22: Angular Momentum

Algebreic Analysis

$L^2|lm\rangle = \hbar^2 l(l+1)|lm\rangle$

$L_z|lm\rangle = \hbar m|lm\rangle$

$\hat{L}^2 = -\hbar\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$

$\hat{L}_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}\left(\frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right)$

$\hat{L}_\pm = \hbar e^{\pm i\phi}\left(i\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\phi} \pm \frac{\partial}{\partial\theta}\right)$

$Y_{lm}(\theta, \phi)$

$\hat{L}^2 Y_{lm} = \hbar^2 l(l+1) Y_{lm}$

$\hat{L}_z Y_{lm} = \hbar m Y_{lm}$

$Y_{lm} \equiv \langle\theta\,\phi|lm\rangle$

$\Psi(x) \equiv \langle x|\Psi\rangle$

$\iiint |\vec{x}\rangle\langle\vec{x}|\,d^3\vec{x} = \mathbb{1}$

$\int |r\theta\phi\rangle\langle r\theta\phi|\,dr\,r\,d\theta\,r\sin\theta\,d\phi = 1$

$\iint |\theta\,\phi\rangle\langle\theta\,\phi|\,d\theta\sin\theta\,d\phi \int r^2|r\rangle\langle r|\,dr = 1$

$\iint |\theta\phi\rangle\langle\theta\,\phi|\,d\theta\sin\theta\,d\phi = 1$

22: Angular Momentum

$$\int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi$$

$$= -\int_1^{-1} d(\cos\theta) \int_0^{2\pi} d\phi$$

$$\int \sin\theta d\theta d\phi = \int_{-1}^{1} d(\cos\theta) \int_0^{2\pi} d\phi$$

$$\equiv \int d\Omega$$

$$\int |\theta\phi\rangle\langle\theta\phi| d\Omega = 1$$

$$\langle l'm' | lm\rangle = \delta_{ll'}\delta_{mm'}$$

$$\int \langle l'm' | \theta\phi\rangle\langle\theta\phi | lm\rangle d\Omega$$

$$= \int Y^*_{l'm'}(\theta,\phi) Y_{lm}(\theta,\phi) d\Omega = \delta_{ll'}\delta_{mm'}$$

Radial Equation

$$H = \frac{\vec{P}^2}{2m} + V(r) = \frac{\hbar^2}{2m} \frac{1}{r} \frac{\partial^2}{\partial r^2} r + \frac{1}{2mr^2} \vec{L}^2 + V(r)$$

$$\Psi_{Elm}(\vec{x}) = f_{Elm}(r) Y_{lm}(\theta,\phi)$$

22: Angular Momentum

$$-\frac{\hbar^2}{2m}\frac{1}{r}\frac{d^2}{dr^2}(rf_{E\ell m})+\frac{\hbar^2\ell(\ell+1)}{2mr^2}f_{E\ell m}+V(r)f_{E\ell m}=Ef_{E\ell m}$$

$$\psi_{E\ell m}=f_{E\ell}(r)Y_{\ell m}(\theta\phi)$$

$$-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}(rf_{E\ell})+\frac{\hbar^2\ell(\ell+1)}{2mr^2}(rf_{E\ell})+V(r)(rf_{E\ell})=E(rf_{E\ell}$$

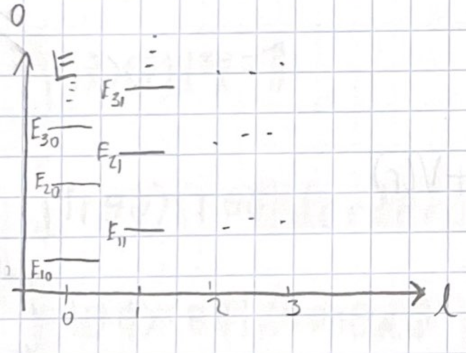
$$u_{E\ell}=rf_{E\ell}$$

$$-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}(u_{E\ell})+\left[V(r)+\frac{\hbar^2\ell(\ell+1)}{2mr^2}\right]u_{E\ell}=Eu_{E\ell}$$

$$\int|\psi_{E\ell m}(\vec{x})|^2d^3x=1$$

$$\iint\frac{|u_{E\ell}|^2}{r^2}Y^*_{\ell m}(\theta,\phi)Y_{\ell m}(\theta,\phi)r^2\,dr\,d\Omega$$

$$\int_0^\infty|u_{E\ell}(r)|^2dr=1$$



$E_{n\ell}$

22: Angular Momentum

$j=2$

$|jm\rangle\, |2m\rangle$

$|2,2\rangle$

$|2,1\rangle$

$|2,0\rangle$

$|2,-1\rangle$

$|2,-2\rangle$

$r \to 0$

$$\lim_{r\to 0} u_{E\ell}(r) = 0$$

$$\lim_{r\to 0} u_{E0}(r) = \text{"}c\text{"}$$

$$\psi_{E0,0} = c\,\frac{u_{E0}(r)}{r}$$

$$r\to 0,\ \psi = \frac{c'}{r}$$

$$H\psi = -\frac{\hbar^2}{2m}\nabla^2\psi + \dots$$

$$\nabla^2\left(\frac{1}{r}\right) = -4\pi\,\delta(\vec{x})$$

22: Angular Momentum

Centrifugal barrier dominates $r \to 0$

$$-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}u_{E\ell}\,\frac{\hbar^2 \ell(\ell+1)}{2mr^2} \approx 0$$

$$\frac{d^2 u_{E\ell}}{dr^2} = \frac{\ell(\ell+1)\,u_{E\ell}}{r^2}$$

$$u_{E\ell} = r^s$$

$$s(s-1) = \ell(\ell+1)$$

$$\begin{cases} s = \ell+1 \\ s = -\ell \end{cases}$$

$$u_{E\ell} = r^{\ell+1}$$

$$u_{E\ell} = \frac{1}{r^{\ell}}$$

not normalizable for $\ell \geq 1$

$$f_{E\ell} \sim c r^{\ell}$$

$r \to \infty$

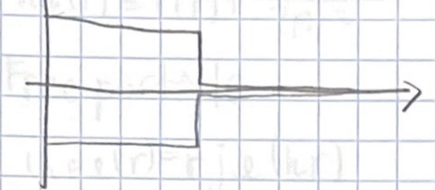
$V(r) = 0$ for $r > r_0$

$rV(r) \to 0$ as $r \to \infty$

22: Angular Momentum

Ignore V_{eff}

$$-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}u_{E\ell} = E u_{E\ell}$$



$E<0 \quad u_{E\ell} = \exp\left(-\sqrt{\frac{2m|E|}{\hbar^2}}\, r\right)$

$E>0 \quad u_E = \exp(\pm ikr) \quad k = \left(\sqrt{\frac{2m|E|}{\hbar^2}}\right)$

Free particles:

$(P_1, P_2, P_3) \to (E, \theta, \phi)$

$\to (E, \ell, m)$

$V=0$

$$-\frac{\hbar^2}{2m}\frac{d^2 u_{E\ell}}{dr^2} + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}u_{E\ell} = E u_{E\ell}$$

$$-\frac{d^2 u_{E\ell}}{dr^2} + \frac{\ell(\ell+1)}{r^2}u_{E\ell} = k^2 u_{E\ell}$$

22: Angular Momentum

$\rho = kr$

$$-\frac{d^2 u_{E\ell}}{d\rho^2} + \frac{\ell(\ell+1)}{\rho^2} u_{E\ell} = u_{E\ell}$$

$$\frac{d^2}{dx^2} \quad \frac{1}{x^2} \quad 1$$

$u_{E\ell} = r j_\ell(kr)$

$\psi_{E\ell m} = j_\ell(kr) Y_{\ell m}(\theta, \phi)$

J, N

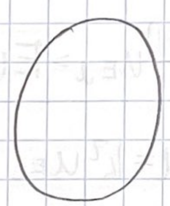
$\rho j_\ell(\rho) \sim \frac{\rho^{\ell+1}}{(2\ell+1)!!}, \rho \to 0$

$\rho j_\ell(\rho) \sim \sin(\rho - \frac{\ell\pi}{2}) \quad \rho \to \infty$

Free particles $u_{E\ell} \sim \sin(kr - \frac{\ell\pi}{2}) \quad r \to \infty$

$u_{E\ell} \sim \sin(kr - \frac{\ell\pi}{2} + \delta_\ell(E))$

$V(r) = \begin{cases} 0 & r < a \\ \infty & r > a \end{cases}$



$u_{E\ell} \sim r j_\ell(kr), j_\ell(ka) = z$

$V(r) = \beta r^2$

23: Angular Momentum

$$\Psi_{E\ell m} = \frac{u_{E\ell}(r)}{r} Y_{\ell m}(\theta, \phi)$$

$$\frac{-\hbar^2}{2m}\frac{d^2 u_{E\ell}}{dr^2} + V_{eff}(r) u_{E\ell} = E u_{E\ell}$$

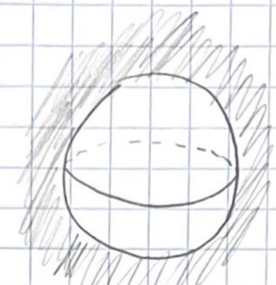
$$V_{eff}(r) \equiv V(r) + \frac{\hbar^2 \ell(\ell+1)}{2mr^2}$$

Free particle

$$u_{E\ell}(r) = r j_\ell(kr)$$

Infinite spherical well

$$V(r) = \begin{cases} 0 & r < a \\ \infty & r \geq a \end{cases}$$



$r < a$

$$-\frac{d^2 u_{E\ell}}{d\rho^2} + \frac{\ell(\ell+1)}{\rho^2} u_{E\ell} = u_{E\ell}$$

$$\rho = kr$$

$$k = \sqrt{\frac{2m|E|}{\hbar^2}}$$

23: Angular Momentum

$l=0$

$$\frac{d^2 u_{E0}}{d\rho^2} = u_{E0}$$

$$u_{E0} = A\sin\rho + B\cos\rho \sim \rho$$

as $\rho \to 0$

$$u_{E0} = \sin\rho = \sin kr$$

$$u_{E0}(r) \sim \sin kr$$

$$u_{E0}(a) = 0$$

$$k_n a = n\pi$$

$$E_n = \frac{\hbar^2 k_n^2}{2m}$$

$$= \frac{\hbar^2}{2ma^2}(k_n a)^2$$

$$= \frac{\hbar^2}{2ma^2}(n\pi)^2$$

$$\mathcal{E}_{nl} \equiv \left(\frac{E_{nl}}{\frac{\hbar^2}{2ma^2}}\right)$$

$$\mathcal{E}_{n0} = (n\pi)^2$$

$$\mathcal{E}_{10} = 9.8696$$

$$\mathcal{E}_{20} = 39.478$$

$$\mathcal{E}_{30} = 88.826$$

23: Angular Momentum

$j_1(\rho) \Rightarrow \tan\rho = \rho$

$z_{n\ell}$ = the nth zero $n=1\ldots$ of j_ℓ

$j_\ell(z_{n\ell}) = 0$

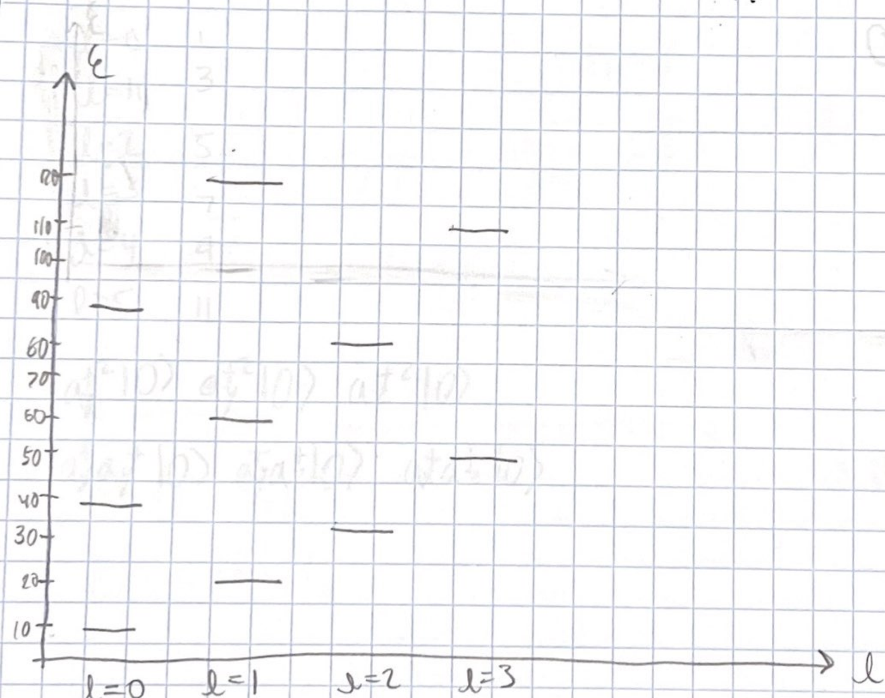
$u_{E\ell}(a) = 0$

$k_{n\ell}\, a = z_{n\ell}$

$E_{n\ell} = \frac{\hbar^2 k_{n\ell}}{2m}$

$= \frac{\hbar^2 (k_{n\ell}\, a)^2}{2ma^2}$

$\mathcal{E}_{n\ell} = (z_{n\ell})^2$



23: Angular Momentum

SHO

$V=\frac{1}{2}m\omega^2(x^2+y^2+z^2)$

$=\frac{1}{2}m\omega^2+r^2$

$H=\hbar\omega(\hat{N}_1+\hat{N}_2+\hat{N}_3+\frac{3}{2})$

H_1 of one dim SHO

a_x^+

a_y^+

a_z^+

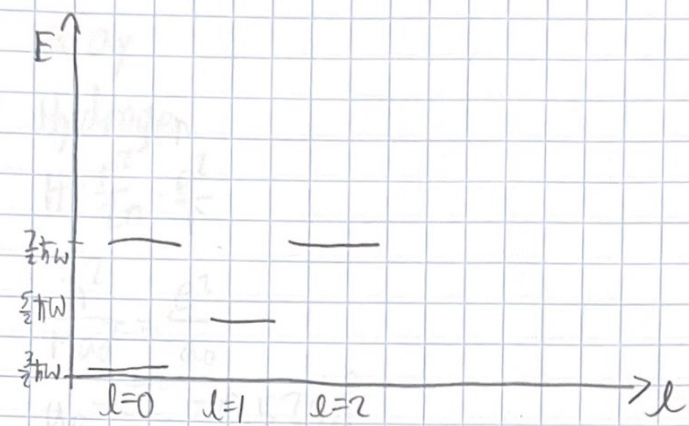
$H_{3D\text{-}SHO}=H_1\otimes H_1\otimes H_1$

$|0\rangle$ $N_1=N_2=N_3=0$

$E=\hbar\omega\left(\frac{3}{2}\right)$

$|0\rangle$

23: Angular Momentum



$a_x^\dagger|0\rangle$

$a_y^\dagger|0\rangle$

$a_z^\dagger|0\rangle$

$\hbar\omega\left(1+\frac{3}{2}\right)=\frac{5}{2}$

$l=0$	1
$l=1$	3
$l=2$	5
$l=3$	7
$l=4$	9
$l=5$	11

$a_x^{\dagger 2}|0\rangle$ $a_y^{\dagger 2}|0\rangle$ $a_z^{\dagger 2}|0\rangle$

$a_x^\dagger a_y^\dagger|0\rangle$ $a_y^\dagger a_z^\dagger|0\rangle$ $a_x^\dagger a_z^\dagger|0\rangle$

23: Angular Momentum

$l=4 \oplus l=2 \oplus l=0$

$a_x, a_y \rightarrow a_R, a_L$

$L_z = \hbar(N_R - N_L)$

$N=1$

$a_R^+|0\rangle \quad \hbar$

$a_z^+|0\rangle \quad 0$

$a_L^+|0\rangle \quad -\hbar$

$N=2$

$a_R^+ a_R^+|0\rangle \quad 2\hbar$

$N=n$

$(a_R^+)^n|0\rangle \quad L_z = n\hbar$

$2n+1$ states

$(a_R^+)^{n-1} a_z^+|0\rangle$

$l=n$

$(a_R^+)^{n-2}(a_z^+)^2|0\rangle$

$(a_R)^{n-1}(a_L^+)|0\rangle$

23: Angular Momentum

$a_x^\dagger a_y$

Hydrogen

$H = \frac{p^2}{2m} - \frac{e^2}{r}$

$\frac{\hbar^2}{m a_0^2} = \frac{e^2}{a_0}$

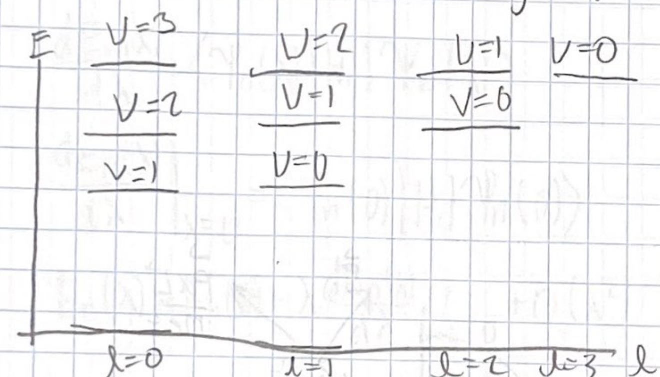
$a_0 = \frac{\hbar^2}{me^2} = 0.529 Å$

Energy scale

$H \equiv \gamma + \frac{1}{2m} \sum_k \left(\hat{p}_k + \frac{i\beta \hat{x}_k}{r}\right)\left(\hat{p}_k - \frac{i\beta \hat{x}_k}{r}\right)$

$\left(\hat{p}_k - \frac{i\beta x_k}{r}\right) |\psi_{gr}\rangle = 0$

$E_{gr} = \gamma$



$n = \nu + l + 1$

$E_{nl} = -\frac{e^2}{2a_0}\frac{1}{n^2}$

23: Angular Momentum

Elliptical orbit

$$H = \frac{p^2}{2m} + V(r)$$

$$\vec{F} = -\nabla V$$

$$= -\frac{V'(r)}{r}\hat{r}$$

$$\frac{d\vec{p}}{dt} = \vec{F} = -\frac{V'(r)}{r}\hat{r}$$

$$\frac{d\vec{L}}{dt} = 0$$

$$\frac{d}{dt}(\vec{p} \times \vec{L}) = m V'(r) r^2 \frac{d}{dt}\left(\frac{\hat{r}}{r}\right)$$

$$V' = \frac{e^2}{r^2}$$

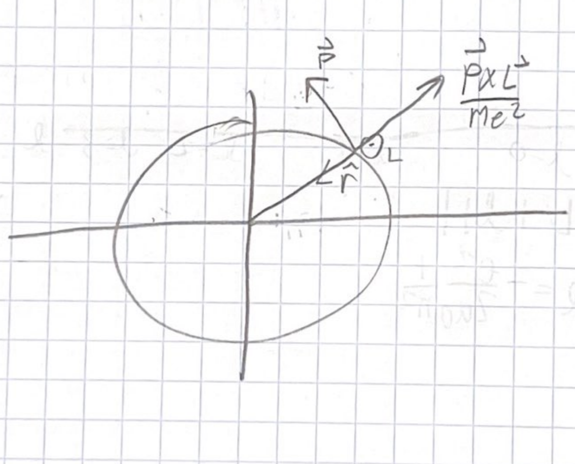
$$V(r) = -\frac{e^2}{r}$$

$$\frac{d}{dt}\left((\vec{p} \times \vec{L}) - me^2 \frac{\hat{r}}{r}\right) = 0$$

$$\frac{\vec{p} \times \vec{L}}{me^2} - \frac{\hat{r}}{r} \equiv R$$

$$\frac{dR}{dt} = 0$$

$$|R| -$$



24: Addition of Angular Momentum

Feynman Hellman Result

$H(\lambda)$ a Hamiltonian with parameter λ

$\Psi_n(\lambda)$ normalized energy eigenstate $E_n(\lambda)$

$$\frac{dE}{d\lambda} = \langle \Psi_n(\lambda) | \frac{dH}{d\lambda}(\lambda) | \Psi_n(\lambda) \rangle$$

$$E_n(\lambda) = \langle \Psi_n(\lambda) | \underbrace{H(\lambda) | \Psi_n(\lambda) \rangle}_{E_n(\lambda) | \Psi_n(\lambda) \rangle}$$

$$\frac{dE_n(\lambda)}{d\lambda} = \langle \Psi_n(\lambda) | \frac{dH}{d\lambda} | \Psi_n(\lambda) \rangle + \frac{d}{d\lambda}(\langle \Psi_n(\lambda) |) + H(\lambda) | \Psi_n(\lambda) \rangle$$
$$+ \langle \Psi_n(\lambda) | H(\lambda) | \frac{d}{d\lambda} \Psi_n(\lambda) \rangle$$

$$H(\lambda) = H_0 + \lambda H_1$$
$$= H_0 + \Delta H$$

$$\frac{dE_n(\lambda)}{d\lambda} = \langle \Psi_n(\lambda) | H_1 | \Psi_n(\lambda) \rangle$$

$$\left.\frac{dE_n(\lambda)}{d\lambda}\right|_{\lambda=0} = \langle \Psi_n(0) | H_1 | \Psi_n(0) \rangle$$

$$E_n(\lambda) = E_n(0) + \lambda \left.\frac{dE_n(\lambda)}{d\lambda}\right|_{\lambda=0} + O(\lambda^2)$$

$$E_n(\lambda) = E_n(0) + \langle \Psi_n(0) | \lambda H_1 | \Psi_n(0) \rangle + O(\lambda^2)$$

24: Addition of Angular Momentum

$E_n(\lambda) \equiv E_n(0) + \langle \psi_n(0) | \Delta H | \psi_n(0) \rangle + O(\Delta H^2)$

J_i^1 algebra of angular momentum

$[J_i^1, J_j^1] = i\hbar \epsilon_{ijk} J_k^1$ on V_1

J_i^2 " " on V_2

$J_i = J_i^1 \otimes \mathbb{1} + \mathbb{1} \otimes J_i^2$ in $V_1 \otimes V_2$

$[J_i, J_j] = [J_i^1 \otimes \mathbb{1} + \mathbb{1} \otimes J_i^2, J_j^1 \otimes \mathbb{1} + \mathbb{1} \otimes J_j^2$

$= [J_i^1 \otimes \mathbb{1} + J_j^1 \otimes \mathbb{1}] + [\mathbb{1} \otimes J_j^2, \mathbb{1} \otimes J_i^2]$

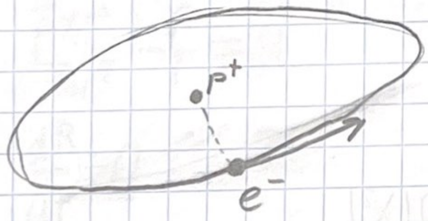
$= [J_i^1, J_j^1] \otimes \mathbb{1} + \mathbb{1} \otimes [J_i^2, J_j^2]$

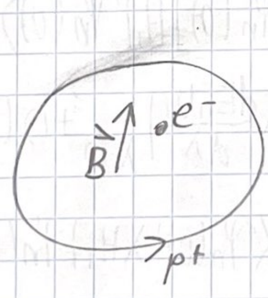
$= i\hbar \epsilon_{ijk} (J_k^1 \otimes \mathbb{1} + \mathbb{1} \otimes J_k^2)$

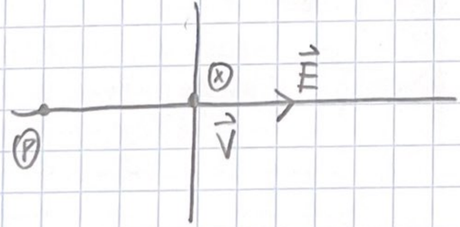
$= i\hbar \epsilon_{ijk} J_k$

$\Delta H = -\vec{\mu} \cdot \vec{B}$

$\vec{\mu} = -\frac{e}{m} \vec{S} \left(-\frac{ge}{mc} \vec{S} \right)$







$\vec{B}' \simeq \frac{-\vec{v} \times \vec{E}}{c}$

24: Addition of Angular momentum

$$\vec{B}' = \frac{\vec{E} \times \vec{v}}{c}$$

$$V(r) = -\frac{e^2}{r}$$

$$\frac{dV}{dr} = \frac{e^2}{r^2}$$

$$\vec{F} = \frac{dV}{dr}\frac{1}{e}\frac{\hat{r}}{r}$$

$$\vec{B}' = \frac{1}{ec}\frac{1}{r}\frac{dV}{dr}\hat{r} \times \vec{v} = \frac{1}{ecm}\frac{1}{r}\frac{dV}{dr}\vec{L}$$

$$P = mv$$

$$\Delta H = -\vec{\mu} \cdot \vec{B}$$

$$= \frac{ge}{2mc}(\vec{S} \cdot \vec{L})\frac{1}{r}\frac{dV}{dr}\frac{1}{ecm}$$

$$g \to g-1$$

$$a_0 = \frac{\hbar^2}{mc^2}$$

$$\alpha \equiv \frac{e^2}{\hbar c} = \frac{1}{137}$$

$$\Delta H = \frac{e}{mc}(\hbar^2)\frac{1}{r}\frac{e^2}{r^2}$$

$$= \frac{g}{2m^2c^2}(\vec{S} \cdot \vec{L})\frac{1}{r}\frac{dV}{dr}$$

$$= \frac{1}{m^2c^2}\frac{e^2}{a_0^3}\hbar^2$$

$$\Delta H = \frac{e^2}{8\pi\epsilon_0}\frac{1}{m^2c^2r^3}(\vec{S} \cdot \vec{L})$$

24: Addition of Angular Momentum

$|\ell m\rangle \otimes |\pm\rangle$
$\quad_3 \qquad\quad _2$

$|\ell m\rangle \otimes |s, m_s\rangle$

$|\frac{1}{2}, \frac{1}{2}\rangle$

$|\frac{1}{2}, -\frac{1}{2}\rangle$

$\vec{L} \cdot \vec{S} \stackrel{?}{=} S_1 \otimes L_1 + S_2 \otimes L_2 + S_3 \otimes L_3$

$= \sum_i S_i \otimes L_i$

$\vec{J} = \vec{L} + \vec{S}$

$\vec{L} \otimes \mathbb{1} + \mathbb{1} \otimes \vec{S}$

$\vec{J}^2 = \vec{L}^2 \otimes \mathbb{1} + \mathbb{1} \otimes \vec{S}^2 + 2\sum_i L_i \otimes S_i$

$\vec{J}^2 = L^2 + S^2 + 2\vec{L} \cdot \vec{S}$

$\vec{L} \cdot \vec{S} = \frac{1}{2}(\vec{J}^2 - \vec{L}^2 - \vec{S}^2)$

Hydrogen

$\{H, L^2, L_z\}$ cs of commuting observables

$\vec{S}_z, \vec{S}^2$

ΔH

24: Addition of Angular Momentum

$H_0 + \Delta H$

$\{H_{Tot}, L^2, J^2, S^2, J_z\}$

$[J, L^2] = [J^2, S^2]$

$|1,1\rangle \otimes |\tfrac{1}{2}, \tfrac{1}{2}\rangle$ (1) $\tfrac{3}{2}$

$|1,0\rangle \otimes |\tfrac{1}{2}, \tfrac{1}{2}\rangle$, $|1,1\rangle \otimes |\tfrac{1}{2}, \tfrac{1}{2}\rangle$ (2,3) $\tfrac{1}{2}$

$|1,0\rangle \otimes |\tfrac{1}{2}, -\tfrac{1}{2}\rangle$, $|1,-1\rangle \otimes |\tfrac{1}{2}, \tfrac{1}{2}\rangle$ (4,5) $-\tfrac{1}{2}$

$|1,-1\rangle \otimes |\tfrac{1}{2}, -\tfrac{1}{2}\rangle$ (6) $-\tfrac{3}{2}$

$$\frac{J_z}{\hbar} = \frac{1}{\hbar}(\vec{L}_z + \vec{S}_z)$$

$$1 \otimes \tfrac{3}{2} = 1 \oplus \tfrac{1}{2}$$

$|J = \tfrac{3}{2}, m = \tfrac{3}{2}\rangle = |1,1\rangle \otimes |\tfrac{1}{2}, \tfrac{1}{2}\rangle$

$|J = \tfrac{3}{2}\ m = -\tfrac{3}{2}\rangle = |1,-1\rangle \otimes |\tfrac{1}{2}, -\tfrac{1}{2}\rangle$

$$J_\pm |jm\rangle = \hbar\sqrt{j(j+1) - m(m+1)}\,|j\,m\pm 1\rangle$$

$$J_- |j = \tfrac{3}{2}, m = \tfrac{3}{2}\rangle = \hbar\sqrt{\tfrac{3}{2}\left(\tfrac{5}{2}\right) - \tfrac{3}{2}\left(\tfrac{1}{2}\right)}\,|\tfrac{3}{2}, \tfrac{1}{2}\rangle$$

$$= \hbar\sqrt{3}\,|\tfrac{3}{2}, \tfrac{1}{2}\rangle$$

$$J_- |1,1\rangle \otimes |\tfrac{1}{2}, \tfrac{1}{2}\rangle = (L_- + S_-)|1,1\rangle \otimes |\tfrac{1}{2}, \tfrac{1}{2}\rangle$$

$$= L_-|1,1\rangle \otimes |\tfrac{1}{2}, \tfrac{1}{2}\rangle + |1,1\rangle \otimes S_-|\tfrac{1}{2}, \tfrac{1}{2}\rangle$$

$$= \hbar\sqrt{1\cdot 2 - 1\cdot 0}\,|1,0\rangle \otimes |\tfrac{1}{2}, \tfrac{1}{2}\rangle + |1,1\rangle\,\hbar\sqrt{\tfrac{1}{2}\tfrac{3}{2} - \tfrac{1}{2}\left(-\tfrac{1}{2}\right)}\,|\tfrac{1}{2}, \tfrac{1}{2}\rangle$$

$$= \hbar\sqrt{2}\,|1,0\rangle \otimes |\tfrac{1}{2}, \tfrac{1}{2}\rangle + |1,1\rangle \otimes |\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$

24: Addition of Angular Momentum

$$\left|\tfrac{3}{2},\tfrac{1}{2}\right\rangle = \sqrt{\tfrac{2}{3}}\,|1,0\rangle \otimes \left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle + \tfrac{1}{\sqrt{3}}\,|1,1\rangle \otimes \left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle$$

$$\left|\tfrac{3}{2},-\tfrac{1}{2}\right\rangle = \sqrt{\tfrac{2}{3}}\,|1,0\rangle \otimes \left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle + \tfrac{1}{\sqrt{3}}\,|1,-1\rangle \otimes \left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle$$

$$\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle = -\tfrac{1}{\sqrt{3}}\,|1,0\rangle \otimes \left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle + \sqrt{\tfrac{2}{3}}\,|1,1\rangle \otimes \left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle$$

$$\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle = \tfrac{1}{\sqrt{3}}\,|1,0\rangle \otimes \left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle \sqrt{\tfrac{2}{3}}\,|1,-1\rangle \otimes \left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle$$

$$L\cdot S = \tfrac{1}{2}(J^2 - L^2 - S^2)$$

$$= \tfrac{\hbar}{2}\left(j(j+1) - \lambda(\lambda+1) - s(s+1)\right)$$

$$= \tfrac{\hbar}{2}\left(j(j+1) - 2 - \tfrac{3}{4}\right)$$