q_i P_i$$

In a quantum system $\Psi^*(x,t)\Psi(x,t)\,dx$ is the probubility that the particle is in $(x, x+dx)$

$$\langle \hat{x} \rangle \equiv \int x \Psi^*(x,t)\Psi(x,t)\,dx$$

↑
expectation value of position

May depend on time

$|\Phi(p)|^2 dp$ is probability to find the particle with momentum in the range $(p, p+dp)$

$$\langle p \rangle \equiv \int p |\Phi(p)|^2 dp$$

$$\equiv \int p \underbrace{\Phi^*(p)}_{x'} \underbrace{\Phi(p)}_{x} dp$$

$$= \int_{-\infty}^{\infty} p\,dp = \int \frac{dx'}{\sqrt{2\pi\hbar}} e^{\frac{ipx'}{\hbar}} \psi^*(x')$$

8.4: Expectation Values of Operators (continued)

$$\int \frac{dx}{\sqrt{2\pi\hbar}} e^{\frac{-ipx}{\hbar}} \Psi(x)$$

$$= \frac{1}{2\pi\hbar} \int \Psi^*(x')\,dx' \int \Psi(x)\,dx \int_{-\infty}^{\infty} p\, e^{\frac{ipx'}{\hbar}} e^{\frac{-ipx}{\hbar}}\,dp$$

$$\int \left(-\frac{\hbar}{i}\frac{\partial}{\partial x}\right) e^{\frac{ipx'}{\hbar}} e^{-\frac{ipx}{\hbar}}\,dp$$

$$= \int \Psi^*(x')\,dx' \int \Psi(x) \left(-\frac{\hbar}{i}\frac{\partial}{\partial x}\right) dx \underbrace{\frac{1}{2\pi\hbar}\int e^{ip(x'-x)/\hbar}\,dp}_{\delta(x'-x)}$$

Integrate by parts

$$\int \Psi^*(x')\,dx \int \frac{\hbar}{i}\frac{\partial \Psi}{\partial x}\delta(x'-x)\,dx$$

$$= \int \frac{\hbar}{i}\frac{\partial \Psi}{\partial x}\,dx \int \Psi^*(x')\,\delta(x-x')\,dx'$$

$$\langle p \rangle \equiv \int p |\Phi(p)|^2\,dp$$

$$= \int \frac{\hbar}{i}\frac{\partial \Psi}{\partial x}\Psi^*(x)\,dx$$

$$= \int \Psi^*(x)\frac{\hbar}{i}\frac{\partial}{\partial x}\Psi(x)\,dx$$

$$\langle p \rangle \equiv \int p |\Phi(p,t)|^2\,dp = \int \Psi^*(x)\,\hat{p}\,\Psi(x)\,dx$$

$\hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x}$ = momentum operator

6.4: Expectation Values of Operators (continued)

In general, for an operator $\hat{Q}$:

$$\langle \hat{Q} \rangle = \int \Psi^*(x,t)\left(\hat{Q}\,\Psi(x,t)\right)dx$$

↑
Time dependent

Kinetic operator

$$\hat{T} = \frac{\hat{p}^2}{2m} \qquad \langle T \rangle = ?$$

$$\langle \hat{T} \rangle = \int \Psi^*(x,t)\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t)\right)dx$$

$$\langle \hat{T} \rangle = \int \frac{p^2}{2m}|\Phi(p)|^2 dp$$

$$= \int \Phi^*(p)\,\frac{p^2}{2m}\,\Phi(p)\,dp$$

$$= \int \frac{\hbar^2}{2m}\left|\frac{\partial \Psi}{\partial x}\right|^2 dx$$

6.5: Time Dependence

$$\frac{d}{dt}\langle \hat{Q} \rangle = \frac{d}{dt}\int \Psi^*(x,t)\, Q\, \Psi(x,t)\, dx$$

$$= \int \frac{\partial \Psi^*}{\partial t} Q \Psi(x,t) + \Psi^* Q \frac{\partial \Psi}{\partial t}\, dx$$

$$= \int \frac{i}{\hbar}(\hat{H}\Psi)^* Q \Psi(x,t) - \frac{i}{\hbar}\Psi^* Q \hat{H}\Psi(x,t)\, dx$$

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}\Psi$$

$$i\hbar \frac{d}{dt}\langle \hat{Q} \rangle = \int \Psi^* Q \hat{H}\Psi - (\hat{H}\Psi)^* Q \Psi\, dx$$

$$= \int \Psi^* Q \hat{H}\Psi - \Psi^* \hat{H} Q \Psi\, dx$$

$$i\hbar \frac{d}{dt}\langle \hat{Q} \rangle = \int \Psi^* \underbrace{(Q\hat{H} - \hat{H}Q)}_{[Q,H]} \Psi(x,t)\, dx$$

$$i\hbar \frac{d}{dt}\langle \hat{Q} \rangle = \langle [Q,H] \rangle$$

$$\hat{H} = \frac{\hat{p}^2}{2m}$$