4: Spin 1/2

$$|x;+\rangle = \frac{1}{\sqrt{2}}\left(|z;+\rangle + |z;-\rangle\right) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

$$|x;-\rangle = \frac{1}{\sqrt{2}}\left(|z;+\rangle - |z;-\rangle\right) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

$$|z;+\rangle = \frac{1}{\sqrt{2}}\left(|x;+\rangle + |x;-\rangle\right)$$

$$|z;-\rangle = \frac{1}{\sqrt{2}}\left(|x;+\rangle - |x;-\rangle\right)$$

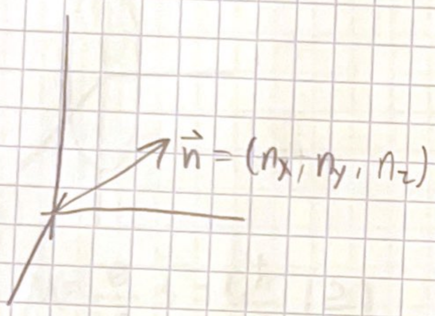
$$\langle x;+|z;+\rangle = \frac{1}{\sqrt{2}}$$

$$\langle x;-|z;+\rangle = \frac{1}{\sqrt{2}}$$

$$\hat{S}_y|y;\pm\rangle = \pm\frac{\hbar}{2}|y;\pm\rangle$$

$$|y;+\rangle = \frac{1}{\sqrt{2}}\left(|z;+\rangle + i|z;-\rangle\right)$$

$$|y;-\rangle = \frac{1}{\sqrt{2}}\left(|z;+\rangle - i|z;-\rangle\right)$$



$$\vec{n} = n_x\vec{e}_x + n_y\vec{e}_y + n_z\vec{e}_z$$

$$\vec{S} = (\hat{S}_x, \hat{S}_y, \hat{S}_z)$$

$$\vec{S} = \hat{S}_x\vec{e}_x + \hat{S}_y\vec{e}_y + \hat{S}_z\vec{e}_z$$

4: Spin 1/2

$$\hat{S}_n \equiv \vec{n}\cdot\vec{S}$$

$$\hat{S}_{\vec{n}} = n_x\hat{S}_x + n_y\hat{S}_y + n_z\hat{S}_z$$

$n_x = \cos\phi\sin\theta$
$n_y = \sin\phi\sin\theta$
$n_z = \cos\theta$

$$\hat{S}_{\vec{n}} = \frac{\hbar}{2}(n_x\sigma_1 + n_y\sigma_2 + n_z\sigma_3)$$

$$= \frac{\hbar}{2}\begin{pmatrix} n_z & n_x - in_y \\ n_x + in_y & -n_z \end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}$$

$\det(A - \lambda\mathbb{1}) = 0$ EV of $\hat{S}_{\vec{n}}$

$$\det\begin{pmatrix} \frac{\hbar}{2}\cos\theta - \lambda & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\frac{\hbar}{2}\cos\theta - \lambda \end{pmatrix} = 0$$

$(A - \lambda\underline{\mathbb{1}})v = 0$

$|\vec{n};+\rangle$

$\hat{S}_{\vec{n}}|\vec{n};\pm\rangle = \pm\frac{\hbar}{2}|n;\pm\rangle$

$|n;+\rangle = c_1|z;+\rangle + c_2|z;-\rangle = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

$(\hat{S}_n - \frac{\hbar}{2}\mathbb{1})\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0$

$$\begin{pmatrix} \cos\theta - 1 & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta - 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0$$

$$c_2 = e^{i\phi}\frac{1-\cos\theta}{\sin\theta}c_1$$

$|\vec{n};+\rangle = \cos\frac{\theta}{2}|z;+\rangle + \sin\frac{\theta}{2}e^{i\phi}|z;-\rangle$, $|\vec{n};-\rangle = -\sin\frac{\theta}{2}e^{-i\phi}|+\rangle + \cos\frac{\theta}{2}|-\rangle$

4: Spin 1/2

$|z;+\rangle$ and $|z;-\rangle$

$S_z = \frac{\hbar}{2}$ $\qquad$ $S_z = -\frac{\hbar}{2}$

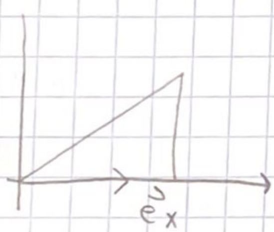
there is an operator $\hat{S}_z$

$\hat{S}_z |z;+\rangle = \frac{\hbar}{2} |z;+\rangle$

$\hat{S}_z |z;-\rangle = -\frac{\hbar}{2} |z;-\rangle$

$$|\psi\rangle = c_1 \underbrace{|z;+\rangle}_{\substack{\text{first}\\\text{basis}\\\text{state}\\|1\rangle}} + c_2 \underbrace{|z;-\rangle}_{\substack{\text{second}\\\text{basis}\\\text{state}\\|2\rangle}} \quad c_1, c_2 \in \mathbb{C}$$

$|z;+\rangle = |1\rangle \leftrightarrow \begin{pmatrix}1\\0\end{pmatrix}$



$\begin{pmatrix}a\\b\end{pmatrix}$

$\begin{pmatrix}1\\0\end{pmatrix} = e_x$

$\begin{pmatrix}0\\1\end{pmatrix} = e_y$

$|z;-\rangle = |2\rangle \leftrightarrow \begin{pmatrix}0\\1\end{pmatrix}$

$|\psi\rangle = c_1 |z;+\rangle + c_2 |z;-\rangle = c_1 \begin{pmatrix}1\\0\end{pmatrix} + c_2 \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}c_1\\c_2\end{pmatrix}$

4: Spin 1/2

$\langle z;-|z;+\rangle = 0 \leftrightarrow \langle z;+|z;-\rangle = 0$

$\langle z;+|z;+\rangle = 1 \leftrightarrow \langle z;-|z;-\rangle = 1$

$\langle i|j\rangle = \delta_{ij}$

$|1\rangle \leftrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\langle 1| \leftrightarrow (1,0)$

$|2\rangle \leftrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\langle 2| \leftrightarrow (0,1)$

$|\alpha\rangle = \alpha_1|1\rangle + \alpha_2|2\rangle \leftrightarrow \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$

$|\beta\rangle = \beta_1|1\rangle + \beta_2|2\rangle \leftrightarrow \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$

$\langle\alpha| \equiv \alpha_1^*\langle 1| + \alpha_2^*\langle 2| \leftrightarrow \alpha_1^*\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_2^*\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha_1^* \\ \alpha_2^* \end{pmatrix}$

$\langle\alpha|\beta\rangle \equiv (\alpha_1^*, \alpha_2^*)\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \alpha_1^*\beta_1 + \alpha_2^*\beta_2$

$\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}|1\rangle$

$\hat{S}_z|2\rangle = -\frac{\hbar}{2}|2\rangle$

4: Spin 1/2

$\hat{L}_z = \hat{x} p_y - \hat{y} p_x$

$[\hat{L}_i, \hat{L}_j] = i\hbar \epsilon_{ijk} \hat{L}_k$

$\epsilon_{123} = 1$

$$\begin{cases} [\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z \\ [\hat{L}_y, \hat{L}_z] = i\hbar \hat{L}_x \\ [\hat{L}_x, \hat{L}_z] = i\hbar \hat{L}_y \end{cases}$$

$$\begin{cases} [\hat{S}_x, \hat{S}_y] = i\hbar \hat{S}_z \\ [\hat{S}_y, \hat{S}_z] = i\hbar \hat{S}_x \\ [\hat{S}_x, \hat{S}_z] = i\hbar \hat{S}_y \end{cases}$$

$\begin{pmatrix} 2c & a-ib \\ a+ib & 2d \end{pmatrix}$ $a, b, c, d \in \mathbb{R} \Rightarrow$ hermitian

remove $(c+d) \mathbb{1}_{2\times 2} \rightarrow \begin{pmatrix} c-d & a-ib \\ a+ib & d-c \end{pmatrix}$

add $\hat{S}_z$ multiple

$$\begin{pmatrix} 0 & a-ib \\ a+ib & 0 \end{pmatrix} = a \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

4: Spin 1/2

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : \lambda = 1 \ \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ \lambda = -1 \ \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} : \lambda = 1 \ \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\lambda = -1 \ \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$[\hat{S}_x, \hat{S}_y] = \frac{\hbar}{2}\frac{\hbar}{2}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]$$

$$= \frac{\hbar}{2}\frac{\hbar}{2}\left[\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\right]$$

$$= \frac{\hbar}{2}\frac{\hbar}{2} 2i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\hat{S}_i = \frac{\hbar}{i}\sigma_i$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\hat{S}_x |x;\pm\rangle = \pm\frac{\hbar}{2}|x;\pm\rangle$$

5: Vector Spaces

$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1$

$M^2 + \alpha M + \beta \mathbb{1} = 0$

$Mv = \lambda v$

$\alpha \lambda v + \beta v + \lambda^2 v = 0$

$(\lambda^2 + \alpha \lambda + \beta) v = 0$

$\lambda^2 + \alpha \lambda + \beta = 0$

$(\lambda)^2 = 1$

$\lambda = \pm 1$

$\mathrm{tr}\, \sigma_1 = 0 =$ sum of eigenvalues

$\hat{S}_i = \frac{\hbar}{2} \sigma_i$

$[\hat{S}_i, \hat{S}_j] = i\hbar \epsilon_{ijk} \hat{S}_k$

$[\sigma_i \sigma_j] = 2i \epsilon_{ijk} \sigma_k$

$\sigma_1 \sigma_2 = -\sigma_2 \sigma_1$

$\sigma_1 \sigma_2 + \sigma_2 \sigma_1 = 0 \rightarrow \{\sigma_1, \sigma_2\} = 0$

$\{A, B\} = AB + BA$

$\{\sigma_i, \sigma_j\} = 2\delta_{ij} \mathbb{1}$

$AB = \frac{1}{2}[A, B] + \frac{1}{2}\{A, B\}$

5: Vector Spaces

$\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$

$\vec{a} = (a_1, a_2, a_3)$

$\vec{a} \cdot \vec{\sigma} \equiv a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3 = a_i\sigma_i$

$(\vec{a} \cdot \vec{\sigma})(\vec{b} \cdot \sigma) = (\vec{a} \cdot \vec{b})\mathbb{1} + i(\vec{a} \times \vec{b}) \cdot \vec{\sigma}$

$(\vec{n} \cdot \vec{\sigma})^2 = \mathbb{1}$

$\hat{S}_{\vec{n}} = \vec{n} \cdot \vec{S} = \frac{\hbar}{2}\vec{n} \cdot \vec{\sigma}$

$(\hat{S}_{\vec{n}})^2 = \left(\frac{\hbar}{2}\right)^2 \mathbb{1}$

$tr(\hat{S}_{\vec{n}}) = 0$

$\vec{X} \cdot \vec{y} \equiv \hat{X}_i \hat{y}_i$

$(\vec{X} \times \vec{Y})_k \equiv \epsilon_{ijk} \hat{X}_i \hat{Y}_j$

vectors and numbers $(\mathbb{R}, \mathbb{C})$ $\mathbb{F}$ field

Vector space V is a set of vectors with an addition $(+)$ that assigns a vector $u+v \in V$ when $u, v \in V$. and a scalar multiplication by elements of $\mathbb{F}$ such that $av \in V$ when $a \in \mathbb{F}$ and $v \in V$

5: Vector Spaces

1) $u+v=v+u$, $u,v\in V$

2) $u+(v+w)=(u+v)+w$

3) $(ab)v=a(bv)$

4) $1\in\mathbb{F}$ satisfies $1v=v$

5) For each $v\in V$, there is a $u\in V$ such that $v+u=0$

6) $a(u+v)=au+av$, $(a+b)v=av+bv$, $a,b\in\mathbb{F}$, $u,v\in V$

The set of N component vectors

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix} \quad a_i\in\mathbb{R}, \; i=1,\dots,N$$

Vector space over $\mathbb{R}$

The space of 2x2 Hermitian matrices is real vector space

$$\begin{pmatrix} c+d & a+ib \\ a-ib & c-d \end{pmatrix} \quad \{a,b,c,d\}\in\mathbb{R}$$

5: Vector Spaces

If the set $P(z)$ has $z \in \mathbb{F}$, $P(z) \in \mathbb{F}$

$p(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n \quad a_i \in \mathbb{F}$

The space $\mathcal{P}(\mathbb{F})$ of all polynomials is a vector space over $\mathbb{F}$

The set $\mathbb{F}^\infty$ of infinite sequences $(x_1, x_2, \dots, x_i)$ $x_i \in \mathbb{F}$ is a vector space over $\mathbb{F}$

The set of complex functions in an interval $x \in [0, L]$ is a complex vector space

A subspace of V is a subset of V that is a vector space.

$V = U_1 \oplus U_2 \oplus U_3 \oplus \dots \oplus U_m$ (direct sum)

U_i's are subspaces of V and any V in the vector space can be written as

$a_1 U_1 + a_2 U_2 + \dots a_n U_n$ with $a_i \in U_i$

5: Vector Spaces

List of vectors $(v_1, v_2, \dots, v_n)$

Any list of vectors has finite length

$\text{span}(v_1, v_2 \dots v_n) = \{a_1 v_1 + a_2 v_2 + \dots + a_n v_n \mid a_i \in \mathbb{F}\}$

V is finite dimensional if V is spuned by some list

V is infinite dimensional if it is not finite dimensional

A list of vectors is linearly independant $(v_1 \dots v_n)$ if $a_1 v_1 + a_2 v_2 + \dots + a_n v_n = 0$ has the unique solutron $a_1 = a_2 = \dots = a_n = 0$

A basis of V is a list of vectors in V that spans V and is linearly independent.

The dimension of a vector space is the length of any basis of V.

$$V \xrightarrow[\text{linear map}]{} W$$

$$V \xrightarrow[\text{linear operator}]{} V$$

functron $T : V \to V$

$$\left.\begin{aligned} T(u+v) &= Tu + Tv \\ T(av) &= aT(v) \end{aligned}\right\} \text{linear}$$

$T_{ij} = \langle i | T | j \rangle$

6: Vector Spaces

$\mathcal{L}(V,W)$ = linear maps from vector space V to vector space W

$\mathcal{L}(V) = \mathcal{L}(V,V)$

1) $V = \mathbb{R}[x]$ all real polynomials in one variable

$T = \frac{\hat{\partial}}{\partial x}$ $Tp = p'$

$S = \hat{x}$ $Sp = xp$

2) $V = \mathbb{F}^{\infty} = \{(x_1, x_2, x_3 \ldots) \mid x_i \in \mathbb{F}\}$

L = left shift operator $L(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$
R = right shift operator $R(x_1, x_2, \ldots) = (0, x_1, \ldots)$

3) 0 : $\hat{0}\vec{v} = \vec{0}$

4) I : $Iv = v$

$\mathcal{L}(V)$ is a vector space with multiplication

Properties
- associative ✓ $A(BC) = (AB)C$
  $(AB)v \equiv A(Bv)$
  $A(B(C(v)))$
- identity ✓
- inverses X
- commutative X

6: Vector Spaces

$STx^n = Snx^{n-1} = nx^n$

$TSx^n = Tx^{n+1} = (n+1)x^n$

$[T,S]x^n = x^n = Ix^n$

$[T,S] = I$

$\text{null } T = \{v \in V \mid Tv = 0\} \supseteq \{0\}$
a subspace of V

T injective $\Leftrightarrow u \neq v$ then $Tu \neq Tv$
$\Leftrightarrow u \neq v$ then $T(u-v) \neq 0$
$\Leftrightarrow \text{null } T = \{0\}$

$\text{range}(T) = T(V) = \{Tv \mid v \in V\}$
a subspace of V

$\text{range}(T) = V \Leftrightarrow T$ is surjective

$\dim(V) = \dim(\text{null}(T)) + \dim(\text{range}(T))$

$$T = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \quad \begin{aligned} &\text{null}(T) = \text{span}\{e_2\} \\ &\text{range}(T) = \text{span}\{e_1, e_3, e_4\} \end{aligned}$$

$$T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow \text{null}(T) = \left\{\begin{pmatrix} * \\ 0 \end{pmatrix}\right\} = \text{span}\{e_1\}$$
$\text{range}(T) = \text{span}\{e_1\}$

6: Vector Spaces

Invertability

T has a left inverse S if $ST = I$

T has a right inverse S' if $TS' = I$

If both exist, then

$S = SI = (ST)S' = S'$

T is invertable then define $T^{-1} = S$

$\exists$ left inverse if T is injective

$\exists$ right inverse if T is surjective

$\dim(V) < \infty \Rightarrow T$ is injective

$\iff T$ is surjective

$\iff T$ is invertable

6: Vector Spaces

Matrices

Choose basis $V_1 \cdots V_n$

Any V can be written a $v = a_1 v_1 + \dots + a_n v_n$

$Tv = a_1 T v_1 + \dots + a_n T v_n$

$$T v_j = T_{1j} v_1 + T_{2j} v_2 + \dots + T_{nj} v_n$$

$$= \sum_{i=1}^{n} T_{ij} v_i$$

$$Tv = \sum_j a_j v_j$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} T_{ij} a_j v_i$$

$$= \sum_{i=1}^{n} b_i v_i$$

$$b_i = \sum_{j=1}^{n} T_{ij} a_j$$

$$\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} T_{1n} & \cdots & T_{1n} \\ \vdots & & \\ T_{i1} & & T_{in} \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

6: Vector Spaces

$$(TS)v_k = T\sum_j S_{jk} v_j$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} T_{ij} S_{jk} v_i$$

$$(TS)_{ik} = \sum_{j=1}^{n} T_{ij} S_{jk}$$

$$I = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \end{pmatrix}, \quad 0 = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \\ \vdots & & & \ddots \end{pmatrix}$$

$T(\{v_i\}_{1\le i\le n})$

Basis independent quantities

tr eigenvalues

det

U is a T-invariant subspace if $T(U) = \{Tu \mid u \in U\} \subseteq U$

$\{0\}, V$

1-d invariant subspaces

$U = \text{span}\{u\}$ U, T-invariant $\iff Tu \in U$

$\iff Tu = \lambda u$ for some u

λ is an eigenvalue

u is an eigenvector

$\text{Spectrum}(T) = \{\lambda \mid \lambda \text{ is an eigenvalue of } T\}$