We can do more than this, though. At the optimum, we can find the variance of profits by replacing N in Equation (12.4) by N^*, which gives

$$\begin{aligned}\sigma_V^{*2} &= \sigma_S^2 + \left(\frac{\sigma_{SF}}{\sigma_F^2}\right)^2 \sigma_F^2 + 2\left(\frac{-\sigma_{SF}}{\sigma_F^2}\right)\sigma_{SF} \\ &= \sigma_S^2 + \frac{\sigma_{SF}^2}{\sigma_F^2} + 2\frac{-\sigma_{SF}^2}{\sigma_F^2} = \sigma_S^2 - \frac{\sigma_{SF}^2}{\sigma_F^2}\end{aligned} \tag{12.10}$$

We can measure the quality of the optimal hedge ratio in terms of the amount by which we decreased the variance of the original portfolio:

$$R^2 = \frac{(\sigma_S^2 - \sigma_V^{*2})}{\sigma_S^2} \tag{12.11}$$

After substitution of Equation (12.10), we find that $R^2 = (\sigma_S^2 - \sigma_S^2 + \sigma_{SF}^2/\sigma_F^2)/\sigma_S^2 = \sigma_{SF}^2/(\sigma_F^2\sigma_S^2) = \rho_{SF}^2$. This unitless number is also the coefficient of determination, or the percentage of variance in $\Delta s/s$ explained by the independent variable $\Delta f/f$. Thus this regression also gives us the **effectiveness** of the hedge, which is measured by the proportion of variance eliminated.

We can also express the volatility of the hedged position from Equation (12.10) using the R^2 as

$$\sigma_V^* = \sigma_S\sqrt{(1 - R^2)} \tag{12.12}$$

This shows that if $R^2 = 1$, the regression fit is perfect, and the resulting portfolio has zero risk. In this situation, the portfolio has no basis risk. However, if the R^2 is very low, the hedge is not effective.

12.2.2 Example

An airline knows that it will need to purchase 10,000 metric tons of jet fuel in three months. It wants some protection against an upturn in prices using futures contracts.

The company can hedge using heating oil futures contracts traded on NYMEX. The notional for one contract is 42,000 gallons. As there is no futures contract on jet fuel, the risk manager wants to check if heating oil could provide an efficient hedge instead. The current price of jet fuel is \$277/metric ton. The futures price of heating oil is \$0.6903/gallon. The standard deviation of the rate of change in jet fuel prices over three months is 21.17%, that of futures is 18.59%, and the correlation is 0.8243.

Compute

a. The notional and standard deviation of the unhedged fuel cost in dollars
b. The optimal number of futures contract to buy/sell, rounded to the closest integer
c. The standard deviation of the hedged fuel cost in dollars

Answer

a. The position notional is Qs = \$2,770,000. The standard deviation in dollars is

$$\sigma(\Delta s/s)sQ = 0.2117 \times \$277 \times 10{,}000 = \$586{,}409$$

For reference, that of one futures contract is

$$\sigma(\Delta f/f)fQ_f = 0.1859 \times \$0.6903 \times 42{,}000 = \$5{,}389.72$$

with a futures notional of $fQ_f = \$0.6903 \times 42{,}000 = \$28{,}992.60$.

b. The cash position corresponds to a payment, or liability. Hence, the company will have to *buy* futures as protection. First, we compute beta, which is $\beta_{sf} = 0.8243(0.2117/0.1859) = 0.9387$. The corresponding covariance term is $\sigma_{sf} = 0.8243 \times 0.2117 \times 0.1859 = 0.03244$. Adjusting for the notionals, this is $\sigma_{SF} = 0.03244 \times \$2{,}770{,}000 \times \$28{,}993 = 2{,}605{,}268{,}452$. The optimal hedge ratio is, using Equation (12.7),

$$N^* = \beta_{sf}\frac{Q \times s}{Q_f \times f} = 0.9387\frac{10{,}000 \times \$277}{42{,}000 \times \$0.69} = 89.7$$

or 90 contracts after rounding (which we ignore in what follows).

c. To find the risk of the hedged position, we use Equation (12.10). The volatility of the unhedged position is $\sigma_S = \$586{,}409$. The variance of the hedged position is

$$\begin{array}{rll} \sigma_S^2 = & (\$586{,}409)^2 & = +343{,}875{,}515{,}281 \\ -\sigma_{SF}^2/\sigma_F^2 = & -(2{,}605{,}268{,}452/5{,}390)^2 & = -233{,}653{,}264{,}867 \\ \hline \text{V(hedged)} & & = +110{,}222{,}250{,}414 \end{array}$$

Taking the square root, the volatility of the hedged position is $\sigma_V^* = \$331{,}997$. Thus the hedge has reduced the risk from \$586,409 to \$331,997. Computing the R^2, we find that one minus the ratio of the hedged and unhedged variances is $(1 - 110{,}222{,}250{,}414/343{,}875{,}515{,}281) = 67.95\%$. This is exactly the square of the correlation coefficient, $0.8243^2 = 0.6795$, or effectiveness of the hedge.

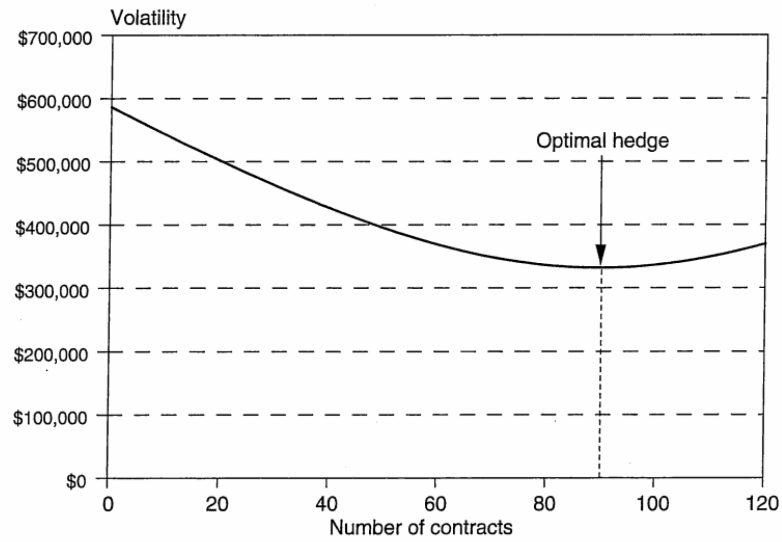

FIGURE 12.2 Risk of Hedged Position and Number of Contracts

Figure 12.2 displays the relationship between the risk of the hedged position and the number of contracts. With no hedging, the volatility is \$586,409. As N increases, the risk decreases, reaching a minimum for $N^* = 90$ contracts. The graph also shows that the quadratic relationship is relatively flat for a range of values around the minimum. Choosing anywhere between 80 and 100 contracts will have little effect on the total risk. Given the substantial reduction in risk, the risk manager could choose to implement the hedge.

12.2.3 Liquidity Issues

Although futures hedging can be successful at mitigating market risk, it can create other risks. Futures contracts are marked to market daily. Hence they can involve large cash inflows or outflows. Cash outflows, in particular, can create liquidity problems, especially when they are not offset by cash inflows from the underlying position.

EXAMPLE 12.3: FRM EXAM 2001—QUESTION 86

If two securities have the same volatility and a correlation equal to -0.5, their minimum variance hedge ratio is

a. 1:1
b. 2:1
c. 4:1
d. 16:1

EXAMPLE 12.4: FRM EXAM 2007—QUESTION 125

A firm is going to buy 10,000 barrels of West Texas Intermediate Crude Oil. It plans to hedge the purchase using the Brent Crude Oil futures contract. The correlation between the spot and futures prices is 0.72. The volatility of the spot price is 0.35 per year. The volatility of the Brent Crude Oil futures price is 0.27 per year. What is the hedge ratio for the firm?

a. 0.9333
b. 0.5554
c. 0.8198
d. 1.2099

EXAMPLE 12.5: FRM EXAM 2003—QUESTION 14

A bronze producer will sell 1,000 mt (metric tons) of bronze in three months at the prevailing market price at that time. The standard deviation of the price of bronze over a three-month period is 2.6%. The company decides to use three-month futures on copper to hedge. The copper futures contract is for 25 mt of copper. The standard deviation of the futures price is 3.2%. The correlation between three-month changes in the futures price and the price of bronze is 0.77. To hedge its price exposure, how many futures contracts should the company buy/sell?

a. Sell 38 futures
b. Buy 25 futures
c. Buy 63 futures
d. Sell 25 futures

12.3 APPLICATIONS OF OPTIMAL HEDGING

The linear framework presented here is completely general. We now specialize it to two important cases, duration and beta hedging. The first applies to the bond market, the second to the stock market.

12.3.1 Duration Hedging

Modified duration can be viewed as a measure of the exposure of relative changes in prices to movements in yields. Using the definitions in Chapter 1, we can write

$$\Delta P = (-D^* P)\Delta y \tag{12.13}$$

where D^* is the modified duration. The **dollar duration** is defined as $(D^* P)$.

Assuming the duration model holds, which implies that the change in yield Δy does not depend on maturity, we can rewrite this expression for the cash and futures positions

$$\Delta S = (-D_S^* S)\Delta y \qquad \Delta F = (-D_F^* F)\Delta y$$

where D_S^* and D_F^* are the modified durations of S and F, respectively. Note that these relationships are supposed to be perfect, without an error term. The variances and covariance are then

$$\sigma_S^2 = (D_S^* S)^2 \sigma^2(\Delta y) \qquad \sigma_F^2 = (D_F^* F)^2 \sigma^2(\Delta y) \qquad \sigma_{SF} = (D_F^* F)(D_S^* S)\sigma^2(\Delta y)$$

We can replace these in Equation (12.6):

$$N^* = -\frac{\sigma_{SF}}{\sigma_F^2} = -\frac{(D_F^* F)(D_S^* S)}{(D_F^* F)^2} = -\frac{(D_S^* S)}{(D_F^* F)} \tag{12.14}$$

Alternatively, this can be derived as follows. Write the total portfolio payoff as

$$\begin{aligned} \Delta V &= \Delta S + N\Delta F \\ &= (-D_S^* S)\Delta y + N(-D_F^* F)\Delta y \\ &= -[(D_S^* S) + N(D_F^* F)] \times \Delta y \end{aligned}$$

which is zero when the net exposure, represented by the term between brackets, is zero. In other words, the optimal hedge ratio is simply minus the ratio of the dollar duration of cash relative to the dollar duration of the hedge. This ratio can also be expressed in dollar value of a basis point (DVBP).

More generally, we can use N as a tool to modify the total duration of the portfolio. If we have a target duration of D_V, this can be achieved by setting $[(D_S^* S) + N(D_F^* F)] = D_V^* V$, or

$$N = \frac{(D_V^* V - D_S^* S)}{(D_F^* F)} \tag{12.15}$$

of which Equation (12.14) is a special case.

KEY CONCEPT

The optimal duration hedge is given by the ratio of the dollar duration of the position to that of the hedging instrument.

Example 1

A portfolio manager holds a bond portfolio worth \$10 million with a modified duration of 6.8 years, to be hedged for three months. The current futures price is 93-02, with a notional of \$100,000. We assume that its duration can be measured by that of the cheapest-to-deliver, which is 9.2 years.

Compute

a. The notional of the futures contract
b. The number of contracts to buy/sell for optimal protection

Answer

a. The notional is $[93 + (2/32)]/100 \times \$100{,}000 = \$93{,}062.5$.
b. The optimal number to *sell* is from Equation (12.14)

$$N^* = -\frac{(D_S^* S)}{(D_F^* F)} = -\frac{6.8 \times \$10{,}000{,}000}{9.2 \times \$93{,}062.5} = -79.4$$

or 79 contracts after rounding. Note that the DVBP of the futures is about $9.2 \times \$93{,}000 \times 0.01\% = \$85$.

Example 2

On February 2, a corporate Treasurer wants to hedge a July 17 issue of \$5 million of commercial paper with a maturity of 180 days, leading to anticipated proceeds of \$4.52 million. The September Eurodollar futures trades at 92, and has a notional amount of \$1 million.

Compute

a. The current dollar value of the futures contract
b. The number of contracts to buy/sell for optimal protection

Answer

a. The current dollar price is given by $\$10{,}000[100 - 0.25(100 - 92)] = \$980{,}000$. Note that the duration of the futures is always three months (90 days), since the contract refers to three-month LIBOR.
b. If rates increase, the cost of borrowing will be higher. We need to offset this by a gain, or a short position in the futures. The optimal number is from Equation (12.14)

$$N^* = -\frac{(D_S^* S)}{(D_F^* F)} = -\frac{180 \times \$4{,}520{,}000}{90 \times \$980{,}000} = -9.2$$

or nine contracts after rounding. Note that the DVBP of the futures is about $0.25 \times \$1{,}000{,}000 \times 0.01\% = \$25$.

EXAMPLE 12.6: FRM EXAM 2007—QUESTION 17

On June 2, a fund manager with USD 10 million invested in government bonds is concerned that interest rates will be highly volatile over the next three months. The manager decides to use the September Treasury bond futures contract to hedge the portfolio. The current futures price is USD 95.0625. Each contract is for the delivery of USD 100,000 face value of bonds. The duration of the manager's bond portfolio in three months will be 7.8 years. The cheapest-to-deliver bond in the Treasury bond futures contract is expected to have a duration of 8.4 years at maturity of the contract. At the maturity of the Treasury bond futures contract, the duration of the underlying benchmark Treasury bond is nine years. What position should the fund manager undertake to mitigate his interest rate risk exposure?

a. Short 94 contracts
b. Short 98 contracts
c. Short 105 contracts
d. Short 113 contracts

EXAMPLE 12.7: DURATION HEDGING

What assumptions does a duration-based hedging scheme make about the way in which interest rates move?

a. All interest rates change by the same amount.
b. A small parallel shift occurs in the yield curve.
c. Any parallel shift occurs in the term structure.
d. Interest rates movements are highly correlated.

EXAMPLE 12.8: HEDGING WITH EURODOLLAR FUTURES

If all spot interest rates are increased by one basis point, a value of a portfolio of swaps will increase by $1,100. How many Eurodollar futures contracts are needed to hedge the portfolio?

a. 44
b. 22
c. 11
d. 1,100

EXAMPLE 12.9: FRM EXAM 2004—QUESTION 4

Albert Henri is the fixed-income manager of a large Canadian pension fund. The present value of the pension fund's portfolio of assets is CAD 4 billion while the expected present value of the fund's liabilities is CAD 5 billion. The respective modified durations are 8.254 and 6.825 years. The fund currently has an actuarial deficit (assets < liabilities) and Albert must avoid widening this gap. There are currently two scenarios for the yield curve: the first scenario is an upward shift of 25 bps, with the second scenario a downward shift of 25 bps. The most liquid interest rate futures contract has a present value of CAD 68,336 and a duration of 2.1468 years. Analyzing both scenarios separately, what should Albert Henry do to avoid widening the pension fund gap? Choose the best option.

First Scenario	Second Scenario
a. Do nothing.	Buy 7,559 contracts.
b. Do nothing.	Sell 7,559 contracts.
c. Buy 7,559 contracts.	Do nothing.
d. Do nothing.	Do nothing.

12.3.2 Beta Hedging

We now turn to equity hedging using stock index futures. **Beta**, or **systematic risk** can be viewed as a measure of the exposure of the rate of return on a portfolio i to movements in the "market" m:

$$R_{it} = \alpha_i + \beta_i R_{mt} + \epsilon_{it} \tag{12.16}$$

where β represents the systematic risk, α the intercept (which is not a source of risk and therefore ignored for risk management purposes), and ϵ the residual component, which is uncorrelated with the market. We can also write, in line with the previous sections and ignoring the residual and intercept,

$$(\Delta S/S) \approx \beta(\Delta M/M) \tag{12.17}$$

Now, assume that we have at our disposal a stock-index futures contract, which has a beta of unity $(\Delta F/F) = 1(\Delta M/M)$. For options, the beta is replaced by the net delta, $(\Delta C) = \delta(\Delta M)$.

As in the case of bond duration, we can write the total portfolio payoff as

$$\begin{aligned}\Delta V &= \Delta S + N\Delta F \\ &= (\beta S)(\Delta M/M) + NF(\Delta M/M) \\ &= [(\beta S) + NF] \times (\Delta M/M)\end{aligned}$$

which is set to zero when the net exposure, represented by the term between brackets is zero. The optimal number of contracts to short is

$$N^* = -\frac{\beta S}{F} \tag{12.18}$$

KEY CONCEPT

The optimal hedge with stock index futures is given by the the beta of the cash position times its value divided by the notional of the futures contract.

Example

A portfolio manager holds a stock portfolio worth \$10 million with a beta of 1.5 relative to the S&P 500. The current futures price is 1,400, with a multiplier of \$250.

Compute

a. The notional of the futures contract
b. The number of contracts to sell short for optimal protection

Answer

a. The notional amount of the futures contract is $\$250 \times 1400 = \$350{,}000$.
b. The optimal number of contract to short is, from Equation (12.18),

$$N^* = -\frac{\beta S}{F} = -\frac{1.5 \times \$10{,}000{,}000}{1 \times \$350{,}000} = -42.9$$

or 43 contracts after rounding.

The quality of the hedge will depend on the size of the residual risk in the market model of Equation (12.16). For large portfolios, the approximation may be good. In contrast, hedging an individual stock with stock index futures may give poor results.

For instance, the correlation of a typical U.S. stock with the S&P 500 is 0.50. For an industry index, it is typically 0.75. Using the regression effectiveness in Equation (12.12), we find that the volatility of the hedged portfolio is still about $\sqrt{1-0.5^2} = 87\%$ of the unhedged volatility for a typical stock and about 66% of the unhedged volatility for a typical industry. The lower number shows that hedging with general stock index futures is more effective for large portfolios. To obtain finer coverage of equity risks, hedgers could use futures contracts on industrial sectors, or exchange-traded funds (ETFs), or even single stock futures.

EXAMPLE 12.10: FRM EXAM 2005—QUESTION 97

Suppose that the benchmark for an equity portfolio of USD 12 million is the S&P 500. Also suppose the current value of the S&P 500 is 1,040 and the portfolio beta relative to the S&P 500 is 1.4. If the portfolio manager wants to completely hedge the portfolio over the next three months using the S&P 500 index futures (that has a multiplier of 250), which of the following is the correct hedging strategy?

a. Long 46 contracts
b. Short 46 contracts
c. Long 65 contracts
d. Short 65 contracts

EXAMPLE 12.11: FRM EXAM 2007—QUESTION 107

The current value of the S&P 500 index is 1,457, and each S&P futures contract is for delivery of 250 times the index. A long-only equity portfolio with market value of USD 300,100,000 has beta of 1.1. To reduce the portfolio beta to 0.75, how many S&P futures contract should you sell?

a. 288 contracts
b. 618 contracts
c. 906 contracts
d. 574 contracts

12.4 IMPORTANT FORMULAS

Profit on position with unit hedge: $Q[(S_2 - S_1) - (F_2 - F_1)] = Q[b_2 - b_1]$
Optimal hedge ratio: $N^* = -\beta_{sf}\frac{Q \times s}{Q_f \times f}$
Optimal hedge ratio (unitless): $\beta_{sf} = \frac{\sigma_{sf}}{\sigma_f^2} = \rho_{sf}\frac{\sigma_s}{\sigma_f}$

Volatility of the hedged position: $\sigma_V^* = \sigma_S\sqrt{(1 - R^2)}$

Duration hedge: $N^* = -\frac{(D_S^* S)}{(D_F^* F)}$

$N^* = -\beta\frac{S}{F}$

12.5 ANSWERS TO CHAPTER EXAMPLES

Example 12.1: FRM Exam 2000—Question 79

d. Basis risk occurs if movements in the value of the cash and hedged positions do not offset each other perfectly. This can happen if the instruments are dissimilar or if the correlation is not unity. Even with similar instruments, if the hedge is lifted before the maturity of the underlying, there is some basis risk.

Example 12.2: FRM Exam 2007—Question 99

c. There is mainly basis risk for positions that are both long and short either different months or contracts. Position II) is long twice the same contract and thus has no basis risk (but a lot of directional risk).

Example 12.3: FRM Exam 2001—Question 86

b. Set x as the amount to invest in the second security, relative to that in the first (or the hedge ratio). The variance is then proportional to $1 + x^2 + 2x\rho$. Taking the derivative and setting to zero, we have $x = -rho = 0.5$. Thus, one security must have twice the amount in the other. Alternatively, the hedge ratio is given by $N^* = -\rho\frac{\sigma_S}{\sigma_F}$, which gives 0.5. Answer b. is the only one which is consistent with this number or its inverse.

Example 12.4: FRM Exam 2007—Question 125

a. The optimal hedge ratio is $\beta_{sf} = \rho_{sf}\frac{\sigma_s}{\sigma_f} = 0.72\ 0.35/0.27 = 0.933$.

Example 12.5: FRM Exam 2003—Question 14

b. The optimal hedge ratio is $\rho\sigma_s/\sigma_f = 0.77 \times 2.6/3.2 = 0.626$. Taking into account the size of the position, the number of contracts to buy is $0.626 \times 1{,}000/25 = 15.03$.

Example 12.6: FRM Exam 2007—Question 17

b. The number of contracts to short is $N^* = -\frac{(D_S^* S)}{(D_F^* F)} = -(7.8 \times 10{,}000{,}000)/(8.4 \times (95.0625) \times 1{,}000) = -97.7$, or 98 contracts. Note that the relevant duration for the futures is that of the CTD; other numbers are irrelevant.

Example 12.7: Duration Hedging

b. The assumption is that of (1) parallel and (2) small moves in the yield curve. Answers a. and c. are the same, and omit the size of the move. Answer d. would require perfect, not high, correlation plus small moves.

Example 12.8: Hedging with Eurodollar Futures

a. The DVBP of the portfolio is \$1100. That of the futures is \$25. Hence the ratio is 1100/25 = 44.

Example 12.9: FRM Exam 2004—Question 4

a. We first have to compute the dollar duration of assets and liabilities, which gives, in millions, $4{,}000 \times 8.254 = 33{,}016$ and $5{,}000 \times 6.825 = 34{,}125$, respectively. Because the DD of liabilities exceeds that of assets, a decrease in rates will increase the liabilities more than the assets, leading to a worsening deficit. Mr. Henri needs to buy interest rate futures as an offset. The number of contracts is $(34{,}125 - 33{,}016)/(68{,}336 \times 2.1468/1{,}000{,}000) = 7{,}559$.

Example 12.10: FRM Exam 2005—Question 97

d. To hedge, the portfolio manager should sell index futures, to create a profit if the portfolio loses value. The number of contracts is $N^* = -\beta S/F = -(1.4 \times 12{,}000{,}000)/(1,400 \times 250) = -64.6$, or 65 contracts.

Example 12.11: FRM Exam 2007—Question 107

a. This is as in the previous question, but the hedge is partial, i.e. for a change of 1.10 to 0.75. So, $N^* = -\beta S/F = -(1.10 - 0.75)300{,}100{,}000/(1457 \times 250) = -288.3$ contracts.

CHAPTER 13

Nonlinear Risk: Options

The previous chapter focused on "linear" hedging, using contracts such as forwards and futures whose values are linearly related to the underlying risk factors. Positions in these contracts are fixed over the hedge horizon. Because linear combinations of normal random variables are also normally distributed, linear hedging maintains normal distributions, albeit with lower variances.

Hedging nonlinear risks, however, is much more complex. Because options have nonlinear payoffs, the distribution of option values can be sharply asymmetrical. Due to the ubiquitous nature of options, risk managers need to be able to evaluate the risk of positions with options. Since options can be replicated by dynamic trading of the underlying instruments, this also provides insights into the risks of active trading strategies.

In a previous chapter, we have seen that market losses can be ascribed to the combination of two factors: exposure and adverse movements in the risk factor. Thus a large loss could occur because of the risk factor, which is bad luck. Too often, however, losses occur because the exposure profile is similar to a short option position. This is less forgivable, because exposure is under the control of the portfolio manager.

The challenge is to develop measures that provide an intuitive understanding of the exposure profile. Section 13.1 introduces option pricing and the Taylor approximation.[1] It starts from the Black–Scholes formula that was presented in Chapter 6. Partial derivatives, also known as "Greeks," are analyzed in Section 13.2. Section 13.3 then turns to the interpretation of dynamic hedging and discusses the distribution profile of option positions.

13.1 EVALUATING OPTIONS

13.1.1 Definitions

We consider a **derivative** instrument whose value depends on an underlying asset, which can be a price, an index, or a rate. As an example, consider a call option

[1] The reader should be forewarned that this chapter is more technical than others. It presupposes some exposure to option pricing and hedging.

where the underlying asset is a foreign currency. We use these definitions:

S_t = current spot price of the asset in dollars
F_t = current forward price of the asset
K = exercise price of option contract
f_t = current value of derivative instrument
r_t = domestic risk-free rate
r_t^* = foreign risk-free rate (also written as y)
σ_t = annual volatility of the rate of change in S
τ = time to maturity.

More generally, r^* represents the income payment on the asset, which represents the *annual rate* of dividend or coupon payments on a stock index or bond.

For most options, we can write the value of the derivative as the function

$$f_t = f(S_t, r_t, r_t^*, \sigma_t, \ K, \tau) \tag{13.1}$$

The contract specifications are represented by K and the time to maturity τ. The other factors are affected by market movements, creating volatility in the value of the derivative. For simplicity, we drop the time subscripts in what follows.

Derivatives pricing is all about finding the value of f, given the characteristics of the option at expiration and some assumptions about the behavior of markets. For a forward contract, for instance, the expression is very simple. It reduces to

$$f = Se^{-r^*\tau} - Ke^{-r\tau} \tag{13.2}$$

More generally, we may not be able to derive an analytical expression for the function f, requiring numerical methods.

13.1.2 Taylor Expansion

We are interested in describing the movements in f. The exposure profile of the derivative can be described *locally* by taking a Taylor expansion,

$$df = \frac{\partial f}{\partial S}dS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}dS^2 + \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial r^*}dr^* + \frac{\partial f}{\partial \sigma}d\sigma + \frac{\partial f}{\partial \tau}d\tau + \cdots \tag{13.3}$$

Because the value depends on S in a nonlinear fashion, we added a quadratic term for S. The terms in Equation (13.3) approximate a nonlinear function by linear and quadratic polynomials.

Option pricing is about finding f. **Option hedging** uses the partial derivatives. **Risk management** is about combining those with the movements in the risk factors.

Figure 13.1 describes the relationship between the value of a European call and the underlying asset. The actual price is the solid line. The straight thin line is the linear (delta) estimate, which is the tangent at the initial point. The other

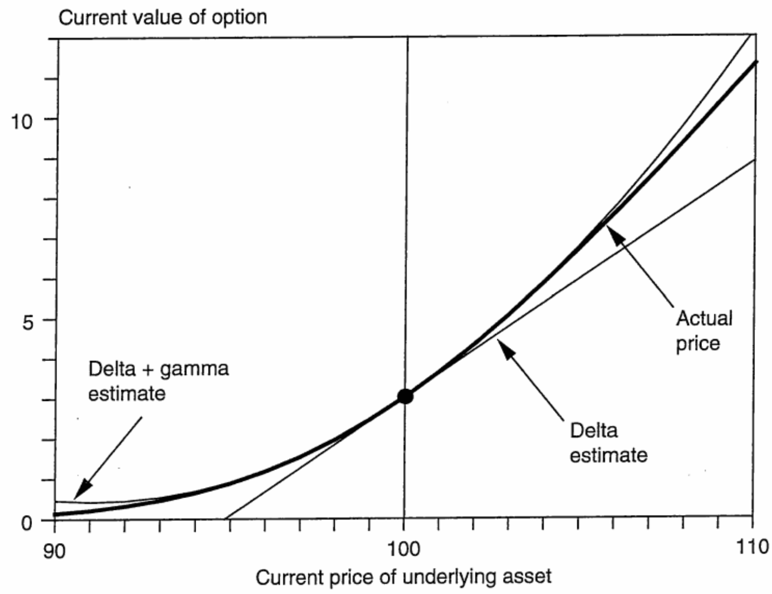

FIGURE 13.1 Delta–Gamma Approximation for a Long Call

line is the quadratic (delta plus gamma) estimates, which gives a much better fit because it has more parameters.

Note that, because we are dealing with sums of local price movements, we can aggregate the sensitivities at the portfolio level. This is similar to computing the portfolio duration from the sum of durations of individual securities, appropriately weighted.

Defining $\Delta = \frac{\partial f}{\partial S}$, for example, we can summarize the portfolio, or "book" Δ_P in terms of the total sensitivity,

$$\Delta_P = \sum_{i=1}^{N} x_i \Delta_i \tag{13.4}$$

where x_i is the number of options of type i in the portfolio. To hedge against first-order price risk, it is sufficient to hedge the *net* portfolio delta. This is more efficient than trying to hedge every single instrument individually.

The Taylor expansion will provide a bad approximation in a number of cases:

- *Large movements in the underlying risk factor*
- *Highly nonlinear exposures*, such as options near expiry or exotic options
- *Cross-partial effects*, such as σ changing in relation with S

If this is the case, we need to turn to a **full revaluation** of the instrument. Using the subscripts 0 and 1 as the initial and final values, the change in the option value is

$$f_1 - f_0 = f(S_1, r_1, r_1^*, \sigma_1,\ K, \tau_1) - f(S_0, r_0, r_0^*, \sigma_0,\ K, \tau_0) \tag{13.5}$$

13.1.3 Option Pricing

We now present the various partial derivatives for conventional European call and put options. As we have seen in Chapter 6, the **Black–Scholes** (BS) model provides a closed-form solution, from which these derivatives can be computed analytically.

The key point of the BS derivation is that a position in the option can be replicated by a "delta" position in the underlying asset. Hence, a portfolio combining the asset and the option in appropriate proportions is risk-free "locally," that is, for small movements in prices. To avoid arbitrage, this portfolio must return the risk-free rate. The option value is the discounted expected payoff:

$$f_t = E_{\mathrm{RN}}[e^{-r\tau} F(S_T)] \tag{13.6}$$

where E_{RN} represents the expectation of the future payoff in a "risk-neutral" world, that is, assuming the underlying asset grows at the risk-free rate and the discounting also employs the risk-free rate.

In the case of a European call, the final payoff is $F(S_T) = \mathrm{Max}(S_T - K, 0)$, and the current value of the call is given by

$$c = Se^{-r^*\tau} N(d_1) - Ke^{-r\tau} N(d_2) \tag{13.7}$$

where $N(d)$ is the cumulative distribution function for the standard normal distribution:

$$N(d) = \int_{-\infty}^{d} \Phi(x)dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{1}{2}x^2} dx$$

with Φ defined as the standard normal density function. $N(d)$ is also the area to the left of a standard normal variable with value equal to d. The values of d_1 and d_2 are

$$d_1 = \frac{ln(Se^{-r^*\tau}/Ke^{-r\tau})}{\sigma\sqrt{\tau}} + \frac{\sigma\sqrt{\tau}}{2}, \quad d_2 = d_1 - \sigma\sqrt{\tau}$$

By put–call parity, the European put option value is

$$p = Se^{-r^*\tau}[N(d_1) - 1] - Ke^{-r\tau}[N(d_2) - 1] \tag{13.8}$$

13.2 OPTION "GREEKS"

13.2.1 Option Sensitivities: Delta and Gamma

Given these closed-form solutions for European options, we can derive all partial derivatives. The most important sensitivity is the **delta**, which is the first partial

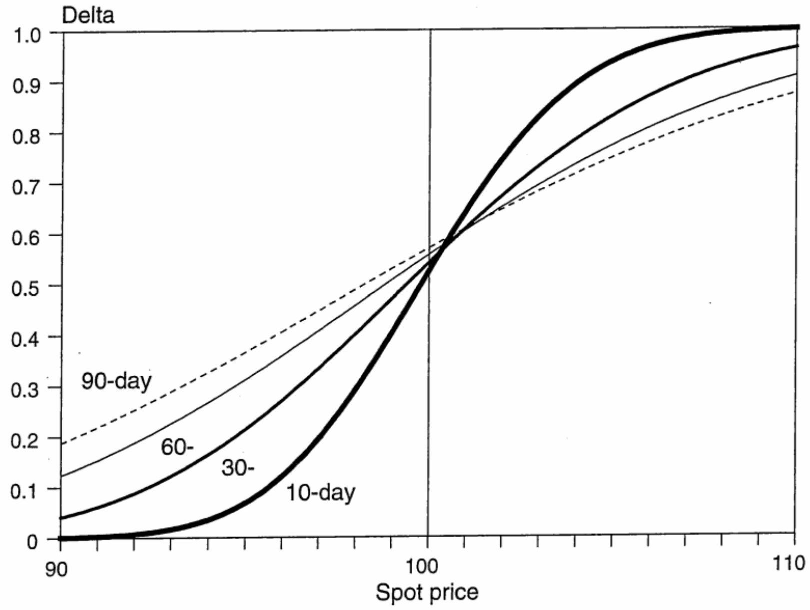

FIGURE 13.2 Option Delta

derivative with respect to the price. For a call option, this can be written explicitly as:

$$\Delta_c = \frac{\partial c}{\partial S} = e^{-r^*\tau} N(d_1) \tag{13.9}$$

which is always positive and below unity.

Figure 13.2 relates delta to the current value of S, for various maturities. The essential feature of this figure is that Δ varies substantially with the spot price and with time. As the spot price increases, d_1 and d_2 become very large, and Δ tends toward $e^{-r^*\tau}$, close to one for short maturities. In this situation, the option behaves like an outright position in the asset. Indeed the limit of Equation (13.7) is $c = Se^{-r^*\tau} - Ke^{-r\tau}$, which is exactly the value of our forward contract, Equation (13.2).

At the other extreme, if S is very low, Δ is close to zero and the option is not very sensitive to S. When S is close to the strike price K, Δ is close to 0.5, and the option behaves like a position of 0.5 in the underlying asset.

KEY CONCEPT

The delta of an at-the-money call option is close to 0.5. Delta moves to 1 as the call goes deep in-the-money. It moves to zero as the call goes deep out-of-the-money.

The delta of a put option is

$$\Delta_p = \frac{\partial p}{\partial S} = e^{-r^*\tau}[N(d_1) - 1] \tag{13.10}$$

which is always negative. It behaves similarly to the call Δ, except for the sign. The delta of an at-the-money put is about -0.5.

KEY CONCEPT

The delta of an at-the-money put option is close to -0.5. Delta moves to -1 as the put goes deep in-the-money. It moves to zero as the put goes deep out-of-the-money.

The figure also shows that, as the option nears maturity, the Δ function becomes more curved. The function converges to a step function, i.e., 0 when $S < K$, and 1 otherwise. Close-to-maturity options have unstable deltas.

For a European call or put, gamma (Γ) is the second-order term,

$$\Gamma = \frac{\partial^2 c}{\partial S^2} = \frac{e^{-r^*\tau}\Phi(d_1)}{S\sigma\sqrt{\tau}} \tag{13.11}$$

which has the "bell shape" of the normal density function Φ. This is also the derivative of Δ with respect to S. Thus Γ measures the "instability" in Δ. Note that gamma is identical for a call and put with identical characteristics.

Figure 13.3 plots the call option gamma. At-the-money options have the highest gamma, which indicates that Δ changes very fast as S changes. In contrast, both in-the-money options and out-of-the-money options have low gammas because their delta is constant, close to one or zero, respectively. The figure also shows that as the maturity nears, the option gamma increases. This leads to the useful rule in the box.

KEY CONCEPT

For vanilla options, gamma is the highest, or nonlinearities are most pronounced, for short-term at-the-money options.

Thus, gamma is similar to the concept of convexity developed for bonds. Fixed-coupon bonds, however, always have positive convexity, whereas options can create positive or negative convexity. Positive convexity or gamma is beneficial, as it implies that the value of the asset drops more slowly and increases more

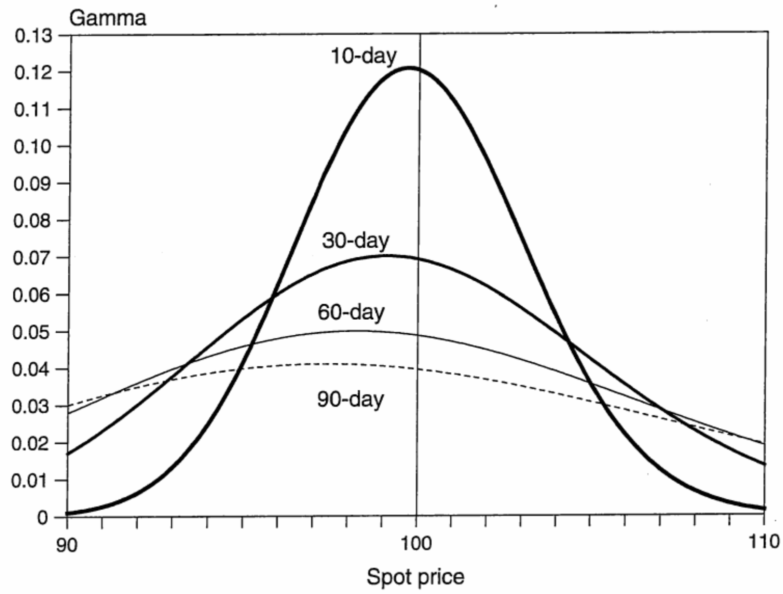

FIGURE 13.3 Option Gamma

quickly than otherwise. In contrast, negative convexity can be dangerous because it implies faster price falls and slower price increases.

Figure 13.4 summarizes the delta and gamma exposures of positions in options. Long positions in options, whether calls or puts, create positive convexity. Short positions create negative convexity. In exchange for assuming the harmful effect of this negative convexity, option sellers receive the premium.

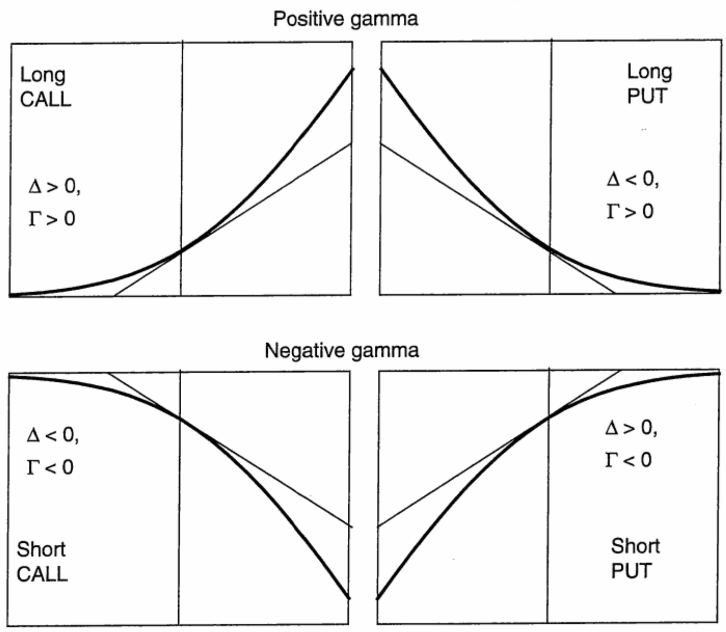

FIGURE 13.4 Delta and Gamma of Option Positions

EXAMPLE 13.1: FRM EXAM 2006—QUESTION 91

The dividend yield of an asset is 10% per annum. What is the delta of a long forward contract on the asset with 6 months to maturity?

a. 0.95
b. 1.00
c. 1.05
d. Cannot determine without additional information

EXAMPLE 13.2: FRM EXAM 2004—QUESTION 21

A 90-day European put option on Microsoft has an exercise price of $30. The current market price for Microsoft is $30. The delta for this option is close to

a. −1
b. −0.5
c. 0.5
d. 1

EXAMPLE 13.3: FRM EXAM 2006—QUESTION 80

You are given the following information about a European call option: Time to maturity = two years; continuous risk-free rate = 4%; continuous dividend yield = 1%; N(d1) = 0.64. Calculate the delta of this option.

a. −0.64
b. 0.36
c. 0.63
d. 0.64

EXAMPLE 13.4: FRM EXAM 2003—QUESTION 94

Which of the following IBM options has the highest gamma with the current market price of IBM common stock at USD 68?

a. Call option expiring in 10 days with strike USD 70
b. Call option expiring in 10 days with strike USD 50
c. Put option expiring in 10 days with strike USD 50
d. Put option expiring in two months with strike USD 70

EXAMPLE 13.5: FRM EXAM 2001—QUESTION 79

A bank has sold USD 300,000 of call options on 100,000 equities. The equities trade at 50, the option strike price is 49, the maturity is in three months, volatility is 20%, and the interest rate is 5%. How does the bank delta hedge?

a. Buy 65,000 shares
b. Buy 100,000 shares
c. Buy 21,000 shares
d. Sell 100,000 shares

EXAMPLE 13.6: FRM EXAM 2006—QUESTION 106

Suppose an existing short option position is delta-neutral, but has a gamma of −600. Also assume that there exists a traded option with a delta of 0.75 and a gamma of 1.50. In order to maintain the position gamma-neutral and delta-neutral, which of the following is the appropriate strategy to implement?

a. Buy 400 options and sell 300 shares of the underlying asset.
b. Buy 300 options and sell 400 shares of the underlying asset.
c. Sell 400 options and buy 300 shares of the underlying asset.
d. Sell 300 options and buy 400 shares of the underlying asset.

13.2.2 Option Sensitivities: Vega

Unlike linear contracts, options are exposed not only to movements in the direction of the spot price, but also in its volatility. Options therefore can be viewed as "volatility bets."

The sensitivity of an option to volatility is called the option **vega** (sometimes also called lambda, or kappa). For European calls and puts, this is

$$\Lambda = \frac{\partial c}{\partial \sigma} = Se^{-r^*\tau}\sqrt{\tau}\ \Phi(d_1) \tag{13.12}$$

which also has the "bell shape" of the normal density function Φ. As with gamma, vega is identical for similar call and put positions. Vega must be positive for long option positions.

Figure 13.5 plots the call option vega. The graph shows that at-the-money options are the most sensitive to volatility. The time effect, however, is different from that for gamma, because the term $\sqrt{\tau}$ appears in the numerator instead of

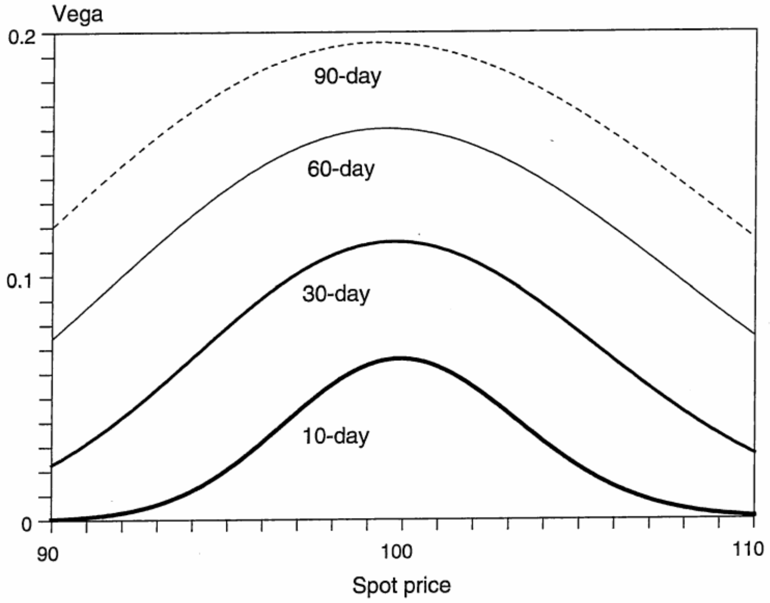

FIGURE 13.5 Option Vega

denominator. Thus, vega decreases with maturity, unlike gamma, which increases with maturity.

> **KEY CONCEPT**
>
> Vega is highest for long-term at-the-money options.

Changes in the volatility parameter can be a substantial source of risk. Figure 13.6 illustrates the time-variation in the **implied volatility** for options on the S&P stock index, also known as **Volatility Index** (VIX).[2] Over this period, the average value of the VIX index was 21%. The volatility in the daily change in VIX was about 2.4%.[3]

The VIX index experiences sharp spikes on a regular basis, however, reflecting increased uncertainty. In particular, VIX came close to or exceeded 50% during the crash of October 1987, during the LTCM crisis of September 1998, after the World Trade Center attack of September 2001, at the bottom of the 2000–2002 bear market in July 2002, and during the credit crisis that suddenly worsened in September 2008.

[2] The implied volatility is derived from the market prices of at-the-money near-term options on the S&P100 index and is calculated by the Chicago Board Options Exchange. In 2003, the methodology was changed; the new VIX index is derived from the prices of S&P500 index options across a wide range of strike prices.

[3] There is strong mean reversion in these data, so that daily volatilities cannot be extrapolated to annual data.

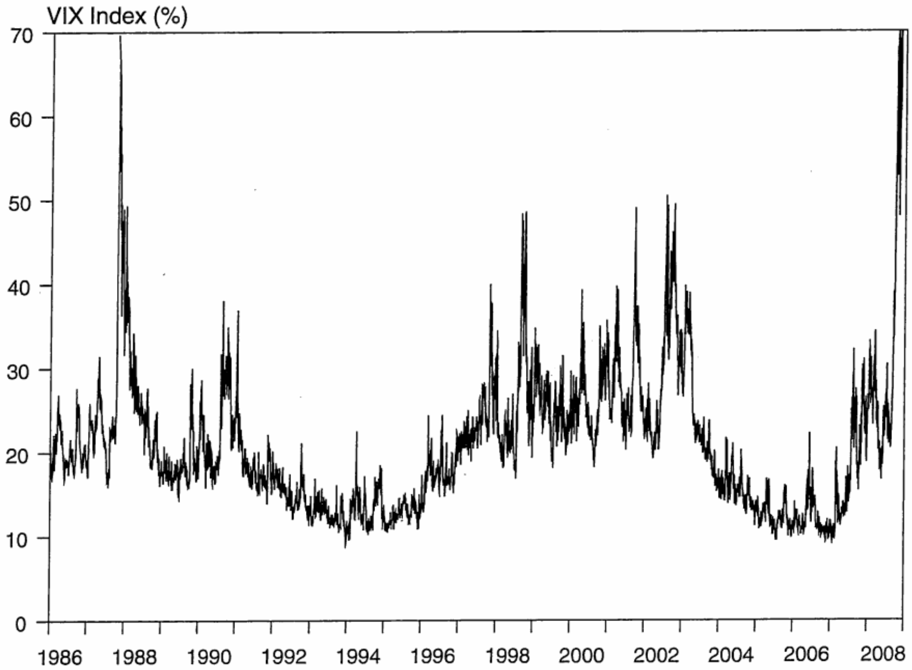

FIGURE 13.6 Movements in Implied Volatility

13.2.3 Option Sensitivities: Rho

The sensitivity to the domestic interest rate, also called **rho**, is

$$\rho_c = \frac{\partial c}{\partial r} = Ke^{-r\tau}\tau N(d_2) \tag{13.13}$$

For a put,

$$\rho_p = \frac{\partial p}{\partial r} = -Ke^{-r\tau}\tau N(-d_2) \tag{13.14}$$

An increase in the rate of interest increases the value of the call, as the underlying asset grows at a higher rate, which increases the probability of exercising the call, with a fixed strike price K. In the limit, for an infinite interest rate, the probability of exercise is one and the call option is equivalent to the stock itself. The reasoning is opposite for a put option.

The exposure to the yield on the asset is, for calls and puts, respectively,

$$\rho_C^* = \frac{\partial c}{\partial r^*} = -Se^{-r^*\tau}\tau N(d_1) \tag{13.15}$$

$$\rho_P^* = \frac{\partial p}{\partial r^*} = Se^{-r^*\tau}\tau N(-d_1) \tag{13.16}$$

An increase in the dividend yield decreases the growth rate of the underlying asset, which is harmful to the value of the call. Again, the reasoning is opposite for a put option.

13.2.4 Option Sensitivities: Theta

Finally, the variation in option value due to the passage of time is also called **theta.** This is also the **time decay**. Unlike other factors, however, the movement in remaining maturity is perfectly predictable. Time is not a risk factor.

For a European call, this is

$$\Theta_c = \frac{\partial c}{\partial t} = -\frac{\partial c}{\partial \tau} = -\frac{Se^{-r^*\tau}\sigma\Phi(d_1)}{2\sqrt{\tau}} + r^*Se^{-r^*\tau}N(d_1) - rKe^{-r\tau}N(d_2) \tag{13.17}$$

For a European put, this is

$$\Theta_p = \frac{\partial p}{\partial t} = -\frac{\partial p}{\partial \tau} = -\frac{Se^{-r^*\tau}\sigma\Phi(d_1)}{2\sqrt{\tau}} - r^*Se^{-r^*\tau}N(-d_1) + rKe^{-r\tau}N(-d_2) \tag{13.18}$$

Theta is generally negative for long positions in both calls and puts. This means that the option loses value as time goes by.

For American options, however, Θ is *always* negative. Because they give their holder the choice to exercise early, shorter-term American options are unambiguously less valuable than longer-term options.

Figure 13.7 displays the behavior of a call option theta for various prices of the underlying asset and maturities. For long positions in options, theta is negative,

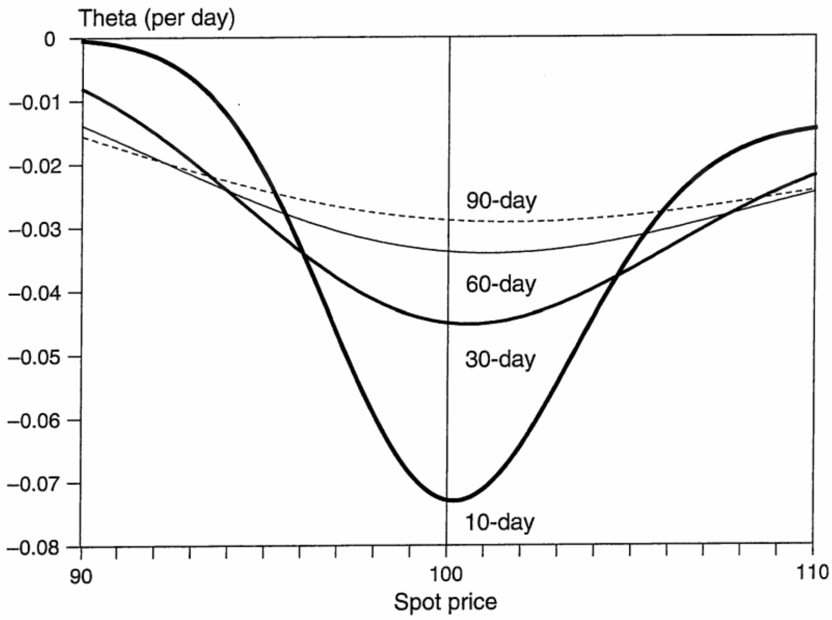

FIGURE 13.7 Option Theta

which reflects the fact that the option is a wasting asset. Like gamma, theta is greatest for short-term at-the-money options, when measured in absolute value. At-the-money options lose a great proportion of their value when the maturity is near.

13.2.5 Option Pricing and the "Greeks"

Having defined the option sensitivities, we can illustrate an alternative approach to the derivation of the Black–Scholes formula. Recall that the underlying process for the asset follows a stochastic process known as a **geometric Brownian motion** (GBM),

$$dS = \mu S dt + \sigma S dz \tag{13.19}$$

where dz has a normal distribution with mean zero and variance dt.

Considering only this *single* source of risk, we can return to the Taylor expansion in Equation (13.3). The value of the derivative is a function of S and time, which we can write as $f(S,t)$. The question is, how does f evolve over time?

We can relate the stochastic process of f to that of S using **Ito's lemma**, named after its creator. This can be viewed as an extension of the Taylor approximation to a stochastic environment. Applied to the GBM, this gives

$$df = \left(\tfrac{\partial f}{\partial S}\mu S + \tfrac{1}{2}\tfrac{\partial^2 f}{\partial S^2}\sigma^2 S^2 + \tfrac{\partial f}{\partial \tau}\right) dt + \left(\tfrac{\partial f}{\partial S}\sigma S\right) dz \tag{13.20}$$

This is also

$$df = (\Delta \mu S + \frac{1}{2}\Gamma\sigma^2 S^2 + \Theta)dt + (\Delta\sigma S)dz \tag{13.21}$$

The first term, including dt, is the trend. The second, including dz, is the stochastic component.

Next, we construct a portfolio delicately balanced between S and f that has no exposure to dz. Define this portfolio as

$$\Pi = f - \Delta S \tag{13.22}$$

Using (13.19) and (13.21), its stochastic process is

$$\begin{aligned} d\Pi &= [(\Delta\mu S + \tfrac{1}{2}\Gamma\sigma^2 S^2 + \Theta)dt + (\Delta\sigma S)dz] - \Delta[\mu S dt + \sigma S dz] \\ &= (\Delta\mu S)dt + (\tfrac{1}{2}\Gamma\sigma^2 S^2)dt + \Theta dt + (\Delta\sigma S)dz - (\Delta\mu S)dt - (\Delta\sigma S)dz \\ &= (\tfrac{1}{2}\Gamma\sigma^2 S^2 + \Theta)dt \end{aligned} \tag{13.23}$$

This simplification is extremely important. Note how the terms involving dz cancel out each other. The portfolio has been immunized against this source of risk. At the same time, the terms in μS also happened to cancel out each other.

The fact that μ disappears from the trend in the portfolio is important, as it explains why the trend of the underlying asset does not appear in the Black–Scholes formula.

Continuing, we note that the portfolio Π has no risk. To avoid arbitrage, it must return the risk-free rate:

$$d\Pi = [r\Pi]dt = r(f - \Delta S)dt \tag{13.24}$$

If the underlying asset has a dividend yield of y, this must be adjusted to

$$d\Pi = (r\Pi)dt + y\Delta S dt = r(f - \Delta S)dt + y\Delta S dt \tag{13.25}$$

Setting the trends in Equations (13.23) and (15.24) equal to each other, we must have

$$(r - y)\Delta S + \frac{1}{2}\Gamma\sigma^2 S^2 + \Theta = rf \tag{13.26}$$

This is the Black–Scholes **partial differential equation (PDE)**, which applies to any contract, or portfolio, that derives its value from S. The solution of this equation, with appropriate boundary conditions, leads to the BS formula for a European call, Equation (13.7).

We can use this relationship to understand how the sensitivities relate to each other. Consider a portfolio of derivatives, all on the same underlying asset, that is delta-hedged. Setting $\Delta = 0$ in Equation (13.26), we have

$$\frac{1}{2}\Gamma\sigma^2 S^2 + \Theta = rf \tag{13.27}$$

This shows that, for such portfolio, when Γ is large and positive, Θ must be negative if rf is small. In other words, a delta-hedged position with positive gamma, which is beneficial in terms of price risk, must have negative theta, or time decay. An example is the long straddle examined in Chapter 6. Such position is delta-neutral and has large gamma or convexity. It would benefit from a large move in S, whether up or down. This portfolio, however, involves buying options whose value decay very quickly with time. Thus, there is an intrinsic trade-off between Γ and Θ.

KEY CONCEPT

For delta-hedged portfolios, Γ and Θ must have opposite signs. Portfolios with positive convexity, for example, must experience time decay.

13.2.6 Option Sensitivities: Summary

We now summarize the sensitivities of option positions with some illustrative data in Table 13.1. Three strike prices are considered, $K = 90, 100$, and 110. We verify

TABLE 13.1 Derivatives for a European Call Parameters: $S = \$100, \sigma = 20\%, r = 5\%, y = 3\%, \tau = 3$ month

	Variable	Unit	Strike K = 90	K = 100	K = 110	Worst Loss Variable	Loss
c		Dollars	\$11.02	\$4.22	\$1.05		
		Change per:					
Δ	spot price	dollar	0.868	0.536	0.197	−\$2.08	−\$1.114
Γ	spot price	dollar	0.020	0.039	0.028	4.33	\$0.084
Λ	volatility	(% pa)	0.103	0.198	0.139	−2.5	−\$0.495
ρ	interest rate	(% pa)	0.191	0.124	0.047	−0.10	−\$0.013
ρ^*	asset yield	(% pa)	−0.220	−0.135	−0.049	0.10	−\$0.014
Θ	time	day	−0.014	−0.024	−0.016		

that the Γ, Λ, and Θ measures are all highest when the option is at-the-money ($K = 100$). Such options have the most nonlinear patterns.

The table also shows the loss for the worst daily movement in each risk factor at the 95% confidence level. For S, this is $dS = -1.645 \times 20\% \times \$100/\sqrt{252} = -\$2.08$. We combine this with delta, which gives a potential loss of $\Delta \times dS = -\$1.114$, or about a fourth of the option value.

Next, we examine the second order term, S^2. The worst squared daily movement is $dS^2 = 2.08^2 = 4.33$ in the risk factor at the 95% confidence level. We combine this with gamma, which gives a potential gain of $\frac{1}{2}\Gamma \times dS^2 = 0.5 \times 0.039 \times 4.33 = \$0.084$. Note that this is a gain because gamma is positive, but much smaller than the first-order effect. Thus the worst loss due to S would be $-\$1.114 + \$0.084 = -\$1.030$ using the linear and quadratic effects.

For σ, we observe a volatility of daily changes in σ on the order of 1.5%. The worst daily move is therefore $-1.645 \times 1.5 = -2.5$, expressed in percent, which gives a worst loss of $-\$0.495$. Finally, for r, we assume an annual volatility of changes in rates of 1%. The worst daily move is then $-1.645 \times 1/\sqrt{252} = -0.10$, in percent, which gives a worst loss of $-\$0.013$. So, most of the risk originates from S. In this case, a linear approximation using Δ only would capture most of the downside risk. For near-term at-the-money options, however, the quadratic effect is more important.

EXAMPLE 13.7: FRM EXAM 2004—QUESTION 65

Which of the following statements is *true* regarding options' Greeks?

a. Theta tends to be large and positive when buying at-the-money options.
b. Gamma is greatest for in-the-money options with long maturities.
c. Vega is greatest for at-the-money options with long maturities.
d. Delta of deep in-the-money put options tends towards +1.

EXAMPLE 13.8: FRM EXAM 2006—QUESTION 33

Steve, a market risk manager at Marcat Securities, is analyzing the risk of its S&P 500 index options trading desk. His risk report shows the desk is net long gamma and short vega. Which of the following portfolios of options shows exposures consistent with this report?

a. The desk has substantial long-expiry long call positions and substantial short-expiry short put positions.
b. The desk has substantial long-expiry long put positions and substantial long-expiry short call positions.
c. The desk has substantial long-expiry long call positions and substantial short-expiry short call positions.
d. The desk has substantial short-expiry long call positions and substantial long-expiry short call positions.

EXAMPLE 13.9: FRM EXAM 2006—QUESTION 54

Which of the following statements is *incorrect*?

a. The vega of a European call option is highest when the option is at-the-money.
b. The delta of a European-styled put option on an underlying stock moves toward zero as the price of the underlying stock rises.
c. The gamma of an at-the-money European-styled option tends to increase as the remaining maturity of the option decreases.
d. Compared to an at-the-money European-styled call option, an out-of-the-money European-styled option with the same strike price and remaining maturity has a greater negative value for theta.

EXAMPLE 13.10: FRM EXAM 2000—QUESTION 76

How can a trader produce a short vega, long gamma position?

a. Buy short-maturity options, sell long-maturity options.
b. Buy long-maturity options, sell short-maturity options.
c. Buy and sell options of long maturity.
d. Buy and sell options of short maturity.

EXAMPLE 13.11: FRM EXAM 2001—QUESTION 113

An option portfolio exhibits high unfavorable sensitivity to increases in implied volatility and while experiencing significant daily losses with the passage of time. Which strategy would the trader most likely employ to hedge his portfolio?

a. Sell short dated options and buy long dated options
b. Buy short dated options and sell long dated options
c. Sell short dated options and sell long dated options
d. Buy short dated options and buy long dated options

13.3 DYNAMIC HEDGING

The BS derivation taught us how to price and hedge options. Perhaps even more importantly, it showed that holding a call option is equivalent to holding a fraction of the underlying asset, where the fraction dynamically changes over time.

13.3.1 Delta and Dynamic Hedging

This equivalence is illustrated in Figure 13.8, which displays the current value of a call as a function of the current spot price. The long position in a call is replicated

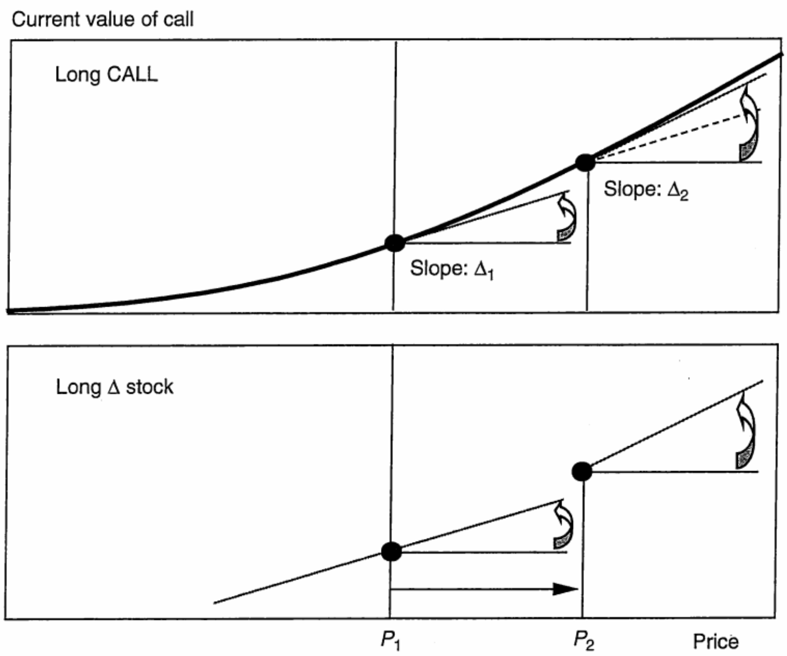

FIGURE 13.8 Dynamic Replication of a Call Option

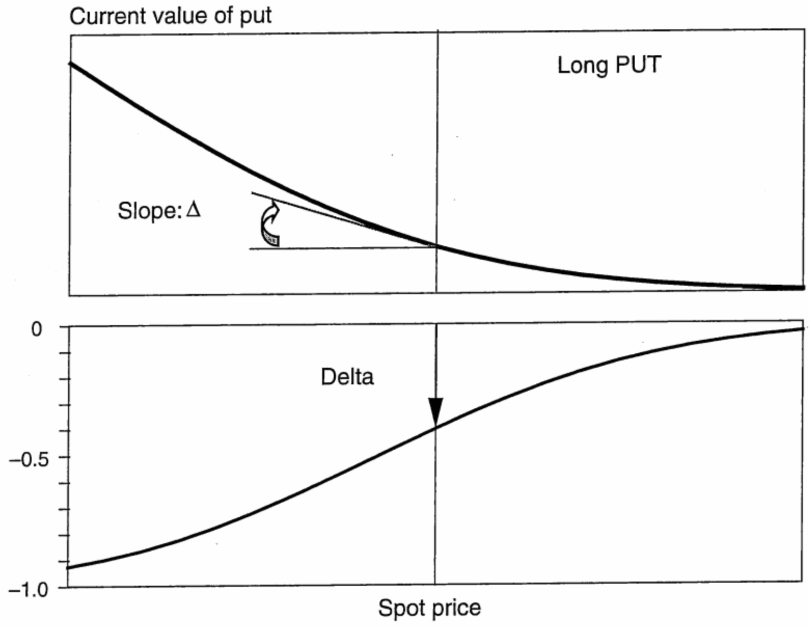

FIGURE 13.9 Dynamic Replication of a Put Option

by a partial position in the underlying asset. For an at-the-money position, the initial delta is about 0.5.

As the stock price increases from P_1 to P_2, the slope of the option curve, or delta, increases from Δ_1 to Δ_2. As a result, the option can be replicated by a larger position in the underlying asset. Conversely, when the stock price decreases, the size of the position is cut, as in a graduated stop-loss order. Thus the dynamic adjustment buys more of the asset as its price goes up, and conversely, sells it after a fall.

Figure 13.9 shows the dynamic replication of a put. We start at-the-money with Δ close to -0.5. As the price S goes up, Δ increases toward 0. Note that this is an increase since the initial delta was negative. As with the long call position, we *buy* more of the asset *after* its price has gone up. In contrast, short positions in calls and puts imply opposite patterns. Dynamic replication of a short option position implies buying more of the asset after its price has gone down.

13.3.2 Implications

For risk managers, these patterns are extremely important for a number of reasons. First, dynamic replication of a long option position is bound to lose money. This is because it buys the asset *after* the price has gone up—in other words, too late. Each transaction loses a small amount of money, which will accumulate precisely to the option premium.

A second point is that these automatic trading systems, if applied on large scale, have the potential to be destabilizing. Selling on a downturn in price can exacerbate the downside move. Some have argued that the crash of 1987 was due to the large-scale selling of portfolio insurers in a falling market. These portfolio insurers were, in effect, replicating a long position in puts, blindly selling when the market was falling.[4]

A third point is that this pattern of selling an asset after its price went down is similar to prudent risk-management practices. Typically, traders must cut down their positions after they incur large losses. This is similar to decreasing Δ when S drops. Thus, loss-limit policies bear some resemblance to a long position in an option.

Finally, the success of this replication strategy critically hinges on the assumption of a continuous GBM price process. With this process, it is theoretically possible to rebalance the portfolio as often as needed. In practice, the replication may fail if prices experience drastic jumps.

13.3.3 Distribution of Option Payoffs

Unlike linear derivatives such as forwards and futures, payoffs on options are intrinsically asymmetric. This is not necessarily because of the distribution of the underlying factor, which is often symmetric, but rather is due to the exposure profile. Long positions in options, whether calls or puts, have positive gamma, positive skewness, or long right tails. In contrast, short positions in options are short gamma and hence have negative skewness or long left tails. This is illustrated in Figure 13.10.

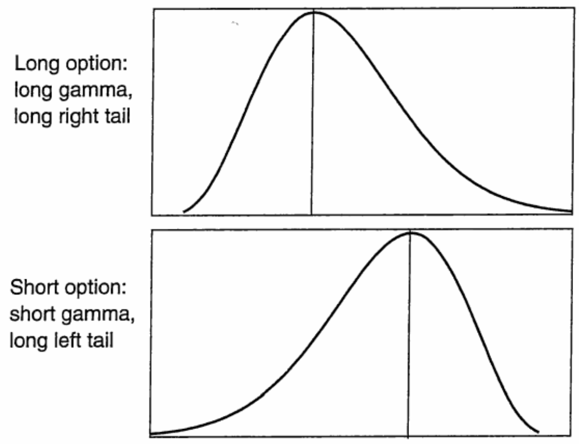

FIGURE 13.10 Distributions of Payoffs on Long and Short Options

[4]The exact role of portfolio insurance, however, is still hotly debated. Others have argued that the crash was aggravated by a breakdown in market structures, i.e., the additional uncertainty due to the inability of the stock exchanges to handle abnormal trading volumes.

We now summarize VAR formulas for simple option positions. Assuming a normal distribution, the VAR of the underlying asset is

$$\mathrm{VAR}(dS) = \alpha S\sigma(dS/S) \tag{13.28}$$

where α corresponds to the desired confidence level, e.g. $\alpha = 1.645$ for a 95% confidence level. The linear VAR for an option is

$$\mathrm{VAR}_1(dc) = \Delta \times \mathrm{VAR}(dS) \tag{13.29}$$

The quadratic VAR for an option is

$$\mathrm{VAR}_2(dc) = \Delta \times \mathrm{VAR}(dS) - \frac{1}{2}\Gamma \times \mathrm{VAR}(dS)^2 \tag{13.30}$$

Long option positions have positive gammas and hence lower risk than using a linear model. Conversely, negative gammas translate into higher VARs.

Lest we think that such options require sophisticated risk management methods, what matters is the *extent* of nonlinearity. Figure 13.11 illustrates the risk of a call option with a maturity of three months. It shows that the degree of nonlinearity also depends on the horizon. With a VAR horizon of two weeks, the range

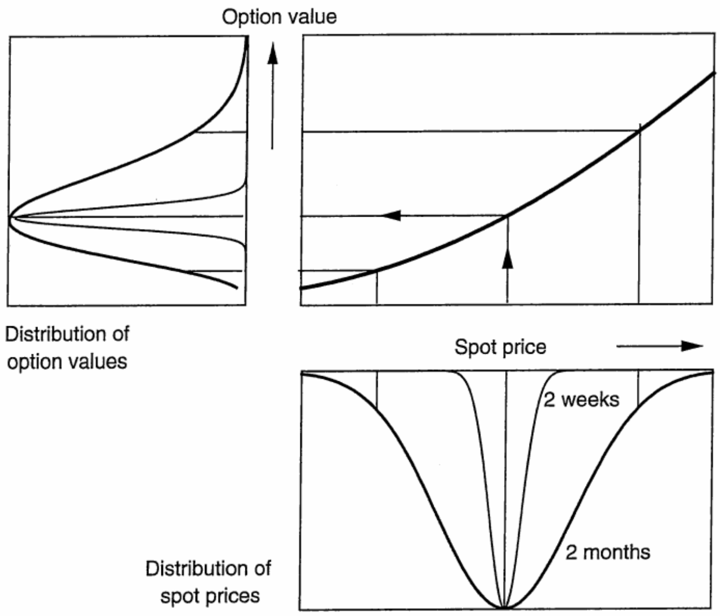

FIGURE 13.11 Skewness and VAR Horizon

of possible values for S is quite narrow. If S follows a normal distribution, the option value will be approximately normal. However, if the VAR horizon is set at two months, the nonlinearities in the exposure combine with the greater range of price movements to create a heavily skewed distribution.

So, for plain-vanilla options, the linear approximation may be adequate as long as the VAR horizon is kept short. For more exotic options, or longer VAR horizons, risk managers must account for nonlinearities.

EXAMPLE 13.12: FRM EXAM 2006—QUESTION 31

You are implementing a portfolio insurance strategy using index futures designed to protect the value of a portfolio of stocks not paying any dividends. Assuming the value of your stock portfolio decreases, which trade would you make to protect your portfolio?

a. Buy an amount of index futures equivalent to the change in the call delta times the original portfolio value.
b. Sell an amount of index futures equivalent to the change in the call delta times the original portfolio value.
c. Buy an amount of index futures equivalent to the change in the put delta times the original portfolio value.
d. Sell an amount of index futures equivalent to the change in the put delta times the original portfolio value.

EXAMPLE 13.13: FRM EXAM 2000—QUESTION 97

A trader buys an at-the-money call option with the intention of delta-hedging it to maturity. Which one of the following is likely to be the most profitable over the life of the option?

a. An increase in implied volatility
b. The underlying price steadily rising over the life of the option
c. The underlying price steadily decreasing over the life of the option
d. The underlying price drifting back and forth around the strike over the life of the option

EXAMPLE 13.14: FRM EXAM 2004—QUESTION 26

A non-dividend-paying stock has a current price of \$100 per share. You have just sold a six-month European call option contract on 100 shares of this stock at a strike price of \$101 per share. You want to implement a dynamic delta hedging scheme to hedge the risk of having sold the option. The option has a delta of 0.50. You believe that delta would fall to 0.44 if the stock price falls to \$99 per share. Identify what action you should take **now** (i.e., when you have just written the option contract) to make your position delta neutral. After the option is written, if the stock price falls to \$99 per share, identify what action should be taken at that time, i.e., **later**, to rebalance your delta-hedged position.

a. **Now**: buy 50 shares of stock; **later**: buy 6 shares of stock.
b. **Now**: buy 50 shares of stock; **later**: sell 6 shares of stock.
c. **Now**: sell 50 shares of stock; **later**: buy 6 shares of stock.
d. **Now**: sell 50 shares of stock; **later**: sell 6 shares of stock.

EXAMPLE 13.15: FRM EXAM 2005—QUESTION 130

An option on the Bovespa stock index is struck on 3,000 Brazilian Real (BRL). The delta of the option is 0.6, and the annual volatility of the index is 24%. Using delta-normal assumptions, what is the 10-day VAR at the 95% confidence level? Assume 260 days per year.

a. 44 BRL
b. 139 BRL
c. 2240 BRL
d. 278 BRL

13.4 IMPORTANT FORMULAS

Black–Scholes option pricing model: $c = Se^{-r^*\tau}N(d_1) - Ke^{-r\tau}N(d_2)$

Taylor series expansion:

$$df = \frac{\partial f}{\partial S}dS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}dS^2 + \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial r^*}dr^* + \frac{\partial f}{\partial \sigma}d\sigma + \frac{\partial f}{\partial \tau}d\tau + \cdots$$

$$df = \Delta dS + \frac{1}{2}\Gamma dS^2 + \rho dr + \rho^* dr^* + \Lambda d\sigma + \Theta d\tau + \cdots$$

Delta: $\Delta_c = \frac{\partial c}{\partial S} = e^{-r^*\tau}N(d_1)$, $\Delta_p = \frac{\partial p}{\partial S} = e^{-r^*\tau}[N(d_1) - 1]$

Gamma (for calls and puts): $\Gamma = \frac{\partial^2 c}{\partial S^2} = \frac{e^{-r^*\tau}}{S\sigma\sqrt{\tau}}\Phi(d_1)$
Vega (for calls and puts): $\Lambda = \frac{\partial c}{\partial \sigma} = Se^{-r^*\tau}\sqrt{\tau}\ \Phi(d_1)$

	Long Call			Long Put		
	OTM	ATM	ITM	ITM	ATM	OTM
Δ	$\to 0$	0.5	$\to 1$	$\to -1$	-0.5	$\to 0$
Γ	Low	High, > 0 esp. short-term	Low	Low	High, > 0 esp. short-term	Low
Λ	Low	High, > 0 esp. long-term	Low	Low	High, > 0 esp. long-term	Low
Θ	Low	High, < 0 esp. short-term	Low	Low	High, < 0 esp. short-term	Low

Black–Scholes PDE: $(r - y)\Delta S + \frac{1}{2}\Gamma\sigma^2 S^2 + \Theta = rf$
Linear VAR for an option: $\text{VAR}_1(dc) = \Delta \times \text{VAR}(dS)$
Quadratic VAR for an option: $\text{VAR}_2(dc) = \Delta \times \text{VAR}(dS) - \frac{1}{2}\Gamma \times \text{VAR}(dS)^2$

13.5 ANSWERS TO CHAPTER EXAMPLES

Example 13.1: FRM Exam 2006—Question 91

a. The delta of a long forward contract is $e^{-r^*\tau} = \exp(-0.10 \times 0.5) = 0.95$.

Example 13.2: FRM Exam 2004—Question 21

b. The option is ATM because the strike price is close to the spot price. This is a put, so the delta must be close to -0.5.

Example 13.3: FRM Exam 2006—Question 80

c. This is a call option, so delta must be positive. This is given by $\Delta = \exp(-r^*\tau)$ $N(d_1) = \exp(-0.01 \times 2) \times 0.64 = 0.63$.

Example 13.4: FRM Exam 2003—Question 94

a. Gamma is highest for short-term ATM options. The first answer has a strike price close to $S = 78$ and short maturity.

Example 13.5: FRM Exam 2001—Question 79

a. This is an at-the-money option with a delta of about 0.5. Since the bank sold calls, it needs to delta-hedge by buying the shares. With a delta of 0.54, it would need to buy approximately 50,000 shares. Answer a. is the closest. Note that most other information is superfluous.

Example 13.6: FRM Exam 2006—Question 106

a. Because gamma is negative, we need to buy a call to increase the portfolio gamma back to zero. The number is $600/1.5 = 400$ calls. This, however, will increase the delta from zero to $400 \times 0.75 = 300$. Hence, we must sell 300 shares to bring back the delta to zero. Note that positions in shares have zero gamma.

Example 13.7: FRM Exam 2004—Question 65

c. Theta is negative for long positions in ATM options, so a. is incorrect. Gamma is small for ITM options, so b. is incorrect. Delta of ITM puts tends to -1, so d. is incorrect.

Example 13.8: FRM Exam 2006—Question 33

d. Long gamma means that the portfolio is long options with high gamma, typically short-term (short-expiry) ATM options. Short vega means that the portfolio is short options with high vega, typically long-term (long-expiry) ATM options.

Example 13.9: FRM Exam 2006—Question 54

d. Vega is highest for ATM European options, so answer a. is correct. Delta is negative and moves to zero as S increases, so answer b. is correct. Gamma increases as the maturity of an ATM option decreases, so answer c. is correct. Theta is greater (in absolute value) for short-term ATM options, so statement d. is incorrect.

Example 13.10: FRM Exam 2000—Question 76

a. Long positions in options have positive gamma and vega. Gamma (or instability in delta) increases near maturity; vega decreases near maturity. So, to obtain positive gamma and negative vega, we need to buy short-maturity options and sell long-maturity options.

Example 13.11: FRM Exam 2001—Question 113

a. Such a portfolio is short vega (volatility) and short theta (time). We need to implement a hedge that is delta-neutral and involves buying and selling options with different maturities. Long positions in short-dated options have high negative theta and low positive vega. Hedging can be achieved by selling short-term options and buying long-term options.

Example 13.12: FRM Exam 2006—Question 31

d. Portfolio insurance is a form of dynamic hedging that replicates a long position in a put option. If the value of the portfolio decreases, one should sell the index futures in the amount that represents the change in the put delta.

Example 13.13: FRM Exam 2000—Question 97

d. An important aspect of the question is the fact that the option is held to maturity. Answer a. is incorrect because changes in the implied volatility would change the value of the option, but this has no effect when holding to maturity. The profit from the dynamic portfolio will depend on whether the actual volatility differs from the initial implied volatility. It does not depend on whether the option ends up in-the-money or not, so answers b. and c. are incorrect. The portfolio will be profitable if the actual volatility is small, which implies small moves around the strike price.

Example 13.14: FRM Exam 2004—Question 26

b. The dynamic hedge should replicate a long position in the call. Due to the positive delta, this implies a long position of $\Delta \times 100 = 50$ shares. If the delta falls, the position needs to be adjusted by selling $(0.5 - 0.44) \times 100 = 6$ shares.

Example 13.15: FRM Exam 2005—Question 130

b. The linear VAR is derived from the worst move in the index value, which is $\alpha S\sigma\sqrt{T} = 1.645 \times 3{,}000(24\%/\sqrt{260})\sqrt{10} = 232.3$. Multiplying by the delta of 0.6 gives 139.

CHAPTER 14

Modeling Risk Factors

We now turn to an analysis of the distribution of risk factors used in risk measurement. A previous chapter has described the major risk factors, including fixed-income, equity, currency, and commodity price risk. The emphasis was on the volatility as a measure of dispersion. More generally, risk managers need to consider the whole shape of the distribution, which is not necessarily normal, as well as potential time variation in this distribution.

In fact, most financial time series are characterized by fatter tails than the normal distribution. In addition, there is ample empirical evidence that, over short horizons, risk changes in a predictable fashion. This time variation could potentially explain the observed high frequency of extreme observations, which could be drawn from distributions with temporarily higher volatility.

Section 14.1 starts by describing the normal distribution. We compare the normal and lognormal distributions and explain why this choice is so popular. A major failing of this distribution, however, is its inability to represent the frequency of large observations found in financial data. Section 14.2 discusses other distributions that have fatter tails than the normal.

Section 14.3 then turns to time-variation in risk. We describe the generalized autoregressive conditional heteroskedastic (GARCH) model and a special case, which is RiskMetrics' exponentially weighted moving average (EWMA).

14.1 NORMAL AND LOGNORMAL DISTRIBUTIONS

14.1.1 Why the Normal?

The normal, or Gaussian, distribution is usually the first choice when modeling asset returns. This distribution plays a special role in statistics, as it is easy to handle and is stable under addition, meaning that a sum of normal variables is itself normal. It also provides the limiting distribution of the average of *independent* random variables (through the central limit theorem).

Empirically, the normal distribution provides a rough, first-order approximation to the distribution of many random variables: rates of changes in currency prices, rates of changes in stock prices, rates of changes in bond prices, changes in yields, and rates of changes in commodity prices. All of these are characterized by many cases of small moves and fewer cases of large moves, which provides a rationale for the bell-shaped, normal distribution.

14.1.2 Computing Returns

In what follows, the random variable is the new price P_1, given the current price P_0. Defining $r = (P_1 - P_0)/P_0$ as the rate of return in the price, we can start with the assumption is that this random variable is drawn from a normal distribution,

$$r \sim \Phi(\mu, \sigma) \tag{14.1}$$

with some mean μ and standard deviation σ. Turning to prices, we have $P_1 = P_0(1 + r)$ and

$$P_1 \sim P_0 + \Phi(P_0\mu, P_0\sigma) \tag{14.2}$$

For instance, starting from a stock price of \$100, if $\mu = 0\%$ and $\sigma = 15\%$, we have $P_1 \sim \$100 + \Phi(\$0, \$15)$.

In this case, however, the normal distribution cannot even be theoretically correct. Because of limited liability, stock prices cannot go below zero. Similarly, commodity prices and yields cannot turn negative. This is why another popular distribution is the **lognormal distribution**, which is such that

$$R = \ln(P_1/P_0) \sim \Phi(\mu, \sigma) \tag{14.3}$$

By taking the logarithm, the price is given by $P_1 = P_0 \exp(R)$, which precludes prices from turning negative as the exponential function is always positive. Figure 14.1 compares the normal and lognormal distributions over a one-year

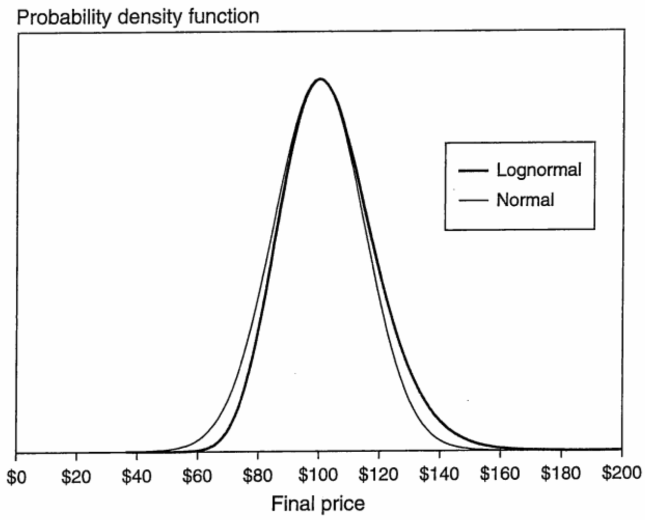

FIGURE 14.1 Normal and Lognormal Distributions—Annual Horizon

horizon with $\sigma = 15\%$ annually. The distributions are very similar, except for the tails. The lognormal is skewed to the right.

The difference between the two distributions is driven by the size of the volatility parameter over the horizon. Small values of this parameter imply that the distributions are virtually identical. This can happen either when the asset is not very risky, that is, when the annual volatility is small, or when the horizon is very short. In this situation, there is very little chance of prices turning negative. The limited liability constraint is not important.

KEY CONCEPT

The normal and lognormal distributions are very similar for short horizons or low volatilities.

As an example, Table 14.1 compares the computation of returns over a one-day and one-year horizon. The one-day returns are 1.000% and 0.995% for discrete and log-returns, respectively, which translates into a relative difference of 0.5%, which is minor. In contrast, the difference is more significant over longer horizons.

14.1.3 Time Aggregation

Longer horizons can be accommodated assuming a constant lognormal distribution across horizons. Over two periods, for instance, the price movement can be described as the sum of the price movements over each day:

$$R_{t,2} = \ln(P_t/P_{t-2}) = \ln(P_t/P_{t-1}) + \ln(P_{t-1}/P_{t-2}) = R_{t-1} + R_t \tag{14.4}$$

More generally, define T as the number of steps. If returns are identically and independently distributed (i.i.d.), the variance of multiple-period returns is

$$V[R(0,T)] = V[R(0,1)] + V[R(1,2)] + \cdots + V[R(T-1,T)] = V[R(0,1)]T \tag{14.5}$$

TABLE 14.1 Comparison between Discrete and Log Returns

	Daily	Annual
Initial Price	100	100
Ending Price	101	115
Discrete Return	1.0000	15.0000
Log Return	0.9950	13.9762
Relative Difference	0.50%	7.33%

since the variances are all the same and all the covariance terms are zero because of the independence assumption. Similarly, the mean of multiple-period returns is

$$E[R(0,T)] = E[R(0,1)] + E[R(1,2)] + \cdots + E[R(T-1,T)] = E[R(0,1)]T \tag{14.6}$$

assuming expected returns are the same for each day.

Thus the multiple-period volatility is

$$\sigma_T = \sigma\sqrt{T} \tag{14.7}$$

If the distribution is stable under addition, i.e., we can use the same multiplier for a one-period and T-period return, we have a multiple-period VAR of

$$\mathrm{VAR} = \alpha(\sigma\sqrt{T})W \tag{14.8}$$

In other words, extension to a multiple period follows a square root of time rule. Figure 14.2 shows how VAR grows with the length of the horizon and for various confidence levels. This is scaled to an annual standard deviation of 1, which is a 84.1% VAR. The figure shows that VAR increases more slowly than time. The one-month 99% VAR is 0.67, but increases only to 2.33 at a one-year horizon.

In summary, the square root of time rule applies under the following conditions:

- The distribution is the same at each period (i.e., there is no predictable time variation in expected return nor in risk).
- Returns are uncorrelated/independent across each period, so that all covariances terms disappear.
- The distribution is the same for one- or T-period, or is stable under addition, such as the normal.

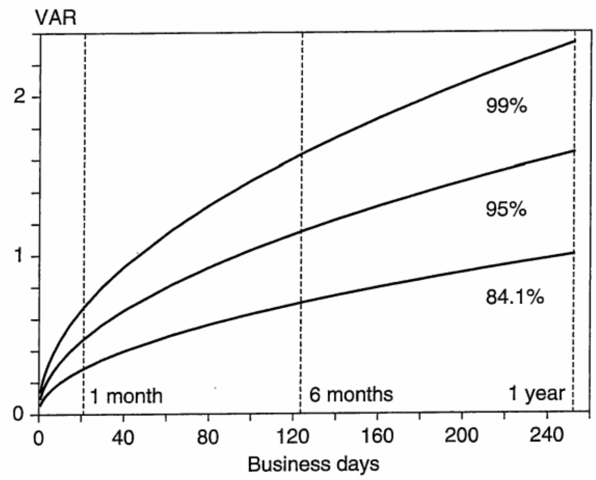

FIGURE 14.2 VAR at Increasing Horizons

If returns are not independent, we may be able to characterize the risk in some cases. For instance, when returns follow a first-order autoregressive process,

$$R_t = \rho R_{t-1} + u_t \tag{14.9}$$

we can write the variance of two-day returns as

$$V[R_t + R_{t-1}] = V[R_t] + V[R_{t-1}] + 2\mathrm{Cov}[R_t, R_{t-1}] = \sigma^2 + \sigma^2 + 2\rho\sigma^2 \tag{14.10}$$

or

$$V[R_t + R_{t-1}] = \sigma^2 \times 2[1 + \rho] \tag{14.11}$$

A positive value for ρ describes a situation where a movement in one direction is likely to be followed by another in the same direction. This implies that markets are trending. In this case, the longer-term volatility increases faster than with the usual square root of time rule.

On the other hand, a negative value for ρ describes a situation where a movement in one direction is likely to be reversed later. In this mean-reversion case, the longer-term volatility increases more slowly than with the usual square root of time rule.

EXAMPLE 14.1: TIME SCALING

Consider a portfolio with a one-day VAR of \$1 million. Assume that the market is trending with an autocorrelation of 0.1. Under this scenario, what would you expect the two-day VAR to be?

a. \$2 million
b. \$1.414 million
c. \$1.483 million
d. \$1.449 million

14.2 FAT TAILS

Perhaps the most serious problem with the normal distribution is the fact that its tails "disappear" too fast, at least faster than what is empirically observed in financial data. We typically observe that every market experiences one or more daily moves of 4 standard deviations or more per year. Such frequency is incompatible with a normal distribution. With a normal distribution, the probability of this happening is 0.0032% for one day, which implies a frequency of once every 125 years.

KEY RULE OF THUMB

Every financial market experiences one or more daily price moves of 4 standard deviations or more each year. And in any year, there is usually at least one market that has a daily move greater than 10 standard deviations.

This empirical observation can be explained in a number of ways: (1) The true distribution has fatter tails (e.g., the Student's t), or (2) the observations are drawn from a mix of distributions (e.g., a mix of two normals, one with low risk, the other with high risk), or (3) the distribution is nonstationary.

The first explanation is certainly a possibility. Figure 14.3 displays the density function of the normal and Student's t distribution, with four and six degrees of freedom (df). The student density has fatter tails, which better reflect the occurrences of extreme observations in empirical financial data.

The distributions are further compared in Table 14.2. The left-side panel reports the tail probability of an observation lower than the deviate. For instance, the probability of observing a draw less than -3 is 0.001, or 0.1% for the normal, 0.012 for the Student's t with six degrees of freedom, and 0.020 for the Student's t with four degrees of freedom. There is a greater probability of observing an extreme move when the data is drawn from a Student's t rather than a normal distribution.

We can transform these into an expected number of occurrences in one year, or 250 business days. The right-side panel shows that the corresponding numbers are 0.34, 3.00, and 4.99 for the respective distributions. In other words, with a normal distribution, we should expect that this extreme movement below $z = -3$ will occur on average one day or less. With a Student's t with df $= 4$, the expected number is five in a year, which is closer to reality.

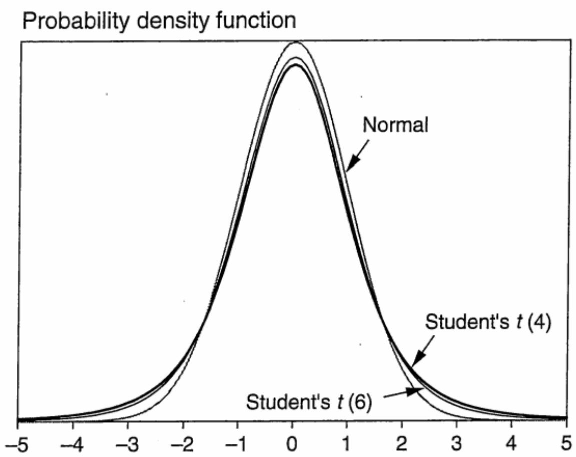

FIGURE 14.3 Normal and Student Distributions

TABLE 14.2 Comparison of the Normal and Student's t Distributions

	Tail probability			Expected Number in 250 days		
Deviate	Normal	t df = 6	t df = 4	Normal	t df = 6	t df = 4
−5	0.00000	0.00123	0.00375	0.00	0.31	0.94
−4	0.00003	0.00356	0.00807	0.01	0.89	2.02
−3	0.00135	0.01200	0.01997	0.34	3.00	4.99
−2	0.02275	0.04621	0.05806	5.69	11.55	14.51
−1	0.15866	0.17796	0.18695	39.66	44.49	46.74
					Deviate (alpha)	
Probability = 1%				2.33	3.14	3.75
Ratio to normal				1.00	1.35	1.61

The bottom panel reports the deviate that corresponds to a 99% right-tail confidence level, or 1% left tail. For the normal distribution, this is the usual 2.33. For the Student's t with df $= 4$, α is 3.75, much higher. The ratio of the two is 1.61. Thus a rule of thumb would be to correct the VAR measure from a normal distribution by a ratio of 1.61 to achieve the desired coverage in the presence of fat tails. More generally, this explains why "safety factors" are used to multiply VAR measures, such as the Basel multiplicative factor of three.

14.3 TIME-VARIATION IN RISK

An alternative class of explanation is that financial data can be viewed as drawn from a normal distribution with time-varying parameters. This is useful only if this time variation has some predictability.

14.3.1 GARCH

A specification that has proved quite successful in practice is the **generalized autoregressive conditional heteroskedastic** (GARCH) model developed by Engle (1982) and Bollerslev (1986).

This class of models assumes that the return at time t has a normal distribution, for example, conditional on parameters μ_t and σ_t:

$$r_t \sim \Phi(\mu_t, \sigma_t) \tag{14.12}$$

The important point is that σ is indexed by time. In this context, we define the **conditional variance** as that conditional on current information. This may differ from the **unconditional variance**, which is the same for the whole sample. Thus the average variance is unconditional, whereas a time-varying variance is conditional.

There is substantial empirical evidence that conditional volatility models successfully forecast risk. The general assumption is that the conditional returns have a normal distribution, although this could be extended to other distributions such as the Student's t.

The GARCH model assumes that the conditional variance depends on the latest innovation, and on the previous conditional variance. Define $h_t = \sigma_t^2$ as the conditional variance, using information up to time $t-1$, and r_{t-1} as the previous day's return, also called innovation. The simplest such model is the GARCH(1,1) process,

$$h_t = \alpha_0 + \alpha_1 r_{t-1}^2 + \beta h_{t-1} \tag{14.13}$$

which involves one lag of the innovation and one lag of the previous forecast. The β term is important because it allows persistence in the variance, which is a realistic feature of the data.

The average, unconditional variance is found by setting $E[r_{t-1}^2] = h_t = h_{t-1} = h$. Solving for h, we find

$$h = \frac{\alpha_0}{1 - \alpha_1 - \beta} \tag{14.14}$$

This model is stationary when the sum of parameters $\gamma = \alpha_1 + \beta$ are less than unity. This sum is also called the **Persistence**, as it defines the speed at which shocks to the variance revert to their long-run values.

To understand how the process works, consider Table 14.3. The parameters are $\alpha_0 = 0.01$, $\alpha_1 = 0.03$, $\beta = 0.95$. The unconditional variance is $0.01/(1 - 0.03 - 0.95) = 0.7$ daily, which is typical of a currency series, as it translates into an annualized volatility of 11%. The process is stationary because $\alpha_1 + \beta = 0.98 < 1$.

At time 0, we start with the variance at $h_0 = 1.1$ (expressed in percent squared). The conditional volatility is $\sqrt{h_0} = 1.05\%$. The next day, there is a large return of 3%. The new variance forecast is then $h_1 = 0.01 + 0.03 \times 3^2 + 0.95 \times 1.1 = 1.32$. The conditional volatility just went up to 1.15%. If nothing happens the following days, the next variance forecast is $h_2 = 0.01 + 0.03 \times 0^2 + 0.95 \times 1.32 = 1.27$. And so on.

TABLE 14.3 Building a GARCH Forecast

Time	Return	Conditional Variance	Conditional Risk	Conditional 95% Limit
$t-1$	r_{t-1}	h_t	$\sqrt{h_t}$	$2\sqrt{h_t}$
0	0.0	1.10	1.05	±2.10
1	3.0	1.32	1.15	±2.30
2	0.0	1.27	1.13	±2.25
3	0.0	1.22	1.10	±2.20

The GARCH process can be extrapolated to later days. For the next-day forecast,

$$E_{t-1}(r_{t+1}^2) = \alpha_0 + \alpha_1 E_{t-1}(r_t^2) + \beta h_t = \alpha_0 + \alpha_1 h_t + \beta h_t = \alpha_0 + \gamma h_t$$

For the following day,

$$E_{t-1}(r_{t+2}^2) = \alpha_0 + \alpha_1 E_{t-1}(r_{t+1}^2) + \beta E_{t-1}(h_{t+1}) = \alpha_0 + (\alpha_1 + \beta) E_{t-1}(r_{t+1}^2)$$
$$E_{t-1}(r_{t+2}^2) = \alpha_0 + \gamma(\alpha_0 + \gamma h_t)$$

Generally,

$$E_{t-1}(r_{t+n}^2) = \alpha_0(1 + \gamma + \gamma^2 + \cdots + \gamma^{n-1}) + \gamma^n h_t$$

Figure 14.4 illustrates the dynamics of shocks to a GARCH process for various values of the persistence parameter. As the conditional variance deviates from the starting value, it slowly reverts to the long-run value at a speed determined by $\alpha_1 + \beta$.

Note that these are forecasts of one-day variances at forward points in time. The total variance over the horizon is the sum of one-day variances. The *average* variance is marked with a black rectangle on the graph.

The graph also shows why the square root of time rule for extrapolating returns does not apply when risk is time-varying. If the initial value of the variance is greater than the long-run average, simply extrapolating the one-day variance to a longer horizon will overstate the average variance. Conversely, starting from a lower value and applying the square root of time rule will understate risk.

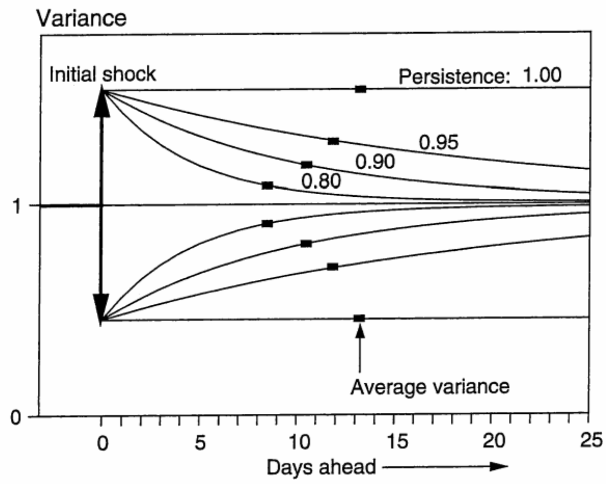

FIGURE 14.4 Shocks to a GARCH Process

KEY CONCEPT

The square root of time rule used to scale one-day returns into longer horizons is generally inappropriate when risk is time-varying.

EXAMPLE 14.2: FRM EXAM 2006—QUESTION 36

Which of the following GARCH models will take the shortest time to revert to its long-run value?

a. $h_t = 0.05 + 0.03r_{t-1}^2 + 0.96h_{t-1}$
b. $h_t = 0.03 + 0.02r_{t-1}^2 + 0.95h_{t-1}$
c. $h_t = 0.02 + 0.01r_{t-1}^2 + 0.97h_{t-1}$
d. $h_t = 0.01 + 0.01r_{t-1}^2 + 0.98h_{t-1}$

EXAMPLE 14.3: FRM EXAM 2006—QUESTION 132

Assume you are using a GARCH model to forecast volatility that you use to calculate the one-day VAR. If volatility is mean reverting, what can you say about the T-day VAR?

a. It is less than the $\sqrt{T}\times$ one-day VAR.
b. It is equal to $\sqrt{T}\times$ one-day VAR.
c. It is greater than the $\sqrt{T}\times$ one-day VAR.
d. It could be greater or less than the $\sqrt{T}\times$ one-day VAR.

EXAMPLE 14.4: FRM EXAM 2007—QUESTION 34

A risk manager estimates daily variance h_t using a GARCH model on daily returns r_t: $h_t = \alpha_0 + \alpha_1 r_{t-1}^2 + \beta h_{t-1}$, with $\alpha_0 = 0.005$, $\alpha_1 = 0.04$, $\beta = 0.94$. The long-run *annualized* volatility is approximately

a. 13.54%
b. 7.94%
c. 72.72%
d. 25.00%

14.3.2 EWMA

The RiskMetrics approach is a specific case of the GARCH process and is particularly simple and convenient to use. Variances are modeled using an **exponentially weighted moving average (EWMA)** forecast. The forecast is a weighted average of the previous forecast, with weight λ, and of the latest squared innovation, with weight $(1-\lambda)$:

$$h_t = \lambda h_{t-1} + (1-\lambda) r_{t-1}^2 \tag{14.15}$$

The λ parameter, also called the **decay factor**, determines the relative weights placed on previous observations. The EWMA model places geometrically declining weights on past observations, assigning greater importance to recent observations. By recursively replacing h_{t-1} in Equation (14.15), we have

$$h_t = (1-\lambda)[r_{t-1}^2 + \lambda r_{t-2}^2 + \lambda^2 r_{t-3}^2 + \cdots] \tag{14.16}$$

The weights therefore decrease at a geometric rate. The lower λ, the more quickly older observations are forgotten. RiskMetrics has chosen $\lambda = 0.94$ for daily data and $\lambda = 0.97$ for monthly data.

Table 14.4 shows how to build the EWMA forecast using a parameter of $\lambda = 0.95$, which is consistent with the previous GARCH example. At time 0, we start with the variance at $h_0 = 1.1$, as before. The next day, we have a return of 3%. The new variance forecast is then $h_1 = 0.05 \times 3^2 + 0.95 \times 1.1 = 1.50$. The next day, this moves to $h_2 = 0.05 \times 0^2 + 0.95 \times 1.50 = 1.42$. And so on.

This model is a special case of the GARCH process, where α_0 is set to 0, and α_1 and β sum to unity. The model therefore has permanent persistence. Shocks to the volatility do not decay, as shown in Figure 14.4 when the persistence is 1.00. Thus longer-term extrapolation from the GARCH and EWMA models may give quite different forecasts. Over a one-day horizon, however, the two models are quite similar and often indistinguishable from each other.

Figure 14.5 displays the pattern of weights for previous observations. With $\lambda = 0.94$, the weights decay quickly. The weight on the last day is $(1-\lambda) = (1-0.94) = 0.06$. The weight on the previous day is $(1-\lambda)\lambda = 0.0564$, and so on. The weight drops below 0.00012 for data more than 100 days old. With $\lambda = 0.97$, the weights start at a lower level but decay more slowly. In comparison, moving average (MA) models have a fixed window, with equal weights within

TABLE 14.4 Building an EWMA Forecast

Time	Return	Conditional Variance	Conditional Risk	Conditional 95% Limit
$t-1$	r_{t-1}	h_t	$\sqrt{h_t}$	$2\sqrt{h_t}$
0	0.0	1.10	1.05	±2.1
1	3.0	1.50	1.22	±2.4
2	0.0	1.42	1.19	±2.4
3	0.0	1.35	1.16	±2.3

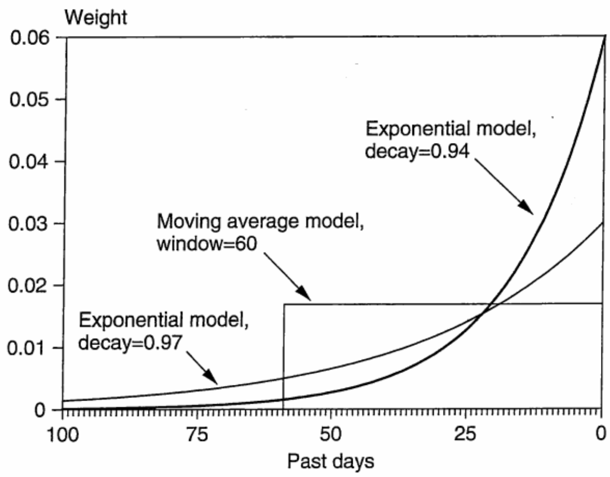

FIGURE 14.5 Weights on Past Observations

the window but otherwise zero. MA models with shorter windows give a greater weight to recent observations. As a result, they are more responsive to current events and more volatile.

EXAMPLE 14.5: FRM EXAM 2002—QUESTION 13

The GARCH model is useful for simulating asset returns. Which of the following statements about this model is *false*?

a. The Exponentially Weighted Moving Average (EWMA) approach of RiskMetrics is a particular case of a GARCH process.
b. The GARCH allows for time-varying volatility.
c. The GARCH can produce fat tails in the return distribution.
d. The GARCH imposes a positive conditional mean return.

EXAMPLE 14.6: FRM EXAM 2007—QUESTION 46

A bank uses the Exponentially Weighted Moving Average (EWMA) technique with λ of 0.9 to model the daily volatility of a security. The current estimate of the daily volatility is 1.5%. The closing price of the security is USD 20 yesterday and USD 18 today. Using continuously compounded returns, what is the updated estimate of the volatility?

a. 3.62%
b. 1.31%
c. 2.96%
d. 5.44%

EXAMPLE 14.7: FRM EXAM 2006—QUESTION 40

Using a daily RiskMetrics EWMA model with a decay factor $\lambda = 0.95$ to develop a forecast of the conditional variance, which weight will be applied to the return that is four days old?

a. 0.000
b. 0.043
c. 0.048
d. 0.950

EXAMPLE 14.8: EFFECT OF WEIGHTS ON OBSERVATIONS

Until January 1999 the historical volatility for the Brazilian real versus the U.S. dollar had been very small for several years. On January 13, Brazil abandoned the defense of the currency peg. Using the data from the close of business on January 13, which of the following methods for calculating volatility would have shown the greatest jump in measured historical volatility?

a. 250 day equal weight
b. Exponentially weighted with a daily decay factor of 0.94
c. 60 day equal weight
d. All of the above

14.3.3 Option Data

All the previous models were based on historical data. Although conditional volatility models are a substantial improvement over models that assume constant risk, they are always, by definition, one step too late.

These models start to react *after* a big shock has occurred. In many situations, this may be too late—hence the quest for forward-looking risk measures.

Such forward-looking measures are contained in option implied standard deviations (ISD). ISD are obtained by, first, assuming an option pricing model and, next, inverting the model, that is, solving for the parameter that will make the model price equal to the observed market price.

Define $f(\cdot)$ as an option pricing function, such as the Black–Scholes model for European options. Normally, we input σ into f along with other parameters and then solve for the option price. However, if the market trades these options and if all the other inputs are observable, we can recover σ_{ISD} by setting the model price equal to the market price:

$$c_{\mathrm{MARKET}} = f(\sigma_{\mathrm{ISD}}) \tag{14.17}$$

This assumes that the model fits the data perfectly, which may not be the case for out-of-the-money options. Hence, this method works best for short-term (two weeks to three months) at-the-money options.

This approach can even be generalized to implied correlations. For this, we need triplets of options, such as \$/yen, \$/euro, yen/euro. The first one can be used to recover σ_1, the second σ_2, and the third the covariance σ_{12}, from which the implied correlation ρ_{12} can be recovered.

There is much empirical evidence that ISD provide superior forecasts of future risk. This is expected, as the essence of option trading is to place volatility bets. The main drawback of these methods is that risk measures recovered from market prices are defined in a risk-neutral space. For forecasting risk, we need actual, real-world, **physical distributions**. Implied volatility may be systematically higher than forecast volatility due to a risk premium. If this risk premium is stable, however, changes in ISDs should prove informative for predicting changes in risk.

In practice, while historical time-series models can be applied systematically to all series for which we have data, we do not have actively traded options for all risk factors. In addition, we have even fewer combinations of options that permit us to compute implied correlations. Thus, it is difficult to integrate ISD with time-series models.

14.3.4 Implied Distributions

Options can be used to derive more information about future distributions than the volatility alone. Recently, option watchers have observed some inconsistencies in the pricing of options, especially for stock index options. In particular, options that differ only by their strike prices are characterized by different ISDs. Options that are out-of-the-money have higher ISDs than at-the-money options. This has become known as the **smile effect** in ISDs, which is shown in Figure 14.6, which

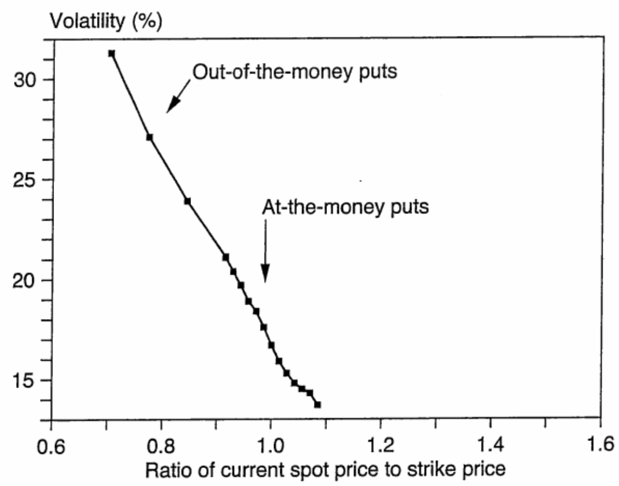

FIGURE 14.6 Smile Effect

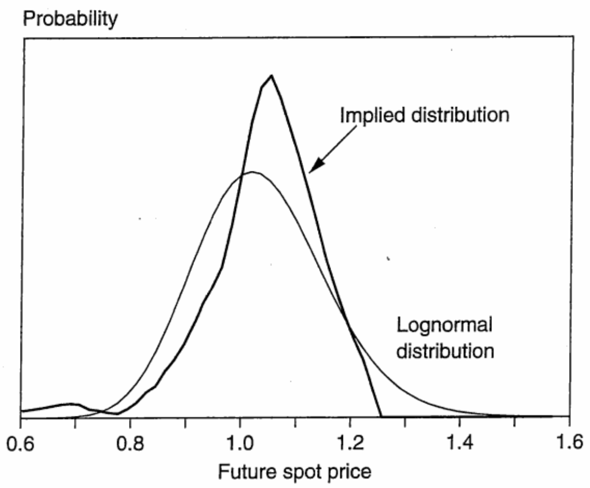

FIGURE 14.7 Implied Distribution

plots equity ISDs against the ratio of the strike price over the current spot price. In this case, the smile is totally asymmetric (more like a smirk).

Low values of the ratio, describing out-of-the-money puts, are associated with high ISDs. In other words, out-of-the-money puts appear overpriced relative to others. Different ISDs are clearly inconsistent with the joint assumption of a lognormal distribution for prices and efficient markets. Perhaps the data are trying to tell a story. This effect became most pronounced after the stock market crash of 1987, raising the possibility that the market expected another crash, although with low probability.

Recently, Rubinstein (1994) has extended the concept of ISD to the whole **implied distribution** of future prices. By judiciously choosing options with sufficiently spaced strike prices, one can recover the entire implied distribution that is consistent with option prices. This distribution, shown in Figure 14.7, displays a hump for values of the future price 30% below the current price. This hump is nowhere apparent from the usual log-normal distribution.

We can give two interpretations to this result. The first is that the market indeed predicts a small probability of a future crash. The second has to do with the fact that this distribution derived from option prices assumes risk-neutrality, since the Black–Scholes approach assumes that investors are risk neutral. Thus this distribution may differ from the true, objective distribution due to a **risk premium**. Intuitively, investors may be very averse to a situation where they have to suffer a large fall in the value of their stock portfolios. As a result, they will bid up the price of put options, which is reflected in a higher than otherwise implied volatility.

This is currently an area of active research. The consensus, however, is that options should contain valuable information about future distributions since, after all, option traders bet good money on their forecasts.

EXAMPLE 14.9: FRM EXAM 2006—QUESTION 29

Risk-neutral default probability and real-world (or physical) default probability are used in the analysis of credit risk. Which one of the following statements on their uses is correct?

a. Real-world default probability should be used in scenario analyses of potential future losses from defaults, and real-world default probability should also be used in valuing credit derivatives.
b. Real-world default probability should be used in scenario analyses of potential future losses from defaults, but risk-neutral default probability should be used in valuing credit derivatives.
c. Risk-neutral default probability should be used in scenario analyses of potential future losses from defaults, and risk-neutral default probability should be used in valuing credit derivatives.
d. Risk-neutral default probability should be used in scenario analyses of potential future losses from defaults, but real-world default probability should be used in valuing credit derivatives.

14.4 IMPORTANT FORMULAS

VAR assuming i.i.d. returns: $\mathrm{VAR} = \alpha(\sigma\sqrt{T})W$

Time aggregation of variance with nonzero autocorrelation:
$V[R_t + R_{t-1}] = \sigma^2 \times 2[1+\rho]$

GARCH process: $h_t = \alpha_0 + \alpha_1 r_{t-1}^2 + \beta h_{t-1}$

GARCH long-run mean: $h = \frac{\alpha_0}{(1-\alpha_1-\beta)}$

EWMA process: $h_t = \lambda h_{t-1} + (1-\lambda) r_{t-1}^2$

14.5 ANSWERS TO CHAPTER EXAMPLES

Example 14.1: Time Scaling

c. Knowing that the variance is $V(\text{2-day}) = V(\text{1-day})\ [2+2\rho]$, we find $\mathrm{VAR}(\text{2-day}) = \mathrm{VAR}(\text{1-day})\sqrt{2+2\rho} = \$1\sqrt{2+0.2} = \$1.483$, assuming the same distribution for the different horizons.

Example 14.2: FRM Exam 2006—Question 36

b. The persistence parameter $\alpha_1 + \beta$ is, respectively, 0.99, 0.97, 0.98, 0.99. Model b. has the lowest parameter and hence will revert the fastest to the mean.

Example 14.3: FRM Exam 2006—Question 132

d. If the initial volatility were equal to the long-run volatility, then the T-day VAR could be computed using the square root of time rule, assuming normal distributions. If the starting volatility were higher, then the T-day VAR should be less than the $\sqrt{T} \times$ one-day VAR. Conversely if the starting volatility were lower than the long-run value. However, the question does not indicate the starting point. Hence, answer d. is correct.

Example 14.4: FRM Exam 2007—Question 34

b. The long-run mean variance is $h = \alpha_0/(1 - \alpha_1 - \beta) = 0.006/(1 - 0.04 - 0.94) = 0.25$. Taking the square root, this gives 0.5 for daily volatility. Multiplying by $\sqrt{252}$, we have an annualized volatility of 7.937%.

Example 14.5: FRM Exam 2002—Question 13

d. The GARCH model allows for time variation in volatility and includes the EWMA model as a special case. It can also induce fat tails in the return distribution, but says nothing about the mean, so answer d. is false.

Example 14.6: FRM Exam 2007—Question 46

a. The log return is $\ln(18/20) = -10.54\%$. The new variance forecasts is $h = 0.90 \times (1.5^2) + (1 - 0.90) \times 10.54^2 = 0.001313$, or taking the square root, 3.62%.

Example 14.7: FRM Exam 2006—Question 40

b. The weight of the last day is $(1 - 0.95) = 0.050$. For the day before, this is 0.05×0.95, and for four days ago, $0.05 \times 0.95^3 = 0.04287$.

Example 14.8: Effect of Weights on Observations

b. The EWMA model puts a weight of 0.06 on the latest observation, which is higher than the weight of $(1/60) = 0.0167$ for the 60-day MA and $(1/250) = 0.004$ for the 250-day MA.

Example 14.9: FRM Exam 2007—Question 29

b. Real-world probabilities should be used for risk management, or to devise scenarios. In contrast, risk-neutral probabilities should be used to price assets, such as credit derivatives.

CHAPTER 15

VAR Methods

So far, we have considered sources of risk in isolation. This approach reflects the state of the art up to the beginning of the 1990s. Until then, risk was measured and managed at the level of a desk or business unit. The finance profession was basically compartmentalized. This approach, however, fails to take advantage of portfolio theory, which has taught us that risk should be measured at the level of the portfolio.

This chapter turns to firm-wide VAR methods. These can be separated into local valuation and full valuation methods. **Local valuation methods** make use of the valuation of the instruments at the current point, along with the first and perhaps the second partial derivatives. **Full valuation methods**, in contrast, reprice the instruments over a broad range of values for the risk factors.

These methods are discussed in Section 15.1. Section 15.2 describes the three main VAR methods. The first step in all methods is **mapping**, which consists of replacing each instrument by its exposures on selected risk factors.

This considerably simplifies the risk measurement process. It would be infeasible to model all instruments individually, because there are too many. The art of risk management consists of choosing a set of limited risk factors that will adequately cover the spectrum of risks for the portfolio at hand. Thus, risk management is truly the art of the approximation. Sometimes, however, these approximations fail, as shown in Section 15.3, which discusses the performance of risk systems during the recent credit crisis.

Finally, Section 15.4 works through a detailed example, which is a forward currency contract.

15.1 VAR: LOCAL VERSUS FULL VALUATION

The various approaches to VAR described in Figure 15.1. The left branch describes local valuation methods, also known as **analytical methods**. These include linear models and nonlinear models. Linear models are based on the covariance matrix approach. This matrix can be simplified using factor models, or even a diagonal model, which is a one-factor model.

Nonlinear models take into account the first and second partial derivatives. The latter are called gamma or convexity. Next, the right branch describes full valuation methods and include historical or Monte Carlo simulations.

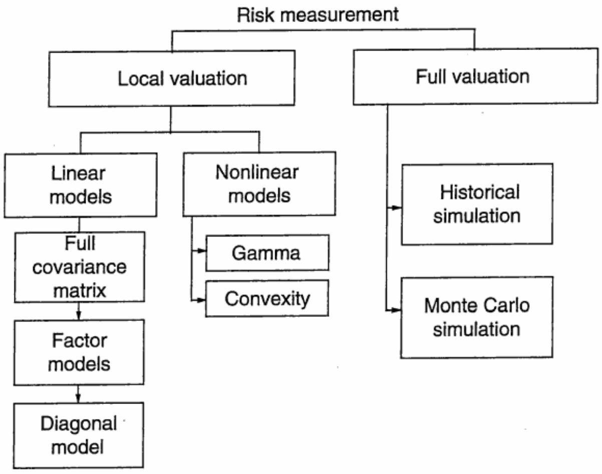

FIGURE 15.1 VAR Methods

15.1.1 Local Valuation

VAR was born from the recognition that we need an estimate that accounts for various sources of risk and expresses loss in terms of probability. Extending the duration equation to the worst change in yield at some confidence level dy, we have

$$(\text{Worst } dP) = (-D^* P) \times (\text{Worst } dy) \tag{15.1}$$

where D^* is modified duration. For a long position in the bond, the worst movement in yield is an increase at say, the 95% confidence level. This will lead to a fall in the bond value at the same confidence level. We call this approach **local valuation**, because it uses information about the initial price and the exposure at the initial point. As a result, the VAR for the bond is given by

$$\text{VAR}(dP) = | -D^* P \mid \times \text{VAR}(dy) \tag{15.2}$$

The main advantage of this approach is its simplicity: The distribution of the price is the same as that of the change in yield. This is particularly convenient for portfolios with numerous sources of risks, because linear combinations of normal distributions are normally distributed. Figure 15.2, for example, shows how the linear exposure combined with the normal density (in the right panel) combines to create a normal density.

15.1.2 Full Valuation

More generally, to take into account nonlinear relationships, one would have to reprice the bond under different scenarios for the yield. Defining y_0 as the initial yield,

$$(\text{Worst } dP) = P[y_0 + (\text{Worst } dy)] - P[y_0] \tag{15.3}$$

We call this approach **full valuation**, because it requires repricing the asset.

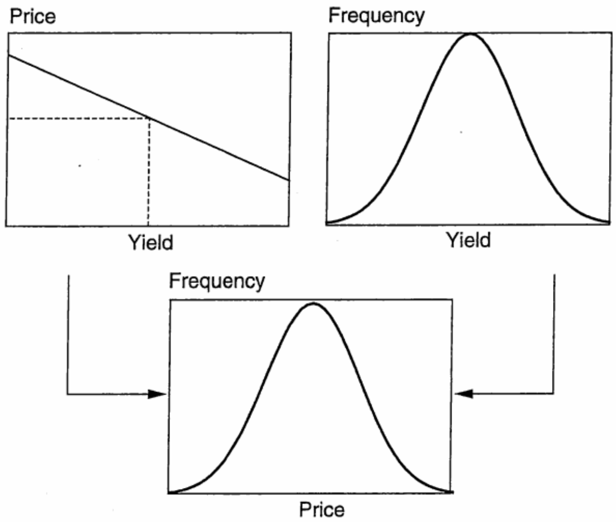

FIGURE 15.2 Distribution with Linear Exposures

This approach is illustrated in Figure 15.3, where the nonlinear exposure combined with the normal density creates a distribution that is not symmetrical any more, but skewed to the right. This is more precise but, unfortunately, is more complex than a simple, linear valuation method.

15.1.3 Delta–Gamma Method

Ideally, we would like to keep the simplicity of the local valuation while accounting for nonlinearities in the payoffs patterns. Using the Taylor expansion,

$$dP \approx \frac{\partial P}{\partial y}dy + (1/2)\frac{\partial^2 P}{\partial y^2}(dy)^2 = (-D^* P)dy + (1/2)CP(dy)^2 \tag{15.4}$$

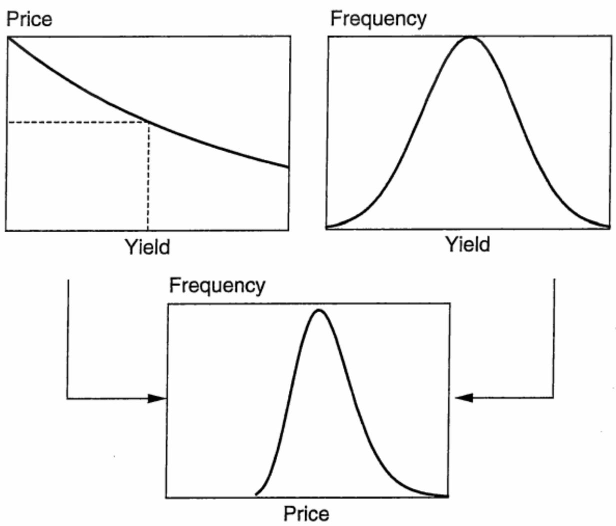

FIGURE 15.3 Distribution with Nonlinear Exposures

where the second-order term involves convexity C. Note that the valuation is still local because we only value the bond once, at the original point. The first and second derivatives are also evaluated at the local point.

Because the price is a monotonic function of the underlying yield, we can use the Taylor expansion to find the worst downmove in the bond price from the worst move in the yield. Calling this $dy^* = \text{VAR}(dy)$, we have

$$(\text{Worst } dP) = P(y_0 + dy^*) - P(y_0) \approx (-D^* P)(dy^*) + (1/2)(C\ P)(dy^*)^2 \qquad (15.5)$$

This leads to a simple adjustment for VAR

$$\text{VAR}(dP) = \mid -D^* P \mid \times \text{VAR}(dy) - (1/2)(C\ P) \times \text{VAR}(dy)^2 \qquad (15.6)$$

More generally, this method can be applied to derivatives, for which we write the Taylor approximation as

$$df \approx \frac{\partial f}{\partial S} dS + (1/2)\frac{\partial^2 f}{\partial S^2} dS^2 = \Delta dS + (1/2)\Gamma dS^2 \qquad (15.7)$$

where Γ is now the second derivative, or gamma, like convexity.

For a long call option, the worst value is achieved as the underlying price moves down by $\text{VAR}(dS)$. With $\Delta > 0$ and $\Gamma > 0$, the VAR for the derivative is now

$$\text{VAR}(df) = \mid \Delta \mid \times \text{VAR}(dS) - (1/2)\Gamma \times \text{VAR}(dS)^2 \qquad (15.8)$$

This method is called **delta–gamma** because it provides an analytical, second-order correction to the delta–normal VAR. This explains why long positions in options, with positive gamma, have less risk than with a linear model. Conversely, short positions in options have greater risk than implied by a linear model.

This simple adjustment, unfortunately, only works when the payoff function is monotonic, that is, involves a one-to-one relationship between the option value f and S. More generally, the **delta–gamma–delta** VAR method involves, first, computing the moments of df using Equation (15.7) and, second, choosing the normal distribution that provides the best fit to these moments.

The improvement brought about by this method depends on the size of the second-order coefficient, as well as the size of the worst move in the risk factor. For forward contracts, for instance, $\Gamma = 0$, and there is no point in adding second-order terms. Similarly, for most fixed-income instruments over a short horizon, the convexity effect is relatively small and can be ignored.

EXAMPLE 15.1: FRM EXAM 2004—QUESTION 60

Which of the following methodologies would be most appropriate for stress testing your portfolio?

a. Delta–gamma valuation
b. Full revaluation
c. Marked to market
d. Delta–normal VAR

EXAMPLE 15.2: FRM EXAM 2002—QUESTION 38

If you use delta–VAR for a portfolio of options, which of the following statements is *always* correct?

a. It necessarily understates VAR because it uses a linear approximation.
b. It can sometimes overstate VAR.
c. It performs most poorly for a portfolio of deep in-the-money options.
d. It performs most poorly for a portfolio of deep out-of-the-money options.

15.2 VAR METHODS: OVERVIEW

15.2.1 Mapping

This section provides an introduction to the three VAR methods. The portfolio could consist of a large number of instruments, say M. Because it would be too complex to model each instrument separately, the first step is **mapping**, which consists of replacing the instruments by positions on a limited number of risk factors. Say we have N risk factors. The positions are then aggregated across instruments, which yields dollar exposures x_i.

For instance, we could reduce the large spectrum of maturities in the U.S. fixed-income market by 14 maturities. We then replace the positions in every bond by exposures on these 14 risk factors. Perhaps this can be reduced further. For some portfolios, one interest rate risk factor may be sufficient.

Figure 15.4 displays the mapping process. We have six instruments, say different forward contracts on the same currency but with different maturities. These can be replaced by positions on three risk factors only. In the next section, we provide a fully worked-out example.

The distribution of the portfolio return $R_{p,t+1}$ is then derived from the exposures and movements in risk factors, Δf. Some care has to be taken defining the risk factors (in gross return, change in yield, rate of return, and so on); the

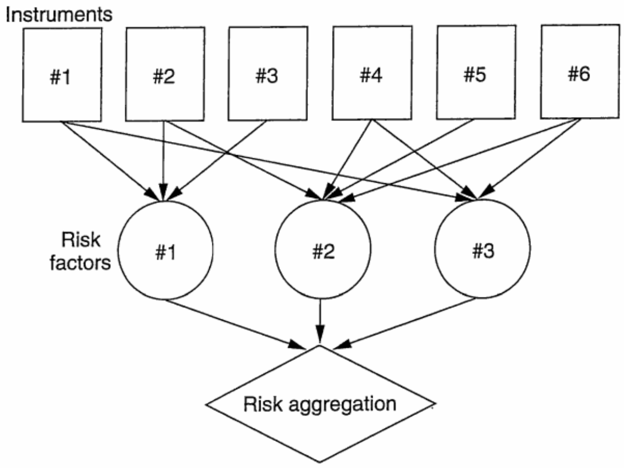

FIGURE 15.4 Mapping Approach

exposures x have to be consistently defined. Here, R_p must be measured as the change in *dollar* value of the portfolio (or whichever base currency is used).

15.2.2 Delta–Normal Method

The **delta–normal method** is the simplest VAR approach. It assumes that the portfolio exposures are linear and that the risk factors are jointly normally distributed. As such, it is a local valuation method.

Because the portfolio return is a linear combination of normal variables, it is normally distributed. Using matrix notations, the portfolio variance is given by

$$\sigma^2(R_{p,t+1}) = x_t' \Sigma_{t+1} x_t \tag{15.9}$$

where Σ_{t+1} is the forecast of the covariance matrix over the horizon.

If the portfolio volatility is measured in dollars, VAR is directly obtained from the standard normal deviate α that corresponds to the confidence level c:

$$\text{VAR} = \alpha\sigma(R_{p,t+1}) \tag{15.10}$$

This is called the **diversified VAR**, because it accounts for diversification effects. In contrast, the **undiversified VAR** is simply the sum of the individual VARs for each risk factor. It assumes that all prices will move in the worst direction simultaneously, which is unrealistic.

The RiskMetrics approach is similar to the delta–normal approach. The only difference is that the risk factor returns are measured as logarithms of the price ratios, instead of rates of returns.

The main benefit of this approach is its appealing simplicity. This is also its drawback. The delta–normal method cannot account for nonlinear effects such as encountered with options. It may also underestimate the occurrence of large observations because of its reliance on a normal distribution.

15.2.3 Historical Simulation Method

The **historical simulation** (HS) method is a full valuation method. It consists of going back in time, e.g., over the last 250 days, and applying current weights to a time-series of historical asset returns. It replays a "tape" of history with current weights.

Define the current time as t; we observe data from 1 to t. The current portfolio value is P_t, which is a function of the current risk factors

$$P_t = P[f_{1,t}, f_{2,t}, \ldots, f_{N,t}] \tag{15.11}$$

We sample the factor movements from the historical distribution, without replacement

$$\Delta f_i^k = \{\Delta f_{i,1}, \Delta f_{i,2}, \ldots, \Delta f_{i,t}\} \tag{15.12}$$

From this we can construct hypothetical factor values, starting from the current one

$$f_i^k = f_{i,t} + \Delta f_i^k \tag{15.13}$$

which are used to construct a hypothetical value of the current portfolio under the new scenario, using Equation (15.11)

$$P^k = P\left[f_1^k, f_2^k, \ldots, f_N^k\right] \tag{15.14}$$

We can now compute changes in portfolio values from the current position $R^k = (P^k - P_t)/P_t$.

We sort the t returns and pick the one that corresponds to the cth quantile, $R_p(c)$. VAR is obtained from the difference between the average and the quantile,

$$\text{VAR} = \text{AVE}[R_p] - R_p(c) \tag{15.15}$$

The advantage of this method is that it makes no specific distributional assumption about return distribution, other than relying on historical data. This is an improvement over the normal distribution because historical data typically contain fat tails. The main drawback of the method is its reliance on a short historical moving window to infer movements in market prices. If this window does not contain some market moves that are likely, it may miss some risks.

15.2.4 Monte Carlo Simulation Method

The **Monte Carlo simulation method** is basically similar to the historical simulation, except that the movements in risk factors are generated by drawings from some prespecified distribution. Instead of Equation (15.12), we have

$$\Delta f^k \sim g(\theta), \quad k = 1, \ldots K \tag{15.16}$$

where g is the joint distribution (e.g. a normal or Student's t) and θ the required parameters. The risk manager samples **pseudo-random numbers** from this distribution and then generates pseudo-dollar returns as before. Finally, the returns are sorted to produce the desired VAR.

This method is the most flexible, but also carries an enormous computational burden. It requires users to make assumptions about the stochastic process and to understand the sensitivity of the results to these assumptions. Thus, it is subject to **model risk.**

Monte Carlo methods also create inherent sampling variability because of the randomization. Different random numbers will lead to different results. It may take a large number of iterations to converge to a stable VAR measure. It should be noted that when all risk factors have a normal distribution and exposures are linear, the method should converge to the VAR produced by the delta–normal VAR.

15.2.5 Comparison of Methods

Table 15.1 provides a summary comparison of the three mainstream VAR methods. Among these methods, the delta–normal is by far the easiest to implement and communicate. For simple portfolios with little optionality, this may be perfectly appropriate. In contrast, the presence of options may require a full valuation method.

TABLE 15.1 Comparison of Approaches to VAR

Features	Delta–normal	Historical simulation	Monte Carlo simulation
Valuation	Linear	Full	Full
Distribution			
Shape	Normal	Actual	General
Extreme events	Low probability	In recent data	Possible
Implementation			
Ease of computation	Yes	Intermediate	No
Communicability	Easy	Easy	Difficult
VAR precision	Excellent	Poor with short window	Good with many iterations
Major pitfalls	Nonlinearities, fat tails	Time variation in risk, unusual events	Model risk

EXAMPLE 15.3: FRM EXAM 2001—QUESTION 92

Under usually accepted rules of market behavior, the relationship between parametric delta–normal VAR and historical VAR will tend to be

a. Parametric VAR will be higher.
b. Parametric VAR will be lower.
c. It depends on the correlations.
d. None of the above are correct.

EXAMPLE 15.4: FRM EXAM 2004—QUESTION 51

In early 2000, a risk manager calculates the VAR for a technology stock fund based on the last three years of data. The strategy of the fund is to buy stocks and write out-of-the-money puts. The manager needs to compute VAR. Which of the following methods would yield results that are *least* representative of the risks inherent in the portfolio?

a. Historical simulation with full repricing
b. Delta–normal VAR assuming zero drift
c. Monte Carlo style VAR assuming zero drift with full repricing
d. Historical simulation using delta–equivalents for all positions

EXAMPLE 15.5: FRM EXAM 2006—QUESTION 114

Which of the following is most accurate with respect to delta–normal VAR?

a. The delta–normal method provides accurate estimates of VAR for assets that can be expressed as a linear or nonlinear combination of normally distributed risk factors.
b. The delta-normal method provides accurate estimates of VAR for options that are at or near-the-money and close to expiration.
c. The delta–normal method provides estimates of VAR by generating a covariance matrix and measuring VAR using relatively simple matrix multiplication.
d. The delta–normal method provides accurate estimates of VAR for options and other derivatives over ranges even if deltas are unstable.

EXAMPLE 15.6: FRM EXAM 2005—QUESTION 94

Which of the following statements about VAR estimation methods is *wrong*?

a. The delta–normal VAR method is more reliable for portfolios that implement portfolio insurance through dynamic hedging than for portfolios that implement portfolio insurance through the purchase of put options.
b. The full valuation VAR method based on historical data is more reliable for large portfolios that contain significant option-like investments than the delta–normal VAR method.
c. The delta–normal VAR method can understate the true VAR for stock portfolios when the distribution of the return of the stocks has high kurtosis.
d. Full valuation VAR methods based on historical data take into account nonlinear relationships between risk factors and security prices.

EXAMPLE 15.7: FRM EXAM 2005—QUESTION 128

Natural gas prices exhibit seasonal volatility. Specifically the entire forward curve is more volatile during the wintertime. Which of the following statements concerning VAR is correct if the VAR is estimated using unweighted historical simulation and a three-year sample period?

a. We will overstate VAR in the summer and understate VAR in the winter.
b. We will overstate VAR in the summer and overstate VAR in the winter.
c. We will understate VAR in the summer and understate VAR in the winter.
d. We will understate VAR in the summer and overstate VAR in the winter.

EXAMPLE 15.8: FRM EXAM 2004—QUESTION 30

You are given the following information about the returns of stock P and stock Q: Variance of return of stock P = 100.0. Variance of return of stock Q = 225.0. Covariance between the return of stock P and the return of stock Q = 53.2. At the end of 1999, you are holding USD 4 million in stock P. You are considering a strategy of shifting USD 1 million into stock Q and keeping USD 3 million in stock P. What percentage of risk, as measured by standard deviation of return, can be reduced by this strategy?

a. 0.5%
b. 5.0%
c. 7.4%
d. 9.7%

15.3 LIMITATIONS OF VAR SYSTEMS

The goal of risk measurement systems is to describe the distribution of potential losses on the portfolio. VAR is a single summary measure of dispersion in portfolio returns and consequently has limitations that should be obvious.

As explained in Chapter 10, VAR cannot be viewed as a worst-loss measure. Instead, it should be viewed as a measure of dispersion that should be exceeded with some regularity, e.g., in 1% of the cases with the usual 99% confidence level. In addition, VAR does not describe the extent of losses in the left tail. Instruments such as short position in options could generate infrequent but extreme losses when they occur. To detect such vulnerabilities, the distribution of losses beyond VAR should be examined as well.

The traditional application of VAR models, such as historical simulation, involves **moving windows**, typically using one to three years of historical data. Such windows may not represent the range of potential movements in the risk factors, however, which is why stress tests are needed.

In addition, as we have seen in this chapter, the implementation of VAR systems often requires simplifications, obtained by mapping the positions on the selected risk factors. Thus, risk managers should be cognizant of weaknesses in their risk systems.

During the credit crisis that started in 2007, risk management systems failed at many banks. Some banks suffered losses that were much more frequent and much worse than they had anticipated. In 2007 alone, for example, UBS suffered losses of $19 billion from positions in mortgage-backed securities. Instead of experiencing the expected number of 2 or 3 exceptions (i.e., 1% of 250 days), this bank suffered 29 exceptions, or losses worse than VAR.[1]

This was due to a number of factors. First, markets were extremely volatile after an extended period of relative stability. Figure 15.5, for example, plots the daily volatility forecast for the S&P stock index using an Exponentially Weighed Moving Average (EWMA) with decay of 0.94. This model shows that during 2004 to 2006, the volatility was very low, averaging 0.7% daily. As a result, many financial institutions entered 2007 with high levels of leverage. When volatility started to spike during 2007, risk models experienced many exceptions. The graph also shows a volatility forecast derived from the usual moving average (MA) model with a window of one year, which is typical of most VAR models based on historical simulation. The figure shows that the MA model systematically underestimated the EWMA volatility starting in mid-2007, which is when banks' risk models started to slip.

In addition to the effects of heightened volatility in all financial markets, many banks experienced large losses on super senior, triple-A rated, tranches of securities backed by subprime mortgages. As will be seen in Chapter 22, these

[1] For a lucid explanation of risk management weaknesses, see the *Shareholder Report on UBS's Write-Downs* (April 2008).

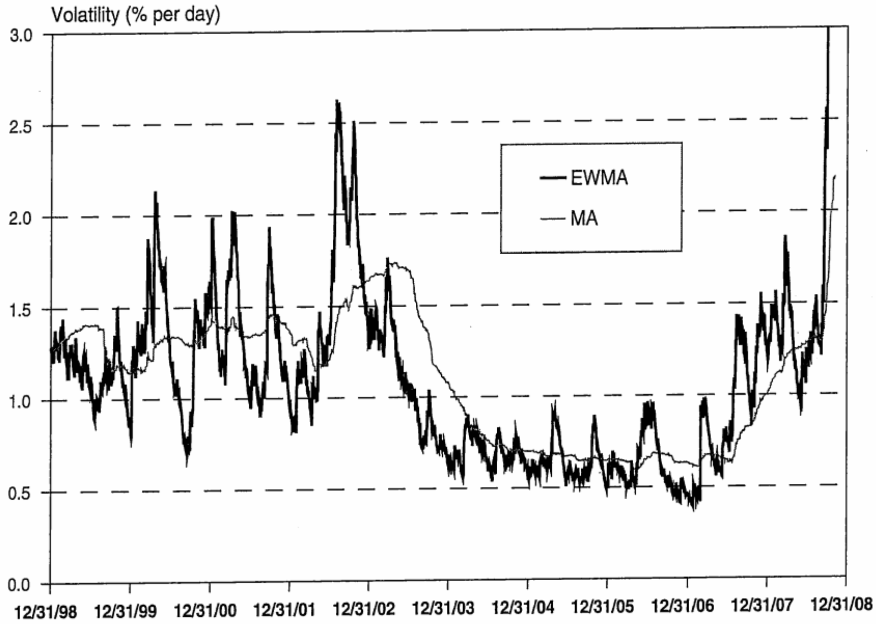

FIGURE 15.5 Volatility of S&P Stock Index (EWMA)

structures are fairly complex to model due to the need to estimate joint default probabilities. Investing in super senior tranches can be viewed as selling out-of-the-money put options, which, as we have seen, involve nonlinear payoffs. As long as the real estate market continued to go up, the default rate on subprime debt was relatively low and the super senior debt was safe, experiencing no price volatility. However, as the real estate market corrected sharply, the put options moved in-the-money, which led to large losses on the super senior debt. Of course, none of these movements showed up in the recent historical data because this only reflected a sustained appreciation in the housing market but also because of the inherent nonlinearity in these securities.

Instead of modeling these complexities, some banks simply mapped the super senior debt on AAA-rated corporate bond curves. This ignored the nonlinearities in the securities and was an act of blind faith in the credit rating. In this case, the mapping process was flawed and gave no warning sign of the impending risks.

By now, a number of reports have been written on the risk management practices at major financial institutions.[2] A striking observation is the range of quality of risk management practices. The characteristics of winners and losers are compared in Table 15.2.

[2] See Senior Supervisor Group (March 6, 2008), *Observations on Risk Management Practices during the Recent Market Turbulence.*

TABLE 15.2 Differences in Risk Management Practices

Practice	Winners	Losers
Business model	▪ Avoided CDOs, SIVs	▪ Exposed to CDOs, SIVs
Organizational structure	▪ Cooperative	▪ Hierarchical
Firm-wide risk analysis	▪ Shared information across the firm	▪ No prompt discussion of risks across the firm
Valuations	▪ Developed in-house expertise	▪ Relied on credit ratings
Management of liquidity	▪ Charged business lines for liquidity risk	▪ Did not consider contingent exposures
Risk measurement	▪ Qualitative and quantitative analysis	▪ Strict model application
	▪ Varied assumptions	▪ Mapped to corporate AAA
	▪ Tested correlations	▪ No test of correlations

In general, institutions that lost the most had a hierarchical business structure where top management wanted to expand the structured credit business, which involved collateralized debt obligations (CDOs) and structured investment vehicles (SIVs), due to their perceived profitability. Top management did not encourage feedback and often did not pay attention to warning signals given by risk managers. Many of these institutions failed to develop their own valuations models for these complex structures and instead relied on credit ratings. In addition, they did not consider contingent exposures and did not charge business lines for potential claims on the bank's balance sheet, which encouraged expansion into structured credit. These institutions blindly applied models without consideration of their weaknesses and typically did not perform stress tests of correlations.

15.4 EXAMPLE

15.4.1 Mark-to-Market

We now illustrate the computation of VAR for a simple example. The problem at hand is to evaluate the one-day downside risk of a currency forward contract. We will show that to compute VAR we need first to value the portfolio, mapping the value of the portfolio on fundamental risk factors, then to generate movements in these risk factors, and finally, to combine the risk factors with the valuation model to simulate movements in the contract value.

Assume that on December 31, 1998, we have a forward contract to buy £10 million in exchange for delivering $16.5 million in three months.

As before, we use these definitions:

$$
\begin{aligned}
S_t &= \text{current spot price of the pound in dollars} \\
F_t &= \text{current forward price} \\
K &= \text{purchase price set in contract}
\end{aligned}
$$

f_t = current value of contract
r_t = domestic risk-free rate
r_t^* = foreign risk-free rate
τ = time to maturity

To be consistent with conventions in the foreign exchange market, we define the present value factors using discrete compounding

$$P_t = \mathrm{PV}(\$1) = \frac{1}{1+r_t\tau} \qquad P_t^* = \mathrm{PV}(\$1) = \frac{1}{1+r_t^*\tau} \tag{15.17}$$

The current market value of a forward contract to buy one pound is given by

$$f_t = S_t\frac{1}{1+r_t^*\tau} - K\frac{1}{1+r_t\tau} = S_tP_t^* - KP_t \tag{15.18}$$

which is exposed to three risk factors: the spot rate and the two interest rates. In addition, we can use this equation to derive the exposures on the risk factors. After differentiation, we have

$$df = \frac{\partial f}{\partial S}dS + \frac{\partial f}{\partial P^*}dP^* + \frac{\partial f}{\partial P}dP = P^*dS + SdP^* - KdP \tag{15.19}$$

Alternatively,

$$df = (SP^*)\frac{dS}{S} + (SP^*)\frac{dP^*}{P^*} - (KP)\frac{dP}{P} \tag{15.20}$$

Intuitively, the forward contract is equivalent to

- A long position of (SP^*) on the spot rate
- A long position of (SP^*) in the foreign bill
- A short position of (KP) in the domestic bill (borrowing)

We can now mark to market our contract. If Q represents our quantity, £10 million, the current market value of our contract is

$$V_t = Qf_t = \$10{,}000{,}000\ S_t\frac{1}{1+r_t^*\tau} - \$16{,}500{,}000\frac{1}{1+r_t\tau} \tag{15.21}$$

On the valuation date, we have $S_t = 1.6637, r_t = 4.9375\%$, and $r_t^* = 5.9688\%$. Hence

$$P_t = \frac{1}{1 + r_t \tau} = \frac{1}{(1 + 4.9375\% \times 90/360)} = 0.9879$$

and similarly, $P_t^* = 0.9854$. The current market value of our contract is

$$V_t = \$10{,}000{,}000 \times 1.6637 \times 0.9854 - \$16{,}500{,}000 \times 0.9879 = \$93{,}581$$

which is slightly in-the-money. We are going to use this formula to derive the distribution of contract values under different scenarios for the risk factors.

15.4.2 Risk Factors

Assume now that we only consider the last 100 days to be representative of movements in market prices. Table 15.3 displays quotations on the spot and 3-month rates for the last 100 business days, starting on August 10.

We first need to convert these quotes into true random variables, that is, with zero mean and constant dispersion. Table 15.4 displays the one-day changes in

TABLE 15.3 Historical Market Factors

		Market Factors		
Date	\$ Eurorate (3mo-%pa)	£ Eurorate (3mo-%pa)	Spot Rate S(\$/£)	Number
8/10/98	5.5938	7.4375	1.6341	
8/11/98	5.5625	7.5938	1.6315	1
8/12/98	6.0000	7.5625	1.6287	2
8/13/98	5.5625	7.4688	1.6267	3
8/14/98	5.5625	7.6562	1.6191	4
8/17/98	5.5625	7.6562	1.6177	5
8/18/98	5.5625	7.6562	1.6165	6
8/19/98	5.5625	7.5625	1.6239	7
8/20/98	5.5625	7.6562	1.6277	8
8/21/98	5.5625	7.6562	1.6387	9
8/24/98	5.5625	7.6562	1.6407	10
...				
12/15/98	5.1875	6.3125	1.6849	90
12/16/98	5.1250	6.2188	1.6759	91
12/17/98	5.0938	6.3438	1.6755	92
12/18/98	5.1250	6.1250	1.6801	93
12/21/98	5.1250	6.2812	1.6807	94
12/22/98	5.2500	6.1875	1.6789	95
12/23/98	5.2500	6.1875	1.6769	96
12/24/98	5.1562	6.1875	1.6737	97
12/29/98	5.1875	6.1250	1.6835	98
12/30/98	4.9688	6.0000	1.6667	99
12/31/98	4.9375	5.9688	1.6637	100

TABLE 15.4 Movements in Market Factors

	Movements in Market Factors				
Number	$dr(\$1)$	$dr(£1)$	$dP/P(\$1)$	$dP/P(£1)$	$dS(\$/£)/S$
1	−0.0313	0.1563	0.00000	−0.00046	−0.0016
2	0.4375	−0.0313	−0.00116	0.00000	−0.0017
3	−0.4375	−0.0937	0.00100	0.00015	−0.0012
4	0.0000	0.1874	−0.00008	−0.00054	−0.0047
5	0.0000	0.0000	−0.00008	−0.00008	−0.0009
6	0.0000	0.0000	−0.00008	−0.00008	−0.0007
7	0.0000	−0.0937	−0.00008	0.00015	0.0046
8	0.0000	0.0937	−0.00008	−0.00031	0.0023
9	0.0000	0.0000	−0.00008	−0.00008	0.0068
10	0.0000	0.0000	−0.00008	−0.00008	0.0012
...					
90	0.0937	0.0625	−0.00031	−0.00023	−0.0044
91	−0.0625	−0.0937	0.00008	0.00015	−0.0053
92	−0.0312	0.1250	0.00000	−0.00038	−0.0002
93	0.0312	0.2188	−0.00015	0.00046	0.0027
94	0.0000	0.1562	−0.00008	−0.00046	0.0004
95	0.1250	−0.0937	−0.00039	0.00015	−0.0011
96	0.0000	0.0000	−0.00008	−0.00008	−0.0012
97	−0.0938	0.0000	0.00015	−0.00008	−0.0019
98	0.0313	−0.0625	−0.00015	0.00008	0.0059
99	−0.2187	−0.1250	0.00046	0.00023	−0.0100
100	−0.0313	−0.0312	0.00000	0.00000	−0.0018

interest rates dr, as well as the relative changes in the associated present value factors dP/P and in spot rates dS/S. For instance, for the first day,

This information is now used to construct the distribution of risk factors.

$$dr_1 = 5.5625 - 5.5938 = -0.0313 \text{ and}$$

$$dS/S_1 = (1.6315 - 1.6341)/1.6341 = -0.0016$$

15.4.3 VAR: Historical Simulation

The **historical-simulation method** takes historical movements in the risk factors to simulate potential future movements. For instance, one possible scenario for the U.S. interest rate is that, starting from the current value $r_0 = 4.9375$, the movement the next day could be similar to that observed on August 11, which is a decrease of $dr_1 = -0.0313$. The new value is $r(1) = 4.9062$.

We compute the simulated values of other variables as

$$r^*(1) = 5.9688 + 0.1563 = 6.1251 \text{ and}$$

$$S(1) = 1.6637 \times (1 - 0.0016) = 1.6611.$$

Armed with these new values, we can reprice the forward contract, now worth

$$V_t = \$10{,}000{,}000 \times 1.6611 \times 0.9849 - \$16{,}500{,}000 \times 0.9879 = \$59{,}941$$

TABLE 15.5 Simulated Market Factors

	Simulated Market Factors					Hypothetical
Number	$r(\$1)$	$r(£1)$	$S(\$/£)$	PV($1)	PV(£1)	MTM Contract
1	4.9062	6.1251	1.6611	0.9879	0.9849	$59,941
2	5.3750	5.9375	1.6608	0.9867	0.9854	$84,301
3	4.5000	5.8751	1.6617	0.9889	0.9855	$59,603
4	4.9375	5.1562	1.6559	0.9878	0.9848	$9,467
5	4.9375	5.9688	1.6623	0.9878	0.9853	$79,407
6	4.9375	5.9688	1.6625	0.9878	0.9853	$81,421
7	4.9375	5.8751	1.6713	0.9878	0.9855	$172,424
8	4.9375	6.0625	1.6676	0.9878	0.9851	$128,149
9	4.9375	5.9688	1.6749	0.9878	0.9853	$204,361
10	4.9375	5.9688	1.6657	0.9878	0.9853	$113,588
...						
90	5.0312	6.0313	1.6564	0.9876	0.9851	$23,160
91	4.8750	5.8751	1.6548	0.9880	0.9855	$7,268
92	4.9063	6.0938	1.6633	0.9879	0.9850	$83,368
93	4.9687	5.7500	1.6683	0.9877	0.9858	$148,705
94	4.9375	6.1250	1.6643	0.9878	0.9849	$93,128
95	5.0625	5.8751	1.6619	0.9875	0.9855	$84,835
96	4.9375	5.9688	1.6617	0.9878	0.9853	$74,054
97	4.8437	5.9688	1.6605	0.9880	0.9853	$58,524
98	4.9688	5.9063	1.6734	0.9877	0.9854	$193,362
99	4.7188	5.8438	1.6471	0.9883	0.9856	−$73,811
100	4.9062	5.9376	1.6607	0.9879	0.9854	$64,073
	4.9375	5.9688	1.6637	0.9879	0.9854	$93,581

Note that, because the contract is long the pound that fell in value, the current value of the contract has decreased relative to the initial value of $93,581.

We record the new contract value and repeat this process for all the movements from day 1 to day 100. This creates a distribution of contract values, which is reported in the last column of Table 15.5.

The final step consists of sorting the contract values, as shown in Table 15.6. Suppose we want to report VAR relative to the initial value (instead of relative to the average on the target date.) The last column in the table reports the *change* in the portfolio value, i.e., $V(k) - V_0$. These range from a loss of $200,752 to a gain of $280,074.

We can now characterize the risk of the forward contract by its entire distribution, which is shown in Figure 15.6. The purpose of VAR is to report a single number as a downside risk measure. Let us take, for instance, the 95% lower quantile. From Table 15.6, we identify the fifth-lowest value out of 100, which is $127,232. Ignoring the mean, the 95% VAR is $\text{VAR}_{\text{HS}} = \$127,232$.

15.4.4 VAR: Delta-Normal Method

The **delta-normal** approach takes a different approach to constructing the distribution of the portfolio value. We assume that the three risk factors (dS/S), (dP/P), (dP^*/P^*) are jointly normally distributed.

TABLE 15.6 Distribution of Portfolio Values

	Sorted Values	
Number	Hypothetical MTM	Change in MTM
1	−$107,171	−$200,752
2	−$73,811	−$167,392
3	−$46,294	−$139,875
4	−$37,357	−$130,938
5	−$33,651	−$127,232
6	−$22,304	−$115,885
7	−$11,694	−$105,275
8	$7,268	−$86,313
9	$9,467	−$84,114
10	$13,744	−$79,837
...		
90	$193,362	$99,781
91	$194,405	$100,824
92	$204,361	$110,780
93	$221,097	$127,515
94	$225,101	$131,520
95	$228,272	$134,691
96	$233,479	$139,897
97	$241,007	$147,426
98	$279,672	$186,091
99	$297,028	$203,447
100	$373,655	$280,074

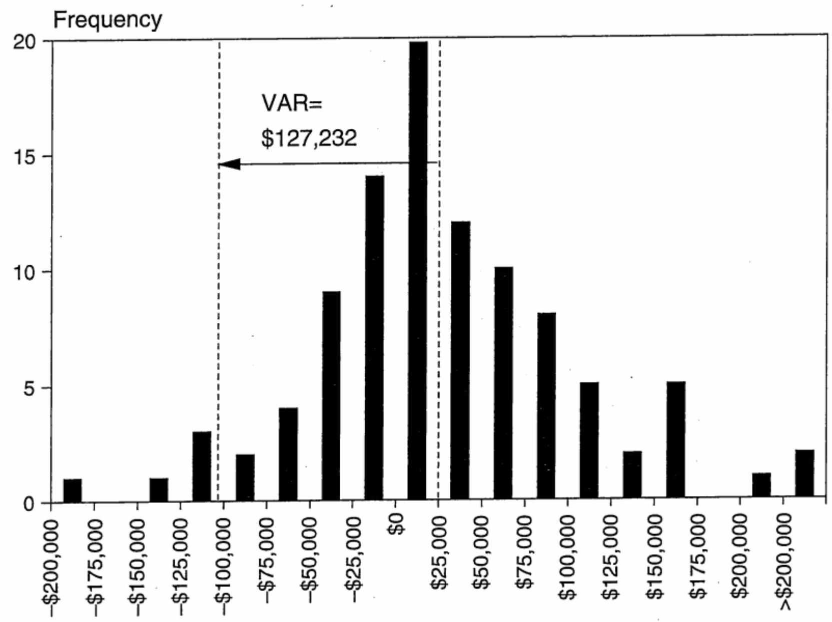

FIGURE 15.6 Empirical Distribution of Value Changes

TABLE 15.7 Covariance Matrix Approach

	$dP/P(\$1)$	$dP/P(£1)$	$dS(\$/£)/S$
Standard Deviation:	0.022%	0.026%	0.473%
Correlation Matrix:	$dP/P(\$1)$	$dP/P(£1)$	$dS(\$/£)/S$
$dP/P(\$1)$	1.000	0.137	0.040
$dP/P(£1)$	0.137	1.000	−0.063
$dS(\$/£)/S$	0.040	−0.063	1.000
Covariance Matrix:	$dP/P(\$1)$	$dP/P(£1)$	$dS(\$/£)/S$
Σ $dP/P(\$1)$	4.839E-08	7.809E-09	4.155E-08
dP/P (£1)	7.809E-09	6.720E-08	−7.688E-08
$dS(\$/£)/S$	4.155E-08	−7.688E-08	2.237E-08
Exposure: x'	−\$16,300,071	\$16,393,653	\$16,393,653

Σx

$$\begin{bmatrix} 4.839\text{E-}08 & 7.809\text{E-}09 & 4.155\text{E-}08 \\ 7.809\text{E-}09 & 6.720\text{E-}08 & -7.688\text{E-}08 \\ 4.155\text{E-}08 & -7.688\text{E-}08 & 2.237\text{E-}05 \end{bmatrix} \times \begin{bmatrix} -\$16{,}300{,}071 \\ \$16{,}393{,}653 \\ \$16{,}393{,}653 \end{bmatrix} = \begin{bmatrix} \$0.020 \\ -\$0.286 \\ \$364.852 \end{bmatrix}$$

$\sigma^2 = x'(\Sigma x)$ Variance:

$$\begin{bmatrix} -\$16{,}300{,}071 & \$16{,}393{,}653 & \$16{,}393{,}653 \end{bmatrix} \times \begin{bmatrix} \$0.020 \\ -\$0.286 \\ \$364.852 \end{bmatrix} = \boxed{\$5{,}976{,}242{,}188}$$

σ Standard deviation \$77,306

We can write Equation (15.20) as

$$df = (SP^*)\frac{dS}{S} + (SP^*)\frac{dP^*}{P^*} - (KP)\frac{dP}{P} = x_1 dz_1 + x_2 dz_2 + x_3 dz_3 \qquad (15.22)$$

where the dz are normal variables and x are exposures.

Define Σ as the (3 by 3) covariance matrix of the dz, and x as the vector of exposures. We compute VAR from $\sigma^2(df) = x'\Sigma x$. Table 15.7 details the steps. First, we compute the covariance matrix of the three risk factors. The top of the table shows the standard deviation of daily returns as well as correlations. From these, we construct the covariance matrix.

Next, the table shows the vector of exposures, x'. The matrix multiplication Σx is shown on the following lines. After that, we compute $x'(\Sigma x)$, which yields the variance. Taking the square root, we have $\sigma(df) = \$77{,}306$. Finally, we transform into a 95% quantile by multiplying by 1.645, which gives $\text{VAR}_{\text{DN}} = \$127{,}169$.

Note how close this number is to the VAR_{HS} of \$127,232 we found previously. This suggests that the distribution of these variables is close to a normal distribution. Indeed, the empirical distribution in Figure 15.6 roughly looks like a normal. The fitted distribution is shown in Figure 15.7.

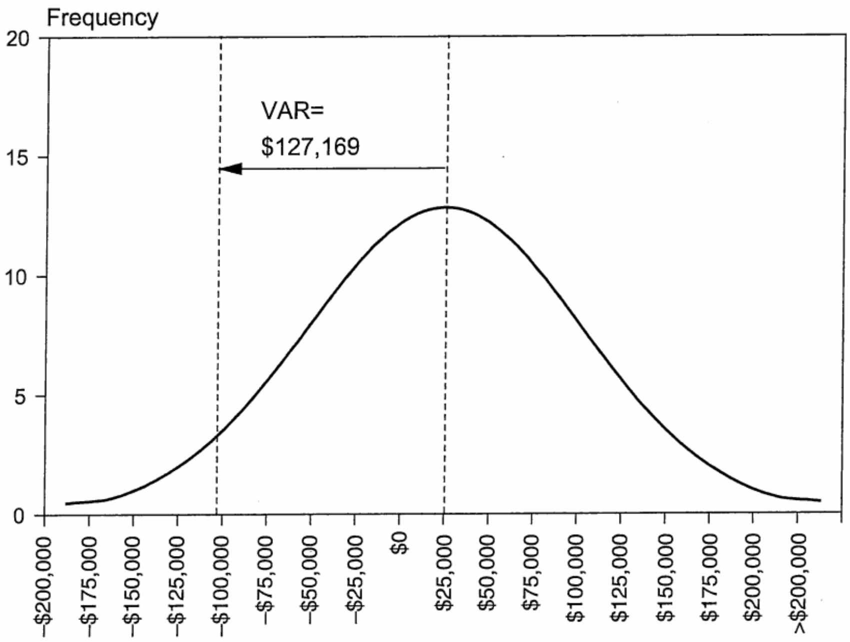

FIGURE 15.7 Normal Distribution of Value Changes

15.5 IMPORTANT FORMULAS

Linear VAR, fixed-income: $\text{VAR}(dP) =\mid -D^*P \mid \times \text{VAR}(dy)$
Quadratic VAR, fixed-income:

$$\text{VAR}(dP) = \mid -D^*P \mid \times \text{VAR}(dy) - (1/2)(C\ P) \times \text{VAR}(dy)^2$$

Full-valuation VAR, fixed-income: $(\text{Worst } dP) = P[y_0 + (\text{Worst } dy)] - P[y_0]$
Delta–VAR: $\text{VAR}(df) = \mid \Delta \mid \text{VAR}(dS)$
Delta–gamma VAR: $\text{VAR}(df) = \mid \Delta \mid \text{VAR}(dS) - (1/2)\Gamma \times \text{VAR}(dS)^2$
Delta–normal VAR: $\text{VAR} = \alpha\sigma(R_{p,t+1}),\ \sigma^2(R_{p,t+1}) = x_t'\Sigma_{t+1}x_t$
Historical simulation VAR: $\Delta f_i^k = \{\Delta f_{i,1}, \Delta f_{i,2}, \ldots, \Delta f_{i,t}\}$
Monte Carlo simulation VAR: $\Delta f^k \sim g(\theta)$

15.6 ANSWERS TO CHAPTER EXAMPLES

Example 15.1: FRM Exam 2004—Question 60

b. By definition, stress-testing involves large movements in the risk factors. This requires a full revaluation of the portfolio.

Example 15.2: FRM Exam 2002—Question 38

b. This question has to be read very carefully in view of the "always" characterization. The Delta–VAR could understate or overstate the true VAR, depending on whether the position is net long or short options, so a. is incorrect. The Delta–VAR is generally better for in-the-money options, because these have low gamma, so c. is false. For out-of-the-money options, delta is close to zero, so the Delta–VAR method would predict zero risk. The risk could indeed be very small, so d. is incorrect. So, b. is the most general statement.

Example 15.3: FRM Exam 2001—Question 92

b. Parametric VAR usually assumes a normal distribution. Given that actual distributions of financial variables have fatter tails than the normal distribution, parametric VAR at high confidence levels will generally underestimate VAR.

Example 15.4: FRM Exam 2004—Question 51

d. Because the portfolio has options, methods a. or c. based on full repricing would be appropriate. Next, recall that technology stocks have had a big increase in price until March 2000. From 1996 to 1999, the NASDAQ index went from 1300 to 4000. This creates a positive drift in the series of returns. So, historical simulation without an adjustment for this drift would bias the simulated returns upward, thereby underestimating VAR.

Example 15.5: FRM Exam 2006—Question 114

c. The delta–normal approach will perform poorly with nonlinear payoffs, so answer a. is false. Similarly, the approach will fail to measure risk properly for options if the delta changes, which is the case for at-the-money options, so answers b. and d. are false.

Example 15.6: FRM Exam 2005—Question 94

a. Full valuation methods are more precise for portfolios with options, so answers b. and d. are correct. The delta–normal VAR understates the risk when distributions have fat tails, so answer c. is correct. Answer a. is indeed wrong. The delta–normal method will be poor for outright positions in options, or their dynamic replication.

Example 15.7: FRM Exam 2005—Question 128

a. This method essentially estimates the average volatility over a three-year window, ignoring seasonality. As a result, if the conditional volatility is higher during the winter, the method will understate the true risk, and conversely for the summer.

Example 15.8: FRM Exam 2004—Question 30

b. The variance of the original portfolio is 1,600, implying a volatility of 40. The new portfolio has variance of $3^2 \times 100 + 1^2 \times 225 + 2 \times 53.2 \times 3 \times 1 = 1{,}444$. This gives a volatility of 38, which is a reduction of 5%.

PART Four

Investment Risk Management

CHAPTER 16

Portfolio Management

Value at Risk techniques were developed in the early 1990s as position-based risk measures to control the risk of proprietary trading desks of commercial banks. The advent of these methods was spurred by commercial bank regulation but quickly spread to investment banks, which also have large trading operations. These techniques have been incorporated in the panoply of risk measurement tools used in the investment management industry. Institutional investors pay particular attention to the control of risk in their investment portfolio.

Risk that can be measured can be managed better. Even so, **risk management** accounts for one facet of the investment process only, which is risk. Investors only assume risk because they expect to be compensated for it in the form of higher returns. The real issue is how to balance risk against expected return.

This trade-off is the subject of **portfolio management**. So, this is much broader than risk management. Once a broad portfolio allocation into asset classes is selected, reflecting the best trade-off between risk and return, the total fund risk can be allocated to various managers using a process called **risk budgeting**.

At the end of the investment process, it is important to assess whether realized returns were in line with the risks assumed. The purpose of **performance attribution** methods is to decompose the investment performance into various components, where the goal is to identify whether the active manager really adds value. Part of the returns represents general market factors, also called "beta bets"; the remainder represents true value added, or "alpha bets."

The purpose of this chapter is to present risk and performance measurement tools in the investment management industry. Section 16.1 gives a brief introduction to institutional investors. Risk and performance measurement techniques are developed in Section 16.2. Finally, Section 16.3 discusses risk budgeting. Hedge funds, because of their importance, will be covered in the next chapter.

16.1 INSTITUTIONAL INVESTORS

Institutional investors are entities that have large amounts of funds to invest for an organization, or on behalf of others. This is in contrast with *private* investors.[1] As shown in Table 16.1, institutional investors can be classified into investment

[1] The SEC has formal definitions of, e.g., "qualified institutional buyers" under Rule 144a.

TABLE 16.1 Classification of Institutional Investors

Investment companies	Open-end funds
	Closed-end funds
Pension funds	Defined-benefit
	Defined-contribution
Insurance funds	Life
	Nonlife
Others	Foundations and endowment funds
	Non-pension funds managed by banks
	Private partnerships

companies, pension funds, insurance funds, and others. The latter category includes endowment funds, bank-managed funds, and private partnerships, also known as hedge funds. **Hedge funds** are private partnership funds that can take long and short positions in various markets and are accessible only to large investors.

Even though institutional investors and bank proprietary desks are generally exposed to similar risk factors, their philosophy is quite different. Bank trading desks employ high leverage and are aggressive investors. They typically have short horizons and engage in active trading in generally liquid markets. Financial institutions, such as commercial banks, investment banks, and broker-dealers, are sometimes called the **sell side** because they are primarily geared toward selling financial services.

On the other hand, institutional investors are part of the **buy side** because they are buying financial services from the sell side, in other words Wall Street for the United States. In contrast to the sell side, institutional investors have little or no leverage and are more conservative. Most have longer time horizons and can invest in less liquid markets. Many hedge funds, however, have greater leverage and trade actively.

16.2 PERFORMANCE MEASUREMENT

Performance measurement should properly adjust for the risks taken. This can be done with a number of metrics, typically based on the standard deviation and regression coefficients. At an even more basic level, however, the first question is how to define the risks that matter to the investor or the manager. In particular, should risk be measured in absolute terms or relative to some **benchmark**?

16.2.1 Risk Measurement

- **Absolute risk** is measured in terms of shortfall relative to the initial value of the investment, or perhaps an investment in cash. It can be expressed in dollar terms (or in the relevant base currency). Let us use the standard deviation as the risk measure and define P as the initial portfolio value and R_P as the rate of return. Absolute risk in dollar terms is

$$\sigma(\Delta P) = \sigma(\Delta P/P) \times P = \sigma(R_P) \times P \tag{16.1}$$

- **Relative risk** is measured relative to a benchmark index and represents active management risk. Defining B as the benchmark, the deviation is $e = R_P - R_B$, which is also known as the **tracking error.** In dollar terms, this is $e \times P$. The risk is

$$\sigma(e)P = [\sigma(R_P - R_B)] \times P = [\sigma(\Delta P/P - \Delta B/B)] \times P = \omega \times P \qquad (16.2)$$

where ω is called **tracking error volatility** (TEV). Defining σ_P and σ_B as the volatility of the portfolio and the benchmark and ρ as their correlation, the variance of the difference is

$$\omega^2 = \sigma_P^2 - 2\rho\sigma_P\sigma_B + \sigma_B^2 \qquad (16.3)$$

For instance, if $\sigma_P = 25\%$, $\sigma_B = 20\%$, $\rho = 0.961$, we have $\omega^2 = 25\%^2 - 2 \times 0.961 \times 25\% \times 20\% + 20\%^2 = 0.0064$, giving $\omega = 8\%$.

To compare these two approaches, take the case of an active equity portfolio manager who is given the task of beating a benchmark, perhaps the S&P 500 index for large U.S. stocks or the MSCI world index for global stock.[2] As an example, if an active portfolio return is -6% over the year but the benchmark dropped by -10%, the excess return is positive: $e = -6\% - (-10\%) = 4\%$. So, in relative terms, the portfolio has done well even though the absolute performance is negative. Another example could be one where the portfolio returns $+6\%$, which is good using absolute measures, but not so good if the benchmark went up by $+10\%$.

Using absolute or relative risk depends on how the trading or investment operation is judged. For bank trading portfolios or hedge funds, market risk is measured in absolute terms. These are sometimes called **total return funds.** On the other hand, portfolio managers that are given the task of beating a benchmark or peer group measure risk in relative terms.

EXAMPLE 16.1: ABSOLUTE AND RELATIVE RISK

An investment manager is given the task of beating a benchmark. Hence the risk should be measured

a. In terms of loss relative to the initial investment
b. In terms of loss relative to the expected portfolio value
c. In terms of loss relative to the benchmark
d. In terms of loss attributed to the benchmark

[2] This refers to a *Morgan Stanley Capital International* (MSCI) index. MSCI provides a battery of country, industry, and global stock indices that are widely used as benchmarks.

16.2.2 Surplus Risk

As is sometimes said, "risk is in the eye of the beholder." For investors with fixed future liabilities, the risk is not being able to perform on these liabilities. For pension funds with **defined benefits,** these liabilities consist of promised payments to current and future pensioners, and are called **defined benefit obligations.** In this case, the investment risk falls on the entity promising the benefits. In contrast, employees covered by a **defined contribution** plan are subject to investment risk.

For life insurance companies, these liabilities represent the likely pattern of future claim payments. These liabilities can be represented by their net present value. In general, the present value of long-term fixed payments behaves very much like a *short position in a fixed-rate bond.* If the payments are indexed to inflation, the analogous instrument is an inflation-protected bond.

The difference between the current values of assets and liabilities is called the **surplus,** S, defined as the difference between the value of assets A and liabilities L. The change is then $\Delta S = \Delta A - \Delta L$. Normalizing by the initial value of assets, we have

$$R_S = \frac{\Delta S}{A} = \frac{\Delta A}{A} - \frac{\Delta L}{L}\frac{L}{A} = R_{\text{asset}} - R_{\text{liabilities}}\frac{L}{A} \tag{16.4}$$

The duration of liabilities is long, typically 12 years. Using the duration approximation, the return on liabilities can be measured from changes in yields y, as $R_{\text{liabilities}} = -D^* \Delta y$. The worst combination of movements in market values is when asset fall due to a fall in equities, in a year when yields decrease. **Immunization** occurs when the asset portfolio, or part of it, provides a perfect hedge against changes in the value of the liabilities. Thus, investments in long-term bonds help to hedge movements in liabilities.

In this case, risk should be measured as the potential shortfall in surplus over the horizon. This is sometimes called **surplus at risk.** This VAR-type measure is an application of relative risk, where the benchmark is the present value of liabilities.

EXAMPLE 16.2: PENSION LIABILITIES

The AT&T pension plan reports a projected benefit obligation of \$17.4 billion. If the discount rate decreases by 0.5%, this liability will increase by \$0.8 billion. Based on this information, the liabilities behave like a

a. Short position in the stock market
b. Short position in cash
c. Short position in a bond with maturity of about nine years
d. Short position in a bond with duration of about nine years

EXAMPLE 16.3: FRM EXAM 2006—QUESTION 25

The DataSoft Corporation has an employee pension scheme with fixed liabilities and a long time horizon reflecting its young workforce. The fund's assets are $9 billion and the present value of its liabilities is $8.8 billion. Which of the following statements are incorrect?

I. The present value of long-term fixed payments behaves very much like a long position in a fixed-rate bond.
II. Surplus at Risk is a measure of relative risk.
III. The DataSoft Corporation will be able to immunize its liabilities by investing $8 billion in long-term fixed-rate bonds.

a. I and II
b. II and III
c. I and III
d. I, II, and III

16.2.3 Risk-Adjusted Performance Measurement

This dichotomy, absolute versus relative returns, carries through performance measurement, which evaluates the risk-adjusted performance of the fund. The **Sharpe ratio** (SR) measures the ratio of the average rate of return, $\mu(R_P)$, in excess of the risk-free rate R_F, to the absolute risk

$$\mathrm{SR} = \frac{[\mu(R_P) - R_F]}{\sigma(R_P)} \tag{16.5}$$

This approach can be extended to include VAR, or the quantile of returns, in the denominator instead of the volatility of returns. The Sharpe ratio focuses on total risk measured in absolute terms. Because total risk includes both systematic and idiosyncratic risk, this measures is appropriate for portfolios that are not very diversified, i.e., which have large idiosyncratic risk.

A related measure is the **Sortino ratio** (SOR). This replaces the standard deviation in the denominator by the semi-standard deviation, $\sigma_L(R_P)$, which considers only data points that represent a loss. The ratio is

$$\mathrm{SOR} = \frac{[\mu(R_P) - R_F]}{\sigma_L(R_P)} \tag{16.6}$$

where $\sigma_L(R_P) = \sqrt{\frac{1}{(N_L)} \sum_{i=1}^{N} [\mathrm{Min}(R_{P,i}, 0)]^2}$, and N_L is the number of observed losses. The Sortino ratio is more relevant than the Sharpe ratio when the return distribution is skewed to the left. It is much less widely used, however.

In contrast, the **information ratio** (IR) measures the ratio of the average rate of return in excess of the benchmark to the TEV

$$\mathrm{IR} = \frac{[\mu(R_P) - \mu(R_B)]}{\omega} \tag{16.7}$$

Table 16.2 presents an illustration. The risk-free interest rate is $R_F = 3\%$ and the portfolio average return is -6%, with volatility of 25%. Hence, the Sharpe Ratio of the portfolio is $\mathrm{SR} = [(-6\%) - (3\%)]/25\% = -0.36$. Because this is negative, the absolute performance is poor.

Assume now that the benchmark returned -10% over the same period and that the tracking error volatility was 8%. Hence, the Information Ratio is $\mathrm{IR} = [(-6\%) - (-10\%)]/8\% = 0.50$, which is positive. The relative performance is good even though the absolute performance is poor. Note that this information ratio of 0.50 is typical of the performance of the top 25th percentile of money managers and is considered "good."[3]

Dealing with ratios, however, is rather abstract. It is more intuitive to express performance in terms of a rate of return, adjusted for risk. Suppose we use a reference benchmark, R_B, for which we measure first its average return and risk. We can leverage up or down the portfolio P so as to bring its volatility in line with B. The **risk-adjusted performance** (RAP) is then[4]

$$\mathrm{RAP}_P = R_F + \frac{\sigma_B}{\sigma_P}[\mu(R_P) - R_F] \tag{16.8}$$

This is illustrated in Figure 16.1. The average return on portfolio P is greater than that of B. Its volatility, however, is much higher. The straight line going from R_F to R_P represents portfolios that mix the risk-free asset with P. For example, an investment of 50% in each will give an average return that is the mean of R_F and $\mu(R_P)$, and a volatility that is half that of P. The slope of this line represents the Sharpe ratio, given by Equation (16.5).

Portfolio P^* has the same level of risk as B; its performance is given by Equation (16.8). We can then compare directly RAP_P and $\mu(R_B)$. In this case, portfolio P underperforms B on a risk-adjusted basis. We obtain the same ranking between P and B, however, using the Sharpe ratio.

TABLE 16.2 Absolute and Relative Performance

	Average	Volatility	Performance
Cash	3%	0%	
Portfolio P	−6%	25%	$SR = -0.36$
Benchmark B	−10%	20%	$SR = -0.65$
Deviation e	4%	8%	$IR = 0.50$

[3] See Grinold and Kahn (2000), *Active Portfolio Management*, McGraw-Hill, New York.
[4] This performance measure is sometimes called M-square.

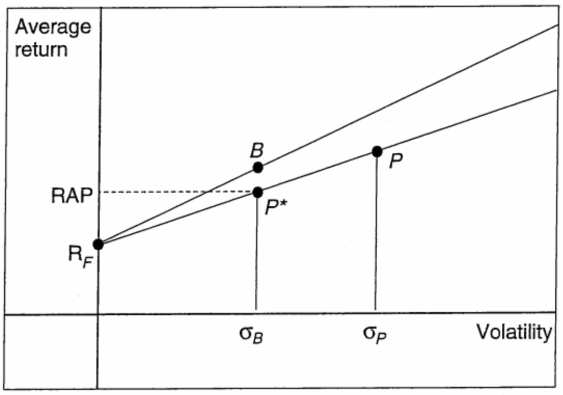

FIGURE 16.1 Risk-Adjusted Performance

EXAMPLE 16.4: SHARPE AND INFORMATION RATIOS

A portfolio manager returns 10% with a volatility of 20%. The benchmark returns 8% with risk of 14%. The correlation between the two is 0.98. The risk-free rate is 3%. Which of the following statement is *correct*?

a. The portfolio has higher SR than the benchmark.
b. The portfolio has negative IR.
c. The IR is 0.35.
d. The IR is 0.29.

16.2.4 Performance Attribution

So far, we have implemented a simple adjustment for risk that takes into account a volatility measure. To evaluate the performance of investment managers, however, it is crucial to decompose the total return into a component due to market risk premia and to other factors. Exposure to the stock market is widely believed to reward investors with a long-term premium, called the **equity premium**. Assume that this premium is EP = 4% annually. This is the expected return in excess of the risk-free rate. For simplicity, it is usually assumed that the same rate applies to lending and borrowing.

Now take the example of an investment fund of $1 million. A long position of $1.5 million, or 150% in passive equities financed by 50% cash borrowing should have an *excess return* composed of the total return on the 150% equity position, minus the cost of borrowing 50%, minus the risk-free rate. This gives

$$[150\% \times (\text{EP} + R_F) - 50\% R_F] - R_F = 1.5 \times \text{EP} = 6\%$$

This could be also achieved by taking a notional position of \$1.5 million in stock index futures and parking the investment in cash, including the margin. So, an investment manager who returns 6% in excess of the risk-free rate in this way is not really delivering any value added because this extra amount is simply due to exposure to the market. Therefore, it is crucial to account for factors that are known to generate risk premia.

Define $R_{M,t}$ as the rate of return in period t on the stock market, say the S&P 500 for U.S. equities, $R_{F,t}$ as the risk-free rate, and $R_{P,t}$ is the return on the portfolio. The general specification for this adjustment consists of estimating the regression

$$R_{P,t} - R_{F,t} = \alpha_P + \beta_P[R_{M,t} - R_{F,t}] + \epsilon_{P,t}, \quad t = 1, \cdots, T \tag{16.9}$$

where β_P is the exposure of portfolio P to the market factor, or **systematic risk,** and α_P is the abnormal performance after taking into account the exposure to the market.

"Abnormal" can only be defined in terms of a "normal" performance. One such definition is the **capital asset pricing model** (CAPM), developed by Professor William Sharpe. Under some conditions, he demonstrated that equilibrium in capital markets implies that the market portfolio is mean-variance efficient. In other words, it has the highest Sharpe ratio of any feasible portfolio. In Figure 16.1, taking B as the market M, the line passing through B is also known as the **capital market line.** It can be shown that the efficiency of the market implies a linear relationship between expected excess returns and systematic risk. For stock or portfolio i, we must have

$$E(R_i) - R_F = 0 + \beta_i[E(R_M) - R_F] \tag{16.10}$$

Comparing with Equation (16.9), this requires all α's to be zero in equilibrium.[5] A related measure is the **Treynor ratio,** which is

$$\text{TR} = \frac{[\mu(R_P) - R_F]}{\beta_P} \tag{16.11}$$

Taking the average of the two sides of Equation (16.9) shows that this represents α_P/β_P, plus the excess return on the market. Instead of focusing on total risk, as in the Sharpe ratio, the Treynor ratio focuses on systematic risk. Thus, this measure is appropriate for well-diversified portfolios.

This specification can be generalized to multiple factors. Assume we believe that in addition to the market premium, a premium is earned for *value* (or for low price-to-book companies) and *size* (of for small firms). We need to take this

[5] The CAPM is based on equilibrium in capital markets, which requires that the demand for securities from risk-averse investors matches the available supply. It also assumes that asset returns have a normal distribution. A major problem with this theory is that it may not be testable unless the "market" is exactly identified.

information into account in evaluating the manager, otherwise he or she may load up on factors that are priced but not recorded in the performance attribution system.

With K factors, Equation (16.9) can be generalized to

$$R_i = \alpha_i + \beta_{i1} y_1 + \cdots + \beta_{iK} y_K + \epsilon_i \tag{16.12}$$

As in the case of the CAPM, the **arbitrage pricing theory** (APT), developed by Professor Stephen Ross, shows that there is a a relationship between α_i and the factor exposures.[6]

In Equation (16.9) or (16.12), the intercept is also known as **Jensen's alpha**. This term is widely used in the investment management industry to describe the performance adjusted for market factors.

This decomposition is also useful to detect **timing ability**, which consists of adding value by changing exposures on risk factors.[7] A manager could, for example, move into stocks with higher betas in anticipation of the market going up. Timing ability can be detected by adding a quadratic term to Equation (16.9)

$$R_{P,t} - R_{F,t} = \alpha_P + \beta_P [R_{M,t} - R_{F,t}] + \delta_P [R_{M,t} - R_{F,t}]^2 + \epsilon_{P,t} \tag{16.13}$$

A positive coefficient δ_P indicates that the manager has added value from market timing, implying that beta is positively correlated with the market.

Return to the estimation of Equation (16.9). Denoting $\overline{R} = (1/T)\sum_{t=1}^{T}(R_t - R_{F,t})$ as the average over the sample period, the estimated alpha is

$$\widehat{\alpha} = \overline{R} - \widehat{\beta}\,\overline{R}_M \tag{16.14}$$

If there is no exposure to the market ($\beta = 0$), Equation (16.14) shows that alpha is the sample average of the investment returns. More generally, Equation (16.14) properly accounts for the exposure to the systematic risk factor. In the case of our investment fund, we have $\overline{R} = 6\%$, and $\beta = 1.5\%$. So, the alpha is

$$\widehat{\alpha} = 6\% - 1.5 \times 4\% = 0$$

which correctly indicates that there is no value added.

[6] The theory does not rely on equilibrium but simply on the assumption that there should be no arbitrage opportunities in capital markets, a much weaker requirement. It does not even need the factor model to hold strictly. Instead, it requires only that the residual risk is very small. This must be the case if a sufficient number of common factors is identified and in a well-diversified portfolio. The APT model does not require the market to be identified, which is an advantage. Like the CAPM, however, tests of this model are ambiguous since the theory provides no guidance as to what the factors should be.

[7] See Treynor and Mazuy (1966), Can Mutual Funds Outguess the Market? *Harvard Business Review* 44, 131–136.

KEY CONCEPT

Performance evaluation must take into account the component of returns that can be attributed to exposures on general market factors (or risk premia). An investment manager only adds value if the residual return, called alpha, is positive.

EXAMPLE 16.5: FRM EXAM 2007—QUESTION 132

Which of the following statements about the Sharpe ratio is *false*?

a. The Sharpe ratio considers both the systematic and unsystematic risks of a portfolio.
b. The Sharpe ratio is equal to the excess return of a portfolio over the risk-free rate divided by the total risk of the portfolio.
c. The Sharpe ratio cannot be used to evaluate relative performance of undiversified portfolios.
d. The Sharpe ratio is derived from the Capital Market Line.

EXAMPLE 16.6: PERFORMANCE EVALUATION

Assume that a hedge fund provides a large positive alpha. The fund can take leveraged long and short positions in stocks. The market went up over the period. Based on this information,

a. If the fund has net positive beta, all of the alpha must come from the market
b. If the fund has net negative beta, part of the alpha comes from the market
c. If the fund has net positive beta, part of the alpha comes from the market
d. If the fund has net negative beta, all of the alpha must come from the market

16.2.5 Performance Evaluation and Survivorship

Another key issue when evaluating the performance of a group of investment managers is **survivorship**. This occurs when funds are dropped from the investment universe for reasons related to poor performance and "survivors" only are considered. Commercial databases often give information on funds that are "alive" only, because clients are no longer interested in "dead" funds.

The problem is that the average performance of the group of funds under examination becomes subject to **survivorship bias**. In other words, the apparent performance of the existing funds is too high, or biased upward relative to the true performance of the underlying population, due to the omission of some poor performing funds.

The extent of this bias depends on the attrition rate of the funds and can be very severe. Mutual fund studies, for example, report an **attrition rate** of 3.6% per year. This represents the fraction of funds existing at the beginning of the year that becomes "dead" during the year. In this sample, the survivorship bias is estimated at approximately 0.70% per annum.[8] This represents the difference between the performance of the survived sample and that of the true population. This is a significant number because it is on the order of management fees, which are around 1% of assets per annum. Samples with higher attrition rates have larger biases. For example, **Commodity Trading Advisors** (CTA), a category of hedge funds, are reported to have an attrition rate of 16% per year, leading to survivorship biases on the order of 5.2% per annum, which is very high.[9]

Other sources of bias can be introduced, due to the inclusion criteria and the voluntary reporting of returns. A fund with excellent performance is more likely to be chosen for inclusion by the database vendor. Or, the investment manager of such a fund may be more inclined to submit the fund returns to the database. Consequently, there is a bias toward adding funds with better returns. Or, a fund may decide to stop reporting returns if its performance drops. This is called **selection bias**. This bias differs from the previous one because it also exists when dead funds are included in the sample.

Finally, another subtle bias arises when firms "incubate" different types of funds before making them available to outsiders. Say 10 different funds are started by the same company over a two-year period. Some will do well and others will not, partly due to chance. The best performing fund is then open to the public, with its performance instantly backfilled for the previous two years. The other funds are ignored or disbanded. As a result, the performance of the public fund is not representative of the entire sample. This is called **instant-history bias**. The difference between this bias and selection bias is that the fund was not open to investors during the reported period.

KEY CONCEPT

Performance evaluation can be overly optimistic if based on a sample of funds affected by survivorship, selection, or instant-history bias. The extent of survivorship bias increases with the attrition rate.

[8] Carhart, Mark, Jennifer Carpenter, Anthony Lynch, and David Musto (2002), Mutual Fund Survivorship, *Review of Financial Studies* 15, 1355–1381.

[9] CTAs are investment managers who trade futures and options. In the United States, they are regulated by the Commodity Futures Trading Commission (CTFC).

EXAMPLE 16.7: FRM EXAM 2005—QUESTION 103

A database of hedge fund returns is constructed as follows. The first year of the database is 1994. All funds existing as of the end of 1994 that were willing to report their verified returns for that year are included in that year.

The database was extended by asking the funds for verified returns before 1994. Subsequently, funds are added as they are willing to report verified returns to the database. If a fund stops reporting returns, its returns are deleted from the database, but the database has an agreement with funds that they will keep reporting verified returns even if they stop being open to new investors.

Consider the four following statements:

I. The database suffers from backfilling bias.
II. The database suffers from survivorship bias.
III. The database suffers from an errors-in-variables bias.
IV. The equally-weighted annual return average of fund returns will underestimate the performance one would expect from a hedge fund.

Which one of the following is correct?

a. All the above statements are correct.
b. Statements I and II are correct.
c. Statements I, II, and III are correct.
d. Statements II and IV are correct.

16.3 RISK BUDGETING

The revolution in risk management reflects the recognition that risk should be measured at the highest level—that is, firmwide or portfoliowide. This ability to measure total risk has led to a top-down allocation of risk, called **risk budgeting**. Risk budgeting is the process of parceling out the total risk of the fund, or risk budget, to various assets classes and managers.

This concept is being implemented by institutional investors as a follow-up to their **asset allocation process**. Asset allocation consists of finding the optimal allocation into major asset classes (i.e., the allocation that provides the best risk/return trade-off for the investor). This choice defines the total risk profile of the portfolio.

16.3.1 Illustration

Consider for instance an investor having to decide how much to invest in U.S. stocks, in U.S. bonds, and in non-U.S. bonds. Risk is measured in absolute terms, assuming returns have a joint normal distribution. More generally, this could be extended to other distributions or to a historical simulation method. The allocation will depend on the expected return and volatility of each asset class, as well as well as their correlations. Table 16.3 illustrates these data, which are based on historical dollar returns measured over the period 1978 to 2003.

Say the investor decides that the portfolio with the best risk/return trade-off has an expected return of 12.0% with total risk of 10.3%. Table 16.3 shows a portfolio allocation of 60.0%, 7.7%, and 32.3% to U.S. stocks, U.S. bonds, and non-U.S. bonds, respectively.

The volatility can be measured in terms of a 95% annual VAR. This defines a total risk budget of $\text{VAR} = \alpha\sigma W = 1.645 \times 10.3\% \times \$100 = \$16.9$ million. This VAR budget can then be parceled out to various asset classes and active managers within asset classes.

Risk budgeting is the process by which these efficient portfolio allocations are transformed into VAR assignments. At the asset class level, the individual VARs are $15.3, $0.9, and $5.9 million, respectively. For instance, the VAR budget for U.S. stocks is $60.0\% \times (1.645 \times 15.50\% \times \$100) = \$15.3$ million. Note that the sum of individual VARs is $22.1 million, which is greater than the portfolio VAR of $16.9 million due to diversification effects.

The process can be repeated at the next level. The fund has a risk budget of $15.3 million devoted to U.S. equities, with an allocation of $60 million. This allocation could be split equally between two active equity managers. Assume that the two managers are equally good, with a correlation of returns of 0.5. The optimal risk budget for each is then $8.83 million. We can verify that the total risk budget is

$$\sqrt{8.83^2 + 8.83^2 + 2 \times 0.5 \times 8.83 \times 8.83} = \sqrt{233.91} = \$15.3$$

Note that, as in the previous step, the sum of the risk budgets, which is $\$8.83 + \$8.83 = \$17.66$ million, is greater than the total risk budget of $15.3 million. This is because the latter takes into account diversification effects. If the two managers were perfectly correlated with each other, the risk budget would have to be $\$15.3/2 = \$7.65$ million for each. This higher risk budget is beneficial for the investor because it creates more opportunities to take advantage of the managers' positive alphas.

TABLE 16.3 Risk Budgeting

Asset		Expected Return	Volatility	Correlations 1	2	3	Percentage Allocation	VAR
U.S. stocks	1	13.80%	15.50%	1.00			60.0	$15.3
U.S. bonds	2	8.40%	7.40%	0.20	1.00		7.7	$0.9
Non-U.S. bonds	3	9.60%	11.10%	0.04	0.40	1.00	32.3	$5.9
Portfolio		12.00%	10.30%				100.0	$16.9

The risk budgeting process highlights the importance of correlations across managers. To control their risk better, institutional investors often choose equity managers that follow different market segments or strategies. For example, the first manager could invest in small growth stocks, the second in medium-size value stocks. Or the first manager could follow momentum-based strategies, the second value-based strategies. The first type tends to buy more of a stock after its price has gone up, and the second after the price has become more attractive (i.e., low). Different styles lead to low correlations across managers. For a given total risk budget, low correlations mean that each manager can be assigned a higher risk budget, leading to a greater value added for the fund.

These low correlations explain why investors much watch for **style drift**, which refers to a situation where an investment manager changes investment style. This is a problem for the investor because it can change the total portfolio risk. If all the managers, for instance, drift into the small-growth category, the total risk of the fund will increase. Style drift is controlled by the choice of benchmarks with different characteristics, such as small-growth and medium-value indices, and by controls on the tracking error volatility for each manager.

In conclusion, this risk budgeting approach is spreading rapidly to the field of investment management. This approach provides a consistent measure of risk across all subportfolios. It forces managers and investors to confront squarely the amount of risk they are willing to assume. It gives them tools to monitor their risk in real time.

16.3.2 Marginal Risk and Contribution to Risk

A well-designed risk system should also provide tools to understand how to manage risk. A risk report should display measures of **marginal risk**. This represents the change in risk due to a small increase in one of the allocations. Using the volatility of returns are the risk measure, this is

$$\text{MRISK} = \frac{\partial \sigma_p}{\partial w_i} = \frac{\text{cov}(R_i, R_p)}{\sigma_P} = \beta_{i,P}\sigma_P \tag{16.15}$$

Thus, beta represents the marginal contribution to the risk of the total portfolio P. A large value for β indicates that a small addition to this position will have a relatively large effect on the portfolio risk. Conversely, positions with large betas should be cut first because they will lead to the greatest reduction in risk.

This can be expanded to measure contributions to the portfolio risk. The **risk contribution**, or **risk allocation**, is obtained by multiplying the marginal risk for position i by its weight i

$$\text{CRISK} = w_i \beta_{i,P}\sigma_P \tag{16.16}$$

Because the beta of a portfolio with itself is one, the sum of $w_i\beta_{i,P}$ is guaranteed to be one. Hence, the sum of the risk contributions adds up exactly to the total

TABLE 16.4 Risk Analysis

Asset	Volatility	Market Allocation	Marginal Risk	Risk Allocation
U.S. stocks	15.50%	60.0%	0.1438	8.63%
U.S. bonds	7.40%	7.7%	0.0278	0.21%
Non-U.S. bonds	11.10%	32.3%	0.0451	1.46%
Portfolio	10.30%	100.0%		10.30%

portfolio risk, σ_P. When risk is expressed in terms of VAR, this measure is called **component VAR.**

Table 16.4 gives an example, expanding on the previous table. The marginal risk column shows that U.S. stocks are the asset class with the greatest marginal contribution to the risk of the portfolio. As an example, increasing the allocation from 60% to 61% increases the portfolio risk from 10.30% to 10.44%, which is an increase of 0.14%. This is precisely the marginal risk number of 0.14 multiplied by the 1% weight increase.

The last column shows the risk contribution, or allocation. Out of a total portfolio risk of 10.30%, 8.63% is attributed to U.S. stocks. This high number reflects the high volatility of this asset class, its high weight in the portfolio, as well as correlations. Reporting systems should therefore display not only the conventional weights, or market allocations, but also risk allocations.

Such analysis provides useful insights into the structure of the portfolio. Given a scarce risk budget, high risk allocations can only be justified by expected returns that are high relative to other assets. In fact, an exact relationship holds for portfolios that are mean-variance efficient, i.e., maximize the Sharpe ratio. If this is the case with portfolio P, then the ratio of excess returns on all assets to their marginal risk, which is also proportional to the Treynor ratio, must be the same. On the other hand, if P is not efficient, then we should be able to improve its performance by tilting toward assets that provide a greater ratio of expected return to their contribution to risk. Thus, this top-down analysis of portfolio risk can help investors improve the performance of their portfolios, given a set of risk measures and asset class forecasts.

EXAMPLE 16.8: PENSION FUND RISK

The AT&T pension fund reports total assets worth \$19.6 billion and liabilities of \$17.4 billion. Assume the surplus has a normal distribution and volatility of 10% per annum. The 95% Surplus at Risk over the next year is

a. \$360 million
b. \$513 million
c. \$2,860 million
d. \$3,220 million

EXAMPLE 16.9: RISK BUDGETING

The AT&T pension fund has 68%, or about $13 billion invested in equities. Assume a normal distribution and volatility of 15% per annum. The fund measures absolute risk with a 95%, one-year VAR, which gives $3.2 billion. The pension plan wants to allocate this risk to two equity managers, each with the same VAR budget. Given that the correlation between managers is 0.5, the VAR budget for each should be

a. $3.2 billion
b. $2.4 billion
c. $1.9 billion
d. $1.6 billion

EXAMPLE 16.10: FRM EXAM 2005—QUESTION 140

Suppose a portfolio consists of four assets. The risk contribution of each asset is as follows: UK Large Cap, 3.9%; UK Small Cap, 4.2%; UK Bonds, 0.9%; Non-UK Bonds, 1.1%. Which of the following explanations would not be a possible explanation for the relatively high risk contribution values for UK equities?

a. High expected returns on UK equities
b. High weights on UK equities
c. High volatilities of UK equities
d. High correlation of UK equities with all other assets in the portfolio

16.4 IMPORTANT FORMULAS

Absolute risk: $\sigma(\Delta P) = \sigma(\Delta P/P) \times P = \sigma(R_P) \times P$

Relative risk: $\sigma(e)P = [\sigma(R_P - R_B)] \times P = [\sigma(\Delta P/P - \Delta B/B)] \times P = \omega \times P$

Tracking error volatility (TEV): $\omega = \sigma(\Delta P/P - \Delta B/B)$

Sharpe ratio (SR): $\mathrm{SR} = \frac{[\mu(R_P) - R_F]}{\sigma(R_P)}$

Risk-adjusted performance (RAP): $\mathrm{RAP}_P = R_F + \frac{\sigma_B}{\sigma_P}[\mu(R_P) - R_F]$

Information ratio (IR): $\mathrm{IR} = \frac{[\mu(R_P) - \mu(R_B)]}{\omega}$

Alpha, from the intercept in: $R_{P,t} - R_{F,t} = \alpha + \beta_P[R_{M,t} - R_{F,t}] + \epsilon_{P,t}$

Treynor ratio (TR): $\mathrm{TR} = \frac{[\mu(R_P) - R_F]}{\beta_P}$

Market timing skill, positive δ in:
$R_{P,t} - R_{F,t} = \alpha_P + \beta_P[R_{M,t} - R_{F,t}] + \delta_P[R_{M,t} - R_{F,t}]^2 + \epsilon_{P,t}$
Marginal risk: The change in total portfolio risk due to a small change in position i
$\text{MRISK} = \frac{\partial \sigma_P}{\partial w_i} = \frac{\text{cov}(R_i, R_P)}{\sigma_P} = \beta_{i,P}\sigma_P$
Risk contribution: A component of total portfolio risk due to one position
$\text{CRISK} = w_i \beta_{i,P}\sigma_P$

16.5 ANSWERS TO CHAPTER EXAMPLES

Example 16.1: Absolute and Relative Risk

c. This is an example of risk measured in terms of deviations of the active portfolio relative to the benchmark. Answers a. and b. are incorrect because they refer to absolute risk. Answer d. is also incorrect because it refers to the absolute risk of the benchmark.

Example 16.2: Pension Liabilities

d. We can compute the modified duration of the liabilities as $D^* = -(\Delta P/P)/\Delta y = -(0.8/17.4)/0.0005 = 9.2$ years. So, the liabilities behave like a short position in a bond with a duration around 9 years. Answers a. and b. are incorrect because the liabilities have fixed future payoffs, which do not resemble cash flow patterns on equities nor cash. Answer c. is incorrect because the duration of a bond with a nine-year maturity is less than nine years. For example, the duration of a 6% coupon par bond with nine-year maturity is seven years only.

Example 16.3: FRM Exam 2006—Question 25

c. Answer I. is incorrect because this liability is similar to a *short* (not long) position in a bond. Answer II. is correct because surplus at risk is a relative risk measure, assets minus liabilities. Answer III. is incorrect because it needs to invest \$8.8 billion, not \$8 billion.

Example 16.4: Sharpe and Information Ratios

d. The Sharpe ratios of the portfolio and benchmark are $(10\% - 3\%)/20\% = 0.35$, and $(8\% - 3\%)/14\% = 0.36$, respectively. So, the SR of the portfolio is lower than that of the benchmark. Answer a. is incorrect. The TEV is the square root of $20\%^2 + 14\%^2 - 2 \times 0.98 \times 20\% \times 14\%$, which is $\sqrt{0.00472} = 6.87\%$. So, the IR of the portfolio is $(10\% - 8\%)/6.87\% = 0.29$. This is positive, so answer b. is incorrect. Answer c. is the SR of the portfolio, not the IR, so it is incorrect.

Example 16.5: FRM Exam 2007—Question 132

c. The SR considers total risk, which includes systematic and unsystematic risks, so a. and b. are correct statements, and incorrect answers. Similarly, the SR is derived from the CML, which states that the market is mean-variance efficient and hence has the highest Sharpe ratio of any feasible portfolio. Finally, the SR can be used to evaluate undiversified portfolios, precisely because it includes idiosyncratic risk.

Example 16.6: Performance Evaluation

c. Because the market went up, a portfolio with positive beta will have part of its positive performance due to the market effect. A portfolio with negative beta will have in part a negative performance due to the market. Answer a. is incorrect because the fund manager could still have generated some of its alpha through judicious stock-picking. Answers b. and d. are incorrect because a negative beta combined with a market going up should lead to a decrease, not an increase, in the alpha.

Example 16.7: FRM Exam 2005—Question 103

b. The database includes histories before 1994 and therefore suffers from backfill bias. Next, funds that stop reporting are deleted from the database, so this has survival bias. Errors-in-variables biases arise in other contexts, such as regression. Finally, the average of fund returns will be too high (not too low) because of these two biases. Hence, I. and II. are correct.

Example 16.8: Pension Fund Risk

a. The fund's surplus is the excess of assets over liabilities, which $19.6 − $17.4 = $2.2 billion. The Surplus at Risk at the 95% level over one year is, assuming a normal distribution, 1.645 × 10% × $2,200 = $360 million. Answer b. is incorrect because it uses a 99% confidence level. Answers c. and d. are incorrect because they apply the risk to the liabilities and assets instead of the surplus.

Example 16.9: Risk Budgeting

c. Call x the risk budget allocation to each manager. This should be such that $x^2 + x^2 + 2\rho\, xx = \$3.2^2$. Solving for $x\sqrt{1+1+2\rho} = x\sqrt{3} = \$3.2$, we find $x =$ $1.85 billion. Answer a. is incorrect because it refers to the total VAR. Answer b. is incorrect because it assumes a correlation of zero. Answer d. is incorrect because it simply divides the $3.2 billion VAR by 2, which ignores diversification effects.

Example 16.10: FRM Exam 2005—Question 140

a. The risk contribution is proportional to the weight times the beta. The latter involves the correlation between the asset and the portfolio, as well as the volatility of the asset. Higher weight, correlation, and volatility would create higher risk contribution. On the other hand, high expected returns would explain a high weight, but not risk contribution.

CHAPTER 17

Hedge Fund Risk Management

The first hedge fund was started by A.W. Jones in 1949. Unlike the typical equity mutual fund, the fund took long *and* short positions in equities. Over the last decades, the hedge fund industry has undergone exponential growth. As of December 2007, hedge funds accounted for more than $1,900 billion in equity capital, called **assets under management** (AUM). **Hedge funds** are private partnership funds that have very few limitations on investment strategy. As a result, they can take long and short positions in various markets and also allow the use of leverage. Due to this leverage, the assets they control are greater than their AUM. Hedge funds have become an important force in financial markets, accounting for the bulk of trading in some markets.

Unlike mutual funds, who are open to any investor, hedge funds are accessible only to accredited investors. This is because of their perceived risk, which can be traced to their use of leverage and short positions. To control their risk, most hedge funds have also adopted risk controls using position-based, VAR-type techniques. Because some types of hedge fund strategies are very similar to those of proprietary trading desks of commercial banks, it was only natural for hedge funds to adopt similar risk management tools.

The purpose of this chapter is to provide an overview of risk management for the hedge fund industry. Section 17.1 gives an introduction to the hedge fund industry. Section 17.2 presents the mechanics of shorting and various measures of leverage. Section 17.3 then analyzes commonly used strategies for hedge funds and shows how to identify and measure their risk. The risk factors that are largely specific to hedge funds are presented in Section 17.4. Section 17.5 shows how to deal with hedge fund risk. Finally, Section 17.6 discusses the general role of hedge funds in financial markets.

17.1 THE HEDGE FUND INDUSTRY

The growth of the hedge fund industry is described in Figure 17.1. By now, there are close to 10,000 hedge fund managers controlling close to $1,900 billion in equity capital, also called net assets, up from $30 billion in 1990. This represents

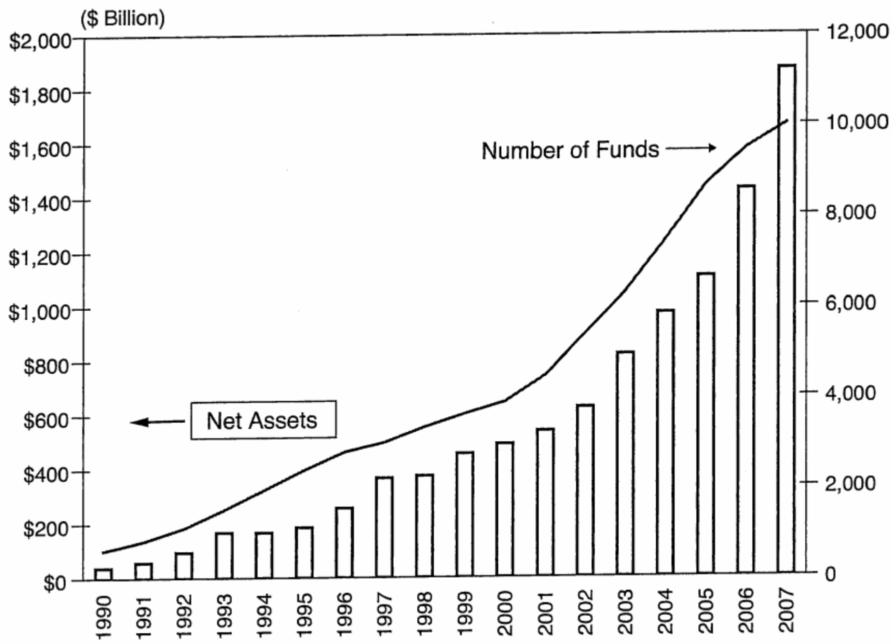

FIGURE 17.1 Growth of Hedge Fund Industry
Source: Hedge Fund Research. Data as of December of each year.

an annualized rate of growth of 25%. In comparison, U.S. mutual funds currently manage $12 trillion, up from $1.1 trillion in 1991. This represents an annualized rate of growth of 15%. Thus, hedge funds have grown at twice the rate of mutual funds over the same period. In 2008, however, the credit crisis will cause a contraction of the entire asset management industry.

The growth of this industry is due to a number of factors. On the investor side, the performance of hedge funds has been attractive, especially compared to the poor record of stock markets during the 2000–2002 period. Hedge funds also claim to have low beta, which makes them useful as diversifiers.

On the manager side, hedge funds provide greater remuneration than traditional investment funds. Typical investment management fees for mutual funds range from a fixed 0.5% to 2% of AUM. In contrast, hedge funds commonly charge a fixed management fee of 1% to 2% of assets plus an incentive fee of 20% of profits.

Hedge funds also typically have fewer restrictions on their investment strategy and are less regulated, giving more leeway to portfolio managers. More flexible investment opportunities include the ability to short securities, to leverage the portfolio, to invest in derivatives, and generally to invest a cross a broader pool of assets. The lighter regulatory environment creates an ability to set performance fees, lockups periods, or other forms of managerial discretion.

The unprecedented turbulence of 2008, however, has hit the hedge fund industry hard. Many funds posted poor performance and suffered widespread investor redemptions, leading to many hedge fund closures. Even so, hedge funds in general suffered only half the loss of equities.