10: Second Derivative Test

Critical point:

$f(x,y)$ where $f_x = 0, f_y = 0$

Decide between $\rightarrow \begin{cases} \text{local max} \\ \text{local min} \\ \text{saddle} \end{cases}$ Global max/min? (critical point / boundry)

Consider $W = ax^2 + bxy + cy^2$

Example: $w = x^2 + 2xy + 3y^2 = (x+y)^2 + 2y$

If $a \neq 0 \rightarrow w = a\left(x^2 + \frac{b}{a}xy\right) + cy^2 = a\left(x + \frac{b}{2a}y\right)^2 + \left(c - \frac{b^2}{4a}\right)y^2$

$w = \frac{1}{4a}\left(4a^2\left(x - \frac{b}{2a}y\right)^2 + \left(4ac - b^2\right)y^2\right)$

3 cases: 1) $4ac - b^2 < 0 \Rightarrow \_ \geq 0, \_ \leq 0$

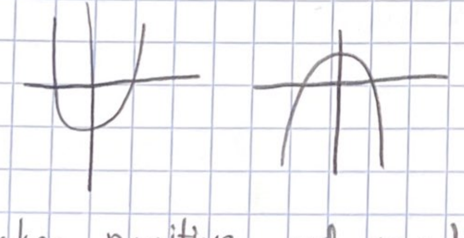
2) $4ac - b^2 = 0$

3) $4ac - b^2 > 0 \Rightarrow w = \frac{1}{4a}\left(+(\ldots)^2 + (\ldots)^2\right)$

if $a > 0$, minimum

if $a < 0$, maximum

$W = ax^2 + bxy + cy^2 = y^2\left(a\left(\frac{x}{y}\right)^2 + b\left(\frac{x}{y}\right) + c\right)$



W takes positive and negative values $\Rightarrow$ saddle

10: second derivative test

$$\frac{\partial^2 f}{\partial x^2} = f_{xx} \quad \frac{\partial^2 f}{\partial x \partial y} = f_{xy}$$

$$\frac{\partial^2 f}{\partial y \partial x} = f_{yx} = f_{xy} \quad \frac{\partial^2 f}{\partial y^2} = f_{yy}$$

At a critical point (x_0, y_0) of f

$A \equiv f_{xx}(x_0, y_0)$, $B \equiv f_{xy}(x_0, y_0)$, $C \equiv f_{yy}(x_0, y_0)$

$$AC - B^2 > 0 \begin{cases} \text{local minimum, } A > 0 \\ \text{local maximum, } A < 0 \end{cases}$$

$AC - B^2 < 0 \Rightarrow$ saddle point

$AC - B^2 = 0 \Rightarrow$ inconclusive

$w = ax^2 + bxy + cy^2$

$f_{xx} = \frac{\partial w}{\partial x}(2ax + by) = 2a$

$f_{xy} = \frac{\partial w}{\partial y}(2ax + by) = b$

$f_{yy} = \frac{\partial w}{\partial y}(bx + 2cy) = 2c$

$A = 2a \quad B = b \quad C = 2c$

$AC - B^2 = 4ac - b^2$

10: Second Derivative Test

Quadratic Approximation

$$\Delta f \approx \underbrace{f_y (x-x_0)}_{0 \text{ at crit}} + \underbrace{f_y(y-y_0)}_{0 \text{ at crit}} + \frac{1}{2} f_{xx}(x-x_0)^2 + f_{xy}(x-x_0)(y-y_0) + \frac{1}{2} f_{yy}(y-y_0)^2$$

In degenerate case, what happens depends on higher order derivatives.

$f(x,y) = x + y + \frac{1}{xy}$ $\{x, y\} > 0$

$f_x = 1 - \frac{1}{x^2 y} = 0$ $\quad f_y = 1 - \frac{1}{xy^2} = 0$

$\begin{cases} x^2 y = 1 \\ xy^2 = 1 \end{cases} \Rightarrow (1,1)$

$f_{xx} = \frac{2}{x^3 y} \Rightarrow A = 2$ $\quad f_{xy} = \frac{1}{x^2 y^2} \Rightarrow B = 1$ $\quad f_{yy} = \frac{2}{xy^3} \Rightarrow C = 2$

$AC - B^2 = 3 > 0 \Rightarrow$ local min/max $(A > 0) \Rightarrow$ local minimum

Maximum $f \to \infty$ when $x \to \infty$ or $y \to \infty$ or $(x,y) \to 0$

11: Differentials and Chain Rule

More tools to study functions

Implicit differentiation

$y=f(x) \to dy=f'(x)dx$

$y=\sin^{-1}(x)$

$x=\sin y \to dx=\cos(y)dy$

$\frac{dy}{dx}=\frac{1}{\cos y}=\frac{1}{\sqrt{1-x^2}}$

Total Differentials

$f(x,y,z)$

$df=f_x dx+f_y dy+f_z dz$

$df \neq \Delta f$

encode how change in x,y,z affect f

placeholder for small variations to get approx formula

$\Delta f \approx f_x \Delta x+f_y \Delta y+f_z \Delta z$

divide by something like dt to get instant rate of change

$\frac{df}{dt}=f_x\frac{dx}{dt}+f_y\frac{dy}{dt}+f_z\frac{dz}{dt} \leftarrow$ Chain Rule

11: Differentials and Chain Rule

$df = f_x\,dx + f_y\,dy + f_z\,dz$

$dx = x'(t)\,dt,\ dy = y'(t)\,dt,\ dz = z'(t)\,dt$

$df = f_x \cdot x'(t)\,dt + f_y y'(t)\,dt + f_z z'(t)\,dt$

$\frac{\Delta f}{\Delta t} \to \frac{df}{dt}$

so $\frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt}$

$w = x^2 y + z$, $x = t,\ y = e^t,\ z = \sin t$

$\frac{dw}{dt} = 2xy \frac{dx}{dt} + x^2 \frac{dy}{dt} + \frac{dz}{dt}$

$= 2te^t + t^2 e^t + \cos t$

$w = x^2 y + z$

$w(t) = t^2 e^t + \sin t$

$\frac{dw}{dt} = 2te^t + t^2 e^t + \cos t$

$f = uv \quad u = u(t) \quad v = v(t)$

$\frac{d(uv)}{dt} = f_u \frac{du}{dt} + f_v \frac{dv}{dt}$

$= v\frac{du}{dt} + u\frac{dv}{dt}$

$g = \frac{u}{v},\ u = u(t) \quad v = v(t)$

$\frac{d\left(\frac{u}{v}\right)}{dt} = \frac{1}{v}\frac{du}{dt} + -\frac{u}{v^2}\frac{dv}{dt} = \frac{u'v - v'u}{v^2}$

11: Differentials and Chain Rule

$W = f(x,y)$

$x = x(u,v)$
$y = y(u,v)$

$= f(x(u,v), y(u,v))$

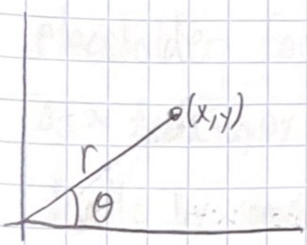
$dw = f_x\,dx + f_y\,dy$

$= f_x(x_u\,du + x_v\,dv) + f_y(y_u\,du + y_v\,dv)$

$= (f_x x_u + f_y y_u)\,du + (f_x x_v + f_y y_v)\,dv$

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$



$x = r\cos\theta$
$y = r\sin\theta$

$f = f(x,y)$

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial f}{\partial r} = f_x\cos\theta + f_y\sin\theta$$

Gradient:

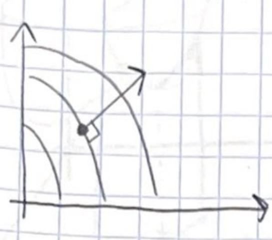
$\nabla f = \langle f_x, f_y, f_z \rangle$

12: Gradient

$\nabla w = \langle w_x, w_y, w_z \rangle$

$\frac{d\vec{r}}{dt} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$

$\nabla w \perp$ to level surface $\{w = \text{constant}\}$

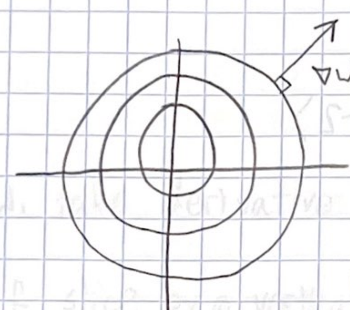


$w = a_1 x + a_2 y + a_3 z$

$\nabla = \langle a_1, a_2, a_3 \rangle$

$w = x^2 + y^2$

$\nabla w = \langle 2x, 2y \rangle$



Take $\vec{r} = \vec{r}(t)$ that stays on level surface $w = c$

$\vec{v} = \frac{d\vec{r}}{dt}$ is tangent to the level $w = c$

12: Gradient

$$\frac{dw}{dt} = \nabla w \cdot \frac{d\vec{r}}{dt}$$
$$= \nabla w \cdot \vec{v} = 0$$

$\nabla w \perp \vec{v}$

Given any vector tangent to the level

$\nabla w \perp \vec{v}$ always

$\nabla w \perp$ tangent plane to the level

Tangent plane to $x^2+y^2-z^2=4$ at $(2,1,1)$:

$w=4$, $w=x^2+y^2-z^2$

$\nabla w = \langle 2x, 2y, -2z \rangle = \langle 4, 2, -2 \rangle$

$4x+2y-2z=8$

or:

$dw = 2x\,dx + 2y\,dy - 2z\,dz$

$(2,1,1) = 4dx + 2dy - 2dz$

$\Delta w \approx 4\Delta x + 2\Delta y - 2\Delta z$

Level at $\Delta w = 0 \Rightarrow 4\Delta x + 2\Delta y - 2\Delta z = 0$

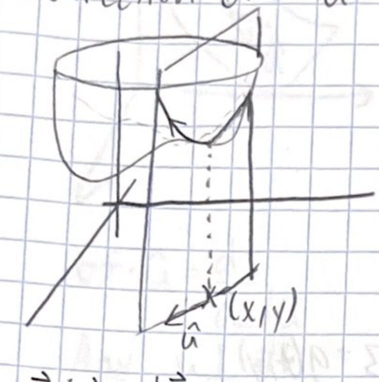
$\Rightarrow 4(x-2) + 2(y-1) - 2(z-1) = 0$

12: Gradient

Directional derivative:

$w(x,y) \to$ know $\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}$ = derivatives in $\hat{\imath}, \hat{\jmath}$

direction of $\hat{u}$ = unit vector



$\vec{r}(s)$, $\frac{d\vec{r}}{ds} = \hat{u}$

If $\hat{u} = \langle a, b \rangle$

$$\begin{cases} x(s) = x_0 + as \\ y(s) = y_0 + bs \end{cases}$$

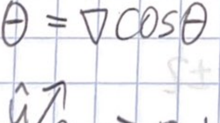
$\to$ plug into w, take derivative $\frac{dw}{ds}$

$\left.\frac{dw}{ds}\right|_{\hat{u}}$ = slope of slice by a vertical plane containing $\hat{u}$

$$\frac{dw}{ds} = \nabla w \cdot \frac{d\vec{r}}{ds} = \nabla w \cdot \hat{u}$$

$$\left.\frac{dw}{ds}\right|_{\hat{u}} = \nabla \cdot \hat{u}$$

$$\left.\frac{dw}{ds}\right|_{\hat{u}} = \nabla \cdot \hat{u} = |\nabla w||\hat{u}|\cos\theta = \nabla \cos\theta$$



when $\cos\theta = 1 \Rightarrow \theta = 0 \Rightarrow \hat{u} = \text{dir}(\nabla)$

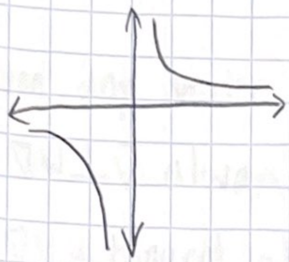
$$|\nabla w| = \left.\frac{dw}{ds}\right|_{\hat{u} = \text{dir}(\nabla w)}$$

13: Lagrange Multipliers

min/max of $f(x,y,z)$ where x,y,z are not independent

$g(x,y,z)=c$

Point closest to origin on hyperbola $xy=3$



$f(x,y)=x^2+y^2$ subject to constraint $xy=3=g(x,y)$

$\nabla f=\lambda \nabla g$

$$\begin{cases} f_x=\lambda g_x \\ f_y=\lambda g_y \\ g=c \end{cases} \rightarrow \begin{cases} 2x=\lambda y \\ 2y=\lambda x \\ xy=3 \end{cases} \rightarrow \begin{cases} 2x-\lambda y=0 \\ \lambda x-2y=0 \\ xy=3 \end{cases}$$

$$\begin{pmatrix} 2 & -\lambda \\ \lambda & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$ Trivial $(0,0) \neq$ good
Others exist if $\det(M)=0$

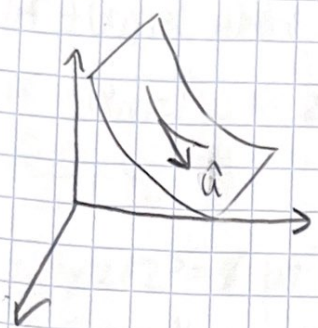
$$\begin{vmatrix} 2 & -\lambda \\ \lambda & -2 \end{vmatrix}=-4+\lambda^2=0 \Rightarrow \lambda=\pm 2$$

$\lambda=2$
$x=y$
$(\sqrt{3},\sqrt{3})=(x,y)$ or $(-\sqrt{3},-\sqrt{3})$

$\lambda=-2$
$x=-y$
no solutions here

13: Lagrange Multipliers

For $\hat{u}$ to be tangent to $g=c \Rightarrow \frac{df}{ds}\Big|_{\hat{u}} = 0$



$\nabla f \cdot \hat{u} = \frac{df}{ds}\Big|_{\hat{u}}$

Any $u \perp \nabla f$

So $\nabla f \perp$ level set of g

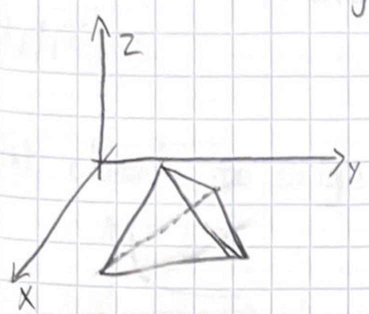
$\nabla f \parallel \nabla g$

Method does not tell whether it is max or min and second derivative test cannot be used.

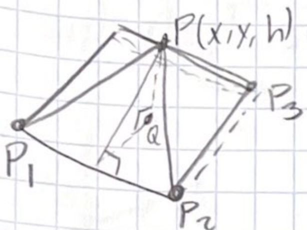
To find min or max we compare values of f at the various solutions to the multiplier equations

13: Lagrange Multipliers

Pyramid with triangular base and given volume



$V = \frac{1}{3} A(b)\, h$



$Q(x,y,0)$

$\nabla f = \lambda \nabla g$

$\frac{\partial f}{\partial u_1} = \frac{1}{2} a_1 \frac{u_1}{\sqrt{u_1^2+h^2}} = \lambda \frac{1}{2} a_1$

$\frac{\partial f}{\partial u_2} = \frac{1}{2} a_2 \frac{u_2}{\sqrt{u_2^2+h^2}} = \lambda \frac{1}{2} a_2$

$\frac{\partial f}{\partial u_3} = \frac{1}{2} a_3 \frac{u_3}{\sqrt{u_3^2+h^2}} = \lambda \frac{1}{2} a_3$

$u_1 = u_2 = u_3$

Q = in center

14: Non-independent Variables

$f(P,V,T)$ where $PV=nRT$

of $f(x,y,z)$ where $g(x,y,z)=c$

If $g(x,y,z)=c \Rightarrow z(x,y)$

$\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

$x^2+yz+z^3=8$ at $(2,3,1)$

$2x\,dx+z\,dy+(y+3z^2)dz=0$

at $(2,3,1)$: $4dx+dy+6dz=0$

$dz=-\frac{1}{6}(4dx+dy)$

$\frac{\partial z}{\partial x}=-\frac{4}{6}=-\frac{2}{3}$, $\frac{\partial z}{\partial y}=-\frac{1}{6}$

set $dy=0$ set $dx=0$

$dg=g_x\,dx+g_y\,dy+g_z\,dz=0$

$dz=-\frac{g_x}{g_z}dx-\frac{g_y}{g_z}dy$

$\frac{\partial z}{\partial x}=\frac{-g_x}{g_z}$

14: Non-Independent variables

$f(x,y)=x+y$

$\frac{\partial f}{\partial x}=1$

$\begin{cases} x=u \\ y=u+v \end{cases}$

$f(u,v)=2u+v$

$\frac{\partial f}{\partial u}=2, \frac{\partial f}{\partial v}=1$

$x=u$ but $\frac{\partial f}{\partial x} \neq \frac{\partial f}{\partial u}$

$\left(\frac{\partial f}{\partial x}\right)_y$ = keep y constant

$\left(\frac{\partial f}{\partial u}\right)_v$ = keep v constant

$$\underbrace{\left(\frac{\partial f}{\partial x}\right)_y}_{1} \neq \underbrace{\left(\frac{\partial f}{\partial x}\right)_v = \left(\frac{\partial f}{\partial u}\right)_v}_{2}$$

$A=\frac{1}{2}ab\sin\theta$

$a=b\cos\theta$

$\frac{dA}{d\theta}=\frac{1}{2}ab\cos\theta$

$b=b(a,\theta)=\frac{a}{\cos\theta}$

14: Non-independent Variables

$\left(\frac{\partial A}{\partial \theta}\right)_a$: $a = b\cos\theta \Rightarrow b = \frac{a}{\cos\theta} = a\sec\theta \Rightarrow \frac{1}{2}a^2\tan\theta$

$= \frac{1}{2}a^2\sec^2\theta$

$dA = \frac{1}{2}a\sin\theta(b\tan\theta\, d\theta) + \frac{1}{2}ab\cos\theta\, d\theta$

$= \frac{1}{2}ab(\sin\theta\tan\theta + \cos\theta)d\theta$

$\left(\frac{\partial A}{\partial \theta}\right)_a = \frac{1}{2}ab\sec\theta$

- Write dA in terms of $da, db, d\theta$
- a is constant $\Rightarrow da = 0$
- differentiate constraint $\rightarrow$ solve for db in terms of $d\theta$
- plug into dA for answer

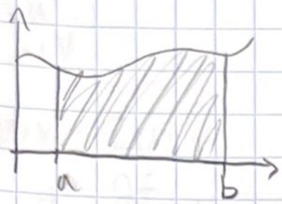
$\left(\frac{\partial}{\partial \theta}\right)_a$ in formula for A

$\left(\frac{\partial A}{\partial \theta}\right)_a = A_\theta\left(\frac{\partial \theta}{\partial \theta}\right)_a + A_a\left(\frac{\partial a}{\partial \theta}\right)_a + A_b\left(\frac{\partial b}{\partial \theta}\right)_a$

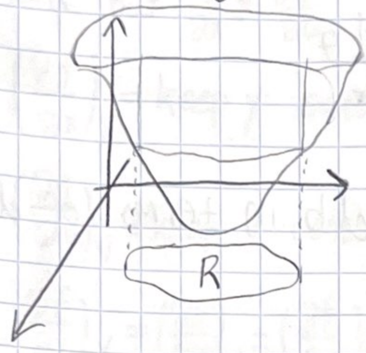
16: Double Integrals

Remember function of one variable:

$\int_a^b f(x)\,dx$ = area below the graph of f over $[a,b]$

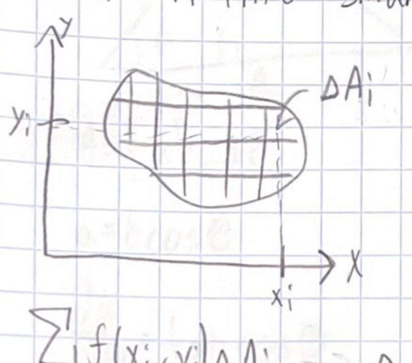


Double integral: Volume $z=f(x,y)$ over region R in xy



$$\iint_R f(x,y)\,dA$$

Cut R into small pieces ΔA_i



$\sum_i f(x_i,y_i)\Delta A_i \rightarrow \Delta A_i \rightarrow 0 \Rightarrow \iint dA$

16: Double Integrals

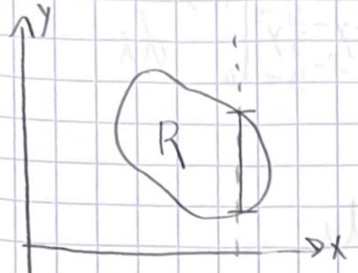
To compute $\iint_R f(x,y)\,dA$: take slices

$S(x)$ = area of slice by plane parallel yz

$V = \int S(x)\,dx$

For a given x:

$S(x) = \int f(x,y)\,dy$



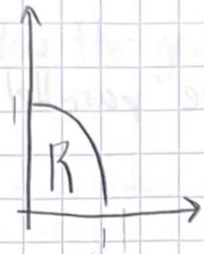
$$\iint_R f(x,y)\,dA = \int_{x_{min}}^{x_{max}} \left(\int_{y_{min}(x)}^{y_{max}(x)} f(x,y)\,dy \right) dx$$

$z = 1 - x^2 - y^2$; $R: \begin{Bmatrix} 0 \le x \le 1 \\ 0 \le y \le 1 \end{Bmatrix}$

$$\int_0^1\int_0^1 1 - x^2 - y^2\,dy\,dx = \int_0^1 \left[y - x^2y - \frac{y^3}{3} \right]_0^1 dx = \int_0^1 \frac{2}{3} - x^2\,dx$$

$$= \left[\frac{2}{3}x - \frac{1}{3}x^3 \right]_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

10: Double Integrals



$R = \{ x \geq 0, y \geq 0 \}$

$$\iint_R 1-x^2-y^2\,dA = \int_0^1 \int_0^{\sqrt{1-x^2}} 1-x^2-y^2\,dy\,dx = \int_0^1 \left[y - x^2y - \frac{1}{3}y^3 \right|_0^{\sqrt{1-x^2}} dx$$

$$= \int_0^1 \sqrt{1-x^2} - x^2\sqrt{1-x^2} - \frac{1}{3}(1-x^2)^{\frac{3}{2}}\,dx$$

$$= \int_0^1 \frac{2}{3}(1-x^2)^{\frac{3}{2}}\,dx \Rightarrow \begin{cases} x = \sin\theta \\ dx = \cos\theta \end{cases} = \int_0^{\frac{\pi}{2}} \frac{2}{3}\cos^3\theta\cos\theta\,d\theta$$

$$= \frac{2}{3}\int_0^{\frac{\pi}{2}} \cos^4\theta\,d\theta = \frac{2}{3}\int_0^{\frac{\pi}{2}} \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$

$$= \frac{2}{3}\int_0^{\frac{\pi}{2}} \frac{1}{4} + \frac{1}{2}\cos 2\theta + \frac{1}{4}\cos^2 2\theta)\,d\theta = \frac{\pi}{8}$$

16: Double Integrals

Exchange order of integratron

$$\int_0^1\int_0^2 dx\,dy = \int_0^2\int_0^1 dy\,dx$$

$$\int_0^1\int_x^{\sqrt{x}} \frac{e^x}{y}\,dy\,dx = \int_0^1\int_{y^2}^{y} \frac{e^y}{y}\,dx\,dy = \int_0^1 \left[\frac{e^x}{y}x\right]_{y^2}^{y} dy$$

$$= \int_0^1 e^y - ye^y\,dy = \left[-ye^y + 2e^y\right]_0^1$$

$$= e^{-2}$$