19: Tensor Products

$V \otimes W$

$\sum_{ij} a_{ij} \, v_i \otimes w_j$

The vector $0 \in V \otimes W$

$0 \otimes w_i \quad 0 \in W$

$v_i \otimes 0 \quad 0 \in V$

$av \otimes w = a(v \otimes w)$

$a = 0$

$0 \otimes w = 0 \in V \otimes W$

Inner Product

$\langle \sum a_{ij} v_i \otimes w_j \sum b_{pq} v_p \otimes w_q \rangle$

$\equiv \sum a_{ij}^* \sum b_{pq} \langle v_i \otimes w_j , v_p \otimes w_q \rangle$

$\sum \sum a_{ij}^* b_{pq} = \langle v_i, v_p \rangle_V \langle w_j , w_q \rangle_W$

19: Tensor Products

2 spin $\frac{1}{2}$ $|\Psi\rangle = \alpha(|+\rangle \otimes |-\rangle - |-\rangle \otimes |+\rangle)$

$\langle \Psi, \Psi \rangle = \alpha^* \alpha \langle |+\rangle \otimes |-\rangle - |-\rangle \otimes |+\rangle, |+\rangle \otimes |-\rangle - |-\rangle \otimes |+\rangle \rangle$

$= |\alpha|^2 (1+1) = 2|\alpha|^2$

$= 1$

$|\Psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle_1 \otimes |-\rangle_2 - |-\rangle_1 \otimes |+\rangle_2)$

Entangled states

$V \otimes W$ includes superpositions.

$\sum_{i} \alpha_{ij} v_i \otimes w_j$

$= v_* \otimes w^*$

$\underset{V}{\cap} \quad \underset{W}{\cap}$

If so the state is not entangled

V, W

$\downarrow \quad \searrow$

$(e_1, e_2) \quad (f_1, f_2)$

Ground state $a_{11} e_1 \otimes f_1 + a_{12} e_1 \otimes f_2 + a_{21} e_2 \otimes f_1 + a_{22} e_2 \otimes f_2$

$\overset{?}{=} (a_1 e_1 + a_2 e_2) \otimes (b_1 f_1 + b_2 f_2)$

To have a solution, $a_{11} = a_1 b_1$

$a_{12} = a_1 b_2$

$a_{21} = a_2 b_1$

$a_{22} = a_2 b_2$

$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

19: Tensor Products

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$\det(A) = 0 \iff$ Not entangled

V state of spin $\frac{1}{2}$

$$\underset{\substack{\uparrow \\ \text{1st}}}{V} \otimes \underset{\substack{\uparrow \\ \text{2nd}}}{V}$$

$$|\Phi_0\rangle \equiv \frac{1}{\sqrt{2}}\left(|+\rangle|+\rangle + |-\rangle|-\rangle\right)$$

$$\langle \Phi_0 | \Phi_0 \rangle = 1$$

$$i = 1,2,3 \quad |\Phi_i\rangle = (\mathbb{1} \otimes \sigma_i)|\Phi_0\rangle$$

$$|\Phi_1\rangle = (\mathbb{1} \otimes \sigma_1) = \frac{1}{\sqrt{2}}\left(|+\rangle|+\rangle + |-\rangle|-\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|+\rangle\sigma_1|+\rangle + |-\rangle\sigma_1|-\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|+\rangle|-\rangle + |-\rangle|+\rangle\right)$$

$$|\Phi_2\rangle = 1 \otimes \sigma_2 |\Phi_0\rangle$$

$$= \frac{i}{\sqrt{2}}\left(|+\rangle|-\rangle - |-\rangle|+\rangle\right)$$

$$|\Phi_3\rangle = 1 \otimes \sigma_3 |\Phi_0\rangle$$

$$= \frac{1}{\sqrt{2}}\left(|+\rangle|+\rangle - |-\rangle|-\rangle\right)$$

$$\langle \Phi_0 | \mathbb{1} \otimes \sigma_i | \Phi_0 \rangle = \langle \Phi_0 | \Phi_i \rangle$$

19: Tensor Products

$\langle \Phi_0 | \Phi_i \rangle = 0$

$\langle \Phi_i | \Phi_j \rangle = \langle \Phi_0 | (\mathbb{1} \otimes \sigma_i)(\mathbb{1} \otimes \sigma_j) | \Phi_0 \rangle$

$= \langle \Phi_0 | \mathbb{1} \otimes \sigma_i \sigma_j | \Phi_0 \rangle$

$= \langle \Phi_0 | \mathbb{1} \otimes (\mathbb{1} \delta_{ij} + i\epsilon_{ijk} \sigma_k) | \Phi_0 \rangle$

$= \delta_{ij} \langle \Phi_0 | \Phi_0 \rangle + i\epsilon_{ijk} \langle \Phi_0 | \Phi_k \rangle$

$\delta_{ij} = \langle \Phi_i | \Phi_j \rangle$

$|+\rangle|+\rangle = \frac{1}{\sqrt{2}}(|\Phi_0\rangle + |\Phi_3\rangle)$

$|+\rangle|-\rangle = \frac{1}{\sqrt{2}}(|\Phi_1\rangle - i|\Phi_2\rangle)$

$|-\rangle|+\rangle = \frac{1}{\sqrt{2}}(|\Phi_1\rangle + i|\Phi_2\rangle)$

$|-\rangle|-\rangle \quad \frac{1}{\sqrt{2}}(|\Phi_0\rangle - i|\Phi_3\rangle)$

Given an orthonormal basis $|e_1\rangle \ldots |e_n\rangle$ we measure a state $|\Psi\rangle$ so probability $|\langle e_i|\Psi\rangle|^2$

$\sigma_1\ \sigma_2\ \sigma_3$ are unitary as operators they can be realized by time evolution with a suitable Hamiltonian

$$\underbrace{e^{i\frac{\pi}{2}(-1+\sigma_1)}}_{\frac{tH}{\hbar}} = e^{-i\frac{\pi}{2}} e^{\frac{i\pi\sigma_1}{2}} = (-i)(\mathbb{1}\cos\tfrac{\pi}{2} + i\sigma_1 \sin\tfrac{\pi}{2})$$

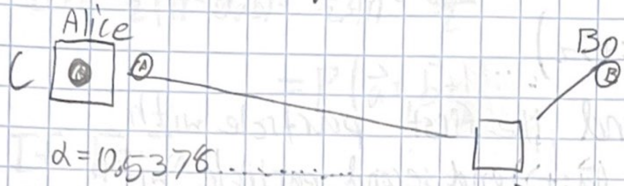
$$= (-i)(i\sigma_1) = \sigma_1$$

19: Tensor Products

Quantum Teleportation

Bennet, Brassard, Creppeau, Jozsa (1993)

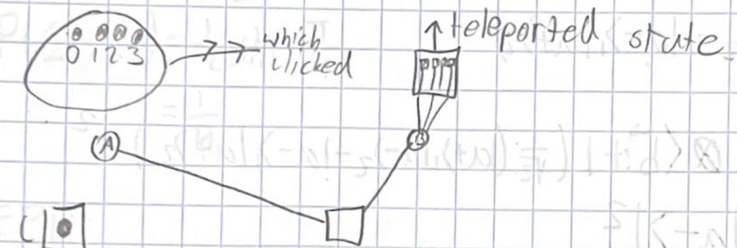
Alice has a quantum state $\alpha|+\rangle + \beta|-\rangle$



$\alpha = 0.5378 \ldots\ldots$

$|\psi\rangle = \alpha|+\rangle_c + \beta|-\rangle_c$

entangled state source



AB pair: $|\Phi_0\rangle_{AB} = \frac{1}{\sqrt{2}}(|+\rangle_A|+\rangle_B + |-\rangle_A|-\rangle_B)$

Total space $|\Phi_0\rangle_{AB} \otimes (\alpha|+\rangle_C + \beta|-\rangle_C)$

$$|\psi_{tot}\rangle = \frac{1}{\sqrt{2}}(|+\rangle_A|+\rangle_B + |-\rangle_A|-\rangle_B) \otimes (\alpha|+\rangle_C + \beta|-\rangle_C)$$

$$= \frac{1}{\sqrt{2}}(\alpha|+\rangle_A|+\rangle_C|+\rangle_B + \beta|+\rangle_A|-\rangle_C|+\rangle_B + \alpha|-\rangle_A|+\rangle_C|-\rangle_B + \beta|-\rangle_A|-\rangle_C|-\rangle_B)$$

$$= \frac{1}{2}(|\Phi_0\rangle_{AC} + |\Phi_3\rangle_{AC})\alpha|+\rangle_B + \frac{1}{2}(|\Phi_1\rangle_{AC} - i|\Phi_2\rangle_{AC})\beta|+\rangle_B$$

$$+ \frac{1}{2}(|\Phi_1\rangle_{AC} + i|\Phi_2\rangle_{AC})\alpha|-\rangle_B + \frac{1}{2}(|\Phi_0\rangle_{AC} - |\Phi_3\rangle_{AC})\beta|-\rangle_B$$

20: Tensor Products

$$\frac{1}{\sqrt{2}}\left(|+\rangle_1|-\rangle_2-|-\rangle_1|+\rangle_2\right)\equiv|\psi\rangle$$

$$\eta_0\to\mu^++\mu^-$$

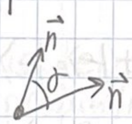
$$\frac{1}{\sqrt{2}}\left(|\vec{n};+\rangle_1|\vec{n};-\rangle_2-|\vec{n};-\rangle_1|\vec{n};+\rangle_2\right)$$

$P(\vec{a}+,\vec{b}+)$ probability to find the first particle with spin along $\vec{a}$ ($|\vec{a};+\rangle$) and second particle with spin along $\vec{b}$ ($|\vec{b};+\rangle$)

$$|\psi\rangle=\frac{1}{\sqrt{2}}\left(|a+\rangle_1|a-\rangle_2-|a-\rangle_1|a+\rangle_2\right)$$

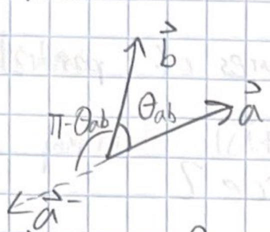
$$P(\vec{a}+,\vec{b}+)=\left|\langle\vec{a};+|\otimes\langle\vec{b};+|\left(\frac{1}{\sqrt{2}}(|a+\rangle_1|a-\rangle_2-|a-\rangle_1|a+\rangle_2)\right)\right|^2$$

$$P(\vec{a}+,\vec{b}+)=\frac{1}{2}|\langle b+|a-\rangle|^2$$



$$|\langle\vec{n}|\vec{n}'\rangle|^2=\cos^2\frac{\gamma}{2}$$

20: Tensor Products



$$\frac{1}{2}\cos^2\frac{1}{2}(\pi-\theta_{ab}) = \frac{1}{2}\sin^2\frac{\theta_{ab}}{2}$$

$$= P(\vec{a}+, \vec{b}+)$$

If $\vec{b} = -\vec{a}$

$\Rightarrow \theta_{ab} = \pi$

$\Rightarrow P(a,b) = \frac{1}{2}$

$$P(\hat{z}+, \hat{x}+) = \frac{1}{2}\sin^2\frac{\pi}{4}$$

$$= \frac{1}{4}$$

EPR:

Local realism

Two assumptions about measurment results:

$\underline{m}$ correspond to some aspect of reality

$\underline{m}$ is independent of actions preformed at a distant location at the same time

20: Tensor Products

ERR says entangled pairs are pares of particles with definite spins

Particle 1	Particle 2
50% $Z+$	$Z-$
50% $Z-$	$Z+$

$A(\lambda)$

Both A and B can measure in two directions $\hat{z}$ and $\hat{x}$

Label for a particle

$(z+, x-)$

measured in $\hat{z} \Rightarrow \uparrow$

measured in $\hat{x} \Rightarrow \downarrow$

Particle 1	Particle 2
25% $(\hat{z}+, \hat{x}+)$	$(\hat{z}-, \hat{x}-)$
25% $(\hat{z}+, \hat{x}-)$	$(\hat{z}-, \hat{x}+)$
25% $(\hat{z}-, \hat{x}+)$	$(\hat{z}+, \hat{x}-)$
25% $(\hat{z}-, \hat{x}-)$	$(\hat{z}+, \hat{x}-)$

$P(z+ \text{ on } 1,\ z- \text{ on } 2) = \frac{1}{2}$

$P(z+ \text{ on } 1,\ x+ \text{ on } 2) = \frac{1}{4}$

20: Tensor Products

Three Directions

Label for a particle $(\vec{a}+\ \vec{b}-\ \vec{c}+)$

$S_{\vec{a}} = \frac{\hbar}{2}$

$S_{\vec{b}} = -\frac{\hbar}{2}$

$S_{\vec{c}} = \frac{\hbar}{2}$

Particle 1	Particle 2
$(a+, b+, c+)$	$(a-, b-, c-)$
$(a+, b+, c-)$	$(a-, b-, c+)$
$(a+, b-, c+)$	$(a-, b+, c-)$
$(a+, b-, c-)$	$(a-, b+, c+)$
$(a-, b+, c+)$	$(a+, b-, c-)$
$(a-, b+, c-)$	$(a+, b-, c+)$
$(a-, b-, c+)$	$(a+, b+, c-)$
$(a-, b-, c-)$	$(a+, b-, c+)$

10: Tensor Products

$N = N_1 + \ldots + N_8$

$P(a+, b+) = \frac{N_3 + N_4}{N}$

$P(a+, c+) = \frac{N_2 + N_4}{N}$

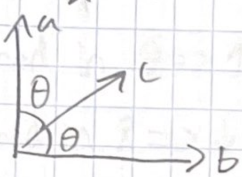
$P(c+, b+) = \frac{N_3 + N_7}{N}$

$\frac{N_3 + N_4}{N} \leq \frac{N_3 + N_7}{N} + \frac{N_4 + N_2}{N}$

$P(a+, b+) \leq P(a+, c+) + P(c+, b+)$

If QM is true:

$\frac{1}{2}\sin^2\frac{\theta_{ab}}{2} \leq \frac{1}{2}\sin^2\frac{\theta_{ac}}{2} + \frac{1}{2}\sin^2\frac{\theta_{bc}}{2}$



$\theta_{ab} = 2\theta$

$\theta_{ac} = \theta_{bc} = \theta$

$\frac{1}{2}\sin^2\theta \leq \frac{1}{2}\sin^2\frac{\theta}{2}$

$\frac{1}{2}\theta^2 \leq \frac{\theta^2}{4}$

$\frac{1}{2} \leq \left(\frac{1}{\sqrt{2}}\right)^2$

20: Tensor Products

Angular Momentum

$L_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y$

$L_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z$

$L_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$

$(\hat{x}, \hat{y}, \hat{z}) \to (\hat{x}_1, \hat{x}_2, \hat{x}_3)$

$(\hat{p}_x, \hat{p}_y, \hat{p}_z) \to (\hat{p}_1, \hat{p}_2, \hat{p}_3)$

$[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$

$\vec{r} = \hat{x}\vec{e}_1 + \hat{y}\vec{e}_2 + \hat{z}\vec{e}_3$

$\vec{a}\cdot\vec{b} \equiv a_i b_i$

$(\vec{a}\times\vec{b})_i \equiv \varepsilon_{ijk} a_j b_k$

$\vec{a}^2 = \vec{a}\cdot\vec{a} = a_i a_i$

$\vec{a}\cdot\vec{b} \neq \vec{b}\cdot\vec{a}$

$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a} + [a_i, b_i]$

$\vec{L}\times\vec{L} = i\hbar\vec{L}$

$\vec{p}\cdot\vec{L} = p_i \epsilon_{ijk} x_j p_k$

$\vec{p}\cdot\vec{L} = 0$

21: Angular Momentum

$\vec{L} = \vec{r} \times \vec{p} = -\vec{p} \times \vec{r}$

$[\vec{L}_i, u_j] = i\hbar \epsilon_{ijk} u_k$

$\left.\begin{array}{l} \vec{u} \cdot \vec{v} = \text{scalar} \\ \vec{u} \times \vec{v} = \text{vector} \end{array}\right\}$ under rotations

$[\hat{L}_i, u \cdot v] = 0$

$[\hat{L}_i, \vec{r}^2] = [L_i, p^2]$

$= [L_i, \vec{r} \cdot \vec{p}] = 0$

$[\hat{L}_i, (u \times v)_j] = i\hbar \epsilon_{ijk} (u \times v)_k$

$\vec{u} = \vec{s}$
$\vec{v} = \vec{p}$

$[L_i, L_j] = i\hbar \epsilon_{ijk} L_k$

$[L_i, \vec{L}^2] = 0$

$[S_i, S_j] = i\hbar \epsilon_{ijk} S_k$

$[S_i, \vec{S}^2] = 3\left(\frac{\hbar}{2}\right)^2 \mathbb{1}$

$[J_i, J_k] = i\hbar \epsilon_{ijk} J_k$

$[J_i, \vec{J}^2] = 0$

$\vec{J} \times \vec{J} = i\hbar \vec{J}$

21: Angular Momentum

$$\vec{p}^2 = \frac{1}{r^2}\left[(r\cdot p)^2 - i\hbar(\vec{r}\cdot\vec{p})\right] + \frac{1}{r^2}\vec{L}^2$$

$$\vec{p} = \frac{\hbar}{i}\nabla$$

$$\vec{r}\cdot\vec{p} = \frac{\hbar}{i} r\frac{\partial}{\partial r}$$

$$-\hbar^2 \frac{1}{r}\frac{\partial^2}{\partial r^2} r$$

$$p^2 = -\hbar^2 \nabla^2$$

$$= -\hbar^2\left(\frac{1}{r}\frac{\partial^2}{\partial r^2} r + \frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin\theta}\frac{\partial^2}{\partial\phi^2}\right)\right)$$

$$\vec{L}^2 = -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

$$H = \frac{\vec{p}^2}{2m} + V(r)$$

$$H = -\frac{\hbar^2}{2m}\frac{1}{r}\frac{\partial^2}{\partial r^2} r + \frac{1}{2mr^2}\vec{L}^2 + V(r)$$

$$[H, \vec{L}] = 0$$

$$\frac{d}{dt}\langle L_i\rangle = \langle [H, L_i]\rangle$$
$$= 0$$

$$\hat{J}_\pm = J_1 \pm iJ_2$$

$$J_+J_- = J_1^2 + J_2^2 - i[J_1, J_2]$$

$$J_+J_- = J_1^2 + J_2^2 + \hbar J_3$$

$$J_-J_+ = J_1^2 + J_2^2 - \hbar J_3$$

$$J_\pm J_\mp = J_1^2 + J_2^2 \pm \hbar J_3$$

$$[J_+, J_-] = 2\hbar J_3$$

21: Angular Momentum

$$\vec{J}^2 = J_+J_- + J_3^2 - \hbar J_3$$

$$= J_-J_+ + J_3^2 + \hbar J_3$$

$$[J_2, J_+] = [J_3, J_1 + iJ_2]$$

$$= i\hbar J_2 + i(-i\hbar J_1)$$

$$= \hbar(J_1 + iJ_2)$$

$$= \hbar J_+$$

$$[J_3, J_\pm] = \pm \hbar J_\pm$$

$$[N, a^+] = a^+$$

$$[N, a] = -a$$

$$\vec{J}^2 |j;m\rangle = \hbar^2 j^2 |j;m\rangle$$

$$J_2 |j;m\rangle = \hbar m |j;m\rangle$$

$J, M \in \mathbb{R}$

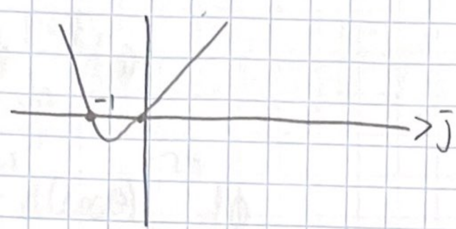
$$\langle jm|J^2|jm\rangle = \hbar^2 j(j+1)$$

$$= \sum_i \langle jm|J_iJ_i|jm\rangle$$

$$= \sum_i \| J_i |jm\rangle \|^2 \geq 0$$

21: Angular Momentum

$j(j+1)$



$j \geq 0$

$$[J_\pm, J^2] = 0$$

$$J^2(J_\pm|jm\rangle) = J_\pm J^2|jm\rangle$$
$$= \hbar^2 j(j+1)(J_\pm|jm\rangle)$$

$$J_\pm|j\,m\rangle \sim |j\,m'\rangle$$

$$J_\pm|j\,m\rangle = ([J_z, J_\pm] + J_\pm J_z)|jm\rangle$$
$$= (\pm\hbar J_\pm + \hbar m J_\pm|jm\rangle)$$
$$= \hbar(m\pm1)(J_\pm|jm\rangle)$$

$$J_\pm|jm\rangle = c_\pm(j,m)|j\,m\pm1\rangle$$

$$\langle jm|J_\mp = c_\pm^*(j,m)\langle j; m\pm1|$$

$$\langle jm|J_\mp J_\pm|jm\rangle = |c_\pm(j,m)|^2$$
$$= \hbar^2(j(j+1) - m(m\pm1))$$