23.1: Energy Levels

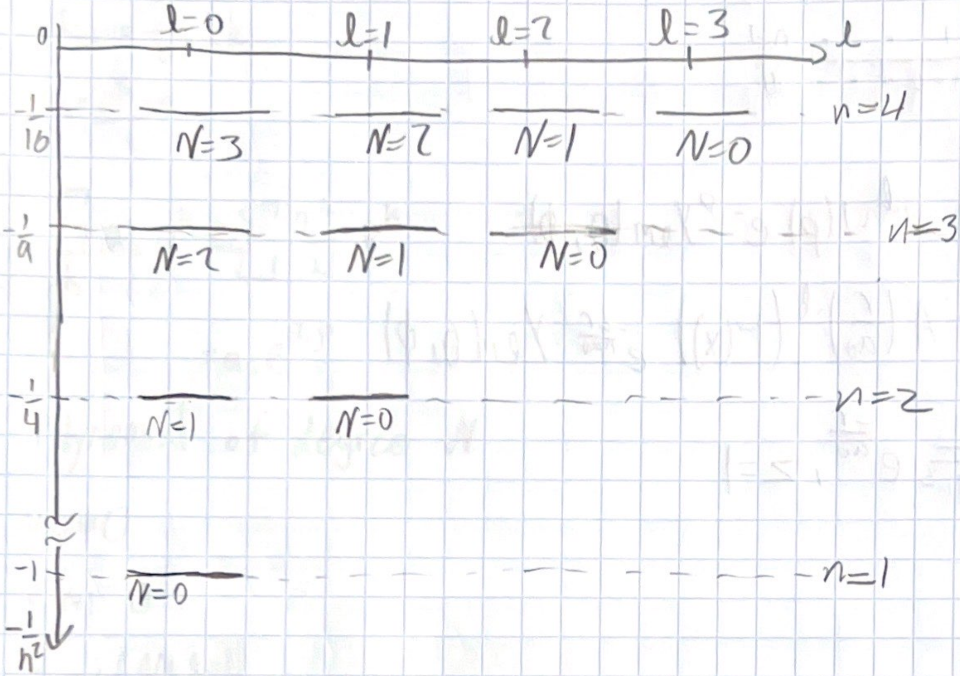
$$\kappa^2 = -\frac{E}{\frac{2Z^2e^2}{a_0}} > 0$$

$$\downarrow$$

$$\frac{1}{2\kappa} = N + l + 1 \equiv n$$

$$E_n = -\frac{Z^2e^2}{2a_0}\frac{1}{n^2}$$

$$\Psi_{nlm}(r,\theta,\phi) = A\left(\frac{r}{a_0}\right)^{l} \left(P(x) \text{ in } \frac{r}{a_0} \text{ in deg}(N-(l+1))\right) e^{-\frac{Zr}{na_0}} Y_{lm}(\theta,\phi)$$



23.2: Degeneracy spectrum

$$V_{eff}(r) = V(r) + \frac{\hbar^2 l(l+1)}{2mr^2}$$

$$Z=1 \quad \psi_{100} = \frac{1}{\sqrt{\pi a_0^3}} e^{-\frac{r}{a_0}}$$

$$Z \neq 1 ? \quad \left(V(r) - \frac{e^2}{r} \rightarrow V(r) = -\frac{Ze^2}{r}\right)$$

$$a_0 = \frac{\hbar^2}{me^2}$$

$$\frac{1}{Z}\frac{\hbar^2}{me^2} = \frac{a_0}{Z}$$

$$a_0 \rightarrow \frac{a_0}{Z}$$

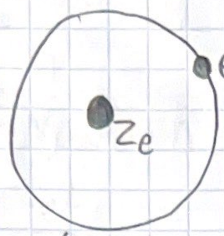
$$\psi_{100} = \sqrt{\frac{Z^3}{\pi a_0^3}} e^{-\frac{Zr}{a_0}}$$

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \frac{\# u}{r^2} - \frac{\#}{r}u = Eu$$

$$r \rightarrow \infty \Rightarrow \frac{\hbar^2}{2m}\frac{d^2u}{dr^2} = Eu$$

$$\frac{d^2u}{dr^2} = -\frac{2mE}{\hbar^2}u$$

23.3: Rydberg Atoms

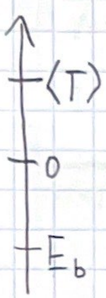


$(z-1)(e^-)$

If n is large, what is the size of the of the atom?

Virial: Theorem

$$\langle T\rangle = -\frac{1}{2}\langle V\rangle$$



$$\langle V\rangle = -2\langle T\rangle$$

$$E_b = \langle T\rangle + \langle V\rangle = -\langle T\rangle$$

$$\langle V\rangle = -2\langle T\rangle = -2(-E_b)$$

$$= E_b$$

$$\left\langle -\frac{e^2}{r}\right\rangle = 2\left(-\frac{e^2}{2a_0}\frac{1}{n^2}\right) \to \left\langle\frac{1}{r}\right\rangle = \frac{1}{n^2 a_0}$$

23.3: Rydberg Atoms

$$\langle r\rangle = n^2 a_0\left(1+\frac{1}{2}\left(1-\frac{\ell(\ell+1)}{n^2}\right)\right)$$

$$\begin{cases} n^2 a_0 \frac{3}{2} & \ell=0 \\ \approx n^2 a_0 & \ell = n-1 \end{cases}$$

$$\psi_{n\ell m} = A r^{\ell} W_{n\ell}\, e^{\frac{-r}{n a_0}}\, Y_{\ell m}(\theta,\phi)$$
$$= f_{n\ell}(r)\, Y_{\ell m}(\Omega)$$

$W_{n\ell}$ deg $(n-(n+1))$

$P(r)dr$

$$P(r)dr = |\psi|^2 d^3x$$
$$= r^2 dr \int |\psi|^2 d\Omega$$
$$= r^2 dr\, |f_{n\ell}|^2 \underbrace{\int Y^*_{\ell m} Y_{\ell m}\, d\Omega}_{1}$$

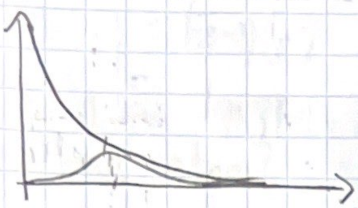
$$P(r) = r^2 |f_{n\ell}(r)|^2$$

23.3: Rydberg Atoms

$$f_{n\ell}(r) \sim r^{\ell}(a_0 + \dots a' r^{n-(\ell-1)}) e^{-\frac{r}{na_0}}$$

$$\sim r^{n-1} e^{-\frac{r}{na_0}}$$

$$P(r) \sim r^{2n} e^{-\frac{2r}{na_0}}$$



$$\max P(r) = \left(\frac{2n}{r} - \frac{2}{na_0}\right) r^{2n} e^{-\frac{2r}{na_0}}$$

$$\frac{n}{r} = \frac{1}{na_0}$$

$$r \cong n^2 a_0$$

$$n = 350$$

$$r = (0.53 \cdot 10^{-10} m)(350)^2 = 6.5 \mu m$$

$$E = \frac{1}{n^2}$$

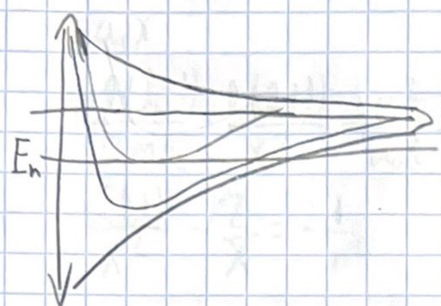
$$\Delta E \sim \frac{1}{n^3}$$

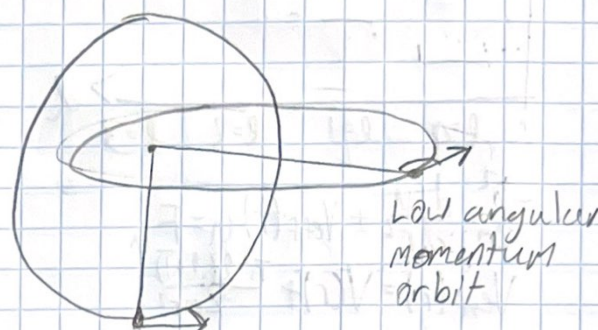
23.4: Orbits of Hydrogen

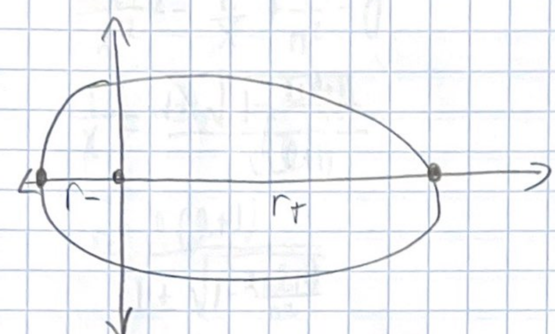
$n=100$, $l=0,\ldots,99$

$l=0$

$l=99$



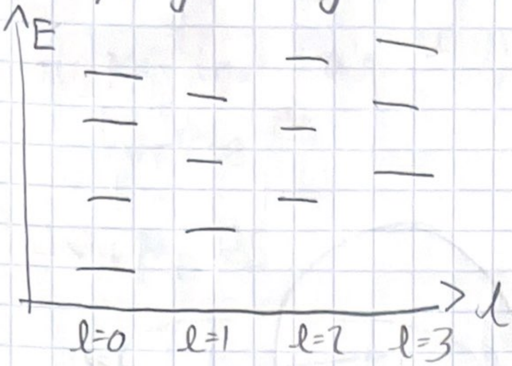




$r_+ \quad r_-$

$r_+ + r_- = \frac{n^2 a_0}{2}$

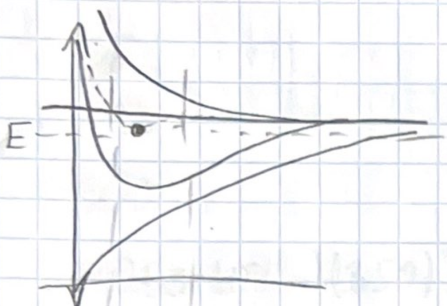
24.1: Hydrogen Degenaracies

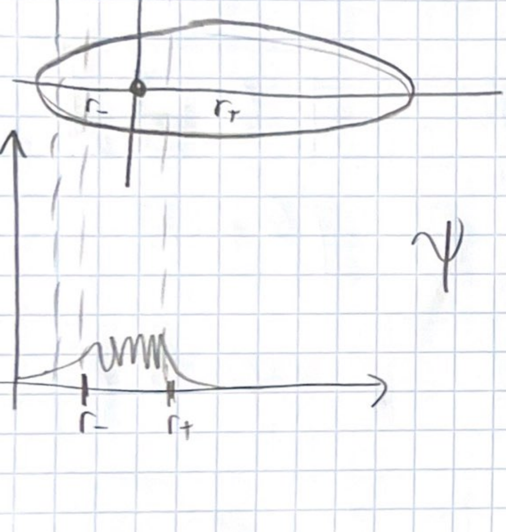


$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + V_{eff}(r)\,u = E\,u$$

$$V_{eff}(r) = V(r) + \frac{\hbar^2 l(l+1)}{2mr^2}$$

$$\psi = \frac{u(r)}{r} Y_{lm}$$





24.1: Hydrogen Degeneracies (continued)

$$\frac{\hbar^2 \ell(\ell+1)}{2mr} - \frac{e^2}{r} = \frac{e^2}{2a_0}\frac{1}{n^2}$$

$$\equiv E_n$$

$r \equiv a_0 x$

$$\frac{\hbar^2 \ell(\ell+1)}{2ma_0}\frac{\ell(\ell+1)}{x^2} - \frac{e^2}{a_0 x} = -\frac{e^2}{2a_0}\frac{1}{n^2}$$

$$\frac{\ell(\ell+1)}{x^2} - \frac{2}{x} = -\frac{1}{n^2}$$

$$\frac{\ell(\ell+1)}{x^2} - \frac{2}{x} + \frac{1}{n^2} = 0$$

$$\frac{1}{x} = \frac{1 \pm \sqrt{1 - \frac{\ell(\ell+1)}{n^2}}}{\ell(\ell+1)}$$

$$x = \frac{\ell(\ell+1)}{1 \pm \sqrt{1 - \frac{\ell(\ell+1)}{n^2}}}$$

$$r_\pm = n^2 a_0 \left| 1 \mp \sqrt{1 - \frac{\ell(\ell+1)}{n^2}}\right.$$

$$\frac{r_+ + r_-}{2} = n^2 a_0$$

24.2: Simple Quantum System

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}\Psi$$

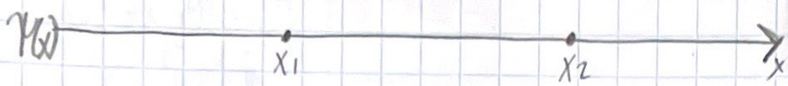
$$\rightarrow \Psi = e^{-\frac{iEt}{\hbar}}\psi$$

$$\rightarrow \hat{H}\psi = E\psi$$

$$(\varphi, \psi) \in \mathbb{C}$$

$$(A\varphi, \psi) = (\varphi, A^{\dagger}\psi)$$

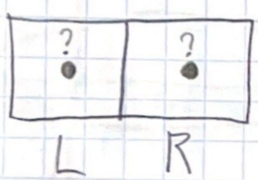
$$\hat{H}^{\dagger} = H$$



$$\psi(x) \rightarrow \begin{pmatrix} \psi(x_1) \\ \psi(x_2) \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$|\alpha|^2$ = probability x_1

$|\beta|^2$ = probability x_2



$$\begin{pmatrix} \text{amplitude L} \\ \text{amplitude R} \end{pmatrix}$$

24.2: Simple Quantum System (continued)

$$(\varphi, \psi) = \int \varphi^*(x)\psi(x)\,dx$$

$$\psi_1 = \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}$$

$$\psi_2 = \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}$$

$$(\psi_1, \psi_2) = \alpha_1^*\alpha_2 + \beta_1^*\beta_2$$

$$= (\alpha_1^*, \beta_1^*)\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}$$

$$\psi(x) = \begin{pmatrix} \vdots \\ \psi(-\epsilon) \\ \psi(0) \\ \psi(\epsilon) \\ \vdots \end{pmatrix}$$

H: Hermitian:

$$(H^T)^* = H$$

24.3: Hamiltonians and Spin

$$H=\begin{pmatrix} a_0+a_3 & a_1-ia_2 \\ a_1+ia_2 & a_0-a_3 \end{pmatrix}$$

$\{a_0, a_1, a_2, a_3\} \in \mathbb{R}$

$$a_0\begin{pmatrix}1&0\\0&1\end{pmatrix}+a_1\underbrace{\begin{pmatrix}0&1\\1&0\end{pmatrix}}_{\sigma_1}+a_2\underbrace{\begin{pmatrix}0&-i\\i&0\end{pmatrix}}_{\sigma_2}+a_3\underbrace{\begin{pmatrix}1&0\\0&-1\end{pmatrix}}_{\sigma_3}$$

Pauli Matrices: $\{\sigma_1, \sigma_2, \sigma_3\}$

$$H=\frac{\hbar\omega_1}{2}\sigma_1+\frac{\hbar\omega_2}{2}\sigma_2+\frac{\hbar\omega_3}{2}\sigma_3$$

$$H=\hbar\omega(\hat{N}+\tfrac{1}{2})$$

$$H=\omega_1(\tfrac{\hbar}{2}\sigma_1)+\omega_2(\tfrac{\hbar}{2}\sigma_2)+\omega_3(\tfrac{\hbar}{2}\sigma_3)$$

$$S_x=\frac{\hbar}{2}\sigma_1$$

$$S_y=\frac{\hbar}{2}\sigma_2$$

$$S_z=\frac{\hbar}{2}\sigma_3$$

$$[S_x,S_y]=\left[\frac{\hbar}{2}\sigma_1,\frac{\hbar}{2}\sigma_2\right]=\frac{\hbar}{2}\frac{\hbar}{2}(\sigma_1\sigma_2-\sigma_2\sigma_1)$$

$$=\frac{\hbar}{2}\frac{\hbar}{2}\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}-\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\right)$$

24.3: Hamiltonians and Spin (continued)

$$[S_x, S_y] = \frac{\hbar}{2}\frac{\hbar}{2}\left[\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\right]$$

$$= \frac{\hbar}{2}\frac{\hbar}{2}\begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix}$$

$$= i\hbar\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{cases} [S_x, S_y] = i\hbar S_z \\ [S_y, S_z] = i\hbar S_x \\ [S_x, S_z] = i\hbar S_y \end{cases}$$

$$H = \omega_1 \hat{S}_x + \omega_2 \hat{S}_y + \omega_3 \hat{S}_z$$
$$= \vec{\omega} \cdot \hat{S}$$

24.4: Eigenstates of the Hamiltonian

L^2, L_z

S^2, S_z

$$\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv |\uparrow\rangle$$

$$S_z|\uparrow\rangle = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{\hbar}{2}|\uparrow\rangle$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv |\downarrow\rangle$$

$$S_z|\downarrow\rangle = \frac{-\hbar}{2}|\downarrow\rangle$$

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$$

$$S_x \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{\hbar}{2}\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{\hbar}{2}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle) = |\uparrow : \hat{x}\rangle$$

24.4: Eigenstates of the Hamiltonian (continued)

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle - |\downarrow\rangle\right) = |\downarrow;\hat{x}\rangle$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + i|\downarrow\rangle\right)$$

$$\hat{S}_y \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}1\\i\end{pmatrix} = \frac{\hbar}{2}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} = |\uparrow;\hat{y}\rangle$$

$$\frac{1}{\sqrt{2}}\left(|\uparrow\rangle - i|\downarrow\rangle\right) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix} = |\downarrow;\hat{y}\rangle$$