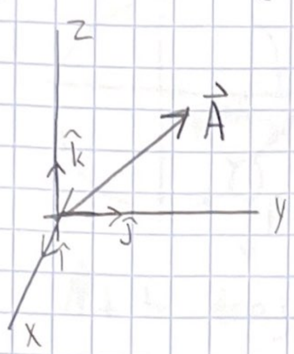
1: Dot Product

Vectors $\begin{cases} \text{direction} \\ \text{magntilde} \end{cases}$



$\vec{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k} = \langle a_1, a_2, a_3 \rangle$

Length $= |\vec{A}|$ (scalar); direction: $\mathrm{dir}(\vec{A})$

$P \longrightarrow Q$

$\vec{PQ} = \vec{A}$

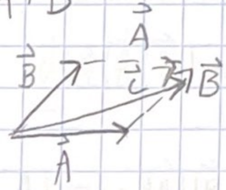
$\vec{A} = 3\hat{i} + 2\hat{j} + \hat{k} = \langle 3, 2, 1 \rangle \Rightarrow |\vec{A}| = \sqrt{9+4+1} = \sqrt{14} = \sqrt{|\vec{B}|^2 + 1}$

$\vec{B} = 3\hat{i} + 2\hat{j} \Rightarrow |\vec{B}| = \sqrt{13}$

$|\vec{A}|^2 + |\vec{B}|^2 = 1$

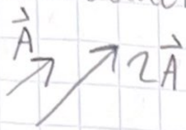
$|\vec{A}| = \sqrt{a_1^2 + a_2^2 + a_3^3}$

$\vec{A} + \vec{B}$



$\vec{A} + \vec{B} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$

1: Dot Product



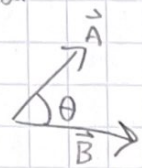
(scalar Multiplication)

The dot product:

$\vec{A}\cdot\vec{B} = \sum a_i b_i = a_1 b_1 + a_2 b_2 + \dots + a_i b_i$

produces a scalar

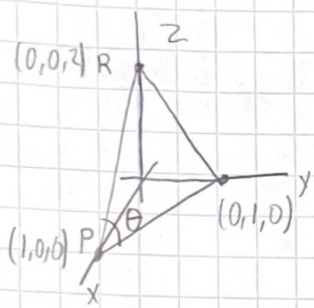
$\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta$



$|\vec{c}|^2 = \vec{c}\cdot\vec{c} = (\vec{A}-\vec{B})\cdot(\vec{A}-\vec{B}) = |\vec{A}|^2 + |\vec{B}|^2 - 2\vec{A}\cdot\vec{B}$

Applications:

1) compute lengths and angles



$\vec{PQ}\cdot\vec{PR} = |\vec{PQ}||\vec{PR}|\cos\theta$

$\cos\theta = \dfrac{\vec{PQ}\cdot\vec{PR}}{|\vec{PQ}||\vec{PR}|} = \dfrac{\langle -1,1,0\rangle\cdot\langle -1,0,2\rangle}{\sqrt{2}\cdot\sqrt{5}} = \dfrac{1}{\sqrt{10}}$

$\cos^{-1}\left(\frac{1}{\sqrt{10}}\right) \approx 71.5^\circ$

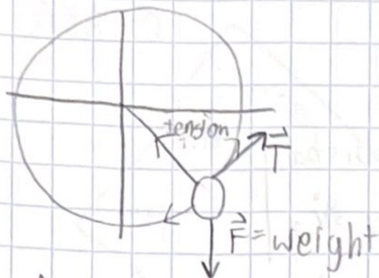
$>0 \rightarrow \theta < 90^\circ$

$=0 \rightarrow \theta = 90^\circ$

$<0 \rightarrow \theta > 90^\circ$

2: Determinants

$\vec{A}\cdot\vec{B}=\sum a_i b_i$

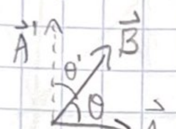


$\hat{N} \perp$ to trajectory

Componant of $\vec{F}$ along $\vec{T}$

Area:

$A=\frac{1}{2}|A||B|\sin\theta$



$\vec{A}'=\vec{A}$ rotated $90°$

$\theta'=\frac{\pi}{2}-\theta$

$\cos(\theta')=\sin(\theta)$

$|\vec{A}||\vec{B}|\sin\theta=\vec{A}'\cdot\vec{B}$

$=a_1b_2-a_2b_1$

The determinant

$$\det(\vec{A},\vec{B})=\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$A(\square)=|\vec{A}||\vec{B}|\sin\theta=\det(\vec{A},\vec{B})$

2: Determinants

Determinant in space: $\vec{A}, \vec{B}, \vec{C}$

$$\det(\vec{A},\vec{B},\vec{C}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Geometrically: $\det(\vec{A},\vec{B},\vec{C}) = \pm$ volume of parallelepiped

The Cross Product

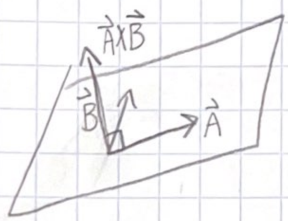
2 vectors in 3D space

$\vec{A} \times \vec{B} =$ vector

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$|\vec{A} \times \vec{B}|$ is area of parallelogram

$\text{dir}(\vec{A} \times \vec{B}) = \perp$ to plane of parallelogram



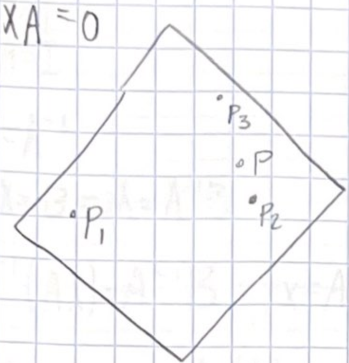
$\hat{\imath} \times \hat{\jmath} = \hat{k}$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 0\hat{\imath} + 0\hat{\jmath} + 1\hat{k}$$

3: Matrices

$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$

$\vec{A}\times\vec{A} = 0$

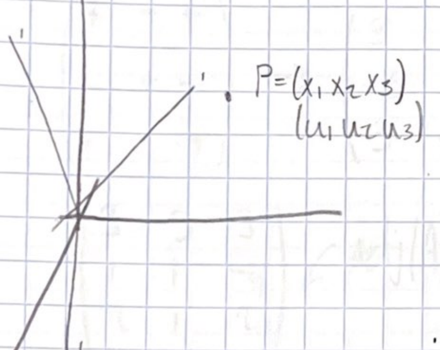


Plane $P_1 P_2 P_3$

$\det(\overrightarrow{P_1P}, \overrightarrow{P_1P_2}, \overrightarrow{P_1P_3}) = 0$ $\quad \overrightarrow{P_1P} \perp \vec{N}$

$\Leftrightarrow \overrightarrow{P_1P}\cdot\vec{N} = 0$

$\vec{N} = \overrightarrow{P_1P_2}\times\overrightarrow{P_1P_3}$



$$\begin{cases} u_1 = 2x_1 + 3x_2 + 3x_3 \\ u_2 = 2x_1 + 4x_2 + 5x_3 \\ u_3 = x_1 + x_2 + 2x_3 \end{cases} \Longrightarrow \begin{pmatrix} 2 & 3 & 3 \\ 2 & 4 & 5 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

$$A \quad X = U$$

3: Matrices

Entries AB

$$\underbrace{\begin{pmatrix} 1 & 2 & 3 & 4 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}}_{A} \begin{pmatrix} \cdot & 0 \\ \cdot & 3 \\ \cdot & 0 \\ \cdot & 2 \end{pmatrix} = \underbrace{\begin{pmatrix} \cdot & 14 \\ \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}}_{A}$$

$$(B)$$

$$(A)\,(X)$$

$$(AB)X = A(BX)$$

$$AB \neq BA$$

$$IX = X$$

$$I_{3\times3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$R\hat{i} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \hat{j}$$

$$R\hat{j} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} = -\hat{i}$$

$$R \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$$

4: Matrix Inverses

Inverse of A: matrix M

$AM=I$
$MA=I$

$M=A^{-1}$

$AX=B \Rightarrow X=A^{-1}B$

$A^{-1}(AX)=A^{-1}B \Rightarrow x=A^{-1}B$

$$A^{-1}=\frac{1}{\det(A)}\operatorname{adj}(A)$$

$$A=\begin{pmatrix}2&3&3\\2&4&5\\1&1&2\end{pmatrix}\rightarrow\begin{pmatrix}3&-1&-2\\3&1&-1\\3&4&2\end{pmatrix}$$

$$\begin{pmatrix}+&-&+\\-&+&-\\+&-&+\end{pmatrix}\rightarrow\begin{pmatrix}3&1&-2\\-3&1&1\\3&-4&2\end{pmatrix}$$

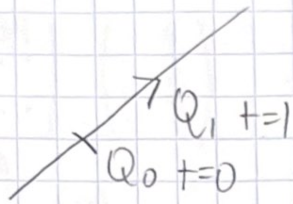
$$\begin{pmatrix}3&-3&3\\1&1&-4\\-2&1&2\end{pmatrix}\leftarrow \operatorname{adj}(A)$$

$$\begin{vmatrix}2&3&3\\2&4&5\\1&1&2\end{vmatrix}=3$$

$$A^{-1}=\frac{1}{3}\begin{pmatrix}3&-3&3\\1&1&-4\\-2&1&2\end{pmatrix}$$

5: Parametric Equations

line = intersection of two planes



$Q_0 = (-1, 2, 2)$, $Q_1 (1, 3, -1)$

$\overrightarrow{Q_0 Q(t)} \sim \overrightarrow{Q_0 Q_1}$

$= t\,\overrightarrow{Q_0 Q_1}$

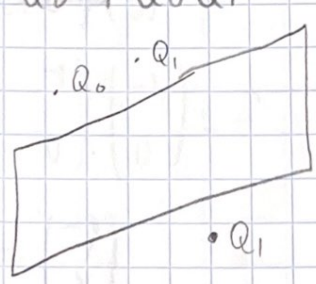
$= t\langle 2, 1, -3\rangle$

$Q(t) = (x(t), y(t), z(t))$

$$\begin{cases} x(t)+1 = t\cdot 2 \\ y(t)-2 = t \\ z(t)-2 = -3t \end{cases}$$

$$\begin{cases} x(t) = -1+2t \\ y(t) = 2+t \\ z(t) = 2-3t \end{cases}$$

$Q(t) = Q_0 + t\,\overrightarrow{Q_0 Q_1}$



$x + 2y + 4z = 7$

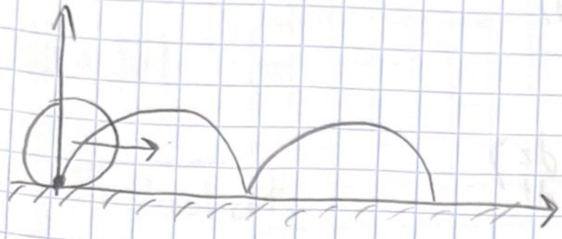
$Q(-1, 2, 2) \to (-1) + 2\cdot 2 + 4\cdot 2 = 11 > 7$

$Q_1(1, 3, -1) \to (1) + 2\cdot 3 + 4\cdot -1 = 3 < 7$

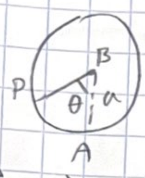
Opposite

5: Parametric Equations

They can be used for arbitrary motion on a plane or in space.



$(x(t), y(t))$ at P



$\vec{OP} = \vec{OA} + \vec{AB} + \vec{BP}$

$\vec{OA} = \langle a\theta, 0 \rangle$

$\vec{AB} = \langle 0, a \rangle$

$\vec{BP} = \langle -a\sin\theta, -a\cos\theta \rangle$

$$\vec{OP} = \langle \underbrace{a\theta - a\sin\theta}_{x(\theta)}, \underbrace{a - a\cos\theta}_{y(\theta)} \rangle$$

Taylor approximation:

t small

$f(t) \approx f(0) + tf'(0) + \frac{t^2}{2}f''(0) + \frac{t^3}{6}f'''(0) + \ldots$

$\sin\theta \approx \theta - \frac{\theta^3}{6}$, $\cos\theta \approx 1 - \frac{\theta^2}{2}$

$x(\theta) \approx \theta - (\theta - \frac{\theta^3}{6}) \approx \frac{\theta^3}{6}$

$y(\theta) \approx 1 - (1 - \frac{\theta^2}{2}) \approx \frac{\theta^2}{2}$

$\frac{3}{\theta} \to \infty$ when $\theta \to 0$

6: Physics Applications

$(x(t), y(t), z(t))$ = position of moving point

Position vector: $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

$\vec{r}(t) = \langle t - \sin t, 1 - \cos t \rangle$

Velocity vector: $\vec{v} = \frac{d\vec{r}}{dt} = \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \rangle$

$\vec{v} = \langle 1 - \cos t, \sin t \rangle$

$t = 0: \vec{v} = 0$

$|\vec{v}| = \sqrt{(1-\cos t)^2 + \sin^2 t}$

$= \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}$

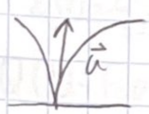
$= \sqrt{2 - 2\cos t}$

Acceleration: $\vec{a} = \frac{d\vec{v}}{dt}$

$t = 0: \vec{a} = 0$

$\vec{a} = \langle \cos t, \cos t \rangle$

$\vec{a} = \langle 0, 1 \rangle$



$\left|\frac{d\vec{r}}{dt}\right| \neq \frac{d|\vec{r}|}{dt}$

6: Physics Applications

Arclength

s = distance traveled

$$\frac{ds}{dt} = |\vec{v}|$$

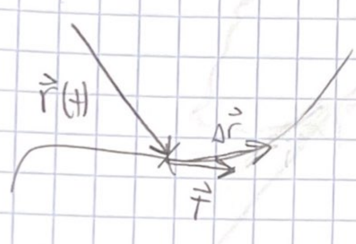
$$\int_0^{2\pi} \sqrt{2-2\cos t}\, dt$$

$\vec{T}$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}\frac{ds}{dt}$$

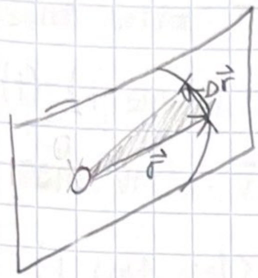
$$= \vec{T}\frac{ds}{dt}$$

$$\frac{d\vec{r}}{dt} = \vec{T}\frac{ds}{dt}$$



6: Physics Examples

Kepler's 2nd Law

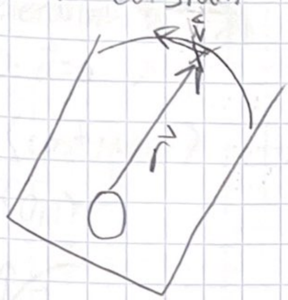


Newton explained this using gravitational attraction.

$\text{Area} \approx \frac{1}{2}|\vec{r} \times \Delta\vec{r}| \approx \frac{1}{2}|\vec{r} \times \vec{v}|\Delta t$

time Δt (small

$|\vec{r} \times \vec{v}| = \text{constant}$



Plane contains $\vec{r}, \vec{v}$

$\vec{r} \times \vec{v}$ is normal to plane

$\frac{d}{dt}(\vec{r} \times \vec{v}) = 0$

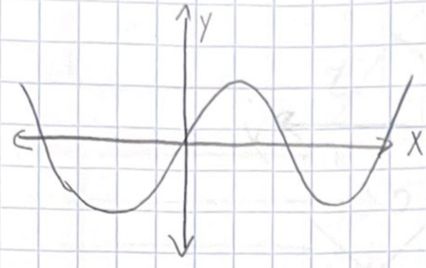
$\frac{d}{dt}(\vec{a} \cdot \vec{b})$ $\quad \frac{d}{dt}(\vec{a} \times \vec{b}) \Rightarrow$ product rule

$\frac{d\vec{r}}{dt} \times \vec{v} + \vec{r} \times \frac{d\vec{v}}{dt} = \vec{v} \times \vec{v} + \vec{r} \times \vec{a} = 0 \Leftrightarrow \vec{r} \times \vec{a} = 0 \Leftrightarrow \vec{a} \parallel \vec{r}$

7: 3D Functions

Function of 1 variable: $f(x)$

$f(x) = \sin x$



Function of two variables:

given $(x,y) \to$ get $f(x,y)$

$f(x,y) = x^2 + y^2$

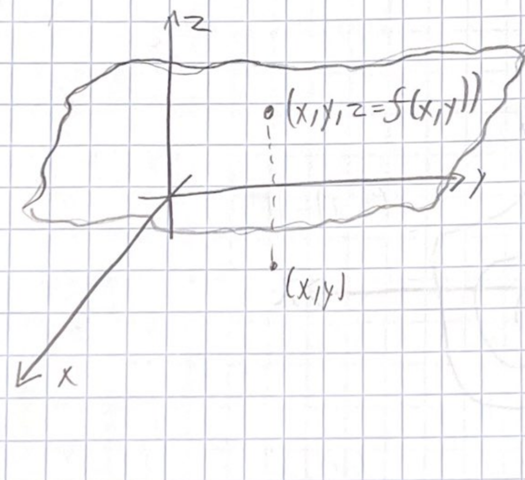
$f(x,y) = \sqrt{y}$ ∴ $y \geq 0$

$f(x,y) = \frac{1}{x+y}$ ∴ $x+y \neq 0$

3 or more parameters exist as well with more functions

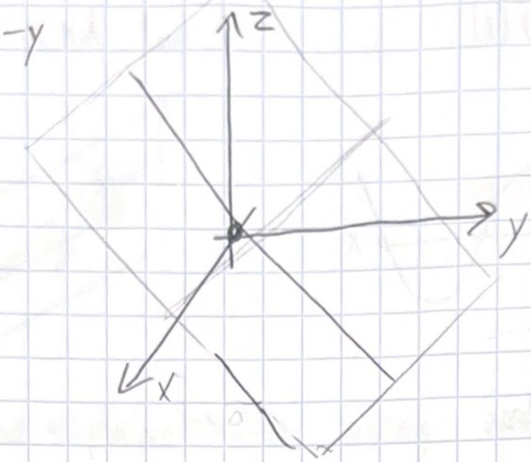
For simplicity, focus on 2 or 3 variables

$z = f(x,y)$

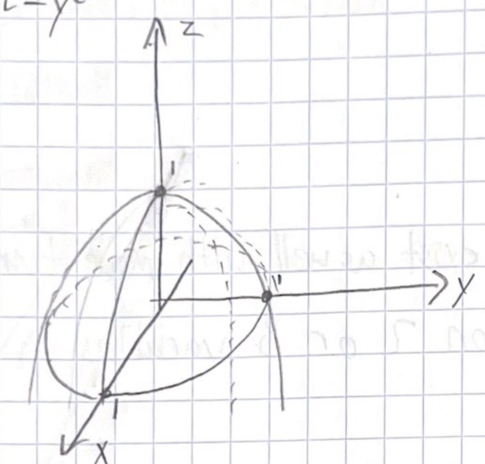


7: 3D Functions

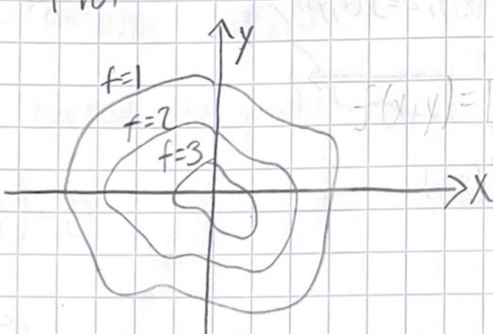
$f(x,y) = -y$



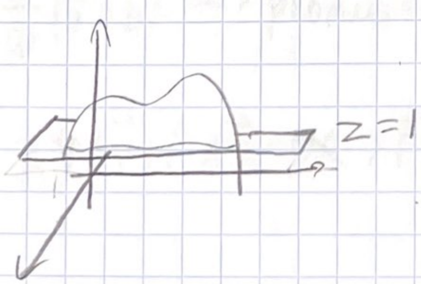
$f(x,y) = 1 - x^2 - y^2$



Contour plot

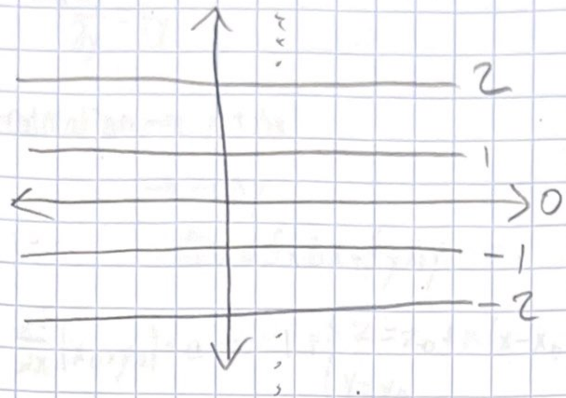


Level Curve

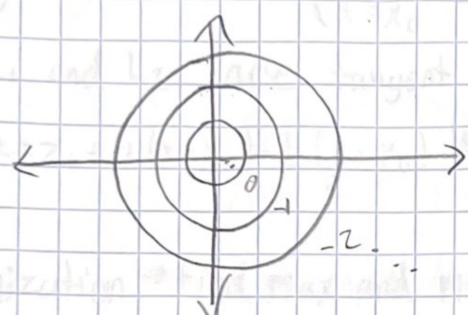


7: 3D Functions

$f(x,y) = -y$

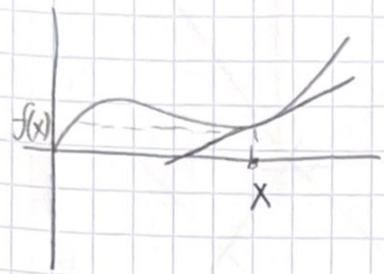


$f(x,y) = 1-(x^2+y^2)$



8: Partial Derivatives

Function $f(x)$: $f'(x) = \frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$

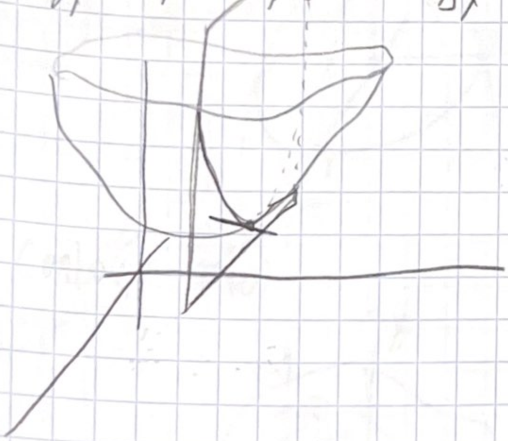


$x_0 \to f(x_0)$

$f(x) \approx f(x_0) + f'(x_0)(x-x_0)$

$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y_0) - f(x_0 y_0)}{\Delta x}$

$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0+\Delta y) - f(x_0, y_0)}{\Delta y}$



$\frac{\partial f}{\partial x} = f_x$

treat y as constant, x as variable

$f(x,y) = x^3 y + y^2$

$\frac{\partial f}{\partial x} = 3x^2 y$, $\frac{\partial f}{\partial y} = x^3 + 2y$

9: Maxima and Minima

$f(x,y) \to \frac{\partial f}{\partial x} = f_x$

$\to \frac{\partial f}{\partial y} = f_y$

Approximation $\to x + \Delta x$

$\to y + \Delta y$

$\Delta z = f_x \Delta x + f_y \Delta y$

If $\frac{\partial f}{\partial x}(x_0, y_0) = a \Rightarrow L_1 = \begin{cases} z = z_0 + a(x - x_0) \\ y = y_0 \end{cases}$

$\frac{\partial f}{\partial y}(x_0 y_0) = b \Rightarrow L_2 = \begin{cases} z = z_0 + b(y - y_0) \\ x = x_0 \end{cases}$

L_1 and L_2 are tangent to surface $z = f(x,y)$

$z = z_0 + a(x - x_0) + b(y - y_0) = \text{plane}$

Optomization → find max and min

At a local min/may → $f_x = 0$ and $f_y = 0$

tangent plane is horizontal

$(x_0 y_0)$ is critical point of f if $f_x(x_0 y_0)$ and $f_y(x_0 y_0) = 0$

$f(x,y) = x^2 - 2xy + 3y^2 + 2x - 2y$

$$\begin{cases} \frac{\partial f}{\partial x} = 2x - 2y + 2 = 0 \Rightarrow 2x + 2 = 0 \Rightarrow x = -1 \\ \frac{\partial f}{\partial y} = -2x + 6y - 2 = 0 \end{cases}$$

$4y = 0 \Rightarrow y = 0$

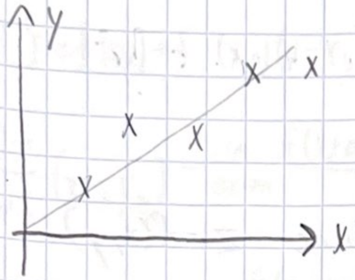
$(x,y) = (-1, 0)$

9: Maxima and Minima

Possibilities
- local min
- local max
- saddle

$f(x,y) = (x-y)^2 + 2y^2 + 2x^2 - 2y$

$= \underbrace{((x-y)+1)^2}_{\geq 0} + \underbrace{2y^2}_{\geq 0} - 1$ so minimum $\Rightarrow f(-1,0)$



Given $(x_1,y_1) \ldots (x_n,y_n) \Rightarrow$ best fit $y = ax+b$

minimize total square deviation

deviation: $y_i - (ax_i + b)$

$$D(a,b) = \sum_{i=1}^{n} \left(y_i - (ax_i + b)\right)^2$$

$$\begin{cases} \dfrac{\partial D}{\partial a} = 0 = \displaystyle\sum_{i=1}^{n} 2(y_i - (ay_i + b))(-x_i) \\ \dfrac{\partial D}{\partial b} = 0 = \displaystyle\sum_{i=1}^{n} 2(y_i - (ay_i + b)(-1) \end{cases}$$

Best exponential fit: $y = ce^{ax} \Leftrightarrow \ln(y) = \ln(c) + ax$