when beliefs are κ^{t+2} independent of the history leading to those beliefs.) But, by definition of the cutoff valuations,

$$\kappa^{t+1} - m^t = \delta(\kappa^{t+1} - m^{t+1}). \tag{10.10}$$

Combining equations 10.9 and 10.10 and using the fact that $t + 2 \leq \varepsilon/\Delta$ and that posteriors are decreasing, we get

$$\begin{aligned}&[P(\kappa^t) - P(\kappa^{t+1})]\kappa^{t+1} - [P(\kappa^t) - P(\kappa^{t+2})]m^{t+1} \\ &\quad > \delta P(\kappa^{t+2})U_s(\kappa^{t+2}) \geq \delta P(\kappa^{\varepsilon/\Delta})U_s(\kappa^{\varepsilon/\Delta}).\end{aligned} \tag{10.11}$$

The left-hand side of equation 10.11 is very small if η is small, and therefore $P(\kappa^{\varepsilon/\Delta})U_s(\kappa^{\varepsilon/\Delta})$ is very small. Thus, for any real time ε, the seller's profit from ε on tends to 0. This does not yet imply that the price at any time $\varepsilon > 0$ tends to 0, as small profits can arise either from low prices or from slow (i.e., delayed) sales. But it can be shown that, for any real time $\varepsilon > 0$, the price tends to 0 as Δ tends to 0.[19] This, in turn, implies trivially that profit at date 0, and not only profit as positive dates, tends to 0 (the buyers do not buy at date 0 at a noninfinitesimal price if they expect infinitesimal prices in the near future).

Ausubel and Deneckere (1989a) show that, without the stationarity assumption, the Coase conjecture does not hold, and, worse still, "anything" is an equilibrium of the no-gap case when $\delta \to 1$:

Theorem 10.3 (Ausubel and Deneckere 1989a)[20] Assume NG and that there exists $L > M > 0$ such that, for all $v \in [0, \bar{v}]$, $Lv \leq P(v) \leq Mv$. Let $U_s^* \equiv \sup_m[m(1 - P(m))]$ denote the monopoly profit. Then, for any $\varepsilon > 0$, there exists $\Delta(\varepsilon) > 0$ such that for any $\Delta \leq \Delta(\varepsilon)$, for any $U_s \in [\varepsilon, U_s^* - \varepsilon]$,

19. A sketch of the argument follows: Fix a real time $\varepsilon > 0$, and suppose that there exists a sequence $\Delta \to 0$ such that the price at date ε does not converge to 0: $m^{\varepsilon/\Delta} \geq m \geq 0$. The proof that this is impossible has four steps: (1) Because prices decrease over time and because the profit from real time Δ (i.e., period 1) on tends to 0, the probability of a sale between Δ and ε must converge to 0 with Δ. (2) Unless he has charged 0 in the past, the seller sells with strictly positive probability in each period. If not, his continuation valuation would be 0, as under assumption S the seller's optimal strategy can be taken to be stationary and therefore to generate no sale forever. However, he would be better off charging a price slightly above 0, as such a price would be accepted with positive probability because of discounting and because of the fact that the seller never charges prices below 0. (3) Because the probability of sale between Δ and ε tends to 0, $m^1 - m^{\varepsilon/\Delta}$ tends to 0. (The price schedule is almost flat, as one can see by using equation 10.10 at periods 1 and ε/Δ.) Let m^* denote the common limit of m^0 and $m^{\varepsilon/\Delta}$. ($m^0 - m^{\varepsilon/\Delta}$ tends to 0 by equation 10.10 applied to $t = 0$.) (4) Because $\bar{v}$ purchases at date 0, $\bar{v} \geq m^*$. Also, $e^{-r\varepsilon}(\kappa^{\varepsilon/\Delta+1} - m^{\varepsilon/\Delta}) \geq \kappa^{\varepsilon/\Delta+1} - m^0$, where $\kappa^{\varepsilon/\Delta+1}$ is the cutoff type at period $\varepsilon/\Delta + 1$, and hence $m^* - \bar{v}$, as $\kappa^{\varepsilon/\Delta+1} \to \bar{v}$ in the limit. This means that type $\bar{v}$ has utility 0 in the limit, and so have the other types (who have lower utility than type $\bar{v}$). Therefore, the (stationary) strategy of each type is to accept any offer less than their type in the limit, and the profit of the seller from any date on cannot go to 0.

20. Ausubel and Deneckere (1989b, theorem 2) show not only that any profit is an equilibrium profit when $\Delta \to 0$, but also that any vector $\{U_s, U_b(\cdot)\}$ of expected utilities for the seller and each type of buyer that is feasible (i.e., is individually rational for all, is incentive compatible for the buyer, and corresponds to a feasible trade policy) can be approximately obtained as a perfect-equilibrium payoff vector of the sale model when $\Delta \to 0$.

Ausubel and Deneckere (1987) and Gul (1987) derive theorems similar to theorem 10.3 for durable-good oligopolists.

there exists a perfect Bayesian equilibrium of the sale model such that the seller's profit is equal to U_s.

The intuition for this theorem is that one can use a Coase path associated with some equilibrium satisfying condition S as a "threat" that prevents the seller from deviating from a given price path. To illustrate the idea, take the linear-demand case of the example in subsection 10.2.3. There, we derived an equilibrium satisfying condition S. The seller's valuation for current cutoff κ was $U_s^C(\kappa) = \gamma(\delta)\kappa^2/2$, with $\lim_{\delta\to 1}\gamma(\delta) = 0$ (where C stands for "Coase"). (Because subsection 10.2.3 was interpreted with a continuum of buyers, the valuations $U_s^*(\cdot)$ and $U_s^C(\cdot)$ in this proof correspond to the continuum-of-buyers case. See subsection 10.2.3 for the link with the single-buyer case.) Let us show that there is a perfect Bayesian equilibrium in which the seller makes a profit close to the monopoly profit $\frac{1}{4}$ (which is obtained by charging the monopoly price $\frac{1}{2}$). Consider the following exponential price path in *real* time (where τ denotes real time and t, as before, denotes periods):

$$m^\tau = \tfrac{1}{2}e^{-\eta\tau}.$$

That is, the price starts at the monopoly price $\frac{1}{2}$ and decreases exponentially. In period t in discrete time, one has

$$m^t = \tfrac{1}{2}e^{-\eta t\Delta}.$$

Note that if η is close to 0, and if m^τ is the equilibrium path, all types $v \geq \frac{1}{2} + \varepsilon$ buy at date 0 for ε small (as $v - \frac{1}{2} = \sup_\tau e^{-r\tau}(v - \frac{1}{2}e^{-\eta\tau})$). Hence, the seller makes almost his monopoly profit (as we noted in chapter 7, he cannot make more than the monopoly profit, because the equilibrium must respect the buyers' incentive-compatibility and individual-rationality constraints).

Consider the following strategies: "The seller charges $m^t = \frac{1}{2}e^{-\eta t\Delta}$; type v chooses t so as to maximize $e^{-rt\Delta}(v - \frac{1}{2}e^{-\eta t\Delta})$. If the seller deviates from the above path, the equilibrium switches immediately to the Coase equilibrium obtained in subsection 10.2.3." Clearly, the buyers' behavior is optimal in view of the seller's. Would the seller ever deviate? Clearly not at date 0; but if we assume that η is very small (as we want to ensure that types above $\frac{1}{2}$ buy at date 0), it might be the case that sales are so slow later that the seller may want to switch to the Coase path. In this case, the high-price path would not be credible and the buyers would refrain from buying because they would expect a switch to the low-price Coase path. To make sure that this does not occur, let us fix η small, take Δ to 0, and show that the seller never wants to deviate from the exponential path.

Note that along the equilibrium path the cutoff valuation κ^{t+1} is given by indifference between buying today and buying tomorrow:

$$\kappa^{t+1} - \tfrac{1}{2}e^{-\eta t\Delta} = e^{-r\Delta}(\kappa^{t+1} - \tfrac{1}{2}e^{-\eta(t+1)\Delta}), \tag{10.12}$$

and therefore

$$\kappa^t - \kappa^{t+1} \simeq \frac{\eta(r+\eta)}{2r} e^{-\eta t \Delta}\Delta \tag{10.13}$$

for Δ small. That is, sales per unit of time decline exponentially at rate η with real time $(t\Delta)$. Furthermore, κ^t is approximately proportional to $e^{-\eta(t\Delta)}$. Integrating price times sales over the remaining horizon, one can see that the expected profit from date t on, $U_s^*(\kappa^t)$, is approximately $\nu(\Delta)(\kappa^t)^2$, where $\lim_{\Delta\to 0}\nu(\Delta) = \nu > 0$. Therefore, for Δ sufficiently small, $U_s^*(\kappa^t) > U_s^C(\kappa^t)$, so deviations to the Coase path are not profitable to the seller.

Clearly, the derivations for profits smaller than the monopoly profit follow the same lines.

Remark This equilibrium has the same bootstrap feature as the equilibria derived in the folk theorem for infinitely repeated games (see chapter 5): A player is threatened by a move to an equilibrium that is bad for him if he deviates. It should, therefore, not come as a surprise that with a finite "real-time" horizon, the Coase conjecture holds in the limit of shorter time periods even without assumption S: See exercise 10.1.

10.2.7 Gap vs. No Gap and Extensions of the Single-Sale Model†††

After this detailed examination of the equilibrium set, it is time to step back and discuss some modeling issues.

Subsections 10.2.4 and 10.2.5 showed how the gap and no-gap cases have different implications. In the gap case negotiations end after a fixed finite number of offers; in the no-gap case negotiations can continue forever. As we suggest in subsection 10.2.2, finite negotiations seem the more reasonable prediction in many cases. If the seller has an "outside option"—an opportunity to sell to another buyer he will take it when he becomes sufficiently pessimistic about the current buyer's willingness to trade. Although this may be an argument for the gap case, it seems unnatural to suppose that all potential buyer types have gains from trade. A more descriptive model might have a positive probability that $v < c$, i.e., that gains from trade are negative, coupled with a decision by the buyers about whether to enter negotiations.[21]

21. The simplest version of such a model, which supposes that all buyers have the same positive cost of entering negotiations, runs into a kind of market shutdown; the seller will never charge a price below the lowest valuation that chooses to enter negotiations, so this lowest-valuation type has a negative payoff from negotiations, and there is an equilibrium where *no* types pay the entry fee. (See Fudenberg and Tirole 1983, Perry 1986, Cramton 1990, and exercise 10.6.) To avoid this market closure, a good model of endogenous entry into negotiations must be even more complicated; one possibility is to make the buyer's entry cost private information as well, with a positive probability that his cost is negative. We would be interested in knowing what such a model implies.

When one interprets the single-sale model as representing sales by a durable-good monopolist, one supposes that all of the monopolist's potential customers are present and actively "shopping" at the beginning of the first period. Sobel (1990) extends the model to allow for a regular flow of new consumers. Because of this inflow, the distribution of consumer types does not deteriorate monotonically as in the equilibria with a fixed stock of consumers. Sobel's model is a generalization of the two-type model discussed in subsection 10.2.5. At each period $t = 0, 1, \ldots$, a new group of buyers enters the market; a proportion $\bar{p}$ of them have valuation $\bar{v}$ and a proportion p have valuation $\underline{v}$. (There is a continuum of "small" consumers of each type, and the aggregate system is modeled as deterministic—if the high-value buyers play a mixed strategy of accepting with probability $\frac{1}{3}$, then a fraction of exactly $\frac{1}{3}$ will accept and the others will reject.) The size of the inflow is the same in each period. The buyers who entered earlier and have not purchased yet are still in the market.

Because the inflow of new consumers prevents the distribution from becoming concentrated at $\underline{v}$, the backward-induction argument that yields uniqueness in the gap case of theorem 10.1 cannot be applied. And indeed there are multiple equilibria in this model; in fact, any payoff between v and the monopoly profit can be sustained by a perfect Bayesian equilibrium.

This leads Sobel to consider a restriction to stationary strategies, similar to that proposed by Gul et al. (1986). Here the state of the system includes the numbers of low- and high-valuation buyers currently in the market, so a stationary strategy for the buyer can depend on this aggregate state as well as on the current price and his own valuation.[22]

Sobel shows that equilibria in stationary strategies exist, and provides a partial characterization. With stationary strategies, the seller cannot sell at prices significantly greater than $\underline{v}$ when the period length shrinks to 0. Moreover, in a stationary equilibrium prices must cycle: As long as the seller charges prices above v, the number of low-valuation buyers in the market increases both in absolute and in relative terms. At some point, it pays the seller to have a "sale," that is, to charge $\underline{v}$ (in that case, there are only new buyers in the following period). The price path is thus an infinite replica of the one obtained in subsection 10.2.5. The price decreases within each cycle according to the formula $m = \bar{v} - \delta^n(\bar{v} - v)$, n periods before the next sale. Sobel shows that there is an upper bound n^* independent of the discount factor such that there are never more than n^* periods until the next sale, so that as the discount factor tends to 1 (i.e., the time intervals

22. Contrast this with the continuum case in the traditional model, where the current price determines the next cutoff valuation and thus makes the information about the current cutoff "irrelevant."

shrink to 0) the next sale is always "soon," and prices converge to $\underline{v}$, as in the Coase conjecture.[23]

The variants of the single-sale model we have discussed so far all suppose that the seller has no private information. This may account for the striking force of the Coase conjecture, which shows that the buyer (who does have private information) has all of the bargaining power when the discount factor is close to 1, at least when attention is restricted to stationary strategies. Since this conclusion seems unrealistic, researchers have studied models in which the seller has private information as well.

One way to introduce private information is to suppose that the buyer does not know the seller's cost, or, more generally, the seller's willingness to sell at a given price. As in the models of chapters 8 and 9, the seller may have an incentive to build a reputation for high cost or "toughness" by charging a high price, just as the buyer has an incentive to build a reputation for a low valuation by rejecting offers. Since the Coase conjecture implies that bargaining is at least approximately efficient and bargaining with two-sided incomplete information tends to be inefficient (subsection 7.4.4), the Coase conjecture should not be expected to obtain here. We say more about this case in section 10.4.

An alternative way to introduce private information for the seller is to allow him to have private information about the quality of the good, as in Evans 1989 and Vincent 1989. If the sellers' cost of production is increasing in quality, or if the good has already been produced and the seller receives a flow of utility from owning the good, the seller's willingness to sell at a given price decreases with quality, which leads to inefficiency in the same way as in Akerlof's (1970) lemons model.

10.3 Intertemporal Price Discrimination: The Rental or Repeated-Sale Model[†††]

Now consider a seller who wishes to rent a service or a good to a buyer. The buyer is free to rent the good today but not rent it tomorrow, so the game is not over the first time the buyer accepts an offer. In this model, it is more convenient to define v and c as flow variables, i.e., the per-period valuation and cost. We consider two variants of the rental model. In the short-term-contracts variant, the seller makes offers for rental in the current

23. There are other interesting variants. Bond and Samuelson (1984, 1987) allow the good to depreciate. Buyers therefore come back to the seller after a while. Under a stationarity assumption, a form of the Coase conjecture holds, but monopoly profits can be sustained with nonstationary perfect Bayesian equilibria. The results thus have the same flavor as those of Sobel (1990) and those of subsections 10.2.5 and 10.2.6.

Durable-good monopolists with either decreasing returns in each period (Kahn 1986) or learning by doing (Olsen 1988; see exercise 10.5 below) have also been studied. With decreasing returns, the Coase conjecture does not hold (an extreme case of decreasing returns is a capacity constraint in each period, which acts as a commitment not to "flood the market"). With learning by doing, only part of the Coase conjecture holds.

period only, and is not allowed to propose prices for rentals in future periods. The other variant supposes that the seller is allowed to propose long-term contracts, but that the seller cannot commit not to try to renegotiate these contracts in the future.

Before treating these two variants, it is interesting to note that the formulations of a single buyer and a continuum of buyers, which coincide in the sale model, are dramatically different in the rental model. A continuum of buyers who are treated anonymously (i.e., cannot be told apart by the seller) do not make strategic use of their information in the rental model, and so the outcome is the usual monopoly outcome. In contrast, we will see that a single buyer with a continuum of possible valuations zealously guards knowledge of his valuation in the rental model. We will pursue the single-buyer interpretation in this section.

10.3.1 Short-Term Contracts

Suppose first that the seller can make offers only for the current period. Thus, at date $t = 0, 1, \ldots, T$, the seller offers a rental price r^t. The buyer with valuation v has utility $v - r^t$ during period t if he accepts the offer, and 0 otherwise, where v is now a flow utility. The history of the game at date t is the sequence of the previous price offers and whether those were accepted. The price and the acceptance decision at date t depend on the history at that date (and the acceptance decision depends also on the current price). We again normalize the production cost at 0.[24]

When T is large, the seller gets almost no profit in this game:

Theorem 10.4 (no price discrimination) Suppose that the buyer has n possible valuations $0 < v_1 = \underline{v} < v_2 < \cdots < v_n = \bar{v}$. And assume $\delta > \frac{1}{2}$ and $T < +\infty$.

- Let $n = 2$. There exist T_0 and T_1 such that, for any $T \geq T_0$ and any perfect Bayesian equilibrium of the rental game, the seller charges $r^t = \underline{v}$ for all $t = 0, 1, \ldots, T - T_1$. (Hart and Tirole 1988)
- Let $n \geq 2$. There exist T_0 and T_1 such that, for any $T \geq T_0$ and any Markov perfect Bayesian equilibrium[25] of the rental game, the seller charges $r^t = \underline{v}$ for all $t = 0, 1, \ldots, T - T_1$. (Schmidt 1990)

Theorem 10.4 shows that when the horizon is long, the seller charges the low price $\underline{v}$ for all periods but the last. Consequently, his expected

24. With positive costs, one must distinguish variable or flow costs from fixed or one-time costs. The analysis below extends immediately to the case of a flow cost. With a one-time cost, the continuation equilibrium after the first period of actual rental is the one described below but the analysis must be extended to include the game before the first rental.

25. Schmidt (1990) uses the Markov perfect Bayesian equilibrium (MPBE) concept defined by Maskin and Tirole (1989): An MPBE is a limit of a sequence of approximate strong MPBE, in which the strategies of the players depend on their private information and the common beliefs. Strong MPBE do not always exist, but MPBE always exist. For example, the unique equilibrium of subsection 10.2.2 is not strong Markov.

discounted profit converges to $\underline{v}/(1-\delta)$ as T goes to infinity. Theorem 10.4 is similar to the Coase conjecture in that the seller cannot price discriminate and makes a profit approximately equal to the lowest (per-period) valuation. It differs from the Coase conjecture in that the result holds for any $\delta > \frac{1}{2}$, whereas the Coase conjecture requires δ near 1. The additional strength comes from a fixed finite horizon. Indeed, the rental model can give weaker conclusions than the sale model with the infinite horizon considered in the last section. For example, with only one valuation for the buyer, this is a standard repeated game, and the folk theorem obtains.

We should also comment on the Markov assumption with more than two potential valuations. A step in the proof that the seller does not price discriminate shows that the seller never offers $r^t < \underline{v}$ for any t, because this offer is dominated by $r^t = \underline{v}$. The difficulty in showing this property comes from the fact that there might be continuation equilibria from $t+1$ on that yield different continuation payoffs for the seller. Therefore, it might be the case that the seller charges $r^t < \underline{v}$ rather than $r^t = \underline{v}$ because the continuation equilibria (which correspond to the same posterior beliefs) differ. Now, with $n = 2$, one can show that (perfect Bayesian equilibrium) continuation payoffs for the seller are unique for given posterior beliefs (see exercise 10.4) and therefore the issue does not arise. On the other hand, for any n, the Markov assumption guarantees that the seller's continuation valuation is the same whether $r^t < v$ or $r^t = \underline{v}$, so that he strictly prefers $r^t = v$. It is not known whether the Markov assumption is needed in general.

To obtain some intuition about the result, consider the case of an infinite horizon and two types, and the following strategies and beliefs: "At any t, the seller offers $r^t = \underline{v}$ if the posterior probability of a high-valuation buyer, μ^t, is less than 1, and $r^t = \bar{v}$ if $\mu^t = 1$; when $\bar{\mu}^t < 1$, the buyer accepts all offers below or at $\underline{v}$, and rejects all other offers whatever his type; if $\bar{\mu}^t = 1$, the v-type accepts $r^t \le \bar{v}$ and rejects higher offers and the $\underline{v}$-type accepts $r^t \le v$ and rejects higher offers. Last, if $\bar{\mu}^t = 1$, then $\bar{\mu}^{t+1} = 1$; if $\bar{\mu}^t < 1$ and $r^t \le v$, then $\mu^{t+1} = \bar{\mu}^t$; if $r^t > v$, then $\bar{\mu}^{t+1} = 1$ if r^t is accepted and $\bar{\mu}^{t+1} = \bar{\mu}^t$ if r^t is rejected." In words, the buyer always refuses any offer above $\underline{v}$ whatever his type, and if he were to accept such an offer he would be identified as a high-valuation buyer. The seller always charges $\underline{v}$. It is easy to see that these strategies form a perfect Bayesian equilibrium of the infinite-horizon game: If type v accepts an offer above $\underline{v}$, he gets current payoffs at most equal to $\bar{v} - \underline{v}$. However, the seller learns his identity and charges v forever. In contrast, if $\bar{v}$ rejects the offer, he will be able to continue buying at price $\underline{v}$ forever. Hence, if

$$v - v < (\delta + \delta^2 + \cdots)(v - \underline{v}) \Leftrightarrow \delta > \tfrac{1}{2},$$

type $\bar{v}$ prefers to reject offers above $\underline{v}$. Knowing this, the seller never tries to offer prices above v.

When $T < \infty$, the unique equilibrium is close to the infinite-horizon one described above.[26] The high-valuation buyer's payoff is at most $\bar{v} - \underline{v}$ if he accepts today, and is close to $[\delta/(1 - \delta)](\bar{v} - \underline{v})$ if he rejects and the horizon is sufficiently long. The seller does not attempt to price discriminate (charge prices above $\underline{v}$) until the final periods of their relationship.

The method of proof used by Hart and Tirole is quite different from that of Schmidt. Hart and Tirole use a reasoning of upward induction on beliefs very similar to that of subsection 10.2.5. Schmidt's method of proof resembles the one used by Fudenberg and Levine (1989) in their paper on the reputation of a long-term player facing a sequence of short-term players (see theorem 9.1).[27] He first shows that there is a strictly positive minimum probability of acceptance of offers $r^t > v$ that are made on the equilibrium path. He then shows that, because types $v > v$ can build a reputation for being of type v, they will do so, as the revelation of their type would be very costly.

Remark Because the seller is a long-run player, he may have an incentive to maintain a reputation of his own if the prior distribution makes this possible. For example, if the seller has private information about his cost, he may try to maintain a reputation for high cost by charging high prices. Thus, the results here about limits of equilibria with a long, finite horizon may be sensitive to the introduction of a small probability that the seller has high cost. An interesting question is whether a small probability of high cost would be swamped by a larger probability that the buyer is type $\underline{v}$, so that the equilibrium outcome would be close to that when the seller's cost is known. This has not been worked out, to the best of our knowledge.

10.3.2 Long-Term Contracts and Renegotiation

Now we suppose that the seller is able to offer long-term rental agreements to the buyer, and that the seller cannot commit not to renegotiate a contract later; that is, although a long-term contract is enforced if one of the parties wants it to be enforced, the parties can agree to replace an old contract with a new one if they both benefit from doing so.[28] As before, we suppose that the seller makes all the offers, including offers to renegotiate existing contracts. We also restrict our attention to the case where the buyer has

26. There are many other perfect Bayesian equilibria in the infinite-horizon game, as the "threat" of reverting to this equilibrium can be used to support price paths where the seller's profit is higher.

27. Fudenberg and Levine's "Stackelberg type," i.e., the type that the buyer would like the seller be convinced of, is type $\underline{v}$ here. Note that theorem 9.1 does not apply for two reasons. First, the rental game has two long-term players rather than a long-term player facing a sequence of short-term players. Second, theorem 9.1 covers both finite-horizon and infinite-horizon games, and has force only in the limit of discount factors $\delta \to 1$. The assumption of a long but finite horizon permits a backward-induction argument that yields a strong conclusion for any $\delta > \frac{1}{2}$.

28. Equivalently, they can keep the old contract and "cancel" its effects through an additional contract.

two possible types, $\underline{v}$ and $\bar{v}$. The space of contracts that can be signed at date t is quite large: A long-term contract signed at date t specifies probabilities of consumption $\{x^{t+\tau}\}_{\tau=0}^{T-t}$ by the buyer from t to T and transfers $\{r^{t+\tau}\}_{\tau=0}^{T-t}$ from the buyer to the seller. The numbers $x^{t+\tau}$ and $r^{t+\tau}$ depend on messages sent by the buyer at each date up to $t + \tau$.[29] Note that the short-term contracts of subsection 10.3.1 are long-term contracts with $x^{t+\tau} = r^{t+\tau} = 0$ for $\tau > 0$.

Although the space of feasible long-term contracts is very large, only one kind of long-term contract is actually used in equilibrium: long-term leasing contracts signed at t in which the seller commits to supply from t to T. (Such contracts can be called "sale contracts" even though the good is a rental good.) As a consequence, the pattern of consumption is the same as if the good were durable. Another way to describe the result is as follows: If the only feasible long-term contract were a sale contract, the equilibrium would be the one of the sale model described in section 10.2, and introducing other contracts does not affect the equilibrium allocation (the time pattern of consumption and utilities).

Theorem 10.5 (Hart and Tirole 1988) Suppose that the buyer has valuation $\underline{v}$ or $\bar{v}$. Then the outcome of the rental model with (possibly renegotiated) long-term contracts coincides with that of the durable-good model.

To obtain some intuition about this result,[30] it is useful to recall the basic conflict between efficiency and rent extraction in mechanism design (see section 7.3 and the price-discrimination example in subsection 7.1.1). Let $\underline{x}^t$ and $\bar{x}^t$ denote the probabilities of consumption at date t of types $\underline{v}$ and $\bar{v}$, and let $\underline{X} \equiv \mathrm{E}(\sum_{t=0}^{T} \delta^t \underline{x}^t)$ and $\bar{X} \equiv \mathrm{E}(\sum_{t=0}^{T} \delta^t \bar{x}^t)$ denote the expected discounted consumptions of the two types, where $0 \leq \underline{X}$ and $\bar{X} \leq 1 + \delta + \cdots + \delta^T$. The total social surplus is $\underline{p}\underline{X}\underline{v} + \bar{p}\bar{X}\bar{v}$. If $\underline{U}_b$ and $\bar{U}_b$ denote the expected utilities of the two types, and U_s the seller's expected profit, that the social surplus equals the buyer surplus plus the seller surplus implies that

$$U_s = \underline{p}(\underline{X}\,\underline{v} - \underline{U}_b) + \bar{p}(\bar{X}\bar{v} - \bar{U}_b).$$

But, in any mechanism or game, type $\bar{v}$ can always pretend to be type $\underline{v}$:

$$\bar{U}_b \geq \underline{X}(\bar{v} - \underline{v}) + \underline{U}_b.$$

Hence,

29. This is the appropriate version of the revelation principle for this model. The standard revelation principle, that the buyer announces his type truthfully to the seller when signing a contract, does not hold because of the possibility of renegotiation.

30. Laffont and Tirole (1990) extend this result to continuous consumption per period in the two-period case.

$$U_{\text{s}} \le U_{\text{b}} + \bar{p}X\bar{v} + \underline{X}(\underline{v} - \bar{p}\,\bar{v}).$$

If the seller could commit to an allocation at date 0, he would clearly choose $U_{\text{b}} = 0$, $\bar{X} = 1 + \delta + \cdots + \delta^T$ (efficient consumption for type $\bar{v}$), and $\underline{X} = 0$ if $\underline{v} < \bar{p}\,\bar{v} \Leftrightarrow \underline{v}/\bar{v} \equiv \alpha < \bar{p}$, or $\underline{X} = 1 + \delta + \cdots + \delta^T$ if $\alpha > \bar{p}$. That is, the seller would either "sell" at date 0 or "not sell" at all.

Assume that the seller would like to price discriminate, i.e., that $\bar{p} > \alpha$. (If $\alpha \geq p$, the seller can guarantee himself the monopoly profit by offering the sale contract at date 0 at price $r^t = \underline{v}$ for all t. The interesting case, and the general one with a general distribution, is that in which profit maximization conflicts with efficiency.) Suppose that the seller tries to get his monopoly profit by offering the two long-term contracts ($x^t = 1$ and $r^t = v$ for all t) and ($x^t = 0$ and $r^t = 0$ for all t) in period 0, and by claiming that no other contract will be offered in the future. If in equilibrium type $\bar{v}$ chooses the first contract and type $\underline{v}$ chooses the second contract (as the seller would want them to do), then if the buyer chooses the second contract at period 0, revealing that he is type $\underline{v}$, in period 1, seller would want to offer ($x^t = 1$ and $r^t = \underline{v}$ for all $t \geq 1$). Anticipating this, type $\bar{v}$ would not want to take the first contract.

As in static mechanism design, the binding incentive constraint here is to induce the high-valuation buyer to reveal his type. When he reveals his type, he must consume with probability 1 in all subsequent periods, because only the efficient contracts are renegotiation-proof under symmetric information. This suggests that a sale contract will be offered in each period (the game being over if it is accepted, as efficient contracts are always renegotiation-proof). Consider now the contract that is chosen at date t by type $\underline{v}$ (one can show that it is unique). Either it is efficient and the game is over; or it is inefficient because the seller prefers to extract rent, and the linearity of the tradeoff between efficiency and rent extraction suggests that $x^t = 0$. That is, the seller can wait at least one more period to offer a contract that is accepted by type $\underline{v}$. In equilibrium, the seller offers only one contract, a sale contract, in each period, and type $\bar{v}$ randomizes between accepting and refusing it, until the seller offers the sale contract at per-period price $\underline{v}$. The nature of renegotiation-proof contracts with several types is still unknown.

10.4 Price Offers by an Informed Player[†††]

In sections 10.2 and 10.3 we obtained a coherent set of results by making the strong assumption that only one party has private information and that the other party has all the bargaining power. Because we have little information about the extensive form that is played in practice, we must consider alternative bargaining processes. Furthermore, both parties may have private information. Changing the model in either of these directions introduces a multiplicity of equilibria. For instance, even in a two-period model, if the informed party makes the first-period offer (and the uninformed party makes the second-period offer) or if both parties have private information, there may exist continua of pooling, separating, and hybrid equilibria.[31] Needless to say, this feature carries over to any horizon. The reason for the large multiplicity of equilibria is the same as in the case of the Spence signaling game. Here an off-the-equilibrium-path offer may be interpreted by the other party as stemming from a "weak type" (high-valuation buyer, low-cost seller) who is eager to reach an agreement and will concede fast in the future; such beliefs, in turn, make the offers unattractive to the party that proposes because they are unlikely to be accepted and will induce the other party to take a tough stance in the future.

The literature on bargaining with price offers by an informed player is large, and because of the limited scope of this chapter we do not attempt to review it thoroughly. It is divided along several lines: whether one player makes all the offers (as in the sale and rental models) or another bargaining process is used (typically, the alternating-move model), whether there is asymmetric information on one side or both, and whether the paper tries to characterize the equilibrium set or uses a refinement to reduce its size. We start with the one-sided-offers, bilateral-asymmetric-information model to highlight some new features relative to the previous sections, and then move on to the alternative-offer, one-sided-asymmetric-information model. Last, we give a few results using mechanism design to characterize equilibria of bargaining processes.

10.4.1 One-Sided Offers and Bilateral Asymmetric Information

Consider the durable-good model of section 10.2, but let the seller have private information about his cost $c \in [\underline{c}, \bar{c}]$. As before, the buyer has private information about his valuation $v \in [\underline{v}, \bar{v}]$. Assume that the horizon is infinite and the seller makes all offers, denoted $\{m^t\}_{t=0}^{\infty}$.

There are many perfect Bayesian equilibria for this model. Assume that the distributions over types are continuous and that $\underline{v} < \bar{c}$. Cramton (1984) constructs an equilibrium for the double-uniform case in which the seller sells only after having revealed his cost, and, as the time Δ between successive offers converges to 0, the initial revealing offer converges to his cost. Thus, the first part of the Coase conjecture (that the seller's profit converges to 0) is satisfied. Cho (1990) obtains an equilibrium in which this property applies only to the seller with type $\underline{c}$, who reveals his type in the first period and makes an offer that converges to c as $\Delta \to 0$. Buyers reject offers much above $\underline{c}$, so sellers with costs much above $\underline{c}$ are unlikely to sell, and, when $\Delta \to 0$, the probability of sale goes to 0 as well! This conclusion may seem perverse, but Cho shows that it applies to any perfect Bayesian equilibrium

31. See Fudenberg and Tirole 1983.

in which the strategies satisfy a form of stationarity and in which the seller's offer in any period, on or off the equilibrium path, is taken to be a perfectly revealing signal of his type.[32]

But one can also construct equilibria in which the seller makes approximately the monopoly profit corresponding to his cost if Δ is close to 0 (Ausubel and Deneckere 1990a). The idea is clear from theorem 10.3: If the seller detectably deviates from the path that gives him almost his monopoly profit, he is thought of as being of type $\underline{c}$, and the continuing path is the Coase-conjecture path corresponding to the buyer's being informed that $c = \underline{c}$.

Ausubel and Deneckere (1990a) give two properties of equilibria of the one-sided-offer, bilateral-asymmetric-information model. First, they show that trade never occurs between a seller of type c and a buyer of type $v < c$. This intuitive property can be obtained as follows: If type c made an offer m^t that was accepted with positive probability by type $v < c$ (and was therefore no higher than c), the seller would lose money on this offer. Furthermore, only types less than v would remain from $t + 1$ on (from the successive-skimming lemma 10.1). Because the buyer never accepts offers above his valuation, type c does not make a profit from $t + 1$ on; therefore, he strictly loses money from date t on, and he would do better to offer prices above v (even though these would be rejected). Second, and more important, Ausubel and Deneckere show that if the supports of the distributions are common ($\underline{c} = \underline{v}$, $\bar{c} = \bar{v}$) and if the Coase conjecture holds for type $\underline{c}$ (that is, his initial offer converges to $\underline{c}$ as $\Delta \to 0$), then the expected amount of "discounted trade" converges to 0 as $\Delta \to 0$.[33] More precisely, fix an equilibrium, and let $x(c, v)$ be the expected discounted value of the indicator function, which has value 1 when trade occurs. Then the expectation of $x(c, v)$ over c and v—the *ex ante* expected discounted trade—converges to 0. The intuition for this result is clear. Let $X(c)$ denote the expected discounted trade for type c ($X(c) = E_v(x(c, v))$), and let $U_s(c)$ denote the expected utility of type c in the equilibrium. Because type $c - dc$ can always mimic type c, $dU_s/dc = -X(c)$ (see chapter 7). Hence,

$$U_s(\underline{c}) = U_s(c) + \int_{\underline{c}}^{c} X(\gamma)\, d\gamma \text{ for all } c.$$

Now, if the Coase conjecture holds for type $\underline{c}$, $U_s(\underline{c}) \to 0$ and $U_s(c) \geq 0$ for all c implies that $X(c) \to 0$ for all c. Thus, there is an inherent conflict between the Coase conjecture and the existence of trade in equilibrium.[34]

32. More precisely, the period-t offer m^t reveals that the seller's type is $c^{-1}(m^t)$ *even if* the seller has previously been revealed to be a different type.
33. More generally, if $\underline{c} < \underline{v}$, the *ex ante* expected probability of trade is bounded above by the probability that $c \leq \underline{v}$. (The case $\underline{c} > \underline{v}$ is, of course, equivalent to the case $\underline{c} = \underline{v}$.)
34. Thus, in the Cramton and Cho equilibria, all trade is deferred far into the future.

10.4.2 Alternating Offers and One-Sided Asymmetric Information

Ausubel and Deneckere (1989b), Chatterjee and Samuelson (1987), Grossman and Perry (1986a), Gul and Sonnenschein (1988), and Rubinstein (1985) have studied the alternating-offer model with one-sided asymmetric information, which has many equilibria for the now-familiar reason. The model is that of Rubinstein and Ståhl (see chapter 4 above), except that one party (the buyer, say) has private information.

What is the equilibrium set for this game? This question is easiest to answer in the case of time period $\Delta \to 0$. (If $\Delta = +\infty$, i.e., $\delta = 0$, the first player to make an offer obtains his monopoly profit.) As is noted above, a perfect Bayesian equilibrium of the bargaining game gives rise to two functions of the buyer's valuation, v: $M(\cdot)$ and $X(\cdot)$. First,

$$X(v) = \mathrm{E}\left(\sum_{t=0}^{\infty} \delta^t x^t(h^t, m^t, v) \right)$$

is the expected discounted trade of type v in this equilibrium (where $x^t(h^t, m^t, v)$ is the probability of trade at date t, and $0 \le x^t(h^t, m^t, v) \le 1$ for all t implies $0 \le X(v) \le 1$). And

$$M(v) = \mathrm{E}\left(\sum_{t=0}^{\infty} \delta^t m^t x^t(h^t, m^t, v) \right)$$

is the expected discounted transfer from the buyer to the seller. One can view $\{M(\cdot), X(\cdot)\}$ as a *mechanism* in the sense of chapter 7.

The mechanism $\{M(\cdot), X(\cdot)\}$ is *feasible* if it satisfies

$$0 \le X(v) \le 1 \text{ for all } v$$

$$(\mathrm{IR_B}) \quad X(v)v - M(v) \ge 0 \text{ for all } v$$

$$(\mathrm{IR_S}) \quad \mathrm{E}_v M(v) \ge 0$$

$$(\mathrm{IC}) \quad X(v)v - M(v) \ge X(\hat{v})v - M(\hat{v}) \text{ for all } (v, \hat{v}).$$

That is, a feasible mechanism is individually rational and incentive compatible. Note that we keep normalizing the seller's cost to be 0.

Any perfect Bayesian equilibrium of the alternating-offer bargaining game (or, more generally, of any sequential bargaining game) with equal discount factors must give rise to a feasible mechanism. First, it must satisfy individual rationality, because each party would not bargain (or would reject offers and make outrageous offers himself) if he expected a negative payoff from bargaining. Second, any type v of buyer can always adopt the strategy of another type $\hat{v}$, and therefore the equilibrium must satisfy IC.

Conversely, we may ask: When is a feasible mechanism the outcome (or approximate outcome) of a perfect Bayesian equilibrium of the alternating-offer game? An answer is found in the following theorem.

Theorem 10.6 (Ausubel and Deneckere 1989b) Assume $\underline{v} = 0$. A feasible mechanism $\{M(\cdot), X(\cdot)\}$ is implementable by a perfect Bayesian equilibrium of the alternating-offer bargaining game with one-sided asymmetric information (in the sense that for any $\varepsilon > 0$ there exists $\Delta_0 > 0$ such that for any $\Delta \leq \Delta_0$ there exists a perfect Bayesian equilibrium with payoffs U_s for the seller and $U_b(\cdot)$ for the buyer such that $|[X(v)v - M(v)] - U_b(v)| < \varepsilon$ and $|E_v M(v) - U_s| < \varepsilon$) if and only if

$$\bar{v}\, X(\bar{v}) - M(\bar{v}) \geq \bar{v}/2.$$

That is, any feasible mechanism is an equilibrium outcome as long as the highest-valuation buyer obtains at least his full information payoff for Δ close to 0 (which is approximately the Nash bargaining solution $\bar{v}/2$; see chapter 4). The intuition for this necessary condition is that buyer $\bar{v}$ is at worst thought of as being weak, i.e., as having type $\bar{v}$. Even in this case, he can guarantee himself the symmetric-information payoff.

Sketch of Proof Let us first show that in equilibrium $X(\bar{v})\bar{v} - M(\bar{v}) \geq \bar{v}/2$. (The following reasoning is due to Grossman and Perry (1986a).) Let $\bar{m}$ denote the highest price that the seller gets (which is either accepted or offered by some type of buyer with some positive probability) in *any* equilibrium after *any* history. Suppose for instance that, for some equilibrium and history, the seller offers $m^t = \bar{m}$ (or close to $\bar{m}$ if $\bar{m}$ is a supremum rather than a maximum) and that at least some type v of buyer accepts $\bar{m}$. This type obtains utility $v - \bar{m}$ from date t on. But he could reject the offer and offer price $m^{t+1} = \delta\bar{m} + \varepsilon$ next period to the seller. The seller would accept m^{t+1} with probability 1, because he would never get more than $\bar{m}$ in the future. Therefore, this deviation yields type v utility $\delta(v - \delta\bar{m} - \varepsilon)$. Now, if $\bar{m} > \bar{v}/(1 + \delta)$,

$$\begin{aligned} v - \bar{m} - \delta(v - \delta\bar{m} - \varepsilon) &= v(1 - \delta) - \bar{m}(1 - \delta^2) + \delta\varepsilon \\ &< (1 - \delta)(\bar{v} - (1 + \delta)\bar{m}) + \delta\varepsilon \\ &< 0 \end{aligned}$$

if ε is close to 0. Hence, $\bar{m} \leq \bar{v}/(1 + \delta)$. Therefore, the type-$v$ buyer can always guarantee himself $v - (\bar{v}/(1 + \delta))$ when it is his turn to make an offer. In particular, type $\bar{v}$ gets at least $\delta\bar{v}/(1 + \delta) \simeq \bar{v}/2$ for δ close to 1. The same reasoning applies if $\bar{m}$ is attained for an offer by the buyer.

Second, to show that any feasible outcome with utility at least $\bar{v}/2$ for type $\bar{v}$ can occur, one can "embed" the seller-offer equilibria of theorem 10.3 with time interval 2Δ between the seller's offers into the alternating-move game with time interval Δ. The idea is to look for equilibria in which the buyer offers nonserious (negative) prices when it is his turn to make an offer. In such periods, beliefs remain the same. However, if the buyer were to deviate and offer a positive price, he would be thought of as the

v type, and the only feasible agreement in the future would be the "full-information" one at price approximately equal to $\bar{v}/2$. Such optimistic beliefs of the seller prevent the buyer from making serious offers. When it is the seller's turn to make offers, the strategies are those of the proof of theorem 10.3. The seller offers a price on an exponential path, say, and if he deviates the equilibrium switches to one in which the Coase conjecture holds. ■

Several authors have imposed additional restrictions to narrow down the set of equilibria with alternating offers. In the two-type case, Rubinstein (1985) looks for equilibria with the following properties:

(i) If the buyer rejects the seller's offer, and if his next counteroffer, when accepted, yields less utility than the acceptance of the seller's offer to the high-valuation type and more utility to the low-valuation type, then the seller assigns probability 1 to the buyer's having a low valuation.[35]

(ii) If the counteroffer, when accepted, yields more utility to both types, then the seller's belief that the buyer has low valuation does not increase.

Grossman and Perry (1986a) restrict the set of equilibria by imposing the notion of perfect sequential equilibrium developed in their 1986b paper to the alternating-offer model. They require that if the seller receives an out-of-equilibrium offer m^t he attempts to find a set of valuations with the property that, if the seller believes that the buyer's type is in this set, then this set is indeed the set of buyer's types that are better off than had the equilibrium path been followed. Gul and Sonnenschein (1988), under assumption G, show that the Coase conjecture holds in the class of pure-strategy equilibria that have the property of stationarity of the buyer's strategy, have the monotonicity property that the possibility of additional high-valuation buyers does not lead a low-valuation buyer to lower his acceptance price, and are such that nonserious offers convey no information beyond the fact that the seller is unwilling to trade in the current period.[36,37]

Admati and Perry (1987) consider a different extensive form for alternating offers, in which a player who receives an offer chooses how much time elapses before he makes a counteroffer, subject to the constraint that the time between the two offers exceeds some fixed number (the other party

35. This property has some of the flavor of the forward-induction idea developed in chapter 11, but is different.

36. That is, the offer of a buyer can influence the seller's beliefs only if the strategies specify that the offer be accepted (with probability 1 because of pure strategies).

37. Ausubel and Deneckere (1990c) show that, under similar assumptions (stationarity, monotonicity, pure strategies, and nonserious offers do not affect relative beliefs over the set of types making them), the buyer never makes a serious offer. That is, all information revelation occurs through passive responses by the informed party to offers of the uninformed party. Ausubel and Deneckere argue that this result provides a justification for the one-sided incomplete-information model of section 10.2, in which only the uninformed party is permitted to make offers.

cannot make another offer in the meantime).[38] Thus, the low-valuation buyer can use delay to signal his valuation to the seller. Admati and Perry use a variation on the Cho-Kreps and Banks-Sobel refinements studied in chapter 11 below, and find that the equilibrium delay does not disappear as the minimum time between offers tends to 0.[39]

There is a small literature on bargaining with two-sided asymmetric information and alternating moves: Chatterjee and Samuelson (1987, 1988), Cho (1990), and Cramton (1987) discuss bargaining with an infinite horizon, and Fudenberg and Tirole (1983) discuss the case of two periods.

10.4.3 Mechanism Design and Bargaining

Cramton (1985), Wilson (1987a,b) and Ausubel and Deneckere (1989b; 1990a,b) have emphasized the link between mechanism design and bargaining. Recall from chapter 7 that Myerson and Satterthwaite showed that if the seller's cost c and the buyer's valuation v are continuously distributed on $[\underline{c}, \bar{c}]$ and $[\underline{v}, \bar{v}]$, respectively, and if $\underline{v} < \bar{c}$, there exists no incentive-compatible and individually rational mechanism that realizes all gains from trade. This result implies that perfect Bayesian equilibria of bargaining games are, in general, inefficient. However, one may ask whether they can be "constrained efficient"—that is, can their outcomes coincide with those of static mechanisms that maximize a convex combination of the seller's and the buyer's *ex ante* payoffs subject to the incentive-compatibility and individual-rationality constraints? Ausubel and Deneckere (1990b) pose this question and provide a partial answer. They show that, under certain distributional assumptions when $\underline{c} = \underline{v}$ and $\bar{c} = \bar{v}$ (common support), all the *ex ante* (constrained) efficient allocations can be attained by perfect Bayesian equilibria of the two infinite-horizon, frequent-one-sided-offers bargaining games (i.e., the seller makes all offers or the buyer makes all offers and Δ goes to zero).[40] Of course, there is no guarantee that the players will coordinate on an *ex ante* efficient equilibrium if such an equilibrium

38. A party can also delay the bargaining process in the models of Perry and Reny (1989) and Stahl (1990). (Unlike the model of Admati and Perry, these are complete-information models.)

39. Cramton (1987) extends the Admati-Perry model to two-sided uncertainty. In his equilibrium, both players delay making an initial offer. Eventually, either the players realize that there are no gains from trade, and so terminate negotiations, or the more patient player makes a revealing offer, at which point the equilibrium path is the same as with one-sided uncertainty.

40. A mechanism specifies discounted probabilities of trade $x(\cdot,\cdot)$ and expected discounted transfers $m(\cdot,\cdot)$. Let $X_s(c) = E_v x(c,v)$, $X_b(v) = E_c x(c,v)$, $M_s(c) = E_v m(c,v)$, and $M_b(v) = E_c m(c,v)$. A mechanism is feasible if it satisfies individual rationality and incentive compatibility:

$$M_s(c) - cX_s(c) \geq 0$$
$$\geq M_s(\hat{c}) - cX_s(\hat{c}) \text{ for all } (c, \hat{c}),$$

$$vX_b(v) - M_b(v) \geq 0$$
$$\geq vX_b(\hat{v}) - M_b(\hat{v}) \text{ for all } (v, \hat{v}).$$

exists, but it is interesting to know that simultaneous offers, as is implied by the revelation principle, may not be necessary in order to obtain *ex ante* efficient outcomes.

Exercises

Exercise 10.1** Consider the sale model with $c = 0$ and v uniformly distributed on $[0, 1]$. Prove the Coase conjecture without a stationarity assumption when the real time horizon τ is finite (so there are $T + 1$ periods of price offers where $T = \tau/\Delta$). Hint: Solve for the last-period price as a function of the beliefs in that period. Prove by backward induction that the seller's expected profit at date t, $U_s^t(\kappa^t)$, is quadratic in the current cutoff valuation κ^t; that the seller charges $\gamma^t\kappa^t$; that the buyer accepts m^t if his valuation exceeds $\lambda^t\kappa^t$; and that the coefficients $\gamma^{w/\Delta}$ (for real time w less than τ) tend to 0.

Exercise 10.2***

(a) Solve the rental model of subsection 10.3.1 with two periods when the buyer has two types, and when he has a continuum of types uniformly distributed on $[0, 1]$, say. In particular show that, in the continuum case, there is a "truncation equilibrium" (in which the buyer accepts in period 0 if and only if his valuation exceeds some number).

(b) Show that with a continuum of types and three periods, no truncation equilibrium exists if δ is sufficiently large.

For more on this, see Fernandez-Arias and Kofman 1989.

Exercise 10.3** Use theorem 10.4 to show that in the rental model with $\delta > \frac{1}{2}$, T large, two types, and $\bar{p} > \alpha = \underline{v}/\bar{v}$ there is positive probability in equilibrium that the buyer does not consume in a period even though he consumed in all earlier periods.

Exercise 10.4*** Show that in the two-type rental model the seller never charges less than $\underline{v}$ and his expected payoff is unique. Hint: Prove by induction that (1) the seller never charges $m^t < \underline{v}$ and type $\underline{v}$ always has payoff 0, (2) the high-valuation buyer's continuation utility at t, $\bar{U}^t(\bar{\mu}^t)$, is a step function ($\exists\, 0 < \mu_1^t < \cdots < \bar{\mu}_K^t < 1$ and $\bar{U}_1^t > \bar{U}_2^t > \cdots > \bar{U}_{K+1}^t$ such that $\mu_k^t < \mu^t < \mu_{k+1}^t \Rightarrow \bar{U}^t(\bar{\mu}^t) = \bar{U}_{k+1}^t$ and $\mu^t = \bar{\mu}_k^t \Rightarrow \bar{U}^t(\bar{\mu}^t) \in [\bar{U}_{k+1}^t, \bar{U}_k^t]$, (3) the seller's continuation valuation at t is unique and is weakly increasing in μ^t, and (4) the expected discounted trades $X^t(\bar{\mu}_k^t)$, $\underline{X}^t(\bar{\mu}_k^t)$ and type $\bar{v}$'s

A mechanism is *ex ante* efficient if it maximizes

$$\lambda \mathrm{E}_c[M_s(c) - cX_s(c)] + (1 - \lambda)\mathrm{E}_v[vX_b(v) - M_b(v)]$$

over the set of feasible mechanisms, for some $\lambda \in [0, 1]$.

continuation payoff from t on are weakly decreasing in $\bar{\mu}_k^t$. Use a revealed preference argument.

Exercise 10.5** Generalize the Sobel-Takahashi infinite-horizon linear equilibrium of the sale model (subsection 10.2.3) to learning by doing (this makes sense only for the durable-good interpretation of the model). The buyers' valuations are uniformly distributed on $[0, 1]$. The horizon is infinite: $t = 0, 1, \ldots$. The seller's unit cost in a given period is constant with respect to the period's output and decreases proportionally with the volume of past sales: $c = c_0 v$, where v is the cutoff valuation at the beginning of the period.

(a) Look for a linear equilibrium in which the seller charges γv when the cutoff valuation is v (where $\gamma < 1$) and the buyers accept m when their valuation exceeds λm (where $\lambda > 1$). Show that

$$\lambda - 1 = \delta\lambda(1 - \gamma)$$

and

$$1 - 2\lambda\gamma + \delta\lambda^2\gamma^2 = c_0[\delta\lambda^2\gamma(2 - \delta\lambda\gamma) - \lambda].$$

(b) Show that only part of the Coase conjecture holds. When $\delta \to 1$, $\gamma \to c_0$ (price converges to marginal cost), and $\lambda \to 1/c_0$. Show that the market is not dissipated immediately (neither would it be if the seller could commit to an intertemporal price path). Compute the rate of sale per unit of time in the limit as the time between offers tends to 0.

(This exercise is from Olsen 1988.)

Exercise 10.6* Consider the sale model (in which the seller makes all the offers). Suppose that the buyer has cost $\varepsilon > 0$ of starting to bargain, which must be paid before the seller makes the first offer. Show that in a perfect Bayesian equilibrium bargaining never takes place, i.e., no buyer pays the entry fee. (Hint: Consider the lowest type who enters the bargaining process.) Is this conclusion robust to the extensive form? (Hint: Suppose that the buyer makes an offer every other period, and take ε to 0.)

Exercise 10.7** The bargaining games we have considered in this chapter have private values: The players do not care about their opponents' private information *per se* (but they try to learn about it to predict their opponents' future behavior). Some interesting bargaining games exhibit common values (see also exercise 10.8). Consider the following model of out-of-court settlement, due to Spier (1989).[41] Player 1, the plaintiff, is uninformed and makes settlement offers m^t at dates $t = 0, \ldots, T - 1$. Player 2, the defendant, knows the expected damage x that he will have to pay player 1 if the two

41. Other models of bargaining before litigation are offered by Bebchuk (1984), Nalebuff (1987), Ordover and Rubinstein (1986), Reinganum and Wilde (1986), and Spulber (1989).

parties do not settle out of court, which results in litigation at date T. The discount factor is δ, and the parties have fixed costs $c_1 > 0$ and $c_2 > 0$ of going to court. Thus, if the parties agree on damage m^t at date t, the payoffs are $u_1 = \delta^t m^t$ for the plaintiff and $u_2 = -\delta^t m^t$ for the defendant; if the defendant rejects all the plaintiff's offers $m^0, \ldots, m^{T-1}$, the payoffs are $u_1 = \delta^T(x - c_1)$ and $u_2 = \delta^T(-x - c_2)$. Assume $c_1 + c_2 < 1$.

(a) Suppose that $T = 1$ (single offer), and that x is uniformly distributed on $[0, 1]$. Show that the plaintiff offers $m^0 = \delta(1 - c_1)$ and that there is positive probability that the case is litigated.

(b) Assume that $x = \underline{x}$ with probability $\underline{p}$ and $x = \bar{x}$ with probability $\bar{p}$. Show that for any T there exists a unique perfect Bayesian equilibrium of the pretrial-bargaining game.

Exercise 10.8** Consider bargaining between a seller and a buyer over one unit of good, and suppose that the buyer's valuation is perfectly correlated with the seller's cost[42]: The cost c can take one of two values, $\underline{c}$ and $\bar{c}$, with equal probabilities, where $\underline{c} = 0$. The buyer's valuation is $v = 0$ if $c = \underline{c}$ and $\bar{v}$ if $c = \bar{c}$. Assume $\bar{v} > \bar{c}$, so that there exist potential gains from trade. The seller knows c, but the buyer does not (and therefore does not know his valuation). The interpretation is that the seller's cost is related to the quality of his good (the cost may be either a production cost or an opportunity cost). Both parties are risk neutral, and $\bar{v} < 2\bar{c}$.

Show that for *any* bargaining process in which each party can refuse to trade, and both parties have the same discount factor, there is no trade in equilibrium. Hint: Use the mechanism-design approach mentioned in section 10.4. Let $\underline{X}$ and $\bar{X}$ denote the expected discounted volumes of trade of the two types of sellers, and let $\underline{M}$ and $\overline{M}$ denote their expected discounted revenues. Write the seller's incentive-compatibility and individual-rationality constraints. Show that no trade ($\underline{X} = \bar{X} = \underline{M} = \overline{M} = 0$) is interim efficient, i.e., that no trade maximizes the buyer's expected utility subject to the above constraints. Conclude that equilibrium involves no trade.

Exercise 10.9** In subsection 10.2.2 it was mentioned that if the seller has the opportunity to switch and bargain with another buyer, he will in general quit the relationship in finite time before having realized all gains from trade with this buyer. The same phenomenon occurs when the seller can break off the negotiations and consume the good himself. Here is an example of the multiplicity of equilibria associated with an endogenous finite horizon.

Consider the sale model. The seller makes offers at $t = 0, 1, \ldots$. The seller's cost is 0. The seller decides in each period whether to consume the good (in which case he does not get to make an offer in this period or any

42. Evans (1989) and Vincent (1989) present such models.

subsequent period) or to make an offer to the buyer. Consumption gives him current utility w. The buyer's valuation, which is private information, takes one of three values, v_1, v_2, and v_3, with probabilities p_1, p_2, and p_3, where $0 \leq v_1 < w < v_2 < v_3$ and $p_1 + p_2 + p_3 = 1$. Assume that $w > (p_2 v_2 + \delta p_1 w)/(p_2 + p_1)$; that $p_3 \tilde{v} + \delta p_2 w < (p_3 + p_2) v_2 < p_3 v_3 + \delta p_2 w$, where $\tilde{v}$ satisfies $v_3 - \tilde{v} = \delta(v_3 - v_2)$; and that $(1 - p_1)v_2 \geq w(1 - \delta p_1)$. Show that the following are equilibria:

(a) Type v_3 accepts any offer $m^t \leq v_3$. The seller charges $m^t = v_3$ if $\min_{\tau<t} m^\tau > v_3$, and consumes otherwise. (Hence, the equilibrium path has an offer at v_3 at $t = 0$ followed by consumption at $t = 1$ if the offer is rejected.)

(b) Type v_3 accepts any offer $m^t \leq \tilde{v}$ (and rejects offers $m^t > \tilde{v}$ if $\min_{\tau<t} m^\tau > \tilde{v}$). The seller charges $m^t = v_2$ if $\min_{\tau<t} m^\tau > \tilde{v}$, and consumes otherwise. (Hence, the equilibrium path has one offer at v_2 at $t = 0$ followed by consumption at $t = 1$ if the offer is rejected.)

Explain the multiplicity of equilibria.

Exercise 10.10** The point of this exercise[43] is to show that part b of Coasian dynamics for the sale model (that the seller's price decreases over time) does not necessarily hold once the seller as well as the buyer has private information. Suppose there are two equally likely types of buyer, $\underline{v} < \bar{v}$, and two equally likely types of seller, $\underline{c} < \bar{c}$. Suppose that (in a one-period model) one type of seller is soft and the other tough: $\underline{v} - \underline{c} \geq (\bar{v} - \underline{c})/2$ and $\underline{v} - \bar{c} < (\bar{v} - \bar{c})/2$.

The bargaining process has two periods, and the seller makes both offers, as in subsection 10.2.2. Let $\tilde{m}$ and $\hat{m} > \tilde{m}$ be defined by $\tilde{m} = \bar{v} - \delta(\bar{v} - \underline{v})$ and $\hat{m} = \bar{v} - (\delta/2)(\bar{v} - \underline{v})$, where δ is the discount factor. Let $x \equiv (\underline{v} + \bar{c} - 2\underline{v})/(\bar{v} - \underline{v})$. Assume that, for $c = \underline{c}, \bar{c}$,

$$\frac{x}{2}(\hat{m} - c) + \delta(1 - x/2)(\underline{v} - c)$$

$$\geq \max\left(\tfrac{1}{2}(\tilde{m} - c) + \frac{\delta}{2}(\underline{v} - c),\ \underline{v} - c,\ \frac{\delta}{2}(\bar{v} - c)\right).^{44}$$

Show that there exists a pooling equilibrium in which both types of seller charge $\hat{m}$ at $t = 0$, which is rejected by type $\underline{v}$ and accepted with probability x by type $\bar{v}$. After rejection of $\hat{m}$, type $\underline{c}$ charges $\underline{v}$ and type $\bar{c}$ raises his price to $\bar{v}$ at $t = 1$. Comparing this equilibrium with the one-sided-asymmetric-information one of subsection 10.2.2, show that the seller gains from having private information about cost when he is of type $\underline{c}$ and loses when he is of type $\bar{c}$.

43. Drawn from Fudenberg and Tirole 1983.
44. For instance, $\{\underline{v} = 1, \bar{v} = 2, \underline{c} = 0, \bar{c} = 1, \delta = 1\}$ satisfies these assumptions.

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V ADVANCED TOPICS

11 More Equilibrium Refinements: Stability, Forward Induction, and Iterated Weak Dominance

A succession of authors have argued for equilibrium refinements that capture some aspect of what is loosely called "forward induction." Forward induction plays a key role in the equilibrium refinements developed for signaling games, and it underlies the nonequilibrium notion of the iterated deletion of weakly dominated strategies. This chapter discusses these equilibrium refinements in some detail, and then presents the "burning money" game as a striking example of the power of iterated weak dominance. The chapter concludes by examining the argument that equilibrium refinements are too strong because they are not robust to certain changes in the information players have about one another's payoffs.

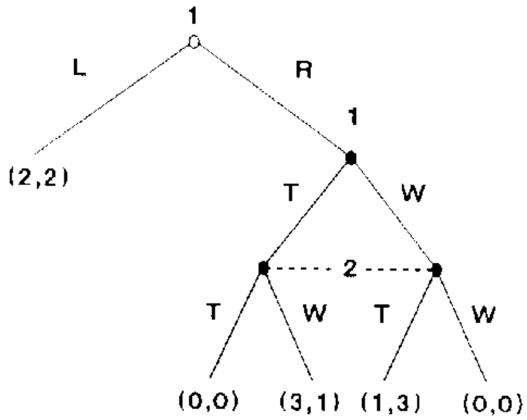
The idea of forward induction is that when an off-the-equilibrium-path information set is reached, the player on move at this information set should not suppose that it was reached by "accident" and then use the equilibrium strategies to look "down" the tree as in backward induction. Instead, the player should take into account what could have happened but did not in forming his beliefs about the nodes in the information set and about what is likely to happen next. Thus, players should reason "forward" from the beginning of the tree as well as backward from its end. For example, if you expected that with probability 1 your opponent in a bargaining game would accept your offer of $10 yet she refused it, and if you knew that her delay costs were positive, you might conclude that she expects to receive a higher offer in the future.[1] This idea is clearly at odds with the interpretation that deviations from equilibrium play are due to unintended errors, for if the refusal was an unintended error it would contain no information about your opponent's likely future play.[2]

11.1 Strategic Stability†††

The idea of forward induction is one of Kohlberg and Mertens' key reasons for introducing their notion of strategic stability. A second motivation is the idea that solution concepts should be invariant to "inessential" transformations of the extensive form.

As a first example of forward induction, consider the game in figure 11.1a, which is from van Damme 1989. Here player 1 has the choice of playing L, which ends the game, or playing R, in which case he and player 2 play a "battle of the sexes" subgame in which the players simultaneously choose between T ("tough") and W ("weak"). This subgame has three Nash equilibria: (T, W), (W, T), and a mixed-strategy equilibrium in which each player plays W with probability $\frac{1}{4}$.

1. Dekel (1990) studies the implications of strategic stability in a two-period simultaneous-move bargaining game.
2. For a similar reason, forward induction conflicts with the concept of Markov equilibrium discussed in chapter 13.



a

	T	W
L	2,2	2,2
RT	0,0	3,1
RW	1,3	0,0

b

Figure 11.1

The profile (LW, T) is a subgame-perfect equilibrium of this game. It is clearly a sequential-equilibrium profile, and it is also trembling-hand perfect in the agent strategic form. However, the equilibrium is not consistent with the following "forward induction" argument: There is no reason for player 1 to play RW, since this gives a payoff of at most 1 while L gives a payoff of 2. However, playing R followed by T would be rational if player 1 expected player 2 to play W. Hence, the argument goes, player 2 should expect that if player 1 plays R he will then play T and not W, so player 2 should play W. And player 1 should foresee this reasoning of player 2, so player 1 should play R instead of L. (This argument does not apply if we suppose that player 1 meant to play L and played R by "mistake," as is implicitly supposed by sequential equilibrium and perfect equilibrium.)

Figure 11.1b gives the reduced strategic form of this game. Note that the strategy RW is strictly dominated by L, and that if RW is removed the only trembling-hand perfect equilibrium is (RT, W), since if player 1 plays RT with positive probability (and RW is not included) then player 2 strictly prefers W to T. Note also that (L, T) is trembling-hand perfect in the strategic form with RW included: Even though RW is strictly dominated, player 1 might play it by "mistake," and this leads player 2 to play T if RW is as likely as RT. Note finally that if player 2 plays T, RW is better for player 1 than RT, so (L, T) is a proper equilibrium.

The link in similar examples between forward induction and removing strictly dominated strategies leads Kohlberg and Mertens to formulate the following requirement for their solution concept which they call "strategic stability." Unlike previous solution concepts, strategic stability is a set-valued concept. That is, instead of each solution being a single equilibrium profile, and the set of solutions being a set of equilibria, each solution is itself a "strategically stable set," and the set of solutions is the set of all such sets. As we will see, the reason Kohlberg and Mertens use a set-valued concept is that the conditions that they wish to impose, taken together, cannot be satisfied by a point-valued concept.

ID (Iterated Dominance) Every strategically stable set of equilibria of a game G should contain a strategically stable set of equilibria of any game G′ obtained from G by deleting strictly dominated strategies.

Note that the example of figure 11.1 shows that proper equilibria do not satisfy condition ID: (L, T) is a proper equilibrium, but it is no longer proper once RW is deleted. Note also that condition ID indeed implies *iterated* strict dominance: If G′ is obtained by deleting dominated strategies from G, and G″ by deleting dominated strategies from G′, then the stable set of G contains a stable set of G′, which in turn contains a stable set of G″.

Although this definition does select the (RT, W) equilibrium in figure 11.1, it may not be clear whether it captures all that one might mean by "forward induction." We will return to this point after we have developed the Kohlberg-Mertens definition of strategic stability. Next, Kohlberg and Mertens wish their solution concept to satisfy the following condition:

A (Admissibility) No mixed strategy in a strategically stable set can assign positive probability to any pure strategy that is weakly dominated.

Figure 11.2 shows that conditions ID and A are inconsistent with the existence of a point-valued solution concept. Here D is strictly dominated, so condition ID requires that the strategically stable set contain a stable set for the game illustrated in figure 11.3a. In that game L weakly dominates R for player 2, so condition A requires that the unique solution be (U, L). But in the original game M is strictly dominated as well, and deleting M instead of D yields the game in figure 11.3b, where (U, R) must be the unique solution from condition A. Hence, the solution to the original game must contain both (U, L) and (U, R), and so it cannot be single valued. Using this example, we can also see that it is not possible to strengthen condition ID to the requirement that a stable set be contained in a stable set of a game obtained by adding dominated strategies: In the game where player 1's only strategy is U, both (U, L) and (U, R) are stable; but if we then add the dominated strategy D, the unique stable solution is (U, R). An alternate way of saying this is that condition ID allows for the possibility that

	L	R
U	3,2	2,2
M	1,1	0,0
D	0,0	1,1

Figure 11.2

a

	L	R
U	3,2	2,2
M	1,1	0,0

b

	L	R
U	3,2	2,2
D	0,0	1,1

Figure 11.3

deleting dominated strategies can make stable profiles that were not stable previously.

Kohlberg and Mertens were the first to propose making sets of equilibria the objects of a theory of equilibrium refinements. Having a set of equilibria as the prediction of a theory is particularly troubling if different equilibria in the set involve different play along the equilibrium path. (Even if all the equilibria in the prediction agree on the path, one would like to know just what off-path play the players expect to see, but this concern may be less troubling.)

Here, Kohlberg and Mertens make use of the following result of Kreps and Wilson (see chapter 8 above):

Theorem 11.1 (Kreps and Wilson 1982) In a fixed tree, for generic assignments of payoffs to terminal nodes, the set of Nash-equilibrium probability distributions over terminal nodes is finite.

Since the distribution over terminal nodes is a continuous function of the strategy profile, when there is only a finite number of equilibrium distributions, every equilibrium in the same connected component[3] must have the same probability distribution over endpoints, and hence the same play at every information set that is reached with positive probability. This avoids one potential drawback of using a set-valued solution concept.

The third main condition Kohlberg and Mertens want their solution concept to satisfy is invariance to certain transformations of the extensive

3. A topological space X is said to be connected if there do not exist two nonempty, disjoint open sets O_1 and O_2 such that $X \subset (O_1 \cup O_2)$, and $X \cap O_1$ and $X \cap O_2$ are both nonempty. The intuitive idea of a connected set is that one should be able to draw a path linking any two points of the set that itself lies in the set.

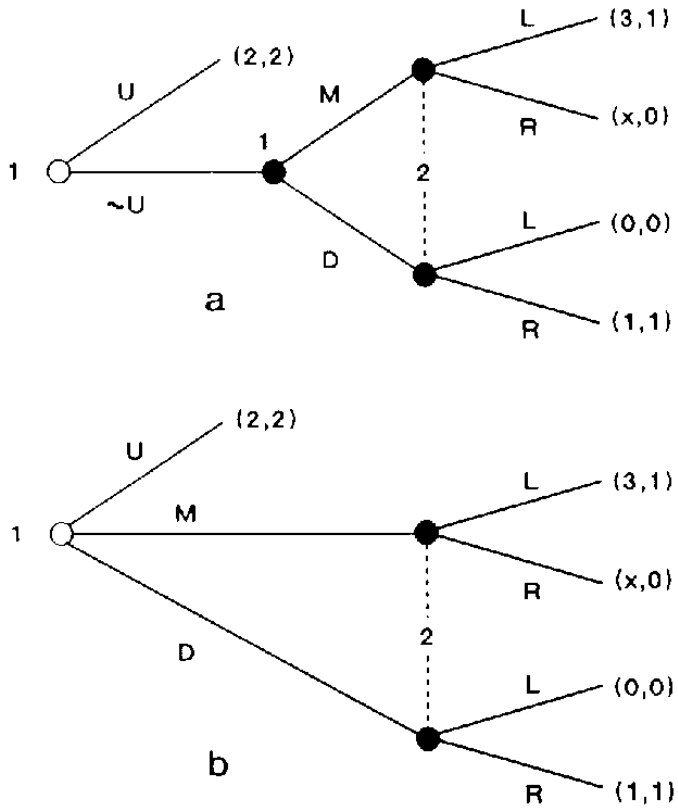


Figure 11.4

form. Consider the games illustrated in figure 11.4, where we set x to lie between 1 and 2. Kohlberg and Mertens argue that figures 11.4a and 11.4b are merely two different representations of the same game, because "the transformed tree is merely a different representation of the same decision problem." In a one-player game, the two alternate representations of player 1's decision are clearly equivalent: The choice of a best element from the set (U, M, D) is exactly the same as the two-stage choice where player 1 first decides whether he would choose M or D if he couldn't go U, and then decides whether he prefers U or the better of M and D.

Although the two games are arguably equivalent, the concept of sequential equilibrium gives different solutions for them, just as in the case of figures 8.6a and 8.6b. In figure 11.4b, (U, R) is sequential but not proper for $x \in (1, 2)$. In figure 11.4a, it is not even subgame perfect for player 1 to choose U: Player 1's second information set begins a proper subgame in which M strictly dominates D, so the unique Nash equilibrium in this subgame is (M, L). Given that player 2 will play L, player 1 will not play U, so sequential equilibrium does not satisfy the invariance property.

As we suggest in chapter 8, the analogy with decision theory is attractive but not fully convincing. Games and decisions differ in one key respect: Probability-0 events are both exogenous and irrelevant in decision problems, whereas what *would* happen if a player played differently in a game is both important and endogenously determined. This point is related to a possible difficulty with the Kohlberg-Mertens position that "the game under consideration fully describes the real situation," so that in particular "any probabilities of error have been modeled in the game tree." As we noted when we introduced backward induction, one can argue to the contrary that this "classical" point of view is not compatible with an attempt to refine Nash equilibrium by restricting the actions players take at information sets that they expect will never be reached, and that the "right" way to develop equilibrium refinements is in the context of a complete theory that provides an explanation (or several) for every conceivable sequence of observations. From the complete-theory viewpoint, every extensive form that does not explain every possible observation is simply a simplified representation of a more complex game in which all observations do have positive probability.

One such complete theory (not necessarily our favorite) is precisely the "trembles" story that players make "errors" with small probability. When this theory is used, the extensive forms in the two figures represent two distinct situations. In figure 11.4b, where player 1 chooses which of three actions to take, he plays either M or D only by mistake, and if the relative probability of mistakes is arbitrary (as in trembling-hand perfection) he may well play D more often than M. In figure 11.4a, if player 1 mistakenly fails to play U he has a chance to reconsider and choose which alternative action he most prefers, and so M will be much more likely than D.

This is not to say that one should be completely indifferent to the argument that a "good" theory would make the same predictions in the two cases. In fact, this example is troubling for economic applications of trembling-hand perfection, because the analyst will seldom know which of the two extensive forms is more descriptive. (One solution concept we develop at the end of this chapter, "c-perfection," allows (U, R) in both extensive forms.) Rather, the point is that arguing about which extensive forms are equivalent before deciding on a (complete) theory of how players behave may be putting the cart before the horse.

With these caveats in mind, we now state the invariance condition.

I (Invariance) The set of strategically stable equilibria should depend only on the reduced strategic form of the game; i.e., all extensive forms with the same reduced strategic form should have the same set of stable equilibria.

(Recall our definitions of the reduced strategic form in chapter 3, which identify "equivalent" strategic-form strategies. Kohlberg and Mertens use the stronger of the two definitions, which requires that a pure strategy s_i be deleted if there is a mixed strategy σ_i with support excluding s_i such that, for all s_{-i} and all players j, $u_j(s_i, s_{-i}) = u_j(\sigma_i, s_{-i})$.)

Kohlberg and Mertens appeal to the results of Thompson (1952) and Dalkey (1953) to argue that all extensive forms with the same reduced strategic form are equivalent. Those authors showed that if two extensive forms have the same reduced strategic form up to identifying equivalent pure strategies, then one of them may be transformed into the other by a sequence of four kinds of transformations:

coalescing and expanding information sets (as in the two extensive forms we considered),

interchange of simultaneous moves (recall that there are two ways of representing a two-player simultaneous-move game),

adding moves that are not revealed to the players and have no influence on payoffs, and

inflation and deflation of information sets (which we do not define, because it is irrelevant in perfect-recall games).

The Kohlberg-Mertens paper has renewed interest in the question of which extensive forms ought to be regarded as equivalent by a solution concept; see Elmes and Reny 1988.

In the 1986 paper, Kohlberg and Mertens go on to develop three increasingly restrictive definitions of sets of equilibria; Mertens (1988) has proposed further definitions. We will focus on the third and most restrictive definition of the 1986 paper, that of a stable set of equilibria. This concept requires that, for any trembles, there exists an equilibrium near the set.

Definition 11.1 (Kohlberg and Mertens 1986) A closed set S of Nash equilibria is *stable* if it is minimal[4] with respect to the property that for each $\eta > 0$ there exists some $\varepsilon' > 0$ such that, for any $\varepsilon < \varepsilon'$ and any numbers

$$\{\varepsilon(s_i)\}_{\substack{i \in \mathscr{I} \\ s_i \in S_i}} \text{ with } 0 < \varepsilon(s_i) \leq \varepsilon,$$

the game where player i is constrained to play each s_i with probability at least $\varepsilon(s_i)$ has an equilibrium σ^ε that is within η (in the space of strategies) of some equilibrium in the set S. If every element of a stable component yields the same probability distribution over endpoints, this distribution is a *stable outcome*.

Comments

(1) From the standard upper-hemi-continuity argument, the set of all Nash equilibria has the property of containing an element that is close to an equilibrium of every perturbed game. Definition 11.1 gains its force from the minimality requirement.

(2) The perturbed game referred to in this definition is the same as that used to define trembling-hand perfection in the strategic form (see definition 8.5A). Thus, all equilibria in a stable set are trembling-hand perfect in the strategic form.

4. A set S is minimal with respect to property P if there does not exist a subset $S' \subset S$ such that S' satisfies P. (The symbol $\subset$ denotes strict inclusion.)

The key difference between trembling-hand perfection in the strategic form and stability is that perfection requires only that there exist a single sequence of perturbed games whose equilibria converge to σ, but a stable set must contain a limit point of the equilibria for every perturbed game. This is related to the fact that stability is defined in terms of sets and not single equilibria, as there may not be a single equilibrium that "works" for every allowed perturbation. If, however, there is a single equilibrium σ such that σ_i is a best response to every sequence $\sigma^n{}_i \to \sigma_{-i}$, then that equilibrium is stable as a one-point set. Kohlberg (1981) calls such equilibria "truly perfect."[5] The game in figure 11.2 has no truly perfect equilibrium: In any perfect equilibrium σ, player 1 plays U with probability 1. If $\sigma_1^n(\mathrm{M}) > \sigma_1^n(\mathrm{D})$, player 2's best response is L; if $\sigma_1^n(\mathrm{M}) < \sigma_1^n(\mathrm{D})$, player 2's best response is R. (Although the payoffs in this strategic form are not generic—i.e., they involve ties—in the extensive-form example of figure 11.5 no truly perfect equilibrium exists for payoffs in the neighborhood of those payoffs that are shown.)

In the case of figure 11.1b, the reason stability eliminates the equilibrium (L, T) is that stability considers all perturbations. If player 1 trembles more onto RT than RW, player 2 responds with W, and this induces player 1 to deviate from the original equilibrium. The equilibrium (RT, W), in contrast, is truly perfect: No small trembles by player 1 will change player 2's optimal response.

(3) It is important that stability considers perturbations of the strategic form and not the agent-strategic form. In the agent-strategic form, a player's "mistakes" at different information sets are independent, and even considering all such independent trembles would not capture the forward-induction argument of figure 11.1a. In the agent-strategic form corresponding to this game, where player 1′ chooses L or R, and player 1″ chooses T or W, the profile where 1′ plays L, 1″ plays W, and 2 plays T is perfect for *any* independent trembles, since with independent trembles 1″ is very likely to play W whether 1′ plays L or R. Forward induction corresponds to the "correlated tremble" where, if 1′ trembles onto R, 1″ is more likely to play T than W. Stability has forward-induction-like properties because it does consider these correlated trembles.

Definition 11.2 A strategy profile s is a *strict equilibrium* if each s_i is a strict best response to s_{-i} in the *reduced* strategic form; that is, $u_i(s_i, s_{-i})$ is strictly greater than $u_i(s_i', s_{-i})$ for $s_i' \neq s_i$.

Clearly any strict equilibrium is stable as a one-point set. But note that in order for s to be a strict equilibrium, it must assign positive probability to every information set of every player, since a player is

5. Okada (1981) independently introduced this concept, which he called "strictly perfect."

indifferent between actions at an information set whose probability is 0. This implies that strict equilibria are less likely to exist in dynamic games than in static games. Note also that mixed-strategy equilibria cannot be strict, and that a totally mixed equilibrium (where every strategy has a positive probability) is stable as a one-point set by definition. (The minimum probability constraints are not binding once they are less than the minimum weight the equilibrium gives any pure strategy.)

Theorem 11.2 (Kohlberg and Mertens 1986) There exists a stable set that is contained in a single connected component of the set of Nash equilibria, and every tree with generic payoffs has a stable payoff (i.e., a payoff that obtains for every equilibrium in a stable set). A stable set contains a stable set of any game obtained by deletion of a weakly dominated strategy, and a stable set contains a stable set of any game obtained by deleting any strategy that is not a weak best response to any of the opponents' strategy profiles in the set.

This last property, called "*never a weak best response*" (NWBR), embodies a version of forward induction not implied by the elimination of weakly dominated strategies: One can prune undominated strategies if they are not best responses to any of the opponents' strategy profiles in the component under consideration, even though they may be best responses to strategies outside the equilibria in the component. To see the force of this property, consider the game illustrated in figure 11.5. In this modification of the Kohlberg-Mertens example of figure 11.4, x has been set equal to $\frac{1}{4}$ and the payoffs (2, 2) if player 1 goes U have been replaced by a simultaneous-move coordination game, one of whose equilibria has payoffs (2, 2). Now the strategy D is not dominated for player 1, and neither is UA or UB. Thus, there are no weakly dominated strategies to be pruned. Yet,

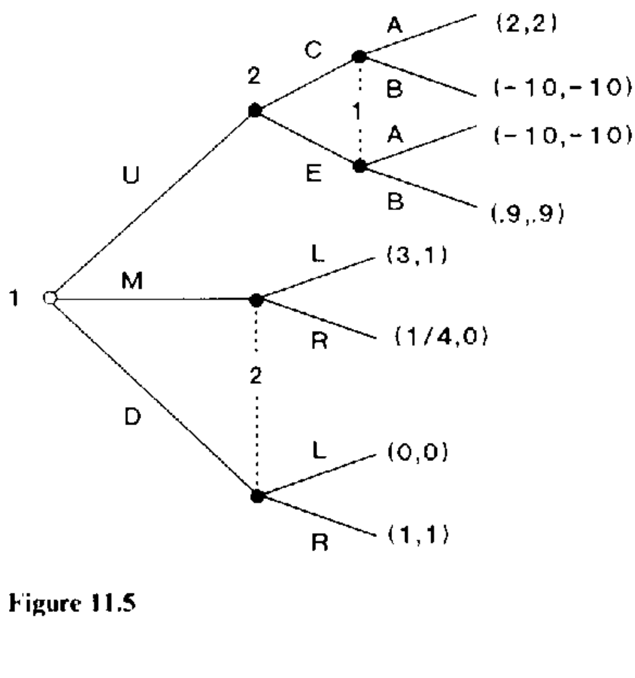


Figure 11.5

in any component of equilibria with outcome (UA, C), E is not a weak best response and can be pruned. Once E is pruned, D is weakly dominated and (2, 2) is not a stable payoff. (Note that (0.9, 0.9) is *not* removed by NWBR, as D is not dominated after C is deleted. One can show that (0.9, 0.9) is a stable payoff.)

The NWBR property is frequently a useful way to show that some components of equilibria are not stable. For example, the refinements for signaling games that we develop below are shown to be weaker than stability by showing that all components they rule out can be ruled out by successive applications of NWBR. Exercise 11.5 gives another example.

One troubling property of stability, established by an example due to Gul, is that stable sets need not be connected and need not contain a sequential equilibrium. (Exercise 11.2 analyzes Gul's example.) This is one reason why alternative definitions of stability have been proposed. Hillas (1990) replaces the trembles in the definition of stability by perturbations of the player's best-response correspondences and says that a set is stable if it is the minimal closed set such that every profile of "nearby" best-response correspondences has a fixed point near the set. (Obviously this definition requires a topology on the space of best-response correspondences.) Hillas shows that stable sets under his definition are connected, and that they satisfy all of Kohlberg and Mertens' other conditions. Mertens (1989, 1990) retains the idea of using trembles as the perturbations but adds a topological requirement on the correspondence from perturbations to the stable set. This alternative definition also satisfies connectedness.

It is also interesting to note that stability and NWBR do not capture all that one might mean by "forward induction." We discuss van Damme's alternative definition of forward induction in section 11.3.

11.2 Signaling Games†††

Kohlberg and Mertens motivate their stability concept by referring to the properties they would like it to have and to its mathematical structure; they do not offer a behavioral argument that players "should" be expected to play as stability predicts. In papers on related equilibrium refinements, Cho and Kreps (1987) and Banks and Sobel (1987) do try to provide at least a heuristic behavioral foundation for stability-like ideas. One goal of these papers is to better understand stability through examining its implications in a simple class of games: the class of signaling games we introduced in chapter 8. (Recall the definition of a signaling game: The informed player, player 1, moves first and chooses an action a_1. Player 2 observes a_1 but not player 1's type θ, and chooses a_2, and then the game ends.)

A second purpose is to develop alternative equilibrium refinements that are weaker and easier to apply. A common theme of these refinements is to take as a behavioral axiom one aspect of the NWBR property described above: that of replacing the equilibrium path by its expected payoff. That is, the solutions suppose that the players are quite sure of the way their opponents will play along the equilibrium path, but that the players are less sure of the off-path play. Thus, if player 1 deviates from the equilibrium, player 2 tries to "explain" the deviation by asking which types of player 1 could do better by making this deviation, if it is met with some response that is "reasonable" in senses to be defined, than by sticking with the equilibrium strategy followed by the equilibrium response.

Before proceeding to the equilibrium concepts used in the above-mentioned papers, we first state two preliminary results which help relate the solution concepts in this section to stability.

Fact In a signaling game, every stable set contains only sequential equilibria.

We observed above that this need not be true for general games. Signaling games have the special property that all strategic-form perfect equilibria are sequential (because each player moves only once, the agent-strategic form coincides with the strategic form), and stable sets contain only strategic-form perfect equilibria by definition. This fact means that if we begin with a stable set, and then, using NWBR, delete a strategy in which type θ plays action a_1, the resulting set must contain a stable component of the reduced game, and hence must contain a sequential equilibrium where beliefs assign probability 0 to type θ following action a_1. Thus, we can infer that sequential equilibria consistent with the deletions exist from the existence of stable components.

The idea of the Cho-Kreps paper is to use the concept of "equilibrium dominance" to argue that certain types should not be expected to use certain strategies. In contrast to stability, which follows Selten in looking at strategies and "trembles" in the strategic form, equilibrium dominance proceeds in the style of sequential equilibrium and develops a further restriction on the beliefs allowed in an extensive-form game. Recall that in a signaling game the only beliefs that aren't pinned down by an equilibrium are those of the receiver when he sees a "message" that has probability 0 according to the equilibrium strategies, and that, since the sender's message is perfectly observed, these beliefs are simply probability distributions over the sender's type. Fix an equilibrium outcome, and let $u_1^*(\theta)$ be type θ's expected payoff.

Definition 11.3 (Equilibrium Dominance) Action a_1 can be eliminated for type θ by equilibrium dominance if

$$u_1^*(\theta) > \max_{a_2} u_1(a_1, a_2, \theta).$$

Note that this test can also be applied to a component of equilibria that all have the same equilibrium payoff. In particular, in generic signaling games, if at a fixed equilibrium the action a_1 is eliminated for type θ by equilibrium dominance, then at the connected component containing this equilibrium, all strategies σ_1 with $\sigma_1(a_1|\theta) > 0$ are eliminated by NWBR. It may seem reasonable to require that player 2 place probability 0 on type θ conditional on a_1 being played, and this restriction may reduce the set of a_2s which are best responses to a_1. Moreover, this restriction on the beliefs, if common knowledge, will lead to further restrictions on which types can reasonably be thought to choose strategy a_1, for now we can eliminate strategy a_1 for type θ' if $u_1^*(\theta')$, the equilibrium payoff of type θ', exceeds the best he could get with a_1, given that player 2 will not play a response that is only justified by beliefs that assign positive probability to type θ. This sort of argument leads to what Cho and Kreps call the "Intuitive Criterion" and the "Equilibrium Domination Test."

Defining these concepts requires some additional notation. For a non-empty subset T of Θ, let $\mathrm{BR}(T, a_1)$ be the set of all pure-strategy best responses for player 2 to action a_1 for beliefs $\mu(\cdot|a_1)$ such that $\mu(T|a_1) = 1$:

$$\mathrm{BR}(T, a_1) = \bigcup_{\mu:\mu(T|a_1)=1} \mathrm{BR}(\mu, a_1),$$

where

$$\mathrm{BR}(\mu, a_1) = \arg\max_{a_2} \sum_{\theta\in\Theta} \mu(\theta|a_1) u_2(a_1, a_2, \theta).$$

Let $\mathrm{MBR}(\mu, a_1)$ be the set of mixed best responses to a_1 given μ, that is, the set of all probability distributions over $\mathrm{BR}(\mu, a_1)$. Now let

$$\mathrm{MBR}(T, a_1) = \bigcup_{\mu:\mu(T|a_1)=1} \mathrm{MBR}(\mu, a_1).$$

(For $T = \varnothing$, set $\mathrm{BR}(\varnothing, a_1) = \mathrm{BR}(\Theta, a_1)$.) This is the set of all mixed best responses to some beliefs with support in T. It will be important in the following discussion that $\mathrm{MBR}(T, a_1)$ need not include every probability distribution over $\mathrm{BR}(T, a_1)$. As in figure 11.7 below, it may be that action a_2' is a best response for some beliefs about player 1, and a_2'' is a best response for other beliefs, but for *no* beliefs is it a best response to randomize between a_2' and a_2''. Let $T\backslash W$ denote the set-theoretic difference between T and W.

Definition 11.4 (Intuitive Criterion) Fix a vector of equilibrium payoffs $u_1^*(\cdot)$ for the sender. For each strategy a_1, let $J(a_1)$ be the set of all θ such that

$$u_1^*(\theta) > \max_{a_2\in \mathrm{BR}(\Theta, a_1)} u_1(a_1, a_2, \theta).$$

If for some a_1 there exists a $\theta' \in \Theta$ such that

$$u_1^*(\theta') < \min_{a_2\in \mathrm{BR}(\Theta\backslash J(a_1), a_1)} u_1(a_1, a_2, \theta'),$$

then the equilibrium fails the Intuitive Criterion.

In words, $J(a_1)$ is the set of types who get less than their equilibrium payoff by choosing a_1, provided the receiver plays an undominated strategy. The equilibrium fails the Intuitive Criterion if there exists a type who would necessarily do better by choosing a_1 than in equilibrium as long as the receiver's beliefs assign probability 0 to types in $J(a_1)$.

Cho and Kreps discuss the idea of iterating this criterion, which leads to a concept we call the Iterated Intuitive Criterion:

Definition 11.5 (Iterated Intuitive Criterion) Fix a vector of equilibrium payoffs $u_1^*(\cdot)$ for the sender. Set $\Theta^0(a_1) = \Theta$ for all a_1. For each strategy a_1 and subset $\Theta^k(a_1)$ of types, let $J(\Theta^k(a_1), a_1)$ be the set of all $\theta \in \Theta^k(a_1)$ such that

$$\text{(i)} \quad u_1^*(\theta) > \max_{a_2 \in \mathrm{BR}(\Theta^k(a_1), a_1)} u_1(a_1, a_2, \theta).$$

$J(\Theta^k(a_1), a_1)$ are the "types who are deleted for strategy a_1 at the kth round of iteration." Set

$$\Theta^{k+1}(a_1) = \Theta^k(a_1) \setminus J(\Theta^k(a_1), a_1).$$

This is the set of types who "could reasonably" choose strategy a_1 when they are certain of their equilibrium payoff, and who believe that player 2 will play some best response to beliefs concentrated on $\Theta^k(a_1)$. If for some a_1 there is a $\theta' \in \Theta^{k+1}(a_1)$ such that

$$\text{(ii)} \quad u_1^*(\theta') < \min_{a_2 \in \mathrm{BR}(\Theta^{k+1}(a_1), a_1)} u_1(a_1, a_2, \theta'),$$

then the equilibrium is said to fail the $(k+1)$st round of the Iterated Intuitive Criterion (note that if the equilibrium fails in round 1 then it fails the Intuitive Criterion). The equilibrium fails the Iterated Intuitive Criterion if it fails the $(k+1)$st round for some k.

Cho and Kreps also offer a modified version of the Iterated Intuitive Criterion. The equilibrium-domination test is defined by replacing condition ii of definition 11.5 with the following:

(ii′) If for some a_1 and all $a_2 \in \mathrm{BR}(\Theta^{k+1}(a_1), a_1)$ there is a $\theta' \in \Theta^{k+1}(a_1)$ such that

$$u_1^*(\theta') < u_1(a_1, a_2, \theta'),$$

then the equilibrium fails the $(k+1)$st round of the equilibrium domination test.

The difference between the Intuitive Criterion and the equilibrium-domination test is in the order of quantifiers: Condition ii asks that there be a single type θ' who prefers a_1 for all responses in $\mathrm{BR}(\Theta^{k+1}(a_1), a_1)$; condition ii′ asks only that for each response in $\mathrm{BR}(\Theta^{k+1}(a_1), a_1)$ there be some type who prefers to deviate.

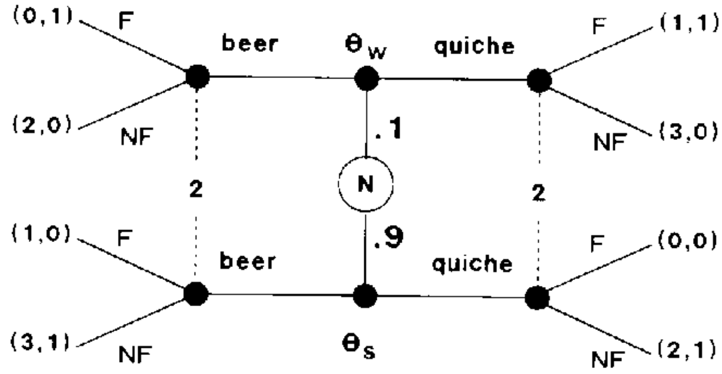


Figure 11.6

Cho and Kreps illustrate the Intuitive Criterion with the beer-quiche game, displayed in figure 11.6. Here, player 1 has two types: θ_w, who is "weak," and θ_s, who is "surly." The prior probability of weak is 0.1. Player 2 prefers to fight if she believes there is probability over $\frac{1}{2}$ that player 1 is weak, but she does not observe player 1's type. However, before deciding whether to fight, player 2 (who is very nosy) observes what player 1 has for breakfast. Player 1 has only two possible breakfasts, "beer" and "quiche"; the surly type prefers beer and the weak type likes quiche. However, regardless of their dietary preferences, either type would have either breakfast in order to avoid being fought. This game has two pooling equilibria, one in which both types have beer and another where they both have quiche; in both cases player 2 must fight with some probability when observing the out-of-equilibrium breakfast in order to make the mismatched type endure gastronomic horror. To support these outcomes as sequential equilibria, we specify that player 2's out-of-equilibrium beliefs are that if the unexpected breakfast is observed, there is probability at least $\frac{1}{2}$ that player 1 is weak.

Cho and Kreps argue that the pooling equilibrium where both types have quiche is not reasonable, and indeed it is eliminated by their Intuitive Criterion: In this equilibrium, the weak type is getting its highest possible payoff, and, so long as it is "convinced" that its equilibrium action will give the equilibrium payoff, it has no incentive to switch to drinking beer, regardless of how it expects player 2 to respond to beer. Once type θ_w is removed for a_1 = beer, the set $\Theta^1(\text{beer}) = \Theta \backslash J(\Theta, \text{beer})$ is simply $\{\theta_s\}$, and player 2's unique best response to beliefs concentrated on θ_s is to not fight, which gives θ_s more than its equilibrium payoff.

Cho and Kreps offer the following heuristic justification of this process of elimination: Suppose that player 1 has the (unmodeled) chance to make a speech to player 2 at the same time he eats his breakfast. Then θ_s could say, "I'm having beer, and you should infer from this that I'm surly, for so long as it is common knowledge that you will not fight if I eat quiche, I would have no incentive to drink beer and make this speech if I were type θ_w."

This heuristic is suggestive but only partially compelling. One would prefer the communication stage to be explicitly modeled into the game, but then one runs into the difficulty that an equilibrium would have to specify how player 2 would respond to each possible speech, and also how player 2 would respond to the *absence* of a speech that would have been made had player 1's type been different. This implies that if only type θ_s is expected to make the speech, and the speech is not made, player 2 should infer that player 1 is type θ_w, which in turn would reduce type θ_w's incentive to remain silent. (Cho and Kreps attribute this reasoning to Stiglitz.) Once again, the problem arises from trying to refine the set of equilibria without specifying a complete theory of play, so that the discussion of beliefs involves considering counterfactuals.

The "both quiche" equilibrium can also be eliminated by applying iterated weak dominance to the strategic form of the corresponding two-player game, where the two types of player 1 are viewed as different information sets of the same player. (The first step is to show that the strategy "beer if weak, quiche if surly" is dominated. Any strategy for player 2 that makes beer optimal when player 1 is weak makes it optimal when he is surly. This kind of monotonicity property is discussed in chapter 6.) As we discuss in chapter 6 in the context of iterated strict dominance, whether the "right" strategic form has two or three players depends on whether we wish to assume that the different types of each player necessarily have the same beliefs about their opponents' strategies.[6]

Finally, Cho and Kreps study the implications of equilibrium refinements in Spence's model of job-market signaling. Cho and Kreps prove that when there are only two types, the only equilibrium not rejected by the Intuitive Criterion is the "Riley outcome," i.e., the separating equilibrium with the least amount of inefficient signaling. With more than two types, selecting the Riley outcome requires the stronger concept of "universal divinity," developed by Banks and Sobel; thus, we will introduce universal divinity before studying the Spence signaling game.

Universal divinity is defined with an iterative process like that in definition 11.5 above; the difference is that more type-strategy pairs may be deleted at each round. As above, we first fix an equilibrium, and let $u_1^*(\theta)$ be the equilibrium payoff of type θ. Define $D(\theta, T, a_1)$ to be the set of mixed-strategy best responses α_2 to action a_1 and beliefs concentrated on T that make type θ strictly prefer a_1 to his equilibrium strategy,[7]

6. Cho and Kreps note that using the "*ex ante*" strategic form that treats the different types as the same player has an additional implication: When one is computing the set of proper equilibria, a tremble by a low-probability type θ' has a low *ex ante* cost, and so is assigned a very small probability, even though the cost of that tremble conditional on the occurrence of type θ' may be quite high.

7. We use α_2 for a probability distribution over A_2 and $\sigma_2(\cdot|\cdot)$ for player 2's overall strategy. Thus, for a given a_1, $\sigma_2(\cdot|a_1)$ is some $\alpha_2 \in \Delta(A_2(a_1))$.

$$D(\theta, T, a_1) = \bigcup_{\mu:\mu(T|a_1)=1} \{\alpha_2 \in \mathrm{MBR}(\mu, a_1) \text{ s.t. } u_1^*(\theta) < u_1(a_1, \alpha_2, \theta)\},$$

and let $D^0(\theta, T, a_1)$ be the set of mixed best responses that make type θ exactly indifferent.[8]

Definition 11.6 A type θ is deleted for strategy a_1 under criterion D1 if there is a θ' such that

$$\{D(\theta, \Theta, a_1) \cup D^0(\theta, \Theta, a_1)\} \subset D(\theta', \Theta, a_1).$$

A type θ is deleted for strategy a_1 under criterion D2 if

$$\{D(\theta, \Theta, a_1) \cup D^0(\theta, \Theta, a_1)\} \subset \bigcup_{\theta' \neq \theta} D(\theta', \Theta, a_1).$$

(The symbol $\subset$ denotes strict inclusion.)

Obviously, having deleted a type for strategy a_1 under either of these conditions, we can impose further restrictions on player 2's responses to a_1; this leads to iterated versions of the two criteria that exactly parallel those of definition 11.5. Banks and Sobel call the iterated version of D2 universal divinity; their "divine equilibrium" results from the iterated application of a criterion slightly weaker than D1.[9]

Criterion D1 says that if the set of player 2's responses that make type θ willing to deviate to a_1 is strictly smaller than the set of responses that make type θ' willing to deviate, then player 2 should believe that type θ' is infinitely more likely to deviate to a_1 than type θ is. This is a strengthening of the Intuitive Criterion, as whenever type θ is removed by the Intuitive Criterion then the sets $D(\theta, \Theta, a_1)$ and $D^0(\theta, \Theta, a_1)$ are empty. Criterion D2 has roughly the same relation to D1 as the equilibrium domination test has to the Intuitive Criterion, as it replaces a single type θ' that deletes θ with the union over all other types.

Note that the "speeches" of Cho and Kreps do not serve to motivate D1 and D2, and Banks and Sobel do not provide a behavioral defense of these criteria. Note in particular that D1 and D2 test a deviation a_1 for a given type θ with respect to each particular mixed best response α_2 of player 2. Type θ is not allowed to be uncertain about which element of $\mathrm{MBR}(T, a_1)$ player 2 will choose, which would correspond to considering all α_2 in the convex hull of the set of best responses.[10] To see the difference this makes,

8. The reader may wonder why the equilibrium-dominance criteria use best responses and the divinity criteria use mixed best responses. Note first that introducing mixed best responses in the equilibrium-dominance criteria would not change those criteria, because the max in condition i and the min in condition ii would be unaffected. Note also that replacing MBR by BR in the divinity criteria would reduce their bite, except in the subclass of signaling games studied in theorem 11.3, where player 2's best response is a singleton. Another possibility is to replace MBR by the convex hull of best responses, as in the discussion of figure 11.7.
9. Divinity does not strike a type completely when it fails D1, but instead requires that if type θ fails D1 for strategy a_1 then the probability of type θ should not increase when a_1 is observed.

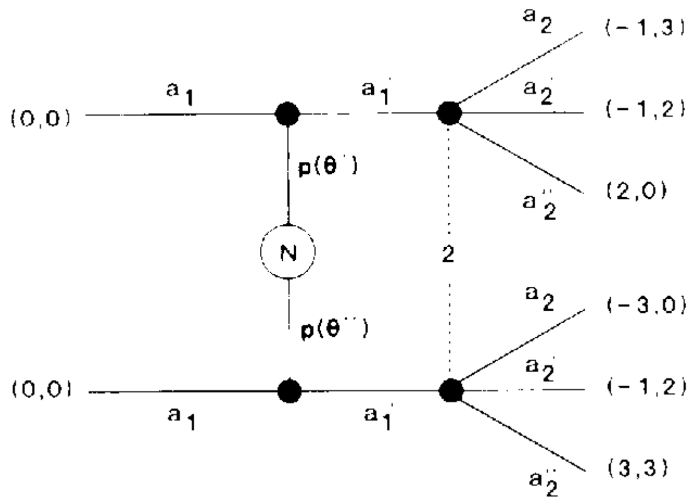


Figure 11.7

consider the game illustrated in figure 11.7. Here the set $\mathrm{MBR}(\mu, a_1')$ of player 2's mixed best responses to a_1' is a_2 if $\mu(\theta'|a_1') > \frac{2}{3}$, any mixture between a_2 and a_2' if $\mu(\theta'|a_1') = \frac{2}{3}$, a_2' if $\frac{1}{3} < \mu(\theta'|a_1') < \frac{2}{3}$, any mixture between a_2' and a_2'' if $\mu(\theta'|a_1') = \frac{1}{3}$, and a_2'' if $\mu(\theta'|a_1') < \frac{1}{3}$. Thus, although a_2 and a_2'' are both best responses to a_1' for some beliefs, there are no beliefs for which a mixture between a_2 and a_2'' is a best response.

This game has a pooling equilibrium where both types of player 1 play a_1, and player 2 plays a_2' in response to a_1', supported by the beliefs $\mu(\theta'|a_1') = \frac{1}{2}$. This equilibrium satisfies the Cho-Kreps conditions, since both types of player 1 would do better with a_1' followed by (undominated response) a_2'' than they do in the equilibrium. Let us check whether the pooling equilibrium satisfies D1 and D2.

To compute the sets $D(\theta', \Theta, a_1')$ and $D(\theta'', \Theta, a_1')$ we first compute which responses $\alpha_2 = \sigma_2(\cdot|a_1')$ would make each type prefer a_1', and then take the intersection with the set of mixed best responses. Simple algebra shows that type θ' prefers a_1' to the equilibrium a_1 if $\alpha_2(a_2'') > \frac{1}{3}$, whereas type θ'' prefers a_1' if $3\alpha_2(a_2'') > 3\alpha_2(a_2) + \alpha_2(a_2')$. Figure 11.8 displays these deviation regions on the probability simplex corresponding to α_2, where the bold edges of the probability simplex correspond to α_2s that put probability 0 on either a_2'' or a_2 and are therefore mixed best responses for some beliefs about player 1.

Inspection of figure 11.8 shows that $D(\theta'', \Theta, a_1')$ strictly contains $D(\theta', \Theta, a_1')$, so that by criterion D1 a deviation to a_1' must be interpreted as coming from θ''. This causes player 2 to respond with a_2'', which induces both types to deviate. However, the deviation regions are not nested when one considers all mixtures over best responses: The mixed strategy

10. This point was made by van Damme (1987) and developed further by Fudenberg and Kreps (1988) and by Sobel, Stole, and Zapater (1990). Substituting the convex hull of (BR) for MBR yields "codivinity," which is not implied by stability in some nongeneric games.

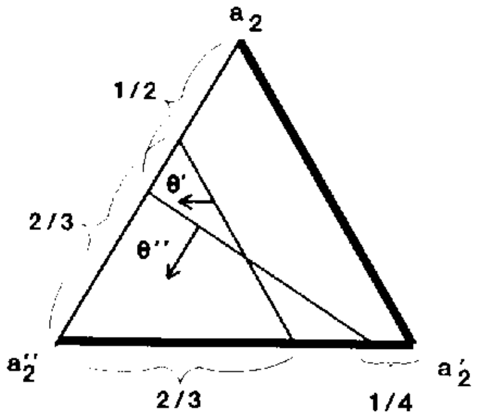


Figure 11.8

$\{\alpha_2(a_2) = \frac{3}{5}, \alpha_2(a_2'') = \frac{2}{5}\}$ would induce type θ' to deviate, but not type θ'', and therefore no type can be eliminated.

In practice, instead of checking condition D2 directly it is often easier to check the following stronger condition. Cho and Kreps call it NWBR, and indeed it is closely related to the NWBR property of Kohlberg and Mertens (i.e., that a stable component remains stable after the deletion of any strategy that is not a weak best response in any of the equilibria in that component). Since the Cho-Kreps version of NWBR is not precisely the same as that of Kohlberg and Mertens, we will call it *NWBR in signaling games.* A type-action pair can be deleted under this criterion if

$$D^0(\theta, \Theta, a_1) \subset \bigcup_{\theta' \neq \theta} D(\theta', \Theta, a_1).$$

Note that any type that is deleted for a_1 under D2 is deleted under NWBR in signaling games.

Since in generic games each stable component consists of equilibria with the same distribution on endpoints, stability implies NWBR in signaling games for generic payoffs. More precisely, fix a signaling game in which each stable component is identified with a stable outcome, and suppose that in one of the equilibria in this component NWBR in signaling games deletes type θ using action a_1. Deleting strategies where θ uses a_1 is consistent with stability if a_1 is not a weak best response for θ in any of the equilibria in the component. If a_1 were a weak best response for θ in some equilibrium in this component, then in that equilibrium player 2's response $\alpha_2(a_1)$ would lie in $D^0(\theta, \Theta, a_1)$. NWBR in signaling games then implies that player 2's response would be in $D(\theta', \Theta, a_1)$ for some other type θ', so that θ' would strictly prefer to deviate and we would not have an equilibrium after all. Thus, deleting equilibria according to NWBR in signaling games, or the weaker D2 or the still weaker condition of equilibrium dominance, does not eliminate any stable outcomes. Since stable outcomes exist in generic signaling games, universally divine (and hence "Intuitive") equilibria exist in the same generic class.

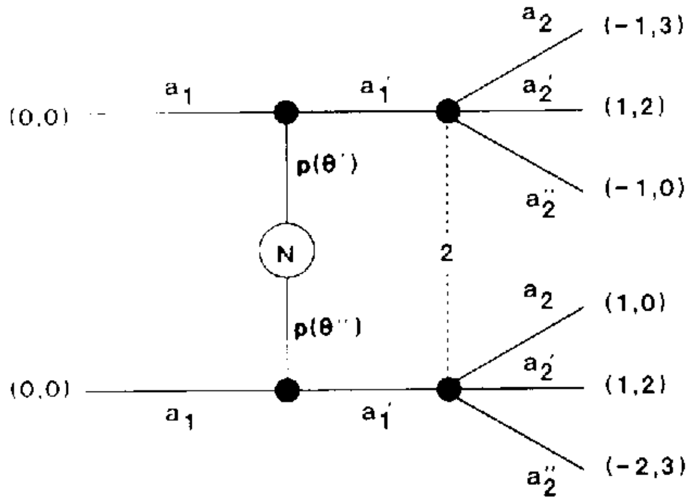


Figure 11.9

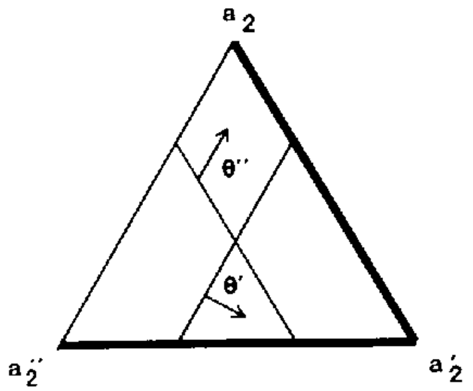


Figure 11.10

To see why NWBR in signaling games is stronger than D2, consider the example illustrated in figure 11.9, which is taken from Cho and Kreps. Player 2's payoffs in this figure are exactly as in figure 11.7; all that has been changed is the payoffs of player 1 when choosing a_1'. In this game, type θ'' strictly prefers a_1' to the equilibrium a_1 if $\alpha_2(a_2'') < \frac{1}{3}$, and type θ' strictly prefers a_1' if $\alpha_2(a_2') > \frac{1}{2}$.

Figure 11.10 displays the intersection of these deviation regions with the set of mixed best responses. (Recall that the mixed best responses for player 2 are the two bold edges in figure 11.10.) Since neither one contains the other, D2 has no bite. But the unique mixed best response $D^0(\theta'', \Theta, a_1')$ that makes θ'' exactly indifferent between the equilibrium and a_1' is contained in $D(\theta', \Theta, a_1')$, so that type θ'' choosing a_1' can be deleted by NWBR in signaling games. This deletion implies that player 2 must respond to a_1' with a_2, which is not an equilibrium response in the original game, so that the equilibrium where both types choose a_1 fails NWBR in signaling games.

Cho and Sobel (1990) have shown that eliminating equilibria that fail D1 is equivalent to stability in the class of "monotonic signaling games."

Definition 11.7 A *monotonic signaling game* has payoffs such that, for all a_1, and for all mixed strategies α_2 and α_2' in the set $\text{MBR}(\Theta, a_1)$ of mixed best responses to a_1, if for some $\theta \in \Theta$

$$u_1(a_1, \alpha_2, \theta) > u_1(a_1, \alpha_2', \theta),$$

then for all $\theta' \in \Theta$

$$u_1(a_1, \alpha_2, \theta') > u_1(a_1, \alpha_2', \theta').$$

Many signaling games in the literature are monotonic. For example, if a_2 is a monetary payment to player 1 and player 1 is risk neutral, then monotonicity follows from the fact that all types of player 1 prefer the α_2 with the highest expected value. This is the case, for example, in Spence's model of job-market signaling. (With a risk-averse player 1, monotonicity is more restrictive.) The Cho-Sobel proof relies on a complex characterization of stability in signaling games due to Banks and Sobel. However, one implication of the monotonicity assumption is easy to obtain:

Lemma 11.1 (Cho and Sobel 1990) In monotonic signaling games, criterion D1 is equivalent to NWBR.

Proof Exercise 11.4.

Example 11.1: Spence's Job-Market Signaling

As an illustration of refinements in signaling games, we now consider a variant of the Spence model we studied in example 8.2. Suppose that there are three types of player 1: θ', θ'', and θ'''. Player 1 moves first, selecting a level of education a_1 from the set $[0, \infty)$. (We work with a continuum of education levels to simplify the analysis, but this does involve some loss of rigor. Note that stability is defined only for finite games, but the Intuitive Criterion and universal divinity can be applied to games with a continuum of actions.) Player 2, the firm, wants to minimize the quadratic difference between the wage a_2 offered to player 1 and player 1's productivity, which is $a_1\theta$, with $\theta' = 2$, $\theta'' = 3$, and $\theta''' = 4$. (This quadratic loss is meant to stand in for a situation of Bertrand competition among several competing firms; allowing several firms would take us out of the signaling-game model.) Player 1's utility is the difference between his wage, a_2, and his disutility of education, which is $a_1{}^2/\theta$. The key aspect of these preferences is that they satisfy the Spence-Mirrlees or sorting or single-crossing condition: The marginal disutility of education is decreasing in player 1's type. This is why there are equilibria where the level of education chosen increases in player 1's productivity: The high-productivity types will be willing to choose higher levels of education than the low-productivity types for a given increment in wages.

As in chapter 8, this game can have a great multiplicity of sequential equilibria. For some parameter values there is a pooling equilibrium, where all three types choose the same education level, supported by the beliefs that the observation of any other education level implies that player 1 is the low-productivity type θ'. There is typically a continuum of different separating equilibria, where each type chooses a different education level.[11] And there are all sorts of "semi-separating" equilibria, where the supports of the education levels chosen by the different types intersect but do not coincide.[12]

Riley (1979) argued that the following equilibrium is the most reasonable one: The lowest-productivity type chooses the level of education that maximizes his utility on the assumption that his type will be fully revealed, so that his wage will equal his productivity. Call this level $a_1^*(\theta')$; with our parameter values, $a_1^*(\theta') = 2$ and $u_1^*(\theta') = 2$. The next-most-able type, θ'', chooses the level of education, $a_1^*(\theta'')$, that maximizes his utility when he is paid his productivity, $3a_1 - a_1{}^2/3$, subject to the constraint that type θ' should not strictly prefer the combination a_1 and a wage of $3a_1$ to his "own" education level and wage. Thus, $a_1^*(\theta'')$ must satisfy

$$3a_1^*(\theta'') - \frac{(a_1^*(\theta''))^2}{2} \leq 2 \Rightarrow a_1^*(\theta'') \simeq 5.2.$$

For $\theta = \theta'''$ (and subsequent types, if any), $a_1^*(\theta)$ is defined as the minimum of the education level that type would choose under perfect information and the education level required to prevent the next-lowest type from "pretending" to be type θ. (One can show that these adjacent incentive constraints are the binding ones: If no type wishes to pretend that it is the next-highest type, then no type prefers any deviation.) The Riley outcome is Pareto efficient in the class of separating equilibria, as it involves the minimum amount of education needed for separation. However, other equilibria can be more efficient in terms of the players' *ex ante* payoffs.

Cho and Kreps show that the Intuitive Criterion selects the Riley outcome if there are only two possible types of player 1: Suppose that in equilibrium types θ' and θ'' both assign positive probability to action $\hat{a}_1$, and let $\tilde{a}_1$ be the highest education level such that type θ' at least weakly prefers education $\tilde{a}_1$ and wage $\tilde{a}_1\theta''$ to his equilibrium action. Since there are only two types, the wage $a_2(\hat{a}_1)$ paid to the pooling action is at most $\hat{a}_1\theta''$, and $\tilde{a}_1 \geq \hat{a}_1$. From single crossing, type θ'' will then strictly prefer actions a_1 just above $\tilde{a}_1$ and wage $a_1\theta''$ to his equilibrium choice of $\hat{a}_1$. Since wage offers greater than $a_1\theta''$ are weakly equilibrium dominated for player 2, the Intuitive Criterion requires that player 2 assign probability 0 to type θ' after all actions just above $\tilde{a}_1$, so that the wage paid for these education levels must be $a_1\theta''$, and player θ'' will strictly prefer to deviate from the equilibrium because of single crossing.

11. With a continuum of types, there is a unique separating equilibrium; see Mailath 1987.
12. The reader will check that the supports cannot coincide because of the sorting condition.

An interesting feature of the Riley outcome selected by the Intuitive Criterion is that this outcome is independent of player 2's beliefs about player 1 as long as the support of the distribution is kept constant, but varies discontinuously when a type is added or deleted. To illustrate this, suppose that initially there is only one possible type: θ''. Type θ'' chooses $a_1(\theta'')$ so as to maximize $3a_1 - a_1{}^2/3$, so $a_1(\theta'') = 4.5$. Suppose now that player 1 has type θ'' with probability $1 - \varepsilon$ and type θ' with probability ε, where ε is small. The Riley outcome predicts that type θ'' chooses $a_1^*(\theta'') \simeq 5.2$. It seems extreme that allocations would be so sensitive to beliefs. Indeed, in the case of a small ε a pooling allocation at a_1 close to 4.5 seems more reasonable.

The Cho-Kreps argument fails when there are three or more types, for in order to delete action a_1 for type θ', a_1 must be large enough that type θ' would not gain by choosing it even if he were paid the wage of type θ''' whose productivity is two steps higher. If type θ'' picks this high an a_1, he is sure to get $a_1\theta''$ or more, but this will no longer guarantee that type θ'' gets more from the deviation than he did from the equilibrium. The point is that, to rule out type θ', the Intuitive Criterion asks us to consider the best possible response, which is $a_1\theta'''$, whereas to conclude that type θ'' would deviate once θ' is ruled out we must allow for the possibility that when type θ'' deviates he is paid his own productivity. With only two types, the best possible response θ' could hope for and the response θ'' could guarantee once θ' is ruled out are the same, which is why in general the Intuitive Criterion has more power in this case. In fact, with only two types it selects the Riley outcome.

Cho and Kreps observe that D1 picks out the Riley outcome with three types.

Cho and Sobel characterize the implications of D1 in a larger class of signaling games: Let $A_1 = [0, 1]^N$ for some N and $A_2 = [0, 1]$, and suppose the set of types Θ is the set of integers from 1 to $\#\Theta$.

Theorem 11.3 (Cho and Sobel 1990) Suppose that a signaling game satisfies the following conditions:
(i) (Monotonicity) If $a_2' > a_2$, then all types θ prefer a_2' to a_2.
(ii) For each $\mu \in \Delta(\Theta)$, $\mathrm{MBR}(\mu, a_1)$ is a single point; MBR is continuous in μ, and if μ' is greater than μ in the sense of first-order stochastic dominance, then $\mathrm{MBR}(\mu', a_1) > \mathrm{MBR}(\mu, a_1)$, so that player 2's response is more favorable to player 1 when player 2 thinks player 1's type is higher.
(iii) Player 1's utility function is differentiable and satisfies the Spence-Mirrlees sorting condition: $-(\partial u_1/\partial a_{1j})/(\partial u_1/\partial a_2)$ is decreasing in θ for each component a_{1j} of a_1.
Then there exists a unique equilibrium satisfying D1.

Since Cho and Sobel require player 1's action space to be bounded above, one possible equilibrium configuration has a set of types pooling at the highest possible action. Cho and Sobel show that this is the only possible kind of pooling, so if no type chooses to send the highest action then the equilibrium must be fully separating, and corresponds to a generalized version of the Riley outcome. Say that $a_1'' > a_1'$ if every component of a_1'' is at least as large as the corresponding component of a_1', and a_1'' is strictly larger in at least one component. A key step in the proof is the following lemma:

Lemma 11.2 Under the hypotheses of theorem 11.3, if type θ'' chooses action a_1' with positive probability in equilibrium, then D1 implies that $\mu(\theta' \mid a_1'') = 0$ if $\theta'' > \theta'$ and $a_1'' > a_1'$.

Proof Fix an equilibrium (σ_1^*, σ_2^*) such that type θ'' chooses a_1' with positive probability. Let $a_2^*(a_1)$ be the action prescribed by $\sigma_2^*(\cdot \mid a_1)$. For each $a_1'' > a_1'$ and θ, let $\hat{a}_2(\theta) \in \mathrm{BR}(\Theta, a_1'')$ satisfy $u_1(a_1'', \hat{a}_2(\theta), \theta) = u_1^*(\theta)$; if no such $\hat{a}_2$ exists, set $\hat{a}_2(\theta) = +\infty$. We claim that single crossing implies that $\hat{a}_2(\theta') > \hat{a}_2(\theta'')$. To see this, consider figure 11.11, which displays the situation $\hat{a}_2(\theta') \le \hat{a}_2(\theta'')$ for the case of a single-dimensional a_1. By definition, type θ'' is indifferent between $A = (a_1', a_2^*(a_1'))$ and $B = (a_1'', \hat{a}_2(\theta''))$. But single crossing means that in the (a_1, a_2) space the indifference curve of type θ' is steeper than that of type θ'' at any point, and therefore the two indifference curves depicted in figure 11.11 cannot intersect at any $a_1 < a_1''$. Therefore, type θ' strictly prefers $(a_1', a_2^*(a_1'))$ to his equilibrium strategy—a contradiction. We leave it to the reader to provide the algebraic proof for a multi-dimensional a_1 (see also chapter 7).

Because $\hat{a}_2(\theta') > \hat{a}_2(\theta'')$,

$$\begin{aligned} &D(\theta', \Theta, a_1'') \cup D^0(\theta', \Theta, a_1'') \\ &\quad = \{a_2 \in \mathrm{BR}(\Theta, a_1'') \text{ such that } a_2 \ge \hat{a}_2(\theta')\} \subset D(\theta'', \Theta, a_1'') \\ &\quad = \{a_2 \in \mathrm{BR}(\Theta, a_1'') \text{ such that } a_2 > \hat{a}_2(\theta'')\}. \end{aligned}$$

Hence, θ' is eliminated by D1 for a_1''. ■

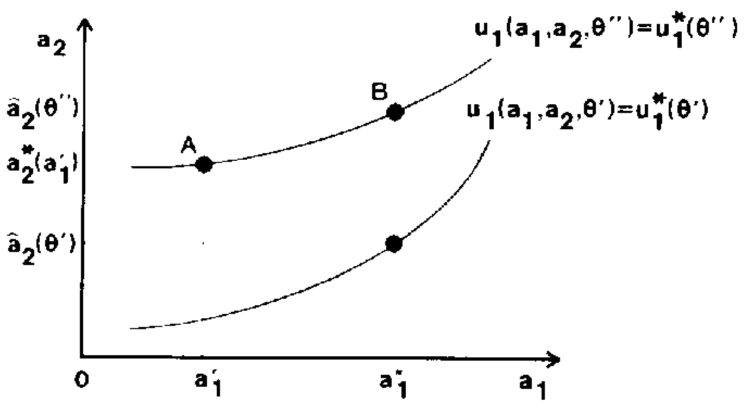


Figure 11.11

With this lemma it is easy to see that any equilibrium where two or more types assign positive probability to the same a_1^* must fail criterion D1. Let θ^* be the highest-productivity type that plays a_1^*, and let a_2^* be player 2's equilibrium response to a_1^*. From the sorting condition, for each a_1' greater than but sufficiently close to a_1^*, there are undominated responses that make type θ^* prefer to deviate to a_1' but do not tempt the lower-productivity types. Thus, type θ^*'s "deviation region" (in the sense of definition 11.6) strictly includes the deviation regions of the other types who play a_1^*, and so, if player 1 chooses any education level "just above" a_1^*, player 2 must assign probability 0 to all types with productivity less than θ^*'s. Then, since player 2's equilibrium response is continuously increasing in the beliefs about player 1, type θ^* can induce a nonnegligible increase in a_2 by an infinitesimal increase in a_1.

Not only does D1 select the Riley outcome, it imposes the following restrictions on the beliefs which are used to support it: Player 2 must assign probability 1 to type θ' after any action in the interval $[a_1^*(\theta'), a_1^*(\theta''))$, probability 1 to type θ'' after any action in $[a_1^*(\theta''), a_1^*(\theta'''))$, etc. Since the motivation for D1 is to refine the set of equilibria using "reasonable" restrictions on beliefs, to the extent that the above 0-1 restrictions are implausible they may cast doubt on D1 as an equilibrium concept.

11.3 Forward Induction, Iterated Weak Dominance, and "Burning Money"††††

Just as iterated strict dominance and rationalizability can be used to narrow down the set of predictions without invoking equilibrium refinements by using rationality arguments alone, the concept of iterated weak dominance (IWD) can be used to capture some of the force of forward and backward induction without assuming that players will coordinate their expectations on a particular equilibrium.[13] Since the idea of forward induction is that players interpret a deviation as a signal of how their opponent intends to player in the future, forward induction seems more compatible with a situation of considerable strategic uncertainty—i.e., a nonequilibrium situation—than with a situation where the strategic uncertainty has been resolved and all players are certain that they know their opponents' strategies. (This is another version of our argument that probability-0 events are best thought of as events whose probability is low.)

One difficulty with iterated weak dominance, as opposed to iterated strict dominance (see section 2.1), is that different orders of deletion can give different solutions, as is shown by the game in figure 11.12. Here, if we first eliminate player 1's weakly dominated strategy D at the first round, the

13. Pearce's (1984) extensive-form rationalizability and the notion of iterated conditional dominance are other ways to make "refined" predictions in a nonequilibrium context.

	L	R
U	1,0	0,1
D	0,0	0,2

Figure 11.12

solution is (U, R) because L is dominated for player 2; if we eliminate L at the first round, player 1 becomes indifferent between U and D, and so the solution set is both (U, R) and (D, R). The standard response to this problem is to specify the maximal amount of deletion at each round, i.e., that at each round all weakly dominated strategies of all players are deleted.[14]

Iterated weak dominance incorporates backward induction in games of perfect information: The suboptimal choices at the last information sets are weakly dominated; once these are removed, all subgame-imperfect choices at the next-to-last information sets are removed at the next round of iteration; and so on. Iterated weak dominance also captures part of the forward-induction notions implicit in stability, as a stable component contains a stable component of the game obtained by deleting a weakly dominated strategy. For example, the stable outcome of the Kohlberg-Mertens example in figure 11.1 can be obtained by iterated weak dominance: The play of RW for player 1 is strictly dominated, and player 2's playing T is weakly dominated once RW has been removed.

The most striking example we have seen of the power of iterated weak dominance is Ben-Porath and Dekel's (1988) study of the following class of

14. Rochet (1980) provides a partial answer to the question of when every order of deleting weakly dominated strategies gives the same solution. His answer is partial in two respects: First, instead of considering weak dominance as we have defined it, he looks only at "pure-strategy dominance"—the process he considers does not delete all weakly dominated strategies, only those that are dominated by another *pure* strategy. (See chapter 1 above for an example in which a mixed strategy strictly dominates a pure strategy that is not (pure-strategy) dominated.) Second, Rochet considers only games in which *some* order of deletion yields a unique prediction. He shows that, if any order of iterated pure-strategy weak dominance yields a unique solution, then any order of deletion yields this same unique solution, under the following assumption: If for some player i and strategy profiles s and s'

$$u_i(s) = u_i(s'),$$

then

$$u_j(s) = u_j(s') \text{ for all } j.$$

(Note that this condition is not satisfied in the strategic form in figure 11.12: (D, L) and (D, R) yield the same payoff for player 1, but not for player 2. In an extensive-form game, a sufficient condition for Rochet's assumption to be satisfied is that there not exist a player i and two terminal nodes z and z' such that $u_i(z) = u_i(z')$. This sufficient condition is satisfied in generic extensive-form games.) Moulin (1986) uses Rochet's theorem to show that backward induction and (any order of) iterated weak dominance give the same unique solution for generic payoffs in finite games of perfect information. Iterated weak dominance can be stronger than backward induction if some player has the same payoff at two distinct terminal nodes.

games[15]: Players 1 and 2 are going to play a simultaneous-move game of coordination, which has several pure-strategy equilibria, all of which are better than not coordinating (so that the mixed equilibria are Pareto dominated) and one of which gives player 1 his highest possible payoff. Before they play, however, player 1 has the option of publicly "burning" a small amount of utility. If the maximum amount player 1 can burn is sufficiently large, and the amount to be burned can be specified sufficiently finely, then the unique outcome according to iterated weak dominance is that player 1 burns no utility, and then the players play the stage-game equilibrium that gives player 1 his highest payoff. This strong conclusion can be viewed either as an argument about how players can arrange to coordinate in a particular way or as evidence that iterated weak dominance (and hence stability) is too restrictive.

Rather than state the theorem of Ben-Porath and Dekel formally, we give an illustrative example: In the first period, player 1 can either "not burn" or "burn" 2.5 utils. After this choice is observed, he and player 2 will play the simultaneous-move game at the top of figure 11.13. Note that without the possibility of burning there is no way to distinguish between the equilibria (U, L) and (D, R); player 1 prefers the first equilibrium and

	L	R
U	9,6	0,4
D	4,0	6,9

a

	L,L	L,R	R,L	R,R
Burn, U	6.5,6	6.5,6	−2.5,4	−2.5,4
Burn, D	1.5,0	1.5,0	3.5,9	3.5,9
Not Burn, U	9,6	0,4	9,6	0,4
Not Burn, D	4,0	6,9	4,0	6,9

b

Figure 11.13

15. Van Damme (1989) independently discovered the power of forward induction in a game in which players can "burn utility." He develops other examples of the power of forward induction and of its relation to stability. See exercise 11.5.

player 2 prefers the second. This creates the game depicted in the bottom of the figure, where the first component of player 2's strategy is how to play if player 1 burns and the second component is how to play if player 1 doesn't burn. In this extended game, the strategy (Burn, D) is strictly dominated for player 1 by (Not burn, D), and so at the next round of iteration player 2's best response to Burn is to play L. (That is, once (Burn, D) is deleted, any strategy s_2 with $s_2(\text{Burn}) = \text{R}$ is weakly dominated for player 2 by the strategy $\hat{s}_2$, where $\hat{s}_2(\text{Not burn}) = s_2(\text{Not burn})$ and $\hat{s}_2(\text{Burn}) = \text{L}$.) Therefore, after two rounds of iteration (Burn, U) guarantees player 1 a payoff of 6.5 and strictly dominates (Not burn, D). Hence, after three rounds of iteration, player 2 should conclude that even if player 1 does not burn he is certain to play U, and so player 1 can use the strategy (Not burn, U) and be sure of a payoff of 9! That is, the mere fact that player 1 could have chosen to burn utility but did not do so ensures that he obtains the equilibrium he most prefers.

Even for a 2×2 second-stage game, for general payoffs the result requires that player 1 have a number of different possible levels of burning. To see this, suppose that the second-stage payoffs are (90, 90) to (U, L), (72, 72) to (D, R), and (0, 0) otherwise, and denote the cost of burning by b. Player 1's maximin strategy in the second stage is $(\frac{4}{9}\text{U}, \frac{5}{9}\text{D})$, which guarantees a payoff of 40; this is also player 1's minmax payoff. If $b \geq 50$, then the best player 1 can obtain by burning is less than 40, so (Burn, U) and (Burn, D) are both weakly dominated by not burning (followed by playing $(\frac{4}{5}\text{U}, \frac{5}{9}\text{D})$) and after one round of elimination the game with burning reduces to the original game. If $b < 32$, then (Burn, D) is a best response to the strategy of player 2, "minmax if player 1 does not burn, and play R if he burns," so no strategies are even weakly dominated. If $b \in [32, 50]$, then only (Burn, D) is weakly dominated (by Not burn, $(\frac{4}{5}\text{U}, \frac{5}{9}\text{D})$.) Once this strategy is removed, all of player 2's strategies that play R after Burn are weakly dominated; but this is as far as the iteration goes. (Burn, U) gives player 1 a payoff of $90 - b \leq 58$; (Not burn, D) could give as much as 72. In this game, IWD is not powerful. The point of the Ben-Porath–Dekel paper is that when there is a sufficiently fine grid of burning levels, player 1 can ensure his most preferred equilibrium *without* burning.[16]

When there is a sufficiently fine grid, the result of Ben-Porath and Dekel is surprisingly strong. Ben-Porath and Dekel respond to the unease that

16. The argument fails when the amount to be burned is chosen from an interval. Thus, if we regard the case of infinitely divisible money as the limit of increasingly fine discrete grids, the set of profiles satisfying iterated weak dominance fails to be lower hemi-continuous. This is related to the familiar observation that the reaction and equilibrium correspondences need not be lower hemi-continuous. From the literature on ε-equilibrium, one suspects that there are various ways of perturbing the game or the solution concept to obtain the solutions of the continuum case in the model with a fine discrete grid, so that the result of Ben-Porath and Dekel may be least compelling when a very fine grid is required.

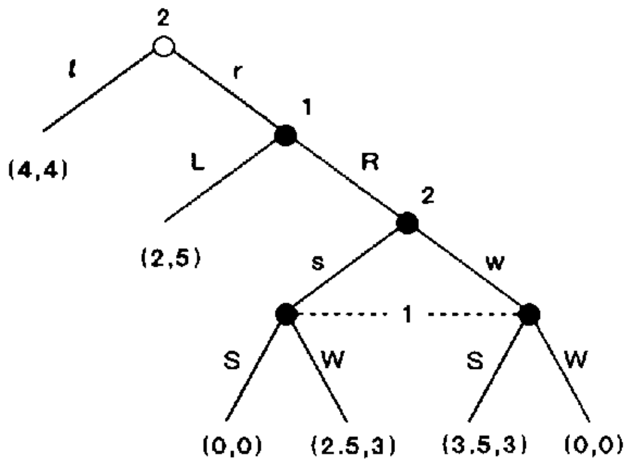
this strength may inspire by suggesting that attention be restricted to games of "common interest," where a single equilibrium of the stage game gives both players their most preferred payoff, so that player 2 is not disadvantaged when player 1 is the only one to burn. They argue that if the stage game is not one of common interest, then both players would "try to be able to burn," and once players can burn or not simultaneously, iterated weak dominance gives much weaker predictions. But we believe that it is still instructive to look at the power of IWD in games like our example where only player 1 burns and the game is not of common interest. *If* one did one's best to ensure that the "real" extensive form was as close as possible to that of the burning-utility game, what outcome should one expect? The process of iteration in the example is sufficiently involved that we have little confidence that the outcome would be as predicted. In part this is because the iteration requires a large number of steps (four), and, like chains of backward induction, chains of forward induction become more suspicious as they grow longer. One way of formalizing this suspicion is to recall that, as with backward induction, each step of the forward-induction process requires another level of the assumption that "player 1 knows that player 2 knows . . . that no one will play a strategy that is weakly dominated if the payoffs are as specified." Another way to justify the suspicion is to ask if even the second step of the induction is plausible: If player 2 sees that player 1 "burned utility," will she reason in the style of forward induction that this is a rational decision by a player 1 whose payoffs are exactly as originally believed? Or will player 2 decide that player 1 is "crazy" and derives positive utility from what was supposed to be a costly act? This latter explanation is at the heart of the Fudenberg-Kreps-Levine (1988) and Dekel-Fudenberg (1990) papers on the robustness of refined predictions to the possibility that players always assign a small but nonzero probability to their opponents' payoffs being very different than originally supposed.

Before developing these papers, we would like to discuss an alternative viewpoint. Van Damme (1989) argues that stability is too weak to capture all the implications of forward induction in an equilibrium context. He proposes that one implication of forward induction ought to be the following:

Definition 11.8 (van Damme 1989) A solution concept S is *consistent with forward induction in the class of generic two-person extensive forms* if there is no equilibrium in S such that some player i, by deviating at a node along the equilibrium path, can ensure (with probability 1) that a proper subgame Γ is reached where (according to S) all solutions but one give the player strictly less than the equilibrium, and where exactly one solution gives the player strictly more.

This definition combines a sort of backward-induction notion with the idea that deviations should be interpreted as a signal of how the deviator intends to play in the future. If player i does deviate in a way that ensures that a proper subgame Γ satisfying the definition is reached, and if it is common knowledge that S gives the set of expected solutions in each subgame, then player i's opponent "should" conclude that player i will play in Γ according to the unique solution that gives him a higher payoff than the equilibrium path. (This idea that players' actions can signal which of several equilibria they expect was first proposed by McLennan 1985, who developed it in a different way.) The reason that the definition covers only two-player games is to ensure that player i has a nontrivial choice in the subgame Γ: In a three-player game, if player 1 deviates but will not play again, there is no particular reason to expect players 2 and 3 to choose the equilibrium that player 1 most prefers. (If only player j moves in Γ, a generic Γ will have a unique solution from backward induction.) An alternative definition would say that if player i deviates in a way that satisfies the definition, then all other players should expect player i to play according to the unique solution that justifies the deviation; this would impose no restrictions if player i did not move again.

Van Damme uses the "outside-option" game illustrated in figure 11.14 to show that stability does not satisfy his definition of forward induction.



a. Extensive Form

	l	rs	rw
L	4,4	2,5	2,5
RS	4,4	0,0	3.5,3
RW	4,4	2.5,3	0,0

b. Reduced Strategic Form

Figure 11.14

We will go through his argument both for its own sake and to help illustrate the mechanics of checking for stability.

To begin, let us analyze the stable equilibria of the subgame $\Gamma(r)$ after player 2's choice of r. Here player 1 can either choose his "outside option" of L, resulting in payoffs (2, 5), or play a "battle of the sexes" game with player 2. There are three subgame-perfect equilibrium outcomes: (RS, w), (RW, s), and L, with payoffs (3.5, 3), (2.5, 3), and (2, 5); the last outcome is supported by the mixed-strategy equilibrium if the "battle of the sexes" game is reached. Since both pure equilibria of this game give player 1 more than his outside option, the fact that player 1 didn't choose L does not "signal" his intentions, and we would expect that stability would not reduce the set of subgame-perfect equilibria. (Note the contrast with the burning-utility game, where stability picked out a unique outcome.)

To verify this intuition, we must identify a component of equilibria of $\Gamma(r)$ with outcome L such that for every perturbation of the game there is an equilibrium near some element of the component. In figure 11.14b, the game $\Gamma(r)$ simply corresponds to deleting player 2's strategy ℓ.

Let q denote the probability that player 2 plays rs, and consider the component $\{(L, (q, (1 - q)))\}$, when $q \in \{\frac{3}{7}, \frac{4}{5}\}$. (Note that this component is not connected, but that both equilibria do have the same outcome.) For either q, player 1 at least weakly prefers L to RS and RW, so both profiles are Nash equilibria. Now perturb $\Gamma(r)$ by requiring that player i place probability at least $\varepsilon(s_i)$ on strategy s_i. If $\varepsilon(\text{RS}) \geq \varepsilon(\text{RW})$, an equilibrium of the perturbed game is for player 1 to play L with probability $1 - 2\varepsilon(\text{RS})$, and play RS and RW with equal probability $\varepsilon(\text{RS})$, and for player 2 to play rs with probability $q = \frac{4}{5}$. Given that $q = \frac{4}{5}$, player 1 is indifferent between L and RW, and so is willing to give RW more than the minimum required probability. Given that RS and RW are equally likely, player 2 is willing to randomize; for small εs, player 2's strategy clearly meets the minimum-probability constraint. As $\varepsilon_1(\cdot) \to 0$, this profile converges to $(L, (\frac{4}{5}, \frac{1}{5}))$, which belongs to the component we constructed. If $\varepsilon(\text{RS}) \leq \varepsilon(\text{RW})$, an equilibrium is for player 1 to give both RS and RW probability $\varepsilon(\text{RW})$ and for player 2 to set $q = \frac{3}{7}$.

Note that the stable component does not include the sequential-equilibrium strategies $(L, (\frac{1}{2}, \frac{1}{2}))$, for these strategies do not make player 1 indifferent between L and R. As we saw above, it is important that player 1 be indifferent, so that if (for example) the perturbations make him tremble more onto RS than RW, he is willing to play RW with greater than the minimum probability in order to restore player 2's indifference between rs and rw.

Next we claim that, in the overall game, player 2 playing ℓ is a stable outcome. Exercise 11.7 asks you to prove this; the starting point is to consider the component

$\{((\frac{3}{5}\text{L}, \frac{2}{5}\text{RS}, 0\,\text{RW}), \ell), ((\frac{1}{3}\text{L}, 0\,\text{RS}, \frac{2}{3}\text{RW}), \ell)\}.$

In both of the equilibria in this component, player 2 is indifferent between ℓ and an alternative. In the first equilibrium the alternative is rw; in the second it is rs. In any perturbed game, player 2 will be willing to play either rs or rw with enough probability to make player 1 indifferent between L and R.

The stable outcome ℓ shows that stability does not satisfy definition 11.8. Fix only equilibria with outcome ℓ, and suppose that player 2 deviates to r. Since $\Gamma(r)$ has a unique stable outcome that makes playing r rational for player 2, namely "1 plays L," definition 11.8 requires that if player 2 plays r she receives payoff 5, which eliminates the equilibrium where 2 plays ℓ.

11.4 Robust Predictions under Payoff Uncertainty†††

Fudenberg, Kreps, and Levine (1988) and Dekel and Fudenberg (1990) discuss what kinds of refined predictions are possible if the main story the players use to explain unexpected deviations is that payoffs are different from what had been originally supposed. The typical game-theoretic assumption is that the payoffs (as functions of the terminal nodes) are correctly specified and indeed are common knowledge. Fudenberg, Kreps, and Levine suggest that this assumption is best viewed as an approximation, as neither the game theorist analyzing the game nor the players in it should be completely certain that the payoffs are as in the "most likely" case depicted by the extensive form.

Allowing for even a small probability of different payoffs has very strong implications, because a small *ex ante* probability can become quite large if there is an unexpected observation. We already observed this in our discussion of Spence's model of job-market signaling model, in section 11.2. Let us give two further examples of this before developing the formal results.

First consider the game illustrated in figure 11.15a. Here the unique subgame-perfect equilibrium is for 1 to play (D, u) and 2 to play R; the profile (U, L) is Nash but not subgame perfect. However, if player 2 expects player 1 to usually play U, and interprets D as a signal that player 1's payoffs are such that player 1 would play d at his second information set, then player 2 is justified in playing L. The extensive form that goes with this story is shown in figure 11.15b: Here player 1 has two possible types, θ and θ'; type θ's payoffs are as in figure 11.15a and those of type θ' make D_2d_2 a weakly dominant strategy. In this game, regardless of the prior probability of θ', the profiles where θ plays U_1u_1, θ' plays D_2d_2, and player 2 plays L are sequential, and indeed are stable as a singleton set, as each player's strategy is a strict best response to the strategy of his opponent. Thus, a "small amount" of payoff uncertainty—i.e., a small probability that a player's payoffs are very different than had been supposed—can justify

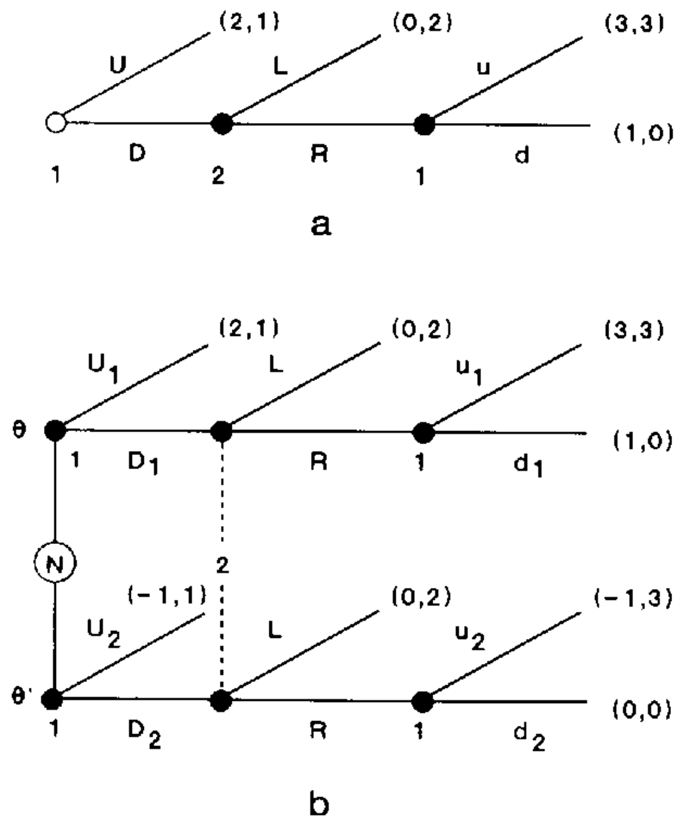


Figure 11.15

subgame-imperfect outcomes. Note that the imperfect equilibrium (Ud, L) of figure 11.15a is trembling-hand perfect in the associated strategic form: If player 1 mostly plays U and "trembles" onto Dd much more than onto Du, then player 2's choice of L is optimal. As discussed in chapter 8 above, Selten introduced the agent-strategic form precisely to rule out this sort of "correlation" in player 1's deviations. If deviations are due to different payoffs rather than to trembles, the argument that a player's deviations should be independent is less convincing.

The game depicted in figure 11.16 illustrates another implication of interpreting deviations as due to payoff uncertainty. In the subgame in which players 1 and 3 have played R and r, the payoffs of players 1 and 2 are independent of player 3's choice between A and B, and they play a "matching pennies" game between themselves. Thus, any Nash equilibrium of the subgame has players 1 and 2 randomize $\frac{1}{2}$-$\frac{1}{2}$ between their two actions. Player 3 thus gets more than 0 in the subgame, and must play r. There is also an imperfect Nash equilibrium where player 1 chooses L and player 3 chooses d. This choice by player 3 can be "justified" if he interprets a deviation by player 1 to R as meaning that players 1 and 2 are going to *correlate* their play in the simultaneous-move subgame, particularly if player 3 expects to face the joint distribution $(\frac{1}{2}(H,h), \frac{1}{2}(T,t))$. This would be the case if player 3 attached a small but positive *ex ante* probability to both of his opponents' having different payoffs, and if he believed that

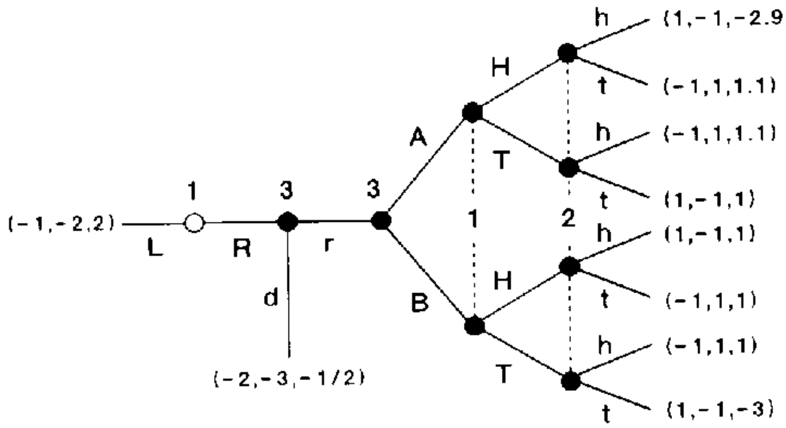


Figure 11.16

these probabilities were correlated, so that player 1's deviation could signal information about player 2's future play. For instance, there could be three states of the world. In state ω_1 (which has probability close to 1), the payoffs are those of figure 11.16. In state ω_2, the payoffs for players 1 and 2 in the subgame following Rr are such that they both have a dominant strategy: to play H and h respectively and get 1. In state ω_3, which is as likely as ω_2, the payoffs for players 1 and 2 in the subgame following Rr are such that T and t are dominant strategies for players 1 and 2, who get 1. Player 3 does not know which state prevails, but players 1 and 2 do. The introduction of states ω_2 and ω_3 models the above correlation of the strategies of players 1 and 2 in the subgame. Such correlations may not seem reasonable in the case where deviations are "trembles"; they seem more natural when deviations are due to payoff uncertainty.

As a special case of correlated types introducing correlated trembles, suppose that the opponents' payoffs do have probability 1 of being as originally supposed, but that there is a small chance the opponents have access to a correlating device. Then, although along the equilibrium path the players will proceed as if no correlating devices were available, they may interpret some unexpected observations as a signal that correlating devices were available after all. Jean-François Mertens has created an example in which this seems particularly apt: Suppose that players 1 and 2, who cannot communicate, have the option of whether or not to play a "battle of the sexes" game in which the payoffs to coordinating are $(1, 2)$ and $(2, 1)$, the payoffs to not coordinating are $(-10, -10)$, and the payoffs for not playing are $(0, 0)$. Player 3 believes there are no correlating devices available, and thus might well predict that players 1 and 2 would choose not to play. However, if, contrary to expectations, players 1 and 2 do agree to play the game, then player 3 might well conclude that a correlating device was available after all.

The examples above show that allowing for small payoff uncertainty can have a "large" effect. (Van Damme (1983) and Myerson (1986) give other examples.) Fudenberg, Kreps, and Levine (1988) give precise definitions of various kinds of "small" uncertainties and characterize the implications of each. Fudenberg et al. consider only the case of payoff uncertainty. That is, they assume that there is no doubt at all about the physical rules of the game (who moves when, what their choices are, and which previous actions they have observed), and that the only "additional" uncertainty concerns other players' payoffs. This leads to the notion of an *elaboration* $\tilde{E}$ of an extensive-form game E.

Definition 11.9 An *elaboration* $\tilde{E}$ of an extensive-form game E is formed as follows. An integer N is given, along with a probability distribution μ on $\mathcal{N} = \{1, 2, \ldots, N\}$. The game tree $\tilde{T}$ of $\tilde{E}$ is an N-fold replication of the tree T of E: $\tilde{T} = T \times \mathcal{N}$; each $n \in \mathcal{N}$ corresponds to a "version" of the game. If player i moves at x in T, he moves at (x, n) in $\tilde{T}$ for all n; the probability distribution over initial nodes $\tilde{w} = (w, n)$ of $\tilde{T}$ is $\rho(w)\mu(n)$, where ρ is the distribution over initial nodes of T. Each player i has a partition $P_i(n)$ over n, and information sets of $\tilde{E}$ take the form $h(x) \times P_i(n)$, where $h(x)$ is the information set of E containing x. Actions at information sets are inherited in the obvious way. Finally, the payoff to player i at terminal node (z, n) is $u_i(z, n)$. The elaboration has *personal types* if player i's payoff is a constant over all $n \in P_i(n)$, so that each player's payoff depends only on z and his own information about nature's choice of version. In this case we identify $P_i(n)$ with player i's "type."[17]

The incomplete-information game depicted in figure 11.15b is an elaboration with personal types of the game in figure 11.15a. Note that the definition of personal types does not require that the distributions over types be independent.

The next step is to specify when an elaboration is "close" to the game upon which it is based.

Definition 11.10 (convergence criterion) The following are sufficient conditions for a sequence of elaborations $\tilde{E}^k$ of an extensive-form game E to *approach* E:
(i) there is a uniform bound on the absolute values of the payoffs in each version and on the number of versions per elaboration,
(ii) there is a single version 1 such that $\lim_{k \to \infty} \mu^k(1) = 1$, and
(iii) for each i and z, $\lim_{k \to \infty} \mu_i^k(z, 1) = u_i(z)$.

Conditions ii and iii require that there be a single version whose probability tends to 1 and in which the payoffs converge to those of the original game. If one replaced these conditions by the requirement that the total probability of all versions with payoffs close to the original game converge

17. The original game E can be a game of incomplete information, so that $P_i(n)$ is not a full description of player i's type in the usual sense—it is his "meta-type."

to 1, then the definition would allow a sequence of games with a correlating device to approach a game in which the device is not present (correlated equilibrium corresponds to elaborations where all versions have the same payoffs as the original game). This seems too loose a notion of "closeness"; we argued above that a game with a small probability of a correlating device should be close to a game without one, but a game in which correlating devices are certain to be present is a different matter.

The first part of condition i ensures that the small probability of different payoffs makes only a small difference in *ex ante* payoffs. Without the uniform bound, the payoffs in other versions can inflate as their probability decreases, so that the limiting values of the *ex ante* payoffs can be quite different than in the original game. Fudenberg, Kreps, and Levine assert that the bound on the number of versions is "probably unnecessary to support a notion of closeness." (Note that the definition gives sufficient conditions for convergence, but not necessary conditions. This is because the definition as stated does not generate a topology on the space of elaborations of a given extensive form.) To characterize the implications of allowing "small" perturbations of the kind defined in the convergence criterion, one must specify an equilibrium refinement to be used in the perturbed games. Fudenberg, Kreps, and Levine use the concept of strict equilibrium, which is very demanding: Any strict equilibrium is stable as a singleton set. They use such a strong concept because their critique of refinements such as stability is most forceful when the equilibria rejected by the refinement can be shown to satisfy a strong version of the refinement in the perturbed game.

Definition 11.11 An equilibrium σ of the strategic form corresponding to an extensive-form game E is *near-strict with personal types* if there is a sequence of elaborations with personal types $\tilde{E}^k$ of E that approaches E in the sense of the convergence criterion, and a sequence of strict equilibria $\tilde{\sigma}^k$ of the reduced strategic forms corresponding to $\tilde{E}^k$, such that the behavior prescribed by $\tilde{\sigma}^k$ at all nodes $(x, 1)$ converges to the behavior prescribed by σ at x.

Recall that the definition of personal elaborations allows the distributions over types to be correlated, which we argued was natural. The equilibrium concept that characterizes the set of near-strict equilibria thus also involves correlation.

Definition 11.12 A strategy profile σ of a strategic form is *c-perfect* if for each player i there is a sequence ϕ^k_{-i} of totally mixed probability distributions over S_{-i} such that $\phi^k_{-i} \to \sigma_{-i}$ and σ_i is a best response to each ϕ^k_{-i}.

C-perfection weakens trembling-hand perfection in the strategic form in two ways. First, instead of using a common $\sigma^k \to \sigma$, each player is allowed to have his own beliefs about the "trembles" of his opponents. If trembles are thought of as occurring very rarely, it seems plausible that the players could have such differing beliefs. Second, the beliefs about the opponents' trembles need not take the form of a mixed strategy, but can be any probability distribution over joint actions by the opponents, so that correlated trembles are allowed. (The c in the term c-perfection is meant to represent this correlation.) Both of these considerations are irrelevant in two-player games, where c-perfection reduces to trembling-hand perfection in the strategic form.

Theorem 11.4 (Fudenberg, Kreps, and Levine 1988)[18] A pure-strategy profile s of an extensive-form game E with payoffs u is near-strict with respect to personal types if and only if there is a sequence $u^k \to u$ such that s is c-perfect in the corresponding strategic-form games.

Theorem 11.4 implies that any c-perfect pure-strategy[19] equilibrium is near-strict; the set of near-strict equilibria also includes equilibria that are c-perfect in games where the payoffs are slightly different. That is, to obtain an "if and only if" characterization, one takes the closure of the c-perfect set with respect to payoff perturbations in the extensive form. Note that these payoff perturbations, where payoffs are certain to be close to those of the original game, are much more restrictive than the perturbations considered in the definition of near-strict.

Fudenberg, Kreps, and Levine argue that when payoff uncertainty is the dominant explanation for deviations, one should not use concepts more restrictive than c-perfection unless one is prepared to argue which forms of payoff uncertainty—that is, which versions of the game—are viewed as more likely. In our view there is always payoff uncertainty: There are no economically interesting situations where the players are completely sure of their opponents' payoffs, and it may not even be reasonable to suppose that this is true as a thought experiment. However, this does not mean that we view the results of Fudenberg, Kreps, and Levine as relevant to all situations. They analyze the effects of small payoff uncertainties, ignoring all other explanations for deviations such as mistakes and experimentation. Thus, their results describe situations where payoff uncertainty is "large" relative to these other explanations. The right model for a given situation depends on which explanation(s) is most likely, and this information is not captured by the usual extensive form. Thus, we fear that it may not be possible to have a single theory of refinements that is appropriate for all extensive-form games.

18. Fudenberg et al. (1988) state the theorem as an equivalence between the set of near-strict equilibria and those which are "quasi c-perfect." As Dekel and Fudenberg (1990) explain, the theorem can be stated in the simpler form given here.

19. A mixed-strategy equilibrium can be made near-strict by first transforming it to a pure-strategy equilibrium with private information à la Harsanyi (see chapter 6 above).

Exercises

Exercise 11.1** Apply the Intuitive Criterion, the Iterated Intuitive Criterion, and universal divinity to the games illustrated in Figure 11.17.

Exercise 11.2*** Show that the game depicted in figure 11.18 has a stable component that does not contain a sequential equilibrium.

Exercise 11.3** Apply iterated weak dominance to the two-player strategic form of the beer-quiche game depicted in figure 11.6.

Exercise 11.4*** Show that D1 and NWBR are equivalent in monotonic signaling games. Hint: Suppose that θ is deleted for a_1 under NWBR in signaling games, so that every response that makes θ indifferent between a_1 and his equilibrium action makes some other types strictly prefer to deviate. Then, since all types have the "same" preferences over responses to a given a_1, all responses that make θ strictly prefer a_1 must make the other types strictly prefer a_1 as well. (This sketch has to be sharpened a bit to get the strict inclusion required by D1.)

Exercise 11.5** Use NWBR to find the stable components of the following game (taken from van Damme 1989). In period 1, players 1 and 2 simultaneously decide whether to burn 0 or 1 util; in period 2, they play the "battle of the sexes" game illustrated in figure 11.19.

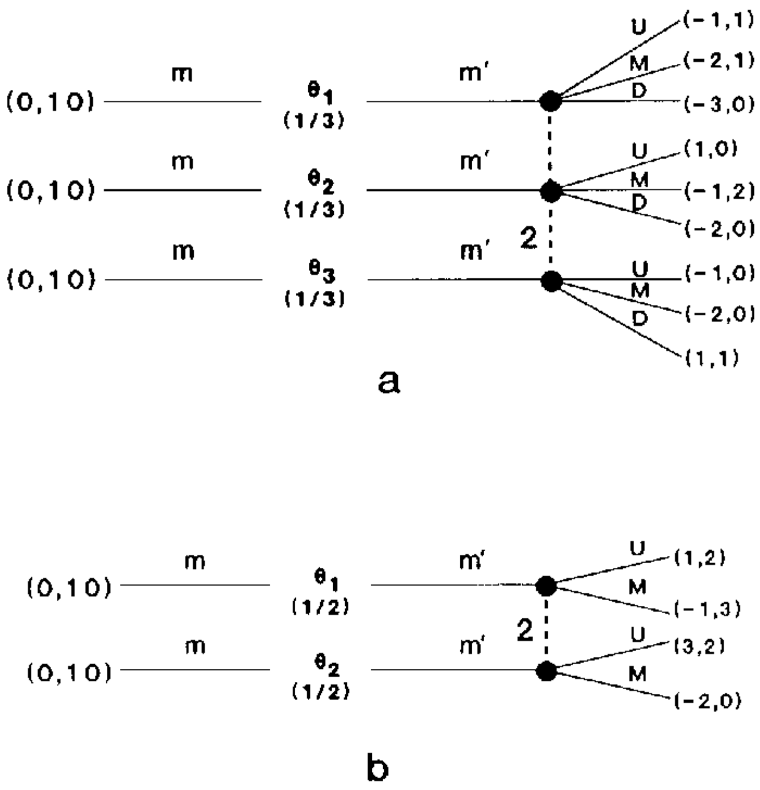


Figure 11.17

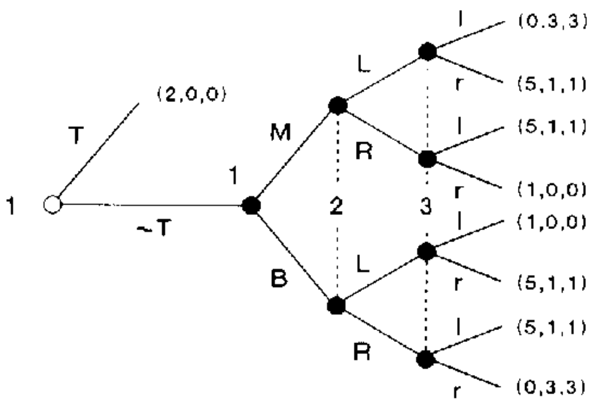


Figure 11.18

	S	W
S	0,0	3,1
W	1,3	0,0

Figure 11.19

(a) Show that strategic stability requires both players to burn utility, so that all stable equilibria are inefficient.

(b) Now suppose that the cost of burning is $1 + \varepsilon$, and that there is a publicly observed random variable ω at the beginning of period 2, where ω has the uniform distribution on the first 100 positive integers. Does the NWBR argument still work, and (***) are there stable components where the players do not burn utility?

Exercise 11.6**

(a) Show that there is a stable component where player 1 plays U with probability 1 in Figure 11.20.

(b) This strategic form is consistent with the extensive form depicted in figure 11.21. In this extensive form, there are three equilibria in the subgame following $\sim$U, only one of which, (M, L), gives player 1 a payoff greater than the 2 he gets from choosing U. What do you think "forward induction" should imply here?

Exercise 11.7** Verify that $\{((\frac{3}{5}\text{L}, \frac{2}{5}\text{RS}, 0\,\text{RW}), \ell), ((\frac{1}{3}\text{L}, 0\,\text{RS}, \frac{2}{3}\text{RW}), \ell)\}$ is a stable component of the game in figure 11.14.

Exercise 11.8** Consider the twice-repeated version of the "battle of the sexes" stage game shown in figure 11.19, where the players maximize the sum of their per-period payoffs.

(a) Show that, although the path where players choose (S, W) in both periods can be supported by a subgame-perfect equilibrium, the component

	L	M	R
U	2,2	2,2	2,2
M	3,3	0,2	3,0
D	0,0	3,2	0,3

Figure 11.20

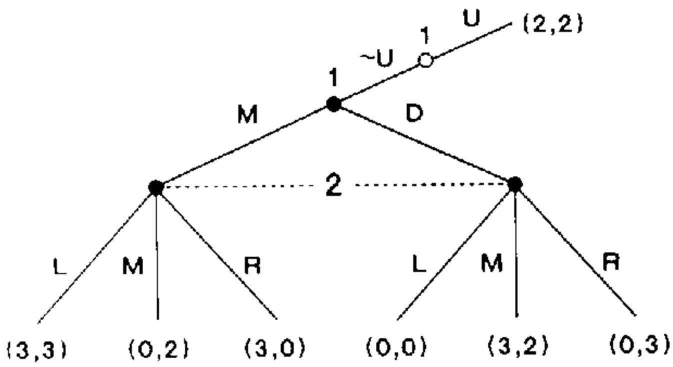


Figure 11.21

where (S, W) is played in both periods does not contain a stable set. (Hint: Use iterated applications of NWBR on the game's reduced strategic form.)

(b) Characterize the stable paths that are in pure strategies.

(c) Construct a stable component where both players randomize $(\frac{1}{2}, \frac{1}{2})$ in the first stage. (Van Damme (1989) and Osborne (1987) discuss forward induction in repeated games.)

Exercise 11.9*** Show that an essential equilibrium (see chapter 12) is "truly perfect."

Exercise 11.10** Consider Farqharson's (1969) model of plurality voting with ties (example 2, p. 73, in Moulin 1986). An election has three candidates or policies, A, B, and C, and three voters, $i = 1, 2, 3$. The voting rule is plurality voting, and player 1 breaks ties: The elected candidate is the one chosen by voters 2 and 3 if they vote for the same candidate, and the one chosen by voter 1 otherwise. Suppose that $u_1(A) > u_1(B) > u_1(C)$, $u_2(C) > u_2(A) > u_2(B)$, and $u_3(B) > u_3(C) > u_3(A)$. Show that iterated weak dominance predicts that candidate C will be elected! Comment.

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12 Advanced Topics in Strategic-Form Games

This chapter collects several classes of results on strategic-form games that require more apparatus than those in chapter 1. Readers lacking mathematical training are advised to skim through the issues, ideas, and results and ignore technical details. Some of the results are stated only for reference, with no attempt made at explaining the proofs.

Section 12.1 develops properties of finite strategic-form games that hold for generic strategic forms. Generically, strategic forms have a finite and odd number of equilibria, and these equilibria are robust in the sense that any perturbed game with nearby payoffs has equilibria that are nearby.

Section 12.2 extends the existence analysis of subsection 1.3.3 to games with "continuous" action spaces (i.e., convex subsets of $\mathbb{R}^n$) and discontinuous payoff functions.

Section 12.3 analyzes the properties of "supermodular games." Roughly speaking, in supermodular games each player's strategies are ordered, and each player's best response is increasing in his opponents' strategies. Supermodular games have pure-strategy Nash equilibria even if the payoffs are neither quasi-concave nor continuous. The sets of Nash-equilibrium strategies and of rationalizable strategies have upper and lower bounds, which furthermore coincide. Also, the learning and comparative-statics properties of supermodular games are straightforward.

12.1 Generic Properties of Nash Equilibria†††

Although Nash equilibria exist in every finite strategic-form game, some other interesting properties of the Nash concept hold only for "almost all finite strategic forms." This section examines two such properties: the finiteness and oddness of the number of equilibria, and the robustness of equilibria to small perturbations of the payoffs.

By "almost all" we mean the following: A finite game with I players in which each player i has $\# S_i$ strategies can be seen as a payoff vector $\{u_i(s)\}_{i \in \mathscr{I}, s \in S}$ in the Euclidean space of dimension $I \cdot \Pi_{i=1}^{I} \# S_i$. For a fixed set of I strategy spaces, "game u" is the game with the fixed strategy spaces and payoff vector u. "Almost all games" satisfy a property if the set of games (i.e., payoff vectors, with the number of players and the strategy spaces kept fixed) that satisfy this property is open and dense in the above Euclidean space. A property is satisfied for "generic games" if it is satisfied for "almost all games."

12.1.1 Number of Nash Equilibria

As Debreu (1970) showed (see Mas-Colell 1985), competitive economies "in general" have a finite and odd number of Walrasian equilibria. "In general" refers to the fact that oddness does not hold for any economy, but rather for almost all of them (more precisely, for an open and dense set of

	L	R
U	1,1	0,0
D	0,0	0,0

Figure 12.1

economies). We may wonder whether a similar result holds for the set of Nash equilibria of a game. It is easy to find games with an even number of Nash equilibria. For example, the game illustrated in figure 12.1 has two equilibria: the pure-strategy profiles (U, L) and (D, R). Wilson (1971) has shown that this game is "exceptional"[1]:

Theorem 12.1 (Wilson's (1971) Oddness Theorem) Almost all finite games have a finite and odd number of equilibria.

12.1.2 Robustness of Equilibria to Payoff Perturbations

In practice it is unlikely that the modeler will have specified payoff functions that are exactly correct. The issue is then whether the Nash predictions of the original game with payoffs u are approximate Nash predictions of the real game with nearby payoffs $\tilde{u}$.

The issue of robustness has many facets. In this subsection, we fix the strategic form (the set of players and their strategy spaces) and relax the assumption that the modeler has specified the correct payoffs, but we maintain the hypothesis that the payoffs are common knowledge among the players themselves. In chapters 11 and 14 we discuss other robustness issues by relaxing the assumption that payoffs are common knowledge among the players.

To define the notion of proximity in finite games, we introduce distances between payoff vectors and between strategy profiles. Let

$$u = \{u_i(s)\}_{i \in \mathscr{I}, s \in S}$$

and

$$\tilde{u} = \{\tilde{u}_i(s)\}_{i \in \mathscr{I}, s \in S}$$

denote two payoff vectors, and let

$$\sigma = \{\sigma_i(s_i)\}_{i \in \mathscr{I}, s_i \in S_i}$$

and

$$\tilde{\sigma} = \{\tilde{\sigma}_i(s_i)\}_{i \in \mathscr{I}, s_i \in S_i}$$

denote two mixed strategy profiles. Let

1. For further odd-number theorems see Eaves 1971, 1973, 1976 and Harsanyi 1973.

	L	R
U	1,1	0,0
D	0,0	$-\eta,-\eta$

Figure 12.2

$$D(u,\tilde{u}) = \max_{i \in \mathscr{I}, s \in S} |u_i(s) - \tilde{u}_i(s)|$$

and

$$d(\sigma,\tilde{\sigma}) = \max_{i \in \mathscr{I}, s_i \in S_i} |\sigma_i(s_i) - \tilde{\sigma}_i(s_i)|.$$

A Nash equilibrium of a game is "essential" or "robust" if there exists a nearby Nash equilibrium for any nearby game:

Definition 12.1 A Nash equilibrium σ of game u is *essential* or *robust* if for any $\varepsilon > 0$ there exists $\eta > 0$, such that for any $\tilde{u}$ such that $D(u,\tilde{u}) < \eta$ there exists a Nash equilibrium $\tilde{\sigma}$ of game $\tilde{u}$ such that $d(\sigma,\tilde{\sigma}) < \varepsilon$. A game u is essential if all its equilibrium points are essential.

Figure 12.1 gives an example of a nonessential game. The strategy profile $\sigma = (\mathrm{D},\mathrm{R})$ is a Nash equilibrium of game u in that figure. However, the only Nash equilibrium of the slightly perturbed game $\tilde{u}$ in figure 12.2 is $\tilde{\sigma} = (\mathrm{U},\mathrm{L})$. Note that $D(u,\tilde{u}) = \eta$ and $d(\sigma,\tilde{\sigma}) = 1$, and so one of the Nash equilibria of game u, viz. $\sigma = (\mathrm{D},\mathrm{R})$, is far away from the nearest (and only) Nash equilibrium, $\tilde{\sigma} = (\mathrm{U},\mathrm{L})$, of game $\tilde{u}$. Again, the game depicted in figure 12.1 is exceptional, as the following theorem demonstrates.

Theorem 12.2 (Wu and Jiang 1962) Almost all finite strategic-form games are essential.

The proof of this theorem relies on the essential fixed-point theorem of Fort (1950). Consider a compact metric space Σ with distance d. A fixed point σ of a continuous mapping f from Σ into itself is essential if for any $\varepsilon > 0$ there exists $\eta > 0$ such that, for any continuous mapping $\tilde{f}$ such that $d(f,\tilde{f}) = \max_{\sigma \in \Sigma} d(f(\sigma),\tilde{f}(\sigma)) < \eta$, there exists a fixed point $\tilde{\sigma}$ of $\tilde{f}$ with $d(\sigma,\tilde{\sigma}) < \varepsilon$. A mapping is essential if all its fixed points are essential. Fort's essential fixed-point theorem asserts that the set of essential mappings is dense in the set of continuous mappings. (This set is also open from its definition.)

Fort's theorem compares fixed points of nearby mappings. It does not quite yield a comparison of equilibria of nearby games. Recalling that Nash equilibria can be obtained as fixed points of certain continuous mappings, Wu and Jiang identify a game u with the "Nash mapping associated with game u." This Nash mapping is the function f_u from Σ into itself, where

$$f_u = \{f_u^{s_i}\}_{i \in \mathscr{I}, s_i \in S_i}$$

and

$$f_u^{s_i}(\sigma) = \frac{\sigma_i(s_i) + \max\{0, u_i(s_i, \sigma_{-i}) - u_i(\sigma)\}}{1 + \sum_{s_i' \in S_i} \max\{0, u_i(s_i', \sigma_{-i}) - u_i(\sigma)\}}.$$

f_u is continuous in σ, and σ is a fixed point of f_u if and only if σ is a Nash equilibrium of game u.[2]

The correspondence from payoffs u to Nash mappings f_u is not one-to-one. If u is replaced by $\tilde{u}$ such that $\tilde{u}_i(s) \equiv u_i(s) + v_i(s_{-i})$ for all i and s, where v_i is an arbitrary function from S_{-i} into $\mathbb{R}$, then $f_u = f_{\tilde{u}}$. More generally, one would like to consider equivalence classes of games. Two games u and $\tilde{u}$ having the same von Neumann-Morgenstern utility functions for all players (i.e., satisfying $\tilde{u}_i(s) = \lambda_i u_i(s) + v_i(s_{-i})$ for some $\lambda_i > 0$ and all i and s) are equivalent. It is easily seen that two equivalent games have not only the same set of Nash equilibria but also the same set of essential equilibria. To identify equivalent games, Wu and Jiang normalize games by requiring that

(i) $\sum_{s_i \in S_i} u_i(s_i, s_{-i}) = 0$ for all s_{-i}

and

(ii) $\sum_{\substack{s_{-i} \in S_{-i} \\ s_i, s_i' \in S_i}} |u_i(s_i, s_{-i}) - u_i(s_i', s_{-i})| =$ either 0 or 1.

(The 0 on the right-hand side of constraint ii is meant to accommodate the payoff function that is constant in player i's strategy; this payoff function is nongeneric anyway. Any payoff function that is not constant in player

2. That a Nash equilibrium is a fixed point of f_u is trivial. Conversely, a fixed point of f_u must satisfy, for all $s_i \in S_i$,

$$\sigma_i(s_i)\left(\sum_{s_i' \in S_i} \max\{0, u_i(s_i', \sigma_{-i}) - u_i(\sigma)\}\right) = \max\{0, u_i(s_i, \sigma_{-i}) - u_i(\sigma)\}.$$

Let $\tilde{S}_i \subseteq S_i$ denote the support of σ_i. Because $\sum_{s_i \in \tilde{S}_i} \sigma_i(s_i) = 1$,

$$\sum_{s_i' \in S_i} \max\{0, u_i(s_i', \sigma_{-i}) - u_i(\sigma)\} = \sum_{s_i' \in \tilde{S}_i} \max\{0, u_i(s_i', \sigma_{-i}) - u_i(\sigma)\},$$

which implies that, for all $s_i' \notin \tilde{S}_i$,

$$u_i(s_i', \sigma_{-i}) \leq u_i(\sigma).$$

If $\sigma_i(s_i) > 0$, then either

$$u_i(s_i, \sigma_{-i}) \leq u_i(\sigma) \text{ for all } s_i \in S_i,$$

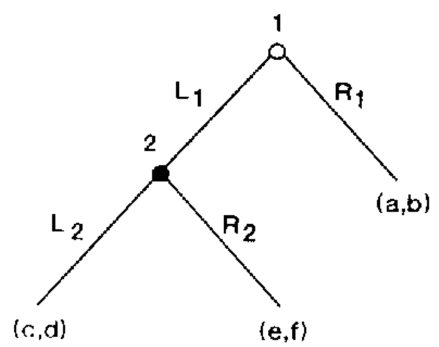
and then σ_i is a best response to σ_{-i}, or

$$u_i(s_i, \sigma_{-i}) > u_i(\sigma) \text{ for all } s_i \in \tilde{S}_i,$$

which is impossible.

i's strategy can be scaled up or down so that the sum in constraint ii is equal to 1.) Constraints i and ii eliminate the $1 + \Pi_{j \neq i}(\# S_j)$ degrees of freedom left in the specification of player i's preferences before normalization for each i.[3] It is straightforward to show that, in the compact metric space of normalized games, two games are equivalent if and only if they are identical. Wu and Jiang then apply Fort's theorem to the subspace of all Nash mappings corresponding to normalized games.

Remark The Wu-Jiang theorem states that the Nash equilibria of generic strategic-form games are robust to perturbations of the payoffs. This is of interest only for simultaneous-move games. To see this, consider the extensive form in figure 12.3a. It depicts a sequential-move game in which player 1 first chooses between L_1 and R_1. If player 1 chooses R_1, the game stops and the payoffs are (a, b). If he chooses L_1, player 2 gets to play. The payoffs are (c, d) if player 2 chooses L_2, and (e, f) if he chooses R_2. Figure 12.3b depicts the associated strategic form, which prescribes payoffs for any pair of strategies. (Strategies L_2 and R_2 for player 2 are shorthand for "L_2 if L_1" and "R_2 if L_1"). It is clear that *genericity in the tree is not equivalent to genericity in the strategic form*: For a given game tree (extensive form), generic extensive-form payoffs can lead to nongeneric payoffs in the strategic form. In figure 12.3b, the payoffs corresponding to the lower row of the strategic form cannot be perturbed independently. That is, for a given game tree, the payoffs are constrained to belong to a subspace of the space of



a. Extensive form

	L_2	R_2
L_1	c,d	e,f
R_1	a,b	a,b

b. Strategic form

Figure 12.3

3. There are $\Pi_{j \neq i}(\# S_j)$ equations in constraint i, and 1 in constraint ii.

strategic-form payoffs $\mathbb{R}^{I \cdot \Pi_i(\# S_i)}$ which in general has measure 0 in that space. (For instance, in the game of figure 12.3, the (unnormalized) payoffs in the extensive form belong to $\mathbb{R}^6$ and the set of (unnormalized) payoffs in the strategic form is $\mathbb{R}^8$.) Generic results in the set of strategic-form payoffs are then meaningless.

12.2 Existence of Nash Equilibrium in Games with Continuous Action Spaces and Discontinuous Payoffs†††

A number of games in the economic literature have discontinuous and/or non-quasi-concave payoff functions. Economic models often have payoffs that are not quasi-concave. On the other hand, one might argue that discontinuities are sometimes built in by the modeler and that small perturbations of the game "smooth" the payoff functions. For instance, in the Hotelling model discussed below, which assumes that products are differentiated only by "location," at some price profiles a small cut in price allows one firm to corner the other firm's "backyard market." When another parameter of differentiation—such as different tastes for quality (if qualities differ) or different transportation costs among consumers—is introduced, the discontinuity may disappear (see, e.g., De Palma et al. 1985). Yet discontinuous games are sometimes of interest. First, smoothing usually requires a more complex model. Second, mechanism design (such as the design of an optimal auction) leads to the consideration of discontinuous games. For instance, a seller with an object for sale may want to offer it to the highest bidder, creating a discontinuous game for the buyers.

Consider the Hotelling model of competition on the line developed in example 1.4. Consumers are distributed uniformly along the interval [0, 1] and have unit transportation cost t. Suppose, in contrast with example 1.4, that firms are located in the interior of the interval, firm 1 at $x = \frac{1}{3}$ and firm 2 at $x = \frac{2}{3}$. Again, we assume that the buyers' valuation for the good supplied by the firms is sufficiently large that we do not have to worry about the buyers' not purchasing in the relevant price range. Consider a consumer located at $x \leq \frac{1}{3}$. This consumer belongs to firm 1's "back yard." His choice between the two firms is determined by the comparison between the generalized prices $p_1 + t(\frac{1}{3} - x)$ and $p_2 + t(\frac{2}{3} - x)$, i.e., between p_1 and $p_2 + t/3$. Thus, all consumers located to the left of firm 1 always make the same brand choice as the consumer located at $x = \frac{1}{3}$. The firms' demands are thus discontinuous at $p_2 = p_1 - t/3$. Figure 12.4 depicts firm 2's profit function u_2 for $p_1 \in (c + t/3, c + 5t/3)$, which is both discontinuous and non-quasi-concave.[4] D'Aspremont, Gabszewicz, and Thisse (1979) showed that there exists no pure-strategy equilibrium for this game.

4. Figure 12.4 assumes that when $p_2 = p_1 - t/3$ or $p_2 = p_1 + t/3$, so that the consumers in the back yard of one of the firms are indifferent between the two firms, these consumers go to the nearer firm. Of course, alternative conventions could be made.

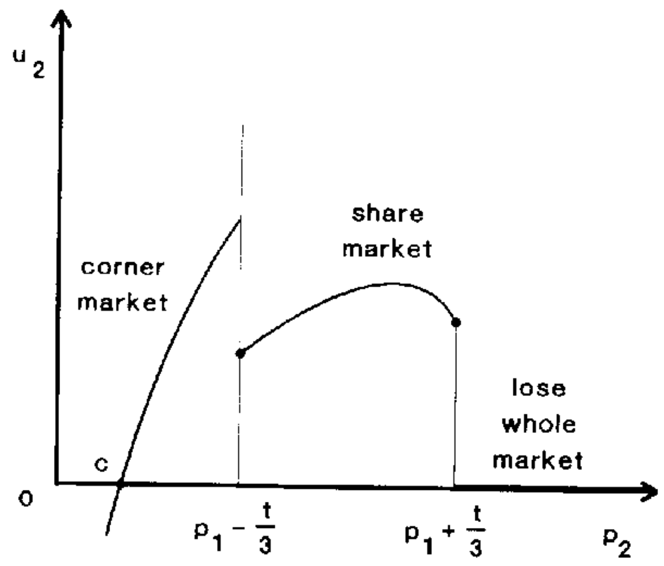


Figure 12.4

Dasgupta and Maskin (1986a) supply two existence theorems for discontinuous games. First, assuming quasi-concavity, they provide conditions (upper semi-continuity and continuous maximum) that are weaker than continuity and allow the use of Kakutani's theorem to guarantee the existence of a pure-strategy equilibrium. Second, they provide conditions for the existence of mixed-strategy equilibria in games without quasi-concave payoffs.

12.2.1 Existence of a Pure-Strategy Equilibrium

With discontinuous payoffs, a compact strategy space no longer ensures that a player's optimal reaction to his opponents' strategies exists. To guarantee existence, we assume that payoff functions are upper semi-continuous. An upper semi-continuous function is a function that has no jumps down.

Definition 12.2 A function $u_i(\cdot)$ on S is *upper semi-continuous* at s, if, for any sequence s^n converging to s,

$$\limsup_{n \to +\infty} u_i(s^n) \le u_i(s).$$ [5]

Note that the function u_2 depicted in figure 12.4 fails to be upper semi-continuous at $p_2 = p_1 - t/3$. (We will show that this example has a mixed-strategy equilibrium as it satisfies a weak form of upper semi-continuity; however, there is no point in trying to "patch" upper semi-continuity to prove the existence of a pure-strategy equilibrium for this game, because payoffs are not quasi-concave.)

5. The limit superior or "lim sup" of a sequence $x^n \in \mathbb{R}$ is the smallest x such that, for all $\varepsilon > 0$, there is an N such that $x^n \le x + \varepsilon$ for all $n > N$. Similarly, the limit inferior or "lim inf" of a sequence $x^n \in \mathbb{R}$ is the largest x such that, for all $\varepsilon > 0$, there is an N such that $x^n \ge x - \varepsilon$ for all $n > N$.

Let

$$r_i^*(s_{-i}) \equiv \left\{ s_i \in S_i \,|\, u_i(s_i, s_{-i}) = \max_{s_i' \in S_i} u_i(s_i', s_{-i}) \right\}$$

denote the set of player i's optimal *pure-strategy reactions* to pure strategies s_{-i}. If the strategy set S_i is compact, a maximand exists and $r_i^*(s_{-i})$ is indeed non-empty-valued for all s_{-i}. (To see this, consider a sequence s_i^n such that $\lim_{n \to \infty} u_i(s_i^n, s_{-i}) = \sup_{s_i' \in S_i} u_i(s_i', s_{-i})$. Because S_i is compact, s_i^n has a converging subsequence, with limit $\bar{s}_i \in S_i$, say. But upper semi-continuity of u_i implies that $u_i(\bar{s}_i, s_{-i}) \geq \sup_{s_i' \in S_i} u_i(s_i', s_{-i})$, and thus an optimal reaction exists.) Note that r_i^* differs from the best response or *reaction correspondence*, r_i, which for a given s_{-i} is the convex hull of the points in r_i^* (i.e., includes the mixed best responses).

To prove the existence of a pure-strategy equilibrium, we follow the method of theorem 1.2. That is, we use Kakutani's theorem to prove that the pure-strategy reaction correspondence $r^*: S \rightrightarrows S$ (defined by $[r^*(s)]_i = r_i^*(s)$) has a fixed point. We will assume that the strategy spaces are compact, convex, nonempty subsets of finite Euclidean spaces. Because r^* is thus nonempty (as noted above), it remains to make assumptions that ensure that r^* is convex valued and has a closed graph.

As in theorem 1.2, to guarantee convex-valuedness we require the payoff functions to be quasi-concave in their own strategy. That is, for all s_{-i}, the set of s_i such that $u_i(s_i, s_{-i}) \geq k$ is convex for all k and thus, in particular, is convex for $k = \max_{s_i \in S_i} u_i(s_i, s_{-i})$.

To ensure that the pure-strategy reaction correspondence has a closed graph (that is, if

$$(s_i^n, s_{-i}^n) \xrightarrow[n \to \infty]{} (s_i, s_{-i})$$

and

$$s_i^n \in r_i^*(s_{-i}^n) \text{ for all } n,$$

then $s_i \in r_i^*(s_{-i})$), we need an assumption that, together with upper semi-continuity of payoffs, ensures closed graph:

Definition 12.3 A function u_i has a *continuous maximum*[6] if $u_i^*(s_{-i}) \equiv \max_{s_i} u_i(s_i, s_{-i})$ is continuous in s_{-i}.

It is easy to see that a continuous maximum and upper semi-continuity imply that r_i^* has a closed graph. If not, there is a sequence $(s_i^n, s_{-i}^n) \to (\bar{s}_i, s_{-i})$ with $s_i^n \in r_i^*(s_{-i}^n)$, but $\bar{s}_i \notin r_i^*(\bar{s}_{-i})$. Then

6. Dasgupta and Maskin impose instead the stronger condition of "graph continuity." A function u_i is graph continuous if, for all $\bar{s}$, there exists a function $f_i: S_{-i} \to S_i$ such that $\bar{s}_i = f_i(\bar{s}_{-i})$ and such that $u_i(f_i(s_{-i}), s_{-i})$ is continuous at $s_{-i} = \bar{s}_{-i}$.

$$\max_{s_i} u_i(s_i, s_{-i}) > u_i(s_i, s_{-i})$$

$$> \limsup_{n \to \infty} u_i(s_i^n, s_{-i}^n) = \limsup_{n \to \infty} \left[\max_{s_i} u_i(s_i, s_{-i}^n) \right],$$

contradicting the assumption of a continuous maximum. We have therefore proved the following:

Theorem 12.3 (Dasgupta and Maskin 1986a) Let S_i be a nonempty, convex, and compact subset of a finite-dimensional Euclidean space, for all i. If, for all i, u_i is quasi-concave in s_i, is upper semi-continuous in s, and has a continuous maximum, there exists a pure-strategy Nash equilibrium.

12.2.2 Existence of a Mixed-Strategy Equilibrium

The idea of the Dasgupta-Maskin result on the existence of a mixed-strategy equilibrium is to approximate the strategy spaces (which are closed intervals of $\mathbb{R}$) by finite grids, and to provide conditions ensuring that the limits of the Nash equilibria of the discretized games do not have "atoms" (nonnegligible probability) on any of the discontinuity points of the payoff functions.[7]

Consider a sequence of finite approximations S_i^n of S_i converging to S_i for all i. By Nash's existence theorem, each discretized game with strategy sets $\times_i S_i^n$ has a mixed-strategy equilibrium $\sigma^n \equiv (\sigma_1^n, \ldots, \sigma_I^n)$; that is,

$$u_i(\sigma_i^n, \sigma_{-i}^n) \geq u_i(s_i, \sigma_{-i}^n) \text{ for all } s_i \in S_i^n \text{ and for all } i. \tag{12.1}$$

Because the space of probability measures on S_i is compact under the topology of weak convergence, there is a subsequence of Nash-equilibrium mixed-strategy profiles, which without loss of generality can be taken to be the sequence itself, that converges to some mixed strategy σ^* on S. Now, if payoffs were continuous, we would be finished: $u_i(\sigma_i^n, \sigma_{-i}^n)$ and $u_i(s_i, \sigma_{-i}^n)$ would converge to $u_i(\sigma_i^*, \sigma_{-i}^*)$ and $u_i(s_i, \sigma_{-i}^*)$, respectively, and the limit strategies σ^* would form a Nash equilibrium of the limit game (this is the essence of theorem 1.3). More generally, if the equilibria σ^n put vanishingly small probability on the discontinuity points of the payoff functions, σ^* would be a Nash equilibrium. Thus, the challenge is to find conditions that ensure that discontinuity points do not matter in the limit game.

Dasgupta and Maskin introduce two assumptions. First, they require that the *sum* of payoffs ($\sum_i u_i$) be upper semi-continuous.[8] (This assumption

7. Simon (1987) relaxes this condition by requiring only that at least one limit has this no-atom property, instead of all of them.

8. Dasgupta and Maskin give the following example of a game that does not satisfy upper semi-continuity of the sum of the payoffs (but satisfies the other assumptions of theorem 12.4) and does not have a mixed-strategy equilibrium: Let $I = 2$, $S_i = [0, 1]$, and $u_i(s_1, s_2) = 0$ if $s_1 = s_2 = 1$ and $-s_i$ otherwise. If one player puts positive weight on 1, then the other player has no optimal strategy, as he wants to play as close as possible to 1 but doesn't want to play 1. And if both players put zero weight on 1, then each wants to play 1—a contradiction.

is satisfied in the Hotelling game.) In particular, they require that this sum not jump down in the limit of the equilibrium strategies

$$\limsup_{n \to \infty} \sum_{i=1}^{I} u_i(\sigma^n) \leq \sum_{i=1}^{I} u_i(\sigma^*). \tag{12.2}$$

Next, they make the assumption of weakly lower semi-continuous payoffs: Let $S^{**}(i)$ denote the set of s such that u_i is discontinuous at s and

$$S_{-i}^{**}(s_i) \equiv \{s_{-i} \in S_{-i} | (s_i, s_{-i}) \in S^{**}(i)\}.$$

Assume that discontinuities occur only on a subset (of measure 0) in which a player's strategy is "related" to another player's. That is, for any two players i and j, there exist a finite number of functions $f_{ij}^d: S_i \to S_j$, where d is an index, that are one-to-one and continuous[9] such that, for each i,

$$S^{**}(i) \subseteq S^*(i) = \{s \in S | \exists j \neq \text{i}, \exists d \text{ such that } s_j = f_{ij}^d(s_i)\}.$$

In the Hotelling example above, discontinuities arose when $p_1 = p_2 - t/3$ or $p_1 = p_2 + t/3$.

$u_i(s)$ is *weakly lower semi-continuous* in s_i if for all s_i there exists $\lambda \in [0, 1]$ such that, for all $s_{-i} \in S_{-i}^{**}(s_i)$,

$$\lambda \liminf_{s_i' \uparrow s_i} u_i(s_i', s_{-i}) + (1 - \lambda) \liminf_{s_i' \downarrow s_i} u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i}).$$

In a sense, this says that u_i does not jump up when s_i' tends to s_i either from the left, or from the right, or both. Roughly, this assumption implies that player i can do about as well with strategies near s_i as with s_i, even if player i's rivals' strategies put weight on the discontinuity points of u_i.[10] Weak lower semi-continuity holds in the Hotelling game.[11] The proof of existence of a mixed-strategy equilibrium under these assumptions is involved, and we refer the reader to the original paper.

9. The assumption that these functions are one-to-one prevents discontinuity curves that are "vertical" or "horizontal" in the Cartesian product space of strategies. The following example (inspired by example 4 of Dasgupta and Maskin) demonstrates the possibility of nonexistence if this assumption is not satisfied. Let $I = 2$ and $S_i = [0, 1]$. Let $u_i(s_1, s_2) = -(s_i - \frac{1}{2})^2$ if $s_i, s_j \neq \frac{1}{2}$; $= -1$ if $s_i = \frac{1}{2}$ and $s_j \neq \frac{1}{2}$; $= +1$ if $s_i \neq \frac{1}{2}$ and $s_j = \frac{1}{2}$; $= 0$ if $s_i = s_j = \frac{1}{2}$. That is, each player wants to be as close as possible to $\frac{1}{2}$, but not to play $\frac{1}{2}$ (which would transfer payoff to his rival). The sum of the payoffs is upper semi-continuous: If anything, Σu_i jumps up at the horizontal and vertical lines corresponding to $s_1 - \frac{1}{2}$ and $s_2 = \frac{1}{2}$. Furthermore, u_i is weakly lower semi-continuous; indeed, for any s_j, player i does better by playing close to $\frac{1}{2}$ than by playing $\frac{1}{2}$. But there is no mixed-strategy equilibrium: For any mixed strategy of his rival, a player wants to play as close as possible to $\frac{1}{2}$, but not to play $\frac{1}{2}$.

10. That weak lower semi-continuity is needed for existence is demonstrated in exercise 12.2.

11. Note that $S_1^{**}(p_2) = \{p_2 - t/3, p_2 + t/3\}$, and, for all $p_1 \in S_1^{**}(p_2)$,

$$\liminf_{p_2' \uparrow p_2} u_2(p_2', p_1) \geq u_2(p_2, p_1)$$

(because firm 2, by undercutting a little, can make sure that it sells to its own back yard or that it invades the other's, depending on the case).

Theorem 12.4 (Dasgupta and Maskin 1986a) Let S_i be a closed interval of $\mathbb{R}$. Suppose that u_i is continuous except on a subset $S^{**}(i)$ of $S^*(i)$, where $S^*(i)$ is defined above; that $\sum_{i=1}^{I} u_i(s)$ is upper semi-continuous; and that $u_i(s_i, s_{-i})$ is bounded and weakly lower semi-continuous in s_i. Then the game has a mixed-strategy equilibrium.

Dasgupta and Maskin prove other existence theorems, in particular for the case in which a discontinuity in a player's payoff occurs independent of discontinuities in the other players' payoffs (as is the case when firms must incur a fixed cost to be in the market). They also prove that symmetric games satisfying the assumptions of theorem 12.4 possess a symmetric mixed-strategy equilibrium. They apply theorem 12.4 to examples such as the above Hotelling game, price competition with capacity constraints, and the insurance market with adverse selection (Dasgupta and Maskin 1986b).

12.3 Supermodular Games†††

Supermodular games, developed by Topkis (1979), were applied to economic problems first by Vives (1990) and then by Milgrom and Roberts (1990). Roughly, they are games in which each player's marginal utility of increasing his strategy rises with increases in his rivals' strategies. In such games the best response correspondences are increasing, so that the players' strategies are "strategic complements." When there are two players, a change in variables allows this framework to also accommodate the case of decreasing best responses (that is, "strategic substitutes").[12]

Supermodular games are particularly well behaved. They have pure-strategy Nash equilibria. The upper bound (defined below) of player i's Nash-equilibrium strategies exists (which is not trivial if the strategy sets are not one-dimensional) and is a best response to the upper bounds of his rivals' sets of Nash-equilibrium strategies, and similarly for the lower bounds. Furthermore, the upper and lower bounds of the sets of Nash equilibria and rationalizable strategies coincide.

The simplicity of supermodular games makes convexity and differentiability assumptions unnecessary, although they are satisfied in most applications. What is needed for the theory is an order structure on strategy spaces and a weak continuity requirement on payoffs, in addition to the above-mentioned property that the marginal utility of each player's strategy is monotonic in the strategies of his rivals, and a "supermodularity requirement."

Suppose that each player i's strategy set S_i is a subset (not necessarily compact and convex) of a finite-dimensional Euclidean space $\mathbb{R}^{m_i}$. Then

12. See Bulow et al. 1985 and Fudenberg and Tirole (1984) for discussions of the use of these concepts in industrial organization. (Bulow et al. coined the strategic complements/substitutes terminology.)

$S \equiv \times_{i=1}^{I} S_i$ is a subset of $\mathbb{R}^m$, where $m \equiv \sum_{i=1}^{I} m_i$. Let x and y denote two vectors in some Euclidean space $\mathbb{R}^K$. Let $x \geq y$ if $x_k \geq y_k$ for all $k = 1, \ldots, K$, and let $x > y$ if $x \geq y$ and there exists k such that $x_k > y_k$. The order $>$ is only a partial order: If a vector dominates another in one component but is dominated in another component, the vectors cannot be compared. Next we define the "meet" $x \wedge y$ and the "join" $x \vee y$ of x and y:

$$x \wedge y \equiv (\min(x_1, y_1), \ldots, \min(x_K, y_K)),$$

$$x \vee y \equiv (\max(x_1, y_1), \ldots, \max(x_K, y_K)).$$

S is a *sublattice* of $\mathbb{R}^m$ if $s \in S$ and $\tilde{s} \in S$ imply that $s \wedge \tilde{s} \in S$ and $s \vee \tilde{s} \in S$.[13]

A set S has a *greatest* element $\bar{s}$ (respectively, a *least* element $\underline{s}$) if $s \leq \bar{s}$ (respectively, $s \geq \underline{s}$) for all $s \in S$. A topological result of Birkhoff (1967) says that if S is a nonempty, compact sublattice of $\mathbb{R}^m$, it has a greatest element and a least element.

The following notion formalizes the notion of strategic complementarity:

Definition 12.4 $u_i(s_i, s_{-i})$ has *increasing differences in* (s_i, s_{-i}) if, for all $(s_i, \tilde{s}_i) \in S_i^2$ and $(s_{-i}, \tilde{s}_{-i}) \in S_{-i}^2$ such that $s_i \geq \tilde{s}_i$ and $s_{-i} \geq \tilde{s}_{-i}$,

$$u_i(s_i, s_{-i}) - u_i(\tilde{s}_i, s_{-i}) \geq u_i(s_i, \tilde{s}_{-i}) - u_i(\tilde{s}_i, \tilde{s}_{-i}).$$

$u_i(s_i, s_{-i})$ has *strictly increasing differences in* (s_i, s_{-i}) if, for all $(s_i, \tilde{s}_i) \in S_i^2$ and $(s_{-i}, \tilde{s}_{-i}) \in S_{-i}^2$ such that $s_i > \tilde{s}_i$ and $s_{-i} > \tilde{s}_{-i}$,

$$u_i(s_i, s_{-i}) - u_i(\tilde{s}_i, s_{-i}) > u_i(s_i, \tilde{s}_{-i}) - u_i(\tilde{s}_i, \tilde{s}_{-i}).$$

Increasing differences says that an increase in the strategies of player i's rivals raises the desirability of playing a high strategy for player i (see theorem 12.7).

Definition 12.5 $u_i(s_i, s_{-i})$ is *supermodular* in s_i if for each s_{-i}

$$u_i(s_i, s_{-i}) + u_i(\tilde{s}_i, s_{-i}) \leq u_i(s_i \wedge \tilde{s}_i, s_{-i}) + u_i(s_i \vee \tilde{s}_i, s_{-i})$$

for all $(s_i, \tilde{s}_i) \in S_i^2$. u_i is *strictly supermodular* in s_i if this inequality is strict whenever s_i and $\tilde{s}_i$ cannot be compared with respect to $\geq$.

Note that supermodularity is automatically satisfied if S_i is single-dimensional. We will need supermodularity in the case of multi-dimensional strategy spaces to prove that each player's best responses are increasing with his rivals' strategies. To see why, suppose that $m_i = 2$. From increasing differences, if s_{-i} increases, the optimal $s_{i,1}$ for a given $s_{i,2}$ increases and so does the optimal $s_{i,2}$ for a given $s_{i,1}$. However, if $\partial^2 u_i / \partial s_{i,1} \partial s_{i,2} < 0$ (with u_i assumed differentiable), a higher $s_{i,2}$ makes a lower $s_{i,1}$ desirable, and

13. Note that $\mathbb{R}^m$ is a lattice in that any two vectors x and y have a meet and a join in $\mathbb{R}^m$.

conversely. This indirect effect of an increase in s_{-i} may outweigh the direct effect, which means that the effect of an increase in s_{-i} on $s_{i,k}$ $(k = 1, 2)$ is ambiguous; all we can say is that $s_{i,1}$ and $s_{i,2}$ cannot both decrease, because this would contradict increasing differences. The supermodularity assumption is thus an assumption of complementarity among the components of a player's strategies; it ensures that these components move together when the rivals' strategies (or the exogenous environment) change.

As Topkis has shown, if $S_i = \mathbb{R}^{m_i}$ and if u_i is twice continuously differentiable in s_i, then u_i is supermodular in s_i if and only if, for any two components s_{ik} and $s_{i\ell}$ of s_i (with $k \neq \ell$), $\partial^2 u_i/\partial s_{ik}\partial s_{i\ell} \geq 0$.

Definition 12.6 A *supermodular game* (respectively, a strictly supermodular game) is such that, for each i, S_i is a sublattice of $\mathbb{R}^{m_i}$, u_i has increasing differences (strictly increasing differences) in (s_i, s_{-i}), and u_i is supermodular (strictly supermodular) in s_i.

Remark Increasing differences in (s_i, s_{-i}) and supermodularity in s_i are both implied by *supermodularity in s*, which requires that, for all s and $\tilde{s}$,

$$u_i(s \vee \tilde{s}) + u_i(s \wedge \tilde{s}) \geq u_i(s) + u_i(\tilde{s}).$$

Supermodularity in s clearly implies supermodularity in s_i (take s and $\tilde{s}$ to differ only in s_i in the above definition). One can see that it implies increasing differences by considering $s_i \geq \tilde{s}_i$ and $s_{-i} \geq \tilde{s}_{-i}$ and letting $u \equiv (\tilde{s}_i, s_{-i})$ and $v \equiv (s_i, \tilde{s}_{-i})$. Then $u \vee v = (s_i, s_{-i})$ and $u \wedge v = (\tilde{s}_i, \tilde{s}_{-i})$. Applying the supermodularity definition to u and v yields increasing differences. In practice, it is often easier to recognize increasing differences than to recognize supermodularity.

If u_i is twice continuously differentiable, u_i is supermodular if and only if, for any two components s_ℓ and s_k of s, $\partial^2 u_i/\partial s_\ell \partial s_k \geq 0$.[14]

Examples[15]

Bertrand game Consider an oligopoly with demand functions

$$D_i(p_i, p_{-i}) = a_i - b_i p_i + \sum_{j \neq i} d_{ij} p_j,$$

14. To see this, let e_ℓ be the vector equal to 1 for the ℓth component and 0 for the other components. Let ε and η be the two positive infinitesimals. Then supermodularity means that, for all s,

$$u_i(s + e_\ell \varepsilon) + u_i(s + e_k \eta) \leq u_i(s) + u_i(s + e_\ell \varepsilon + e_k \eta),$$

or

$$(\varepsilon\eta)\frac{\partial^2 u_i}{\partial s_\ell \partial s_k} \geq 0.$$

The proof of the converse is omitted.

15. See Topkis 1979, Vives 1990, and Milgrom and Roberts 1990 for other applications.

where $b_i > 0$ and $d_{ij} > 0$. Let

$$u_i(p_i, p_{-i}) = (p_i - c_i)D_i(p_i, p_{-i}).$$

Then $\partial^2 u_i/\partial p_i \partial p_j > 0$ for all $i, j \neq i$, so the game in which firms choose prices simultaneously has increasing differences. But many Bertrand games are not supermodular. For instance, the Hotelling game described in section 12.2 does not have increasing differences: Though firm i's best response to p_j is increasing in p_j as long as it is optimal for firm i to share the market (the demand function has the above linear form in the range of prices for which both firms have positive market share), an increase in p_j may make it more attractive for firm i to corner the whole market i.e., to lower its price to $(p_j - t/3)$.[16]

Cournot game Consider a duopoly. Firm i ($i \in \{1, 2\}$) chooses a quantity $q_i \in [0, \bar{q}_i]$. Suppose that the inverse demand functions $P_i(q_i, q_j)$ are twice continuously differentiable, and that P_i and firm i's marginal revenue (i.e., $P_i + q_i \partial P_i/\partial q_i$) are decreasing in q_j. Firm i's cost, $C_i(q_i)$, is assumed differentiable. The payoffs are

$$u_i(q_i, q_j) = q_i P_i(q_i, q_j) - C_i(q_i).$$

If $s_1 \equiv q_1$ and $s_2 \equiv -q_2$, the transformed payoffs satisfy $\partial^2 u_i/\partial s_i \partial s_j \geq 0$ for all $i \neq j$ (note that this transformation works only for $I = 2$). Thus, the game is supermodular.

Aggregate-demand externalities The stag-hunt game of chapter 1 is supermodular. Let "hunt the hare" be action 1 and "hunt the stag" be action 2. The game exhibits increasing differences in that, if a player hunts the stag instead of the hare, hunting the stag becomes more attractive to the other players.

As we note in chapter 1, aggregate-demand-externality models in macroeconomics have a similar flavor. For example, a simple search model *à la* Diamond (1982) has payoff functions

$$u_i(s) = \alpha s_i \sum_{j \neq i} s_j - c(s_i),$$

where s_i is player i's search intensity, $c(s_i)$ is the cost of search, $s_i \sum_{j \neq i} s_j$ is the probability of finding a trading partner, and α is the gain when a partner is found. Note that $\partial^2 u_i/\partial s_i \partial s_j = \alpha > 0$ for $j \neq i$. The game is supermodular (and in general has multiple equilibria, some with high search activity and some with low search activity). See also exercise 12.3.

16. For instance, for $c_2 = 0$, firm 2 charges $p_2 = (p_1 + t)/2$ for $p_1 \in [0, (3 - 4/\sqrt{3})t)$ and undercuts to $p_1 - t/3 < (p_1 + t)/2$ for p_1 a bit above $(3 - 4/\sqrt{3})t$.

Remark Vives (1990, section 6) noted that the theory of supermodular games can also be applied to games in which players have private information. We invite the reader to think about why this is so.[17]

From the point of view of existence of a pure-strategy Nash equilibrium, supermodular games derive their interest from the following result:

Theorem 12.5 (Tarski 1955) If S is a nonempty, compact sublattice of $\mathbb{R}^m$ and $f: S \to S$ is increasing ($f(x) \leq f(y)$ if $x \leq y$), f has a fixed point in S.

To obtain intuition about this fixed-point theorem, consider the single-dimensional case $S = [0, 1]$, which is depicted in figure 12.5. In order not to have a fixed point, the function f in figure 12.5 would need to "escape" the area above the diagonal and "jump into" the area below the diagonal; but increasing functions do not jump down. In the multi-dimensional case the intuition is the same, as no component of $f(x)$ jumps down when an arbitrary component of x increases.[18]

Tarski's theorem is relevant here because the set $r_i^*(s_{-i})$ of s_i that maximize $u_i(\cdot, s_{-i})$ turns out to be a sublattice and to "increase" with s_{-i}, as we now show.

If S_i is compact and u_i is upper semi-continuous in s_i, r_i^* is nonempty since $u_i(s_i, s_{-i})$ attains a maximum in s_i on S_i. (Consider a sequence s_i^n such

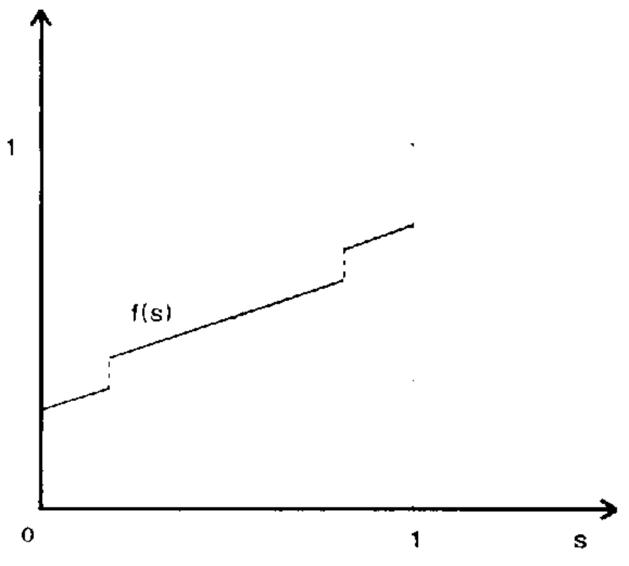


Figure 12.5

17. For an interesting application, see, in Milgrom and Roberts 1990, the Hendricks-Kovenock (1989) oil-drilling game, in which each firm would like the other to explore the other's tract in order to learn about the profitability of its own tract.

18. A related result of Tarski (in the single-dimensional case) is that a function f from $[0, 1]$ to $[0, 1]$ which has no downward jump has a fixed point, even if the function is not everywhere nondecreasing. (Figure 12.5 again yields the intuition for this result.) Vives (1990) uses this second result of Tarski to give a simple proof of the result (originally due to McManus (1964) and Roberts and Sonnenschein (1977)) that a (symmetric) pure-strategy equilibrium exists in symmetric homogeneous-good Cournot games with convex cost functions.

that $\sup_{s_i' \in S_i} u_i(s_i', s_{-i}) = \lim_{n\to+\infty} u_i(s_i^n, s_{-i})$. Compactness implies that there exists a converging subsequence $s_i^n \to s_i$. Upper semi-continuity implies that $u_i(s_i, s_{-i}) \geq \limsup_{n\to+\infty} u_i(s_i^n, s_{-i})$, so that s_i is indeed a best response to s_{-i}.)

To show that $r_i^*(s_{-i})$ is a sublattice for each s_{-i}, suppose that s_i and $\tilde{s}_i$ are both elements of $r_i^*(s_{-i})$ and that $u_i(s_i \wedge \tilde{s}_i, s_{-i}) < u_i(s_i, s_{-i}) = u_i(\tilde{s}_i, s_{-i})$. Supermodularity of $u_i(\cdot, s_{-i})$ then implies that $u_i(s_i \vee \tilde{s}_i, s_{-i}) > u_i(s_i, s_{-i}) = u_i(\tilde{s}_i, s_{-i})$, which contradicts the assumption that s_i and $\tilde{s}_i$ are best responses to s_{-i}. The same reasoning applies to the join.

Because $r_i^*(s_{-i})$ is a nonempty compact sublattice of $\mathbb{R}^{m_i}$, it has a greatest element $\bar{s}_i(s_{-i})$. We leave it to the reader to check that increasing differences of u_i implies that $\bar{s}_i(\cdot)$ is nondecreasing:

$$s_{-i} \geq \tilde{s}_{-i} \Rightarrow \bar{s}_i(s_{-i}) \geq \bar{s}_i(\tilde{s}_{-i}).$$

We can now apply Tarski's theorem to $f(\tilde{s}) = (\bar{s}_1(\tilde{s}), \ldots, \bar{s}_I(\tilde{s}))$. By construction, a fixed point $\bar{s}$ of f (which exists) is a pure-strategy Nash equilibrium. It can be shown (see also the proof of theorem 12.8 below) that $\bar{s}$ is the greatest element in the set of Nash equilibria. The intuition is again that a higher strategy triggers a higher best response. Last, by symmetry, the analysis applies to lower bounds as well. This proves part a of theorem 12.6:

Theorem 12.6

(a) (Topkis 1979) If, for each i, S_i is compact and u_i is upper semi-continuous in s_i for each s_{-i}, and if the game is supermodular, the set of pure-strategy Nash equilibria is nonempty and possesses greatest and least equilibrium points $\bar{s}$ and $\underline{s}$.

(b) (Vives 1990) If furthermore the game is *strictly* supermodular, the set of Nash equilibria is a nonempty complete sublattice. ("Complete" means that the sup and the inf of any subset belongs to the set.)

The proof of part a of theorem 12.6 relies on the monotonicity of the upper bound on the reaction correspondence for supermodular games. For supermodular games with *strictly* increasing differences, one can prove monotonicity of the entire reaction correspondence—a fact of considerable economic interest:

Theorem 12.7 (Topkis 1979) Consider a supermodular game with strictly increasing differences. If $s_i \in r_i^*(s_{-i})$, $\tilde{s}_i \in r_i^*(\tilde{s}_{-i})$, and $s_{-i} \geq \tilde{s}_{-i}$, then $s_i \geq \tilde{s}_i$.

Proof Theorem 12.7 results from the following chain of inequalities:

$$\begin{aligned} 0 &\leq u_i(s_i, s_{-i}) - u_i(s_i \vee \tilde{s}_i, s_{-i}) \\ &\leq u_i(s_i, \tilde{s}_{-i}) - u_i(s_i \vee \tilde{s}_i, \tilde{s}_{-i}) \\ &\leq u_i(s_i \wedge \tilde{s}_i, \tilde{s}_{-i}) - u_i(\tilde{s}_i, \tilde{s}_{-i}) \leq 0. \end{aligned}$$

The first and fourth inequalities result from optimality of s_i against s_{-i} and $\tilde{s}_i$ against $\tilde{s}_{-i}$, the second from increasing differences and $s_i \leq s_i \vee \tilde{s}_i$, and the third from supermodularity in player i's strategy. Last, note that if $s_i \ngtr \tilde{s}_i$ then $s_i < s_i \vee \tilde{s}_i$, and strictly increasing differences implies that the second inequality is strict. ■

After this investigation of the Nash set, we study iterated strict dominance and learning processes of supermodular games. Vives (1990) notes that such games have nice stability properties. He analyzes Cournot tâtonnement (the sequence of strategies starting from some arbitrary pure-strategy profile s^0 is given by $s^n \in r^*(s^{n-1})$, as explained in section 1.2) and makes the convention that if player i's rivals choose the same strategies at steps n and $n+1$ then player i also chooses the same strategy at steps $n+1$ and $n+2$ (that is, $s_{-i}^{n+1} = s_{-i}^n \Rightarrow s_i^{n+2} = s_i^{n+1}$). He shows that the tâtonnement process converges monotonically to an equilibrium point of the game when the starting point, s^0, is "below" or "above" the best reply correspondences of the players. A related result is proved by Milgrom and Roberts (1990). Consider a model of learning in which players repeatedly play the same game, play myopically (maximize current payoff at each stage), and form expectations about their rivals' play on the basis of their previous behavior in such a way that they assign small probabilities to strategies that they have not observed for a long time (see section 1.2). In a stage game with simultaneous moves, models in which players learn their opponents' strategies predict that the players will not use strictly dominated strategies, that their opponents will learn this, and so on. Thus, strategic learning therefore rules out all strategies ruled out by iterated strict dominance.

Milgrom and Roberts also use the monotonic sequences of strategies studied in learning processes to analyze rationalizable strategies. They show that the greatest and the least Nash equilibria, $\bar{s}$ and $\underline{s}$ (whose existence was ascertained in theorem 12.6a), are also the greatest and the least elements in the set of strategies that survive iterated deletion of strictly dominated strategies:

Theorem 12.8 (Milgrom and Roberts 1990) Consider a supermodular game such that, for each i, S_i is a complete sublattice and is bounded, and such that u_i is continuous and is bounded above. Then the iterated deletion of strictly dominated strategies yields a set of strategies in which the greatest and the least elements are the Nash equilibrium $\bar{s}$ and $\underline{s}$.

Proof The proof of theorem 12.8 is both simple and instructive. Start from the upper bounds $s^0 = (s_1^0, \ldots, s_I^0)$ on the strategy sets. Let s_i and s_i' denote two elements of $r_i^*(s_{-i}^0)$ such that there exists no s_i'' in $r_i^*(s_{-i}^0)$ such that either $s_i'' > s_i$ or $s_i'' > s_i'$. Suppose that $s_i \neq s_i'$ (that is, s_i exceeds s_i' along some dimension and s_i' exceeds s_i along another). We claim that $s_i \wedge s_i'$ is a strictly better response to s_{-i}^0 than s_i, a contradiction: Supermodularity implies that

$$u_i(s_i, s_{-i}^0) - u_i(s_i \wedge s_i', s_{-i}^0) \leq u_i(s_i \vee s_i', s_{-i}^0) - u_i(s_i', s_{-i}^0) < 0,$$

where the strict inequality results from the fact that $s_i \vee s_i' > s_i'$ and therefore cannot be a best response to s_{-i}^0. We thus conclude that $r_i^*(s_{-i}^0)$ has a greatest element s_i^1. Let $s^1 \equiv (s_1^1, \ldots, s_I^1) \leq s^0$. One then defines s^n by induction; any s_i that does not satisfy $s_i \leq s_i^n$ is strictly dominated by $s_i \wedge s_i^n < s_i$: Because all strategies s_{-i} remaining after $n - 1$ rounds of deletion satisfy $s_{-i} < s_{-i}^{n-1}$ from the induction hypothesis,

$$\begin{aligned} u_i(s_i, s_{-i}) - u_i(s_i \wedge s_i^n, s_{-i}) &\leq u_i(s_i, s_{-i}^{n-1}) - u_i(s_i \wedge s_i^n, s_{-i}^{n-1}) \\ &\leq u_i(s_i \vee s_i^n, s_{-i}^{n-1}) - u_i(s_i^n, s_{-i}^{n-1}) \\ &< 0, \end{aligned}$$

where the first inequality results from increasing differences, the second from supermodularity, and the third from the facts that s_i^n is the greatest best response to s_{-i}^{n-1} and that $s_i \vee s_i^n > s_i$.

Because the sequence s^n is bounded below and decreasing, it converges to some $\bar{s}$. To show that $\bar{s}$ is a Nash equilibrium, fix an arbitrary s_i; by optimality of s_i^{n+1} against s_{-i}^n,

$$u_i(s_i^{n+1}, s_{-i}^n) \geq u_i(s_i, s_{-i}^n);$$

by continuity, $u_i(s_i^{n+1}, s_{-i}^n)$ converges to $u_i(\bar{s}_i, \bar{s}_{-i})$ and $u_i(s_i, s_{-i}^n)$ converges to $u_i(s_i, \bar{s}_{-i})$. Because weak inequalities are preserved in the limit, $\bar{s}_i$ is a better response to $\bar{s}_{-i}$ than s_i, for each s_i.

By symmetry, the least Nash equilibrium $\underline{s}$ is also the lower bound on the set of strategies that survive iterated deletion of strictly dominated strategies. ■

Note that theorem 12.8 implies that, if there exists a unique Nash equilibrium, the game is solvable by iterated strict dominance.

Supermodular games have several other convenient features. First, comparative-statics exercises are straightforward. Suppose that payoffs are indexed by a parameter α, $u_i(s_i, s_{-i}, \alpha)$, and that u_i has increasing differences in $((s_i, \alpha), s_{-i})$. Then the greatest and least Nash strategies, $\bar{s}_i(\alpha)$ and $\underline{s}_i(\alpha)$, are nondecreasing functions of α (Milgrom and Roberts 1990, theorem 6). (This result is particularly useful when there is a unique Nash equilibrium. Special versions of it have been used in the papers mentioned in footnote 12.)

Second, one can compare payoffs in two Nash equilibria s and $\tilde{s}$ that satisfy $s \geq \tilde{s}$ (Milgrom and Roberts 1990, theorem 7). If $u_i(s_i, s_{-i})$ is increasing in s_{-i}, then $u_i(s) \geq u_i(\tilde{s})$ (for instance, Bertrand oligopolists prefer an equilibrium with high prices for all firms). If $u_i(s_i, s_{-i})$ is decreasing in s_{-i}, then $u_i(s) \leq u_i(\tilde{s})$ (for instance, in Cournot duopoly, a firm prefers the equilibrium in which it produces the highest output—i.e., in which its rival produces the lowest output).

Exercises

Exercise 12.1* Are the games in figure 1.10a (matching pennies), and figure 1.18 for $\lambda = 0$, essential?

Exercise 12.2*** The Sion-Wolfe (1957) example of nonexistence of a mixed-strategy equilibrium is a two-person, zero-sum game with strategy sets $S_1 = S_2 = [0, 1]$ and with the following payoffs:

$$u_1(s_1, s_2) = \begin{cases} -1 \text{ if } s_1 < s_2 < s_1 + \frac{1}{2} \\ 0 \text{ if } s_1 = s_2 \text{ or } s_2 = s_1 + \frac{1}{2} \\ 1 \text{ otherwise.} \end{cases}$$

(a) Show that

$$\sup_{\sigma_1} \inf_{\sigma_2} u_1(\sigma_1, \sigma_2) = \tfrac{1}{3} < \inf_{\sigma_2} \sup_{\sigma_1} u_1(\sigma_1, \sigma_2) = \tfrac{3}{7},$$

where the sup inf is obtained by player 1 putting weight on 0, $\frac{1}{2}$, and 1 and the inf sup is obtained by player 2 putting weight on $\frac{1}{4}$, $\frac{1}{2}$, and 1).

(b) Conclude that there exists no Nash equilibrium.

(c) What assumption in theorem 12.4 is violated here?

Exercise 12.3* Games with strategic complementarities can often be studied simply by using standard techniques, although the theory of supermodularity offers a more elegant and general method. Bulow et al. (1985) and Fudenberg and Tirole (1984) offer industrial-organization examples. This exercise develops a macroeconomic example. As Cooper and John (1988) argue, many models with "aggregate demand externalities," "spillovers," or "Keynesian effects" have a common structure. Consider a symmetric I-player game in which player i's payoff is $u(s_i, s_{-i})$, where $s_i \geq 0$. Assume that $u(\cdot, s_{-i})$ is strictly concave in s_i. When player i's opponents choose the same action s^*, player i's payoff is written

$$U(s_i, s^*) \equiv u(s_i, (s^*, \ldots, s^*)).$$

Since all players have the same payoff function, they have the same reaction function. Let $r^*(s^*)$ be the optimal response (for any player) to the profile in which all of his opponents play s^*. Assume that $\partial U/\partial s^* > 0$ (that is, that the game exhibits "positive spillovers," in the terminology of Cooper and John), and that $\partial^2 U/\partial s^{*2} < 0$. We will focus on *symmetric* Nash equilibria.

(a) Show on a diagram why there may exist multiple symmetric equilibria.

(b) Show that any symmetric Nash equilibrium involves action $s^* < \hat{s}$, where $\hat{s}$ is the optimal symmetric level of activity. Show that the symmetric Nash equilibria are Pareto ranked.

(c) Let $r^{*\prime} \equiv -(\partial^2 U/\partial s_i \partial s^*)/(\partial^2 U/\partial s_i^2)$ denote the slope of the reaction function. A "stable" equilibrium (see chapter 1) is such that $r^{*\prime} < 1$. Index the utility functions by a parameter γ_i, $U(s_i, s^*, \gamma_i)$, such that $\partial^2 U/\partial s_i \partial \gamma_i > 0$. Show that a symmetric stable equilibrium (corresponding to $\gamma_i = \gamma$ for all i) exhibits "multiplier effects":

$$\frac{d(\sum s_j^*)}{d\gamma_i} > \frac{\partial r_i^*}{\partial \gamma_i}.$$

(d) Think of reasons why spillovers may be relevant. (Hint: Consider monopolistic competition, search, learning spillovers, and so on.)

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13 Payoff-Relevant Strategies and Markov Equilibrium

Chapter 5 addressed repeated games, in which the "physical environment" is the same in every period. Now we study environments in which the past has a direct influence on current opportunities, say by determining the level of installed capacity or the quantity of discovered but unexploited natural resources. Such environments can be modeled as discrete-time games (sections 13.1 and 13.2) or with their continuous-time analogue of differential games (section 13.3).

In studying repeated games, we considered strategies in which past play influences current and future strategies, not because it had a direct effect on the environment, but rather because all players believe that the past play matters. When studying more complex environments, economists often focus attention on equilibria in a smaller class of "Markov" or "state-space" strategies in which the past influences current play only through its effect on a state variable that summarizes the direct effect of the past on the current environment. A *Markov perfect equilibrium* (MPE) is a profile of Markov strategies that yields a Nash equilibrium in every proper subgame. Since the state captures the influence of past play on the strategies and payoff functions for each subgame, if a player's opponents use Markov strategies, that player has a best response that is Markov as well. Thus, a Markov perfect equilibrium is still a perfect equilibrium when the Markov restriction is not imposed. However, there can be many other equilibria. The simplest example is that of infinitely repeated games, where the state variable is null, so the only Markov equilibria correspond to the infinite repetition of one of the equilibria of the stage game. For instance, the only Markov equilibrium of the infinitely repeated prisoner's dilemma (figure 4.1) has both players defect in every period.[1] Section 13.1 presents less trivial examples in which the Markov restriction has bite, and studies MPE in specific classes of games.

1. We do not assert that this is the most likely outcome when players are patient. The inadequacy of the Markov equilibrium here can be interpreted either as a critique of the Markov assumption or as a sign that the complete-information model omits some important features. Chapter 9, on reputation effects, shows how small amounts of certain kinds of incomplete information lead to Markov equilibria with more intuitive outcomes.

Chapter 9 perturbs the information structure of the game. Another way of avoiding the strong implication of Markov perfection in repeated games is to relax the requirement that players always move simultaneously. Maskin and Tirole (1988b), using the Markov restriction, obtain collusion in a repeated price game in which prices are locked in for two periods. They argue that what is meant by "reaction" is often an attempt by firms to react to a state that affects their *current* profits; for instance, when facing a low price by their opponents, they may want to regain market share. In the classic repeated-game model, firms move simultaneously, and there is no physical state to react to. If, however, one allows firms to alternate moves, they can react to their opponent's price. (Maskin and Tirole derive asynchronicity as the (equilibrium) result of the two-period commitments.) Gertner (1986) formalizes collusion with Markov strategies when commitment (inertia) takes the form of a fixed cost of changing prices. Halperin (1990) obtains an elegant characterization of the set of MPE when firms face such "menu costs" with or without inflation. He shows the existence of staggered and synchronized price cycles, some of which follow an (S, s) rule.

The MPE restriction pushes the notion that "bygones are bygones" further than perfect equilibrium does. It also runs counter to the equilibrium restrictions based on forward induction that we developed in chapter 11: The idea of forward induction is that past actions will be interpreted as signals of future intentions even though those actions may not influence payoffs in the continuation game.

Although some dynamic games are not presented with an explicit state variable, the notion of state is implicit in the way that the past influences the present. Subsection 13.2.1 shows how to construct an explicit state variable and thus extend the definition of MPE to general games of complete information.

Subsection 13.2.3 considers the robustness of MPE (in the sense of lower hemi-continuity) to small changes in the payoff functions. That is, start from an "original game" where the state space is small and so the MPE has bite; for example, one might think that the payoff-relevant state in an investment game is the current capacities of the players. Now consider a perturbation of the payoff functions that makes more aspects of the past relevant to the payoff; for example, learning by doing might imply that the exact time sequence of investments has a small effect as well. Lower hemi-continuity means that, for each MPE of the original game, there is an MPE of the perturbed game that is close to it, in that strategies depend "mostly" on the state variables of the original game. Lower hemi-continuity obtains for generic games. This is reassuring, as it is difficult to be certain that some past variables have absolutely no effect on the environment.

Despite this lower hemi-continuity, the set of Markov perfect equilibria can change discontinuously when the payoffs are perturbed. This is because MPE allows strategies to depend "a lot" on any state variable, even one whose influence on payoffs is small, but requires that strategies not depend at all on variables that have exactly zero influence. For example, if we modify the prisoner's dilemma by adding a "state variable" that keeps track of the number of times a player has defected, and allow this variable to have an arbitrarily small influence on payoffs, then "always cooperate" becomes the outcome of an MPE. This shows that the set of MPE is not upper hemi-continuous. (In the spirit of Markov perfection, it is then natural to select MPE in which variables that have a small effect on payoffs also have a small effect on strategies. The lower-hemi-continuity result guarantees that this can be done generically.)

Section 13.3 analyzes the class of differential games, which are the continuous-time analogues of stochastic games in which the state evolves according to a (deterministic) differential equation.[2] Since the optimal

2. There is also a literature on stochastic differential games, in which the state follows a stochastic differential equation.

solution to a one-player differential game (i.e., a control problem) can be chosen to be Markov, the MPE of differential games may correspond to the multi-player versions of control optima.[3] Rightly or wrongly, the analogy with control optima has led control theorists to study a class of MPE in differential games. Case (1969) and Starr and Ho (1969) have analyzed smooth perfect equilibria in Markov strategies in differential games. Section 13.4 treats the capital-accumulation game as an example of a differential game.

Because we develop few applications of the Markov concept in this chapter, we refer to other contributions for further examples.[4]

13.1 Markov Equilibria in Specific Classes of Games†††

13.1.1 Stochastic Games: Definition and Existence of MPE

Our first application of the Markov concept is to stochastic games.[5] The idea behind a stochastic game is that the history at each period can be summarized by a "state" (e.g., capital levels, goodwills). Current payoffs depend on this state and on current actions (e.g., investments, prices, advertising levels). The state follows a Markov process; that is, the probability distribution on tomorrow's state is determined by today's state and actions.

A stochastic game is defined by state variables $k \in K$, action spaces $A_i(k)$ with mixed actions $\mathscr{A}_i(k)$, a transition function $q(k^{t+1}|k^t, a^t)$ which gives the probability that the next period's state is k^{t+1} conditional on its being k^t at date t and on the playing of action a^t, and payoff functions $u_i = \sum_{t=0}^{\infty} \delta^t g_i(k^t, a^t)$. (Note that we are abusing notation when we define action spaces. Formally, the sets A_i and $\mathscr{A}_i$ of pure and mixed stage-game strategies, respectively, are functions of the whole history h^t. However, if, as we assume, they are measurable with respect to the state, it is notationally simpler to make them functions of the state only.) The game starts in some

3. In a control problem, there is a Markov optimum whenever an optimum exists. However, in a game with a continuum of actions, the conditions required for existence of a *Markov* perfect equilibrium are stronger than those for the existence of a perfect equilibrium. Harris (1990) discusses this. See also subsection 13.2.2.

4. See the literatures on resource extraction (Amir 1989; Amit and Halperin 1989; Dutta and Sundaram 1988; Lancaster 1973; Levhari and Mirman 1980; Loury 1990; Sundaram 1989), on bequest equilibria (Bernheim and Ray 1989; Harris 1985; Kohlberg 1976; Leininger 1986), on research and development (Harris and Vickers 1987), and on dynamic monopoly or oligopoly (Bénabou 1989; Dana and Montrucchio 1987; Eaton and Engers 1990; Gertner 1986; Harris 1988; Judd 1990; Kirman and Sobel 1974; Maskin and Tirole 1987, 1988a,b; Villas-Boas 1990). The Markov concept is also often used in games of incomplete information.

5. For notational simplicity, the definition of stochastic games in this section is more restrictive than that in much of the literature. For more on stochastic games, see Friedman 1986, Shapley 1953, and Sobel 1971.

state k^0 at date 0. The players know the entire history of play, $h^t = (k^0, a^0, k^1, a^1, \ldots, k^{t-1}, a^{t-1}, k^t)$, when they choose their period-t actions. An oligopoly example of a stochastic game is found in the paper by Kirman and Sobel (1974), where k^t is a vector of individual states k_i^t which represent firm i's stock of goodwill, and a_i^t is firm i's choices of price and advertising level at date t. A perfect equilibrium of this game allows the strategies $\sigma_i(h^t)$ to be functions of the entire history h^t. MPE requires that, for each player i and time t, $\sigma_i(h^t) = \sigma_i(\hat{h}^t)$ if the two histories have the same value of the state variable k^t. Another way of putting this is that the set M_i of Markov strategies for player i can be identified with the set of all maps σ_i: with $\sigma_i(k) \in \mathscr{A}_i(k)$. (Again, we abuse notation by defining strategies as functions of the state.)

Theorem 13.1 Markov perfect equilibria exist in stochastic games with a finite number of states and actions.

Proof

In the spirit of the agent strategic form of section 8.4, construct a *Markov strategic form* in which each agent (i, k) chooses a mixed action in $\mathscr{A}_i(k)$, and each agent (i, k) has the payoff function of player i at state k. Since there are finitely many states, the Markov strategic form has a finite number of players, each of whom has a finite number of pure actions. Thus theorem 1.1 implies the game has a Nash equilibrium. Moreover, σ^* is a Markov profile of the original game, and for each player i, σ_i^* is a best response in Markov strategies to σ_{-i}^*. Hence, as we remarked above, σ^* is a Nash equilibrium. Finally, this Markov equilibrium is perfect because player i optimizes in each state by construction. ■

This existence theorem has been extended to countable state spaces (see e.g., Parthasarathy 1982 and Rieder 1979). Existence theorems for uncountable (e.g., continuous) state spaces are much harder to obtain. Whitt (1980) proves the existence of an ε-equilibrium; Duffie et al. (1988) extend the basic game by adding a sequence of independently and identically distributed non-payoff-relevant public randomizations, and prove the existence of an extended concept of MPE in which the strategies in any given period depend (at least) on the current state, on the continuation payoff that players in the preceding stage anticipated would obtain if that state were realized, and on the current public signal. Mertens and Parthasarathy (1987) prove the existence of an extended concept of MPE in which strategies in any given period depend on the current state and on the anticipated continuation payoff (but not on any public signal). Harris (1990) proves the existence of an extended concept of MPE in which strategies in any given period depend on the current state and on the current public signal (but not on any continuation payoff). Thus, Harris' extended concept coincides with MPE, except for the dependence on the payoff-irrelevant public randomization at the beginning of the period.

13.1.2 Separable Sequential Games

A class of games for which one can obtain a general characterization of Markov-equilibrium strategies is the class of games of perfect information with separable payoffs.

Definition 13.1 A *separable sequential game* is defined by the following:
(i) a countable set of players, $t = 0, 1, \ldots,$
(ii) a state variable $k^t \in K \subseteq \mathbb{R}$ with evolution equation $k^{t+1} = f_{t+1}(a^t)$,
(iii) a sequence of action spaces $A^t(k^t) \subseteq \mathbb{R}$,
(iv) an objective function for each player of the form

$$u_t = g_t(k^t, a^t) + w_t(k^{t+1}, a^{t+1}, a^{t+2}, \ldots),$$

(v) perfect information (player t knows $h^t = (a^0, \ldots, a^{t-1})$ before choosing action a^t).

This class is more general than it might appear. First, the evolution equation could depend on both a^t and k^t; it suffices to relabel a^t to identify it with k^{t+1}.[6] Second, finite-horizon games belong to this class. Third, a given player may play in different periods; it suffices to distinguish his various "incarnations." Thus, player i playing at date t and player i playing at date t' can be formalized as two distinct players whose objective functions are derived from the same preferences. For instance, the class includes alternating-move duopoly games in which each firm's action is committed for two periods and players take turns.[7] If firm 1 plays in odd periods and has current payoff function $g_1(a_1, a_2)$, its objective function is

$$u_1 = g_1(a^{-1}, a^0) + \delta g_1(a^1, a^0) + \cdots + \delta^{2k+1} g_1(a^{2k+1}, a^{2k}) + \delta^{2k+2} g_1(a^{2k+1}, a^{2k+2}) + \cdots.$$

We leave it to the reader to transform the alternating-move duopoly game into a separable sequential game.

A (pure) Markov strategy in a separable sequential game is a strategy $s: K \times T \to A$ (where T and A are the time and action spaces). If the functions g_t and w_t and the action spaces A^t are time independent, a (pure) Markov strategy is a time-invariant map, $s: K \to A$. The strategy can be interpreted as a "reaction function." (In contrast with the reaction

6. For instance, suppose $k^{t+1} = f(k^t, a^t)$ is the date-$(t+1)$ capital given date-t capital and savings, $g_t(k^t, a^t) = g(k^t - a^t)$, and $\gamma(k^t, k^{t+1})$ is the amount of savings needed to obtain capital k^{t+1} from capital k^t. Then, redefine the action as the choice of tomorrow's capital, $a^t = k^{t+1}$, the transition equation as the identity, and the current-payoffs function as $g(k^t - \gamma(k^t, a^t))$.
7. See Cyert and DeGroot 1970; Dana and Montrucchio 1987; Eaton and Engers 1990; Gertner 1986; Maskin and Tirole 1987, 1988a,b.

functions defined in chapter 1, these are "real" reaction functions, i.e., parts of equilibrium strategies.)

A nice feature of separable sequential games is that the reaction functions are monotonic under the standard sorting condition.[8]

Definition 13.2 The function g_t satisfies the *sorting condition* if it is twice differentiable and either

$$(\mathrm{CS}^+)\ \frac{\partial^2 g_t}{\partial k^t \partial a^t} \geq 0$$

or

$$(\mathrm{CS}^-)\ \frac{\partial^2 g_t}{\partial k^t \partial a^t} \leq 0.$$

The sorting condition CS^+ means that a higher state variable makes a higher action more desirable. And conversely for CS^-. We now borrow from the mechanism-design literature (see chapter 7) the simple proof of the following proposition:

Theorem 13.2 Consider a separable sequential game satisfying the sorting condition. Suppose that the action spaces are state independent. Then the equilibrium Markov strategies $s^t(k^t)$ are nondecreasing (under CS^+) or nonincreasing (under CS^-).

Proof Fix Markov strategies for players $(t + 1), (t + 2), \ldots$, and let

$$v_t(k^{t+1}) \equiv w_t(k^{t+1}, s^{t+1}(k^{t+1}), s^{t+2}(f_{t+2}(s^{t+1}(k^{t+1}))), \ldots)$$

denote the continuation valuation of player t for state variable k^{t+1}. Now consider two possible states, k and $\tilde{k}$, at date t, and let $a = s^t(k)$ and $\tilde{a} = s^t(\tilde{k})$. By definition of equilibrium, player t prefers action a to action $\tilde{a}$ when the state is k:

$$g_t(k, a) + v_t(f_{t+1}(a)) \geq g_t(k, \tilde{a}) + v_t(f_{t+1}(\tilde{a})).$$

Similarly, in state $\tilde{k}$, player t prefers action $\tilde{a}$ to action a:

$$g_t(\tilde{k}, \tilde{a}) + v_t(f_{t+1}(\tilde{a})) \geq g_t(\tilde{k}, a) + v_t(f_{t+1}(a)).$$

These inequalities are called the *incentive-compatibility constraints*. Adding them up eliminates the continuation valuations:

$$g_t(k, a) + g_t(\tilde{k}, \tilde{a}) - g_t(k, \tilde{a}) - g_t(\tilde{k}, a) \geq 0,$$

which can be rewritten as

8. This condition is also called "single crossing condition" or "Spence-Mirrlees condition" or "constant sign partial derivatives" (Guesnerie and Laffont 1984).

$$\int_{a}^{\tilde{a}} \int_{k}^{\tilde{k}} \frac{\partial^2 g_t}{\partial x \partial y}\, dx dy \geq 0.$$

Hence, if $\tilde{k} > k$, then $\tilde{a} \geq a$ if CS^+ holds and $\tilde{a} \leq a$ if CS^- holds. ■

Theorem 13.2 extends trivially to the case in which the players play mixed (Markov) strategies. The result is then that the supports of the mixed strategies are ordered; for instance, under CS^+, if $k^t > \tilde{k}^t$, then

$$\min\{a \mid \sigma^t(a \mid k^t) > 0\} \geq \max\{a \mid \sigma^t(a \mid \tilde{k}^t) > 0\}.$$

Also, it is easy to see where the proof of monotonicity breaks down if preferences are nonseparable or if players $(t+1), (t+2), \ldots$ use non-Markov strategies. For instance, in the latter case, the continuation valuation v_t may depend on a^t and k^t (in a nonseparable way), so that the addition of the two incentive-compatibility constraints does not eliminate the continuation valuations.

Maskin and Tirole (1987, 1988a) make heavy use of the monotonicity property in alternating-move Cournot duopoly games. If the cross-partial derivatives of the firms' per-period payoffs with respect to the two outputs are negative (the two outputs are strategic substitutes in the terminology of subsection 12.3), then the Markov strategies or reaction curves are downward sloping, like the (fictitious) reaction curves of the static Cournot game (see chapter 1).

13.1.3 Examples from Economics

We now give two economic applications of Markov equilibrium. These applications differ slightly from the framework of subsection 13.1.1 in that strategy and state spaces are continuous rather than finite and in that the number of players is infinite in the first application. Furthermore, the transition function for the state is deterministic in both applications. These applications are developed in some detail, and readers familiar with examples of stochastic games may want to skip them.

Example 1: Bequest Games

Intergenerational family transfers give rise to games among successive generations. Suppose that each generation cares about the consumption of the next generation, which cares about the consumption of the following generation, and so on. This succession of generations does not behave like a single decision maker: A generation wants to leave a bequest to the next generation, but the two generations have different preferences about what to do with the bequest. A simple class of bequest games that has been much studied in the literature is the following:

There is a single good, used both for consumption and as productive capital. Generation t $(t = 0, 1, \ldots)$ lives for a single period (period t) and inherits an amount of good $k^t \geq 0$ from generation $t-1$. Generation t's utility depends on its own consumption, c^t, and on the next generation's consumption, c^{t+1}:

$$u_t = u(c^t, c^{t+1}).$$

Generation t saves $a^t \in [0, k^t]$ and consumes $c^t = k^t - a^t$. The next generation's inheritance or capital is $k^{t+1} = f(a^t)$, where f is an increasing function with $f(0) = 0$. A pure-strategy MPE of the bequest game is a strategy $s(k)$ such that

$$s(k) \in \arg\max_{x \in [0,k]} u(k - x, f(x) - s(f(x))).$$

We first derive an MPE for a parametric example. We then investigate general properties of MPE and study existence.

A Parametric Example Suppose that

$$u(c^t, c^{t+1}) = \ln c^t + \delta \ln c^{t+1}$$

and

$$f(a) = a^\alpha,$$

where δ and α belong to $(0, 1)$. Let us look for an MPE in which the stationary savings strategy $s(\cdot)$ is differentiable. The first-order condition for the program

$$\max_{x \in [0,k]} \{\ln(k - x) + \delta \ln[x^\alpha - s(x^\alpha)]\}$$

is

$$\frac{1}{k - x} = \frac{\delta \alpha x^{\alpha-1}[1 - s'(x^\alpha)]}{x^\alpha - s(x^\alpha)}.$$

This suggests looking for a linear strategy, $s(k) = sk$, where $s \in (0, 1)$, so

$$(1 - s)x = \delta\alpha(1 - s)(k - x)$$

or

$$x = \frac{\delta\alpha}{1 + \delta\alpha} k, \text{ so that } s \equiv \frac{\delta\alpha}{1 + \delta\alpha}.$$

There thus exists an MPE with a constant savings ratio; this savings ratio grows with the discount factor and with the productivity of savings.

Characterization of Equilibrium Markov Strategies When the utility function is separable, it is possible to characterize the slope of any equilibrium strategy $s(\cdot)$ as in subsection 13.1.2.

Theorem 13.2 as stated requires action spaces to be state independent, but its result extends trivially to some situations where action spaces are state dependent (such as separable bequest games). In these games,

$$u_t = u(c^t) + z(c^{t+1});$$

using the general notation,

$$g_t(k^t, a^t) = u(k^t - a^t)$$

and

$$w_t(k^{t+1}, a^{t+1}, a^{t+2}, \ldots) = z(k^{t+1} - a^{t+1}).$$

The action space $A^t(k^t) = [0, k^t]$ is state dependent. Thus, if $k > \tilde{k}$, $s(k) \in [0, k]$ may not be feasible for state $\tilde{k}$ (although $s(\tilde{k}) \in [0, \tilde{k}]$ is feasible for state k). But if $s(k) > \tilde{k}$, then $s(k) > s(\tilde{k})$ and hence monotonicity holds anyway. If $s(k) \leq \tilde{k}$, the proof of theorem 13.2 shows that $s(k) \geq s(\tilde{k})$ if u is concave (so that $\partial^2 g_t/\partial k^t \partial a^t \geq 0$). We conclude that equilibrium strategies are nondecreasing in the bequest game.

Existence Proving the existence of a pure-strategy MPE in the bequest game is much harder than showing monotonicity. Bernheim and Ray (1989) and Leininger (1986) have obtained existence results.

It is instructive to understand why the "natural method" of proving existence of an MPE does not work. Suppose that a generation inherits k and chooses $x = s(k)$ so as to maximize $u(k - s, f(x) - \bar{s}(f(x)))$, where $\bar{s}(\cdot)$ is the strategy of the following generation. If $\bar{s}(\cdot)$ is continuous, a maximum exists. One can thus map the continuous function $\bar{s}(\cdot)$ into a function $s(\cdot)$. A fixed point of this mapping is an MPE. However, $s(\cdot)$ need not be continuous, so one cannot use a fixed-point theorem on the space of continuous functions on some bounded interval $[0, \bar{k}]$.

Leininger shows that if f is continuous and increasing and u belongs to some class of utility functions (containing the separable function $u(c^t, c^{t+1}) = v(c^t) + \delta v(c^{t+1})$, where v is strictly concave and increasing),[9] there exists a monotonic, pure-strategy equilibrium in the finite-horizon version of the bequest game ($s(\cdot)$ nondecreasing). Similarly, Bernheim and Ray and Leininger show the existence of a monotonic MPE in the infinite-horizon version.

9. More precisely, consider the optimal-choice correspondence $\Phi(k|\bar{s})$, that is, the set of maximizers of $u(k - x, f(x) - \bar{s}(f(x)))$. (This correspondence is nonempty, compact valued, and upper hemi-continuous if u is continuous.) It is required that u be continuous and increasing in both variables, and that any selection $s^*(k)$ in $\Phi(k|\bar{s})$ be such that s^* is nondecreasing. Bernheim and Ray show that continuous and increasing utility functions that satisfy, for all $c^t \geq \tilde{c}^t \geq 0$ and $c^{t+1} \geq \tilde{c}^{t+1} \geq 0$,

$$u(c^t, c^{t+1}) + u(\tilde{c}^t, \tilde{c}^{t+1}) - u(\tilde{c}^t, c^{t+1}) - u(c^t, \tilde{c}^{t+1}) \geq 0,$$

a weaker form of the sorting condition, belong to the class of utility functions defined by Leininger.

Example 2: Extraction of a Common Resource

Several economists have looked at the extraction of a renewable resource by competing players. The classical fishing game (Lancaster 1973; Levhari and Mirman 1980) has two commercial fisheries at each period, fishing simultaneously in the same pool. Because the number of fish in the pool in the next fishing season depends on how many are left by the fisheries at the end of the current season, the fisheries exert a negative externality on each other that usually prevents a socially efficient fishing policy.

Consider the following model: Let $k^t \geq 0$ denote the current stock of the common resource. At date t, players 1 and 2 simultaneously choose how much to extract ($a_1^t \geq 0$ and $a_2^t \geq 0$). If $k^t \geq a_1^t + a_2^t$, player i gets instantaneous payoff $g_i(a_i^t)$, and the stock at the beginning of date $t+1$ is $k^{t+1} = f(k^t - a_1^t - a_2^t)$, where f is the transition or reproduction function. (Note that f is assumed to be deterministic, and that it depends on the state variable k^t and the actions a^t in a specific way.) If $k^t < a_1^t + a_2^t$, some rule allocates the limited stock to the players; let us assume for instance that each player gets $k^t/2$, which yields payoff $g_i(k^t/2)$ and that $k^{t+1} = f(0) = 0$. We take the state space and the action spaces to be the intervals $[0, \bar{k}]$ and $[0, \bar{a}]$, respectively, where $\bar{k}$ and $\bar{a}$ are "sufficiently large" (see Dutta and Sundaram 1988 for details). We thus work with continuous spaces, rather than with discrete spaces as in the above existence proof. Assume further that $g_i(\cdot)$ is continuously differentiable and strictly concave with $\lim_{x \to 0} g_i'(x) = +\infty$ (this assumption prevents a corner solution at zero extraction when $k^t > 0$), and that f is continuously differentiable, strictly increasing, and strictly concave with $f'(0) > 1/\delta$ and $f'(+\infty) < 1$.

For a profile $(s_1(\cdot), s_2(\cdot))$ of pure Markov strategies, let $\psi(k) = k - s_1(k) - s_2(k)$ denote the remaining stock at the end of the period. Note that, without loss of generality, one can restrict s_i to belong to $[0, k]$.

A natural research procedure is to look for continuously differentiable strategies $s_i(\cdot)$ (assuming that a differentiable MPE exists). An MPE must then satisfy the Bellman equation : For all k,

$$g_i'(s_i(k)) = \delta g_i'(s_i(f(\psi(k))))f'(\psi(k))[1 - s_j'(f(\psi(k)))]. \tag{13.1}$$

To obtain equation 13.1, suppose that player i extracts one more unit of the common resource at date t when the stock is k. He increases his current payoff by $g_i'(s_i(k))$. Assume further that he also modifies his date-$(t+1)$ strategy so as to get back to the original, equilibrium stock k^{t+2} at date $t+2$. The reduction of the stock at date $t+1$ resulting from a unit increase in the date-t extraction is equal to $f'(\psi(k))$. Because player j's extraction at $t+1$ depends on k^{t+1}, the total reduction in stock at the end of $t+1$ is

$$f'(\psi(k))[1 - s_j'(f(\psi(k)))].$$

Player i must reduce his date-$(t+1)$ extraction by this amount. Equation 13.1 simply asserts that the increase in a_i^t and the decrease in a_i^{t+1} (or the converse) does not affect player i's intertemporal welfare. (Of course, the above reasoning is valid only if one can reduce extraction slightly in each period whenever $k \neq 0$. The assumption on $g_i'(0)$ is meant to ensure that this is feasible.)

Dutta and Sundaram (1988) note that equation 13.1 has strong implications. First, tomorrow's stock is a strictly monotonic function of today's stock (and is thus strictly increasing as $\psi(0) = 0$): Because $\psi(\cdot)$ is continuous, nonmonotonicity would imply the existence of a pair $(k, \tilde{k})$, with $k \neq \tilde{k}$, such that $\psi(k) = \psi(\tilde{k})$. Equation 13.1 would then imply that $s_i(k) = s_i(\tilde{k})$ for $i = 1, 2$. But because

$$k - s_1(k) - s_2(k) = \tilde{k} - s_1(\tilde{k}) - s_2(\tilde{k}),$$

$k = \tilde{k}$ after all. Second, let us compare a steady state of the game and the steady state of a centrally planned economy, in which a social planner would choose extraction rates so as to maximize a weighted sum of the two players' intertemporal utilities. (It can be shown that in both the game and the centrally planned situation the stock converges monotonically to a steady state. In the case of the game, this results from the fact that tomorrow's equilibrium stock is an increasing function of today's stock. Because strategies are continuous in the current stock, from any initial level the stock converges monotonically to a steady state level. The proof for a centrally planned economy is similar.) In a steady state $\hat{k}$ of the game, $\hat{k} = f(\psi(\hat{k}))$. Then equation 13.1 implies that for all i

$$\delta f'(\psi(\hat{k}))[1 - s_i'(\hat{k})] = 1.$$

Consider now a *stable* steady state $\hat{k}$ (that is, the stock converges to $\hat{k}$ for any initial level in a neighborhood of $\hat{k}$). Stability implies that

$$\left.\frac{d(k - f(\psi(k)))}{dk}\right|_{k=\hat{k}} > 0,$$

or

$$f'(\psi(\hat{k}))(1 - s_1'(\hat{k}) - s_2'(\hat{k})) < 1.$$

Using the Bellman equation at $\hat{k}$ then yields, for all i,

$$\delta[1 - s_i'(\hat{k})] > 1 - s_1'(\hat{k}) - s_2'(\hat{k}).$$

We thus conclude that $s_1'(\hat{k}) = s_2'(\hat{k}) > 0$, so that

$$\delta f'(\psi(\hat{k})) > 1.$$

In contrast, the steady state of the centrally planned economy, k^*, must be the "golden rule" level, $\delta f'(\psi^*(k^*)) = 1$, because the central planner must be indifferent between sacrificing one unit of consumption today and having $f'(\psi^*(k^*))$ more tomorrow. ($\psi^*(\cdot)$ is the central planner's savings function.) Because f is strictly concave, $\psi^*(k^*) > \psi(\hat{k})$, and therefore $\hat{k} = f(\psi(\hat{k})) < k^* = f(\psi^*(k^*))$. There is always a "tragedy of the commons" in a stable steady state of a differentiable MPE. (Dutta and Sundaram (1990b) give examples of *over*-accumulation in MPEs not satisfying the conditions above.)

Levhari and Mirman (1980) found that a differentiable MPE exists for the specification $f(k) = k^\alpha$ $(0 < \alpha < 1)$ and $g_i(x) = \ln x$. If a linear solution $(s_i(k) = sk)$ is postulated, equation 13.1 yields

$$s_i(k) = \frac{1 - \alpha\delta}{2 - \alpha\delta} k$$

and

$$\psi(k) = \frac{\alpha\delta}{2 - \alpha\delta} k,$$

so that

$$\hat{k} = \left(\frac{\alpha\delta}{2 - \alpha\delta}\right)^{\alpha/(1-\alpha)} < k^* = (\alpha\delta)^{\alpha/(1-\alpha)}.$$

(Levhari and Mirman actually use a different method to derive this infinite-horizon equilibrium. They compute the finite-horizon equilibrium and take its limit when the horizon tends to infinity.)

Dutta and Sundaram (1988) also present a more general analysis of this problem in which it is not assumed that the Markov strategies are differentiable. They consider the broader class of pure Markov strategies that (a) satisfy $\sum_i s_i(k) \le k$ for all k and (b) have $s_i(\cdot)$ lower semi-continuous in k.[10] Rather than studying existence (Sundaram (1989) shows that if $g_1 = g_2$ there exists a symmetric MPE in this class, and Dutta and Sundaram (1990a) generalize this result to stochastic reproduction functions), Dutta and Sundaram (1988) show that, if an equilibrium in this class exists, the path of k^t is monotonic and the steady state can be below or above the golden rule.

Amir (1989) provides a general existence theorem using a lattice-theoretic approach. (To the best of our knowledge, this is the first application to dynamic games of the theory of supermodular games developed in section 12.3 above.) Under the assumptions described above (duopoly, increasing and concave production function, compact action space), Amir shows that

10. $s_i(\cdot)$ is lower semi-continuous at point k if, for any sequence $k^n \to k$, $\liminf_{n\to\infty} s_i(k^n) \ge s_i(k)$. If $s_j(\cdot)$ is not lower semi-continuous, player i's best response may not be well defined. Suppose that, given k^t, player j plays $s_j(k^t)$. Suppose further that $s_j(\cdot)$ jumps up at $\tilde{k}$; then player i's payoff jumps down at $\tilde{s}_i^t = k^t - s_j(k^t) - f^{-1}(\tilde{k})$, because player j's date-$(t+1)$ reaction jumps up. If player i's intertemporal objective function increases to the left of $\tilde{s}_i^t$, player i may face an "openness problem," so that his best response may not exist. Sundaram (1989) shows that, if the strategies are lower semi-continuous (and belong to $[0, k]$), each player has an optimal Markov strategy and his value function is an upper semi-continuous function of the state.

there exist MPE equilibrium strategies satisfying conditions a and b above and also (c) having Lipschitz constant equal to 1 (for all i, k, and $\tilde{k}$, $|s_i(\tilde{k}) - s_i(k)| \leq |\tilde{k} - k|$.)

Amit and Halperin (1989) study the I-player continuous-time version of this game (see section 13.3 for the definition of differential games). They demonstrate the existence of a family of MPE in the class of strategies that are continuous and continuously differentiable (except perhaps at isolated levels) functions of k. One of these MPE Pareto dominates the others.

13.2 Markov Perfect Equilibrium in General Games: Definition and Properties†††

The definition and the characterization of MPE follow those of Maskin and Tirole (1989).

13.2.1 Definition

Whereas the payoff-relevant variable is usually an explicit part of models that use the MPE concept, MPE can be defined starting from any extensive game without an explicit state variable. This subsection shows how to extend MPE to general multi-stage games with observed actions, in which at each date t all players know all actions chosen before date t. There are $T+1$ periods ($t = 0, \ldots, T$) where T can be finite or infinite. At date t, player i ($i = 1, \ldots, I$) knows the *history* $h^t = (a^0, \ldots, a^{t-1})$ (where $a^\tau \equiv (a_1^\tau, \ldots, a_I^\tau)$) and chooses an action a_i^t in a finite action set $A_i^t(h^t)$. (This formalism allows stochastic games (by letting one of the players be nature) as well as perfect-information or sequential games (in which in each period all players but one have a singleton action space).) The *future* f^t at date t is the vector of current and future actions: $f^t \equiv (a^t, \ldots, a^T)$. Player i has the von Neumann-Morgenstern payoff function

$$u_i(a^0, a^1, \ldots, a^T) \equiv u_i(h^t, f^t)$$

for all t.

In chapter 3 we defined a (subgame-) perfect equilibrium as a profile of strategies $\sigma_i^* = \{\sigma_i^{t*}(h^t)\}_{t=0,\ldots,T}$ that form a Nash equilibrium for any history: For all t, h^t, i, and σ_i,

$$\mathrm{E}_{f^t}(u_i(h^t, f^t)|(\sigma_i^*, \sigma_{-i}^*)) \geq \mathrm{E}_{f^t}(u_i(h^t, f^t)|(\sigma_i, \sigma_{-i}^*)).$$

(E_{f^t} indicates the expectation over the futures f^t, where the distribution over the futures is determined by the mixed-strategy profile in the conditional.)

As discussed in the introduction to this chapter, a Markov strategy for player i may be conditioned on less than player i's information. We are thus led to consider *summaries* or *partitions* of the history $\{H^t(h^t)\}_{t=0,\ldots,T}$, which, for each date, are mappings from the set of histories into a set of disjoint and exhaustive subsets of the set of possible histories at that date. Suppose for instance that there are four possible histories, h, h', h'', and h''', at the beginning of date 2. One partition is $H^2(h) = H^2(h') = A$, $H^2(h'') = B$, and $H^2(h''') = C$, in which the first two histories are lumped in the same summary. The partition can also be written $\{(h,h'),(h''),(h''')\}$.

While summarizing the history, a partition must not be too coarse. That is, at each date, the players must be able to recover the strategic elements of the ensuing subgame from the element of the partition to which h^t belongs:

Definition 13.3 A partition $\{H^t(\cdot)\}_{t=0,\dots,T}$ is *sufficient* if, for all t, h^t, and $\tilde{h}^t$ such that $H^t(h^t) = H^t(\tilde{h}^t)$, the subgames starting at date t after histories h^t and $\tilde{h}^t$ are strategically equivalent:
(i) The action spaces (defined conditionally on actions taken from date t on) are identical: For all i, $\tau \geq 0$, and $a^t,\dots,a^{t+\tau-1}$,

$$A_i^{t+\tau}(h^t, a^t, \dots, a^{t+\tau-1}) = A_i^{t+\tau}(\tilde{h}^t, a^t, \dots, a^{t+\tau-1}).$$

(ii) The players' von Neumann-Morgenstern utility functions conditional on h^t and $\tilde{h}^t$ are representations of the same preferences: $\exists\, \lambda_i(\cdot,\cdot) > 0$ and $\mu_i(\cdot,\cdot,\cdot)$ such that, for all f^t,

$$u_i(h^t, f^t) = \lambda_i(h^t, \tilde{h}^t)u_i(\tilde{h}^t, f^t) + \mu_i(h^t, \tilde{h}^t, f_{-i}^t),$$

where $f_{-i}^t \equiv (a_{-i}^t, \dots, a_{-i}^T)$.

Of course, the entire history ($H^t(h^t) = (h^t)$) is a sufficient partition, but it may be too fine in that it contains information that is not relevant to the subgame.

Definition 13.4 The *payoff-relevant history* is the minimal (i.e., coarsest) sufficient partition.

Note that, by construction, the payoff-relevant history is uniquely defined. In our example, if the subgames starting at date 2 after histories h, h', and h'' (but not h''') are strategically equivalent, the partition $\{(\mathrm{h},\mathrm{h}'),(\mathrm{h}''),(\mathrm{h}''')\}$ is sufficient but not minimal. The coarsest sufficient partition is $\{(\mathrm{h},\mathrm{h}',\mathrm{h}''),(\mathrm{h}''')\}$.

Remark on Infinite-Horizon Games The above definition of payoff-relevant history is not quite restrictive enough for infinite-horizon games. A Markov strategy of a stationary game should be independent of the calendar time, t. To achieve this, it suffices to include calendar time in the history. Markov strategies are then independent of time if the state, but not time, affects current and future payoffs and action spaces. The analysis then carries through trivially.[11]

11. Similarly, if the game is cyclical, Markov strategies are independent of calendar time except for the location within the cycle.

Definition 13.5 A Markov perfect equilibrium (MPE) is a profile of strategies σ that are a perfect equilibrium and are measurable with respect to the payoff-relevant history $(H^t(h^t) = H^t(\tilde{h}^t) \Rightarrow \forall i, \sigma_i^t(h^t) = \sigma_i^t(\tilde{h}^t))$.[12]

13.2.2 Existence

Theorem 13.2 Suppose either that $T < \infty$, or that $T = \infty$ and the objective functions are continuous at infinity (see chapter 4—for instance, the present discounted value of per-period payoffs is continuous at infinity if the discount factor is less than 1 and if per-period payoffs are uniformly bounded). Then there exists an MPE.

Proof The proof for a finite horizon is trivial: At date T, select a Nash equilibrium that is the same for all histories h^T in the same payoff-relevant history $H^T(h^T)$ (because, for all histories with the same payoff-relevant history, the last-period subgames are strategically equivalent and the sets of Nash equilibria are the same). Folding back, the subgame at $T-1$ becomes a one-period game, and one can select a Nash equilibrium that depends only on the payoff-relevant history. And so on, by backward induction.

The proof for $T = \infty$ has two steps. The first step was developed in chapter 4: Associate with G^∞ the T-period truncation game G^T in which the players are forced to take a fixed ("null") action after date T. Such games are finite-horizon games and, from the finite-horizon proof admit an MPE $\{\sigma_i^{t,T}\}$. Then take a subsequence converging to some

$$\sigma = \{\sigma_i^t\}_{\substack{i=1,\ldots,I \\ t=0,\ldots,+\infty}}$$

when T goes to ∞ (see chapter 4), and check that σ is a perfect equilibrium of G^∞. The second step consists of showing that σ is an MPE of G^∞. We will shortly see that limits of MPE are not always MPE. Whether this holds depends on whether the payoff-relevant history becomes coarser or finer in the limit. Here we take the limit of date-t strategies $\sigma_i^{t,T}$ as the horizon T goes to ∞. Intuitively, when T grows to $T' > T$, there is "at least as much relevant history to remember" at date $t \le T$, because parts of the history might have an influence on the game at dates $T+1, \ldots, T'$ even though they did not have any at dates $t, \ldots, T$. That is, the payoff-relevant history at t cannot become coarser in the limit, so the limit of Markov strategies is itself Markov. ∎

12. One can use additional considerations to obtain stronger versions of MPE. For example, the Markov restriction can be combined with the iterated elimination of strictly dominated strategies. The point here is that a past variable that is payoff relevant only if some player plays a strictly dominated strategy in the subgame ought not to be treated as part of the state. The elimination of a (conditionally) strictly dominated strategy results in fewer Markov strategies, which in turn may lead to a new round of deletion of strictly dominated strategies, and so forth.

Existence of a Pure-Strategy MPE in Games of Perfect Information (technical)

The previous existence result, as in the Nash case, allows mixed strategies. Some researchers have tried to find classes of games in which a pure-strategy MPE exists. One such class is the class of finite games of perfect information (in which all information sets are singletons, which in particular implies that players play sequentially). Let $t = 0, 1, \ldots, T$, where $T < \infty$, and let $i(t)$ denote the player who has the move at date t (knowing the moves $h^t = (a^0, \ldots, a^{t-1})$ up to date t). Proving existence when action spaces are finite is straightforward: At date T, let player $i(T)$ pick one of his optimal actions (the same for each history with the same payoff-relevant history). Folding back, do the same for player $i(T-1)$ and so forth.

Unfortunately, existence of a pure-strategy MPE does not extend to infinite games. Gurvich (1986) exhibits an infinite-horizon game of perfect information without a pure-strategy MPE. Furthermore, even with a finite horizon, infinite action spaces create existence problems. In the case of action spaces that are compact subsets of a Euclidean space, Hellwig and Leininger (1989) identify the following "openness problem": Consider a three-player game; player $i = 1, 2, 3$ chooses at date i an action a_i in the interval $[0, 1]$. Let the preferences of players 2 and 3 be given by, respectively,

$$u_2 = -(a_2 - \tfrac{1}{2})^2 + a_3(a_1 - \tfrac{1}{2})$$

and

$$u_3 = a_3[1 - (a_2 + \tfrac{1}{2})].$$

(Player 1's preferences turn out to be irrelevant.) Note that the payoff-relevant history at dates 2 and 3 is a_1 and a_2, respectively. Player 3's optimal pure-strategy response is

$$s_3^*(a_2) = \begin{cases} 1 & \text{if } a_2 < \tfrac{1}{2} \\ \in [0, 1] & \text{if } a_2 = \tfrac{1}{2} \\ 0 & \text{if } a_2 > \tfrac{1}{2}. \end{cases}$$

Now consider player 2. From the first term in his objective function, he wants to choose a_2 as close as possible to $\frac{1}{2}$. But the second term and player 3's reaction imply that whether $a_2 = \frac{1}{2}$ or $\frac{1}{2} + \varepsilon$ or $\frac{1}{2} - \varepsilon$ matters considerably. Figure 13.1 depicts player 2's payoff as a function of a_2 for $a_1 > \frac{1}{2}$ and for $a_1 < \frac{1}{2}$. Player 2's payoff for $a_2 = \frac{1}{2}$ depends on how player 3's indifference is resolved. If $s_3^*(\frac{1}{2}) = 0$, then B in figure 13.1 belongs to the bold and broken lines. If $s_3^*(\frac{1}{2}) = 1$, then A belongs to the bold line and C to the broken line.

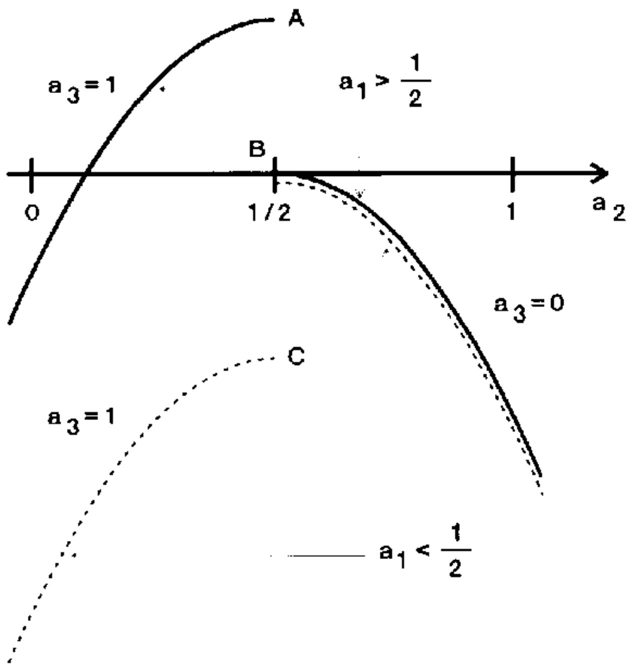

Figure 13.1

Now, for player 2's optimal choice to be well defined (i.e., not face an openness problem) when $a_1 > \frac{1}{2}$, one needs $s_3^*(\frac{1}{2}) = 1$; similarly, for the optimal choice to be well defined when $a_1 < \frac{1}{2}$, one needs $s_3^*(\frac{1}{2}) = 0$. Thus, existence of a pure-strategy equilibrium requires that player 3's response to $a_2 = \frac{1}{2}$ depend on the date-3 payoff-irrelevant variable a_1. We thus conclude that no pure-strategy MPE exists. (The same argument should convince the reader that there is no mixed-strategy MPE either.)

On the other hand, there exists a pure-strategy perfect equilibrium. (As an exercise, find it.) This is actually a more general result: a finite-horizon game of perfect information that satisfies the following conditions for each t has a pure-strategy perfect equilibrium (Goldman 1980; Harris 1985; Hellwig and Leininger 1987):

player $i(t)$'s action space is a compact subset of a Euclidean space,

player $i(t)$'s utility function is continuous in $(a^0, a^1, \ldots, a^T)$, and

player $i(t)$'s action space is a closed-valued and continuous correspondence $A_{i(t)}(a^0, \ldots, a^{t-1})$.

We leave it to the reader to check that, with finite action spaces that approximate the interval $[0, 1]$, there exists a pure-strategy MPE that is close to (converges to when the grid converges to the continuum) the perfect equilibrium mentioned above.

To avoid the nonexistence problem, Hellwig and Leininger (1989) assume that the state variable k^{t+1} at date $t+1$ depends only on k^t and $a_{i(t)}$, and they make a strong assumption on preferences—namely, that they be (forwardly) recursively separable in state-action pairs,

$$u_{i(t)} = v_t^t(k^t, a_{i(t)}, v_{t+1}^t(k^{t+1}, a_{i(t+1)}, v_{t+2}^t(\cdots))),$$

and that v_τ^t be either independent or strictly increasing in $v_{\tau+1}^t$ $(\tau \geq t)$.[13] This class of preferences generalizes the altruistic preferences found in bequest games (Phelps and Pollak 1968) in which the players are generations t $(i(t) = t)$ that live for one period, choose the level of capital k^{t+1} (that is, $a^t = k^{t+1}$) to give to their heirs, and have utility

$$u_t = v_t(f(k^t) - k^{t+1}, v_{t+1}(f(k^{t+1}) - k^{t+2}, v_{t+2}(\cdots))).$$

They also make a regularity assumption that rules out another cause of nonexistence associated with the action spaces. To paraphrase them, this regularity assumption guarantees that the effect of a small change in the state variable k^t on the subsequent state variable k^{t+1} can be exactly neutralized by a suitable small change in the choice variable $a_{i(t)}$. This assumption, together with recursive separability of preferences, guarantees existence of a pure-strategy MPE.

13.2.3 **Robustness to Payoff Perturbations** (technical)

The Markov concept emphasizes the influence of a few key variables to the exclusion of minor ones, and has bite only if a small number of past variables affect the current and future action spaces and objective functions. Perturbing the objective functions is likely to make the entire history payoff relevant and to deprive the Markov concept of its power, as discussed in the introduction to this chapter.

Consider a finite extensive form and identify games with the vector u of payoffs of all players at the terminal nodes of the tree. Let U denote the set of possible games (i.e., payoffs). Define distances between two games u and $\tilde{u}$ as

$$\|\tilde{u} - u\| = \max_{i,s} |\tilde{u}_i(s) - u_i(s)|$$

and distances between two strategies σ and $\tilde{\sigma}$ as

$$\|\tilde{\sigma} - \sigma\| = \max_{i,t,h^t,a_i^t} |\tilde{\sigma}_i^t(a_i^t \mid h^t) - \sigma_i^t(a_i^t \mid h^t)|.$$

With a game u is associated a payoff-relevant history H^u. Let $U(H)$ denote the subset of payoffs in U that give rise to payoff-relevant history H.

Consider a game u and an MPE σ of u; and let u^n denote a sequence of small perturbations of game u $(\lim_{n\to+\infty} \|u^n - u\| = 0)$. Note that the payoff-relevant history of u^n can be finer than that of u. (Indeed, it is easy to see that "for almost all payoffs v" the payoff-relevant history for game v is the identity, i.e., the entire history itself.) Does there exist a sequence of

13. With recursively separable utilities, all interactions between future actions and past ones take place through tomorrow's state variable. Such utility functions have received much attention in (one-player) optimal-growth theory (see, e.g., Beals and Koopmans 1969).

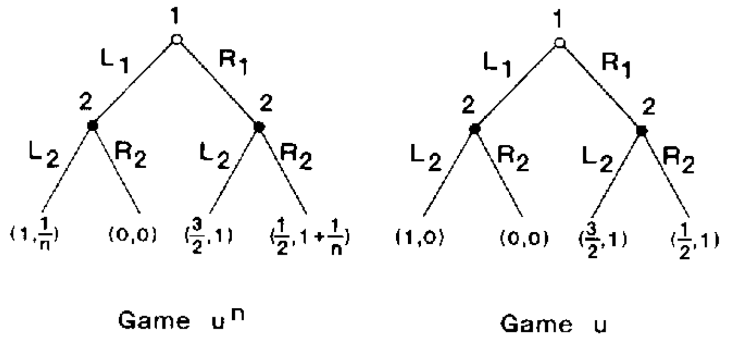

Figure 13.2

MPE σ^n of games u^n such that $\lim_{n \to +\infty} \|\sigma^n - \sigma\| = 0$? Not necessarily, as figure 13.2 demonstrates. In game u^n, the only perfect (and therefore Markov perfect) equilibrium has player 2 playing L_2 if L_1 and playing R_2 if R_1, and player 1 playing L_1. However, in the limit game u, player 1's action a_1 is payoff irrelevant in period 2. Hence, in an MPE of the limit game, player 2's strategy is the same whether player 1 plays L_1 or R_1, and hence player 1 plays R_1.

Game u is, in a sense, exceptional. Not only is a_1 payoff irrelevant; also, for any a_1, player 2 is indifferent between his pure strategies. (Note that the Markov concept has less appeal than usual in such a game. For example, one might argue that player 2, who is indifferent, would seek revenge by playing R_2 if player 1 played L_1 and would reward player 1 by playing L_2 if player 1 played R_1.)

Definition 13.6 A game u is *essential* with respect to MPE if, for any $\varepsilon > 0$, there exists $\zeta > 0$ such that, for all $\tilde{u}$ satisfying $\|\tilde{u} - u\| < \zeta$ and all MPE σ of game u, there exists an MPE $\tilde{\sigma}$ of game $\tilde{u}$ satisfying $\|\tilde{\sigma} - \sigma\| < \varepsilon$.

In other words, a game is essential with respect to MPE if its Markov perfect equilibria are robust to payoff perturbations. We already encountered the notion of essentiality in chapter 12. It was noted there that almost all strategic-form games are essential with respect to Nash equilibrium. We cannot make direct use of this result, however, because Markov perfect equilibrium is a refinement of Nash equilibrium and also because the genericity concepts in strategic-form and extensive-form games do not coincide (see section 12.1). However, it can be shown that the game u in figure 13.2 is indeed exceptional:

Theorem 13.4 (Maskin and Tirole 1989) Fix a finite, multi-stage tree and consider the set $U(H)$ of games with payoff-relevant history H. Almost all games in $U(H)$ (that is, almost all payoffs consistent with payoff-relevant history H) are essential with respect to the MPE concept and have a finite number of MPE.

Theorem 13.4 shows that, starting from a game in which some past variables have no influence on future payoffs and perturbing the game slightly, one can almost always choose equilibria of the perturbed game in which past variables that (now) have a small influence on future payoffs have only a small influence on equilibrium strategies. (Maskin and Tirole extend the proposition to an infinite horizon and define Markov equilibrium for games of incomplete or imperfect information.)

13.3 Differential Games†††

13.3.1 Definition

Differential games are continuous-time stochastic games for which control theorists have analyzed a subclass of solutions in the set of Markov perfect equilibria. Introduced by Isaacs (1954), these games have (unfortunately for social scientists) been developed mainly in the context of zero-sum two-person games. That the sum of the payoffs equals 0 was perceived as a decent approximation for various tactical problems studied in that literature, but it considerably limits the scope of their application to economics. Some progress has been made, however, on the non-zero-sum front. (For more material on differential games, see Basar and Olsder 1982, Blaquiere 1971, and Isaacs 1965. Applications to economics include Levine and Thepot 1982, Reinganum 1982, Simaan and Takayama 1978, and the papers quoted in note 14.) Let us first described the framework.

Time t varies continuously from 0 to $T \leq \infty$. Let $k^t \equiv (k_1^t, \ldots, k_n^t)$, a vector in real Euclidean space $\mathbb{R}^n$, denote the position, or state, or payoff-relevant history of the game at date t, and let it obey a system of first-order differential equations,

$$\frac{dk_j^t}{dt} = h_j^t(k^t, a^t), \quad j = 1, \ldots, n, \tag{13.2}$$

where $a^t = (a_1^t, \ldots, a_I^t)$ is the vector of actions chosen by all players at date t. Player i's action a_i^t belongs to some Euclidean space. (One can think of the continuous-time game as a limit of discrete-time games in which the players know the state at the beginning of date t and simultaneously choose actions.) The state at date 0 is given and is equal to $k(0)$.

The payoff functions are

$$u_i = \int_0^T g_i^t(k^t, a^t)\, dt + v_i^T(k^T), \tag{13.3}$$

where the terminal payoff, v_i^T, may depend on the state at the end of the game. A two-player game is zero-sum if $u_1 + u_2 \equiv 0$.

Differential game theorists impose the Markov restriction that strategies depend only on time and the state, and, more strongly, only on the state for stationary games, i.e., games in which h_j^t does not depend on t and the g_i^t take the form $e^{-rt}g_i(\cdot, \cdot)$.

The path of the state is then given by the differential equations

$$\frac{dk_j^t}{dt} = h_j^t(k^t, s^t(k^t)) \equiv \tilde{h}_j^t(k^t), \tag{13.4}$$

with initial condition

$$k^0 = k(0). \tag{13.5}$$

We postpone the discussion of whether these differential equations have a single solution until subsection 13.3.4.

To characterize Nash or perfect equilibria in Markov strategies of a differential game, it is tempting to borrow the techniques of dynamic programming, namely the maximum principle and the Pontryagin conditions (again, we postpone the discussion about whether it is proper to do so). With $s_{-i} = \{s_{-i}^t(k^t)\}$ given as the strategies of player i's rivals, the evolution of the state variables as a function of player i's actions is

$$\frac{dk_j^t}{dt} = h_j^t(k^t, a_i^t, s_{-i}^t(k^t)) \equiv \hat{h}_j^t(k^t, a_i^t). \tag{13.6}$$

If $\hat{g}_i^t(k^t, a_i^t) \equiv g_i^t(k^t, a_i^t, s_{-i}^t(k^t))$, player i's control problem is to maximize

$$\int_0^T \hat{g}_i^t(k^t, a_i^t)\, dt + v_i^T(k^T)$$

subject to equations 13.6 and 13.5 and to $a_i^t \in A_i^t(k^t)$ (player i's date-t action set).

13.3.2 Equilibrium Conditions

As mentioned above, each player's choice of an optimal strategy is a control problem in which the player takes into account the influence of his actions on the state, both directly and indirectly through the influence of the state on the strategies of the player's opponents. Subject to the technical caveats of subsection 13.3.4, it is easy to extend the conditions of Pontryagin et al. (1962) to multi-player situations. Starr and Ho (1969) restrict their attention to equilibria in which the equilibrium payoffs are continuous and almost-everywhere-differentiable functions of the state variables. This restriction obtains naturally for control problems in smooth environments, but it imposes a significant restriction in games: It might be that each player's strategy, and thus each player's payoff, changes discontinuously with the state because of the self-fulfilling expectation that the other players use discontinuous strategies, as we will see in section 13.4. Perhaps the continuity restriction can be justified by the claim that the "endogenous discontinuities" that it prohibits require excessive coordination, or are not robust to the addition of a small amount of noise in the players' observations. We are unaware of formal arguments along these lines.

The technical advantage of restricting attention to smooth equilibria is that necessary conditions can then be derived by means of the variational methods of optimal-control theory. Assume that player i wishes to choose s_i to maximize u_i, subject to the state-evolution equation (13.6) and the initial condition (13.5).

Introducing co-state variables λ_i^t, which are the vectors of $\{\lambda_{ij}^t\}_{j=1,\ldots,n}$, we define $\mathscr{H}_i^t$, the Hamiltonian for player i, as

$$\mathscr{H}_i^t(k^t, a^t, \lambda_i^t) = g_i^t(k^t, a^t) + \sum_j \lambda_{ij}^t h_j^t(k^t, a^t). \tag{13.7}$$

MPE strategies $s_i = \{s_i^t(k^t)\}$ must satisfy the generalized Hamilton-Jacobi equation

$$s_i^t(k^t) \in \arg\max_{a_i} \mathscr{H}_i^t(k^t, a_i, a_{-i}^t, \lambda_i^t) \tag{13.8}$$

as well as, for all j,

$$\frac{d\lambda_{ij}^t}{dt} = -\frac{\partial \mathscr{H}_i^t}{\partial k_j^t} - \sum_{\substack{\ell=1,\ldots,I\\ \ell \neq i}} \left(\frac{\partial \mathscr{H}_i^t}{\partial a_\ell^t}\right)' \frac{\partial s_\ell^t}{\partial k_j^t}, \tag{13.9}$$

where $\partial s_\ell^t/\partial k_j^t$ is the vector of partial derivatives of player ℓ's strategy (which is assumed to be piecewise C^1) with respect to the jth component of the state, with the convention that the derivative of the scalar $\mathscr{H}_i^t$ with respect to the vector a_ℓ^t (i.e., $\partial \mathscr{H}_i^t/\partial a_\ell^t$) is a column vector. They must also satisfy the appropriate transversality condition (e.g., when $T < \infty$, $\lambda_{ij}^T = \partial v_i^T/\partial k_j^T$). Notice that for a one-player game the second term in equation 13.9 vanishes, and the conditions reduce to the familiar ones. In the multi-player case, this second term captures the fact that player i cares about how his opponents will react to changes in the state. Because of the cross-influence term, the evolution of the shadow price of the jth state variable for player i, λ_{ij}^t, is determined by a system of partial differential equations, instead of by ordinary differential equations as in the one-player case. As a result, very few differential games can be solved in closed form. An exception is the linear-quadratic case; see subsection 13.3.3.

Another way of approaching the problem is to work with the value functions $V_i^t(k)$. One has

$$\frac{\partial V_i^t}{\partial k_j^t} = \lambda_{ij}^t$$

and

$$\frac{\partial V_i^t}{\partial t} = \max_{a_i^t} \left\{ \mathscr{H}_i^t\left(k^t, a_i^t, a_{-i}^t, \frac{\partial V_i^t}{\partial k^t}\right)\right\}.$$

A differential game is said to be *normal* (see Starr and Ho 1969) if it is possible to find a unique instantaneous Nash equilibrium $\hat{a}^t$ for the payoffs $\mathscr{H}_i^t$ for all k, λ, and t and if integrating the equations

$$\frac{\partial V_i^t}{\partial t} = \mathscr{H}_i^t\left(k^t, \hat{a}^t, \frac{\partial V_i^t}{\partial k^t}\right)$$

and

$$\frac{dk_j^t}{dt} = h_j^t(k^t, \hat{a}^t)$$

backward from all points on the terminal surface yields feasible trajectories. Starr and Ho prove that linear-quadratic differential games are normal.

13.3.3 Linear-Quadratic Differential Games

Linear-quadratic games are games for which the equations of motion are linear in the state and control variables and the objective functions are quadratic in the state and control variables. They were first studied by Case (1969) and by Starr and Ho (1969). The MPE strategies given by the first-order conditions 13.8 and 13.9 can be obtained numerically. Hoping that a linear-quadratic model is a good Taylor approximation to more general games, many researchers have computed the differential-game equilibria of such models.[14]

For notational simplicity, we assume that the payoffs and the state-evolution equation are independent of time, so that the system is autonomous. The linear-quadratic case is then

$$u_i = \int_0^T \left(\tfrac{1}{2}k'Q_i k + \tfrac{1}{2}\sum_{\ell=1}^{I} a_\ell' R_{i\ell} a_\ell + \sum_{\ell=1}^{I} r_{i\ell}' a_\ell + q_i' k + f_i\right) e^{-rt}\, dt$$
$$+ \tfrac{1}{2} k^{T\prime} S_i k^T, \tag{13.10}$$

and

$$\frac{dk}{dt} = Ak + \sum_{\ell=1}^{I} B_\ell a_\ell, \tag{13.11}$$

where the dependence of k and $a_i = s_i(k)$ on time is suppressed. Here, a prime denotes a transpose. Q_i, S_i, and A are $n \times n$ matrices, where n is the

14. For instance, Pindyck (1977) analyzes a game between the Federal Reserve System and the U.S. Government. Fershtman and Muller (1984), Fershtman and Kamien (1987), and Hanig (1986, chapter 4) apply linear-quadratic games to duopoly markets with slow adjustment of capacity, goodwill, or price variables. Other applications study the arms race. Clemhout and Wan (1979) present further examples.

dimension of the state variable. $R_{i\ell}$ is an $m_\ell \times m_\ell$ matrix where m_ℓ is the dimension of player ℓ's action space. B_ℓ is an $n \times m_\ell$ matrix, $r_{i\ell}$ is an m_ℓ-vector, q_i is an n-vector, and f_i is a real number. r is the instantaneous rate of interest. Matrix R_{ii} is assumed to be negative definite to ensure that player i's optimal control is well defined. Since the quadratic form $x'Cx$ equals $x'[(C + C')/2]x$, we can take the matrices Q_i, $R_{i\ell}$, and S_i to be symmetric without loss of generality.

The "current Hamiltonian" for player i is[15]

$$\mathscr{H}_i = \tfrac{1}{2}k'Q_ik + \tfrac{1}{2}\sum_{\ell=1}^{I} a'_\ell R_{i\ell}a_\ell + \sum_{\ell=1}^{I} r'_{i\ell}a_\ell + q'_ik + f_i$$
$$+ \lambda'_i\left(Ak + \sum_{\ell=1}^{I} B_\ell a_\ell\right).$$

The optimal-control vector a_i for player i maximizes $\mathscr{H}_i$:

$$a_i \equiv -R_{ii}^{-1}(r_{ii} + B'_i\lambda_i). \tag{13.12}$$

The co-state variables vary according to the current version of equation 13.9:

$$\frac{d\lambda_i}{dt} \equiv r\lambda_i - (Q_ik + q_i + A'\lambda_i) - \sum_{\ell \neq i}\frac{\partial s_\ell}{\partial k}(R_{i\ell}a_\ell + r_{i\ell} + B'_\ell\lambda_i), \tag{13.13}$$

where $\partial s_\ell/\partial k$ is the matrix whose jth row is $\partial s_\ell/\partial k_j$.

One then tries to find a solution for which the co-state variables, and thus the strategies from equation 13.12, are affine functions of the state variables. That is, one looks for $n \times n$ matrices Λ_i and n-vectors γ_i such that, for all i,

$$\lambda_i = \Lambda_ik + \gamma_i. \tag{13.14}$$

Equation 13.14 then yields

$$a_i = (-R_{ii}^{-1}B'_i\Lambda_i)k - R_{ii}^{-1}(r_{ii} + B'_i\gamma_i). \tag{13.15}$$

Differentiating equation 13.14, eliminating $\dot{k}$ using equation 13.11, and using equation 13.15 yields an affine function of k in the left-hand side of equation 13.13. Similarly, substituting equations 13.14 and 13.15 into the right-hand side of equation 13.13 yields another affine function of k. Identifying the coefficients of k and the constant coefficients on the two sides of the resulting identity, one finds that, if there exists a solution of the form $\lambda_i = \Lambda_ik + \gamma_i$, then Λ_i and γ_i must satisfy the "Riccati equations":

15. The Hamiltonian in equation 13.7 is in terms of present value. Accordingly, we will adjust equation 13.9 to account for the fact that λ_i is a vector of current—not present discounted shadow prices. Equation 13.13 will thus include a term that represents the interest on the shadow price. Arrow and Kurz (1970) discuss this formulation in a single-player context.

$$\Lambda_i A + A'\Lambda_i + Q_i - r\Lambda_i + \sum_{\ell \neq i} \Lambda'_\ell B_\ell (R_{\ell\ell}^{-1}) R_{i\ell} R_{\ell\ell}^{-1} B'_\ell \Lambda_\ell$$
$$- \sum_{\ell \neq i} \Lambda'_\ell B_\ell (R_{\ell\ell}^{-1})' B'_\ell \Lambda_i - \sum_{\ell} \Lambda_i B_\ell R_{\ell\ell}^{-1} B'_\ell \Lambda_\ell = 0 \tag{13.16}$$

and

$$r\gamma_i - A'\gamma_i - q_i + \sum_{\ell} \Lambda_i B_\ell R_{\ell\ell}^{-1} (r_{\ell\ell} + B'_\ell \gamma_\ell)$$
$$+ \sum_{\ell \neq i} \Lambda'_\ell B_\ell (R_{\ell\ell})^{-1} B'_\ell \gamma_i$$
$$- \sum_{\ell \neq i} \Lambda'_\ell B_\ell (R_{\ell\ell})^{-1} (r_{i\ell} - R_{i\ell}^{-1} R_{\ell\ell}^{-1} (r_{\ell\ell} + B'_\ell \gamma_\ell)) = 0. \tag{13.17}$$

These quadratic equations can be solved numerically. In many applications, they are actually much less complex than they appear. For instance, many games have no cross-terms in actions $R_{i\ell}$ $(i \neq \ell)$; each player i chooses a one-dimensional action, which affects "his" level of capital k_i, and so forth.

One may wonder whether the linear-quadratic solution derived above indeed forms a perfect equilibrium. In the infinite-horizon case ($T = +\infty$), Papavassilopoulos et al. (1979) have shown that if a stability condition on the matrices is satisfied then the strategies are indeed perfect—for more detail see the original article or chapter 2 of Hanig 1986.[16]

Judd (1985) offers an alternative to the strong functional-form assumptions typically invoked to obtain closed-form solutions to differential games. His method is to analyze the game in the neighborhood of a parameter value that leads to a unique and easily computed equilibrium. In his examples of patent races, he looks at patents with values of almost 0. Obviously, if the value of a patent is exactly 0, in the unique equilibrium the players do no research and development and their values are 0. Judd proceeds to expand the system about this point, neglecting all terms over third order in the value of the patent. Judd's method gives only local results, but it solves an "open set" in the space of games, as opposed to conventional techniques that can be thought of as solving a lower-dimensional subset of games.

13.3.4 Technical Issues

The focus on Markov strategies is guided not only by a subjective notion of what strategies are reasonable, but also by technical considerations associated with continuous time. As Anderson (1985) observes, "general" (that is, functions of the whole history) continuous-time strategies need not lead to a well-defined outcome path for the game, even if the strategies and

16. In the finite-horizon case, the above linear-quadratic solution is unique in the space of strategies that are analytic functions of the state variables. See Papavassilopoulos and Cruz 1979.

the outcome path are restricted to be continuous functions of time. Anderson offers the example of a continuous-time game in which two players simultaneously choose actions and there is no state variable. Consider the continuous-time strategy "play at each time t the limit as $\tau \to t$ of what the opponent has played at times τ previous to t." This limit is the natural analogue of the discrete-time strategies "match the opponent's last action." If the players have chosen matching actions at all times before t and the history is continuous, there is no problem in computing what should be played at t. However, there is not a unique way of extending the outcome path beyond time t. Knowing play before t determines the outcome at t but is not sufficient to extend the outcome path to any open interval beyond t. (As a result of this problem, Anderson opts to study the limits of discrete-time equilibria instead of working with continuous time.)

Restricting strategies to be Markov does not avoid the following difficulty: To make things simple, assume that the function g_i^t and h_j^t are defined and continuously differentiable over the whole Euclidean space. Even so, the differential equations 13.4 with the initial condition 13.5 may not have a unique solution. The right-hand side of equation 13.4 may not be Lipschitz-continuous in k^t unless the strategies are continuously differentiable.[17] Alas, the class of continuously differentiable strategies may be too small even in a single-player environment (i.e., in a control problem), as continuously differentiable strategies may be dominated by piecewise continuously differentiable (piecewise-C^1) ones, for instance. Control theorists thus often restrict their attention to piecewise-C^1 strategies; to make sure that the differential equations define a unique path of the state variables, they must verify that the optimal response to a piecewise-C^1 strategy can be chosen piecewise C^1.

In order for the Pontryagin conditions for player i to be applied, the class of allowable strategies must be "sufficiently large" to include all the perturbations required to obtain these conditions. In particular, piecewise-C^1 strategies must be allowed. But, as we already noted, allowing a large class of controls conflicts with the traditional assumption in control theory that the right-hand side of the differential equations governing the evolution of the state variables is C^1. Since at least the class of piecewise-C^1 functions must be allowed for player i's strategy s_i, $\hat{h}_j^t$ may be discontinuous, and applying extensions of traditional control theory to discontinuous evolution equations requires some assumptions about the relationship between the manifolds of discontinuity (which are not always satisfied in examples).

17. The function $\tilde{h}_j^t(k^t)$ is Lipschitz-continuous in k^t if, for any k^t and any $\hat{k}^t$ in a neighborhood of k^t,

$$|\tilde{h}_j^t(k^t) \quad \tilde{h}_j^t(\hat{k}^t)| \leq L|k^t - \hat{k}^t| \text{ for some } L > 0.$$

The Lipschitz property plays a crucial role in the existence of a unique solution to differential equations. See Smart 1974.

For sufficient conditions for piecewise-C^1 strategies satisfying the first-order conditions of the players' control problems to form a Nash equilibrium in a zero-sum differential game, see (e.g.) Berkovitz 1971.

Once the class of allowable strategies is fixed, the issue of the existence of an MPE in this class arises. As we mentioned, one may verify that the solution to the first-order conditions satisfies some sufficient conditions. Sufficient conditions to obtain existence without characterizing the equilibrium strategies are known only for special differential games, such as the linear-quadratic games discussed in subsection 13.3.3 and the zero-sum games to which we will turn shortly.

13.3.5 Zero-Sum Differential Games (technical)

Two-player zero-sum games have the convenient property that the sets of perfect-equilibrium and Nash-equilibrium outcomes coincide (see exercise 4.10). The existence of a Markov perfect equilibrium in a class of strategies (e.g., piecewise-C^1 strategies) then results from standard theorems of the existence of Nash equilibria for zero-sum games if such theorems apply. (In particular, a Nash equilibrium exists if the strategy spaces are compact, convex subsets of linear topological spaces and if u_i is upper semi-continuous and concave in s_i and lower semi-continuous and convex in s_j.)

A first approach to proving the existence of a Nash equilibrium relies on the fact that a pure-strategy Nash equilibrium in open-loop strategies (i.e., strategies that depend on time but not on the state) is also a pure-strategy Nash equilibrium in closed-loop strategies (i.e., strategies that are contingent both on time and the state).[18] Sufficient conditions for the existence of a pure-strategy open-loop Nash equilibrium are that the functions h_j^t and g_i^t are linear in the state and actions, that the game has fixed finite duration T, and that the action spaces are compact, convex, and independent of time and state (see, e.g., Fichefet 1970).

This simple method of proving the existence of a Nash equilibrium in closed-loop strategies via the existence of a Nash equilibrium in open-loop strategies breaks down in stochastic differential games, because, with a random evolution of the state variables, a player is not able to perfectly predict the value of the state variables at each date, and thus the best response to even an open-loop strategy of one's opponent is a closed-loop strategy. For instance, researchers have studied differential games with deterministic payoffs as in equation 13.3 but with stochastic differential equations for the evolution of the state variables:

$$dk_j^t = h_j^t(k^t, a^t)\,dt + \gamma^t(k^t)\,dB_j^t,$$

18. The point is that player i can perfectly foresee the evolution of the state as a function of s_i since the other player is using a pure strategy. This property also holds for more than two players and for non-zero-sum games.

where B^t is a Brownian motion in $\mathbb{R}^n$. To prove the existence of equilibrium, they use Hamiltonian methods. See, e.g., Elliott 1976, and, for extensions to non-zero-sum games, Uchida 1978 (also with a Brownian perturbation of the differential equations). Wernerfelt (1988) discusses piecewise-constant jump processes.

13.4 Capital-Accumulation Games†††

Capital-accumulation games offer a useful illustration of differential-game techniques. Following Spence (1979), we will consider continuous-time, infinite-horizon, stationary, duopoly versions of the capital-accumulation games. Here the control variable a_i is firm i's rate of investment in its own capital k_i. With current capital stocks $k = (k_1, k_2)$ (we continue to suppress the time superscripts), there is an equilibrium in the product market—i.e., outputs $q_i(k)$ and prices $p_i(k)$—with firm i's profit, net of production costs, equal to $R_i(k_1, k_2)$. (We implicitly make a Markov assumption that current pricing and production are not affected by payoff-irrelevant aspects of past play.) The firms also pay a maintenance cost m_i per unit of capital. If we let $C_i(k_i, a_i)$ denote the firm's cost of investing at rate a_i when its capital stock is k_i, the firm's instantaneous net profit is

$$g_i(k, a) = R_i(k) - m_i k_i - C_i(k_i, a_i).$$

We assume that $\partial^2 R_i/\partial k_i^2 < 0$ (concave profit function), $\partial R_i/\partial k_j < 0$ (firms dislike increases in their opponent's capital), and $\partial^2 R_i/\partial k_i \partial k_j < 0$ (the marginal productivity of capital decreases with the rival's capital that is, the capital levels are strategic substitutes as defined in section 12.3). We also assume that the revenue functions R_i and their derivatives are bounded above and below. For tractability, one of two specifications—irreversible investment and reversible investment—has been adopted for the investment function in the literature.

Irreversible investment The levels of capital never decrease: $dk_i/dt = a_i$. The unit cost of investment is assumed to be 1 up to a maximum investment rate a_i and infinite thereafter (so, overall, the investment cost is convex). In other words, the instantaneous action space is $A_i = [0, \bar{a}_i]$ and the cost is $C_i(k_i, a_i) = a_i$. The upper bound on investment prevents firms from investing at an "infinite speed," which allows the game to be truly dynamic.

Reversible investment Firm i's capital level depreciates at rate ρ_i: $dk_i/dt = a_i - \rho_i k_i$. For tractability, the reversible-investment case is studied with a quadratic, rather than a discontinuous, cost of investment: Either $C_i(k_i, a_i) = c_i a_i^2/2$ (the cost depends on gross investment) or $C_i(k_i, a_i) = c_i(a_i - \rho_i k_i)^2/2$ (the cost depends on net investment).

13.4.1 Open-Loop, Closed-Loop, and Markov Strategies

We begin with the case of irreversible investment. The per-period payoffs are assumed to be $[R_i(k_i, k_j) - m_i k_i - a_i]$, and we impose an upper bound on investment a_i. Thus, $dk_i/dt = a_i \in [0, \bar{a}_i]$. For concreteness, let both firms enter the market at time $t = 0$ without any capital (but in a perfect equilibrium, equilibrium behavior is defined from any initial levels of capital).

In a first step, we assume that the firms maximize their time-average payoffs, so that only the eventual steady-state capital levels matter. Because the marginal productivity of capital is bounded, and capital requires maintenance, no firm will choose an infinite capital stock. The time-averaging specification has the peculiar feature that the investment path leading to a given steady state does not affect the payoffs, but it allows a simple analysis of strategic investment. (We will discuss the case of a strictly positive rate of interest below.)

We define the "Cournot reaction curves" $r_i(\cdot)$ for this time-averaging specification by $\partial R_i(r_i(k_j), k_j)/\partial k_i = m_i$. Under our assumptions, the reaction curves are downward sloping. We will assume that they have a unique (stable) intersection $C = (k_1^c, k_2^c)$ as in figure 13.3 (which depicts the symmetric case).

Let us first examine the *"precommitment"* or *"open-loop"* equilibria. (See chapter 4 for the notion of open-loop equilibrium.) In a precommitment equilibrium, firms simultaneously commit themselves to entire time paths of investment. Thus, the precommitment equilibria are really static, in that there is only one decision point for each firm. The precommitment equilibria are just like Cournot-Nash equilibria, but with a larger strategy space. In the capital-accumulation game, the precommitment equilibrium is exactly the same as if both firms built their entire capital stocks at the start (because of no discounting). In the resulting "Cournot" equilibrium, each firm invests to the point at which the marginal productivity of capital equals m_i, given the steady-state capital level of its opponent. There are many different paths which lead to this steady state, all of which are precommitment equilibria. For example, each firm's strategy could be to invest as quickly as possible to its Cournot level. We can highlight the similarity of this solution to a Cournot equilibrium by defining the "steady-state reaction curves" that give each firm's desired steady-state capital level as a function of the steady-state capital level of the opposing firm. The precommitment equilibrium is at C in figure 13.3. As we saw in chapter 4, the use of the precommitment concept transforms an apparently dynamic game into a static one. As a modeling strategy, this transformation is ill advised. As Kreps and Spence (1984) note, "one should not allow precommitment to enter by the back door.... If it is possible, it should be explicitly modeled... as a formal choice in the game."

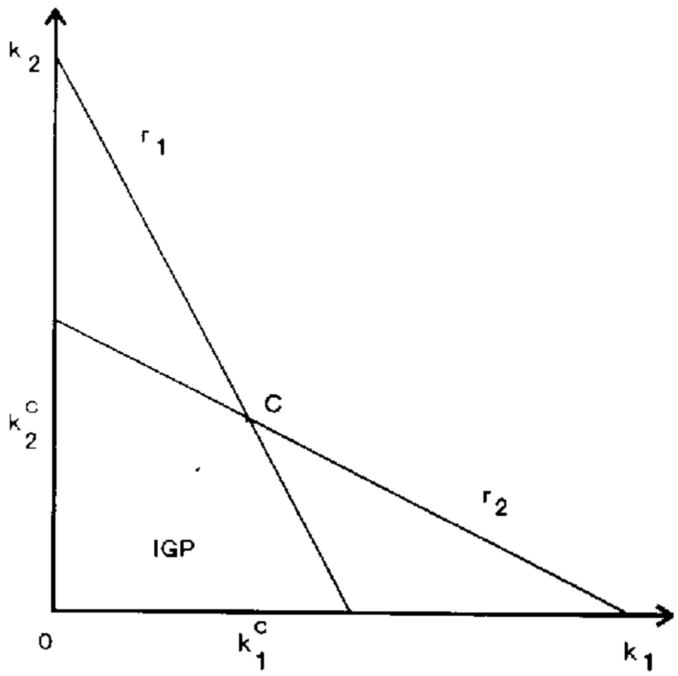

Figure 13.3

Let us next allow firm i's investment at time t to depend on the capital stocks as well as on time. The capital stocks are the state variables. A "closed-loop" equilibrium is a Nash equilibrium in state-dependent ("closed-loop") strategies. The first thing we should point out is that precommitment ("open-loop") equilibria are closed-loop equilibria. If the strategy of a firm's opponent depends only on time (and there are no random disturbances in the system), the firm can without loss restrict itself to a strategy that depends only on time: If the firm's optimal closed-loop strategy is given, the path of the system (the capital stocks at every time) is completely determined. We can then construct an open-loop strategy that calls for the same rate of investment at every point in time as the closed-loop strategy does. In the jargon of optimal control, this is called "synthesizing the feedback control."

Thus simply expanding the strategy space to allow dependence on history does not remove the "static" precommitment equilibria. Moreover, many other implausible outcomes are closed-loop equilibria. For example, firm 1 can threaten to "blow the game up" by building huge amounts of capacity if firm 2 dares to invest beyond some small level. In view of firm 1's threat, firm 2's best response may well be to acquiesce and accept a very small long-run market share. This is the now-familiar "perfection" problem—firm 1 is making a threat it would not choose to carry out were its bluff to be called. Of course, a firm may be willing and able to commit itself to such a threat, using a "doomsday machine" (perhaps a contract with a third player—see Schelling 1960 and Gelman and Salop 1983) preset to inflict some terrible harm on itself if it backs down. The point is the same here as with the open-loop strategies: To the extent that such commitments are possible, they should be included in the formal model. Given such a model, we would expect that each decision a firm makes is part of an optimal plan for the remainder of the game. We will further require that bygones be bygones except to the extent that past choices influence the current and future competitive environment. To impose these requirements we restrict attention to Markov perfect equilibria.

Neither of the two equilibria discussed so far is perfect. In the second equilibrium, if firm 2 were to disregard firm 1's threat and undertake nonnegligible investment, firm 1 would not wish to build the threatened level of capacity. In other words, the given strategies do not form an equilibrium starting from states with nonnegligible k_2. The first (precommitment) equilibrium is imperfect in a less obvious way: If both firms invest as quickly as possible, then generally one of them (say firm 1) will get to its Cournot capital level before the other. The specified strategies then say that firm 1 should stop investing while firm 2 continues on to its Cournot level. But consider what would happen if firm 1 were to deviate by investing past its Cournot level by a small amount before stopping. Firm 2's strategy says that, "no matter what," firm 2 will invest up to its Cournot level. But if firm 1 has already invested past k_1^c, then the best thing for firm 2 to do would be to stop on its reaction curve. The given strategies do not form a Nash equilibrium starting from states in which $k_1 > k_1^c$; thus, they are not perfect.

The above argument suggests that the firm with the greater investment speed (or the firm with a "head start" in investing—the model can be extended to allow unequal entry times) can invest "strategically" (that is, "overinvest" relative to C) in order to reduce the investment of the other firm. Because we have assumed that such "overinvestment" is locked in (there is no depreciation or disinvestment), the best the follower can do when presented with the *fait accompli* of overinvestment is take it as given when making its own decisions. With a large enough discrepancy in speeds, the "leader" can act as a Stackelberg leader and choose its preferred point on the "follower's" reaction curve.

To see this, consider figure 13.4, which depicts a perfect equilibrium. The arrows indicate the direction of motion of the state: horizontal if only firm 1 is investing, vertical if only firm 2 is investing, diagonal if each firm is investing as quickly as possible, and "+" if neither firm invests (because of the linearities, the optimal strategies are "bang-bang"). Note that we have defined choices at every state, and not just those along the equilibrium path this is necessary in order to test for perfection. Looking at figure 13.4, we see that unless firm 1 has a head start, it cannot enforce its Stackelberg level. But if firm 1 starts with more capital than firm 2, it invests as fast as it can until either it reaches its Stackelberg level of capital or firm 2 reaches its reaction curve (the Stackelberg level is defined as in chapter

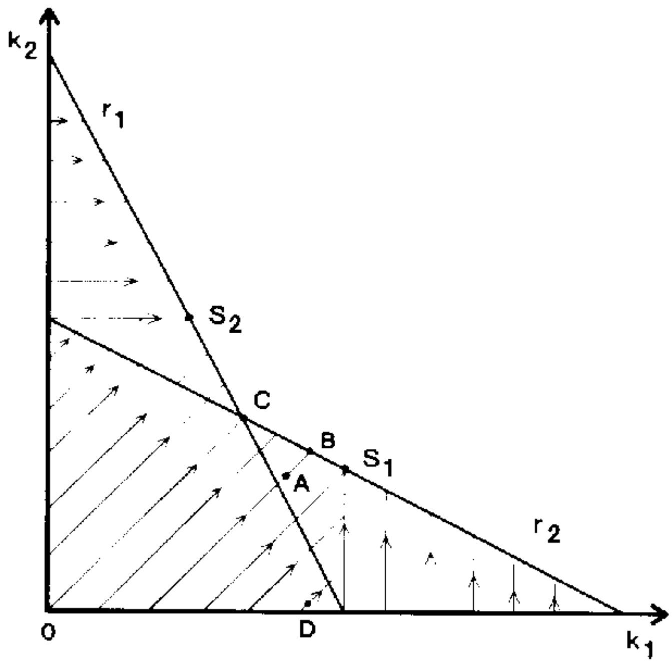

Figure 13.4

3: it maximizes $R_1(k_1, r_2(k_1)) - m_1 k_1$ with respect to k_1. The Stackelberg point is denoted S_1 in figure 13.4). If it reaches the Stackelberg level before firm 2 reaches its reaction curve, firm 2 then continues investing up to r_2. If for some reason firm 1's capital stock already exceeds its Stackelberg level, it stops immediately. The situation is symmetric above the 45° line in figure 13.4, which corresponds to states in which firm 2 has a head start. Thus, this equilibrium demonstrates how an advantage in investment speed or initial conditions can be exploited. The conditions of the growth phase (whose got there first, the costs of adjustment, etc.) have a permanent impact on the structure of the industry. This model also illustrates the importance of using the concept of perfect equilibrium to rule out empty threats.

It turns out that the equilibrium pictured in figure 13.4 is not unique; there are many others. To understand why, consider point A in figure 13.4, which is close to firm 2's reaction curve and past firm 1's reaction curve. The strategies specify that, from A on, both firms invest until r_2 is reached. However, both firms would prefer the *status quo* at A. Firm 1 in particular would not want to invest even if firm 2 stopped investing; it just invests in self-defense to reduce firm 2's eventual capital level. Both firms' stopping at A is an equilibrium in the subgame starting at A, enforced by the credible threat of going to B (or close to B) if anyone continues investing past A. Thus, the Markov restriction does not greatly restrict the set of equilibria in the investment game.

The existence of these early-stopping equilibria naturally depends on each firm's being able to respond quickly to its rival's investment. Sklivas (1986) studies the capital-accumulation game in (infinite-horizon) discrete time.[19] He shows that the early-stopping equilibria do exist as in the continuous-time model, but that the set of such equilibria shrinks and eventually vanishes when the upper bounds on the investment levels $\bar{a}_1$ and $\bar{a}_2$ become large. With the possibility of quick investment, a firm can get very close to its reaction curve within a period, which implies that its rival does not have time to react. The existence of early-stopping equilibria thus relies on the information lag being short relative to the speed of investment.

McLean and Sklivas (1988) consider the finite-horizon, discrete-time game. They show that backward induction has very strong implications: There exists an essentially unique MPE outcome (there may exist two MPE, but they differ only in the last move of one firm). This unique equilibrium converges to the Spence solution (stopping on the upper envelope of the reaction curves) as the horizon goes to ∞ and the discount factor goes to 1. In the last period, a firm that has not reached its reaction curve invests. In the penultimate period, firms invest if they are under their reaction curves as they know that the last period will have investment no matter what, and so forth. Studying the finite-horizon case thus highlights the bootstrap nature of the early-stopping equilibria and illustrates a similarity with the infinite-horizon, possibly cooperative equilibria and the finite-horizon noncooperative equilibria of repeated games (see chapter 5).

The early-stopping equilibria suggest two further refinements which may be in the Markov spirit. First, in the early-stopping equilibria, small causes do not have small effects. A lack of coordination or a small investment mistake drives the equilibrium away from the stopping curve to the upper envelope of the reaction curves. In the spirit of subsection 13.2.3 (which proved that, generically, MPE are robust to small payoff perturbations), one might require that the strategies be fairly insensitive to small variations in the state. However, in general games one cannot find a continuous equilibrium selection. Second, one might require infinite-horizon MPE to be limits of finite-horizon MPE. The idea would be that the modeler may know that the players know that the game ends at some date T, and T is known to the players but not to the modeler. It is not known what conditions imply that all infinite-horizon MPE are limits of finite-horizon MPE.

A word on the case of discounting with irreversible investment: Fudenberg and Tirole (1983) note that when the follower invests as quickly as possible to its reaction curve, it is no longer optimal for the leader to invest as quickly as possible. This is a simple point in optimal control. Suppose, for instance, that in figure 13.4 both firms were engaged in the investment path from D to S_1. Then firm 1 would be better off staying a bit

19. In discrete time, with discount factor δ between the periods, the reaction curves are defined by $\partial R_i(r_i(k_j), k_j)/\partial k_i = 1 - \delta + m_i$.

longer near its reaction curve. Its optimal control, in view of firm 2's strategy, is to adopt a "two-switchpoints" or "S-curve" investment strategy. That is, firm 1 may invest as fast as possible, then stop investing, and finally resume investment until the state reaches firm 2's reaction curve. The state then follows an S curve. Naturally, preemption motives may dominate, so that the optimal path may involve zero or one switch points. (For a more thorough analysis, see Nguyen 1986.)

13.4.2 Differential-Game Strategies

Hanig (1986, chapter 3) and Reynolds (1987a,b) have obtained interesting results in the reversible-investment case by applying differential-game techniques. They use the linear-quadratic specification of subsection 13.3.3:

$$R_i(k_i, k_j) = [d - b(k_i + k_j)]k_i,^{20} \tag{13.18}$$

$$\frac{dk_i}{dt} = a_i - \rho k_i, \tag{13.19}$$

and either

$$C_i(k_i, a_i) = \tfrac{1}{2}c(a_i)^2 + \tilde{c}a_i \quad \text{(Reynolds)} \tag{13.20}$$

or

$$C_i(k_i, a_i) = \tfrac{1}{2}c(a_i - \rho k_i)^2 + \tilde{c}a_i \quad \text{(Hanig)} \tag{13.21}$$

(where d, b, c, and $\tilde{c}$ are all strictly positive).

Let us assume that $m_i = 0$ for simplicity (a symmetric maintenance cost can be included in d).

The Cournot-Nash level can be computed in the following heuristic way for the specification 13.21 (we leave it to the reader to check that $k^c = [d - \tilde{c}(r + \rho)]/[3b + c\rho(r + \rho)]$ for specification 13.20): Suppose that the firms are in a steady state at Cournot level (k^c, k^c), and let firm 1, say, increase its investment rate by 1 during a period of time dt, and then revert to its previous investment policy once time dt has elapsed. Because in a steady state $a_i = \rho k_i$, the cost of this investment is $\tilde{c}(dt)$. *If this investment does not affect firm 2's investment policy* (this is the open-loop assumption underlying the steady-state Cournot levels), the extra revenue for firm 1 is

$$\int_0^{\infty} (d - 3bk^c)(e^{-\rho s}dt)e^{-rs}\,ds = (d - 3bk^c)\frac{dt}{r + \rho},$$

since firm 1's marginal revenue is $d - bk_2 - 2bk_1 = d - 3bk^c$, and the remaining proportion of the extra investment s units of time after it is

20. For conditions under which such revenue functions may emerge from Cournot competition, see Fudenberg and Tirole [1983]; on their emerging from price competition under capacity constraints k_i, see chapter 5 of Tirole 1988.

made is $e^{-\rho s}$. At the Cournot level, one must have $\tilde{c} = (d - 3bk^c)/(r + \rho)$ or $k^c = (d - \tilde{c}(r + \rho))/3b$.

Hanig and Reynolds solve equations 13.16 and 13.17 for this game. They find that there exists a unique linear equilibrium, and that the investment strategy $a_i = s_i(k_i, k_j)$ decreases (linearly) with firm j's capital level, k_j. They show that the capital levels (k_1, k_2) converge to a steady state (k_1^*, k_2^*).[21] In the symmetric case, the symmetric steady-state level k^* strictly exceeds the Cournot level derived above. Thus, the two firms engage in a sort of "symmetric Stackelberg behavior," as at the Cournot level each firm has an incentive to increase its capital at least a bit. If its rival didn't react, such a move would affect the firm's profit only to the second order from the previous reasoning; but in the perfect equilibrium, the rival responds to a higher level of capital by reducing its investment. Because of the existence of adjustment costs, the current increase in capital has commitment power, because it would be very costly for the firm to move back quickly to its Cournot level. (Interestingly enough, k^* does not converge to k^c when the adjustment cost c tends to 0. There is thus a "discontinuity" at $c = 0$. In contrast, k^* converges to k^c when the adjustment cost tends to infinity. Reader: Find out why.)

The investment path to the steady state also has interesting features. In particular, a firm may "overshoot" its steady-state capital level (see figure 13.5, which is drawn from Hanig 1986). This result is along the lines of the

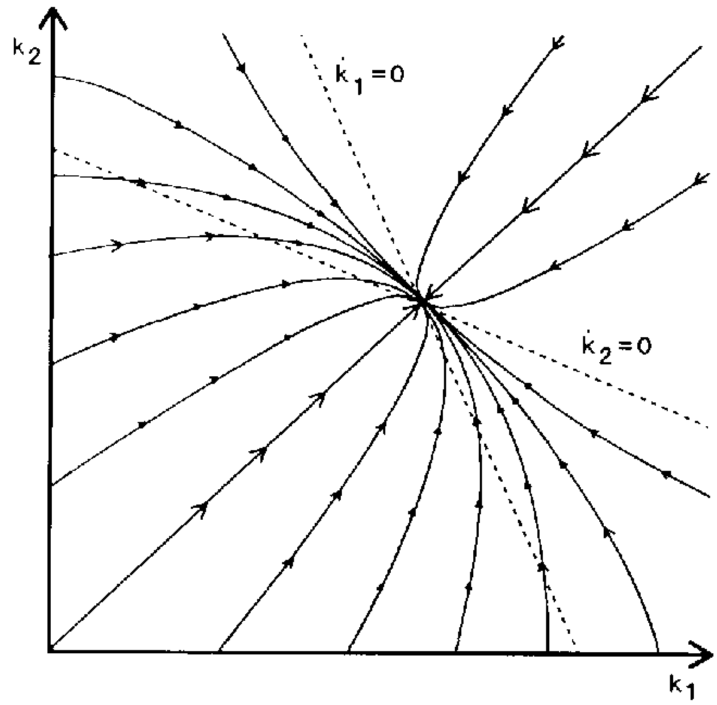

Figure 13.5

21. As in the open-loop case (Fershtman and Muller 1984), the steady state is independent of the initial levels of capital.

result found in the irreversible-investment case that a firm may invest beyond its Cournot level to reduce its rival's capital level. Here the firm does not reduce its rival's capital level in the long run, but reduces it in the medium run.

Last, Hanig shows that, in the asymmetric case in which c_1 is very large and c_2 very small, the perfect-equilibrium steady state is close to the Stackelberg level with firm 1 as the leader (as firm 1's capital level has more commitment power).

Exercises

Exercise 13.1** Figure 13.2 shows that, unlike the Nash-equilibrium and perfect-equilibrium correspondences, the MPE correspondence does not have a closed graph in U. Does it have a closed graph in $U(H)$? (Hint: Think of an MPE as a perfect equilibrium of a game with a different information structure.)

Exercise 13.2** Consider the nonmyopic Cournot tâtonnement model. Firms 1 and 2 alternate in choosing quantities q_1 and q_2. Quantities are fixed for two periods, i.e., $q_1^{2n+2} = q_1^{2n+1}$ and $q_2^{2n+1} = q_2^{2n}$ for all n. Payoffs are $\sum_{t=0}^{\infty} \delta^t g_i(q_i^t, q_j^t)$ where $\partial^2 g_i/\partial q_i^2 < 0$, $\partial g_i/\partial q_j < 0$, and $\partial^2 g_i/\partial q_j\, \partial q_i < 0$. Firms adopt Markov strategies: $q_i = r_i(q_j)$.

(a) Check that the "reaction functions" r_i are nonincreasing.

(b) Suppose that $g_i = q_i(1 - q_i - q_j)$. Look for a linear MPE $r_i(q_j) = a - bq_j$. Compare its steady state to the Cournot level ($\frac{1}{3}$). How does it vary with δ? Qualitatively compare the solution against that obtained by Hanig and Reynolds in the capital-accumulation game (see subsection 13.4.2).

(c) Still in the quadratic model of question b, note by backward induction that the reaction functions are linear in the *finite-horizon* (T) model, and that the function that maps the slope and the intercept of r_{t+1}^T and r_{t+2}^T into the slope and intercept of r_{t-1}^T and r_t^T is a contraction mapping in the space of linear functions with slopes in $[-\frac{1}{2}, 0]$ and intercepts in $[0, 1]$. Conclude that the reactions functions r_t^T converge to the infinite-horizon solution derived in part b of this exercise when T tends to $+\infty$. (See Maskin and Tirole 1987 for the answers.)

Exercise 13.3** Two firms play the Cournot game (see chapter 1) repeatedly. Let a_i^t denote firm i's output at date t, and let $a^t \equiv a_1^t + a_2^t$. A fraction $\varepsilon > 0$ of the good sold at date t is recycled (once). The consumers do not receive income when the good they consumed is recycled (a recycling industry purchases the old units at price 0—this ensures that consumers are myopic). The inverse demand curve at date t is $p^t = 1 - a^t - \varepsilon a^{t-1}$. Production by the duopolists is costless. Assume that ε is "small."

(a) What is the payoff-relevant variable in this game?

(b) Write the first-order condition for an MPE. (Use dynamic programming, and introduce valuation functions $V_i(a^{t-1})$.)

(c) Look for a symmetric equilibrium with quadratic valuation function, so $dV/da^{t-1} = -\alpha + \beta a^{t-1}$. Show that $\beta(3 - 2\delta\beta) = \varepsilon^2$ and $\alpha\varepsilon = \beta(1 - \delta\alpha)$ (where δ is the discount factor).

Exercise 13.4** Consider the following three-period game, which is due to Harris (1990): In period 1, two gamblers A and B pick a $\in [0, 1]$ and b $\in [0, 1]$ respectively. In period 2, two greyhounds C and D each receive an injection of size $a + b$, which changes their attitude toward the race. They pick $c \in [0, 1]$ and $d \in [0, 1]$, which are the times in which they complete the course. In period 3 each of two referees E and F must declare a winner, picking $e \in \{C, D\}$ and $f \in \{C, D\}$ respectively. In each period choices are made simultaneously, and players in later periods observe the actions taken by players in earlier periods.

Gambler A obtains a payoff of $1 - a$ if greyhound C is declared the winner by both referees, and $-1 - a$ otherwise. Similarly, gambler B obtains $1 - b$ if both referees declare D to be the winner, and $-1 - b$ otherwise. In other words, A wants C to win, B wants D to win, and both want a result. They would also like to keep their contributions to the injection as small as possible. The payoff to greyhound C is $2c$ if $e = C$ and $1 - (a + b)(1 - c)$ if $e = D$. That is, the form of his payoff depends on whether he or the other greyhound is declared the winner by referee E, but either way he would prefer to run the race as slowly as possible. Also, he would prefer to be first rather than second provided that he does not have to run too fast. The payoff to greyhound D is $2d$ if $e = D$ and $1 - (a + b)(1 - d)$ if $e = C$; like greyhound C, he is interested only in the verdict of referee E. Lastly, referee E gets payoff d if he declares C to be the winner, and c if he declares D to be the winner. Referee F's payoffs are identical. Show that there exists no MPE. (There actually exists no subgame-perfect equilibrium.)

Exercise 13.5** Two firms costlessly produce an infinitely durable good at dates 0, 1, ... At the beginning of date t, the existing stock of the good is X^t. The firms simultaneously choose outputs a_1^t and a_2^t and the date-t rental price is $1 - X^{t+1}$, where $X^{t+1} \equiv X^t + a_1^t + a_2^t$. The discount factor is δ. Look for a symmetric MPE in which the buyers have rational expectations (date-t price is equal to date-t rental value plus discounted date-$(t + 1)$ price), the strategies are linear in the state variable, and the valuation functions are quadratic. (For more details, see Carlton and Gertner 1989.)

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14 Common Knowledge and Games

14.1 Introduction††

The idea of common knowledge   that players know that their opponents know that they know...—is a useful tool for understanding how the equilibria of a game depend on its information structure. This chapter gives a formal definition of common knowledge of an event, and illustrates its implications in a few examples of games.

Section 14.2 gives two equivalent definitions of common knowledge. The first is the recursive one just mentioned: An event is common knowledge if players know this event, know that the other players know this event, and so on *ad infinitum*. The second is perhaps less natural but is simpler to apply: To be common knowledge, an event must be implied by (that is, be a superset of) the element in the finest common coarsening of the players' partitions that contains the state of nature. We apply the definitions to the celebrated "dirty face" example.

Section 14.3 illustrates the difference between knowledge (that everyone knows) and common knowledge (that everyone knows that everyone knows that...) of the payoff functions in games. The point of that section is to emphasize some strong implications of common knowledge. We first add strategies and payoffs for the players to the "dirty face" information structure introduced in section 14.2, and show how the set of equilibria is changed by the introduction of an extra player who publicly announces something that everyone knew was true but that was not common knowledge. This example illustrates the strikingly different implications of knowledge and common knowledge for games. The second illustration is the archetypal result in the theory of asset pricing that trade among asymmetrically informed players should not occur if the allocation determined before they receive their private information is Pareto optimal. This result is shown to be closely related to the fact that, starting from a common prior on states of nature, players cannot disagree on the posterior probability of an event if the posterior probabilities of the event are common knowledge.

Section 14.4 asks whether a Nash equilibrium of a given game is close to one of a perturbed game with an information structure very "close" to that of the original game. Because this is a lower-hemicontinuity property, it can hold at most for generic games. (See sections 1.3, 12.1, and 13.2.) But even for generic games, one must use an appropriate definition of what it means for two information structures to be close. The "electronic mail" example developed in section 14.4 shows that such a definition must be stringent. However, it turns out that a simple generalization of common knowledge, "almost common knowledge," ensures generic lower hemicontinuity. Take a given game with payoffs u and a Nash equilibrium σ, and consider a perturbed game in which it is very likely that payoffs are u but in which payoffs may differ from u with small probability. Roughly, game u is almost common knowledge if each player puts high probability on payoffs' being u, puts high probability on other players' putting high probability on payoffs' being u, and so on *ad infinitum*. For generic games u, the perturbed game has an equilibrium near σ if payoffs u are almost common knowledge.[1]

Our focus throughout is on the players' knowledge about payoff functions and other exogenous data of the game. There is also an extensive literature characterizing various equilibrium concepts in terms of the players' beliefs about the others' *strategies*, about their opponents' beliefs, and so forth (see, e.g., Aumann 1987, Brandenburger and Dekel 1987, and Tan and Werlang 1988; Brandenburger 1990 and Brandenburger and Dekel 1990 are recent surveys).

14.2 Knowledge and Common Knowledge[2] ††

Before defining common knowledge we must give a definition of knowledge. That is, when will we say that an agent "knows" something? As throughout the book, we represent agent i's beliefs by an information partition H_i. Formally, we suppose that the exogenous uncertainty in the model is represented by a finite set Ω of nature's moves and a common prior distribution p, and that all of player i's information about ω is represented by the element (or event) $h_i(\omega)$ of H_i that contains ω. The interpretation is that player i knows that the true state is some $\omega' \in h_i(\omega)$, but he does not know which one it is. In particular, player i's information partition represents all the information he has about the information of other players, about their information about his information, and so on. (See the discussion of the notion of type in chapter 6.) We suppose that all states in Ω have positive prior probability; states with probability 0 are dropped from the description of the state space.

Player i's posterior beliefs about the state when knowing that $h_i(\omega) = h_i$ are given by

$$p(\omega \mid h_i) = p(\omega) \Big/ \sum_{\omega' \in h_i} p(\omega') = p(\omega)/p(h_i).$$

We then say that player i knows event E at ω if he knows that the true state lies in E that is, if $h_i(\omega) \subseteq E$. The event "player i knows E," denoted $K_i(E)$,

1. These perturbations are closely related to those in the Fudenberg-Kreps-Levine paper discussed in section 11.4, except that here we do not restrict our attention to "personal types." Fudenberg, Kreps, and Levine show that with the more general elaborations discussed in this chapter, any Nash equilibrium is near-strict. We should also point out that in a sequence of elaborations $\tilde{E}$ that approach E in the sense of section 11.4, the payoff functions of E become almost common knowledge.

2. Our presentation of this material draws heavily on the excellent survey by Binmore and Brandenburger (1989).

is then $\{\omega | h_i(\omega) \subseteq E\}$. Since information partitions must satisfy $\omega \in h_i(\omega)$, if player i knows E, then E is true.[3,4] With this formulation of knowledge, more precise information corresponds to knowing a *smaller* set: Knowledge here is the ability to rule out some of the states that were possible *ex ante*. In particular, if a player knows that the true state is in E, he knows that the true state is in any superset of E.

The event "everyone knows E," denoted $K_{\mathscr{I}}(E)$, is then the set

$$\left\{\omega \,\middle|\, \bigcup_{i \in \mathscr{I}} h_i(\omega) \subseteq E\right\}.$$

Then, because all players know the information partitions, player i knows that everyone knows E if $h_i(\omega) \subseteq K_{\mathscr{I}}(E)$, and the event that everyone knows that everyone knows E is

$$K^2_{\mathscr{I}}(E) = \left\{\omega \,\middle|\, \bigcup_{i \in \mathscr{I}} h_i(\omega) \subseteq K_{\mathscr{I}}(E)\right\}.$$

Then event $K^{\infty}_{\mathscr{I}}(E)$ is the intersection of all sets of the form $K^n_{\mathscr{I}}(E)$, which is a decreasing sequence of events in the sense that $K^{n+1}_{\mathscr{I}}(E) \subseteq K^n_{\mathscr{I}}(E)$ for all n.

Definition 14.1 Event E is common knowledge at ω if $\omega \in K^{\infty}_{\mathscr{I}}(E)$.

If E is common knowledge, then any statement of the form "player i knows that players j and k know that player i knows that m knows... E" is true. The term "common knowledge" was first used to describe the infinite regress of "I know that you know" by Lewis (1969), who attributed the basic idea to Schelling (1960). Aumann (1976) proposed the notion independently, and gave a characterization of common knowledge in terms of the *meet* of the individual agents' partitions; we discuss this below. It is interesting to note that Littlewood (1953) developed some examples of common-knowledge-type reasoning without developing a formal definition of the concept.

The definition of common knowledge takes for granted a state space and information partitions that incorporate *all* of the agents' initial uncertainty about the structure of the game. This framework makes the information

3. This is called the "axiom of knowledge." Other axioms implied by the partition formulation are $K_i(E) = K_iK_i(E)$ (player i knows E if and only if he knows that he knows it) and $\sim K_i(\sim K_i(E)) \subseteq K_i(E)$ (if player i does not know that he does not know E, then he knows E). Although the partition model of knowledge is standard for decision theory, with other interpretations of knowledge this model may be too strong. Bacharach (1985), Brown and Geanakoplos (1988), Geanakoplos (1989), Rubinstein and Wolinsky (1989), Samet (1987), and Shin (1987) discuss common knowledge when knowledge is modeled by more general "knowledge operators" K_i that need not be derived from partitions. This work is discussed in the survey by Binmore and Brandenburger (1989).

4. If some states had probability 0, then it might be that $h_i(\omega)$ is not contained in E, yet player i assigns E posterior probability 1. See Brandenbuger and Dekel 1987a for a discussion of how to extend the definitions of knowledge and common knowledge to models in which some states have prior probability 0.

partitions common knowledge in an informal sense. Otherwise, if (say) player 2 did not know whether player 1's information when ω occurred was h_1' or h_1'', we would need to add additional states of the world to Ω to model the different beliefs player 2 might have; since the players' beliefs derive from a common prior, there would have to be positive probability that each of player 2's beliefs was in fact correct.

Returning to the formal definition of common knowledge, we can easily check that if everyone knows E, then E must be true, that is, $K_{\mathscr{I}}(E) \subseteq E$. Thus, as we iterate the (everyone knows) operator, the set of states included cannot grow, and if E is common knowledge and the state space is finite, there must be a finite n such that $K_{\mathscr{I}}^n(E) = K_{\mathscr{I}}^{\infty}(E)$.

As an illustration of this definition, consider the following examples, which are variants of an example originally developed by Littlewood (1953).

Example 14.1: Dirty Face without a Sage

Suppose there are three players and eight states, written in binary notation as 000, 001, 010, 011, etc., and that all states have equal prior probability. Player 1 knows the second and third components of the state but not the first, player 2 knows the first and third but not the second, and player 3 knows the first and second but not the third. In terms of the information partitions, this means that H_1 has the four elements $\{000, 100\}$, $\{001, 101\}$, $\{010, 110\}$, and $\{011, 111\}$. In a famous story we develop below, the ith component of the state is 0 if player i's face is clean and 1 if player i's face is dirty; each player observes the others' faces but not his own.

With this information structure, everyone knows the event E^* = "at least one player's face is dirty"—that is, "not 000"—if there are at least two dirty faces; if there is only one dirty face, then the player whose face is dirty doesn't know if there are zero or one dirty faces. So

$$K_{\mathscr{I}}(E^*) = \{111, 110, 101, 011\} \equiv E^{**}.$$

Then $K_{\mathscr{I}}^2(E^*) = K_{\mathscr{I}}(E^{**}) = 111$: At 101, for example, player 1 cannot rule out the state 001, which is not in E^{**}. Finally, $K_{\mathscr{I}}^3(E^*) = K_{\mathscr{I}}(111) = \varnothing$, since no player can distinguish 111 from the state in which his face is clean and the others are dirty. Thus, there is no ω at which E^* is common knowledge. In fact, theorem 14.1 below makes it easy to check that the only event that is common knowledge is the whole state space Ω.

Example 14.2: Dirty Face with a Sage

Next, we modify the information partitions in example 14.1 so that if all three faces are clean, all players are informed of this in public by a sage. Then player 1's partition H_1 has the five elements $\{000\}$, $\{100\}$, $\{001, 101\}$, $\{010, 110\}$, and $\{011, 111\}$. Now the event 000 is common knowledge when it occurs, as is the complementary event $K_{\mathscr{I}}(E^*) = E^*$, so $K_{\mathscr{I}}^2(E^*) = E^*$, and so on.

In examples 14.1 and 14.2 it was fairly easy to determine when a state was common knowledge by applying definition 14.1 directly. This is not always the case. Aumann (1976) gives an equivalent definition of common knowledge that provides a simple algorithm for determining the commonly known information without explicitly iterating the (everyone knows) operator. To present this definition, we first recall that the *meet* $\mathcal{M}$ of a collection of partitions H_i is the finest common coarsening of the partitions. We let $M(\omega)$ be the element of $\mathcal{M}$ containing ω. The requirement that $\mathcal{M}$ be a common coarsening means that it is not more informative than any of the H_i; that is, for all players i and all ω,

$$h_i(\omega) \subseteq M(\omega).$$

$\mathcal{M}$ is the finest common coarsening if there does not exist another common coarsening $\mathcal{M}'$ with $M'(\omega) \subseteq M(\omega)$ for all ω and strict inclusion $M'(\hat{\omega}) \subset M(\hat{\omega})$—for at least some $\hat{\omega}$.[5]

The idea of "reachability" provides a simple algorithm for computing the meet, and some intuition for understanding theorem 14.1 below. It is easy to see that $\omega' \in M(\omega)$ if there exists a chain $\omega_0 \equiv \omega, \omega_1, \omega_2, \ldots, \omega_m \equiv \omega'$ such that for all $k \in \{0, \ldots, m-1\}$ there exists a player $i(k)$ such that $h_{i(k)}(\omega_k) = h_{i(k)}(\omega_{k+1})$. In words, there exists a chain of states from ω to ω' such that two consecutive states are in the same information set of some player. One can also check that ω' belongs to $M(\omega)$ only if ω' is *reachable* from ω in the above sense.

Theorem 14.1 (Aumann 1976) Let $\mathcal{M}$ be the meet of the individual players' partitions. Event E is common knowledge at ω if and only if $M(\omega) \subseteq E$.

The intuition for theorem 14.1 is as follows: If there exists ω' reachable from ω through $\omega_1, \ldots, \omega_{m-1}$, then player $i(0)$ cannot exclude the possibility that ω_2 is consistent with the information of player $i(1)$, who in turn cannot exclude the possibility that ω_3 is consistent with the information of player $i(2)$, who.... Hence, someone believes that someone believes that...that someone believes that ω' is possible, and thus no event E that excludes ω' can be common knowledge. In contrast, if any chain of "i knows that j knows that..." is "trapped" in E (that is, if $M(\omega) \subseteq E$), everyone knows that everyone knows that...the state is in E.

Figure 14.1 gives an example, where $\Omega = (1, 2, 3, 4)$, $H_1 = \{(1, 2), (3, 4)\}$, $H_2 = \{(1), (2, 3), (4)\}$, and $\omega = (2)$. Then player 2 cannot rule out state 3, and at state 3 player 1 cannot rule out state 4. Since player 1 cannot rule out state 1,

$$M(2) = \Omega = (1, 2, 3, 4),$$

5. Actually, if this inclusion holds for some $\hat{\omega}$ it must also hold for some $\tilde{\omega} \in \Omega \backslash M'(\hat{\omega})$. Proving this is exercise 14.4.

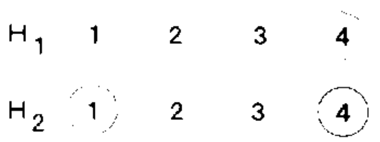

Figure 14.1

and so all that is commonly known at state 2 is the original set of possible states.

Proof of Theorem 14.1 First we claim that $M(\omega)$ is common knowledge at every $\omega' \in M(\omega)$:

$$K_{\mathscr{I}}(M(\omega)) = \left\{ \omega \,\middle|\, \bigcup_{i \in \mathscr{I}} h_i(\omega) \subseteq M(\omega) \right\} = M(\omega),$$

since $\mathscr{M}$ is a coarsening of each H_i, and so $K^n_{\mathscr{I}}(M(\omega)) = M(\omega)$ for all n, and $K^{\infty}_{\mathscr{I}}(M(\omega)) = M(\omega)$. Next if E contains $M(\omega)$, then, since $M(\omega)$ is common knowledge, so is E.

Conversely, if E is common knowledge at ω,

$M(\omega) \subseteq E.$

To see this, suppose that there exists $\omega' \in M(\omega)$ such that $\omega' \notin E$. Because $\omega' \in M(\omega)$, and because of the reachability criterion, there exists a sequence $k = 0, \ldots, m$ with associated states of nature $\omega_0, \ldots, \omega_m$ and information sets $h_{i(k)}(\omega_k)$ such that $\omega_0 = \omega$, $\omega_m = \omega'$, and $\omega_k \in h_{i(k)}(\omega_{k+1})$. But at information set $h_{i(m)}(\omega_{m-1})$, player $i(m)$ does not know event E; working backward on k, we see that event E cannot be common knowledge. ■

If $E = K_{\mathscr{I}}(E)$, we say that event E is a *common truism* (Binmore and Brandenburger 1989), a *public event* (Milgrom 1981), or *self-evident* (Samet 1987). Clearly a common truism is common knowledge whenever it occurs.[6] Moreover, the proof above shows that the common truisms are precisely the elements of $\mathscr{M}$ and unions of elements of $\mathscr{M}$, and so any commonly known event must be the consequence of a common truism. Note how easy this makes it to determine which events can be common knowledge in example 14.1: Since no player knows the state of his own face, the only common truism is the whole state space.

14.3 Common Knowledge and Equilibrium[††]

We now analyze two games in which common knowledge has strong implications. Throughout our discussion, we assume that the structure of the game is common knowledge in an informal sense. Applying formal

6. Iterating the (everybody knows) operator: $K_{\mathscr{I}}(E) = K^2_{\mathscr{I}}(E) = \cdots$ and therefore $E = K^{\infty}_{\mathscr{I}}(E)$.

definitions of common knowledge to the structure of the game leads to technical and philosophical problems that we prefer not to address.

14.3.1 The Dirty Faces and the Sage

To illustrate how the equilibrium strategies in a state of nature (more precisely, the projection of the set of equilibrium profiles onto the state) can vary with what is common knowledge, we return to the examples 14.1 and 14.2, which gave the story of the clean and dirty faces. To make this into a game, suppose that in each of $T+1$ periods ($t = 0, 1, \ldots, T$, where $T \geq 2$) the three players simultaneously decide whether or not to blush, and that their actions are revealed at the end of each period (so this is a multi-stage game with observed actions). Each player can blush in at most one period. Each player receives payoff δ^t if he blushes in period t when his face is dirty, 1 if he does not blush when his face is clean, -100 if he blushes when his face is clean, and -1 if he does not blush when his face is dirty. Thus, no player will blush unless he is quite certain that his face is dirty. We assume that the discount factor δ is smaller than 1 (so that a player blushes immediately when he knows that his face is dirty), that each player's face is equally likely to be clean or dirty, and that the states of the three faces are independently distributed.

We begin with the information structure of example 14.1, in which each player's information is the state of the other two players' faces. We claim that the unique Nash equilibrium is for no player ever to blush, even if all faces are actually dirty. To see that this is an equilibrium, note that players will learn nothing from their opponents' play and that, as long as no player deviates, each player's posterior beliefs about his own face equals the prior, which is that clean and dirty are equally likely. To see that not blushing is the only Nash equilibrium of the game, let t_0 denote the first date at which at least one player (player i, say) blushes with positive probability. At date t_0 player i has not learned anything, because no one blushes before t_0. Therefore, his posterior belief that his face is dirty is still 0.5, and he gets a very negative expected payoff by blushing at t_0.

Now consider the information structure of example 14.2. This can be explained by saying that there is a sage who will announce at the beginning of the first period that at least one face is dirty if and only if this is in fact the case. With this information structure there is no longer a Nash equilibrium where players never blush when all their faces are dirty.

To see this, we proceed by induction on the number of dirty faces. When exactly one face is dirty, the sage announces that there is at least one dirty face; the player with a dirty face sees two clean faces and thus will blush in the first period (because of discounting). Since all players know that their opponents know the game structure, all players know that a player with a dirty face would blush if there were exactly one dirty face. Thus, if no one blushes in the first period, everyone knows that there are at least two dirty faces; this fact is common knowledge in the formal sense, because the structure of the game is common knowledge in the informal sense. Continuing with the induction: If there are exactly two dirty faces, two players each see one clean face, and these two players will blush in the second period. Thus, if no one blushes in the second period, all players know that all three faces are dirty, and all players blush in the third period. More generally, it is easy to see that, with I players, if all faces are dirty, all blush at date $I - 1$.

This example is sometimes analyzed by considering only the change the announcement makes in the state where all faces are dirty. Since even without the sage's announcement all the players know that there is at least one dirty face when all faces are dirty, the only change the announcement makes *in this state* is to make common knowledge a fact that was previously known to all the players. An alternative interpretation is that, in any state where only one face is dirty, the sage's announcement gives the player with the dirty face payoff-relevant information he did not have before. This observation can be generalized: With a fixed state space Ω and a prior p, if E is not common knowledge at ω with partitions $\{H_i\}$ but E is common knowledge at ω with partitions $\{\hat{H}_i\}$, there must be a player j and a state $\hat{\omega}$ such that player j's *knowledge* at $\hat{\omega}$ is different under H_j and $\hat{H}_j$. That is, there must be an event $\tilde{E}$ such that $h_j(\hat{\omega}) \not\subseteq \tilde{E}$ but $\hat{h}_j(\hat{\omega}) \subseteq \tilde{E}$. (Proving this is exercise 14.5.)

14.3.2 Agreeing to Disagree†††

The first and best-known result obtained with the formal definition of common knowledge is Aumann's proof that rational players cannot "agree to disagree" about the probability of a given event. The intuition for this is that if a player knows that his opponents' beliefs are different from his own, he should revise his beliefs to take the opponents' information into account. Of course, this intuition doesn't make sense if the player thinks his opponents are simply crazy; it requires that he believe that his opponents process information correctly and that the difference in the beliefs reflects some objective information. More formally, Aumann's result requires that the players' beliefs be derived by Bayesian updating from a common prior distribution.

Theorem 14.2 (Aumann 1976) Suppose that it is common knowledge at ω that player i's posterior probability of event E is q_i and that player j's posterior probability of E is q_j. Then $q_i = q_j$.

Proof Let $\mathcal{M}$ be the meet of all the players' partitions, and let $M(\omega)$ be the element of $\mathcal{M}$ that contains ω. Write $M(\omega) = \bigcup_k h_i^k$, where each h_i^k is an element of player i's partition H_i. Since player i's posterior probability of event E is common knowledge, it is constant on $M(\omega)$, and hence

$$q_i = p(E \cap h_i^k)/p(h_i^k) \text{ for all } k.$$

Hence,

$$p(E \cap h_i^k) = q_i p(h_i^k),$$

and summing over k yields

$$p(E \cap M(\omega)) = q_i p(M(\omega)).$$

Applying the same reasoning to player j shows that

$$p(E \cap M(\omega)) = q_j p(M(\omega)),$$

so $q_i = q_j$. ∎

The proof of the theorem makes explicit use of the assumption of a common prior over Ω. It should be clear that this assumption is necessary: When the priors differ, each player, rather than ascribe any difference in an opponent's beliefs to "real" information, is free to ascribe such a difference to the opponent's having used the "wrong" prior.[7]

As Aumann observes, the theorem also requires the assumption that the players' partitions are common knowledge in the informal sense that the model fully describes the players' information. Intuitively, if player i did not know how player j arrived at his posterior beliefs, player i would not know how to evaluate the fact that player j's beliefs differed from his own. Somewhat more formally, if the partitions were not common knowledge (say, because player i did not know player j's partition), we would need to enlarge the state space to assign positive probability to each H_j that player i believes has positive probability. On this new expanded state space, the theorem would hold as before, again with the supposition that the players have a common prior.

Aumann also gives an example to show that the result fails if the players merely know each other's posteriors, as opposed to the posteriors' being common knowledge. In his example, Ω has four equally likely elements, ω_1, ω_2, ω_3, and ω_4, players 1's partition is $H_1 = \{(\omega_1, \omega_2), (\omega_3, \omega_4)\}$, and player 2's partition is $H_2 = \{(\omega_1, \omega_2, \omega_3), (\omega_4)\}$. Let E be the event (ω_1, ω_4). Then at ω_1, player 1's posterior probability of E is

$$q_1(E) = p[(\omega_1, \omega_4) | (\omega_1, \omega_2)] = \tfrac{1}{2},$$

and player 2's posterior probability of E is

$$q_2(E) = p[(\omega_1, \omega_4) | (\omega_1, \omega_2, \omega_3)] = \tfrac{1}{3}.$$

Moreover, player 1 knows that player 2's information is the set $(\omega_1, \omega_2, \omega_3)$, so player 1 knows $q_2(E)$. Player 2 knows that player 1's information is either (ω_1, ω_2) or (ω_3, ω_4), and either way player 1's posterior probability of E is

7. Here and in his 1987 paper, Aumann appeals to the "Harsanyi doctrine" to support the assumption of a common prior.

$\frac{1}{2}$, so player 2 knows $q_1(E)$. Thus, each player knows the other player's posterior, yet the two players' posteriors differ. The explanation is that the posteriors are not common knowledge. In particular, player 2 does not know what player 1 thinks $q_2(E)$ is, as $\omega = \omega_3$ is consistent with player 2's information, and in this case player 1 believes there is probability $\frac{1}{2}$ that $q_2(E) = \frac{1}{3}$ (if $\omega = \omega_3$) and probability $\frac{1}{2}$ that $q_2(E) = 1$ (if $\omega = \omega_4$).

Geanakoplos and Polemarchakis (1982) observe that Aumann's result does not address the issue of how and when the players' posterior beliefs might come to be common knowledge in the first place. They assume that the players communicate only by announcing their posterior beliefs. (In particular, players are not allowed to communicate information sets.) Geanakoplos and Polemarchakis analyze the process in which two agents take turns announcing their posterior distributions cooperatively (i.e., truthfully) to each other; if the state space is finite, this process converges in finite time. Or, as in the title of their paper, "we can't disagree forever."

Geanakoplos and Polemarchakis note that the fact that agents' posteriors about an event E eventually converge (and thus become common knowledge) does not imply that the players know as much about the event as they would know if they pooled their information. That is, even though the agents' posteriors are equal, they need not have the same information. A counterexample has four equally likely states, $\omega = 00$, 10, 01, and 11; player 1 knows the first component of ω, and player 2 knows the second component. Consider the posterior probabilities for the event $E = \{00, 11\}$ when the state is 00. On the basis of only his own information, each player's posterior probability for E is always $\frac{1}{2}$. When player 1 announces that his posterior for E is $\frac{1}{2}$, this does not give any information to player 2—for instance, when the second component is 0, player 2 knows that ω is either 00 or 10; if it is 00, player 1 knows that ω is either $00 \in E$ or $01 \notin E$, and so assigns E probability $\frac{1}{2}$; if it is 10, player 1 knows that ω is either $10 \notin E$ or $11 \in E$, and again player 1's posterior for E is $\frac{1}{2}$. Thus, player 2's revised posterior for E is still $\frac{1}{2}$, and his announcing this gives no information to player 1.

14.3.3 No-Speculation Theorems†††

Results on agreeing to disagree are closely related to the results about the impossibility of risk-averse agents' taking opposing sides of the same purely speculative bet.[8] Intuitively, if player 1 is risk averse and accepts an even-odds bet that a coin will come up heads, he must assign heads a probability greater than $\frac{1}{2}$, whereas if player 2 is risk averse and accepts the other side of this bet that the coin will come up tails then player 2 thinks the probability of heads is less than $\frac{1}{2}$, so the players would be "agreeing to disagree."

8. Rubinstein and Wolinsky (1989) present a theorem that encompasses both types of results.

There are two sorts of no-speculation theorems in the literature: "equilibrium" theorems, which assert that speculation cannot occur in equilibria of various games, and "common-knowledge" theorems, which assert that it cannot be common knowledge that all players expect to gain from speculation. Although the first no-speculation theorems were equilibrium theorems, we will present a common-knowledge theorem first, as it is more closely related to the theme of this chapter.

To distinguish speculative trading from trading for other purposes, we decompose the state of nature ω into two parts: $\omega = (x, z)$, where the players' *ex post* utilities and initial endowments depend only on x and where z is a vector of signals $z = (z_1, \ldots, z_I)$, possibly correlated with the payoff-relevant uncertainty x. Thus, player i's information is

$$h_i(\omega) = h_i(x, z) = \{(x', z') \text{ such that } z_i' = z_i\}.$$

A *net trade* is a map y from the set of states Ω to consumption bundles in a set B, where $y_i(\omega)$ is player i's net trade at ω; y is *feasible* if it lies in the (exogenous) set Y. Player i's endowment is $e_i(x)$, and his utility of $y(\cdot)$ at ω is

$$\tilde{u}_i(y_i(\omega) + e_i(x), x) \equiv u_i(y_i(\omega), x).$$

Suppose as usual that the players have a common prior distribution $p(\cdot)$ on Ω.

Theorem 14.3 (Milgrom and Stokey 1982) Suppose that traders are weakly risk averse (i.e., u_i is concave in y_i) and that $\hat{y} \equiv 0$ is Pareto optimal in the set of all feasible net trades. If y is feasible, and if it is common knowledge at ω and that each player weakly prefers y to $\hat{y}$, then every player is indifferent between y and $\hat{y}$; if all players are strictly risk averse (that is, u_i is strictly concave), then $y = \hat{y}$.

Proof If it is common knowledge at ω' that all players at least weakly prefer y to $\hat{y}$, then for all ω'' in $M(\omega')$

$$\mathrm{E}[u_i(y_i(\omega), x) \mid h_i(\omega'')] \geq \mathrm{E}[u_i(\hat{y}_i(\omega), x) \mid h_i(\omega'')]. \tag{14.1}$$

We claim that equation 14.1 must hold with exact equality for all players i. To see this, define $y^*(\omega)$ by

$$\begin{aligned} y^*(\omega) &= y(\omega) \text{ if } \omega \in M(\omega') \\ y^*(\omega) &= \hat{y}(\omega) \text{ if } \omega \in M^c(\omega'), \end{aligned} \tag{14.2}$$

where $M^c(\omega') = \Omega \backslash M(\omega')$ is the complement of $M(\omega')$. Now, since $h_i(\omega'') \subseteq M(\omega')$ for every $\omega'' \in M(\omega')$, each player can deduce whether $y^*(\omega) = y(\omega)$ or $y^*(\omega) = \hat{y}(\omega)$. Player i's *ex ante* expected utility of y^* is

$$\begin{aligned} &\mathrm{E}[u_i(y^*(\omega), x)] \\ &= \sum_{h_i \in M(\omega')} p(h_i)\mathrm{E}[u_i(y_i(\omega), x)|h_i] \\ &\quad + \sum_{h_i \in M^c(\omega')} p(h_i)\mathrm{E}[u_i(\hat{y}_i(\omega), x)|h_i] \\ &\quad\quad > \mathrm{E}[u_i(y(\omega), x)], \end{aligned} \tag{14.3}$$

where the last inequality comes from substituting equation 14.1 for player i's utility conditional on $h_i(\omega'') \in M(\omega')$. Moreover, if equation 14.1 holds strictly for some player j, so does equation 14.3, which contradicts the assumption that $\hat{y}$ was Pareto optimal *ex ante*. If traders are strictly risk averse and the endowments are Pareto optimal, then no other allocation is Pareto indifferent for all players. ■

To see the role of common knowledge in the theorem, consider an economy in which there are two equally likely states, ω_1 and ω_2, and two players, 1 and 2, with $H_1 = \{(\omega_1), (\omega_2)\}$ and $H_2 = \{(\omega_1, \omega_2)\}$. There is a single consumption good, and endowments and utility functions are independent of the state (so that, in our previous notation, $\omega \equiv z$). Players are risk neutral, so $\mathrm{E}u_i(y_i(\omega), x) = \mathrm{E}y_i(\omega)$, and $\hat{y}(\omega) \equiv 0$ for all ω is Pareto optimal. Consider the feasible net trade given by

$$y_1(\omega_1) = 1,\ y_2(\omega_1) = -1,\ y_1(\omega_2) = -2,\ y_2(\omega_2) = 2.$$

At state ω_1 both players strictly prefer y to $\hat{y}$ on the basis of their information ($\mathrm{E}[y_1(\omega)|h_1(\omega_1)] = 1$, and $\mathrm{E}[y_2(\omega)|h_2(\omega_1)] = \frac{1}{2}(-1) + \frac{1}{2}(2) = \frac{1}{2}$), and so the conclusion of the theorem does not hold. Of course, it is not common knowledge at ω_1 that both players prefer y to $\hat{y}$, as $M(\omega_1) = (\omega_1, \omega_2)$ contains $h_1(\omega_2)$ and, conditional on $h_1(\omega_2)$, player 1 prefers $\hat{y}$ to y. Thus, inequality 14.1 does not apply, which is why the proof fails.

Intuitively, a situation like this could not arise in equilibrium, as it involves player 1's "fooling" player 2. For example, if we suppose that the players vote on moving from $\hat{y}$ to y after receiving their information, and that either player can veto a move, then it is not an equilibrium for player 1 to vote for y in state ω_1 and for $\hat{y}$ in state ω_2, and for player 2 to vote for y: The outcome of this profile would be y in state ω_1 and $\hat{y}$ in state ω_2, and player 2 would do better to vote for $\hat{y}$.

The first formal result along these lines was reported by Kreps (1977), who gave credit to Stiglitz (1971). Kreps shows that in a rational-expectations equilibrium of an asset market with asymmetric information but common priors on the distribution of uncertainty, and with risk-neutral traders, traders cannot be better off than if they held onto their initial shares and refused to participate in the market. In other words, a trader with superior information cannot benefit from his information, because other traders anticipate that he will buy when the asset is likely to do well and sell when prospects are less favorable. That is, with common priors, the trading game is a zero-sum game. This result can be called the "no-speculation" result. The slightly stronger "no-trade" result does not quite hold here; risk-neutral traders can trade at commonly agreed fair prices, and thus we can assert only that traders can do as well by not trading, not that they will not trade.

One way to obtain "equilibrium" no-speculation results is to suppose that the equilibrium strategies are common knowledge in an informal sense, as are the structure (information and strategy spaces) of the game and the fact that each player's equilibrium strategy maximizes his expected payoff. Consider some arbitrary "trading game" of announcements, bids, etc., supposing only that there is a fixed "no-trade" outcome, independent of the sequence of play, and that each player i can, for each $h_i \in H_i$, play a strategy that ensures the no-trade outcome (for himself) when his information is h_i. Let s^* be an equilibrium profile in the (unmodeled) trading game, and let $y = y(s^*(\cdot))$ be the corresponding net trade. Then, since s_i^* is a best response to each s^*_{-i}, inequality 14.1 is satisfied at all $\omega \in \Omega$, so that equations 14.2 and 14.3 hold where $M(\omega')$ is replaced by Ω; hence the conclusion that all players must be indifferent between y and the null trade $\hat{y}$.

Although the absence of speculation in equilibrium can thus be inferred from no-common-knowledge-of-speculation by assuming that the strategies are themselves common knowledge, we should point out that the equilibrium version of the result can also be obtained directly without formal use of the concept of common knowledge: In any Nash equilibrium, equation 14.1 holds for all $\omega \in \Omega$.

Tirole (1982) extends the Kreps-Stiglitz result in another direction than Milgrom and Stokey. Studying an intertemporal (finite or infinite horizon) asset market where traders are risk neutral and their time-varying private information stems from a filtration, he shows that with a finite number of traders the price of the asset must be at each instant equal to the expected present discounted value of dividends for any "interior trader," that is, for any trader who is not constrained by short-sale constraints or by the impossibility of buying more than 100 percent of the shares. Thus, there cannot exist any "bubble" (difference between market price and market fundamental) for any interior trader. The link with the no-speculation result is that it can be shown that if a bubble existed in a given period, at least one (interior) trader would have an intertemporal trading strategy that would make him strictly better off than if no trade occurred. And, again, the asset market is a zero-sum game and each trader can guarantee himself his no-trade payoff.[9]

9. The no-bubble result fails if there are an infinite number of overlapping finite-lived generations.

14.3.4 Interim Efficiency and Incomplete Contracts[†††]

The version of the Milgrom-Stokey result we gave allowed the set of feasible contracts Y to be arbitrary; in particular, the feasible set need not include all complete contingent contracts. In this case, though, it is not always natural to suppose that, as theorem 14.3 requires, the initial allocation is *ex ante* Pareto optimal, as the players may not be able to meet and contract before receiving any private information.

It is interesting, then, to note that the conclusion of theorem 14.1 holds on the weaker assumption that the initial allocation $\hat{y}$ is *interim efficient* in the sense of Holmström and Myerson (1983). If v is the vector of *ex post* verifiable variables (which might be a subvector of x, for instance), let $e(z, v)$ denote the contingent endowment if no further trading occurs[10] and let $y(z, v)$ denote a v-contingent trade when players receive private information z_i. The no-trade allocation is interim efficient if there exists no $y(\cdot,\cdot)$ such that for all i, z, and z_i'

$$\mathrm{E}(u_i(y(z,v),x)\,|\,z_i) \geq \mathrm{E}(u_i(0,x)\,|\,z_i) \tag{14.4}$$

and

$$\mathrm{E}(u_i(y(z,v),x)\,|\,z_i) \geq \mathrm{E}(u_i(y((z_i',z_{-i}),v),x)\,|\,z_i). \tag{14.5}$$

Inequalities 14.4 and 14.5 express individual rationality and incentive compatibility. In particular, inequality 14.5 reflects the fact that information z_i is private and must be elicited. It is clear that, for any trading or bargaining process, the no-trade allocation will not be altered by further contracting as long as each trader can guarantee himself the no-trade allocation. In other words, interim efficient allocations are "strongly renegotiation-proof."[11]

14.4 Common Knowledge, Almost Common Knowledge, and the Sensitivity of Equilibria to the Information Structure[†††]

This section discusses how the Nash equilibria of a game can vary with "small" changes in its information structure. Changes in the information

10. $e(z, v)$ may stem from a previous contract that, for instance, sets up a revelation mechanism for eliciting the private signals z (see chapter 7). But note the following subtlety: We implicitly assume that $e(z, v)$ is not affected by the process of bargaining for further trade. For instance, the revelation mechanism that elicits z may have several equilibria; although this is not an issue when only one player has private information, unique implementation with several informed players requires some care (see the references cited in section 7.2). If the revelation game has multiple equilibria, then even if one equilibrium is interim efficient, it might be the case that trade occurs, i.e., that the initial contract is renegotiated. Renegotiation may be enforced by the threat that a "bad equilibrium" prevails if the initial contract is not replaced by a new one. Also, if the revelation game is not in dominant strategies, the change in beliefs during the bargaining phase may destroy incentive compatibility of the revelation game.

11. They are strongly renegotiation-proof (at least if traders are strictly risk averse) in two ways. First, for a given bargaining process for renegotiation, they are not renegotiated in *any*

structure can change what each player knows, and thus change what is common knowledge, so the notion of exact common knowledge will not be very useful here. As we will see, though, the closely related notion of "almost common knowledge" is very useful indeed.

We start with two examples to illustrate the possibility that apparently small perturbations in the information structures starting from common knowledge of the payoffs may change the equilibrium set considerably. That is, some equilibria of the game in which payoffs are common knowledge are not near any equilibrium of the perturbed game, even if with high probability all players know that the payoffs are as in the original game. This lack of lower hemi-continuity should not be surprising: We saw that small changes in the payoffs of players may eliminate equilibria; a small probability that the payoffs might be different can have the same effect. The simplest version of this point is that, if a player is slightly uncertain about his opponent's payoffs, he may be unwilling to play a strategy that is weakly dominated. As a slightly more complicated example, the game illustrated in figure 14.2a has a component of equilibria where player 2 plays A with probability 1 and player 1 plays A with probability at least $\frac{5}{6}$, and also a pure-strategy equilibrium where both players play B; the equilibrium where both play A with probability 1 has the highest payoffs. However, playing A is weakly dominated for player 1. Now consider a perturbed version of the game in which with probability $1 - \varepsilon$ the state is ω_1 and the payoffs are as in figure 14.2a, and with probability ε the state is ω_2 and the payoffs are as in figure 14.2b. Moreover, player 1 is uncertain which state prevails - he has the trivial partition $H_1 = \{\omega_1, \omega_2\}$—but player 2 knows the state. Then, in state ω_2, player 2 plays B, as B strictly dominates A, and so player 1, not knowing the state, will not play the weakly dominated strategy A. Hence, although (A, A) is an equilibrium if it is common knowledge the payoffs are as in figure 14.2a, (A, A) is not an equilibrium of the perturbed game.

	A	B
A	8,8	-10,6
B	8,-10	0,0

a

	A	B
A	0,0	-10,1
B	1,-10	8,8

b

Figure 14.2

Nash equilibrium of the renegotiation game. Second, this holds for any renegotiation process. Interim efficiency is necessary and sufficient for strong renegotiation-proofness. (However, it is not in general necessary for the allocation to be "weakly renegotiation-proof" (in that there exists *some* equilibrium of the renegotiation game for which it is not renegotiated). One can define a weaker notion of efficiency—"weak interim efficiency"—to characterize weakly renegotiation-proof allocations. See Maskin and Tirole 1989.)

Subsection 14.4.1 develops two subtler examples of this lack of lower hemi-continuity. Example 14.3 considers a situation where, as above, the payoffs are given by either figure 14.2a or figure 14.2b, but the information structure considered is more complex. In particular, there is a state with high prior probability in which *both* players know that the payoffs are as in figure 14.2a. Nevertheless, even in this state, both players must play **B** in equilibrium: Although both players know that the payoffs are as in figure 14.2a, this is not common knowledge.

The payoffs in figure 14.2a are not generic, and the argument concerning this figure focused on eliminating equilibria where the weakly dominated strategy A is played. This raises the question of whether a similar lack of lower hemi-continuity can arise when each payoff matrix being considered is generic in the space of strategic forms. In example 14.4, the payoff matrices considered are generic and the perturbation in the information structure eliminates a strict equilibrium. Whether this is viewed as a failure of lower hemi-continuity for generic payoffs depends on whether the perturbation in the information structure is indeed small. This turns out to be a subtle question (as the example has an infinite state space), and there are several seemingly reasonable ways to define what it means for two information structures on an infinite state space to be close. Subsection 14.4.2 develops one such notion, based on the idea of almost common knowledge, and shows that it yields a result with generic lower hemi-continuity.

14.4.1 The Lack of Lower Hemi-Continuity

Example 14.3

Suppose that players 1 and 2 play the game depicted in figure 14.2a. As remarked above, this game has two components of Nash equilibria: the pure-strategy equilibrium (B, B) and any profile where player 2 plays A with probability 1 and player 1 plays A with probability at least $\frac{5}{6}$. Now suppose we want to model a situation in which players 1 and 2 both know that the payoffs are as in figure 14.2a but player 1 assigns positive probability to player 2's believing that the payoffs may actually be as in figure 14.2b. Then (since players 1 and 2 have a common prior over nature's moves) nature must assign positive probability to player 1's not being fully informed of nature's move. One such game is depicted in figure 14.3.

In this game, nature has four possible moves: ω_1, ω_2, ω_3, and ω_4. In ω_1 and ω_2 the payoffs are as in figure 14.2a; in ω_3 and ω_4 the payoffs are as in figure 14.2b. The players' information partitions are $H_1 = \{(\omega_1, \omega_2), (\omega_3, \omega_4)\}$ and $H_2 = \{\omega_1, (\omega_2, \omega_3), \omega_4\}$. Player 1 always knows the payoffs, and player 2 may or may not know them. Furthermore, player 1 does not know whether player 2 knows. In state ω_1 the players know that the payoffs are as in figure 14.2a, and player 2 can infer that player 1 knows the payoffs, but player 1 does not know whether player 2 knows the payoffs. In ω_2 the

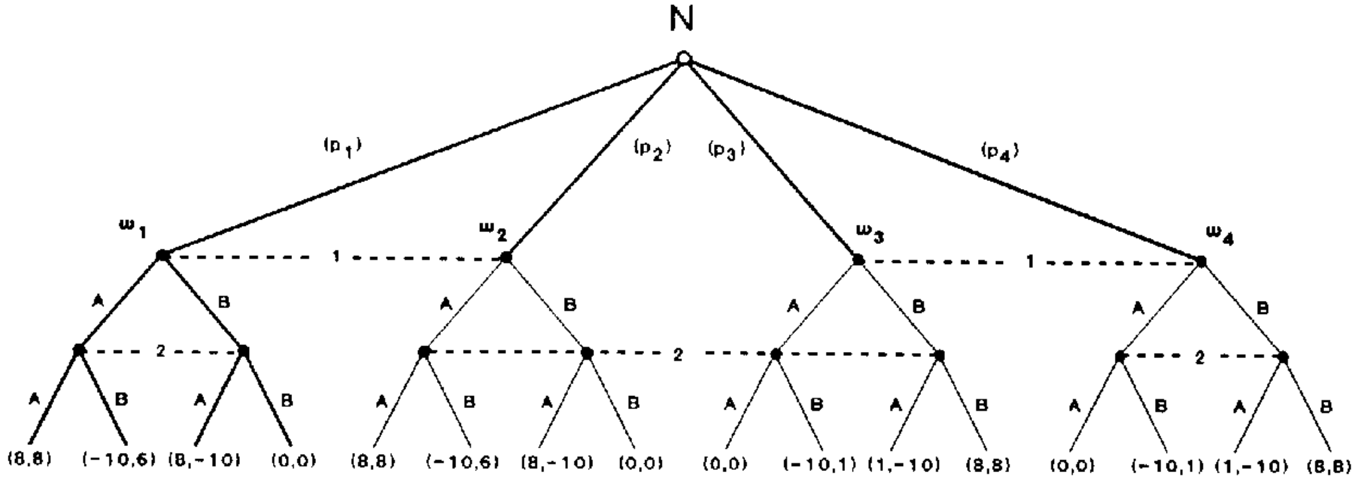

Figure 14.3

payoffs are as in figure 14.2a, and player 1 is informed of this but player 2 is not; again player 1 does not know whether player 2 is informed. In state ω_3 the payoffs are as in figure 14.2b, player 1 is informed, and player 2 is not; in ω_4 the payoffs are as in figure 14.2b and both players know this; in ω_3 and ω_4 player 1 does not know whether player 2 is informed. Thus, if all states have positive prior probability, the only common-knowledge event is the whole state space.

Note well that there are more states in Ω than there are possible payoff matrices: The state must describe not only the payoffs, and each player's information about the payoffs, but also each player's information about his opponent's information, and so on. Note as well how we changed the model to incorporate the uncertainty of player 1: We added additional states until the players' beliefs could again be described by a common prior over a common state space. To precisely describe a real-world game in this way could easily require a large (or even infinite) state space. The hope is that small-state-space models can be a good approximation to the game of interest.

Returning to the example, let p_i denote the prior probability of ω_i. Then, if $p_2 < p_3$, the only Nash equilibrium of the game is for both players to play B in every state. The proof of this proceeds state by state. First, in state ω_4, both players know the payoffs, and for each of them it is a dominant strategy to play B, as it is for player 1 in state ω_3. Let $q = p_2/(p_2 + p_3)$ be player 2's posterior probability of ω_2 when his information is (ω_2, ω_3). Given that player 1 plays B in state ω_3, when player 2 is told (ω_2, ω_3) he receives at most $q8 + (1-q)(-10)$ from playing A and at least $q(-10) + (1-q)8$ from playing B. Since $p_2 < p_3$, $q < \frac{1}{2}$, and player 2 must play B. Next, when player 1 is told (ω_1, ω_2) he knows that the payoffs are as in figure 14.2a, so that (A, A) is Pareto optimal, yet he also knows that there is positive probability that the state is ω_2 and so player

2 will play B, and hence player 1 plays B. Finally, given that player 1 plays B in ω_1 and ω_2, player 2 plays B when told ω_1.

The foregoing is true regardless of the absolute probabilities of ω_2 and ω_3, and regardless of the relative probability of ω_1 and ω_2. In particular, the conclusions hold at each p^n in the sequence $p_1^n = 1 - 4/n$, $p_2^n = 1/n$, $p_3^n = 2/n$, and $p_4^n = 1/n$, which converges to the limit $p_1 = 1$ where (A, A) *is* a Nash equilibrium. Moreover, when n is large, in state ω_1 both players know that the payoffs are as in figure 14.2a, player 2 knows that player 1 knows this, and player 1 is "almost certain" that player 2 knows this, yet the set of equilibria differs from what it would be if state ω_1 had probability 1 and hence were common knowledge. Put differently: The set of equilibria with "almost common knowledge" of the payoffs (defined formally in subsection 14.4.2) differs from that where the payoffs are known with certainty. Though this may be troubling, it should not be a surprise: We saw in chapter 1 that the Nash correspondence need not be lower hemi-continuous in the payoffs, and this example simply illustrates a lack of lower hemi-continuity in the prior distribution. Note also that the payoffs in figure 14.2a are not generic, which is why a very small chance of ω_2 could force player 1 to play B. From the result on *generic* lower hemi-continuity of the Nash correspondence in finite games (see section 12.1), we would expect that if the players were uncertain which of a finite number of payoff matrices prevailed, and if the underlying state space Ω were finite, the Nash correspondence would be lower hemi-continuous in the neighborhood of common knowledge of a generic game, so that for generic payoffs "almost common knowledge" and common knowledge should have the same implications. Subsection 14.4.2 presents a version of this result. First, though, we give an example to show that the preceding intuition relies on the restriction to finite games.

Example 14.4: Electronic-Mail Game

Consider Rubinstein's (1989) updated version of Gray's (1978) "coordinated attack problem." Here the payoff matrices are as in figure 14.4. Note that in figure 14.4a the Pareto-optimal equilibrium (A, A) is strict. The information structure (represented in figure 14.5) is as follows: In state 0 the payoffs are as in figure 14.4b. In states 1, 2, ..., the payoffs are as in

	A	B
A	8,8	−10,1
B	1,−10	0,0

a

	A	B
A	0,0	−10,1
B	1,−10	8,8

b

Figure 14.4

figure 14.4a. Player 1's partition is the sets

$(0), (1, 2), (3, 4), \ldots, (2n-1, 2n), \ldots.$

Player 2's partition is

$(0, 1), (2, 3), \ldots, (2n, 2n+1), \ldots.$

The prior probability of state 0 is $\frac{2}{3}$, and the probability of state $n \geq 1$ is $\varepsilon(1-\varepsilon)^{n-1}/3$. The interpretation of this information structure is that if player 1 learns that the payoffs are as in figure 14.4a he sends a message to player 2, who does not know *a priori* which matrix is relevant. The message has probability ε of not being received; if player 2 does receive the message, he sends a response, which in turn has probability ε of not being received; if player 1 receives player 2's response, he sends a second response, acknowledging his receipt of player 2's response, and so on. In the original version of this game the messages were carried by horsemen through a valley occupied by enemy forces who might intercept them[12]; the updated version has the messages sent by an electronic-mail system that sometimes fails. It is important that sending the messages is not a strategic decision on the part of the players, but rather an exogenous process that determines their initial information.

H_1 0 1 2 3 4 5 6 • • •

H_2 0 1 2 3 4 5 • • •

Figure 14.5

12. Halpern (1986) gives the following account of the coordinated-attack problem:

Two divisions of an army are camped on two hilltops overlooking a common valley. In the valley awaits the enemy. It is clear that if both divisions attack the enemy simultaneously they will win the battle, whereas if only one division attacks it will be defeated. The divisions do not initially have plans for launching an attack on the enemy, and the commanding general of the first division wishes to coordinate a simultaneous attack (at some time the next day). Neither general will decide to attack unless he is sure that the other will attack with him. The generals can only communicate by means of a messenger. Normally, it takes the messenger one hour to get from one encampment to the other. However, it is possible that he will get lost in the dark or, worse yet, be captured by the enemy. Fortunately, on this particular night, everything goes smoothly. How long will it take them to coordinate an attack?

Suppose the messenger sent by general A makes it to general B with a message saying "Let's attack at dawn." Will general B attack? Of course not, since general A does not know he got the message, and thus may not attack. So general B sends the messenger back with an acknowledgement. Suppose the messenger makes it. Will general A attack? No, because now general B does not know he got the message, so he thinks general A may think that he (B) didn't get the original message, and thus not attack. So A sends the messenger back with an acknowledgement. But of course, this is not enough either. I will leave it to the reader to convince himself that no amount of acknowledgements sent back and forth will ever guarantee agreement. Note that this is true even if the messenger succeeds in delivering the message every time. All that is required in this reasoning is the *possibility* that each messenger doesn't succeed.

If the state is $n > 0$, then n messages have been sent and $n - 1$ of them received. For instance, if $n = 2k$, player 2 knows that the payoffs are as in figure 14.4a and that player 1 knows the payoffs, player 1 knows that player 2 knows this, player 2 knows that player 1 knows, and so on, for any string of length less than n. That is, $n \in K_{\mathscr{G}}^{n-1}$ ($n > 0$). Nevertheless, there is no finite n for which "$n > 0$" is common knowledge, as figure 14.5 and the reachability criterion make clear.

One might think that for small ε it would be possible for the players to coordinate on (A, A) when the payoff matrix is that of figure 14.4a (that is, when $n > 0$); however, this is not the case. The argument is an extension of the argument in the last example. Player 1 plays **B** in state 0, as this is a dominant strategy. Since player 1 plays **B** in state 0, and the probability of 0 given (0, 1) is greater than $\frac{1}{2}$, playing **B** in states (0, 1) gives player 2 a payoff of at least 4, whereas playing A gives at most -1, so player 2 plays **B**. Since the probability of state 1 given (1, 2) is

$$q = \frac{\varepsilon}{\varepsilon + \varepsilon(1 - \varepsilon)} > \tfrac{1}{2},$$

player 1, who in states (1, 2) knows that player 2 plays **B** with probability greater than $\frac{1}{2}$, plays **B** in states (1, 2), as he obtains at most $(1 - q)8 + q(-10) < 0$ from playing A and at least 0 from playing **B**. Given that player 1 plays **B** in states (1, 2), in states (2, 3) player 2 plays **B**, as the probability of 2 given (2, 3) exceeds $\frac{1}{2}$, so player 1 plays **B** in states (3, 4), and the proof continues by induction to show that the two players play **B** in any state.

If ε is very small, then conditional on $n \neq 0$ a large number of messages are likely to be sent and received. However, for any $\varepsilon > 0$, there is no equilibrium where players play (A, A) whenever the payoffs are as in figure 14.4a, even through (A, A) is a strict equilibrium when payoffs are certain to be as in figure 14.4a.

Whether example 14.4 represents a failure of lower hemi-continuity depends on whether we view the ε error probability as a "small" change in the information structure, and also on whether we test for convergence of the strategies as maps from the state space to actions or instead test for convergence of the probability distribution over the payoff-relevant outcomes. We begin with the second point. Eddie Dekel-Tabak has suggested a way of defining convergence of strategies under which the equilibrium strategies in the perturbed games are in fact the *same* as the strategies when $\varepsilon = 0$. First, compactify the state space by adding the point ∞, corresponding to an infinite number of messages, so that the state space is $\bar{\Omega} = \Omega \cup \{\infty\}$. When an infinite number of messages are sent and received (which has probability 0 under the prior distribution), it is common knowledge that the payoffs are as in figure 14.4a. Thus, on this expanded state space, one equilibrium is for both players to play B when ω is finite and A when $\omega = \infty$. On this view, since the set of equilibrium strategies in the example does not vary with the parameter ε, there is no failure of lower hemi-continuity regardless of whether or not the information structures converge as $\varepsilon \to 0$. (But recall that in example 14.3 the set of equilibrium *strategies* in state ω_1 did change.)

Although this viewpoint may help illuminate the structure of the problem, it does not address the fact that equilibrium *payoffs* when $\varepsilon > 0$ are strictly lower than the payoffs when $\varepsilon = 0$. This suggests looking at convergence in the space of probability distributions over outcomes. Since the equilibrium probability distributions are changed by introducing the error probability, the question of lower hemi-continuity then depends on whether the error probability represents a small perturbation.

The next subsection introduces the concept of "almost common knowledge," and shows that the set of equilibria is lower hemi-continuous when the perturbations are deemed to be small only if the unperturbed payoffs remain almost common knowledge. With that approach, example 14.4 does not display lack of lower hemi-continuity, because the perturbation is not judged to be small.

But there is an equally intuitive way of defining a small perturbation so that example 14.4 does qualify. This definition is based on extending the topology of weak convergence to the probability distributions in the compactified state space Ω. (To extend this topology, note that $\bar{\Omega}$ is isomorphic to the set $\{2, 1, \frac{1}{2}, \frac{1}{3}, \ldots, 1/n, \ldots, 0\}$ by the transformation $x(n) = 1/n$ for $n > 0$, $x(\infty) = 0$, and $x(0) = 2$. Probability distributions over Ω are thus a subset of the set $\mathscr{P}$ of probability distributions over the interval $[0, 2]$, which we endow with the weak topology on distributions over $[0, 2]$.[13]) Many sequences of distributions on $\bar{\Omega}$ converge in the weak topology to the distribution $\{p(0) = \frac{2}{3}, p(\infty) = \frac{1}{3}\}$, which corresponds to common knowledge. One sequence is that of the example. Another is defined by

$$p^{\varepsilon}(0) = \tfrac{2}{3},$$

$$p^{\varepsilon}(2k + 1) = \varepsilon(1 - \varepsilon)^{k}(1 - 2\varepsilon)^{k}/3$$

and

$$p^{\varepsilon}(2k + 2) = 2\varepsilon(1 - \varepsilon)^{k+1}(1 - 2\varepsilon)^{k}/3$$

13. p^n converges to p in the weak topology if, for each continuous function f on $[0, 2]$,

$$\int_0^2 f(x)\,dp^n(x)$$

converges to

$$\int_0^2 f(x)\,dp(x).$$

for $k \geq 0$. The interpretation of this information structure is that player 2's messages are twice as likely to be intercepted as player 1's.

With this notion of closeness, the reason the example displays a lack of lower hemi-continuity is that the set of equilibria of the "limit" game where payoffs are common knowledge contains both the limits of equilibria of perturbed games with equal error probabilities and the limits of equilibria of games where player 2's error probability is twice as large as player 1's. And with this latter information structure, it is a Nash equilibrium for player 1 to play A in all states $n > 0$, and for player 2 to play A in all states $n > 1$: Now, when player 1 has partition $(1, 2)$, his posterior probability of state 2 is

$$\frac{2\varepsilon(1-\varepsilon)}{\varepsilon + 2\varepsilon(1-\varepsilon)} \cong \tfrac{2}{3}$$

for small ε, and player 1 is willing to play A if he expects player 2 to play A in state 2, even though he expects player 2 to play B in state 1. Thus, player 2 is willing to play A when his partition is $(2, 3)$, and so on.

14.4.2 Lower Hemi-Continuity and Almost Common Knowledge (technical)

Subsection 14.4.1 presented examples of ways in which the equilibrium correspondence can fail to be lower hemi-continuous. One response is to parallel chapter 1 and ask for lower hemi-continuity of the ε-equilibrium correspondence; another is to identify conditions for the Nash correspondence itself to be lower hemi-continuous. We will consider the two responses in turn, beginning with the idea of using ε equilibrium. As we will see, there are two distinct versions of ε equilibrium: "*ex ante*" and "interim." Stronger conditions are required for lower hemi-continuity of the interim version.

The ε equilibrium we discussed in chapter 1 corresponds to *ex ante* ε equilibrium in the present setting. Recall from chapter 1 that, for a family of (finite) strategic-form games with the same strategy space S and with payoff functions $u_i(\cdot, \lambda)$ that vary continuously in λ, a Nash equilibrium for payoffs λ is an ε Nash equilibrium for payoffs λ^n, where $\varepsilon \to 0$ as $\lambda^n \to \lambda$. (Recall that an ε equilibrium is a profile where no player can increase his payoff by more than ε by deviating.) That is, lower hemi-continuity is restored by looking at ε equilibria. Since changing the prior distribution on a fixed state space changes only the payoffs to each strategy and not the strategy space itself, this result extends immediately to changes in the prior distribution. Formally, fix a finite state space Ω, partitions H_i, H_i-measurable strategies $\sigma_i \in \mathscr{S}_i$ from Ω to the space of (probability mixtures) on S_i, and payoff functions u_i on $S \times \Omega$, and let $G(p)$ be the strategic-form game corresponding to prior distribution p over Ω. If profile σ is a Nash equilibrium of $G(p)$, then σ is an ε Nash equilibrium for $G(p^n)$, with $\varepsilon \to 0$ as

$p^n \to p$. In particular, if under prior p there is probability 1 that $u_i(s, \omega)$ equals some fixed $\bar{u}_i(s)$ for all s and all i, so that the payoff functions are common knowledge under p, and under p^n each player assigns high probability to the payoffs being given by $\bar{u}$, then any Nash equilibrium for $G(p)$ is an ε Nash equilibrium of $G(p^n)$. The reason is that if the probability that u differs from $\bar{u}$ is very small, then each player loses very little in terms of expected payoff (in the usual game) by playing a response that is suboptimal when u differs from $\bar{u}$. Thus, in an ε equilibrium of $G(p)$, players can make "big" mistakes at unlikely information sets. To emphasize this point, ε equilibria of the strategic-form game are called *ex ante* ε equilibria.

The electronic-mail game of example 14.4 has an infinite number of states, so the observation in the preceding paragraph does not apply. However, the following profile is an *ex ante* ε' equilibrium of that game, where ε' is of the same order as the probability ε that a message is lost: "Player 1 plays B when $h_1 = 0$ and plays A in all other states. Player 2 plays B when $h_2 = (0, 1)$ and plays A in all other states." Given player 1's strategy, player 2's strategy is exactly optimal: When player 2's information is $(0, 1)$, player 1 is likely to play B; when player 2's information is anything else, player 1 is certain to play A. Player 1's strategy is exactly optimal when his information is state 0 or when player 2 knows that the state is greater than 2. Player 1's choice of A given information $h_1 = (1, 2)$ is not optimal; however, since this event has probability $[\varepsilon + \varepsilon(1 - \varepsilon)]/3$, player 1's strategy is almost optimal when ε is small.

More generally, if s is a Nash-equilibrium profile when payoffs are known to be given by $\bar{u}(\cdot)$, then s is an *ex ante* ε Nash equilibrium, with ε small, if there is probability near 1 that each player believes payoffs are very likely to be given by $\bar{u}(\cdot)$. This is true whether the state space is finite or infinite (Monderer and Samet 1988; Stinchcombe 1988).

Though this result provides one resolution of the lack of lower hemicontinuity, it is not completely satisfying, because the notion of *ex ante* ε equilibrium is so weak. Instead, one might wish to use the concept of interim ε equilibrium: Profile s is an *interim* ε equilibrium[14] if, for all players i and all states ω, strategy $s_i(\omega)$ comes within ε of maximizing player i's expected payoff conditional on $\omega' \in h_i(\omega)$. If we let E denote the expectation operator, the formal condition is

$$\mathrm{E}[u_i(s_i(\omega'), s_{-i}(\omega')) \mid h_i(\omega)] \geq \mathrm{E}[u_i(s_i, s_{-i}(\omega')) \mid h_i(\omega)] - \varepsilon$$

for all ω and i, and for all $s_i \in S_i$. Clearly, every interim ε equilibrium is an *ex ante* ε equilibrium; the converse is true for $\varepsilon = 0$ (so long as all states have positive probability) but not for positive ε. As we noted above, a player might make a big mistake in an unlikely state in an *ex ante* ε equilibrium.

14. Monderer and Samet call it an *ex post* ε equilibrium. We prefer "interim" because in information economics "*ex post*" refers to the situation in which the state of the world is revealed to all.

Monderer and Samet (1989) provide conditions for the lower hemicontinuity of the interim ε equilibrium in terms of "almost common knowledge" of the payoffs, which requires that all players be "pretty sure" that their opponents are "pretty sure" about the payoffs, and so on..., as opposed to knowing that their opponents know them.

Monderer and Samet say that player i "r-believes E" at ω if his posterior probability $p(\mathrm{E}|h_i(\omega))$ is greater than or equal to r. The event "player i r-believes E" is denoted $B_i^r(\mathrm{E})$; this is the set $\{\omega|p(\mathrm{E}|h_i(\omega)) \geq r\}$. When all states have strictly positive prior probability, 1-belief is equivalent to knowledge.[15] Event E is "common r-belief" if everyone believes E has probability at least r, everyone believes there is probability at least r that everyone believes E has probability at least r, and so on.[16] Common 1-belief is the same as common knowledge; common r-belief for r large corresponds to almost common knowledge. Stinchcombe (1988) independently proposed a closely related notion of almost common knowledge that is slightly weaker. His approach can be interpreted as defining common (r, n)-belief in E to require that statements of the form "everyone believes there is probability at least r that there is probability at least r... that E is true" hold so long as they involve n or fewer instances of "everyone believes." For instance, if all players know E but no player knows that his opponents know it, then E is common (1, 1)-belief; common r-belief in the Monderer-Samet sense is common (r, ∞)-belief in Stinchcombe's sense. Stinchcombe then says that payoffs are almost common knowledge if they are common (r, n)-belief for r near 1 and n near infinity.

The two definitions of almost common knowledge are equivalent with a finite state space (because the r-knowledge operator stops after a finite number of steps), but they can differ in games with a countable state space, as is shown by example 14.4. Here, payoffs become almost common knowledge in Stinchcombe's sense as the number of messages sent increases: If $n + 1$ messages have been sent, it is common $(1, n)$-belief that the state is not 0. However, no matter how many messages have been sent, the payoffs

15. With a continuum of states and a smooth prior, no individual state has positive probability, and one may wish to distinguish between knowledge and 1-belief. For instance, if a number is picked at random from the interval [0, 1], players 1-believe that the number is irrational, but they do not "know" it in the sense we have defined.

16. Starting from ${}^1B_i^r(E) \equiv B_i^r(E)$ and ${}^1B_{\mathcal{I}}^r(E) \equiv \bigcap_{i \in \mathcal{I}} {}^1B_i^r(E)$, let

$$ {}^nB_i^r(E) \equiv \{\omega|p({}^{n-1}B_{\mathcal{I}}^r(E)|h_i(\omega)) \geq r\} $$

and

$$ {}^nB_{\mathcal{I}}^r(E) \equiv \bigcap_{i \in \mathcal{I}} {}^nB_i^r(E). $$

Then E is common r-belief at ω if $\omega \in {}^{\infty}B_{\mathcal{I}}^r(E)$.

As with common knowledge, there is an equivalent, "Aumann-style" definition of common r-belief. An event E is a common r-truism if $E \subseteq B_{\mathcal{I}}^r(E)$. That is, when E occurs, every player assigns a probability of at least r to its occurrence. An event E' is a common r-belief at ω if there exists a common r-truism E such that $\omega \in E$ and $E \subseteq B_{\mathcal{I}}^r(E')$.

never become almost common knowledge in the Monderer-Samet sense, as for no $r > (1 - \varepsilon)/(2 - \varepsilon)$ are the payoffs common r-belief.[17]

In contrast, the payoffs in example 14.3 become almost common knowledge as the probability of ω_1 goes to 1: Let $p_1^n \to 1$ be the prior probability of ω_1. At state ω_1, the event $E = \{\omega_1\}$ is common p_1^n-belief (because $E \subseteq B_{\mathscr{I}}^{p_1^n}(E)$), so the game is almost common knowledge for n large. (Theorem 14.5 extends this observation by showing that as the prior probability of an event E goes to 1, the probability that the event is almost common knowledge goes to 1 as well.)

Monderer and Samet generalize theorem 14.2 by showing that if the posterior probabilities of event E are common r-belief, then any two posteriors can differ by at most $2(1 - r)$.[18] As we said above, they also use their notion of almost common knowledge to extend the result about the lower hemi-continuity of *ex ante* ε equilibria to the interim version.

Monderer and Samet consider games G with a finite number of payoff functions $u^\ell(\cdot)$, where $\ell = 1, \ldots, L$. The payoffs in state ω are $u(\cdot, \omega) = u^{\lambda(\omega)}(\cdot)$, where Ω is either finite or countably infinite. Let $G^\ell = \{\omega | \lambda(\omega) = \ell\}$ be the set of all states ω at which the payoffs are given by u^ℓ. Payoffs u^ℓ are common r-belief at ω if the event G^ℓ is common r-belief at ω. For each ℓ, let σ^ℓ be a Nash equilibrium for common-knowledge payoffs u^ℓ, and define $\sigma^*: \Omega \to \Sigma$ by $\sigma^*(\omega) = \sigma^{\lambda(\omega)}$. This function assigns each ω a Nash equilibrium for the payoffs $\lambda(\omega)$. If the payoffs are common knowledge at each ω, then σ^* is a Nash equilibrium of the overall game G.[19]

Monderer and Samet show that for each Nash profile σ^ℓ of a common-knowledge game u^ℓ there exists an interim ε equilibrium of any game in which it is almost common knowledge that payoffs are u^ℓ, such that the players play strategy profile σ^ℓ with probability close to 1.

17. $(1 - \varepsilon)/(2 - \varepsilon)$ is the conditional probability that a player attaches to his not receiving a message that was sent by the other player (as opposed to the other player's not receiving his previous message), given that he did not receive a new message. One way of showing that no $r > (1 - \varepsilon)/(2 - \varepsilon)$ makes the payoffs common r-belief is to use the iterative definition of common r-belief. Another is to note that for E to be a common r-truism (that is, $E \subseteq B_{\mathscr{I}}^r(E)$), r must be less than $(1 - \varepsilon)/(2 - \varepsilon)$: Let n_0 denote the lowest element in E; if $n_0 > 1$, one of the players at state n_0 puts probability $1/(2 - \varepsilon)$ on E being false; and at $n_0 = 1$, player 2 puts probability $2/(2 + \varepsilon)$ on E being false.

18. The precise meaning of the posterior probabilities that an event E is common r-belief is the following: Fix I posterior beliefs, $q_1, \ldots, q_I$, for event E. Let

$$E' = \{\omega | p(E | h_i(\omega)) = q_i \text{ for all } i \in \mathscr{I}\}.$$

Posterior beliefs $(q_1, \ldots, q_I)$ are common r-beliefs at ω if E' is common r-belief at ω. The result is then

$$\max_{i,j} |q_i - q_j| \leq 2(1 - r).$$

The method of proof is to show that if E'' is a common r-truism contained in E' (whose existence is ascertained in note 16), each q_i cannot differ from the probability of E conditional on E'' by more than $1 - r$.

19. Every Nash equilibrium of G is not necessarily such an σ^*, unless there is a single ω in each G^ℓ; otherwise, correlation over the equilibria can be introduced through the public signal ω.

Theorem 14.4 (Monderer and Samet 1989) Fix an $r \in (0.5, 1]$, and set $q = p[\omega |$ for some ℓ, payoffs u^ℓ are common r-belief at $\omega]$. Then for any selection s^* from the set of common-knowledge-of-payoffs Nash equilibria, there is a profile s of G such that

$$p[\omega \,|\, s(\omega) = s^*(\omega)] \geq q$$

and such that s is an interim ε equilibrium for all

$$\varepsilon > 4(1 - r) \max_{i, \ell, \ell', \sigma, \sigma'} |u_i^\ell(\sigma) - u_i^{\ell'}(\sigma')|.$$

In particular, for any $\varepsilon > 0$, there are $\bar{r} < 1$ and $\bar{q} < 1$ such that for all $r \geq \bar{r}$ and $q \geq \bar{q}$ there exists an interim ε equilibrium s such that $p[\omega \,|\, s(\omega) = s^*(\omega)] > 1 - \varepsilon$.

Proof[20] Let $E^\ell = \{\omega \,|\, G^\ell$ is common r-belief at $\omega\}$. If $r > \frac{1}{2}$, there can be at most one E^ℓ that player i r-believes has occurred. Set $\Omega_i = \bigcup_\ell B_i^r(E^\ell)$, and let Ω_i^c be the complement of Ω_i. For $\omega \in \Omega_i$, specify that player i plays the strategy σ_i^ℓ corresponding to the payoffs he r-believes to be true. Let

$$K = \max_{i, \ell, \ell', \sigma, \sigma'} |u_i^\ell(\sigma) - u_i^{\ell'}(\sigma')|.$$

We claim first that, at any $\omega \in B_i^r(E^\ell)$, σ_i^ℓ is a $4K(1 - r)$-optimal interim response to any s_{-i} with $s_j(\omega) = \sigma_j^\ell$ for all $\omega \in B_j^r(E^\ell)$, all ℓ, and all j. To see this, note that at $\omega \in B_i^r(E^\ell)$ player i assigns probability at least r to the payoffs' being given by u^ℓ (he himself r-believes G^ℓ), and he also assigns probability at least r to $\omega \in B_j^r(E^\ell)$ for all $j \neq i$ (for $\omega \in E^\ell$, player j r-believes E^ℓ), so he assigns probability at least r to $s_{-i}(\omega) = \sigma_{-i}^\ell$. Since σ_i^ℓ is a best response to σ_{-i}^ℓ, we have

$$\mathrm{E}(u_i(\sigma_i^\ell, s_{-i}(\omega)) | h_i(\omega))$$

$$\geq u_i^\ell(\sigma_i^\ell, \sigma_{-i}^\ell) - 2K(1 - r) \geq u_i^\ell(\sigma_i', \sigma_{-i}^\ell) - 2K(1 - r)$$

(since σ_i^ℓ is a best response to σ_{-i}^ℓ in game G^ℓ)

$$\geq \mathrm{E}(u_i(\sigma_i', s_{-i}(\omega)) | h_i(\omega)) - 4K(1 - r).$$

Note also that $p[\omega \,|\, s(\omega) = s^*(\omega)] \geq q$, as required.

It remains only to define $s_i(\omega)$ for $\omega \notin \Omega_i$. For this, it suffices to look for a Bayesian equilibrium in the game where players are constrained to follow the strategy s_i defined above in Ω_i. (Such an equilibrium exists from Glicksberg's existence theorem in chapter 1.) ■

Corollary If for each ℓ σ^ℓ is a *strict* equilibrium, then for any $\varepsilon > 0$ there are r and q such that, for all $r > \bar{r}$ and $q > q$, if the payoffs are com-

20. Monderer and Samet use a slightly different proof.

mon r-belief with probability q, there is an *exact* equilibrium σ with $p[\omega | \sigma(\omega) = \sigma^*(\omega)] > 1 - \varepsilon$.

As we remarked above, the almost-common-knowledge condition is not satisfied in the electronic-mail example, so the theorem and its corollary do not apply. However, in the truncated version of the game where player 2 does not respond after receiving n messages, the state $2n$ is common $(1 - \varepsilon)$-belief where it occurs: Player 2 knows that the state is $2n$, and player 1 assigns it probability $1 - \varepsilon$. Hence, from the corollary, there is an exact equilibrium where both players play A when the state is $2n$.

This truncated example brings us to our final point: On a finite state space, if event C has probability close to 1 there is high probability that it is a common r-belief for r large.

Theorem 14.5 Consider an event C on a finite state space Ω and a sequence of prior distributions p^n on Ω such that $p^n(C) \to 1$. Then there are sequences $q^n \to 1$ and $r^n \to 1$ such that under p^n there is probability q^n that C is a common r^n-belief.

Remark To apply this theorem to the existence of interim ε Nash equilibria, fix an $\hat{r}$ close to 1 and let the event C be "all players $\hat{r}$-believe they know the true payoffs."

The proof of the theorem uses the following lemma:

Lemma 14.1 If $p(C) \geq q$, then

$$p(B_i^r(C)) \geq \frac{q - r}{1 - r}.$$

Thus, if event C is likely *ex ante*, there is high probability that player i will believe that it is likely conditional on his information.

Proof of Lemma Let $\mu_i(\omega) = p(C | h_i(\omega))$. Then

$$p(C) = \mathrm{E}\mu_i(\omega)$$

$$= p[\mu_i(\omega) \geq r]\mathrm{E}[\mu_i | \mu_i \geq r] + (1 - p[\mu_i(\omega) \geq r])\mathrm{E}[\mu_i | \mu_i < r].$$

Then

$$p[\mu_i(\omega) \geq r] = \frac{p(C) - \mathrm{E}[\mu_i | \mu_i < r]}{\mathrm{E}[\mu_i | \mu_i \geq r] - \mathrm{E}[\mu_i | \mu_i < r]}$$

$$\geq \frac{q - r}{1 - r}. \qquad \blacksquare$$

Proof of Theorem 14.5 For $q \in (0, 1)$, define the function $\imath(q) = q - \sqrt{1 - q}$. For fixed n, set $D^0 = C^0 = C$ and $q^0 = p^n(C)$. Recursively define

$$C^m = \bigcap_{i=1}^{I} B_i^{\varepsilon(q^{m-1})}(D^{m-1}),$$

$$D^m = C^m \cap D^{m-1},$$

$$q^m = q^{m-1} - \frac{I\sqrt{1-q^{m-1}}}{1+\sqrt{1-q^{m-1}}}.$$

(Note that q^m is a decreasing sequence.) We claim that $p^n(D^m) \geq q^m$; that $D^m \subseteq D^{m-1}$; that for some M, $D^{M+1} = D^M$; and that with probability q^M the event D^M is common $\varepsilon(q^M)$-belief.

The first claim is true for D^0. For $m > 0$, use lemma 14.1 to conclude that if $p^n(D^{m-1}) \geq q^{m-1}$ then, for each player i,

$$p^n(B_i^{\varepsilon(q^{m-1})}(D^{m-1})) \geq \frac{q^{m-1} - \varepsilon(q^{m-1})}{1 - \varepsilon(q^{m-1})} = \frac{1}{1+\sqrt{1-q^{m-1}}},$$

and so the probability of $[\bigcap_{i=1}^{I} B_i^{\varepsilon(q^{m-1})}(D^{m-1})] \cap D^{m-1}$ is at least

$$[1 - I(1 - p^n(B_i^{\varepsilon(q^{m-1})}(D^{m-1})))] + q^{m-1} - 1 = q^{m-1} - \frac{I\sqrt{1-q^{m-1}}}{1+\sqrt{1-q^{m-1}}}.$$

(Recall that for any sets A and B, $p(A \cap B) = p(A) + p(B) - p(A \cup B) \geq p(A) + p(B) - 1$). That $D^m \subseteq D^{m-1}$ results from the definition of D^m. Since the D^m are nested, and Ω is finite, there exists an M such that $D^{M+1} = D^M$. Let $r = \varepsilon(q^M)$. By definition of D^{M+1},

$$D^{M+1} = D^M = \bigcap_{i=1}^{I} B_i^r(D^M),$$

so that D^M is evident r-belief. Because $D^M \subseteq C$ and because D^M has probability at least q^M, C is r-common belief with probability q^M. Finally, note that with a finite state space, M is bounded above by the number of states M, and that from the difference equation giving q^m, $q^{\bar{M}}$ $(\leq q^M)$ converges to 1 when $p^n(C)$ goes to 1; similarly, $r = q^M - \sqrt{1-q^M}$ converges to 1.

■

Because many games do not have strict equilibria, we now turn to the question of lower hemi-continuity (with exact equilibria) for general games. We will consider a finite state space Ω, and a finite pure-strategy set S_i for each player. As in chapter 12, we can define a distance between two strategy profiles σ and $\tilde{\sigma}$:

$$\|\tilde{\sigma} - \sigma\| = \max_{\substack{i \in \mathscr{I} \\ s_i \in S_i}} |\sigma_i(s_i) - \tilde{\sigma}_i(s_i)|.$$

We consider a sequence of priors $p^n(\cdot)$ on Ω, and a finite collection of games $\{G^\ell\}_{\ell=1}^{L}$ with payoffs $\{u^\ell\}_{\ell=1}^{L}$, and assume that, as $n \to \infty$, it becomes very likely that one of these games is almost common knowledge.

Theorem 14.6 Consider a game with a finite state space Ω, partitions H_i, and L possible payoff functions u^ℓ. Consider a sequence of priors p^n such that, for some sequence $r^n \to 1$,

$$p^n(\omega \mid \exists G^\ell \text{ with } G^\ell \text{ common } r^n\text{-belief at } \omega) \to 1.$$

Suppose further that, for each player i and each $\ell \in \{1, \dots, L\}$, $B_i^{r^n}(G^\ell)$ is a single information set h_i^ℓ. Then, for a generic choice of $\{G^\ell\}_{\ell=1}^L$, for any $\varepsilon > 0$ and any selection of equilibria $\sigma^{\ell *}$ of (common-knowledge) games G^ℓ, there exists N such that, for $n > N$, there exists a Bayesian equilibrium of the game with prior p^n such that

$$\omega \in \bigcap_{i \in \mathscr{I}} B_i^{r^n}(G^\ell) \Rightarrow \|\sigma(\cdot \mid \omega) - \sigma^{\ell *}\| < \varepsilon.$$

Remark We have no reason to believe that the restriction to a single information set is necessary; we use it to simplify the proof.

Proof Assume that $B_i^{r^n}(G^\ell)$ is a single information set for all i, ℓ. Let σ_i^ℓ denote the strategy of player i when $\omega \in B_i^{r^n}(G^\ell)$, let $\sigma^\ell = (\sigma_i^\ell)_{i \in \mathscr{I}}$, let $\sigma_i^c(\omega)$ denote player i's strategy for $\omega \notin \bigcup_\ell B_i^{r^n}(G^\ell)$, let $\sigma^c = (\sigma_i^c)_{i \in \mathscr{I}}$, and let $\sigma^{-\ell} = (\sigma^1, \dots, \sigma^{\ell-1}, \sigma^{\ell+1}, \dots, \sigma^L, \sigma^c)$. Fixing $\sigma^{-\ell}$, we define a I-player common-knowledge game $G(\sigma^{-\ell})$ among the types who are almost sure that the game is G^ℓ. Because at information set $B_i^{r^n}(G^\ell)$ player i is almost sure that the payoffs are u^ℓ and is almost sure that $\omega \in \bigcap_{j \neq i} B_j^{r^n}(G^\ell)$, player i's payoff in $G(\sigma^{-\ell})$, as a function of σ^ℓ, is very close to $u_i^\ell(\sigma^\ell)$ for n large. From Wu and Jiang's theorem (section 12.1), for a generic choice of u^ℓ, for any equilibrium $\sigma^{\ell *}$ of u^ℓ, and for n sufficiently large, there exists an equilibrium $\sigma^\ell = \hat{\sigma}^\ell(\sigma^{-\ell})$ of the common-knowledge game $G(\sigma^{-\ell})$ such that $\|\sigma^\ell - \sigma^{\ell *}\| < \varepsilon$. Furthermore, σ^ℓ is continuous in $\sigma^{-\ell}$. We now extend $\hat{\sigma}^\ell$ to a function on the space of strategy profiles Σ by defining $\hat{\sigma}^\ell(\sigma) = \hat{\sigma}^\ell(\sigma^{-\ell})$.

After constructing $\hat{\sigma}^\ell(\cdot)$ for each ℓ, we define $\hat{\sigma}_i$ on $C_i \equiv \Omega \backslash \bigcup_\ell B_i^{r^n}(G^\ell)$ (sets which have vanishingly small probability). There we simply require player i to optimize against $(\sigma^1, \dots, \sigma^L, \sigma_{-i}^c)$. Let $\sigma_i^c = \hat{\sigma}_i^c(\sigma)$ denote player i's optimal response in those states; it is nonempty, it is compact-and-convex-valued, and it has a closed graph.

Let $\Sigma_\varepsilon^\ell = \{\sigma^\ell \mid \|\sigma^\ell - \sigma^{\ell *}\| < \varepsilon\}$ and $\Sigma^c = \{\sigma^c\}$. The correspondence $\hat{\sigma}$: $(\sigma^1, \dots, \sigma^L, \sigma^c) \to \hat{\sigma}(\sigma)$ maps $\Sigma_\varepsilon^1 \times \cdots \times \Sigma_\varepsilon^L \times \Sigma^c$ into itself, and has a fixed point from Kakutani's theorem. By construction, this fixed point is a Bayesian equilibrium and lies within ε of $\sigma^{\ell *}$ when $\omega \in \bigcap_{i \in \mathscr{I}} B_i^{r^n}(G^\ell)$. ■

Remark The hypotheses of the theorem are satisfied if each G^ℓ is common r^n-belief at exactly one ω^ℓ, as the total probability of $\{\omega^\ell\}_{\ell \in L}$ converges to 1. Thus, the theorem applies to the information structure of the truncated (finite-message) version of the electronic-mail game, and so the equilibrium correspondence there is lower hemi-continuous at the common-knowledge limit even if the payoff matrices given in figure 14.4 (which have strict equilibria) are replaced by matrices with essential equilibria.

The nice mathematical properties of games with a finite state space do not mean that infinite-state-space models are irrelevant—indeed, the standard requirement that the state space represent all the players' uncertainty about one another's information may naturally lead to state spaces that are even uncountably infinite. Mertens and Zamir (1985) and Brandenberger and Dekel (1987b) explicitly construct the "universal type space" required to capture general uncertainty of this kind, and find that it is quite large indeed. Mertens and Zamir observe that the universal type space can be approximated by a finite one when closeness is measured by the weak topology. However, as our discussion of example 14.4 shows, the set of equilibria is not continuous in that topology, so that the finite-state-space "approximation" can have a very different set of equilibria. In practical applications one works with finite type spaces for reasons of tractability. The sensitivity of even the Nash-equilibrium set to low-probability infinite-state perturbations is another reason to think seriously about the robustness of one's conclusions to the information structure of the game.

Exercises

Exercise 14.1** Consider the following version of the undiscounted, three-times-repeated prisoner's dilemma. There are three equally likely states of the world, ω_1, ω_2, and ω_3, with $H_1 = \{(\omega_1, \omega_2), (\omega_3)\}$ and $H_2 = \{(\omega_1), (\omega_2, \omega_3)\}$. In state ω_1 both players' payoffs are the sum of the stage-game payoffs shown in figure 14.6. In states ω_2 and ω_3, player 1's payoff is still the sum of the stage-game payoffs above, but player 2 must play the strategy "tit for tat" (which is to play C in the first period, and in periods 2 and 3 to play the same way player 1 did the period before). Show that the profile "both players play D every period," which is the unique equilibrium when it is common knowledge that the state is ω_1, is *not* a Nash-equilibrium outcome of the overall game—i.e., that there is no Nash equilibrium when in state ω_1 both players play D every period. Show that there is an equilibrium where both players play C in the first period in *every* state.

Exercise 14.2* Let Ω be the set of integers from 1 to 11. (a) Let $H_1 = \{(0,3,6,9), (1,4,7,10), (2,5,8,11)\}$ and $H_2 = \{(0,4,8), (1,5,9), (2,6,10), (3,7,11)\}$. What is the meet of these partitions?
(b) What if $h_1(\omega)$ is the set of all ω' such that $[\omega'/2] = [\omega/2]$, where $[x]$ denotes the greatest integer less than or equal to x, and $h_2(\omega)$ is the set of all ω' such that $[\omega'/4] = [\omega/4]$?