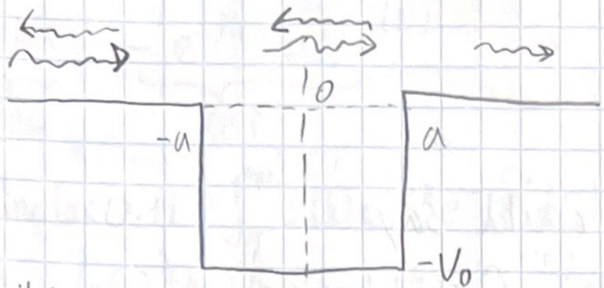
17.1: Waves on the Finite Square well

Ramsauer-Townsend Effect



$Ae^{ikx} + Be^{-ikx}$

$E > 0$

$k^2 = \frac{2mE}{\hbar^2}$

$Ce^{ik_2x} + De^{-ik_2x}$

$k_2^2 = \frac{2m(E+V_0)}{\hbar^2}$

$$\Psi = \begin{cases} \text{—} & x < -a \\ \text{—} & -a < x < a \\ \text{—} & x > a \end{cases}$$

$R = \left|\frac{B}{A}\right|^2$

$T = \left|\frac{F}{A}\right|^2$

$J_L \sim |A|^2 - |B|^2$

$J_R \sim |F|^2$

17.1: Waves on the Finite Square Well (continued)

$$|A|^2 - |B|^2 = |F|^2$$

$$|A|^2 = |B|^2 + |F|^2$$

$$1 = \left|\frac{B}{A}\right|^2 + \left|\frac{F}{A}\right|^2 = R + T$$

$$\frac{1}{T} = 1 + \frac{1}{4}\frac{V_0^2}{E(E+V_0)} \sin^2(2k_2 a)$$

17.2: Resonant Transmission

$$\frac{1}{T} = 1 + \frac{1}{4} \frac{V_0}{E(E+V_0)} \sin^2(2k_2 a)$$

$$\Rightarrow T \leq 1$$

$$E \to 0 \Leftrightarrow T \to 0$$

$$E \to \infty \Leftrightarrow T \to 1$$

$$2k_2 a = 2\sqrt{\frac{2ma(E+V_0)}{\hbar^2}}$$

$$= 2\sqrt{\frac{2ma^2 V_0 \left(1+\frac{E}{V_0}\right)}{\hbar^2}}$$

$$e \equiv \frac{E}{V_0}$$

$$z_0^2 \equiv \frac{2ma^2 V_0}{\hbar^2}$$

$$= 2 z_0 \sqrt{1+e}$$

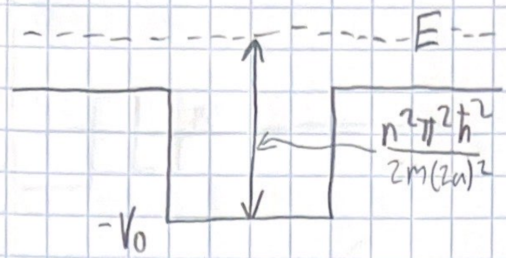
$$\frac{1}{T} = 1 + \frac{1}{4e(1+e)} \sin^2\left(2 z_0 \sqrt{1+e}\right)$$

$$e_n = -1 + \frac{n^2 \pi^2}{4 z_0^2}$$

$$E = -V_0 + \frac{n^2 \pi^2 V_0}{4 \cdot \frac{2ma^2 V_0}{\hbar^2}}$$

$$E = -V_0 + \frac{n^2 \pi^2 \hbar^2}{2m(2a)^2}$$

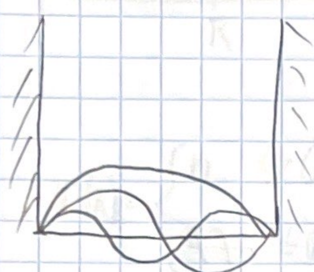
17.2: Resonant Transmission (continued)



$k_2(2a) = n\pi$

$k_2 = \frac{2\pi}{\lambda}(2a) = n\pi$

$\frac{(2a)}{\lambda} = \frac{n}{2}$



$\frac{2a}{\left(\frac{\lambda}{2}\right)} = n$

17.3: Ramsauer Townsend

$T_1 \quad z_0 = \frac{13\pi}{4}$

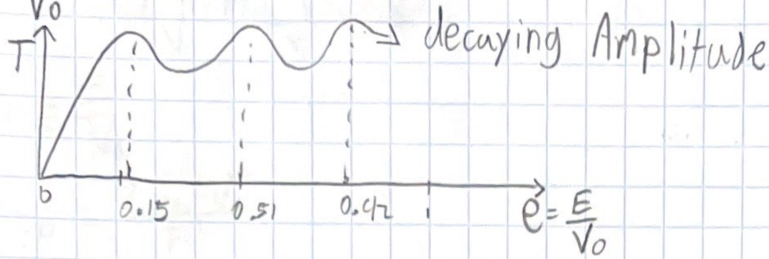
$n \geq \frac{2z_0}{\pi} = \frac{13}{2}$

$n = 7, 8, 9, \ldots$

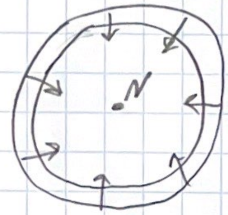
$\frac{E_7}{V_0} = 0.15976$

$\frac{E_8}{V_0} = 0.514793$

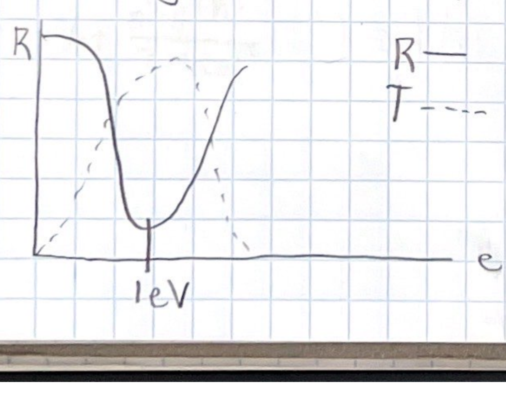
$\frac{E_9}{V_0} = 0.41716$



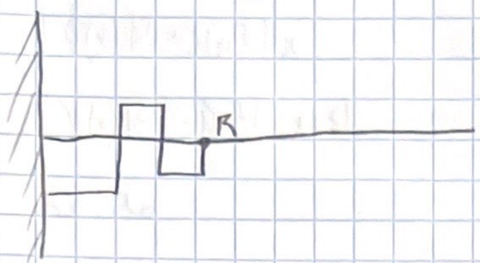
Scattered elasticity low energy electrons off of rare gas atoms



The reflection coefficient is a proxy for the scattering cross section

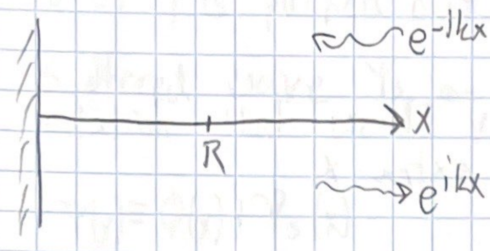


17.4: Scattering in 1D



$$V(x) = \begin{cases} 0 & x > R \\ V(x) & R > x < 0 \\ \infty & x \geq 0 \end{cases}$$ R=range

No potential



$$V(x) = \begin{cases} 0 & x > 0 \\ \infty & x \leq 0 \end{cases}$$

$\phi(x) \sim e^{ikx} - e^{-ikx}$

$\phi(x) = -\frac{e^{-ikx} + e^{ikx}}{2i}$

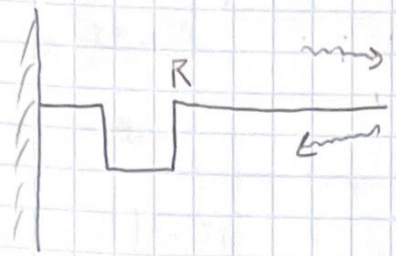
$= \sin kx$

$= -\frac{e^{-ikx}}{2i}$

$= \frac{e^{ikx}}{2i}$

17.4: Scattering in 1D (continued)

Potential exists



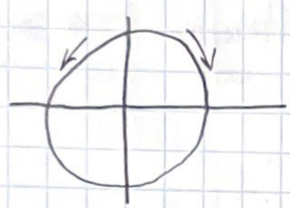
$-\frac{e^{-ikx}}{2i} \quad x>R$

$\frac{1}{2i}e^{ikx}e^{2i\delta(k)} \quad x>R$

$Ae^{-ikx}+Be^{ikx} \quad |A|^2-|B|^2$

Due to current conservation, no extra x dependence is allowed

$J_{inc}=J_{ref}$



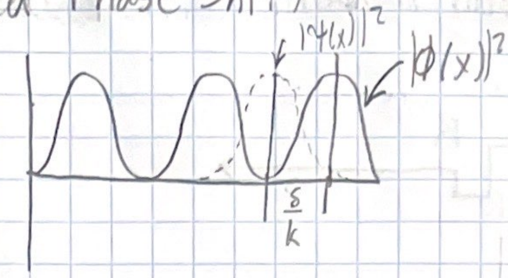
$$\psi(x)=\frac{1}{2i}\left(e^{ikx+2i\delta}-e^{-ikx}\right)$$

$$=\frac{e^{i\delta}}{2i}\left(e^{i(kx+\delta)}-e^{-i(kx+\delta)}\right)$$

$$=e^{i\delta}\sin(kx+\delta)\equiv\psi(x)$$

17.5: Scattered Wave and Phase Shift

$|\phi(x)|^2 = \sin^2 kx$



$|\psi(x)|^2 = \sin^2(kx+\delta)$

$kx = a_0$

$k\bar{x} + \delta = a_0$

$\bar{x} = \frac{a_0}{k} - \frac{\delta}{k}$

$\delta > 0$, ψ is pulled in (potential attractive)

$\delta < 0$, ψ is pushed out (potential repulsive)

Scattered wave ψ_s as extra piece in the ψ solution that would vanish without a potential

$\psi(x) \equiv \phi(x) + \psi_s(x)$

$\psi_s = \psi - \phi$

$= \frac{1}{2i}\left(e^{ikx+2i\delta} - e^{-ikx}\right) - \frac{1}{2i}\left(e^{ikx} - e^{-ikx}\right)$

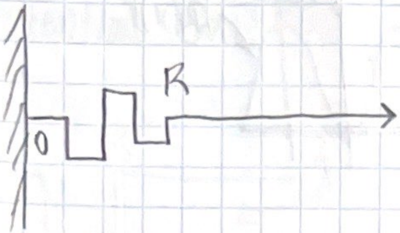
$= \frac{e^{ikx}}{2i}\left(e^{2i\delta} - 1\right)$

$= e^{ikx} e^{i\delta} \sin\delta$

$\psi_s = A_s e^{ikx}$

$A_s = e^{i\delta} \sin\delta$

18.1: Incident Packet



$$V(x)=\begin{cases} V(x) & 0<x<R \\ 0 & x>R \\ \infty & x\leq 0 \end{cases}$$

$$E=\frac{\hbar^2k^2}{2m}$$

$$\phi(x)=\sin kx$$
$$=\frac{-e^{-ikx}}{2i}+\frac{e^{ikx}}{2i}$$

$$\psi(x)=e^{i\delta}\sin(kx+\delta)$$
$$=\frac{-e^{-ikx}}{2i}+\frac{e^{2i\delta}e^{ikx}}{2i} \quad x>R$$

$$\psi=\phi+\psi_s$$

$$\psi_s(x)=e^{i\delta}\sin\delta\, e^{ikx}$$

$$\Psi_{inc}(x,t)=\int_0^{\infty}f(k)e^{-ikx}e^{\frac{iE(k)t}{\hbar}}dk \quad x>R$$

$$\Psi_{ref}(x,t)=-\int_0^{\infty}f(k)e^{ikx}e^{2i\delta(E)}e^{-\frac{iE(k)t}{\hbar}}dk$$

18.1: Stationary Packet (continued)

Stationary phase $k = k_0$

ψ_{inc}: $x = -\frac{\hbar k_0 t}{m} = v_g t$

ψ_{ref}: $x = v_g (t - 2\hbar \delta'(E))$

$$\Delta t = 2\hbar \delta'(E)$$

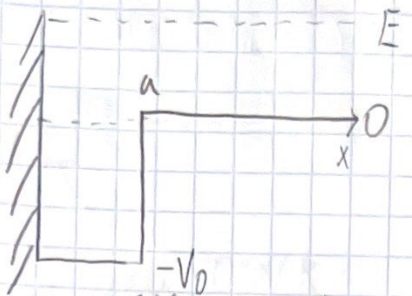
$$= 2\hbar \frac{d\delta}{dk} \frac{dk}{dE}$$

$$= \frac{2}{\left(\frac{1}{\hbar}\frac{dE}{dk}\Big|_{k_0}\right)} \frac{d\delta}{dk}\Big|_{k_0}$$

$$= \frac{2}{v_g} \frac{d\delta}{dk}$$

$$= \frac{1}{R}\frac{d\delta}{dk} = \frac{\Delta t}{\left(\frac{2}{v_g}R\right)} = \frac{\text{delay}}{\text{transit time}}$$

18.2: Phase Shift for a Potential Well



$$\Psi(x) = \begin{cases} e^{i\delta}\sin(kx+\delta), & x>a \\ A\sin k'x, & x<a \end{cases}$$

$$k^2 = \frac{2mE}{\hbar^2}$$

$$k'^2 = \frac{2m(E+V_0)}{\hbar^2}$$

Ψ continuous at $x=a$

$$A\sin k'a = e^{i\delta}\sin(ka+\delta)$$

Ψ' continuous at $x=a$

$$Ak'\cos k'a = ke^{i\delta}\cos(ka+\delta)$$

$$k\cot(ka+\delta) = k'\cot(k'a)$$

$$\cot(ka+\delta) = \frac{k'}{k}\cot k'a$$

$$\left.\cot(A+B) = \frac{\cot A\cot B - 1}{\cot A + \cot B}\right\} \text{Formula}$$

$$\frac{\cot(ka)\cot\delta - 1}{\cot(ka)+\cot\delta}$$

$$\cot\delta = \frac{\tan ka + \frac{k'}{k}\cot(k'a)}{1 - \frac{k'}{k}\cot(k'a)\tan ka}$$

10.3: Excursion of Phase Shift

$$\cot\delta = \frac{\tan ka + \frac{k'}{k}\cot(k'a)}{1 - \frac{k'}{k}\cot(k'a)\tan ka}$$

$$(ka)^2 = \frac{2mEa^2}{\hbar^2}$$

$$\equiv u^2 \} \text{ definition}$$

$$(k'a)^2 = \frac{2mEa^2}{\hbar^2} + \frac{2mV_0a^2}{\hbar^2} \equiv u^2 + z_0^2$$

$$\tan\delta = \frac{1 - \frac{k'a}{ka}\cot(k'a)\tan ka}{\tan ka + \frac{k'a}{ka}\cot(k'a)}$$

$$k'a = \sqrt{u^2 + z_0^2}\,, \quad \frac{k'a}{ka} = \sqrt{1 + \frac{z_0^2}{u^2}}$$

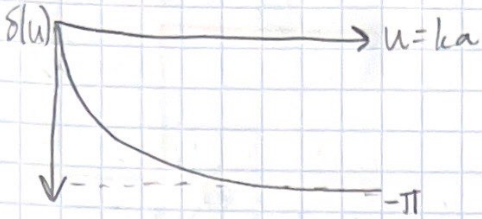
$$\frac{1 - \sqrt{1+\frac{z_0^2}{u^2}}\cot\sqrt{z_0^2+u^2}\,\tan u}{\tan u + \sqrt{1+\frac{z_0^2}{u^2}}\cot\sqrt{z_0^2+u^2}} = \tan\delta$$

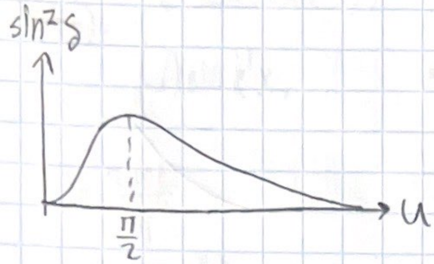
as $u \to 0 \Rightarrow \frac{\text{finite}}{\infty} \to 0$

$\tan\delta \to 0 \to \delta(ka=0) \to 0$

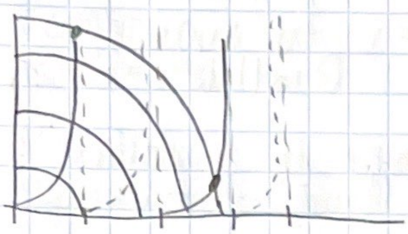
18.3: Excursion of the Phase Shift (continued)

$z_0^2 = 3.4 \; (0.59\pi = z_0)$

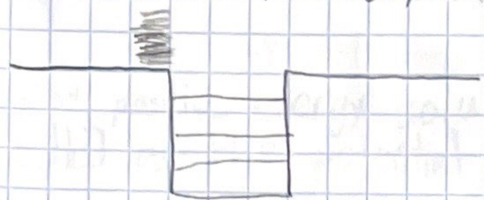




delay $= \frac{1}{a}\frac{d\delta}{dk}$

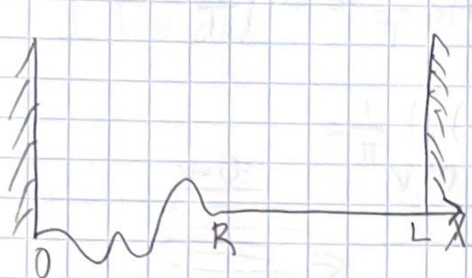


18.4: Levinson's Theorem II



Recall the # N of bound states of the potential to the excursion of the phase from $E=0$ to $E=\infty$

$$N=\frac{1}{\pi}(\delta(0)-\delta(\infty))$$



$L \gg R$

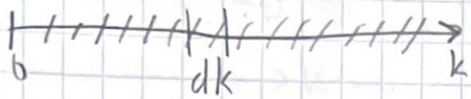
L is a regulator to avoid a continuum of states

Let $V=0$, positive energy states

$\phi(x)=\sin kx \quad \phi(L)=0 \Rightarrow \sin kL=0$

$kL=n\pi \quad (n=1,2,3,\ldots\infty)$

18.4: Levinson's Theorem I (continued)



$dk\,L = dn\,\pi$

$dn = \frac{L}{\pi} dk$ = # of positive energy states in dk

Repeat for $V \neq 0$ $x > R$

$\psi(x) = e^{i\delta} \sin(kx + \delta)$

$\psi(x = L) = 0$

$kL + \delta(k) = n' \pi$

$dk\,L + \frac{d\delta}{dk} dk = dn' \pi$

$dn' = \frac{L}{\pi} dk + \frac{1}{\pi} \frac{d\delta}{dk} dk$

18.5: Levinson's Theorem II

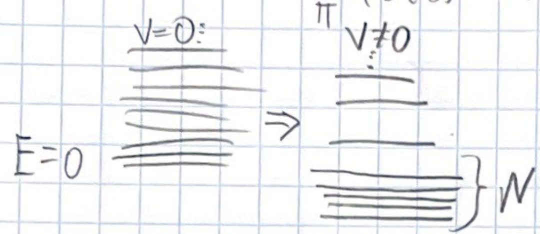
of positive energy solutions lost in the interval [dk] as the potential is turned on is:

$$dn - dn' = \frac{-1}{\pi}\frac{d\delta}{dk}dk$$

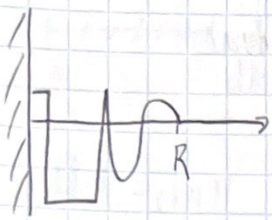
# of positive energy solutions lost as the potential is turned on:

$$\int_0^\infty \left(-\frac{1}{\pi}\frac{d\delta}{dk}\right)dk = -\frac{1}{\pi}\,\delta(k)\Big|_0^\infty$$

$$= \frac{1}{\pi}\left(\delta(0) - \delta(\infty)\right)$$



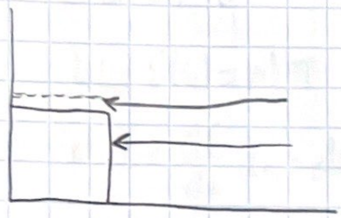
19.1: Resonances

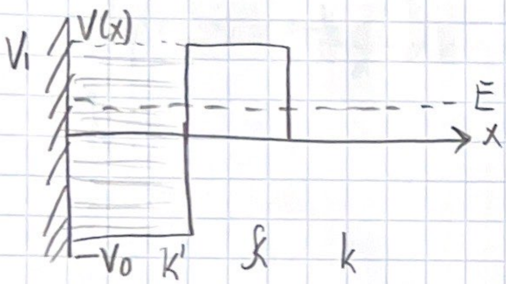


time delay $= 2\hbar \dfrac{d\delta(E)}{dE} \gtrsim -\dfrac{2R}{v}$

$$= 2\hbar \frac{1}{\left(\frac{dE}{dk}\right)} \frac{d\delta}{dk} \gtrsim -\frac{2R}{v}$$

$$\frac{d\delta}{dk} \gtrsim -R$$





$0 < E < V_1$

$$k^2 = \frac{2mE}{\hbar^2}$$

$$k'^2 = \frac{2m(E+V_0)}{\hbar^2}$$

$$\kappa^2 = \frac{2m(V_1-E)}{\hbar^2}$$

19.1: Resonances (continued)

$$\psi(x) = \begin{cases} e^{i\delta}\sin(kx+\delta), & x \geq 2a \\ A\sin k'x, & 0 < x < a \\ A\sin k'a\cosh(\kappa(x-a)) + B\sinh(\kappa(x-a)) \end{cases}$$

$e^{i\delta}\sin(kx+\delta)$, $x > R$

$e^{\kappa x}$ $e^{-\kappa x}$

$\sinh\kappa x$ $\cosh\kappa x$

$\sinh(\kappa(x-a))$

$\cosh(\kappa(x-a))$

19.2: Effects of Resonance on Phase Shifts

$$\tan(2k+\delta) = \left(\frac{k}{\kappa}\right)\left(\frac{\sin k'a \cosh \kappa a + \frac{k'}{k}\cos k'a \sinh \kappa a}{\sin k'a \sinh \kappa a + \frac{k'}{\kappa}\cos k'a \cosh \kappa a}\right)$$

$ka \equiv u$

$z_0^2 \equiv \frac{2mV_0 a^2}{\hbar^2}$

$z_1^2 \equiv \frac{2mV_1 a^2}{\hbar^2}$

$\frac{E}{V_1} \equiv e$

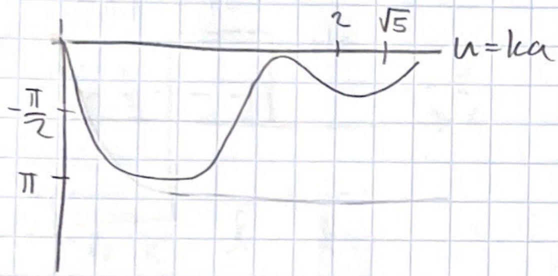
$= \frac{\hbar^2 k^2 a^2}{2mV_1 a^2} = \frac{u^2}{z_1^2}$

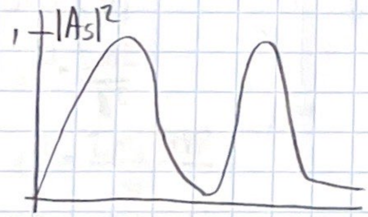
$(k'a)^2 = u^2 + z_0^2$

$(\kappa a) = z_1^2 - u^2$

$z_0^2 = 1$

$z_1^2 = 5$





$|A_s|^2 = \sin^2\delta$

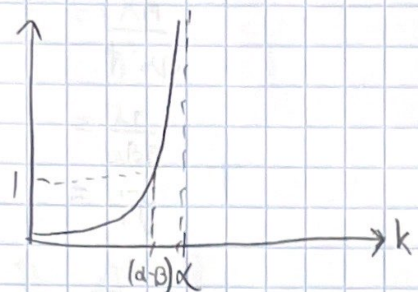
19.3: Modeling Resonance

Finding Resonance

$k = \alpha$

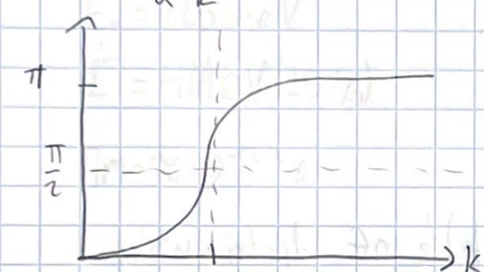
$$\tan \delta = \frac{\beta}{\alpha - k}$$

$$\delta = \tan^{-1}\left(\frac{\beta}{\alpha - k}\right)$$



β small for sharp behavior

$$\tan^{-1}\frac{\beta}{\alpha - k}$$



$$\left.\frac{d\delta}{dk}\right|_{k=\alpha} = \frac{1}{\beta}$$

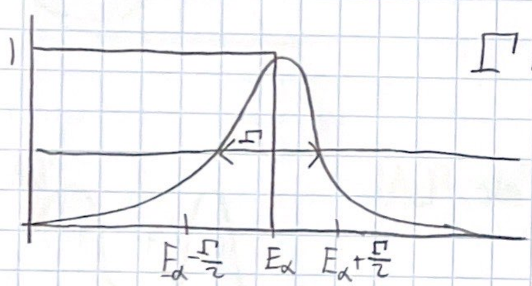
19.3: Modeling Resonance (continued

$$|\psi_s|^2 = \sin^2\delta$$
$$= |A_s|^2$$
$$= \frac{\beta^2}{\beta^2 + (\alpha - k)^2}$$
$$= \text{Breit Wigner distribution}$$

$$E - E_\alpha = \frac{\hbar^2 k^2}{2m} - \frac{\hbar^2 \alpha^2}{2m}$$
$$= \frac{\hbar^2}{2m}(k^2 - \alpha^2)$$
$$= \frac{\hbar^2}{2m}(k-\alpha)(k+\alpha)$$
$$= \frac{\hbar^2 \alpha}{m}(k - \alpha)$$

$$|\psi_s|^2 = \frac{\frac{1}{4}\Gamma^2}{(E-E_\alpha)^2 + \frac{1}{4}\Gamma^2}$$

$$\Gamma = \frac{2\alpha\beta\hbar^2}{m}$$



Γ is (width) $\frac{1}{2}$ of distribution

19.4: Half Width

$\tau = \frac{\hbar}{\Gamma}$ [+]

$\tau \equiv \frac{m}{2\alpha\beta\hbar}$

$\Delta t = 2\hbar \frac{d\delta}{dE} = 2\hbar \frac{dk}{dE} \frac{d\delta}{dk}$

$= \frac{2km}{\hbar^2 \alpha\beta}$

$= \frac{2m}{\alpha\beta\hbar}$

$= 4\tau$

$\frac{\hbar}{\Gamma} = \frac{\Delta t}{4}$

Higgs-Boson

$E_\alpha = 125\,GeV$

$\Gamma = 4\,MeV\,(\pm 5\%)$

$Time \approx 10^{-22}\,s$

19.5: Resonances in Complex Plane

$\Psi_s = A_s e^{ikx}$

$A_s = \sin\delta e^{i\delta}$

$= \frac{\sin\delta}{e^{-i\delta}}$

$= \frac{\sin\delta}{\cos\delta - i\sin\delta}$

$A_s = \frac{\tan\delta}{1 - i\tan\delta}$

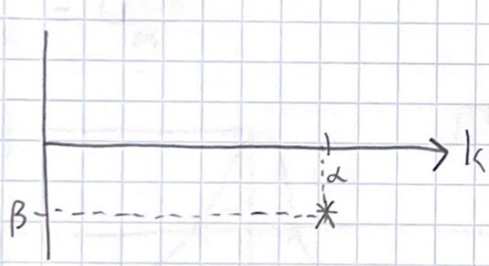
$\tan\delta = -i$

A_s will become infinite

$A_s = \frac{\frac{\beta}{\alpha - k}}{1 - \frac{i\beta}{(\alpha - k)}}$

$= \frac{\beta}{\alpha - k - i\beta}$

$= \frac{\beta}{(\alpha - i\beta) - k}$



$\alpha - i\beta$ is a pole of A_s

$\tan\delta(k) = -i$