Uncertainty and Probability Theory

(Uncertainty, Basic Probability Notation, Independence of Events, Bayes' theorem, Problems in Probability)

Uncertainty: Example 1 (Automated taxi)

- Suppose that an automated taxi has the goal of delivering a passenger to the airport on time.
- The agent forms a plan, A90, that involves leaving home 90 minutes before the flight departs and driving at a reasonable speed.
- Even though the airport is only about 5 miles away, a logical taxi agent will not be able to conclude with certainty that "Plan A90 will get us to the airport in time." Instead, it reaches the weaker conclusion *"Plan A90 will get us to the airport in time, as long as the car doesn't break down or run out of gas, and I don't get into an accident, and there are no accidents on the bridge, and the plane doesn't leave early, and no meteorite hits the car, and . . . ."*
- Since none of the above conditions can be deduced for sure, the plan's success cannot be inferred.

Uncertainty: Example 2 (Medical diagnosis)

- Consider a case of diagnosing a patient's toothache.
- A simple rule could be:
  - toothache $\rightarrow$ cavity
- But there can be many reasons for toothache, so the rule could be:
  - toothache $\rightarrow$ cavity $\vee$ gum-problem $\vee$ abscess … *(Note: the number of possibilities could be unlimited!)*
- Not possible to list out an exhaustive list of possibilities in the rule above
- Trying to use logic to cope with a domain like medical diagnosis thus fails for three main reasons:
  - Laziness: It is too much work to list the complete set of antecedents or consequents needed to ensure an exception-less rule and too hard to use such rules
  - Theoretical ignorance: Medical science has no complete theory for the domain
  - Practical ignorance: Even if we know all the rules, we might be uncertain about a particular patient because not all the necessary tests have been or can be run.

Uncertainty

- What to do when there is uncertainty?
  - An agent's knowledge can at best provide only a **degree of belief** in the relevant sentences
  - Such degree of belief can be expressed using probability theory

What is a random variable?

- In probability and statistics, a random variable is described informally as a variable whose values depend on outcomes of a random phenomenon.
- As a function, a random variable is required to be measurable, which allows for probabilities to be assigned to sets of its potential values.
- Example: When tossing a fair coin, the final outcome of heads or tails depends on the uncertain physical conditions, so the outcome being observed is uncertain. Outcome= {Head, Tail}.

Discrete Random Variables

Let x be a discrete random variable that can assume any of the finite number m of different values in the set $\mathcal{X} = \{v_1, v_2, \ldots, v_m\}$. We denote by p_i the probability that x assumes the value v_i:

$$p_i = \Pr\{x = v_i\}, \quad i = 1, \ldots, m.$$

Then the probabilities p_i must satisfy the following two conditions:

$$\begin{aligned} p_i &\geq 0 \quad \text{and} \\ \sum_{i=1}^{m} p_i &= 1. \end{aligned}$$

Joint Probability

- Let x and y be two random variables which can take on values from $X=\{v_1, v_2.. v_m\}$ and $Y=\{w_1, w_2.. w_n\}$. For each possible pair of values (v_i, w_j) we have a joint probability $p_{ij}=P\{x=v_i, y=w_j\}$. These mn joint probabilities are non-negative and sum to 1. This is indicated by the equations below:

$$\begin{aligned} P(x,y) &\geq 0 \quad \text{and} \\ \sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} P(x,y) &= 1. \end{aligned}$$

Independence of Events

- Two events are independent or statistically independent if the occurrence of one does not affect the probability of occurrence of the other.
- Mathematically put, variables x and y are said to be statistically independent if and only if:

$$P(x, y) = P(x)P(y)$$

where $P(x, y)$ is the joint probability of x and y and $P(x)$and $P(y)$ are the independent (marginal) probabilities of x and y.

Independent and Dependent Events: Some Examples

- Example 1: Drawing cards
- **Independent Events:**
- If two cards are drawn with replacement from a deck of cards, the event of drawing a red card on the first trial and that of drawing a red card on the second trial are independent.
- **Dependent Events:**
- By contrast, if two cards are drawn without replacement from a deck of cards, the event of drawing a red card on the first trial and that of drawing a red card on the second trial are not independent, because a deck that has had a red card removed has proportionately fewer red cards than before.

Independent and Dependent Events: Some Examples

- **Example 2: Independent Event: Tossing a coin**
  - Each toss of a coin is a perfect isolated thing.
  - What it did in the past will not affect the current toss.
  - The chance is simply 1-in-2, or 50%, just like ANY toss of the coin.
  - So each toss is an Independent Event.
- **Example 3: Dependent Event: Drawing marbles from a bag without replacement**
  - Suppose there are 2 blue and 3 red marbles in a bag.
  - If drawn, what are the chances of getting a blue marble the first time? (It is 2/5).
  - What are the chances of getting a blue marble the second time? (It is 1/4).. and so on

Conditional Probability

When two variables are statistically dependent, knowing the value of one of them lets us get a better estimate of the value of the other one. For any propositions a and b, we can express the conditional probability of *a* given *b* as:

$$P(a \mid b) = \frac{P(a \wedge b)}{P(b)}$$

Alternatively, we can write this as:

$$P(a \mid b) = \frac{P(a, b)}{P(b)}$$

Product Rule: $P(a \wedge b) = P(a \mid b) P(b)$

Simply put,

- Joint probability is the probability of two events occurring simultaneously
- Marginal probability is the probability of an event irrespective of the outcome of another variable
- Conditional probability is the probability of one event occurring in the presence of a second event (i.e. given the second event has already occurred)

Bayes' Rule

- The Bayes' Rule can be stated as:
  - $P(x|y) = \frac{P(y|x)P(x)}{P(y)}$

- This can be interpreted as:
  - $posterior = \frac{likelihood \times prior}{evidence}$

Deriving Bayes' Rule

Let us revisit the equation for conditional probability:

$$P(x|y) = \frac{P(x,y)}{P(y)} \qquad (1)$$

Similarly, we can say that $P(y|x) = \frac{P(x,y)}{P(x)}$ (2)

Thus, $P(x,y) = P(y|x)P(x)$ (3)

Substituting eq(3) into eq(1), we get:

$$P(x|y) = \frac{P(y|x)P(x)}{P(y)} \qquad (4)$$

[Bayes' Rule]

Note that $P(y)$ is obtained by summing the numerator over all possible values of x . Hence, we also have:

$$P(x|y) = \frac{P(y|x)P(x)}{\sum_{x \in \mathcal{X}} P(y|x)P(x)} \qquad (5)$$

Example: Application of conditional probability (Bayes' rule)

- Box P has 2 red balls and 3 blue balls, and Box Q has 3 red balls and 1 blue ball. A ball is selected as follows: i) Select a box ii)Choose a ball from the selected box such that each ball in the box is equally likely to be chosen. The probability of selecting boxes P and Q are 1/3 and 2/3 respectively. Given that a ball selected in the above process is a red ball, the probability that it came from the box P is?
  - A) 11/19
  - B) 20/43
  - C) 4/19
  - D) 14/45

Solution

It is given that:

- Probability of selecting box P is P(boxP) = 1/3 (1)
- Probability of selecting box Q is P(boxQ)= 2/3 (2)
- We are required to find: P(boxP/Red)=?
- Applying Bayes' Rule:

$$P(boxP|Red) = \frac{P(Red|boxP)\,P(boxP)}{P(Red)}$$

Solution (cont..)

$$P(boxP|Red) = \frac{P(Red|boxP)\ P(boxP)}{P(Red)} \qquad (3)$$

Now, box P has 2 red balls out of a total 5. So, $P(Red|boxP) = 2/5$ (4)

Also, the denominator P(Red) can be computed as: $P(Red) = P(Red|boxP)P(boxP) + P(Red|boxQ)P(boxQ)$ (5)

$$= \left(\frac{2}{5} \times \frac{1}{3}\right) + \left(\frac{3}{4} \times \frac{2}{3}\right) = \frac{38}{60}$$

Substituting values from (1), (4) and (5) into equation (3), we get: $P(boxP|Red) = \frac{4}{19}$ (6)

Thus, the answer to Example (5) is Option C *i.e.* 4/19