10: Uncertainty

$\Delta A(\Psi) \equiv \| (A - \langle A \rangle \mathbb{1}) \|$

$(\Delta A(\Psi))^2 = \langle A^2 \rangle_\Psi - \langle A \rangle_\Psi^2 \geq 0$

$|\Psi\rangle = |+\rangle = |z;+\rangle$

$\Delta S_x = ?$

$\langle S_x \rangle = \langle + | S_x | + \rangle$

$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$S_x|+\rangle = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{\hbar}{2}|-\rangle$

$\langle S_x^2 \rangle = \left(\frac{\hbar}{2}\right)\langle \mathbb{1} \rangle = \left(\frac{\hbar}{2}\right)^2$

$(\Delta S_x)^2 = \left(\frac{\hbar}{2}\right)^2$

$\Rightarrow \Delta S_x = \frac{\hbar}{2}$

Uncertainty Principle

Given two hermitian operators A, B and a state ψ (normalized)

$(\Delta A(\psi))^2(\Delta B(\psi))^2 \geq \left(\langle \psi | \underbrace{\frac{1}{2i}[A,B]}_{\text{operator}} | \psi \rangle\right)^2$

$[\hat{x}, \hat{p}] = i\hbar \mathbb{1}$

$\left(\frac{1}{i}[A,B]\right)^\dagger = \left(\frac{1}{i}(AB - BA)\right)^\dagger = \frac{1}{-i}(B^\dagger A^\dagger - A^\dagger B^\dagger)$

$= -\frac{1}{i}(BA - AB)$

$= \frac{1}{i}[A,B] \Rightarrow$ Hermitian

10: Uncertainty

$\Delta A\,\Delta B \geq |\langle\psi|\frac{1}{2i}[A,B]|\psi\rangle|$

$A=\hat{x}\,,\ B=\hat{p}$

$[\hat{x},\hat{p}]=i\hbar\mathbb{1}$

$(\Delta x)^2(\Delta p)^2 \geq (\langle\psi|\frac{1}{2i}i\hbar\mathbb{1}|\psi\rangle)^2=(\frac{\hbar}{2})^2$

$\Delta x\,\Delta p \geq \frac{\hbar}{2}$

$|f\rangle \equiv (A-\langle A\rangle\mathbb{1})|\psi\rangle$

$|g\rangle \equiv (B-\langle B\rangle\mathbb{1})|\psi\rangle$

$(\Delta A)^2=\langle f|f\rangle,\quad (\Delta B)^2=\langle g|g\rangle$

$|f||g| \geq |\langle f|g\rangle|$

$\langle f|f\rangle\langle g|g\rangle \geq |\langle f|g\rangle|^2 \equiv$ schwarz

$(\Delta A)^2(\Delta B)^2 \geq (Re(\langle f|g\rangle))^2+(Im(\langle f|g\rangle))^2$

$\langle f|g\rangle=\langle\psi|\underbrace{(A-\langle A\rangle)}_{\check{A}}\underbrace{(B-\langle B\rangle)}_{\check{B}}|\psi\rangle$

$=\langle\psi|\check{A}\,\check{B}|\psi\rangle-\langle A\rangle\langle B\rangle$

$\langle g|f\rangle=\langle\psi|BA|\psi\rangle-\langle A\rangle\langle B\rangle$

10: Uncertainty

$\mathrm{Im}\langle f, g\rangle = \frac{1}{2i}(\langle f|g\rangle - \langle g|f\rangle)$

$= \frac{1}{2i}\langle\Psi|[A,B]|\Psi\rangle$

$\mathrm{Re}\langle f, g\rangle = \frac{1}{2}(\langle f|g\rangle + \langle g|f\rangle)$

$= \frac{1}{2}\langle\Psi|\{\check{A},\check{B}\}|\Psi\rangle$

$(\Delta A)^2(\Delta B)^2 \geq (\langle\Psi|\frac{1}{2i}[A,B]|\Psi\rangle)^2 + (\langle\Psi|\frac{1}{2}\{\check{A},\check{B}\}|\Psi\rangle)^2$

$(\Delta A)^2(\Delta B)^2 \geq (\langle\Psi|\frac{1}{2i}[A,B]|\Psi\rangle)^2$

Need schwarz saturated $\Rightarrow |g\rangle = \beta|f\rangle,\ \beta \in \mathbb{C}$

$\mathrm{Re}\langle f|g\rangle + \langle g|f\rangle = 0$

$\beta\langle f|f\rangle + \beta^*\langle f|f\rangle = 0$

$\beta + \beta^* = 0 \Rightarrow \mathrm{Re}(\beta) = 0$

β is purely imaginary $\Rightarrow \beta = i\lambda,\ \lambda \in \mathbb{R}$

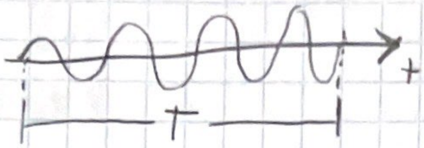
$(B - \langle B\rangle)|\Psi\rangle = i\lambda(A - \langle A\rangle)|\Psi\rangle$

Take norm: $\Delta B = |\lambda|\Delta A(\Psi)$

$\lambda = \pm\frac{\Delta B(\Psi)}{\Delta A(\Psi)}$

10: Uncertainty

Energy time uncertainty



$N = \#$ of waves $= \dfrac{T}{\left(\frac{2\pi}{\omega}\right)} = \dfrac{\omega T}{2\pi}$

$\Delta N \cong 1 \rightarrow \dfrac{\Delta\omega T}{2\pi} = 1$

$\Delta \omega T = 2\pi$

$E = \hbar\omega$

$\hbar \Delta\omega = \Delta E$

$\Delta E T = 2\pi\hbar$

$A = H,\ B = Q(\hat{x}, \hat{p})$

Q has no explicit time dependence

$$(\Delta H)^2 (\Delta Q)^2 \geq \left(\langle \Psi | \tfrac{1}{2i} \underbrace{[H, Q]}_{\frac{dQ}{dt}} | \Psi \rangle \right)^2$$

$\langle Q \rangle = \langle \Psi, Q\Psi \rangle$

$\dfrac{d}{dt}\langle Q \rangle = \left\langle \dfrac{\partial\Psi}{\partial t}, Q\Psi \right\rangle + \left\langle \Psi, Q\dfrac{\partial\Psi}{\partial t} \right\rangle$

10. Uncertainty

$i\hbar \frac{\partial \Psi}{\partial t} = H\Psi$

$$= \langle \tfrac{1}{i\hbar} H\Psi, Q\Psi \rangle + \langle \Psi, Q \tfrac{1}{i\hbar} H\Psi \rangle$$

$$= \tfrac{1}{-i\hbar} \langle \Psi, HQ\Psi \rangle + \tfrac{1}{i\hbar} \langle \Psi, QH\Psi \rangle$$

$$= \tfrac{i}{\hbar} \langle \Psi, (HQ - QH)\Psi \rangle$$

$$\frac{d}{dt}\langle Q \rangle = \frac{i}{\hbar} \langle \Psi | [H, Q] | \Psi \rangle$$

$$(\Delta H)^2 (\Delta Q)^2 \geq \left(\frac{1}{2i} \frac{\hbar}{i} \frac{d}{dt} \langle Q \rangle \right)^2$$

$$\geq \left(\frac{\hbar}{2}\right)^2 \left(\frac{d\langle Q \rangle}{dt} \right)^2$$

$$\Delta H \Delta Q \geq \frac{\hbar}{2} \left| \frac{d\langle Q \rangle}{dt} \right|$$

$$\frac{\Delta Q}{\left| \frac{d\langle Q \rangle}{dt} \right|} = \Delta t$$

$\approx$ time needed for $\langle Q \rangle$ to change by ΔQ

$$\Delta H \Delta t \geq \frac{\hbar}{2}$$

Δt takes $\Psi(x,t)$ to become orthogonal to $\Psi(x,0)$:

$$\Delta t \Delta E \geq \frac{\hbar}{4}$$

10: Uncertainty

$Q = H$

$$\frac{d}{dt}\langle H\rangle = \frac{i}{\hbar}\langle [H, H]\rangle = 0$$

$Q = H^2$

$$\frac{d}{dt}\langle H^2\rangle = \frac{i}{\hbar}\langle [H, H^2]\rangle = 0$$

$$\frac{d}{dt}(\Delta H^2) = \frac{d}{dt}(\langle H^2\rangle - \langle H\rangle^2)$$

$$= 0$$

ΔH is constant

Excited state of an atom $\rightarrow$ ground state (photon emits)

Lifetime τ

$\Delta E \tau \sim \frac{\hbar}{2}$

1420 MHz

$\frac{\Delta\lambda}{\lambda} \sim 10^{-8}$

II: Uncertainty

$\langle \Psi(0) | \Psi(t) \rangle$

$\left(\int \Psi^*(0,\vec{x}) \Psi(t,\vec{x}) d\vec{x} \right)$

$|\langle \Psi(0) | \Psi(t) \rangle|^2$

time independent Hamiltonian

$\Rightarrow \Delta H, t \ldots \rightarrow + O(t^3)$

$H = \frac{p^2}{2m} + \alpha x^4$

$\langle H \rangle_g = \frac{\langle p^2 \rangle_g}{2m} + \alpha \langle x^4 \rangle_g$

$\langle p \rangle_g = \frac{\hbar}{i} \int \psi \frac{\partial}{\partial x} \psi dx = 0$

$\langle x \rangle_g = 0$

$(\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2$

$(\Delta p)^2_g = \langle p^2 \rangle_g$

$(\Delta x)^2_g = \langle x^2 \rangle_g$

$\langle x^4 \rangle_g \geq \langle x^2 \rangle^2_g = (\Delta x)^4_g$

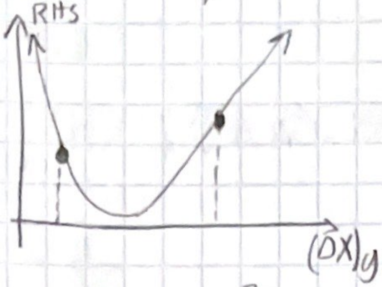
$\langle H \rangle_{gs} = \frac{(\Delta p)^2_g}{2m} + \alpha \langle x^4 \rangle_g \geq \frac{(\Delta p)^2_g}{2m} + \alpha (\Delta x)^4_g$

$\Delta p \Delta x \geq \frac{\hbar}{2}$

$(\Delta p)_g (\Delta x)_g \geq \frac{\hbar}{2} \Rightarrow (\Delta p)_g \geq \frac{\hbar}{2(\Delta x)_g}$

$\langle H \rangle_g \geq \frac{\hbar^2}{8m(\Delta x)^2_g} + \alpha (\Delta x)^4_g$

11: Uncertainty



$$\langle H\rangle_g \geq \min_{\Delta X}\left(\frac{\hbar^2}{8m(\Delta X)^2}+\alpha(\Delta X)^4\right)$$

$\frac{A}{x^2}+Bx^4$ is minimized $x^2 = \frac{1}{2^{\frac{1}{3}}}\left(\frac{A}{B}\right)^{\frac{1}{3}}$

at $2^{\frac{1}{3}}\left(\frac{3}{2}\right)A^{\frac{2}{3}}B^{\frac{1}{3}}$

$$\langle H\rangle_g \geq 2^{\frac{1}{3}}\frac{3}{8}\left(\frac{\hbar^2\sqrt{\alpha}}{m}\right)^{\frac{2}{3}} \approx 0.4724\left(\frac{\hbar\sqrt{\alpha}}{m}\right)^{\frac{2}{3}}$$

↓

0.668

$$\langle\psi, p\psi\rangle = \int \psi(x)\frac{\hbar}{i}\frac{\partial}{\partial x}\psi\, dx$$

$$=\frac{\hbar}{2i}\int\frac{\partial}{\partial x}(\psi^2)\, dx$$

$$=\frac{\hbar}{2i}(\psi^2)\Big|_{-\infty}^{\infty}$$

11: Uncertainty

Diagonalization of Operators

If can find a basis $\in V$ where the matrix representing the operator is diagonal, the operator is diagonalizable

T is a diagonal in $u_1 \cdots u_n$

$T u_i = \sum_k T_{ki} u_i$

$T u_1 = \lambda_1 u_1$
$T u_2 = \lambda_2 u_2$
$\vdots$
$T u_n = \lambda_n u_n$

T is diagonalizable if it has a set of eigenvectors that span V

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\lambda = 0$ $T\begin{pmatrix} a \\ b \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix} \Rightarrow b = 0$

$\lambda = 0 \Rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$V \to$ basis $(u_1 \cdots u_n)$ of $T \to T_{ij}(\{v\})$

$u_k = A v_k$

$T(\{u\}) = A^{-1} T(\{v\}) A$

$T_{ij}(\{u\}) = (A^{-1})_{ik} T_{kp}(\{v\}) A_{pj}$

$|$ TA

11: Uncertainty

$A^{-1}TA$ is diagonal in origional basis

$Tu_i = \lambda_i u_i$ (i is not summed)

$TAv_i = \lambda_i Av_i$

$A^{-1}TAv_i = \lambda_i v_i$

u_k are eigenvectors

$u = Av_k = \sum_i A_{ik} v_i$

$u_k = A_{1k}v_1 + \dots + A_{nk}v_n = \begin{pmatrix} A_{1k} \\ A_{2k} \\ \vdots \\ A_{nk} \end{pmatrix} \quad v_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$

Unitarily Diagonalizeable

T has an orthonormal basis of eigenvectors

$T(\{v\}) = U^{+}T(\{v\})U$

M is normal if $[M^{+}, M] = 0$

$U^{+}U = UU^{+} = 1$

If M is normal, W is an eigenvector such that $MW = \lambda W$ $(\lambda \in \mathbb{C})$

Then $M^{+}W = \lambda^{*}W$

Let M be an operator in $\mathbb{C}$ vector space the vector space has an orthonormal basis of eigenvectors of M if and only if M is normal

11: Uncertainty

Suppose unitarily diagonalizable

$$V^{+} M V = D_m$$

$$M = V D_m V^{+}$$

$$M^{+} M = V D_m^{+} D_m V^{+}$$

$$M M^{+} = V D_m D_m^{+} V^{+}$$

$$[M^{+}, M] = V(D_m^{+} D_m - D_m D_m^{+}) V^{+}$$

$$\left(\begin{array}{c|c} \lambda & 0 \\ \hline 0 & \end{array}\right)$$

Simultaneous diagonalization of hermitian operators

$$S, T \in \mathcal{L}(V)$$

$$[S, T] = 0$$

If S and T are commuting hermitian operators they can be simultaneously diagonalized.

$$S \to S T u_i = \lambda S u_i$$

$$U_k = \{u \mid S u = \lambda_k u\} \quad \dim(U_k) = d_k$$

$$(u_1^k, u_2^k, \ldots u_{d_k}^k)$$

$$V = U_1 \oplus U_2 \oplus \ldots \oplus U_m$$

$$S = \mathrm{diag}(\underbrace{\lambda_1 \ldots \lambda_1}_{d_1} \ldots \underbrace{\lambda_m \ldots \lambda_m}_{d_m})$$

12: Quantum Dynamics

Harmonic Oscilator

$E = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$

$H = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2 \hat{x}^2$

$[\hat{x}, \hat{p}] = i\hbar$

Wavelength $\mathcal{H}$

$\mathcal{H}$ = intragrable functions $\in \mathbb{C}$ on real line

$H = \frac{1}{2}m\omega^2\left(\hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2}\right)$

$V^\dagger V$

$H = \frac{1}{2}m\omega^2\left(\underbrace{\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right)}_{V^\dagger}\underbrace{\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right)}_{V} + \frac{\hbar}{m\omega} + \mathbb{1}\right)$

$H = \frac{1}{2}m\omega^2 V^\dagger V + \frac{1}{2}\hbar\omega \mathbb{1}$

$\langle\psi|H|\psi\rangle = \frac{1}{2}m\omega^2 \underbrace{\langle\psi|V^\dagger V|\psi\rangle}_{|V\psi|^2} + \frac{1}{2}\hbar\omega$

$\langle\psi|H|\psi\rangle \geq \frac{1}{2}\hbar\omega$

ψ: $H|\psi\rangle = E|\psi\rangle$

$E \geq \frac{1}{2}\hbar\omega$

$a \equiv \sqrt{\frac{m\omega}{2\hbar}}\, V$, $a^\dagger \equiv \sqrt{\frac{m\omega}{2\hbar}}\, V^\dagger$

$[a, a^\dagger] = 1$

12: Quantum Dynamics

$x = \sqrt{\frac{\hbar}{2m\omega}}(a+a^{+})$

$\hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}(a^{+}-a)$

$H = \hbar\omega(a^{+}a+\frac{1}{2})$

$N \equiv a^{+}a$ = number operator (eigenvalues of numbers

$H = \hbar\omega(\hat{N}+\frac{1}{2})$

$E = \hbar\omega(N+\frac{1}{2})$

$[\hat{H},\hat{a}] = -\hbar\omega a$

$[\hat{H},\hat{a}^{+}] = \hbar\omega a^{+}$

Assume $|E\rangle$

$\hat{H}|E\rangle = E|E\rangle \Rightarrow \langle E|E\rangle > 0$

a^{+} = creation a = distruction

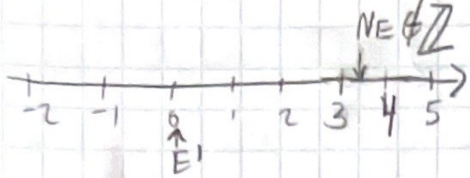
$|E_{+}\rangle = a^{+}|E\rangle$, $|E_{-}\rangle = a|E\rangle$

$E_{+} = E + \hbar\omega \quad E_{-} = E - \hbar\omega$

$\langle a^{+}E|a^{+}E\rangle = \langle E|aa^{+}|E\rangle = \langle E|1+\hat{N}|E\rangle = (1+N_E)\langle E|E\rangle$

$\langle aE|aE\rangle = \langle E|a^{+}a|E\rangle = N_E\langle E|E\rangle$

12: Quantum Dynamics

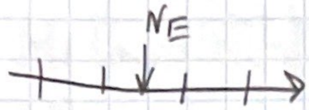


$\langle aE'|aE'\rangle = N_{E'}\langle E'|E'\rangle$

$\langle aE'|aE'\rangle = 0$

$a|E'\rangle = 0$

$N_E \notin \mathbb{Z}$



$a|E\rangle = 0$

$\rightarrow a^{\dagger}a|E\rangle = 0$

$N_E = 0$

$\langle x|a|E\rangle = 0$

$\sqrt{\frac{m\omega}{2\hbar}}\langle x|\left(x + \frac{i\hat{p}}{m\omega}\right)|E\rangle = 0$

$x\psi_E(x) + \frac{i}{m\omega}\frac{\hbar}{i}\frac{d}{dx}\psi_E(x) = 0$

$\left(x + \frac{\hbar}{m\omega}\frac{d}{dx}\right)\psi_E(x) = 0$

$\psi_E(x) = N_0 \exp\left(-\frac{m\omega}{2\hbar}x^2\right)$

$|0\rangle$

$\uparrow$

$N = 0$

$\psi_E(x) = \langle x|0\rangle$

12: Quantum Dynamics

$E = \frac{1}{2}\hbar\omega$

$|1\rangle = a^\dagger|0\rangle$

$\hat{N}a^\dagger|0\rangle = a^\dagger a a^\dagger|0\rangle$

$= a^\dagger|0\rangle$

$\Rightarrow \hat{N} = 1$

$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle$

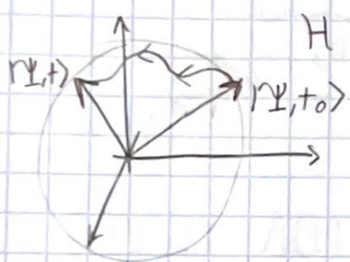
$\langle m|n\rangle = \delta_{mn}$

$H = u_0 \oplus u_1 \oplus u_2 \oplus \cdots$

$u_k = \{\alpha|n\rangle ; |\hat{N}|n\rangle = n|n\rangle\}$

$(\Delta x)^2_{|n\rangle} = ??$

$(\Delta p)^2_{|n\rangle} = ??$



12: Quantum Dynamics

$$|\Psi,t\rangle = \underbrace{U(t,t_0)}_{\text{unitary}}|\Psi,t_0\rangle \quad \forall t,t_0$$

$$(U(t,t_0))^{\dagger}(U(t,t_0)) = \mathbb{1}$$

$$U^{\dagger}(t,t_0)\,U(t,t_0) = \mathbb{1}$$

$$U(t_0,t_0) = \mathbb{1} \quad \forall t_0$$

$$|\Psi,t_2\rangle = U(t_2,t_1)|\Psi,t_1\rangle = U(t_2,t_1)U(t_1,t_0)|\Psi,t_0\rangle$$

$$U(t_2,t_0) = U(t_2,t_1)\,U(t_1,t_0)$$

$$t_2 = t_0,\ t_1 = t$$

$$\mathbb{1} = U(t_0,t)\,U(t,t_0) = \mathbb{1}$$

$$U^{-1}(t,t_0) = U(t_0,t)$$
$$= U^{\dagger}(t,t_0)$$

$$\frac{\partial}{\partial t}|\Psi,t\rangle = \frac{\partial}{\partial t}U(t,t_0)|\Psi,t_0\rangle$$
$$= \frac{\partial U(t,t_0)}{\partial t}U(t_0,t)|\Psi,t\rangle$$
$$= \underbrace{\frac{\partial U}{\partial t}(t,t_0)\,U^{\dagger}(t,t_0)}_{\Lambda(t,t_0)}|\Psi,t\rangle$$

$$\frac{\partial}{\partial t}|\Psi,t\rangle = \Lambda(t,t_0)|\Psi,t\rangle$$

12: Quantum Dynamics

$\Lambda(t,t_0)$ is antihermitian

$$\Lambda^+ = u(t,t_0)\frac{\partial u^+(t,t_0)}{\partial t}$$

$$u(t,t_0)u^+(t,t_0) = \mathbb{1}$$

$$\underbrace{\frac{\partial u}{\partial t}(t,t_0)u^+(t,t_0)}_{\Lambda} + \underbrace{u(t,t_0)\frac{\partial u^+}{\partial t}(t,t_0)}_{\Lambda} = 0$$

$$\Lambda(t,t_0) = \Lambda(t,t_1) \Rightarrow \frac{\partial \Lambda}{\partial t_0} = 0$$

$$= \frac{\partial u(t,t_0)}{\partial t}u^+(t,t_0)$$

$$= \frac{\partial u}{\partial t}(t,t_0)u(t_0,t_1)u^+(t_0,t_1)u^+(t,t_0)$$

$$= \frac{\partial}{\partial t}(u(t,t_0)u(t_0,t_1))u(t_1,t_0)u(t_0,t)$$

$$= \frac{\partial}{\partial t}u(t,t_1)\underbrace{u(t_1,t)}_{u^+(t,t_1)}$$

$$= \Lambda(t,t_1)$$

$$\frac{\partial}{\partial t}|\Psi,t\rangle = \Lambda(t)|\Psi,t\rangle$$

$$i\hbar\Lambda(t) \equiv H(t)$$

$$i\hbar\frac{\partial}{\partial t}|\Psi\rangle = H(t)|\Psi,t\rangle$$