1: Wave Mechanics

Schrödinger Equation

$$i\hbar \frac{\partial}{\partial t}\Psi(x,t) = \left(-\frac{\hbar^2}{2m}\frac{\partial}{\partial x^2} + V(x,t)\right)\Psi(x,t)$$

$\Psi(x,t)$ (wave function) $\in \mathbb{C}$

E, M

$\vec{E}(z,t) = Re\left(E_0(\hat{x}+i\hat{y})e^{i(kz-\omega t)}\right)$

$\vec{E}$ is real and all "i"s are auxiliary

1st order differential equation in time

$\Psi(x,t_0)$ determines Ψ at all times

Equation is linear

$\Psi_1, \Psi_2, \Rightarrow a_1\Psi_1 + a_2\Psi_2$ is solution
$a_1, a_2 \in \mathbb{C}$

$z = a + ib$

$a, b \in \mathbb{R}$

$z^* = a - ib$
$|z| = \sqrt{a^2+b^2} = \sqrt{zz^*}$

$P(x,t) \equiv \rho(x,t) = \Psi^*(x,t)\Psi(x,t)$

1: Wave Mechanics

$P(x,t)dx$ = probability to find particle in interval $[x, x+dx]$ in time t

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = 1 \quad \forall t$$

$$[\Psi] = \frac{1}{\sqrt{L}}$$

Suppose there exists a wave function such that:

$$\int_{-\infty}^{\infty} |\Psi(x,t_0)|^2 dx = 1$$

Quick calculation to show:

$$\frac{d}{dt}\int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = 0$$

$$\Psi(x) = Nx\exp(-\tfrac{1}{2}\alpha x^2),\ \alpha > 0$$

$$|\Psi(x)|^2 = \int_{-\infty}^{\infty} |Nx\exp(-\tfrac{1}{2}\alpha x^2)|^2 dx$$

1: Wave Mechanics

Probability Current:

$J(x,t) = \frac{\hbar}{m} \text{Im}\left(\Psi^* \frac{\partial \Psi}{\partial x}\right)$

$\rho(x,t)$ = probability density

E, M

$\vec{J}, \rho$

$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$

$\frac{\partial J}{\partial x} + \frac{\partial \rho}{\partial t} = 0$

$P_{ab}(t) = \int_a^b |\Psi|^2 dx$

$\frac{dP_{ab}}{dt} = J(a,t) - J(b,t)$

Ψ_1, Ψ_2 are equivilant if $\Psi_1(x,t) = \alpha \Psi_2(x,t)$

$\alpha \in \mathbb{C}$

1: Wave Mechanics

Stationary Solution:

Assume V is time independent

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}\Psi$$

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)$$

$$\Psi(x,t) = e^{-\frac{iEt}{\hbar}}\psi$$

$\psi(x) \in \mathbb{C}$ but is time independant

$$\rightarrow i\hbar \frac{\partial}{\partial t} e^{-\frac{iEt}{\hbar}}\psi(x) = e^{-\frac{iEt}{\hbar}}\hat{H}\psi$$

$$E\psi(x) = \hat{H}\psi$$

Time independent Schrödinger Equation

2nd order differential equation in space

$$\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi(x) = E\psi(x) = \hat{H}\psi$$

$\psi(x_0)$, $\psi'(x_0)$ suffise for solution

1: Wave Mechanics

$$\begin{array}{c} \Psi(x_0)=0 \\ \Psi'(x_0)=0 \\ \Downarrow \\ \Psi(x)=0 \end{array}$$

Solution for Ψ is an energy eigenstates

set of values of E is the spectrum

A degenerate spectrum is more than 1 Ψ for a given E

$$\text{Potentials} = \begin{cases} \text{failure of continuity} \\ \text{failure to be bounded} \\ \text{inclusion of } \delta \text{ functions } (\notin \delta'' \ \delta') \end{cases}$$

$\Psi(x)$ is continuous and bounded and its derivative Ψ' is bounded.

Eigenstates of $\hat{H}$:

$E_0 \leq E_1 \leq E_2 \cdots$

$\Psi_0, \Psi_1, \cdots$

$\hat{H}\Psi_n = E\Psi_n$

1: Wave Mechanics

$\psi_m \psi_n$

$$\int_{-\infty}^{\infty} \psi_m^*(x)\psi_n(x)\,dx = \delta_{mn} \Rightarrow \text{Orthonormal}$$

Completeness:

$$\psi(x) = \sum_{n=0}^{\infty} b_n \psi_n(x)$$

$b_n \in \mathbb{C}$

$$\Psi \rightarrow \Psi(x,t=0) = \psi(x) = \sum b_n \psi_n(x)$$

$$\Psi(x,t) = \sum_{n=0}^{\infty} b_n e^{-\frac{iE_n t}{\hbar}} \psi_n(x)$$

$$b_m = \int_{-\infty}^{\infty} \psi_m^*(x)\,\psi(x)\,dx$$

$$\psi(x) = \sum_{n=0}^{\infty} \left(\int_{-\infty}^{\infty} \psi_n^*(x')\psi(x')\,dx'\right)\psi_n(x)$$

$$= \int_{-\infty}^{\infty} dx' \sum_{n=0}^{\infty} \psi_n^*(x')\psi_n(x))\,\psi(x')$$

$$= \int K(x,x')\,\psi(x')\,dx'$$

1: Wave Mechanics

$$\sum_{n=0}^{\infty} \psi_n^*(x') \psi_n(x) = \delta(x-x')$$

$$\psi(x)$$

m

$$\langle \hat{A} \rangle_\Psi (t) \equiv \int_{-\infty}^{\infty} \Psi^*(x,t) (\hat{A} \Psi(x,t)) dx$$

Ψ is normalized

$$\langle \hat{H} \rangle_\Psi (t) = \int_{-\infty}^{\infty} \Psi^*(x,t) (\hat{H} \Psi(x,t)) dx$$

$$\Psi = \sum_{n=0}^{\infty} b_n \psi_n$$

$$= \sum_{n=0}^{\infty} |b_n|^2 E_n$$

Ψ not normalized

$\dfrac{\Psi(x,t)}{\sqrt{\int \Psi^* \Psi dx}}$ will normalize it

$$\langle \hat{A} \rangle_\Psi = \frac{\int_{-\infty}^{\infty} \Psi^* \hat{A} \Psi dx}{\int_{-\infty}^{\infty} \Psi^* \Psi dx}$$

2: Wave Mechanics

Energy Eigenstates

if $\Psi(x) \to 0$ when $|x| \to \infty$

$\hookrightarrow \frac{d^2\Psi}{dx^2} + \frac{2m}{\hbar^2}(E - V(x))\Psi = 0$

$$\mathcal{E} \equiv \frac{2m}{\hbar^2} E$$

$$\mathcal{V}(x) \equiv \frac{2m}{\hbar} V(x)$$

$$\Psi''(x) + (\mathcal{E} - \mathcal{V}(x))\Psi = 0$$

There is no degeneracy for bound states of one dimensional Potentials

Assume a degeneracy $\Psi_1 \Psi_2$ different with the same energy

$$\Psi_1(x) = c\Psi_2(x)$$

Energy eigenstates of $\Psi(x)$ can be chosen to be real

complex solution $\Psi(x)$

$\rightarrow$ implies existence of two real solutions (degenerate)

$\Psi_{re}(x) = \frac{1}{2}(\Psi + \Psi^*)$ $\qquad$ $\Psi_{im} = \frac{1}{2i}(\Psi - \Psi^*)$

2: Wave Mechanics

Corrilary for bound states of one dimensional potentials, any solution is up to a phase equal to a real solution

$\psi_{im}(x) = c\psi_{re}(x)$

$c \in \mathbb{R}$

$\psi = \psi_{re} + i\psi_{im} = (1+ic)\psi_{re}(x)$

$= |1+ic| e^{i\beta} \psi_{re}(x)$

If the potential $V(x)$ is even $V(-x) = V(x)$, the eigenstates can be chosen to be even or odd under $x \to -x$

begin with wavefunction $\psi(x)$ that is neither even or odd.

$\psi(-x)$ is a solution with same energy

$\psi_s \equiv \frac{1}{2}(\psi(x) + \psi(-x))$

$\psi_a \equiv \frac{1}{2}(\psi(x) - \psi(-x))$

Corrilary for bound states in one dimension the solutions are either odd or even.

(with $V(x) = V(-x)$)

2: Wave Mechanics

$\psi(-x) = c\psi(x)$

$c \in \mathbb{R}$

$= c(c\psi(-x))$

$c = \pm 1$

Nature of the spectrum:

$\psi'' = -\frac{2m}{\hbar}(E - V(x))\psi$

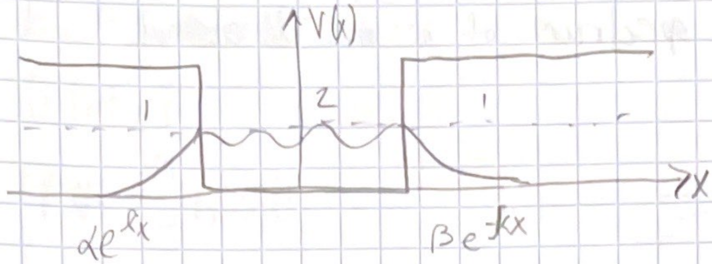
ψ must be continuous

Possibilities for $V(x)$

$$\begin{cases} V \text{ is continuous, } \psi' \text{ is continuous} \\ V \text{ has finite jumps, } \psi' \text{ is continuous} \\ V \text{ has } \delta \text{ functions, } \psi' \text{ is discontinuous} \\ V \text{ has hard wall, } \psi' \text{ is discontinuous} \end{cases}$$

ψ and ψ' are continuous unless V has δ or hard walls in which case ψ' can have finite jumps

2: Wave Mechanics



4 parameters → 3 constant

Discrete nondegenerate spectrum

E continuous spectrum, nondegenerate

E continuous spectrum, doubly degenerate

2: Wave Mechanics

Discrete bound state spectrum of a one dimensional potential:

$$E_1 < E_2 < E_3 < \ldots$$

$$\psi_1(x), \psi_2(x), \psi_3(x) \ldots$$

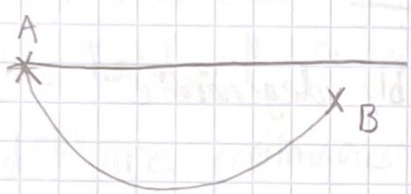
ψ_1 has no nodes

ψ_2 has one node

$\vdots$

ψ_n has $n-1$ nodes

Johann Bernoulli (1696)



Brachistochrome minimizes time

... Eigenfunctions?

$\hat{H}\psi = E\psi(\vec{x})$ $\quad \int |\psi|^2 d\vec{x} = 1$

$$\int \psi^*(x)\hat{H}\psi(x)\,dx = \langle \hat{H} \rangle_\psi \geq E_{gs} \quad \forall \psi$$

2: Wave Mechanics

$$E_{gs} = E_1 \le E_2 \le E_3 \le \dots$$

$$\hat{H}\psi_n = E_n \psi_n$$

$$\psi(\vec{x}) = \sum_{n=0}^{\infty} b_n \psi_n(\vec{x})$$

$$\int |\psi|^2 dx = 1$$

$$\sum_{n=0}^{\infty} |b_n|^2 = 1$$

$$\int \psi^* \hat{H} \psi = \sum_{n=1}^{\infty} |b_n|^2 E_n$$

$$\sum_{n=1}^{\infty} |b_n|^2 E_1 = E_1 = E_{gs}$$

If $\psi(x)$ is not normalized

then $\dfrac{\psi(x)}{\sqrt{\int |\psi|^2 dx}}$ is normalized

$$\frac{\int \psi^* \hat{H} \psi dx}{\int \psi^* \psi dx} \geq E_{gs} = F[\psi]$$

functional

2: Wave Mechanics

$V(x) = -\delta(x) \quad \alpha > 0$

$E_{gs} = -\frac{m\alpha^2}{2\hbar^2}$

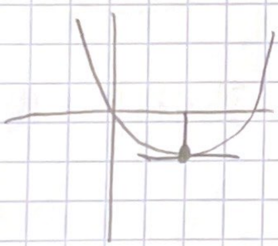
$\psi(x) = e^{-\frac{1}{2}\beta^2 x^2}$ trial

$\int |\psi|^2 dx = \sqrt{\frac{\pi}{\beta}}$

$$\frac{\int \psi H \psi}{\int \psi \psi} = \frac{\beta}{\sqrt{\pi}} \int e^{-\frac{1}{2}\beta^2 x^2}\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \alpha\delta(x)\right) e^{-\frac{1}{2}\beta^2 x^2} dx$$

$$= -\frac{\beta\alpha}{\sqrt{\pi}} + \frac{\beta}{\sqrt{\pi}}\frac{\hbar^2}{2m}\int \left(\frac{d}{dx} e^{-\frac{1}{2}\beta^2 x^2}\right)^2 dx$$

$$= -\frac{\beta}{\sqrt{\pi}}\alpha + \frac{\beta^2\hbar^2}{4m}$$



$$E_{gs} \leq -\frac{m\alpha^2}{\pi\hbar^2} = \frac{2}{\pi}\left(-\frac{m\alpha^2}{2\hbar}\right)^2$$

3: Stern-Gerlach

Position, Momentum = $\hat{x}, \hat{p}$

$[\hat{x}, \hat{p}] = i\hbar$

$\psi(x),\ \hat{x}\psi(x) \equiv x\psi(x)$

$\psi(x): 0 \le x \le a$

$N\epsilon = a$

$\psi(x) \longrightarrow \begin{pmatrix} \psi(0) \\ \psi(\epsilon) \\ \vdots \\ \psi(N\epsilon) \end{pmatrix}$ $(N+1)$ column vector

$(N+1)\times(N+1)$

$\hat{x} \longleftrightarrow \begin{pmatrix} 0 & & & 0 \\ \vdots & \epsilon & & \vdots \\ \vdots & & 2\epsilon & \vdots \\ 0 & & \ddots & N\epsilon \end{pmatrix} \rightarrow \hat{x}\psi(x) = \begin{pmatrix} 0\cdot\psi(0) \\ \epsilon\cdot\psi(\epsilon) \\ \vdots \\ N\epsilon\cdot\psi(N\epsilon) \end{pmatrix}$

$\langle \hat{x} \rangle_\psi = \int \psi^*(x)(\hat{x}\psi(x))\,dx = \int \psi^*(x)\,x\,\psi(x)\,dx$

Eigenstates of $\hat{x}$: $\psi_{x_0}(x) \equiv \delta(x - x_0)$

$$\hat{x}\psi_{x_0}(x) = x\psi_{x_0}(x)$$
$$= x\delta(x - x_0)$$
$$= x_0\delta(x - x_0)$$
$$= x_0\psi_{x_0}(x)$$

3: Stern Gerlach

$\hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x}$

$\hat{p}\,\Psi(x) = \frac{\hbar}{i}\frac{\partial \Psi(x)}{\partial x}$

$[\hat{x},\hat{p}]\Psi(x) \stackrel{?}{=} i\hbar\Psi(x)$

$(\hat{x}\hat{p}-\hat{p}\hat{x})\Psi(x) = \hat{x}\hat{p}\Psi - \hat{p}\hat{x}\Psi$

$= x\left(\frac{\hbar}{i}\frac{\partial \Psi}{\partial x}\right) - \hat{p}(x\Psi)$

$= \frac{\hbar}{i}x\frac{\partial \Psi}{\partial x} - \frac{\hbar}{i}\frac{\partial}{\partial x}(x\Psi)$

$= -\frac{\hbar}{i}\Psi = i\hbar\Psi$

Eigenstates of $\hat{p}$

$\Psi_p(x) \equiv \frac{e^{ipx/\hbar}}{\sqrt{2\pi\hbar}}$

$\hat{p}\,\Psi_p(x) = \frac{\hbar}{i}\frac{\partial}{\partial x}\Psi_p$

$= p\,\Psi_p(x)$

$\Psi(x) = \int_{-\infty}^{\infty}\frac{e^{ipx/\hbar}}{\sqrt{2\pi\hbar}}\tilde{\Psi}(p)\,dp$

$\tilde{\Psi}(p) = \int_{-\infty}^{\infty}\frac{e^{-ipx/\hbar}}{\sqrt{2\pi\hbar}}\Psi(x)\,dx$

$\tilde{\Psi}_p = M_{px}\Psi_x$

3: Stern Gerlach

$\Psi(x) \leftrightarrow \tilde{\Psi}(p)$

$$\frac{\hbar}{i}\frac{\partial}{\partial x}\Psi(x) = \frac{\hbar}{i}\frac{\partial}{\partial x}\int_{-\infty}^{\infty}\frac{e^{ipx/\hbar}}{\sqrt{2\pi\hbar}}\tilde{\Psi}(p)\,dp$$

$$= \int_{-\infty}^{\infty}\frac{e^{ipx/\hbar}}{\sqrt{2\pi\hbar}}\,p\tilde{\Psi}(p)\,dp$$

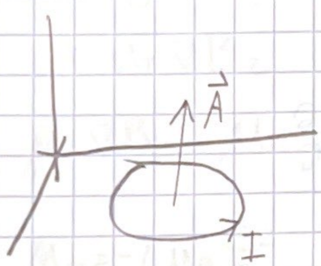
$$\frac{\hbar}{i}\frac{\partial}{\partial x}\Psi \leftrightarrow p\tilde{\Psi}(p) = \hat{p}\Psi(x)$$

$$\hat{p}\tilde{\Psi}(p) \equiv p\tilde{\Psi}(p)$$

Stern-Gerlach Experiment

Magnetic moment μ

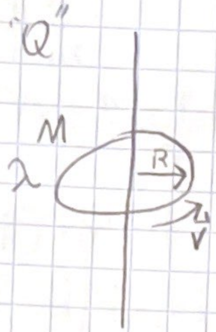
$$\vec{\mu} = I\vec{A}$$



Units

$$[\mu] = \frac{[J]}{[T]}$$

3: Stern Gerlach



$I = \lambda v = \frac{Q}{2\pi R} v$

$M = IA = \frac{Q}{2\pi R} v \cdot \pi R^2 = \frac{1}{2} Q v R$

$L = M v R$

$M = \frac{1}{2} \frac{Q}{m} L$

$M = \frac{e}{2m_e} s = \frac{e\hbar}{2m_e} \left(\frac{s}{\hbar}\right)$

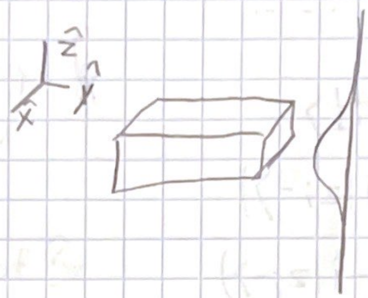
$M = g \frac{e\hbar}{2m} \frac{s}{\hbar}$

$g_l = 2$

3: Stern Gerlach

$\vec{\mu} = -2\mu_B \frac{\vec{S}}{\hbar}$

$\vec{F} = \nabla(\vec{\mu} \cdot \vec{B})$



$\vec{F} \approx \nabla(\mu_z B_z)$

$\approx \mu_z \nabla B_z$

$\mu_z \nabla B_z \approx \mu_z \frac{\partial B_z}{\partial z}$

$\mu_z = -2\mu_B \frac{S_z}{\hbar}$

$\left(\frac{S_z}{\hbar}\right) = \pm\frac{1}{2}$ spin $\frac{1}{2}$ particle

$|z,+\rangle$

$|z,-\rangle$