25: Addition of Angular Momentum

$n=2,\ l=1 \Rightarrow (3\text{ states}) \otimes s\tfrac{1}{2}\ (2\text{ states})$

$j=\frac{3}{2}\ (4\text{ states}) \oplus j=\frac{1}{2}\ (2\text{ states})$

$\vec{L}\cdot\vec{S}_e \equiv$ $j=\frac{3}{2}$: $\frac{\hbar^2}{2}$; $j=\frac{1}{2}$: $-\hbar^2$

Hyperfine Structure

proton spin $\frac{1}{2}$

$\vec{J}=\vec{L}+\vec{S}_e+\vec{S}_p$ conserved

$CSCO=\{H_{Tot}, L^2, S_e^2, S_p^2, J^2, J_z\}$

$\vec{\mu}_e=\frac{-e}{m_e}\vec{S}_e$

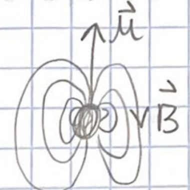
$\vec{\mu}_p=g_p\frac{e}{2m_p}\vec{S}_p$

$g_p\approx 5.6$

$H_{HF}=-\vec{\mu}_e\cdot\vec{B}_p(r)$

$=\frac{e}{m}\vec{S}_e\cdot\vec{B}_p$

$\vec{B}_p(\vec{r})=\frac{1}{c^2r^3}[3\vec{\mu}_p\cdot\hat{r}\hat{r}-\mu_p]+\frac{8\pi}{3c^2}\mu_p\,\delta^3(r)$



25: Addition of Angular Momentum

$$H_{HF} = \underbrace{\frac{ge^2}{2m_e m_p c^2}}_{Q} \left\{ \frac{1}{r^3}(S_e \cdot \hat{r} S_p \cdot \hat{r} - S_e \cdot S_p) + \frac{8\pi}{3} S_e \cdot S_p \delta^3(\vec{r}) \right\}$$

$$H_{HF} = Q S_e^i S_p^j \left\{ \frac{\hat{r}_i \hat{r}_j - \delta_{ij}}{r^3} + \frac{8\pi}{3} \delta_{ij} \delta^3(\vec{r}) \right\}$$

$$\langle \Psi_{100} | H_{HF} | \Psi_{100} \rangle \equiv \langle H_{HF} \rangle_{100}$$

$$= Q S_e^i S_p^j \int |\Psi_{100}(r^2)|^2 \left\{ \frac{3\hat{r}_i \hat{r}_j}{r^3} + \frac{8\pi}{3} \delta_{ij} \delta_3(\vec{r}) \right\} d^3\vec{r}$$

$$\int r_i \, d^3 r = 0$$

$$\int r_i r_j \, d^3 r = \# \delta_{ij}$$

$$\int r_i r_j f(r^2) \, d^3 r = A_f \delta_{ij}$$

$$\approx \vec{S}_e \vec{S}_p \frac{8\pi}{3} Q |\Psi_{100}(0)|^2$$

$$\uparrow \frac{1}{\pi a_0}$$

$$= \frac{4}{3} g_p \frac{m_e}{m_p} m_e c^2 \alpha^4 \vec{S}_e \vec{S}_p$$

25: Addition of Angular Momentum

$(J_1 = \frac{1}{2}) \otimes (J_2 = \frac{1}{2})$

$\vec{J} = \vec{J}_1 + \vec{J}_2$

$|\frac{1}{2}, \frac{1}{2}\rangle \otimes |\frac{1}{2}, \frac{1}{2}\rangle \quad m=1 \quad j=1$

$m = m_1 + m_2$

$|\frac{1}{2}, \frac{1}{2}\rangle \otimes |\frac{1}{2}, -\frac{1}{2}\rangle, |\frac{1}{2}, -\frac{1}{2}\rangle \otimes |\frac{1}{2}, \frac{1}{2}\rangle \; m=0$

$|\frac{1}{2}, -\frac{1}{2}\rangle \otimes |\frac{1}{2}, \frac{1}{2}\rangle \quad m=-1 \quad j=1$

$J_\pm |j\, m\rangle = \sqrt{j(j+1) - m(m\pm1)}\; |j, m\pm1\rangle$

$J_- |j=1, m=1\rangle = \sqrt{1\cdot2 - 1\cdot0}\, |j, m-1\rangle = \sqrt{2}\, |j=1, m=0\rangle$

$J_- = J_{1-} + J_{2-}$

$J_- |\frac{1}{2}, \frac{1}{2}\rangle \otimes |\frac{1}{2}, \frac{1}{2}\rangle = (J_{1-} |\frac{1}{2}, \frac{1}{2}\rangle) \otimes |\frac{1}{2}, \frac{1}{2}\rangle + |\frac{1}{2}, \frac{1}{2}\rangle \otimes (J_{2-} |\frac{1}{2}, \frac{1}{2}\rangle)$

$= |\frac{1}{2}, -\frac{1}{2}\rangle \otimes |\frac{1}{2}, \frac{1}{2}\rangle + |\frac{1}{2}, \frac{1}{2}\rangle \otimes |\frac{1}{2}, -\frac{1}{2}\rangle$

$|1, 0\rangle = \frac{1}{\sqrt{2}} (|\frac{1}{2}, \frac{1}{2}\rangle \otimes |\frac{1}{2}, -\frac{1}{2}\rangle + |\frac{1}{2}, -\frac{1}{2}\rangle \otimes |\frac{1}{2}, \frac{1}{2}\rangle)$

$|J=0, m=0\rangle = \frac{1}{\sqrt{2}} (|\frac{1}{2}, \frac{1}{2}\rangle \otimes |\frac{1}{2}, -\frac{1}{2}\rangle - |\frac{1}{2}, -\frac{1}{2}\rangle \otimes |\frac{1}{2}, \frac{1}{2}\rangle)$

$|\frac{1}{2}, \frac{1}{2}\rangle = |\uparrow\rangle$

$|\frac{1}{2}, -\frac{1}{2}\rangle = |\downarrow\rangle$

25: Addition of Angular Momentum

$|\frac{1}{2},\frac{1}{2}\rangle \otimes |\frac{1}{2},-\frac{1}{2}\rangle = |\uparrow\downarrow\rangle$

$j=1$:

$|\uparrow\uparrow\rangle$

$\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$

$|\downarrow\downarrow\rangle$

$j=0$:

$\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$

$\langle H_{HF}\rangle_{100} = \Delta E_{HF}\, \vec{S}_e \cdot \vec{S}_p$

$= \frac{\Delta E_{HF}}{2\hbar^2}(\vec{J}^2 - S_e^2 - S_p^2)$

$\vec{L} = 0 \Rightarrow \vec{J} = \vec{S}_e + \vec{S}_p$

$= \frac{\Delta E_{HF}}{2\hbar^2}(J^2 - \frac{3}{2}\hbar^2)$

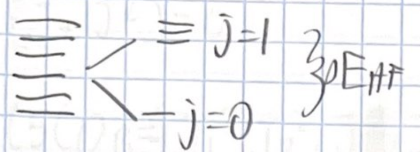
$\langle j=1,m|\langle H_{HF}\rangle_{100}|j=1,m\rangle \quad m=-1,0,1$

$= \frac{\Delta E_{HF}}{2}(1(1+1) - \frac{3}{2}) = \frac{1}{4}\Delta E_{HF}$

$\langle j=0,m|\langle H_{HF}\rangle_{100}|j=0,m\rangle$

$= \frac{\Delta E_{HF}}{2}(0 \cdot 1 - \frac{3}{2}) = -\frac{3}{4}\Delta E_{HF}$

25: Addition of Angular Momentum



$\Delta E_{HF} = 5.9 \cdot 10^{6}$ eV

Bohr energy $= \alpha^2 m_e c^2$

Spin orbit $= \alpha^4 m_e c^2$

Hyperfine $= \alpha^4 m_e c^2 \left(\frac{m_e}{m_p}\right)$

Lamb Shift $= \alpha^5 m_e c^2$

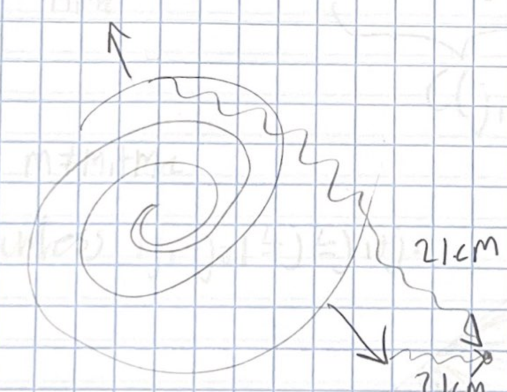
decay $j=1 .. j=0$ by a photon

$\lambda = \frac{hc}{\Delta E_{HF}} \simeq 21.1$ cm

$\nu = 1420$ MHz

$E kT = 0.2 \cdot 10^{-3}$ eV

$T \sim 27$ K



25: Addition of Angular Momentum

$\vec{J}_1 \quad |j_1, m_1\rangle \quad m_1 = j_1 \dots j$

$\mathcal{H}_1^{(j_1)} = \{|j_1, m_1\rangle\}$

$\vec{J} = \vec{J}_1 + \vec{J}_2 = \vec{J}_1 \otimes \mathbb{1} + \mathbb{1} \otimes \vec{J}_2$

acts on vectors $\in \mathcal{H}_1^{(j_1)} \otimes \mathcal{H}_2^{(j_2)}$

Uncoupled basis

$|j_1 j_2, m_1 m_2\rangle = |j_1\, m_1\rangle \otimes |j_2\, m_2\rangle$

$$\begin{Bmatrix} J_1^2 \\ J_{1z} \\ J_2^2 \\ J_{2z} \end{Bmatrix} |j_1 j_2\, m_1 m_2\rangle = \begin{Bmatrix} \hbar^2 j_1(j_1+1) \\ \hbar m_1 \\ \hbar^2 j_2(j_2+1) \\ \hbar m_2 \end{Bmatrix} |j_1 j_2\, m_1 m_2\rangle$$

CSCO$_A$

$$J^2 = (J_1 \otimes \mathbb{1} + \mathbb{1} \otimes J_2)(J_1 \otimes \mathbb{1} + \mathbb{1} \otimes J_2)$$

$$J_1^2 \otimes \mathbb{1} + \mathbb{1} \otimes J_2^2 + \sum_k J_{1k} \otimes J_{2k}$$

$[J_{1z}, J^2] \neq 0$

25: Addition of Angular Momentum

$[J_z, J^2] = 0$

CSCO $B = \{J_1^2, J_2^2, J^2, J_z\}$

$|j_1 j_2 j m\rangle$

both buses are orthonormal

$$\sum_{j_1 j_2} \sum_{m_1 m_2} |j_1 j_2 m_1 m_2\rangle\langle j_1 j_2 m_1 m_2| = 1$$

on $H_1 \otimes H_2$

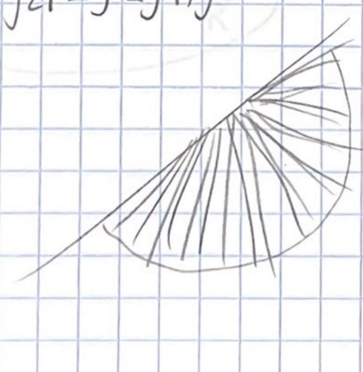
$$\sum_{m_1 m_2} |j_1 j_2 m_1 m_2\rangle\langle j_1 j_2 m_1 m_2| = 1$$

on $H_1^{(j_1)} \otimes H_2^{(j_2)}$

$$|j_1 j_2 j m\rangle = \sum_{m_1 m_2} |j_1 j_2 m_1 m_2\rangle \underbrace{\langle j_1 j_2 m_1 m_2 | j_1 j_2 m_1 m_2\rangle}_{C(j_1, j_2):}$$

vanish if $m \neq m_1 + m_2$

vanish unless $|j_1 - j_2| \leq j \leq j_1 + j_2$



25: Addition of Angular Momentum

$$N_t = \sum_{j=|j_1-j_2|}^{j_1+j_2} (2j+1)$$

$$= \sum_{j=0}^{j_1+j_2} (2j+1) - \sum_{j=0}^{j_1-j_2-1} (2j+1)$$

$$= (2j_1+1)(2j_2+1)$$

$$J^2 = J_1^2 + J_2^2 + 2J_1 \cdot J_2$$

$$= J_1^2 + J_2^2 + 2J_{1z}J_{2z} + J_{1+}J_{2-} + J_{1-}J_{2+}$$

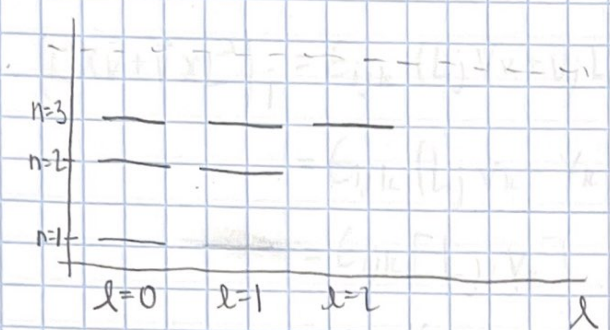
$$J^2 |j_1 j_2 \, j_1+j_2 \, j_1+j_2\rangle = (J_1^2 + J_2^2 + 2J_{1z}J_{2z} + J_{1+}J_{2-} + J_{1-}J_{2+}) |j_1 j_2 \, j_1 j_2\rangle$$

$$= \hbar^2 (j_1(j_1+1) + j_2(j_2+1) + 2j_1 j_2) |\cdots\rangle$$

26: Addition of Angular Momentum

$$\underset{\vec{J}_1}{\overset{\uparrow}{j_1}} \otimes \underset{\vec{J}_2}{\overset{\uparrow}{j_2}} = (j_1+j_2) \oplus (j_1+j_2-1) \oplus \cdots \oplus |j_1-j_2|$$

$$\vec{J} = \vec{J}_1 + \vec{J}_2$$

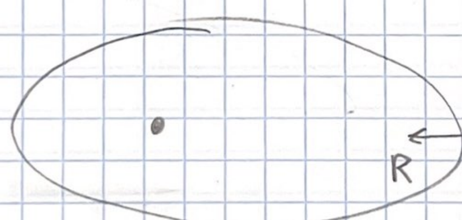


$$E_n = -\frac{e^2}{2a_0}\frac{1}{n^2}$$

Each n: $l = 0 \ldots n-1$

$$H = \frac{\vec{p}^2}{2m} - \frac{e^2}{r}$$

$$\vec{R} = \frac{1}{2me^2}(\vec{p}\times\vec{L} - \vec{L}\times\vec{p}) - \frac{\vec{r}}{r}$$



26: Addition of Angular Momentum

$[H, \vec{R}] = 0$

$\vec{p} \times \vec{L} = -\vec{L} \times \vec{p} + 2i\hbar \vec{p}$

$\vec{R} = \frac{1}{me^2}(\vec{p} \times \vec{L} - i\hbar \vec{p}) - \frac{\vec{r}}{r}$

$= \frac{1}{me^2}(-\vec{L} \times \vec{p} + i\hbar \vec{p}) - \frac{\vec{r}}{r}$

$\vec{R}^2 = 1 + \frac{2H}{me^4}(\vec{L}^2 + \hbar^2)$

$\vec{r} \cdot \vec{L} = 0$

$\vec{p} \cdot \vec{L} = 0$

$(\vec{p} \times \vec{L}) \cdot \vec{L} = \epsilon_{ijk} \vec{p}_j \vec{L}_k \vec{L}_i$

$= \epsilon_{jki} p_j L_k L_i$

$= p_j (\vec{L} \times \vec{L})_j$

$= \vec{p} \cdot (i\hbar \vec{L})$

26: Addition of Angular Momentum

$\vec{R}\cdot\vec{L}=0$

Vector under rotations:

$[L_i, V_j]=i\hbar\epsilon_{ijk}V_k$

$$\left(\vec{L}\times\vec{V}+\vec{V}\times\vec{L}\right)_i=\epsilon_{ijk}(L_jV_k+V_jL_k)$$
$$=\epsilon_{ijk}(L_jV_k-V_kL_j)$$
$$=\epsilon_{ijk}[L_j,V_k]$$
$$=\epsilon_{ijk}\epsilon_{jkl}V_l\,i\hbar$$
$$=2\delta_{il}\,i\hbar V_l$$
$$=2i\hbar V_i$$

$\vec{L}\times\vec{R}+\vec{R}\times\vec{L}=2i\hbar\vec{R}$

$[L_i,R_j]=i\hbar\epsilon_{ijk}R_k$

$[R_i,R_j]=\vec{R}\times\vec{R}$

26: Addition of Angular Momentum

$[[S_1, S_2], H] = 0$

$\vec{R} \times \vec{R} = (\ldots)\vec{L}$

$\vec{r} \to -\vec{r}$

$\vec{p} \to -\vec{p}$

$\vec{L} \to \vec{L}$

$\vec{R} \to -\vec{R}$

$\vec{R} \times \vec{R} = i\hbar\left(\frac{-2H}{me^4}\right)\vec{L}$

Whole values in a degenerate subspace with eigenvalues $H = H'$

$H' = -\frac{me^4}{2\hbar^2}\frac{1}{\nu^2} \quad \nu \in \mathbb{R}$

$\frac{-2H}{me^4} = \frac{1}{\hbar^2\nu^2}$

$\vec{R} \times \vec{R} = \frac{i}{\hbar}\frac{1}{\nu^2}\vec{L}$

$\vec{R}^2 = 1 - \frac{1}{\hbar^2\nu^2}(\vec{L}^2 + \hbar^2)$

$(\hbar\nu\vec{R}) \times (\hbar\nu\vec{R}) = i\hbar\vec{L}$

$\hbar^2\nu^2\vec{R}^2 = \hbar^2(\nu^2 - 1) - \vec{L}^2$

26: Addition of Angular Momentum

$$J_1 \equiv \frac{1}{2}(\vec{L} + \hbar\nu\vec{R})$$

$$J_2 \equiv \frac{1}{2}(\vec{L} - \hbar\nu\vec{R})$$

$$\vec{L} = \vec{J}_1 + \vec{J}_2$$

$$\hbar\nu\vec{R} = \vec{J}_1 - \vec{J}_2$$

$$[J_{1i}, J_{2j}] = \frac{1}{4}[L_i + \hbar\nu\vec{R}_i, L_j - \hbar\nu\vec{R}_j]$$

$$= \frac{1}{4}(i\hbar\epsilon_{ijk}L_k - \hbar\nu[L_i, R_j] - \hbar\nu[L_j, R_i] - i\hbar\epsilon_{ijk}L_k)$$

$$\left.\begin{matrix}\vec{J}_1 \times \vec{J}_1 \\ \vec{J}_2 \times \vec{J}_2\end{matrix}\right\} = \frac{1}{4}(\vec{L} \pm \hbar\nu\vec{R}) \times (\vec{L} \pm \hbar\nu\vec{R})$$

$$= \frac{1}{4}(i\hbar\vec{L} \pm (\vec{L} \times \hbar\nu\vec{R} + \hbar\nu\vec{R} \times \vec{L}))$$

$$= \frac{1}{4}(2i\hbar\vec{L} \pm 2i\hbar\nu\vec{R})$$

$$= i\hbar\frac{1}{2}(L \pm \hbar\nu R) = \left\{\begin{matrix}J_1 \\ J_2\end{matrix}\right\}$$

$$(\vec{J}_1 + \vec{J}_2)(\vec{J}_1 - \vec{J}_2) = 0 \Rightarrow \vec{J}_1^2 = \vec{J}_2^2$$

$$\vec{R} \cdot \vec{L} = 0$$

26: Addition of Angular Momentum

$J_1^2 = J_2^2 = \frac{1}{4}\hbar^2(v^2-1) = \hbar^2 j(j+1)$

$v^2 = 1+4j(j+1)$

$= 4j^2+4j+1$

$= (2j+1)^2$

$|j, m_1\rangle \otimes |j, m_2\rangle$

$j \otimes j = 2j \oplus 2j-1 \oplus \ldots \oplus 0 = (l=n-1) \oplus (l=n-2) \oplus \ldots \oplus (l=0)$