When Time Varies

Some of the most important problems of the calculus are those where time is the independent variable, and we have to think about the values of some other quantity that varies when the time varies. Some things grow larger as time goes on; some other things grow smaller. The distance that a train has got from its starting place goes on ever increasing as time goes on. Trees grow taller as the years go by. Which is growing at the greater rate; a plant 12 inches high which in one month becomes 14 inches high, or a tree 12 feet high which in a year becomes 14 feet high?

In this chapter we are going to make much use of the word rate. Nothing to do with poor-rate, or water-rate (except that even here the word suggests a proportion-a ratio-so many pence in the pound). Nothing to do even with birth-rate or death-rate, though these words suggest so many births or deaths per thousand of the population. When a motor-car whizzes by us, we say: What a terrific rate! When a spendthrift is flinging about his money, we remark that that young man is living at a prodigious rate.

What do we mean by rate? In both these cases we are making a mental comparison of something that is happening, and the length of time that it takes to happen. If the motor-car flies past us going 10 yards per second, a simple bit of mental arithmetic will show us that this is equivalent-while it lasts-to a rate of 600 yards per minute, or over 20 miles per hour.

Now in what sense is it true that a speed of 10 yards per second is the same as 600 yards per minute? Ten yards is not the same as 600 yards, nor is one second the same thing as one minute. What we mean by saying that the rate is the same, is this: that the proportion borne between distance passed over and time taken to pass over it, is the same in both cases.

Take another example. A man may have only a few pounds in his possession, and yet be able to spend money at the rate of millions a year-provided he goes on spending money at that rate for a few minutes only. Suppose you hand a shilling over the counter to pay for some goods; and suppose the operation lasts exactly one second. Then, during that brief operation, you are parting with your money at the rate of 1 shilling per second, which is the same rate as $£3$ per minute, or $£180$ per hour, or $£4320$ per day, or $£1,576,800$ per year! If you have $£10$ in your pocket, you can go on spending money at the rate of a million a year for just $5\frac{1}{4}$ minutes.

It is said that Sandy had not been in London

Learning outcomes:
Author(s):

above five minutes when "bang went saxpence." If he were to spend money at that rate all day long, say for 12 hours, he would be spending 6 shillings

an hour, or £3. 12s . per day, or £21. 12s . a week, not counting the Sabbath.

Now try to put some of these ideas into differential notation.

Let y in this case stand for money, and let t stand for time.

If you are spending money, and the amount you spend in a short time dt be called dy, the rate of spending it will be $\frac{dy}{dt}$, or rather, should be written with a minus sign, as $-\frac{dy}{dt}$, because dy is a decrement, not an increment. But money is not a good example for the calculus, because it generally comes and goes by jumps, not by a continuous flow-you may earn £200 a year, but it does not keep running in all day long in a thin stream; it comes in only weekly, or monthly, or quarterly, in lumps: and your expenditure also goes out in sudden payments.

A more apt illustration of the idea of a rate is furnished by the speed of a moving body. From London (Euston station) to Liverpool is 200 miles. If a train leaves London at 7 o'clock, and reaches Liverpool at 11 o'clock, you know that, since it has travelled 200 miles in 4 hours, its average rate must have been 50 miles per hour; because $\frac{200}{4} = \frac{50}{1}$. Here you are really making a mental comparison between

the distance passed over and the time taken to pass over it. You are dividing one by the other. If y is the whole distance, and t the whole time, clearly the average rate is $\frac{y}{t}$. Now the speed was not actually constant all the way: at starting, and during the slowing up at the end of the journey, the speed was less. Probably at some part, when running downhill, the speed was over 60 miles an hour. If, during any particular element of time dt, the corresponding element of distance passed over was dy, then at that part of the journey the speed was $\frac{dy}{dt}$. The rate at which one quantity (in the present instance, distance) is changing in relation to the other quantity (in this case, time) is properly expressed, then, by stating the differential coefficient of one with respect to the other. A velocity, scientifically expressed, is the rate at which a very small distance in any given direction is being passed over; and may therefore be written

$$v = \frac{dy}{dt}.$$

But if the velocity v is not uniform, then it must be either increasing or else decreasing. The rate at which a velocity is increasing is called the acceleration. If a moving body is, at any particular instant, gaining an additional velocity dv in an element of time dt, then the acceleration a at that instant may be written

$$a = \frac{dv}{dt};$$

but dv is itself $d\left(\frac{dy}{dt}\right)$. Hence we may put

$$a = \frac{d\left(\frac{dy}{dt}\right)}{dt};$$

and this is usually written $a = \frac{d^2y}{dt^2}$; or the acceleration is the second differential coefficient of the distance, with respect to time. Acceleration is expressed as a change of velocity in unit time, for instance, as being so many feet per second per second; the notation used being feet ÷ second^2.

When a railway train has just begun to move, its velocity v is small; but it is rapidly gaining speed-it is being hurried up, or accelerated, by the effort of the engine. So its $\frac{d^2y}{dt^2}$ is large. When it has got up its top speed it is no longer being accelerated, so that then $\frac{d^2y}{dt^2}$ has fallen to zero. But when it nears its stopping place its speed begins to slow down; may, indeed, slow down very quickly if the brakes are put on, and during this period of deceleration or slackening of pace, the value of $\frac{dv}{dt}$, that is, of $\frac{d^2y}{dt^2}$ will be negative.

To accelerate a mass m requires the continuous application of force. The force necessary to accelerate a mass is proportional to the mass, and it is also proportional to the acceleration which is being imparted. Hence we may write for the force f, the expression

$$f = ma \tag{1}$$

$$\text{or} \tag{2}$$

$$f = m\frac{dv}{dt}\text{or} \tag{3}$$

$$f = m\frac{d^2y}{dt^2} \tag{4}$$

The product of a mass by the speed at which it is going is called its momentum, and is in symbols mv. If we differentiate momentum with respect to time we shall get $\frac{d(mv)}{dt}$ for the rate of change of momentum. But, since m is a constant quantity, this may be written $m\frac{dv}{dt}$, which we see above is the same as f. That is to say, force may be expressed either as mass times acceleration, or as rate of change of momentum.

Again, if a force is employed to move something (against an equal and opposite counter-force), it does work; and the amount of work done is measured by the product of the force into the distance (in its own direction) through which its point of application moves forward. So if a force f moves forward through a length y, the work done (which we may call w) will be

$w = f * y$

where we take f as a constant force. If the force varies at different parts of the range y, then we must find an expression for its value from point to point. If f be the force along the small element of length dy, the amount of work done will be $f * dy$. But as dy is only an element of length, only an element of work will be done. If we write w for work, then an element of work will be dw; and we have

$dw = f * dy$

which may be written

$dw = ma * dy \text{ or} dw = m\frac{d^2y}{dt^2} * dy \text{or} dw = m\frac{dv}{dt} * dy$

Further, we may transpose the expression and write

$\frac{dw}{dy} = f.$

This gives us yet a third definition of force; that if it is being used to produce a displacement in any direction, the force (in that direction) is equal to the rate at which work is being done per unit of length in that direction. In this last sentence the word rate is clearly not used in its time-sense, but in its meaning as ratio or proportion.

Sir Isaac Newton, who was (along with Leibniz) an inventor of the methods of the calculus, regarded all quantities that were varying as flowing; and the ratio which we nowadays call the differential coefficient he regarded as the rate of flowing, or the fluxion of the quantity in question. He did not use the notation of the dy and dx, and dt (this was due to Leibnitz), but had instead a notation of his own. If y was a quantity that varied, or "flowed," then his symbol for its rate of variation (or "fluxion") was

$\dot{y}$. If x was the variable, then its fluxion was called $\dot{x}$. The dot over the letter indicated that it had been differentiated. But this notation does not tell us what is the independent variable with respect to which the differentiation has been effected. When we see $\frac{dy}{dt}$ we know that y is to be differentiated with respect to t. If we see $\frac{dy}{dx}$ we know that y is to be differentiated with respect to x. But if we see merely $\dot{y}$, we cannot tell without looking at the context whether this is to mean $\frac{dy}{dx}$ or $\frac{dy}{dt}$ or $\frac{dy}{dz}$, or what is the other variable. So, therefore, this fluxional notation is less informing than the differential notation, and has in consequence largely dropped out of use. But its simplicity gives it an advantage if only we will agree to use it for those cases exclusively where time is the independent variable. In that case $\dot{y}$ will mean $\frac{dy}{dt}$ and $\dot{u}$ will mean $\frac{du}{dt}$; and $\ddot{x}$ will mean $\frac{d^2x}{dt^2}$.

Adopting this fluxional notation we may write the mechanical equations considered in the paragraphs above, as follows:

distance	x
velocity	$v = \dot{x}$
acceleration	$a = \dot{v} = \ddot{x}$
force	$f = m\dot{v} = m\ddot{x}$
work	$w = x * m\ddot{x}$

Examples (1) A body moves so that the distance x (in feet), which it travels from a certain point O, is given by the relation $x = 0.2t^2 + 10.4$, where t is the time in seconds elapsed since a certain instant. Find the velocity and acceleration 5 seconds after the body began to move, and also find the corresponding values when the distance covered is 100 feet. Find also the average velocity during the first 10 seconds of its motion. (Suppose distances and motion to the right to be positive.)

Now

$$x = 0.2t^2 + 10.4v = \dot{x} = \frac{dx}{dt} = 0.4t; \quad \text{and} \quad a = \ddot{x} = \frac{d^2x}{dt^2} = 0.4 = \text{constant}.$$

When $t = 0$, $x = 10.4$ and $v = 0$. The body started from a point 10.4 feet to the right of the point O; and the time was reckoned from the instant the body started.

When $t = 5$, $v = 0.4 * 5 = 2\text{ft./sec.}$; $a = 0.4\text{ft./sec}^2$.

When $x = 100$, $100 = 0.2t^2 + 10.4$, or $t^2 = 448$, and $t = 21.17\text{sec.}$; $v = 0.4 * 21.17 = 8.468\text{ft./sec.}$

When $t = 10$,

$$\text{distance travelled} = 0.2 * 10^2 + 10.4 - 10.4 = 20\text{ft.}$$
$$\text{Average velocity} = \tfrac{20}{10} = 2\text{ft./sec.}$$

(It is the same velocity as the velocity at the middle of the interval, $t = 5$; for, the acceleration being constant, the velocity has varied uniformly from zero when $t = 0$ to 4ft./sec. when $t = 10$.)

(2) In the above problem let us suppose

$$x = 0.2t^2 + 3t + 10.4.$$
$$v = \dot{x} = \frac{dx}{dt} = 0.4t + 3; \quad a = \ddot{x} = \frac{d^2x}{dt^2} = 0.4 = \text{constant}.$$

When $t = 0$, $x = 10.4$ and $v = 3$ ft./sec, the time is reckoned from the instant at which the body passed a point 10.4 ft. from the point O, its velocity being then already 3 ft./sec. To find the time elapsed since it began moving, let $v = 0$; then $0.4t + 3 = 0$, $t = -\dfrac{3}{.4} = -7.5$ sec. The body began moving 7.5 sec.

before time was begun to be observed; 5 seconds after this gives $t = -2.5$ and $v = 0.4 * -2.5 + 3 = 2$ ft./sec.

When $x = 100$ ft.,

$$100 = 0.2t^2 + 3t + 10.4; \quad \text{or } t^2 + 15t - 448 = 0;$$

hence $t = 14.95$ sec., $v = 0.4 * 14.95 + 3 = 8.98$ ft./sec.

To find the distance travelled during the 10 first seconds of the motion one must know how far the body was from the point O when it started.

When $t = -7.5$,

$$x = 0.2 * (-7.5)^2 - 3 * 7.5 + 10.4 = -0.85\text{ft.},$$

that is 0.85 ft. to the left of the point O.

Now, when $t = 2.5$,

$$x = 0.2 * 2.5^2 + 3 * 2.5 + 10.4 = 19.15.$$

So, in 10 seconds, the distance travelled was $19.15 + 0.85 = 20$ ft., and

$$\text{the average velocity } = \tfrac{20}{10} = 2 \text{ ft./sec.}$$

(3) Consider a similar problem when the distance is given by $x = 0.2t^2 - 3t + 10.4$. Then $v = 0.4t - 3$, $a = 0.4 =$ constant. When $t = 0$, $x = 10.4$ as before, and

$v = -3$; so that the body was moving in the direction opposite to its motion in the previous cases. As the acceleration is positive, however, we see that this velocity will decrease as time goes on, until it becomes zero, when $v = 0$ or $0.4t - 3 = 0$; or $t = 7.5$ sec. After this, the velocity becomes positive; and 5 seconds after the body started, $t = 12.5$, and

$$v = 0.4 * 12.5 - 3 = 2 \text{ ft./sec.}$$

When $x = 100$,

$$100 = 0.2t^2 - 3t + 10.4, \quad \text{or } t^2 - 15t - 448 = 0, \text{and } t = 29.95;\ v = 0.4 * 29.95 - 3 = 8.98\text{ft./sec.}$$

When v is zero, $x = 0.2 * 7.5^2 - 3 * 7.5 + 10.4 = -0.85$, informing us that the body moves back to 0.85 ft. beyond the point O before it stops. Ten seconds later

$$t = 17.5 \text{ and } x = 0.2 * 17.5^2 - 3 * 17.5 + 10.4 = 19.15.$$

The distance travelled $= .85 + 19.15 = 20.0$, and the average velocity is again 2 ft./sec.

(4) Consider yet another problem of the same sort with $x = 0.2t^3 - 3t^2 + 10.4$; $v = 0.6t^2 - 6t$; $a = 1.2t - 6$. The acceleration is no more constant.

When $t = 0$, $x = 10.4$, $v = 0$, $a = -6$. The body is at rest, but just ready to move with a negative acceleration, that is to gain a velocity towards the point O.

(5) If we have $x = 0.2t^3 - 3t + 10.4$, then $v = 0.6t^2 - 3$, and $a = 1.2t$.

When $t = 0$, $x = 10.4$; $v = -3$; $a = 0$.

The body is moving towards the point O with

a velocity of 3 ft./sec., and just at that instant the velocity is uniform.

We see that the conditions of the motion can always be at once ascertained from the time-distance equation and its first and second derived functions. In the last two cases the mean velocity during the first 10 seconds and the velocity 5 seconds after the start will no more be the same, because the velocity is not increasing uniformly, the acceleration being no longer constant.

(6) The angle θ (in radians) turned through by a wheel is given by $\theta = 3 + 2t - 0.1t^3$, where t is the time in seconds from a certain instant; find the angular velocity ω and the angular acceleration α, (a) after 1 second; (b) after it has performed one revolution. At what time is it at rest, and how many revolutions has it performed up to that instant?

Writing for the acceleration

$$\omega = \dot{\theta} = \frac{d\theta}{dt} = 2 - 0.3t^2, \quad \alpha = \ddot{\theta} = \frac{d^2\theta}{dt^2} = -0.6t.$$

When $t = 0$, $\theta = 3$; $\omega = 2$ rad./sec.; $\alpha = 0$.

When $t = 1$,

$$\omega = 2 - 0.3 = 1.7\text{rad./sec.}; \quad \alpha = -0.6\text{rad./sec}^2.$$

This is a retardation; the wheel is slowing down.

After 1 revolution

$$\theta = 2\pi = 6.28; \quad 6.28 = 3 + 2t - 0.1t^3.$$

By plotting the graph, $\theta = 3 + 2t - 0.1t^3$, we can get the value or values of t for which $\theta = 6.28$; these are 2.11 and 3.03 (there is a third negative value).

When $t = 2.11$,

$$\theta = 6.28; \quad \omega = 2 - 1.34 = 0.66 \text{ rad./sec.};$$
$$\alpha = -1.27 \text{ rad./sec}^2.$$

When $t = 3.03$,

$$\theta = 6.28; \quad \omega = 2 - 2.754 = -0.754 \text{ rad./sec.};$$
$$\alpha = -1.82 \text{ rad./sec}^2.$$

The velocity is reversed. The wheel is evidently at rest between these two instants; it is at rest when $\omega = 0$, that is when $0 = 2 - 0.3t^3$, or when $t = 2.58$sec., it has performed

$$\frac{\theta}{2\pi} = \frac{3 + 2 * 2.58 - 0.1 * 2.58^3}{6.28} = 1.025 \text{ revolutions.}$$

Exercises V

Problem 1 *If* $y = a + bt^2 + ct^4$; *find* $\frac{dy}{dt} = \boxed{2bt + 4ct^3}$ *and* $\frac{d^2y}{dt^2} = \boxed{\frac{d^2y}{dt^2} = 2b + 12ct^2}$.

Problem 2 *A body falling freely in space describes in t seconds a space s, in feet, expressed by the equation $s = 16t^2$. Draw a curve showing the relation between s and t. Also determine the velocity of the body at the following times from its being let drop: $t = 2$ seconds; Answer:* $\boxed{64}$ *$t = 4.6$ seconds; Answer:* $\boxed{0.32}$ *$t = 0.01$ second. Answer =* $\boxed{147.2}$

Problem 3 *If* $x = at - \frac{1}{2}gt^2$; *find* $\dot{x} = \boxed{x = a - gt}$ *and* $\ddot{x} = \boxed{\ddot{x} = -g}$.

Problem 4 *If a body move according to the law*

$s = 12 - 4.5t + 6.2t^2$,

find its velocity when $t = 4$ seconds; s being in feet. $\boxed{45.1}$

Problem 5 *Find the acceleration of the body mentioned in the preceding example.* $\boxed{12.4}$

Question 5.1 *Is the acceleration the same for all values of t?*

Multiple Choice:

(a) *yes* ✓

(b) *no*

Problem 6 *The angle θ (in radians) turned through by a revolving wheel is connected with the time t (in seconds) that has elapsed since starting; by the law*

$$\theta = 2.1 - 3.2t + 4.8t^2.$$

Problem 6.1 *Find the angular velocity (in radians per second) of that wheel when* $1\frac{1}{2}$ *seconds have elapsed. Angular velocity* $= \boxed{11.2}$ *radians per second.*

Problem 6.1.1 *Find also its angular acceleration. Angular acceleration* $= \boxed{9.6}$ *radians per second per second*

Problem 6.2 *A slider moves so that, during the first part of its motion, its distance s in inches from its starting point is given by the expression*

$s = 6.8t^3 - 10.8t$;*t being in seconds.*

Find the expression for the velocity at any time: $v = \boxed{20.4t^2 - 10.8}$

Problem 6.3 *and the acceleration at any time:* $a = \boxed{40.8t}$

Problem 6.3.1 *and hence find the velocity and the acceleration after 3 seconds. Answer:* $\boxed{122.4}$ *in./sec*^{2}

Problem 7 *The motion of a rising balloon is such that its height h, in miles, is given at any instant by the expression* $h = 0.5 + \frac{1}{10}\sqrt[3]{t-125}$; *$t$ being in seconds.*

Find an expression for the velocity and the acceleration at any time. Draw curves to show the variation of height, velocity and acceleration during the first ten minutes of the ascent.

Velocity: $v = \boxed{\frac{1}{30\sqrt[3]{(t-125)^2}}}$ *Acceleration:* $a = \boxed{-\frac{1}{45\sqrt[3]{(t-125)^5}}}$

Problem 8 *A stone is thrown downwards into water and its depth p in metres at any instant t seconds after reaching the surface of the water is given by the expression*

$$p = \frac{4}{4+t^2} + 0.8t - 1.$$

Find an expression for the velocity and the acceleration at any time.

Velocity: $v = \boxed{0.8 - \frac{8t}{(4+t^2)^2}}$ *Acceleration:* $v = \boxed{0.8 - \frac{8t}{(4+t^2)^2}}$

Problem 8.1 *Find the velocity and acceleration after 10 seconds. Velocity:* $\boxed{0.7926}$ *Acceleration:* $\boxed{0.00211}$

Problem 9 *A body moves in such a way that the spaces described in the time t from starting is given by $s = t^n$, where n is a constant. Find the value of n when the velocity is doubled from the 5th to the 10th second:* $\boxed{2}$

Problem 9.1 *find it also when the velocity is numerically equal to the acceleration at the end of the 10th second.* $\boxed{11}$