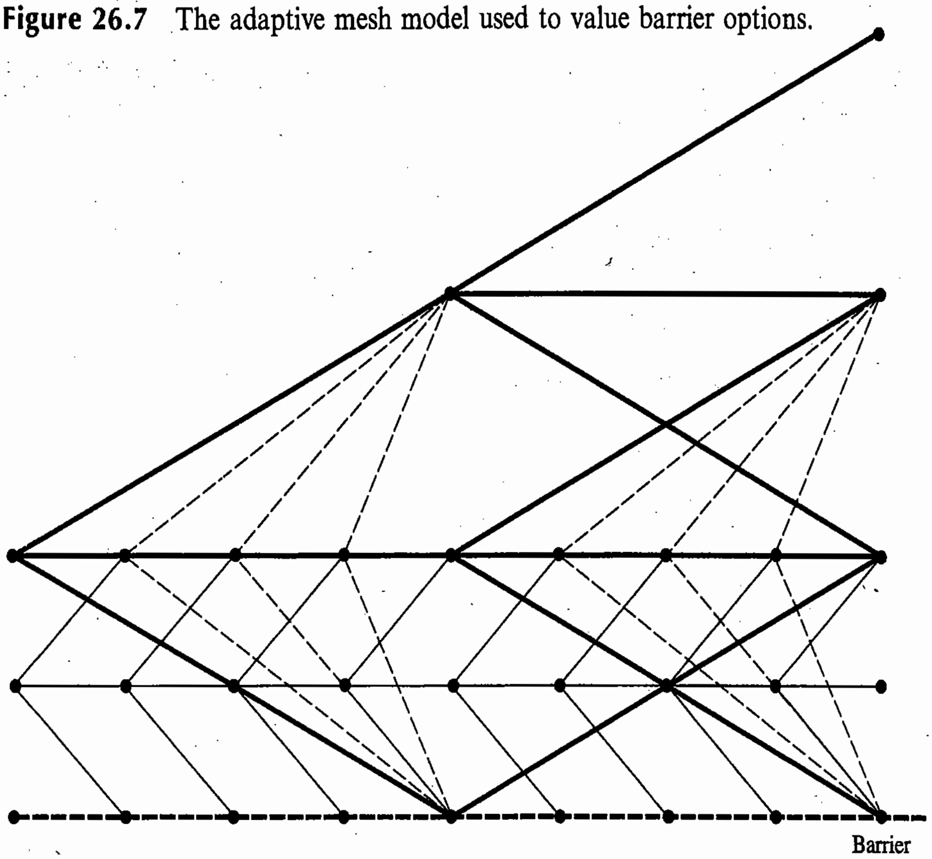
Figure 26.7 The adaptive mesh model used to value barrier options.



tree to achieve a more detailed modeling of the asset price in the regions of the tree where it is needed most.

To value a barrier option, it is useful to have a fine tree close to barriers. Figure 26.7 illustrates the design of the tree. The geometry of the tree is arranged so that nodes lie on the barriers. The probabilities on branches are chosen, as usual, to match the first two moments of the process followed by the underlying asset. The heavy lines in Figure 26.7 are the branches of the coarse tree. The light solid line are the fine tree. We first roll back through the coarse tree in the usual way. We then calculate the value at additional nodes using the branches indicated by the dotted lines. Finally we roll back through the fine tree.

26.7 OPTIONS ON TWO CORRELATED ASSETS

Another tricky numerical problem is that of valuing American options dependent on two assets whose prices are correlated. A number of alternative approaches have been suggested. This section will explain three of these.

Transforming Variables

It is relatively easy to construct a tree in three dimensions to represent the movements of two *uncorrelated* variables. The procedure is as follows. First, construct a two-dimensional tree for each variable, and then combine these trees into a single three-dimensional tree. The probabilities on the branches of the three-dimensional tree are the product of the corresponding probabilities on the two-dimensional trees. Suppose, for example, that the variables are stock prices, S_1 and S_2. Each can be represented in two dimensions by a Cox, Ross, and Rubinstein binomial tree. Assume that S_1 has a probability p_1 of moving up by a proportional amount u_1 and a probability $1 - p_1$ of moving down by a proportional amount d_1. Suppose further that S_2 has a probability p_2 of moving up by a proportional amount u_2 and a probability $1 - p_2$ of moving down by a proportional amount d_2. In the three-dimensional tree there are four branches emanating from each node. The probabilities are:

$$\begin{aligned} p_1 p_2&: S_1 \text{ increases; } S_2 \text{ increases} \\ p_1(1-p_2)&: S_1 \text{ increases; } S_2 \text{ decreases} \\ (1-p_1)p_2&: S_1 \text{ decreases; } S_2 \text{ increases} \\ (1-p_1)(1-p_2)&: S_1 \text{ decreases; } S_2 \text{ decreases} \end{aligned}$$

Consider next the situation where S_1 and S_2 are correlated. Suppose that the risk-neutral processes are:

$$dS_1 = (r - q_1)S_1\,dt + \sigma_1 S_1\,dz_1$$

$$dS_2 = (r - q_2)S_2\,dt + \sigma_2 S_2\,dz_2$$

and the instantaneous correlation between the Wiener processes, dz_1 and dz_2, is ρ. This means that

$$d\ln S_1 = (r - q_1 - \sigma_1^2/2)\,dt + \sigma_1\,dz_1$$

$$d\ln S_2 = (r - q_2 - \sigma_2^2/2)\,dt + \sigma_2\,dz_2$$

Two new uncorrelated variables can be defined:[23]

$$x_1 = \sigma_2 \ln S_1 + \sigma_1 \ln S_2$$

$$x_2 = \sigma_2 \ln S_1 - \sigma_1 \ln S_2$$

These variables follow the processes

$$dx_1 = [\sigma_2(r - q_1 - \sigma_1^2/2) + \sigma_1(r - q_2 - \sigma_2^2/2)]\,dt + \sigma_1\sigma_2\sqrt{2(1+\rho)}\,dz_A$$

$$dx_2 = [\sigma_2(r - q_1 - \sigma_1^2/2) - \sigma_1(r - q_2 - \sigma_2^2/2)]\,dt + \sigma_1\sigma_2\sqrt{2(1-\rho)}\,dz_B$$

where dz_A and dz_B are uncorrelated Wiener processes.

The variables x_1 and x_2 can be modeled using two separate binomial trees. In time Δt, x_i has a probability p_i of increasing by h_i and a probability $1 - p_i$ of decreasing by h_i. The variables h_i and p_i are chosen so that the tree gives correct values for the first two moments of the distribution of x_1 and x_2. Because they are uncorrelated, the two trees can be combined into a single three-dimensional tree, as already described. At each node of the tree, S_1 and S_2 can be calculated from x_1 and x_2 using the inverse

[23] This idea was suggested in J. Hull and A. White, "Valuing Derivative Securities Using the Explicit Finite Difference Method," *Journal of Financial and Quantitative Analysis*, 25 (1990): 87–100.

relationships

$$S_1 = \exp\left[\frac{x_1 + x_2}{2\sigma_2}\right] \quad \text{and} \quad S_2 = \exp\left[\frac{x_1 - x_2}{2\sigma_1}\right]$$

The procedure for rolling back through a three-dimensional tree to value a derivative is analogous to that for a two-dimensional tree.

Using a Nonrectangular Tree

Rubinstein has suggested a way of building a three-dimensional tree for two correlated stock prices by using a nonrectangular arrangement of the nodes.[24] From a node (S_1, S_2), where the first stock price is S_1 and the second stock price is S_2, there is a 0.25 chance of moving to each of the following:

$$(S_1u_1, S_2A), \quad (S_1u_1, S_2B), \quad (S_1d_1, S_2C), \quad (S_2d_1, S_2D)$$

where

$$u_1 = \exp[(r - q_1 - \sigma_1^2/2)\,\Delta t + \sigma_1\sqrt{\Delta t}\,]$$
$$d_1 = \exp[(r - q_1 - \sigma_1^2/2)\,\Delta t - \sigma_1\sqrt{\Delta t}\,]$$

and

$$A = \exp[(r - q_2 - \sigma_2^2/2)\Delta t + \sigma_2\sqrt{\Delta t}\,(\rho + \sqrt{1-\rho^2}\,)]$$
$$B = \exp[(r - q_2 - \sigma_2^2/2)\Delta t + \sigma_2\sqrt{\Delta t}\,(\rho - \sqrt{1-\rho^2}\,)]$$
$$C = \exp[(r - q_2 - \sigma_2^2/2)\Delta t - \sigma_2\sqrt{\Delta t}\,(\rho - \sqrt{1-\rho^2}\,)]$$
$$D = \exp[(r - q_2 - \sigma_2^2/2)\Delta t - \sigma_2\sqrt{\Delta t}\,(\rho + \sqrt{1-\rho^2}\,)]$$

When the correlation is zero, this method is equivalent to constructing separate trees for S_1 and S_2 using the alternative binomial tree construction method in Section 19.4.

Adjusting the Probabilities

A third approach to building a three-dimensional tree for S_1 and S_2 involves first assuming no correlation and then adjusting the probabilities at each node to reflect the correlation.[25] The alternative binomial tree construction method for each of S_1 and S_2 in Section 19.4 is used. This method has the property that all probabilities are 0.5. When the two binomial trees are combined on the assumption that there is no correlation, the probabilities are as shown in Table 26.1. When the probabilities are adjusted to reflect the correlation, they become those shown in Table 26.2.

26.8 MONTE CARLO SIMULATION AND AMERICAN OPTIONS

Monte Carlo simulation is well suited to valuing path-dependent options and options where there are many stochastic variables. Trees and finite difference methods are well

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[24] See M. Rubinstein, "Return to Oz," *Risk*, November (1994): 67–70.

[25] This approach was suggested in the context of interest rate trees in J. Hull and A. White, "Numerical Procedures for Implementing Term Structure Models II: Two-Factor Models," *Journal of Derivatives*, Winter (1994): 37–48.

Table 26.1 Combination of binomials assuming no correlation.

S_2-move	S_1-move	
	Down	*Up*
Up	0.25	0.25
Down	0.25	0.25

suited to valuing American-style options. What happens if an option is both path dependent and American? What happens if an American option depends on several stochastic variables? Section 26.5 explained a way in which the binomial tree approach can be modified to value path-dependent options in some situations. A number of researchers have adopted a different approach by searching for a way in which Monte Carlo simulation can be used to value American-style options.[26] This section explains two alternative ways of proceeding.

The Least-Squares Approach

In order to value an American-style option it is necessary to choose between exercising and continuing at each early exercise point. The value of exercising is normally easy to determine. A number of researchers including Longstaff and Schwartz provide a way of determining the value of continuing when Monte Carlo simulation is used.[27] Their approach involves using a least-squares analysis to determine the best-fit relationship between the value of continuing and the values of relevant variables at each time an early exercise decision has to be made. The approach is best illustrated with a numerical example. We use the one in the Longstaff–Schwartz paper.

Consider a 3-year American put option on a non-dividend-paying stock that can be exercised at the end of year 1, the end of year 2, and the end of year 3. The risk-free rate is 6% per annum (continuously compounded). The current stock price is 1.00 and the strike price is 1.10. Assume that the eight paths shown in Table 26.3 are sampled for the stock price. (This example is for illustration only; in practice many more paths would be

Table 26.2 Combination of binomials assuming correlation of ρ.

S_2-move	S_1-move	
	Down	*Up*
Up	$0.25(1-\rho)$	$0.25(1+\rho)$
Down	$0.25(1+\rho)$	$0.25(1-\rho)$

[26] Tilley was the first researcher to publish a solution to the problem. See J. A. Tilley, "Valuing American Options in a Path Simulation Model," *Transactions of the Society of Actuaries*, 45 (1993): 83–104.

[27] See F. A. Longstaff and E. S. Schwartz, "Valuing American Options by Simulation: A Simple Least-Squares Approach," *Review of Financial Studies*, 14, 1 (Spring 2001): 113–47.

Table 26.3 Sample paths for put option example.

Path	$t=0$	$t=1$	$t=2$	$t=3$
1	1.00	1.09	1.08	1.34
2	1.00	1.16	1.26	1.54
3	1.00	1.22	1.07	1.03
4	1.00	0.93	0.97	0.92
5	1.00	1.11	1.56	1.52
6	1.00	0.76	0.77	0.90
7	1.00	0.92	0.84	1.01
8	1.00	0.88	1.22	1.34

sampled.) If the option can be exercised only at the 3-year point, it provides a cash flow equal to its intrinsic value at that point. This is shown in the last column of Table 26.4.

If the put option is in the money at the 2-year point, the option holder must decide whether to exercise. Table 26.3 shows that the option is in the money at the 2-year point for paths 1, 3, 4, 6, and 7. For these paths, we assume an approximate relationship:

$$V = a + bS + cS^2$$

where S is the stock price at the 2-year point and V is the value of continuing, discounted back to the 2-year point. Our five observations on S are: 1.08, 1.07, 0.97, 0.77, and 0.84. From Table 26.4 the corresponding values for V are: 0.00, $0.07e^{-0.06\times 1}$, $0.18e^{-0.06\times 1}$, $0.20e^{-0.06\times 1}$, and $0.09e^{-0.06\times 1}$. The values of a, b, and c that minimize

$$\sum_{i=1}^{5}(V_i - a - bS_i - cS_i^2)^2$$

where S_i and V_i are the ith observation on S and V, respectively, are $a = -1.070$, $b = 2.983$ and $c = -1.813$, so that the best-fit relationship is

$$V = -1.070 + 2.983S - 1.813S^2$$

Table 26.4 Cash flows if exercise only at the 3-year point.

Path	$t=1$	$t=2$	$t=3$
1	0.00	0.00	0.00
2	0.00	0.00	0.00
3	0.00	0.00	0.07
4	0.00	0.00	0.18
5	0.00	0.00	0.00
6	0.00	0.00	0.20
7	0.00	0.00	0.09
8	0.00	0.00	0.00

Table 26.5 Cash flows if exercise only possible at 2- and 3-year point.

Path	$t=1$	$t=2$	$t=3$
1	0.00	0.00	0.00
2	0.00	0.00	0.00
3	0.00	0.00	0.07
4	0.00	0.13	0.00
5	0.00	0.00	0.00
6	0.00	0.33	0.00
7	0.00	0.26	0.00
8	0.00	0.00	0.00

This gives the value at the 2-year point of continuing for paths 1, 3, 4, 6, and 7 of 0.0369, 0.0461, 0.1176, 0.1520, and 0.1565, respectively. From Table 26.3 the value of exercising is 0.02, 0.03, 0.13, 0.33, and 0.26. This means that we should exercise at the 2-year point for paths 4, 6, and 7. Table 26.5 summarizes the cash flows assuming exercise at either the 2-year point or the 3-year point for the eight paths.

Consider next the paths that are in the money at the 1-year point. These are paths 1, 4, 6, 7, and 8. From Table 26.3 the values of S for the paths are 1.09, 0.93, 0.76, 0.92, and 0.88, respectively. From Table 26.5, the corresponding continuation values discounted back to $t=1$ are 0.00, $0.13e^{-0.06\times 1}$, $0.33e^{-0.06\times 1}$, $0.26e^{-0.06\times 1}$, and 0.00, respectively. The least-squares relationship is

$$V = 2.038 - 3.335S + 1.356S^2$$

This gives the value of continuing at the 1-year point for paths 1, 4, 6, 7, 8 as 0.0139, 0.1092, 0.2866, 0.1175, and 0.1533, respectively. From Table 26.3 the value of exercising is 0.01, 0.17, 0.34, 0.18, and 0.22, respectively. This means that we should exercise at the 1-year point for paths 4, 6, 7, and 8. Table 26.6 summarizes the cash flows assuming that early exercise is possible at all three times. The value of the option is determined by discounting each cash flow back to time zero at the risk-free rate and calculating the

Table 26.6 Cash flows from option.

Path	$t=1$	$t=2$	$t=3$
1	0.00	0.00	0.00
2	0.00	0.00	0.00
3	0.00	0.00	0.07
4	0.17	0.00	0.00
5	0.00	0.00	0.00
6	0.34	0.00	0.00
7	0.18	0.00	0.00
8	0.22	0.00	0.00

mean of the results. It is

$$\tfrac{1}{8}(0.07e^{-0.06\times 3}+0.17e^{-0.06\times 1}+0.34e^{-0.06\times 1}+0.18e^{-0.06\times 1}+0.22e^{-0.06\times 1})=0.1144$$

Because this is greater than 0.10, it is not optimal to exercise the option immediately.

This method can be extended in a number of ways. If the option can be exercised at any time we can approximate its value by considering a large number of exercise points (just as a binomial tree does). The relationship between V and S can be assumed to be more complicated. For example we could assume that V is a cubic rather than a quadratic function of S. The method can be used where the early exercise decision depends on several state variables. A functional form for the relationship between V and the variables is assumed and the parameters are estimated using the least-squares approach, as in the example just considered.

The Exercise Boundary Parameterization Approach

A number of researchers, such as Andersen, have proposed an alternative approach where the early exercise boundary is parameterized and the optimal values of the parameters are determined iteratively by starting at the end of the life of the option and working backward.[28] To illustrate the approach, we continue with the put option example and assume that the eight paths shown in Table 26.3 have been sampled. In this case, the early exercise boundary at time t can be parameterized by a critical value of S, $S^*(t)$. If the asset price at time t is below $S^*(t)$ we exercise at time t; if it is above $S^*(t)$ we do not exercise at time t. The value of $S^*(3)$ is 1.10. If the stock price is above 1.10 when $t=3$ (the end of the option's life) we do not exercise; if it is below 1.10 we exercise. We now consider the determination of $S^*(2)$.

Suppose that we choose a value of $S^*(2)$ less than 0.77. The option is not exercised at the 2-year point for any of the paths. The value of the option at the 2-year point for the eight paths is then 0.00, 0.00, $0.07e^{-0.06\times 1}$, $0.18e^{-0.06\times 1}$, 0.00, $0.20e^{-0.06\times 1}$, $0.09e^{-0.06\times 1}$, and 0.00, respectively. The average of these is 0.0636. Suppose next that $S^*(2)=0.77$. The value of the option at the 2-year point for the eight paths is then 0.00, 0.00, $0.07e^{-0.06\times 1}$, $0.18e^{-0.06\times 1}$, 0.00, 0.33, $0.09e^{-0.06\times 1}$, and 0.00, respectively. The average of these is 0.0813. Similarly when $S^*(2)$ equals 0.84, 0.97, 1.07, and 1.08, the average value of the option at the 2-year point is 0.1032, 0.0982, 0.0938, and 0.0963, respectively. This analysis shows that the optimal value of $S^*(2)$ (i.e., the one that maximizes the average value of the option) is 0.84. (More precisely, it is optimal to choose $0.84 \leqslant S^*(2) < 0.97$.) When we choose this optimal value for $S^*(2)$, the value of the option at the 2-year point for the eight paths is 0.00, 0.00, 0.0659, 0.1695, 0.00, 0.33, 0.26, and 0.00, respectively. The average value is 0.1032.

We now move on to calculate $S^*(1)$. If $S^*(1) < 0.76$ the option is not exercised at the 1-year point for any of the paths and the value at the option at the 1-year point is $0.1032e^{-0.06\times 1}=0.0972$. If $S^*(1)=0.76$, the value of the option for each of the eight paths at the 1-year point is 0.00, 0.00, $0.0659e^{-0.06\times 1}$, $0.1695e^{-0.06\times 1}$, 0.0, 0.34, $0.26e^{-0.06\times 1}$, and 0.00, respectively. The average value of the option is 0.1008. Similarly when $S^*(1)$ equals 0.88, 0.92, 0.93, and 1.09 the average value of the option is 0.1283, 0.1202, 0.1215, and 0.1228, respectively. The analysis therefore shows that the optimal

[28] See L. Andersen, "A Simple Approach to the Pricing of Bermudan Swaptions in the Multifactor LIBOR Market Model," *Journal of Computational Finance*, 3, 2 (Winter 2000): 1–32.

value of $S^*(1)$ is 0.88. (More precisely, it is optimal to choose $0.88 \leqslant S^*(1) < 0.92$.) The value of the option at time zero with no early exercise is $0.1283e^{-0.06\times 1} = 0.1208$. This is greater than the value of 0.10 obtained by exercising at time zero.

In practice, tens of thousands of simulations are carried out to determine the early exercise boundary in the way we have described. Once the early exercise boundary has been obtained, the paths for the variables are discarded and a new Monte Carlo simulation using the early exercise boundary is carried out to value the option. Our American put option example is simple in that we know that the early exercise boundary at a time can be defined entirely in terms of the value of the stock price at that time. In more complicated situations it is necessary to make assumptions about how the early exercise boundary should be parameterized.

Upper Bounds

The two approaches we have outlined tend to underprice American-style options because they assume a suboptimal early exercise boundary. This has led Andersen and Broadie to propose a procedure that provides an upper bound to the price.[29] This procedure can be used in conjunction with any algorithm that generates a lower bound and pinpoints the true value of an American-style option more precisely than the algorithm does by itself.

SUMMARY

A number of models have been developed to fit the volatility smiles that are observed in practice. The constant elasticity of variance model leads to a volatility smile similar to that observed for equity options. The jump–diffusion model leads to a volatility smile similar to that observed for currency options. Variance-gamma and stochastic volatility models are more flexible in that they can lead to either the type of volatility smile observed for equity options or the type of volatility smile observed for currency options. The implied volatility function model provides even more flexibility than this. It is designed to provide an exact fit to any pattern of European option prices observed in the market.

The natural technique to use for valuing path-dependent options is Monte Carlo simulation. This has the disadvantage that it is fairly slow and unable to handle American-style derivatives easily. Luckily, trees can be used to value many types of path-dependent derivatives. The approach is to choose representative values for the underlying path function at each node of the tree and calculate the value of the derivative for each of these values as we roll back through the tree.

The binomial tree methodology can be extended to value convertible bonds. Extra branches corresponding to a default by the company are added to the tree. The roll-back calculations then reflect the holder's option to convert and the issuer's option to call.

Trees can be used to value many types of barrier options, but the convergence of the option value to the correct value as the number of time steps is increased tends to be slow. One approach for improving convergence is to arrange the geometry of the tree so

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[29] See L. Andersen and M. Broadie, "A Primal-Dual Simulation Algorithm for Pricing Multi-Dimensional American Options," *Management Science*, 50, 9 (2004), 1222–34.

that nodes always lie on the barriers. Another is to use an interpolation scheme to adjust for the fact that the barrier being assumed by the tree is different from the true barrier. A third is to design the tree so that it provides a finer representation of movements in the underlying asset price near the barrier.

One way of valuing options dependent on the prices of two correlated assets is to apply a transformation to the asset price to create two new uncorrelated variables. These two variables are each modeled with trees and the trees are then combined to form a single three-dimensional tree. At each node of the tree, the inverse of the transformation gives the asset prices. A second approach is to arrange the positions of nodes on the three-dimensional tree to reflect the correlation. A third approach is to start with a tree that assumes no correlation between the variables and then adjust the probabilities on the tree to reflect the correlation.

Monte Carlo simulation is not naturally suited to valuing American-style options, but there are two ways it can be adapted to handle them. The first involves using a least-squares analysis to relate the value of continuing (i.e, not exercising) to the values of relevant variables. The second involves parameterizing the early exercise boundary and determining it iteratively by working back from the end of the life of the option to the beginning.

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Questions and Problems (Answers in Solutions Manual)

26.1. Confirm that the CEV model formulas satisfy put–call parity.

26.2. Explain how you would use Monte Carlo simulation to sample paths for the asset price when Merton's jump–diffusion model is used.

26.3. Confirm that Merton's jump–diffusion model satisfies put–call parity when the jump size is lognormal.

26.4. Suppose that the volatility of an asset will be 20% from month 0 to month 6, 22% from month 6 to month 12, and 24% from month 12 to month 24. What volatility should be used in Black–Scholes to value a 2-year option?

26.5. Consider the case of Merton's jump–diffusion model where jumps always reduce the asset price to zero. Assume that the average number of jumps per year is λ. Show that the price of a European call option is the same as in a world with no jumps except that the risk-free rate is $r+\lambda$ rather than r. Does the possibility of jumps increase or reduce the value of the call option in this case? (*Hint*: Value the option assuming no jumps and assuming one or more jumps. The probability of no jumps in time T is $e^{-\lambda T}$).

26.6. At time 0 the price of a non-dividend-paying stock is S_0. Suppose that the time interval between 0 and T is divided into two subintervals of length t_1 and t_2. During the first subinterval, the risk-free interest rate and volatility are r_1 and σ_1, respectively. During the second subinterval, they are r_2 and σ_2, respectively. Assume that the world is risk neutral.

(a) Use the results in Chapter 13 to determine the stock price distribution at time T in terms of r_1, r_2, σ_1, σ_2, t_1, t_2, and S_0.

(b) Suppose that $\bar{r}$ is the average interest rate between time zero and T and that $\bar{V}$ is the average variance rate between times zero and T. What is the stock price distribution as a function of T in terms of $\bar{r}$, $\bar{V}$, T, and S_0?

(c) What are the results corresponding to (a) and (b) when there are three subintervals with different interest rates and volatilities?

(d) Show that if the risk-free rate, r, and the volatility, σ, are known functions of time, the stock price distribution at time T in a risk-neutral world is

$$\ln S_T \sim \phi\left[\ln S_0 + \left(\bar{r} - \frac{\bar{V}}{2}\right)T,\ \sqrt{\bar{V}T}\right]$$

where $\bar{r}$ is the average value of r, $\bar{V}$ is equal to the average value of σ^2, and S_0 is the stock price today.

26.7. Write down the equations for simulating the path followed by the asset price in the stochastic volatility model in equations (26.2) and (26.3).

26.8. "The IVF model does not necessarily get the evolution of the volatility surface correct." Explain this statement.

26.9. "When interest rates are constant the IVF model correctly values any derivative whose payoff depends on the value of the underlying asset at only one time." Explain this statement.

26.10. Use a three-time-step tree to value an American floating lookback call option on a currency when the initial exchange rate is 1.6, the domestic risk-free rate is 5% per annum, the foreign risk-free interest rate is 8% per annum, the exchange rate volatility is 15%, and the time to maturity is 18 months. Use the approach in Section 26.5.

26.11. What happens to the variance-gamma model as the parameter v tends to zero?

26.12. Use a three-time-step tree to value an American put option on the geometric average of the price of a non-dividend-paying stock when the stock price is $40, the strike price is $40, the risk-free interest rate is 10% per annum, the volatility is 35% per annum, and the time to maturity is three months. The geometric average is measured from today until the option matures.

26.13. Can the approach for valuing path-dependent options in Section 26.5 be used for a 2-year American-style option that provides a payoff equal to $\max(S_{\text{ave}} - K, 0)$, where S_{ave} is the average asset price over the three months preceding exercise? Explain your answer.

26.14. Verify that the 6.492 number in Figure 26.4 is correct.

26.15. Examine the early exercise policy for the eight paths considered in the example in Section 26.8. What is the difference between the early exercise policy given by the least squares approach and the exercise boundary parameterization approach? Which gives a higher option price for the paths sampled?

26.16. Consider a European put option on a non-dividend paying stock when the stock price is $100, the strike price is $110, the risk-free rate is 5% per annum, and the time to maturity is one year. Suppose that the average variance rate during the life of an option has a 0.20 probability of being 0.06, a 0.5 probability of being 0.09, and a 0.3 probability of being 0.12. The volatility is uncorrelated with the stock price. Estimate the value of the option. Use DerivaGem.

26.17. When there are two barriers how can a tree be designed so that nodes lie on both barriers?

26.18. Consider an 18-month zero-coupon bond with a face value of $100 that can be converted into five shares of the company's stock at any time during its life. Suppose that the current share price is $20, no dividends are paid on the stock, the risk-free rate for all maturities is 6% per annum with continuous compounding, and the share price volatility is 25% per annum. Assume that the default intensity is 3% per year and the recovery rate is 35%. The bond is callable at $110. Use a three-time-step tree to calculate the value of the bond. What is the value of the conversion option (net of the issuer's call option)?

Assignment Questions

26.19. A new European-style floating lookback call option on a stock index has a maturity of 9 months. The current level of the index is 400, the risk-free rate is 6% per annum, the dividend yield on the index is 4% per annum, and the volatility of the index is 20%. Use the approach in Section 26.5 to value the option and compare your answer to the result given by DerivaGem using the analytic valuation formula.

26.20. Suppose that the volatilities used to price a 6-month currency option are as in Table 18.2. Assume that the domestic and foreign risk-free rates are 5% per annum and the current exchange rate is 1.00. Consider a bull spread that consists of a long position in a 6-month call option with strike price 1.05 and a short position in a 6-month call option with a strike price 1.10.
(a) What is the value of the spread?
(b) What single volatility if used for both options gives the correct value of the bull spread? (Use the DerivaGem Application Builder in conjunction with Goal Seek or Solver.)
(c) Does your answer support the assertion at the beginning of the chapter that the correct volatility to use when pricing exotic options can be counterintuitive?
(d) Does the IVF model give the correct price for the bull spread?

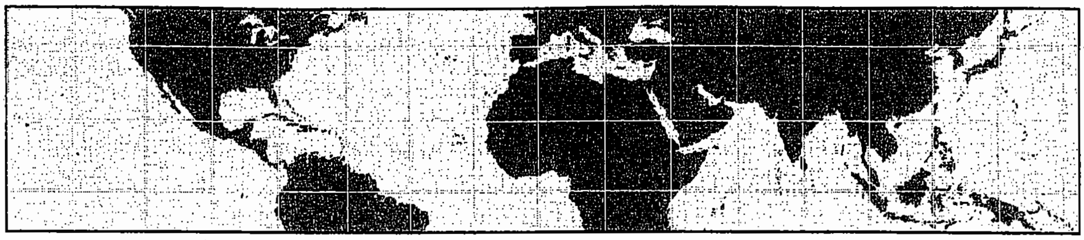
26.21. Repeat the analysis in Section 26.8 for the put option example on the assumption that the strike price is 1.13. Use both the least squares approach and the exercise boundary parameterization approach.

26.22. Consider the situation in Merton's jump–diffusion model where the underlying asset is a non-dividend-paying stock. The average frequency of jumps is one per year. The average percentage jump size is 2% and the standard deviation of the logarithm of the percentage jump size is 20%. The stock price is 100, the risk-free rate is 5%, the volatility, σ provided by the diffusion part of the process is 15%, and the time to maturity is six months. Use the DerivaGem Application Builder to calculate an implied volatility when the strike price is 80, 90, 100, 110, and 120. What does the volatility smile or skew that you obtain imply about the probability distribution of the stock price.

26.23. A 3-year convertible bond with a face value of $100 has been issued by company ABC. It pays a coupon of $5 at the end of each year. It can be converted into ABC's equity at the end of the first year or at the end of the second year. At the end of the first year, it can be exchanged for 3.6 shares immediately after the coupon date. At the end of the second year, it can be exchanged for 3.5 shares immediately after the coupon date. The current stock price is $25 and the stock price volatility is 25%. No dividends are paid on the stock. The risk-free interest rate is 5% with continuous compounding. The yield on bonds issued by ABC is 7% with continuous compounding and the recovery rate is 30%.
(a) Use a three-step tree to calculate the value of the bond.
(b) How much is the conversion option worth?

(c) What difference does it make to the value of the bond and the value of the conversion option if the bond is callable any time within the first 2 years for \$115?
(d) Explain how your analysis would change if there were a dividend payment of \$1 on the equity at the 6-month, 18-month, and 30-month points. Detailed calculations are not required.

(*Hint*: Use equation (22.2) to estimate the average default intensity.)



CHAPTER 27

Martingales and Measures

Up to now interest rates have been assumed to be constant when valuing options. In this chapter, this assumption is relaxed in preparation for valuing interest rate derivatives in Chapters 28 to 32.

The risk-neutral valuation principle states that a derivative can be valued by (a) calculating the expected payoff on the assumption that the expected return from the underlying asset equals the risk-free interest rate and (b) discounting the expected payoff at the risk-free interest rate. When interest rates are constant, risk-neutral valuation provides a well-defined and unambiguous valuation tool. When interest rates are stochastic, it is less clear-cut. What does it mean to assume that the expected return on the underlying asset equals to the risk-free rate? Does it mean (a) that each day the expected return is the one-day risk-free rate, or (b) that each year the expected return is the 1-year risk-free rate, or (c) that over a 5-year period the expected return is the 5-year rate at the beginning of the period? What does it mean to discount expected payoffs at the risk-free rate? Can we, for example, discount an expected payoff realized in year 5 at today's 5-year risk-free rate?

In this chapter we explain the theoretical underpinnings of risk-neutral valuation when interest rates are stochastic and show that there are many different risk-neutral worlds that can be assumed in any given situation. We first define a parameter known as the *market price of risk* and show that the excess return over the risk-free interest rate earned by any derivative in a short period of time is linearly related to the market prices of risk of the underlying stochastic variables. What we will refer to as the *traditional risk-neutral world* assumes that all market prices of risk are zero, but we will find that other assumptions about the market price of risk are useful in some situations.

Martingales and *measures* are critical to a full understanding of risk neutral valuation. A martingale is a zero-drift stochastic process. A measure is the unit in which we value security prices. A key result in this chapter will be the *equivalent martingale measure result*. This states that if we use the price of a traded security as the unit of measurement then there is some market price of risk for which all security prices follow martingales.

This chapter illustrates the power of the equivalent martingale measure result by using it to extend Black's model (see Section 16.8) to the situation where interest rates are stochastic and to value options to exchange one asset for another. Chapter 28 uses the result to understand the standard market models for valuing interest rate derivatives, Chapter 29 uses it to value some nonstandard derivatives, and Chapter 31 uses it to develop the LIBOR market model.

27.1 THE MARKET PRICE OF RISK

We start by considering the properties of derivatives dependent on the value of a single variable θ. Assume that the process followed by θ is

$$\frac{d\theta}{\theta} = m\,dt + s\,dz \tag{27.1}$$

where dz is a Wiener process. The parameters m and s are the expected growth rate in θ and the volatility of θ, respectively. We assume that they depend only on θ and time t. The variable θ need not be the price of an investment asset. It could be something as far removed from financial markets as the temperature in the center of New Orleans.

Suppose that f_1 and f_2 are the prices of two derivatives dependent only on θ and t. These can be options or other instruments that provide a payoff equal to some function of θ at some future time. Assume that during the time period under consideration f_1 and f_2 provide no income.[1]

Suppose that the processes followed by f_1 and f_2 are

$$\frac{df_1}{f_1} = \mu_1\,dt + \sigma_1\,dz$$

and

$$\frac{df_2}{f_2} = \mu_2\,dt + \sigma_2\,dz$$

where μ_1, μ_2, σ_1, and σ_2 are functions of θ and t. The "dz" in these processes must be the same dz as in equation (27.1) because it is the only source of the uncertainty in the prices of f_1 and f_2.

The prices f_1 and f_2 can be related using an analysis similar to the Black–Scholes analysis described in Section 13.6. The discrete versions of the processes for f_1 and f_2 are

$$\Delta f_1 = \mu_1 f_1\,\Delta t + \sigma_1 f_1\,\Delta z \tag{27.2}$$

$$\Delta f_2 = \mu_2 f_2\,\Delta t + \sigma_2 f_2\,\Delta z \tag{27.3}$$

We can eliminate the Δz by forming an instantaneously riskless portfolio consisting of $\sigma_2 f_2$ of the first derivative and $-\sigma_1 f_1$ of the second derivative. If Π is the value of the portfolio, then

$$\Pi = (\sigma_2 f_2) f_1 - (\sigma_1 f_1) f_2 \tag{27.4}$$

and

$$\Delta\Pi = \sigma_2 f_2\,\Delta f_1 - \sigma_1 f_1\,\Delta f_2$$

Substituting from equations (27.2) and (27.3), this becomes

$$\Delta\Pi = (\mu_1\sigma_2 f_1 f_2 - \mu_2\sigma_1 f_1 f_2)\Delta t \tag{27.5}$$

[1] The analysis can be extended to derivatives that provide income (see Problem 27.7).

Because the portfolio is instantaneously riskless, it must earn the risk-free rate. Hence,

$$\Delta\Pi = r\Pi\,\Delta t$$

Substituting into this equation from equations (27.4) and (27.5) gives

$$\mu_1\sigma_2 - \mu_2\sigma_1 = r\sigma_2 - r\sigma_1$$

or

$$\frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2} \tag{27.6}$$

Note that the left-hand side of equation (27.6) depends only on the parameters of the process followed by f_1 and the right-hand side depends only on the parameters of the process followed by f_2. Define λ as the value of each side in equation (27.6), so that

$$\frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2} = \lambda$$

Dropping subscripts, equation (27.6) shows that if f is the price of a derivative dependent only on θ and t with

$$\frac{df}{f} = \mu\,dt + \sigma\,dz \tag{27.7}$$

then

$$\frac{\mu - r}{\sigma} = \lambda \tag{27.8}$$

The parameter λ is known as the *market price of risk* of θ. It can be dependent on both θ and t, but it is not dependent on the nature of the derivative f. Our analysis shows that, for no arbitrage, $(\mu - r)/\sigma$ must at any given time be the same for all derivatives that are dependent only on θ and t.

It is worth noting that σ, which we will refer to as the volatility of f, can be either positive or negative. It is the coefficient of dz in equation (27.7). If the volatility, s, of θ is positive and f is positively related to θ (so that $\partial f/\partial\theta$ is positive), σ is positive. But if f is negatively related to θ, then σ is negative. The volatility of f, as it is traditionally defined, is $|\sigma|$.

The market price of risk of θ measures the trade-offs between risk and return that are made for securities dependent on θ. Equation (27.8) can be written

$$\mu - r = \lambda\sigma \tag{27.9}$$

The variable σ can be loosely interpreted as the quantity of θ-risk present in f. On the right-hand side of the equation, the quantity of θ-risk is multiplied by the price of θ-risk. The left-hand side is the expected return, in excess of the risk-free interest rate, that is required to compensate for this risk. Equation (27.9) is analogous to the capital asset pricing model, which relates the expected excess return on a stock to its risk. This chapter will not be concerned with the measurement of the market price of risk. This will be discussed in Chapter 33 when the evaluation of real options is considered.

Chapter 5 distinguished between investment assets and consumption assets. An investment asset is an asset that is bought or sold purely for investment purposes by a significant number of investors. Consumption assets are held primarily for consumption.

Equation (27.8) is true for all investment assets that provide no income and depend only on θ. If the variable θ itself happens to be such an asset, then

$$\frac{m-r}{s}=\lambda$$

But, in other circumstances, this relationship is not necessarily true.

Example 27.1

Consider a derivative whose price is positively related to the price of oil and depends on no other stochastic variables. Suppose that it provides an expected return of 12% per annum and has a volatility of 20% per annum. Assume that the risk-free interest rate is 8% per annum. It follows that the market price of risk of oil is

$$\frac{0.12-0.08}{0.2}=0.2$$

Note that oil is a consumption asset rather than an investment asset, so its market price of risk cannot be calculated from equation (27.8) by setting μ equal to the expected return from an investment in oil and σ equal to the volatility of oil prices.

Example 27.2

Consider two securities, both of which are positively dependent on the 90-day interest rate. Suppose that the first one has an expected return of 3% per annum and a volatility of 20% per annum, and the second one has a volatility of 30% per annum. Assume that the instantaneous risk-free rate of interest is 6% per annum. The market price of interest rate risk is, using the expected return and volatility for the first security,

$$\frac{0.03-0.06}{0.2}=-0.15$$

From a rearrangement of equation (27.9), the expected return from the second security is, therefore,

$$0.06-0.15\times 0.3=0.015$$

or 1.5% per annum.

Alternative Worlds

The process followed by derivative price f is

$$df=\mu f\,dt+\sigma f\,dz$$

The value of μ depends on the risk preferences of investors. In a world where the market price of risk is zero, λ equals zero. From equation (27.9) $\mu=r$, so that the process followed by f is

$$df=rf\,dt+\sigma f\,dz$$

We will refer to this as the *traditional risk-neutral world.*

Other assumptions about the market price of risk, λ, enable other worlds that are internally consistent to be defined. From equation (27.9),

$$\mu=r+\lambda\sigma$$

so that

$$df = (r + \lambda\sigma) f\,dt + \sigma f\,dz \tag{27.10}$$

The market price of risk of a variable determines the growth rates of all securities dependent on the variable. As we move from one market price of risk to another, the expected growth rates of security prices change, but their volatilities remain the same. This is a general property of variables following diffusion processes and was illustrated in Section 11.7. Choosing a particular market price of risk is also referred to as defining the *probability measure*. Some value of the market price of risk corresponds to the "real world" and the growth rates of security prices that are observed in practice.

27.2 SEVERAL STATE VARIABLES

Suppose that n variables, $\theta_1, \theta_2, \dots, \theta_n$, follow stochastic processes of the form

$$\frac{d\theta_i}{\theta_i} = m_i\,dt + s_i\,dz_i \tag{27.11}$$

for $i = 1, 2, \dots, n$, where the dz_i are Wiener processes. The parameters m_i and s_i are expected growth rates and volatilities and may be functions of the θ_i and time. The appendix at the end of this chapter provides a version of Itô's lemma that covers functions of several variables. It shows that the process for the price, f, of a security that is dependent on the θ_i has the form

$$\frac{df}{f} = \mu\,dt + \sum_{i=1}^{n} \sigma_i\,dz_i \tag{27.12}$$

In this equation, μ is the expected return from the security and $\sigma_i\,dz_i$ is the component of the risk of this return attributable to θ_i.

The appendix at the end of the chapter shows that

$$\mu - r = \sum_{i=1}^{n} \lambda_i \sigma_i \tag{27.13}$$

where λ_i is the market price of risk for θ_i. This equation relates the expected excess return that investors require on the security to the λ_i and σ_i. Equation (27.9) is the particular case of this equation when $n = 1$. The term $\lambda_i\sigma_i$ on the right-hand side measures the extent that the excess return required by investors on a security is affected by the dependence of the security on θ_i. If $\lambda_i\sigma_i = 0$, there is no effect; if $\lambda_i\sigma_i > 0$, investors require a higher return to compensate them for the risk arising from θ_i; if $\lambda_i\sigma_i < 0$, the dependence of the security on θ_i causes investors to require a lower return than would otherwise be the case. The $\lambda_i\sigma_i < 0$ situation occurs when the variable has the effect of reducing rather than increasing the risks in the portfolio of a typical investor.

Example 27.3

A stock price depends on three underlying variables: the price of oil, the price of gold, and the performance of a stock index. Suppose that the market prices of risk for these variables are 0.2, −0.1, and 0.4, respectively. Suppose also that the σ_i

factors in equation (27.12) corresponding to the three variables have been estimated as 0.05, 0.1, and 0.15, respectively. The excess return on the stock over the risk-free rate is

$$0.2 \times 0.05 - 0.1 \times 0.1 + 0.4 \times 0.15 = 0.06$$

or 6.0% per annum. If variables other than those considered affect the stock price, this result is still true provided that the market price of risk for each of these other variables is zero.

Equation (27.13) is closely related to arbitrage pricing theory, developed by Stephen Ross in 1976.[2] The continuous-time version of the capital asset pricing model (CAPM) can be regarded as a particular case of the equation. CAPM argues that an investor requires excess returns to compensate for any risk that is correlated to the risk in the return from the stock market, but requires no excess return for other risks. Risks that are correlated with the return from the stock market are referred to as *systematic*; other risks are referred to as *nonsystematic*. If CAPM is true, then λ_i is proportional to the correlation between changes in θ_i and the return from the market. When θ_i is uncorrelated with the return from the market, λ_i is zero.

27.3 MARTINGALES

A *martingale* is a zero-drift stochastic process.[3] A variable θ follows a martingale if its process has the form

$$d\theta = \sigma\, dz$$

where dz is a Wiener process. The variable σ may itself be stochastic. It can depend on θ and other stochastic variables. A martingale has the convenient property that its expected value at any future time is equal to its value today. This means that

$$E(\theta_T) = \theta_0$$

where θ_0 and θ_T denote the values of θ at times zero and T, respectively. To understand this result, note that over a very small time interval the change in θ is normally distributed with zero mean. The expected change in θ over any very small time interval is therefore zero. The change in θ between time 0 and time T is the sum of its changes over many small time intervals. It follows that the expected change in θ between time 0 and time T must also be zero.

The Equivalent Martingale Measure Result

Suppose that f and g are the prices of traded securities dependent on a single source of uncertainty. Assume that the securities provide no income during the time period under consideration and define $\phi = f/g$.[4] The variable ϕ is the relative price of f with respect

[2] See S.A. Ross, "The Arbitrage Theory of Capital Asset Pricing," *Journal of Economic Theory*, 13 (December 1976): 343–62.

[3] More formally, a sequence of random variables $X_0, X_1, \ldots$ is a martingale if, for all $i > 0$,

$$E(X_i \mid X_{i-1}, X_{i-2}, , X_0) = X_{i-1}$$

where E denotes expectation.

[4] Problem 27.8 extends the analysis to situations where the securities provide income.

to g. It can be thought of as measuring the price of f in units of g rather than dollars. The security price g is referred to as the *numeraire*.

The *equivalent martingale measure* result shows that, when there are no arbitrage opportunities, ϕ is a martingale for some choice of the market price of risk. What is more, for a given numeraire security g, the same choice of the market price of risk makes ϕ a martingale for all securities f. This choice of the market price of risk is the volatility of g. In other words, when the market price of risk is set equal to the volatility of g, the ratio f/g is a martingale for all security prices f.

To prove this result, suppose that the volatilities of f and g are σ_f and σ_g. From equation (27.10), in a world where the market price of risk is σ_g,

$$df = (r + \sigma_g\sigma_f)f\,dt + \sigma_f f\,dz$$

$$dg = (r + \sigma_g^2)g\,dt + \sigma_g g\,dz$$

Using Itô's lemma gives

$$d\ln f = (r + \sigma_g\sigma_f - \sigma_f^2/2)\,dt + \sigma_f\,dz$$

$$d\ln g = (r + \sigma_g^2/2)\,dt + \sigma_g\,dz$$

so that

$$d(\ln f - \ln g) = (\sigma_g\sigma_f - \sigma_f^2/2 - \sigma_g^2/2)\,dt + (\sigma_f - \sigma_g)\,dz$$

or

$$d\left(\ln\frac{f}{g}\right) = -\frac{(\sigma_f - \sigma_g)^2}{2}\,dt + (\sigma_f - \sigma_g)\,dz$$

Itô's lemma can be used to determine the process for f/g from the process for $\ln(f/g)$:

$$d\left(\frac{f}{g}\right) = (\sigma_f - \sigma_g)\frac{f}{g}\,dz \tag{27.14}$$

This shows that f/g is a martingale and proves the equivalent martingale measure result. We will refer to a world where the market price of risk is the volatility σ_g of g as a world that is *forward risk neutral* with respect to g.

Because f/g is a martingale in a world that is forward risk neutral with respect to g, it follows from the result at the beginning of this section that

$$\frac{f_0}{g_0} = E_g\left(\frac{f_T}{g_T}\right)$$

or

$$f_0 = g_0 E_g\left(\frac{f_T}{g_T}\right) \tag{27.15}$$

where E_g denotes the expected value in a world that is forward risk neutral with respect to g.

27.4 ALTERNATIVE CHOICES FOR THE NUMERAIRE

We now present a number of examples of the equivalent martingale measure result. The first example shows that it is consistent with the traditional risk-neutral valuation result used in earlier chapters. The other examples prepare the way for the valuation of bond options, interest rate caps, and swap options in Chapter 28.

Money Market Account as the Numeraire

The dollar money market account is a security that is worth \$1 at time zero and earns the instantaneous risk-free rate r at any given time.[5] The variable r may be stochastic. If we set g equal to the money market account, it grows at rate r so that

$$dg = rg\,dt \tag{27.16}$$

The drift of g is stochastic, but the volatility of g is zero. It follows from the results in Section 27.3 that f/g is a martingale in a world where the market price of risk is zero. This is the world we defined earlier as the traditional risk-neutral world. From equation (27.15),

$$f_0 = g_0\hat{E}\left(\frac{f_T}{g_T}\right) \tag{27.17}$$

where $\hat{E}$ denotes expectations in the traditional risk-neutral world.

In this case, $g_0 = 1$ and

$$g_T = e^{\int_0^T r\,dt}$$

so that equation (27.17) reduces to

$$f_0 = \hat{E}\left(e^{-\int_0^T r\,dt} f_T\right) \tag{27.18}$$

or

$$f_0 = \hat{E}\left(e^{-\bar{r}T} f_T\right) \tag{27.19}$$

where $\bar{r}$ is the average value of r between time 0 and time T. This equation shows that one way of valuing an interest rate derivative is to simulate the short-term interest rate r in the traditional risk-neutral world. On each trial the expected payoff is calculated and discounted at the average value of the short rate on the sampled path.

When the short-term interest rate r is assumed to be constant, equation (27.19) reduces to

$$f_0 = e^{-rT}\hat{E}(f_T)$$

or the risk-neutral valuation relationship used in earlier chapters.

Zero-Coupon Bond Price as the Numeraire

Define $P(t, T)$ as the price at time t of a zero-coupon bond that pays off \$1 at time T. We now explore the implications of setting g equal to $P(t, T)$. Let E_T denote expectations in a world that is forward risk neutral with respect to $P(t, T)$. Because $g_T = P(T, T) = 1$ and $g_0 = P(0, T)$, equation (27.15) gives

$$f_0 = P(0, T)E_T(f_T) \tag{27.20}$$

[5] The money account is the limit as Δt approaches zero of the following security. For the first short period of time of length Δt, it is invested at the initial Δt period rate; at time Δt, it is reinvested for a further period of time Δt at the new Δt period rate; at time $2\Delta t$, it is again reinvested for a further period of time Δt at the new Δt period rate; and so on. The money market accounts in other currencies are defined analogously to the dollar money market account.

Notice the difference between equations (27.20) and (27.19). In equation (27.19), the discounting is inside the expectations operator. In equation (27.20) the discounting, as represented by the $P(0, T)$ term, is outside the expectations operator. The use of $P(t, T)$ as the numeraire therefore considerably simplifies things for a security that provides a payoff solely at time T.

Consider any variable θ that is not an interest rate.[6] A forward contract on θ with maturity T is defined as a contract that pays off $\theta_T - K$ at time T, where θ_T is the value θ at time T. Define f as the value of this forward contract. From equation (27.20),

$$f_0 = P(0, T)[E_T(\theta_T) - K]$$

The forward price, F, of θ is the value of K for which f_0 equals zero. It therefore follows that

$$P(0, T)[E_T(\theta_T) - F] = 0$$

or

$$F = E_T(\theta_T) \tag{27.21}$$

Equation (27.21) shows that the forward price of any variable (except an interest rate) is its expected future spot price in a world that is forward risk neutral with respect to $P(t, T)$. Note the difference here between forward prices and futures prices. The argument in Section 16.7 shows that the futures price of a variable is the expected future spot price in the traditional risk-neutral world.

Equation (27.20) shows that any security that provides a payoff at time T can be valued by calculating its expected payoff in a world that is forward risk neutral with respect to a bond maturing at time T and discounting at the risk-free rate for maturity T. Equation (27.21) shows that it is correct to assume that the expected value of the underlying variables equal their forward values when computing the expected payoff.

Interest Rates when a Bond Price is the Numeraire

For the next result, define $R(t, T, T^*)$ as the forward interest rate as seen at time t for the period between T and T^* expressed with a compounding period of $T^* - T$. (For example, if $T^* - T = 0.5$, the interest rate is expressed with semiannual compounding; if $T^* - T = 0.25$, it is expressed with quarterly compounding; and so on.) The forward price, as seen at time t, of a zero-coupon bond lasting between times T and T^* is

$$\frac{P(t, T^*)}{P(t, T)}$$

A forward interest rate is defined differently from the forward value of most variables. A forward interest rate is the interest rate implied by the corresponding forward bond price. It follows that

$$\frac{1}{[1 + (T^* - T)R(t, T, T^*)]} = \frac{P(t, T^*)}{P(t, T)}$$

[6] The analysis given here does not apply to interest rates because forward contracts for interest rates are defined differently from forward contracts for other variables. A forward interest rate is the interest rate implied by the corresponding forward bond price.

so that

$$R(t, T, T^*) = \frac{1}{T^* - T}\left[\frac{P(t, T)}{P(t, T^*)} - 1\right]$$

or

$$R(t, T, T^*) = \frac{1}{T^* - T}\left[\frac{P(t, T) - P(t, T^*)}{P(t, T^*)}\right]$$

Setting

$$f = \frac{1}{T^* - T}[P(t, T) - P(t, T^*)]$$

and $g = P(t, T^*)$, the equivalent martingale measure result shows that $R(t, T, T^*)$ is a martingale in a world that is forward risk neutral with respect to $P(t, T^*)$. This means that

$$R(0, T, T^*) = E_{T^*}[R(T, T, T^*)] \tag{27.22}$$

where E_{T^*} denotes expectations in a world that is forward risk neutral with respect to $P(t, T^*)$.

The variable $R(0, T, T^*)$ is the forward interest rate between times T and T^* as seen at time 0, whereas $R(T, T, T^*)$ is the realized interest rate between times T and T^*. Equation (27.22) therefore shows that the forward interest rate between times T and T^* equals the expected future interest rate in a world that is forward risk neutral with respect to a zero-coupon bond maturing at time T^*. This result, when combined with that in equation (27.20), will be critical to an understanding of the standard market model for interest rate caps in the next chapter.

Annuity Factor as the Numeraire

For the next application of equivalent martingale measure arguments, consider a swap starting at a future time T with payment dates at times $T_1, T_2, \dots, T_N$. Define $T_0 = T$. Assume that the principal underlying the swap is \$1. Suppose that the forward swap rate (i.e., the interest rate on the fixed side that makes the swap have a value of zero) is $s(t)$ at time t ($t \leqslant T$). The value of the fixed side of the swap is

$$s(t)A(t)$$

where

$$A(t) = \sum_{i=0}^{N-1}(T_{i+1} - T_i)P(t, T_{i+1})$$

Chapter 7 showed that, when the principal is added to the payment on the last payment date swap, the value of the floating side of the swap on the initiation date equals the underlying principal. It follows that if \$1 is added at time T_N, the floating side is worth \$1 at time T. The value of \$1 received at time T_N is $P(t, T_N)$. The value of \$1 at time T_0 is $P(t, T_0)$. The value of the floating side at time t is, therefore,

$$P(t, T_0) - P(t, T_N)$$

Equating the values of the fixed and floating sides gives

$$s(t)A(t) = P(t, T_0) - P(t, T_N)$$

or

$$s(t) = \frac{P(t, T_0) - P(t, T_N)}{A(t)} \tag{27.23}$$

The equivalent martingale measure result can be applied by setting f equal to $P(t, T_0) - P(t, T_N)$ and g equal to $A(t)$. This leads to

$$s(t) = E_A[s(T)] \tag{27.24}$$

where E_A denotes expectations in a world that is forward risk neutral with respect to $A(t)$. Therefore, in a world that is forward risk neutral with respect to $A(t)$, the expected future swap rate is the current swap rate.

For any security, f, the result in equation (27.15) shows that

$$f_0 = A(0)E_A\left[\frac{f_T}{A(T)}\right] \tag{27.25}$$

This result, when combined with the result in equation (27.24), will be critical to an understanding of the standard market model for European swap options in the next chapter.

27.5 EXTENSION TO SEVERAL FACTORS

The results presented in Sections 27.3 and 27.4 can be extended to cover the situation when there are many independent factors.[7] Assume that there are n independent factors and that the processes for f and g in the traditional risk-neutral world are

$$df = rf\,dt + \sum_{i=1}^{n} \sigma_{f,i} f\,dz_i$$

and

$$dg = rg\,dt + \sum_{i=1}^{n} \sigma_{g,i} g\,dz_i$$

It follows from Section 27.2 that other internally consistent worlds can be defined by setting

$$df = \left[r + \sum_{i=1}^{n} \lambda_i \sigma_{f,i}\right] f\,dt + \sum_{i=1}^{n} \sigma_{f,i} f\,dz_i$$

and

$$dg = \left[r + \sum_{i=1}^{n} \lambda_i \sigma_{g,i}\right] g\,dt + \sum_{i=1}^{n} \sigma_{g,i} g\,dz_i$$

where the λ_i $(1 \leqslant i \leqslant n)$ are the n market prices of risk. One of these other worlds is the real world.

The definition of forward risk neutrality can be extended so that a world is forward risk neutral with respect to g, where $\lambda_i = \sigma_{g,i}$ for all i. It can be shown from Itô's lemma, using the fact that the dz_i are uncorrelated, that the process followed by f/g in

[7] The independence condition is not critical. If factors are not independent they can be orthogonalized.

this world has zero drift (see Problem 27.12). The rest of the results in the last two sections (from equation (27.15) onward) are therefore still true.

27.6 BLACK'S MODEL REVISITED

Section 16.8 explained that Black's model is a popular tool for pricing European options in terms of the forward or futures price of the underlying asset when interest rates are constant. We are now in a position to relax the constant interest rate assumption and show that Black's model can be used to price European options in terms of the forward price of the underlying asset when interest rates are stochastic.

Consider a European call option on an asset with strike price K that lasts until time T. From equation (27.20), the option's price is given by

$$c = P(0, T)E_T[\max(S_T - K, 0)] \tag{27.26}$$

where S_T is the asset price at time T and E_T denotes expectations in a world that is forward risk neutral with respect to $P(t, T)$. Define F_0 and F_T as the forward price of the asset at time 0 and time T for a contract maturing at time T. Because $S_T = F_T$,

$$c = P(0, T)E_T[\max(F_T - K, 0)]$$

Assume that F_T is lognormal in the world being considered, with the standard deviation of $\ln(F_T)$ equal to $\sigma_F\sqrt{T}$. This could be because the forward price follows a stochastic process with constant volatility σ_F. The appendix at the end of Chapter 13 shows that

$$E_T[\max(F_T - K, 0)] = E_T(F_T)N(d_1) - KN(d_2) \tag{27.27}$$

where

$$d_1 = \frac{\ln[E_T(F_T)/K] + \sigma_F^2 T}{\sigma_F\sqrt{T}}$$

$$d_2 = \frac{\ln[E_T(F_T)/K] - \sigma_F^2 T}{\sigma_F\sqrt{T}}$$

From equation (27.21), $E_T(F_T) = E_T(S_T) = F_0$. Hence,

$$c = P(0, T)[F_0 N(d_1) - KN(d_2)] \tag{27.28}$$

where

$$d_1 = \frac{\ln[F_0/K] + \sigma_F^2 T}{\sigma_F\sqrt{T}}$$

$$d_2 = \frac{\ln[F_0/K] - \sigma_F^2 T}{\sigma_F\sqrt{T}}$$

Similarly,

$$p = P(0, T)[KN(-d_2) - F_0 N(-d_1)] \tag{27.29}$$

where p is the price of a European put option on the asset with strike price K and time to maturity T. This is Black's model. It applies to both investment and consumption assets and, as we have just shown, is true when interest rates are stochastic provided that F_0 is the forward asset price. The variable σ_F can be interpreted as the (constant) volatility of the forward asset price.

27.7 OPTION TO EXCHANGE ONE ASSET FOR ANOTHER

Consider next an option to exchange an investment asset worth U for an investment asset worth V. This has already been discussed in Section 24.11. Suppose that the volatilities of U and V are σ_U and σ_V and the coefficient of correlation between them is ρ.

Assume first that the assets provide no income and choose the numeraire security g to be U. Setting $f = V$ in equation (27.15) gives

$$V_0 = U_0 E_U\left(\frac{V_T}{U_T}\right) \tag{27.30}$$

where E_U denotes expectations in a world that is forward risk neutral with respect to U.

The variable f in equation (27.15) can be set equal to the value of the option under consideration, so that $f_T = \max(V_T - U_T, 0)$. It follows that

$$f_0 = U_0\, E_U\left[\frac{\max(V_T - U_T, 0)}{U_T}\right]$$

or

$$f_0 = U_0\, E_U\left[\max\left(\frac{V_T}{U_T} - 1, 0\right)\right] \tag{27.31}$$

The volatility of V/U is $\hat{\sigma}$ (see Problem 27.14), where

$$\hat{\sigma}^2 = \sigma_U^2 + \sigma_V^2 - 2\rho\sigma_U\sigma_V$$

From the appendix at the end of Chapter 13, equation (27.31) becomes

$$f_0 = U_0\left[E_U\left(\frac{V_T}{U_T}\right)N(d_1) - N(d_2)\right]$$

where

$$d_1 = \frac{\ln(V_0/U_0) + \hat{\sigma}^2 T/2}{\hat{\sigma}\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \hat{\sigma}\sqrt{T}$$

Substituting from equation (27.30) gives

$$f_0 = V_0 N(d_1) - U_0 N(d_2) \tag{27.32}$$

This is the value of an option to exchange one asset for another when the assets provide no income.

Problem 27.8 shows that, when f and g provide income at rate q_f and q_g, equation (27.15) becomes

$$f_0 = g_0 e^{(q_f - q_g)T} E_g\left(\frac{f_T}{g_T}\right)$$

This means that equations (27.30) and (27.31) become

$$E_U\left(\frac{V_T}{U_T}\right) = e^{(q_U - q_V)T}\frac{V_0}{U_0}$$

and

$$f_0 = e^{-q_U T} U_0 E_U \left[\max\left(\frac{V_T}{U_T} - 1, 0 \right) \right]$$

and equation (27.32) becomes

$$f_0 = e^{-q_V T} V_0 N(d_1) - e^{-q_U T} U_0 N(d_2)$$

with d_1 and d_2 being redefined as

$$d_1 = \frac{\ln(V_0/U_0) + (q_U - q_V + \hat{\sigma}^2/2)T}{\hat{\sigma}\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \hat{\sigma}\sqrt{T}$$

This is the result given in equation (24.3) for the value of an option to exchange one asset for another.

27.8 CHANGE OF NUMERAIRE

In this section we consider the impact of a change in numeraire on the process followed by a market variable. In a world that is forward risk neutral with respect to g, the process followed by a traded security f is

$$df = \left[r + \sum_{i=1}^{n} \sigma_{g,i}\sigma_{f,i} \right] f\,dt + \sum_{i=1}^{n} \sigma_{f,i} f\,dz_i$$

Similarly, in a world that is forward risk neutral with respect to another security h, the process followed by f is

$$df = \left[r + \sum_{i=1}^{n} \sigma_{h,i}\sigma_{f,i} \right] f\,dt + \sum_{i=1}^{n} \sigma_{f,i} f\,dz_i$$

where $\sigma_{h,i}$ is the ith component of the volatility of h.

The effect of moving from a world that is forward risk neutral with respect to g to one that is forward risk neutral with respect to h (i.e., of changing the numeraire from g to h) is therefore to increase the expected growth rate of the price of any traded security f by

$$\sum_{i=1}^{n} (\sigma_{h,i} - \sigma_{g,i})\sigma_{f,i}$$

Consider next a variable v that is a function of the prices of traded securities (where v is not necessarily the price of a traded security itself). Define $\sigma_{v,i}$ as the ith component of the volatility of v. From Itô's lemma in the appendix at the end of this chapter, it is possible to calculate what happens to the process followed by v when there is a change in numeraire causing the expected growth rate of the underlying traded securities to change. It turns out that the expected growth rate of v responds to a change in numeraire in the same way as the expected growth rate of the prices of traded securities (see Problem 12.6 for the situation where there is only one stochastic variable and

Problem 27.13 for the general case). It increases by

$$\alpha_v = \sum_{i=1}^{n} (\sigma_{h,i} - \sigma_{g,i})\sigma_{v,i} \tag{27.33}$$

Define $w = h/g$ and $\sigma_{w,i}$ as the ith component of the volatility of w. From Itô's lemma (see Problem 27.14),

$$\sigma_{w,i} = \sigma_{h,i} - \sigma_{g,i}$$

so that equation (27.33) becomes

$$\alpha_v = \sum_{i=1}^{n} \sigma_{w,i}\,\sigma_{v,i} \tag{27.34}$$

We will refer to w as the *numeraire ratio*. Equation (27.34) is equivalent to

$$\alpha_v = \rho\sigma_v\sigma_w \tag{27.35}$$

where σ_v is the total volatility of v, σ_w is the total volatility of w, and ρ is the instantaneous correlation between changes in v and w.[8]

This is a surprisingly simple result. The adjustment to the expected growth rate of a variable v when we change from one numeraire to another is the instantaneous covariance between the percentage change in v and the percentage change in the numeraire ratio. This result will be used when timing and quanto adjustments are considered in Chapter 29.

27.9 GENERALIZATION OF TRADITIONAL VALUATION METHODS

When a derivative depends on the values of variables at more than one point in time, it is usually necessary to work in the traditional risk-neutral world where the numeraire is the money market account. Technical Note 20 on the author's website considers the situation where a derivative depends on variables θ_i following the processes in equation (27.11). It extends the material in Chapter 13 by producing the differential equation that must be satisfied by the derivative. It shows that traditional risk-neutral valuation methods can be implemented by changing the growth rate of each θ_i from m_i to $m_i - \lambda_i s_i$ and using the short-term risk-free rate at time t as the discount rate at time t. If θ_i is the price of a traded security that provides no income, equation (27.9) shows that changing the growth rate from m_i to $m_i - \lambda_i s_i$ is equivalent to setting the return on the security equal to the short-term risk-free rate. (This is as expected.) However, the θ_i need not be the prices of traded securities and some may be interest rates.

[8] To see this, note that the changes Δv and Δw in v and w in a short period of time Δt are given by

$$\Delta v = \cdots + \sum \sigma_{v,i}\, v\epsilon_i\sqrt{\Delta t}$$
$$\Delta w = \cdots + \sum \sigma_{w,i}\, w\epsilon_i\sqrt{\Delta t}$$

Since the dz_i are uncorrelated, it follows that $E(\epsilon_i\epsilon_j) = 0$ when $i \neq j$. Also, from the definition of ρ, we have

$$\rho v\sigma_v w\sigma_w = E(\Delta v\,\Delta w) - E(\Delta v)\,E(\Delta w)$$

When terms of higher order than Δt are ignored this leads to

$$\rho\sigma_v\sigma_w = \sum \sigma_{w,i}\,\sigma_{v,i}$$

In the traditional risk-neutral world, the expected price of θ_i at time T is its futures price for a contract maturing at time T. When futures contracts on θ_i are available, it is therefore possible to estimate the process followed by θ_i in the traditional risk-neutral world without estimating the λ_i explicitly. This will be discussed further in the context of real options in Chapter 33.

SUMMARY

The market price of risk of a variable defines the trade-offs between risk and return for traded securities dependent on the variable. When there is one underlying variable, a derivative's excess return over the risk-free rate equals the market price of risk multiplied by the variable's volatility. When there are many underlying variables, the excess return is the sum of the market price of risk multiplied by the volatility for each variable.

A powerful tool in the valuation of derivatives is risk-neutral valuation. This was introduced in Chapters 11 and 13. The principle of risk-neutral valuation shows that, if we assume that the world is risk neutral when valuing derivatives, we get the right answer—not just in a risk-neutral world, but in all other worlds as well. In the traditional risk-neutral world, the market price of risk of all variables is zero. This chapter has extended the principle of risk-neutral valuation. It has shown that, when interest rates are stochastic, there are many interesting and useful alternatives to the traditional risk-neutral world.

A martingale is a zero drift stochastic process. Any variable following a martingale has the simplifying property that its expected value at any future time equals its value today. The equivalent martingale measure result shows that, if g is a security price, there is a world in which the ratio f/g is a martingale for all security prices f. It turns out that, by appropriately choosing the numeraire security g, the valuation of many interest rate dependent derivatives can be simplified.

This chapter has used the equivalent martingale measure result to extend Black's model to the situation where interest rates are stochastic and to value an option to exchange one asset for another. In Chapters 28 to 32, it will be useful in valuing interest rate derivatives.

FURTHER READING

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Duffie, D., *Dynamic Asset Pricing Theory*, 3rd edn. Princeton University Press, 2001.

Garman, M., "A General Theory of Asset Valuation Under Diffusion State Processes," Working Paper 50, University of California, Berkeley, 1976.

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Questions and Problems (Answers in the Solutions Manual)

27.1. How is the market price of risk defined for a variable that is not the price of an investment asset?

27.2. Suppose that the market price of risk for gold is zero. If the storage costs are 1% per annum and the risk-free rate of interest is 6% per annum, what is the expected growth rate in the price of gold? Assume that gold provides no income.

27.3. Consider two securities both of which are dependent on the same market variable. The expected returns from the securities are 8% and 12%. The volatility of the first security is 15%. The instantaneous risk-free rate is 4%. What is the volatility of the second security?

27.4. An oil company is set up solely for the purpose of exploring for oil in a certain small area of Texas. Its value depends primarily on two stochastic variables: the price of oil and the quantity of proven oil reserves. Discuss whether the market price of risk for the second of these two variables is likely to be positive, negative, or zero.

27.5. Deduce the differential equation for a derivative dependent on the prices of two non-dividend-paying traded securities by forming a riskless portfolio consisting of the derivative and the two traded securities.

27.6. Suppose that an interest rate x follows the process

$$dx = a(x_0 - x)\,dt + c\sqrt{x}\,dz$$

where a, x_0, and c are positive constants. Suppose further that the market price of risk for x is λ. What is the process for x in the traditional risk-neutral world?

27.7. Prove that, when the security f provides income at rate q, equation (27.9) becomes $\mu + q - r = \lambda\sigma$. (*Hint*: Form a new security f^* that provides no income by assuming that all the income from f is reinvested in f.)

27.8. Show that when f and g provide income at rates q_f and q_g, respectively, equation (27.15) becomes

$$f_0 = g_0 e^{(q_f - q_g)T} E_g\left(\frac{f_T}{g_T}\right)$$

(*Hint*: Form new securities f^* and g^* that provide no income by assuming that all the income from f is reinvested in f and all the income in g is reinvested in g.)

27.9. "The expected future value of an interest rate in a risk-neutral world is greater than it is in the real world." What does this statement imply about the market price of risk for (a) an interest rate and (b) a bond price. Do you think the statement is likely to be true? Give reasons.

27.10. The variable S is an investment asset providing income at rate q measured in currency A. It follows the process

$$dS = \mu_S S\,dt + \sigma_S S\,dz$$

in the real world. Defining new variables as necessary, give the process followed by S, and the corresponding market price of risk, in:

(a) A world that is the traditional risk-neutral world for currency A
(b) A world that is the traditional risk-neutral world for currency B
(c) A world that is forward risk neutral with respect to a zero-coupon currency A bond maturing at time T

(d) A world that is forward risk neutral with respect to a zero coupon currency B bond maturing at time T.

27.11. Explain the difference between the way a forward interest rate is defined and the way the forward values of other variables such as stock prices, commodity prices, and exchange rates are defined.

27.12. Prove the result in Section 27.5 that when

$$df = \left[r + \sum_{i=1}^{n} \lambda_i \sigma_{f,i}\right] f\,dt + \sum_{i=1}^{n} \sigma_{f,i} f\,dz_i$$

and

$$dg = \left[r + \sum_{i=1}^{n} \lambda_i \sigma_{g,i}\right] g\,dt + \sum_{i=1}^{n} \sigma_{g,i} g\,dz_i$$

with the dz_i uncorrelated, f/g is a martingale for $\lambda_i = \sigma_{g,i}$.

27.13. Prove equation (27.33) in Section 27.7.

27.14. Show that when $w = h/g$ and h and g are each dependent on n Wiener processes, the ith component of the volatility of w is the ith component of the volatility of h minus the ith component of the volatility of g. Use this to prove the result that if σ_U is the volatility of U and σ_V is the volatility of V then the volatility of U/V is $\sqrt{\sigma_U^2 + \sigma_V^2 - 2\rho\sigma_U\sigma_V}$. (*Hint*: Use the result in Footnote 7.)

Assignment Questions

27.15. A security's price is positively dependent on two variables: the price of copper and the yen/dollar exchange rate. Suppose that the market price of risk for these variables is 0.5 and 0.1, respectively. If the price of copper were held fixed, the volatility of the security would be 8% per annum; if the yen/dollar exchange rate were held fixed, the volatility of the security would be 12% per annum. The risk-free interest rate is 7% per annum. What is the expected rate of return from the security? If the two variables are uncorrelated with each other, what is the volatility of the security?

27.16. Suppose that the price of a zero-coupon bond maturing at time T follows the process

$$dP(t,T) = \mu_P P(t,T)\,dt + \sigma_P P(t,T)\,dz$$

and the price of a derivative dependent on the bond follows the process

$$df = \mu_f f\,dt + \sigma_f f\,dz$$

Assume only one source of uncertainty and that f provides no income.

(a) What is the forward price F of f for a contract maturing at time T?

(b) What is the process followed by F in a world that is forward risk neutral with respect to $P(t,T)$?

(c) What is the process followed by F in the traditional risk-neutral world?

(d) What is the process followed by f in a world that is forward risk neutral with respect to a bond maturing at time T^*, where $T^* \neq T$? Assume that σ_P^* is the volatility of this bond.

27.17. Consider a variable that is not an interest rate:

(a) In what world is the futures price of the variable a martingale?
(b) In what world is the forward price of the variable a martingale?
(c) Defining variables as necessary, derive an expression for the difference between the drift of the futures price and the drift of the forward price in the traditional risk-neutral world.
(d) Show that your result is consistent with the points made in Section 5.8 about the circumstances when the futures price is above the forward price.

APPENDIX

HANDLING MULTIPLE SOURCES OF UNCERTAINTY

This appendix extends Itô's lemma to cover situations where there are multiple sources of uncertainty and proves the result in equation (27.13) relating the excess return to market prices of risk when there are multiple sources of uncertainty.

Itô's Lemma for a Function of Several Variables

Itô's lemma, as presented in the appendix to Chapter 12, provides the process followed by a function of a single stochastic variable. Here we present a generalized version of Itô's lemma for the process followed by a function of several stochastic variables.

Suppose that a function f depends on the n variables $x_1, x_2, \dots, x_n$ and time t. Suppose further that x_i follows an Itô process with instantaneous drift a_i and instantaneous variance b_i^2 $(1 \leqslant i \leqslant n)$, that is,

$$dx_i = a_i\,dt + b_i\,dz_i \tag{27A.1}$$

where dz_i $(1 \leqslant i \leqslant n)$ is a Wiener process. Each a_i and b_i may be any function of all the x_i and t. A Taylor series expansion of Δf gives

$$\Delta f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\Delta x_i + \frac{\partial f}{\partial t}\Delta t + \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i\,\partial x_j}\Delta x_i\,\Delta x_j + \tfrac{1}{2}\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i\,\partial t}\Delta x_i\,\Delta t + \cdots \tag{27A.2}$$

Equation (27A.1) can be discretized as

$$\Delta x_i = a_i\,\Delta t + b_i\epsilon_i\sqrt{\Delta t}$$

where ϵ_i is a random sample from a standardized normal distribution. The correlation ρ_{ij} between dz_i and dz_j is defined as the correlation between ϵ_i and ϵ_j. In the appendix to Chapter 12 it was argued that

$$\lim_{\Delta t\to 0} \Delta x_i^2 = b_i^2\,dt$$

Similarly,

$$\lim_{\Delta t\to 0} \Delta x_i\,\Delta x_j = b_i b_j \rho_{ij}\,dt$$

As $\Delta t \to 0$, the first three terms in the expansion of Δf in equation (27A.2) are of order Δt. All other terms are of higher order. Hence,

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}dx_i + \frac{\partial f}{\partial t}dt + \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i\,\partial x_j}b_i b_j \rho_{ij}\,dt$$

This is the generalized version of Itô's lemma. Substituting for dx_i from equation (27A.1) gives

$$df = \left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}a_i + \frac{\partial f}{\partial t} + \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i\,\partial x_j}b_i b_j \rho_{ij}\right)dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}b_i\,dz_i \tag{27A.3}$$

For an alternative generalization of Itô's lemma, suppose that f depends on a single variable x and that the process for x involves more than one Wiener process:

$$dx = a\,dt + \sum_{i=1}^{m} b_i\,dz_i$$

In this case,

$$\Delta f = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial t}\Delta t + \tfrac{1}{2}\frac{\partial^2 f}{\partial x^2}\Delta x^2 + \tfrac{1}{2}\frac{\partial^2 f}{\partial x\,\partial t}\Delta x\,\Delta t + \cdots$$

$$\Delta x = a\,\Delta t + \sum_{i=1}^{m} b_i\epsilon_i\sqrt{\Delta t}$$

and

$$\lim_{\Delta t\to 0}\Delta x_i^2 = \sum_{i=1}^{m}\sum_{j=1}^{m} b_i b_j \rho_{ij}\,dt$$

where, as before, ρ_{ij} is the correlation between dz_i and dz_j This leads to

$$df = \left(\frac{\partial f}{\partial x}a + \frac{\partial f}{\partial t} + \tfrac{1}{2}\frac{\partial^2 f}{\partial x^2}\sum_{i=1}^{m}\sum_{j=1}^{m} b_i b_j \rho_{ij}\right)dt + \frac{\partial f}{\partial x}\sum_{i=1}^{m} b_i\,dz_i \tag{27A.4}$$

Finally, consider the more general case where f depends on variables x_i $(1 \leqslant i \leqslant n)$ and

$$dx_i = a_i\,dt + \sum_{k=1}^{m} b_{ik}\,dz_k$$

A similar analysis shows that

$$df = \left(\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}a_i + \frac{\partial f}{\partial t} + \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 f}{\partial x_i\,\partial x_j}\sum_{k=1}^{m}\sum_{l=1}^{m} b_{ik}b_{jl}\rho_{kl}\right)dt + \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}\sum_{k=1}^{m} b_{ik}\,dz_k \tag{27A.5}$$

The Return for a Security Dependent on Multiple Sources of Uncertainty

Section 27.1 proved a result relating return to risk when there is one source of uncertainty. We now prove the result in equation (27.13) for the situation where there are multiple sources of uncertainty.

Suppose that there are n stochastic variables following Wiener processes. Consider $n + 1$ traded securities whose prices depend on some or all of the n stochastic variables. Define f_j as the price of the jth security $(1 \leqslant j \leqslant n + 1)$. Assume that no dividends or other income is paid by the $n + 1$ traded securities.[9] It follows from the previous section

[9] This is not restrictive. A non-dividend-paying security can always be obtained from a dividend-paying security by reinvesting the dividends in the security.

that the securities follow processes of the form

$$df_j = \mu_j f_j \, dt + \sum_{i=1}^{n} \sigma_{ij} f_j \, dz_i \tag{27A.6}$$

Since there are $n+1$ traded securities and n Wiener processes, it is possible to form an instantaneously riskless portfolio Π using the securities. Define k_j as the amount of the jth security in the portfolio, so that

$$\Pi = \sum_{j=1}^{n+1} k_j f_j \tag{27A.7}$$

The k_j must be chosen so that the stochastic components of the returns from the securities are eliminated. From equation (27A.6), this means that

$$\sum_{j=1}^{n+1} k_j \sigma_{ij} f_j = 0 \tag{27A.8}$$

for $1 \leqslant i \leqslant n$. Equation (27A.8) consists of n equations in $n+1$ unknowns $(k_1, k_2, \dots, k_{n+1})$. From linear algebra we know that this set of equations always has a solution where not all of the k_j are zero. This shows that the risk-free portfolio Π can always be created.

The return from the portfolio is given by

$$d\Pi = \sum_{j=1}^{n+1} k_j \mu_j f_j \, dt$$

The cost of setting up the portfolio is

$$\sum_{j=1}^{n+1} k_j f_j$$

If there are no arbitrage opportunities, the portfolio must earn the risk-free interest rate, so that

$$\sum_{j=1}^{n+1} k_j \mu_j f_j = r \sum_{j=1}^{n+1} k_j f_j \tag{27A.9}$$

or

$$\sum_{j=1}^{n+1} k_j f_j (\mu_j - r) = 0 \tag{27A.10}$$

Equations (27A.8) and (27A.10) can be regarded as $n+1$ homogeneous linear equations in the k_j. The k_j are not all zero. From a well-known theorem in linear algebra, equations (27A.8) and (27A.10) can be consistent only if the left-hand side of equation (27A.10) is a linear combination of the left-hand side of equation (27A.8). This means that, for all j,

$$f_j(\mu_j - r) = \sum_{i=1}^{n} \lambda_i \sigma_{ij} f_j \tag{27A.11}$$

or

$$\mu_j - r = \sum_{i=1}^{n} \lambda_i \sigma_{ij} \tag{27A.12}$$

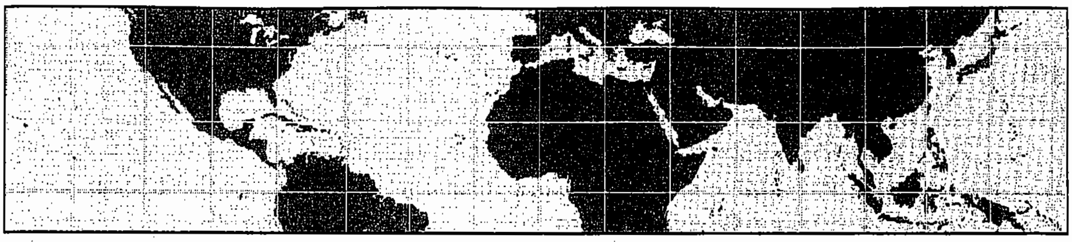
for some λ_i ($1 \leqslant i \leqslant n$) that are dependent only on the state variables and time. Dropping the j subscript, this shows that, for any security f dependent on the n stochastic variables,

$$df = \mu f\, dt + \sum_{i=1}^{n} \sigma_i f\, dz_i$$

where

$$\mu - r = \sum_{i=1}^{n} \lambda_i \sigma_i$$

This proves the result in equation (27.13).



CHAPTER 28

Interest Rate Derivatives: The Standard Market Models

Interest rate derivatives are instruments whose payoffs are dependent in some way on the level of interest rates. In the 1980s and 1990s, the volume of trading in interest rate derivatives in both the over-the-counter and exchange-traded markets increased rapidly. Many new products were developed to meet particular needs of end users. A key challenge for derivatives traders was to find good, robust procedures for pricing and hedging these products. Interest rate derivatives are more difficult to value than equity and foreign exchange derivatives for the following reasons:

1. The behavior of an individual interest rate is more complicated than that of a stock price or an exchange rate.
2. For the valuation of many products it is necessary to develop a model describing the behavior of the entire zero-coupon yield curve.
3. The volatilities of different points on the yield curve are different.
4. Interest rates are used for discounting the derivative as well as defining its payoff.

This chapter considers the three most popular over-the-counter interest rate option products: bond options, interest rate caps/floors, and swap options. It explains how the products work and the standard market models used to value them.

28.1 BOND OPTIONS

A bond option is an option to buy or sell a particular bond by a particular date for a particular price. In addition to trading in the over-the-counter market, bond options are frequently embedded in bonds when they are issued to make them more attractive to either the issuer or potential purchasers.

Embedded Bond Options

One example of a bond with an embedded bond option is a *callable bond.* This is a bond that contains provisions allowing the issuing firm to buy back the bond at a predetermined price at certain times in the future. The holder of such a bond has sold a call option to the issuer. The strike price or call price in the option is the predetermined price that must be paid by the issuer to the holder. Callable bonds cannot usually be called for the first few years of their life. (This is known as the lock-out period.) After that, the call price is usually a decreasing function of time. For example, in a 10-year callable bond, there might be no call privileges for the first 2 years. After that, the issuer might have the right to buy the bond back at a price of 110 in years 3 and 4 of its life, at a price of 107.5 in years 5 and 6, at a price of 106 in years 7 and 8, and at a price of 103 in years 9 and 10. The value of the call option is reflected in the quoted yields on bonds. Bonds with call features generally offer higher yields than bonds with no call features.

Another type of bond with an embedded option is a *puttable bond*. This contains provisions that allow the holder to demand early redemption at a predetermined price at certain times in the future. The holder of such a bond has purchased a put option on the bond as well as the bond itself. Because the put option increases the value of the bond to the holder, bonds with put features provide lower yields than bonds with no put features. A simple example of a puttable bond is a 10-year bond where the holder has the right to be repaid at the end of 5 years. (This is sometimes referred to as a *retractable bond*.)

Loan and deposit instruments also often contain embedded bond options. For example, a 5-year fixed-rate deposit with a financial institution that can be redeemed without penalty at any time contains an American put option on a bond. (The deposit instrument is a bond that the investor has the right to put back to the financial institution at its face value at any time.) Prepayment privileges on loans and mortgages are similarly call options on bonds.

Finally, a loan commitment made by a bank or other financial institution is a put option on a bond. Consider, for example, the situation where a bank quotes a 5-year interest rate of 5% per annum to a potential borrower and states that the rate is good for the next 2 months. The client has, in effect, obtained the right to sell a 5-year bond with a 5% coupon to the financial institution for its face value any time within the next 2 months. The option will be exercised if rates increase.

European Bond Options

Many over-the-counter bond options and some embedded bond options are European. The assumption made in the standard market model for valuing European bond options is that the forward bond price has a constant volatility σ_B. This allows Black's model in Section 27.6 to be used. In equations (27.28) and (27.29), σ_F is set equal to σ_B and F_0 is set equal to the forward bond price F_B, so that

$$c = P(0,T)[F_B N(d_1) - KN(d_2)] \tag{28.1}$$

$$p = P(0,T)[KN(-d_2) - F_B N(-d_1)] \tag{28.2}$$

where

$$d_1 = \frac{\ln(F_B/K) + \sigma_B^2 T/2}{\sigma_B\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma_B\sqrt{T}$$

with K the strike price of the bond option and T its time to maturity.

From Section 5.5, F_B can be calculated using the formula

$$F_B = \frac{B_0 - I}{P(0, T)} \tag{28.3}$$

where B_0 is the bond price at time zero and I is the present value of the coupons that will be paid during the life of the option. In this formula, both the spot bond price and the forward bond price are cash prices rather than quoted prices. The relationship between cash and quoted bond prices is explained in Section 6.1.

The strike price K in equations (28.1) and (28.2) should be the cash strike price. In choosing the correct value for K, the precise terms of the option are therefore important. If the strike price is defined as the cash amount that is exchanged for the bond when the option is exercised, K should be set equal to this strike price. If, as is more common, the strike price is the quoted price applicable when the option is exercised, K should be set equal to the strike price plus accrued interest at the expiration date of the option. Traders refer to the quoted price of a bond as the *clean price* and the cash price as the *dirty price*.

Example 28.1

Consider a 10-month European call option on a 9.75-year bond with a face value of \$1,000. (When the option matures, the bond will have 8 years and 11 months remaining.) Suppose that the current cash bond price is \$960, the strike price is \$1,000, the 10-month risk-free interest rate is 10% per annum, and the volatility of the forward bond price for a contract maturing in 10 months is 9% per annum. The bond pays a coupon of 10% per year (with payments made semiannually). Coupon payments of \$50 are expected in 3 months and 9 months. (This means that the accrued interest is \$25 and the quoted bond price is \$935.) We suppose that the 3-month and 9-month risk-free interest rates are 9.0% and 9.5% per annum, respectively. The present value of the coupon payments is, therefore,

$$50e^{-0.25\times0.09} + 50e^{-0.75\times0.095} = 95.45$$

or \$95.45. The bond forward price is from equation (28.3) given by

$$F_B = (960 - 95.45)e^{0.1\times0.8333} = 939.68$$

(a) If the strike price is the cash price that would be paid for the bond on exercise, the parameters for equation (28.1) are $F_B = 939.68$, $K = 1000$, $P(0, T) = e^{-0.1\times(10/12)} = 0.9200$, $\sigma_B = 0.09$, and $T = 10/12$. The price of the call option is \$9.49.

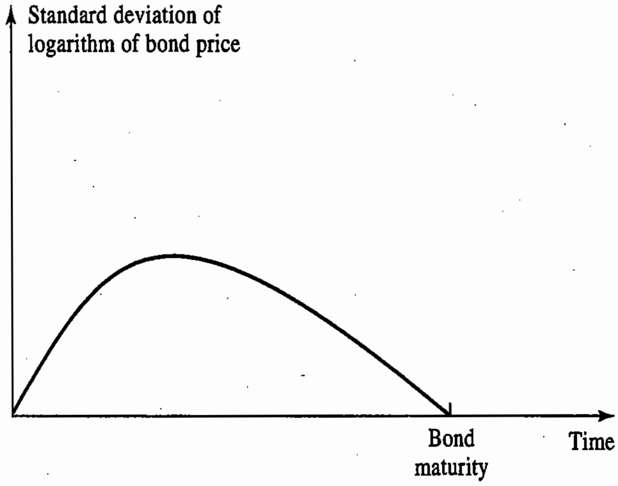
(b) If the strike price is the quoted price that would be paid for the bond on exercise, 1 month's accrued interest must be added to K because the maturity of the option is 1 month after a coupon date. This produces a value for K of

$$1{,}000 + 100 \times 0.08333 = 1{,}008.33$$

The values for the other parameters in equation (28.1) are unchanged (i.e., $F_B = 939.68$, $P(0, T) = 0.9200$, $\sigma_B = 0.09$, and $T = 0.8333$). The price of the option is \$7.97.

Figure 28.1 shows how the standard deviation of the logarithm of a bond's price

Figure 28.1 Standard deviation of logarithm of bond price at future times.



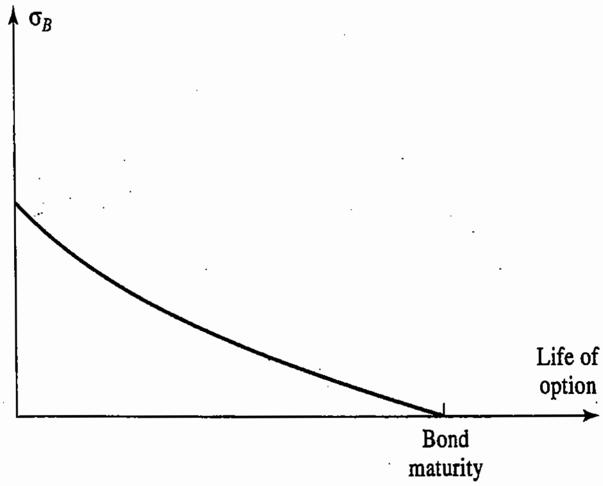
changes as we look further ahead. The standard deviation is zero today because there is no uncertainty about the bond's price today. It is also zero at the bond's maturity because we know that the bond's price will equal its face value at maturity. Between today and the maturity of the bond, the standard deviation first increases and then decreases.

The volatility σ_B that should be used when a European option on the bond is valued is

$$\frac{\text{Standard deviation of logarithm of bond price at maturity of option}}{\sqrt{\text{Time to maturity of option}}}$$

What happens when, for a particular underlying bond, the life of the option is increased? Figure 28.2 shows a typical pattern for σ_B as a function of the life of the option. In general, σ_B declines as the life of the option increases.

Figure 28.2 Variation of forward bond price volatility σ_B with life of option when bond is kept fixed.



Yield Volatilities

The volatilities that are quoted for bond options are often yield volatilities rather than price volatilities. The duration concept, introduced in Chapter 4, is used by the market to convert a quoted yield volatility into a price volatility. Suppose that D is the modified duration of the bond underlying the option at the option maturity, as defined in Chapter 4. The relationship between the change ΔF_B in the forward bond price F_B and the change Δy_F in the forward yield y_F is

$$\frac{\Delta F_B}{F_B} \approx -D\Delta y_F$$

or

$$\frac{\Delta F_B}{F_B} \approx -Dy_F\frac{\Delta y_F}{y_F}$$

Volatility is a measure of the standard deviation of percentage changes in the value of a variable. This equation therefore suggests that the volatility of the forward bond price σ_B used in Black's model can be approximately related to the volatility of the forward bond yield σ_y by

$$\sigma_B = Dy_0\sigma_y \tag{28.4}$$

where y_0 is the initial value of y_F. When a yield volatility is quoted for a bond option, the implicit assumption is usually that it will be converted to a price volatility using equation (28.4), and that this volatility will then be used in conjunction with equation (28.1) or (28.2) to obtain the option's price. Suppose that the bond underlying a call option will have a modified duration of 5 years at option maturity, the forward yield is 8%, and the forward yield volatility quoted by a broker is 20%. This means that the market price of the option corresponding to the broker quote is the price given by equation (28.1) when the volatility variable σ_B is

$$5 \times 0.08 \times 0.2 = 0.08$$

or 8% per annum. Figure 28.2 shows that forward bond volatilities depend on the option considered. Forward yield volatilities as we have just defined them are more constant. This is why traders prefer them.

The Bond_Options worksheet of the software DerivaGem accompanying this book can be used to price European bond options using Black's model by selecting Black-European as the Pricing Model. The user inputs a yield volatility, which is handled in the way just described. The strike price can be the cash or quoted strike price.

Example 28.2

Consider a European put option on a 10-year bond with a principal of 100. The coupon is 8% per year payable semiannually. The life of the option is 2.25 years and the strike price of the option is 115. The forward yield volatility is 20%. The zero curve is flat at 5% with continuous compounding. DerivaGem shows that the quoted price of the bond is 122.84. The price of the option when the strike price is a quoted price is \$2.37. When the strike price is a cash price, the price of the option is \$1.74. (Note that DerivaGem's prices may not exactly agree with manually calculated prices because DerivaGem assumes 365 days per year and rounds times to the nearest whole number of days. See Problem 28.16 for the manual calculation.)

28.2 INTEREST RATE CAPS AND FLOORS

A popular interest rate option offered by financial institutions in the over-the-counter market is an *interest rate cap*. Interest rate caps can best be understood by first considering a floating-rate note where the interest rate is reset periodically equal to LIBOR. The time between resets is known as the *tenor*. Suppose the tenor is 3 months. The interest rate on the note for the first 3 months is the initial 3-month LIBOR rate; the interest rate for the next 3 months is set equal to the 3-month LIBOR rate prevailing in the market at the 3-month point; and so on.

An interest rate cap is designed to provide insurance against the rate of interest on the floating-rate note rising above a certain level. This level is known as the *cap rate*. Suppose that the principal amount is \$10 million, the tenor is 3 months, the life of the cap is 5 years, and the cap rate is 4%. (Because the payments are made quarterly, this cap rate is expressed with quarterly compounding.) The cap provides insurance against the interest on the floating rate note rising above 4%.

For the moment we ignore day count issues and assume that there is exactly 0.25 year between each payment date. (We will cover day count issues at the end of this section.) Suppose that on a particular reset date the 3-month LIBOR interest rate is 5%. The floating rate note would require

$$0.25 \times 0.05 \times \$10{,}000{,}000 = \$125{,}000$$

of interest to be paid 3 months later. With a 3-month LIBOR rate of 4% the interest payment would be

$$0.25 \times 0.04 \times \$10{,}000{,}000 = \$100{,}000$$

The cap therefore provides a payoff of \$25,000. The payoff does not occur on the reset date when the 5% is observed: it occurs 3 months later. This reflects the usual time lag between an interest rate being observed and the corresponding payment being required.

At each reset date during the life of the cap, LIBOR is observed. If LIBOR is less than 4%, there is no payoff from the cap three months later. If LIBOR is greater than 4%, the payoff is one quarter of the excess applied to the principal of \$10 million. Note that caps are usually defined so that the initial LIBOR rate, even if it is greater than the cap rate, does not lead to a payoff on the first reset date. In our example, the cap lasts for 5 years. There are, therefore, a total of 19 reset dates (at times 0.25, 0.50, 0.75, …, 4.75 years) and 19 potential payoffs from the caps (at times 0.50, 0.75, 1.00, …, 5.00 years).

The Cap as a Portfolio of Interest Rate Options

Consider a cap with a total life of T, a principal of L, and a cap rate of R_K. Suppose that the reset dates are $t_1, t_2, \dots, t_n$ and define $t_{n+1} = T$. Define R_k as the LIBOR interest rate for the period between time t_k and t_{k+1} observed at time t_k $(1 \leqslant k \leqslant n)$. The cap leads to a payoff at time t_{k+1} $(k = 1, 2, \dots, n)$ of

$$L\delta_k \max(R_k - R_K, 0) \tag{28.5}$$

where $\delta_k = t_{k+1} - t_k$.[1] Both R_k and R_K are expressed with a compounding frequency equal to the frequency of resets.

[0] Day count issues are discussed at the end of this section.

Expression (28.5) is the payoff from a call option on the LIBOR rate observed at time t_k with the payoff occurring at time t_{k+1}. The cap is a portfolio of n such options. LIBOR rates are observed at times $t_1, t_2, t_3, \ldots, t_n$ and the corresponding payoffs occur at times $t_2, t_3, t_4, \ldots, t_{n+1}$. The n call options underlying the cap are known as *caplets*.

A Cap as a Portfolio of Bond Options

An interest rate cap can also be characterized as a portfolio of put options on zero-coupon bonds with payoffs on the puts occurring at the time they are calculated. The payoff in equation (28.5) at time t_{k+1} is equivalent to

$$\frac{L\delta_k}{1+R_k\delta_k}\max(R_k - R_K,\ 0)$$

at time t_k. A few lines of algebra show that this reduces to

$$\max\left[L - \frac{L(1+R_K\delta_k)}{1+R_k\delta_k},\ 0\right] \tag{28.6}$$

The expression

$$\frac{L(1+R_K\delta_k)}{1+R_k\delta_k}$$

is the value at time t_k of a zero-coupon bond that pays off $L(1+R_K\delta_k)$ at time t_{k+1}. The expression in equation (28.6) is therefore the payoff from a put option with maturity t_k on a zero-coupon bond with maturity t_{k+1} when the face value of the bond is $L(1+R_K\delta_k)$ and the strike price is L. It follows that an interest rate cap can be regarded as a portfolio of European put options on zero-coupon bonds.

Floors and Collars

Interest rate floors and interest rate collars (sometimes called floor–ceiling agreements) are defined analogously to caps. A *floor* provides a payoff when the interest rate on the underlying floating-rate note falls below a certain rate. With the notation already introduced, a floor provides a payoff at time t_{k+1} ($k = 1, 2, \ldots, n$) of

$$L\delta_k \max(R_K - R_k,\ 0)$$

Analogously to an interest rate cap, an interest rate floor is a portfolio of put options on interest rates or a portfolio of call options on zero-coupon bonds. Each of the individual options comprising a floor is known as a *floorlet*. A *collar* is an instrument designed to guarantee that the interest rate on the underlying LIBOR floating-rate note always lies between two levels. A collar is a combination of a long position in a cap and a short position in a floor. It is usually constructed so that the price of the cap is initially equal to the price of the floor. The cost of entering into the collar is then zero.

Business Snapshot 28.1 gives the put–call parity relationship between caps and floors.

Valuation of Caps and Floors

As shown in equation (28.5), the caplet corresponding to the rate observed at time t_k provides a payoff at time t_{k+1} of

$$L\delta_k \max(R_k - R_K,\ 0)$$

Business Snapshot 28.1 Put–Call Parity for Caps and Floors

There is a put–call parity relationship between the prices of caps and floors. This is

$$\text{Value of cap} = \text{Value of floor} + \text{Value of swap}$$

In this relationship, the cap and floor have the same strike price, R_K. The swap is an agreement to receive LIBOR and pay a fixed rate of R_K with no exchange of payments on the first reset date. All three instruments have the same life and the same frequency of payments.

To see that the result is true, consider a long position in the cap combined with a short position in the floor. The cap provides a cash flow of LIBOR $- R_K$ for periods when LIBOR is greater than R_K. The short floor provides a cash flow of $-(R_K - \text{LIBOR}) = \text{LIBOR} - R_K$ for periods when LIBOR is less than R_K. There is therefore a cash flow of LIBOR $- R_K$ in all circumstances. This is the cash flow on the swap. It follows that the value of the cap minus the value of the floor must equal the value of the swap.

Note that swaps are usually structured so that LIBOR at time zero determines a payment on the first reset date. Caps and floors are usually structured so that there is no payoff on the first reset date. This is why put–call parity involves a nonstandard swap where there is no payment on the first reset date.

Under the standard market model, the value of the caplet is

$$L\delta_k P(0, t_{k+1})[F_k N(d_1) - R_K N(d_2)] \tag{28.7}$$

where

$$d_1 = \frac{\ln(F_k/R_K) + \sigma_k^2 t_k/2}{\sigma_k\sqrt{t_k}}$$

$$d_2 = \frac{\ln(F_k/R_K) - \sigma_k^2 t_k/2}{\sigma_k\sqrt{t_k}} = d_1 - \sigma_k\sqrt{t_k}$$

F_k is the forward interest rate at time 0 for the period between time t_k and t_{k+1}, and σ_k is the volatility of this forward interest rate. This is a natural extension of Black's model. The volatility σ_k is multiplied by $\sqrt{t_k}$ because the interest rate R_k is observed at time t_k, but the discount factor $P(0, t_{k+1})$ refects the fact that the payoff is at time t_{k+1}, not time t_k. The value of the corresponding floorlet is

$$L\delta_k P(0, t_{k+1})[R_K N(-d_2) - F_k N(-d_1)] \tag{28.8}$$

Example 28.3

Consider a contract that caps the LIBOR interest rate on \$10,000 at 8% per annum (with quarterly compounding) for 3 months starting in 1 year. This is a caplet and could be one element of a cap. Suppose that the LIBOR/swap zero curve is flat at 7% per annum with quarterly compounding and the volatility of the 3-month forward rate underlying the caplet is 20% per annum. The continuously compounded zero rate for all maturities is 6.9394%. In equation (28.7), $F_k = 0.07$, $\delta_k = 0.25$, $L = 10{,}000$, $R_K = 0.08$, $t_k = 1.0$, $t_{k+1} = 1.25$, $P(0, t_{k+1}) =$

$e^{-0.069394\times 1.25} = 0.9169$, and $\sigma_k = 0.20$. Also,

$$d_1 = \frac{\ln(0.07/0.08) + 0.2^2 \times 1/2}{0.20 \times 1} = -0.5677$$

$$d_2 = d_1 - 0.20 = -0.7677$$

so that the caplet price is

$$0.25 \times 10{,}000 \times 0.9169[0.07N(-0.5677) - 0.08N(-0.7677)] = \$5.162$$

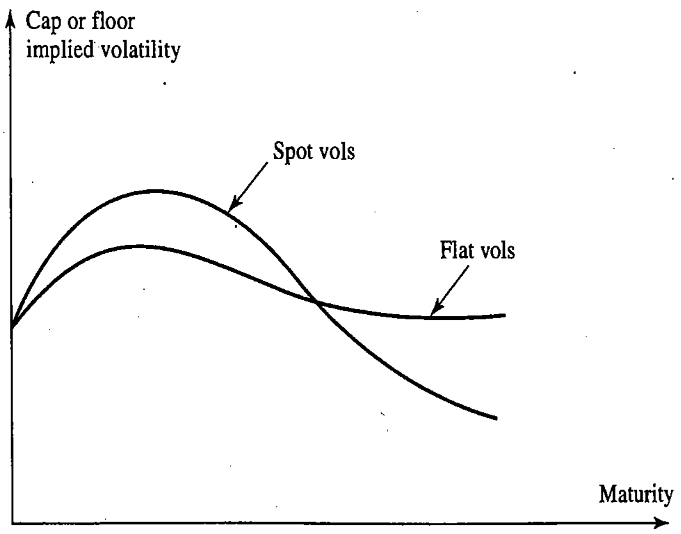
(Note that DerivaGem gives $5.146 for the price of this caplet. This is because it assumes 365 days per year and rounds times to the nearest whole number of days.)

Each caplet of a cap must be valued separately using equation (28.7). Similarly, each floorlet of a floor must be valued separately using equation (28.8). One approach is to use a different volatility for each caplet (or floorlet). The volatilities are then referred to as *spot volatilities*. An alternative approach is to use the same volatility for all the caplets (floorlets) comprising any particular cap (floor) but to vary this volatility according to the life of the cap (floor). The volatilities used are then referred to as *flat volatilities*.[2] The volatilities quoted in the market are usually flat volatilities. However, many traders like to estimate spot volatilities because this allows them to identify underpriced and overpriced caplets (floorlets). The put (call) options on Eurodollar futures are very similar to caplets (floorlets) and the spot volatilities used for caplets and floorlets on 3-month LIBOR are frequently compared with those calculated from the prices of Eurodollar futures options.

Spot Volatilities vs. Flat Volatilities

Figure 28.3 shows a typical pattern for spot volatilities and flat volatilities as a function of maturity. (In the case of a spot volatility, the maturity is the maturity of a caplet or floorlet; in the case of a flat volatility, it is the maturity of a cap or floor.) The flat

Figure 28.3 The volatility hump.



[0] Flat volatilities can be calculated from spot volatilities and vice versa (see Problem 28.20).

Table 28.1 Typical broker implied flat volatility quotes for US dollar caps and floors (% per annum).

Life	*Cap bid*	*Cap offer*	*Floor bid*	*Floor offer*
1 year	18.00	20.00	18.00	20.00
2 years	23.25	24.25	23.75	24.75
3 years	24.00	25.00	24.50	25.50
4 years	23.75	24.75	24.25	25.25
5 years	23.50	24.50	24.00	25.00
7 years	21.75	22.75	22.00	23.00
10 years	20.00	21.00	20.25	21.25

volatilities are akin to cumulative averages of the spot volatilities and therefore exhibit less variability. As indicated by Figure 28.3, a "hump" in the volatilities is usually observed. The peak of the hump is at about the 2- to 3-year point. This hump is observed both when the volatilities are implied from option prices and when they are calculated from historical data. There is no general agreement on the reason for the existence of the hump. One possible explanation is as follows. Rates at the short end of the zero curve are controlled by central banks. By contrast, 2- and 3-year interest rates are determined to a large extent by the activities of traders. These traders may be overreacting to the changes observed in the short rate and causing the volatility of these rates to be higher than the volatlity of short rates. For maturities beyond 2 to 3 years, the mean reversion of interest rates, which is discussed in Chapter 30, causes volatilities to decline.

Brokers provide tables of implied flat volatilities for caps and floors. The instruments underlying the quotes are usually at the money. This means that the cap/floor rate equals the swap rate for a swap that has the same payment dates as the cap. Table 28.1 shows typical broker quotes for the US dollar market. The tenor of the cap is 3 months and the cap life varies from 1 to 10 years. The data exhibits the type of "hump" shown in Figure 28.3.

Theoretical Justification for the Model

The extension of Black's model used to value a caplet can be shown to be internally consistent by considering a world that is forward risk neutral with respect to a zero-coupon bond maturing at time t_{k+1}. The analysis in Section 27.4 shows that:

1. The current value of any security is its expected value at time t_{k+1} in this world multiplied by the price of a zero-coupon bond maturing at time t_{k+1} (see equation (27.20)).
2. The expected value of an interest rate lasting between times t_k and t_{k+1} equals the forward interest rate in this world (see equation (27.22)).

The first of these results shows that, with the notation introduced earlier, the price of a caplet that provides a payoff at time t_{k+1} is

$$L\delta_k P(0, t_{k+1}) E_{k+1}[\max(R_k - R_K, 0)] \tag{28.9}$$

where E_{k+1} denotes expected value in a world that is forward risk neutral with respect to a zero-coupon bond maturing at time t_{k+1}. When the forward interest rate underlying the cap (initially F_k) is assumed to have a constant volatility σ_k, R_k is lognormal in the world we are considering, with $\ln(R_k) = \sigma_k\sqrt{t_k}$. From the appendix at the end of Chapter 13, equation (28.9) becomes

$$L\delta_k P(0, t_{k+1})[E_{k+1}(R_k)N(d_1) - R_K N(d_2)]$$

where

$$d_1 = \frac{\ln[E_{k+1}(R_k)/R_K] + \sigma_k^2 t_k/2}{\sigma_k\sqrt{t_k}}$$

$$d_2 = \frac{\ln[E_{k+1}(R_k)/R_K] - \sigma_k^2 t_k/2}{\sigma_k\sqrt{t_k}} = d_1 - \sigma\sqrt{t_k}$$

The second result implies that

$$E_{k+1}(R_k) = F_k$$

Together the results lead to the cap pricing model in equation (28.7). They show that we can discount at the t_{k+1}-maturity interest rate observed in the market today providing we set the expected interest rate equal to the forward interest rate.

Use of DerivaGem

The software DerivaGem accompanying this book can be used to price interest rate caps and floors using Black's model. In the Cap_and_Swap_Option worksheet select Cap/Floor as the Underlying Type and Black-European as the Pricing Model. The zero curve is input using continuously compounded rates. The inputs include the start and end date of the period covered by the cap, the flat volatility, and the cap settlement frequency (i.e., the tenor). The software calculates the payment dates by working back from the end of period covered by the cap to the beginning. The initial caplet/floorlet is assumed to cover a period of length between 0.5 and 1.5 times a regular period. Suppose, for example, that the period covered by the cap is 1.22 years to 2.80 years and the settlement frequency is quarterly. There are six caplets covering the periods 2.55 to 2.80 years, 2.30 to 2.55 years, 2.05 to 2.30 years, 1.80 to 2.05 years, 1.55 to 1.80 years, and 1.22 to 1.55 years.

The Impact of Day Count Conventions

The formulas we have presented so far in this section do not reflect day count conventions (see Section 6.1 for an explanation of day count conventions). Suppose that the cap rate R_K is expressed with an actual/360 day count (as would be normal in the United States). This means that the time interval δ_k in the formulas should be replaced by a_k, the *accrual fraction* for the time period between t_k and t_{k+1}. Suppose, for example, that t_k is May 1 and t_{k+1} is August 1. Under actual/360 there are 92 days between these payment dates so that $a_k = 92/360 = 0.2521$. The forward rate F_k must be expressed with an actual/360 day count. This means that we must set it by solving

$$1 + a_k F_k = \frac{P(0, t_k)}{P(0, t_{k+1})}$$

The impact of all this is much the same as calculating δ_k on an actual/actual basis converting R_K from actual/360 to actual/actual, and calculating F_k on an actual/actual basis by solving

$$1 + \delta_k F_k = \frac{P(0, t_k)}{P(0, t_{k+1})}$$

28.3 EUROPEAN SWAP OPTIONS

Swap options, or *swaptions*, are options on interest rate swaps and are another popular type of interest rate option. They give the holder the right to enter into a certain interest rate swap at a certain time in the future. (The holder does not, of course, have to exercise this right.) Many large financial institutions that offer interest rate swap contracts to their corporate clients are also prepared to sell them swaptions or buy swaptions from them. As shown in Business Snapshot 28.2, a swaption can be viewed as a type of bond option. To give an example of how a swaption might be used, consider a company that knows that in 6 months it will enter into a 5-year floating-rate loan agreement and knows that it will wish to swap the floating interest payments for fixed interest payments to convert the loan into a fixed-rate loan (see Chapter 7 for a discussion of how swaps can be used in this way). At a cost, the company could enter into a swaption giving it the right to receive 6-month LIBOR and pay a certain fixed rate of interest, say 8% per annum, for a 5-year period starting in 6 months. If the fixed rate exchanged for floating on a regular 5-year swap in 6 months turns out to be less than 8% per annum, the company will choose not to exercise the swaption and will enter into a swap agreement in the usual way. However, if it turns out to be greater than 8% per annum, the company will choose to exercise the swaption and will obtain a swap at more favorable terms than those available in the market.

Swaptions, when used in the way just described, provide companies with a guarantee that the fixed rate of interest they will pay on a loan at some future time will not exceed some level. They are an alternative to forward swaps (sometimes called *deferred swaps*). Forward swaps involve no up-front cost but have the disadvantage of obligating the company to enter into a swap agreement. With a swaption, the company is able to benefit from favorable interest rate movements while acquiring protection from unfavorable interest rate movements. The difference between a swaption and a forward swap is analogous to the difference between an option on a foreign currency and a forward contract on the currency.

Valuation of European Swaptions

As explained in Chapter 7 the swap rate for a particular maturity at a particular time is the (mid-market) fixed rate that would be exchanged for LIBOR in a newly issued swap with that maturity. The model usually used to value a European option on a swap assumes that the underlying swap rate at the maturity of the option is lognormal. Consider a swaption where the holder has the right to pay a rate s_K and receive LIBOR on a swap that will last n years starting in T years. We suppose that there are m payments per year under the swap and that the notional principal is L.

Chapter 7 showed that day count conventions may lead to the fixed payments under a swap being slightly different on each payment date. For now we will ignore the effect of

Business Snapshot 28.2 Swaptions and Bond Options

As explained in Chapter 7, an interest rate swap can be regarded as an agreement to exchange a fixed-rate bond for a floating-rate bond. At the start of a swap, the value of the floating-rate bond always equals the principal amount of the swap. A swaption can therefore be regarded as an option to exchange a fixed-rate bond for the principal amount of the swap—that is, a type of bond option.

If a swaption gives the holder the right to pay fixed and receive floating, it is a put option on the fixed-rate bond with strike price equal to the principal. If a swaption gives the holder the right to pay floating and receive fixed, it is a call option on the fixed-rate bond with a strike price equal to the principal.

day count conventions and assume that each fixed payment on the swap is the fixed rate times L/m. The impact of day count conventions is considered at the end of this section.

Suppose that the swap rate for an n-year swap starting at time T proves to be s_T. By comparing the cash flows on a swap where the fixed rate is s_T to the cash flows on a swap where the fixed rate is s_K, it can be seen that the payoff from the swaption consists of a series of cash flows equal to

$$\frac{L}{m}\max(s_T - s_K,\ 0)$$

The cash flows are received m times per year for the n years of the life of the swap. Suppose that the swap payment dates are $T_1, T_2, \dots, T_{mn}$, measured in years from today. (It is approximately true that $T_i = T + i/m$.) Each cash flow is the payoff from a call option on s_T with strike price s_K.

Whereas a cap is a portfolio of options on interest rates, a swaption is a single option on the swap rate with repeated payoffs. The standard market model gives the value of a swaption where the holder has the right to pay s_K as

$$\sum_{i=1}^{mn}\frac{L}{m}P(0, T_i)[s_0N(d_1) - s_KN(d_2)]$$

where

$$d_1 = \frac{\ln(s_0/s_K) + \sigma^2T/2}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(s_0/s_K) - \sigma^2T/2}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

s_0 is the forward swap rate at time zero calculated as indicated in equation (27.23), and σ is the volatility of the forward swap rate (so that $\sigma\sqrt{T}$ is the standard deviation of $\ln s_T$).

This is a natural extension of Black's model. The volatility σ is multiplied by $\sqrt{T}$. The $\sum_{i=1}^{mn} P(0, T_i)$ term is the discount factor for the mn payoffs. Defining A as the value of a contract that pays $1/m$ at times T_i $(1 \leqslant i \leqslant mn)$, the value of the swaption becomes

$$LA[s_0N(d_1) - s_KN(d_2)] \tag{28.10}$$

where

$$A = \frac{1}{m}\sum_{i=1}^{mn} P(0, T_i)$$

If the swaption gives the holder the right to receive a fixed rate of s_K instead of paying it, the payoff from the swaption is

$$\frac{L}{m}\max(s_K - s_T,\ 0)$$

This is a put option on s_T. As before, the payoffs are received at times T_i $(1 \leqslant i \leqslant mn)$. The standard market model gives the value of the swaption as

$$LA[s_K N(-d_2) - s_0 N(-d_1)] \tag{28.11}$$

Example 28.4

Suppose that the LIBOR yield curve is flat at 6% per annum with continuous compounding. Consider a swaption that gives the holder the right to pay 6.2% in a 3-year swap starting in 5 years. The volatility of the forward swap rate is 20%. Payments are made semiannually and the principal is \$100. In this case,

$$A = \tfrac{1}{2}(e^{-0.06\times5.5} + e^{-0.06\times6} + e^{-0.06\times6.5} + e^{-0.06\times7} + e^{-0.06\times7.5} + e^{-0.06\times8}) = 2.0035$$

A rate of 6% per annum with continuous compounding translates into 6.09% with semiannual compounding. It follows that, in this example, $s_0 = 0.0609$, $s_K = 0.062$, $T = 5$, and $\sigma = 0.2$, so that

$$d_1 = \frac{\ln(0.0609/0.062) + 0.2^2 \times 5/2}{0.2\sqrt{5}} = 0.1836 \quad \text{and} \quad d_2 = d_1 - 0.2\sqrt{5} = -0.2636$$

From equation (28.10), the value of the swaption is

$$100 \times 2.0035 \times [0.0609 \times N(0.1836) - 0.062 \times N(-0.2636)] = 2.07$$

or \$2.07. (This is in agreement with the price given by DerivaGem.)

Broker Quotes

Brokers provide tables of implied volatilities for European swaptions (i.e., values of σ implied by market prices when equations (28.10) and (28.11) are used). The instruments underlying the quotes are usually at the money. This means that the strike swap rate equals the forward swap rate. Table 28.2 shows typical broker quotes provided for the

Table 28.2 Typical broker quotes for US European swaptions (mid-market volatilities percent per annum).

Expiration	*Swap length (years)*						
	1	*2*	*3*	*4*	*5*	*7*	*10*
1 month	17.75	17.75	17.75	17.50	17.00	17.00	16.00
3 months	19.50	19.00	19.00	18.00	17.50	17.00	16.00
6 months	20.00	20.00	19.25	18.50	18.75	17.75	16.75
1 year	22.50	21.75	20.50	20.00	19.50	18.25	16.75
2 years	22.00	22.00	20.75	19.50	19.75	18.25	16.75
3 years	21.50	21.00	20.00	19.25	19.00	17.75	16.50
4 years	20.75	20.25	19.25	18.50	18.25	17.50	16.00
5 years	20.00	19.50	18.50	17.75	17.50	17.00	15.50

US dollar market. The life of the option is shown on the vertical scale. This varies from 1 month to 5 years. The life of the underlying swap at the maturity of the option is shown on the horizontal scale. This varies from 1 to 10 years. The volatilities in the 1-year column of the table exhibit a hump similar to that discussed for caps earlier. As we move to the columns corresponding to options on longer-lived swaps, the hump persists but it becomes less pronounced.

Theoretical Justification for the Swaption Model

The extension of Black's model used for swaptions can be shown to be internally consistent by considering a world that is forward risk neutral with respect to the annuity A. The analysis in Section 27.4 shows that:

1. The current value of any security is the current value of the annuity multiplied by the expected value of
$$\frac{\text{Security price at time } T}{\text{Value of the annuity at time } T}$$
in this world (see equation (27.25)).
2. The expected value of the swap rate at time T in this world equals the forward swap rate (see equation (27.24)).

The first result shows that the value of the swaption is

$$LAE_A[\max(s_T - s_K, 0)]$$

From the appendix at the end of Chapter 13, this is

$$LA[E_A(s_T)N(d_1) - s_K N(d_2)]$$

where

$$d_1 = \frac{\ln[E_A(s_T)/s_K] + \sigma^2 T/2}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln[E_A(s_T)/s_K] - \sigma^2 T/2}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

The second result shows that $E_A(s_T)$ equals s_0. Taken together, the results lead to the swap option pricing formula in equation (28.10). They show that interest rates can be treated as constant for the purposes of discounting provided that the expected swap rate is set equal to the forward swap rate.

The Impact of Day Count Conventions

The above formulas can be made more precise by considering day count conventions. The fixed rate for the swap underlying the swap option is expressed with a day count convention such as actual/365 or 30/360. Suppose that $T_0 = T$ and that, for the applicable day count convention, the accrual fraction corresponding to the time period between T_{i-1} and T_i is a_i. (For example, if T_{i-1} corresponds to March 1 and T_i corresponds to September 1 and the day count is actual/365, $a_i = 184/365 = 0.5041$.)

The formulas that have been presented are then correct with the annuity factor A being defined as

$$A = \sum_{i=1}^{mn} a_i P(0, T_i)$$

As indicated by equation (27.23) the forward swap rate s_0 is given by solving

$$s_0 A = P(0, T) - P(0, T_{mn})$$

28.4 GENERALIZATIONS

We have presented three different versions of Black's model: one for bond options, one for caps, and one for swap options. Each of the models is internally consistent, but they are not consistent with each other. For example, when future bond prices are lognormal, future zero rates and swap rates are not lognormal; when future zero rates are lognormal, future bond prices and swap rates are not lognormal.

The results can be generalized as follows:

1. Consider any instrument that provides a payoff at time T dependent on the value of a bond observed at time T. Its current value is $P(0, T)$ times the expected payoff provided that expectations are calculated in a world where the expected price of the bond equals its forward price.
2. Consider any instrument that provides a payoff at time T^* dependent on the interest rate observed at time T for the period between T and T^*. Its current value is $P(0, T^*)$ times the expected payoff provided that expectations are calculated in a world where the expected value of the underlying interest rate equals the forward interest rate.
3. Consider any instrument that provides a payoff in the form of an annuity. Suppose that the size of the annuity is determined at time T as a function of the n-year swap rate at time T. Suppose also that annuity lasts for n years and payment dates for the annuity are the same as those for the swap. The value of the instrument is A times the expected payoff per year where (a) A is the current value of the annuity when payments are at the rate $1 per year and (b) expectations are taken in a world where the expected future swap rate equals the forward swap rate.

The first of these results is a generalization of the European bond option model; the second is a generalization of the cap/floor model; the third is a generalization of the swaption model.

28.5 HEDGING INTEREST RATE DERIVATIVES

This section discusses how the material on Greek letters in Chapter 17 can be extended to cover interest rate derivatives.

In the context of interest rate derivatives, delta risk is the risk associated with a shift in the zero curve. Because there are many ways in which the zero curve can shift, many deltas can be calculated. Some alternatives are:

1. Calculate the impact of a 1-basis-point parallel shift in the zero curve. This is sometimes termed a DV01.
2. Calculate the impact of small changes in the quotes for each of the instruments used to construct the zero curve.
3. Divide the zero curve (or the forward curve) into a number of sections (or buckets). Calculate the impact of shifting the rates in one bucket by 1 basis point, keeping the rest of the initial term structure unchanged. (This is described in Business Snapshot 6.3.)
4. Carry out a principal components analysis as outlined in Section 20.9. Calculate a delta with respect to the changes in each of the first few factors. The first delta then measures the impact of a small, approximately parallel, shift in the zero curve; the second delta measures the impact of a small twist in the zero curve; and so on.

In practice, traders tend to prefer the second approach. They argue that the only way the zero curve can change is if the quote for one of the instruments used to compute the zero curve changes. They therefore feel that it makes sense to focus on the exposures arising from changes in the prices of these instruments.

When several delta measures are calculated, there are many possible gamma measures. Suppose that 10 instruments are used to compute the zero curve and that deltas are calculated by considering the impact of changes in the quotes for each of these. Gamma is a second partial derivative of the form $\partial^2\Pi/\partial x_i\,\partial x_j$, where Π is the portfolio value. There are 10 choices for x_i and 10 choices for x_j and a total of 55 different gamma measures. This may be "information overload". One approach is ignore cross-gammas and focus on the 10 partial derivatives where $i = j$. Another is to calculate a single gamma measure as the second partial derivative of the value of the portfolio with respect to a parallel shift in the zero curve. A further possibility is to calculate gammas with respect to the first two factors in a principal components analysis.

The vega of a portfolio of interest rate derivatives measures its exposure to volatility changes. One approach is to calculate the impact on the portfolio of making the same small change to the Black volatilities of all caps and European swap options. However, this assumes that one factor drives all volatilities and may be too simplistic. A better idea is to carry out a principal components analysis on the volatilities of caps and swap options and calculate vega measures corresponding to the first 2 or 3 factors.

SUMMARY

Black's model and its extensions provide a popular approach for valuing European-style interest rate options. The essence of Black's model is that the value of the variable underlying the option is assumed to be lognormal at the maturity of the option. In the case of a European bond option, Black's model assumes that the underlying bond price is lognormal at the option's maturity. For a cap, the model assumes that the interest rates underlying each of the constituent caplets are lognormally distributed. In the case of a swap option, the model assumes that the underlying swap rate is lognormally distributed. Each of these models is internally consistent, but they are not consistent with each other.

Black's model involves calculating the expected payoff based on the assumption that the expected value of a variable equals its forward value and then discounting the expected payoff at the zero rate observed in the market today. This is the correct procedure for the "plain vanilla" instruments we have considered in this chapter. However, as we shall see in the next chapter, it is not correct in all situations.

FURTHER READING

Black, F., "The Pricing of Commodity Contracts," *Journal of Financial Economics*, 3 (March 1976): 167–79.

Questions and Problems (Answers in Solutions Manual)

28.1. A company caps 3-month LIBOR at 10% per annum. The principal amount is \$20 million. On a reset date, 3-month LIBOR is 12% per annum. What payment would this lead to under the cap? When would the payment be made?

28.2. Explain why a swap option can be regarded as a type of bond option.

28.3. Use the Black's model to value a 1-year European put option on a 10-year bond. Assume that the current value of the bond is \$125, the strike price is \$110, the 1-year interest rate is 10% per annum, the bond's forward price volatility is 8% per annum, and the present value of the coupons to be paid during the life of the option is \$10.

28.4. Explain carefully how you would use (a) spot volatilities and (b) flat volatilities to value a 5-year cap.

28.5. Calculate the price of an option that caps the 3-month rate, starting in 15 months' time, at 13% (quoted with quarterly compounding) on a principal amount of \$1,000. The forward interest rate for the period in question is 12% per annum (quoted with quarterly compounding), the 18-month risk-free interest rate (continuously compounded) is 11.5% per annum, and the volatility of the forward rate is 12% per annum.

28.6. A bank uses Black's model to price European bond options. Suppose that an implied price volatility for a 5-year option on a bond maturing in 10 years is used to price a 9-year option on the bond. Would you expect the resultant price to be too high or too low? Explain.

28.7. Calculate the value of a 4-year European call option on bond that will mature 5 years from today using Black's model. The 5-year cash bond price is \$105, the cash price of a 4-year bond with the same coupon is \$102, the strike price is \$100, the 4-year risk-free interest rate is 10% per annum with continuous compounding, and the volatility for the bond price in 4 years is 2% per annum.

28.8. If the yield volatility for a 5-year put option on a bond maturing in 10 years time is specified as 22%, how should the option be valued? Assume that, based on today's interest rates the modified duration of the bond at the maturity of the option will be 4.2 years and the forward yield on the bond is 7%.

28.9. What other instrument is the same as a 5-year zero-cost collar where the strike price of the cap equals the strike price of the floor? What does the common strike price equal?

28.10. Derive a put–call parity relationship for European bond options.

28.11. Derive a put–call parity relationship for European swap options.

28.12. Explain why there is an arbitrage opportunity if the implied Black (flat) volatility of a cap is different from that of a floor. Do the broker quotes in Table 28.1 present an arbitrage opportunity?

28.13. When a bond's price is lognormal can the bond's yield be negative? Explain your answer.

28.14. What is the value of a European swap option that gives the holder the right to enter into a 3-year annual-pay swap in 4 years where a fixed rate of 5% is paid and LIBOR is received? The swap principal is \$10 million. Assume that the yield curve is flat at 5% per annum with annual compounding and the volatility of the swap rate is 20%. Compare your answer with that given by DerivaGem.

28.15. Suppose that the yield R on a zero-coupon bond follows the process

$$dR = \mu\, dt + \sigma\, dz$$

where μ and σ are functions of R and t, and dz is a Wiener process. Use Itô's lemma to show that the volatility of the zero-coupon bond price declines to zero as it approaches maturity.

28.16. Carry out a manual calculation to verify the option prices in Example 28.2.

28.17. Suppose that the 1-year, 2-year, 3-year, 4-year, and 5-year zero rates are 6%, 6.4%, 6.7%, 6.9%, and 7%. The price of a 5-year semiannual cap with a principal of \$100 at a cap rate of 8% is \$3. Use DerivaGem to determine:
(a) The 5-year flat volatility for caps and floors
(b) The floor rate in a zero-cost 5-year collar when the cap rate is 8%

28.18. Show that $V_1 + f = V_2$, where V_1 is the value of a swap option to pay a fixed rate of s_K and receive LIBOR between times T_1 and T_2, f is the value of a forward swap to receive a fixed rate of s_K and pay LIBOR between times T_1 and T_2, and V_2 is the value of a swap option to receive a fixed rate of s_K between times T_1 and T_2. Deduce that $V_1 = V_2$ when s_K equals the current forward swap rate.

28.19. Suppose that zero rates are as in Problem 28.17. Use DerivaGem to determine the value of an option to pay a fixed rate of 6% and receive LIBOR on a 5-year swap starting in 1 year. Assume that the principal is \$100 million, payments are exchanged semiannually, and the swap rate volatility is 21%.

28.20. Describe how you would (a) calculate cap flat volatilities from cap spot volatilities and (b) calculate cap spot volatilities from cap flat volatilities.

Assignment Questions

28.21. Consider an 8-month European put option on a Treasury bond that currently has 14.25 years to maturity. The current cash bond price is \$910, the exercise price is \$900, and the volatility for the bond price is 10% per annum. A coupon of \$35 will be paid by the bond in 3 months. The risk-free interest rate is 8% for all maturities up to 1 year. Use Black's model to determine the price of the option. Consider both the case where the strike price corresponds to the cash price of the bond and the case where it corresponds to the quoted price.

28.22. Calculate the price of a cap on the 90-day LIBOR rate in 9 months' time when the principal amount is $1,000. Use Black's model and the following information:

(a) The quoted 9-month Eurodollar futures price = 92. (Ignore differences between futures and forward rates.)

(b) The interest rate volatility implied by a 9-month Eurodollar option = 15% per annum.

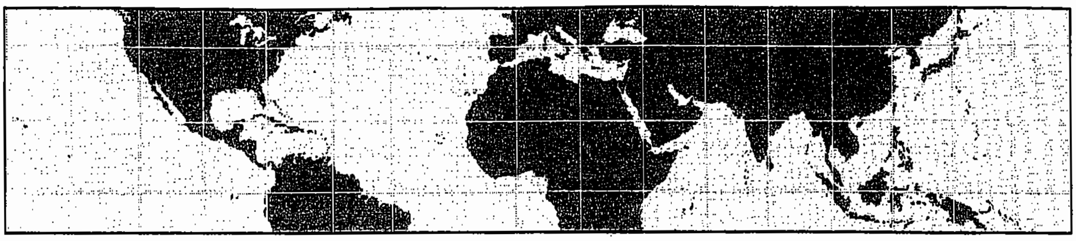
(c) The current 12-month interest rate with continuous compounding = 7.5% per annum.

(d) The cap rate = 8% per annum. (Assume an actual/360 day count.)

28.23. Suppose that the LIBOR yield curve is flat at 8% with annual compounding. A swaption gives the holder the right to receive 7.6% in a 5-year swap starting in 4 years. Payments are made annually. The volatility of the forward swap rate is 25% per annum and the principal is $1 million. Use Black's model to price the swaption. Compare your answer with that given by DerivaGem.

28.24. Use the DerivaGem software to value a 5-year collar that guarantees that the maximum and minimum interest rates on a LIBOR-based loan (with quarterly resets) are 7% and 5%, respectively. The LIBOR zero curve (continuously compounded) is currently flat at 6%. Use a flat volatility of 20%. Assume that the principal is $100.

28.25. Use the DerivaGem software to value a European swap option that gives you the right in 2 years to enter into a 5-year swap in which you pay a fixed rate of 6% and receive floating. Cash flows are exchanged semiannually on the swap. The 1-year, 2-year, 5-year, and 10-year zero-coupon interest rates (continuously compounded) are 5%, 6%, 6.5%, and 7%, respectively. Assume a principal of $100 and a volatility of 15% per annum. Give an example of how the swap option might be used by a corporation. What bond option is equivalent to the swap option?



CHAPTER 29

Convexity, Timing, and Quanto Adjustments

A popular two-step procedure for valuing a European-style derivative is:

1. Calculate the expected payoff by assuming that the expected value of each underlying variable equals its forward value.
2. Discount the expected payoff at the risk-free rate applicable for the time period between the valuation date and the payoff date.

We first used this procedure when valuing FRAs and swaps. Chapter 4 shows that an FRA can be valued by calculating the payoff on the assumption that the forward interest rate will be realized and then discounting the payoff at the risk-free rate. Similarly, Chapter 7 shows that swaps can be valued by calculating cash flows on the assumption that forward rates will be realized and discounting the cash flows at risk-free rates. Chapters 16 and 27 show that Black's model provides a general approach to valuing a wide range of European options—and Black's model is an application of the two-step procedure. The models presented in Chapter 28 for bond options, caps/floors, and swap options are all examples of the two-step procedure.

This raises the issue of whether it is always correct to value European-style interest rate derivatives by using the two-step procedure. The answer is no! For nonstandard interest rate derivatives, it is sometimes necessary to modify the two-step procedure so that an adjustment is made to the forward value of the variable in the first step. This chapter considers three types of adjustments: convexity adjustments, timing adjustments, and quanto adjustments.

29.1 CONVEXITY ADJUSTMENTS

Consider first an instrument that provides a payoff dependent on a bond yield observed at the time of the payoff.

Usually the forward value of a variable S is calculated with reference to a forward contract that pays off $S_T - K$ at time T. It is the value of K that causes the contract to have zero value. As discussed in Section 27.4, forward interest rates and forward yields are defined differently. A forward interest rate is the rate implied by a forward zero-coupon bond. More generally, a forward bond yield is the yield implied by the forward bond price.

Suppose that B_T is the price of a bond at time T, y_T is its yield, and the (bond pricing) relationship between B_T and y_T is

$$B_T = G(y_T)$$

Define F_0 as the forward bond price at time zero for a contract maturing at time T and y_0 as the forward bond yield at time zero. The definition of a forward bond yield means that

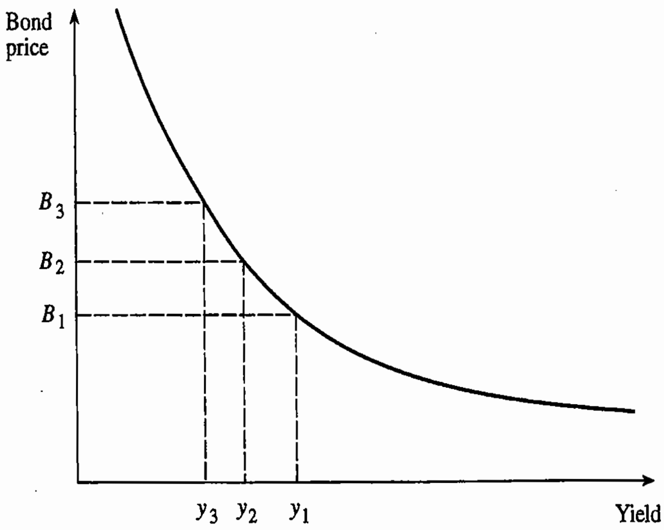
$$F_0 = G(y_0)$$

The function G is nonlinear. This means that, when the expected future bond price equals the forward bond price (so that we are in a world that is forward risk neutral with respect to a zero-coupon bond maturing at time T), the expected future bond yield does not equal the forward bond yield.

This is illustrated in Figure 29.1, which shows the relationship between bond prices and bond yields at time T. For simplicity, suppose that there are only three possible bond prices, B_1, B_2, and B_3 and that they are equally likely in a world that is forward risk neutral with respect to $P(t, T)$. Assume that the bond prices are equally spaced, so that $B_2 - B_1 = B_3 - B_2$. The forward bond price is the expected bond price B_2. The bond prices translate into three equally likely bond yields: y_1, y_2, and y_3. These are not equally spaced. The variable y_2 is the forward bond yield because it is the yield corresponding to the forward bond price. The expected bond yield is the average of y_1, y_2, and y_3 and is clearly greater than y_2.

Consider a derivative that provides a payoff dependent on the bond yield at time T. From equation (27.20), it can be valued by (a) calculating the expected payoff in a world that is forward risk neutral with respect to a zero-coupon bond maturing at time T and (b) discounting at the current risk-free rate for maturity T. We know that the expected bond price equals the forward price in the world being considered. We

Figure 29.1 Relationship between bond prices and bond yields at time T.



therefore need to know the value of the expected bond yield when the expected bond price equals the forward bond price. The analysis in the appendix at the end of this chapter shows that an approximate expression for the required expected bond yield is

$$E_T(y_T) = y_0 - \tfrac{1}{2}y_0^2\sigma_y^2 T \frac{G''(y_0)}{G'(y_0)} \tag{29.1}$$

where G' and G'' denote the first and second partial derivatives of G, E_T denotes expectations in a world that is forward risk neutral with respect to $P(t, T)$, and σ_y is the forward yield volatility. It follows that the expected payoff can be discounted at the current risk-free rate for maturity T provided the expected bond yield is assumed to be

$$y_0 - \tfrac{1}{2}y_0^2\sigma_y^2 T \frac{G''(y_0)}{G'(y_0)}$$

rather than y_0. The difference between the expected bond yield and the forward bond yield

$$-\tfrac{1}{2}y_0^2\sigma_y^2 T \frac{G''(y_0)}{G'(y_0)}$$

is known as a *convexity adjustment*. It corresponds to the difference between y_2 and the expected yield in Figure 29.1. (The convexity adjustment is positive because $G'(y_0) < 0$ and $G''(y_0) > 0$.)

Application 1: Interest Rates

For a first application of equation (29.1), consider an instrument that provides a cash flow at time T equal to the interest rate between times T and T^* applied to a principal of L. (This example will be useful when we consider LIBOR-in-arrears swaps in Chapter 32.) Note that the interest rate applicable to the time period between times T and T^* is normally paid at time T^*; here it is assumed that it is paid early, at time T.

The cash flow at time T is $LR_T\tau$, where $\tau = T^* - T$ and R_T is the zero-coupon interest rate applicable to the period between T and T^* (expressed with a compounding period of τ).[1] The variable R_T can be viewed as the yield at time T on a zero-coupon bond maturing at time T^*. The relationship between the price of this bond and its yield is

$$G(y) = \frac{1}{1 + y\tau}$$

From equation (29.1),

$$E_T(R_T) = R_0 - \tfrac{1}{2}R_0^2\sigma_R^2 T \frac{G''(R_0)}{G'(R_0)}$$

or

$$E_T(R_T) = R_0 + \frac{R_0^2\sigma_R^2\tau T}{1 + R_0\tau} \tag{29.2}$$

where R_0 is the forward rate applicable to the period between T and T^* and σ_R is the volatility of the forward rate.

[1] As usual, for ease of exposition we assume actual/actual day counts in our examples.

The value of the instrument is therefore

$$P(0,T)L\tau\left[R_0 + \frac{R_0^2\sigma_R^2\tau T}{1+R_0\tau}\right]$$

Example 29.1

Consider a derivative that provides a payoff in 3 years equal to the 1-year zero-coupon rate (annually compounded) at that time multiplied by \$1000. Suppose that the zero rate for all maturities is 10% per annum with annual compounding and the volatility of the forward rate applicable to the time period between year 3 and year 4 is 20%. In this case, $R_0 = 0.10$, $\sigma_R = 0.20$, $T = 3$, $\tau = 1$, and $P(0,3) = 1/1.10^3 = 0.7513$. The value of the derivative is

$$0.7513 \times 1000 \times 1 \times \left[0.10 + \frac{0.10^2 \times 0.20^2 \times 1 \times 3}{1 + 0.10 \times 1}\right]$$

or \$75.95. (This compares with a price of \$75.13 when no convexity adjustment is made.)

Application 2: Swap Rates

Consider next a derivative providing a payoff at time T equal to a swap rate observed at that time. A swap rate is a par yield. For the purposes of calculating a convexity adjustment we can make an approximation and assume that the N-year swap rate at time T equals the yield at that time on an N-year bond with a coupon equal to today's forward swap rate. This enables equation (29.1) to be used.

Example 29.2

Consider an instrument that provides a payoff in 3 years equal to the 3-year swap rate at that time multiplied by \$100. Suppose that payments are made annually on the swap, the zero rate for all maturities is 12% per annum with annual compounding, the volatility for the 3-year forward swap rate in 3 years (implied from swap option prices) is 22%. When the swap rate is approximated as the yield on a 12% bond, the relevant function $G(y)$ is

$$G(y) = \frac{0.12}{1+y} + \frac{0.12}{(1+y)^2} + \frac{1.12}{(1+y)^3}$$

$$G'(y) = -\frac{0.12}{(1+y)^2} - \frac{0.24}{(1+y)^3} - \frac{3.36}{(1+y)^4}$$

$$G''(y) = \frac{0.24}{(1+y)^3} + \frac{0.72}{(1+y)^4} + \frac{13.44}{(1+y)^5}$$

In this case the forward yield y_0 is 0.12, so that $G'(y_0) = -2.4018$ and $G''(y_0) = 8.2546$. From equation (29.1),

$$E_T(y_T) = 0.12 + \tfrac{1}{2} \times 0.12^2 \times 0.22^2 \times 3 \times \frac{8.2546}{2.4018} = 0.1236$$

A forward swap rate of 0.1236 (= 12.36%) rather than 0.12 should therefore be assumed when valuing the instrument. The instrument is worth

$$\frac{100 \times 0.1236}{1.12^3} = 8.80$$

or \$8.80. (This compares with a price of 8.54 obtained without any convexity adjustment.)

29.2 TIMING ADJUSTMENTS

In this section consider the situation where a market variable V is observed at time T and its value is used to calculate a payoff that occurs at a later time T^*. Define:

V_T: Value of V at time T

$E_T(V_T)$: Expected value of V_T in a world that is forward risk-neutral with respect to $P(t, T)$

$E_{T^*}(V_T)$: Expected value of V_T in a world that is forward risk-neutral with respect to $P(t, T^*)$

The numeraire ratio when we move from the $P(t, T)$ numeraire to the $P(t, T^*)$ numeraire (see Section 27.8) is

$$W = \frac{P(t, T^*)}{P(t, T)}$$

This is the forward price of a zero-coupon bond lasting between times T and T^*. Define:

σ_V: Volatility of V

σ_W: Volatility of W

ρ_{VW}: Correlation between V and W

From equation (27.35) the change of numeraire increases the growth rate of V by α_V, where

$$\alpha_V = \rho_{VW}\sigma_V\sigma_W \tag{29.3}$$

This result can be expressed in terms of the forward interest rate between times T and T^*. Define:

R: Forward interest rate for period between T and T^*, expressed with a compounding frequency of m

σ_R: Volatility of R

The relationship between W and R is

$$W = \frac{1}{(1 + R/m)^{m(T^*-T)}}$$

The relationship between the volatility of W and the volatility of R can be calculated from Itô's lemma as

$$\sigma_W = -\frac{\sigma_R R(T^* - T)}{1 + R/m}$$

Hence equation (29.3) becomes[2]

$$\alpha_V = -\frac{\rho_{VR}\sigma_V\sigma_R R(T^* - T)}{1 + R/m}$$

where $\rho_{VR} = -\rho_{VW}$ is the instantaneous correlation between V and R. As an approximation, it can be assumed that R remains constant at its initial value, R_0, and that the volatilities and correlation in this expression are constant to get, at time zero,

$$E_{T^*}(V_T) = E_T(V_T)\exp\left[-\frac{\rho_{VR}\sigma_V\sigma_R R_0(T^* - T)}{1 + R_0/m}T\right] \tag{29.4}$$

Example 29.3

Consider a derivative that provides a payoff in 6 years equal to the value of a stock index observed in 5 years. Suppose that 1,200 is the forward value of the stock index for a contract maturing in 5 years. Suppose that the volatility of the index is 20%, the volatility of the forward interest rate between years 5 and 6 is 18%, and the correlation between the two is –0.4. Suppose further that the zero curve is flat at 8% with annual compounding. The results just produced can be used with V defined as the value of the index, $T = 5$, $T^* = 6$, $m = 1$, $R_0 = 0.08$, $\rho_{VR} = -0.4$, $\sigma_V = 0.20$, and $\sigma_R = 0.18$, so that

$$E_{T^*}(V_T) = E_T(V_T)\exp\left[-\frac{-0.4 \times 0.20 \times 0.18 \times 0.08 \times 1}{1 + 0.08} \times 5\right]$$

or $E_{T^*}(V_T) = 1.00535E_T(V_T)$. From the arguments in Chapter 27, $E_T(V_T)$ is the forward price of the index, or 1,200. It follows that $E_{T^*}(V_T) = 1{,}200 \times 1.00535 = 1206.42$. Using again the arguments in Chapter 27, it follows from equation (27.20) that the value of the derivative is $1206.42 \times P(0, 6)$. In this case, $P(0, 6) = 1/1.08^6 = 0.6302$, so that the value of the derivative is 760.25.

Application 1 Revisited

The analysis just given provides a different way of producing the result in Application 1 of Section 29.1. Using the notation from that application, R_T is the interest rate between T and T^* and R_0 as the forward rate for the period between time T and T^*. From equation (27.22),

$$E_{T^*}(R_T) = R_0$$

Applying equation (29.4) with V equal to R gives

$$E_{T^*}(R_T) = E_T(R_T)\exp\left[-\frac{\sigma_R^2 R_0\tau}{1 + R_0\tau}T\right]$$

[2] Variables R and W are negatively correlated. We can reflect this by setting $\sigma_W = -\sigma_R(T^* - T)/(1 + R/m)$, which is a negative number, and setting $\rho_{VW} = \rho_{VR}$. Alternatively we can change the sign of σ_W so that it is positive and set $\rho_{VW} = -\rho_{VR}$. In either case, we end up with the same formula for α_V.

where $\tau = T^* - T$ (note that $m = 1/\tau$). It follows that

$$R_0 = E_T(R_T)\exp\left[-\frac{\sigma_R^2 R_0 T\tau}{1+R_0\tau}\right]$$

or

$$E_T(R_T) = R_0 \exp\left[\frac{\sigma_R^2 R_0 T\tau}{1+R_0\tau}\right]$$

Approximating the exponential function gives

$$E_T(R_T) = R_0 + \frac{R_0^2\sigma_R^2\tau T}{1+R_0\tau}$$

This is the same result as equation (29.2).

29.3 QUANTOS

A *quanto* or *cross-currency derivative* is an instrument where two currencies are involved. The payoff is defined in terms of a variable that is measured in one of the currencies and the payoff is made in the other currency. One example of a quanto is the CME futures contract on the Nikkei discussed in Business Snapshot 5.3. The market variable underlying this contract is the Nikkei 225 index (which is measured in yen), but the contract is settled in US dollars.

Consider a quanto that provides a payoff in currency X at time T. Assume that the payoff depends on the value V of a variable that is observed in currency Y at time T. Define:

$P_X(t,T)$: Value at time t in currency X of a zero-coupon bond paying off 1 unit of currency X at time T

$P_Y(t,T)$: Value at time t in currency Y of a zero-coupon bond paying off 1 unit of currency Y at time T

V_T: Value of V at time T

$E_X(V_T)$: Expected value of V_T in a world that is forward risk neutral with respect to $P_X(t,T)$

$E_Y(V_T)$: Expected value of V_T in a world that is forward risk neutral with respect to $P_Y(t,T)$

The numeraire ratio when we move from the $P_Y(t,T)$ numeraire to the $P_X(t,T)$ numeraire is

$$W(t) = \frac{P_X(t,T)}{P_Y(t,T)}S(t)$$

where $S(t)$ is the spot exchange rate (units of Y per unit of X) at time t. It follows from this that the numeraire ratio $W(t)$ is the forward exchange rate (units of Y per unit of X) for a contract maturing at time T. Define:

σ_W Volatility of W

σ_V: Volatility of V

ρ_{VW}: Instantaneous correlation between V and W.

From equation (27.35), the change of numeraire increases the growth rate of V by α_V, where

$$\alpha_V = \rho_{VW}\sigma_V\sigma_W \tag{29.5}$$

If it is assumed that the volatilities and correlation are constant, this means that

$$E_X(V_T) = E_Y(V_T)e^{\rho_{VW}\sigma_V\sigma_W T}$$

or as an approximation

$$E_X(V_T) = E_Y(V_T)(1 + \rho_{VW}\sigma_V\sigma_W T) \tag{29.6}$$

This equation will be used for the valuation of what are known as diff swaps in Chapter 32.

Example 29.4

Suppose that the current value of the Nikkei stock index is 15,000 yen, the 1-year dollar risk-free rate is 5%, the 1-year yen risk-free rate is 2%, and the Nikkei dividend yield is 1%. The forward price of the Nikkei for a 1-year contract denominated in yen can be calculated in the usual way from equation (5.8) as

$$15{,}000e^{(0.02-0.01)\times 1} = 15{,}150.75$$

Suppose that the volatility of the index is 20%, the volatility of the 1-year forward yen per dollar exchange rate is 12%, and the correlation between the two is 0.3. In this case $E_Y(V_T) = 15{,}150.75$, $\sigma_F = 0.20$, $\sigma_W = 0.12$ and $\rho = 0.3$. From equation (29.6), the expected value of the Nikkei in a world that is forward risk neutral with respect to a dollar bond maturing in 1 year is

$$15{,}150.75e^{0.3\times 0.2\times 0.12\times 1} = 15{,}260.23$$

This is the forward price of the Nikkei for a contract that provides a payoff in dollars rather than yen. (As an approximation, it is also the futures price of such a contract.)

Using Traditional Risk-Neutral Measures

The forward risk-neutral measure works well when payoffs occur at only one time. In other situations, it is often more appropriate to use the traditional risk-neutral measure. Suppose the process followed by a variable V in the traditional currency-Y risk-neutral world is known and we wish to estimate its process in the traditional currency-X risk-neutral world. Define:

S: Spot exchange rate (units of Y per unit of X)

σ_S: Volatility of S

σ_V: Volatility of V

ρ: Instantaneous correlation between S and V

In this case, the change of numeraire is from the money market account in currency Y to the money market account in currency X (with both money market accounts being denominated in currency X). Define g_X as the value of the money market account in

Business Snapshot 29.1 Siegel's Paradox

Consider two currencies, X and Y. Suppose that the interest rates in the two currencies, r_X and r_Y, are constant. Define S as the number of units of currency Y per unit of currency X. As explained in Chapter 5, a currency is an asset that provides a yield at the foreign risk-free rate. The traditional risk-neutral process for S is therefore

$$dS = (r_Y - r_X)S\,dt + \sigma_S S\,dz$$

From Itô's lemma, this implies that the process for $1/S$ is

$$d(1/S) = (r_X - r_Y + \sigma_S^2)(1/S)\,dt - \sigma_S(1/S)\,dz$$

This leads to what is known as *Siegel's paradox*. Since the expected growth rate of S is $r_Y - r_X$ in a risk-neutral world, symmetry suggests that the expected growth rate of $1/S$ should be $r_X - r_Y$ rather than $r_X - r_Y + \sigma_S^2$.

To understand Siegel's paradox it is necessary to appreciate that the process we have given for S is the risk-neutral process for S in a world where the numeraire is the money market account in currency Y. The process for $1/S$, because it is deduced from the process for S, therefore also assumes that this is the numeraire. Because $1/S$ is the number of units of X per unit of Y, to be symmetrical we should measure the process for $1/S$ in a world where the numeraire is the money market account in currency X. Equation (29.7) shows that when we change the numeraire, from the money market account in currency Y to the money market account in currency X, the growth rate of a variable V increases by $\rho\sigma_V\sigma_S$, where ρ is the correlation between S and V. In this case, $V = 1/S$, so that $\rho = -1$ and $\sigma_V = \sigma_S$. It follows that the change of numeraire causes the growth rate of $1/S$ to increase by $-\sigma_S^2$. This neutralizes the $+\sigma_S^2$ in the process given above for $1/S$. The process for $1/S$ in a world where the numeraire is the money market account in currency X is therefore

$$d(1/S) = (r_X - r_Y)(1/S)\,dt - \sigma_S(1/S)\,dz$$

This is symmetrical with the process we started with for S. The paradox has been resolved!

currency X and g_Y as the value of the money market account in currency Y. The numeraire ratio is

$$\frac{g_X}{g_Y}S$$

The variables $g_X(t)$ and $g_Y(t)$ have a stochastic drift but zero volatility as explained in Section 27.4. From Itô's lemma it follows that the volatility of the numeraire ratio is σ_S. The change of numeraire therefore involves increasing the expected growth rate of V by

$$\rho\sigma_V\sigma_S \tag{29.7}$$

The market price of risk changes from zero to $\rho\sigma_S$. An application of this result is to Siegel's paradox (see Business Snapshot 29.1).

Example 29.5

A 2-year American option provides a payoff of $S - K$ pounds sterling where S is the level of the S&P 500 at the time of exercise and K is the strike price. The current level of the S&P 500 is 1,200. The risk-free interest rates in sterling and dollars are both constant at 5% and 3%, respectively, the correlation between the dollars/sterling exchange rate and the S&P 500 is 0.2, the volatility of the S&P 500 is 25%, and the volatility of the exchange rate is 12%. The dividend yield on the S&P 500 is 1.5%.

This option can be valued by constructing a binomial tree for the S&P 500 using as the numeraire the money market account in the UK (i.e., using the traditional risk-neutral world as seen from the perspective of a UK investor). From equation (29.7), the change in numeraire from the US to UK money market account leads to an increase in the expected growth rate in the S&P 500 of

$$0.2 \times 0.25 \times 0.12 = 0.006$$

or 0.6%. The growth rate of the S&P 500 using a US dollar numeraire is $3\% - 1.5\% = 1.5\%$. The growth rate using the sterling numeraire is therefore 2.1%. The risk-free interest rate in sterling is 5%. The S&P 500 therefore behaves like an asset providing a dividend yield of $5\% - 2.1\% = 2.9\%$ under the sterling numeraire. Using the parameter values of $S = 1{,}200$, $K = 1{,}200$, $r = 0.05$, $q = 0.029$, $\sigma = 0.25$, and $T = 2$ with 100 time steps, DerivaGem estimates the value of the option as £179.83.

SUMMARY

When valuing a derivative providing a payoff at a particular future time it is natural to assume that the variables underlying the derivative equal their forward values and discount at the rate of interest applicable from the valuation date to the payoff date. This chapter has shown that this is not always the correct procedure.

When a payoff depends on a bond yield y observed at time T the expected yield should be assumed to be higher than the forward yield as indicated by equation (29.1). This result can be adapted for situations where a payoff depends on a swap rate. When a variable is observed at time T but the payoff occurs at a later time T^* the forward value of the variable should be adjusted as indicated by equation (29.4). When a variable is observed in one currency but leads to a payoff in another currency the forward value of the variable should also be adjusted. In this case the adjustment is shown in equation (29.6).

These results will be used when nonstandard swaps are considered in Chapter 32.

FURTHER READING

Brotherton-Ratcliffe, R., and B. Iben, "Yield Curve Applications of Swap Products," in *Advanced Strategies in Financial Risk Management* (R. Schwartz and C. Smith, eds.). New York Institute of Finance, 1993.

Jamshidian, F., "Corralling Quantos," *Risk*, March (1994): 71–75.

Reiner, E., "Quanto Mechanics," *Risk*, March (1992), 59–63.

Questions and Problems (Answers in Solutions Manual)

29.1. Explain how you would value a derivative that pays off $100R$ in 5 years, where R is the 1-year interest rate (annually compounded) observed in 4 years. What difference would it make if the payoff were in 4 years? What difference would it make if the payoff were in 6 years?

29.2. Explain whether any convexity or timing adjustments are necessary when:

(a) We wish to value a spread option that pays off every quarter the excess (if any) of the 5-year swap rate over the 3-month LIBOR rate applied to a principal of \$100. The payoff occurs 90 days after the rates are observed.

(b) We wish to value a derivative that pays off every quarter the 3-month LIBOR rate minus the 3-month Treasury bill rate. The payoff occurs 90 days after the rates are observed.

29.3. Suppose that in Example 28.3 of Section 28.2 the payoff occurs after 1 year (i.e., when the interest rate is observed) rather than in 15 months. What difference does this make to the inputs to Black's models?

29.4. The yield curve is flat at 10% per annum with annual compounding. Calculate the value of an instrument where, in 5 years' time, the 2-year swap rate (with annual compounding) is received and a fixed rate of 10% is paid. Both are applied to a notional principal of \$100. Assume that the volatility of the swap rate is 20% per annum. Explain why the value of the instrument is different from zero.

29.5. What difference does it make in Problem 29.4 if the swap rate is observed in 5 years, but the exchange of payments takes place in (a) 6 years, and (b) 7 years? Assume that the volatilities of all forward rates are 20%. Assume also that the forward swap rate for the period between years 5 and 7 has a correlation of 0.8 with the forward interest rate between years 5 and 6 and a correlation of 0.95 with the forward interest rate between years 5 and 7.

29.6. The price of a bond at time T, measured in terms of its yield, is $G(y_T)$. Assume geometric Brownian motion for the forward bond yield y in a world that is forward risk neutral with respect to a bond maturing at time T. Suppose that the growth rate of the forward bond yield is α and its volatility σ_y.

(a) Use Itô's lemma to calculate the process for the forward bond price in terms of α, σ_y, y, and $G(y)$.

(b) The forward bond price should follow a martingale in the world considered. Use this fact to calculate an expression for α.

(c) Show that the expression for α is, to a first approximation, consistent with equation (29.1).

29.7. The variable S is an investment asset providing income at rate q measured in currency A. It follows the process

$$dS = \mu_S S\,dt + \sigma_S S\,dz$$

in the real world. Defining new variables as necessary, give the process followed by S, and the corresponding market price of risk, in:

(a) A world that is the traditional risk-neutral world for currency A

(b) A world that is the traditional risk-neutral world for currency B

(c) A world that is forward risk neutral with respect to a zero-coupon currency A bond maturing at time T

(d) A world that is forward risk neutral with respect to a zero-coupon currency B bond maturing at time T

29.8. A call option provides a payoff at time T of $\max(S_T - K, 0)$ yen, where S_T is the dollar price of gold at time T and K is the strike price. Assuming that the storage costs of gold are zero and defining other variables as necessary, calculate the value of the contract.

29.9. Suppose that an index of Canadian stocks currently stands at 400. The Canadian dollar is currently worth 0.70 US dollars. The risk-free interest rates in Canada and the US are constant at 6% and 4%, respectively. The dividend yield on the index is 3%. Define Q as the number of Canadian dollars per U.S dollar and S as the value of the index. The volatility of S is 20%, the volatility of Q is 6%, and the correlation between S and Q is 0.4. Use DerivaGem to determine the value of a 2-year American-style call option on the index if:

(a) It pays off in Canadian dollars the amount by which the index exceeds 400.

(b) It pays off in US dollars the amount by which the index exceeds 400.

Assignment Questions

29.10. Consider an instrument that will pay off S dollars in 2 years, where S is the value of the Nikkei index. The index is currently 20,000. The yen/dollar exchange rate is 100 (yen per dollar). The correlation between the exchange rate and the index is 0.3 and the dividend yield on the index is 1% per annum. The volatility of the Nikkei index is 20% and the volatility of the yen/dollar exchange rate is 12%. The interest rates (assumed constant) in the US and Japan are 4% and 2%, respectively.

(a) What is the value of the instrument?

(b) Suppose that the exchange rate at some point during the life of the instrument is Q and the level of the index is S. Show that a US investor can create a portfolio that changes in value by approximately ΔS dollar when the index changes in value by ΔS yen by investing S dollars in the Nikkei and shorting SQ yen.

(c) Confirm that this is correct by supposing that the index changes from 20,000 to 20,050 and the exchange rate changes from 100 to 99.7.

(d) How would you delta hedge the instrument under consideration?

29.11. Suppose that the LIBOR yield curve is flat at 8% (with continuous compounding). The payoff from a derivative occurs in 4 years. It is equal to the 5-year rate minus the 2-year rate at this time, applied to a principal of \$100 with both rates being continuously compounded. (The payoff can be positive or negative.) Calculate the value of the derivative. Assume that the volatility for all rates is 25%. What difference does it make if the payoff occurs in 5 years instead of 4 years? Assume all rates are perfectly correlated.

29.12. Suppose that the payoff from a derivative will occur in 10 years and will equal the 3-year US dollar swap rate for a semiannual-pay swap observed at that time applied to a certain principal. Assume that the yield curve is flat at 8% (semiannually compounded) per annum in dollars and 3% (semiannually compounded) in yen. The forward swap rate volatility is 18%, the volatility of the 10-year "yen per dollar" forward exchange rate is 12%, and the correlation between this exchange rate and US dollar interest rates is 0.25.

(a) What is the value of the derivative if the swap rate is applied to a principal of \$100 million so that the payoff is in dollars?

(b) What is its value of the derivative if the swap rate is applied to a principal of 100 million yen so that the payoff is in yen?

29.13. The payoff from a derivative will occur in 8 years. It will equal the average of the 1-year interest rates observed at times 5, 6, 7, and 8 years applied to a principal of \$1,000. The yield curve is flat at 6% with annual compounding and the volatilities of all rates are 16%. Assume perfect correlation between all rates. What is the value of the derivative?

APPENDIX

PROOF OF THE CONVEXITY ADJUSTMENT FORMULA

This appendix calculates a convexity adjustment for forward bond yields. Suppose that the payoff from a derivative at time T depends on a bond yield observed at that time. Define:

y_0: Forward bond yield observed today for a forward contract with maturity T

y_T: Bond yield at time T

B_T: Price of the bond at time T

σ_y: Volatility of the forward bond yield

Suppose that

$$B_T = G(y_T)$$

Expanding $G(y_T)$ in a Taylor series about $y_T = y_0$ yields the following approximation:

$$B_T = G(y_0) + (y_T - y_0)G'(y_0) + 0.5(y_T - y_0)^2 G''(y_0)$$

where G' and G'' are the first and second partial derivatives of G. Taking expectations in a world that is forward risk neutral with respect to a zero-coupon bond maturing at time T gives

$$E_T(B_T) = G(y_0) + E_T(y_T - y_0)G'(y_0) + \tfrac{1}{2}E_T[(y_T - y_0)^2]G''(y_0)$$

where E_T denotes expectations in this world. The expression $G(y_0)$ is by definition the forward bond price. Also, because of the particular world we are working in, $E_T(B_T)$ equals the forward bond price. Hence $E_T(B_T) = G(y_0)$, so that

$$E_T(y_T - y_0)G'(y_0) + \tfrac{1}{2}E_T[(y_T - y_0)^2]G''(y_0) = 0$$

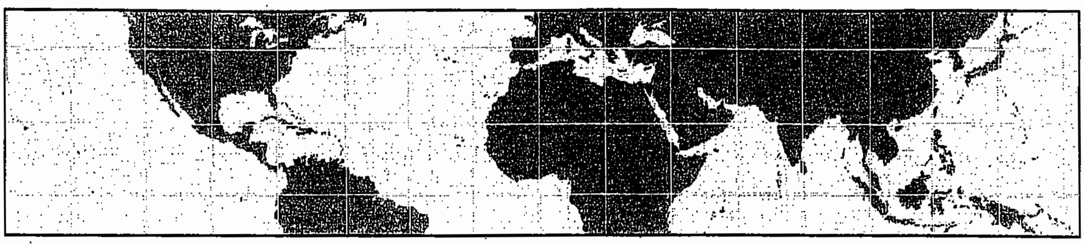
The expression $E_T[(y_T - y_0)^2]$ is approximately $\sigma_y^2 y_0^2 T$. Hence it is approximately true that

$$E_T(y_T) = y_0 - \tfrac{1}{2}y_0^2\sigma_y^2 T \frac{G''(y_0)}{G'(y_0)}$$

This shows that, to obtain the expected bond yield in a world that is forward risk neutral with respect to a zero-coupon bond maturing at time T, the term

$$-\tfrac{1}{2}y_0^2\sigma_y^2 T \frac{G''(y_0)}{G'(y_0)}$$

should be added to the forward bond yield. This is the result in equation (29.1). For an alternative proof, see Problem 29.6.



CHAPTER 30

Interest Rate Derivatives: Models of the Short Rate

The models for pricing interest rate options that we have presented so far make the assumption that the probability distribution of an interest rate, a bond price, or some other variable at a future point in time is lognormal. They are widely used for valuing instruments such as caps, European bond options, and European swap options. However, they have limitations. They do not provide a description of how interest rates evolve through time. Consequently, they cannot be used for valuing interest rate derivatives such as American-style swap options, callable bonds, and structured notes.

This chapter and the next discuss alternative approaches for overcoming these limitations. These involve building what is known as a *term structure model.* This is a model describing the evolution of all zero-coupon interest rates.[1] This chapter focuses on term structure models constructed by specifying the behavior of the short-term interest rate, r.

30.1 BACKGROUND

The short rate, r, at time t is the rate that applies to an infinitesimally short period of time at time t. It is sometimes referred to as the *instantaneous short rate.* Bond prices, option prices, and other derivative prices depend only on the process followed by r in a risk-neutral world. The process for r in the real world is irrelevant. The relevant risk-neutral world for this chapter is the traditional risk-neutral world where, in a very short time period between t and $t+\Delta t$, investors earn on average $r(t)\,\Delta t$. All processes for r that will be considered here are processes in this risk-neutral world.

From equation (27.19), the value at time t of an interest rate derivative that provides a payoff of f_T at time T is

$$\hat{E}[e^{-\bar{r}(T-t)} f_T] \tag{30.1}$$

where $\bar{r}$ is the average value of r in the time interval between t and T, and $\hat{E}$ denotes expected value in the traditional risk-neutral world.

[1] Note that when a term structure model is used we do not need to make the convexity, timing, and quanto adjustments discussed in the previous chapter.

As usual, define $P(t, T)$ as the price at time t of a zero-coupon bond that pays off \$1 at time T. From equation (30.1),

$$P(t, T) = \hat{E}[e^{-\bar{r}(T-t)}] \tag{30.2}$$

If $R(t, T)$ is the continuously compounded interest rate at time t for a term of $T - t$, then

$$P(t, T) = e^{-R(t,T)(T-t)} \tag{30.3}$$

so that

$$R(t, T) = -\frac{1}{T-t} \ln P(t, T) \tag{30.4}$$

and, from equation (30.2),

$$R(t, T) = -\frac{1}{T-t} \ln \hat{E}[e^{-\bar{r}(T-t)}] \tag{30.5}$$

This equation enables the term structure of interest rates at any given time to be obtained from the value of r at that time and the risk-neutral process for r. It shows that, once the process for r has been defined, everything about the initial zero curve and its evolution through time can be determined.

30.2 EQUILIBRIUM MODELS

Equilibrium models usually start with assumptions about economic variables and derive a process for the short rate, r. They then explore what the process for r implies about bond prices and option prices.

In a one-factor equilibrium model, the process for r involves only one source of uncertainty. Usually the risk-neutral process for the short rate is described by an Itô process of the form

$$dr = m(r)\,dt + s(r)\,dz$$

The instantaneous drift, m, and instantaneous standard deviation, s, are assumed to be functions of r, but are independent of time. The assumption of a single factor is not as restrictive as it might appear. A one-factor model implies that all rates move in the same direction over any short time interval, but not that they all move by the same amount. The shape of the zero curve can therefore change with the passage of time.

This section considers three one-factor equilibrium models:

$m(r) = \mu r;\; s(r) = \sigma r$ (Rendleman and Bartter model)

$m(r) = a(b - r);\; s(r) = \sigma$ (Vasicek model)

$m(r) = a(b - r);\; s(r) = \sigma\sqrt{r}$ (Cox, Ingersoll, and Ross model)

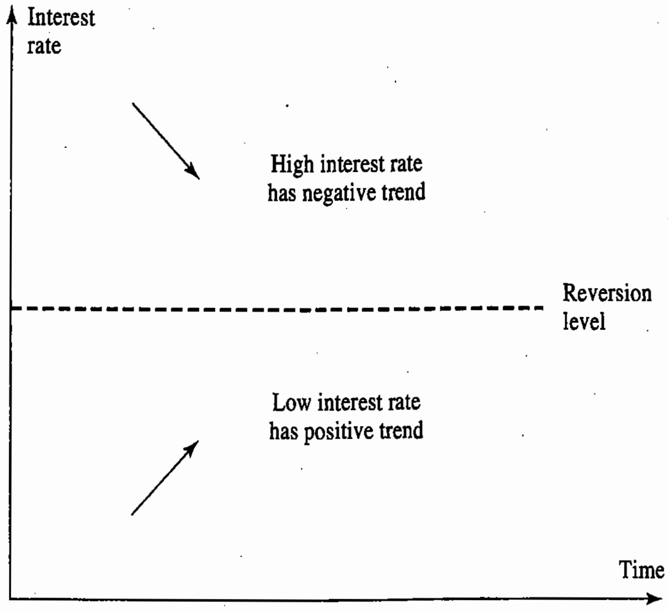
The Rendleman and Bartter Model

In Rendleman and Bartter's model, the risk-neutral process for r is[2]

$$dr = \mu r\,dt + \sigma r\,dz$$

[2] See R. Rendleman and B. Bartter, "The Pricing of Options on Debt Securities," *Journal of Financial and Quantitative Analysis*, 15 (March 1980): 11–24.

Figure 30.1 Mean reversion.



where μ and σ are constants. This means that r follows geometric Brownian motion. The process for r is of the same type as that assumed for a stock price in Chapter 13. It can be represented using a binomial tree similar to the one used for stocks in Chapter 11.[3]

The assumption that the short-term interest rate behaves like a stock price is a natural starting point but is less than ideal. One important difference between interest rates and stock prices is that interest rates appear to be pulled back to some long-run average level over time. This phenomenon is known as *mean reversion*. When r is high, mean reversion tends to cause it to have a negative drift; when r is low, mean reversion tends to cause it to have a positive drift. Mean reversion is illustrated in Figure 30.1. The Rendleman and Bartter model does not incorporate mean reversion.

There are compelling economic arguments in favor of mean reversion. When rates are high, the economy tends to slow down and there is low demand for funds from borrowers. As a result, rates decline. When rates are low, there tends to be a high demand for funds on the part of borrowers and rates tend to rise.

The Vasicek Model

In Vasicek's model, the risk-neutral process for r is

$$dr = a(b - r)\,dt + \sigma\,dz$$

where a, b, and σ are constants.[4] This model incorporates mean reversion. The short rate is pulled to a level b at rate a. Superimposed upon this "pull" is a normally distributed stochastic term $\sigma\,dz$.

[3] The way that the interest rate tree is used is explained later in the chapter.

[4] See O.A. Vasicek, "An Equilibrium Characterization of the Term Structure," *Journal of Financial Economics*, 5 (1977): 177–88.

Vasicek shows that equation (30.2) can be used to obtain the following expression for the price at time t of a zero-coupon bond that pays \$1 at time T:

$$P(t, T) = A(t, T)e^{-B(t,T)r(t)} \tag{30.6}$$

In this equation $r(t)$ is the value of r at time t,

$$B(t, T) = \frac{1 - e^{-a(T-t)}}{a} \tag{30.7}$$

and

$$A(t, T) = \exp\left[\frac{(B(t, T) - T + t)(a^2b - \sigma^2/2)}{a^2} - \frac{\sigma^2 B(t, T)^2}{4a}\right] \tag{30.8}$$

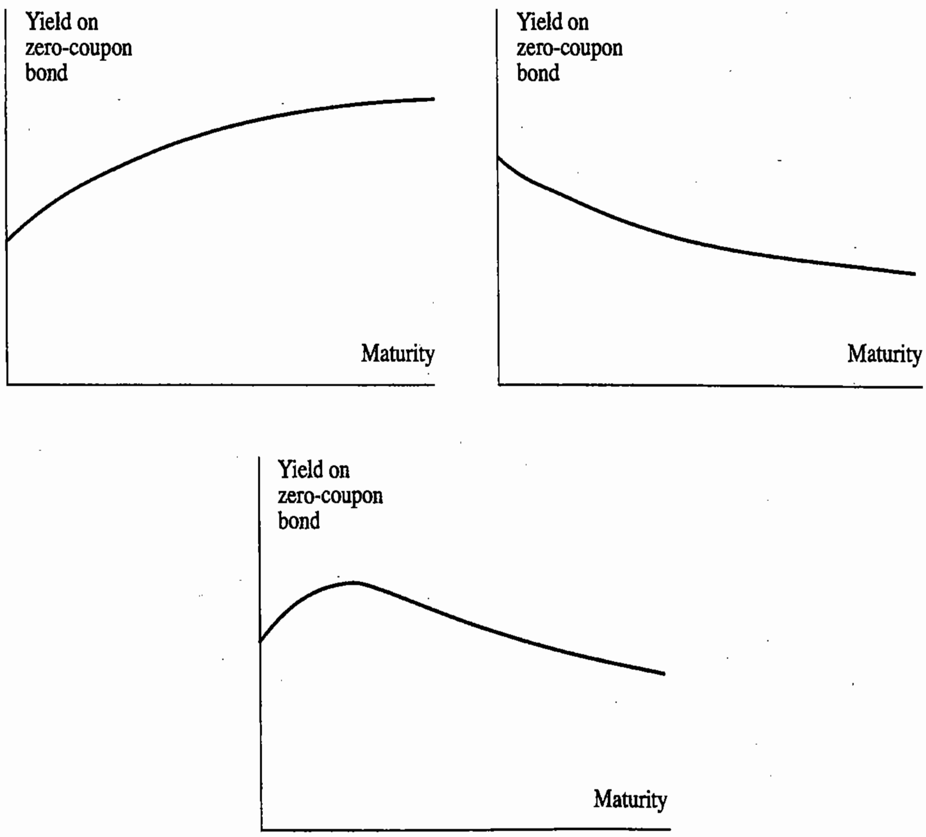
When $a = 0$, $B(t, T) = T - t$ and $A(t, T) = \exp[\sigma^2(T - t)^3/6]$.

From equation (30.4),

$$R(t, T) = -\frac{1}{T-t}\ln A(t, T) + \frac{1}{T-t}B(t, T)r(t) \tag{30.9}$$

showing that the entire term structure can be determined as a function of $r(t)$ once a, b, and σ are chosen. The shape can be upward-sloping, downward-sloping, or slightly "humped" (see Figure 30.2).

Figure 30.2 Possible shapes of term structure when Vasicek's model is used.



The Cox, Ingersoll, and Ross Model

In Vasicek's model the short-term interest rate, r, can become negative. Cox, Ingersoll, and Ross have proposed an alternative model where rates are always non-negative.[5] The risk-neutral process for r in their model is

$$dr = a(b - r)\,dt + \sigma\sqrt{r}\,dz$$

This has the same mean-reverting drift as Vasicek, but the standard deviation of the change in the short rate in a short period of time is proportional to $\sqrt{r}$. This means that, as the short-term interest rate increases, its standard deviation increases.

Cox, Ingersoll, and Ross show that, in their model, bond prices have the same general form as those in Vasicek's model,

$$P(t, T) = A(t, T)e^{-B(t,T)r}$$

but the functions $B(t, T)$ and $A(t, T)$ are different,

$$B(t, T) = \frac{2(e^{\gamma(T-t)} - 1)}{(\gamma + a)(e^{\gamma(T-t)} - 1) + 2\gamma}$$

and

$$A(t, T) = \left[\frac{2\gamma e^{(a+\gamma)(T-t)/2}}{(\gamma + a)(e^{\gamma(T-t)} - 1) + 2\gamma}\right]^{2ab/\sigma^2}$$

with $\gamma = \sqrt{a^2 + 2\sigma^2}$. Upward-sloping, downward-sloping, and slightly humped yield curves are possible. As in the case of Vasicek's model, the long rate, $R(t, T)$, is linearly dependent on $r(t)$. This means that the value of $r(t)$ determines the level of the term structure at time t. The general shape of the term structure at time t is independent of $r(t)$, but does depend on t.

Two-Factor Models

A number of researchers have investigated the properties of two-factor models. For example, Brennan and Schwartz have developed a model where the process for the short rate reverts to a long rate, which in turn follows a stochastic process.[6] The long rate is chosen as the yield on a perpetual bond that pays \$1 per year. Because the yield on this bond is the reciprocal of its price, Itô's lemma can be used to calculate the process followed by the yield from the process followed by the price of the bond. The bond is a traded security. This simplifies the analysis because the expected return on the bond in a risk-neutral world must be the risk-free interest rate.

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[5] See J.C. Cox, J.E. Ingersoll, and S.A. Ross, "A Theory of the Term Structure of Interest Rates," *Econometrica*, 53 (1985): 385–407.

[6] See M.J. Brennan and E.S. Schwartz, "A Continuous Time Approach to Pricing Bonds," *Journal of Banking and Finance*, 3 (July 1979): 133–55; M.J. Brennan and E.S. Schwartz, "An Equilibrium Model of Bond Pricing and a Test of Market Efficiency," *Journal of Financial and Quantitative Analysis*, 21, 3 (September 1982): 301–29.

Another two-factor model, proposed by Longstaff and Schwartz, starts with a general equilibrium model of the economy and derives a term structure model where there is stochastic volatility.[7] The model proves to be analytically quite tractable.

30.3 NO-ARBITRAGE MODELS

The disadvantage of the equilibrium models we have presented is that they do not automatically fit today's term structure of interest rates. By choosing the parameters judiciously, they can be made to provide an approximate fit to many of the term structures that are encountered in practice. But the fit is not usually an exact one and, in some cases, no reasonable fit can be found. Most traders find this unsatisfactory. Not unreasonably, they argue that they can have very little confidence in the price of a bond option when the model does not price the underlying bond correctly. A 1% error in the price of the underlying bond may lead to a 25% error in an option price.

A *no-arbitrage model* is a model designed to be exactly consistent with today's term structure of interest rates. The essential difference between an equilibrium and a no-arbitrage model is therefore as follows. In an equilibrium model, today's term structure of interest rates is an output. In a no-arbitrage model, today's term structure of interest rates is an input.

In an equilibrium model, the drift of the short rate (i.e., the coefficient of dt) is not usually a function of time. In a no-arbitrage model, the drift is, in general, dependent on time. This is because the shape of the initial zero curve governs the average path taken by the short rate in the future in a no-arbitrage model. If the zero curve is steeply upward-sloping for maturities between t_1 and t_2, then r has a positive drift between these times; if it is steeply downward-sloping for these maturities, then r has a negative drift between these times.

It turns out that some equilibrium models can be converted to no-arbitrage models by including a function of time in the drift of the short rate. We now consider the Ho–Lee, Hull–White (one- and two-factor), and Black–Karasinski models.

The Ho–Lee Model

Ho and Lee proposed the first no-arbitrage model of the term structure in a paper in 1986.[8] They presented the model in the form of a binomial tree of bond prices with two parameters: the short-rate standard deviation and the market price of risk of the short rate. It has since been shown that the continuous-time limit of the model is

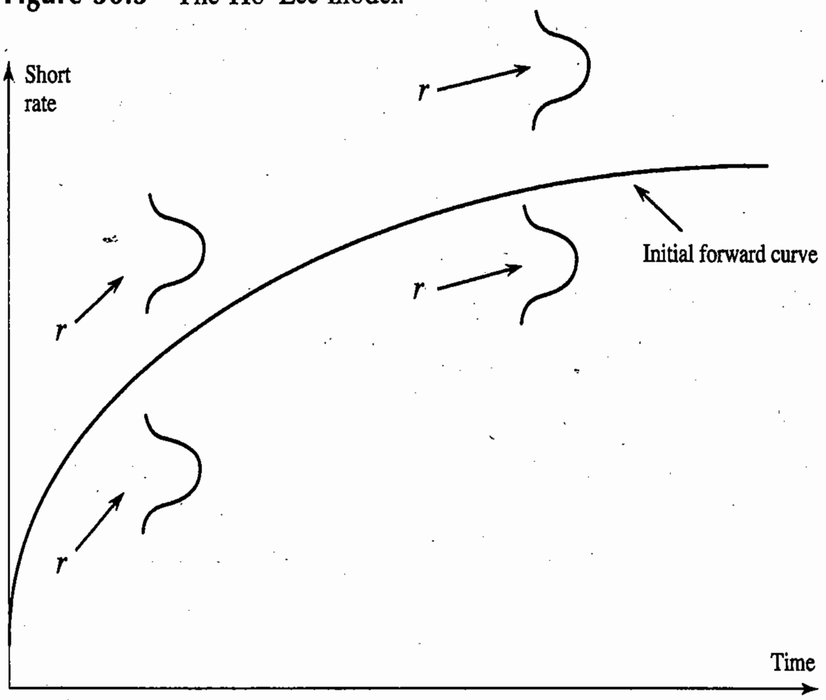
$$dr = \theta(t)\,dt + \sigma\,dz \qquad \textbf{(30.10)}$$

where σ, the instantaneous standard deviation of the short rate, is constant and $\theta(t)$ is a function of time chosen to ensure that the model fits the initial term structure. The variable $\theta(t)$ defines the average direction that r moves at time t. This is independent of the level of r. Interestingly, Ho and Lee's parameter that concerns the market price of

[7] See F. A. Longstaff and E. S. Schwartz, "Interest Rate Volatility and the Term Structure: A Two Factor General Equilibrium Model," *Journal of Finance*, 47, 4 (September 1992): 1259–82.

[8] See T. S. Y. Ho and S.-B. Lee, "Term Structure Movements and Pricing Interest Rate Contingent Claims," *Journal of Finance*, 41 (December 1986): 1011–29.

Figure 30.3 The Ho–Lee model.



risk proves to be irrelevant when the model is used to price interest rate derivatives. This is analogous to risk preferences being irrelevant in the pricing of stock options.

The variable $\theta(t)$ can be calculated analytically (see Problem 30.13). It is

$$\theta(t) = F_t(0, t) + \sigma^2 t \tag{30.11}$$

where the $F(0, t)$ is the instantaneous forward rate for a maturity t as seen at time zero and the subscript t denotes a partial derivative with respect to t. As an approximation, $\theta(t)$ equals $F_t(0, t)$. This means that the average direction that the short rate will be moving in the future is approximately equal to the slope of the instantaneous forward curve. The Ho–Lee model is illustrated in Figure 30.3. The slope of the forward curve defines the average direction that the short rate is moving at any given time. Superimposed on this slope is the normally distributed random outcome.

In the Ho–Lee model, zero-coupon bonds and European options on zero-coupon bonds can be valued analytically. The expression for the price of a zero-coupon bond at time t in terms of the short rate is

$$P(t, T) = A(t, T)e^{-r(t)(T-t)} \tag{30.12}$$

where

$$\ln A(t, T) = \ln \frac{P(0, T)}{P(0, t)} + (T - t)F(0, t) - \tfrac{1}{2}\sigma^2 t(T - t)^2$$

In these equations, time zero is today. Times t and T are general times in the future with $T \geqslant t$. The equations, therefore, define the price of a zero-coupon bond at a future time t in terms of the short rate at time t and the prices of bonds today. The latter can be calculated from today's term structure.

The Hull–White (One-Factor) Model

In a paper published in 1990, Hull and White explored extensions of the Vasicek model that provide an exact fit to the initial term structure.[9] One version of the extended Vasicek model that they consider is

$$dr = [\theta(t) - ar]\,dt + \sigma\,dz \tag{30.13}$$

or

$$dr = a\left[\frac{\theta(t)}{a} - r\right]dt + \sigma\,dz$$

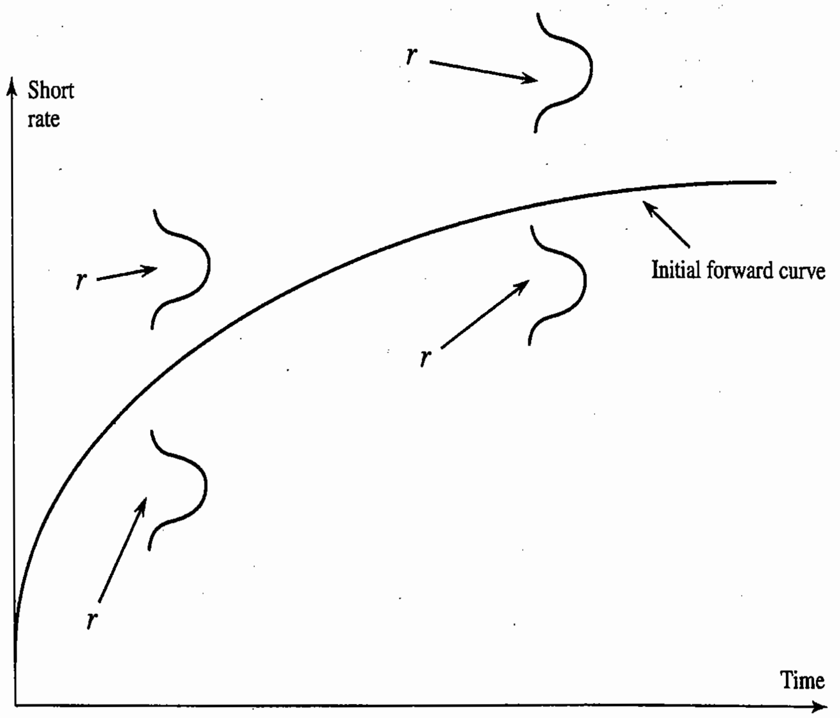
where a and σ are constants. This is known as the Hull–White model. It can be characterized as the Ho–Lee model with mean reversion at rate a. Alternatively, it can be characterized as the Vasicek model with a time-dependent reversion level. At time t, the short rate reverts to $\theta(t)/a$ at rate a. The Ho–Lee model is a particular case of the Hull–White model with $a = 0$.

The model has the same amount of analytic tractability as Ho–Lee. The $\theta(t)$ function can be calculated from the initial term structure (see Problem 30.14):

$$\theta(t) = F_t(0, t) + aF(0, t) + \frac{\sigma^2}{2a}(1 - e^{-2at}) \tag{30.14}$$

The last term in this equation is usually fairly small. If we ignore it, the equation implies that the drift of the process for r at time t is $F_t(0, t) + a[F(0, t) - r]$. This shows that, on

Figure 30.4 The Hull–White model.



[9] See J. Hull and A. White, "Pricing Interest Rate Derivative Securities," *Review of Financial Studies*, 3, 4 (1990): 573–92.

average, r follows the slope of the initial instantaneous forward rate curve. When it deviates from that curve, it reverts back to it at rate a. The model is illustrated in Figure 30.4.

Bond prices at time t in the Hull–White model are given by

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)} \tag{30.15}$$

where

$$B(t,T) = \frac{1-e^{-a(T-t)}}{a} \tag{30.16}$$

and

$$\ln A(t,T) = \ln\frac{P(0,T)}{P(0,t)} + B(t,T)F(0,t) - \frac{1}{4a^3}\sigma^2(e^{-aT}-e^{-at})^2(e^{2at}-1) \tag{30.17}$$

Equations (30.15), (30.16), and (30.17) define the price of a zero-coupon bond at a future time t in terms of the short rate at time t and the prices of bonds today. The latter can be calculated from today's term structure.

The Black–Karasinski Model

The Ho–Lee and Hull–White models have the disadvantage that the short-term interest rate, r, can become negative. A model that allows only positive interest rates is a model proposed by Black and Karasinski:[10]

$$d\ln r = [\theta(t) - a(t)\ln(r)]\,dt + \sigma(t)\,dz \tag{30.18}$$

The variable $\ln r$ follows the same process as r in the Hull–White model. Whereas the value of r at a future time is normal in the Ho–Lee and Hull–White models, it is lognormal in the Black–Karasinski model.

The Black–Karasinski model does not have as much analytic tractability as Ho–Lee or Hull–White. For example, it is not possible to produce formulas for valuing bonds in terms of r using the model.

The Hull–White Two-Factor Model

A model that involves a similar idea to the two-factor model suggested by Brennan and Schwartz, but is arbitrage-free, is[11]

$$df(r) = [\theta(t) + u - af(r)]\,dt + \sigma_1\,dz_1 \tag{30.19}$$

where u has an initial value of zero and follows the process

$$du = -bu\,dt + \sigma_2\,dz_2$$

As in the one-factor models just considered, the parameter $\theta(t)$ is chosen to make the model consistent with the initial term structure. The stochastic variable u is a component of the reversion level of r and itself reverts to a level of zero at rate b. The

[10] See F. Black and P. Karasinski, "Bond and Option Pricing When Short Rates Are Lognormal," *Financial Analysts Journal*, July/August (1991), 52–59.

[11] See J. Hull and A. White, "Numerical Procedures for Implementing Term Structure Models II: Two-Factor Models," *Journal of Derivatives*, 2, 2 (Winter 1994): 37–48.

parameters a, b, σ_1, and σ_2 are constants and dz_1 and dz_2 are Wiener processes with instantaneous correlation ρ.

This model provides a richer pattern of term structure movements and a richer pattern of volatilities than one-factor models of r. For more information on the model, see Technical Note 14 on the author's website.

30.4 OPTIONS ON BONDS

Some of the models just presented allow options on zero-coupon bonds to be valued analytically. For the Vasicek, Ho–Lee, and Hull–White models, the price at time zero of a call option that matures at time T on a zero-coupon bond maturing at time s is

$$LP(0, s)N(h) - KP(0, T)N(h - \sigma_P) \tag{30.20}$$

where L is the principal of the bond, K is its strike price, and

$$h = \frac{1}{\sigma_P}\ln\frac{LP(0, s)}{P(0, T)K} + \frac{\sigma_P}{2}$$

The price of a put option on the bond is

$$KP(0, T)N(-h + \sigma_P) - LP(0, s)N(-h)$$

In the case of the Vasicek and Hull–White models,

$$\sigma_P = \frac{\sigma}{a}[1 - e^{-a(s-T)}]\sqrt{\frac{1 - e^{-2aT}}{2a}}$$

In the case of the Ho–Lee model,

$$\sigma_P = \sigma(s - T)\sqrt{T}$$

Equation (30.20) is essentially the same as Black's model for pricing bond options in Section 28.2. The forward bond price volatility is $\sigma_P/\sqrt{T}$ and the standard deviation of the logarithm of the bond price at time T is σ_P. As explained in Section 28.3, an interest rate cap or floor can be expressed as a portfolio of options on zero-coupon bonds. It can, therefore, be valued analytically using the equations just presented.

There are also formulas for valuing options on zero-coupon bonds in the Cox, Ingersoll, and Ross model, which we presented in Section 30.2. These involve integrals of the noncentral chi-square distribution.

Options on Coupon-Bearing Bonds

In a one-factor model of r, all zero-coupon bonds move up in price when r decreases and all zero-coupon bonds move down in price when r increases. As a result, a one-factor model allows a European option on a coupon-bearing bond to be expressed as the sum of European options on zero-coupon bonds. The procedure is as follows:

1. Calculate r^*, the critical value of r for which the price of the coupon-bearing bond equals the strike price of the option on the bond at option maturity.

2. Calculate the prices of options on the zero-coupon bonds that comprise the coupon-bearing bond. Set the strike price of each option equal to the value the corresponding zero-coupon bond will have at time T when $r = r^*$.
3. Set the price of the option on the coupon-bearing bond equal to the sum of the prices on the options on zero-coupon bonds calculated in step 2.

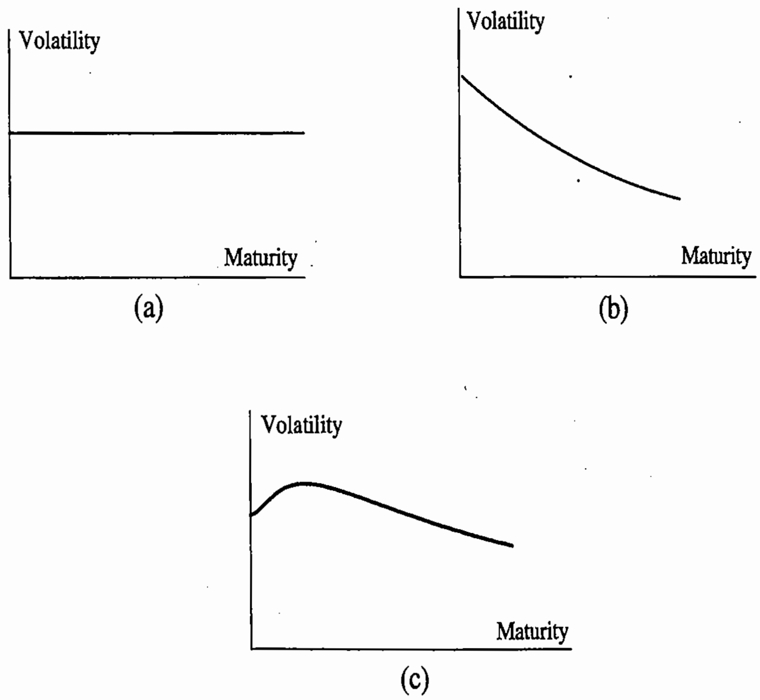
This allows options on coupon-bearing bonds to be valued for the Vasicek, Cox, Ingersoll, and Ross, Ho–Lee, and Hull–White models As explained in Business Snapshot 28.2, a European swap option can be viewed as an option on a coupon-bearing bond. It can, therefore, be valued using this procedure. For more details on the procedure, see Technical Note 15 on the author's website.

30.5 VOLATILITY STRUCTURES

The models we have looked at give rise to different volatility environments. Figure 30.5 shows the volatility of the 3-month forward rate as a function of maturity for Ho–Lee, Hull–White one-factor and Hull–White two-factor models. The term structure of interest rates is assumed to be flat.

For Ho–Lee the volatility of the 3-month forward rate is the same for all maturities. In the one-factor Hull–White model the effect of mean reversion is to cause the volatility of the 3-month forward rate to be a declining function of maturity. In the Hull–White two-factor model when parameters are chosen appropriately, the volatility

Figure 30.5 Volatility of 3-month forward rate as a function of maturity for (a) the Ho–Lee model, (b) the Hull–White one-factor model, and (c) the Hull–White two-factor model (when parameters are chosen appropriately).



of the 3-month forward rate has a "humped" look. The latter is consistent with empirical evidence and implied cap volatilities discussed in Section 28.3.

30.6 INTEREST RATE TREES

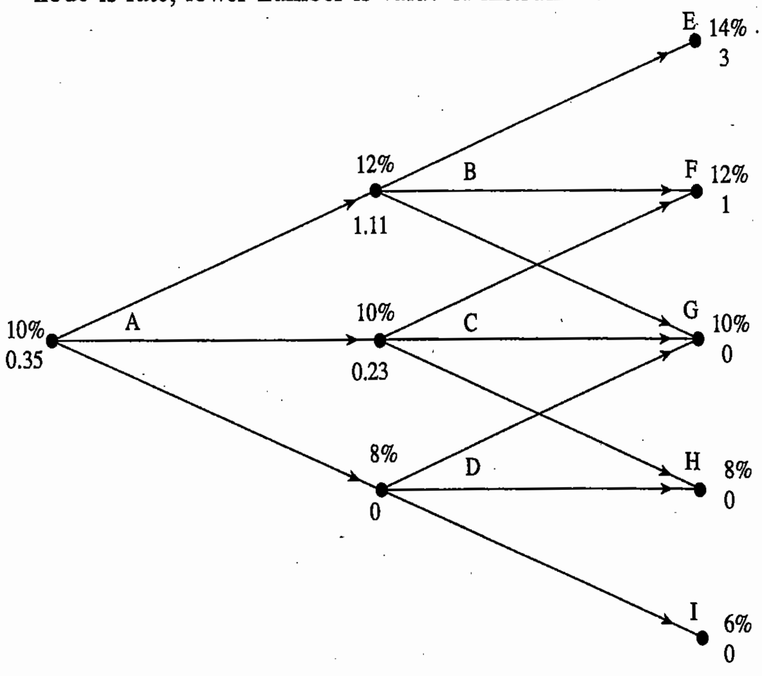
An interest rate tree is a discrete-time representation of the stochastic process for the short rate in much the same way as a stock price tree is a discrete-time representation of the process followed by a stock price. If the time step on the tree is Δt, the rates on the tree are the continuously compounded Δt-period rates. The usual assumption when a tree is constructed is that the Δt-period rate, R, follows the same stochastic process as the instantaneous rate, r, in the corresponding continuous-time model. The main difference between interest rate trees and stock price trees is in the way that discounting is done. In a stock price tree, the discount rate is usually assumed to be the same at each node (or a function of time). In an interest rate tree, the discount rate varies from node to node.

It often proves to be convenient to use a trinomial rather than a binomial tree for interest rates. The main advantage of a trinomial tree is that it provides an extra degree of freedom, making it easier for the tree to represent features of the interest rate process such as mean reversion. As mentioned in Section 19.8, using a trinomial tree is equivalent to using the explicit finite difference method.

Illustration of Use of Trinomial Trees

To illustrate how trinomial interest rate trees are used to value derivatives, consider the simple example shown in Figure 30.6. This is a two-step tree with each time step equal to 1 year in length so that $\Delta t = 1$ year. Assume that the up, middle, and down

Figure 30.6 Example of the use of trinomial interest rate trees. Upper number at each node is rate; lower number is value of instrument.



probabilities are 0.25, 0.50, and 0.25, respectively, at each node. The assumed Δt-period rate is shown as the upper number at each node.[12]

The tree is used to value a derivative that provides a payoff at the end of the second time step of

$$\max[100(R - 0.11), 0]$$

where R is the Δt-period rate. The calculated value of this derivative is the lower number at each node. At the final nodes, the value of the derivative equals the payoff. For example, at node E, the value is $100 \times (0.14 - 0.11) = 3$. At earlier nodes, the value of the derivative is calculated using the rollback procedure explained in Chapters 11 and 19. At node B, the 1-year interest rate is 12%. This is used for discounting to obtain the value of the derivative at node B from its values at nodes E, F, and G as

$$[0.25 \times 3 + 0.5 \times 1 + 0.25 \times 0]e^{-0.12\times 1} = 1.11$$

At node C, the 1-year interest rate is 10%. This is used for discounting to obtain the value of the derivative at node C as

$$(0.25 \times 1 + 0.5 \times 0 + 0.25 \times 0)e^{-0.1\times 1} = 0.23$$

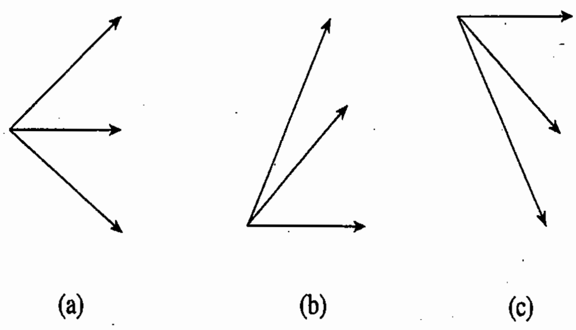
At the initial node, A, the interest rate is also 10% and the value of the derivative is

$$(0.25 \times 1.11 + 0.5 \times 0.23 + 0.25 \times 0)e^{-0.1\times 1} = 0.35$$

Nonstandard Branching

It sometimes proves convenient to modify the standard trinomial branching pattern that is used at all nodes in Figure 30.6. Three alternative branching possibilities are shown in Figure 30.7. The usual branching is shown in Figure 30.7(a). It is "up one/straight along/down one". One alternative to this is "up two/up one/straight along", as shown in Figure 30.7(b). This proves useful for incorporating mean reversion when interest rates are very low. A third branching pattern shown in Figure 30.7(c) is "straight along/down one/down two". This is useful for incorporating mean reversion when interest rates are very high. The use of different branching patterns is illustrated in the following section.

Figure 30.7 Alternative branching methods in a trinomial tree.



[12] We explain later how the probabilities and rates on an interest rate tree are determined.

30.7 A GENERAL TREE-BUILDING PROCEDURE

Hull and White have proposed a robust two-stage procedure for constructing trinomial trees to represent a wide range of one-factor models.[13] This section first explains how the procedure can be used for the Hull–White model in equation (30.13) and then shows how it can be extended to represent other models.

First Stage

The Hull–White model for the instantaneous short rate r is

$$dr = [\theta(t) - ar]\,dt + \sigma\,dz$$

We suppose that the time step on the tree is constant and equal to Δt.[14]

Assume that the Δt rate, R, follows the same process as r.

$$dR = [\theta(t) - aR]\,dt + \sigma\,dz$$

Clearly, this is reasonable in the limit as Δt tends to zero. The first stage in building a tree for this model is to construct a tree for a variable R^* that is initially zero and follows the process

$$dR^* = -aR^*\,dt + \sigma\,dz$$

This process is symmetrical about $R^* = 0$. The variable $R^*(t + \Delta t) - R^*(t)$ is normally distributed. If terms of higher order than Δt are ignored, the expected value of $R^*(t + \Delta t) - R^*(t)$ is $-aR^*(t)\Delta t$ and the variance of $R^*(t + \Delta t) - R^*(t)$ is $\sigma^2 \Delta t$.

The spacing between interest rates on the tree, ΔR, is set as

$$\Delta R = \sigma\sqrt{3\Delta t}$$

This proves to be a good choice of ΔR from the viewpoint of error minimization.

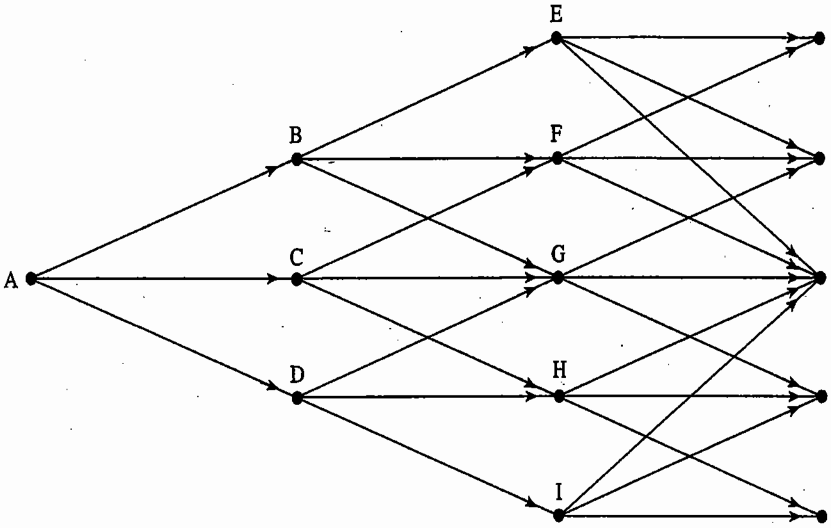
The objective of the first stage of the procedure is to build a tree similar to that shown in Figure 30.8 for R^*. To do this, it is first necessary to resolve which of the three branching methods shown in Figure 30.7 will apply at each node. This will determine the overall geometry of the tree. Once this is done, the branching probabilities must also be calculated.

Define (i, j) as the node where $t = i\,\Delta t$ and $R^* = j\,\Delta R$. (The variable i is a positive integer and j is a positive or negative integer.) The branching method used at a node must lead to the probabilities on all three branches being positive. Most of the time, the branching shown in Figure 30.7(a) is appropriate. When $a > 0$, it is necessary to switch from the branching in Figure 30.7(a) to the branching in Figure 30.7(c) for a sufficiently large j. Similarly, it is necessary to switch from the branching in Figure 30.7(a) to the branching in Figure 30.7(b) when j is sufficiently negative. Define j_{max} as the value of j where we switch from the Figure 30.7(a) branching to the Figure 30.7(c) branching and j_{min} as the value of j where we switch from the Figure 30.7(a) branching to the Figure 30.7(b) branching. Hull and White show that probabilities are always positive

[13] See J. Hull and A. White, "Numerical Procedures for Implementing Term Structure Models I: Single-Factor Models," *Journal of Derivatives*, 2, 1 (1994): 7–16; and J. Hull and A. White, "Using Hull-White Interest Rate Trees," *Journal of Derivatives*, (Spring 1996): 26–36.

[14] See Technical Note 16 on the author's website for a discussion of how nonconstant time steps can be used.

Figure 30.8 Tree for R^* in Hull–White model (first stage).



Node:	A	B	C	D	E	F	G	H	I
R(%)	0.000	1.732	0.000	−1.732	3.464	1.732	0.000	−1.732	−3.464
p_u	0.1667	0.1217	0.1667	0.2217	0.8867	0.1217	0.1667	0.2217	0.0867
p_m	0.6666	0.6566	0.6666	0.6566	0.0266	0.6566	0.6666	0.6566	0.0266
p_d	0.1667	0.2217	0.1667	0.1217	0.0867	0.2217	0.1667	0.1217	0.8867

if $j_{\max}$ is set equal to the smallest integer greater than $0.184/(a\,\Delta t)$ and $j_{\min}$ is set equal to $-j_{\max}$.[15] Define p_u, p_m, and p_d as the probabilities of the highest, middle, and lowest branches emanating from the node. The probabilities are chosen to match the expected change and variance of the change in R^* over the next time interval Δt. The probabilities must also sum to unity. This leads to three equations in the three probabilities.

As already mentioned, the mean change in R^* in time Δt is $-aR^*\Delta t$ and the variance of the change is $\sigma^2\Delta t$. At node (i, j), $R^* = j\,\Delta r$. If the branching has the form shown in Figure 30.7(a), the p_u, p_m, and p_d at node (i, j) must satisfy the following three equations to match the mean and standard deviation:

$$p_u\Delta R - p_d\Delta R = -aj\,\Delta R\,\Delta t$$
$$p_u\Delta R^2 + p_d\Delta R^2 = \sigma^2\Delta t + a^2 j^2\Delta R^2\Delta t^2$$
$$p_u + p_m + p_d = 1$$

Using $\Delta R = \sigma\sqrt{3\Delta t}$, the solution to these equations is

$$p_u = \tfrac{1}{6} + \tfrac{1}{2}(a^2 j^2\Delta t^2 - aj\,\Delta t)$$
$$p_m = \tfrac{2}{3} - a^2 j^2\Delta t^2$$
$$p_d = \tfrac{1}{6} + \tfrac{1}{2}(a^2 j^2\Delta t^2 + aj\,\Delta t)$$

[15] The probabilities are positive for any value of $j_{\max}$ between $0.184/(a\,\Delta t)$ and $0.816/(a\,\Delta t)$ and for any value of $j_{\min}$ between $-0.184/(a\,\Delta t)$ and $-0.816/(a\,\Delta t)$. Changing the branching at the first possible node proves to be computationally most efficient.

Similarly, if the branching has the form shown in Figure 30.7(b), the probabilities are

$$p_u = \tfrac{1}{6} + \tfrac{1}{2}(a^2j^2\Delta t^2 + aj\,\Delta t)$$
$$p_m = -\tfrac{1}{3} - a^2j^2\Delta t^2 - 2aj\,\Delta t$$
$$p_d = \tfrac{7}{6} + \tfrac{1}{2}(a^2j^2\Delta t^2 + 3aj\,\Delta t)$$

Finally, if the branching has the form shown in Figure 30.7(c), the probabilities are

$$p_u = \tfrac{7}{6} + \tfrac{1}{2}(a^2j^2\Delta t^2 - 3aj\,\Delta t)$$
$$p_m = -\tfrac{1}{3} - a^2j^2\Delta t^2 + 2aj\,\Delta t$$
$$p_d = \tfrac{1}{6} + \tfrac{1}{2}(a^2j^2\Delta t^2 - aj\,\Delta t)$$

To illustrate the first stage of the tree construction, suppose that $\sigma = 0.01$, $a = 0.1$, and $\Delta t = 1$ year. In this case, $\Delta R = 0.01\sqrt{3} = 0.0173$, $j_{\max}$ is set equal to the smallest integer greater than 0.184/0.1, and $j_{\min} = -j_{\max}$. This means that $j_{\max} = 2$ and $j_{\min} = -2$ and the tree is as shown in Figure 30.8. The probabilities on the branches emanating from each node are shown below the tree and are calculated using the equations above for p_u, p_m, and p_d.

Note that the probabilities at each node in Figure 30.8 depend only on j. For example, the probabilities at node B are the same as the probabilities at node F. Furthermore, the tree is symmetrical. The probabilities at node D are the mirror image of the probabilities at node B.

Second Stage

The second stage in the tree construction is to convert the tree for R^* into a tree for R. This is accomplished by displacing the nodes on the R^*-tree so that the initial term structure of interest rates is exactly matched. Define

$$\alpha(t) = R(t) - R^*(t)$$

The α's are calculated iteratively so that the initial term structure is matched exactly.[16] Define α_i as $\alpha(i\,\Delta t)$, the value of R at time $i\,\Delta t$ on the R-tree minus the corresponding value of R^* at time $i\,\Delta t$ on the r^*-tree. Define $Q_{i,j}$ as the present value of a security that pays off \$1 if node (i, j) is reached and zero otherwise. The α_i and $Q_{i,j}$ can be calculated using forward induction in such a way that the initial term structure is matched exactly.

Illustration of Second Stage

Suppose that the continuously compounded zero rates in the example in Figure 30.8 are as shown in Table 30.1. The value of $Q_{0,0}$ is 1.0. The value of α_0 is chosen to give the

[16] It is possible to estimate $\alpha(t)$ analytically. Since

$$dR = [\theta(t) - aR]\,dt + \sigma\,dz \quad \text{and} \quad dR^* = -aR^*\,dt + \sigma\,dz$$

it follows that

$$d\alpha = [\theta(t) - a\alpha(t)]\,dt$$

If we ignore the distinction between r and R, the solution to this is

$$\alpha(t) = F(0, t) + \frac{\sigma^2}{2a^2}(1 - e^{-at})^2$$

However, these are instantaneous α's and do not lead to the tree calculations exactly matching the term structure of interest rates.

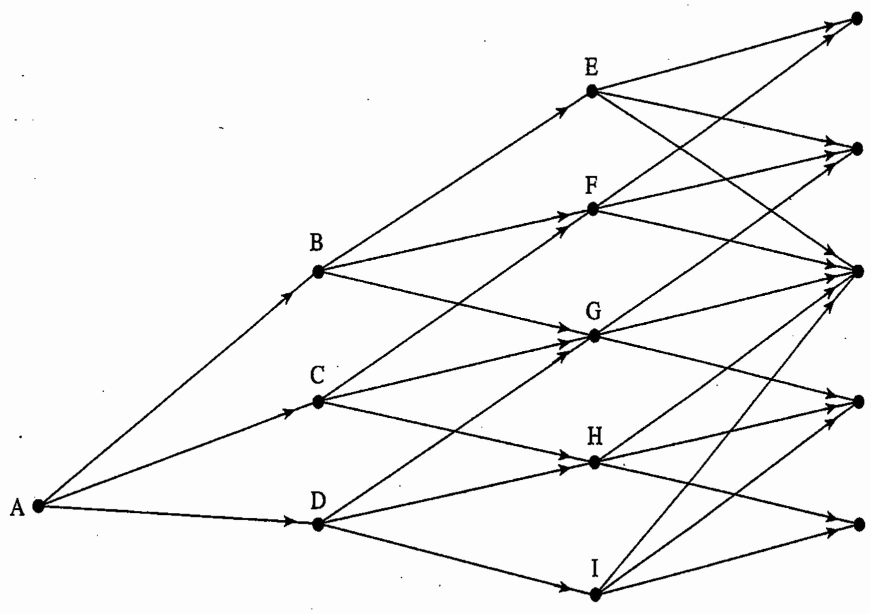
Table 30.1 Zero rates for example in Figures 30.8 and 30.9.

Maturity	*Rate (%)*
0.5	3.430
1.0	3.824
1.5	4.183
2.0	4.512
2.5	4.812
3.0	5.086

right price for a zero-coupon bond maturing at time Δt. That is, α_0 is set equal to the initial Δt-period interest rate. Because $\Delta t = 1$ in this example, $\alpha_0 = 0.03824$. This defines the position of the initial node on the R-tree in Figure 30.9. The next step is to calculate the values of $Q_{1,1}$, $Q_{1,0}$, and $Q_{1,-1}$. There is a probability of 0.1667 that the (1, 1) node is reached and the discount rate for the first time step is 3.82%. The value of $Q_{1,1}$ is therefore $0.1667e^{-0.0382} = 0.1604$. Similarly, $Q_{1,0} = 0.6417$ and $Q_{1,-1} = 0.1604$.

Once $Q_{1,1}$, $Q_{1,0}$, and $Q_{1,-1}$ have been calculated, α_1 can be determined. It is chosen to give the right price for a zero-coupon bond maturing at time $2\Delta t$. Because $\Delta R = 0.01732$ and $\Delta t = 1$, the price of this bond as seen at node B is $e^{-(\alpha_1+0.01732)}$. Similarly, the price as seen at node C is $e^{-\alpha_1}$ and the price as seen at node D is $e^{-(\alpha_1-0.01732)}$. The price as seen at

Figure 30.9 Tree for R in Hull–White model (the second stage).



Node:	A	B	C	D	E	F	G	H	I
R(%)	3.824	6.937	5.205	3.473	9.716	7.984	6.252	4.520	2.788
p_u	0.1667	0.1217	0.1667	0.2217	0.8867	0.1217	0.1667	0.2217	0.0867
p_m	0.6666	0.6566	0.6666	0.6566	0.0266	0.6566	0.6666	0.6566	0.0266
p_d	0.1667	0.2217	0.1667	0.1217	0.0867	0.2217	0.1667	0.1217	0.8867

the initial node A is therefore

$$Q_{1,1}e^{-(\alpha_1+0.01732)} + Q_{1,0}e^{-\alpha_1} + Q_{1,-1}e^{-(\alpha_1-0.01732)} \tag{30.21}$$

From the initial term structure, this bond price should be $e^{-0.04512\times2} = 0.9137$. Substituting for the Q's in equation (30.21),

$$0.1604e^{-(\alpha_1+0.01732)} + 0.6417e^{-\alpha_1} + 0.1604e^{-(\alpha_1-0.01732)} = 0.9137$$

or

$$e^{-\alpha_1}(0.1604e^{-0.01732} + 0.6417 + 0.1604e^{0.01732}) = 0.9137$$

or

$$\alpha_1 = \ln\left[\frac{0.1604e^{-0.01732} + 0.6417 + 0.1604e^{0.01732}}{0.9137}\right] = 0.05205$$

This means that the central node at time Δt in the tree for R corresponds to an interest rate of 5.205% (see Figure 30.9).

The next step is to calculate $Q_{2,2}$, $Q_{2,1}$, $Q_{2,0}$, $Q_{2,-1}$, and $Q_{2,-2}$. The calculations can be shortened by using previously determined Q values. Consider $Q_{2,1}$ as an example. This is the value of a security that pays off \$1 if node F is reached and zero otherwise. Node F can be reached only from nodes B and C. The interest rates at these nodes are 6.937% and 5.205%, respectively. The probabilities associated with the B–F and C–F branches are 0.6566 and 0.1667. The value at node B of a security that pays \$1 at node F is therefore $0.6566e^{-0.06937}$. The value at node C is $0.1667e^{-0.05205}$. The variable $Q_{2,1}$ is $0.6566e^{-0.06937}$ times the present value of \$1 received at node B plus $0.1667e^{-0.05205}$ times the present value of \$1 received at node C; that is,

$$Q_{2,1} = 0.6566e^{-0.0693} \times 0.1604 + 0.1667e^{-0.05205} \times 0.6417 = 0.1998$$

Similarly, $Q_{2,2} = 0.0182$, $Q_{2,0} = 0.4736$, $Q_{2,-1} = 0.2033$, and $Q_{2,-2} = 0.0189$.

The next step in producing the R-tree in Figure 30.9 is to calculate α_2. After that, the $Q_{3,j}$'s can then be computed. The variable α_3 can then be calculated, and so on.

Formulas for α's and Q's

To express the approach more formally, suppose that the $Q_{i,j}$ have been determined for $i \leqslant m$ ($m \geqslant 0$). The next step is to determine α_m so that the tree correctly prices a zero-coupon bond maturing at $(m+1)\,\Delta t$. The interest rate at node (m, j) is $\alpha_m + j\,\Delta R$, so that the price of a zero-coupon bond maturing at time $(m+1)\Delta t$ is given by

$$P_{m+1} = \sum_{j=-n_m}^{n_m} Q_{m,j}\exp[-(\alpha_m + j\,\Delta R)\Delta t] \tag{30.22}$$

where n_m is the number of nodes on each side of the central node at time $m\,\Delta t$. The solution to this equation is

$$\alpha_m = \frac{\ln\sum_{j=-n_m}^{n_m} Q_{m,j}e^{-j\Delta R\Delta t} - \ln P_{m+1}}{\Delta t}$$

Once α_m has been determined, the $Q_{i,j}$ for $i = m+1$ can be calculated using

$$Q_{m+1,j} = \sum_k Q_{m,k} q(k,j) \exp[-(\alpha_m + k\,\Delta R)\,\Delta t]$$

where $q(k, j)$ is the probability of moving from node (m, k) to node $(m+1, j)$ and the summation is taken over all values of k for which this is nonzero.

Extension to Other Models

The procedure that has just been outlined can be extended to more general models of the form

$$df(r) = [\theta(t) - af(r)]\,dt + \sigma\,dz \tag{30.23}$$

This family of models has the property that they can fit any term structure.[17]

As before, we assume that the Δt period rate, R, follows the same process as r:

$$df(R) = [\theta(t) - af(R)]\,dt + \sigma\,dz$$

We start by setting $x = f(R)$, so that

$$dx = [\theta(t) - ax]\,dt + \sigma\,dz$$

The first stage is to build a tree for a variable x^* that follows the same process as x except that $\theta(t) = 0$ and the initial value is zero. The procedure here is identical to the procedure already outlined for building a tree such as that in Figure 30.8.

As in Figure 30.9, the nodes at time $i\,\Delta t$ are then displaced by an amount α_i to provide an exact fit to the initial term structure. The equations for determining α_i and $Q_{i,j}$ inductively are slightly different from those for the $f(R) = R$ case. The value of Q at the first node, $Q_{0,0}$, is set equal to 1. Suppose that the $Q_{i,j}$ have been determined for $i \leqslant m$ $(m \geqslant 0)$. The next step is to determine α_m so that the tree correctly prices an $(m+1)\Delta t$ zero-coupon bond. Define g as the inverse function of f so that the Δt-period interest rate at the jth node at time $m\,\Delta t$ is

$$g(\alpha_m + j\,\Delta x)$$

The price of a zero-coupon bond maturing at time $(m+1)\Delta t$ is given by

$$P_{m+1} = \sum_{j=-n_m}^{n_m} Q_{m,j} \exp[-g(\alpha_m + j\,\Delta x)\Delta t] \tag{30.24}$$

This equation can be solved using a numerical procedure such as Newton–Raphson. The value α_0 of α when $m = 0$, is $f\big(R(0)\big)$.

Once α_m has been determined, the $Q_{i,j}$ for $i = m+1$ can be calculated using

$$Q_{m+1,j} = \sum_k Q_{m,k} q(k,j) \exp[-g(\alpha_m + k\,\Delta x)\Delta t]$$

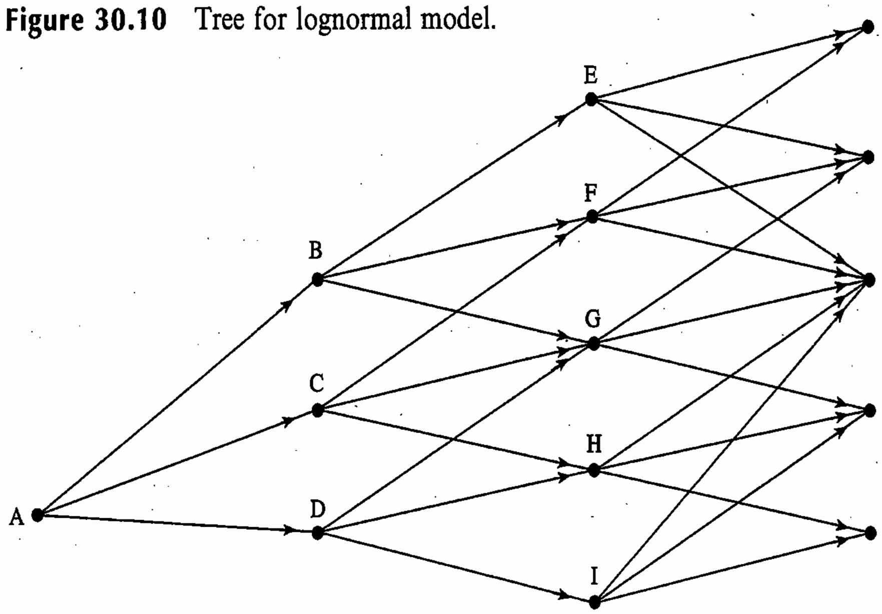
---

[17] Not all no-arbitrage models have this property. For example, the extended-CIR model, considered by Cox, Ingersoll, and Ross (1985) and Hull and White (1990), which has the form

$$dr = [\theta(t) - ar]\,dt + \sigma\sqrt{r}\,dz$$

cannot fit yield curves where the forward rate declines sharply. This is because the process is not well defined when $\theta(t)$ is negative.

Figure 30.10 Tree for lognormal model.



Node:	A	B	C	D	E	F	G	H	I
x	−3.373	−2.875	−3.181	−3.487	−2.430	−2.736	−3.042	−3.349	−3.655
R(%)	3.430	5.642	4.154	3.058	8.803	6.481	4.772	3.513	2.587
p_u	0.1667	0.1177	0.1667	0.2277	0.8609	0.1177	0.1667	0.2277	0.0809
p_m	0.6666	0.6546	0.6666	0.6546	0.0582	0.6546	0.6666	0.6546	0.0582
p_d	0.1667	0.2277	0.1667	0.1177	0.0809	0.2277	0.1667	0.1177	0.8609

where $q(k, j)$ is the probability of moving from node (m, k) to node $(m + 1, j)$ and the summation is taken over all values of k where this is nonzero.

Figure 30.10 shows the results of applying the procedure to the model

$$d\ln(r) = [\theta(t) - a\ln(r)]\,dt + \sigma\,dz$$

when $a = 0.22$, $\sigma = 0.25$, $\Delta t = 0.5$, and the zero rates are as in Table 30.1.

Choosing $f(r)$

Setting $f(r) = r$ leads to the Hull–White model in equation (30.13); setting $f(r) = \ln(r)$ leads to the Black–Karasinksi model in equation (30.18). In most circumstances these two models appear to perform about the same in fitting market data on actively traded instruments such as caps and European swap options. The main advantage of the $f(r) = r$ model is its analytic tractability. Its main disadvantage is that negative interest rates are possible. In most circumstances, the probability of negative interest rates occurring under the model is very small, but some analysts are reluctant to use a model where there is any chance at all of negative interest rates. The $f(r) = \ln r$ model has no analytic tractability, but has the advantage that interest rates are always positive. Another advantage is that traders naturally think in terms of σ's arising from a lognormal model rather than σ's arising from a normal model.

There is a problem in choosing a satisfactory model for countries with low interest rates. The normal model is unsatisfactory because, when the initial short rate is low, the probability of negative interest rates in the future is no longer negligible. The lognormal model is unsatisfactory because the volatility of rates (i.e., the σ parameter in the lognormal model) is usually much greater when rates are low than when they are high. (For example, a volatility of 100% might be appropriate when the short rate is very low, while 20% might be appropriate when it is 4% or more.) A model that appears to work well is one where $f(r)$ is chosen so that rates are lognormal for r less than 1% and normal for r greater than 1%.[18]

Using Analytic Results in Conjunction with Trees

When a tree is constructed for the $f(r) = r$ version of the Hull–White model, the analytic results in Section 30.3 can be used to provide the complete term structure and European option prices at each node. It is important to recognize that the interest rate on the tree is the Δt-period rate R. It is not the instantaneous short rate r.

From equations (30.15), (30.16), and (30.17) it can be shown (see Problem 30.21) that

$$P(t,T) = \hat{A}(t,T)e^{-\hat{B}(t,T)R} \tag{30.25}$$

where

$$\ln \hat{A}(t,T) = \ln\frac{P(0,T)}{P(0,t)} - \frac{B(t,T)}{B(t,t+\Delta t)}\ln\frac{P(0,t+\Delta t)}{P(0,t)} - \frac{\sigma^2}{4a}(1-e^{-2at})B(t,T)[B(t,T) - B(t,t+\Delta t)] \tag{30.26}$$

and

$$\hat{B}(t,T) = \frac{B(t,T)}{B(t,t+\Delta t)}\Delta t \tag{30.27}$$

(In the case of the Ho–Lee model, we set $\hat{B}(t,T) = T - t$ in these equations.)

Bond prices should therefore be calculated with equation (30.25), and not with equation (30.15).

Example 30.1

Suppose zero rates are as in Table 30.2. The rates for maturities between those indicated are generated using linear interpolation.

Consider a 3-year ($= 3 \times 365$ days) European put option on a zero-coupon bond that will expire in 9 years ($= 9 \times 365$ days). Interest rates are assumed to follow the Hull–White ($f(r) = r$) model. The strike price is 63, $a = 0.1$, and $\sigma = 0.01$. A 3-year tree is constructed and zero-coupon bond prices are calculated analytically at the final nodes as just described. As shown in Table 30.3, the results from the tree are consistent with the analytic price of the option.

This example provides a good test of the implementation of the model because the gradient of the zero curve changes sharply immediately after the expiration of the option. Small errors in the construction and use of the tree are liable to have a big effect on the option values obtained. (The example is used in Sample Application G of the DerivaGem Application Builder software.)

[18] See J. Hull and A. White "Taking Rates to the Limit," *Risk*, December (1997): 168–69.

Table 30.2 Zero curve with all rates continuously compounded.

Maturity	*Days*	*Rate* (%)
3 days	3	5.01772
1 month	31	4.98284
2 months	62	4.97234
3 months	94	4.96157
6 months	185	4.99058
1 year	367	5.09389
2 years	731	5.79733
3 years	1,096	6.30595
4 years	1,461	6.73464
5 years	1,826	6.94816
6 years	2,194	7.08807
7 years	2,558	7.27527
8 years	2,922	7.30852
9 years	3,287	7.39790
10 years	3,653	7.49015

Tree for American Bond Options

The DerivaGem software accompanying this book implements the normal and the lognormal model for valuing European and American bond options, caps/floors, and European swap options. Figure 30.11 shows the tree produced by the software when it is used to value a 1.5-year American call option on a 10-year bond using four time steps and the lognormal model. The parameters used in the lognormal model are $a = 5\%$ and $\sigma = 20\%$. The underlying bond lasts 10 years, has a principal of 100, and pays a coupon of 5% per annum semiannually. The yield curve is flat at 5% per annum. The strike price is 105. As explained in Section 28.1 the strike price can be a cash strike price or a quoted strike price. In this case it is a quoted strike price. The bond price shown on the tree is the cash bond price. The accrued interest at each node is shown below the tree. The cash strike price is calculated as the quoted strike price plus accrued interest. The quoted bond price is the cash bond price minus accrued interest. The

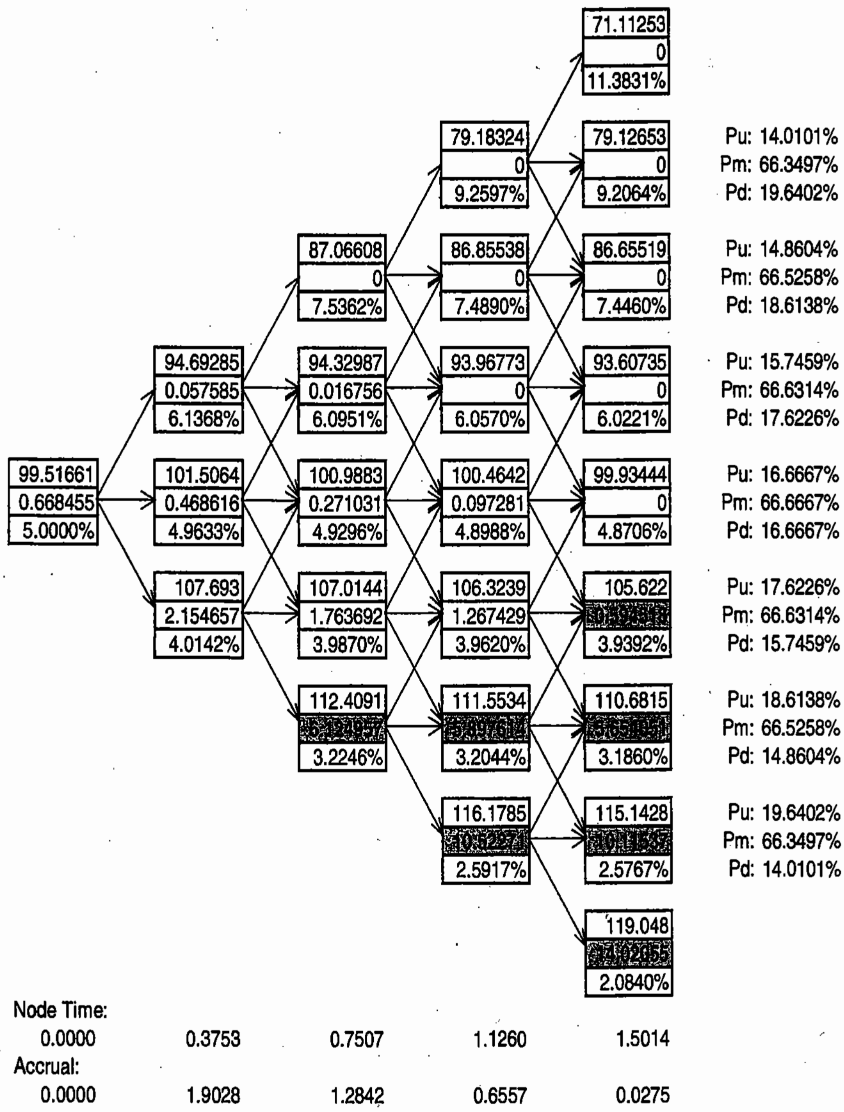
Table 30.3 Value of a three-year put option on a nine-year zero-coupon bond with a strike price of 63: $a = 0.1$ and $\sigma = 0.01$; zero curve as in Table 30.2.

Steps	*Tree*	*Analytic*
10	1.8658	1.8093
30	1.8234	1.8093
50	1.8093	1.8093
100	1.8144	1.8093
200	1.8097	1.8093
500	1.8093	1.8093

Figure 30.11 Tree, produced by DerivaGem, for valuing an American bond option.

At each node:
Upper value = Cash Bond Price
Middle value = Option Price
Lower value = dt-period Rate
Shaded values are as a result of early exercise

Strike price = 105
Time step, dt = 0.3753 years, 137.00 days



payoff from the option is the cash bond price minus the cash strike price. Equivalently it is the quoted bond price minus the quoted strike price.

The tree gives the price of the option as 0.668. A much larger tree with 100 time steps gives the price of the option as 0.699. Two points should be noted about Figure 30.11:

1. The software measures the time to option maturity as a whole number of days. For example, when an option maturity of 1.5 years is input, the life of the option is assumed to be 1.5014 years (or 1 year and 183 days). Coupon dates (and therefore accruals) depend on the valuation date, taken from the computer's clock.

2. The price of the 10-year bond cannot be computed analytically when the lognormal model is assumed. It is computed numerically by rolling back through a much larger tree than that shown.

30.8 CALIBRATION

Up to now, we have assumed that the volatility parameters a and σ are known. We now discuss how they are determined. This is known as calibrating the model.

The volatility parameters are determined from market data on actively traded options (e.g., broker quotes on caps and swap options such as those in Tables 28.1 and 28.2). These will be referred to as the *calibrating instruments*. The first stage is to choose a "goodness-of-fit" measure. Suppose there are n calibrating instruments. A popular goodness-of-fit measure is

$$\sum_{i=1}^{n}(U_i - V_i)^2$$

where U_i is the market price of the ith calibrating instrument and V_i is the price given by the model for this instrument. The objective of calibration is to choose the model parameters so that this goodness-of-fit measure is minimized.

The number of volatility parameters should not be greater than the number of calibrating instruments. If a and σ are constant, there are only two volatility parameters. The models can be extended so that a or σ, or both, are functions of time. Step functions can be used. Suppose, for example, that a is constant and σ is a function of time. We might choose times $t_1, t_2, \dots, t_n$ and assume $\sigma(t) = \sigma_0$ for $t \leqslant t_1$, $\sigma(t) = \sigma_i$ for $t_i < t \leqslant t_{i+1}$ $(1 \leqslant i \leqslant n-1)$, and $\sigma(t) = \sigma_n$ for $t > t_n$. There would then be a total of $n+2$ volatility parameters: a, σ_0, $\sigma_1, \dots,$ and σ_n.

The minimization of the goodness-of-fit measure can be accomplished using the Levenberg–Marquardt procedure.[19] When a or σ, or both, are functions of time, a penalty function is often added to the goodness-of-fit measure so that the functions are "well behaved". In the example just mentioned, where σ is a step function, an appropriate objective function is

$$\sum_{i=1}^{n}(U_i - V_i)^2 + \sum_{i=1}^{n} w_{1,i}(\sigma_i - \sigma_{i-1})^2 + \sum_{i=1}^{n-1} w_{2,i}(\sigma_{i-1} + \sigma_{i+1} - 2\sigma_i)^2$$

The second term provides a penalty for large changes in σ between one step and the next. The third term provides a penalty for high curvature in σ. Appropriate values for $w_{1,i}$ and $w_{2,i}$ are based on experimentation and are chosen to provide a reasonable level of smoothness in the σ function.

The calibrating instruments chosen should be as similar as possible to the instrument being valued. Suppose, for example, that the model is to be used to value a Bermudan-style swap option that lasts 10 years and can be exercised on any payment date between year 5 and year 9 into a swap maturing 10 years from today. The most relevant calibrating instruments are 5×5, 6×4, 7×3, 8×2, and 9×1 European swap options. (An $n \times m$ European swap option is an n-year option to enter into a swap lasting for m years beyond the maturity of the option.)

[19] For a good description of this procedure, see W.H. Press, B.P. Flannery, S.A. Teukolsky, and W.T. Vetterling, *Numerical Recipes in C: The Art of Scientific Computing*. Cambridge University Press, 1988.

The advantage of making a or σ, or both, functions of time is that the models can be fitted more precisely to the prices of instruments that trade actively in the market. The disadvantage is that the volatility structure becomes nonstationary. The volatility term structure given by the model in the future is liable to be quite different from that existing in the market today.[20]

A somewhat different approach to calibration is to use all available calibrating instruments to calculate "global-best-fit" a and σ parameters. The parameter a is held fixed at its best-fit value. The model can then be used in the same way as Black–Scholes. There is a one-to-one relationship between options prices and the σ parameter. The model can be used to convert tables such as Tables 28.1 and 28.2 into tables of implied σ's.[21] These tables can be used to assess the σ most appropriate for pricing the instrument under consideration.

30.9 HEDGING USING A ONE-FACTOR MODEL

Section 28.6 outlined some general approaches to hedging a portfolio of interest rate derivatives. These approaches can be used with the term structure models in this chapter. The calculation of deltas, gammas, and vegas involves making small changes to either the zero curve or the volatility environment and recomputing the value of the portfolio.

Note that, although one factor is often assumed when pricing interest rate derivatives, it is not appropriate to assume only one factor when hedging. For example, the deltas calculated should allow for many different movements in the yield curve, not just those that are possible under the model chosen. The practice of taking account of changes that cannot happen under the model considered, as well as those that can, is known as *outside model hedging* and is standard practice for traders.[22] The reality is that relatively simple one-factor models if used carefully usually give reasonable prices for instruments, but good hedging procedures must explicitly or implicitly assume many factors.

SUMMARY

The traditional models of the term structure used in finance are known as equilibrium models. These are useful for understanding potential relationships between variables in the economy, but have the disadvantage that the initial term structure is an output from the model rather than an input to it. When valuing derivatives, it is important that the model used be consistent with the initial term structure observed in the market. No-arbitrage models are designed to have this property. They take the initial term structure as given and define how it can evolve.

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[20] For a discussion of the implementation of a model where a and σ are functions of time, see Technical Note 16 on the author's website.

[21] Note that in a term structure model the implied σ's are not the same as the implied volatilities calculated from Black's model in Tables 28.1 and 28.2. The procedure for computing implied σ's is as follows. The Black volatilities are converted to prices using Black's model. An iterative procedure is then used to imply the σ parameter in the term structure model from the price.

[22] A simple example of outside model hedging is in the way that the Black–Scholes model is used. The Black–Scholes model assumes that volatility is constant—but traders regularly calculate vega and hedge against volatility changes.

This chapter has provided a description of a number of one-factor no-arbitrage models of the short rate. These are robust and can be used in conjunction with any set of initial zero rates. The simplest model is the Ho–Lee model. This has the advantage that it is analytically tractable. Its chief disadvantage is that it implies that all rates are equally variable at all times. The Hull–White model is a version of the Ho–Lee model that includes mean reversion. It allows a richer description of the volatility environment while preserving its analytic tractability. Lognormal one-factor models avoid the possibility of negative interest rates, but have no analytic tractability.

FURTHER READING

Equilibrium Models

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No-Arbitrage Models

Black, F., and P. Karasinski, "Bond and Option Pricing When Short Rates Are Lognormal," *Financial Analysts Journal*, July/August (1991): 52–59.

Ho, T. S. Y., and S.-B. Lee, "Term Structure Movements and Pricing Interest Rate Contingent Claims," *Journal of Finance*, 41 (December 1986): 1011–29.

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Li, A., P. Ritchken, and L. Sankarasubramanian, "Lattice Models for Pricing American Interest Rate Claims," *Journal of Finance*, 50, 2 (June 1995): 719–37.

Rebonato, R., *Interest Rate Option Models*. Chichester: Wiley, 1998.

Questions and Problems (Answers in Solutions Manual)

30.1. What is the difference between an equilibrium model and a no-arbitrage model?

30.2. Suppose that the short rate is currently 4% and its standard deviation is 1% per annum. What happens to the standard deviation when the short rate increases to 8% in (a) Vasicek's model; (b) Rendleman and Bartter's model; and (c) the Cox, Ingersoll, and Ross model?

30.3. If a stock price were mean reverting or followed a path-dependent process there would be market inefficiency. Why is there not a market inefficiency when the short-term interest rate does so?

30.4. Explain the difference between a one-factor and a two-factor interest rate model.

30.5. Can the approach described in Section 30.4 for decomposing an option on a coupon-bearing bond into a portfolio of options on zero-coupon bonds be used in conjunction with a two-factor model? Explain your answer.

30.6. Suppose that $a = 0.1$ and $b = 0.1$ in both the Vasicek and the Cox, Ingersoll, Ross model. In both models, the initial short rate is 10% and the initial standard deviation of the short-rate change in a short time Δt is $0.02\sqrt{\Delta t}$. Compare the prices given by the models for a zero-coupon bond that matures in year 10.

30.7. Suppose that $a = 0.1$, $b = 0.08$, and $\sigma = 0.015$ in Vasicek's model, with the initial value of the short rate being 5%. Calculate the price of a 1-year European call option on a zero-coupon bond with a principal of \$100 that matures in 3 years when the strike price is \$87.

30.8. Repeat Problem 30.7 valuing a European put option with a strike of \$87. What is the put–call parity relationship between the prices of European call and put options? Show that the put and call option prices satisfy put–call parity in this case.

30.9. Suppose that $a = 0.05$, $b = 0.08$, and $\sigma = 0.015$ in Vasicek's model with the initial short-term interest rate being 6%. Calculate the price of a 2.1-year European call option on a bond that will mature in 3 years. Suppose that the bond pays a coupon of 5% semiannually. The principal of the bond is 100 and the strike price of the option is 99. The strike price is the cash price (not the quoted price) that will be paid for the bond.

30.10. Use the answer to Problem 30.9 and put–call parity arguments to calculate the price of a put option that has the same terms as the call option in Problem 30.9.

30.11. In the Hull–White model, $a = 0.08$ and $\sigma = 0.01$. Calculate the price of a 1-year European call option on a zero-coupon bond that will mature in 5 years when the term structure is flat at 10%, the principal of the bond is \$100, and the strike price is \$68.

30.12. Suppose that $a = 0.05$ and $\sigma = 0.015$ in the Hull–White model with the initial term structure being flat at 6% with semiannual compounding. Calculate the price of a 2.1-year European call option on a bond that will mature in 3 years. Suppose that the bond pays a coupon of 5% per annum semiannually. The principal of the bond is 100 and the strike price of the option is 99. The strike price is the cash price (not the quoted price) that will be paid for the bond.

30.13. Use a change of numeraire argument to show that the relationship between the futures rate and forward rate for the Ho–Lee model is as shown in Section 6.4. Use the relationship to verify the expression for $\theta(t)$ given for the Ho–Lee model in equation (30.11). (*Hint*: The futures price is a martingale when the market price of risk is zero. The forward price is a martingale when the market price of risk is a zero-coupon bond maturing at the same time as the forward contract.)

30.14. Use a similar approach to that in Problem 30.13 to derive the relationship between the futures rate and the forward rate for the Hull–White model. Use the relationship to verify the expression for $\theta(t)$ given for the Hull–White model in equation (30.14).

30.15. Suppose $a = 0.05$, $\sigma = 0.015$, and the term structure is flat at 10%. Construct a trinomial tree for the Hull–White model where there are two time steps, each 1 year in length.

30.16. Calculate the price of a 2-year zero-coupon bond from the tree in Figure 30.6.

30.17. Calculate the price of a 2-year zero-coupon bond from the tree in Figure 30.9 and verify that it agrees with the initial term structure.

30.18. Calculate the price of an 18-month zero-coupon bond from the tree in Figure 30.10 and verify that it agrees with the initial term structure.

30.19. What does the calibration of a one-factor term structure model involve?

30.20. Use the DerivaGem software to value 1×4, 2×3, 3×2, and 4×1 European swap options to receive fixed and pay floating. Assume that the 1-, 2-, 3-, 4-, and 5-year interest rates are 6%, 5.5%, 6%, 6.5%, and 7%, respectively. The payment frequency on the swap is semiannual and the fixed rate is 6% per annum with semiannual compounding. Use the Hull–White model with $a = 3\%$ and $\sigma = 1\%$. Calculate the volatility implied by Black's model for each option.

30.21. Prove equations (30.25), (30.26), and (30.27).

Assignment Questions

30.22. Construct a trinomial tree for the Ho–Lee model where $\sigma = 0.02$. Suppose that the the initial zero-coupon interest rate for a maturities of 0.5, 1.0, and 1.5 years are 7.5%, 8%, and 8.5%. Use two time steps, each 6 months long. Calculate the value of a zero-coupon bond with a face value of \$100 and a remaining life of 6 months at the ends of the final nodes of the tree. Use the tree to value a 1-year European put option with a strike price of 95 on the bond. Compare the price given by your tree with the analytic price given by DerivaGem.

30.23. A trader wishes to compute the price of a 1-year American call option on a 5-year bond with a face value of 100. The bond pays a coupon of 6% semiannually and the (quoted) strike price of the option is \$100. The continuously compounded zero rates for maturities of 6 months, 1 year, 2 years, 3 years, 4 years, and 5 years are 4.5%, 5%, 5.5%, 5.8%, 6.1%, and 6.3%. The best-fit reversion rate for either the normal or the lognormal model has been estimated as 5%.

A 1-year European call option with a (quoted) strike price of 100 on the bond is actively traded. Its market price is \$0.50. The trader decides to use this option for calibration. Use the DerivaGem software with 10 time steps to answer the following questions:

(a) Assuming a normal model, imply the σ parameter from the price of the European option.

(b) Use the σ parameter to calculate the price of the option when it is American.

(c) Repeat (a) and (b) for the lognormal model. Show that the model used does not significantly affect the price obtained providing it is calibrated to the known European price.

(d) Display the tree for the normal model and calculate the probability of a negative interest rate occurring.

(e) Display the tree for the lognormal model and verify that the option price is correctly calculated at the node where, with the notation of Section 30.7, $i = 9$ and $j = -1$.

30.24. Use the DerivaGem software to value 1×4, 2×3, 3×2, and 4×1 European swap options to receive floating and pay fixed. Assume that the 1-, 2-, 3-, 3-, and 5-year interest rates are 3%, 3.5%, 3.8%, 4.0%, and 4.1%, respectively. The payment frequency on the swap is semiannual and the fixed rate is 4% per annum with semiannual compounding. Use the lognormal model with $a = 5\%$, $\sigma = 15\%$, and 50 time steps. Calculate the volatility implied by Black's model for each option.

30.25. Verify that the DerivaGem software gives Figure 30.11 for the example considered. Use the software to calculate the price of the American bond option for the lognormal and normal models when the strike price is 95, 100, and 105. In the case of the normal model, assume that $a = 5\%$ and $\sigma = 1\%$. Discuss the results in the context of the heaviness of the tails arguments of Chapter 18.

30.26. Modify Sample Application G in the DerivaGem Application Builder software to test the convergence of the price of the trinomial tree when it is used to price a 2-year call option on a 5-year bond with a face value of 100. Suppose that the strike price (quoted) is 100, the coupon rate is 7% with coupons being paid twice a year. Assume that the zero curve is as in Table 30.2. Compare results for the following cases:
(a) Option is European; normal model with $\sigma = 0.01$ and $a = 0.05$
(b) Option is European; lognormal model with $\sigma = 0.15$ and $a = 0.05$
(c) Option is American; normal model with $\sigma = 0.01$ and $a = 0.05$
(d) Option is American; lognormal model with $\sigma = 0.15$ and $a = 0.05$

CHAPTER 31

Interest Rate Derivatives: HJM and LMM

The interest rate models discussed in Chapter 30 are widely used for pricing instruments when the simpler models in Chapter 28 are inappropriate. They are easy to implement and, if used carefully, can ensure that most nonstandard interest rate derivatives are priced consistently with actively traded instruments such as interest rate caps, European swap options, and European bond options. Two limitations of the models are:

1. Most involve only one factor (i.e., one source of uncertainty).
2. They do not give the user complete freedom in choosing the volatility structure.

By making the parameters a and σ functions of time, an analyst can use the models so that they fit the volatilities observed in the market today, but as mentioned in Section 30.8 the volatility term structure is then nonstationary. The volatility structure in the future is liable to be quite different from that observed in the market today.

This chapter discusses some general approaches to building term structure models that give the user more flexibility in specifying the volatility environment and allow several factors to be used. The models require much more computation time than the models in Chapter 30. As a result, they are often used for research and development rather than routine pricing.

This chapter also covers the mortgage-backed security market in the United States and describes how some of the ideas presented in the chapter can be used to price instruments in that market.

31.1 THE HEATH, JARROW, AND MORTON MODEL

In 1990 David Heath, Bob Jarrow, and Andy Morton (HJM) published an important paper describing the no-arbitrage conditions that must be satisfied by a model of the yield curve.[1] To describe their model, we will use the following notation:

$P(t,T)$: Price at time t of a zero-coupon bond with principal \$1 maturing at time T

[1] See D. Heath, R.A. Jarrow, and A. Morton, "Bond Pricing and the Term Structure of Interest Rates: A New Methodology," *Econometrica*, 60, 1 (1992): 77–105.

Ω_t: Vector of past and present values of interest rates and bond prices at time t that are relevant for determining bond price volatilities at that time

$v(t, T, \Omega_t)$: Volatility of $P(t, T)$

$f(t, T_1, T_2)$: Forward rate as seen at time t for the period between time T_1 and time T_2

$F(t, T)$: Instantaneous forward rate as seen at time t for a contract maturing at time T

$r(t)$: Short-term risk-free interest rate at time t

$dz(t)$: Wiener process driving term structure movements

Processes for Zero-Coupon Bond Prices and Forward Rates

We start by assuming there is just one factor and will use the traditional risk-neutral world. A zero-coupon bond is a traded security providing no income. Its return in the traditional risk-neutral world must therefore be r. This means that its stochastic process has the form

$$dP(t, T) = r(t)P(t, T)\,dt + v(t, T, \Omega_t)P(t, T)\,dz(t) \tag{31.1}$$

As the argument Ω_t indicates, the zero-coupon bond's volatility v can be, in the most general form of the model, any well-behaved function of past and present interest rates and bond prices. Because a bond's price volatility declines to zero at maturity, we must have[2]

$$v(t, t, \Omega_t) = 0$$

From equation (4.5), the forward rate $f(t, T_1, T_2)$ can be related to zero-coupon bond prices as follows:

$$f(t, T_1, T_2) = \frac{\ln[P(t, T_1)] - \ln[P(t, T_2)]}{T_2 - T_1} \tag{31.2}$$

From equation (31.1) and Itô's lemma,

$$d\ln[P(t, T_1)] = \left[r(t) - \frac{v(t, T_1, \Omega_t)^2}{2}\right]dt + v(t, T_1, \Omega_t)\,dz(t)$$

and

$$d\ln[P(t, T_2)] = \left[r(t) - \frac{v(t, T_2, \Omega_t)^2}{2}\right]dt + v(t, T_2, \Omega_t)\,dz(t)$$

so that

$$df(t, T_1, T_2) = \frac{v(t, T_2, \Omega_t)^2 - v(t, T_1, \Omega_t)^2}{2(T_2 - T_1)}\,dt + \frac{v(t, T_1, \Omega_t) - v(t, T_2, \Omega_t)}{T_2 - T_1}\,dz(t) \tag{31.3}$$

Equation (31.3) shows that the risk-neutral process for f depends solely on the v's. It depends on r and the P's only to the extent that the v's themselves depend on these variables.

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[2] The $v(t, t, \Omega_t) = 0$ condition is equivalent to the assumption that all discount bonds have finite drifts at all times. If the volatility of the bond does not decline to zero at maturity, an infinite drift may be necessary to ensure that the bond's price equals its face value at maturity.

When we put $T_1 = T$ and $T_2 = T + \Delta T$ in equation (31.3) and then take limits as ΔT tends to zero, $f(t, T_1, T_2)$ becomes $F(t, T)$, the coefficient of $dz(t)$ becomes $v_T(t, T, \Omega_t)$, and the coefficient of dt becomes

$$\frac{1}{2}\frac{\partial[v(t, T, \Omega_t)^2]}{\partial T} = v(t, T, \Omega_t)\, v_T(t, T, \Omega_t)$$

where the subscript to v denotes a partial derivative. It follows that

$$dF(t, T) = v(t, T, \Omega_t)\, v_T(t, T, \Omega_t)\, dt - v_T(t, T, \Omega_t)\, dz(t) \tag{31.4}$$

Once the function $v(t, T, \Omega_t)$ has been specified, the risk-neutral processes for the $F(t, T)$'s are known.

Equation (31.4) shows that there is a link between the drift and standard deviation of an instantaneous forward rate. This is the key HJM result. Integrating $v_\tau(t, \tau, \Omega_t)$ between $\tau = t$ and $\tau = T$ leads to

$$v(t, T, \Omega_t) - v(t, t, \Omega_t) = \int_t^T v_\tau(t, \tau, \Omega_t)\, d\tau$$

Because $v(t, t, \Omega_t) = 0$, this becomes

$$v(t, T, \Omega_t) = \int_t^T v_\tau(t, \tau, \Omega_t)\, d\tau$$

If $m(t, T, \Omega_t)$ and $s(t, T, \Omega_t)$ are the instantaneous drift and standard deviation of $F(t, T)$, so that

$$dF(t, T) = m(t, T, \Omega_t)\, dt + s(t, T, \Omega_t)\, dz$$

then it follows from equation (31.4) that

$$m(t, T, \Omega_t) = s(t, T, \Omega_t) \int_t^T s(t, \tau, \Omega_t)\, d\tau \tag{31.5}$$

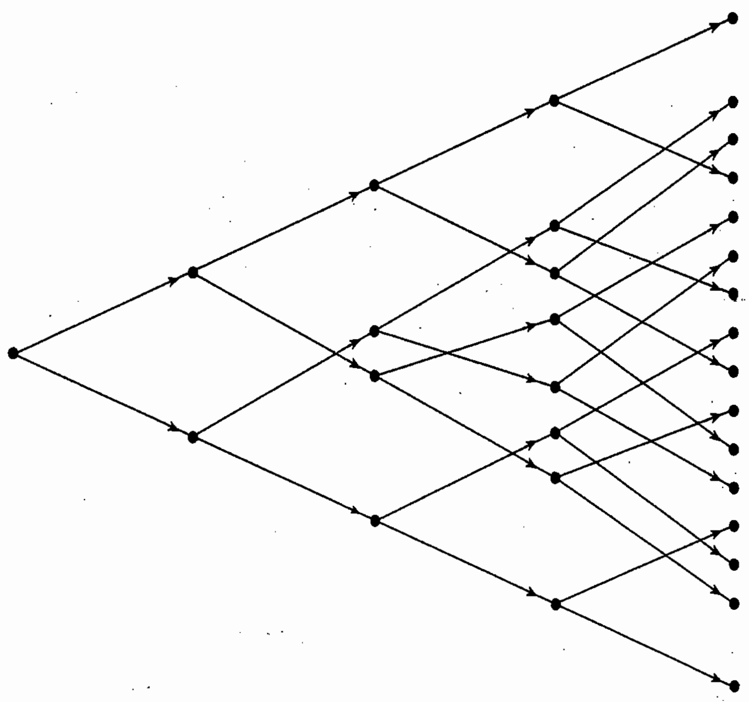
This is the HJM result.

The process for the short rate r in the general HJM model is non-Markov. This means that the process for r at a future time t depends on the path followed by r between now and time t as well as on the the value of r at time t.[3] This is the key problem in implementing a general HJM model. Monte Carlo simulation has to be used. It is difficult to use a tree to represent term structure movements because the tree is usually nonrecombining. Assuming the model has one factor and the tree is binomial as in Figure 31.1, there are 2^n nodes after n time steps (when $n = 30$, 2^n is about 1 billion).

The HJM model in equation (31.4) is deceptively complex. A particular forward rate $F(t, T)$ is Markov in most applications of the model and can be represented by a recombining tree. However, the same tree cannot be used for all forward rates.

[3] For more details, see Technical Note 17 on the author's website.

Figure 31.1 A nonrecombining tree such as that arising from the general HJM model.



Extension to Several Factors

The HJM result can be extended to the situation where there are several independent factors. Suppose

$$dF(t, T) = m(t, T, \Omega_t)\, dt + \sum_k s_k(t, T, \Omega_t)\, dz_k$$

A similar analysis to that just given (see Problem 31.2) shows that

$$m(t, T, \Omega_t) = \sum_k s_k(t, T, \Omega_t) \int_t^T s_k(t, \tau, \Omega_t)\, d\tau \tag{31.6}$$

31.2 THE LIBOR MARKET MODEL

One drawback of the HJM model is that it is expressed in terms of instantaneous forward rates and these are not directly observable in the market. Another drawback is that it is difficult to calibrate the model to prices of actively traded instruments. This has led Brace, Gatarek, and Musiela (BGM), Jamshidian, and Miltersen, Sandmann, and Sondermann to propose an alternative.[4] It is known as the *LIBOR market model* (LMM) or the *BGM model* and it is expressed in terms of the forward rates that traders are used to working with.

[4] See A. Brace, D. Gatarek, and M. Musiela "The Market Model of Interest Rate Dynamics," *Mathematical Finance* 7, 2 (1997): 127–55; F. Jamshidian, "LIBOR and Swap Market Models and Measures," *Finance and Stochastics*, 1 (1997): 293–330; and K. Miltersen, K. Sandmann, and D. Sondermann, "Closed Form Solutions for Term Structure Derivatives with LogNormal Interest Rate," *Journal of Finance*, 52, 1 (March 1997): 409–30.

The Model

Define $t_0 = 0$ and let $t_1, t_2, \ldots$ be the reset times for caps that trade in the market today. In the United States, the most popular caps have quarterly resets, so that it is approximately true that $t_1 = 0.25$, $t_2 = 0.5$, $t_3 = 0.75$, and so on. Define $\delta_k = t_{k+1} - t_k$, and

$F_k(t)$: Forward rate between times t_k and t_{k+1} as seen at time t, expressed with a compounding period of δ_k and an actual/actual day count

$m(t)$: Index for the next reset date at time t; this means that $m(t)$ is the smallest integer such that $t \leqslant t_{m(t)}$

$\zeta_k(t)$: Volatility of $F_k(t)$ at time t

$v_k(t)$: Volatility of the zero-coupon bond price $P(t, t_k)$ at time t

Initially, we will assume that there is only one factor. As shown in Section 27.4, in a world that is forward risk neutral with respect to $P(t, t_{k+1})$, $F_k(t)$ is a martingale and follows the process

$$dF_k(t) = \zeta_k(t) F_k(t)\, dz \tag{31.7}$$

where dz is a Wiener process.

In practice, it is often most convenient to value interest rate derivatives by working in a world that is always forward risk neutral with respect to a bond maturing at the next reset date. We refer to this as a *rolling forward risk-neutral world.*[5] In this world we can discount from time t_{k+1} to time t_k using the zero rate observed at time t_k for a maturity t_{k+1}. We do not have to worry about what happens to interest rates between times t_k and t_{k+1}.

At time t the rolling forward risk-neutral world is a world that is forward risk neutral with respect to the bond price, $P(t, t_{m(t)})$. Equation (31.7) gives the process followed by $F_k(t)$ in a world that is forward risk neutral with respect to $P(t, t_{k+1})$. From Section 27.8, it follows that the process followed by $F_k(t)$ in the rolling forward risk-neutral world is

$$dF_k(t) = \zeta_k(t)[v_{m(t)}(t) - v_{k+1}(t)] F_k(t)\, dt + \zeta_k(t) F_k(t)\, dz \tag{31.8}$$

The relationship between forward rates and bond prices is

$$\frac{P(t, t_i)}{P(t, t_{i+1})} = 1 + \delta_i F_i(t)$$

or

$$\ln P(t, t_i) - \ln P(t, t_{i+1}) = \ln[1 + \delta_i F_i(t)]$$

Itô's lemma can be used to calculate the process followed by the left-hand side and the right-hand side of this equation. Equating the coefficients of dz gives

$$v_i(t) - v_{i+1}(t) = \frac{\delta_i F_i(t) \zeta_i(t)}{1 + \delta_i F_i(t)} \tag{31.9}$$

[5] In the terminology of Section 27.4, this world corresponds to using a "rolling CD" as the numeraire. A rolling CD (certificate of deposit) is one where we start with \$1, buy a bond maturing at time t_1, reinvest the proceeds at time t_1 in a bond maturing at time t_2, reinvest the proceeds at time t_2 in a bond maturing at time t_3, and so on. (Strictly speaking, the interest rate trees we constructed in Chapter 30 are in a rolling forward risk-neutral world rather than the traditional risk-neutral world.) The numeraire is a CD rolled over at the end of each time step.

so that from equation (31.8) the process followed by $F_k(t)$ in the rolling forward risk-neutral world is

$$\frac{dF_k(t)}{F_k(t)} = \sum_{i=m(t)}^{k} \frac{\delta_i F_i(t)\zeta_i(t)\zeta_k(t)}{1+\delta_i F_i(t)}\, dt + \zeta_k(t)\, dz \tag{31.10}$$

The HJM result in equation (31.4) is the limiting case of this as the δ_i tend to zero (see Problem 31.7).

Forward Rate Volatilities

The model can be simplified by assuming that $\zeta_k(t)$ is a function only of the number of whole accrual periods between the next reset date and time t_k. Define Λ_i as the value of $\zeta_k(t)$ when there are i such accrual periods. This means that $\zeta_k(t) = \Lambda_{k-m(t)}$ is a step function.

The Λ_i can (at least in theory) be estimated from the volatilities used to value caplets in Black's model (i.e., from the spot volatilities in Figure 28.3).[6] Suppose that σ_k is the Black volatility for the caplet that corresponds to the period between times t_k and t_{k+1}. Equating variances, we must have

$$\sigma_k^2 t_k = \sum_{i=1}^{k} \Lambda_{k-i}^2\, \delta_{i-1} \tag{31.11}$$

This equation can be used to obtain the Λ's iteratively.

Example 31.1

Assume that the δ_i are all equal and the Black caplet spot volatilities for the first three caplets are 24%, 22%, and 20%. This means that $\Lambda_0 = 24\%$. Since

$$\Lambda_0^2 + \Lambda_1^2 = 2 \times 0.22^2$$

Λ_1 is 19.80%. Also, since

$$\Lambda_0^2 + \Lambda_1^2 + \Lambda_2^2 = 3 \times 0.20^2$$

Λ_2 is 15.23%.

Example 31.2

Consider the data in Table 31.1 on caplet volatilities σ_k. These exhibit the hump discussed in Section 28.3. The Λ's are shown in the second row. Notice that the hump in the Λ's is more pronounced than the hump in the σ's.

Table 31.1 Volatility data; accrual period = 1 year.

Year, k:	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
σ_k (%):	15.50	18.25	17.91	17.74	17.27	16.79	16.30	16.01	15.76	15.54
Λ_{k-1} (%):	15.50	20.64	17.21	17.22	15.25	14.15	12.98	13.81	13.60	13.40

[6] In practice the Λ's are determined using a least-squares calibration that we will discuss later.

Implementation of the Model

The LIBOR market model can be implemented using Monte Carlo simulation. Expressed in terms of the Λ_i's, equation (31.10) is

$$\frac{dF_k(t)}{F_k(t)} = \sum_{i=m(t)}^{k} \frac{\delta_i F_i(t)\Lambda_{i-m(t)}\Lambda_{k-m(t)}}{1+\delta_i F_i(t)}\,dt + \Lambda_{k-m(t)}\,dz \tag{31.12}$$

or

$$d\ln F_k(t) = \left[\sum_{i=m(t)}^{k} \frac{\delta_i F_i(t)\Lambda_{i-m(t)}\Lambda_{k-m(t)}}{1+\delta_i F_i(t)} - \frac{(\Lambda_{k-m(t)})^2}{2}\right] dt + \Lambda_{k-m(t)}\,dz \tag{31.13}$$

If, as an approximation, we assume in the calculation of the drift of $\ln F_k(t)$ that $F_i(t) = F_i(t_j)$ for $t_j < t < t_{j+1}$, then

$$F_k(t_{j+1}) = F_k(t_j)\exp\left[\left(\sum_{i=j+1}^{k} \frac{\delta_i F_i(t_j)\Lambda_{i-j-1}\Lambda_{k-j-1}}{1+\delta_i F_i(t_j)} - \frac{\Lambda_{k-j-1}^2}{2}\right)\delta_j + \Lambda_{k-j-1}\epsilon\sqrt{\delta_j}\right] \tag{31.14}$$

where ϵ is a random sample from a normal distribution with mean equal to zero and standard deviation equal to one.

Extension to Several Factors

The LIBOR market model can be extended to incorporate several independent factors. Suppose that there are p factors and $\zeta_{k,q}$ is the component of the volatility of $F_k(t)$ attributable to the qth factor. Equation (31.10) becomes (see Problem 31.11)

$$\frac{dF_k(t)}{F_k(t)} = \sum_{i=m(t)}^{k} \frac{\delta_i F_i(t)\sum_{q=1}^{p}\zeta_{i,q}(t)\zeta_{k,q}(t)}{1+\delta_i F_i(t)}\,dt + \sum_{q=1}^{p}\zeta_{k,q}(t)\,dz_q \tag{31.15}$$

Define $\lambda_{i,q}$ as the qth component of the volatility when there are i accrual periods between the next reset date and the maturity of the forward contract. Equation (31.14) then becomes

$$F_k(t_{j+1}) = F_k(t_j)\exp\left[\left(\sum_{i=j+1}^{k} \frac{\delta_i F_i(t_j)\sum_{q=1}^{p}\lambda_{i-j-1,q}\lambda_{k-j-1,q}}{1+\delta_i F_i(t_j)} - \frac{\sum_{q=1}^{p}\lambda_{k-j-1,q}^2}{2}\right)\delta_j + \sum_{q=1}^{p}\lambda_{k-j-1,q}\,\epsilon_q\sqrt{\delta_j}\right] \tag{31.16}$$

where the ϵ_q are random samples from a normal distribution with mean equal to zero and standard deviation equal to one.

The approximation that the drift of a forward rate remains constant within each accrual period allows us to jump from one reset date to the next in the simulation. This is convenient because as already mentioned the rolling forward risk-neutral world allows us to discount from one reset date to the next. Suppose that we wish to simulate a zero curve for N accrual periods. On each trial we start with the forward rates at time zero. These are $F_0(0), F_1(0), \dots, F_{N-1}(0)$ and are calculated from the initial zero curve. Equation (31.16) is used to calculate $F_1(t_1), F_2(t_1), \dots, F_{N-1}(t_1)$. Equation (31.16) is then used again to calculate $F_2(t_2), F_3(t_2), \dots, F_{N-1}(t_2)$, and so on, until $F_{N-1}(t_{N-1})$ is obtained. Note that as we move through time the zero curve gets shorter and shorter. For example, suppose each accrual period is 3 months and $N = 40$. We start with a 10-year zero curve. At the 6-year point (at time t_{24}), the simulation gives us information on a 4-year zero curve.

The drift approximation can be tested by valuing caplets using equation (31.16) and comparing the prices to those given by Black's model. The value of $F_k(t_k)$ is the realized rate for the time period between t_k and t_{k+1} and enables the caplet payoff at time t_{k+1} to be calculated. This payoff is discounted back to time zero, one accrual period at a time. The caplet value is the average of the discounted payoffs. The results of this type of analysis show that the cap values from Monte Carlo simulation are not significantly different from those given by Black's model. This is true even when the accrual periods are 1 year in length and a very large number of trials is used.[7] This suggests that the drift approximation is innocuous in most situations.

Ratchet Caps, Sticky Caps, and Flexi Caps

The LIBOR market model can be used to value some types of nonstandard caps. Consider ratchet caps and sticky caps. These incorporate rules for determining how the cap rate for each caplet is set. In a *ratchet cap* it equals the LIBOR rate at the previous reset date plus a spread. In a *sticky cap* it equals the previous capped rate plus a spread. Suppose that the cap rate at time t_j is K_j, the LIBOR rate at time t_j is R_j, and the spread is s. In a ratchet cap, $K_{j+1} = R_j + s$. In a sticky cap, $K_{j+1} = \min(R_j, K_j) + s$.

Tables 31.2 and 31.3 provide valuations of a ratchet cap and sticky cap using the LIBOR market model with one, two, and three factors. The principal is $100. The term

Table 31.2 Valuation of ratchet caplets.

Caplet start time (years)	*One factor*	*Two factors*	*Three factors*
1	0.196	0.194	0.195
2	0.207	0.207	0.209
3	0.201	0.205	0.210
4	0.194	0.198	0.205
5	0.187	0.193	0.201
6	0.180	0.189	0.193
7	0.172	0.180	0.188
8	0.167	0.174	0.182
9	0.160	0.168	0.175
10	0.153	0.162	0.169

[7] See J. C. Hull and A. White, "Forward Rate Volatilities, Swap Rate Volatilities, and the Implementation of the LIBOR Market Model," *Journal of Fixed Income*, 10, 2 (September 2000): 46–62. The only exception is when the cap volatilities are very high.

Table 31.3 Valuation of sticky caplets.

Caplet start time (years)	*One factor*	*Two factors*	*Three factors*
1	0.196	0.194	0.195
2	0.336	0.334	0.336
3	0.412	0.413	0.418
4	0.458	0.462	0.472
5	0.484	0.492	0.506
6	0.498	0.512	0.524
7	0.502	0.520	0.533
8	0.501	0.523	0.537
9	0.497	0.523	0.537
10	0.488	0.519	0.534

structure is assumed to be flat at 5% per annum and the caplet volatilities are as in Table 31.1. The interest rate is reset annually. The spread is 25 basis points. Tables 31.4 and 31.5 show how the volatility was split into components when two- and three-factor models were used. The results are based on 100,000 Monte Carlo simulations incorporating the antithetic variable technique described in Section 19.7. The standard error of each price is about 0.001.

A third type of nonstandard cap is a *flexi cap*. This is like a regular cap except that there is a limit on the total number of caplets that can be exercised. Consider an annual-pay flexi cap when the principal is $100, the term structure is flat at 5%, and the cap volatilities are as in Tables 31.1, 31.4, and 31.5. Suppose that all in-the-money caplets are exercised up to a maximum of five. With one, two, and three factors, the LIBOR market model gives the price of the instrument as 3.43, 3.58, and 3.61, respectively (see Problem 31.15 for other types of flexi caps).

The pricing of a plain vanilla cap depends only on the total volatility and is independent of the number of factors. This is because the price of a plain vanilla caplet depends on the behavior of only one forward rate. The prices of caplets in the nonstandard instruments we have looked at are different in that they depend on the joint probability distribution of several different forward rates. As a result they do depend on the number of factors.

Table 31.4 Volatility components in two-factor model.

Year, k:	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
$\lambda_{k-1,1}$ (%):	14.10	19.52	16.78	17.11	15.25	14.06	12.65	13.06	12.36	11.63
$\lambda_{k-1,2}$ (%):	−6.45	−6.70	−3.84	−1.96	0.00	1.61	2.89	4.48	5.65	6.65
Total volatility (%):	15.50	20.64	17.21	17.22	15.25	14.15	12.98	13.81	13.60	13.40

Table 31.5 Volatility components in a three-factor model.

Year, k:	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
$\lambda_{k-1,1}$ (%):	13.65	19.28	16.72	16.98	14.85	13.95	12.61	12.90	11.97	10.97
$\lambda_{k-1,2}$ (%):	−6.62	−7.02	−4.06	−2.06	0.00	1.69	3.06	4.70	5.81	6.66
$\lambda_{k-1,3}$ (%):	3.19	2.25	0.00	−1.98	−3.47	−1.63	0.00	1.51	2.80	3.84
Total volatility (%):	15.50	20.64	17.21	17.22	15.25	14.15	12.98	13.81	13.60	13.40

Valuing European Swap Options

As shown by Hull and White, there is an analytic approximation for valuing European swap options in the LIBOR market model.[8] Let T_0 be the maturity of the swap option and assume that the payment dates for the swap are $T_1, T_2, \dots, T_N$. Define $\tau_i = T_{i+1} - T_i$. From equation (27.23), the swap rate at time t is given by

$$s(t) = \frac{P(t, T_0) - P(t, T_N)}{\sum_{i=0}^{N-1} \tau_i P(t, T_{i+1})}$$

It is also true that

$$\frac{P(t, T_i)}{P(t, T_0)} = \prod_{j=0}^{i-1} \frac{1}{1 + \tau_j G_j(t)}$$

for $1 \leqslant i \leqslant N$, where $G_j(t)$ is the forward rate at time t for the period between T_j and T_{j+1}. These two equations together define a relationship between $s(t)$ and the $G_j(t)$. Applying Itô's lemma (see Problem 31.12), the variance $V(t)$ of the swap rate $s(t)$ is given by

$$V(t) = \sum_{q=1}^{p} \left[\sum_{k=0}^{N-1} \frac{\tau_k \beta_{k,q}(t) G_k(t) \gamma_k(t)}{1 + \tau_k G_k(t)} \right]^2 \tag{31.17}$$

where

$$\gamma_k(t) = \frac{\prod_{j=0}^{N-1}[1 + \tau_j G_j(t)]}{\prod_{j=0}^{N-1}[1 + \tau_j G_j(t)] - 1} - \frac{\sum_{i=0}^{k-1} \tau_i \prod_{j=i+1}^{N-1}[1 + \tau_j G_j(t)]}{\sum_{i=0}^{N-1} \tau_i \prod_{j=i+1}^{N}[1 + \tau_j G_j(t)]}$$

and $\beta_{j,q}(t)$ is the qth component of the volatility of $G_j(t)$. We approximate $V(t)$ by setting $G_j(t) = G_j(0)$ for all j and t. The swap volatility that is substituted into the

[8] See J. C. Hull and A. White, "Forward Rate Volatilities, Swap Rate Volatilities, and the Implementation of the LIBOR Market Model," *Journal of Fixed Income*, 10, 2 (September 2000): 46–62. Other analytic approximations have been suggested by A. Brace, D. Gatarek, and M. Musiela "The Market Model of Interest Rate Dynamics," *Mathematical Finance*, 7, 2 (1997): 127–55 and L. Andersen and J. Andreasen, "Volatility Skews and Extensions of the LIBOR Market Model," *Applied Mathematical Finance*, 7, 1 (March 2000), 1–32.

standard market model for valuing a swaption is then

$$\sqrt{\frac{1}{T_0}\int_{t=0}^{T_0} V(t)\,dt}$$

or

$$\sqrt{\frac{1}{T_0}\int_{t=0}^{T_0}\sum_{q=1}^{p}\left[\sum_{k=0}^{N-1}\frac{\tau_k\beta_{k,q}(t)G_k(0)\gamma_k(0)}{1+\tau_kG_k(0)}\right]^2 dt} \qquad \textbf{(31.18)}$$

In the situation where the length of the accrual period for the swap underlying the swaption is the same as the length of the accrual period for a cap, $\beta_{k,q}(t)$ is the qth component of volatility of a cap forward rate when the time to maturity is $T_k - t$. This can be looked up in a table such as Table 31.5

The accrual periods for the swaps underlying broker quotes for European swap options do not always match the accrual periods for the caps and floors underlying broker quotes. For example, in the United States, the benchmark caps and floors have quarterly resets, while the swaps underlying the benchmark European swap options have semiannual resets on the fixed side. Fortunately, the valuation result for European swap options can be extended to the situation where each swap accrual period includes M subperiods that could be accrual periods in a typical cap. Define $\tau_{j,m}$ as the length of the mth subperiod in the jth accrual period so that

$$\tau_j = \sum_{m=1}^{M}\tau_{j,m}$$

Define $G_{j,m}(t)$ as the forward rate observed at time t for the $\tau_{j,m}$ accrual period. Because

$$1+\tau_jG_j(t) = \prod_{m=1}^{M}[1+\tau_{j,m}G_{j,m}(t)]$$

the analysis leading to equation (31.18) can be modified so that the volatility of $s(t)$ is obtained in terms of the volatilities of the $G_{j,m}(t)$ rather than the volatilities of the $G_j(t)$. The swap volatility to be substituted into the standard market model for valuing a swap option proves to be (see Problem 31.13)

$$\sqrt{\frac{1}{T_0}\int_{t=0}^{T_0}\sum_{q=1}^{p}\left[\sum_{k=n}^{N-1}\sum_{m=1}^{M}\frac{\tau_{k,m}\beta_{k,m,q}(t)G_{k,m}(0)\gamma_k(0)}{1+\tau_{k,m}G_{k,m}(0)}\right]^2 dt} \qquad \textbf{(31.19)}$$

Here $\beta_{j,m,q}(t)$ is the qth component of the volatility of $G_{j,m}(t)$. It is the qth component of the volatility of a cap forward rate when the time to maturity is from t to the beginning of the mth subperiod in the (T_j, T_{j+1}) swap accrual period.

The expressions in equations (31.18) and (31.19) for the swap volatility do involve the approximations that $G_j(t) = G_j(0)$ and $G_{j,m}(t) = G_{j,m}(0)$. Hull and White compared the prices of European swap options calculated using equations (31.18) and (31.19) with the prices calculated from a Monte Carlo simulation and found the two to be very close. Once the LIBOR market model has been calibrated, equations (31.18) and (31.19) therefore provide a quick way of valuing European swap options. Analysts can determine whether European swap options are overpriced or underpriced relative to caps. As we will see shortly, they can also use the results to calibrate the model to the market prices of swap options.

Calibrating the Model

To calibrate the LIBOR market model, it is necessary to determine the Λ_j and how they are split into $\lambda_{j,q}$. The Λ's are usually determined from current market data, whereas the split into λ's is determined from historical data.

A principal components analysis (see Section 20.9) can determine the way the Λ's are split into λ's. The model is

$$\Delta F_j = \sum_{q=1}^{M} \alpha_{j,q} x_q$$

where M is the total number of factors, ΔF_j is the change in the forward rate for a forward contract maturing in j accrual periods, $\alpha_{j,q}$ is the factor loading for the jth forward rate and the qth factor, x_q is the factor score for the qth factor, and

$$\sum_{j=1}^{M} \alpha_{j,q_1} \alpha_{j,q_2}$$

equals 1 when $q_1 = q_2$ and 0 when $q_1 \neq q_2$. Define s_q as the standard deviation of the qth factor score. If the number of factors used in the LIBOR market model, p, is equal to the total number of factors, M, it is correct to set

$$\lambda_{j,q} = \alpha_{j,q} s_q$$

for $1 \leqslant j,q \leqslant M$. When $p < M$, the $\lambda_{j,q}$ must be scaled so that

$$\Lambda_j = \sqrt{\sum_{q=1}^{p} \lambda_{j,q}^2}$$

This involves setting

$$\lambda_{j,q} = \frac{\Lambda_j s_q \alpha_{j,q}}{\sqrt{\sum_{q=1}^{p} s_q^2 \alpha_{j,q}^2}} \tag{31.20}$$

Equation (31.11) provides one way to determine the Λ's so that they are consistent with caplet prices. In practice, it is not usually used because it often leads to wild swings in the Λ's.[9] Also, although the LIBOR market model is designed to be consistent with the prices of caplets, analysts sometimes like to calibrate it to European swaptions. A commonly used calibration procedure is similar to that described for one-factor models in Section 30.8. Suppose that U_i is the market price of the ith calibrating instrument and V_i is the model price. The Λ's are chosen to minimize

$$\sum_i (U_i - V_i)^2 + P$$

[9] Sometimes there is no set of Λ's consistent with a set of cap quotes.

where P is a penalty function chosen to ensure that the Λ's are "well behaved". Similarly to Section 30.8, P has the form

$$P = \sum_i w_{1,i}(\Lambda_{i+1} - \Lambda_i)^2 + \sum_i w_{2,i}(\Lambda_{i+1} + \Lambda_{i-1} - 2\Lambda_i)^2$$

When some calibrating instruments are European swaptions the formulas in equations (31.18) and (31.19) make the minimization feasible using the Levenberg–Marquardt procedure. Equation (31.20) is used to determine the λ's from the Λ's.

Volatility Skews

Brokers provide quotes on caps that are not at the money as well as on caps that are at the money. In some markets a volatility skew is observed, that is, the quoted (Black) volatility for a cap or a floor is a declining function of the strike price. This can be handled using the CEV model. (See Section 26.1 for the application of the CEV model to equities.) The model is

$$dF_i(t) = \cdots + \sum_{q=1}^{p} \zeta_{i,q}(t) F_i(t)^{\alpha}\, dz_q \tag{31.21}$$

where α is a constant $(0 < \alpha < 1)$. It turns out that this model can be handled very similarly to the lognormal model. Caps and floors can be valued analytically using the cumulative noncentral χ^2 distribution. There are similar analytic approximations to those given above for the prices of European swap options.[10]

Bermudan Swap Options

A popular interest rate derivative is a Bermudan swap option. This is a swap option that can be exercised on some or all of the payment dates of the underlying swap. Bermudan swap options are difficult to value using the LIBOR market model because the LIBOR market model relies on Monte Carlo simulation and it is difficult to evaluate early exercise decisions when Monte Carlo simulation is used. Fortunately, the procedures described in Section 26.8 can be used. Longstaff and Schwartz apply the least-squares approach when there are a large number of factors. The value of not exercising on a particular payment date is assumed to be a polynomial function of the values of the factors.[11] Andersen shows that the optimal early exercise boundary approach can be used. He experiments with a number of ways of parameterizing the early exercise boundary and finds that good results are obtained when the early exercise decision is assumed to depend only on the intrinsic value of the option.[12]. Most traders value Bermudan options using one of the one-factor no-arbitrage models discussed in

[10] For details, see L. Andersen and J. Andreasen, "Volatility Skews and Extensions of the LIBOR Market Model," *Applied Mathematical Finance*, 7, 1 (2000): 1–32; J.C. Hull and A. White, "Forward Rate Volatilities, Swap Rate Volatilities, and the Implementation of the LIBOR Market Model," *Journal of Fixed Income*, 10, 2 (September 2000): 46–62.

[11] See F.A. Longstaff and E.S. Schwartz, "Valuing American Options by Simulation: A Simple Least Squares Approach," *Review of Financial Studies*, 14, 1 (2001): 113–47.

[12] L. Andersen, "A Simple Approach to the Pricing of Bermudan Swaptions in the Multifactor LIBOR Market Model," *Journal of Computational Finance*, 3, 2 (Winter 2000): 5–32.

Chapter 30. However, the accuracy of one-factor models for pricing Bermudan swap options has become a controversial issue.[13]

31.3 AGENCY MORTGAGE-BACKED SECURITIES

One application of the models presented in this chapter is to the agency mortgage-backed security (agency MBS) market in the United States.

An agency MBS is similar to the ABS considered in Section 23.7 except that payments are guaranteed by a government-related agency such as the Government National Mortgage Association (GNMA) or the Federal National Mortgage Association (FNMA) so that investors are protected against defaults. This makes an agency MBS sound like a regular fixed-income security issued by the government. In fact, there is a critical difference between an agency MBS and a regular fixed-income investment. This difference is that the mortgages in an agency MBS pool have prepayment privileges. These prepayment privileges can be quite valuable to the householder. In the United States, mortgages typically last for 25 years and can be prepaid at any time. This means that the householder has a 25-year American-style option to put the mortgage back to the lender at its face value.

Prepayments on mortgages occur for a variety of reasons. Sometimes interest rates fall and the owner of the house decides to refinance at a lower rate. On other occasions, a mortgage is prepaid simply because the house is being sold. A critical element in valuing an agency MBS is the determination of what is known as the *prepayment function*. This is a function describing expected prepayments on the underlying pool of mortgages at a time t in terms of the yield curve at time t and other relevant variables.

A prepayment function is very unreliable as a predictor of actual prepayment experience for an individual mortgage. When many similar mortgage loans are combined in the same pool, there is a "law of large numbers" effect at work and prepayments can be predicted more accurately from an analysis of historical data. As mentioned, prepayments are not always motivated by pure interest rate considerations. Nevertheless, there is a tendency for prepayments to be more likely when interest rates are low than when they are high. This means that investors require a higher rate of interest on an agency MBS than on other fixed-income securities to compensate for the prepayment options they have written.

Collateralized Mortgage Obligations

The simplest type of agency MBS is referred to as a *pass-through*. All investors receive the same return and bear the same prepayment risk. Not all mortgage-backed securities work in this way. In a *collateralized mortgage obligation* (CMO) the investors are divided into a number of classes and rules are developed for determining how principal repayments are channeled to different classes. A CMO creates classes of securities that bear different amounts of prepayment risk in the same way that an ABS creates classes of securities bearing different amounts of credit risk (see Section 23.7).

[13] For opposing viewpoints, see "Factor Dependence of Bermudan Swaptions: Fact or Fiction," by L. Andersen and J. Andreasen, and "Throwing Away a Billion Dollars: The Cost of Suboptimal Exercise Strategies in the Swaption Market," by F. A. Longstaff, P. Santa-Clara, and E. S. Schwartz. Both articles are in *Journal of Financial Economics*, 62, 1 (October 2001).

Business Snapshot 31.1 IOs and POs

In what is known as a *stripped MBS*, principal payments are separated from interest payments. All principal payments are channeled to one class of security, known as a *principal only* (PO). All interest payments are channeled to another class of security known as an *interest only* (IO). Both IOs and POs are risky investments. As prepayment rates increase, a PO becomes more valuable and an IO becomes less valuable. As prepayment rates decrease, the reverse happens. In a PO, a fixed amount of principal is returned to the investor, but the timing is uncertain. A high rate of prepayments on the underlying pool leads to the principal being received early (which is, of course, good news for the holder of the PO). A low rate of prepayments on the underlying pool delays the return of the principal and reduces the yield provided by the PO. In the case of an IO, the total of the cash flows received by the investor is uncertain. The higher the rate of prepayments, the lower the total cash flows received by the investor, and vice versa.

As an example of a CMO, consider an agency MBS where investors are divided into three classes: class A, class B, and class C. All the principal repayments (both those that are scheduled and those that are prepayments) are channeled to class A investors until investors in this class have been completely paid off. Principal repayments are then channeled to class B investors until these investors have been completely paid off. Finally, principal repayments are channeled to class C investors. In this situation, class A investors bear the most prepayment risk. The class A securities can be expected to last for a shorter time than the class B securities, and these, in turn, can be expected to last less long than the class C securities.

The objective of this type of structure is to create classes of securities that are more attractive to institutional investors than those created by a simpler pass-through MBS. The prepayment risks assumed by the different classes depend on the par value in each class. For example, class C bears very little prepayment risk if the par values in classes A, B, and C are 400, 300, and 100, respectively. Class C bears rather more prepayment risk in the situation where the par values in the classes are 100, 200, and 500.

The creators of mortgage-backed securities have created many more exotic structures than the one we have just described. Business Snapshot 31.1 gives an example.

Valuing Agency Mortgage-Backed Securities

Agency MBSs are usually valued using Monte Carlo simulation. Either the HJM or LIBOR market models can be used to simulate the behavior of interest rates month by month throughout the life of an agency MBS. Consider what happens on one simulation trial. Each month, expected prepayments are calculated from the current yield curve and the history of yield curve movements. These prepayments determine the expected cash flows to the holder of the agency MBS and the cash flows are discounted to time zero to obtain a sample value for the agency MBS. An estimate of the value of the agency MBS is the average of the sample values over many simulation trials.

Option-Adjusted Spread

In addition to calculating theoretical prices for mortgage-backed securities and other bonds with embedded options, traders also like to compute what is known as the *option-adjusted spread* (OAS). This is a measure of the spread over the yields on government Treasury bonds provided by the instrument when all options have been taken into account.

An input to any term structure model is the initial zero-coupon yield curve. Usually this is the LIBOR zero curve. However, to calculate an OAS for an instrument, it is first priced using the zero-coupon government Treasury curve. The price of the instrument given by the model is compared to the price in the market. A series of iterations is then used to determine the parallel shift to the input Treasury curve that causes the model price to be equal to the market price. This parallel shift is the OAS.

To illustrate the nature of the calculations, suppose that the market price is \$102.00 and that the price calculated using the Treasury curve is \$103.27. As a first trial we might choose to try a 60-basis-point parallel shift to the Treasury zero curve. Suppose that this gives a price of \$101.20 for the instrument. This is less than the market price of \$102.00 and means that a parallel shift somewhere between 0 and 60 basis points will lead to the model price being equal to the market price. We could use linear interpolation to calculate

$$60 \times \frac{103.27 - 102.00}{103.27 - 101.20} = 36.81$$

or 36.81 basis points as the next trial shift. Suppose that this gives a price of \$101.95. This indicates that the OAS is slightly less than 36.81 basis points. Linear interpolation suggests that the next trial shift be

$$36.81 \times \frac{103.27 - 102.00}{103.27 - 101.95} = 35.41$$

or 35.41 basis points; and so on.

SUMMARY

The HJM and LMM models provide approaches to valuing interest rate derivatives that give the user complete freedom in choosing the volatility term structure. The LMM model has two key advantages over the HJM model. First, it is developed in terms of the forward rates that determine the pricing of caps, rather than in terms of instantaneous forward rates. Second, it is relatively easy to calibrate to the price of caps or European swap options. The HJM and LMM models both have the disadvantage that they cannot be represented as recombining trees. In practice, this means that they must be implemented using Monte Carlo simulation.

The agency mortgage-backed security market in the United States has given birth to many exotic interest rate derivatives: CMOs, IOs, POs, and so on. These instruments provide cash flows to the holder that depend on the prepayments on a pool of mortgages. These prepayments depend on, among other things, the level of interest rates. Because they are heavily path dependent, agency mortgage-backed securities usually have to be valued using Monte Carlo simulation. These are, therefore, ideal candidates for applications of the HJM and LMM models.

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Questions and Problems (Answers in Solutions Manual)

31.1. Explain the difference between a Markov and a non-Markov model of the short rate.

31.2. Prove the relationship between the drift and volatility of the forward rate for the multifactor version of HJM in equation (31.6).

31.3. "When the forward rate volatility $s(t, T)$ in HJM is constant, the Ho–Lee model results." Verify that this is true by showing that HJM gives a process for bond prices that is consistent with the Ho–Lee model in Chapter 30.

31.4. "When the forward rate volatility, $s(t, T)$, in HJM is $\sigma e^{-a(T-t)}$, the Hull–White model results." Verify that this is true by showing that HJM gives a process for bond prices that is consistent with the Hull–White model in Chapter 30.

31.5. What is the advantage of LMM over HJM?

31.6. Provide an intuitive explanation of why a ratchet cap increases in value as the number of factors increase.

31.7. Show that equation (31.10) reduces to (31.4) as the δ_i tend to zero.

31.8. Explain why a sticky cap is more expensive than a similar ratchet cap.

31.9. Explain why IOs and POs have opposite sensitivities to the rate of prepayments.

31.10. "An option adjusted spread is analogous to the yield on a bond." Explain this statement.

31.11. Prove equation (31.15).

31.12. Prove the formula for the variance $V(T)$ of the swap rate in equation (31.17).

31.13. Prove equation (31.19).

Assignment Questions

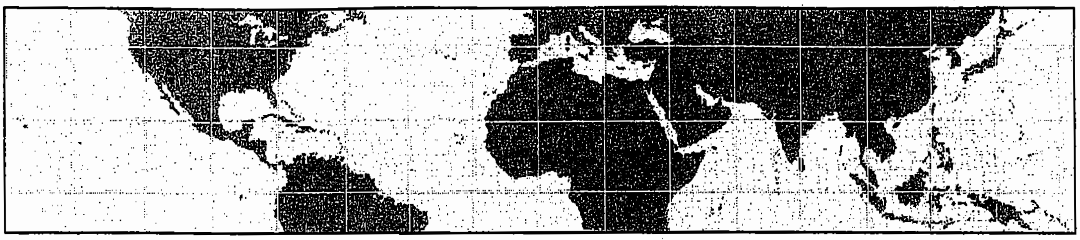
31.14. In an annual-pay cap, the Black volatilities for caplets with maturities 1, 2, 3, and 5 years are 18%, 20%, 22%, and 20%, respectively. Estimate the volatility of a 1-year forward rate in the LIBOR Market Model when the time to maturity is (a) 0 to 1 year, (b) 1 to 2 years, (c) 2 to 3 years, and (d) 3 to 5 years. Assume that the zero curve is flat at 5% per annum (annually compounded). Use DerivaGem to estimate flat volatilities for 2-, 3-, 4-, 5-, and 6-year caps.

31.15. In the flexi cap considered in Section 31.2 the holder is obligated to exercise the first N in-the-money caplets. After that no further caplets can be exercised. (In the example, $N = 5$.) Two other ways that flexi caps are sometimes defined are:

(a) The holder can choose whether any caplet is exercised, but there is a limit of N on the total number of caplets that can be exercised.

(b) Once the holder chooses to exercise a caplet all subsequent in-the-money caplets must be exercised up to a maximum of N.

Discuss the problems in valuing these types of flexi caps. Of the three types of flexi caps, which would you expect to be most expensive? Which would you expect to be least expensive?



CHAPTER 32

Swaps Revisited

Swaps have been central to the success of over-the-counter derivatives markets. They have proved to be very flexible instruments for managing risk. Based on the range of different contracts that now trade and the total volume of business transacted each year, swaps are arguably one of the most successful innovations in financial markets ever.

Chapter 7 discussed how plain vanilla interest rate swaps can be valued. The standard approach can be summarized as: "Assume forward rates will be realized." The steps are as follows:

1. Calculate the swap's net cash flows on the assumption that LIBOR rates in the future equal the forward rates calculated from today's LIBOR/swap zero curve.
2. Set the value of the swap equal to the present value of the net cash flows using the LIBOR/swap zero curve for discounting.

This chapter describes a number of nonstandard swaps. Some can be valued using the "assume forward rates will be realized" approach; some require the application of the convexity, timing, and quanto adjustments we encountered in Chapters 29; some contain embedded options that must be valued using the procedures described in Chapters 28, 30, and 31.

32.1 VARIATIONS ON THE VANILLA DEAL

Many interest rate swaps involve relatively minor variations to the plain vanilla structure discussed in Chapter 7. In some swaps the notional principal changes with time in a predetermined way. Swaps where the notional principal is an increasing function of time are known as *step-up swaps*. Swaps where the notional principal is a decreasing function of time are known as *amortizing swaps*. Step-up swaps could be useful for a construction company that intends to borrow increasing amounts of money at floating rates to finance a particular project and wants to swap to fixed-rate funding. An amortizing swap could be used by a company that has fixed-rate borrowings with a certain prepayment schedule and wants to swap to borrowings at a floating rate.

Business Snapshot 32.1 Hypothetical Confirmation for Nonstandard Swap

Trade date:	5-January, 2007
Effective date:	11-January, 2007
Business day convention (all dates):	Following business day
Holiday calendar:	US
Termination date:	11-January, 2012
Fixed amounts	
Fixed-rate payer:	Microsoft
Fixed-rate notional principal:	USD 100 million
Fixed rate:	6% per annum
Fixed-rate day count convention:	Actual/365
Fixed-rate payment dates	Each 11-July and 11-January commencing 11-July, 2007, up to and including 11-January, 2012
Floating amounts	
Floating-rate payer	Goldman Sachs
Floating-rate notional principal	USD 120 million
Floating rate	USD 1-month LIBOR
Floating-rate day count convention	Actual/360
Floating-rate payment dates	11-July, 2007, and the 11th of each month thereafter up to and including 11-January, 2012

The principal can be different on the two sides of a swap. Also the frequency of payments can be different. Business Snapshot 32.1 illustrates this by showing a hypothetical swap between Microsoft and Goldman Sachs where the notional principal is $120 million on the floating side and $100 million on fixed side. Payments are made every month on the floating side and every 6 months on the fixed side. These type of variations to the basic plain vanilla structure do not affect the valuation methodology. The "assume forward rates are realized" approach can still be used.

The floating reference rate for a swap is not always LIBOR. In some swaps for instance, it is the commercial paper (CP) rate. A *basis swap* involves exchanging cash flows calculated using one floating reference rate for cash flows calculated using another floating reference rate. An example would be a swap where the 3-month CP rate plus 10 basis points is exchanged for 3-month LIBOR with both being applied to a principal of $100 million. A basis swap could be used for risk management by a financial institution whose assets and liabilities are dependent on different floating reference rates.

Swaps where the floating reference rate is not LIBOR can be valued using the "assume forward rates are realized" approach. A zero curve other than LIBOR is necessary to calculate future cash flows on the assumption that forward rates are realized. The cash flows are discounted at LIBOR.

Business Snapshot 32.2 Hypothetical Confirmation for Compounding Swap

Trade date:	5-January, 2007
Effective date:	11-January, 2007
Holiday calendar:	US
Business day convention (all dates):	Following business day
Termination date:	11-January, 2012
Fixed amounts	
Fixed-rate payer:	Microsoft
Fixed-rate notional principal:	USD 100 million
Fixed rate:	6% per annum
Fixed-rate day count convention:	Actual/365
Fixed-rate payment date:	11-January, 2012
Fixed-rate compounding:	Applicable at 6.3%
Fixed-rate compounding dates	Each 11-July and 11-January commencing 11-July, 2007, up to and including 11-July, 2011
Floating amounts	
Floating-rate payer:	Goldman Sachs
Floating-rate notional principal:	USD 100 million
Floating rate:	USD 6-month LIBOR plus 20 basis points
Floating-rate day count convention:	Actual/360
Floating-rate payment date:	11-January, 2012
Floating-rate compounding:	Applicable at LIBOR plus 10 basis points
Floating-rate compounding dates:	Each 11-July and 11-January commencing 11-July, 2007, up to and including 11-July, 2011

32.2 COMPOUNDING SWAPS

Another variation on the plain vanilla swap is a *compounding swap*. A hypothetical confirmation for a compounding swap is in Business Snapshot 32.2. In this example there is only one payment date for both the floating-rate payments and the fixed-rate payments. This is at the end of the life of the swap. The floating rate of interest is LIBOR plus 20 basis points. Instead of being paid, the interest is compounded forward until the end of the life of the swap at a rate of LIBOR plus 10 basis points. The fixed rate of interest is 6%. Instead of being paid this interest is compounded forward at a fixed rate of interest of 6.3% until the end of the swap.

The "assume forward rates are realized" approach can be used for valuing a compounding swap such as that in Business Snapshot 32.2. It is straightforward to deal with the fixed side of the swap because the payment that will be made at maturity is known with certainty. The "assume forward rates are realized" approach for the floating side is justifiable because there exist a series of forward rate agreements (FRAs) where the floating-rate cash flows are exchanged for the values they would have if each floating rate equaled the corresponding forward rate.[1]

Example 32.1

A compounding swap with annual resets has a life of 3 years. A fixed rate is paid and a floating rate is received. The fixed interest rate is 4% and the floating interest rate is 12-month LIBOR. The fixed side compounds at 3.9% and the floating side compounds at 12-month LIBOR minus 20 basis points. The LIBOR zero curve is flat at 5% with annual compounding and the notional principal is \$100 million.

On the fixed side, interest of \$4 million is earned at the end of the first year. This compounds to $4 \times 1.039 = \$4.156$ million at the end of the second year. A second interest amount of \$4 million is added at the end of the second year bringing the total compounded forward amount to \$8.156 million. This compounds to $8.156 \times 1.039 = \$8.474$ million by the end of the third year when there is the third interest amount of \$4 million. The cash flow at the end of the third year on the fixed side of the swap is therefore \$12.474 million.

On the floating side we assume all future interest rates equal the corresponding forward LIBOR rates. Given the LIBOR zero curve, this means that all future interest rates are assumed to be 5% with annual compounding. The interest calculated at the end of the first year is \$5 million. Compounding this forward at 4.8% (forward LIBOR minus 20 basis points) gives $5 \times 1.048 = \$5.24$ million at the end of the second year. Adding in the interest, the compounded forward amount is \$10.24 million. Compounding forward to the end of the third year, we get $10.24 \times 1.048 = \$10.731$ million. Adding in the final interest gives \$15.731 million.

The swap can be valued by assuming that it leads to an inflow of \$15.731 million and an outflow of \$12.474 million at the end of year 3. The value of the swap is therefore

$$\frac{15.731 - 12.474}{1.05^3} = 2.814$$

or \$2.814 million. (This analysis ignores day count issues.)

32.3 CURRENCY SWAPS

Currency swaps were introduced in Chapter 7. These enable an interest rate exposure in one currency to be swapped for an interest rate exposure in another currency. Usually two principals are specified, one in each currency. The principals are exchanged at both the beginning and the end of the life of the swap as described in Section 7.8.

Suppose that the currencies involved in a currency swap are US dollars (USD) and British pounds (GBP). In a fixed-for-fixed currency swap, a fixed rate of interest is specified in each currency. The payments on one side are determined by applying the fixed rate of interest in USD to the USD principal; the payments on the other side are determined by applying the fixed rate of interest in GBP to the GBP principal. Section 7.9 discussed the valuation of this type of swap.

Another popular type of currency swap is floating-for-floating. In this, the payments

[1] See Technical Note 18 on the author's website for the details.

on one side are determined by applying USD LIBOR (possibly with a spread added) to the USD principal; similarly the payments on the other side are determined by applying GBP LIBOR (possibly with a spread added) to the GBP principal. A third type of swap is a cross-currency interest rate swap where a floating rate in one currency is exchanged for a fixed rate in another currency.

Floating-for-floating and cross-currency interest rate swaps can be valued using the "assume forward rates are realized" rule. Future LIBOR rates in each currency are assumed to equal today's forward rates. This enables the cash flows in the currencies to be determined. The USD cash flows are discounted at the USD LIBOR zero rate. The GBP cash flows are discounted at the GBP LIBOR zero rate. The current exchange rate is then used to translate the two present values to a common currency.

An adjustment to this procedure is sometimes made to reflect the realities of the market. In theory, a new floating-for-floating swap should involve exchanging LIBOR in one currency for LIBOR in another currency (with no spreads added). In practice, macroeconomic effects give rise to spreads. Financial institutions often adjust the discount rates they use to allow for this. As an example, suppose that market conditions are such that USD LIBOR is exchanged for Japanese yen (JPY) LIBOR minus 20 basis points in new floating-for-floating swaps of all maturities. In its valuations a US financial institution would discount USD cash flows at USD LIBOR and it would discount JPY cash flows at JPY LIBOR minus 20 basis points.[2] It would do this in all swaps that involved both JPY and USD cash flows.

32.4 MORE COMPLEX SWAPS

We now move on to consider some examples of swaps where the simple rule "assume forward rates will be realized" does not work. In each case, it is assumed that an adjusted forward rate, rather than the actual forward rate, is realized.

LIBOR-in-Arrears Swap

A plain vanilla interest rate swap is designed so that the floating rate of interest observed on one payment date is paid on the next payment date. An alternative instrument that is sometimes traded is a *LIBOR-in-arrears swap*. In this, the floating rate paid on a payment date equals the rate observed on the payment date itself.

Suppose that the reset dates in the swap are t_i for $i = 0, 1, \dots, n$, with $\tau_i = t_{i+1} - t_i$. Define R_i as the LIBOR rate for the period between t_i and t_{i+1}, F_i as the forward value of R_i, and σ_i as the volatility of this forward rate. (The value of σ_i is typically implied from caplet prices.) In a LIBOR-in-arrears swap the payment on the floating side at time t_i is based on R_i rather than R_{i-1}. As explained in Section 29.1, it is necessary to make a convexity adjustment to the forward rate when the payment is valued. The valuation should be based on the assumption that the floating rate paid is

$$F_i + \frac{F_i^2 \sigma_i^2 \tau_i t_i}{1 + F_i \tau_i} \tag{32.1}$$

and not F_i.

[2] This adjustment is *ad hoc*, but, if it is not made, traders make an immediate profit or loss every time they trade a new JPY/USD floating-for-floating swap.

Example 32.2

In a LIBOR-in-arrears swap, the principal is \$100 million. A fixed rate of 5% is received annually and LIBOR is paid. Payments are exchanged at the ends of years 1, 2, 3, 4, and 5. The yield curve is flat at 5% per annum (measured with annual compounding). All caplet volatilities are 22% per annum.

The forward rate for each floating payment is 5%. If this were a regular swap rather than an in-arrears swap, its value would (ignoring day count conventions, etc.) be exactly zero. Because it is an in-arrears swap, convexity adjustments must be made. In equation (32.1), $F_i = 0.05$, $\sigma_i = 0.22$, and $\tau_i = 1$ for all i. The convexity adjustment changes the rate assumed at time t_i from 0.05 to

$$0.05 + \frac{0.05^2 \times 0.22^2 \times 1 \times t_i}{1 + 0.05 \times 1} = 0.05 + 0.000115 t_i$$

The floating rates for the payments at the ends of years 1, 2, 3, 4, and 5 should therefore be assumed to be 5.0115%, 5.0230%, 5.0345%, 5.0460%, and 5.0575%, respectively. The net exchange on the first payment date is equivalent to a cash outflow of 0.0115% of \$100 million or \$11,500. Equivalent net cash flows for other exchanges are calculated similarly. The value of the swap is

$$-\frac{11{,}500}{1.05} - \frac{23{,}000}{1.05^2} - \frac{34{,}500}{1.05^3} - \frac{46{,}000}{1.05^4} - \frac{57{,}500}{1.05^5}$$

or −\$144,514.

CMS and CMT Swaps

A constant maturity swap (CMS) is an interest rate swap where the floating rate equals the swap rate for a swap with a certain life. For example, the floating payments on a CMS swap might be made every 6 months at a rate equal to the 5-year swap rate. Usually there is a lag so that the payment on a particular payment date is equal to the swap rate observed on the previous payment date. Suppose that rates are set at times $t_0, t_1, t_2, \dots$, payments are made at times $t_1, t_2, t_3, \dots$, and L is the notional principal. The floating payment at time t_{i+1} is

$$\tau_i L S_i$$

where $\tau_i = t_{i+1} - t_i$ and S_i is the swap rate at time t_i.

Suppose that y_i is the forward value of the swap rate S_i. To value the payment at time t_{i+1}, it turns out to be correct to make a convexity adjustment to the forward swap rate, so that the realized swap rate is assumed to be

$$y_i - \tfrac{1}{2} y_i^2 \sigma_{y,i}^2 t_i \frac{G_i''(y_i)}{G_i'(y_i)} - \frac{y_i \tau_i F_i \rho_i \sigma_{y,i} \sigma_{F,i} t_i}{1 + F_i \tau_i} \tag{32.2}$$

rather that y_i. In this equation, $\sigma_{y,i}$ is the volatility of the forward swap rate, F_i is the current forward interest rate between times t_i and t_{i+1}, $\sigma_{F,i}$ is the volatility of this forward rate, and ρ_i is the correlation between the forward swap rate and the forward interest rate. $G_i(x)$ is the price at time t_i of a bond as a function of its yield x. The bond pays coupons at rate y_i and has the same life and payment frequency as the swap from which the CMS rate is calculated. $G_i'(x)$ and $G_i''(x)$ are the first and second partial derivatives of G_i with respect to x. The volatilities $\sigma_{y,i}$ can be implied from swaptions;

the volatilities $\sigma_{F,i}$ can be implied from caplet prices; the correlation ρ_i can be estimated from historical data.

Equation (32.2) involves a convexity and a timing adjustment. The term

$$-\tfrac{1}{2}y_i^2\sigma_{y,i}^2 t_i \frac{G_i''(y_i)}{G_i'(y_i)}$$

is an adjustment similar the one in Example 29.2 of Section 29.1. It is based on the assumption that the swap rate S_i leads to only one payment at time t_i rather than to an annuity of payments. The term

$$-\frac{y_i\tau_i F_i\rho_i\sigma_{y,i}\sigma_{F,i}t_i}{1+F_i\tau_i}$$

is similar to the one in Section 29.2 and is an adjustment for the fact that the payment calculated from S_i is made at time t_{i+1} rather than t_i.

Example 32.3

In a 6-year CMS swap, the 5-year swap rate is received and a fixed rate of 5% is paid on a notional principal of \$100 million. The exchange of payments is semiannual (both on the underlying 5-year swap and on the CMS swap itself). The exchange on a payment date is determined from the swap rate on the previous payment date. The term structure is flat at 5% per annum with semiannual compounding. All options on five-year swaps have a 15% implied volatility and all caplets with a 6-month tenor have a 20% implied volatility. The correlation between each cap rate and each swap rate is 0.7.

In this case, $y_i = 0.05$, $\sigma_{y,i} = 0.15$, $\tau_i = 0.5$, $F_i = 0.05$, $\sigma_{F,i} = 0.20$, and $\rho_i = 0.7$ for all i. Also,

$$G_i(x) = \sum_{i=1}^{10}\frac{2.5}{(1+x/2)^i} + \frac{100}{(1+x/2)^{10}}$$

so that $G_i'(y_i) = -437.603$ and $G_i''(y_i) = 2261.23$. Equation (32.2) gives the total convexity/timing adjustment as $0.0001197t_i$, or 1.197 basis points per year until the swap rate is observed. For example, for the purposes of valuing the CMS swap, the 5-year swap rate in 4 years' time should be assumed to be 5.0479% rather than 5% and the net cash flow received at the 4.5-year point should be assumed to be $0.5 \times 0.000479 \times 100{,}000{,}000 = \$23{,}940$. Other net cash flows are calculated similarly. Taking their present value, we find the value of the swap to be \$159,811.

A constant maturity Treasury swap (CMT swap) works similarly to a CMS swap except that the floating rate is the yield on a Treasury bond with a specified life. The analysis of a CMT swap is essentially the same as that for a CMS swap with S_i defined as the par yield on a Treasury bond with the specified life.

Differential Swaps

A *differential swap*, sometimes referred to as a *diff swap*, is an interest rate swap where a floating interest rate is observed in one currency and applied to a principal in another currency. Suppose that the LIBOR rate for the period between t_i and t_{i+1} in currency Y is applied to a principal in currency X with the payment taking place at time t_{i+1}. Define V_i as the forward interest rate between t_i and t_{i+1} in currency Y and W_i as the forward exchange rate for a contract with maturity t_{i+1} (expressed as the number of units of currency Y that equal one unit of currency X). If the LIBOR rate in currency Y were applied to a principal in currency Y, the cash flow at time t_{i+1} would be valued on the assumption that the LIBOR rate at time t_i equals V_i. From the analysis in Section 29.3, a quanto adjustment is necessary when it is applied to a principal in currency X. It is correct to value the cash flow on the assumption that the LIBOR rate equals

$$V_i + V_i\rho_i\sigma_{W,i}\sigma_{V,i}t_i \qquad (32.3)$$

where $\sigma_{V,i}$ is the volatility of V_i, $\sigma_{W,i}$ is the volatility of W_i, and ρ_i is the correlation between V_i and W_i.

Example 32.4

Zero rates in both the US and Britain are flat at 5% per annum with annual compounding. In a 3-year diff swap agreement with annual payments, USD 12-month LIBOR is received and sterling 12-month LIBOR is paid with both being applied to a principal of 10 million pounds sterling. The volatility of all 1-year forward rates in the US is estimated to be 20%, the volatility of the forward USD/sterling exchange rate (dollars per pound) is 12% for all maturities, and the correlation between the two is 0.4.

In this case, $V_i = 0.05$, $\rho_i = 0.4$, $\sigma_{W,i} = 0.12$, $\sigma_{V,i} = 0.2$. The floating-rate cash flows dependent on the 1-year USD rate observed at time t_i should therefore be calculated on the assumption that the rate will be

$$0.05 + 0.05 \times 0.4 \times 0.12 \times 0.2 \times t_i = 0.05 + 0.00048t_i$$

This means that the net cash flows from the swap at times 1, 2, and 3 years should be assumed to be 0, 4,800, and 9,600 pounds sterling for the purposes of valuation. The value of the swap is therefore

$$\frac{0}{1.05} + \frac{4,800}{1.05^2} + \frac{9,600}{1.05^3} = 12,647$$

or 12,647 pounds sterling.

32.5 EQUITY SWAPS

In an equity swap, one party promises to pay the return on an equity index on a notional principal, while the other promises to pay a fixed or floating return on a notional principal. Equity swaps enable a fund managers to increase or reduce their exposure to an index without buying and selling stock. An equity swap is a convenient way of packaging a series of forward contracts on an index to meet the needs of the market.

The equity index is usually a total return index where dividends are reinvested in the stocks comprising the index. An example of an equity swap is in Business Snapshot 32.3. In this, the 6-month return on the S&P 500 is exchanged for LIBOR. The principal on either side of the swap is \$100 million and payments are made every 6 months.

For an equity-for-floating swap such as that in Business Snapshot 32.3 the value at the start of its life is zero. This is because a financial institution can arrange to costlessly

Business Snapshot 32.3 Hypothetical Confirmation for an Equity Swap

Trade date:	5-January, 2007
Effective date:	11-January, 2007
Business day convention (all dates):	Following business day
Holiday calendar:	US
Termination date:	11-January, 2012
Equity amounts	
Equity payer:	Microsoft
Equity principal	USD 100 million
Equity index:	Total Return S&P 500 index
Equity payment:	$100(I_1 - I_0)/I_0$, where I_1 is the index level on the payment date and I_0 is the index level on the immediately preceding payment date. In the case of the first payment date, I_0 is the index level on 11-January, 2007
Equity payment dates:	Each 11-July and 11-January commencing 11-July, 2007, up to and including 11-January, 2012
Floating amounts	
Floating-rate payer:	Goldman Sachs
Floating-rate notional principal:	USD 100 million
Floating rate:	USD 6-month LIBOR
Floating-rate day count convention:	Actual/360
Floating-rate payment dates:	Each 11-July and 11-January commencing 11-July, 2007, up to and including 11-January, 2012

replicate the cash flows to one side by borrowing the principal on each payment date at LIBOR and investing it in the index until the next payment date with any dividends being reinvested. A similar argument shows that the swap is always worth zero immediately after a payment date.

Between payment dates the equity cash flow and the LIBOR cash flow at the next payment date must be valued. The LIBOR cash flow was fixed at the last reset date and so can be valued easily. The value of the equity cash flow is LE/E_0, where L is the principal, E is the current value of the equity index, and E_0 is its value at the last payment date.[3]

32.6 SWAPS WITH EMBEDDED OPTIONS

Some swaps contain embedded options. In this section we consider some commonly encountered examples.

[3] See Technical Note 19 on the author's website for a more detailed discussion of this.

Accrual Swaps

Accrual swaps are swaps where the interest on one side accrues only when the floating reference rate is within a certain range. Sometimes the range remains fixed during the entire life of the swap; sometimes it is reset periodically.

As a simple example of an accrual swap, consider a deal where a fixed rate Q is exchanged for 3-month LIBOR every quarter and the fixed rate accrues only on days when 3-month LIBOR is below 8% per annum. Suppose that the principal is L. In a normal swap the fixed-rate payer would pay QLn_1/n_2 on each payment date where n_1 is the number of days in the preceding quarter and n_2 is the number of days in the year. (This assumes that the day count is actual/actual.) In an accrual swap, this is changed to QLn_3/n_2, where n_3 is the number of days in the preceding quarter that the 3-month LIBOR was below 8%. The fixed-rate payer saves QL/n_2 on each day when 3-month LIBOR is above 8%.[4] The fixed-rate payer's position can therefore be considered equivalent to a regular swap plus a series of binary options, one for each day of the life of the swap. The binary options pay off QL/n_2 when the 3-month LIBOR is above 8%.

To generalize, suppose that the LIBOR cutoff rate (8% in the case just considered) is R_K and that payments are exchanged every τ years. Consider day i during the life of the swap and suppose that t_i is the time until day i. Suppose that the τ-year LIBOR rate on day i is R_i so that interest accrues when $R_i < R_K$. Define F_i as the forward value of R_i and σ_i as the volatility of F_i. (The latter is estimated from spot caplet volatilities.) Using the usual lognormal assumption, the probability that LIBOR is greater than R_K in a world that is forward risk neutral with respect to a zero-coupon bond maturing at time $t_i + \tau$ is $N(d_2)$, where

$$d_2 = \frac{\ln(F_i/R_K) - \sigma_i^2 t_i/2}{\sigma_i\sqrt{t_i}}$$

The payoff from the binary option is realized at the swap payment date following day i. Suppose that this is at time s_i. The probability that LIBOR is greater than R_K in a world that is forward risk neutral with respect to a zero-coupon bond maturing at time s_i is given by $N(d_2^*)$, where d_2^* is calculated using the same formula as d_2, but with a small timing adjustment to F_i reflecting the difference between time $t_i + \tau$ and time s_i.

The value of the binary option corresponding to day i is

$$\frac{QL}{n_2} P(0, s_i) N(d_2^*)$$

The total value of the binary options is obtained by summing this expression for every day in the life of the swap. The timing adjustment (causing d_2 to be replaced by d_2^*) is so small that, in practice, it is frequently ignored.

Cancelable Swap

A cancelable swap is a plain vanilla interest rate swap where one side has the option to terminate on one or more payment dates. Terminating a swap is the same as entering

[4] The usual convention is that, if a day is a holiday, the applicable rate is assumed to be the rate on the immediately preceding business day.

into the offsetting (opposite) swap. Consider a swap between Microsoft and Goldman Sachs. If Microsoft has the option to cancel, it can regard the swap as a regular swap plus a long position in an option to enter into the offsetting swap. If Goldman Sachs has the cancelation option, Microsoft has a regular swap plus a short position in an option to enter into the swap.

If there is only one termination date, a cancelable swap is the same as a regular swap plus a position in a European swaption. Consider, for example, a 10-year swap where Microsoft will receive 6% and pay LIBOR. Suppose that Microsoft has the option to terminate at the end of 6 years. The swap is a regular 10-year swap to receive 6% and pay LIBOR plus long position in a 6-year European option to enter into a 4-year swap where 6% is paid and LIBOR is received. (The latter is referred to as a 6×4 European swaption.) The standard market model for valuing European swaptions is described in Chapter 28.

When the swap can be terminated on a number of different payment dates, it is a regular swap plus a Bermudan-style swaption. Consider, for example, the situation where Microsoft has entered into a 5-year swap with semiannual payments where 6% is received and LIBOR is paid. Suppose that the counterparty has the option to terminate the swap on payment dates between year 2 and year 5. The swap is a regular swap plus a short position in a Bermudan-style swaption, where the Bermudan-style swaption is an option to enter into a swap that matures in 5 years and involves a fixed payment at 6% being received and a floating payment at LIBOR being paid. The swaption can be exercised on any payment date between year 2 and year 5. Methods for valuing Bermudan swaptions are discussed in Chapters 30 and 31.

Cancelable Compounding Swaps

Sometimes compounding swaps can be terminated on specified payment dates. On termination, the floating-rate payer pays the compounded value of the floating amounts up to the time of termination and the fixed-rate payer pays the compounded value of the fixed payments up to the time of termination.

Some tricks can be used to value cancelable compounding swaps. Suppose first that the floating rate is LIBOR and it is compounded at LIBOR. Assume that the principal amount of the swap is paid on both the fixed and floating sides of the swap at the end of its life. This is similar to moving from Table 7.1 to Table 7.2 for a vanilla swap. It does not change the value of the swap and has the effect of ensuring that the value of the floating side is always equals the notional principal on a payment date. To make the cancelation decision, we need only look at the fixed side. We construct an interest rate tree as outlined in Chapter 30. We roll back through the tree in the usual way valuing the fixed side. At each node where the swap can be canceled, we test whether it is optimal to keep the swap or cancel it. Canceling the swap in effect sets the fixed side equal to par. If we are paying fixed and receiving floating, our objective is to minimize the value of the fixed side; if we are receiving fixed and paying floating, our objective is to maximize the value of the fixed side.

When the floating side is LIBOR plus a spread compounded at LIBOR, the cash flows corresponding to the spread rate of interest can be subtracted from the fixed side instead of adding them to the floating side. The option can then be valued as in the case where there is no spread.

When the compounding is at LIBOR plus a spread, an approximate approach is as follows:[5]

1. Calculate the value of the floating side of the swap at each cancelation date assuming forward rates are realized.
2. Calculate the value of the floating side of the swap at each cancelation date assuming that the floating rate is LIBOR and it is compounded at LIBOR.
3. Define the excess of step 1 over step 2 as the "value of spreads" on a cancelation date.
4. Treat the option in the way described above. In deciding whether to exercise the cancelation option, subtract the value of the spreads from the values calculated for the fixed side.

32.7 OTHER SWAPS

This chapter has discussed just a few of the swap structures in the market. In practice, the range of different contracts that trade is limited only by the imagination of financial engineers and the appetite of corporate treasurers for innovative risk management tools.

A swap that was very popular in the United States in the mid-1990s is an *index amortizing rate swap* (also called an *indexed principal swap*). In this, the principal reduces in a way dependent on the level of interest rates. The lower the interest rate, the greater the reduction in the principal. The fixed side of an indexed amortizing swap was originally designed to mirror, at least approximately, the return obtained by an investor on a mortgage-backed security after prepayment options are taken into account. The swap therefore exchanged the return on a mortgage-backed security for a floating-rate return.

Commodity swaps are now becoming increasingly popular. A company that consumes 100,000 barrels of oil per year could agree to pay \$8 million each year for the next 10 years and to receive in return $100{,}000S$, where S is the market price of oil per barrel. The agreement would in effect lock in the company's oil cost at \$80 per barrel. An oil producer might agree to the opposite exchange, thereby locking in the price it realized for its oil at \$80 per barrel. Energy derivatives such as this were discussed in Chapter 25.

A number of other types of swaps are discussed elsewhere in this book. For example, asset swaps are discussed in Chapter 22, total return swaps and various types of credit default swaps are covered in Chapter 23, and volatility and variance swaps are analyzed in Chapter 24.

Bizarre Deals

Some swaps have payoffs that are calculated in quite bizarre ways. An example is a deal entered into between Procter and Gamble and Bankers Trust in 1993 (see Business Snapshot 32.4). The details of this transaction are in the public domain because it later became the subject of litigation.[6]

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[5] This approach is not perfectly accurate in that it assumes that the decision to exercise the cancelation option is not influenced by future payments being compounded at a rate different from LIBOR.

[6] See D. J. Smith, "Aggressive Corporate Finance: A Close Look at the Procter and Gamble–Bankers Trust Leveraged Swap," *Journal of Derivatives* 4, 4 (Summer 1997): 67–79.

Business Snapshot 32.4 Procter and Gamble's Bizarre Deal

A particularly bizarre swap is the so-called "5/30" swap entered into between Bankers Trust (BT) and Procter and Gamble (P&G) on November 2, 1993. This was a 5-year swap with semiannual payments. The notional principal was $200 million. BT paid P&G 5.30% per annum. P&G paid BT the average 30-day CP (commercial paper) rate minus 75 basis points plus a spread. The average commercial paper rate was calculated by taking observations on the 30-day commercial paper rate each day during the preceding accrual period and averaging them.

The spread was zero for the first payment date (May 2, 1994). For the remaining nine payment dates, it was

$$\max\left[0, \frac{98.5\left(\dfrac{\text{5-year CMT\%}}{5.78\%}\right) - (\text{30-year TSY price})}{100}\right]$$

In this, 5-year CMT is the constant maturity Treasury yield (i.e., the yield on a 5-year Treasury note, as reported by the US Federal Reserve). The 30-year TSY price is the midpoint of the bid and offer cash bond prices for the 6.25% Treasury bond maturing on August 2023. Note that the spread calculated from the formula is a decimal interest rate. It is not measured in basis points. If the formula gives 0.1 and the CP rate is 6%, the rate paid by P&G is 15.25%.

P&G were hoping that the spread would be zero and the deal would enable it to exchange fixed-rate funding at 5.30% for funding at 75 basis points less than the commercial paper rate. In fact, interest rates rose sharply in early 1994, bond prices fell, and the swap proved very, very expensive (see Problem 32.10).

SUMMARY

Swaps have proved to be very versatile financial instruments. Many swaps can be valued by (a) assuming that LIBOR (or some other floating reference rate) will equal its forward value and (b) discounting the resulting cash flows at the LIBOR/swap rate. These include plain vanilla interest swaps, most types of currency swaps, swaps where the principal changes in a predetermined way, swaps where the payment dates are different on each side, and compounding swaps.

Some swaps require adjustments to the forward rates when they are valued. These adjustments are termed convexity, timing, or quanto adjustments. Among the swaps that require adjustments are LIBOR-in-arrears swaps, CMS/CMT swaps, and differential swaps.

Equity swaps involve the return on an equity index being exchanged for a fixed or floating rate of interest. They are usually designed so that they are worth zero immediately after a payment date, but they may have nonzero values between payment dates.

Some swaps involve embedded options. An accrual swap is a regular swap plus a large portfolio of binary options (one for each day of the life of the swap). A cancelable swap is a regular swap plus a Bermudan swaption.

FURTHER READING

Chance, D., and Rich, D., "The Pricing of Equity Swap and Swaptions," *Journal of Derivatives* 5, 4 (Summer 1998): 19–31.

Smith D. J., "Aggressive Corporate Finance: A Close Look at the Procter and Gamble–Bankers Trust Leveraged Swap," *Journal of Derivatives*, 4, 4 (Summer 1997): 67–79.

Questions and Problems (Answers in Solutions Manual)

32.1. Calculate all the fixed cash flows and their exact timing for the swap in Business Snapshot 32.1. Assume that the day count conventions are applied using target payment dates rather than actual payment dates.

32.2. Suppose that a swap specifies that a fixed rate is exchanged for twice the LIBOR rate. Can the swap be valued using the "assume forward rates are realized" rule?

32.3. What is the value of a 2-year fixed-for-floating compound swap where the principal is $100 million and payments are made semiannually. Fixed interest is received and floating is paid? The fixed rate is 8% and it is compounded at 8.3% (both semiannually compounded). The floating rate is LIBOR plus 10 basis points and it is compounded at LIBOR plus 20 basis points. The LIBOR zero curve is flat at 8% with semiannual compounding.

32.4. What is the value of a 5-year swap where LIBOR is paid in the usual way and in return LIBOR compounded at LIBOR is received on the other side? The principal on both sides is $100 million. Payment dates on the pay side and compounding dates on the receive side are every 6 months and the yield curve is flat at 5% with semiannual compounding.

32.5. Explain carefully why a bank might choose to discount cash flows on a currency swap at a rate slightly different from LIBOR.

32.6. Calculate the total convexity/timing adjustment in Example 32.3 of Section 32.4 if all cap volatilities are 18% instead of 20% and volatilities for all options on 5-year swaps are 13% instead of 15%. What should the 5-year swap rate in 3 years' time be assumed for the purpose of valuing the swap? What is the value of the swap?

32.7. Explain why a plain vanilla interest rate swap and the compounding swap in Section 32.2 can be valued using the "assume forward rates are realized" rule, but a LIBOR-in-arrears swap in Section 32.4 cannot.

32.8. In the accrual swap discussed in the text, the fixed side accrues only when the floating reference rate lies below a certain level. Discuss how the analysis can be extended to cope with a situation where the fixed side accrues only when the floating reference rate is above one level and below another.

Assignment Questions

32.9. LIBOR zero rates are flat at 5% in the United States and flat at 10% in Australia (both annually compounded). In a 4-year swap Australian LIBOR is received and 9% is paid with both being applied to a USD principal of $10 million. Payments are exchanged annually. The volatility of all 1-year forward rates in Australia is estimated to be 25%, the volatility of the forward USD/AUD exchange rate (AUD per USD) is 15% for all maturities, and the correlation between the two is 0.3. What is the value of the swap?

32.10. Estimate the interest rate paid by P&G on the 5/30 swap in Section 32.7 if (a) the CP rate is 6.5% and the Treasury yield curve is flat at 6% and (b) the CP rate is 7.5% and the Treasury yield curve is flat at 7% with semiannual compounding.

32.11. Suppose that you are trading a LIBOR-in-arrears swap with an unsophisticated counterparty who does not make convexity adjustments. To take advantage of the situation, should you be paying fixed or receiving fixed? How should you try to structure the swap as far as its life and payment frequencies?

Consider the situation where the yield curve is flat at 10% per annum with annual compounding. All cap volatilities are 18%. Estimate the difference between the way a sophisticated trader and an unsophisticated trader would value a LIBOR-in-arrears swap where payments are made annually and the life of the swap is (a) 5 years, (b) 10 years, and (c) 20 years. Assume a notional principal of $1 million.

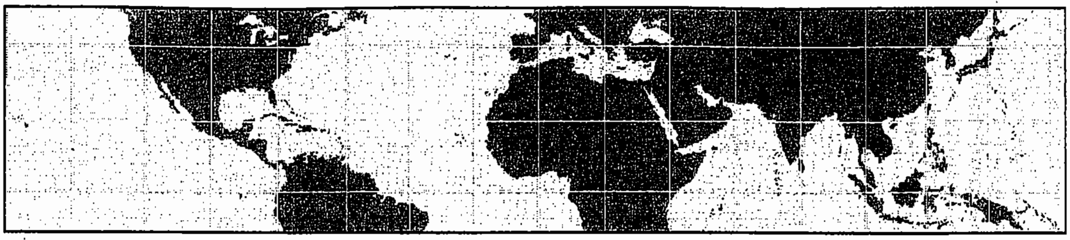
32.12. Suppose that the LIBOR zero rate is flat at 5% with annual compounding. In a 5-year swap, company X pays a fixed rate of 6% and receives LIBOR. The volatility of the 2-year swap rate in 3 years is 20%.

(a) What is the value of the swap?

(b) Use DerivaGem to calculate the value of the swap if company X has the option to cancel after 3 years.

(c) Use DerivaGem to calculate the value of the swap if the counterparty has the option to cancel after 3 years.

(d) What is the value of the swap if either side can cancel at the end of 3 years?



CHAPTER 33 Real Options

Up to now we have been almost entirely concerned with the valuation of financial assets. In this chapter we explore how the ideas we have developed can be extended to assess capital investment opportunities in real assets such as land, buildings, plant, and equipment. Often there are options embedded in these investment opportunities (the option to expand the investment, the option to abandon the investment, the option to defer the investment, and so on.) These options are very difficult to value using traditional capital investment appraisal techniques. The approach known as *real options* attempts to deal with this problem using option pricing theory.

The chapter starts by explaining the traditional approach to evaluating investments in real assets and shows how difficult it is to correctly value embedded options when this approach is used. It then explains how the risk-neutral valuation approach can be extended to handle the valuation of real assets and presents a number of examples illustrating the application of the approach in different situations.

33.1 CAPITAL INVESTMENT APPRAISAL

The traditional approach to valuing a potential capital investment project is known as the "net present value" (NPV) approach. The NPV of a project is the present value of its expected future incremental cash flows. The discount rate used to calculate the present value is a "risk-adjusted" discount rate, chosen to reflect the risk of the project. As the riskiness of the project increases, the discount rate also increases.

As an example, consider an investment that costs \$100 million and will last 5 years. The expected cash inflow in each year (in the real world) is estimated to be \$25 million. If the risk-adjusted discount rate is 12% (with continuous compounding), the net present value of the investment is (in millions of dollars)

$$-100 + 25e^{-0.12\times1} + 25e^{-0.12\times2} + 25e^{-0.12\times3} + 25e^{-0.12\times4} + 25e^{-0.12\times5} = -11.53$$

A negative NPV, such as the one we have just calculated, indicates that the project will reduce the value of the company to its shareholders and should not be undertaken. A positive NPV would indicate that the project should be undertaken because it will increase shareholder wealth.

The risk-adjusted discount rate should be the return required by the company, or the company's shareholders, on the investment. This can be calculated in a number of ways. One approach often recommended involves the capital asset pricing model. The steps are as follows:

1. Take a sample of companies whose main line of business is the same as that of the project being contemplated.
2. Calculate the betas of the companies and average them to obtain a proxy beta for the project.
3. Set the required rate of return equal to the risk-free rate plus the proxy beta times the excess return of the market portfolio over the risk-free rate.

One problem with the traditional NPV approach is that many projects contain embedded options. Consider, for example, a company that is considering building a plant to manufacture a new product. Often the company has the option to abandon the project if things do not work out well. It may also have the option to expand the plant if demand for the output exceeds expectations. These options usually have quite different risk characteristics from the base project and require different discount rates.

To understand the problem here, return to the example at the beginning of Chapter 11. This involved a stock whose current price is \$20. In three months the price will be either \$22 or \$18. Risk-neutral valuation shows that the value of a three-month call option on the stock with a strike price of 21 is 0.633. Footnote 1 of Chapter 11 shows that if the expected return required by investors on the stock in the real world is 16% then the expected return required on the call option is 42.6%. A similar analysis shows that if the option is a put rather than a call the expected return required on the option is −52.5%. These analyses mean that if the traditional NPV approach were used to value the call option the correct discount rate would be 42.6%, and if it were used to value a put option the correct discount rate would be −52.5%. There is no easy way of estimating these discount rates. (We know them only because we are able to value the options another way.) Similarly, there is no easy way of estimating the risk-adjusted discount rates appropriate for cash flows when they arise from abandonment, expansion, and other options. This is the motivation for exploring whether the risk-neutral valuation principle can be applied to options on real assets as well as to options on financial assets.

Another problem with the traditional NPV approach lies in the estimation of the appropriate risk-adjusted discount rate for the base project (i.e., the project without embedded options). The companies that are used to estimate a proxy beta for the project in the three-step procedure above have expansion options and abandonment options of their own. Their betas reflect these options and may not therefore be appropriate for estimating a beta for the base project.

33.2 EXTENSION OF THE RISK-NEUTRAL VALUATION FRAMEWORK

In Section 27.1 the market price of risk for a variable θ was defined as

$$\lambda = \frac{\mu - r}{\sigma} \tag{33.1}$$

where r is the risk-free rate, μ is the return on a traded security dependent only on θ, and σ is its volatility. As shown in Section 27.1, the market price of risk, λ, does not depend on the particular traded security chosen.

Suppose that a real asset depends on several variables θ_i ($i = 1, 2, \dots$). Let m_i and s_i be the expected growth rate and volatility of θ_i so that

$$\frac{d\theta_i}{\theta_i} = m_i\,dt + s_i\,dz_i$$

where z_i is a Wiener process. Define λ_i as the market price of risk of θ_i. As explained in Section 27.9, risk-neutral valuation can be extended to show that any asset dependent on the θ_i can be valued by[1]

1. Reducing the expected growth rate of each θ_i from m_i to $m_i - \lambda_i s_i$
2. Discounting cash flows at the risk-free rate.

Example 33.1

The cost of renting commercial real estate in a certain city is quoted as the amount that would be paid per square foot per year in a new 5-year rental agreement. The current cost is \$30 per square foot. The expected growth rate of the cost is 12% per annum, its volatility is 20% per annum, and its market price of risk is 0.3. A company has the opportunity to pay \$1 million now for the option to rent 100,000 square feet at \$35 per square foot for a 5-year period starting in 2 years. The risk-free rate is 5% per annum (assumed constant). Define V as the quoted cost per square foot of office space in 2 years. Assume that rent is paid annually in advance. The payoff from the option is

$$100{,}000A\max(V - 35,\ 0)$$

where A is an annuity factor given by

$$A = 1 + 1 \times e^{-0.05\times 1} + 1 \times e^{-0.05\times 2} + 1 \times e^{-0.05\times 3} + 1 \times e^{-0.05\times 4} = 4.5355$$

The expected payoff in a risk-neutral world is therefore

$$100{,}000 \times 4.5355 \times \hat{E}[\max(V - 35,\ 0)] = 453{,}550 \times \hat{E}[\max(V - 35,\ 0)]$$

where $\hat{E}$ denotes expectations in a risk-neutral world. Using the result in equation (13A.1), this is

$$453{,}550[\hat{E}(V)N(d_1) - 35N(d_2)]$$

where

$$d_1 = \frac{\ln[\hat{E}(V)/35] + .2^2 \times 2/2}{0.2\sqrt{2}}$$

$$d_2 = \frac{\ln[\hat{E}(V)/35] - .2^2 \times 2/2}{0.2\sqrt{2}}$$

The expected growth rate in the cost of commercial real estate in a risk-neutral

[1] To see that this is consistent with regular risk-neutral valuation, suppose that θ_i is the price of a non-dividend-paying stock. Since this is the price of a traded security, equation (33.1) implies that $(m_i - r)/s_i = \lambda_i$, or $m_i - \lambda_i s_i = r$. The expected growth-rate adjustment is therefore the same as setting the return on the stock equal to the risk-free rate. For a proof of the more general result, see Technical Note 20 on the author's website.

world is $m - \lambda s$, where m is the real-world growth rate, s is the volatility, and λ is the market price of risk. In this case, $m = 0.12$, $s = 0.2$, and $\lambda = 0.3$, so that the expected risk-neutral growth rate is 0.06, or 6%, per year. It follows that $\hat{E}(V) = 30e^{0.06\times 2} = 33.82$. Substituting this in the expression above gives the expected payoff in a risk-neutral world as \$1.5015 million. Discounting at the risk-free rate the value of the option is $1.5015e^{-0.05\times 2} = \$1.3586$ million. This shows that it is worth paying \$1 million for the option.

33.3 ESTIMATING THE MARKET PRICE OF RISK

The real-options approach to evaluating an investment avoids the need to estimate risk-adjusted discount rates in the way described in Section 33.1, but it does require market price of risk parameters for all stochastic variables. When historical data are available for a particular variable, its market price of risk can be estimated using the capital asset pricing model. To show how this is done, we consider an investment asset dependent solely on the variable and define:

μ: Expected return of the investment asset

σ: Volatility of the return of the investment asset

λ: Market price of risk of the variable

ρ: Instantaneous correlation between the percentage changes in the variable and returns on a broad index of stock market prices

μ_m: Expected return on broad index of stock market prices

σ_m: Volatility of return on the broad index of stock market prices

r: Short-term risk-free rate

Because the investment asset is dependent solely on the market variable, the instantaneous correlation between its return and the broad index of stock market prices is also ρ. From the continuous-time version of the capital asset pricing model,

$$\mu - r = \frac{\rho\sigma}{\sigma_m}(\mu_m - r)$$

From equation (33.1), another expression for $\mu - r$ is

$$\mu - r = \lambda\sigma$$

It follows that

$$\lambda = \frac{\rho}{\sigma_m}(\mu_m - r) \tag{33.2}$$

This equation can be used to estimate λ.

Example 33.2

A historical analysis of company's sales, quarter by quarter, show that percentage changes in sales have a correlation of 0.3 with returns on the S&P 500 index. The volatility of the S&P 500 is 20% per annum and based on historical data the expected excess return of the S&P 500 over the risk-free rate is 5%. Equation (33.2) estimates the market price of risk for the company's sales as

$$\frac{0.3}{0.2} \times 0.05 = 0.075$$

When no historical data are available for the particular variable under consideration, other similar variables can sometimes be used as proxies. For example, if a plant is being constructed to manufacture a new product, data can be collected on the sales of other similar products. The correlation of the new product with the market index can then be assumed to be the same as that of these other products. In some cases, the estimate of ρ in equation (33.2) must be based on subjective judgment. If an analyst is convinced that a particular variable is unrelated to the performance of a market index, its market price of risk should be set to zero.

For some variables, it is not necessary to estimate the market price of risk because the process followed by a variable in a risk-neutral world can be estimated directly. For example, if the variable is the price of an investment asset, its total return in a risk-neutral world is the risk-free rate. If the variable is the short-term interest rate r, Chapter 30 shows how a risk-neutral process can be estimated from the initial term structure of interest rates. Later in this chapter we will show how the risk-neutral process for a commodity can be estimated from futures prices.

33.4 APPLICATION TO THE VALUATION OF A BUSINESS

Traditional methods of business valuation, such as applying a price/earnings multiplier to current earnings, do not work well for new businesses. Typically a company's earnings are negative during its early years as it attempts to gain market share and establish relationships with customers. The company must be valued by estimating future earnings and cash flows under different scenarios.

The real options approach can be useful in this situation. A model relating the company's future cash flows to variables such as the sales growth rates, variable costs as a percent of sales, fixed costs, and so on, is developed. For key variables, a risk-neutral stochastic process is estimated as outlined in the previous two sections. A Monte Carlo simulation is then carried out to generate alternative scenarios for the net cash flows per year in a risk-neutral world. It is likely that under some of these scenarios the company does very well and under others it becomes bankrupt and ceases operations. (The simulation must have a built in rule for determining when bankruptcy happens.) The value of the company is the present value of the expected cash flow in each year using the risk-free rate for discounting. Business Snapshot 33.1 gives an example of the application of the approach to Amazon.com.

33.5 COMMODITY PRICES

Many investments involve uncertainties related to future commodity prices. Often futures prices can be used to estimate the risk-neutral stochastic process for a commodity price directly. This avoids the need to explicitly estimate a market price of risk for the commodity.

Business Snapshot 33.1 Valuing Amazon.com

One of the earliest published attempts to value a company using the real options approach was Schwartz and Moon (2000), who considered Amazon.com at the end of 1999. They assumed the following stochastic processes for the company's sales revenue R and its revenue growth rate μ:

$$\frac{dR}{R} = \mu\,dt + \sigma(t)\,dz_1$$

$$d\mu = \kappa(\bar{\mu} - \mu)\,dt + \eta(t)\,dz_2$$

They assumed that the two Wiener processes dz_1 and dz_2 were uncorrelated and made reasonable assumptions about $\sigma(t)$, $\eta(t)$, κ, and $\bar{\mu}$ based on available data.

They assumed the cost of goods sold would be 75% of sales, other variable expenses would be 19% of sales, and fixed expenses would be \$75 million per quarter. The initial sales level was \$356 million, the initial tax loss carry forward was \$559 million, and the tax rate was assumed to be 35%. The market price of risk for R was estimated from historical data using the approach described in the previous section. The market price of risk for μ was assumed to be zero.

The time horizon for the analysis was 25 years and the terminal value of the company was assumed to be ten times pretax operating profit. The initial cash position was \$906 million and the company was assumed to go bankrupt if the cash balance became negative.

Different future scenarios were generated in a risk-neutral world using Monte Carlo simulation. The evaluation of the scenarios involved taking account of the possible exercise of convertible bonds and the possible exercise of employee stock options. The value of the company to the share holders was calculated as the present value of the net cash flows discounted at the risk-free rate.

Using these assumptions, Schwartz and Moon provided an estimate of the value of Amazon.com's shares at the end of 1999 equal to \$12.42. The market price at the time was \$76.125 (although it declined sharply in 2000). One of the key advantages of the real-options approach is that it identifies the key assumptions. Schwartz and Moon found that the estimated share value was very sensitive to $\eta(t)$, the volatility of the growth rate. This was an important source of optionality. A small increase in $\eta(t)$ leads to more optionality and a big increase in the value of Amazon.com shares.

From Section 16.7 the expected future price of a commodity in the traditional risk-neutral world is its futures price. If the expected growth rate in the commodity price is dependent solely on time and that the volatility of the commodity price is constant, then the risk-neutral process for the commodity price S has the form

$$\frac{dS}{S} = \mu(t)\,dt + \sigma\,dz \tag{33.3}$$

and

$$F(t) = \hat{E}[S(t)] = S(0)e^{\int_0^t \mu(\tau)d\tau}$$

where $F(t)$ is the futures price for a contract with maturity t and $\hat{E}$ denotes expected value in a risk-neutral world. It follows that

$$\ln F(t) = \ln S(0) + \int_0^t \mu(\tau)d\tau$$

Differentiating both sides with respect to time gives

$$\mu(t) = \frac{\partial}{\partial t}[\ln F(t)]$$

Example 33.3

Suppose that the futures prices of live cattle at the end of July 2008 are (in cents per pound) as follows:

August 2008	62.20
October 2008	60.60
December 2008	62.70
February 2009	63.37
April 2009	64.42
June 2009	64.40

These can be used to estimate the expected growth rate in live cattle prices in a risk-neutral world. For example, when the model in equation (33.3) is used, the expected growth rate in live cattle prices between October and December 2008, in a risk-neutral world is

$$\ln\left(\frac{62.70}{60.60}\right) = 0.034$$

or 3.4% per 2 months with continuous compounding. On an annualized basis, this is 20.4% per annum.

Example 33.4

Suppose that the futures prices of live cattle are as in Example 33.3. A certain breeding decision would involve an investment of \$100,000 now and expenditures of \$20,000 in 3 months, 6 months, and 9 months. The result is expected to be that an extra cattle will be available for sale at the end of the year. There are two major uncertainties: the number of pounds of extra cattle that will be available for sale and the price per pound. The expected number of pounds is 300,000. The expected price of cattle in 1 year in a risk-neutral world is, from Example 33.3, 64.40 cents per pound. Assuming that the risk-free rate of interest is 10% per annum, the value of the investment (in thousands of dollars) is

$$-100 - 20e^{-0.1\times0.25} - 20e^{-0.1\times0.50} - 20e^{-0.1\times0.75} + 300 \times 0.644e^{-0.1\times1} = 17.729$$

This assumes that any uncertainty about the extra amount of cattle that will be available for sale has zero systematic risk and that there is no correlation between the amount of cattle that will be available for sale and the price.

Mean Reversion

It can be argued that the process in equation (33.3) for commodity prices is too simplistic. In practice, most commodity prices follow mean-reverting processes. They tend to get pulled back to a central value. A more realistic process than equation (33.3) for the risk-neutral process followed by the commodity price S is

$$d \ln S = [\theta(t) - a \ln S]\, dt + \sigma\, dz \tag{33.4}$$

This incorporates mean reversion and is analogous to the lognormal process assumed for the short-term interest rate in Chapter 30. The trinomial tree methodology in Section 30.7 can be adapted to construct a tree for S and determine the value of $\theta(t)$ such that $F(t) = \hat{E}[S(t)]$

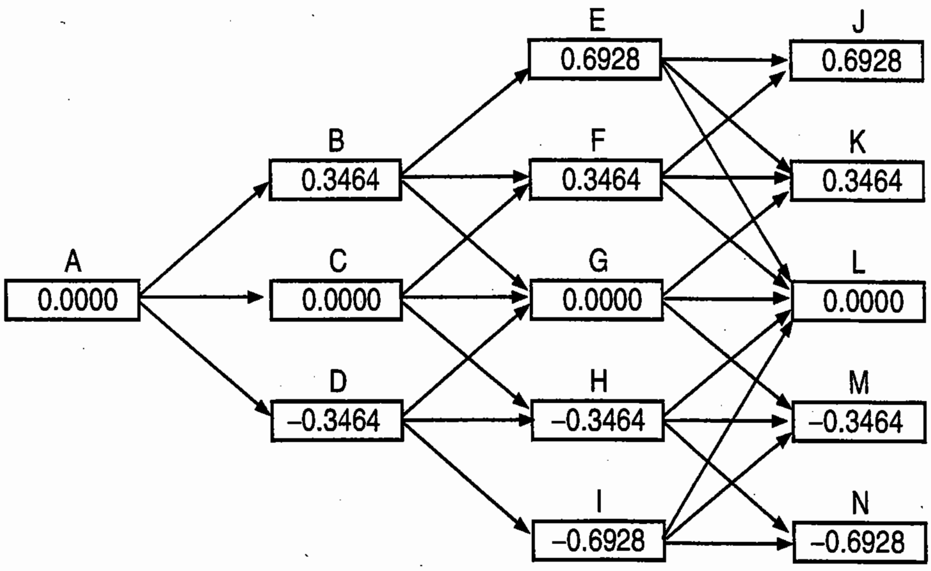
We will illustrate this process by building a three-step tree for the commodity price. Suppose that the current spot price is \$20 and the 1-year, 2-year, and 3-year futures prices are \$22, \$23, and \$24, respectively. Suppose that $a = 0.1$ and $\sigma = 0.2$ in equation (33.4). We first define a variable X that is initially zero and follows the process

$$dX = -aX\, dt + \sigma\, dz \tag{33.5}$$

Using the procedure in Section 30.7, a trinomial tree can be constructed for X. This is shown in Figure 33.1.

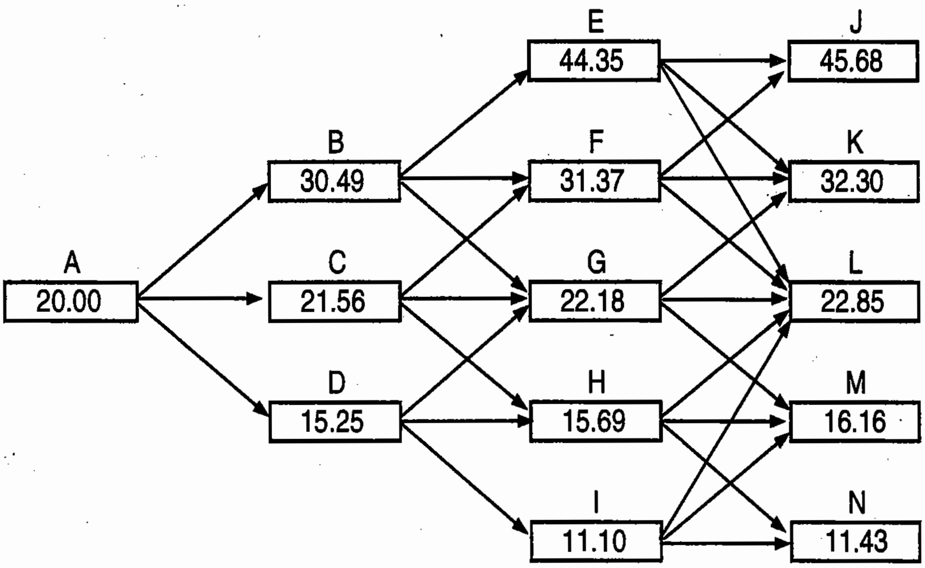
The variable $\ln S$ follows the same process as X except for a time-dependent drift. Analogously to Section 30.7, the tree for X can be converted to a tree for $\ln S$ by displacing the positions of nodes. This tree is shown in Figure 33.2. The initial node corresponds to a price of 20, so the displacement for that node is ln 20. Suppose that the

Figure 33.1 Tree for X. Constructing this tree is the first stage in constructing a tree for the spot price of a commodity, S. Here p_u, p_m, and p_d are the probabilities of "up", "middle", and "down" movements from a node.



Node:	A	B	C	D	E	F	G	H	I
p_u:	0.1667	0.1217	0.1667	0.2217	0.8867	0.1217	0.1667	0.2217	0.0867
p_m:	0.6666	0.6566	0.6666	0.6566	0.0266	0.6566	0.6666	0.6566	0.0266
p_d:	0.1667	0.2217	0.1667	0.1217	0.0867	0.2217	0.1667	0.1217	0.8867

Figure 33.2 Tree for spot price of a commodity: p_u, p_m, and p_d are the probabilities of "up", "middle", and "down" movements from a node.



Node:	A	B	C	D	E	F	G	H	I
p_u:	0.1667	0.1217	0.1667	0.2217	0.8867	0.1217	0.1667	0.2217	0.0867
p_m:	0.6666	0.6566	0.6666	0.6566	0.0266	0.6566	0.6666	0.6566	0.0266
p_d:	0.1667	0.2217	0.1667	0.1217	0.0867	0.2217	0.1667	0.1217	0.8867

displacement of the nodes at 1 year is α_1. The values of the X at the three nodes at the 1-year point are $+0.3464$, 0, and -0.3464. The corresponding values of $\ln S$ are $0.3464+\alpha_1$, α_1, and $-0.3464+\alpha_1$. The values of S are therefore $e^{0.3464+\alpha_1}$, e^{α_1}, and $e^{-0.3464+\alpha_1}$, respectively. We require the expected value of S to equal the futures price. This means that

$$0.1667e^{0.3464+\alpha_1}+0.6666e^{\alpha_1}+0.1667e^{-0.3464+\alpha_1}=22$$

The solution to this is $\alpha_1=3.071$. The values of S at the 1-year point are therefore 30.49, 21.56, and 15.25.

At the 2-year point, we first calculate the probabilities of nodes E, F, G, H, and I being reached from the probabilities of nodes B, C, and D being reached. The probability of reaching node F is the probability of reaching node B times the probability of moving from B to F plus the probability of reaching node C times the probability of moving from C to F. This is

$$0.1667\times 0.6566+0.6666\times 0.1667=0.2206$$

Similarly the probabilities of reaching nodes E, G, H, and I are 0.0203, 0.5183, 0.2206, and 0.0203, respectively. The amount α_2 by which the nodes at time 2 years are displaced must satisfy

$$0.0203e^{0.6928+\alpha_2}+0.2206e^{0.3464+\alpha_2}+0.5183e^{\alpha_2}+0.2206e^{-0.3464+\alpha_2}+0.0203e^{-0.6928+\alpha_2}=23$$

The solution to this is $\alpha_2 = 3.099$. This means that the values of S at the 2-year point are 44.35, 31.37, 22.18, 15.69, and 11.10, respectively.

A similar calculation can be carried out at time 3 years. Figure 33.2 shows the resulting tree for S. The next section illustrates how the tree can be used for the valuation of a real option.

33.6 EVALUATING OPTIONS IN AN INVESTMENT OPPORTUNITY

As already mentioned, most investment projects involve options. These options can add considerable value to the project and are often either ignored or valued incorrectly. Examples of the options embedded in projects are:

1. *Abandonment options*. This is an option to sell or close down a project. It is an American put option on the project's value. The strike price of the option is the liquidation (or resale) value of the project less any closing-down costs. When the liquidation value is low, the strike price can be negative. Abandonment options mitigate the impact of very poor investment outcomes and increase the initial valuation of a project.
2. *Expansion options*. This is the option to make further investments and increase the output if conditions are favorable. It is an American call option on the value of additional capacity. The strike price of the call option is the cost of creating this additional capacity discounted to the time of option exercise. The strike price often depends on the initial investment. If management initially choose to build capacity in excess of the expected level of output, the strike price can be relatively small.
3. *Contraction options*. This is the option to reduce the scale of a project's operation. It is an American put option on the value of the lost capacity. The strike price is the present value of the future expenditures saved as seen at the time of exercise of the option.
4. *Options to defer*. One of the most important options open to a manager is the option to defer a project. This is an American call option on the value of the project.
5. *Options to extend*. Sometimes it is possible to extend the life of an asset by paying a fixed amount. This is a European call option on the asset's future value.

As a simple example of the evaluation of an investment with an embedded option, consider a company that has to decide whether to invest \$15 million to extract 6 million units of a commodity from a certain source at the rate of 2 million units per year for 3 years. The fixed costs of operating the equipment are \$6 million per year and the variable costs are \$17 per unit of the commodity extracted. We assume that the risk-free interest rate is 10% per annum for all maturities, that the spot price of the commodity is \$20, and that the 1-, 2-, and 3-year futures prices are \$22, \$23, and \$24, respectively. We assume that the stochastic process for the commodity price has been estimated as equation (33.4) with $a = 0.1$ and $\sigma = 0.2$. This means that the tree in Figure 33.2 describes the behavior of the commodity price in a risk-neutral world.

First assume that the project has no embedded options. The expected prices of the commodity in 1, 2, and 3 years' time in a risk-neutral world are \$22, \$23, and \$24, respectively. The expected payoff from the project (in millions of dollars) in a risk-neutral world can be calculated from the cost data as 4.0, 6.0, and 8.0 in years 1, 2, and 3, respectively. The value of the project is therefore

$$-15.0 + 4.0e^{-0.1\times 1} + 6.0e^{-0.1\times 2} + 8.0e^{0.1\times 3} = -0.54$$

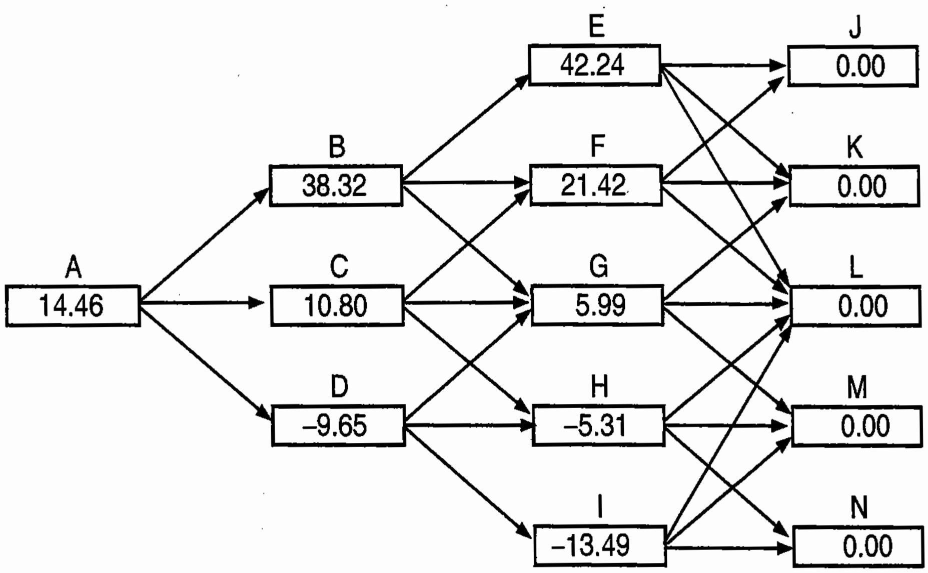
This analysis indicates that the project should not be undertaken because it would reduce shareholder wealth by 0.54 million.

Figure 33.3 shows the value of the project at each node of Figure 33.2. This is calculated from Figure 33.2. Consider, for example, node H. There is a 0.2217 probability that the commodity price at the end of the third year is 22.85, so that the third-year profit is $2 \times 22.85 - 2 \times 17 - 6 = 5.70$. Similarly, there is a 0.6566 probability that the commodity price at the end of the third year is 16.16, so that the profit is -7.68 and there is a 0.1217 probability that the commodity price at the end of the third year is 11.43, so that the profit is -17.14. The value of the project at node H in Figure 33.3 is therefore

$$[0.2217 \times 5.70 + 0.6566 \times (-7.68) + 0.1217 \times (-17.14)]e^{-0.1\times 1} = -5.31$$

As another example, consider node C. There is a 0.1667 chance of moving to node F where the commodity price is 31.37. The second year cash flow is then $2 \times 31.37 - 2 \times 17 - 6 = 22.74$. The value of subsequent cash flows at node F is 21.42. The total value of the project if we move to node F is therefore $21.42 + 22.74 = 44.16$. Similarly the total value of the project if we move to nodes G

Figure 33.3 Valuation of base project with no embedded options: p_u, p_m, and p_d are the probabilities of "up", "middle", and "down" movements from a node.



Node:	A	B	C	D	E	F	G	H	I
p_u:	0.1667	0.1217	0.1667	0.2217	0.8867	0.1217	0.1667	0.2217	0.0867
p_m:	0.6666	0.6566	0.6666	0.6566	0.0266	0.6566	0.6666	0.6566	0.0266
p_d:	0.1667	0.2217	0.1667	0.1217	0.0867	0.2217	0.1667	0.1217	0.8867

and H are 10.35 and −13.93, respectively. The value of the project at node C is therefore

$$[0.1667 \times 44.16 + 0.6666 \times 10.35 + 0.1667 \times (-13.93)]e^{-0.1\times 1} = 10.80$$

Figure 33.3 shows that the value of the project at the initial node A is 14.46. When the initial investment is taken into account the value of the project is therefore −0.54. This is in agreement with our earlier calculations.

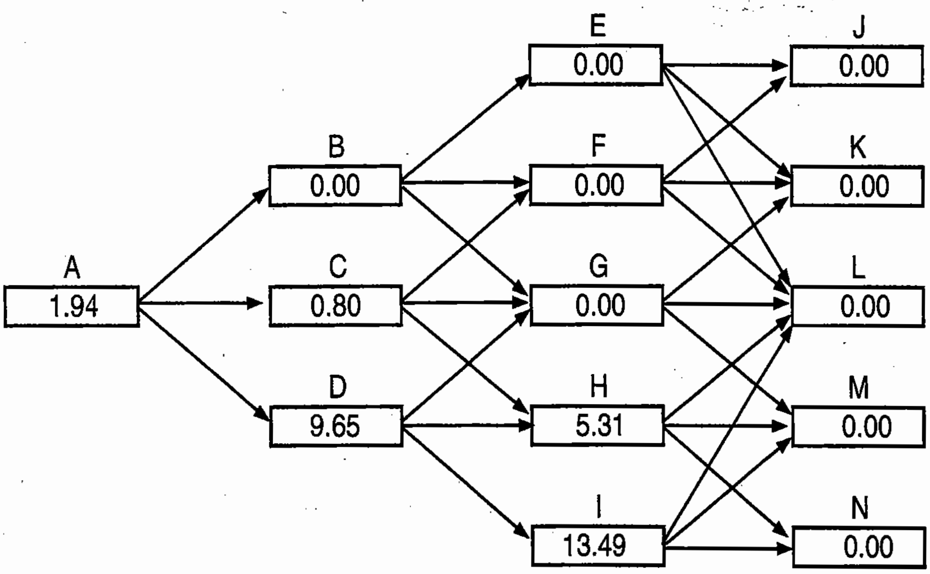
Suppose now that the company has the option to abandon the project at any time. We suppose that there is no salvage value and no further payments are required once the project has been abandoned. Abandonment is an American put option with a strike price of zero and is valued in Figure 33.4. The put option should not be exercised at nodes E, F, and G because the value of the project is positive at these nodes. It should be exercised at nodes H and I. The value of the put option is 5.31 and 13.49 at nodes H and I, respectively. Rolling back through the tree, the value of the abandonment put option at node D if it is not exercised is

$$(0.1217 \times 13.49 + 0.6566 \times 5.31 + 0.2217 \times 0)e^{-0.1\times 1} = 4.64$$

The value of exercising the put option at node D is 9.65. This is greater than 4.64, and so the put should be exercised at node D. The value of the put option at node C is

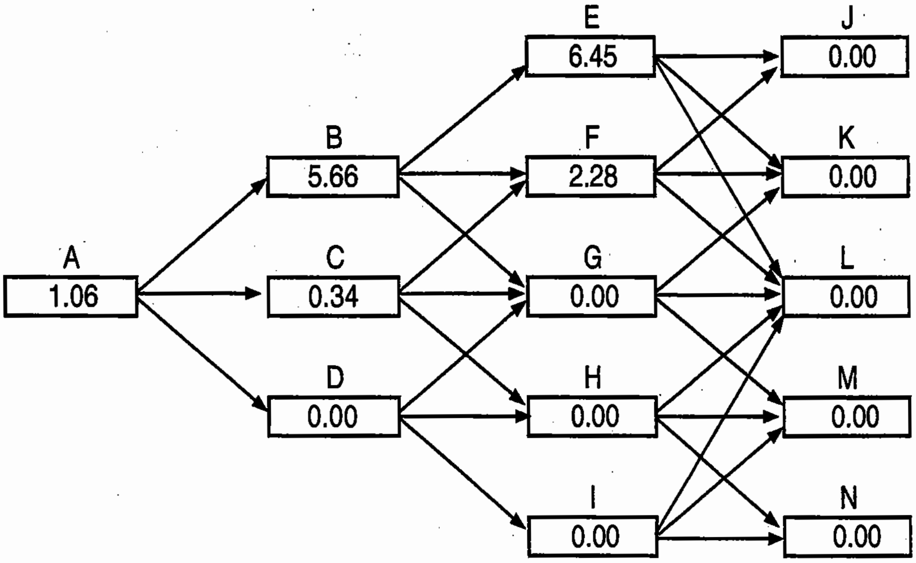
$$[0.1667 \times 0 + 0.6666 \times 0 + 0.1667 \times (5.31)]e^{-0.1\times 1} = 0.80$$

Figure 33.4 Valuation of option to abandon the project: p_u, p_m, and p_d are the probabilities of "up", "middle", and "down" movements from a node.



Node:	A	B	C	D	E	F	G	H	I
p_u:	0.1667	0.1217	0.1667	0.2217	0.8867	0.1217	0.1667	0.2217	0.0867
p_m:	0.6666	0.6566	0.6666	0.6566	0.0266	0.6566	0.6666	0.6566	0.0266
p_d:	0.1667	0.2217	0.1667	0.1217	0.0867	0.2217	0.1667	0.1217	0.8867

Figure 33.5 Valuation of option to expand the project: p_u, p_m, and p_d are the probabilities of "up", "middle", and "down" movements from a node.



Node:	A	B	C	D	E	F	G	H	I
p_u:	0.1667	0.1217	0.1667	0.2217	0.8867	0.1217	0.1667	0.2217	0.0867
p_m:	0.6666	0.6566	0.6666	0.6566	0.0266	0.6566	0.6666	0.6566	0.0266
p_d:	0.1667	0.2217	0.1667	0.1217	0.0867	0.2217	0.1667	0.1217	0.8867

and the value at node A is

$$(0.1667 \times 0 + 0.6666 \times 0.80 + 0.1667 \times 9.65)e^{-0.1\times 1} = 1.94$$

The abandonment option is therefore worth \$1.94 million. It increases the value of the project from −\$0.54 million to +\$1.40 million. A project that was previously unattractive now has a positive value to shareholders.

Suppose next that the company has no abandonment option. Instead it has the option at any time to increase the scale of the project by 20%. The cost of doing this is \$2 million. Production increases from 2.0 to 2.4 million units per year. Variable costs remain \$17 per unit and fixed costs increase by 20% from \$6.0 million to \$7.2 million. This is an American call option to buy 20% of the base project in Figure 33.3 for \$2 million. The option is valued in Figure 33.5. At node E, the option should be exercised. The payoff is $0.2 \times 42.24 - 2 = 6.45$. At node F, it should also be exercised for a payoff of $0.2 \times 21.42 - 2 = 2.28$. At nodes G, H, and I, the option should not be exercised. At node B, exercising is worth more than waiting and the option is worth $0.2 \times 38.32 - 2 = 5.66$. At node C, if the option is not exercised, it is worth

$$(0.1667 \times 2.28 + 0.6666 \times 0.00 + 0.1667 \times 0.00)e^{-0.1\times 1} = 0.34$$

If the option is exercised, it is worth $0.2 \times 10.80 - 2 = 0.16$. The option should therefore not be exercised at node C. At node A, if not exercised, the option is worth

$$(0.1667 \times 5.66 + 0.6666 \times 0.34 + 0.1667 \times 0.00)e^{-0.1\times 1} = 1.06$$

If the option is exercised it is worth $0.2 \times 14.46 - 2 = 0.89$. Early exercise is therefore not optimal at node A. In this case, the option increases the value of the project from -0.54 to $+0.52$. Again we find that a project that previously had a negative value now has a positive value.

The expansion option in Figure 33.5 is relatively easy to value because, once the option has been exercised, all subsequent cash inflows and outflows increase by 20%. In the case where fixed costs remain the same or increase by less than 20%, it is necessary to keep track of more information at the nodes of Figure 33.3. Specifically we need to record the following:

1. The present value of subsequent fixed costs
2. The present value of subsequent revenues net of variable costs

The payoff from exercising the option can then be calculated.

When a project has two or more options, they are typically not independent. The value of having both option A and option B is typically not the sum of the values of the two options. To illustrate this, suppose that the company we have been considering has both abandonment and expansion options. The project cannot be expanded if it has already been abandoned. Moreover, the value of the put option to abandon depends on whether the project has been expanded.[2]

These interactions between the options in our example can be handled by defining four states at each node:

1. Not already abandoned; not already expanded
2. Not already abandoned; already expanded
3. Already abandoned; not already expanded
4. Already abandoned; already expanded

When we roll back through the tree we calculate the combined value of the options at each node for all four alternatives. This approach to valuing path-dependent options is discussed in more detail in Section 26.5.

When there are several stochastic variables, the value of the base project is usually determined by Monte Carlo simulation. The valuation of the project's embedded options is then more difficult because a Monte Carlo simulation works from the beginning to the end of a project. When we reach a certain point, we do not have information on the present value of the project's future cash flows. However, the techniques mentioned in Section 26.8 for valuing American options using Monte Carlo simulation can sometimes be used.

As an illustration of this point, Schwartz and Moon (2000) explain how their Amazon.com analysis outlined in Business Snapshot 33.1 could be extended to take account of the option to abandon (i.e. the option to declare bankruptcy) when the value of future cash flows is negative.[3] At each time step, a polynomial relationship between the value of not abandoning and variables such as the current revenue, revenue growth rate, volatilities, cash balances, and loss carry forwards is assumed. Each simulation trial

[2] As it happens, the two options do not interact in Figures 33.4 and 33.5. However, the interactions between the options would become an issue if a larger tree with smaller time steps were built.

[3] The analysis in Section 33.4 assumed that bankruptcy occurs when the cash balance falls below zero, but this is not necessarily optimal for Amazon.com.

provides an observation for obtaining a least-squares estimate of the relationship at each time. This is the Longstaff and Schwartz approach of Section 26.8.[4]

SUMMARY

This chapter has investigated how the ideas developed earlier in the book can be applied to the valuation of real assets and options on real assets. It has shown how the risk-neutral valuation principle can be used to value a project dependent on any set of variables. The expected growth rate of each variable is adjusted to reflect its market price of risk. The value of the asset is then the present value of its expected cash flows discounted at the risk-free rate.

Risk-neutral valuation provides an internally consistent approach to capital investment appraisal. It also makes it possible to value the options that are embedded in many of the projects that are encountered in practice. This chapter has illustrated the approach by applying it to the valuation of Amazon.com at the end of 1999 and the valuation of a project involving the extraction of a commodity.

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Copeland, T., T. Koller, and J. Murrin, *Valuation: Measuring and Managing the Value of Companies*, 3rd edn. New York: Wiley, 2000.

Copeland, T., and V. Antikarov, *Real Options: A Practitioners Guide*, New York: Texere, 2003.

Schwartz, E. S., and M. Moon, "Rational Pricing of Internet Companies," *Financial Analysts Journal*, May/June (2000): 62–75.

Trigeorgis, L., *Real Options: Managerial Flexibility and Strategy in Resource Allocation*, Cambridge, MA: MIT Press, 1996.

Questions and Problems (Answers in Solutions Manual)

33.1. Explain the difference between the net present value approach and the risk-neutral valuation approach for valuing a new capital investment opportunity. What are the advantages of the risk-neutral valuation approach for valuing real options?

33.2. The market price of risk for copper is 0.5, the volatility of copper prices is 20% per annum, the spot price is 80 cents per pound, and the 6-month futures price is 75 cents per pound. What is the expected percentage growth rate in copper prices over the next 6 months?

33.3. Consider a commodity with constant volatility σ and an expected growth rate that is a function solely of time. Show that, in the traditional risk-neutral world,

$$\ln S_T \sim \phi[(\ln F(T) - \tfrac{1}{2}\sigma^2 T,\ \sigma^2 T]$$

[4] F. A. Longstaff and E. S. Schwartz, "Valuing American Options by Simulation: A Simple Least-Squares Approach," *Review of Financial Studies*, 14, 1 (Spring 2001): 113–47.

where S_T is the value of the commodity at time T, $F(t)$ is the futures price at time 0 for a contract maturing at time t, and $\phi(m, v)$ is a normal distribution with mean m and variance v.

33.4. Derive a relationship between the convenience yield of a commodity and its market price of risk.

33.5. The correlation between a company's gross revenue and the market index is 0.2. The excess return of the market over the risk-free rate is 6% and the volatility of the market index is 18%. What is the market price of risk for the company's revenue?

33.6. A company can buy an option for the delivery of 1 million units of a commodity in 3 years at \$25 per unit. The 3-year futures price is \$24. The risk-free interest rate is 5% per annum with continuous compounding and the volatility of the futures price is 20% per annum. How much is the option worth?

33.7. A driver entering into a car lease agreement can obtain the right to buy the car in 4 years for \$10,000. The current value of the car is \$30,000. The value of the car, S, is expected to follow the process

$$dS = \mu S\,dt + \sigma S\,dz$$

where $\mu = -0.25$, $\sigma = 0.15$, and dz is a Wiener process. The market price of risk for the car price is estimated to be -0.1. What is the value of the option? Assume that the risk-free rate for all maturities is 6%.

Assignment Questions

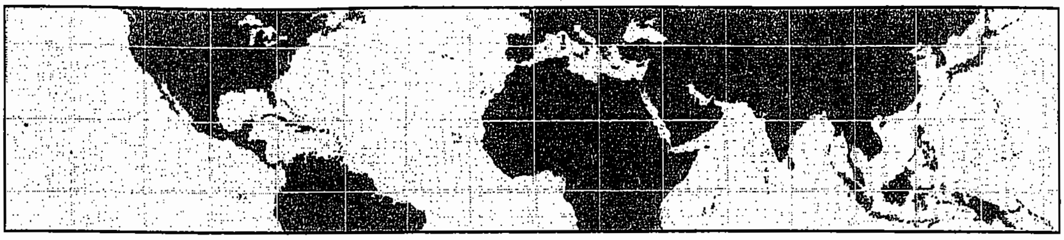
33.8. Suppose that the spot price, 6-month futures price, and 12-month futures price for wheat are 250, 260, and 270 cents per bushel, respectively. Suppose that the price of wheat follows the process in equation (33.4) with $a = 0.05$ and $\sigma = 0.15$. Construct a two-time-step tree for the price of wheat in a risk-neutral world.

A farmer has a project that involves an expenditure of \$10,000 and a further expenditure of \$90,000 in 6 months. It will increase wheat that is harvested and sold by 40,000 bushels in 1 year. What is the value of the project? Suppose that the farmer can abandon the project in 6 months and avoid paying the \$90,000 cost at that time. What is the value of the abandonment option? Assume a risk-free rate of 5% with continuous compounding.

33.9. In the example considered in Section 33.6:

(a) What is the value of the abandonment option if it costs \$3 million rather than zero?

(b) What is the value of the expansion option if it costs \$5 million rather than \$2 million?



CHAPTER 34

Derivatives Mishaps and What We Can Learn from Them

Since the mid-1980s there have been some spectacular losses in derivatives markets. Some of the losses made by financial institutions are listed in Business Snapshot 34.1, and some of those made by nonfinancial organizations in Business Snapshot 34.2. What is remarkable about these lists is the number of situations where huge losses arose from the activities of a single employee. In 1995, Nick Leeson's trading brought a 200-year-old British bank, Barings, to its knees; in 1994, Robert Citron's trading led to Orange County, a municipality in California, losing about \$2 billion. Joseph Jett's trading for Kidder Peabody lost \$350 million. John Rusnak's losses of \$700 million for Allied Irish Bank came to light in 2002. In 2006 the hedge fund Amaranth lost \$6 billion because of trading risks taken by Brian Hunter. In 2008, Jérôme Kerviel lost over \$7 billion trading equity index futures for Société Générale. The huge losses at Daiwa, Shell, and Sumitomo were also each the result of the activities of a single individual.

The losses should not be viewed as an indictment of the whole derivatives industry. The derivatives market is a vast multitrillion dollar market that by most measures has been outstandingly successful and has served the needs of its users well. To quote from Alan Greenspan (May 2003):

> The use of a growing array of derivatives and the related application of more sophisticated methods for measuring and managing risk are key factors underpinning the enhanced resilience of our largest financial intermediaries.

The events listed in Business Snapshots 23.1 and 23.2 represent a tiny proportion of the total trades (both in number and value). Nevertheless, it is worth considering carefully the lessons that can be learned from them.

34.1 LESSONS FOR ALL USERS OF DERIVATIVES

First, we consider the lessons appropriate to all users of derivatives, whether they are financial or nonfinancial companies.

Business Snapshot 34.1 Big Losses by Financial Institutions

Allied Irish bank

This bank lost about $700 million from speculative activities of one of its foreign exchange traders, John Rusnak, that lasted a number of years. Rusnak managed to cover up his losses for a number of years by creating fictitious option trades.

Amaranth

This hedge fund lost $6 billion in 2006 betting on the future direction of natural gas prices.

Barings (see page 15)

This 200-year-old British bank was destroyed in 1995 by the activities of one trader, Nick Leeson, in Singapore, who made big bets on the future direction of the Nikkei 225 using futures and options. The total loss was close to $1 billion.

Daiwa Bank

A trader working in New York for this Japanese bank lost more than $1 billion in the 1990s.

Kidder Peabody (see page 103)

The activities of a single trader, Joseph Jett, led to this New York investment dealer losing $350 million trading US government securities. The loss arose because of a mistake in the way the company's computer system calculated profits.

Long-Term Capital Management (see page 30)

This hedge fund lost about $4 billion in 1998 as a result of Russia's default on its debt and the resultant flight to quality. The New York Federal Reserve organized an orderly liquidation of the fund by arranging for 14 banks to invest in the fund.

Midland Bank

This British bank lost $500 million in the early 1990s largely because of a wrong bet on the direction of interest rates. It was later taken over by the Hong Kong and Shanghai bank.

National Westminster Bank

This British bank lost about $130 million from using an inappropriate model to value swap options in 1997.

Société Générale

Jérôme Kerviel lost over $7 billion speculating on the future direction of equity indices in January 2008.

Subprime Mortgage Losses (see page 539)

In 2007 investors lost confidence in the structured products created from US subprime mortgages. This led to a "credit crunch" and losses of tens of billions of dollars by financial institutions.

Business Snapshot 34.2 Big Losses by Nonfinancial Organizations

Allied Lyons
The treasury department of this drinks and food company lost $150 million in 1991 selling call options on the US dollar–sterling exchange rate.

Gibson Greetings
The treasury department of this greeting card manufacturer in Cincinnati lost about $20 million in 1994 trading highly exotic interest rate derivatives contracts with Bankers Trust. They later sued Bankers Trust and settled out of court.

Hammersmith and Fulham (see page 171)
This British Local Authority lost about $600 million on sterling interest rate swaps and options in 1988. All its contracts were later declared null and void by the British courts, much to the annoyance of the banks on the other side of the transactions.

Metallgesellschaft (see page 66)
This German company entered into long-term contracts to supply oil and gasoline and hedged them by rolling over short-term futures contracts. It lost $1.8 billion when it was forced to discontinue this activity.

Orange County (see page 84)
The activities of the treasurer, Robert Citron, led to this California municipality losing about $2 billion in 1994. The treasurer was using derivatives to speculate that interest rates would not rise.

Procter & Gamble (see page 741)
The treasury department of this large U.S. company lost about $90 million in 1994 trading highly exotic interest rate derivatives contracts with Bankers Trust. It later sued Bankers Trust and settled out of court.

Shell
A single employee working in the Japanese subsidiary of this company lost $1 billion dollars in unauthorized trading of currency futures.

Sumitomo
A single trader working for this Japanese company lost about $2 billion in the copper spot, futures, and options market in the 1990s.

Define Risk Limits

It is essential that all companies define in a clear and unambiguous way limits to the financial risks that can be taken. They should then set up procedures for ensuring that the limits are obeyed. Ideally, overall risk limits should be set at board level. These should then be converted to limits applicable to the individuals responsible for managing particular risks. Daily reports should indicate the gain or loss that will be experienced for particular movements in market variables. These should be checked against the actual gains and losses that are experienced to ensure that the valuation procedures underlying the reports are accurate.

It is particularly important that companies monitor risks carefully when derivatives are used. This is because, as we saw in Chapter 1, derivatives can be used for hedging, speculation, and arbitrage. Without close monitoring, it is impossible to know whether a derivatives trader has switched from being a hedger to a speculator or switched from being an arbitrageur to being a speculator. Barings is a classic example of what can go wrong. Nick Leeson's mandate was to carry out low-risk arbitrage between the Singapore and Osaka markets on Nikkei 225 futures. Unknown to his superiors in London, Leeson switched from being an arbitrageur to taking huge bets on the future direction of the Nikkei 225. Systems within Barings were so inadequate that nobody knew what he was doing.

The argument here is not that no risks should be taken. A treasurer working for a corporation, or a trader in a financial institution, or a fund manager should be allowed to take positions on the future direction of relevant market variables. But the sizes of the positions that can be taken should be limited and the systems in place should accurately report the risks being taken.

Take the Risk Limits Seriously

What happens if an individual exceeds risk limits and makes a profit? This is a tricky issue for senior management. It is tempting to ignore violations of risk limits when profits result. However, this is shortsighted. It leads to a culture where risk limits are not taken seriously, and it paves the way for a disaster. In many of the situations listed in Business Snapshots 34.1 and 34.2, the companies had become complacent about the risks they were taking because they had taken similar risks in previous years and made profits.

The classic example here is Orange County. Robert Citron's activities in 1991–93 had been very profitable for Orange County, and the municipality had come to rely on his trading for additional funding. People chose to ignore the risks he was taking because he had produced profits. Unfortunately, the losses made in 1994 far exceeded the profits from previous years.

The penalties for exceeding risk limits should be just as great when profits result as when losses result. Otherwise, traders who make losses are liable to keep increasing their bets in the hope that eventually a profit will result and all will be forgiven.

Do Not Assume You Can Outguess the Market

Some traders are quite possibly better than others. But no trader gets it right all the time. A trader who correctly predicts the direction in which market variables will move 60% of the time is doing well. If a trader has an outstanding track record (as Robert Citron did in the early 1990s), it is likely to be a result of luck rather than superior trading skill.

Suppose that a financial institution employs 16 traders and one of those traders makes profits in every quarter of a year. Should the trader receive a good bonus? Should the trader's risk limits be increased? The answer to the first question is that inevitably the trader will receive a good bonus. The answer to the second question should be no. The chance of making a profit in four consecutive quarters from random trading is 0.5^4 or 1 in 16. This means that just by chance one of the 16 traders will "get it right" every single quarter of the year. It should not be assumed that the trader's luck will continue and the trader's risk limits should not be increased.

Do Not Underestimate the Benefits of Diversification

When a trader appears good at predicting a particular market variable, there is a tendency to increase the trader's limits. We have just argued that this is a bad idea because it is quite likely that the trader has been lucky rather than clever. However, let us suppose that a fund is really convinced that the trader has special talents. How undiversified should it allow itself to become in order to take advantage of the trader's special skills? The answer is that the benefits from diversification are huge, and it is unlikely that any trader is so good that it is worth foregoing these benefits to speculate heavily on just one market variable.

An example will illustrate the point here. Suppose that there are 20 stocks, each of which have an expected return of 10% per annum and a standard deviation of returns of 30%. The correlation between the returns from any two of the stocks is 0.2. By dividing an investment equally among the 20 stocks, an investor has an expected return of 10% per annum and standard deviation of returns of 14.7%. Diversification enables the investor to reduce risks by over half. Another way of expressing this is that diversification enables an investor to double the expected return per unit of risk taken. The investor would have to be extremely good at stock picking to get a better risk–return tradeoff by investing in just one stock.

Carry out Scenario Analyses and Stress Tests

The calculation of risk measures such as VaR should always be accompanied by scenario analyses and stress testing to obtain an understanding of what can go wrong. These were mentioned in Chapter 20. They are very important. Human beings have an unfortunate tendency to anchor on one or two scenarios when evaluating decisions. In 1993 and 1994, for example, Procter & Gamble and Gibson Greetings may have been so convinced that interest rates would remain low that they ignored the possibility of a 100-basis-point increase in their decision making.

It is important to be creative in the way scenarios are generated. One approach is to look at 10 or 20 years of data and choose the most extreme events as scenarios. Sometimes there is a shortage of data on a key variable. It is then sensible to choose a similar variable for which much more data is available and use historical daily percentage changes in that variable as a proxy for possible daily percentage changes in the key variable. For example, if there is little data on the prices of bonds issued by a particular country, historical data on prices of bonds issued by other similar countries can be used to develop possible scenarios.

34.2 LESSONS FOR FINANCIAL INSTITUTIONS

We now move on to consider lessons that are primarily relevant to financial institutions.

Monitor Traders Carefully

In trading rooms there is a tendency to regard high-performing traders as "untouchable" and to not subject their activities to the same scrutiny as other traders. Apparently Joseph Jett, Kidder Peabody's star trader of Treasury instruments, was often "too busy" to answer questions and discuss his positions with the company's risk managers.

It is important that all traders—particularly those making high profits—be fully accountable. It is important for the financial institution to know whether the high profits are being made by taking unreasonably high risks. It is also important to check that the financial institution's computer systems and pricing models are correct and are not being manipulated in some way.

Separate the Front, Middle, and Back Office

The *front office* in a financial institution consists of the traders who are executing trades, taking positions, and so forth. The *middle office* consists of risk managers who are monitoring the risks being taken. The *back office* is where the record keeping and accounting takes place. Some of the worst derivatives disasters have occurred because these functions were not kept separate. Nick Leeson controlled both the front and back office for Barings in Singapore and was, as a result, able to conceal the disastrous nature of his trades from his superiors in London for some time. Jérôme Kerviel had worked in Société Générale's back office before becoming a trader and took advantage of his knowledge of its systems to hide his positions.

Do Not Blindly Trust Models

Some of the large losses incurred by financial institutions arose because of the models and computer systems being used. We discussed how Kidder Peabody was misled by its own systems on page 103. Another example of an incorrect model leading to losses is provided by National Westminster Bank. This bank had an incorrect model for valuing swap options that led to significant losses.

If large profits are reported when relatively simple trading strategies are followed, there is a good chance that the models underlying the calculation of the profits are wrong. Similarly, if a financial institution appears to be particularly competitive on its quotes for a particular type of deal, there is a good chance that it is using a different model from other market participants, and it should analyze what is going on carefully. To the head of a trading room, getting too much business of a certain type can be just as worrisome as getting too little business of that type.

Be Conservative in Recognizing Inception Profits

When a financial institution sells a highly exotic instrument to a nonfinancial corporation, the valuation can be highly dependent on the underlying model. For example, instruments with long-dated embedded interest rate options can be highly dependent on the interest rate model used. In these circumstances, a phrase used to describe the daily marking to market of the deal is *marking to model*. This is because there are no market prices for similar deals that can be used as a benchmark.

Suppose that a financial institution manages to sell an instrument to a client for \$10 million more than it is worth—or at least \$10 million more than its model says it is worth. The \$10 million is known as an *inception profit*. When should it be recognized? There appears to be quite a variation in what different investment banks do. Some recognize the $10 million immediately, whereas others are much more conservative and recognize it slowly over the life of the deal.

Recognizing inception profits immediately is very dangerous. It encourages traders to use aggressive models, take their bonuses, and leave before the model and the value of the deal come under close scrutiny. It is much better to recognize inception profits slowly, so that traders have the motivation to investigate the impact of several different models and several different sets of assumptions before committing themselves to a deal.

Do Not Sell Clients Inappropriate Products

It is tempting to sell corporate clients inappropriate products, particularly when they appear to have an appetite for the underlying risks. But this is shortsighted. The most dramatic example of this is the activities of Bankers Trust (BT) in the period leading up to the spring of 1994. Many of BT's clients were persuaded to buy high-risk and totally inappropriate products. A typical product (e.g., the 5/30 swap discussed on page 741) would give the client a good chance of saving a few basis points on its borrowings and a small chance of costing a large amount of money. The products worked well for BT's clients in 1992 and 1993, but blew up in 1994 when interest rates rose sharply. The bad publicity that followed hurt BT greatly. The years it had spent building up trust among corporate clients and developing an enviable reputation for innovation in derivatives were largely lost as a result of the activities of a few overly aggressive salesmen. BT was forced to pay large amounts of money to its clients to settle lawsuits out of court. It was taken over by Deutsche Bank in 1999.

Do Not Ignore Liquidity Risk

Financial engineers usually base the pricing of exotic instruments and other instruments that trade relatively infrequently on the prices of actively traded instruments. For example:

1. A financial engineer often calculates a zero curve from actively traded government bonds (known as on-the-run bonds) and uses it to price bonds that trade less frequently (off-the-run bonds).
2. A financial engineer often implies the volatility of an asset from actively traded options and uses it to price less actively traded options.
3. A financial engineer often implies information about the behavior of interest rates from actively traded interest rate caps and swap options and uses it to price products that are highly structured.

These practices are not unreasonable. However, it is dangerous to assume that less actively traded instruments can always be traded at close to their theoretical price. When financial markets experience a shock of one sort or another there is often a "flight to quality." Liquidity becomes very important to investors, and illiquid instruments often sell at a big discount to their theoretical values. This happened in 2007 following the jolt to credit markets caused by lack of confidence in securities backed by subprime mortgages.

Another example of losses arising from liquidity risk is provided by Long-Term Capital Management (LTCM), which was discussed in Business Snapshot 2.2. This hedge fund followed a strategy known as *convergence arbitrage*. It attempted to identify two securities (or portfolios of securities) that should in theory sell for the same price. If the market price of one security was less that of the other, it would buy that security and sell the other. The strategy is based on the idea that if two securities have the same theoretical price their market prices should eventually be the same.

In the summer of 1998 LTCM made a huge loss. This was largely because a default by Russia on its debt caused a flight to quality. LTCM tended to be long illiquid instruments and short the corresponding liquid instruments (for example, it was long off-the-run bonds and short on-the-run bonds). The spreads between the prices of illiquid instruments and the corresponding liquid instruments widened sharply after the Russian default. LTCM was highly leveraged. It experienced huge losses and there were margin calls on its positions that it was unable to meet.

The LTCM story reinforces the importance of carrying out scenario analyses and stress testing to look at what can happen in the worst of all worlds. LTCM could have tried to examine other times in history when there have been extreme flights to quality to quantify the liquidity risks it was facing.

Beware When Everyone Is Following the Same Trading Strategy

It sometimes happens that many market participants are following essentially the same trading strategy. This creates a dangerous environment where there are liable to be big market moves, unstable markets, and large losses for the market participants.

We gave one example of this in Chapter 17 when discussing portfolio insurance and the market crash of October 1987. In the months leading up to the crash, increasing numbers of portfolio managers were attempting to insure their portfolios by creating synthetic put options. They bought stocks or stock index futures after a rise in the market and sold them after a fall. This created an unstable market. A relatively small decline in stock prices could lead to a wave of selling by portfolio insurers. The latter would lead to a further decline in the market, which could give rise to another wave of selling, and so on. There is little doubt that without portfolio insurance the crash of October 1987 would have been much less severe.

Another example is provided by LTCM in 1998. Its position was made more difficult by the fact that many other hedge funds were following similar convergence arbitrage strategies. After the Russian default and the flight to quality, LTCM tried to liquidate part of its portfolio to meet margin calls. Unfortunately, other hedge funds were facing similar problems to LTCM and trying to do similar trades. This exacerbated the situation, causing liquidity spreads to be even higher than they would otherwise have been and reinforcing the flight to quality. Consider, for example, LTCM's position in U.S. Treasury bonds. It was long the illiquid off-the-run bonds and short the liquid on-the-run bonds. When a flight to quality caused spreads between yields on the two types of bonds to widen, LTCM had to liquidate its positions by selling off-the-run bonds and buying on-the-run bonds. Other large hedge funds were doing the same. As a result, the price of on-the-run bonds rose relative to off-the-run bonds and the spread between the two yields widened even more than it had done already.

A further example is provided by the activities of British insurance companies in the late 1990s. These insurance companies had entered into many contracts promising that the rate of interest applicable to an annuity received by an individual on retirement would be the greater of the market rate and a guaranteed rate. At about the same time, all insurance companies decided to hedge part of their risks on these contracts by buying long-dated swap options from financial institutions. The financial institutions they dealt with hedged their risks by buying huge numbers of long-dated sterling bonds. As a result, bond prices rose and long sterling rates declined. More bonds had to be bought to maintain the dynamic hedge, long sterling rates declined further, and so on. Financial institutions lost money and, because long rates declined, insurance companies found themselves in a worse position on the risks that they had chosen not to hedge.

The chief lesson to be learned from these stories is that it is important to see the big picture of what is going on in financial markets and to understand the risks inherent in situations where many market participants are following the same trading strategy.

Do Not Finance Long-Term Assets with Short-Term Liabilities

As discussed in Section 4.10, it is important for a financial institution to match the maturities of assets and liabilities. If it does not do this, it is subjecting itself to significant interest rate risk. Savings and Loans in the United States ran into difficulties in the 1960s, 1970s, and 1980s because they financed long-term mortgages with short-term deposits. Continental Bank failed in 1984 for a similar reason (see Business Snapshot 4.3).

During the period leading up to the credit crunch of 2007, there was a tendency for subprime mortgages and other long-term assets to be financed by commercial paper while they were in a portfolio waiting to be packaged into structured products (see Business Snapshot 23.3). Conduits and special purpose vehicles had an ongoing requirement for this type of financing. The commercial paper would typically be rolled over every month. For example, the purchasers of commercial paper issued on April 1 would be redeemed with the proceeds of a new commercial paper issue on May 1. This new commercial paper issue would in turn be redeemed with another new commercial paper issue on June 1, and so on. When investors lost confidence in subprime mortgages in August 2007, it became impossible to roll over commercial paper. In many instances banks had provided guarantees and had to provide financing. This led to a shortage of liquidity. As a result, the credit crunch was more severe than it would have been if longer-term financing had been arranged.

Market Transparency Is Important

One of the lessons from the credit crunch of 2007 is that market transparency is important. During the period leading up to 2007, investors traded highly structured products without any real knowledge of the underlying assets. All they knew was the credit rating of the security being traded. With hindsight, we can say that investors should have demanded more information about the underlying assets and should have more carefully assessed the risks they were taking—but it is easy to be wise after the event!

The subprime meltdown of August 2007 caused investors to lose confidence in all structured products and withdraw from that market. This led to a market breakdown where tranches of structured products could only be sold at prices well below their theoretical values. There was a flight to quality and credit spreads increased. If there had been market transparency so that investors understood the asset-backed securities they were buying, there would still have been subprime losses, but the flight to quality and disruptions to the market would have been less pronounced.

34.3 LESSONS FOR NONFINANCIAL CORPORATIONS

We now consider lessons primarily applicable to nonfinancial corporations.

Make Sure You Fully Understand the Trades You Are Doing

Corporations should never undertake a trade or a trading strategy that they do not fully understand. This is a somewhat obvious point, but it is surprising how often a trader working for a nonfinancial corporation will, after a big loss, admit to not knowing what was really going on and claim to have been misled by investment bankers. Robert Citron, the treasurer of Orange County did this. So did the traders working for Hammersmith and Fulham, who in spite of their huge positions were surprisingly uninformed about how the swaps and other interest rate derivatives they traded really worked.

If a senior manager in a corporation does not understand a trade proposed by a subordinate, the trade should not be approved. A simple rule of thumb is that if a trade and the rationale for entering into it are so complicated that they cannot be understood by the manager, it is almost certainly inappropriate for the corporation. The trades undertaken by Procter & Gamble and Gibson Greetings would have been vetoed using this criterion.

One way of ensuring that you fully understand a financial instrument is to value it. If a corporation does not have the in-house capability to value an instrument, it should not trade it. In practice, corporations often rely on their derivatives dealers for valuation advice. This is dangerous, as Procter & Gamble and Gibson Greetings found out. When they wanted to unwind their deals, they found they were facing prices produced by Bankers Trust's proprietary models, which they had no way of checking.

Make Sure a Hedger Does Not Become a Speculator

One of the unfortunate facts of life is that hedging is relatively dull, whereas speculation is exciting. When a company hires a trader to manage foreign exchange, commodity price, or interest rate risk, there is a danger that the following might happen. At first, the trader does the job diligently and earns the confidence of top management. He or she assesses the company's exposures and hedges them. As time goes by, the trader becomes convinced that he or she can outguess the market. Slowly the trader becomes a speculator. At first things go well, but then a loss is made. To recover the loss, the trader doubles up the bets. Further losses are made—and so on. The result is likely to be a disaster.

As mentioned earlier, clear limits to the risks that can be taken should be set by senior management. Controls should be put in place to ensure that the limits are obeyed. The trading strategy for a corporation should start with an analysis of the risks facing the corporation in foreign exchange, interest rate, commodity markets, and so on. A decision should then be taken on how the risks are to be reduced to acceptable levels. It is a clear sign that something is wrong within a corporation if the trading strategy is not derived in a very direct way from the company's exposures.

Be Cautious about Making the Treasury Department a Profit Center

In the last 20 years there has been a tendency to make the treasury department within a corporation a profit center. This appears to have much to recommend it. The treasurer is motivated to reduce financing costs and manage risks as profitably as possible. The problem is that the potential for the treasurer to make profits is limited. When raising funds and investing surplus cash, the treasurer is facing an efficient market. The treasurer can usually improve the bottom line only by taking additional risks. The company's hedging program gives the treasurer some scope for making shrewd decisions that increase profits. But it should be remembered that the goal of a hedging program is to reduce risks, not to increase expected profits. As pointed out in Chapter 3, the decision to hedge will lead to a worse outcome than the decision not to hedge roughly 50% of the time. The danger of making the treasury department a profit center is that the treasurer is motivated to become a speculator. This is liable to lead to the type of outcome experienced by Orange County, Procter & Gamble, or Gibson Greetings.

SUMMARY

The huge losses experienced from the use of derivatives have made many treasurers very wary. Following some of the losses, some nonfinancial corporations have announced plans to reduce or even eliminate their use of derivatives. This is unfortunate because derivatives provide treasurers with very efficient ways to manage risks.

The stories behind the losses emphasize the point, made as early as Chapter 1, that derivatives can be used for either hedging or speculation; that is, they can be used either to reduce risks or to take risks. Most losses occurred because derivatives were used inappropriately. Employees who had an implicit or explicit mandate to hedge their company's risks decided instead to speculate.

The key lesson to be learned from the losses is the importance of *internal controls*. Senior management within a company should issue a clear and unambiguous policy statement about how derivatives are to be used and the extent to which it is permissible for employees to take positions on movements in market variables. Management should then institute controls to ensure that the policy is carried out. It is a recipe for disaster to give individuals authority to trade derivatives without a close monitoring of the risks being taken.

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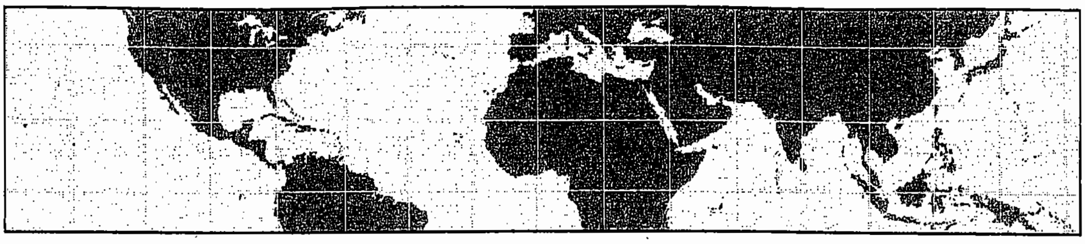

Glossary of Terms

ABS *See* Asset-Backed Security

Accrual Swap An interest rate swap where interest on one side accrues only when a certain condition is met.

Accrued Interest The interest earned on a bond since the last coupon payment date.

Adaptive Mesh Model A model developed by Figlewski and Gao that grafts a high-resolution tree on to a low-resolution tree so that there is more detailed modeling of the asset price in critical regions.

Agency Costs Costs arising from a situation where the agent (e.g., manager) is not motivated to act in the best interests of the principal (e.g., shareholder).

American Option An option that can be exercised at any time during its life.

Amortizing Swap A swap where the notional principal decreases in a predetermined way as time passes.

Analytic Result Result where answer is in the form of an equation.

Arbitrage A trading strategy that takes advantage of two or more securities being mispriced relative to each other.

Arbitrageur An individual engaging in arbitrage.

Asian Option An option with a payoff dependent on the average price of the underlying asset during a specified period.

Ask Price The price that a dealer is offering to sell an asset.

Asked Price *See* Ask Price.

Asset-Backed Security Security created from a portfolio of loans, bonds, credit card receivables, or other assets.

Asset-or-Nothing Call Option An option that provides a payoff equal to the asset price if the asset price is above the strike price and zero otherwise.

Asset-or-Nothing Put Option An option that provides a payoff equal to the asset price if the asset price is below the strike price and zero otherwise.

Asset Swap Exchanges the coupon on a bond for LIBOR plus a spread.

As-You-Like-It Option *See* Chooser Option.

At-the-Money Option An option in which the strike price equals the price of the underlying asset.

Average Price Call Option An option giving a payoff equal to the greater of zero and the amount by which the average price of the asset exceeds the strike price.

Average Price Put Option An option giving a payoff equal to the greater of zero and the amount by which the strike price exceeds the average price of the asset.

Average Strike Option An option that provides a payoff dependent on the difference between the final asset price and the average asset price.

Backdating Practice (often illegal) of marking a document with a date that precedes the current date.

Back Testing Testing a value-at-risk or other model using historical data.

Backwards Induction A procedure for working from the end of a tree to its beginning in order to value an option.

Barrier Option An option whose payoff depends on whether the path of the underlying asset has reached a barrier (i.e., a certain predetermined level).

Base Correlation Correlation that leads to the price of a 0% to X% CDO tranche being consistent with the market for a particular value of X.

Basel II New international regulations for calculating bank capital expected to come into effect in about 2007.

Basis The difference between the spot price and the futures price of a commodity.

Basis Point When used to describe an interest rate, a basis point is one hundredth of one percent (= 0.01%)

Basis Risk The risk to a hedger arising from uncertainty about the basis at a future time.

Basis Swap A swap where cash flows determined by one floating reference rate are exchanged for cash flows determined by another floating reference rate.

Basket Credit Default Swap Credit default swap where there are several reference entities.

Basket Option An option that provides a payoff dependent on the value of a portfolio of assets.

Bear Spread A short position in a put option with strike price K_1 combined with a long position in a put option with strike price K_2 where $K_2 > K_1$. (A bear spread can also be created with call options.)

Bermudan Option An option that can be exercised on specified dates during its life.

Beta A measure of the systematic risk of an asset.

Bid–Ask Spread The amount by which the ask price exceeds the bid price.

Bid–Offer Spread *See* Bid–Ask Spread.

Bid Price The price that a dealer is prepared to pay for an asset.

Binary Credit Default Swap Instrument where there is a fixed dollar payoff in the event of a default by a particular company.

Binary Option Option with a discontinuous payoff, e.g., a cash-or-nothing option or an asset-or-nothing option.

Binomial Model A model where the price of an asset is monitored over successive short periods of time. In each short period it is assumed that only two price movements are possible.

Binomial Tree A tree that represents how an asset price can evolve under the binomial model.

Bivariate Normal Distribution A distribution for two correlated variables, each of which is normal.

Black's Approximation An approximate procedure developed by Fischer Black for valuing a call option on a dividend-paying stock.

Black's Model An extension of the Black–Scholes model for valuing European options on futures contracts. As described in Chapter 26, it is used extensively in practice to value European options when the distribution of the asset price at maturity is assumed to be lognormal.

Black–Scholes Model A model for pricing European options on stocks, developed by Fischer Black, Myron Scholes, and Robert Merton.

Board Broker The individual who handles limit orders in some exchanges. The board broker makes information on outstanding limit orders available to other traders.

Bond Option An option where a bond is the underlying asset.

Bond Yield Discount rate which, when applied to all the cash flows of a bond, causes the present value of the cash flows to equal the bond's market price.

Bootstrap Method A procedure for calculating the zero-coupon yield curve from market data.

Boston Option *See* Deferred Payment Option.

Box Spread A combination of a bull spread created from calls and a bear spread created from puts.

Break Forward *See* Deferred Payment Option.

Brownian Motion *See* Wiener Process.

Bull Spread A long position in a call with strike price K_1 combined with a short position in a call with strike price K_2, where $K_2 > K_1$. (A bull spread can also be created with put options.)

Butterfly Spread A position that is created by taking a long position in a call with strike price K_1, a long position in a call with strike price K_3, and a short position in two calls with strike price K_2, where $K_3 > K_2 > K_1$ and $K_2 = 0.5(K_1 + K_3)$. (A butterfly spread can also be created with put options.)

Calendar Spread A position that is created by taking a long position in a call option that matures at one time and a short position in a similar call option that matures at a different time. (A calendar spread can also be created using put options.)

Calibration Method for implying a model's parameters from the prices of actively traded options.

Callable Bond A bond containing provisions that allow the issuer to buy it back at a predetermined price at certain times during its life.

Call Option An option to buy an asset at a certain price by a certain date.

Cancelable Swap Swap that can be canceled by one side on prespecified dates.

Cap *See* Interest Rate Cap.

Cap Rate The rate determining payoffs in an interest rate cap.

Capital Asset Pricing Model A model relating the expected return on an asset to its beta.

Caplet One component of an interest rate cap.

Cash Flow Mapping A procedure for representing an instrument as a portfolio of zero-coupon bonds for the purpose of calculating value at risk.

Cash-or-Nothing Call Option An option that provides a fixed predetermined payoff if the final asset price is above the strike price and zero otherwise.

Cash-or-Nothing Put Option An option that provides a fixed predetermined payoff if the final asset price is below the strike price and zero otherwise.

Cash Settlement Procedure for settling a futures contract in cash rather than by delivering the underlying asset.

CAT Bond Bond where the interest and, possibly, the principal paid are reduced if a particular category of "catastrophic" insurance claims exceed a certain amount.

CDD Cooling degree days. The maximum of zero and the amount by which the daily average temperature is greater than 65° Fahrenheit. The average temperature is the average of the highest and lowest temperatures (midnight to midnight).

CDO *See* Collateralized Debt Obligation.

CDO Squared An instrument in which the default risks in a portfolio of CDO tranches are allocated to new securities.

CDX NA IG Portfolio of 125 North American companies.

Cheapest-to-Deliver Bond The bond that is cheapest to deliver in the Chicago Board of Trade bond futures contract.

Cholesky Decomposition A method of sampling from a multivariate normal distribution.

Chooser Option An option where the holder has the right to choose whether it is a call or a put at some point during its life.

Class of Options *See* Option Class.

Clean Price of Bond The quoted price of a bond. The cash price paid for the bond (or dirty price) is calculated by adding the accrued interest to the clean price.

Clearinghouse A firm that guarantees the performance of the parties in an exchange-traded derivatives transaction (also referred to as a clearing corporation).

Clearing Margin A margin posted by a member of a clearinghouse.

CMO Collateralized Mortgage Obligation.

Collar *See* Interest Rate Collar.

Collateralization A system for posting collateral by one or both parties in a derivatives transaction.

Collateralized Debt Obligation A way of packaging credit risk. Several classes of securities (known as tranches) are created from a portfolio of bonds and there are rules for determining how the cost of defaults are allocated to classes.

Collateralized Mortgage Obligation (CMO) A mortgage-backed security where investors are divided into classes and there are rules for determining how principal repayments are channeled to the classes.

Combination A position involving both calls and puts on the same underlying asset.

Commission Brokers Individuals who execute trades for other people and charge a commission for doing so.

Commodity Futures Trading Commission A body that regulates trading in futures contracts in the United States.

Commodity Swap A swap where cash flows depend on the price of a commodity.

Compound Correlation Correlation implied from the market price of a CDO tranche.

Compound Option An option on an option.

Compounding Frequency This defines how an interest rate is measured.

Compounding Swap Swap where interest compounds instead of being paid.

Conditional Value at Risk (C-VaR) Expected loss during N days conditional on being in the $(100 - X)\%$ tail of the distribution of profits/losses. The variable N is the time horizon and $X\%$ is the confidence level.

Confirmation Contract confirming verbal agreement between two parties to a trade in the over-the-counter market.

Constant Elasticity of Variance (CEV) Model Model where the variance of the change in a variable in a short period of time is proportional to the value of the variable.

Constant Maturity Swap (CMS) A swap where a swap rate is exchanged for either a fixed rate or a floating rate on each payment date.

Constant Maturity Treasury Swap A swap where the yield on a Treasury bond is exchanged for either a fixed rate or a floating rate on each payment date.

Consumption Asset An asset held for consumption rather than investment.

Contango A situation where the futures price is above the expected future spot price.

Continuous Compounding A way of quoting interest rates. It is the limit as the assumed compounding interval is made smaller and smaller.

Control Variate Technique A technique that can sometimes be used for improving the accuracy of a numerical procedure.

Convenience Yield A measure of the benefits from ownership of an asset that are not obtained by the holder of a long futures contract on the asset.

Conversion Factor A factor used to determine the number of bonds that must be delivered in the Chicago Board of Trade bond futures contract.

Convertible Bond A corporate bond that can be converted into a predetermined amount of the company's equity at certain times during its life.

Convexity A measure of the curvature in the relationship between bond prices and bond yields.

Convexity Adjustment An overworked term. For example, it can refer to the adjustment necessary to convert a futures interest rate to a forward interest rate. It can also refer to the adjustment to a forward rate that is sometimes necessary when Black's model is used.

Copula A way of defining the correlation between variables with known distributions.

Cornish–Fisher Expansion An approximate relationship between the fractiles of a probability distribution and its moments.

Cost of Carry The storage costs plus the cost of financing an asset minus the income earned on the asset.

Counterparty The opposite side in a financial transaction.

Coupon Interest payment made on a bond.

Covariance Measure of the linear relationship between two variables (equals the correlation between the variables times the product of their standard deviations).

Covered Call A short position in a call option on an asset combined with a long position in the asset.

Credit Contagion The tendency of a default by one company to lead to defaults by other companies.

Credit Default Swap An instrument that gives the holder the right to sell a bond for its face value in the event of a default by the issuer.

Credit Derivative A derivative whose payoff depends on the creditworthiness of one or more companies or countries.

Credit Rating A measure of the creditworthiness of a bond issue.

Credit Ratings Transition Matrix A table showing the probability that a company will move from one credit rating to another during a certain period of time.

Credit Risk The risk that a loss will be experienced because of a default by the counterparty in a derivatives transaction.

Credit Spread Option Option whose payoff depends on the spread between the yields earned on two assets.

Credit Value at Risk The credit loss that will not be exceeded at some specified confidence level.

CreditMetrics A procedure for calculating credit value at risk.

Cross Hedging Hedging an exposure to the price of one asset with a contract on another asset.

Cumulative Distribution Function The probability that a variable will be less than x as a function of x.

Currency Swap A swap where interest and principal in one currency are exchanged for interest and principal in another currency.

Day Count A convention for quoting interest rates.

Day Trade A trade that is entered into and closed out on the same day.

Default Correlation Measures the tendency of two companies to default at about the same time.

Default Intensity *See* Hazard Rate.

Default Probability Density Measures the unconditional probability of default in a future short period of time.

Deferred Payment Option An option where the price paid is deferred until the end of the option's life.

Deferred Swap An agreement to enter into a swap at some time in the future (also called a forward swap).

Delivery Price Price agreed to (possibly some time in the past) in a forward contract.

Delta The rate of change of the price of a derivative with the price of the underlying asset.

Delta Hedging A hedging scheme that is designed to make the price of a portfolio of derivatives insensitive to small changes in the price of the underlying asset.

Delta-Neutral Portfolio A portfolio with a delta of zero so that there is no sensitivity to small changes in the price of the underlying asset.

DerivaGem The software accompanying this book.

Derivative An instrument whose price depends on, or is derived from, the price of another asset.

Deterministic Variable A variable whose future value is known.

Diagonal Spread A position in two calls where both the strike prices and times to maturity are different. (A diagonal spread can also be created with put options.)

Differential Swap A swap where a floating rate in one currency is exchanged for a floating rate in another currency and both rates are applied to the same principal.

Diffusion Process Model where value of asset changes continuously (no jumps).

Dirty Price of Bond Cash price of bond.

Discount Bond *See* Zero-Coupon Bond.

Discount Instrument An instrument, such as a Treasury bill, that provides no coupons.

Discount Rate The annualized dollar return on a Treasury bill or similar instrument expressed as a percentage of the final face value.

Dividend A cash payment made to the owner of a stock.

Dividend Yield The dividend as a percentage of the stock price.

Dollar Duration The product of a bond's modified duration and the bond price.

Down-and-In Option An option that comes into existence when the price of the underlying asset declines to a prespecified level.

Down-and-Out Option An option that ceases to exist when the price of the underlying asset declines to a prespecified level.

Downgrade Trigger A clause in a contract that states that the contract will be terminated with a cash settlement if the credit rating of one side falls below a certain level.

Drift Rate The average increase per unit of time in a stochastic variable.

Duration A measure of the average life a bond. It is also an approximation to the ratio of the proportional change in the bond price to the absolute change in its yield.

Duration Matching A procedure for matching the durations of assets and liabilities in a financial institution.

Dynamic Hedging A procedure for hedging an option position by periodically changing the position held in the underlying asset. The objective is usually to maintain a delta-neutral position.

Early Exercise Exercise prior to the maturity date.

Efficient Market Hypothesis A hypothesis that asset prices reflect relevant information.

Electronic Trading System of trading where a computer is used to match buyers and sellers.

Embedded Option An option that is an inseparable part of another instrument.

Empirical Research Research based on historical market data.

Employee Stock Option A stock option issued by company on its own stock and given to its employees as part of their remuneration.

Equilibrium Model A model for the behavior of interest rates derived from a model of the economy.

Equity Swap A swap where the return on an equity portfolio is exchanged for either a fixed or a floating rate of interest.

Eurocurrency A currency that is outside the formal control of the issuing country's monetary authorities.

Eurodollar A dollar held in a bank outside the United States.

Eurodollar Futures Contract A futures contract written on a Eurodollar deposit.

Eurodollar Interest Rate The interest rate on a Eurodollar deposit.

European Option An option that can be exercised only at the end of its life.

EWMA Exponentially weighted moving average.

Exchange Option An option to exchange one asset for another.

Ex-dividend Date When a dividend is declared, an ex-dividend date is specified. Investors who own shares of the stock just before the ex-dividend date receive the dividend.

Exercise Limit Maximum number of option contracts that can be exercised within a five-day period.

Exercise Multiple Ratio of stock price to strike price at time of exercise for employee stock option.

Exercise Price The price at which the underlying asset may be bought or sold in an option contract (also called the strike price).

Exotic Option A nonstandard option.

Expectations Theory The theory that forward interest rates equal expected future spot interest rates.

Expected Shortfall *See* Conditional Value at Risk.

Expected Value of a Variable The average value of the variable obtained by weighting the alternative values by their probabilities.

Expiration Date The end of life of a contract.

Explicit Finite Difference Method A method for valuing a derivative by solving the underlying differential equation. The value of the derivative at time t is related to three values at time $t + \Delta t$. It is essentially the same as the trinomial tree method.

Exponentially Weighted Moving Average Model A model where exponential weighting is used to provide forecasts for a variable from historical data. It is sometimes applied to variances and covariances in value at risk calculations.

Exponential Weighting A weighting scheme where the weight given to an observation depends on how recent it is. The weight given to an observation i time periods ago is λ times the weight given to an observation $i - 1$ time periods ago where $\lambda < 1$.

Exposure The maximum loss from default by a counterparty.

Extendable Bond A bond whose life can be extended at the option of the holder.

Extendable Swap A swap whose life can be extended at the option of one side to the contract.

Factor Source of uncertainty.

Factor analysis An analysis aimed at finding a small number of factors that describe most of the variation in a large number of correlated variables (similar to a principal components analysis).

FAS 123 Accounting standard in United States relating to employee stock options.

FAS 133 Accounting standard in United States relating to instruments used for hedging.

FASB Financial Accounting Standards Board.

Financial Intermediary A bank or other financial institution that facilitates the flow of funds between different entities in the economy.

Finite Difference Method A method for solving a differential equation.

Flat Volatility The name given to volatility used to price a cap when the same volatility is used for each caplet.

Flex Option An option traded on an exchange with terms that are different from the standard options traded by the exchange.

Flexi Cap Interest rate cap where there is a limit on the total number of caplets that can be exercised.

Floor *See* Interest Rate Floor.

Floor–Ceiling Agreement *See* Collar.

Floorlet One component of a floor.

Floor Rate The rate in an interest rate floor agreement.

Foreign Currency Option An option on a foreign exchange rate.

Forward Contract A contract that obligates the holder to buy or sell an asset for a predetermined delivery price at a predetermined future time.

Forward Exchange Rate The forward price of one unit of a foreign currency.

Forward Interest Rate The interest rate for a future period of time implied by the rates prevailing in the market today.

Forward Price The delivery price in a forward contract that causes the contract to be worth zero.

Forward Rate Rate of interest for a period of time in the future implied by today's zero rates.

Forward Rate Agreement (FRA) Agreement that a certain interest rate will apply to a certain principal amount for a certain time period in the future.

Forward Risk-Neutral World A world is forward risk-neutral with respect to a certain asset when the market price of risk equals the volatility of that asset.

Forward Start Option An option designed so that it will be at-the-money at some time in the future.

Forward Swap *See* Deferred Swap.

Futures Contract A contract that obligates the holder to buy or sell an asset at a predetermined delivery price during a specified future time period. The contract is settled daily.

Futures Option An option on a futures contract.

Futures Price The delivery price currently applicable to a futures contract.

Futures-Style Option Futures contract on the payoff from an option.

Gamma The rate of change of delta with respect to the asset price.

Gamma-Neutral Portfolio A portfolio with a gamma of zero.

GARCH Model A model for forecasting volatility where the variance rate follows a mean-reverting process.

Gaussian Copula Model A model for defining a correlation structure between two or more variables. In some credit derivatives models, it is used to define a correlation structure for times to default.

Gaussian Quadrature Procedure for integrating over a normal distribution.

Generalized Wiener Process A stochastic process where the change in a variable in time t has a normal distribution with mean and variance both proportional to t.

Geometric Average The nth root of the product of n numbers.

Geometric Brownian Motion A stochastic process often assumed for asset prices where the logarithm of the underlying variable follows a generalized Wiener process.

Girsanov's Theorem Result showing that when we change the measure (e.g., move from real world to risk-neutral world) the expected return of a variable changes but the volatility remains the same.

Greeks Hedge parameters such as delta, gamma, vega, theta, and rho.

Haircut Discount applied to the value of an asset for collateral purposes.

Hazard Rate Measures probability of default in a short period of time conditional on no earlier default.

HDD Heating degree days. The maximum of zero and the amount by which the daily average temperature is less than 65° Fahrenheit. The average temperature is the average of the highest and lowest temperatures (midnight to midnight).

Hedge A trade designed to reduce risk.

Hedger An individual who enters into hedging trades.

Hedge Ratio The ratio of the size of a position in a hedging instrument to the size of the position being hedged.

Historical Simulation A simulation based on historical data.

Historic Volatility A volatility estimated from historical data.

Holiday Calendar Calendar defining which days are holidays for the purposes of determining payment dates in a swap.

IMM Dates Third Wednesday in March, June, September, and December.

Implicit Finite Difference Method A method for valuing a derivative by solving the underlying differential equation. The value of the derivative at time $t + \Delta t$ is related to three values at time t.

Implied Correlation Correlation number implied from the price of a credit derivative using the Gaussian copula or similar model.

Implied Distribution A distribution for a future asset price implied from option prices.

Implied Tree A tree describing the movements of an asset price that is constructed to be consistent with observed option prices.

Implied Volatility Volatility implied from an option price using the Black–Scholes or a similar model.

Implied Volatility Function (IVF) Model Model designed so that it matches the market prices of all European options.

Inception Profit Profit created by selling a derivative for more than its theoretical value.

Index Amortizing Swap See indexed principal swap.

Index Arbitrage An arbitrage involving a position in the stocks comprising a stock index and a position in a futures contract on the stock index.

Index Futures A futures contract on a stock index or other index.

Index Option An option contract on a stock index or other index.

Indexed Principal Swap A swap where the principal declines over time. The reduction in the principal on a payment date depends on the level of interest rates.

Initial Margin The cash required from a futures trader at the time of the trade.

Instantaneous Forward Rate Forward rate for a very short period of time in the future.

Interest Rate Cap An option that provides a payoff when a specified interest rate is above a certain level. The interest rate is a floating rate that is reset periodically.

Interest Rate Collar A combination of an interest-rate cap and an interest rate floor.

Interest Rate Derivative A derivative whose payoffs are dependent on future interest rates.

Interest Rate Floor An option that provides a payoff when an interest rate is below a certain level. The interest rate is a floating rate that is reset periodically.

Interest Rate Option An option where the payoff is dependent on the level of interest rates.

Interest Rate Swap An exchange of a fixed rate of interest on a certain notional principal for a floating rate of interest on the same notional principal.

International Swaps and Derivatives Association Trade Association for over-the-counter derivatives and developer of master agreements used in over-the-counter contracts.

In-the-Money Option Either (a) a call option where the asset price is greater than the strike price or (b) a put option where the asset price is less than the strike price.

Intrinsic Value For a call option, this is the greater of the excess of the asset price over the strike price and zero. For a put option, it is the greater of the excess of the strike price over the asset price and zero.

Inverted Market A market where futures prices decrease with maturity.

Investment Asset An asset held by at least some individuals for investment purposes.

IO Interest Only. A mortgage-backed security where the holder receives only interest cash flows on the underlying mortgage pool.

ISDA *See* International Swaps and Derivatives Association.

Itô Process A stochastic process where the change in a variable during each short period of time of length Δt has a normal distribution. The mean and variance of the distribution are proportional to Δt and are not necessarily constant.

Itô's Lemma A result that enables the stochastic process for a function of a variable to be calculated from the stochastic process for the variable itself.

ITraxx Europe Portfolio of 125 investment-grade European companies.

Jump–Diffusion Model Model where asset price has jumps superimposed on to a diffusion process such as geometric Brownian motion.

Kurtosis A measure of the fatness of the tails of a distribution.

LEAPS Long-term equity anticipation securities. These are relatively long-term options on individual stocks or stock indices.

LIBID London interbank bid rate. The rate bid by banks on Eurocurrency deposits (i.e., the rate at which a bank is willing to borrow from other banks).

LIBOR London interbank offered rate. The rate offered by banks on Eurocurrency deposits (i.e., the rate at which a bank is willing to lend to other banks).

LIBOR Curve LIBOR zero-coupon interest rates as a function of maturity.

LIBOR-in-Arrears Swap Swap where the interest paid on a date is determined by the interest rate observed on that date (not by the interest rate observed on the previous payment date).

Limit Move The maximum price move permitted by the exchange in a single trading session.

Limit Order An order that can be executed only at a specified price or one more favorable to the investor.

Liquidity Preference Theory A theory leading to the conclusion that forward interest rates are above expected future spot interest rates.

Liquidity Premium The amount that forward interest rates exceed expected future spot interest rates.

Liquidity Risk Risk that it will not be possible to sell a holding of a particular instrument at its theoretical price.

Locals Individuals on the floor of an exchange who trade for their own account rather than for someone else.

Lognormal Distribution A variable has a lognormal distribution when the logarithm of the variable has a normal distribution.

Long Hedge A hedge involving a long futures position.

Long Position A position involving the purchase of an asset.

Lookback Option An option whose payoff is dependent on the maximum or minimum of the asset price achieved during a certain period.

Low Discrepancy Sequence *See* Quasi-random Sequence.

Maintenance Margin When the balance in a trader's margin account falls below the maintenance margin level, the trader receives a margin call requiring the account to be topped up to the initial margin level.

Margin The cash balance (or security deposit) required from a futures or options trader.

Margin Call A request for extra margin when the balance in the margin account falls below the maintenance margin level.

Market Maker A trader who is willing to quote both bid and offer prices for an asset.

Market Model A model most commonly used by traders.

Market Price of Risk A measure of the trade-offs investors make between risk and return.

Market Segmentation Theory A theory that short interest rates are determined independently of long interest rates by the market.

Marking to Market The practice of revaluing an instrument to reflect the current values of the relevant market variables.

Markov Process A stochastic process where the behavior of the variable over a short period of time depends solely on the value of the variable at the beginning of the period, not on its past history.

Martingale A zero drift stochastic process.

Maturity Date The end of the life of a contract.

Maximum Likelihood Method A method for choosing the values of parameters by maximizing the probability of a set of observations occurring.

Mean Reversion The tendency of a market variable (such as an interest rate) to revert back to some long-run average level.

Measure Sometimes also called a probability measure, it defines the market price of risk.

Modified Duration A modification to the standard duration measure so that it more accurately describes the relationship between proportional changes in a bond price and actual changes in its yield. The modification takes account of the compounding frequency with which the yield is quoted.

Money Market Account An investment that is initially equal to $1 and, at time t, increases at the very short-term risk-free interest rate prevailing at that time.

Monte Carlo Simulation A procedure for randomly sampling changes in market variables in order to value a derivative.

Mortgage-Backed Security A security that entitles the owner to a share in the cash flows realized from a pool of mortgages.

Naked Position A short position in a call option that is not combined with a long position in the underlying asset.

Netting The ability to offset contracts with positive and negative values in the event of a default by a counterparty.

Newton–Raphson Method An iterative procedure for solving nonlinear equations.

No-Arbitrage Assumption The assumption that there are no arbitrage opportunities in market prices.

No-Arbitrage Interest Rate Model A model for the behavior of interest rates that is exactly consistent with the initial term structure of interest rates.

Nonstationary Model A model where the volatility parameters are a function of time.

Nonsystematic Risk Risk that can be diversified away.

Normal Backwardation A situation where the futures price is below the expected future spot price.

Normal Distribution The standard bell-shaped distribution of statistics.

Normal Market A market where futures prices increase with maturity.

Notional Principal The principal used to calculate payments in an interest rate swap. The principal is "notional" because it is neither paid nor received.

Numeraire Defines the units in which security prices are measured. For example, if the price of IBM is the numeraire, all security prices are measured relative to IBM. If IBM is \$80 and a particular security price is \$50, the security price is 0.625 when IBM is the numeraire.

Numerical Procedure A method of valuing an option when no formula is available.

OCC Options Clearing Corporation. *See* Clearinghouse.

Offer Price *See* Ask Price.

Open Interest The total number of long positions outstanding in a futures contract (equals the total number of short positions).

Open Outcry System of trading where traders meet on the floor of the exchange

Option The right to buy or sell an asset.

Option-Adjusted Spread The spread over the Treasury curve that makes the theoretical price of an interest rate derivative equal to the market price.

Option Class All options of the same type (call or put) on a particular stock.

Option Series All options of a certain class with the same strike price and expiration date.

Order Book Official *See* Board Broker.

Out-of-the-Money Option Either (a) a call option where the asset price is less than the strike price or (b) a put option where the asset price is greater than the strike price.

Over-the-Counter Market A market where traders deal by phone. The traders are usually financial institutions, corporations, and fund managers.

Package A derivative that is a portfolio of standard calls and puts, possibly combined with a position in forward contracts and the asset itself.

Par Value The principal amount of a bond.

Par Yield The coupon on a bond that makes its price equal the principal.

Parallel Shift A movement in the yield curve where each point on the curve changes by the same amount.

Path-Dependent Option An option whose payoff depends on the whole path followed by the underlying variable—not just its final value.

Payoff The cash realized by the holder of an option or other derivative at the end of its life.

Plain Vanilla A term used to describe a standard deal.

P-Measure Real-world measure.

PO Principal Only. A mortgage-backed security where the holder receives only principal cash flows on the underlying mortgage pool.

Poisson Process A process describing a situation where events happen at random. The probability of an event in time Δt is $\lambda \, \Delta t$, where λ is the intensity of the process.

Portfolio Immunization Making a portfolio relatively insensitive to interest rates.

Portfolio Insurance Entering into trades to ensure that the value of a portfolio will not fall below a certain level.

Position Limit The maximum position a trader (or group of traders acting together) is allowed to hold.

Premium The price of an option.

Prepayment function A function estimating the prepayment of principal on a portfolio of mortgages in terms of other variables.

Principal The par or face value of a debt instrument.

Principal Components Analysis An analysis aimed at finding a small number of factors that describe most of the variation in a large number of correlated variables (similar to a factor analysis).

Program Trading A procedure where trades are automatically generated by a computer and transmitted to the trading floor of an exchange.

Protective Put A put option combined with a long position in the underlying asset.

Pull-to-Par The reversion of a bond's price to its par value at maturity.

Put–Call Parity The relationship between the price of a European call option and the price of a European put option when they have the same strike price and maturity date.

Put Option An option to sell an asset for a certain price by a certain date.

Puttable Bond A bond where the holder has the right to sell it back to the issuer at certain predetermined times for a predetermined price.

Puttable Swap A swap where one side has the right to terminate early.

Q-Measure Risk-neutral measure.

Quanto A derivative where the payoff is defined by variables associated with one currency but is paid in another currency.

Quasi-random Sequences A sequences of numbers used in a Monte Carlo simulation that are representative of alternative outcomes rather than random.

Rainbow Option An option whose payoff is dependent on two or more underlying variables.

Range Forward Contract The combination of a long call and short put or the combination of a short call and long put.

Ratchet Cap Interest rate cap where the cap rate applicable to an accrual period equals the rate for the previous accrual period plus a spread.

Real Option Option involving real (as opposed to financial) assets. Real assets include land, plant, and machinery.

Rebalancing The process of adjusting a trading position periodically. Usually the purpose is to maintain delta neutrality.

Recovery Rate Amount recovered in the event of a default as a percent of the face value.

Reference Entity Company for which default protection is bought in a credit default swap.

Repo Repurchase agreement. A procedure for borrowing money by selling securities to a counterparty and agreeing to buy them back later at a slightly higher price.

Repo Rate The rate of interest in a repo transaction.

Reset Date The date in a swap or cap or floor when the floating rate for the next period is set.

Reversion Level The level that the value of a market variable (e.g., an interest rate) tends to revert.

Rho Rate of change of the price of a derivative with the interest rate.

Rights Issue An issue to existing shareholders of a security giving them the right to buy new shares at a certain price.

Risk-Free Rate The rate of interest that can be earned without assuming any risks.

Risk-Neutral Valuation The valuation of an option or other derivative assuming the world is risk neutral. Risk-neutral valuation gives the correct price for a derivative in all worlds, not just in a risk-neutral world.

Risk-Neutral World A world where investors are assumed to require no extra return on average for bearing risks.

Roll Back *See* Backwards Induction.

Scalper A trader who holds positions for a very short period of time.

Scenario Analysis An analysis of the effects of possible alternative future movements in market variables on the value of a portfolio.

SEC Securities and Exchange Commission.

Settlement Price The average of the prices that a contract trades for immediately before the bell signaling the close of trading for a day. It is used in mark-to-market calculations.

Short Hedge A hedge where a short futures position is taken.

Short Position A position assumed when traders sell shares they do not own.

Short Rate The interest rate applying for a very short period of time.

Short Selling Selling in the market shares that have been borrowed from another investor.

Short-Term Risk-Free Rate *See* Short Rate.

Shout Option An option where the holder has the right to lock in a minimum value for the payoff at one time during its life.

Simulation *See* Monte Carlo Simulation.

Specialist An individual responsible for managing limit orders on some exchanges. The specialist does not make the information on outstanding limit orders available to other traders.

Speculator An individual who is taking a position in the market. Usually the individual is betting that the price of an asset will go up or that the price of an asset will go down.

Spot Interest Rate *See* Zero-Coupon Interest Rate.

Spot Price The price for immediate delivery.

Spot Volatilities The volatilities used to price a cap when a different volatility is used for each caplet.

Spread Option An option where the payoff is dependent on the difference between two market variables.

Spread Transaction A position in two or more options of the same type.

Static Hedge A hedge that does not have to be changed once it is initiated.

Static Options Replication A procedure for hedging a portfolio that involves finding another portfolio of approximately equal value on some boundary.

Step-up Swap A swap where the principal increases over time in a predetermined way.

Sticky Cap Interest rate cap where the cap rate applicable to an accrual period equals the capped rate for the previous accrual period plus a spread.

Stochastic Process An equation describing the probabilistic behavior of a stochastic variable.

Stochastic Variable A variable whose future value is uncertain.

Stock Dividend A dividend paid in the form of additional shares.

Stock Index An index monitoring the value of a portfolio of stocks.

Stock Index Futures Futures on a stock index.

Stock Index Option An option on a stock index.

Stock Option Option on a stock.

Stock Split The conversion of each existing share into more than one new share.

Storage Costs The costs of storing a commodity.

Straddle A long position in a call and a put with the same strike price.

Strangle A long position in a call and a put with different strike prices.

Strap A long position in two call options and one put option with the same strike price.

Stress Testing Testing of the impact of extreme market moves on the value of a portfolio.

Strike Price The price at which the asset may be bought or sold in an option contract (also called the exercise price).

Strip A long position in one call option and two put options with the same strike price.

Strip Bonds Zero-coupon bonds created by selling the coupons on Treasury bonds separately from the principal.

Subprime Mortgage Mortgage granted to borrower with a poor credit history or no credit history.

Swap An agreement to exchange cash flows in the future according to a prearranged formula.

Swap Rate The fixed rate in an interest rate swap that causes the swap to have a value of zero.

Swaption An option to enter into an interest rate swap where a specified fixed rate is exchanged for floating.

Swing Option Energy option in which the rate of consumption must be between a minimum and maximum level. There is usually a limit on the number of times the option holder can change the rate at which the energy is consumed.

Synthetic CDO A CDO created by selling credit default swaps.

Synthetic Option An option created by trading the underlying asset.

Systematic Risk Risk that cannot be diversified away.

Tailing the Hedge A procedure for adjusting the number of futures contracts used for hedging to reflect daily settlement.

Tail Loss *See* Conditional Value at Risk.

Take-and-Pay Option *See* Swing Option.

Term Structure of Interest Rates The relationship between interest rates and their maturities.

Terminal Value The value at maturity.

Theta The rate of change of the price of an option or other derivative with the passage of time.

Time Decay *See* Theta.

Time Value The value of an option arising from the time left to maturity (equals an option's price minus its intrinsic value).

Timing Adjustment Adjustment made to the forward value of a variable to allow for the timing of a payoff from a derivative.

Total Return Swap A swap where the return on an asset such as a bond is exchanged for LIBOR plus a spread. The return on the asset includes income such as coupons and the change in value of the asset.

Tranche One of several securities that have different risk attributes. Examples are the tranches of a CDO or CMO.

Transaction Costs The cost of carrying out a trade (commissions plus the difference between the price obtained and the midpoint of the bid–offer spread).

Treasury Bill A short-term non-coupon-bearing instrument issued by the government to finance its debt.

Treasury Bond A long-term coupon-bearing instrument issued by the government to finance it debt.

Treasury Bond Futures A futures contract on Treasury bonds.

Treasury Note *See* Treasury Bond. (Treasury notes have maturities of less than 10 years.)

Treasury Note Futures A futures contract on Treasury notes.

Tree Representation of the evolution of the value of a market variable for the purposes of valuing an option or other derivative.

Trinomial Tree A tree where there are three branches emanating from each node. It is used in the same way as a binomial tree for valuing derivatives.

Triple Witching Hour A term given to the time when stock index futures, stock index options, and options on stock index futures all expire together.

Underlying Variable A variable on which the price of an option or other derivative depends.

Unsystematic Risk *See* Nonsystematic Risk.

Up-and-In Option An option that comes into existence when the price of the underlying asset increases to a prespecified level.

Up-and-Out Option An option that ceases to exist when the price of the underlying asset increases to a prespecified level.

Uptick An increase in price.

Value at Risk A loss that will not be exceeded at some specified confidence level.

Variance–Covariance Matrix A matrix showing variances of, and covariances between, a number of different market variables.

Variance-Gamma Model A pure jump model where small jumps occur often and large jumps occur infrequently.

Variance Rate The square of volatility.

Variance Reduction Procedures Procedures for reducing the error in a Monte Carlo simulation.

Variance Swap Swap where the realized variance rate during a period is exchanged for a fixed variance rate. Both are applied to a notional principal.

Variation Margin An extra margin required to bring the balance in a margin account up to the initial margin when there is a margin call.

Vega The rate of change in the price of an option or other derivative with volatility.

Vega-Neutral Portfolio A portfolio with a vega of zero.

Vesting Period Period during which an option cannot be exercised.

VIX Index Index of the volatility of the S&P 500.

Volatility A measure of the uncertainty of the return realized on an asset.

Volatility Skew A term used to describe the volatility smile when it is nonsymmetrical.

Volatility Smile The variation of implied volatility with strike price.

Volatility Surface A table showing the variation of implied volatilities with strike price and time to maturity.

Volatility Swap Swap where the realized volatility during a period is exchanged for a fixed volatility. Both percentage volatilities are applied to a notional principal.

Volatility Term Structure The variation of implied volatility with time to maturity.

Warrant An option issued by a company or a financial institution. Call warrants are frequently issued by companies on their own stock.

Weather Derivative Derivative where the payoff depends on the weather.

Wiener Process A stochastic process where the change in a variable during each short period of time of length Δt has a normal distribution with a mean equal to zero and a variance equal to Δt.

Wild Card Play The right to deliver on a futures contract at the closing price for a period of time after the close of trading.

Writing an Option Selling an option.

Yield A return provided by an instrument.

Yield Curve *See* Term Structure.

Zero-Coupon Bond A bond that provides no coupons.

Zero-Coupon Interest Rate The interest rate that would be earned on a bond that provides no coupons.

Zero-Coupon Yield Curve A plot of the zero-coupon interest rate against time to maturity.

Zero Curve *See* Zero-Coupon Yield Curve.

Zero Rate *See* Zero-Coupon Interest Rate.