7: Linear Algebra

U is a T-invarient subspace if $T(U)=\{Tu \mid u\in U\} \subseteq U$

U to be 1-D $U=\{\alpha u \mid \alpha \in \mathbb{F}$

T-invarient $Tu=\lambda u$ for some λ

spectrum $(T)=\{\lambda \mid \lambda$ is an eigenvalue of $T\}$

λ is an eigenvalue $\Rightarrow \underbrace{(T-\lambda I)}_{\text{operator}}u=0$

$\Rightarrow (T-\lambda I)$ not injective

$\Rightarrow (T-\lambda I)$ not invertable

null$(T-\lambda I)=\{$eigenvectors with eigenvalue $\lambda\}$

Let $T\in\mathcal{L}(V)$ and $\lambda_1 \dots \lambda_n$ are distinct eigenvalues of T with corresponding eigenvectors $\{u_1 \dots u_n\}$, then $\{u_1 \dots u_n\}$ are linearly independent

If false: $c_1u_1+c_2u_2+\dots c_ku_k=0$, let there be $k\leq n$ so holds for $c_1\neq 0$. $(T-\lambda_k I)(c_1u_1+\dots+c_ku_k)$
$=c_1\underbrace{(\lambda_1-\lambda_k)}_{\text{number}}u_1+\dots c_{k-1}\underbrace{(\lambda_{k-1}-\lambda_k)}_{\text{number}}u_{k-1}+0u_k \Rightarrow *$

$T\in\mathcal{L}(V)$ with V being finite dimensional and complex, there is at least one eigenvalue

7 Linear Algebra

$(T-\lambda I)u=0 \iff (T-\lambda I)$ not invertable
$\iff$ matrix representation of $(T-\lambda I)$ in any basis is not invertable
$\iff \det(T-\lambda I)=0=f(\lambda)$

$f(\lambda)=(\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n)$

$\lambda_i \in \mathbb{C}$ and can be repeated

Inner product on a vector space V is a map from $V \otimes V \to \mathbb{F}$

$V=\mathbb{R}^n \qquad \vec{a}=(a_1, a_2 \cdots a_n), \vec{b}=(b_1, b_2 \cdots b_n) \in V$

dot product $\vec{a}\cdot\vec{b}=a_1b_1+a_2b_2+\ldots+a_nb_n$

$\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$

$\vec{a}\cdot\vec{a}\geq 0$ with $\vec{a}\cdot\vec{a}=0 \Rightarrow \vec{a}=0$

$\vec{a}\cdot(\beta_1\vec{b_1}+\beta_2\vec{b_2})=\beta_1\vec{a}\cdot\vec{b_1}+\beta_2\vec{a}\cdot\vec{b_2}$

defines length, norm $|\vec{a}|^2=\vec{a}\cdot\vec{a}$

Not unique (consider "$a\cdot b$" $=c_1a_1b_1+c_2a_2b_2+\cdots+c_na_nb_n$)
$c_i>0$

7: Linear Algebra

Schwarz Inequality

$|\vec{a}\cdot\vec{b}| \leq |\vec{a}||\vec{b}|$

$z \in \mathbb{C}^n$, $z = (z_1, z_2 \dots z_n)$ $z_i \in \mathbb{C}$

$|z| = \sqrt{|z_1|^2 + |z_2|^2 + \dots + |z_n|^2} = \sqrt{z_1^* z_1 + z_2^* z_2 + \dots + z_n^* z_n}$

Inner product:

$\langle \cdot | \cdot \rangle : V \otimes V \to \mathbb{C}$

$\langle a|b \rangle \in \mathbb{C}$, $a, b \in V$

$\langle a|b \rangle = \langle b|a \rangle^*$

$\langle a|a \rangle \geq 0$ only 0 if $a = 0$

$\langle a|\beta_1 b_1 + \beta_2 b^2 \rangle = \beta_1 \langle a|b_1 \rangle + \beta_2 \langle a|b_2 \rangle$, $\beta_n \in \mathbb{C}$

$\langle a|b \rangle$ notation of Dirac $\Leftrightarrow$ (a,b) in math

$\langle \alpha_1 a_1 + \alpha_2 a_2 | b \rangle = \alpha_1^* \langle a_1|b \rangle + \alpha_2^* \langle \alpha_2|b \rangle$

$V = \mathbb{C}^n$

$\langle a|b \rangle = a_1^* b_1 + a_2^* b_2 + \dots + a_n^* b_n$

7: Linear Algebra

V = set of all $f(x) \in \mathbb{C}$ with $x \in [0, L]$,

$f, g \in V \Rightarrow \langle f | g \rangle \equiv \int_0^L f^*(x) g(x) dx$

Inner Product

$\langle a | b \rangle \quad a, b \in V$

if $\langle a | b \rangle = 0 \Rightarrow \vec{a}$ and $\vec{b}$ are orthogonal

If have a set of vectors $\{e_1 e_2 \ldots e_n\}$ such that $\langle e_i | e_j \rangle = \delta_{ij} \Rightarrow$ the set is orthonormal

Let $\{e_i\}$ form a basis of V and $\{e_i\}$ is orthonormal

$a \in V, \quad a = \sum_{i=1}^{n} a_i e_i$

$\langle e_k | a \rangle = \sum_{i=1}^{n} a_i \langle e_k | e_i \rangle = a_k$

Norm: $|a|^2 = \langle a | a \rangle$

$|\langle a | b \rangle| \leq |a||b| \Rightarrow$ schwarz inequality

$|a+b| \leq |a| + |b|$

Vector space with an inner product = Hilbert Space

For ∞ dimensional vector space with an inner product = Hilbert Space

7: Linear Algebra

V = space of polynomials on an interval $[0, L]$

$P_N(x) = p_0 + p_1 x + \dots + p_n x^n$

$P_N(x) = \sum_{i=1}^{n} \frac{x^i}{i!}$ $n \to \infty$ $= e^x$

Gram-Schmidt Procedure

list $\{v_1 v_2 \dots v_n\}$ linearly independent

construct $\{e_1 e_2 \dots e_n\}$ is linearly independent, orthonormal

$e_j = v_j - \sum_{i<j} \langle e_i | v_j \rangle e_i$

V, vector space

U is a set of $v \in V$

define $U^\perp$ (orthogonal complement of U)

$U^\perp = \{v \in V \mid \langle v|u \rangle = 0 \ \forall u \in U\}$

$U^\perp$ is a subspace of V

If U is a subspace of V

$V = U \oplus U^\perp$

7: Linear Algebra

$|a\rangle$ = ket $\in V$

$\langle a|b\rangle = \langle a|\,|b\rangle$

$\langle \cdot \quad \cdot \rangle$ bracket $\langle \cdot |\, | \rangle$ bra ket

Dirac Notation

$\langle a|$ is a map from $V \to \mathbb{C} \notin V$

$\langle a|$ belongs to a dual space V^*

$\langle w| = \alpha\langle a| + \beta\langle b| \in V^*$

$\langle w|v\rangle = \alpha\langle a|v\rangle \beta\langle b|v\rangle \ \forall v \in V$

For any $|v\rangle \in V$, there is a unique bra $\langle v| \in V^*$

$\exists \langle v|, \langle v'| \in V^*$ such that $\langle v|w\rangle = \langle v'|w\rangle \ \forall |w\rangle \in V$

$\langle v|w\rangle - \langle v'|w\rangle = 0$

$= (\langle w|v\rangle - \langle w|v'\rangle)^*$

$= (\langle w|v-v'\rangle)^*$

$\Rightarrow |v-v'\rangle = |0\rangle \Rightarrow v = v'$

$|v\rangle = \alpha_1|a_1\rangle + \alpha_2|a_2\rangle \longleftrightarrow \langle v| = \alpha_1^*\langle a_1| + \alpha_2^*\langle a_2|$

8: Linear Algebra

$\langle u, v \rangle = \langle u | v \rangle$

Adjoints or Hermitian Conjugates:

A linear functional on V is a linear map from $V \to F$

$\varphi(v) \in F$

$\varphi(v_1 + v_2) = \varphi(v_1) + (v_2)$

$\varphi(av) = a\varphi(v)$

$\varphi: \mathbb{R}^3 \to \mathbb{R} \quad \varphi(x_1, x_2, x_3) = 3x_1 - x_2 + 7x_3$

$V = \langle u, v \rangle$

$u = (3, -1, 7)$

Let φ be a linear functional on V, then there is a unique vector $u \in V$ such that:

$\varphi(v) = \langle u, v \rangle \equiv \varphi_u(v)$

Say $(e_1 \dots e_n)$ is an orthonormal basis

$v = \langle e_1, v \rangle e_1 + \dots + \langle e_n, v \rangle e_n$

$\varphi(v) = \langle e_1, v \rangle \varphi(e_1) + \dots + \langle e_n, v \rangle \varphi(e_n)$

$= \langle e_1 (\varphi(e_1))^*, v \rangle + \dots + \langle e_n (\varphi(e_n))^*, v \rangle$

$\varphi(v) = \langle e_1 (\varphi(e_1))^* + \dots + e_n (\varphi(e_n))^*, v \rangle$

$\langle u, v \rangle = \langle u', v \rangle \; \forall v$

$\langle u - u', v \rangle = 0 \quad \forall v$

6: Linear Algebra

$\langle x, v\rangle = 0 \ \forall v$

$\langle x, x\rangle = 0 \Rightarrow u = u'$

$T \in \mathcal{L}(v),\ T^{+} \in \mathcal{L}(v)$

$\langle u, T, v\rangle = \varphi(v) = \langle T^{+}u, v\rangle$

Claim $T^{+} \in \mathcal{L}(v)$

$\langle u_1 + u_2, Tv\rangle = \langle T^{+}(u_1 + u_2), v\rangle$

$= \langle T^{+}u_1, v\rangle + \langle T^{+}u_2, v\rangle$

$= \langle T^{+}u_1 + T^{+}u_2, v\rangle$

$\langle au, Tv\rangle = \langle T^{+}(au), v\rangle$

$a^{*}\langle u_1, Tv\rangle = \bar{a}\langle T^{+}u, v\rangle$

$= \langle aT^{+}u, v\rangle$

$T^{+}(au) = aT^{+}(u)$

$(ST)^{+} = T^{+}S^{+}$

$V \in \mathbb{C}^3 \quad v = (v_1, v_2, v_3),\ v_i \in \mathbb{C}$

$T(v_1, v_2, v_3) = (0v_1 + 2v_2 + iv_3,\ v_1 - iv_2 + 0v_3,\ 3iv_1 + v_2 + 7v_3)$

$e_1 = (1,0,0)\ \ e_2 = (0,1,0),\ e_3 = (0,0,1)$

$\langle u, Tv\rangle = \langle T^{+}u, v\rangle$

$T^{+}(u_1, u_2, u_3) = (u_2 - 3i u_3,\ 2u_1 + iu_2 + u_3,\ -iu_1 + 7u_3)$

8: Linear Algebra

$$T = \begin{pmatrix} 0 & 2 & i \\ 1 & -i & 0 \\ 3i & 1 & 7 \end{pmatrix}$$

$$T^{+} = \begin{pmatrix} 0 & 1 & -3i \\ 2 & i & 1 \\ -i & 0 & 7 \end{pmatrix}$$

$$\langle T^{+}u, v\rangle = \langle u, Tv\rangle$$

$$\langle T^{+}e_i, e_j\rangle = \langle e_i, Te_j\rangle$$

$$\langle (T^{+})_{ki} e_k, e_j\rangle = \langle e_i, (T)_{kj} e_k$$

$$(T^{+})_{ij} = (T_{ji})^{*}$$

If the basis is not orthonormal then
$\langle e_i, e_j\rangle = e_{ij}$

$$(T^{+})^{*}_{ki}\, g_{kj} = T_{kj}\, g_{ik}$$

$$\langle T^{+}u, v\rangle = \langle u|Tv\rangle$$

$$= \langle u|T|v\rangle$$

$$\langle u|T^{+}|v\rangle = \langle v|T|u\rangle^{*}$$

$$\langle e_i, Te_j\rangle = \langle e_i, (T)_{kj}\, e_k\rangle$$

$$(T)_{ij} = \langle e_i|T|e_j\rangle$$

4: Linear Algebra

$|e_i\rangle\langle e_i| = |e_1\rangle\langle e_1| + |e_2\rangle\langle e_2| + \ldots$

$|e_i\rangle\langle e_i|a\rangle = |a\rangle$

$|a\rangle = \alpha_i |e_i\rangle$

$\langle e_j|a\rangle = \alpha_j$

$\mathbb{1} = |e_i\rangle\langle e_i|$

$T = \mathbb{1}T\mathbb{1} = |e_i\rangle\langle e_i|T|e_j\rangle\langle e_j|$

$= \sum_{ij} |e_i\rangle (T_{ij}) \langle e_j|$

$\langle v, Tv\rangle = 0 \ \forall v \in V$

Let V be a complex inner product space

$\langle V, TV\rangle = 0 \ \forall v \in V \Rightarrow T = 0$

8: Linear Algebra

Suppose I could prove that $\langle u, Tv\rangle = 0 \ \forall u, v$

then pick $u = Tv \Rightarrow \langle Tv, Tv\rangle = 0 \Rightarrow Tv = 0 \Rightarrow T = 0$

$$\langle u, Tv\rangle \stackrel{?}{=} \langle \ , T\ \rangle + \langle \ , T\ \rangle + \dots$$

$$\stackrel{?}{=} \langle u+v, T(u+v)\rangle - \langle (u-v), T(u-v)\rangle$$

$$\stackrel{?}{=} \langle u+iv, T(u+iv)\rangle - \langle u-iv, T(u-iv)\rangle$$

$$= 2i\langle u, Tv\rangle - 2i\langle v, Tu\rangle$$

$$\frac{1}{4}\Big(\langle u+v, T(u+v)\rangle - \langle u-v, T(u-v)\rangle$$

$$+\frac{1}{i}\langle u+iv, T(u+iv)\rangle - \frac{1}{i}\langle u-iv, T(u-iv)\rangle\Big)$$

$\langle v, Tv\rangle \in \mathbb{R} \ \forall v \Rightarrow T^{\dagger} = T$

$\langle v, Tv\rangle \stackrel{\mathbb{R}}{=} \langle v, Tv\rangle^{*} = \langle Tv, v\rangle$

$\langle v, (T^{\dagger} - T)v\rangle = 0 \ \forall v$

9: Bra and ket

$\langle v, Tv \rangle = 0 \ \forall v \in V \rightarrow \mathbb{C}$ vs
$\Rightarrow T = 0$

$T = T^\dagger \Leftrightarrow \langle v, TV \rangle \in \mathbb{R} \ \forall V$

$\langle v | T | v \rangle = \langle v, Tv \rangle$

The eigenvalues of hermitian operators are real

$\langle v, Tv \rangle \quad Tv = \lambda v$

$= \langle v, \lambda v \rangle = \lambda \langle v, v \rangle$

$= \langle T^\dagger v, v \rangle = \langle \lambda v, v \rangle = \lambda^* \langle v, v \rangle$

$(\lambda - \lambda^*) \langle v, v \rangle = 0$

$\lambda = \lambda^*$

Different eigenvalues of hermitian operators correspond to orthogonal eigenfunctions

$$\left.\begin{array}{l} Tv_1 = \lambda_1 v_1 \\ Tv_2 = \lambda_2 v_2 \\ \lambda_1 \neq \lambda_2 \end{array}\right\} \longrightarrow \langle v_2, Tv_1 \rangle = \lambda_1 \langle v_2, v_1 \rangle$$

$$\begin{aligned} &\langle v_2, Tv_1 \rangle \\ &= \langle Tv_2, v_1 \rangle \\ &= \langle \lambda_2 v_2, v_1 \rangle \\ &= \lambda_2 \langle v_2, v_1 \rangle \end{aligned}$$

$\lambda \in \mathbb{R} \Rightarrow (\lambda_1 - \lambda_2) \langle v_1, v_2 \rangle = 0 \rightarrow \langle v_1, v_2 \rangle = 0$

9: Bra and Ket

If S is unitary ("an isometry")

$|Su| = |u| \ \forall u \in V$

λI is unitary if $|\lambda| = 1$

$|av| = |a||v|$

$|\lambda I u| = |\lambda u| = |\lambda||u|$

$\text{null}(S) = 0$

$\langle Su, Su \rangle = \langle u, u \rangle$

$\langle u, S^{\dagger} S u \rangle = \langle u, u \rangle$

$S^{\dagger} S = \mathbb{1}$

$S^{\dagger}$ is inverse of S

on basis $(e_1 \dots e_n)$

$f_i = U e_i$, U is unitary

$$\langle Su, Sv \rangle = \langle S^{\dagger} S u, v \rangle = \langle u, v \rangle$$

9. Bra and Ket

position states $|x\rangle \quad \forall x \in \mathbb{R}$

$\uparrow$ particle at x

$|ax\rangle \neq a|x\rangle$

$|-x\rangle \neq -|x\rangle$

$|x_1+x_2\rangle \neq |x_1\rangle + |x_2\rangle$

$|\vec{x}\rangle$

$|v\rangle \in V$

$$\langle x|y\rangle = \delta(x-y)$$

$$\mathbb{1} = \int |x\rangle\langle x| \, dx$$

$$\mathbb{1}|y\rangle = |y\rangle = \int |x\rangle\langle x|y\rangle \, dx = |y\rangle$$

$$|\psi\rangle \longrightarrow \psi(x) \equiv \langle x|\psi\rangle \in \mathbb{C}$$

$$|\psi\rangle = \mathbb{1}|\psi\rangle = \int |x\rangle\langle x|\psi\rangle \, dx$$

$$= \int |x\rangle \psi(x) \, dx$$

9: Bra and ket

$$\langle \phi | \psi \rangle = \langle \phi | \int |x\rangle\langle x|\psi\rangle dx$$

$$= \int \langle \phi | x \rangle \langle x | \psi \rangle dx$$

$$= \int \phi^*(x)\, \psi(x)\, dx$$

$$\langle \phi | \hat{x} | \psi \rangle = \int \langle \phi | \hat{x} | x \rangle \langle x | \psi \rangle dx$$

$$= \int x \langle \phi | x \rangle \langle x | \psi \rangle dx$$

$$= \int x\, \phi^*(x) \psi(x)\, dx$$

$|p\rangle \in \mathbb{R}$ $\langle p' | p \rangle = \delta(p - p')$

$\mathbb{1} = \int |p\rangle\langle p|\, dp$

$\hat{p}|p\rangle = p|p\rangle$

$$\psi_p(x) = \frac{e^{\frac{ipx}{\hbar}}}{\sqrt{2\pi\hbar}} = \langle x | p \rangle$$

$$\langle p | \psi \rangle = \int \langle p | x \rangle \langle x | \psi \rangle dx$$

$$= \int \frac{e^{\frac{ipx}{\hbar}}}{\sqrt{2\pi\hbar}} \psi(x)\, dx = \tilde{\psi}(p)$$

9: Bra and Ket

$$\langle x|\hat{p}|\psi\rangle = \int \langle x|\hat{p}|p\rangle\langle p|\psi\rangle dp$$

$$= \int (p\langle x|p\rangle)\langle p|\psi\rangle dp$$

$$= \int \left(\frac{\hbar}{i}\frac{d}{dx}\langle x|p\rangle\right)\langle p|\psi\rangle dp$$

$$= \frac{\hbar}{i}\frac{d}{dx}\int \langle x|p\rangle\langle p|\psi\rangle dp$$

$$= \frac{\hbar}{i}\frac{d}{dx}\langle x|\psi\rangle$$

$$\langle p|\hat{x}|\psi\rangle = i\hbar\frac{d}{dp}\tilde{\psi}(p)$$

A hermitian operator

Ψ state

$$\langle A\rangle = \langle\Psi|A|\Psi\rangle = \langle\Psi, A\Psi\rangle \in \mathbb{R}$$

$\Delta A(\Psi)$ = uncertainty of A in $|\Psi\rangle$

$$\Delta A(\Psi) \equiv |(A - \langle A\rangle \mathbb{1})\Psi|$$

$$(A - \langle A\rangle \mathbb{1})\Psi = 0$$

$$A\Psi = \langle A\rangle\Psi$$

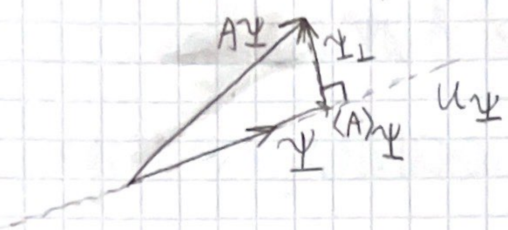
$$A\Psi = \lambda\Psi \Rightarrow \lambda = \langle A\rangle$$

9: Bra and ket

$$(\Delta A(\Psi))^2 = \langle (A-\langle A\rangle \mathbb{1})\Psi, (A-\langle A\rangle \mathbb{1})\Psi\rangle$$

$$= \langle A^2\rangle - \langle A\rangle\langle A\rangle - \langle A\rangle\langle A\rangle + \langle A\rangle^2$$

$$(\Delta A(\Psi))^2 = \langle A^2\rangle - \langle A\rangle^2$$



$$|\Psi_\perp| = \Delta A(\Psi)$$

$$P_{u_\Psi} = |\Psi\rangle\langle\Psi|$$

$$PA|\Psi\rangle = |\Psi\rangle\langle\Psi|A|\Psi\rangle$$

$A|\Psi\rangle - |\Psi\rangle\langle A\rangle$ = orthogonal to $\Psi = \Psi_\perp$

$$\Delta A(\Psi)\Delta B(\Psi) \geq (\ )$$