21: Gradient Fields

Given $\vec{F} = \nabla f$

$$\Rightarrow \int_C \vec{F} \cdot d\vec{r} = f(P_1) - f(P_0)$$

path independent

If $\vec{F} = \nabla f \Rightarrow M = f_x,\ N = f_y \Rightarrow$ then $f_{xy} = f_{yx} \Rightarrow M_y = N_x$

$$\vec{F} = \underbrace{-y}_{M}\hat{\imath} + \underbrace{x}_{N}\hat{\jmath}$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow \text{Not a gradient field}$$

$$\vec{F} = (4x^2 + axy)\hat{\imath} + (3y^2 + 4x^2)\hat{\jmath}$$

$$\left.\begin{array}{l} M_y = ax \\ N_x = 8x \end{array}\right\} a = 8$$

$$\int_C \vec{F} \cdot d\vec{r} = f(x_1, y_1) - f(0,0)$$

$$f(x_1, y_1) = \int_C \vec{F} \cdot d\vec{r} + f(0,0)$$

$$\vec{F} = \langle 4x^2 + 8xy,\ 3y^2 + 4x^2 \rangle$$

$$\int_C (4x^2 + 8xy)\,dx + (3y^2 + 4x^2)\,dy$$

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^{x_1} 4x^2\,dx = \frac{4}{3}x_1^3, \quad \int_{C_2} \vec{F} \cdot d\vec{r} = \int_0^{y_1} 3y^2 + 4x_1^2\,dy = y_1^3 + 4x_1^2 y_1$$

$$f(x,y) = \frac{4}{3}x^3 + y^3 + 4x^2 y + C$$

21: Gradient Fields

Solve $\begin{cases} f_x = 4x^2+8xy \\ f_y = 3y^2+4x^2 \end{cases}$

$f = \frac{4}{3}x^3+4x^2y+y^3+c$

$\vec{F}=\langle M, N\rangle$ is ∇ Field in R

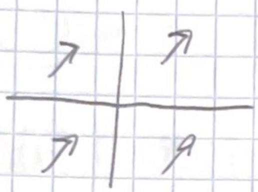
$\oint_C \vec{F}\cdot d\vec{r} = 0$

$\text{curl}(\vec{F}) = N_x - M_y$

$\text{curl}(\vec{F}) = 0 \Rightarrow$ Test for conservativeness

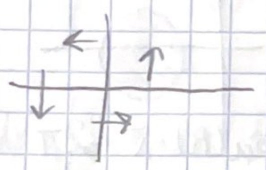
velosity field

curl measures rotation component of motion



$\vec{F}=\langle a, b\rangle, \Rightarrow \text{curl}(\vec{F})=0$

$\text{curl}(\vec{F}) = \frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x) = 0$



$\vec{F}=\langle -y, x\rangle$

$\text{curl}(\vec{F}) = \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y) = 2$

Curl measures (2x) angular velosity of rotation component of a velosity field

22: Green's Theorem

$\oint_C \vec{F} \cdot d\vec{r}$

Green's Theorem

If C is a closed curve enclosing a region R, counterclockwise,
$\vec{F}$ vector field defined and differentiable in R,
then:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \text{curl}(\vec{F})\, dA$$

$$\oint_C M dx + N dy = \iint_R (N_x - M_y)\, dA$$

Only for closed curves

Let C = circle of r=1 at (2,0)

$$\oint_C y e^{-x} dx + \left(\tfrac{1}{2}x^2 - e^{-x}\right) dy$$

$$\iint_R \text{curl}(\vec{F})\, dA = \iint_R (x + e^{-x}) - e^{-x}\, dA = \iint_R x\, dA = \text{Area}(R) \cdot \bar{x} = 2\pi$$

$$\bar{x} = \frac{1}{\text{Area}} \iint_R x\, dA$$

22: Green's Theorem

If $\text{curl}(\vec{F})=0$

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R \text{curl}(\vec{F})\,dA = 0 \text{ if } \text{curl}(\vec{F})=0$$

This proves if $\text{curl}(\vec{F})=0$ everywhere in R, then $\oint_C \vec{F}\cdot d\vec{r}=0$

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R \text{curl}(\vec{F})\,dA = \iint_R 0\,dA = 0$$

$$\oint_C M\,dx + N\,dy = \iint_R (N_x - M_y)\,dA$$

$$\oint_C M\,dx = \iint_R -M_y\,dA \quad (N=0) \Longleftrightarrow \oint_C N\,dy = \iint_R N_x\,dA \quad (M=0)$$

$$\oint_{C_1} M\,dx = \iint_{R_1} -M_y\,dA \Longleftrightarrow \oint_{C_2} M\,dx = \iint_{R_2} -M_y\,dA$$

$$\oint_C M\,dx = \oint_{C_1} + \oint_{C_2} = \iint_{R_1} + \iint_{R_2} = \iint_R -M_y\,dA$$

$\oint_C M\,dx = \iint_R -M_y\,dA$ if R is vertically simple and C is boundary of R counter clockwise

$$\int_{C_1} M\,dx = \int_a^b M(x, f(x))\,dx$$

$y = f_1(x)$

x from a to b

22: Green's Theorem

$$\int_{C_2} M dx = 0 \Rightarrow \int_{C_4} M dx = 0$$

$x = b \Rightarrow dx = 0$

$$\int_{C_3} M dx = \int_b^a M(x, f_2(x))\, dx$$

$y = f_2(x)$

x from b to a

$$\oint_C M dx = \int_a^b M(x, f_1(x))\, dx - \int_a^b M(x, f_2(x))\, dx$$

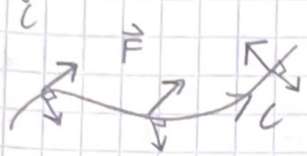
$$\iint_R -M_y\, dA = -\int_a^b \int_{f_1(x)}^{f_2(x)} \frac{\partial M}{\partial y}\, dy\, dx = M(x, f_2(x)) - M(x, f_1(x)) \text{ QED}$$

23: Flux

Another line integral, C plane curve, $\vec{F}$ vector field

Flux of $\vec{F}$ across C:

$$\int_C \vec{F}\cdot\hat{n}\,ds$$



$$\text{Flux} = \lim_{s\to 0}\left(\Sigma\, \vec{F}\cdot\hat{n}\,\Delta s\right)$$

Work:

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot\hat{T}\,ds$$

C = circle of $r=a$ at origin $\vec{F} = x\hat{i} + y\hat{j}$

Along C $\vec{F} \parallel \hat{n} \Rightarrow \vec{F}\cdot\vec{n} = |\vec{F}||\hat{n}| = a$

$$\int_C \vec{F}\cdot\hat{n}\,ds = \int_C a\,ds = a\int_C ds = 2\pi a^2$$

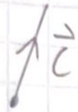
$\vec{F}\cdot\hat{n} = 0 \Rightarrow \text{Flux} = 0$

23: Flux

$d\vec{r} = \hat{T}\,ds = \langle dx, dy\rangle$

$\hat{n} = \hat{T}$ rotated $90°$ clockwise

$\hat{n}\,ds = \langle dy, -dx\rangle$



$\hat{T}\Delta s = \Delta\vec{r} = \langle \Delta x, \Delta y\rangle$

$\hat{n}\Delta s = \langle \Delta y, -\Delta x\rangle$

If $\vec{F} = \langle P, Q\rangle$ then

$$\int_C \vec{F}\cdot\hat{n}\,ds = \int_C \langle P, Q\rangle \cdot \langle dy, -dx\rangle = \int_C -Q\,dx + P\,dy$$

If C encloses R counter clockwise and $\vec{F}$ defined in R

$$\oint_C \vec{F}\cdot\hat{n}\,ds = \iint_R \operatorname{div}(\vec{F})\,dA$$

$\operatorname{div}\langle P, Q\rangle = P_x + Q_y$

Circle $r=a$ $\vec{F} = x\hat{\imath} + y\hat{\jmath}$

$\operatorname{div}(\vec{F}) = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) = 2$

$$\oint_C F\cdot\hat{n}\,ds = \iint_R 2\,dA = 2\,\text{area}(R) = 2\pi a^2$$

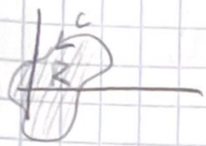
24: Connected Regions

$$\oint_C \vec{F}\cdot\hat{T}\,ds = \iint_R \text{curl}(\vec{F})\,dA$$

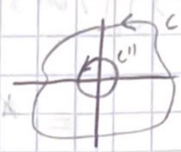
$$\oint_C \vec{F}\cdot\hat{n}\,ds = \iint_R \text{div}(\vec{F})\,dA$$

$$\vec{F} = \frac{-y\hat{\imath}+x\hat{\jmath}}{x^2+y^2}$$

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R \text{curl}(\vec{F})\,dA$$



$$\oint_C \vec{F}\cdot d\vec{r}\,?$$



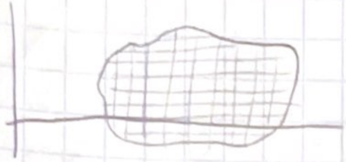
$$\oint_{C'} \vec{F}\cdot d\vec{r} - \oint_{C''} \vec{F}\cdot d\vec{r} = \iint_R \text{curl}(\vec{F})\,dA = 0$$

A connected region in the plain is simply connected if the interior of any closed curve in R is also in R

If curl$(\vec{F}) = 0$ and domain where $\vec{F}$ is defined is simply connected then $\vec{F}$ is conservative and a gradient field.

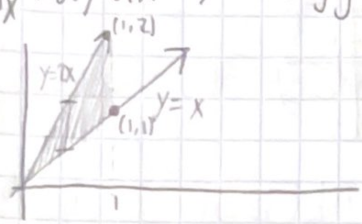
24: Connected Regions

$\iint_R f\,dA \;/\; \int_C \vec{F}\cdot\hat{T}ds$
$\hat{n}ds$



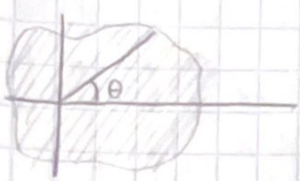
Set up $\iint_R$: draw picture of R
take slices
$\rightarrow$ iterated $\int$

$$\int_0^1\int_x^{2x} f\,dy\,dx \Rightarrow \ldots \Rightarrow \iint f\,dx\,dy$$



$\Rightarrow x=\frac{1}{2}y$

$$\int_0^1\int_{\frac{y}{2}}^{y} f\,dx\,dy + \int_1^2\int_{\frac{y}{2}}^{1} f\,dx\,dy$$



$\text{area}(R)=\iint_R 1\,dA$

24: Connected Regions

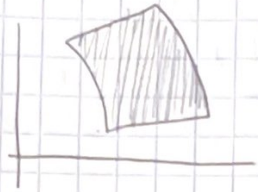
Change of Variables

$u = u(x,y)$
$v = v(x,y)$

- Jacobian

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \quad du\,dv = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| dx\,dy$$

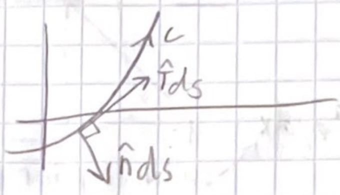
substitute x, y integrand to u, v



Line Integrals

$\vec{F} = \langle M, N \rangle$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C M\,dx + N\,dy$$



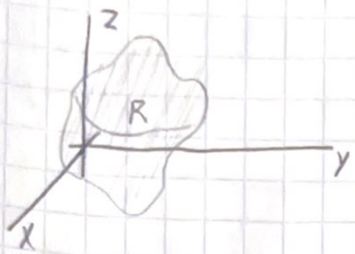
$x = x(t)$
$y = y(t)$

If $\operatorname{curl}(\vec{F}) = N_x - M_y = 0$ and domain is simply connected then $\vec{F}$ is ∇ field

$$\begin{cases} f_x = M \\ f_y = N \end{cases} \iff \vec{F} = \nabla f$$

$$\int_C \nabla f \cdot d\vec{r} = f(P_1) - f(P_0)$$

25: Triple Integrals

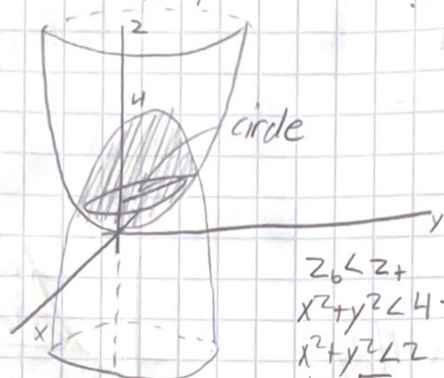


$$\iiint_R f\,\underbrace{dV}$$

volume element $\overset{rec}{\Rightarrow} dx\,dy\,dz$

Region between paraboloids $z = x^2+y^2$ and $z = 4-x^2-y^2$

$$\iiint 1\,dV$$



$$\int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{x^2+y^2}^{4-x^2-y^2} dz\,dy\,dx$$

$z_b < z_t$

$x^2+y^2 < 4-x^2-y^2$

$x^2+y^2 < 2$

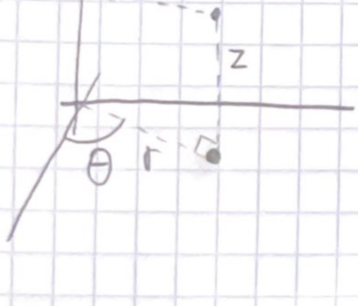
$r = \sqrt{2}$

$$\int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} 4-2x^2-2y^2\,dy\,dx$$

switch to polar (cylindrical)

$$\int_0^{2\pi} \int_0^{\sqrt{2}} \int_{r^2}^{4-r^2} dz \cdot r\,dr\,d\theta = \int_0^{2\pi} \int_0^{\sqrt{2}} 4r\,dr\,d\theta \Rightarrow \ldots$$

Cylindrical coordinates



(r, θ, z)

25: Triple Integrals

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

Volume element

$$dx\,dy\,dz \to r\,dr\,d\theta\,dz$$

$$\text{Mass} \Rightarrow dm = \delta \cdot dV$$

$$\text{Mass} = \iiint_R \delta\, dV$$

Average $f(x,y,z)$

$$\bar{f} = \frac{1}{\text{Vol}(R)} \iiint_R f\, dV$$

with density $\frac{1}{\text{Mass}(R)} \iiint_R f\,\delta\, dV$

COM $(\bar{x}, \bar{y}, \bar{z})$

$$\bar{x} = \frac{1}{\text{Mass}} \iiint x\,\delta\, dV$$

$$\iiint_R (\text{distance to axis})^2\, dV$$

$$I_z = \iiint_R r^2 \delta\, dV = \iiint_R x^2 + y^2 \delta\, dV$$

$$I_x = \iiint_R y^2 + z^2 \delta\, dV$$

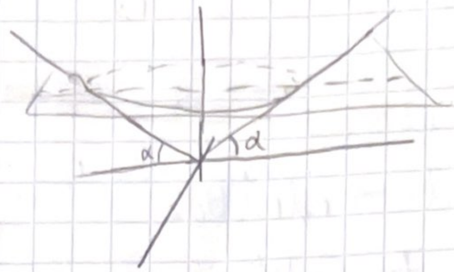
$$I_y = \iiint_R x^2 + z^2 \delta\, dV$$

25: Triple Integrals

I_z of solid cone betwee $z=ar$, $z=b$

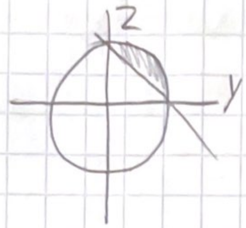
$$I_z=\int_0^b\int_0^{2\pi}\int_0^{z/a} r^2\, r\,dr\,d\theta\,dz$$

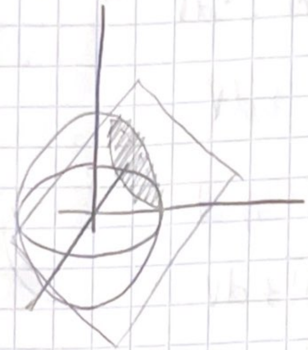
$$=\frac{\pi b^5}{10a^4}$$



$z \geq 1-y$ In unit bowl

$$\int_0^1\int_{-\sqrt{2y-y^2}}^{\sqrt{2y-y^2}}\int_{1-y}^{\sqrt{1-x^2-y^2}} dz\,dx\,dy$$





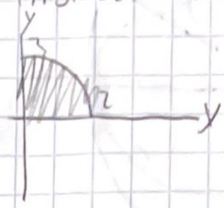
$$\int_0^2\int_0^{\frac{\pi}{2}}\int_0^3 xy^2\cos z\,dy\,dx\,dz=\int_0^2\int_0^{\frac{\pi}{2}} 9x\cos z\,dz\,dx=\int_0^2 9x\,dx=18$$

$$\int_0^4\int_0^1\int_0^x 2\sqrt{y}\,e^{-x^2}\,dz\,dx\,dy=\int_0^4\int_0^1 x2\sqrt{y}\;e^{-x^2}\,dx\,dy \rightarrow \begin{cases} u_x=-x^2 \\ du_x=-2x\,dx \end{cases} \Bigg| \begin{matrix} u=-1 \\ u=0 \end{matrix}$$

$$=\int_0^4\int_{-1}^0 \sqrt{y}\,e^u\,du\,dy=\int_0^4 \sqrt{y}\,(1-e^{-1})\,dy=\frac{16}{3}(1-e^{-1})$$

25: Triple Integrals

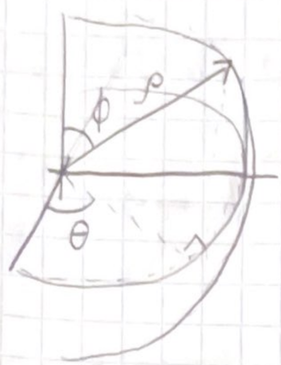
$\iiint_T y\,dV$ T: {solid bound by $z=4-x^2-y^2$ in first octant}



$$\int_0^{\frac{\pi}{2}}\int_0^{2}\int_0^{4-r^2} y\cdot r\,dz\,dr\,d\theta = \int_0^{\frac{\pi}{2}}\int_0^{2} r^2\sin\theta(4-r^2)\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{32}{3}\sin\theta - \frac{32}{5}\sin\theta\,d\theta = \int_0^{\frac{\pi}{2}} \frac{64}{15}\,d\theta = \frac{64}{15}$$

26: Spherical Coordinates



$z = \rho\cos\phi$

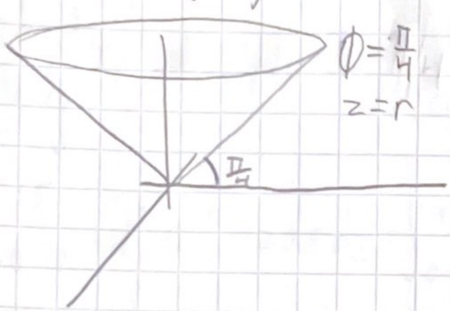
$r = \rho\sin\phi$

$x = r\cos\theta = \rho\sin\phi\cos\theta$

$y = r\sin\theta = \rho\sin\phi\sin\theta$

$z = \rho\cos\phi$

$\rho = \sqrt{r^2+z^2} = \sqrt{x^2+y^2+z^2}$



$\phi = \frac{\pi}{4}$

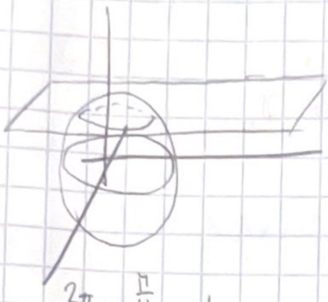
$z = r$

26: Spherical Coordinates

$\iiint_V$ in SC

$dV = ?$

$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$



$$\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_{\frac{1}{\sqrt{2}}\sec\phi}^{1} 1 \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$z = \frac{1}{\sqrt{2}}$

$\rho\cos\phi = \frac{1}{\sqrt{2}}$

$\rho = \frac{1}{\sqrt{2}\cos\phi}$

$\phi = \frac{\pi}{4}$

$f(x,y,z) = \sqrt{x^2+y^2}, \quad y_1 = x^2+z^2, \quad y_2 = 8-x^2-z^2$

$$\int_0^{2\pi} \int_0^{2} \int_{r^2}^{8-r^2} r \, dy \, r\,dr\,d\theta = \int_0^{2\pi} \int_0^{2} \left[ry\right]_{r^2}^{8-r^2} r\,dr\,d\theta = \int_0^{2\pi}\int_0^{2} 8r^2 - 2r^4 \, r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[\frac{8}{3}r^3 - \frac{2}{3}r^5\right]_0^2 d\theta = \int_0^{2\pi} \frac{64}{3} - \frac{64}{5} \, d\theta = \frac{256\pi}{15}$$

26: Spherical Coordinates

$$\iiint_T \sqrt{x^2+y^2+z^2}\,dV \quad T: \{\text{bounded } x^2+y^2+z^2 \leq 1\}$$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{1} \rho \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi}\int_0^{\pi} \frac{1}{4}\left[\rho^4\right]_0^1 \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{1}{4}\sin\phi \, d\phi \, d\theta = \int_0^{2\pi} -\frac{1}{4}\left[\cos\phi\right]_0^{\pi} d\theta = \int_0^{2\pi} \frac{1}{2}\, d\theta = \pi$$