14.1: Recursion Relation to Solution

$$\frac{d^2h}{du^2} - 2u\frac{dh}{du} + (\varepsilon - 1)h = 0$$

$$h(u) = \sum_{k=0}^{\infty} a_k u^k$$

$$(a_j u^j + a_{j+1} u^{j+1} \ldots\ldots)$$

Terms with u^j in differential equation

$$\frac{d^2h}{du^2}: (j+2)(j+1)a_{j+2}u^j$$

$$-2u\frac{dh}{du}: -2ja_ju^j$$

$$(\varepsilon-1)h: (\varepsilon-1)a_ju^j$$

$$\sum_{j=0}^{\infty}\left((j+2)(j+1)a_{j+2} - 2ja_j + (\varepsilon-1)a_j\right)u^j$$

$$(j+2)(j+1)a_{j+2} = (2j+1-\varepsilon)a_j$$

$$a_{j+2} = \frac{(2j+1-\varepsilon)}{(j+2)(j+1)}a_j$$

$j = 0, 1, 2 \ldots$

$a_0 \rightarrow a_2 \rightarrow a_4$

$a_1 \rightarrow a_3 \rightarrow a_5$

$$(a_0, a_1)$$
$$\downarrow \quad \downarrow$$
$$h(0) \; h'(0)$$

14.2: Quantization of Energy

$$\frac{a_{j+2}}{a_j}\ (j\to\infty) \cong \frac{2j}{j^2} = \frac{2}{j}$$

$$e^{u^2} = \sum_{n=0}^{\infty} \frac{1}{n!} u^{2n}$$

$$= \sum_{j=0,2,4,\dots} \underbrace{\frac{1}{\left(\frac{j}{2}\right)!} u^j}_{c_j u^j}$$

$$c_j = \frac{1}{\left(\frac{j}{2}\right)!}$$

$$\frac{c_{j+2}}{c_j} = \frac{\left(\frac{j}{2}\right)!}{\left(\frac{j+2}{2}\right)!} = \frac{1}{\frac{j+2}{2}} = \frac{2}{j+2} \approx \frac{2}{j}$$

If the series does not truncate, then it will diverge

$h(u) \sim e^{u^2}$

$\varphi(u) \sim e^{\frac{u^2}{2}} \Rightarrow$ not a solution

$\exists\ j$ such that $2j+1 = \varepsilon$

$a_{j+2} = 0$

$h(u) = a_j u^j + a_{j-2} u^{j-2} + \dots$

14.2: Quantization of Energy (continued)

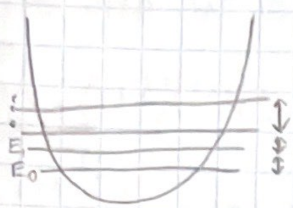
Call $j = n$

$\varepsilon = 2n+1$

$h(u) = a_n u^n + a_{n-2} u^{n-2} + \dots$

$E = \frac{\hbar\omega}{2}(2n+1)$

$E_n = \hbar\omega(n+\frac{1}{2})$



$\varepsilon = 2n+1$

$$\frac{d^2 H_n(u)}{du^2} - 2u\frac{dH_n}{du} + 2nH_n(u) = 0$$

Hernete's Differential Equation

$H_n(u) = 2^n u^n \pm \dots u^{n-2} \dots$

$H_0(u) = 1$

$H_1(u) = 2u$

$H_2(u) = 4u^2 - 2$

$H_3(u) = 8u^3 - 12u$

$$e^{-z^2+2zu} = \sum_{n=0}^{\infty} \frac{z^n}{n!} H_n(u)$$

$u = \frac{x}{a}$

$\varphi_n(x) = H_n(\frac{x}{a})e^{-\frac{x^2}{2a^2}}, \quad a^2 = \frac{\hbar}{m\omega}$

Solutions where $E_n = \hbar\omega(n+\frac{1}{2})$

14.3: Algebraic Solution to SHO

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$

$$= \frac{1}{2}m\omega^2\left(\hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2}\right)$$

$$H = V^\dagger V + \#$$

$$(AB)^\dagger = B^\dagger A^\dagger$$

$$\hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2} \overset{?}{=} \left(\hat{x} - i\frac{\hat{p}}{m\omega}\right)\left(\hat{x} + i\frac{\hat{p}}{m\omega}\right)$$

$$= \hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2} + \frac{i}{m\omega} - \underbrace{[\hat{x}, \hat{p}]}_{i\hbar}$$

$$\hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2} = \underbrace{\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right)}_{V^\dagger}\underbrace{\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right)}_{V} + \frac{\hbar}{m\omega}$$

$$(\varphi, A\psi) = (A^\dagger\varphi, \psi)$$

$$\hat{H} = \frac{1}{2}m\omega^2\left(V^\dagger V + \frac{\hbar}{m\omega}\right)$$

$$\hat{H} = \frac{1}{2}m\omega^2 V^\dagger V + \frac{1}{2}\hbar\omega$$

$$[V, V^\dagger] = \left[x + \frac{i\hat{p}}{m\omega}, \hat{x} - \frac{i\hat{p}}{m\omega}\right]$$

$$= -\frac{i}{m\omega}[\hat{x}, \hat{p}] + \frac{i}{m\omega}[\hat{p}, \hat{x}]$$

14.3: Algebraic Solution to SHO (continued)

$$[V, V^\dagger] = \frac{2\hbar}{m\omega}$$

$$\left[\sqrt{\frac{m\omega}{2\hbar}}V, \sqrt{\frac{m\omega}{2\hbar}}V^\dagger\right] = 1$$

$$\hat{a} \equiv \sqrt{\frac{m\omega}{2\hbar}}V$$

$$\hat{a}^\dagger \equiv \sqrt{\frac{m\omega}{2\hbar}}V^\dagger$$

$$[a, a^\dagger] = 1$$

$$\left.\begin{aligned}
a &= \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right) \\
a^\dagger &= \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right) \\
\hat{x} &= \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger) \\
\hat{p} &= i\sqrt{\frac{m\omega\hbar}{2}}(a^\dagger - a)
\end{aligned}\right\}$$

14.4: Ground State Wavefunction

$$V^\dagger V = \frac{2\hbar}{m\omega} a^\dagger a$$

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right)$$

$$(\varphi, \psi) = \int \varphi^*(x)\psi(x)\,dx$$

$$\langle H \rangle_\Psi = (\psi, \hat{H}\psi)$$

$$= \left(\psi, \hbar\omega a^\dagger a \psi + \frac{\hbar\omega}{2}\psi\right)$$

$$= \hbar\omega(\psi, a^\dagger a\psi) + \frac{\hbar\omega}{2}(\psi, \psi)$$

$$= \hbar\omega(a\psi, a\psi) + \frac{\hbar\omega}{2} \geq \frac{\hbar\omega}{2}$$

$$(\varphi, \varphi) \geq 0$$

$E = \frac{\hbar\omega}{2}$, then $a\psi = 0$

$$\left(x + \frac{ip}{m\omega}\right)\psi(x) = 0$$

$$\left(x + \frac{\hbar}{m\omega}\frac{d}{dx}\right)\psi(x) = 0$$

$$\frac{d\psi_0}{dx} = -\frac{m\omega}{\hbar}x\psi_0$$

$$N e^{-\frac{m\omega}{2\hbar}x^2} = \psi_0(x)$$

$$\hat{H}\psi_0 = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)\psi_0 = \frac{\hbar}{2}\psi_0$$

$\hat{H} = \hbar\omega\left(\hat{N} + \frac{1}{2}\right)$, $\hat{N} \equiv a^\dagger a$ $\quad \hat{N}\psi_0 = 0$

15.1: Number Operators and Commutators

$\hat{H} = \hbar\omega(\hat{a}^{+}\hat{a} + \frac{1}{2}) = \hbar\omega(\hat{N} + \frac{1}{2})$, $\hat{N} = \hat{a}^{+}\hat{a}$

$\hat{a} \sim \hat{x} + \frac{i\hat{p}}{m\omega}$

$\hat{a}^{+} \sim \hat{x} - \frac{i\hat{p}}{m\omega}$

$\hat{a}\varphi_0(x) = 0$ ground state $\hat{N}\varphi_0 = 0$

$E = \hbar\omega(N + \frac{1}{2})$

$$[\hat{N}, \hat{a}] = [\hat{a}^{+}\hat{a}, \hat{a}]$$

$$= [\hat{a}^{+}, \hat{a}]\hat{a}$$

$$= -a$$

$$[\hat{N}, \hat{a}^{+}] = [\hat{a}^{+}\hat{a}, \hat{a}^{+}]$$

$$= \hat{a}^{+}[\hat{a}, \hat{a}^{+}]$$

$$= \hat{a}^{+}$$

$$[\hat{N}, \hat{a}] = -\hat{a}$$

$$[\hat{N}, \hat{a}^{+}] = a^{+}$$

15.1: Number Operators and Commutators (continued)

$$[\hat{a}, (\hat{a}^{+})^{k}] = k(\hat{a}^{+})^{k-1}$$

$$[\hat{a}^{+}, (\hat{a})^{k}] = -k(\hat{a})^{k-1}$$

$$[\hat{N}, (\hat{a})^{k}] = -k(\hat{a})^{k}$$

$$[\hat{N}, (\hat{a}^{+})^{k}] = k(\hat{a}^{+})^{k}$$

Important power commutators

15.2: Exited States of the Harmonic Oscilator

$\varphi_1 \equiv \hat{a}^\dagger \varphi_0$

$\hat{N}\varphi_1 = \hat{N}\varphi_1 = \hat{N}\hat{a}^\dagger\varphi_0$

$= [\hat{N}, \hat{a}^\dagger]\varphi_0$

$= (\hat{N}\hat{a}^\dagger + \underbrace{\hat{a}^\dagger\hat{N}}_{0})\varphi_0$

$= \hat{a}^\dagger\varphi_0 \equiv \varphi_1$

$E = \hbar\omega(1 + \frac{1}{2}) = \frac{3}{2}\hbar\omega$

$(\varphi_0, \varphi_0) = 1$

$(\varphi_1, \varphi_1) = (\hat{a}^\dagger\varphi_0, \hat{a}^\dagger\varphi_0)$

$= (\varphi_0, \hat{a}\hat{a}^\dagger\varphi_0)$

$= (\varphi_0, [\hat{a}, \hat{a}^\dagger]\varphi_0)$

$= (\varphi_0, \varphi_0)$

$= 1$

$\varphi_2' = \hat{a}^\dagger\hat{a}^\dagger\varphi_0$

$= (\hat{a}^\dagger)^2\varphi_0$

$\hat{N}\varphi_2' = \hat{N}(\hat{a}^\dagger)^2\varphi_0$

$= [\hat{N}, (\hat{a}^\dagger)^2]\varphi_0$

$= 2(\hat{a}^\dagger)^2\varphi_0$

$= 2\varphi_2'$

15-2: Energy Eigenstates of the Harmonic Oscilator (continued)

$$(\varphi_2', \varphi_2') = (\hat{a}^+\hat{a}^+\varphi_0, \hat{a}^+\hat{a}^+\varphi_0)$$

$$= (\varphi_0, \hat{a}\hat{a}\hat{a}^+\hat{a}^+\varphi_0)$$

$$= (\varphi_0, \hat{a}[\hat{a}, (\hat{a}^+)^2]\varphi_0)$$

$$= 2(\varphi_0 \hat{a}\hat{a}^+\varphi_0)$$

$$= 2(\varphi_0, [\hat{a}, \hat{a}^+]\varphi_0)$$

$$= 2$$

$$\varphi_2 = \frac{1}{\sqrt{2}}(\hat{a}^+\hat{a}^+)\varphi_0$$

$$\varphi_n = \frac{1}{\sqrt{n!}}\underbrace{\hat{a}^+ \dots \hat{a}^+}_{n}\varphi_0$$

$$\hat{N}\varphi_n = \frac{1}{\sqrt{n!}}\hat{N}\underbrace{\hat{a}^+ \dots \hat{a}^+}_{n}\varphi_0$$

$$= \frac{1}{\sqrt{n!}}[\hat{N}, (\hat{a}^+)^n]\varphi_0$$

$$= \frac{1}{\sqrt{n!}} n(\hat{a}^+)^n\varphi_0$$

$$= n\varphi_n$$

$N = n$, $E = \hbar\omega(n + \frac{1}{2})$

$$(\varphi_n, \varphi_n) = \frac{1}{n!}\left((\underbrace{\hat{a}^+ \dots \hat{a}^+}_{n})\varphi_0 (\underbrace{\hat{a}^+ \dots \hat{a}^+}_{n})\varphi_0\right)$$

15.3: Operators on Energy Eigenstates

$$\hat{a}\varphi_n = \frac{1}{\sqrt{n!}}\hat{a}(\hat{a}^\dagger)^n\varphi_0$$

$$= \frac{1}{\sqrt{n!}}[\hat{a}, (\hat{a}^\dagger)^n]\varphi_0$$

$$= \frac{1}{\sqrt{n!}} n(\hat{a}^\dagger)^{n-1}\varphi_0$$

$$= \frac{n}{\sqrt{n!}} \sqrt{(n-1)!}\,\varphi_{n-1}$$

$$\hat{a}\varphi_n = \sqrt{n}\,\varphi_{n-1}$$

$$\hat{a}^\dagger\varphi_n = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^{n+1}\varphi_0$$

$$= \frac{1}{\sqrt{n!}}\sqrt{(n+1)!}\,\varphi_{n+1}$$

$$\langle\hat{x}\rangle_{\varphi_n} = \int x(\varphi_n(x))^2 dx = 0 = (\varphi_n, \hat{x}\varphi_n)$$

$$\langle\hat{p}\rangle_{\varphi_n} = 0$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger)$$

$$\langle\hat{x}\rangle = \sqrt{\frac{\hbar}{2m\omega}}(\varphi_n, (\hat{a} + \hat{a}^\dagger)\varphi_n)$$

15.3: Operators on Energy Eigenstates (continued)

$(\varphi_3, \varphi_2) \cong (\hat{a}^\dagger \hat{a}^\dagger \hat{a}^\dagger \varphi_0, \hat{a}^\dagger \hat{a}^\dagger \varphi_0) = (\varphi_0, \hat{a}\hat{a}\hat{a}\, \hat{a}^\dagger \hat{a}^\dagger \varphi_0)$

$= 0$

ΔX in φ_n

$(\Delta X)^2 = \langle \hat{X}^2 \rangle_{\varphi_n} - \langle \hat{X} \rangle_{\varphi_n}$

$\langle \hat{X}^2 \rangle_{\varphi_n} = (\varphi_n, (\hat{X})^2 \varphi_n)$

$= \frac{\hbar}{2m\omega} (\varphi_n, (a + a^\dagger)(a + a^\dagger) \varphi_n)$

$= \frac{\hbar}{2m\omega} (\varphi_n, (aa + a^\dagger a^\dagger + aa^\dagger + a^\dagger a) \varphi_n)$

$[a, a^\dagger] = \hat{N}, 1 + \hat{N}$

$\langle \hat{X}^2 \rangle = \frac{\hbar}{2m\omega} (\varphi_n, (1 + 2\hat{N}) \varphi_n)$

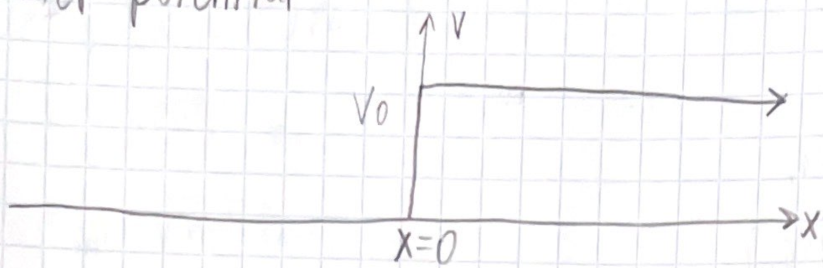
$= \frac{\hbar}{2m\omega} (\varphi_n, \varphi_n)(1 + 2n)$

$\langle \hat{X}^2 \rangle_{\varphi_n} = \frac{\hbar}{m\omega} \left(n + \frac{1}{2}\right) = (\Delta X)^2$

15.4: Scattering States

Non normalizable energy eigenstates

Step potential



$E > V_0$

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} \\ Ce^{i\bar{k}x} \end{cases}$$

$$\left(e^{-\frac{iEt}{\hbar}}\right)$$

$$E = \frac{\hbar^2 k^2}{2m}$$

$$k^2 = \frac{2mE}{\hbar}$$

$$\bar{k}^2 = \frac{2m(E-V_0)}{\hbar^2}$$

ψ must be continuous at $x=0$

$A+B=C$

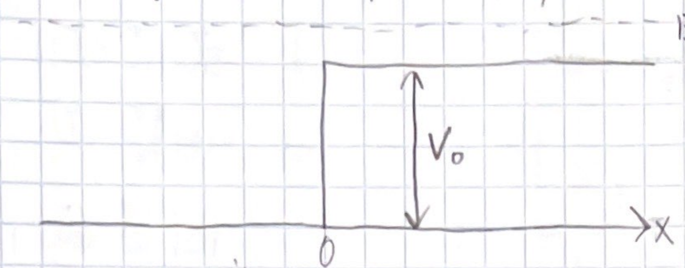
ψ' must be continuous at $x=0$

$$\begin{cases} ikA - ikB = i\bar{k}C \\ A - B = \frac{\bar{k}}{k}C \end{cases}$$

$$\frac{B}{A} = \frac{k-\bar{k}}{k+\bar{k}}$$

$$\frac{C}{A} = \frac{2k}{k+\bar{k}}$$

16.1: Step Potential Probability Current



$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & x<0 \\ Ce^{i\bar{k}x} & x>0 \end{cases}$$

$\bar{k} < k$

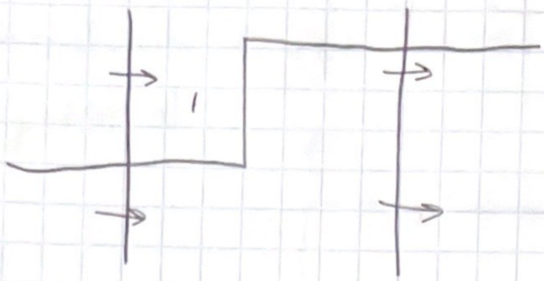
$E = V_0,\ \bar{k} = 0$

$\Downarrow$

$B = A$

$$\psi(x) = \begin{cases} 2A\cos kx \\ 2A \end{cases}$$

16.1: Step Probability Current (continued)



Probability Current

$$J(x) = \frac{\hbar}{m} \operatorname{Im}\left(\Psi^* \frac{\partial \Psi}{\partial x}\right)$$

$x<0$

$$J_L(x) = \frac{\hbar k}{m}\left(|A|^2 - |B|^2\right)$$

$x>0$

$$J_R(x) = \frac{\hbar \bar{k}}{m} |C|^2$$

$$J_L = \frac{\hbar k}{m}\left(1 - \left|\frac{B}{A}\right|^2\right)|A|^2$$

$$= \frac{\hbar k}{m}\left(1 - \left(\frac{k-\bar{k}}{k+\bar{k}}\right)^2\right)|A|^2$$

$$= \frac{\hbar k}{m}\left(\frac{4k\bar{k}}{(k+\bar{k})^2}\right)|A|^2$$

$J_L = J_R$

16.2: Reflection and Transmission

$J_L = J_A - J_B$

$J_A = \frac{\hbar k}{m}|A|^2$

$J_B = \frac{\hbar k}{m}|B|^2$

$R \equiv \frac{J_B}{J_A} \equiv$ reflection coefficient

$= \left|\frac{B}{A}\right|^2$

$T \equiv$ transmission coefficient $\equiv \frac{J_C}{J_A}$

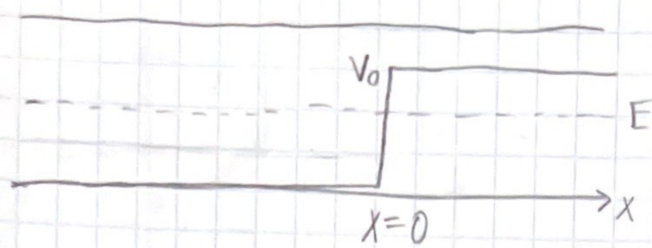
$= \frac{\bar{k}}{k}\left|\frac{C}{A}\right|^2$

$J_L = J_R \rightarrow J_A - J_B = J_C$

$\rightarrow R + T = \frac{J_B}{J_A} + \frac{J_C}{J_A} = \frac{J_B + J_C}{J_A}$

$\rightarrow J_A = J_B + J_C$

16.3: Energy Below Barrier



$x<0 \Rightarrow$ same solution

$k \to i\kappa$

$\kappa^2 = \frac{2m(V_0-E)}{\hbar^2}$

$\psi \to Ce^{i(i\kappa)x} = Ce^{-\kappa x}$

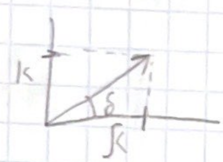
$\frac{B}{A} = \frac{k-i\kappa}{k+i\kappa}$

$= \frac{-i(\kappa+ik)}{i(\kappa-ik)}$

$= -\left(\frac{\kappa+ik}{\kappa-ik}\right)$

$= -e^{2i\delta(E)}$

$\delta(E)$



$\delta(E) = \tan^{-1}\left(\frac{k}{\kappa}\right)$

16.3: Energy Below Barrier (continued)

$\delta(E) = \tan^{-1}\sqrt{\frac{E}{V_0 - E}}$

$J_C = 0$, solution real for $x > 0 \rightarrow 0$ at ∞

$J_A = J_B$

$|A|^2 = |B|^2$

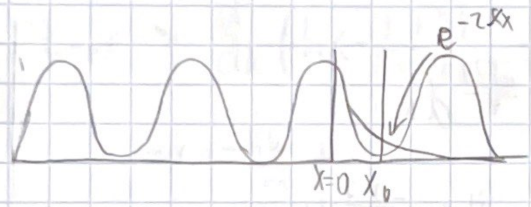
$x < 0 \quad \Psi(x) = Ae^{ikx} - Ae^{2i\delta(E)}e^{-ikx}$

$x > 0 \quad \Psi(x) = Ce^{-\kappa x}$

$\Psi(x) = Ae^{i\delta(E)}\left(e^{i(kx - \delta(E))} + e^{-i(kx - \delta(E))}\right)$

$\Psi(x) = 2iAe^{i\delta(E)}\sin(kx - \delta(E))$

$|\Psi|^2 = 4|A|^2\sin^2(kx - \delta(E))$

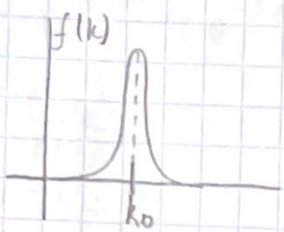


16.4: Wavepackets

With $A=1$, $E > V_0$

$$\left.\begin{array}{ll} e^{ikx} + \left(\frac{k-\bar{k}}{k+\bar{k}}\right)e^{-ikx} & x<0 \\ \frac{2k}{k+\bar{k}} e^{i\bar{k}x} e^{-\frac{iEt}{\hbar}} & x>0 \end{array}\right\} = \Psi(x,t)$$

$$\left.\begin{array}{l} \int_0^\infty f(k)\left(e^{ikx} + \left(\frac{k+\bar{k}}{k-\bar{k}}\right)e^{-ikx}\right)e^{-\frac{iEt}{\hbar}}\,dk \\ \int_0^\infty f(k)\,\frac{2k}{k+\bar{k}}\,e^{i\bar{k}x} e^{-\frac{iEt}{\hbar}}\,dk \end{array}\right\} = \Psi(x,t)$$



$$\Psi_{inc}(x<0,t) = \int_0^\infty f(k) e^{ikx} e^{-\frac{iEt}{\hbar}} dk$$

$$\Psi_{ref}(x<0,t) = \int_0^\infty f(k)\left(\frac{k-\bar{k}}{k+\bar{k}}\right) e^{-ikx} e^{-\frac{iEt}{\hbar}} dk$$

$$\Psi_{trans}(x>0,t) = \int_0^\infty f(k)\left(\frac{2k}{k+\bar{k}}\right) e^{i\bar{k}x} e^{-\frac{iEt}{\hbar}} dk$$

16.4: Wavepackets (continued)

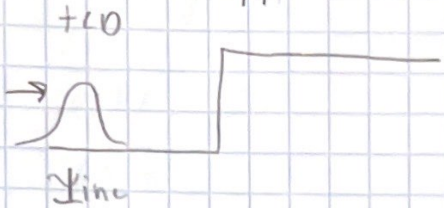
$$\Psi(x,t) = \Psi_{inc} + \Psi_{ref} \quad x<0$$

$$\Psi(x,t) = \Psi_{trans} \quad x>0$$

$$\Psi_{inc} \rightarrow \frac{d}{dk}\left(kx - \frac{Et}{\hbar}\right)\Big|_{k_0} = 0$$

$$\Rightarrow x - \frac{\hbar k t}{m} = 0$$

$$x = \frac{\hbar k_0}{m} t$$



$$\Psi_{ref} \rightarrow \frac{d}{dk}\left(-kx - \frac{Et}{\hbar}\right)\Big|_{k_0} = 0$$

$$\Rightarrow x = -\frac{\hbar k_0}{m} t$$

Ψ_{trans}

$$x = \frac{\hbar k}{m} t$$

16.5: Wavepackets Below Barrier

$E < V_0$

$\underbrace{e^{ikx}}_{inc} \quad \underbrace{-e^{-ikx} e^{2i\delta(E)}}_{ref}$

$\Psi_{inc}(x<0,t) = \int_0^\infty f(k) e^{ikx} e^{-\frac{iEt}{\hbar}} dk$

$\Psi_{ref}(x<0,t) = \int_0^\infty f(k) e^{-ikx} e^{2i\delta(E)} e^{-\frac{iEt}{\hbar}} dk$

$\frac{d}{dk}\left(-kx + 2\delta(E) - \frac{Et}{\hbar}\right)\Big|_{k_0} = 0$

$x = -\frac{\hbar k_0}{m}\left(t - 2\hbar\delta'(E)\right)$

Delay $= 2\hbar\delta'(E)$

16.6: Particle in Forbidden Region

Contradictory if one could say:

1) particle is in the forbidden region
2) particle has energy $< V_0$ which implies negative kinetic energy

$e^{-\kappa x}$

$x \approx \frac{1}{\kappa}$

$\kappa^2 = \frac{2m(V_0-E)}{\hbar^2}$

$\Delta x \leq \frac{1}{\kappa}$

$p \geq \frac{\hbar}{\Delta x} \geq \hbar\kappa$

Total Energy $E + \Delta E = E + V_0 - E \geq V_0$