1 ■ Basic Ideas and Examples

ALL INTRODUCTORY TEXTBOOKS begin by attempting to convince their student readers that the subject is of great importance in the world and therefore merits their attention. The physical sciences and engineering claim to be the basis of modern technology and therefore of modern life; the social sciences discuss big issues of governance, such as democracy and taxation; the humanities claim to revive your soul after it has been deadened by exposure to the physical and social sciences and to engineering. Where does the subject of games of strategy, often called game theory, fit into this picture, and why should you study it?

We offer a practical motivation that is much more individual and probably closer to your personal concerns than most other subjects. You play games of strategy all the time: with your parents, siblings, friends, and enemies, and even with your professors. You have probably acquired a lot of instinctive expertise in playing such games, and we hope you will be able to connect what you have already learned to the discussion that follows. We will build on your experience, systematize it, and develop it to the point where you will be able to improve your strategic skills and use them more methodically. Opportunities for such uses will appear throughout your life; you will go on playing such games with your employers, employees, spouses, children, and even strangers.

Not that the subject lacks wider importance. Similar games are played in business, politics, diplomacy, and war—in fact, whenever people interact to strike mutually agreeable deals or to resolve conflicts. Being able to recognize such games will enrich your understanding of the world around you and will make you a better participant in all its affairs. Understanding games of strategy will also have a more immediate payoff in your study of many other subjects.

Economics and business courses already use a great deal of game-theoretic thinking. Political science, psychology, and philosophy are also using game theory to study interactions, as is biology, which has been importantly influenced by the concepts of evolutionary games and has in turn exported these ideas to economics. Psychology and philosophy also interact with the study of games of strategy. Game theory provides concepts and techniques of analysis for many disciplines—one might even say all disciplines except those dealing with completely inanimate objects.

1 WHAT IS A GAME OF STRATEGY?

The word *game* may convey an impression that our subject is frivolous or unimportant in the larger scheme of things—that it deals with trivial pursuits such as gambling and sports when the world is full of weightier matters such as war and business and your education, career, and relationships. Actually, games of strategy are not “just a game”; all of these weightier matters are instances of games, and game theory helps us understand them all. But it will not hurt to start with game theory as applied to gambling or sports.

Most games include chance, skill, and strategy in varying proportions. Playing double or nothing on the toss of a coin is a game of pure chance, unless you have exceptional skill in doctoring or tossing coins. A hundred-yard dash is a game of pure skill, although some chance elements can creep in; for example, a runner may simply have a slightly off day for no clear reason.

Strategy is a skill of a different kind. In the context of sports, it is a part of the mental skill needed to play well; it is the calculation of how best to use your physical skill. For example, in tennis, you develop physical skill by practicing your serves (first serves hard and flat, second serves with spin or kick) and passing shots (hard, low, and accurate). The strategic skill is knowing where to put your serve (wide, or on the T) or passing shot (crosscourt, or down the line). In football, you develop physical skills such as blocking and tackling, running and catching, and throwing. Then the coach, knowing the physical skills of his own team and those of the opposing team, calls the plays that best exploit his team’s skills and the other team’s weaknesses. The coach’s calculation constitutes the strategy. The physical game of football is played on the gridiron by jocks;

the strategic game is played in the offices and on the sidelines by coaches and by nerdy assistants.

A hundred-yard dash is a matter of exercising your physical skill as best you can; it offers no opportunities to observe and react to what other runners in the race are doing and therefore no scope for strategy. Longer races do entail strategy—whether you should lead to set the pace, how soon before the finish you should try to break away, and so on.

Strategic thinking is essentially about your interactions with others, as they do similar thinking at the same time and about the same situation. Your competitors in a marathon may try to frustrate or facilitate your attempts to lead, given what they think best suits their interests. Your opponent in tennis tries to guess where you will put your serve or passing shot. The opposing coach in football calls the play that will best counter what he thinks your coach will call. Of course, just as you must take into account what the other player is thinking, he is taking into account what you are thinking. Game theory is the analysis, or science, if you like, of such interactive decision making.

When you think carefully before you act—when you are aware of your objectives or preferences and of any limitations or constraints on your actions, and choose your actions in a calculated way to do the best according to your own criteria—you are said to be behaving rationally. Game theory adds another dimension to rational behavior—namely, interaction with other equally rational decision makers. In other words, game theory is the science of rational behavior in interactive situations.

However, rationality does have limits. Recent research in psychology and behavioral economics has shown that many decisions are made instinctively and are based on rules or heuristics. Good strategic thinking will recognize the possibility that other players may not be rational and will also anticipate and prepare for one’s own instinctive departures from rationality. We will include these aspects of thinking as we pursue our analyses of strategy.

We do not claim that game theory will teach you the secrets of perfect play or ensure that you will never lose. For one thing, your opponent can read the same book, and both of you cannot win all the time. More importantly, many games are complex and subtle, and most actual situations include enough idiosyncratic or chance elements that game theory cannot hope to offer surefire recipes for action. What it can do is provide some general principles for thinking about strategic interactions. You have to supplement these principles with details specific to your situation before you can devise a successful strategy for it. Good strategists mix the science of game theory with their own experience; one might say that game playing is as much art as science. We will develop the general ideas of the science but will also point out its limitations and tell you when the art is more important.

You may think that you have already acquired the art of game playing from your experience or instinct, but you will find the study of the science useful nonetheless. The science systematizes general principles that are common to many seemingly different contexts or applications. Without general principles, you would have to figure out from scratch each new situation that requires strategic thinking. That would be especially difficult to do in new areas of application—for example, if you learned your art by playing games against parents and siblings and must now practice strategy against business competitors. The general principles of game theory provide you with a ready reference point. With this foundation in place, you can proceed much more quickly and confidently to acquire and add the situation-specific features or elements of the art to your thinking and action.

2 SOME EXAMPLES AND STORIES OF STRATEGIC GAMES

With the aims announced in Section 1, we will begin by offering you some simple examples, many of them taken from situations that you have probably encountered in your own life, where strategy is of the essence. In each case, we will point out the crucial strategic principle. Each of these principles will be discussed more fully in a later chapter, and after each example we will tell you where the details can be found. But don’t jump to those details right away; just read all the examples here to get a preliminary idea of the whole scope of strategy and of strategic games.

A. Which Passing Shot?

Tennis at its best consists of memorable duels between top players: Roger Federer versus Rafael Nadal, Serena Williams versus Venus Williams, Pete Sampras versus Andre Agassi, and Martina Navratilova versus Chris Evert. Picture the 1983 U.S. Open final between Evert and Navratilova.[1] Navratilova at the net has just volleyed to Evert on the baseline. Evert is about to hit a passing shot. Should she go down the line or crosscourt? And should Navratilova expect a down-the-line shot and lean slightly that way or expect a crosscourt shot and lean the other way?

Conventional wisdom favors the down-the-line shot. The ball has a shorter distance to travel to the net, so the other player has less time to react. But this does not mean that Evert should use that shot all the time. If she did, Navratilova would confidently come to expect it and prepare for it, and the shot would not be so successful. To improve the success of the down-the-line passing shot, Evert has to use the crosscourt shot often enough to keep Navratilova guessing on any single instance. Similarly, in football, with a yard to go on third down, a run up the middle is the percentage play—that is, the one used most often—but the offense must throw a pass occasionally in such situations "to keep the defense honest."

Thus, the most important general principle of such situations is not what Evert *should* do but what she *should not* do: She should not do the same thing all the time or systematically. If she did, then Navratilova would learn to cover that shot, and Evert's chances of success would fall.

Not doing any one thing systematically means more than not playing the same shot in every situation of this kind. Evert should not even mechanically switch back and forth between the two shots. Navratilova would spot and exploit this pattern or, indeed, any other detectable system. Evert must make the choice on each particular occasion *at random* to prevent this exploitation.

This general idea of "mixing one's plays" is well known, even to sports commentators on TV. But there is more to the idea, and these further aspects require analysis in greater depth. Why is down-the-line the percentage shot? Should one play it 80% of the time, or 90%, or 99%? Does it make any difference if the occasion is particularly big; for example, does one throw that pass on third down in the regular season but not in the Super Bowl? In actual practice, just how does one mix one's plays? What happens when a third possibility (the lob) is introduced? We will examine and answer such questions in Chapter 7.

The movie *The Princess Bride* (1987) illustrates the same idea in the "battle of wits" between the hero (Westley) and a villain (Vizzini). Westley is to poison one of two wineglasses out of Vizzini's sight, and Vizzini is to decide who will drink from which glass. Vizzini goes through a number of convoluted arguments as to why Westley should poison one glass. But all of the arguments are innately contradictory, because Westley can anticipate Vizzini's logic and choose to put the poison in the other glass. Conversely, if Westley uses any specific logic or system to choose one glass, Vizzini can anticipate that and drink from the other glass, leaving Westley to drink from the poisoned one. Thus, Westley's strategy has to be random or unsystematic.

The scene illustrates something else as well. In the film, Vizzini loses the game and with it his life. But it turns out that Westley had poisoned both glasses; over the last several years, he had built up immunity to the poison. So Vizzini was actually playing the game under a fatal information disadvantage. Players can sometimes cope with such asymmetries of information; Chapters 9 and 14 examine when and how they can do so.

B. The GPA Rat Race

Imagine that you are enrolled in a course that is graded on a curve. No matter how well you do in absolute terms, only 40% of the students will get As, and only 40% will get Bs. Therefore, you must work hard, not just in absolute terms, but relative to how hard your classmates (actually, “class enemies” seems a more fitting term in this context) work. All of you recognize this, and after the first lecture you hold an impromptu meeting in which all students agree not to work too hard. As weeks pass by, the temptation to get an edge on the rest of the class by working just that little bit harder becomes overwhelming. After all, the others are not able to observe your work in any detail, nor do they have any real hold over you. And the benefits of an improvement in your grade point average are substantial. So you hit the library more often and stay up a little longer.

The trouble is, everyone else in the class is doing the same. Therefore, your grade is no better than it would have been if you and everyone else had abided by the agreement. The only difference is that all of you have spent more time working than you would have liked.

This scenario is an example of the prisoners’ dilemma.[2] In the original story, two suspects in a crime are being separately interrogated and invited to confess. One of them, say A, is told, “If the other suspect, B, does not confess, then you can cut a very good deal for yourself by confessing. But if B does confess, then you would do well to confess, too; otherwise the court will be especially tough on you. So you should confess no matter what B does.” B is urged to confess with the use of similar reasoning. Faced with these choices, both A and B confess. But it would have been better for both if neither had confessed, because the police had no really compelling evidence against them.

Your situation in the class is similar. If the others slack off, then you can get a much better grade by working hard; if the others work hard, then you had better do the same, or else you will get a poor grade. You may even think that the label “prisoner” is fitting for a group of students trapped in a required course.

Professors and schools have their own prisoners’ dilemmas. Each professor can make her course look good or attractive by grading it slightly more liberally, and each school can place its students in better jobs or attract better applicants by grading all of its courses a little more liberally. Of course, when all do this, none has any advantage over the others; the only result is rampant grade inflation, which compresses the spectrum of grades and therefore makes it difficult to distinguish abilities.

People often think that in every game there must be a winner and a loser. The prisoners’ dilemma is different—all players can come out losers. People play, and lose, such games every day, and their losses can range from minor inconveniences to potential disasters. Spectators at a sports event stand up to get a better view, but when all stand, no one has a better view than when they were all sitting. Superpowers acquire more weapons to get an edge over their rivals, but when both do so, the balance of power is unchanged; all that has happened is that both have spent economic resources that they could have used for better purposes, and the risk of accidental war has escalated. The magnitude of the potential cost of such games to all players makes it important to understand the ways in which mutually beneficial cooperation can be achieved and sustained. All of Chapter 10 deals with the study of this game.

The prisoners’ dilemma is potentially a lose-lose game, but there are win-win games, too. International trade is an example: When each country produces more of what it can produce relatively well, all share in the fruits of this international division of labor. But successful bargaining about the division of the pie is needed if the full potential of trade is to be realized. The same applies to many other bargaining situations. We will study these situations in Chapter 17.

C. “We Can’t Take the Exam Because We Had a Flat Tire”

Here is a story, probably apocryphal, that circulates on the undergraduate e-mail networks and which each of us has independently received from our students:

> There were two friends taking chemistry at Duke. Both had done pretty well on all of the quizzes, the labs, and the midterm, so that going into the final they each had a solid A. They were so confident the weekend before the final that they decided to go to a party at the University of Virginia. The party was so good that they slept all day Sunday and got back to Duke too late to study for the chemistry final that was scheduled for Monday morning. Rather than take the final unprepared, they went to the professor with a sob story. They said they had gone up to UVA and had planned to come back in good time to study for the final, but had a flat tire on the way back. Because they didn’t have a spare, they had spent most of Sunday night looking for help. Now they were really too tired, so could they please have a makeup final the next day? The professor thought it over and agreed.
>
> The two studied all of Monday evening and came well prepared on Tuesday morning. The professor placed them in separate rooms and handed the test to each. The first question on the first page, worth 10 points, was very easy. Each of them wrote a good answer and, greatly relieved, turned the page. It had just one question, worth 90 points. It was: “Which tire?”

The story has two important strategic lessons for future partygoers. The first is to recognize that the professor may be an intelligent game player. He may suspect some trickery on the part of the students and may use some device to catch them. Given their excuse, the test question was the likeliest such device. They should have foreseen it and prepared their answer in advance. This idea that one should look ahead to future moves in the game and then reason backward to calculate one’s best current action is a very general principle of strategy, which we will elaborate on in Chapter 3. We will also use it, most notably, in Chapter 8.

But it may not be possible to foresee all such professorial countermoves; after all, professors have much more experience seeing through students’ excuses than students have making up such excuses. If the two students in the story are unprepared, can they independently produce a mutually consistent lie? If each picks a tire at random, the chances are only 25% that the two will pick the same one. (Why?) Can they do better?

You may think that the front tire on the passenger side is the one most likely to suffer a flat because a nail or a shard of glass is more likely to lie closer to that side of the road than to the middle, so the front tire on that side will encounter the nail or glass first. You may think this is good logic, but that is not enough to make your choice a good one. What matters is not the logic of the choice, but making the same choice as your friend does. Therefore, you have to think about whether your friend would use the same logic and would consider that choice equally obvious. But even that is not the end of the chain of reasoning. Would your friend think that the choice would be equally obvious to you? And so on. The point is not whether a choice is obvious or logical, but whether it is obvious to the other that it is obvious to you that it is obvious to the other . . . In other words, what is needed is a convergence of expectations about what should be chosen in such circumstances. Such a commonly expected strategy on which game players can successfully coordinate is called a focal point.

There is nothing general or intrinsic to the structure of these games that creates such convergence. In some games, a focal point may exist because of chance circumstances surrounding the labeling of strategies or because of some experience or knowledge shared by the players. For example, if the front passenger side of a car were for some reason called the Duke side, then two Duke students would be very likely to choose that tire without any need for explicit prior understanding. Or, if the front driver's side of all cars were painted orange (for safety, to be easily visible to oncoming cars), then two Princeton students would be very likely to choose that tire, because orange is the Princeton color. But without some such clue, tacit coordination might not be possible at all.

We will study focal points in more detail in Chapter 4. Here, in closing, we merely point out that when asked to make this choice in classrooms, more than 50% of students choose the front driver's side. They are generally unable to explain why, except to say that it seems the obvious choice.

D. Why Are Professors So Mean?

Many professors have inflexible rules against giving makeup exams and accepting late submission of problem sets or term papers. Students think these professors must be really hardhearted to behave in this way. The true strategic reason is often exactly the opposite. Most professors are kindhearted and would like to give their students every reasonable break and accept any reasonable excuse. The trouble lies in judging what is reasonable. It is hard to distinguish between similar excuses and almost impossible to verify their truth. The professor knows that on each occasion she will end up giving the student the benefit of the doubt. But the professor also knows that this is a slippery slope. As students come to know that the professor is a soft touch, they will procrastinate more and produce ever-flimsier excuses. Deadlines will cease to mean anything, and examinations will become a chaotic mix of postponements and makeup tests.

Often the only way to avoid this slippery slope is to refuse to take even the first step down it. Refusal to accept any excuses is the only realistic alternative to accepting them all. By making an advance commitment to the "no excuses" strategy, the professor avoids the temptation to give in to all.

But how can a softhearted professor maintain such a hardhearted commitment? She must find some way to make a refusal firm and credible. The simplest way is to hide behind an administrative procedure or university-wide policy. "I wish I could accept your excuse, but the university won't let me" not only puts the professor in a nicer light, but also removes the temptation by genuinely leaving her no choice in the matter. Of course, the rules may be made by the same collectivity of professors that hides behind them, but once they are made, no individual professor can unmake the rules in any particular instance.

If the university does not provide such a general shield, then the professor can try to make up commitment devices of her own. For example, she can make a clear and firm announcement of the policy at the beginning of the course. Any time an individual student asks for an exception, the professor can then invoke a fairness principle, saying, “If I do this for you, I would have to do it for everyone.” Or the professor can acquire a reputation for toughness by acting tough a few times. This may be unpleasant and run against her true inclination, but it helps in the long run over her whole career. If a professor is believed to be tough, few students will try excuses on her, so she will actually suffer less pain in denying them.

We will study commitments, and related strategies such as threats and promises, in considerable detail in Chapter 8.

E. Roommates and Families on the Brink

You are sharing an apartment with one or more other students. You notice that the apartment is nearly out of dishwasher detergent, paper towels, cereal, beer, and other items. You have an agreement to share the actual expenses, but the trip to the store takes time. Do you spend your own time going to the store or do you hope that someone else will spend his, leaving you more time to study or relax? Do you go and buy the soap or stay in and watch TV to catch up on the soap operas?[3]

In many situations of this kind, the waiting game goes on for quite a while before someone who is really impatient for one of the items (usually beer) gives in and spends the time for the shopping trip. Things may deteriorate to the point of serious quarrels or even breakups among the roommates.

This game of strategy can be viewed from two perspectives. In one, each of the roommates is regarded as having a simple binary choice—to do the shopping or not. The best outcome for you is that someone else does the shopping and you stay at home; the worst is that you do the shopping while the others get to use their time better. If more than one roommate does the shopping (unknown to one another, on the way home from school or work), there is unnecessary duplication and perhaps some waste of perishables; if no one does the shopping, there can be serious inconvenience or even disaster if the toilet paper runs out at a crucial time.

This binary choice game is analogous to the game of chicken that used to be played by American teenagers. Two of them drove their cars toward each other. The first to swerve to avoid a collision was the loser (chicken); the one who kept driving straight was the winner. We will analyze the game of chicken further in Chapter 4 and in Chapters 7, 11, and 12.

A more interesting dynamic perspective on the same situation regards it as a war of attrition, where each roommate tries to wait out the others, hoping that someone else's patience will run out first. In the meantime, the risk escalates that the apartment will run out of something critical, leading to serious inconvenience or a blowup. Each player lets the risk escalate to the point of his own tolerance; the one revealed to have the least tolerance loses. Each sees how close to the brink of disaster the others will let the situation go. Hence the name *brinkmanship* for this strategy and this game. It is a dynamic version of chicken, offering richer and more interesting possibilities.

One of us (Dixit) was privileged to observe a brilliant example of brinkmanship at a dinner party one Saturday evening. Before dinner, the company was sitting in the living room when the host's 15-year-old daughter appeared at the door and said, "Bye, Dad." The father asked, "Where are you going?" and the daughter replied, "Out." After a pause that was only a couple of seconds but seemed much longer, the host said, "All right, bye."

Your strategic observer of this scene was left thinking how it might have gone differently. The host might have asked, "With whom?" and the daughter might have replied, "Friends." The father could have refused permission unless the daughter told him exactly where and with whom she would be. One or the other might have capitulated at some such later stage of this exchange, or it could have led to a blowup.

This was a risky game for both the father and the daughter to play. The daughter might have been punished or humiliated in front of strangers; an argument could have ruined the father's evening with his friends. Each had to judge how far to push the process, without being fully sure whether and when the other might give in or whether there would be an unpleasant scene. The risk of an explosion would increase as the father tried harder to force the daughter to answer and as she defied each successive demand.

In this respect, the game played by the father and the daughter was just like that between a union and a company’s management who are negotiating a labor contract, or between two superpowers that are encroaching on each other’s sphere of influence in the world. Neither side can be fully sure of the other’s intentions, so each side explores them through a succession of small incremental steps, each of which escalates the risk of mutual disaster. The daughter in our story was exploring previously untested limits on her freedom; the father was exploring previously untested—and perhaps unclear even to himself—limits on his authority.

This exchange was an example of brinkmanship—a game of escalating mutual risk—par excellence. Such games can end in one of two ways. In the first, one of the players reaches the limit of his own tolerance for risk and concedes. (The father in our story conceded quickly, at the very first step. Other fathers might be more successful disciplinarians, and their daughters might not even initiate a game like this.) In the second, before either player has conceded, the risk that they both fear comes about, and the blowup (the strike or the war) occurs. The feud in our host’s family ended “happily”; although the father conceded and the daughter won, a blowup would have been much worse for both.

We will analyze the strategy of brinkmanship and the dynamic version of chicken more fully in Chapter 9; in Chapter 13, we will examine a particularly important instance of it—namely, the Cuban missile crisis of 1962.

F. The Dating Game

When you go on a date, you want to show off the best attributes of your personality to your date and to conceal the worst ones. Of course, you cannot hope to conceal them forever if the relationship progresses, but you are resolved to improve or hope that by that stage the other person will accept the bad things about you with the good ones. And you know that the relationship will not progress at all unless you make a good first impression; you won’t get a second chance to do so.

Of course, you want to find out everything, good and bad, about the other person. But you know that if the other is as good at the dating game as you are, he or she will similarly try to show the best side and hide the worst. You will think through the situation more carefully and try to figure out which signs of good qualities are real and which ones can easily be put on for the sake of making a good impression. Even the worst slob can easily appear well groomed for a big date; ingrained habits of courtesy and manners that are revealed in a hundred minor details may be harder to simulate for a whole evening. Flowers are relatively cheap; more expensive gifts may have value not for intrinsic reasons, but as credible evidence of how much the other person is willing to sacrifice for you. And the currency in which the gift is given may have differing significance depending on the context; from a millionaire, a diamond may be worth less in this regard than the act of giving up valuable time for your company or for time spent on some activity at your request.

You should also recognize that your date will similarly scrutinize your actions for their information content. Therefore, you should take actions that are credible signals of your true good qualities, not actions that anyone can imitate. This is important not only on a first date; revealing, concealing, and eliciting information about each partner's deepest intentions remain important throughout a relationship. Here is a story to illustrate this principle:

> Once upon a time in New York City there lived a man and a woman who had separate rent-controlled apartments, but their relationship had reached the point at which they were using only one of them. The woman suggested to the man that they give up the other apartment. The man, an economist, explained to her a fundamental principle: It is always better to have more choice available. The probability of their splitting up might be small, but given even a small risk, it would be useful to retain the second low-rent apartment. The woman took this very badly and promptly ended the relationship!

Someone who hears this story might say that it just confirms the principle that greater choice is better. But strategic thinking offers a very different and more compelling explanation. The woman was not sure of the man's commitment to the relationship, and her suggestion was a brilliant strategic device to elicit the truth. Words are cheap; anyone can say, "I love you." If the man had put his property where his mouth was and had given up his rent-controlled apartment, this would have been concrete evidence of his love. The fact that he refused to do so constituted hard evidence of the opposite, and the woman did right to end the relationship.

This story, designed to appeal to your immediate experience, is an example of a very important class of games—namely, those where the real strategic issue is manipulation of information. Strategies that convey good information about yourself are called signals; strategies that induce others to act in ways that will credibly reveal private information about themselves, good or bad, are called screening devices.

Thus, the woman’s suggestion of giving up one of the apartments was a screening device, which put the man in the situation of either offering to give up his apartment or revealing his lack of commitment. We will study games of information, as well as signaling and screening, in Chapters 9 and 14.

Endnotes

- Evert and Navratilova's rivalry throughout the 1970s and 1980s is arguably the greatest in women's tennis history and, indeed, one of the greatest in sports history. We introduce an example based on their rivalry in Chapter 4 and return to it in several later chapters, most notably in Chapter 7. Return to reference 1
- There is some disagreement regarding the appropriate grammatical placement of the apostrophe in the term *prisoners' dilemma.* Our placement acknowledges the facts that there must be at least two prisoners in order for there to be any dilemma at all and that the (at least two) prisoners therefore jointly possess the dilemma. Return to reference 2
- This example comes from Michael Grunwald's "At Home" column, "A Game of Chicken," in the *Boston Globe Magazine*, April 28, 1996. Return to reference 3

3 OUR STRATEGY FOR STUDYING GAMES OF STRATEGY

In Section 2, we presented several examples that relate to your experiences as an amateur strategist in real life to illustrate some basic concepts of strategic thinking and strategic games. We could continue, building a stock of dozens of similar stories. The hope would be that when you faced an actual strategic situation, you might recognize a parallel with one of these stories, which would help you decide the appropriate strategy for your own situation. This is the *case study* approach taken by most business schools. It offers a concrete and memorable vehicle for presenting the underlying concepts. However, each new strategic situation typically consists of a unique combination of so many variables that an intolerably large stock of cases would be needed to cover all of them.

An alternative approach focuses on the general principles behind the examples and so constructs a *theory* of strategic action—namely, formal game theory. The hope here would be that when you faced an actual strategic situation, you might recognize which principle or principles apply to it. This is the route taken by the more academic disciplines, such as economics and political science. A drawback to this approach is that the theory may be presented in a very abstract and mathematical manner, without enough cases or examples. This might make it difficult for most beginners to understand or remember the theory and to connect the theory with reality afterward.

But knowing some general theory has an overwhelming compensating advantage: It gives you a deeper understanding of games and of *why* they have the outcomes they do. This understanding will help you play better than you would if you merely read some cases and knew the recipes for *how* to play some specific games. With the knowledge of why, you can think through new and unexpected situations where a mechanical follower of a "how" recipe would be lost. A world champion of checkers, Tom Wiswell, has expressed this beautifully: "The player who knows how will usually draw; the player who knows why will usually win."[4] This statement is not to be taken literally for all games; some games may be hopeless situations for one of the players no matter how knowledgeable he may be. But the statement contains the germ of an important general truth: Knowing why gives you an advantage beyond what you can get if you merely know how. For example, knowing the why of a game can help you foresee a hopeless situation and avoid getting into such a game in the first place.

In this book, we take an intermediate route that combines some of the advantages of both approaches—case studies (how) and theory (why). Chapters 3–7 (Part Two) are organized around the general principles of game theory, but we develop these principles through illustrative cases rather than abstractly. That way, the context and scope of each idea will be clearer and more evident. In other words, we will focus on theory, but build it up through cases. Starting with Chapter 8, we will then apply the theory to several types of strategic situations.

Of course, such an approach requires some compromises of its own. Most importantly, you should remember that each of our examples serves the purpose of conveying some general idea or principle of game theory. Therefore, we will leave out many details of each case that are incidental to the principle at stake. If some examples seem somewhat artificial, please bear with us; we have generally considered the omitted details and left them out for good reasons.

A word of reassurance: Although the examples that motivate the development of our conceptual or theoretical frameworks are deliberately selected for that purpose (even at the cost of leaving out some other features of reality), once the theory has been constructed, we will pay a lot of attention to its connection with reality. Throughout the book, we will examine factual and experimental evidence concerning how well the theory explains reality. The frequent answer—very well in some respects and less well in others—should give you cautious confidence in using the theory and should be a spur to the formulation of better theories. In appropriate places, we will examine in great detail how institutions evolve in practice to solve problems highlighted by the theory. For example, Chapter 10 explores how prisoners' dilemmas arise and are solved in reality, Chapter 11 provides a similar discussion of more general collective-action problems, Chapter 13 examines the use of brinkmanship in the Cuban missile crisis, and Chapter 15 discusses how to design an auction and avoid the winner's curse.

To pursue our approach, in which examples lead to general theories that are then tested against reality and used to interpret reality, we must first identify the general principles that serve to organize the discussion. We will do so in Chapter 2 by classifying games along several key dimensions, such as whether players have aligned or conflicting interests, whether one player acts first (sequential games) or all players act at once (simultaneous games), whether all players have the same information about the game, and so on. Once this general framework has been constructed in Chapter 2, the chapters that follow will build on it, systematically developing ideas and principles that you can deploy when analyzing each player's strategic choice and the interaction of all players' strategies in games.

Endnotes

- Quoted in Victor Niederhoffer, *The Education of a Speculator* (New York: Wiley, 1997), p. 169. We thank Austin Jaffe of Pennsylvania State University for bringing this aphorism to our attention. Return to reference 4

2 ■ How to Think about Strategic Games

CHAPTER 1 GAVE SOME simple examples of strategic games and strategic thinking. In this chapter, we begin a more systematic and analytical approach to the subject. We choose some crucial conceptual categories or dimensions of strategic games, each of which has a dichotomy of types of strategic interactions. For example, one such dimension concerns the timing of the players' actions, and the two pure types are games where the players act in strict turns (sequential moves) and where they act at the same time (simultaneous moves). We consider some matters that arise in thinking about each pure type in this dichotomy as well as in similar dichotomies, such as whether the game is played only once or repeatedly and what the players know about one another.

In Chapters 3–7, we will examine each of these categories or dimensions in more detail; in Chapters 8–17, we will show how game-theoretic analysis can be used in several contexts. Of course, most actual situations to which these analyses are applied are not purely of just one type, but are rather a mixture. Moreover, in each application, two or more of the categories have some relevance. The lessons learned from the study of the pure types must therefore be combined in appropriate ways. We will show how to do this by using the context of our applications.

In this chapter, we present some basic concepts and terms—such as strategies, payoffs, and equilibrium—that are used in game-theoretic analysis, and we briefly describe methods of solving a game. We also provide a brief discussion of the uses of game theory and an overview of the structure of the remainder of the book.

1 STRATEGIC GAMES

The very word *game* conjures up many meanings or connotations. The two that are most common, but misleading in the present context, are that a game involves two players or teams, and that one of them wins and the other loses. Our study of games of strategy does include two-player, win-lose games, but covers much more. Most of you have probably played “massively multiplayer online games,” where hundreds or thousands of players interact simultaneously on the same server. Or you have done battle against your future self when avoiding procrastination or overeating. And you constantly hear or read pundits and analysts who discuss whether international trade can be a win-win game or assert that nuclear war is only a lose-lose game. These other types of games are also within the purview of game theory, as are a broad range of interactions you have on a regular basis.

Our broad concept of a game of strategy (we will usually just say game, because we are not concerned with games of pure skill or pure chance) applies to any situation in which strategies of the participants (players) interact, so that the outcome for each participant depends on the choices of all. In this very broad sense, it is difficult to think of a decision that gets made by any person or team that is not part of a game. You may think that your decision to brush your teeth in the morning is purely personal. But if you are a waiter, it will influence your customers’ satisfaction with the service, and that will affect your tips. Even in your private life, it may have a bearing on your close personal relationships![1] Indeed, strategic interactions are everywhere, but their nature depends on how many players are involved and how those players interact.

Consider a market for bread with thousands of customers and bakers. Each customer decides whether to buy bread, and how much to buy, at the going price; each baker decides how much bread to make. Their demand and supply decisions collectively determine the market price of bread. Thus, there is a feedback loop from all of the customers' and bakers' choices (strategies) to the price, and from the price back to their choices. But each individual has only a tiny effect on the price. Therefore, no one needs to be concerned with whether or how much any other specific individual is buying or selling. Their strategies respond to, and collectively feed back on, only a summary or aggregate measure—namely, the market price.[2] In most multiplayer games, there will be similar summary measures that channel strategic interactions. For example, each driver choosing from alternative routes to work responds to the expected travel time on each route, and their collective choices determine those travel times.

Contrast this situation with a market that has only a few sellers, such as the one for mobile phones. Each firm has to be concerned with what choices each of the other few sellers is making and plan its own strategy in response. Samsung looks at the specific choices of features, prices, and advertising being made by Apple, Nokia, and the others when making its own choices. It does not merely respond to broad market-wide statistics on phone attributes.

Game theory encompasses multiplayer games as well as small-numbers games, but the applicable concepts, the nature of strategies, and the ranges of outcomes can differ across these types of games—as does the usefulness of game theory itself. In games with few players, where each must think about the others' thinking, there is much more scope for manipulating information and actions in the game. In such situations, game theory often yields useful advice on how to play or change a game to your advantage; see, for instance, our analyses of the timing of moves in Chapter 6, strategic moves in Chapter 8, signaling and screening in Chapter 9, repeated games in Chapter 10, and incentive design in Chapter 14.

In multiplayer games, where each player’s decision has a negligible effect on others and the optimal strategy for each player merely involves responding to an overall variable such as a price, there is much less scope for advice or advantage. Even so, game theory can be useful in such games for motivating and coordinating collective action (as we will show in Chapter 11) as well as for generating insights that can help outside observers understand the interactions and guide policy efforts aimed at achieving better outcomes for society as a whole. As such, game theory can be especially useful to leaders and policymakers looking to change a multiplayer game. For example, game theory helps us identify the nature of the game (prisoners’ dilemma) that leads to excessive congestion on roads, pollution of air and water, and depletion of fisheries, and thereby helps society design institutions that can ameliorate these problems.

Sometimes multiplayer and small-numbers games are interwoven. For instance, the game of finding a romantic partner often proceeds in two stages, initially with many players all looking to meet others and then later in a relationship with just one other person. When formulating one’s strategy in the initial multiplayer phase, it pays to think ahead about the two-player game that will unfold with one’s chosen partner. A similar set of considerations arises when building a house. During the earliest planning stage, a customer can choose any of several dozen local contractors, and each contractor can similarly choose from multiple potential customers. Once a customer and contractor choose to work together, however, the customer pays an initial installment and the builder buys materials that can be used only on that customer’s house. The two become tied to each other, separately from the broader housing market—creating a small-numbers game between *one* customer and *one* contractor.

Endnotes

- Alan S. Blinder, “The Economics of Brushing Teeth,” *Journal of Political Economy*, vol. 82, no. 4 (July–August 1974), pp. 887–91. Return to reference 1
- Some multiplayer games may have players of very unequal sizes. For example, in the game among nations to deal with climate change, emissions from China and the United States have a substantial direct effect on smaller nations. When formulating their climate-change strategies, China and the United States should therefore bear in mind how smaller nations will react (collectively) to their individual moves. Return to reference 2

Glossary

game (game of strategy)
: An action situation where there are two or more mutually aware players, and the outcome for each depends on the actions of all.

game (game of strategy)
: An action situation where there are two or more mutually aware players, and the outcome for each depends on the actions of all.

2 CLASSIFYING GAMES

Games of strategy arise in many different contexts and cannot all be analyzed in the same way. That said, some games that seem quite different on the surface share underlying features that allow them to be studied and understood very similarly from a game-theoretic point of view. When faced with a strategic situation, how can you tell what type of game it is and how it should be analyzed? In this section, we show how games can be (roughly but usefully) divided into a relatively small number of categories on the basis of the answers to a few driving questions discussed in the subsections that follow. As Stephen Jay Gould once wrote,[3] “Dichotomies are useful or misleading, not true or false. They are simplifying models for organizing thought, not ways of the world. . . . They do not blend, but dwell together in tension and fruitful interaction.” In the same spirit, we present the categorization of games here as a useful framework for organizing one’s thinking about games and for sparking creative connections.

A. Are Players' Interests in Total Alignment, Total Conflict, or a Mix of Both?

Perhaps the most fundamental question to ask about a game is whether the players have aligned or conflicting individual interests. Players on the same sports team share a broad goal, to win, but also play strategic games with one another during the contest. For example, during a football passing play, the quarterback decides where to throw the ball and the receiver decides where to run. This interaction constitutes a game because each player's choice affects the success of the play, but it is a game in which both players want the same thing, to complete the pass. Players with perfectly aligned interests in such games can get their best outcome relatively easily by communicating and coordinating their moves—for instance, by the quarterback telling the receiver where to run during the huddle before the play.

Competing sports teams can also be viewed as players in games, albeit now with totally conflicting interests, as each team wants to win while the other loses. Similarly, in gambling games, one player's winnings are the others' losses, with the sum of their earnings equal to 0. This is why such situations are called zero-sum games. More generally, players' interests will be in complete conflict whenever they are dividing up any fixed amount of total benefit, whether it be measured in yards, dollars, acres, or scoops of ice cream. Because the available gain need not always be exactly zero, the term constant-sum game is often substituted for *zero-sum game*; we will use the two terms interchangeably.

Most economic and social games are not zero-sum.[4] Trade, or economic activity more generally, offers scope for deals that benefit everyone. Joint ventures can combine the participants' different skills and generate synergy to produce more than the sum of what they could have produced separately. However, the partners' interests are typically not completely aligned either. They can cooperate to create a larger total pie, but they will tend to clash when it comes to deciding how to split this pie among themselves.

Even wars and strikes are not zero-sum games. A nuclear war is the most striking example of a situation where there can only be losers, but the concept is far older. Pyrrhus, the king of Epirus, defeated the Romans at Heraclea in 280 B.C., but at such great cost to his own army that he exclaimed, "Another such victory and we are lost!" Hence the phrase "Pyrrhic victory." In the 1980s, at the height of the frenzy of business takeovers, the battles among rival bidders led to such costly escalation that the successful bidder's victory was often similarly Pyrrhic.

Most games in reality have this tension between conflict and cooperation, and many of the most interesting analyses in game theory come from the need to handle it. The players' attempts to resolve their conflict—for example, over the distribution of territory or profit—are influenced by the knowledge that if they fail to agree, the outcome will be bad for all of them. One side's threat of a war or a strike is its attempt to frighten the other side into conceding to its demands.

B. Are the Moves in the Game Sequential or Simultaneous?

Moves in chess are sequential: White moves first, then Black, then White again, and so on. In contrast, participants in an auction for an oil-drilling lease or a part of the airwave spectrum make their bids simultaneously, without knowing competitors' bids. Most actual games combine aspects of both types of moves. In a race to research and develop a new product, the firms act simultaneously, but each competitor has partial information about the others' progress and can respond. During one play in football, the opposing offensive and defensive coaches simultaneously send out teams with the expectation of carrying out certain plays, but after seeing how the defense has set up, the quarterback can change the play at the line of scrimmage or call a time-out so that the coach can change the play.

The distinction between sequential moves and simultaneous moves is important because sequential- and simultaneous-move games require different sorts of strategic thinking. In a sequential-move game, each player must think: "If I do this, how will my opponent react?" Your current move is governed by your calculation of its *future* consequences. With simultaneous moves, you have the trickier task of trying to figure out what your opponent is going to do *right now*. But you must recognize that, in making his own calculation, your opponent is also trying to figure out your current move, while at the same time recognizing that you are doing the same with him . . . Both of you have to think your way out of this logical circle.

In Chapter 3, we will examine sequential-move games, where you must look ahead to act now. In Chapters 4 and 5, we will move on to simultaneous-move games, where you must square the circle of “He thinks that I think that he thinks . . .” In each case, we will devise some simple tools for thinking through each type of game—trees for sequential-move games and payoff tables for simultaneous-move games—and derive some simple rules to guide our actions.

Studying sequential-move games will also shed light on when it is advantageous to move first or to move second in a game, a topic we will consider in Chapter 6. Roughly speaking, which order a player prefers depends on the relative importance of commitment and flexibility in the game in question. For example, a game of economic competition among rival firms in a market has a first-mover advantage if one firm, by making a commitment to compete aggressively, can get its rivals to back off. On the other hand, in a political competition, a candidate who has taken a firm stand on an issue may give her rivals a clear focus for their attack ads, so that game has a second-mover advantage. Understanding these considerations can help you devise ways to manipulate the order of moves to your own advantage. That in turn leads to the study of strategic moves such as threats and promises, which we will take up in Chapter 8.

C. Are the Rules of the Game Fixed or Manipulable?

The rules of chess, card games, or sports are given, and every player must follow them, no matter how arbitrary or strange they seem. But in games of business, politics, and ordinary life, the players can make their own rules to a greater or lesser extent. For example, in the home, parents constantly try to make the rules, and children constantly look for ways to manipulate or circumvent those rules. In legislatures, rules for the progress of a bill (including the order in which amendments and main motions are voted on) are fixed, but the game that sets the agenda—which amendments are brought to a vote first—can be manipulated. This is where political skill and power have the most scope. We will address these matters in detail in Chapter 16.

In such situations, the real game is the "pregame" where rules are made, and your strategic skill must be deployed at that point. The actual playing out of the subsequent game can be more mechanical; you could even delegate it to someone else. However, if you snooze through the pregame, you might find that you have lost the game before it ever began. For example, American firms ignored the rise of foreign competition in just this way for many years and ultimately paid the price. But some entrepreneurs, such as oil magnate John D. Rockefeller Sr., adopted the strategy of limiting their play to games in which they could also participate in making the rules.[5]

A recurring theme throughout this book is the distinction between how best to *play* a given game and how best to *change the rules* of the game being played. In Chapter 8, we will consider strategic moves (such as promises and threats) whereby one player seeks to shape the outcome of the game by committing to how he will play the game and/or by changing the timing of moves. Similarly, in Chapters 14 and 15, we will consider how one player ("the principal" in Chapter 14 or "the auctioneer" in Chapter 15) can shape the options and/or incentives of other players ("the agent" or "the bidders") to get a better outcome for himself.

In each case, the rules of the game are determined in a pregame. But what determines the rules of the pregame? In some cases, only one player has the opportunity to set the rules. For instance, an auctioneer selling an item gets to decide what sort of auction to conduct, while in an employment relationship, the boss gets to decide what sort of incentive contract to use to motivate an employee. In other cases, the nature of the pregame may depend on players' innate abilities. In business competition, one firm can take preemptive actions that alter subsequent games between it and its rivals—for instance, by expanding a factory or by advertising in a way that affects subsequent price competition. Which firm can do this best or most easily depends on which has the managerial or organizational resources to make the necessary moves, and on which has sufficient understanding of game theory to recognize and seize the opportunity to do so.

Players may also be unsure of their rivals' abilities. This important nuance often makes the pregame one of asymmetric information (see Section 2.D), requiring more subtle strategies and occasionally resulting in big surprises. We will comment on these matters in Chapter 9 and elsewhere in the chapters that follow.

D. Do the Players Have Full or Equal Information?

In chess, each player knows exactly what the current situation is and all the moves that led to it, and each knows that the other aims to win. This situation is exceptional; in most other games, the players face some limitations on information. Such limitations come in two kinds. First, a player may not know all the information that is pertinent to the choices that he has to make at every point in the game. This type of information problem arises because of the player's uncertainty about relevant variables, both internal and external to the game. For example, he may be uncertain about external circumstances, such as the weekend weather or the quality of a product he wishes to purchase; we call this situation one of external uncertainty. Or he may be uncertain about exactly what moves his opponent has made in the past or is making at the same time he makes his own move; we call this situation one of strategic uncertainty. If a game has neither external nor strategic uncertainty, we call it a game with perfect information; otherwise, the game has imperfect information. We will give a more precise technical definition of perfect information in Chapter 6, Section 3.A, after we have introduced the concept of an information set. We will develop the theory of games with imperfect information (uncertainty) in three future chapters. In Chapter 4, we will discuss games with contemporaneous (simultaneous) actions, which entail strategic uncertainty, and we will analyze methods for making choices under external uncertainty in Chapter 9 and its appendix.

Trickier strategic situations arise when one player knows more than another does, what is referred to as asymmetric information. In such situations, the players' attempts to infer, conceal, or sometimes convey their private information become an important part of the game and the strategies. In bridge or poker, each player has only partial knowledge of the cards held by the others. The actions of the players (bidding and play in bridge, the number of cards taken and the betting behavior in poker) give information to their opponents. Each player tries to manipulate her actions to mislead the opponents (and, in bridge, to inform her partner truthfully), but in doing so, she must be aware that the opponents know this and that they will use strategic thinking to interpret her actions.

You may think that if you have superior information, you should always conceal it from other players. But that is not true. For example, suppose you are the CEO of a pharmaceutical firm that is engaged in an R&D competition to develop a new drug. If your scientists make a discovery that is a big step forward, you may want to let your competitors know that, in the hope that they will give up their own searches and you won’t face any future competition. In war, each side wants to keep its tactics and troop deployments secret, but in diplomacy, if your intentions are peaceful, then you desperately want other countries to know and believe that fact.

The general principle here is that you want to release your information selectively. You want to reveal the good information (the kind that will draw responses from the other players that work to your advantage) and conceal the bad (the kind that may work to your disadvantage).

This principle raises a problem. Your opponents in a strategic game are purposive and rational, and they know that you are, too. They will recognize your incentive to exaggerate or even to lie. Therefore, they are not going to accept your unsupported declarations about your progress or capabilities. They can be convinced only by objective evidence or by actions that are credible proof of your information. Such actions on the part of the more informed player are called signals, and strategies that use them are called signaling. Conversely, the less informed party can create situations in which the more informed player will have to take some action that credibly reveals his information; such strategies are called screening, and the methods they use are called screening devices. The word *screening* is used here in the sense of testing in order to sift or separate, not in the sense of concealing.

Sometimes the same action may be used as a signal by the informed player or as a screening device by the uninformed player. Recall that in the dating game in Section 2.F of Chapter 1, the woman was screening the man to test his commitment to their relationship, and her suggestion that the pair give up one of their two rent-controlled apartments was the screening device. If the man had been committed to the relationship, he might have acted first and volunteered to give up his apartment; this action would have been a signal of his commitment.

Now we see how, when different players have different information, the manipulation of information itself becomes a game, and often a vitally important one. Such information games are ubiquitous, and playing them well is essential for success in life. We will study more games of this kind in greater detail in Chapter 9 (games with signaling and/or screening), Chapter 14 (incentive design), and Chapter 15 (auctions).

E. Is the Game Played Once or Repeatedly, and with the Same or Changing Opponents?

A game played just once is in some respects simpler and in others more complicated than one that includes many interactions. You can think about a one-shot game without worrying about its repercussions on other games you might play in the future against the same person or against others who might hear of your actions in this one. Therefore, actions in one-shot games are more likely to be unscrupulous or ruthless. For example, an automobile repair shop is much more likely to overcharge a passing motorist than a regular customer.

In one-shot encounters, no player knows much about the others—for example, what their capabilities and priorities are, whether they are good at calculating their best strategies, or whether they have any weaknesses that can be exploited. Therefore, in one-shot games, secrecy or surprise is likely to be an important component of good strategy.

Games with ongoing relationships require the opposite considerations. You have an opportunity to build a reputation (for example, for toughness, fairness, honesty, or reliability, depending on the circumstances) and to find out more about your opponent. The players together can better exploit mutually beneficial prospects by arranging to divide the spoils over time (taking turns to "win") or to punish a cheater in future plays (an eye for an eye, or tit-for-tat). We will consider these possibilities in Chapter 10 on the prisoners' dilemma.

More generally, a game may be zero-sum in the short run but have scope for mutual benefit in the long run. For example, each football team likes to win, but they all recognize that close competition generates more spectator interest, which benefits all teams in the long run. That is why they agree to a drafting scheme where teams get to pick players in reverse order of their current standing, thereby reducing the inequality of talent. In long-distance races, runners or cyclists often develop a lot of cooperation; two or more of them can help one another by taking turns following in one another's slipstream. But near the end of the race, the cooperation collapses as all of them dash for the finish line.

Here is a useful rule of thumb for your own strategic actions in life: In a game that has some conflict and some scope for cooperation, you will often think up a great strategy for winning big and grinding a rival into the dust but have a nagging worry at the back of your mind that you are behaving badly. In such a situation, the chances are that the game has a repeated or ongoing aspect that you have overlooked. Your aggressive strategy may gain you a short-run advantage, but the cost of its long-run side effects outweigh your gains. Therefore, you should dig deeper, discover the cooperative element, and alter your strategy accordingly. You will be surprised how often niceness, integrity, and the golden rule of doing to others as you would have them do to you turn out to be not just old nostrums, but good strategies as well when you consider the whole complex of games that you will be playing in the course of your life.[6]

F. Are Agreements to Cooperate Enforceable?

We have argued that most strategic interactions consist of a mixture of conflict and common interest. Under these circumstances, there is a case to be made that all participants should get together and reach an agreement about what everyone should do, balancing their mutual interest in maximizing the total benefit with their conflicting interests in the division of gains. Such negotiations can take several rounds in which agreements are made on a tentative basis, better alternatives are explored, and the deal is finalized only when no group of players can find anything better. However, even after the completion of such a process, additional difficulties often arise in putting the final agreement into practice. For instance, all the players must perform, in the end, the actions that were stipulated for them in the agreement. When all others do what they are supposed to do, any one participant can typically get a better outcome for herself by doing something different. And, if any participant suspects that the others may cheat in this way, she would be foolish to adhere to her stipulated cooperative action.

Agreements to cooperate can succeed if all players act immediately and in the presence of the whole group, but agreements with such immediate implementation are quite rare. More often the participants disperse after the agreement has been reached and then take their actions in private. Still, if these actions are observable to the others, and a third party—for example, a court of law—can enforce compliance, then the agreement of joint action can prevail.

However, in many other instances, individual actions are neither directly observable nor enforceable by external forces. Without enforceability, agreements will stand only if it is in all participants' individual interests to abide by them. Games among sovereign countries are of this kind, as are many games with private information or games where the players' actions cannot be proven to the standard of evidence required by courts of law. In fact, games where agreements for joint action are not enforceable constitute a vast majority of strategic interactions.

Game theory uses special terminology to capture the distinction between situations in which agreements are enforceable and those in which they are not. Games in which joint-action agreements are enforceable are called cooperative games; those in which such enforcement is not possible, and individual participants must be allowed to act in their own interests, are called noncooperative games. This terminology has become standard, but it is somewhat unfortunate because it gives the impression that the former will produce cooperative outcomes and the latter will not. In fact, individual actions can be compatible with the achievement of a lot of mutual gain, especially in repeated interactions. The important distinction is that in so-called noncooperative games, cooperation will emerge only if it is in the participants' separate and individual interests to continue to take the prescribed actions. This emergence of cooperative outcomes from noncooperative behavior is one of the most interesting findings of game theory, which we will develop in Chapters 10, 11, and 12.

We will adhere to the standard terminology in this book, but emphasize that the terms *cooperative* and *noncooperative* refer to the way in which actions are implemented or enforced—collectively in the former mode and individually in the latter—and not to the nature of the outcomes. In practice, as we said earlier, most games do not have adequate mechanisms for external enforcement of joint-action agreements. Bearing that in mind, most of our analytical discussion will deal with noncooperative games. The one exception comes in our discussion of bargaining in Chapter 17.

Endnotes

- Stephen Jay Gould, *Time's Arrow, Time's Cycle: Myth and Metaphor in the Discovery of Geologic Time* (Cambridge, Mass.: Harvard University Press, 1987), pp. 199–200. Return to reference 3
- Even when a game is constant-sum for all players, if the game has three (or more) players, two of them may cooperate at the expense of the third. Such situations lead to the study of alliances and coalitions, topics we explore in Chapter 17, on bargaining. Return to reference 4
- For more on Rockefeller's rise to power, see Ron Chernow, *Titan* (New York: Random House, 1998). Return to reference 5
- For more on the strategic advantages of being honest and kind, see David McAdams, "Game Theory and Cooperation: How Putting Others First Can Help Everyone," *Frontiers for Young Minds: Mathematics*, vol. 5, no. 66 (December, 2017), available at https://kids.frontiersin.org/article/10.3389/frym.2017.00066. Return to reference 6

Glossary

zero-sum game
: A game where the sum of the payoffs of all players equals zero for every configuration of their strategy choices. (This is a special case of a *constant-sum game*, but in practice no different because adding a constant to all the payoff numbers of any one player makes no difference to his choices.)

constant-sum game
: A game in which the sum of all players' payoffs is a constant, the same for all their strategy combinations. Thus, there is a strict conflict of interests among the players—a higher payoff to one must mean a lower payoff to the collectivity of all the other players. If the payoff scales can be adjusted to make this constant equal to zero, then we have a *zero-sum game*.

sequential moves
: The moves in a game are sequential if the rules of the game specify a strict order such that at each action node only one player takes an action, with knowledge of the actions taken (by others or himself) at previous nodes.

simultaneous moves
: The moves in a game are simultaneous if each player must take his action without knowledge of the choices of others.

external uncertainty
: A player's uncertainty about external circumstances such as the weather or product quality.

strategic uncertainty
: A player's uncertainty about an opponent's moves made in the past or made at the same time as her own.

perfect information
: A game is said to have perfect information if players face neither strategic nor external uncertainty.

[imperfect information]

A game is said to have perfect information if each player, at each point where it is his turn to act, knows the full history of the game up to that point, including the results of any random actions taken by nature or previous actions of other players in the game, including pure actions as well as the actual outcomes of any mixed strategies they may play. Otherwise, the game is said to have imperfect information.

[asymmetric information]

Information is said to be asymmetric in a game if some aspects of it—rules about what actions are permitted and the order of moves if any, payoffs as functions of the players strategies, outcomes of random choices by "nature," and of previous actions by the actual players in the game—are known to some of the players but are not common knowledge among all players.

[signals]

Devices used for signaling.

[signaling]

Strategy of a more-informed player to convey his "good" information credibly to a less-informed player.

[screening]

Strategy of a less-informed player to elicit information credibly from a more-informed player.

[screening devices]

Methods used for screening.

[cooperative game]

A game in which the players decide and implement their strategy choices jointly, or where joint-action agreements are directly and collectively enforced.

[noncooperative game]

A game where each player chooses and implements his action individually, without any joint-action agreements directly enforced by other players.

3 SOME TERMINOLOGY AND BACKGROUND ASSUMPTIONS

When one thinks about a strategic game, the logical place to begin is by specifying its structure. A game’s structure includes all the strategies available to all the players, their information, and their objectives. The first two aspects will differ from one game to another along the dimensions discussed in the preceding section, and one must locate one’s particular game within that framework. The objectives raise some new and interesting considerations. Here we consider all of these aspects of game structure.

A. Strategies

Strategies are simply the choices available to the players, but even this basic notion requires some further study and elaboration. If a game has purely simultaneous moves made only once, then each player’s strategy is simply the action taken on that single occasion. But if a game has sequential moves, then a player who moves later in the game can respond to what other players have done (or what he himself has done) at earlier points. Therefore, each such player must make a complete plan of action—for example, “If the other does A, then I will do X, but if the other does B, then I will do Y.” This complete plan of action constitutes the strategy in such a game.

A very simple test determines whether your strategy is complete: Does it specify such full detail about how you would play the game—describing your action in every contingency—that if you were to write it all down, hand it to someone else, and go on vacation, this other person, acting as your representative, could play the game just as you would have played it? Would that person know what to do on each occasion that could conceivably arise in the course of play without ever needing to disturb your vacation for instructions on how to deal with some situation that you had not foreseen? The use of this test will become clearer in Chapter 3, when we will develop and apply it in some specific contexts. For now, you should simply remember that a strategy is a complete plan of action.

This notion is similar to the common usage of the word *strategy* to denote a longer-term or larger-scale plan of action as distinct from tactics, which pertain to a shorter term or a smaller scale. For example, generals in the military make strategic plans for a war or a large-scale battle, while lower-level officers devise tactics for a smaller skirmish or a particular theater of battle based on local conditions. But game theory does not use the term *tactics* at all. The term *strategy* covers all situations, meaning a complete plan of action when necessary and meaning a single move if that is all that is needed in the particular game being studied.

The word *strategy* is also commonly used to describe a person’s decisions over a fairly long time span and sequence of choices, even when there is no game in our sense of purposive interaction with other people. Thus, you have probably already chosen a career strategy. When you start earning an income, you will make saving and investment strategies and eventually plan a retirement strategy. This usage of the term *strategy* has the same sense as ours: a plan for a succession of actions in response to evolving circumstances. The only difference is that we are reserving it for a situation—namely, a game—where the circumstances evolve because of actions taken by other purposive players.

B. Payoffs

When asked what a player's objective in a game is, most newcomers to strategic thinking respond that it is "to win," but matters are not always so simple. Sometimes the margin of victory matters; for example, in R&D competition, if your product is only slightly better than your nearest rival's, your patent may be more open to challenge. Sometimes there may be smaller prizes for several participants, so winning isn't everything. Most importantly, very few games of strategy are purely zero-sum or win-lose; most games combine some common interests and some conflict among the players. Thinking about such mixed-motive games requires more refined calculations than the simple dichotomy of winning and losing—for example, comparisons of the gains from cooperating versus cheating.

To enable such comparisons, we will assume that each player is able to assign a *number* to each logically conceivable outcome of the game, corresponding to each logically possible combination of choices made by all the players. The number that a player assigns to a given outcome is that player's payoff for that outcome. Higher payoff numbers are given to outcomes that better achieve that player's objectives.

Sometimes a player's payoffs will be represented as a simple ranking of the outcomes, the worst labeled 1, the next worst 2, and so on, all the way to the best. In other games, there may be other more natural numerical scales—for example, money income or profit for firms, viewer-share ratings for TV networks, and so on. In still other situations, the payoff numbers that we use are only educated guesses. In such cases, we need to make sure that the results of our analysis do not change significantly if we vary these guesses within some reasonable margin of error.

Two important points about payoffs need to be understood clearly. First, a player's payoffs in a game capture everything that the player cares about vis-à-vis that game. In particular, players need not be selfish; their concern about others can be included in their payoffs. Second, we will assume that, if the player faces a random prospect of outcomes, then the number associated with this prospect is the average of the payoffs associated with each potential outcome, each weighted by its probability. Thus, if in one player's ranking, outcome A has payoff 0 and outcome B has payoff 100, then the prospect of a 75% probability of A and a 25% probability of B should have the payoff $0.75 \times 0 + 0.25 \times 100 = 25$. This number is often called the expected payoff from the random prospect. The word *expected* has a special connotation in the jargon of probability theory. It does not mean what you think you will get or expect to get; it is the mathematical or probabilistic or statistical expectation, meaning an average of all possible outcomes, where each is given a weight proportional to its probability.

The second point creates a potential difficulty. Consider a game where players get or lose money and where payoffs are measured simply in money amounts. In reference to the preceding example, if a player has a 75% chance of getting nothing and a 25% chance of getting $100, then the expected payoff, as calculated in that example, is $25. That is also the payoff that the player would get from a simple nonrandom outcome of $25. In other words, in this way of calculating payoffs, a person should be indifferent to whether she receives $25 for sure or faces a risky prospect of which the average amount is $25. But one would think that most people would be averse to risk, preferring a sure $25 to a gamble that yields only $25 on the average.

Aversion to risk can be incorporated into the theory in several ways. One is to measure payoffs not in money terms, but with a nonlinear rescaling of dollar amounts. This approach, called expected utility theory, is widely used, although it is not without its own difficulties; it is also mathematically complex. Therefore, in the text, we adopt a simpler approach that has some support from recent behavioral research: We capture aversion to risk by attaching a higher weight to losses than to gains, both measured from some average or status quo level. We will develop this idea in Chapter 9 and use it later in Chapter 14.

C. Rationality

Each player's aim in a game is to achieve as high a payoff for himself as possible. But how good is each player at pursuing this aim? This question is not about whether and how other players pursuing their own interests will impede him; that is in the very nature of a game of strategic interaction. Rather, achieving a high payoff depends on how good each player is at calculating the strategy that is in his own best interests and at following this strategy in the actual course of play.

Traditional game theory assumes that players are perfect calculators and flawless followers of their best strategies. In other words, it assumes that players will exhibit rational behavior. Observe the precise sense in which the term *rational* is being used here. It means that each player has a consistent payoff ranking of all the logically possible outcomes of the game and calculates the strategy that best serves his interests. Thus, rationality has two essential ingredients: complete knowledge of one's own interests, and flawless calculation of what actions will best serve those interests.

It is equally important to understand what is *not* included in this concept of rational behavior. It does not mean that players are selfish; a player may rank the well-being of some other player(s) highly and incorporate this high ranking into his payoffs. It does not mean that players are short-term thinkers; in fact, calculation of future consequences is an important part of strategic thinking, and actions that seem irrational from an immediate perspective may have valuable long-term strategic roles. Most importantly, being rational does not mean having the same value system that other players, or sensible people, or ethical or moral people would use; it means merely pursuing one’s own value system consistently. Therefore, when one player carries out an analysis of how other players will respond (in a game with sequential moves) or of successive rounds of thinking about thinking (in a game with simultaneous moves), he must recognize that the other players calculate the consequences of their choices by using their own value or ranking system. You must not impute your own value system or standards of rationality to others and assume that they would act as you would in the same situation. Thus, many “experts” commenting on the Persian Gulf conflict in late 1990 and again in 2002–2003 predicted that Saddam Hussein would back down “because he is rational”; they failed to recognize that Saddam’s value system was different from the one held by most Western governments and Western experts.

In most games, no player really knows the other players’ value systems; this is part of the reason that in reality, many games have asymmetric information. In such games, trying to find out the values of others and trying to conceal or convey one’s own values become important components of strategy.

Although the assumption of rational behavior remains the basis for much of game theory and for the majority of our exposition, significant departures from it have been observed and built into modern theories. Research in psychology and behavioral economics has found that people often take actions based on instincts, fixed rules, or heuristics. We will incorporate these considerations throughout this book.

Even for players who have clear preferences and want to pursue optimal strategies, calculation of the optimal strategy is often far from easy. Most games in real life are very complex, and most real players are limited in their thinking and computational abilities. In games such as chess, it is known that the calculation of the best strategy can be performed in a finite number of steps, but that number is mind-bogglingly large. Only in the last few years has the advent of artificial intelligence and neural networks in computing enabled such calculations. We discuss advances in games like chess further in Chapter 3.

The assumption of rationality may be closer to reality when the players are regulars who play the game often. In this case, they benefit from having experienced the different possible outcomes. They understand how the strategic choices of various players lead to the outcomes and how well or badly they themselves fare. Thus we can hope that their choices, even if not made with full and conscious computations, closely approximate the results of such computations. We can think of the players as implicitly choosing the optimal strategy or behaving as if they were perfect calculators. We will offer some experimental evidence in Chapter 5 that the experience of playing the game generates more rational behavior.

The advantage of making a complete calculation of your best strategy, taking into account the corresponding calculations of a similar strategically calculating rival, is that you are not making mistakes that the rival can exploit. In many actual situations, you may have specific knowledge of the ways in which the other players fall short of this standard of rationality, and you can exploit this information in devising your own strategy. We will say something about such calculations, but very often they are a part of the "art" of game playing, not easily codifiable in rules to be followed. You must always beware of the danger that the other players are merely pretending to have poor skills or strategy, losing small sums through bad play and hoping that you will then raise the stakes, at which time they can raise the level of their play and exploit your gullibility. When this risk is real, the safest advice to a player may be to assume that the rivals are perfect and rational calculators and to choose your own best response to their actions. In other words, you should play to your opponents' capabilities instead of their limitations.

D. Common Knowledge of Rules

We assume that, at some level, the players have a common understanding of the rules of the game. In a *Peanuts* cartoon, Lucy thought that body checking was allowed in golf and decked Charlie Brown just as he was about to take his swing. We do not allow this.

The qualification “at some level” is important. We saw how the rules of the immediate game could be manipulated. But this merely admits that there is another game being played at a deeper level—namely, the pregame where the players choose the rules of the superficial game. Then the question is whether the rules of this deeper game are fixed. For example, in the legislative context, what are the rules of the agenda-setting game? They may be that the committee chairs have the power. Then how are the committees and their chairs elected? And so on. At some basic level, the rules are fixed by a nation’s constitution, by the technology of campaigning, or by general social norms of behavior. We ask that all players recognize the given rules of this basic game, and that is the focus of the analysis. Of course, that is an ideal; in practice, we may not be able to proceed to a deep enough level of analysis.

Strictly speaking, the rules of a game consist of (1) the list of players, (2) the strategies available to each player, (3) the payoffs to each player for all possible combinations of strategies pursued by all the players, and (4) the assumption that each player is a rational maximizer.

Game theory cannot properly analyze a situation where one player does not know whether another player is participating in the game, what the entire sets of actions available to the other players are from which they can choose, what their value systems are, or whether they are conscious maximizers of their own payoffs. But in actual strategic interactions, some of the biggest gains are to be made by taking advantage of the element of surprise and doing something that your rivals never thought you capable of. Several vivid examples can be found in historic military conflicts. For example, in 1967, Israel launched a preemptive attack that destroyed the Egyptian air force on the ground; in 1973, it was Egypt’s turn to spring a surprise on Israel by launching a tank attack across the Suez Canal.

It would seem, then, that the strict definition of game theory leaves out a very important aspect of strategic behavior, but in fact the problem is not that serious. The theory can be reformulated so that each player attaches some small probability to the situation in which such dramatically different strategies are available to the other players. Of course, each player knows her own set of available strategies. Therefore, the game becomes one of asymmetric information and can be handled by using the methods developed in Chapter 9.

The concept of common knowledge itself requires some explanation. For some fact or situation X to be common knowledge between two people, A and B, it is not enough for each of them separately to know X. Each should also know that the other knows X; otherwise, for example, A might think that B does not know X and might act under this misapprehension in the midst of a game. But then A should also know that B knows that A knows X, and the other way around, otherwise A might mistakenly try to exploit B’s supposed ignorance of A’s knowledge. Of course, it doesn’t even stop there. A should know that B knows that A knows that B knows, and so on ad infinitum. Philosophers have a lot of fun exploring the fine points of this infinite regress and the intellectual paradoxes that it can generate. For us, the general notion that the players have a common understanding of the rules of their game will suffice.

E. Equilibrium

Finally, what happens when rational players' strategies interact? Our answers will generally apply the framework of equilibrium. This term simply means that each player is using the strategy that is the best response to the strategies of the other players. We will develop game-theoretic concepts of equilibrium in Chapters 3 - 7 and then use them in subsequent chapters.

Equilibrium does not mean that things don't change; in sequential-move games, the players' strategies are complete plans of action and reaction, and any move that one player makes affects the circumstances in which later moves are made and responded to. Nor does equilibrium mean that everything is for the best; the interaction of rational strategic choices by all players can lead to bad outcomes for all, as in the prisoners' dilemma. But we will generally find that the idea of equilibrium is a useful descriptive tool and organizing principle for our analysis. We will consider this idea in greater detail later in this book, in connection with specific equilibrium concepts. We will also see how the concept of equilibrium can be augmented or modified to remove some of its flaws and to incorporate behavior that falls short of full calculating rationality.

Just as the rational behavior of individual players can be the result of experience in playing the game, the fitting of their choices into an overall equilibrium can come about after some plays that involve trial and error and nonequilibrium outcomes. We will look at this matter in Chapter 5.

Defining an equilibrium is not hard, but finding an equilibrium in a particular game—that is, solving the game—

can be a lot harder. Throughout this book, we will solve many simple games in which there are two or three players, each of them having two or three strategies or one move each in turn. Many people believe this to be the limit of the reach of game theory and therefore believe that the theory is useless for the more complex games that take place in reality. That is not true.

Humans are severely limited in their speed of calculation and in their patience for performing long calculations. Therefore, humans can easily solve only simple games with two or three players and strategies. But computers are very good at speedy and lengthy calculations. Many games that are far beyond the power of human calculators are easy for computers. The complexity of many games in business and politics is already within the power of computers. Even for games such as chess that are far too complex to solve completely, computers have reached a level of ability comparable to that of the best human players; we will consider chess in more detail in Chapter 3.

A number of computer programs for solving complex games exist, and more are appearing rapidly. Mathematica and similar program packages contain routines for finding equilibria in simultaneous-move games, even when such equilibria entail strategies in which players incorporate randomness into their moves. Gambit, a National Science Foundation project led by Professors Richard D. McKelvey of the California Institute of Technology and Andrew McLennan of the University of Minnesota, is producing a comprehensive set of routines for finding equilibria in games with sequential and simultaneous moves, with pure and mixed strategies, and with varying degrees of uncertainty and asymmetric information. We will refer to this project again in several places in the next several chapters. The biggest advantage of the project is that its programs are open source and can easily be obtained from its Web site, www.gambit-project.org.

Why, then, do we set up and solve several simple games in detail in this book? Understanding the concepts of game theory is an important prerequisite for making good use of the mechanical solutions that computers can deliver, and understanding comes from doing simple cases yourself. This is exactly how you learned, and now use, arithmetic. You came to understand the ideas of addition, subtraction, multiplication, and division by doing many simple problems mentally or using paper and pencil. With this grasp of basic concepts, you can now use calculators and computers to do far more complicated sums than you would ever have the time or patience to do manually. But if you did not understand the concepts, you would make errors in using calculators; for example, you might solve $3 + 4 \times 5$ by grouping additions and multiplications incorrectly as $(3 + 4) \times 5 = 35$ instead of correctly as $3 + (4 \times 5) = 23$.

Thus, the first step of understanding the concepts and tools is essential. Without it, you would never learn to set up correctly the games that you ask the computer to solve. You would not be able to inspect the solution with any feeling for whether it was reasonable, and if it was not, you would not be able to go back to your original specifications, improve them, and solve the game again until the specifications and the calculations correctly captured the strategic situation that you wanted to study. Therefore, please pay serious attention to the simple examples that we will solve and the exercises that we will ask you to solve, especially in Chapters 3–7.

F. Dynamics and Evolutionary Games

The theory of games described thus far, based on assumptions of rationality and equilibrium, has proved very useful, but it would be a mistake to rely on it totally. When games are played by novices who do not have the necessary experience to perform the calculations to choose their best strategies, explicitly or implicitly, their choices, and therefore the outcome of the games, can differ significantly from the predictions of analyses based on the concept of equilibrium.

However, we should not abandon all notions of good choice; we should recognize the fact that even poor calculators are motivated to do better for their own sakes and will learn from experience and by observing others. We should allow for a dynamic process in which strategies that proved to be better in previous plays of the game are more likely to be chosen in later plays.

The concept of evolutionary_games, derived from the idea of evolution in biology, does just this. Any individual animal’s genes strongly influence its behavior. Some behaviors succeed better in the prevailing environment, in the sense that the animals exhibiting those behaviors are more likely to reproduce successfully and pass their genes to their progeny. An evolutionarily stable state, relative to a given environment, is the ultimate outcome of this process over several generations.

The analogy in games would be to assume that strategies are not chosen by conscious rational maximizers, but instead that each player comes to the game with a particular strategy “hardwired” or “programmed” in. The players then confront other players who may be programmed to apply the same or different strategies. The payoffs to all the players in such games are then obtained. The strategies that fare better—in the sense that the players programmed to play them get higher payoffs in the games—multiply faster, whereas the strategies that fare worse decline. In biology, the mechanism of this growth or decay is purely genetic transmission through reproduction. In the context of strategic games in business and society, the mechanism is much more likely to be social or cultural—observation and imitation, teaching and learning, greater availability of capital for the more successful ventures, and so on.

The object of study is the dynamics of this process. Does it converge to an evolutionarily stable state? Does just one strategy prevail over all others in the end, or can a few strategies coexist? Interestingly, in many games, the evolutionarily stable state is the same as the equilibrium that would result if the players were consciously rational calculators. Therefore, the evolutionary approach gives us a backdoor justification for equilibrium analysis.

The concept of evolutionary games has thus imported biological ideas into game theory, but there has been an influence in the opposite direction, too. Biologists have recognized that significant parts of animal behavior consist of strategic interactions with other animals. Members of a given species compete with one another for space or mates; members of different species relate to one another as predators and prey along a food chain. The payoff in such games in turn contributes to reproductive success and therefore to biological evolution. Just as game theory has benefited by importing ideas from biological evolution for its analysis of choice and dynamics, biology has benefited by importing game-theoretic ideas of strategies and payoffs for its characterization of basic interactions among animals. We have here a true instance of synergy or symbiosis. We will provide an introduction to the study of evolutionary games in Chapter 12.

G. Observation and Experiment

All of Section 3 to this point has concerned how to think about games or how to analyze strategic interactions—in other words, it has been about theory. This book will cover an extremely simple level of theory, developed through cases and illustrations instead of formal mathematics or theorems, but it will be theory just the same. All theory should relate to reality in two ways: Reality should help structure the theory, and reality should provide a check on the results of the theory.

We can find out the reality of strategic interactions in two ways: (1) by observing them as they occur naturally and (2) by conducting special experiments that help us pin down the effects of particular conditions. Both methods have been used, and we will mention several examples of each in the proper contexts.

Many people have studied strategic interactions—the participants' behavior and the outcomes—under experimental conditions, in classrooms among "captive" players, or in special laboratories with volunteers. Auctions, bargaining, the prisoners' dilemma, and several other games have been studied in this way. The results are mixed. Some conclusions of the theoretical analysis are borne out; for example, in games of buying and selling, the participants generally settle quickly on the economic equilibrium. In other contexts, the outcomes differ significantly from the theoretical predictions; for example, prisoners' dilemmas and bargaining games show more cooperation than theory based on the assumption of selfish, maximizing behavior would lead us to expect, whereas auctions show some gross overbidding.

At several points in the chapters that follow, we will review the knowledge that has been gained by observation and experiments, discuss how it relates to the theory, and consider what reinterpretations, extensions, and modifications of the theory have been made or should be made in light of this knowledge.

Glossary

strategy
: A complete plan of action for a player in a game, specifying the action he would take at all nodes where it is his turn to act according to the rules of the game (whether these nodes are on or off the equilibrium path of play). If two or more nodes are grouped into one information set, then the specified action must be the same at all these nodes.

payoff
: The objective, usually numerical, that a player in a game aims to maximize.

expected_payoff
: The probability-weighted average (statistical mean or expectation) of the payoffs of one player in a game, corresponding to all possible realizations of a random choice of nature or mixed strategies of the players.

rational_behavior
: Perfectly calculating pursuit of a complete and internally consistent objective (payoff) function.

equilibrium
: A configuration of strategies where each player's strategy is his best response to the strategies of all the other players.

evolutionary_game
: A situation where the strategy of each player in a population is fixed genetically, and strategies that yield higher payoffs in random matches with others from the same population reproduce faster than those with lower payoffs.

4 THE USES OF GAME THEORY

We began Chapter 1 by saying that games of strategy are everywhere—in your personal and working life; in the functioning of the economy, society, and polity around you; in sports and other serious pursuits; in war and in peace. This should be motivation enough to study such games systematically, and that is what game theory is about. But your study can be better directed if you have a clearer idea of just how you can put game theory to use. We suggest a threefold perspective.

The first use is in *explanation*. Many events and outcomes prompt us to ask, Why did that happen? When a situation requires the interaction of decision makers with different aims, game theory often supplies the key to understanding that situation. For example, cutthroat competition in business is the result of the rivals being trapped in a prisoners' dilemma. At several points in this book, we will mention actual cases where game theory has helped us to understand how and why the events unfolded as they did. These cases include Chapter 13's detailed study of the Cuban missile crisis from the perspective of game theory.

The other two uses evolve naturally from the first. The second use is in *prediction*. When looking ahead to situations where multiple decision makers will interact strategically, we can use game theory to foresee what actions they will take and what outcomes will result. Of course, prediction in a particular context depends on its details, but we will prepare you to use prediction by analyzing several broad classes of games that arise in many applications

The third use is in *advice* or *prescription*. We can act in the service of one participant in a future interaction and tell him which strategies are likely to yield good results and which ones are likely to lead to disaster. Once again, such work is context specific, but we can equip you with several general principles and techniques and show you how to apply them to some general types of contexts. For example, in Chapter 7, we will show how to mix moves; in Chapter 8, we will examine how to make commitments, threats, and promises credible; and in Chapter 10, we will examine alternative ways of overcoming prisoners' dilemmas.

The theory is far from perfect in performing any of these three functions. To explain an outcome, one must first have a correct understanding of the motives and behavior of the participants. As we saw earlier, most of game theory takes a specific approach to these matters—namely, by applying a framework in which individual players make rational choices and those choices constitute an equilibrium of the game. Actual players and interactions in a game might not conform to this framework. But the proof of the pudding is in the eating. Game-theoretic analysis has greatly improved our understanding of many phenomena, as reading this book should convince you. The theory continues to evolve and improve as the result of ongoing research. This book will equip you with the basics so that you can more easily learn and profit from the new advances as they appear.

When explaining a past event, we can often use historical records to get a good idea of the motives and the behavior of the players in the game. When attempting prediction or advice, we have the additional problem of determining what motives will drive the players' actions, what informational or other limitations they will face, and sometimes even who the players will be. Most importantly, if game-theoretic analysis assumes that a player is a rational maximizer of his own objectives when in fact he is unable to do the required calculations or is a clueless person acting at random, the advice based on that analysis may prove wrong. This risk is reduced as more and more players recognize the importance of strategic interaction and think through their strategic choices or get expert advice on the matter, but some risk remains. Even then, the systematic thinking made possible by the framework of game theory helps keep the errors down to this irreducible minimum by eliminating the errors that arise from faulty logical thinking about the strategic interaction. Furthermore, game theory can take into account many kinds of uncertainty and asymmetric information, including uncertainties about the strategic possibilities and the rationality of the players. We will consider a few examples in the chapters to come.

5 THE STRUCTURE OF THE CHAPTERS TO FOLLOW

In this chapter, we introduced several considerations that arise in almost every game in reality. We also introduced some basic concepts that prove useful for game-theoretic analysis, to understand or predict the outcomes of games. However, trying to cope with all of these concepts at once merely leads to confusion and a failure to grasp any of them. Therefore, we will build up the theory one concept at a time. We will develop the appropriate technique for applying each concept and illustrate it.

In Part Two of this book, in Chapters 3 – 7, we will construct and illustrate the most important of these concepts and techniques. We will examine purely sequential-move games in Chapter 3 and introduce the techniques—game trees and rollback reasoning—that are used to analyze and solve such games. In Chapters 4 and 5, we will turn to games with simultaneous moves and develop for them another set of concepts—payoff tables, dominance, and Nash equilibrium. Both chapters will focus on games where players use pure strategies; in Chapter 4, we will restrict players to a finite set of pure strategies, and in Chapter 5, we will allow strategies that are continuous variables. Chapter 5 will also examine some mixed empirical evidence, conceptual criticisms, and counterarguments on Nash equilibrium, as well as a prominent alternative to Nash equilibrium—namely, rationalizability. In Chapter 6, we will show how games that have some sequential moves and some simultaneous moves can be studied by combining the techniques developed in Chapters 3 – 5. In Chapter 7, we will turn to simultaneous-move games that require the use of randomization or mixed strategies. We will start by introducing the basic ideas about mixing in two-by-two games, develop the simplest techniques for finding mixed-strategy Nash equilibria, and then consider more complex examples along with the empirical evidence on mixing.

The ideas and techniques developed in Chapters 3 - 7 are the most basic ones: (1) correct forward-looking reasoning for sequential-move games, and (2) equilibrium strategies—pure and mixed—for simultaneous-move games. Equipped with these concepts and tools, we can apply them to study some broad classes of games and strategies in Part Three (Chapters 8 - 12).

In Chapter 8, we will examine strategies that players use to manipulate the rules of a game, such as seizing a first-mover advantage and/or making a strategic move. Such moves are of three kinds: commitments, threats, and promises. In each case, credibility is essential to the success of the move, and we will outline some ways of making such moves credible. In Chapter 9, we will study what happens in games when players are subject to uncertainty or when they have asymmetric information. We will examine strategies for coping with risk and even for using risk strategically. We will also study the important strategies of signaling and screening that are used for manipulating and eliciting information. We will develop the appropriate generalization of Nash equilibrium in the context of uncertainty—namely, Bayesian Nash equilibrium—and show the different kinds of equilibria that can arise. We will also consider situations in which the asymmetry of information is two-way and the implications of such two-way private information for dynamic games (games of timing).

In Chapter 10, we will move on to study the best-known game of them all: the prisoners' dilemma. We will study whether and how cooperation can be sustained, most importantly in a repeated or ongoing relationship. Then, in Chapter 11, we will turn to situations where large populations, rather than pairs or small groups of players, interact strategically—games that concern problems of collective action. Each person's actions have an effect—in some instances beneficial, in others, harmful—on the others. The outcomes are generally not the best from the aggregate perspective of the society as a whole. We will clarify the nature of these outcomes and describe some simple policies that can lead to better outcomes.

All these theories and applications are based on the supposition that the players in a game fully understand the nature of the game and deploy calculated strategies that best serve their objectives in the game. Such rationally optimal behavior is sometimes too demanding of information and calculating power to be believable as a good description of how people really act. Therefore, Chapter 12 will look at games from a very different perspective. Here, the players are not calculating and do not pursue optimal strategies. Instead, each player is tied, as if genetically programmed, to a particular strategy. The population is diverse, and different players have different predetermined strategies. When players from such a population meet and act out their strategies, which strategies perform better? And if the more successful strategies proliferate better in the population, whether through inheritance or imitation, then what will the eventual structure of the population look like? It turns out that such evolutionary dynamics often favor exactly those strategies that would be used by rational optimizing players. Thus, our study of evolutionary games lends useful indirect support to the theories of optimal strategic choice and equilibrium that we will have studied in the previous chapters.

In Part Four, Chapters 13–17, we will take up specific applications of game theory to situations of strategic interactions. Here, we will use as needed the ideas and methods from all the earlier chapters. Chapter 13 will apply ideas from Chapters 8 and 9 to examine a particularly interesting dynamic version of a threat, known as the strategy of brinkmanship. We will elucidate its nature and apply the model developed in Chapter 9 to study the Cuban missile crisis of 1962. Chapter 14 will analyze strategies that people and firms use to deal with others who have some private information. We will illustrate the screening mechanisms that are used for eliciting information—for example, the multiple fares with different restrictions that airlines use for separating the business travelers who are willing to pay more from the tourists who are more price sensitive. We will also develop the methods for designing incentive payments to elicit effort from workers when direct monitoring is difficult or too costly. Chapter 15 will examine auctions as a means to allocate valuable economic resources, emphasizing the roles of information and attitudes toward risk (developed in Chapter 9) in the formulation of optimal strategies for both bidders and sellers.

Chapter 16 will consider voting in committees and elections. We will look at the variety of voting rules available and some paradoxical results that can arise. In addition, we will address the potential for strategic behavior not only by voters but also by candidates in a variety of election types. Finally, Chapter 17 will present bargaining in both cooperative and noncooperative settings.

All of these chapters together provide a lot of material; how might readers or teachers with more specialized interests choose from it? Chapters 3–7 constitute the core theoretical ideas that are needed throughout the rest of the book. Chapters 8 and 10 are likewise important for the general classes of games and strategies considered therein. Beyond that, there is a lot from which to pick and choose. Section 3 of Chapter 5, Section 6 of Chapter 7, and Section 5 of Chapter 11, for example, all consider more advanced topics. These sections will appeal to those with more scientific and quantitative backgrounds and interests, but those who come from the social sciences or humanities and have less quantitative background can omit them without loss of continuity.

Chapters 8 and 10 are key to understanding many phenomena in the real world, and most teachers will want to include them in their courses. Chapter 9 deals with an important topic in that most games, in practice, have asymmetric information, and the players' attempts to manipulate information is a critical aspect of many strategic interactions. However, the concepts and techniques for analyzing information games are inherently complex. Therefore, some readers and teachers may choose to study just the examples that convey the basic ideas of signaling and screening and leave out the rest. We have placed this chapter early in Part Three, however, in view of the importance of the subject.

Chapters 11 and 12 both look at games with large numbers of players. In Chapter 11, the focus is on social interactions; in Chapter 12, the focus is on evolutionary biology. The topics in Chapter 12 will be of greatest interest to those with interests in biology, but similar themes are emerging in the social sciences, and students from that background should aim to get the gist of the ideas even if they skip the details. Chapter 14 is most important for students of business and organization theories, while Chapters 13 and 16 present topics from political science (international diplomacy and elections, respectively). Chapters 15 and 17 cover topics from economics (auctions and bargaining). Those teaching courses with more specialized audiences may choose a subset from Chapters 11–17 and, indeed, expand on the ideas considered therein.

Whether you come from mathematics, biology, economics, politics, or other sciences, or from history or sociology, the theory and examples of strategic games will stimulate and challenge your intellect. We urge you to enjoy the subject even as you are studying or teaching it.

SUMMARY

A game of strategy is any situation with multiple decision makers, referred to as players, whose choices affect one another. Games are everywhere: at school and at work, on social media or out on a date. There is no escaping them! Being able to recognize what types of games you are playing, and knowing how to play them, will give you an advantage in life. Of course, games differ in many important ways and can be classified along several dimensions, such as whether there are many players or a small number, the timing of play, the extent to which players have common or conflicting interests, the number of times an interaction occurs, what players know about one another, whether the rules can be changed, and whether coordinated action is feasible.

Learning the terminology for a game’s structure is crucial for game-theoretic analysis. Players have *strategies* that lead to different *outcomes* with different associated *payoffs*. Payoffs incorporate everything that is important to a player about a game and are calculated by using probabilistic averages or *expectations* if outcomes are random or include some risk. *Rationality*, or consistent behavior, is assumed of all players, who must also be aware of all of the relevant rules of conduct. *Equilibrium* arises when all players use strategies that are best responses to others’ strategies; some classes of games allow learning from experience and the study of dynamic movements toward equilibrium. The study of behavior in actual game situations provides additional information about the performance of game theory.

Game theory may be used for explanation, prediction, or prescription in various circumstances. Although not perfect in any of these roles, the theory continues to evolve; the importance of strategic interaction and strategic thinking has also become more widely understood and accepted.

KEY TERMS[7]

asymmetric information (24)

constant-sum game (20)

cooperative game (27)

equilibrium (33)

evolutionary game (35)

expected payoff (29)

external uncertainty (23)

game (game of strategy) (18)

imperfect information (24)

noncooperative game (27)

payoff (29)

perfect information (24)

rational behavior (30)

screening (24)

screening device (24)

sequential moves (21)

signal (24)

signaling (24)

Endnotes

- The number in parentheses after each key term is the page on which that term is defined or discussed. Return to reference 7

Glossary

asymmetric information
: Information is said to be asymmetric in a game if some aspects of it—rules about what actions are permitted and the order of moves if any, payoffs as functions of the players strategies, outcomes of random choices by "nature," and of previous actions by the actual players in the game—are known to some of the players but are not common knowledge among all players.

constant-sum game
: A game in which the sum of all players' payoffs is a constant, the same for all their strategy combinations. Thus, there is a strict conflict of interests among the players—a higher payoff to one must mean a lower payoff to the collectivity of all the other players. If the payoff scales can be adjusted to make this constant equal to zero, then we have a *zero-sum game*.

cooperative game
: A game in which the players decide and implement their strategy choices jointly, or where joint-action agreements are directly and collectively enforced.

equilibrium
: A configuration of strategies where each player's strategy is his best response to the strategies of all the other players.

evolutionary game
: A situation where the strategy of each player in a population is fixed genetically, and strategies that yield higher payoffs in random matches with others from the same population reproduce faster than those with lower payoffs.

expected payoff
: The probability-weighted average (statistical mean or expectation) of the payoffs of one player in a game,

corresponding to all possible realizations of a random choice of nature or mixed strategies of the players.

[external uncertainty]

A player’s uncertainty about external circumstances such as the weather or product quality.

[game (game of strategy)]

An action situation where there are two or more mutually aware players, and the outcome for each depends on the actions of all.

[imperfect information]

A game is said to have perfect information if each player, at each point where it is his turn to act, knows the full history of the game up to that point, including the results of any random actions taken by nature or previous actions of other players in the game, including pure actions as well as the actual outcomes of any mixed strategies they may play. Otherwise, the game is said to have imperfect information.

[noncooperative game]

A game where each player chooses and implements his action individually, without any joint-action agreements directly enforced by other players.

[payoff]

The objective, usually numerical, that a player in a game aims to maximize.

[perfect information]

A game is said to have perfect information if players face neither strategic nor external uncertainty.

[rational behavior]

Perfectly calculating pursuit of a complete and internally consistent objective (payoff) function.

[screening]

Strategy of a less-informed player to elicit information credibly from a more-informed player.

[screening devices]

Methods used for screening.

[sequential moves]

The moves in a game are sequential if the rules of the game specify a strict order such that at each action node only one player takes an action, with knowledge of the actions taken (by others or himself) at previous nodes.

signals

Devices used for signaling.

signaling

Strategy of a more-informed player to convey his “good” information credibly to a less-informed player.

simultaneous moves

The moves in a game are simultaneous if each player must take his action without knowledge of the choices of others.

strategic uncertainty

A player’s uncertainty about an opponent’s moves made in the past or made at the same time as her own.

strategy

A complete plan of action for a player in a game, specifying the action he would take at all nodes where it is his turn to act according to the rules of the game (whether these nodes are on or off the equilibrium path of play). If two or more nodes are grouped into one information set, then the specified action must be the same at all these nodes.

zero-sum game

A game where the sum of the payoffs of all players equals zero for every configuration of their strategy choices. (This is a special case of a *constant-sum game*, but in practice no different because adding a constant to all the payoff numbers of any one player makes no difference to his choices.)

SOLVED EXERCISES[8]

1. Determine which of the following scenarios describe decisions, which of them describe games with a small number of players, and which of them describe games with many players. In each case, indicate what specific features of the scenario caused you to classify it as you did.
   1. A group of grocery shoppers in the dairy section, with each shopper choosing a flavor of yogurt to purchase
   2. A pair of teenage girls choosing dresses for their prom
   3. A college student considering what type of postgraduate education to pursue
   4. The *New York Times* and the *Wall Street Journal* choosing the prices for their online subscriptions this year
   5. College students in a class, deciding how hard to study for the final exam
   6. A presidential candidate picking a running mate
2. Consider the strategic games described below. In each case, state how you would classify the game according to the six dimensions outlined in Sections 2.A–2.F of this chapter: (i) Are players' interests totally aligned, totally in conflict, or a mix of both? (ii) Are moves sequential or simultaneous? (iii) Are the rules fixed or not? (iv) Is it a game with imperfect information, and if so, is it one with asymmetric information? (v) Is the game repeated? (vi) Are cooperative agreements possible or not? If you do not have enough information to classify a game in a particular dimension, explain why not.
   1. *Rock-Paper-Scissors:* On the count of three, each player makes the shape of one of the three items with

her hand. Rock beats Scissors, Scissors beats Paper, and Paper beats Rock.
2. *Roll-call voting:* Voters cast their votes orally as their names are called. The choice with the most votes wins.
3. *Sealed-bid auction:* Bidders on a bottle of wine seal their bids in envelopes. The highest bidder wins the item and pays the amount of his bid.

3. “A game player would never prefer an outcome in which every player gets a little profit to an outcome in which he gets all the available profit.” Is this statement true or false? Explain why in one or two sentences.
4. You and a rival are engaged in a game in which there are three possible outcomes: you win, your rival wins (you lose), or the two of you tie. You get a payoff of 50 if you win, a payoff of 20 if you tie, and a payoff of 0 if you lose. What is your expected payoff in each of the following situations?
    1. There is a 50% chance that the game ends in a tie, but only a 10% chance that you win. (There is thus a 40% chance that you lose.)
    2. There is a 50–50 chance that you win or lose. There are no ties.
    3. There is an 80% chance that you lose, a 10% chance that you win, and a 10% chance that you tie.
5. Explain the difference between game theory’s use as a predictive tool and its use as a prescriptive tool. In what types of real-world settings might these two uses be most important?

UNSOLVED EXERCISES

1. Determine which of the following scenarios describe decisions, which of them describe games with a small number of players, and which of them describe games with many players. In each case, indicate what specific features of the scenario caused you to classify it as you did.
    1. A party nominee for president of the United States must choose whether to use private financing or public financing for her campaign.
    2. Frugal Fred receives a $20 gift card for downloadable music and must choose whether to purchase individual songs or whole albums.
    3. Beautiful Belle receives 100 replies to her online dating profile and must choose whether to reply to each of them.
    4. After news that gold has been discovered, Americans living on the East Coast decide whether to join the Gold Rush to California.
    5. NBC chooses how to distribute its TV shows online this season. The executives consider Amazon.com, iTunes, and/or NBC.com. The fee they might pay to Amazon or to iTunes is open to negotiation.
    6. China chooses a level of tariffs to apply to American imports.
2. Consider the strategic games described below. In each case, state how you would classify the game according to the six dimensions outlined in the text. (i) Are players' interests totally aligned, totally in conflict, or a mix of both? (ii) Are moves sequential or simultaneous? (iii) Are the rules fixed or not? (iv) Is there imperfect information, and if so, is there asymmetric information? (v) Is the game repeated? (vi) Are cooperative agreements possible or not? If you do not

have enough information to classify a game in a particular dimension, explain why not.

1. Garry and Ross are sales representatives for the same company. Their manager informs them that of the two of them, whoever sells more this year wins a Cadillac.
2. On the game show *The Price Is Right*, four contestants are asked to guess the price of a TV set. Play starts with the leftmost player, and each player's guess must be different from the guesses of the previous players. The person who comes closest to the real price, without going over it, wins the TV set.
3. Six thousand players each pay $10,000 to enter the World Series of Poker. Each starts the tournament with $10,000 in chips, and they play No-Limit Texas Hold 'Em (a type of poker) until someone wins all the chips. The top 600 players each receive prize money according to their order of finish, with the winner receiving more than $8,000,000.
4. Passengers on Desert Airlines are not assigned seats; passengers choose seats once they board. The airline assigns the order of boarding according to the time the passenger checks in, either on the Web site up to 24 hours before takeoff or in person at the airport.

3. "Any gain by the winner must harm the loser." Is this statement true or false? Explain your reasoning in one or two sentences.
4. Alice, Bob, and Confucius are bored during recess, so they decide to play a new game. Each of them puts a dollar in the pot, and each tosses a quarter. Alice wins if the coins land all heads or all tails. Bob wins if two heads and one tail land, and Confucius wins if one head and two tails land. The quarters are fair. The winner receives a net payment of $2 ($3 − $1 = $2), and the losers lose their $1.
   1. What is the probability that Alice will win and the probability that she will lose?

2. What is Alice’s expected payoff?
3. What is the probability that Confucius will win and the probability that he will lose?
4. What is Confucius’s expected payoff?
5. Is this a constant-sum game? Please explain your answer.

5. “When one player surprises another, this indicates that the players did not have common knowledge of the rules.” Give an example that illustrates this statement and give a counterexample that shows that the statement is not always true.

Endnotes

- **Note to Students:** The solutions to the **Solved Exercises** are found on the Web site **digital.wwnorton.com/gamesofstrategy5**. Instructions on how to download this resource are available at that location. Return to reference 8

PART TWO

Fundamental Concepts and Techniques

3 ■ Games with Sequential Moves

SEQUENTIAL-MOVE GAMES entail strategic situations in which there is a strict order of play. Players take turns making their moves, and they know what the players who have gone before them have done. To play well in such a game, participants must use a particular type of interactive thinking. Each player must consider how her opponent will respond if she makes a particular move. Whenever actions are taken, players need to think about how their current actions will influence future actions, both for their rivals and for themselves. Players thus decide their current moves on the basis of calculations of future consequences.

Most actual games combine aspects of sequential- and simultaneous-move situations. But the concepts and methods of analysis are more easily understood if they are first developed separately for the two pure cases. Therefore, in this chapter, we study purely sequential-move games. Chapters 4 and 5 deal with purely simultaneous-move games, and Chapter 6 and parts of Chapter 7 show how to combine the two types of analysis in more realistic mixed situations. The analysis presented here can be used whenever a game includes sequential decision making. Analysis of sequential-move games also provides information about when it is to a player’s advantage to move first and when it is better to move second. Players can then devise ways, called *strategic moves*, to manipulate the order of play to their advantage. The analysis of such moves is the focus of Chapter 8.

1 GAME TREES

We begin by developing a graphical technique for displaying and analyzing sequential-move games, called a game tree. This tree is referred to as the extensive form of a game. It shows all the component parts of the game that we introduced in Chapter 2: players, actions, and payoffs.

You have probably come across decision trees in other contexts. Such trees show all the successive decision points, or nodes, for a single decision maker in a neutral environment. Decision trees also include branches corresponding to the available choices emerging from each node. Game trees are just joint decision trees for all the players in a game. The trees illustrate all the possible actions that can be taken by all the players and indicate all the possible outcomes of the game.

A. Nodes, Branches, and Paths of Play

Figure 3.1 shows the tree for a particular sequential-move game. We do not supply a story for this game, because we want to omit circumstantial details and help you focus on general concepts. Our game has four players: Ann, Bob, Chris, and Deb. The rules of the game give the first move to Ann; this is shown at the leftmost node, which is called the initial node or root of the game tree. At this node, which may also be called an action node or decision node, Ann has two choices available to her. Ann’s possible choices, labeled Stop and Go (remember that these labels are abstract and have no necessary significance), are shown as branches emerging from the initial node.

If Ann chooses Stop, then it will be Bob’s turn to move. At his action node, he has three available choices, labeled 1, 2, and 3. If Ann chooses Go, then Chris gets the next move, with choices Risky and Safe. Other nodes and branches follow successively; rather than listing them all in words, we draw your attention to a few prominent features.

If Ann chooses Stop and then Bob chooses 1, Ann gets another turn, with new choices, Up and Down. It is quite common in actual sequential-move games for a player to get to move several times and to have her available moves differ at different turns. In chess, for example, two players make alternate moves; each move changes the board, and therefore the available moves are changed at subsequent turns.

B. Uncertainty and "Nature's Moves"

If Ann chooses Go and then Chris chooses Risky, something happens at random: A fair coin is tossed, and the outcome of the game is determined by whether that coin comes up heads or tails. This aspect of the game is an example of external uncertainty and is handled in the tree by introducing an outside player called "Nature." Control over the random event is ceded to the player known as Nature, who chooses, as it were, one of two branches, each with 50% probability. Although Nature makes its choices with the specified probabilities, it is otherwise a passive, neutral player that receives no payoffs of its own. In this example, the probabilities are fixed by the type of random event—a coin toss—but could vary in other circumstances. For example, with the throw of a die, Nature could specify six possible outcomes, each with $16\frac{2}{3}\%$ probability. Use of the player Nature allows us to introduce external uncertainty in a game and gives us a mechanism to allow things to happen that are outside the control of any of the actual players.

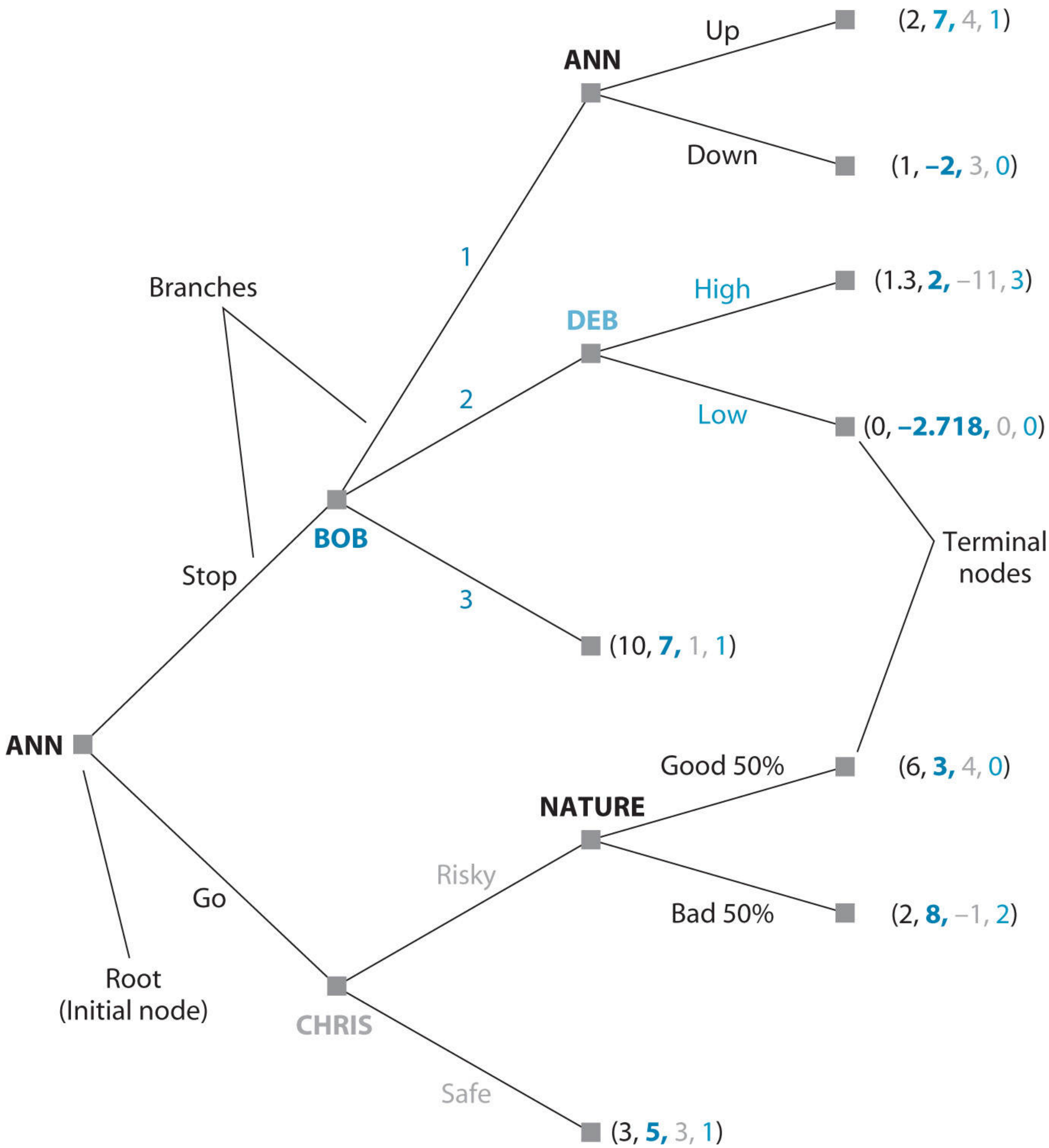


Figure 3.1 An Illustrative Game Tree

You can trace a number of different paths through the game tree by following successive branches. In Figure 3.1, each path leads you to an end point of the game after a finite number of moves. An end point is not a necessary feature of all games; some may, in principle, go on forever. But most applications that we will consider are finite games.

C. Outcomes and Payoffs

At the last node along each path, called a terminal node, no player has another move. (Note that terminal nodes are thus distinguished from *action* nodes.) Instead, we show the outcome of that particular sequence of actions, as measured by the payoffs for the players. For our four players, we list the payoffs in order (Ann, Bob, Chris, Deb). It is important to specify which payoff belongs to which player. The usual convention is to list payoffs in the order in which the players make their moves. But this method may sometimes be ambiguous; in our example, it is not clear whether Bob or Chris should be said to have the second move. Thus, we have used alphabetical order. Further, we have color-coded everything so that Ann's name, choices, and payoffs are all in black; Bob's in dark blue; Chris's in gray; and Deb's in light blue. When drawing trees for any games that you analyze, you can choose any specific convention you like, but you should state and explain it clearly for the reader.

The payoffs are numerical, and generally, for each player, a higher number means a better outcome. Thus, for Ann, the outcome of the bottommost path (payoff 3) is better than that of the topmost path (payoff 2) in Figure 3.1. But there is no necessary comparability across players. Thus, there is no necessary sense in which, at the end of the topmost path, Bob (payoff 7) does better than Ann (payoff 2). Sometimes—if payoffs are dollar amounts, for example—such interpersonal comparisons may be meaningful.

Players use information about payoffs when deciding among the various actions available to them. The inclusion of a random event (a choice made by Nature) means that players need to determine what they get on average when Nature moves. For example, if Ann chooses Go at the game's first move, Chris may then choose Risky, giving rise to the coin toss and Nature's "choice" of Good or Bad. In this situation, Ann could anticipate a payoff of 6 half the time and a payoff of 2 half the time, or a statistical average, or *expected payoff,* of $4 = (0.5 \times 6) + (0.5 \times 2)$.

D. Strategies

Finally, we use the tree in Figure 3.1 to explain the concept of a strategy. A single action taken by a player at a node is called a move. But players can, do, and should make plans for the succession of moves that they expect to make in all the various eventualities that might arise in the course of a game. Such a complete plan of action is called a *strategy*. (Although the term *strategy* always refers to a *complete* plan of action, as we described in Chapter 2, we will sometimes use the term *complete strategy* to remind you of the need to include actions for all possible eventualities in your description of a player's strategy.)

In this tree, Bob, Chris, and Deb each get to move at most once; Chris, for example, gets a move only if Ann chooses Go on her first move. For them, there is no distinction between a move and a strategy. We can qualify their moves by specifying the contingencies in which they get made; thus, a (complete) strategy for Bob might be, "Choose 1 if Ann has chosen Stop." But Ann has two opportunities to move, so her strategy needs a fuller specification. One (complete) strategy for her is, "Choose Stop, and then if Bob chooses 1, choose Down."

In more complex games such as chess, where there are long sequences of moves with many choices available at each, descriptions of strategies get very complicated; we will consider this aspect in more detail later in this chapter. But the general principle for constructing strategies is simple, except for one peculiarity. If Ann chooses Go on her first move, she never gets to make a second move. Should a strategy in which she chooses Go also specify what she would do in the hypothetical case in which she somehow found herself at the action node of her second move? Your first instinct may be to say *no*, but formal game theory says *yes*, for two reasons.

First, Ann's choice of Go at her first move may be influenced by her consideration of what she would have to do at her second move if she were to choose Stop originally instead. For example, if she chooses Stop, Bob may then choose 1; then Ann gets a second move, and her best choice would be Up, giving her a payoff of 2. If she chooses Go at her first move, Chris may then choose Safe (because his payoff of 3 from Safe is better than his expected payoff of 1.5 from Risky), and that outcome would yield Ann a payoff of 3. To make this thought process clearer, we state Ann's strategy as, "Choose Go at the first move, and choose Up if the next move arises."

The second reason for this seemingly pedantic specification of strategies has to do with the stability of equilibrium. When considering this stability, we ask what would happen if players' choices were subjected to small disturbances. One such disturbance is that players make small mistakes. If choices are made by pressing a key, for example, Ann may intend to press the Go key, but there is a small probability that her hand will tremble and she will press the Stop key instead. In such a setting, it is important to specify how Ann will follow up when she discovers her error because Bob chooses 1 and it is Ann's turn to move again. More advanced levels of game theory require such stability analyses, and we want to prepare you for that by insisting on your specifying strategies as *complete* plans of action right from the beginning.

E. Tree Construction

Let’ s sum up the general concepts illustrated by the tree in Figure 3.1. Game trees consist of nodes and branches. The nodes, which are connected to one another by the branches, come in two types. The first type is called a decision node. Each decision node is associated with the player who chooses an action at that node. Every tree has one decision node that is the game’ s initial node, the starting point of the game. The second type of node is called a terminal node. Each terminal node has associated with it a set of outcomes for the players taking part in the game; these outcomes are the payoffs received by each player if the game has followed the branches that lead to this particular terminal node.

The branches of a game tree represent the possible actions that can be taken at any decision node. Each branch leads from a decision node either to another decision node, generally for a different player, or to a terminal node. The tree must account for all the possible choices that could be made by a player at each node, so some game trees include branches associated with the choice Do Nothing. There must be at least one branch leading from each decision node, but there is no maximum number. No decision node can have more than one branch leading to it, however.

Game trees are often drawn from left to right across a page. However, game trees can be drawn in any orientation that suits the game at hand: bottom up, sideways, top down, or even radiating outward from a center. The tree is a metaphor, and its important feature is the idea of successive branching as decisions are made at the tree nodes.

Glossary

game tree
: Representation of a game in the form of nodes, branches, and terminal nodes and their associated payoffs.

extensive form
: Representation of a game by a game tree.

decision tree
: Representation of a sequential decision problem facing one person, shown using nodes, branches, terminal nodes, and their associated payoffs.

node
: This is a point from which branches emerge, or where a branch terminates, in a decision or game tree.

initial node
: The starting point of a sequential-move game. (Also called the root of the tree.)

root
: Same as initial node.

action node
: A node at which one player chooses an action from two or more that are available.

decision node
: A decision node in a decision or game tree represents a point in a game where an action is taken.

terminal node
: This represents an end point in a game tree, where the rules of the game allow no further moves, and payoffs for each player are realized.

move
: An action at one node of a game tree.

branch
: Each branch emerging from a node in a game tree represents one action that can be taken at that node.

2 SOLVING GAMES BY USING TREES

We illustrate the use of trees in finding equilibrium outcomes of sequential-move games in a very simple context that many of you have probably confronted—whether to smoke. As we mentioned in Chapter 2, players in a game need not be physically distinct persons. The question of whether to smoke, like many other similar one-player strategic situations, can be described as a game if we recognize that future choices are made by the player's future self, who will be subject to different influences and have different views about the ideal outcome of the game.

Take, for example, a teenager named Carmen who is deciding whether to smoke. First, her current self, who we call "Today's Carmen," has to decide whether to try smoking at all. If she does try it, she creates a future version of herself—a "Future Carmen"—who will have a different ranking of the alternatives available in the future. Future Carmen is the one who makes the choice of whether to continue smoking. When Today's Carmen makes her choice, she has to look ahead, consider Future Carmen's probable addiction to nicotine, and factor that into her current decision, which she should make on the basis of her current preferences. In other words, when Today's Carmen makes her decision, she has to play a game against Future Carmen.

To illustrate this example, we can use the simple game tree shown in Figure 3.2. At the initial node, Today's Carmen decides whether to try smoking. If her decision is to try, then the addicted Future Carmen comes into being and chooses whether to continue. We show the healthy, nonsmoking Today's Carmen, her actions, and her payoffs in blue and the addicted Future Carmen, her actions, and her payoffs in black, the color that her lungs have become.

In order to solve this game—that is, to determine the strategies that players choose in equilibrium—we need to use the payoffs shown at the terminal nodes of the tree. As shown in Figure 3.2, we have chosen the outcome of never smoking at all as the standard of reference, and we call its payoff 0. There is no special significance to the number 0 in this context; all that matters for comparing outcomes, and thus for making choices, is whether this payoff is bigger or smaller than the others. Suppose Today’s Carmen likes best the outcome in which she tries smoking for a while but Future Carmen chooses not to continue. The reason may be that she just likes to have experienced many things firsthand, or so that she can more convincingly be able to say, “I have been there and know it to be a bad situation,” when she tries in the future to dissuade her children from smoking. We give this outcome the payoff +1 for Today’s Carmen. The outcome in which Today’s Carmen tries smoking and then Future Carmen continues is the worst for Today’s Carmen. In addition to the long-term health hazards, she foresees other problems—her hair and clothes will smell bad, and her friends will avoid her. We give this outcome the payoff −1 for Today’s Carmen. Future Carmen has become addicted to smoking and has different preferences from Today’s Carmen. She will enjoy continuing to smoke and will suffer terrible withdrawal symptoms if she does not continue. In the tree, we show Future Carmen’s payoff from “Continue” as +1 and that from “Not” as −1.

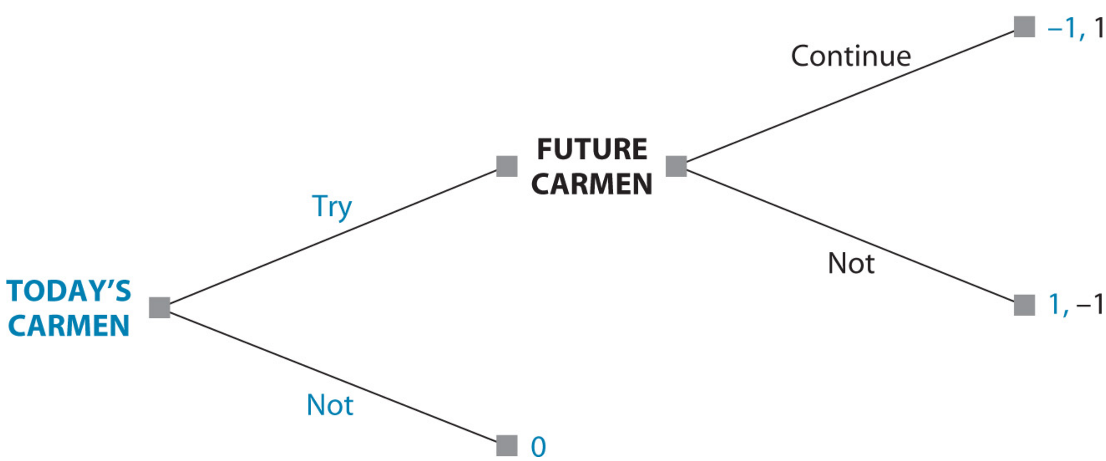


Figure 3.2 The Smoking Game

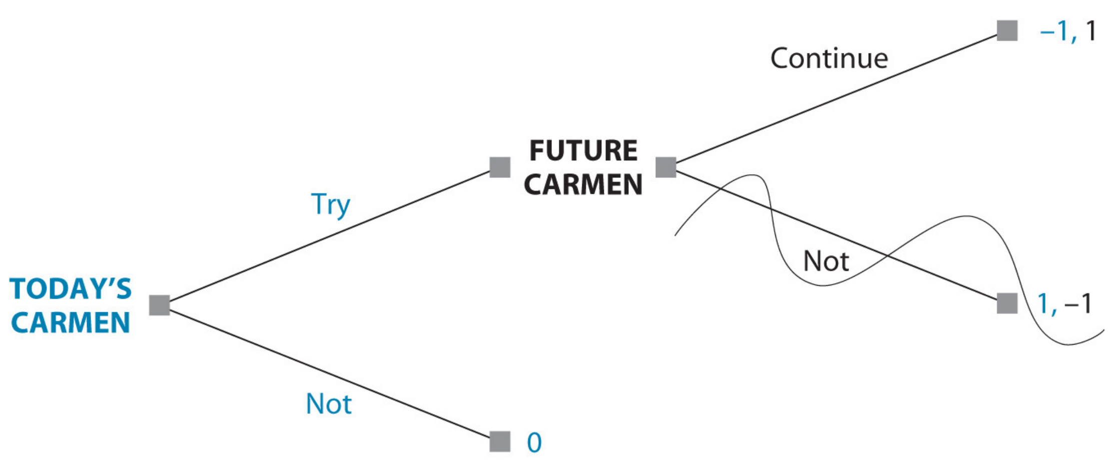
Given the preferences of the addicted Future Carmen, she will choose Continue at her decision node. Today’s Carmen should look ahead to that prospect and fold it into her current decision, recognizing that the choice to try smoking will inevitably lead to continuing to smoke. Even though Today’s Carmen does not want to continue to smoke in the future, given her preferences today, she will not be able to implement her currently preferred choice in the future because Future Carmen, who has different preferences, will make that choice. So Today’s Carmen should foresee that the choice Try will lead to

Continue and get her the payoff −1, as judged by her today, whereas the choice Not will get her the payoff 0. So she should choose Not.

This argument is shown more formally, and with greater visual effect, in Figure 3.3. In Figure 3.3a, we cut off, or prune, the branch Not emerging from the second node. This pruning corresponds to the fact that Future Carmen, who makes the choice at that node, will not choose the action associated with that branch, given her preferences, as shown in black.

The tree that remains has two branches emerging from the first node, where Today's Carmen makes her choice; each of these branches now leads directly to a terminal node. The pruning allows Today's Carmen to forecast completely the eventual consequence of each of her choices. Try will be followed by Continue and will yield a payoff of −1, as judged by the preferences of Today's Carmen, while Not will yield 0. Carmen's choice today should then be Not rather than Try. Therefore, we can prune the Try branch emerging from the first node (along with its foreseeable continuation). This pruning is done in Figure 3.3b. The tree shown there is now fully pruned, leaving only one branch emerging from the initial node and leading to a terminal node. Following the only remaining path through the tree shows what will happen in the game when all players make their best choices with correct forecasting of all future consequences.

(a) Pruning at second node:



(b) Full pruning:

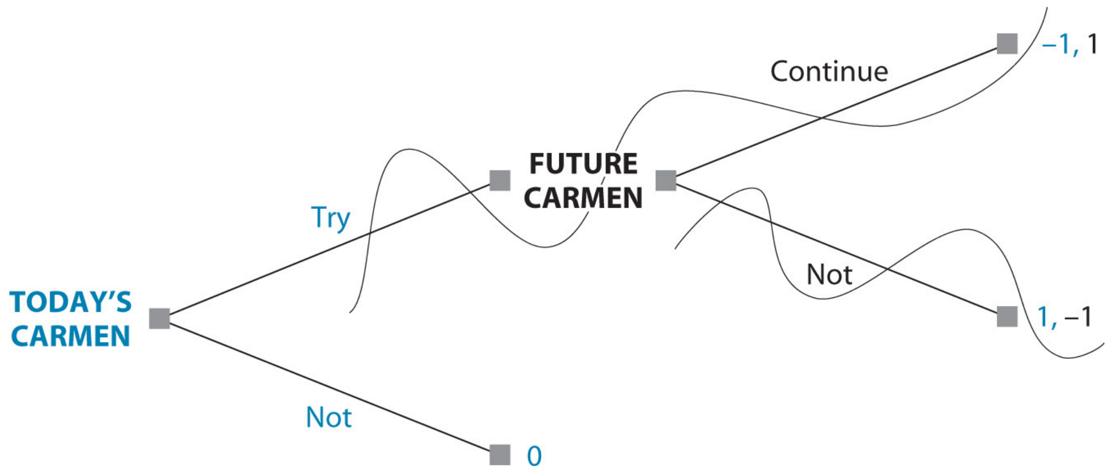


Figure 3.3Pruning the Tree of the Smoking Game

In pruning the tree in Figure 3.3, we crossed out the branches not chosen. An equivalent but alternative way of showing player choices is to highlight the branches that *are* chosen. To do so, we can place check marks or arrowheads on those branches, or show them as thicker lines. Any method will do; Figure 3.4 shows them all. You can choose whether to prune or to highlight, but the latter method, especially in its arrowhead form, has some advantages. First, it produces a cleaner picture. Second, the mess of the pruning picture sometimes does not clearly show the order in which various branches were cut. For example, in Figure 3.3b, a reader may get confused and incorrectly think that the Continue branch at the second node was cut first and that the Try branch at the initial node followed by the Not branch at the second node were cut next. Finally, and most importantly, the arrowheads show the outcome of the sequence of optimal choices most visibly, as a continuous sequence of arrowheads from the initial node to a terminal node. Therefore, in subsequent diagrams of this type, we will generally use highlighting with arrowheads instead of pruning. When you draw game trees, you should practice both methods for a while; when you are comfortable with trees, you can choose either one to suit your taste.

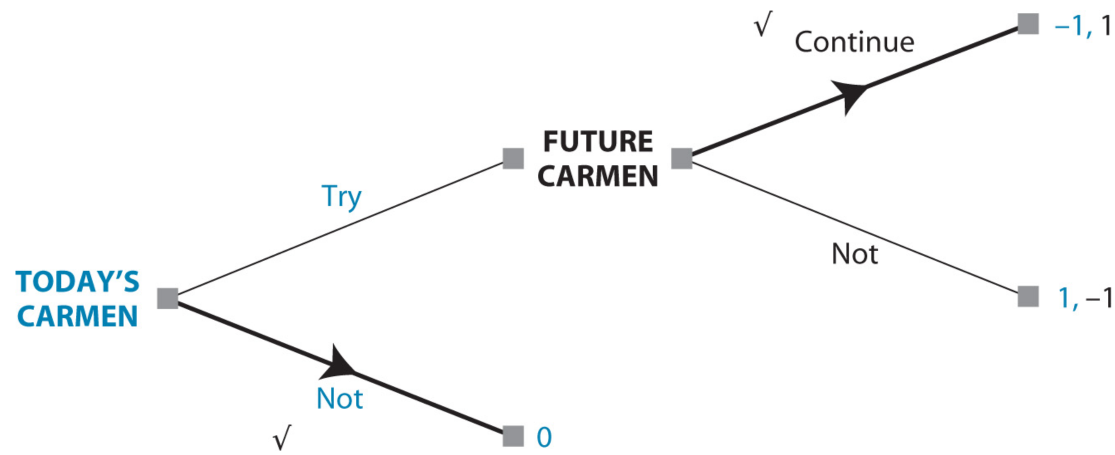


Figure 3.4 Showing Branch Selection on the Tree of the Smoking Game

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No matter how you display your thinking in a game tree, the logic of the analysis is the same. You must start your analysis by considering those action nodes that lead directly to terminal nodes. The optimal choices for a player making a move at such a node can be found immediately by comparing her payoffs at the relevant terminal nodes. By using these end-of-game choices to forecast consequences of earlier actions, we can determine the optimal choices at nodes just preceding the final decision nodes. Then the same can be done for the nodes before them, and so on. By working backward along the tree in this way, we can solve the whole game.

This method of looking ahead and reasoning back to determine optimal behavior in sequential-move games is known as rollback. As the name suggests, using rollback requires starting to think about what will happen at each terminal node and literally "rolling back" through the tree to the initial node as you do your analysis. Because this reasoning requires working backward one step at a time, the method is also called backward induction. We use the term *rollback* because it is simpler and is becoming more widely used, but other sources on game theory will use the older term *backward induction*. Just remember that the two are equivalent.

When all players use rollback to choose their optimal strategies, we call this set of strategies the rollback equilibrium of the game; the outcome that arises from playing these strategies is the *rollback equilibrium outcome*. More advanced game-theory texts refer to this concept as *subgame-perfect equilibrium*, and your instructor may prefer to use that term. We will provide more formal explanation and analysis of subgame-perfect equilibrium in Chapter 6, but we generally prefer the simpler and more intuitive term *rollback equilibrium*. Game theory predicts this outcome as the equilibrium of a sequential-move game in which all players are rational calculators in pursuit of their respective best payoffs. Later in this chapter, we will address how well this prediction is borne out in practice. For now, you should know that all finite sequential-move games presented in this book have at least one rollback equilibrium. In fact, most have exactly one. Only in those exceptional cases where a player gets equal payoffs from two or more different sets of moves, and is therefore indifferent among them, will games have more than one rollback equilibrium.

In the smoking game, the rollback equilibrium is the list of strategies where Today's Carmen chooses the strategy Not and Future Carmen chooses the strategy Continue. When Today's Carmen takes her optimal action, the addicted Future Carmen does not come into being at all, and therefore gets no actual opportunity to make a move. But Future Carmen's shadowy presence, and the strategy that she would choose if Today's Carmen chose Try and gave her an opportunity to move, are important parts of the game. In fact, they are instrumental in determining the optimal move for Today's Carmen.

We have introduced the ideas of the game tree and rollback in a very simple example, where the solution was obvious from verbal argument. Now we will proceed to use these ideas in successively more complex situations, where verbal analysis becomes harder to conduct and visual analysis with the use of the tree becomes more important.

Glossary

pruning

Using rollback analysis to identify and eliminate from a game tree those branches that will not be chosen when the game is rationally played.

rollback

Analyzing the choices that rational players will make at all nodes of a game, starting at the terminal nodes and working backward to the initial node. Also called backward induction.

backward induction

Same as rollback.

rollback equilibrium

The strategies (complete plans of action) for each player that remain after rollback analysis has been used to prune all the branches that can be pruned.

3 ADDING MORE PLAYERS

The techniques developed in Section 2 in the simplest setting of two players and two moves can be readily extended. The game trees get more complex, with more branches and nodes, but the basic concepts and the rollback method remain unchanged. In this section, we consider a game with three players, each of whom has two choices; this game, with slight variations, reappears in many subsequent chapters.

The three players, Emily, Nina, and Talia, all live on the same small street. Each has been asked to contribute toward the creation of a flower garden where their small street intersects with the main highway. The ultimate size and splendor of the garden depends on how many of them contribute. Furthermore, although each player is happy to have the garden—and happier as its size and splendor increase—each is reluctant to contribute because of the cost that she must incur to do so.

Suppose that, if two or all three contribute, there will be sufficient resources for the initial planting and subsequent maintenance of the garden; it will then be quite attractive and pleasant. However, if one or none contribute, it will be too sparse and poorly maintained to be pleasant. From each player's perspective, there are thus four distinguishable outcomes:

- She does not contribute, but both of the others do (resulting in a pleasant garden and saving the cost of her own contribution).
- She contributes, and one or both of the others do as well (resulting in a pleasant garden, but incurring the cost of her own contribution).
- She does not contribute, and only one or neither of the others does (resulting in a sparse garden, but saving the cost of her own contribution).
- She contributes, but neither of the others does (resulting in a sparse garden and incurring the cost of her own contribution).

Of these outcomes, the one listed first is clearly the best, and the one listed last is clearly the worst. We want higher payoff numbers to indicate outcomes that are more highly regarded, so we give the first outcome the payoff 4 and the last one the payoff 1. (Sometimes payoffs are associated with an outcome's rank order, so with four outcomes, 1 would be best and 4 worst, and smaller numbers would denote more preferred outcomes. When reading, you should carefully note which convention the author is using; when writing, you should carefully state which convention you are using.)

There is some ambiguity about the two middle outcomes. Let us suppose that each player regards a pleasant garden more highly than her own contribution. Then the outcome listed second gets payoff 3, and the outcome listed third gets payoff 2.

Suppose the players move sequentially. Emily has the first move, and chooses whether to contribute. Then, after observing what Emily has chosen, Nina chooses between contributing and not contributing. Finally, having observed what Emily and Nina have chosen, Talia makes a similar choice.[1]

Figure 3.5 shows the tree for this game. We have labeled the action nodes for easy reference. Emily moves at the initial node, *a*, and the branches corresponding to her two choices, Contribute and Don't, lead to nodes *b* and *c,* respectively. At each of these nodes, Nina gets to move and to choose between Contribute and Don't. Her choices lead to nodes *d*, *e*, *f*, and *g*, at each of which Talia gets to move. Her choices lead to eight terminal nodes, where we list the payoffs for the players in order (Emily, Nina, Talia).[2] For example, if Emily contributes, then Nina does not, and finally Talia does, then the garden is pleasant, and the two contributors each get a payoff of 3, while the noncontributor gets her top outcome, with a payoff of 4; in this case, the payoff list is (3, 4, 3).

To apply rollback to this game, we begin with the action nodes that come immediately before the terminal nodes—namely, *d*, *e*, *f*, and *g.* Talia moves at each of these nodes. At *d*, she faces the situation where both Emily and Nina have contributed. The garden is already assured to be pleasant, so, if Talia chooses Don't, she gets her best payoff, 4, whereas, if she chooses Contribute, she gets the next best, 3. Her preferred choice at this node is Don't. We show this preference both by thickening the branch for Don't and by adding an arrowhead; either one would suffice to illustrate Talia's choice. At node *e*, Emily has contributed and Nina has not; so Talia's contribution is crucial for a pleasant garden. Talia gets the payoff 3 if she chooses Contribute and 2 if she chooses Don't. Her preferred choice at *e* is Contribute. You can check Talia's choices at the other two nodes similarly.

Now we roll back the analysis to the preceding stage—namely, nodes *b* and *c*, where it is Nina's turn to choose. At *b*, Emily has contributed. Nina's reasoning now goes as follows: "If I choose Contribute, that will take the game to node *d*, where I know that Talia will choose Don't, and my payoff will be 3. (The garden will be pleasant, but I will have incurred the cost of my contribution.) If I choose Don't, the game will go to node *e*, where I know that Talia will choose Contribute, and I will get a payoff of 4. (The garden will be pleasant, and I will have saved the cost of my contribution.) Therefore, I should choose Don't." Similar reasoning shows that at *c*, Nina will choose Contribute.

Finally, consider Emily's choice at the initial node, *a*. She can foresee the subsequent choices of both Nina and Talia. Emily knows that, if she chooses Contribute, these later choices will be Don't for Nina and Contribute for Talia. With two contributors, the garden will be pleasant, but Emily will have incurred a cost; so her payoff will be 3. If Emily chooses Don't, then the subsequent choices will both be Contribute, and, with a pleasant garden and no cost incurred, Emily's payoff will be 4. So her preferred choice at *a* is Don't.

The result of rollback analysis for this street-garden game is now easily summarized. Emily will choose Don't, then Nina will choose Contribute, and finally Talia will choose Contribute. These choices trace a particular path of play through the tree—along the lower branch from the initial node, *a*, and then along the upper branches at each of the two subsequent nodes reached, *c* and *f.* In Figure 3.5, the path of play is easily seen as the continuous sequence of arrowheads joined tail to tip from the initial node to the terminal node fifth from the top of the tree. The payoffs that accrue to the players are shown at this terminal node.

Rollback is a simple and appealing method of analysis. Here, we emphasize some features that emerge from it. First, notice that the equilibrium path of play of a sequential-move game (the one that results in the rollback equilibrium outcome) misses most of the branches and nodes. Calculating the best actions that could be taken if these other nodes were reached, however, is an important part of determining the ultimate equilibrium. Choices made early in the game are affected by players' expectations of what would happen if they chose to do something other than their best actions and by what would happen if any opposing player chose to do something other than what was best for her. These expectations, based on predicted actions at out-of-equilibrium nodes (nodes associated with branches pruned in the process of rollback), keep players choosing optimal actions at each node. For instance, Emily's optimal choice of Don't at the first move is governed by the knowledge that, if she chooses Contribute, then Nina will choose Don't, followed by Talia choosing Contribute; this sequence will give Emily the payoff 3, instead of the 4 that she can get by choosing Don't at the first move.

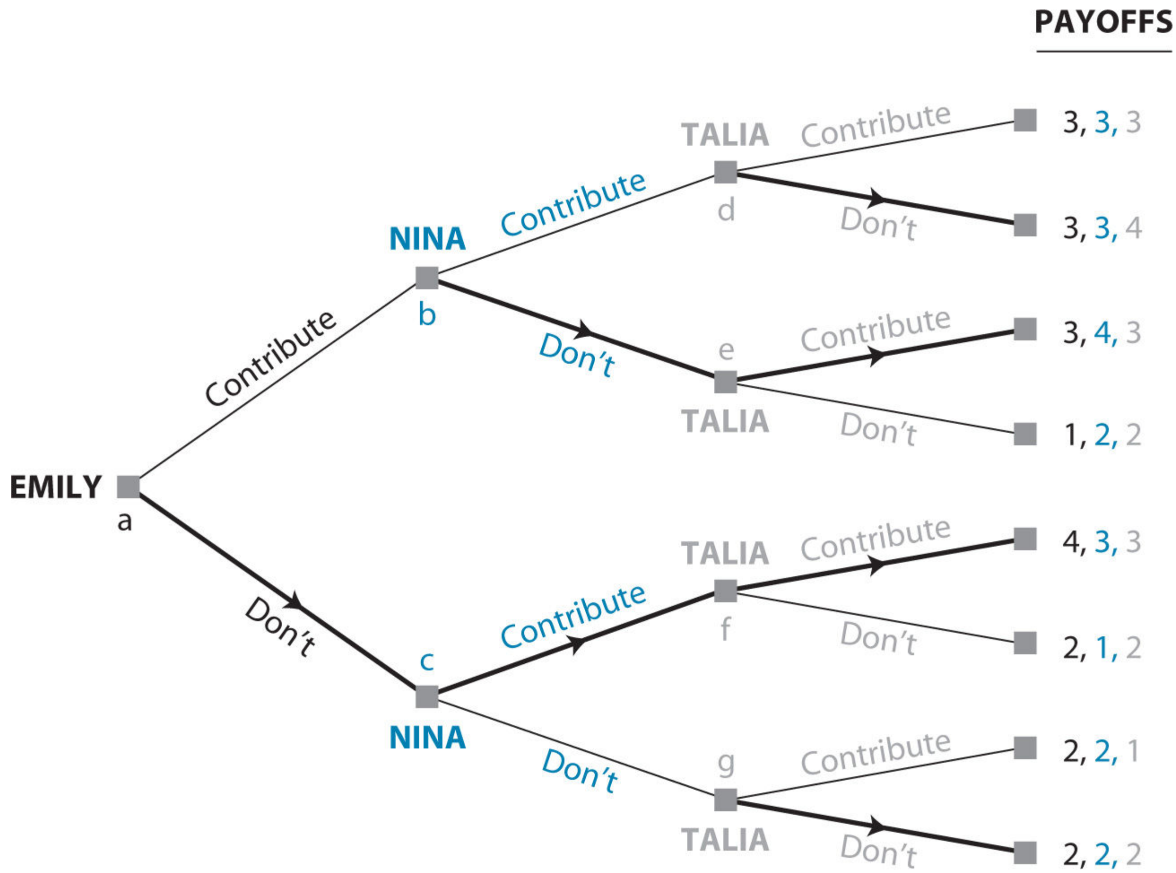


Figure 3.5 The Street-Garden Game

The rollback equilibrium gives a complete statement of all this analysis by specifying the optimal *strategy* for each player. Recall that a strategy is a complete plan of action. Emily moves first and has just two choices, so her strategy is quite simple and is effectively the same thing as her move. Emily has available to her two (complete) strategies, of which her optimal strategy is Don't, as shown in Figure 3.5. But Nina, moving second, acts at one of two nodes: at one if Emily has chosen Contribute and at the other if Emily has chosen Don't. Nina's complete plan of action has to specify what she will do in either case. One such plan, or strategy, might be "Choose Contribute if Emily has chosen Contribute, choose Don't if Emily has chosen Don't." We know from our rollback analysis that Nina will not choose this strategy, but our interest at this point is in describing all the available strategies from which Nina can choose within the rules of the game. We can abbreviate and write C for Contribute and D for Don't; thus this strategy can be written as “C if Emily chooses C so that the game is at node *b*, D if Emily chooses D so that the game is at node *c*,” or, more simply, “C at *b*, D at *c*,” or even “CD” if the circumstances in which each of the stated actions is taken are evident or previously explained. Now it is easy to see that, because Nina has two choices available at each of the two nodes where she might be acting, she has available to her four plans, or (complete) strategies: “C at *b*, C at *c*,” “C at *b*, D at *c*,” “D at *b*, C at *c*,” and “D at *b*, D at *c*”; or “CC,” “CD,” “DC,” and “DD.” Of these strategies, the rollback analysis and the arrowheads at nodes *b* and *c* of Figure 3.5 show that her optimal strategy is DC.

Matters are even more complicated for Talia. When her turn comes, the history of play can, according to the rules of the game, be any one of four possibilities. Talia’s turn to act comes at one of four nodes in the tree: the first after Emily has chosen C and Nina has chosen C (node *d*), the second after Emily’s C and Nina’s D (node *e*), the third after Emily’s D and Nina’s C (node *f*), and the fourth after both Emily and Nina choose D (node *g*). Each of Talia’s strategies, or complete plans of action, must specify one of her two possible actions for each of these four scenarios—that is, at each of her four possible action nodes. With four nodes at which to specify an action and with two actions from which to choose at each node, there are 2 × 2 × 2 × 2, or 16 possible combinations of actions. So Talia has available to her 16 possible (complete) strategies. One of them could be written as

C at *d*, D at *e*, D at *f*, C at *g*,

or CDDC for short, where we have fixed the order of the four scenarios (the histories of moves by Emily and Nina) in the order of nodes *d*, *e*, *f*, and *g*. Then, with the use of the same abbreviations, the full list of 16 strategies available to Talia is

CCCC, CCCD, CCDC, CCDD, CDCC, CDCD, CDDC, CDDD,

DCCC, DCCD, DCDC, DCDD, DDCC, DDCD, DDDC, DDDD.

Of these strategies, the rollback analysis in Figure 3.5 and the arrowheads that follow nodes *d*, *e*, *f*, and *g* show that Talia's optimal strategy is DCCD.

Now we can express the findings of our rollback analysis by stating the strategy choices of each player: Emily chooses D from the 2 strategies available to her, Nina chooses DC from the 4 strategies available to her, and Talia chooses DCCD from the 16 strategies available to her. When each player looks ahead in the tree to forecast the eventual outcomes of her current choices, she is calculating the optimal strategies of the other players. This configuration of strategies—D for Emily, DC for Nina, and DCCD for Talia—then constitutes the rollback equilibrium of the game.

We can put together the optimal strategies of the players to find the equilibrium path of play. Emily will begin by choosing D. Nina, following her strategy DC, chooses the action C in response to Emily's D. (Remember that Nina's DC means "Choose D if Emily has played C, and choose C if Emily has played D.") According to the convention that we have adopted, Talia's actual action after Emily's D and then Nina's C—at node *f*—is the third letter in the four-letter specification of her strategies. Because Talia's optimal strategy is DCCD, her action along the equilibrium path of play is C. Thus the equilibrium path of play consists of Emily playing D, followed successively by Nina and Talia playing C.

To sum up, we have three distinct concepts:

1. The list of available strategies, or complete plans of action, for each player. The list, especially for later players, may be very long because their actions in situations corresponding to all conceivable preceding moves by other players must be specified.
2. The optimal strategy for each player. This strategy must specify the player's *best* choice at each node where the rules of the game specify that she moves, even though many of these nodes will never be reached in the actual path of play. This specification is, in effect, the preceding movers'

forecasting of what would happen if they took different actions and is therefore an important part of their calculations of their own best actions at the earlier nodes. The optimal strategies of all players together yield the rollback equilibrium.

3. The equilibrium path of play, found by putting together the optimal strategies for all the players.

Endnotes

- In later chapters, we vary the rules of this game—the order of moves and payoffs—and examine how such variation changes the outcomes. Return to reference 1
- Recall from the discussion of game trees in Section 1 that the usual convention for sequential-move games is to list payoffs in the order in which the players move; however, in case of ambiguity, or simply for clarity, it is good practice to specify the order explicitly. Return to reference 2

Glossary

path of play
: A route through the game tree (linking a succession of nodes and branches) that results from a configuration of strategies for the players that are within the rules of the game. (See also *equilibrium path of play*.)

equilibrium path of play
: The *path of play* actually followed when players choose their rollback equilibrium strategies in a sequential game.

4 ORDER ADVANTAGES

In the rollback equilibrium of the street-garden game, Emily gets her best outcome (payoff 4) because she can take advantage of the opportunity to make the first move. When she chooses not to contribute, she puts the onus on the other two players—each of whom can get her next best outcome if and only if both of them choose to contribute. Most casual thinkers about strategic games have the preconception that such a first-mover advantage should exist in all games. However, that is not the case. It is easy to think of games in which an opportunity to move second is an advantage. Consider the strategic interaction between two firms that sell similar merchandise from catalogs—say, Lands' End and L.L.Bean. If one firm had to release its catalog first, and the second firm could see what prices the first had set before printing its own catalog, then the second mover could undercut its rival on all items and gain a tremendous competitive edge.

First-mover advantage comes from the ability to commit oneself to an advantageous position and to force the other players to adapt to it; second-mover advantage comes from the flexibility to adapt oneself to the others' choices. Whether commitment or flexibility is more important in a specific game depends on the particular details of the players' strategies and payoffs, and we will come across examples of both kinds of advantages throughout this book. The general point that there need not be a first-mover advantage—a point that runs against much common perception—is so important that we felt it necessary to emphasize at the outset.

When a game has a first- or second-mover advantage, each player may try to manipulate the order of play so as to secure for herself the advantageous position. We will discuss a player's ability to change the order of moves in greater detail in Chapter 6, and we will take up the manipulation of a game's order of play by way of strategic moves in Chapter 8.

Glossary

first-mover advantage
: This exists in a game if, considering a hypothetical choice between moving first and moving second, a player would choose the former.

second-mover advantage
: A game has this if, considering a hypothetical choice between moving first and moving second, a player would choose the latter.

5 ADDING MORE MOVES

We saw in Section 3 that adding more players increases the complexity of the analysis of sequential-move games. In this section, we consider another type of complexity, which arises from adding more moves to the game. We can do so most simply in a two-person game by allowing players to alternate moves more than once. In this case, the tree is enlarged in the same fashion that a multiple-player game tree would be, but later moves in the tree are made by the same players who have made decisions earlier in the game.

Many common board games, such as tic-tac-toe, checkers, chess, and Go, are two-person strategic games with such alternating sequential moves. The use of game trees and rollback should allow us to solve such games—that is, to determine the rollback equilibrium outcome and the optimal strategies leading to that outcome. Unfortunately, as the complexity of the game grows, and as strategies become more and more intricate, the search for optimal strategies and equilibria becomes more and more difficult as well. Fortunately, or perhaps unfortunately for the humans who used to be experts and champions at these games, computer science has evolved to a point where artificial intelligence (AI) methods can go where no human has been before. In this section, we give a brief overview of the search for solutions to such games.

A. Tic-Tac-Toe

Let’ s start with the simplest of the four examples mentioned in the preceding paragraph, tic-tac-toe, and consider an easier-than-usual version in which each of two players (X and O) tries to be the first to get two of their symbols to fill any row, column, or diagonal of a two-by-two game board. The first player has four possible actions, or positions in which to put her X. The second player then has three possible actions at each of four decision nodes. When the first player gets to her second turn, she has two possible actions at each of 12 (4 × 3) decision nodes. As Figure 3.6 shows, even this mini-game of tic-tac-toe has quite a complex game tree. This tree is actually not too complex, because the game is guaranteed to end after the first player moves a second time, but there are still 24 terminal nodes to consider.

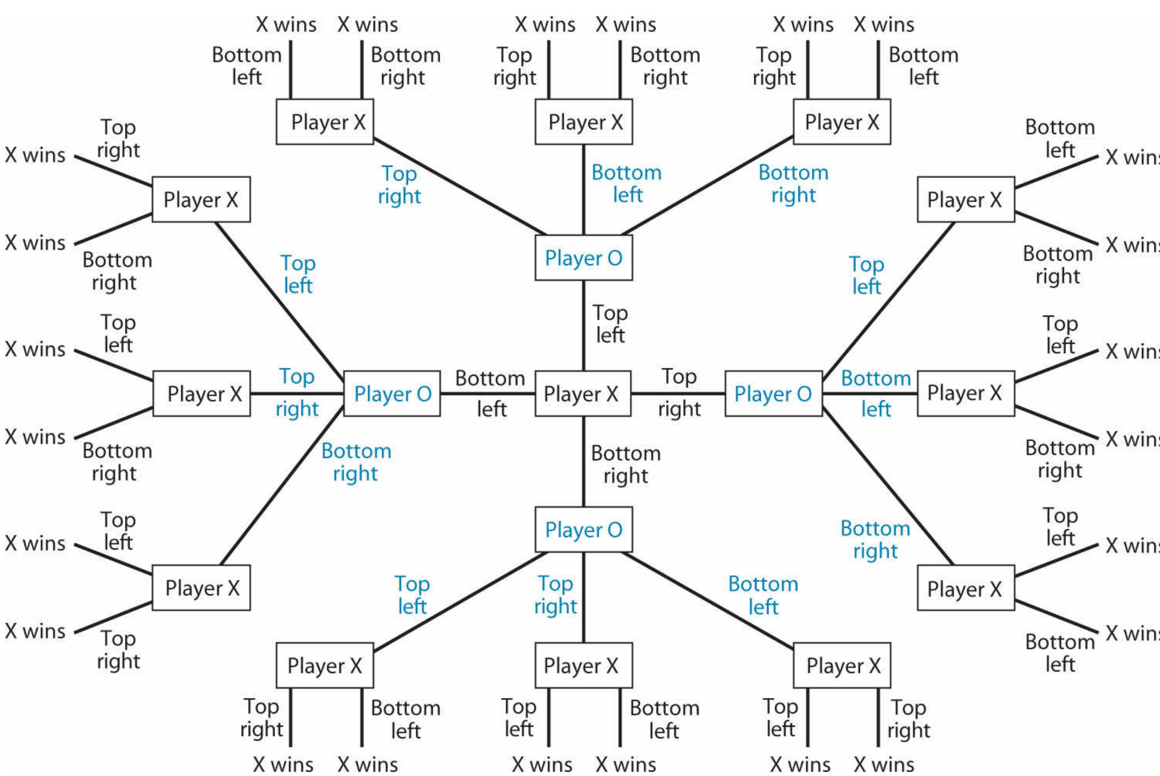


Figure 3.6 The Complex Tree for Simple Two-by-Two Tic-Tac-Toe

We show this tree merely as an illustration of how complex game trees can become even in simple (or simplified) games. As it turns out, using rollback on this mini-game of tic-tac-toe leads us quickly to an equilibrium. Rollback shows that all the choices for the first player at her second move lead to the same outcome. There is no optimal action; any move is as good as any other move. Thus, when the second player makes her first move, she also sees that each possible move yields the same outcome, and she, too, is indifferent among her three choices at each of her four decision nodes. Finally, the same is true for the first player on her first move; any choice is as good as any other, so she is guaranteed to win the game.

Although this version of tic-tac-toe has an interesting tree, its solution is not as interesting. The first mover always wins, so no choices made by either player can affect the ultimate outcome. Most of us are more familiar with the three-by-three version of tic-tac-toe. To illustrate that version with a game tree, we would have to show that the first player has nine possible actions at the initial node; that the second player has eight possible actions at each of nine decision nodes; that the first player, on her second turn, has seven possible actions at each of $8 \times 9 = 72$ nodes; and that the second player, on her second turn, has six possible actions at each of $7 \times 8 \times 9 = 504$ nodes. This pattern continues until eventually the tree stops branching so rapidly because certain combinations of moves lead to a win for one player, and the game ends. But no win is possible until at least the fifth move. Drawing the complete tree for this game requires a very large piece of paper or very tiny handwriting.

Most of you know, however, how to achieve, at worst, a tie when you play three-by-three tic-tac-toe. So there is a simple solution to this game that can be found by rollback, and a learned strategic thinker can reduce the complexity of the game considerably in the quest for such a solution. It turns out that, as in the two-by-two version, many of the possible paths through the game tree are strategically identical. Of the nine possible initial moves, there are only three types: You put your X in a corner position (of which there are four possibilities), a side position (of which there are also four possibilities), or the (one) middle position. Using this method to simplify the tree can help reduce the complexity of the problem and lead you to a rollback equilibrium. Specifically, we can show that the player who moves second can always guarantee at least a tie by choosing an appropriate first move and then by continually blocking the first player’s attempts to get three symbols in a row.[3]

B. Checkers

Although relatively small games, such as tic-tac-toe, can be solved using rollback, we saw in Figure 3.6 how rapidly the complexity of game trees can increase, even in two-player games, as the number of possible moves increases. Thus, when we consider more complex games, finding a complete solution becomes much more difficult. Consider checkers, a two-player game played on an eight-by-eight board in which each player has 12 round game pieces of different colors, as shown in Figure 3.7. Players take turns moving their pieces diagonally on the board, jumping (and capturing) the opponent's pieces when possible. The game ends, and Player A wins, when Player B is either out of pieces or unable to move; the game can also end in a draw if both players agree that neither can win. There are about 5×10^{20} possible arrangements of pieces on the board, so drawing a game tree is out of the question.

Conventional wisdom and evidence from world checkers championships over many years suggested that good play should lead to a draw, but there was no proof. That is, not until 2007, when a computer scientist in Canada proved that a computer program named Chinook could play to a guaranteed tie. Checkers was solved.[4]

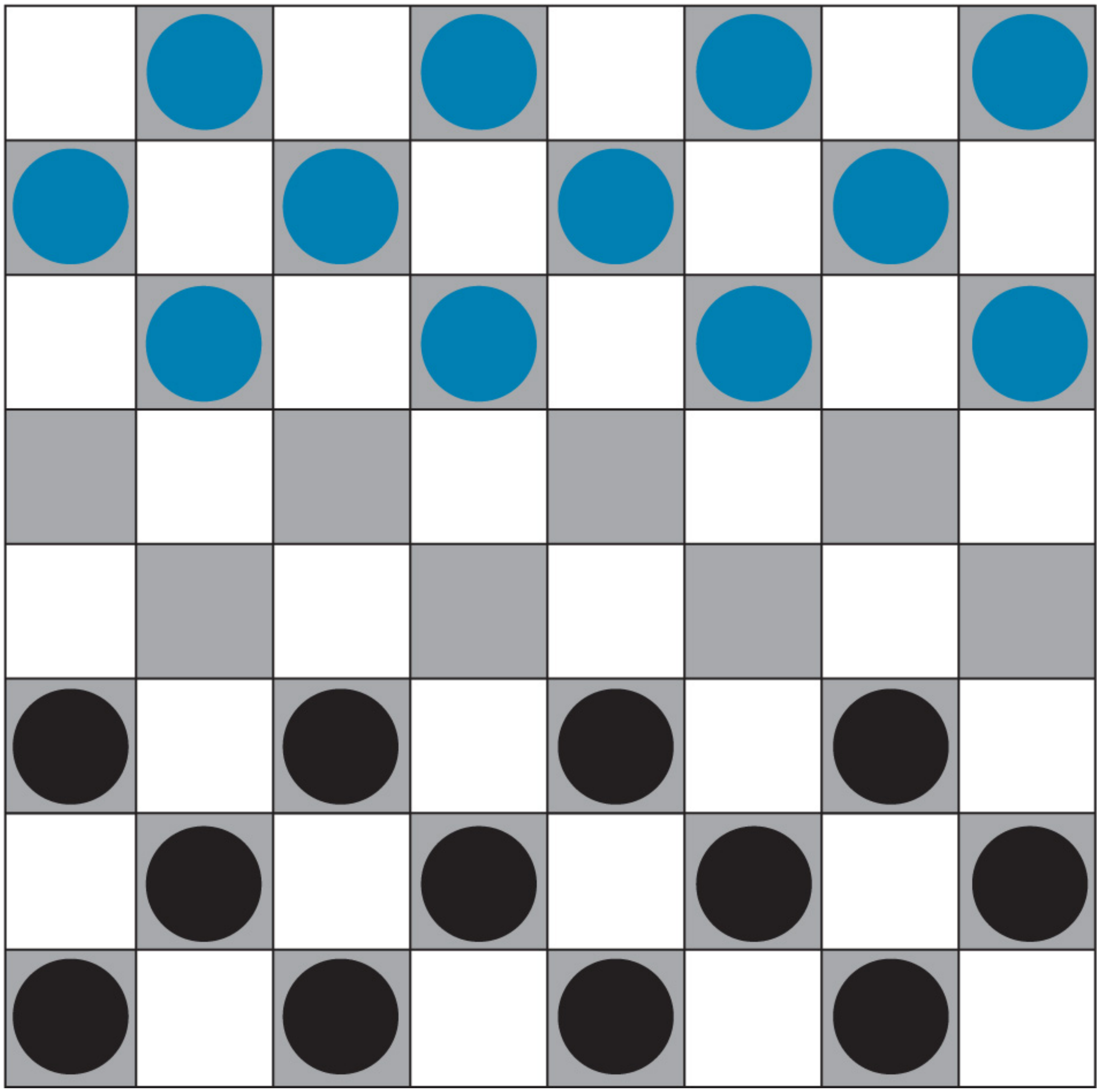


Figure 3.7 Checkerboard

Chinook, first created in 1989, played the world champion, Marion Tinsley, in 1992 (losing 4 to 2, with 33 draws) and again in 1994 (when Tinsley's health failed during a series of draws). It was put on hold between 1997 and 2001 while its creators waited for computer technology to improve. And it finally exhibited a loss-proof algorithm in the spring of 2007. That algorithm uses a combination of endgame rollback analysis and starting position forward analysis, along with calculations designed to determine the value of intermediate positions, to trace out the best moves within a database including all possible positions on the board.

The creators of Chinook describe the full game of checkers as “weakly solved”; they know that they can generate a tie, and they have a strategy for reaching that tie from the start of the game. For all 39×10^{12} possible positions that include 10 or fewer pieces on the board, they describe checkers as “strongly solved”; not only do they know they can play to a tie, but they can reach that tie from any of the possible positions that can arise once only 10 pieces remain. Their algorithm first solved these 10-piece endgames, then went back to the start to search out paths of play in which both players make optimal choices. The search mechanism, involving a complex system of evaluating the value of each intermediate position, invariably led to those 10-piece positions that generate a draw.

Thus, our hope for the future of rollback analysis may not be misplaced. We know that for very simple games, we can find the rollback equilibrium by verbal reasoning without having to draw the game tree explicitly. For games having an intermediate range of complexity, verbal reasoning is too hard, but a complete tree can be drawn and used for rollback. Sometimes we can enlist the aid of a computer to draw and analyze a moderately complicated game tree. For the most complex games, such as checkers (and chess and Go, as we will see), we can draw only a small part of the game tree, and we must use a combination of two methods: (1) calculation based on the logic of rollback, and (2) rules of thumb for valuing intermediate positions on the basis of experience. The computational power of current algorithms has shown that even some games in this category are amenable to solution, provided one has the time and resources to devote to the problem.

C. Chess

In chess, each of the players, White and Black, has a collection of 16 pieces in six distinct shapes, each of which is bound by specified rules of movement on the eight-by-eight game board shown in Figure 3.8.[5] White opens with a move, Black responds with one, and so on, in turns. All the moves are visible to the other player, and nothing is left to chance, as it would be in card games that include shuffling and dealing. Moreover, a chess game must end in a finite number of moves. The rules declare that a game is drawn if a given position on the board is repeated three times in the course of play. Because there are a finite number of ways to place the 32 (or fewer after captures) pieces on 64 squares, a game could not go on infinitely without running up against this rule. Therefore, in principle, chess is amenable to full rollback analysis.[6]

That rollback analysis has not been carried out, however. Chess has not been solved, as tic-tac-toe and checkers have been. And the reason is that, for all its simplicity of rules, chess is a bewilderingly complex game. From the initial set position of the pieces, illustrated in Figure 3.8, White can open with any one of 20 moves,[7] and Black can respond with any of 20. Therefore, 20 branches emerge from the first node of the game tree, each leading to a second node, from each of which 20 more branches emerge. After only two moves, there are already 400 branches, each leading to a node from which many more branches emerge. And the total number of possible moves in chess has been estimated to be 10^{120}, or a one with 120 zeroes after it. A supercomputer a thousand times as fast as your PC, making a trillion calculations a second, would need more than 10^{100} years to check out all these moves.[8] Astronomers offer us less than 10^{10} years before the sun turns into a red giant and swallows the earth.

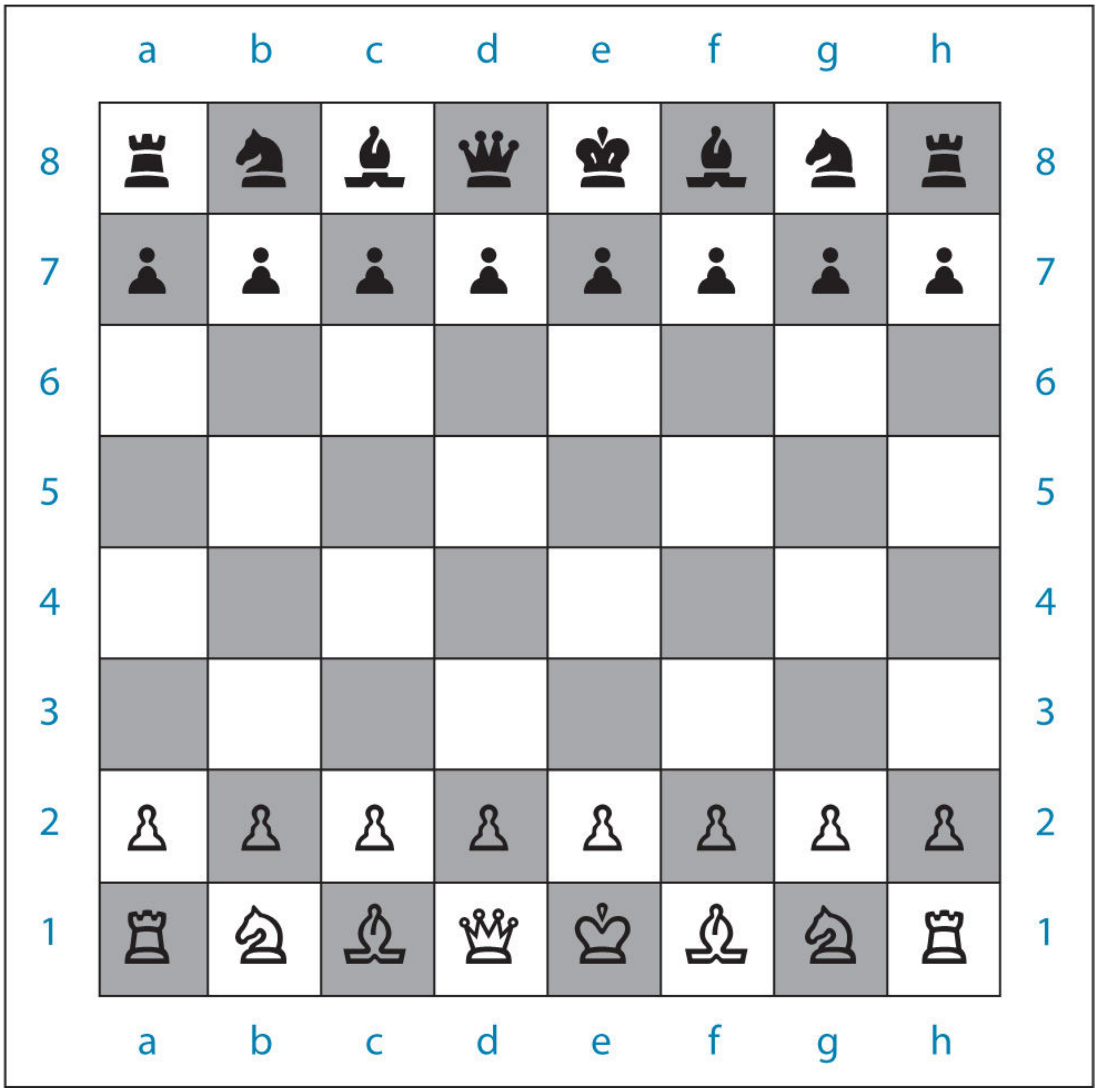


Figure 3.8 Chessboard

The general point is that, although a game may be amenable in principle to solution by rollback, its tree may be too complex to permit such a solution in practice. Faced with such a situation, what is a player to do? We can learn a lot about this problem by reviewing the history of attempts to program computers to play chess.

When computers first started to prove their usefulness for complex calculations in science and business, many mathematicians and computer scientists thought that a chess-playing computer program would soon beat the world champion. It took a lot longer, because even though computer technology improved rapidly, human thought progressed much more slowly. Finally, in December 1992, a German chess program called Fritz2 beat world champion Gary Kasparov in some blitz (high-speed, time-limited) games. Under regular rules, where each player gets two and a half hours to make 40 moves, humans retained their superiority longer. A team sponsored by IBM put a lot of effort and resources into the development of a specialized chess-playing computer and its associated software. In February 1996, this package, called Deep Blue, was pitted against Kasparov in a best-of-six series. Deep Blue caused a sensation by winning the first game, but Kasparov quickly figured out its weaknesses, improved his counterstrategies, and won the series handily. In the next 15 months, the IBM team improved Deep Blue's hardware and software, and the resulting Deeper Blue beat Kasparov in another best-of-six series in May 1997.

To sum up, chess-playing computers progressed in a combination of slow patches and some rapid spurts, while human players held some superiority, but were not able to improve sufficiently fast to keep ahead. Closer examination reveals that the two used quite different approaches to think through the very complex game tree of chess.

When contemplating a move in chess, looking ahead to the end of the whole game is too hard (for humans and computers both). How about looking part of the way—say, 5 or 10 moves ahead—and working back from there? The game need not end within this limited horizon; that is, the nodes that you reach after 5 or 10 moves will not generally be terminal nodes. Only terminal nodes have payoffs specified by the rules of the game. Therefore, you need some indirect way of assigning plausible payoffs to nonterminal nodes if you are not able to explicitly roll back from a full look-ahead. A rule that assigns such payoffs is called an intermediate valuation function.

In chess, humans and computer programs have both used such partial look-ahead in conjunction with an intermediate valuation function. The typical method assigns a numerical value to each piece and to positional and combinational advantages that can arise during play. Values for different positions are generally quantified on the basis of the whole chess-playing community’s experience of play in past games starting from such positions or patterns; this experience is called *knowledge*. The sum of all the numerical values attached to pieces and their combinations in a position is the intermediate value of that position. A move is judged by the value of the position to which it is expected to lead after an explicit forward-looking calculation for a certain number—say, 5 or 6—of moves.

The valuation of intermediate positions has progressed furthest with respect to chess openings—that is, the first dozen or so moves of a game. Each opening can lead to any one of a vast multitude of further moves and positions, but experience enables players to sum up certain openings as being more or less likely to favor one player or the other. This knowledge has been written down in massive books of openings, and all top players and computer programs remember and use this information.

At the end stages of a game, when only a few pieces are left on the board, rollback is often simple enough to yield a complete solution from that point forward in the game. The midgame, when positions have evolved to a level of complexity that will not simplify within a few moves, is the hardest to analyze. To find a good move from a midgame position, a well-built intermediate valuation function is likely to be more valuable than the ability to calculate another few moves further ahead. Gradually, computer scientists improved their chess-playing programs by building on this knowledge. When modifying Deep Blue in 1996 and 1997, IBM enlisted the help of human experts to improve the intermediate valuation function in its software. These consultants played repeatedly against the machine, noted its weaknesses, and suggested how the valuation function should be modified to correct the flaws. Deep Blue benefited from the contributions of the experts and their subtle kind of thinking, which results from long experience and an awareness of complex interconnections among the pieces on the board.

The art of the midgame in chess is also an exercise in recognizing and evaluating patterns, something humans have always excelled at, but computer scientists used to find it difficult to program into exact algorithms. This is where Kasparov had his greatest advantage over Fritz2 or Deep Blue. It also explains why computer programs do better against humans at blitz or limited-time games: The humans do not have the time to marshal their art of the midgame.

This state of affairs has changed dramatically in the last decade with the advent of artificial intelligence, including machine learning and deep learning. Instead of having programs that perform one very specific calculation, computers are programmed to develop knowledge and to learn to perform broader sets of tasks. Machine learning “trains” the computer by feeding it huge quantities of relevant data, allowing its algorithms to draw and test inferences and, gradually, to improve its ability. Perhaps the best example is in facial recognition, where deep learning improves the process by constructing the machine as an artificial neural network, mimicking the neural connection structure of the brain.

Applying these techniques to chess means having a machine play against itself numerous times in order to build its own “knowledge” and “art” of the game and to improve its strategies.[9] So successful is this new method that in December 2017, AlphaZero, a game-playing AI program created by Deep Mind (another subsidiary of Google’s parent company, Alphabet), achieved an amazing feat. Programmers gave it only the formal rules of chess and fed in no prior knowledge at all. Then, after only four hours of playing against itself, AlphaZero beat Stockfish 8 (the then-world-champion specialized chess program) in a 100-game match without losing a single game, winning 28 and drawing 72![10] Similar AI techniques have been applied to other games, including Go, which we consider next.

D. Go

The complexity of chess may be mind-boggling, but it pales in comparison with that of Go. In this two-player sequential-move game, played on a 19-by-19 board (although some simpler variants use smaller boards), each player has a large supply of tokens (“stones”) of one color, usually black for one and white for the other. The players take turns placing one stone at a time at an empty intersection of the grid lines that divide the board. There are additional rules about capturing and removing the opponent’s stones, and about ending and scoring the game. Figure 3.9 shows a board and a game in progress.

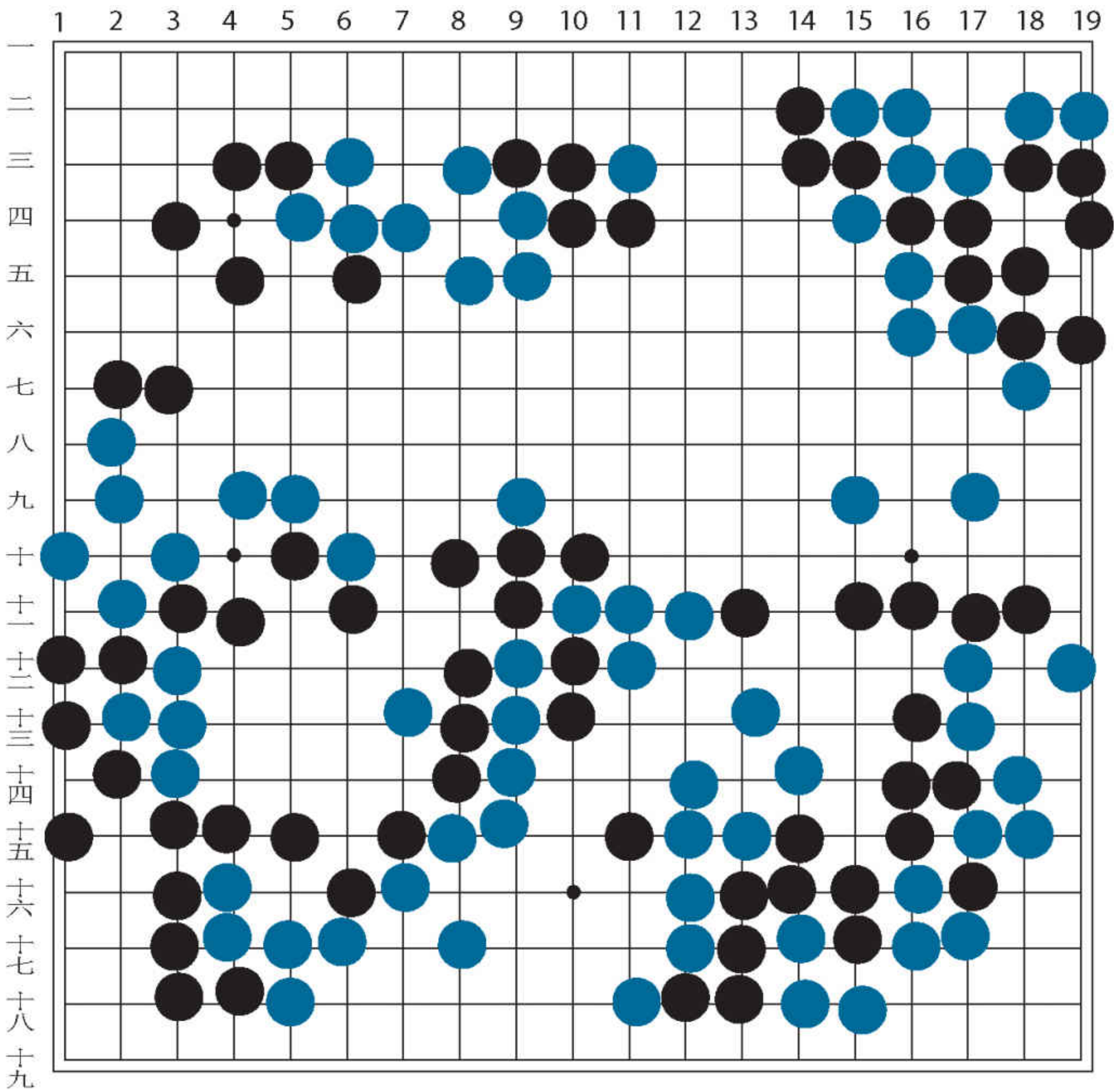


Figure 3.9 Go Game Board, © Natalya Erofeeva/Shutterstock

It has been estimated that the number of board positions in Go is about 10^{172}, and that a game tree would have about 10^{360} nodes. By contrast, chess, with its 10^{120} possible moves, may seem almost trivial! For a long time, it was believed that human skills at pattern recognition and knowledge development would always give people an advantage in this game against the brute-force calculations of a computer. But in the last two or three years, artificial intelligence has progressed to a level where it can successfully challenge top players. Much of this progress has been achieved by AlphaGo, a special deep-learning neural network created by Deep Mind for playing Go. In October 2015, AlphaGo beat Europe’s Go champion, Fan Hui, 5 to 0. In March 2016, it beat Lee Sedol, one of the world’s top players, 4 to 1. And in 2017, it won a three-game match against Ke Jie, who at the time had been the world’s top-ranked player for two successive years. AlphaGo benefited from “reinforcement learning” by being fed data from human play and then improving by playing against itself. Then, later in 2017, an improved version called AlphaGoZero (similar to the chess-playing AlphaZero), using skills attained from playing against itself for a mere eight hours, beat AlphaGo by 60 games to 40 in a 100-game match.[11] It seems that human superiority at Go is Go-ne.

Thankfully, most of the strategic games that we encounter in economics, politics, sports, business, and daily life are far less complex than Go, chess, or even checkers. These games may have a number of players who move a number of times; they may even have a large number of players or a large number of moves. But we have a chance at being able to draw a reasonable-looking tree for those games that are sequential in nature. The logic of rollback remains valid, and once you understand the idea of rollback, you can often carry out the necessary logical thinking and solve the game without explicitly drawing a tree. Moreover, it is precisely at this intermediate level of difficulty, between the simple examples that we solved explicitly in this chapter and the insoluble cases such as chess, that computer software such as Gambit is most likely to be useful; this is indeed fortunate for the prospect of applying the theory to solve many games in practice.

Endnotes

- If the first player puts her first symbol in the middle position, the second player must put her first symbol in a corner position. Then the second player can guarantee a tie by taking the third position in any row, column, or diagonal that the first player tries to fill. If the first player goes to a corner or a side position first, the second player can guarantee a tie by going to the middle first and then following the same blocking technique. Note that if the first player picks a corner, the second player picks the middle, and the first player then picks the corner opposite from her original play, then the second player must not pick one of the remaining corners if she is to ensure at least a tie. For a beautifully detailed picture of the complete contingent strategy in tic-tac-toe, see the online comic strip at http://xkcd.com/832/. Return to reference 3
- Our account is based on two reports in the journal *Science*. See Adrian Cho, "Program Proves That Checkers, Perfectly Played, Is a No-Win Situation," *Science*, vol. 317 (July 20, 2007), pp. 308–9, and Jonathan Schaeffer, Neil Burch, Yngvi Björnsson, Akihiro Kishimoto, Martin Müller, Robert Lake, Paul Lu, and Steve Sutphen, "Checkers Is Solved," *Science*, vol. 317 (September 14, 2007), pp. 1518–22. Return to reference 4
- An easily accessible statement of the rules of chess and much more is at Wikipedia, at http://en.wikipedia.org/wiki/Chess. Return to reference 5
- In fact, John Von Neumann, the polymath who was one of the pioneers of game theory, called chess merely "a well-defined form of computation," and wanted to reserve the name "game theory" for the analysis of "bluffing, . . . little tactics of deception, . . . asking yourself what is the other man going to think I mean to do." See William Poundstone, *Prisoner's Dilemma* (New York: Anchor Books, 1992), p. 6. Return to reference 6
- He can move one of the eight pawns forward either one square or two, or he can move one of the two knights in one of two

ways (to square a3, c3, f3, or h3). Return to reference 7

- This would have to be done only once, because after the game has been solved, anyone will be able to use the solution and no one will actually need to play. Everyone will know whether White has a win or whether Black can force a draw. Players will toss a coin to decide who gets which color. They will then know the outcome, shake hands, and go home. Return to reference 8
- This process was perhaps foreshadowed in the 1983 movie *War Games,* where a military computer, by playing multiple times against itself, learns that in the game of Global Thermonuclear War, both sides always lose. Or, as the computer famously put it, "The only winning move is not to play." Return to reference 9
- "AlphaZero AI Beats Champion Chess Program after Teaching Itself in Four Hours," *The Guardian,* December 7, 2017. Return to reference 10
- For more information on these AI accomplishments, see https://en.wikipedia.org/wiki/DeepMind, and the article referenced in note 10. Return to reference 11

Glossary

intermediate valuation function
: A rule assigning payoffs to nonterminal nodes in a game. In many complex games, this must be based on knowledge or experience of playing similar games, instead of explicit rollback analysis.

6 EVIDENCE CONCERNING ROLLBACK

As we have seen, rollback calculations may be infeasible in some complex multistage games. But do actual participants, even in simpler sequential-move games, act as rollback reasoning predicts? Classroom and research experiments with some games have yielded outcomes that appear to counter the predictions of game theory. Some of these experiments and their outcomes have interesting implications for the strategic analysis of sequential-move games.

Many experimenters have had subjects play a single-round bargaining game in which two players, designated A and B, are chosen from a class or a group of volunteers. The experimenter provides a dollar (or some known total amount of money), which can be divided between the players according to the following procedure: Player A proposes a split—for example, “75% to me, 25% to B.” If Player B accepts this proposal, the dollar is divided as proposed by A. If B rejects the proposal, neither player gets anything. Because A is, in effect, offering B a stark choice of “this or nothing,” the game is known as the *ultimatum game*.

Rollback in this case predicts that B should accept any sum, no matter how small, because the alternative payoff is even worse—namely, $0—and that A, foreseeing this, should propose “99% to me, 1% to B.” This particular split almost never happens. Most players assigned the A role propose a much more equal split. In fact, 50:50 is the single most common proposal. Furthermore, most players assigned the B role turn down proposals that leave them with 25% or less of the total and walk away with nothing; some reject proposals that would give them 40% of the pie.[12]

Many game theorists remain unpersuaded that these findings undermine the theory. They counter with some variant of the following argument: “The sums are so small as to make the whole thing trivial in the players’ minds. The B players lose 25 or 40 cents, which is almost nothing, and perhaps gain some private satisfaction that they walked away from a humiliatingly small award. If the total were a thousand dollars, so that 25% of it amounted to real money, the B players would accept.” But this argument does not seem to be valid. Experiments with much larger stakes show similar results. The findings from experiments conducted in Indonesia with sums that were small in dollars, but amounted to as much as three months’ earnings for the participants, showed no clear tendency on the part of the A players to make less equal offers, although the B players tended to accept somewhat smaller shares as the sums increased; similar experiments conducted in the Slovak Republic found the behavior of inexperienced players unaffected by large changes in payoffs.[13]

The participants in these experiments typically have no prior knowledge of game theory and no special computational abilities. But the game is extremely simple; surely even the most naive player can see through the reasoning, and players’ answers to direct questions after the experiment generally show that most of them do. The results show not so much the failure of rollback as the theorist’s error in supposing that each player cares only about her own money earnings. People have an innate sense of fairness; most societies instill in their members such a sense, which causes the B players to reject anything that is grossly unfair. Anticipating this, the A players offer relatively equal splits.

Supporting evidence comes from the new field of neuroeconomics. Alan Sanfey and his colleagues took MRI readings of players’ brains as they made their choices in the ultimatum game. They found stimulation of “activity in a region well known for its involvement in negative emotion” in the brains of responders (B players) when they rejected “unfair” (less than 50:50) offers. Thus, deep instincts or emotions of anger and disgust seem to be implicated in these rejections. They also found that “unfair” (less than 50:50) offers were rejected less often when responders knew that the offerer was a computer than when they knew that the offerer was human.[14]

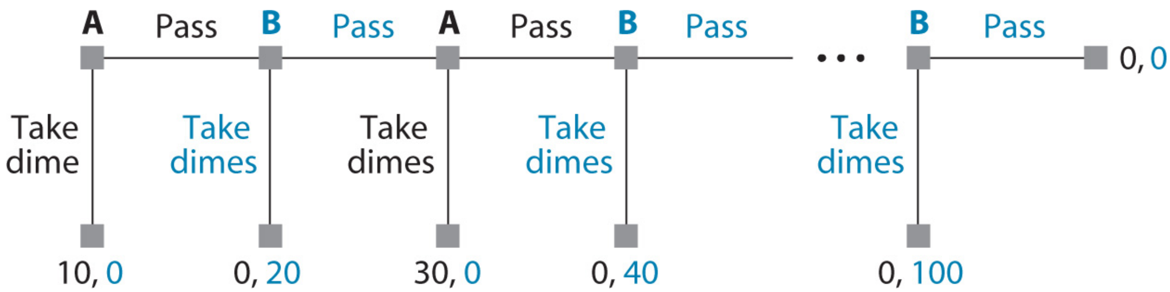


Figure 3.10 The Centipede Game

But this sense of fairness also has a social or cultural component, because different societies exhibit it to different degrees. A group of researchers carried out identical experiments, including the ultimatum game, in 15 societies. In none was the rollback equilibrium with purely selfish behavior observed, but the extent of departure from it varied widely. Most interestingly, "the higher the degree of market integration and the higher the payoffs to cooperation in everyday life, the greater the level of prosociality expressed in experimental games."[15]

Notably, A players have some tendency to be generous even without the threat of retaliation. In a drastic variant of the ultimatum game, called the *dictator game*, where the A player decides on the split and the B player has no choice at all, many As still give significant shares to the Bs, suggesting that the players have some intrinsic preference for relatively equal splits.[16] However, the offers by the A players are noticeably less generous in the dictator game than in the ultimatum game, suggesting that a credible fear of retaliation is also a strong motivator. Other people's perceptions of ourselves also appear to matter. When the experimental design is changed so that not even the experimenter can identify who proposed (or accepted) the split, the extent of sharing drops noticeably.

Another experimental bargaining game with similarly paradoxical outcomes goes as follows: Two players are chosen and designated A and B. The experimenter puts a dime on the table. Player A can take it or pass. If A takes the dime, the game is over, with A getting the 10 cents and B getting nothing. If A passes, the experimenter adds a dime, and now B has the choice of taking the 20 cents or passing. The turns alternate, and the pile of money grows until reaching some limit—say, a dollar—that is known in advance by both players.

We show the tree for this game in Figure 3.10. Because of the appearance of the tree, this type of game is often called the *centipede game*. You may not even need the tree to use rollback on this game. Player B is sure to take the dollar at the last stage, so A should take the 90 cents at the penultimate stage, and so on. Thus, A should take the very first dime and end the game.

In experiments, however, the centipede game typically goes on for at least a few rounds. Remarkably, by behaving "irrationally," the players as a group make more money than they would if they followed the logic of backward reasoning. Sometimes A does better and sometimes B does, but sometimes they even solve this conflict or bargaining problem. In a classroom experiment that one of us (Dixit) conducted, the game went all the way to the end. Player B collected the dollar, and quite voluntarily gave 50 cents to Player A. Dixit asked A, "Did you two conspire? Is B a friend of yours?" and A replied, "No, we didn't even know each other before. But he is a friend now." We will come across some similar evidence of cooperation that seems to contradict backward reasoning when we look at finitely repeated prisoners' dilemma games in Chapter 10.

The centipede game points out a possible problem with the logic of rollback in non-zero-sum games, even for players who care only about their monetary payoffs. Note that if Player A passes in the first round, he has already shown himself not to be playing rollback. So what should Player B expect him to do in round 3? Having passed once, he might pass again, which would make it rational for Player B to pass in round 2. Eventually, someone will take the pile of money, but an initial deviation from rollback equilibrium makes it difficult to predict exactly when this will happen. And because the size of the pie keeps growing, if I see you deviate from rollback equilibrium, I might want to deviate as well, at least for a little while. A player might deliberately pass in an early round in order to signal a willingness to pass in future rounds. This problem does not arise in zero-sum games, where there is no incentive to cooperate by passing.

Steven Levitt, John List, and Sally Sadoff, who conducted experiments with world-class chess players, found more rollback behavior in zero-sum sequential-move games than in the non-zero-sum centipede game. Their centipede game involved 6 nodes, with total payoffs increasing quite steeply across rounds.[17] While there are considerable gains to players who can manage to pass back and forth to each other, the rollback equilibrium specifies playing Take at each node. In stark contrast to the predictions of game theory, only 4% of players played Take at node 1, providing little support for rollback equilibrium even in this simple six-move game. (The fraction of players who played Take increased over the course of the game.[18])

By contrast, in a zero-sum sequential-move game whose rollback equilibrium involves 20 moves (you are invited to solve such a game in Exercise S7), the chess players played the exact rollback equilibrium 10 times as often as in the six-move centipede game.[19] Levitt and his coauthors also experimented with a similar but more difficult zero-sum game (a version of which you are invited to solve in Exercise U5). There, the chess players played the complete rollback equilibrium only 10% of the time (20% for the highest-ranked grandmasters), although by the last few moves, the agreement with rollback equilibrium was nearly 100%. Given that world-class chess players spend tens of thousands of hours trying to win chess games using rollback, these results indicate that even highly experienced players usually cannot immediately carry their experience over to a new game; they need a little experience with the new game before they can figure out the optimal strategy. An advantage of learning game theory is that you can more easily spot underlying similarities between seemingly different situations and thus devise good strategies more quickly in any new games you may face.

Some other instances of non-optimal, short-sighted, or even irrational behavior are observed, not in games involving two or more people, but in one person’s decisions over time, which amount to games between one’s current self and future self. For example, when issuers of credit cards offer favorable initial interest rates or no fees for the first year, many people fall for these offers without realizing that they may have to pay much more later. And when people start new jobs, they may be offered a choice among several savings plans, and fully intend to sign up for one, but keep postponing the action. Psychologists and behavioral economists have offered various explanations for these observations: a tendency to discount the immediate future much more heavily than the longer-term future (so-called *hyperbolic discounting*), or framing of choices that implicitly favors inaction or the status quo. Some of these tendencies can be countered using different framing or “nudges” toward the choice that one’s future self would prefer. For example, the default for new employees could be a modest savings plan, and an employee who actually did not want to save would have to take action to drop out of that plan.[20] Thus, game-theoretic analysis of rollback and rollback equilibria can serve an advisory or prescriptive role as much as it does a descriptive role. People equipped with an understanding of rollback are in a position to make better strategic decisions and to get higher payoffs, no matter what they include in their payoff calculations. And game theorists can use their expertise to give valuable advice to those who are placed in complex strategic situations but lack the skill to determine their own best strategies.

To sum up, we have seen several reasons and explanations for observed departures from rollback behavior. Section 5 argued that rollback equilibrium may be too complex to compute in games like chess and Go, and other look-ahead methods may have to be devised. In this section, we learned that people’s preferences (which determine their payoffs in games) may include additional factors, such as a concern about fairness or about other people’s welfare. People may also procrastinate or choose inaction to favor the status quo. Or they may simply be inexperienced and miscalculate, in which case their actions may come into better conformity with rollback logic as their experience of using it in a game increases.

Thus, the theory has three types of uses: (1) normative, telling us how a player with certain stipulated objectives should behave to achieve those objectives as fully as possible—in other words, what their “rational behavior” is; (2) descriptive, telling us how experienced players with known objectives will behave; and (3) prescriptive, telling players how to behave in light of their own cognitive and other limitations, and yielding suggestions for policies like nudges.

Endnotes

- For a detailed account of this game and related ones, read Richard H. Thaler, "Anomalies: The Ultimate Game," *Journal of Economic Perspectives*, vol. 2, no. 4 (Fall 1988), pp. 195–206; and Douglas D. Davis and Charles A. Holt, *Experimental Economics* (Princeton, NJ: Princeton University Press, 1993), pp. 263–69. Return to reference 12
- The results of the Indonesian experiment are reported in Lisa Cameron, "Raising the Stakes in the Ultimatum Game: Experimental Evidence from Indonesia," *Economic Inquiry*, vol. 37, no. 1 (January 1999), pp. 47–59. Robert Slonim and Alvin Roth report results similar to Cameron's, but they also found that offers (in all rounds of play) were rejected less often as the payoffs were raised. See Robert Slonim and Alvin Roth, "Learning in High Stakes Ultimatum Games: An Experiment in the Slovak Republic," *Econometrica*, vol. 66, no. 3 (May 1998), pp. 569–96. Return to reference 13
- See Alan Sanfey, James Rilling, Jessica Aronson, Leigh Nystrom, and Jonathan Cohen, "The Neural Basis of Economic Decision-Making in the Ultimatum Game," *Science*, vol. 300 (June 13, 2003), pp. 1755–58. Return to reference 14
- Joseph Henrich et al., " 'Economic Man' in Cross-cultural Perspective: Behavioral Experiments in 15 Small-Scale Societies," *Behavioral and Brain Sciences,* vol. 28, no. 6 (December 2005), pp. 795–815. Return to reference 15
- One could argue that this social norm of fairness may actually have value in the ongoing evolutionary game being played by the whole society. Players with a sense of fairness reduce transaction costs and the costs of fights; that can be beneficial to society in the long run. This idea is supported by the findings of the cross-cultural experiments cited in note 15). The correlation between individuals' prosociality and society's state of development may be cause and effect. These matters will be discussed in more detail in Chapters 10 and 11. Return to reference 16
- See Steven D. Levitt, John A. List, and Sally E. Sadoff, "Checkmate: Exploring Backward Induction among Chess

Players," *American Economic Review*, vol. 101, no. 2 (April 2011), pp. 975–90. The details of the game tree are as follows. If A plays Take at node 1, then A receives $4 while B receives $1. If A passes and B plays Take at node 2, then A receives $2 while B receives $8. This pattern of doubling continues until node 6, where if B plays Take, the payoffs are $32 for A and $128 for B, but if B plays Pass, the payoffs are $256 for A and $64 for B. Return to reference 17

- Different results were found in an earlier paper by Ignacio Palacios-Huerta and Oscar Volij, "Field Centipedes," *American Economic Review*, vol. 99, no. 4 (September 2009), pp. 1619–35. Of the chess players they studied, 69% played Take at the first node, with the more highly rated chess players being more likely to play Take at the first opportunity. These results indicated a surprisingly high ability of players to carry experience with them to a new game context, but these results were not reproduced in the later paper discussed in note 17. Return to reference 18
- As you will see in the exercises, another key distinction of this zero-sum game is that there is a way for one player to guarantee victory, regardless of what the other player does. By contrast, a player's best move in the centipede game depends on what she expects the other player to do. Return to reference 19
- See Richard Thaler and Cass Sunstein, *Nudge: Improving Decisions about Health, Wealth, and Happiness* (New Haven, CT: Yale University Press, 2008). Return to reference 20

SUMMARY

Sequential-move games require players to consider the future consequences of their current moves before choosing their actions. Analysis of pure sequential-move games generally requires the creation of a *game tree*. The tree is made up of *nodes* and *branches* that show all the possible actions available to each player at each of her opportunities to move, as well as the payoffs associated with all possible outcomes of the game. Strategies for each player are complete plans that describe actions at each of the player's decision nodes contingent on all possible combinations of actions made by players who acted at earlier nodes. The equilibrium concept employed in sequential-move games is that of *rollback equilibrium*, in which players' optimal strategies are found by looking ahead to subsequent nodes and the actions that would be taken there and by using these forecasts to calculate one's current best action. This process is known as *rollback*, or *backward induction*.

Different types of games entail advantages for different players, such as *first-mover advantages or second-mover advantages.* The inclusion of many players or many moves enlarges the game tree of a sequential-move game but does not change the solution process. In some cases, drawing the full tree for a particular game requires more space or time than is feasible. Such games can often be solved by identifying strategic similarities between actions that reduce the size of the tree, or by simple logical thinking.

When solving games with more moves, verbal reasoning can lead to the rollback equilibrium if the game is simple enough, or a complete tree may be drawn and analyzed. If the game is so complex that verbal reasoning is too difficult and a complete tree is too large to draw, we can enlist the help of a computer program. Checkers has been solved with the use of such a program, and artificial intelligence techniques are rapidly improving computer skills in the play of games like chess and Go.

Tests of the theory of sequential-move games seem to suggest that actual play shows the irrationality of the players or the failure of the theory to predict behavior adequately. The counterargument points out the complexity of actual preferences for different possible outcomes and the usefulness of game theory for identifying optimal actions when actual preferences are known.

KEY TERMS

Glossary

action node
: A node at which one player chooses an action from two or more that are available.

backward induction
: Same as rollback.

branch
: Each branch emerging from a node in a game tree represents one action that can be taken at that node.

decision node
: A decision node in a decision or game tree represents a point in a game where an action is taken.

decision tree
: Representation of a sequential decision problem facing one person, shown using nodes, branches, terminal nodes, and their associated payoffs.

equilibrium path of play
: The *path of play* actually followed when players choose their rollback equilibrium strategies in a sequential game.

extensive form
: Representation of a game by a game tree.

first-mover advantage
: This exists in a game if, considering a hypothetical choice between moving first and moving second, a player would choose the former.

game tree
: Representation of a game in the form of nodes, branches, and terminal nodes and their associated payoffs.

initial node
: The starting point of a sequential-move game. (Also called the root of the tree.)

intermediate valuation function

A rule assigning payoffs to nonterminal nodes in a game. In many complex games, this must be based on knowledge or experience of playing similar games, instead of explicit rollback analysis.

move

An action at one node of a game tree.

node

This is a point from which branches emerge, or where a branch terminates, in a decision or game tree.

path of play

A route through the game tree (linking a succession of nodes and branches) that results from a configuration of strategies for the players that are within the rules of the game. (See also *equilibrium path of play*.)

pruning

Using rollback analysis to identify and eliminate from a game tree those branches that will not be chosen when the game is rationally played.

rollback

Analyzing the choices that rational players will make at all nodes of a game, starting at the terminal nodes and working backward to the initial node. Also called backward induction.

rollback equilibrium

The strategies (complete plans of action) for each player that remain after rollback analysis has been used to prune all the branches that can be pruned.

root

Same as initial node.

second-mover advantage

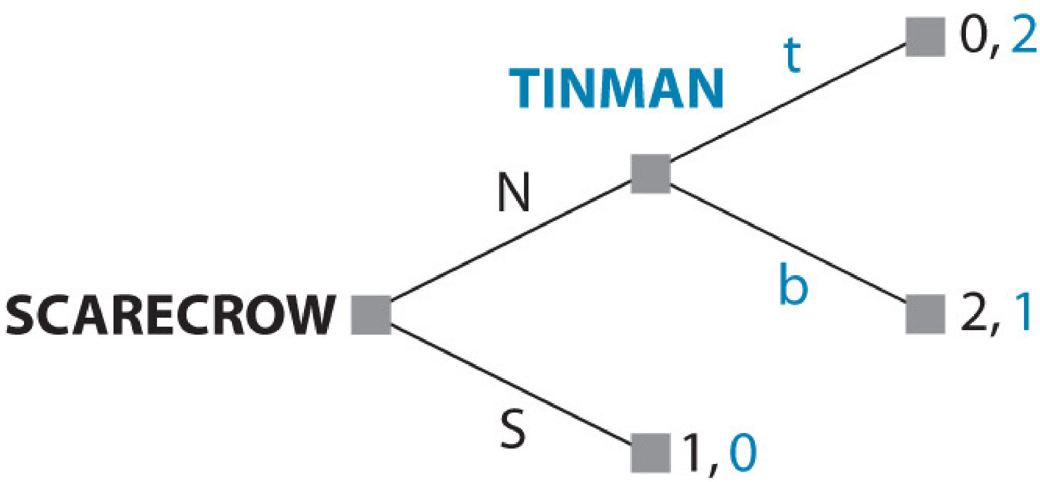
A game has this if, considering a hypothetical choice between moving first and moving second, a player would choose the latter.

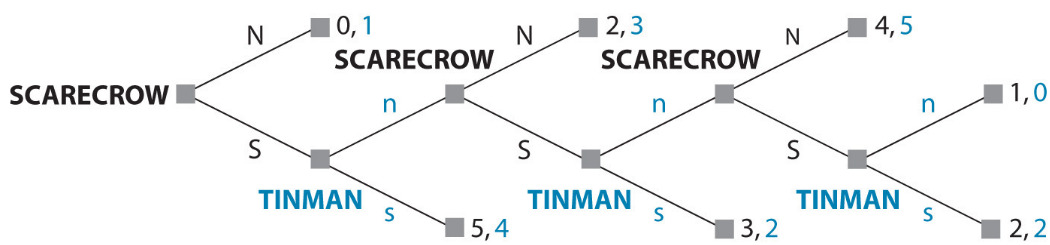
terminal node

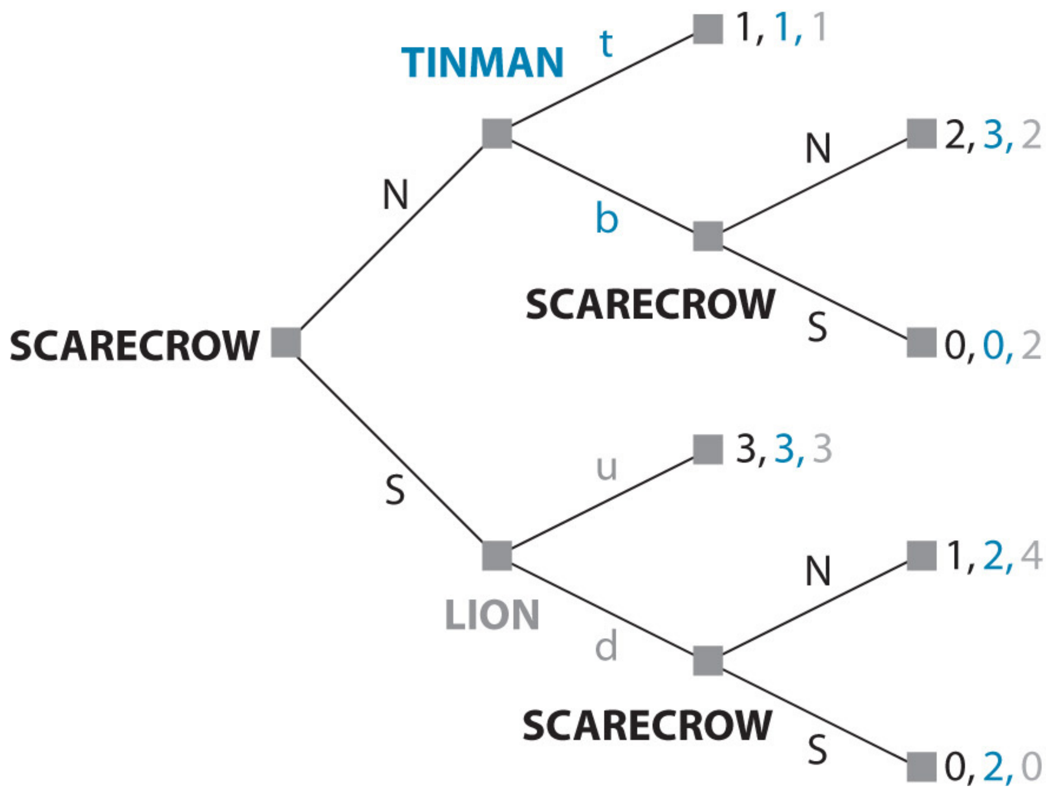
This represents an end point in a game tree, where the rules of the game allow no further moves, and payoffs for each player are realized.

SOLVED EXERCISES

1. Suppose two players, Hansel and Gretel, take part in a sequential-move game. Hansel moves first, Gretel moves second, and each player moves only once.
    1. Draw a game tree for a game in which Hansel has two possible actions (Up or Down) at each node and Gretel has three possible actions (Top, Middle, or Bottom) at each node. How many of each node type—decision and terminal—are there?
    2. Draw a game tree for a game in which Hansel and Gretel each have three possible actions (Sit, Stand, or Jump) at each node. How many of each node type are there?
    3. Draw a game tree for a game in which Hansel has four possible actions (North, South, East, or West) at each node and Gretel has two possible actions (Stay or Go) at each node. How many of each node type are there?
2. In each of the following games, how many strategies (complete plans of action) are available to each player? List all the possible strategies for each player.

1. 

2. 



3.

3. For each of the games illustrated in Exercise S2, identify the rollback equilibrium outcome and the optimal (complete) strategy for each player.
4. Consider the rivalry between Airbus and Boeing to develop a new commercial jet aircraft. Suppose Boeing is ahead in the development process, and Airbus is considering whether to enter the competition. If Airbus stays out, it earns a profit of $0, whereas Boeing enjoys a monopoly and earns a profit of $1 billion. If Airbus decides to enter and develop a rival airplane, then Boeing has to decide whether to accommodate Airbus peaceably or to wage a price war. In the event of peaceful competition, each firm will make a profit of $300 million. If there is a price war, each will lose $100 million because the prices of airplanes will fall so low that neither firm will be able to recoup its development costs.

   Draw the tree for this game. Find the rollback equilibrium and describe each firm's optimal strategy.

5. Consider a game in which two players, Fred and Barney, take turns removing matchsticks from a pile. They start with 21 matchsticks, and Fred goes first. On each turn, each player may remove either 1, 2, 3, or 4 matchsticks. The player to remove the last matchstick wins the game.

1. Suppose there are only 6 matchsticks left, and it is Barney’ s turn. What move should Barney make to guarantee himself victory? Explain your reasoning.
2. Suppose there are 12 matchsticks left, and it is Barney’ s turn. What move should Barney make to guarantee himself victory? (Hint: Use your answer to part (a) and roll back.)
3. Now start from the beginning of the game. If both players play optimally, who will win?
4. What are the optimal (complete) strategies for each player?

6. Consider the game in the previous exercise. Suppose the players have reached a point where it is Fred’ s move and there are just 5 matchsticks left.
   1. Draw the game tree for the game starting with 5 matchsticks.
   2. Find the rollback equilibria for this game starting with 5 matchsticks.
   3. Would you say this 5-matchstick game has a first-mover advantage or a second-mover advantage?
   4. Explain why you found more than one rollback equilibrium. How is your answer related to the optimal strategies you found in part (c) of the previous exercise?
7. Elroy and Judy play a game that Elroy calls “the race to 100.” Elroy goes first, and the two players take turns choosing numbers between 1 and 9. On each turn, they add the new number to a running total. The player who brings the total exactly to 100 wins the game.
   1. If both players play optimally, who will win the game? Does this game have a first-mover advantage? Explain your reasoning.
   2. What are the optimal (complete) strategies for each player?
8. A slave has just been thrown to the lions in the Roman Colosseum. Three lions are chained down in a line, with Lion 1 closest to the slave. Each lion’ s chain is short enough that he can only reach the two players immediately adjacent to him.

   The game proceeds as follows. First, Lion 1 decides whether or not to eat the slave.

   If Lion 1 has eaten the slave, then Lion 2 decides whether or not to eat Lion 1 (who is then too heavy to defend himself). If Lion 1 has not eaten the slave, then Lion 2 has no choice: He cannot try to eat Lion 1, because a fight would kill both lions.

   Similarly, if Lion 2 has eaten Lion 1, then Lion 3 decides whether or not to eat Lion 2.

   Each lion’ s preferences are fairly natural: best (4) is to eat and stay alive, next best (3) is to stay alive but go hungry, next (2) is to eat and be eaten, and worst (1) is to go hungry and be eaten.

1. Draw the game tree, with payoffs, for this three-player game.
2. What is the rollback equilibrium of this game? Make sure to describe the players' optimal strategies, not just the payoffs.
3. Is there a first-mover advantage to this game? Explain why or why not.
4. How many (complete) strategies does each lion have? List them.

9. Consider three major department stores—Big Giant, Titan, and Frieda's—that are contemplating opening a branch in one of two new Boston-area shopping malls. Urban Mall is located close to the large and rich population center of the area; it is relatively small and can accommodate at most two department stores as anchors for the mall. Rural Mall is farther out in a rural and relatively poor area; it can accommodate as many as three anchor stores. None of the three stores wants to have branches in both malls, because there is sufficient overlap of customers between the malls that locating in both would just mean competing with itself. Each store would rather be in a mall with one or more other department stores than be alone in the same mall, because a mall with multiple department stores will attract enough additional customers that each store's profit will be higher. Further, each store prefers Urban Mall to Rural Mall because of the richer customer base. Each store must choose between trying to get a space in Urban Mall (knowing that if the attempt fails, it will try for a space in Rural Mall) and trying to get a space in Rural Mall right away (without even attempting to get into Urban Mall).

   In this case, the stores rank the five possible outcomes as follows: 5 (best), in Urban Mall with one other department store; 4, in Rural Mall with one or two other department stores; 3, alone in Urban Mall; 2, alone in Rural Mall; and 1 (worst), alone in Rural Mall after having attempted to get into Urban Mall and failed, by which time other non-department stores have signed up the best anchor locations in Rural Mall.

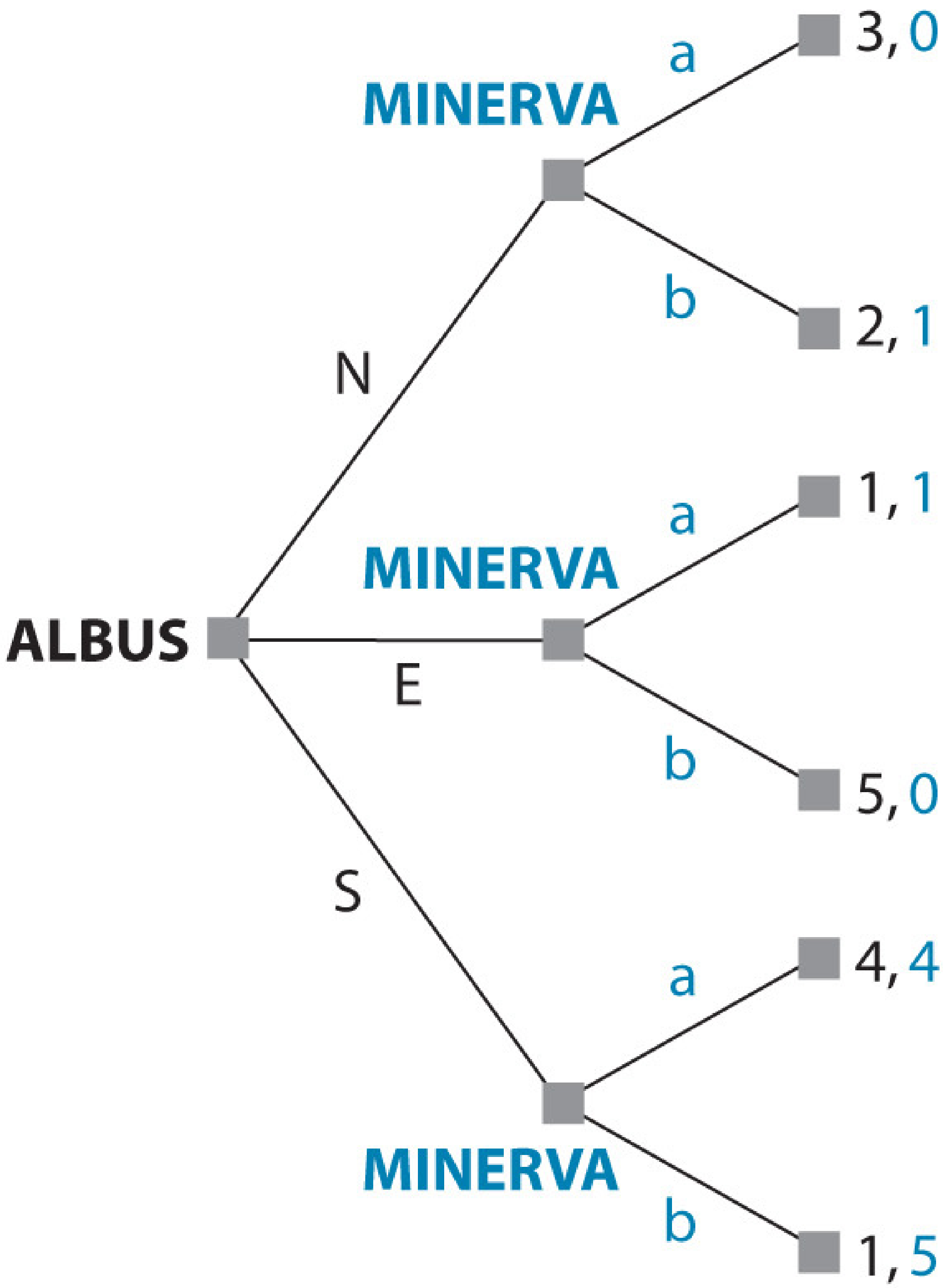
   The three stores are sufficiently different in their managerial structures that they experience different lag times in doing the paperwork required to request a space in a new mall. Frieda's moves quickly, followed by Big Giant, and finally by Titan, which is the least efficient in readying a location plan. When all three have made their requests, the malls decide which stores to let in. Because of the name recognition that both Big Giant and Titan have with potential customers, a mall would take either (or both) of those stores before it took Frieda's. Thus, Frieda's does not get one of the two spaces in Urban Mall if all three stores request those spaces; this is true even though Frieda’s moves first.

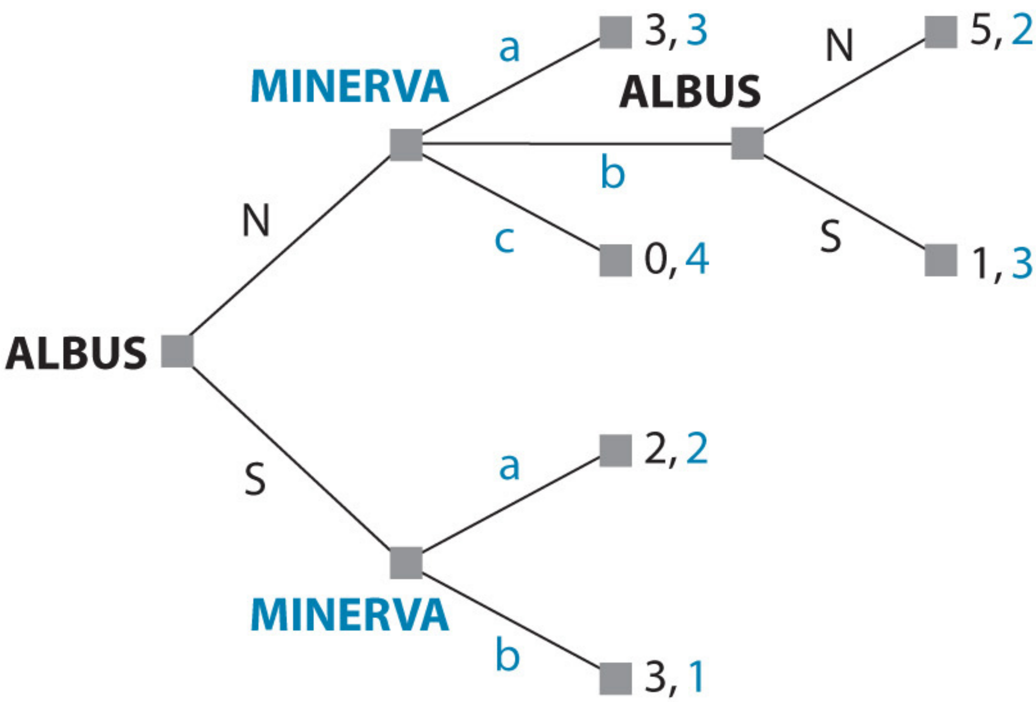
1. Draw the game tree for this mall location game.
2. Illustrate the pruning process on your game tree and use the fully pruned tree to find the rollback equilibrium. Describe the equilibrium by specifying the optimal strategies employed by each department store. What are the payoffs to each store at the rollback equilibrium outcome?

10. (**Optional**) Consider the following ultimatum game, which has been studied in laboratory experiments. The Proposer moves first, and proposes a split of $10 between himself and the Responder. Any whole-dollar split may be proposed. For example, the Proposer may offer to keep the whole $10 for himself, he may propose to keep $9 for himself and give $1 to the Responder, $8 to himself and $2 to the Responder, and so on. (Note that the Proposer therefore has 11 possible choices.) After seeing the split, the Responder can choose to accept the split or reject it. If the Responder accepts, both players get the proposed amounts. If she rejects, both players get $0.
    1. Draw the game tree for this game.
    2. How many complete strategies does each player have?
    3. What is the rollback equilibrium of this game, assuming the players care only about their cash payoffs?
    4. Suppose Rachel the Responder would accept any offer of $3 or more, and reject any offer of $2 or less. Suppose Pete the Proposer knows Rachel’s strategy, and he wants to maximize his cash payoff. What strategy should he use?
    5. Rachel’s true payoff might not be the same as her cash payoff. What other aspects of the game might she care about? Given your answer, propose a set of payoffs for Rachel that would make her strategy optimal.
    6. In laboratory experiments, players typically do not play the rollback equilibrium. Proposers typically offer an amount between $2 and $5 to Responders. Responders often reject offers of $3, $2, and especially $1. Explain why you think this might occur.

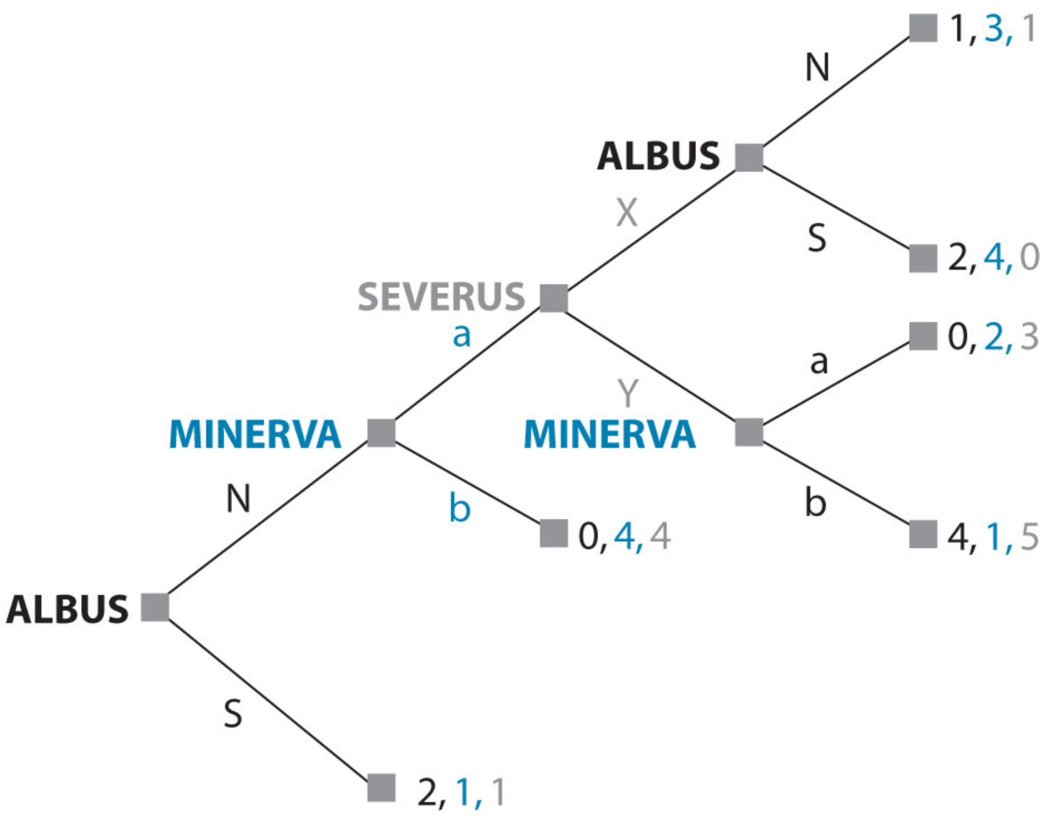
UNSOLVED EXERCISES

1. “In a sequential-move game, the player who moves first is sure to win.” Is this statement true or false? State the reason for your answer in a few brief sentences, and give an example of a game that illustrates your answer.
2. In each of the following games, how many strategies (complete plans of action) are available to each player? List all the possible strategies for each player.

1. 



2.



3.

3. For each of the games illustrated in Exercise U2, identify the rollback equilibrium outcome and the optimal strategy for each

player.

4. Two distinct proposals, A and B, are being debated in Washington. Congress likes Proposal A, and the president likes Proposal B. The proposals are not mutually exclusive; either or both or neither may become law. Thus, there are four possible outcomes, and they are ranked by the two sides are as follows, where a larger number represents a more favored outcome:

Outcome	Congress	President
A becomes law	4	1
B becomes law	1	4
Both A and B become law	3	3
B Neither (status quo prevails)	2	2

   The moves in the game are as follows: First, Congress decides whether to pass a bill and whether the bill is to contain A or B or both. Then the president decides whether to sign or veto the bill. Congress does not have enough votes to override a veto.

   1. Draw a tree for this game and find the rollback equilibrium.
   2. Now suppose the rules of the game are changed in only one respect: The president is given the extra power of a line-item veto. Thus, if Congress passes a bill containing both A and B, the president may choose not only to sign or veto the bill as a whole, but also to veto just one of the two items. Show the new tree and find the rollback equilibrium.
   3. Explain intuitively why the difference between the two equilibria arises.

5. Two players, Amy and Beth, play the following game with a jar containing 100 pennies. The players take turns; Amy goes first. Each time it is a player’s turn, she takes between 1 and 10 pennies out of the jar. The player whose move empties the jar wins.
   1. If both players play optimally, who will win the game? Does this game have a first-mover advantage? Explain your reasoning.

    1. What are the optimal strategies for each player?
6. Consider a slight variant to the game in Exercise U5. Now the player whose move empties the jar loses.
    1. Does this game have a first-mover advantage?
    2. What are the optimal strategies for each player?
7. Kermit and Fozzie play a game with two jars, each containing 100 pennies. The players take turns; Kermit goes first. Each time it is a player's turn, he chooses one of the jars and removes anywhere from 1 to 10 pennies from it. The player whose move leaves both jars empty wins. (Note that when a player empties the second jar, the first jar must already have been emptied in some previous move by one of the players.)
    1. Does this game have a first-mover advantage or a second-mover advantage? Explain which player can guarantee victory and how he can do it. (Hint: Simplify the game by starting with a smaller number of pennies in each jar, and see if you can generalize your finding to the actual game.)
    2. What are the optimal strategies for each player? (Hint: First think of a starting situation in which both jars have equal numbers of pennies. Then consider starting positions in which the two jars differ by 1 to 10 pennies. Finally, consider starting positions in which the jars differ by more than 10 pennies.)
8. Modify Exercise S8 so that there are now four lions.
    1. Draw the game tree, with payoffs, for this four-player game.
    2. What is the rollback equilibrium of this game? Make sure to describe the players' (complete) optimal strategies, not just the payoffs.
    3. Is the additional lion good or bad for the slave? Explain.
9. To give Mom a day of rest, Dad plans to take his two children, Bart and Cassie, on an outing on Sunday. Bart prefers to go to the amusement park (A), whereas Cassie prefers to go to the science museum (S). Each child gets 3 units of value from his/her more preferred activity and only 2 units of value from his/her less preferred activity. Dad gets 2 units of value for either of the two activities.

   To choose their activity, Dad plans first to ask Bart for his preference, then to ask Cassie after she hears Bart's choice. Each child can choose either the amusement park (A) or the science museum (S). If both children choose the same activity, then that is what they will all do. If the children choose different activities, Dad will make a tie-breaking decision. As the parent, Dad has an additional option: He can choose the amusement park, the science museum, or his personal favorite, the mountain hike (M). Bart and

Cassie each get 1 unit of value from the mountain hike, and Dad gets 3 units of value from the mountain hike.

Because Dad wants his children to cooperate with each other, he gets 2 extra units of value if the children choose the same activity (no matter which one of the two it is).

1. Draw the game tree for this three-person game, with payoffs based on the units of value specified for each outcome.
2. What is the rollback equilibrium of this game? Make sure to describe the players' optimal strategies, not just the payoffs.
3. How many different (complete) strategies does Bart have? Explain.
4. How many (complete) strategies does Cassie have? Explain.

4 ■ Simultaneous-Move Games: Discrete Strategies

RECALL FROM CHAPTER 2 that a game is said to have *simultaneous moves* if each player must move without knowing what other players have chosen to do. This is obviously true if players choose their actions at exactly the same time (have *literally* simultaneous moves), but it is also true in any situation in which players make choices at different times but do not have any information about others' moves when deciding what to do. (For this reason, simultaneous-move games have *imperfect information* in the sense defined in Chapter 2, Section 2.D.)

Many familiar strategic situations can be described as simultaneous-move games. Firms that make TV sets, stereos, or automobiles make decisions about product design and features without knowing what rival firms are doing with their own products. Voters in U.S. elections cast their votes without knowing how others are voting, and hence make simultaneous moves when deciding each election. The interaction between a soccer goalie and an opposing striker during a penalty kick requires both players to make their decisions simultaneously—the kicker does not know which way the goalie will be jumping, but the goalie also cannot afford to wait until the ball has actually been kicked to decide which way to go, because the ball travels too fast to watch and then respond to.

In Chapters 2 and 3, we emphasized that a strategy is a complete plan of action. But in a simultaneous-move game, each player has exactly one opportunity to act. Therefore, there is no real distinction between *strategy* and *action* in simultaneous-move games, and the terms are often used as synonyms in this context. There is one more complication: In simultaneous-move games, a strategy can be a probabilistic choice from the basic actions initially specified. For example, in football, a coach may deliberately randomize whether to run or pass the ball in order to keep the other side guessing. Such probabilistic strategies, called mixed strategies, are the focus of Chapter 7. In this chapter, we confine our attention to the initially specified actions available to the players, called pure strategies.

In many games, each player has available to her a finite number of discrete pure strategies—for example, Dribble, Pass, or Shoot in basketball. In other games, each player's pure strategy can be any number from a continuous range—for example, the price charged for a product by a firm.[1] This distinction makes no difference to the general concept of equilibrium in simultaneous-move games, but the key ideas are more easily conveyed with discrete strategies. Therefore, in this chapter, we restrict our analysis to the simpler case of discrete pure strategies, then take up continuously variable strategies in Chapter 5.

When a player in a simultaneous-move game chooses her action, she obviously does so without any knowledge of the choices made by other players. She also cannot look ahead to how they will react to her choice, because they do not know what she is choosing. Rather, each player must figure out what others are choosing to do while the others are figuring out what she is choosing to do. This situation may seem complex, but the analysis is not difficult once you understand the relevant concepts and methods. We begin by showing how to depict simultaneous-move games in a way that makes them especially easy to analyze.

Endnotes

- In fact, prices must be denominated in the minimum unit of coinage—for example, whole cents—and can therefore take on only a finite number of discrete values. But this unit is usually so small that it makes more sense to think of price as a continuous variable. Return to reference 1

Glossary

mixed strategy
: A mixed strategy for a player consists of a random choice, to be made with specified probabilities, from his originally specified pure strategies.

pure strategy
: A rule or plan of action for a player that specifies without any ambiguity or randomness the action to take in each contingency or at each node where it is that player’s turn to act.

1 DEPICTING SIMULTANEOUS-MOVE GAMES WITH DISCRETE STRATEGIES

A simultaneous-move game with discrete strategies is most often depicted using a game table (also called a payoff table or payoff matrix). This table is referred to as the normal form or strategic form of the game. Games with any number of players can be illustrated by using a game table, but its dimensions must equal the number of players. For a two-player game, the table is two-dimensional and appears similar to a spreadsheet. The row and column headings of the table correspond to the strategies available to the first and second players, respectively. The size of the table, then, is determined by the number of strategies available to each player.[2] Each cell within the table lists the payoffs to all players that arise under the configuration of strategies that places players in that cell. Games with three players require three-dimensional tables; we will consider them later in this chapter.

		COLUMN		
		Left	Middle	Right
ROW	Top	3, 1	2, 3	10, 2
	High	4, 5	3, 0	6, 4
	Low	2, 2	5, 4	12, 3
	Bottom	5, 6	4, 5	9, 7

You may need to scroll left and right to see the full figure.

FIGURE 4.1 Representing a Simultaneous-Move Game in a Table

We illustrate the concept of a game table for a simple game in Figure 4.1. The game here has no special interpretation, so we can develop the concepts without the distraction of a "story." The players are named Row and Column. Row has four choices (strategies or actions), labeled Top, High, Low, and Bottom; Column has three choices, labeled Left, Middle, and Right. Each selection by Row and Column generates a potential outcome of the game. The payoffs associated with each outcome are shown in the cell corresponding to that row and that column. By convention, the first number in each cell is Row's payoff in that outcome, while the second number is Column's payoff. For example, if Row chooses High and Column chooses Right, payoffs are 6 to Row and 4 to Column. For additional convenience, we show everything pertaining to Row—player name, strategies, and payoffs—in black, and everything pertaining to Column in blue.

Next, we consider a second example of a game table with more of a story attached. Figure 4.2 represents a very simplified version of a single play in American football. Offense attempts to move the ball forward to improve its chances of kicking a field goal. It has four possible strategies: a run and passes of three different lengths (short, medium, and long). Defense can adopt one of three strategies to try to keep Offense at bay: a run defense, a pass defense, or a blitz of the quarterback. Offense tries to gain yardage while Defense tries to prevent it from doing so. Suppose we have enough information about the underlying strengths of the two teams to work out the probabilities of their completing different plays and to determine the average gain in yardage that could be expected under each combination of strategies. For example, when Offense chooses Medium Pass and Defense counters with its Pass defense, we estimate Offense's payoff to be a gain of 4.5 yards, or +4.5.[3] Defense’s “payoff” is a loss of 4.5 yards, or −4.5. The other cells similarly show our estimates of each team’s gain or loss of yardage.

		DEFENSE		
		Run	Pass	Blitz
OFFENSE	Run	2, −2	5, −5	13, −13
	Short Pass	6, −6	5.6, −5.6	10.5, −10.5
	Medium Pass	6, −6	4.5, −4.5	1, −1
	Long Pass	10, −10	3, −3	−2, 2

You may need to scroll left and right to see the full figure.

Figure 4.2 A Single Play in American Football

Sometimes, for obviously zero-sum games like this one, only the row player’s payoffs are shown in the game table, and the column player’s payoffs are understood to be the negatives of those numbers. We will not adopt that approach here, however. Always showing both payoffs helps reduce the possibility of any confusion.

Endnotes

- In a game where each firm can set its price at any number of cents in a range that extends over a dollar, each firm has 100 distinct strategies, and the size of the table becomes 100 × 100. That is surely too unwieldy to analyze. Algebraic formulas with prices as continuous variables provide a simpler approach—not a more complicated one, as some readers might fear. We will develop this "algebra is our friend" method in Chapter 5. Return to reference 2
- Here is how the payoffs for this case were constructed. When Offense chooses Medium Pass and Defense counters with its Pass defense, our estimate is a probability of 50% that the pass will be completed for a gain of 15 yards, a probability of 40% that the pass will fall incomplete (0 yards), and a probability of 10% that the pass will be intercepted for a loss of 30 yards; this makes an average of $(0.5 \times 15) + (0.4 \times 0) + (0.1 \times (-30)) = 4.5$ yards. The numbers in the table were constructed by a small panel of expert neighbors and friends convened by Dixit on one fall Sunday afternoon. They received a liquid consultancy fee. Return to reference 3

Glossary

game table
: A spreadsheetlike table whose dimension equals the number of players in the game; the strategies available to each player are arrayed along one of the dimensions (row, column, page, . . .); and each cell shows the payoffs of all the players in a specified order, corresponding to the configuration of strategies that yield that cell. Also called payoff table.

payoff table
: Same as game table.

payoff matrix
: Same as payoff table and game table.

normal form
: Representation of a game in a game matrix, showing the strategies (which may be numerous and complicated if the game has several moves) available to each player along a separate dimension (row, column, etc.) of the matrix and the outcomes and payoffs in the multidimensional cells. Also called strategic form.

strategic form
: Same as normal form.

2 NASH EQUILIBRIUM

To analyze simultaneous-move games, we need to consider how players choose their actions. Return to the game table in Figure 4.1. Focus on one specific outcome—namely, the one where Row chooses Low and Column chooses Middle; the payoffs there are 5 to Row and 4 to Column. Each player wants to pick an action that yields her the highest payoff, and in this outcome each indeed makes such a choice, given what her opponent chooses. Given that Row is choosing Low, can Column do any better by choosing something other than Middle? No, because Left would give her the payoff 2, and Right would give her 3, neither of which is better than the 4 she gets from Middle. Thus, Middle is Column's best response to Row's choice of Low. Conversely, given that Column is choosing Middle, can Row do better by choosing something other than Low? Again, no, because the payoffs from switching to Top (2), High (3), or Bottom (4) would all be worse than what Row gets with Low (5). Thus, Low is Row's best response to Column's choice of Middle.

These two choices, Low for Row and Middle for Column, have the property that each is the chooser's best response to the other's action. If the players were making these choices, neither would want to switch to a different response *on her own*. By the definition of a noncooperative game, the players are making their choices independently; therefore, such unilateral changes are all that each player can contemplate. Because neither wants to make such a change, it is natural to call this state of affairs an equilibrium. This is exactly the concept of a Nash equilibrium.

To state it a little more formally, a Nash equilibrium[4] in a game is a list of strategies, one for each player, such that no player can get a better payoff by switching to some other strategy that is available to her while all the other players adhere to the strategies specified for them in that list.

A. Some Further Explanation of the Concept of Nash Equilibrium

To understand the concept of Nash equilibrium better, we take another look at the game table in Figure 4.1. Consider now a cell other than (Low, Middle)—say, the one where Row chooses High and Column chooses Left. Can these strategies be a Nash equilibrium? No, because if Column is choosing Left, Row does better to choose Bottom and get the payoff 5 rather than to choose High, which gives her only 4. Similarly, (Bottom, Left) is not a Nash equilibrium, because Column can do better by switching to Right, thereby improving her payoff from 6 to 7.

The definition of Nash equilibrium does not require equilibrium choices to be strictly better than other available choices. Figure 4.3 is the same as Figure 4.1 except that Row's payoff from (Bottom, Middle) is changed to 5, the same as that from (Low, Middle). It is still true that, given Column's choice of Middle, Row *could not do any better* than she does when choosing Low. So neither player has a reason to change her action, and that qualifies (Low, Middle) as a Nash equilibrium.[5]

More importantly, a Nash equilibrium does not have to be jointly best for the players. In Figure 4.1, the strategy pair (Bottom, Right) gives payoffs (9, 7), which are better for both players than the (5, 4) of the Nash equilibrium. However, playing independently, they cannot sustain (Bottom, Right). Given that Column plays Right, Row would want to deviate from Bottom to Low and get 12 instead of 9. Getting the jointly better payoffs of (9, 7) would require cooperative action that made such "cheating" impossible. We examine how to support cooperative behavior later in this chapter and in more detail in Chapter 10. For now, we merely point out the fact that a Nash equilibrium may not be in the joint interests of the players.

		COLUMN		
		Left	Middle	Right
ROW	Top	3, 1	2, 3	10, 2
	High	4, 5	3, 0	6, 4
	Low	2, 2	5, 4	12, 3
	Bottom	5, 6	4, 5	9, 7

You may need to scroll left and right to see the full figure.

FIGURE 4.3 Variation on Game in Figure 4.1 with a Tie in Payoffs

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To reinforce the concept of Nash equilibrium, look at the football game table in Figure 4.2. If Defense is choosing the Pass defense, then the best choice for Offense is Short Pass (payoff of 5.6 versus 5, 4.5, or 3). Conversely, if Offense is choosing the Short Pass, then Defense's best choice is the Pass defense—it holds Offense down to 5.6 yards, whereas the Run defense and the Blitz would concede 6 and 10.5 yards, respectively. (Remember that the entries in each cell of a zero-sum game are the row player's payoffs; therefore, the best choice for the column player is the one that yields the smallest number, not the largest.) In this game, the strategy combination (Short Pass, Pass) is a Nash equilibrium, and the resulting payoff to Offense is 5.6 yards.

How does one find Nash equilibria in games? One can always check every cell to see if the strategies that generate it satisfy the definition of a Nash equilibrium. Such a systematic analysis is foolproof, but tedious and unmanageable except in simple games or with the help of a good computer program. In the next few sections, we develop other methods (based on the concepts of *dominance* and *best response*) that will not only help us find Nash equilibria much more quickly, but also shed light on how players form beliefs and make choices in games.

B. Nash Equilibrium as a System of Beliefs and Choices

Before we proceed with further study and use of the Nash equilibrium concept, we should try to clarify something that may have bothered some of you. We said that, in a Nash equilibrium, each player chooses her "best response" to the other's choice. But the two choices are made simultaneously. How can one *respond* to something that has not yet happened, at least when one does not *know* what has happened?

People play simultaneous-move games all the time in real life, and they do make such choices. To do so, they must find a substitute for actual knowledge or observation of others' actions. Players can make blind guesses and hope that they turn out to be inspired ones, but luckily there are more systematic ways to try to figure out what the other players are doing. One method is experience and observation: if the players play this game, or similar games, with similar players all the time, they may develop a pretty good idea of what the other players do. Then choices that are not the best will be unlikely to persist for long. Another method is the logical process of thinking through the others' thinking. You put yourself in the position of the other players and think what they are thinking, which of course includes their putting themselves in your position and thinking what you are thinking. The logic seems circular, but there are several ways of breaking into the circle, as we demonstrate through specific examples in the sections that follow. Nash equilibrium can be thought of as a culmination of this process of thinking about thinking where each player has correctly figured out the others' choices.

Whether by observation, logical deduction, or some other method, you, the game player, acquire some notion of what the other players are choosing in simultaneous-move games. It is not easy to find a word to describe this process or its outcome. It is not anticipation, nor is it forecasting, because the others' actions do not lie in the future—they occur simultaneously with your own. The word most frequently used by game theorists is belief. This word is not perfect either, because it seems to connote more confidence or certainty than is intended; in fact, in Chapter 7, we will allow for the possibility that beliefs are held with some uncertainty. But for lack of a better word, it will have to suffice.

This concept of belief also relates to our discussion of uncertainty in Chapter 2, Section 2.D. There, we introduced the concept of strategic uncertainty. Even when a player knows all the rules of a game—the strategies available to all players and the payoffs for each as functions of those strategies—they may be uncertain about what actions the others are taking at the same time. Similarly, if past actions are not observable, each player may be uncertain about what actions the others took in the past. How can players choose in the face of this strategic uncertainty? They must form some subjective views or estimates of the others' actions. That is exactly what the notion of belief captures.

Now think of Nash equilibrium in this light. We defined it as a configuration of strategies such that each player's strategy is her best response to the others' strategies. If she does not know the others' actual choices but has beliefs about them, in Nash equilibrium those beliefs are assumed to be correct—the others' actual actions are just what she believes them to be. Thus, we can define Nash equilibrium in an alternative and equivalent way: It is a set of beliefs and strategies, one for each player, such that (1) each player has correct beliefs about the others' strategies and (2) each player's strategy is best for herself, given her beliefs about the others' strategies.[6]

This way of thinking about Nash equilibrium has two advantages. First, the concept of "best response" is no longer logically flawed. Each player is choosing her best response not to the as yet unobserved actions of the others, but only to her own already formed beliefs about their actions. Second, in Chapter 7, where we will allow mixed strategies, the randomness in one player's strategy may be better interpreted as uncertainty in the other players' beliefs about that player's action. For now, we proceed by using both definitions of Nash equilibrium in parallel.

You might think that forming correct beliefs and calculating best responses is too daunting a task for mere humans. We will discuss some criticisms of this kind, as well as empirical and experimental evidence concerning Nash equilibrium, in Chapter 5 for pure strategies and in Chapter 7 for mixed strategies. For now, we simply say that the proof of the pudding is in the eating. We will develop and illustrate the Nash equilibrium concept by applying it. We hope that seeing it in use will prove a better way to understand its strengths and drawbacks than would an abstract discussion at this point.

Endnotes

- This concept is named for the mathematician and economist John Nash, who developed it in his doctoral dissertation at Princeton in 1949. Nash also proposed a solution to cooperative games, which we consider in Chapter 17. He shared the 1994 Nobel Prize in economics with two other game theorists, Reinhard Selten and John Harsanyi; we will treat some aspects of their work in Chapters 8, 9, and 14. Sylvia Nasar's biography of Nash, *A Beautiful Mind* (New York: Simon & Schuster, 1998), was the (loose) basis for a movie starring Russell Crowe. Unfortunately, the movie failed in its attempt to explain the concept of Nash equilibrium. We explain this failure in Exercise S15 of this chapter and in Exercise S14 of Chapter 7. Return to reference 4
- But note that (Bottom, Middle) with the payoffs of (5, 5) is not itself a Nash equilibrium. If Row was choosing Bottom, Column's own best choice would not be Middle; she could do better by choosing Right. In fact, you can check all the other cells in in Figure 4.3 to verify that none of them can be a Nash equilibrium. Return to reference 5
- In this chapter, we consider only Nash equilibria in pure strategies—namely, those strategies initially listed in the rules of the game, not mixtures of two or more of them. Therefore, in such an equilibrium, each player has certainty about the actions of the others; strategic uncertainty is removed. When we consider mixed-strategy equilibria Chapter 7, the strategic uncertainty for each player will consist of the probabilities with which the various strategies are played in the other players' equilibrium mixtures. Return to reference 6

Glossary

best response
: The strategy that is optimal for one player, given the strategies actually played by the other players, or the belief of this player about the other players’ strategy choices.

Nash equilibrium
: A configuration of strategies (one for each player) such that each player’s strategy is best for him, given those of the other players. (Can be in pure or mixed strategies.)

belief
: The notion held by one player about the strategy choices of the other players and used when choosing his own optimal strategy.

3 DOMINANCE

In some games, one or more players have a strategy that is uniformly better than all their other strategies, no matter what strategies other players adopt. When that is the case for some player, that player’s uniformly best strategy is referred to as her dominant strategy, or more precisely, her *strictly dominant strategy*.[7] Other times, when a player doesn’t have a dominant strategy, there may still be some strategy that is uniformly worse for that player than some other strategy no matter what other players do. The uniformly worse strategy is then referred to as a dominated strategy, and we say that it is *dominated* by the uniformly better strategy. (When a player has a dominant strategy, all other strategies are dominated by the dominant strategy.) The presence of dominant strategies simplifies the search for and interpretation of Nash equilibrium because any player with a dominant strategy has an incentive to play that dominant strategy no matter what her belief about others’ strategies.

A. Both Players Have Dominant Strategies

The game known as the prisoners’ dilemma nicely illustrates the concept of a dominant strategy. Consider a story line of the type that appears regularly in the TV program *Law and Order*. A husband and wife have been arrested under the suspicion that they conspired in the murder of a young woman. Detectives Green and Lupo place the suspects in separate detention rooms and interrogate them one at a time, thereby ensuring that the game between the two suspects, Husband and Wife, has simultaneous moves. There is little concrete evidence linking the pair to the murder, although there is some evidence that they were involved in kidnapping the victim. The detectives explain to each suspect that they are both looking at jail time for the kidnapping charge, probably 3 years, even if there is no confession from either of them. In addition, Husband and Wife are told individually that the detectives “know” what happened and “know” how one was coerced by the other to participate in the crime; it is implied that jail time for a solitary confessor will be significantly reduced if the whole story is committed to paper. (In a scene common to many similar programs, a yellow legal pad and a pencil are produced and placed on the table at this point.) Finally, the suspects are told that if both confess, jail terms could be negotiated down for both, but not as much as they would be if there were one confession and one denial.

	WIFE	
	Confess (Defect)	Deny (Cooperate)

You may need to scroll left and right to see the full figure.

		WIFE	
		Confess (Defect)	Deny (Cooperate)
HUSBAND	Confess (Defect)	10 yr, 10 yr	1 yr, 25 yr
	Deny (Cooperate)	25 yr, 1 yr	3 yr, 3 yr

You may need to scroll left and right to see the full figure.

FIGURE 4.4 Prisoners' Dilemma

In this situation, both Husband and Wife are players in a two-person, simultaneous-move game in which each has to choose between confessing and not confessing to the crime of murder. They both know that no confession leaves them each with a 3-year jail sentence for involvement with the kidnapping. They also know that if one of them confesses, he or she will get a short sentence of 1 year for cooperating with the police, while the other will go to jail for a minimum of 25 years. If both confess, they figure that they can negotiate for jail terms of 10 years each.

The choices and outcomes for this game are summarized by the game table in Figure 4.4. The strategies Confess and Deny can also be called Defect and Cooperate to capture their roles in the relationship between the *two players*; thus, Defect means to defect from any tacit arrangement with the spouse, and Cooperate means to take the action that helps the spouse (not cooperate with the cops).

The payoffs in this game are the lengths of the jail sentences associated with each outcome, so lower numbers are better for each player. In that sense, this example differs from most of the games that we analyze, in which larger payoffs are better and smaller payoffs are worse. We take this opportunity to remind you that "large is good" is not always true. Also, as mentioned in Section 1, in the game table for a zero-sum game that shows only one player's bigger-is-better payoffs, smaller numbers are better for the other player. In the prisoners' dilemma here, smaller numbers are better for both players. Thus, if you ever draw a payoff table where larger numbers are worse, you should alert the reader by pointing it out clearly. And when reading someone else's work, be aware of that possibility.

Now consider the prisoners' dilemma in Figure 4.4 from Husband's perspective. He has to think about what Wife will choose. Suppose he believes that she will confess. Then his best choice is to confess; he gets a sentence of only 10 years, whereas denial would have meant 25 years. What if he believes that Wife will deny? Again, his own best choice is to confess; he gets only 1 year instead of the 3 that his own denial would bring in this case. Thus, in this game, Confess is better than Deny for Husband *regardless of his belief about Wife's choice.* We say that, for Husband, the strategy Confess is a *dominant strategy* or that the strategy Deny is a *dominated strategy*. Equivalently, we could say that the strategy Confess *dominates* the strategy Deny or that the strategy Deny is *dominated* by the strategy Confess.

If an action is clearly best for a player no matter what other players might be doing, then there is compelling reason to think that a rational player would choose it. And if an action is clearly bad for a player no matter what others might be doing, then there is equally compelling reason to think that a rational player would avoid it. When it exists, dominance therefore provides a shortcut in our search for solutions to simultaneous-move games.

Games like the prisoners' dilemma are simple to analyze because the logic that led us to determine that Confess is dominant for Husband applies equally to Wife's choice. Her own strategy Confess dominates her own strategy Deny, so both players should choose Confess. Therefore, (Confess, Confess) is the predicted outcome and the unique Nash equilibrium of this game.

In this game, the best choice for each player is independent of whether his or her beliefs about the other are correct. Indeed, this is the meaning of dominance. But if each player attributes to the other the same rationality that he or she practices, then both of them should be able to form correct beliefs. And the actual action of each is the best response to the actual action of the other. Note that the fact that Confess dominates Deny for both players is completely independent of whether they are actually guilty, as in many episodes of *Law and Order*, or are being framed, as happened in the movie *L.A. Confidential.* It depends only on the pattern of payoffs dictated by the various jail terms.

We will examine the prisoners’ dilemma in great detail throughout the book, including later in this chapter, in all of Chapter 10, and in several sections in other chapters as well. Why is this game so important to the study of game theory that it appears multiple times in this book? There are two main reasons. The first reason is that many important, though seemingly quite different, economic, social, political, and even biological strategic situations can be interpreted as examples of the prisoners’ dilemma. Understanding the prisoners’ dilemma will give you better insight into such games.

The second reason why the prisoners’ dilemma is integral to any discussion of games of strategy is the somewhat curious nature of the equilibrium outcome achieved in such games. Both players choose their dominant strategies, but the resulting equilibrium outcome yields them payoffs that are lower than they could have achieved if they had each chosen their dominated strategies. Thus, the equilibrium outcome in the prisoners’ dilemma is actually a bad outcome for the players. There is another outcome that they both prefer to the equilibrium outcome; the problem is how to guarantee that someone will not cheat. This particular feature of the prisoners’ dilemma has received considerable attention from game theorists who have asked an obvious question: What can players in a prisoners’ dilemma do to achieve the better outcome? We leave this question for now to continue our discussion of simultaneous-move games, but will return to it in later chapters, where we will develop some of the tools that can be used to “change the game” to improve outcomes.[8]

B. One Player Has a Dominant Strategy

In a simultaneous-move game in which only one player has a dominant strategy, we predict that the player who has a dominant strategy will choose it and that the other player will form a correct belief about the first player's strategy and choose her own best response to it.

As an illustration of such a situation, consider a game frequently played between the U.S. Congress, which is responsible for fiscal policy (taxes and government expenditures), and the Federal Reserve (Fed), which is in charge of monetary policy (primarily interest rates).[9] To simplify the game to its essential features, suppose that Congress's fiscal policy can result in either a balanced budget or a deficit, and that the Fed can set interest rates either high or low. In reality, the timing of moves in this game is not entirely clear (and may differ from one country to another). For now, we focus on the simultaneous-move version of the game, then return to it in Chapter 6 to consider how its outcomes may differ if Congress or the Fed moves first.

Almost everyone wants lower taxes. But there is no shortage of good claims on government funds for defense, education, health care, and so on. There are also various politically powerful special interest groups—including farmers and industries hurt by foreign competition—who want government subsidies. Therefore, Congress is under constant pressure both to lower taxes and to increase spending. But such behavior runs the budget into deficit, which can lead to inflation. The Fed's primary task is to prevent inflation. However, it also faces political pressure for lower interest rates from many important groups, especially homeowners, who benefit from lower mortgage rates. Lower interest rates lead to higher demand for automobiles, housing, and capital investment by firms, and that can cause inflation. The Fed is generally happy to lower interest rates, but only so long as inflation is not a threat. And there is less threat of inflation when the government’ s budget is in balance. With all these factors in mind, we construct the payoff matrix for this game in Figure 4.5.

		FEDERAL RESERVE	
		Low interest rates	**High interest rates**
CONGRESS	Budget balance	3, 4	1, 3
	Budget deficit	4, 1	2, 2

FIGURE 4.5 Game of Fiscal and Monetary Policies

Congress’ s best outcome (payoff 4) is a budget deficit and low interest rates, as this pleases all of its immediate political constituents. This outcome may entail trouble for the future, but political time horizons are short. For the same reason, Congress’ s worst outcome (payoff 1) is a balanced budget and high interest rates. Of the other two outcomes, Congress likes balanced budget and low interest rates (payoff 3) better than a budget deficit and high interest rates (payoff 2), as low interest rates please the important home-owning middle classes. Moreover, with low interest rates, less money is needed to service the government debt, leaving room even in a balanced budget for many other expenditures or tax cuts.

The Fed’ s worst outcome (payoff 1) is a budget deficit and low interest rates, because this combination is the most inflationary. The Fed’ s best outcome (payoff 4) is a balanced budget and low interest rates, because this combination can sustain a high level of economic activity without much risk of inflation. Comparing the other two outcomes with high interest rates, the Fed prefers the one with a balanced budget because it reduces the risk of inflation.

We look now for dominant strategies in this game. The Fed does better by choosing low interest rates if it believes that Congress is opting for a balanced budget (the Fed’ s payoff is 4 rather than 3), but it does better by choosing high interest rates if it believes that Congress is choosing to run a budget deficit (the Fed’ s payoff is 2 rather than 1). Therefore, the Fed does not have a dominant strategy. What about Congress? If Congress believes that the Fed is choosing low interest rates, it does better for itself by choosing a budget deficit rather than a balanced budget (Congress’ s payoff is 4 instead of 3). If Congress believes that the Fed is choosing high interest rates, again, it does better for itself by choosing a budget deficit rather than a balanced budget (Congress’ s payoff is 2 instead of 1). So, running a budget deficit is Congress’ s dominant strategy.

The choice for Congress is now clear. No matter what Congress believes the Fed is doing, Congress will choose to run a budget deficit. The Fed can now take this choice into account when making its own decision. The Fed should believe that Congress will choose its dominant strategy (budget deficit) and therefore choose the best strategy for itself, given this belief. That means that the Fed should choose high interest rates. Indeed, the unique Nash equilibrium of this game is a budget deficit and high interest rates.

In the Nash equilibrium, each side gets payoff 2. But an inspection of Figure 4.5 shows that, just as in the prisoners’ dilemma, there is another outcome (namely, a balanced budget and low interest rates) that can give both players higher payoffs (namely, 3 for Congress and 4 for the

Fed). Why is that outcome not achievable as an equilibrium? The problem is that Congress would be tempted to deviate from its stated strategy and sneakily run a budget deficit. The Fed, knowing this temptation, and knowing that it would then get its worst outcome (payoff 1), also deviates from its stated strategy by choosing high interest rates. In Chapters 6 and 8, we consider how the two players can get around this difficulty to achieve their mutually preferred outcome. But we should note that, in many countries, these two policy authorities are indeed often stuck in the worse outcome. In these countries, the fiscal policy is too loose and the monetary policy has to be tightened to keep inflation down.

C. Successive Elimination of Dominated Strategies

The games considered so far have had only two pure strategies available to each player. In such games, if one strategy is dominant, the other is dominated, so choosing the dominant strategy is equivalent to eliminating the dominated one. In games where players have a large number of pure strategies available to them, some of a player’s strategies may be dominated even though no single strategy dominates all the others. If players find themselves in a game of this type, they may be able to reach an equilibrium by removing dominated strategies from consideration as possible choices. Removing dominated strategies reduces the size of the game, and the “new” game that results may have other dominated strategies that can also be removed. Or the “new” game may even have a dominant strategy for one of the players. Successive or iterated elimination of dominated strategies is the process of removing dominated strategies and reducing the size of a game until no further reductions can be made. If this process ends in a single outcome, then the game is said to be dominance solvable, and the strategies that yield that outcome constitute the (unique) Nash equilibrium of the game.

We can use the game in Figure 4.1 to provide an example of this process. Consider first Row’s strategies. If any one of Row’s strategies always provides worse payoffs for Row than another of her strategies, then that strategy is dominated and can be eliminated from consideration as Row’s choice. Here, the only dominated strategy for Row is High, which is dominated by Bottom: If Column plays Left, Row gets 5 from Bottom and only 4 from High; if Column plays Middle, Row gets 4 from Bottom and only 3 from High; and if Column plays Right, Row gets 9 from Bottom and only 6 from High. So we can eliminate High. We now turn to Column’s choices to see if any of them can be eliminated. We find that Column’ s Left is now dominated by Right (with similar reasoning, as 1 < 2, 2 < 3, and 6 < 7). Note that we could not say this before Row’ s High was eliminated because, against Row’ s High, Column would get 5 from Left but only 4 from Right. Thus, the first step of eliminating Row’ s High makes possible the second step of eliminating Column’ s Left. Then, within the remaining set of strategies (Top, Low, and Bottom for Row, and Middle and Right for Column), Row’ s Top and Bottom are both dominated by her Low. When Row is left with only Low, Column chooses her best response—namely, Middle.

The game is thus dominance solvable, and the Nash equilibrium is (Low, Middle), with payoffs (5, 4). We identified this list of strategies as a Nash equilibrium when we first illustrated that concept using this game. Now we can see in better detail the thought processes of the players that lead to the formation of correct beliefs. A rational Row will not choose High. A rational Column will recognize this and, thinking about how her various strategies perform for her against Row’ s remaining strategies, will not choose Left. In turn, Row will recognize this, and therefore will not choose either Top or Bottom. Finally, Column will see through all this, and choose Middle.

Other games may not be dominance solvable, or successive elimination of dominated strategies may not yield a unique Nash equilibrium. Even in such cases, some elimination may reduce the size of the game and make it easier to solve by using one or more of the methods described in the following sections. Thus, eliminating dominated strategies can be a useful step toward solving large simultaneous-move games, even when this method does not completely solve the game.

Endnotes

- A strategy is *strictly dominant* for a player when it is strictly better than any other strategy, no matter what strategies other players adopt. By contrast, a strategy is *weakly dominant* (as discussed below in Section 4) when it is no worse than any other strategy, no matter what others do. When we use the term *dominant* without any additional qualifier, we always mean strictly dominant, and similarly with *dominated*. Return to reference 7
- In his book *Game-Changer: Game Theory and the Art of Transforming Strategic Situations* (New York: W.W. Norton, 2014), McAdams identifies five sorts of "escape routes" from the prisoners' dilemma: "cartelization" (discussed briefly in Chapter 5), "retaliation," "relationships," and "trust," (all discussed at least indirectly in Chapter 10), and "regulation" (which is beyond the scope of our book). Return to reference 8
- Similar games are played in many other countries that have central banks with operational independence in their choice of monetary policy. Fiscal policies may be chosen by different political entities—the executive or the legislature—in different countries. Return to reference 9

Glossary

dominant strategy
: A strategy X is dominant for a player if the outcome when playing X is always better than the outcome when playing any other strategy, no matter what strategies other players adopt.

dominated strategy
: A strategy X is dominated by another strategy Y for a player if the outcome when playing X is always worse than the outcome when playing Y, no matter what strategies other players adopt.

prisoners' dilemma
: A game where each player has two strategies, say Cooperate and Defect, such that [1] for each player, Defect dominates Cooperate, and [2] the outcome (Defect, Defect) is worse for both than the outcome (Cooperate, Cooperate).

successive elimination of dominated strategies
: Same as iterated elimination of dominated strategies.

successive elimination of dominated strategies
: Same as iterated elimination of dominated strategies.

dominance solvable
: A game where iterated elimination of dominated strategies leaves a unique outcome, or just one strategy for each player.

4 STRONGER AND WEAKER FORMS OF DOMINANCE

Recall that a strategy is dominant (or, more precisely, strictly dominant) for a player if it gives that player a higher payoff than any other strategy, no matter what strategies other players employ. This definition seems simple at first glance, but it masks some important subtleties. In this section, we examine dominance in more depth by defining stronger and weaker versions of the concept and exploring their implications.

A. Superdominance

In a simultaneous-move game, a strategy is superdominant for a player if the worst possible outcome when playing that strategy is better than the best possible outcome when playing any other strategy. To see what that means, consider a generic game with Row’ s payoffs as shown in Figure 4.6. (We have omitted Column’ s payoffs for clarity.) For Up to be superdominant for Row, both outcomes that could conceivably occur if she were to play Up must be better for her than both possible outcomes if she were to play Down. In other words, if Row were to rank the four outcomes—with 4 being best and 1 being worst—the outcomes ranked 4 and 3 must be in the Up row and the outcomes ranked 2 and 1 must be in the Down row.

		COLUMN	
		Left	Right
ROW	Up	3 or 4	3 or 4
	Down	1 or 2	1 or 2

FIGURE 4.6 Possible Ranking of Outcomes for Row When Up Is Superdominant

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		COLIN	
		Ask her out	Don’ t ask
ROWAN	Ask her out	50%, 50%	100%, 0%
	Don’ t ask	0%, 100%	0%, 0%

FIGURE 4.7 The Dating Game

Any superdominant strategy must also be dominant, but a dominant strategy need not be superdominant. For instance, in the prisoners’ dilemma shown in Figure 4.4, each player’ s dominant strategy Confess is *not* superdominant because the worst possible outcome when playing Confess (10 years in jail) is worse than the best possible outcome when playing Deny (3 years in jail).

For a simple example of superdominance, consider a game played between two high-school boys, Rowan and Colin, each of whom has a crush on the same girl (Winnie) and would like to take her to the big dance.[10] Each boy decides whether to ask Winnie out, knowing that Winnie likes them equally; so, if both boys ask her out, Winnie will flip a coin to decide whose offer to accept. Each boy wants to maximize his own chance of taking Winnie to the dance. The payoffs to each boy, which correspond to the probability that Winnie will go with him to the dance, are shown in Figure 4.7. Asking Winnie out is superdominant for each of them, because the worst possible outcome after asking her out (50% chance) is better than the best possible outcome after not asking (0% chance).

The distinction between dominance and superdominance can be especially important in games with sequential moves. A player with a dominant strategy in a simultaneous-move game might prefer not to play that strategy if the game were changed to have sequential moves. By contrast, a player with a superdominant strategy will always prefer to play it, no matter what the timing of moves. We will return to discuss this issue in depth later in the book, especially in Chapters 6 and 8.

B. Weak Dominance

In a simultaneous-move game, a strategy is weakly dominant for a player if the outcome when playing that strategy is *never worse* than the outcome when playing any other strategy, no matter what strategies other players adopt. Recall that a strategy is dominant for a player if it is *always better* than any other strategy, no matter what strategies other players adopt. The difference is that weak dominance allows for the possibility that a player may be indifferent between her weakly dominant strategy and some other strategy; that is, that for some subset of choices made by other players, she gets the same payoff from her weakly dominant strategy that she gets from some other strategy available to her, but for the remainder of the other players' choices, the weakly dominant strategy gives her a higher payoff than any other strategy. Dominance (or, more precisely, strict dominance) requires that she always get the highest payoff from her dominant strategy. This distinction may seem insignificant, but it has important implications for how one should analyze games.

Recall from Section 3 that when a rational player in a simultaneous-move game has a dominated strategy, we expect that she will not play that strategy because its outcome is always worse than that of any other strategy. This allows us to eliminate the dominated strategy from further consideration and lets us focus on the resulting simpler game. But what if a player has only a *weakly* dominated strategy? Eliminating such a strategy from consideration entails an extra assumption: that the player with a weakly dominant strategy will always "break ties" in favor of playing it. But we cannot be sure that she would do that.

In fact, in some games, there are Nash equilibria in which one or more players use weakly dominated strategies. Consider the game illustrated in Figure 4.8, in which, for Rowena, Up is weakly dominated by Down. (If Colin plays Left, then Rowena gets a better payoff by playing Down than by playing Up, and, if Colin plays Right, then Rowena gets the same payoffs from her two strategies.) Similarly, for Colin, Right weakly dominates Left. So the game is dominance solvable, and solving it tells us that (Down, Right) is a Nash equilibrium. However, while (Down, Right) might be the predicted outcome in many cases, there are other possible equilibrium outcomes of this game. (Down, Left) and (Up, Right) are also Nash equilibria. When Rowena is playing Down, Colin cannot improve his payoff by switching to Right, and when Colin is playing Left, Rowena's best response is clearly to play Down; similar reasoning verifies that (Up, Right) is also a Nash equilibrium.

		COLIN	
		Left	Right
ROWENA	Up	0, 0	1, 1
	Down	1, 1	1, 1

FIGURE 4.8 Elimination of Weakly Dominated Strategies

Therefore, if you eliminate weakly dominated strategies, it is a good idea to use other methods (such as the one described in the next section) to see if you have missed any other equilibria. The iterated dominance solution seems to be a reasonable outcome to predict as the Nash equilibrium of this simultaneous-move game, but it is also important to consider the significance of the multiple equilibria as well as that of the other equilibria themselves. We will address these issues in later chapters, taking up a discussion of multiple equilibria in Chapter 5 and the interconnections between sequential- and simultaneous-move games in Chapter 6.

Endnotes

- We use these names for players in some of our examples and end-of-chapter exercises, along with Rowena and Collette, in the hope that they will aid you in remembering which player chooses the row and which chooses the column. We acknowledge Robert Aumann, who shared the Nobel Prize in Economics with Thomas Schelling in 2005 (and whose ideas will be prominent in Chapter 8), for inventing this clever naming idea. Return to reference 10

Glossary

superdominant
: A strategy is superdominant for a player if the worst possible outcome when playing that strategy is better than the best possible outcome when playing any other strategy.

weakly dominant
: A strategy is weakly dominant for a player if the outcome when playing that strategy is never worse than the outcome when playing any other strategy, no matter what strategies other players adopt.

5 BEST-RESPONSE ANALYSIS

Many simultaneous-move games have no dominant strategies and no dominated strategies. Others have one or several dominated strategies, but iterated elimination of dominated strategies does not yield a unique outcome. In such cases, we need to take another step in the process of solving the game. We are still looking for a Nash equilibrium in which every player does the best she can given the actions of the other players, but we must now rely on subtler strategic thinking than the simple elimination of dominated strategies requires.

A. Identifying Best Responses

In this section, we develop another systematic method for finding Nash equilibria that will prove very useful in later analyses. We begin without imposing a requirement of correctness of beliefs about other players' actions. We take each player's perspective in turn and ask the following question: For each of the choices that the other players might be making, what is the best choice for this player? Thus, we find the best responses of each player to all available strategies of the others. In mathematical terms, we find each player's best-response strategy depending on, or as a function of, the other players' available strategies.

		COLUMN		
		Left	Middle	Right
ROW	Top	3, 1	2, 3	10, 2
	High	4, 5	3, 0	6, 4
	Low	2, 2	5, 4	12, 3
	Bottom	5, 6	4, 5	9, 7

You may need to scroll left and right to see the full figure.

FIGURE 4.9 Best-Response Analysis

Let's return to the game played by Row and Column in Figure 4.1 and reproduce it as Figure 4.9. We first consider Row's responses. If Column chooses Left, Row's best response is Bottom, yielding 5. We show this best response by circling that payoff in the game table. If Column chooses Middle, Row's best response is Low (also yielding 5). And if Column chooses Right, Row's best choice is again Low (now yielding 12). Again, we show Row's best choices by circling the appropriate payoffs. Similarly, Column’s best responses are shown by circling her payoffs: 3 (Middle as best response to Row’s Top), 5 (Left to Row’s High), 4 (Middle to Row’s Low), and 7 (Right to Row’s Bottom).[11] We see that one cell —namely, (Low, Middle)—has both its payoffs circled. Therefore, the strategies Low for Row and Middle for Column are simultaneously best responses to each other. We have found the Nash equilibrium of this game. (Again.)

Best-response analysis is a comprehensive way of locating *all* possible Nash equilibria of a game. You should improve your understanding of it by trying it out on the other games that have been used as examples in this chapter. The game in Figure 4.8, where both players have a weakly dominant strategy, is one of interest. You will find that there are ties for best responses in that game. Rowena’s Up and Down both yield her a payoff of 1 in response to Colin’s choice of Right. And Colin’s Left and Right both yield him a payoff of 1 in response to Rowena’s choice of Down. In such cases, you should circle both (or all) of the payoffs that tie for best response. Your final analysis will show three cells with both payoffs circled, a confirmation that there are indeed three Nash equilibria in that game.

Comparing best-response analysis with the dominance method is also enlightening. If Row has a dominant strategy, that same strategy is her best response to all of Column’s strategies; therefore, her best responses are all lined up horizontally in the same row. Similarly, if Column has a dominant strategy, her best responses are all lined up vertically in the same column. You should see for yourself how the Nash equilibria in the Husband–Wife prisoners’ dilemma shown in Figure 4.4 and the Congress–Federal Reserve game depicted in Figure 4.5 emerge from such an analysis.

There will be some games for which best-response analysis does not find a Nash equilibrium, just as not all games are dominance solvable. But in this case, we can say something more specific than can be said when the iterated dominance fails. When best-response analysis of a discrete-strategy game does not find a Nash equilibrium, then the game has no equilibrium in pure strategies. We will discuss games of this type in Section 8 of this chapter. In Chapter 5, we will extend best-response analysis to games where the players' strategies are continuous variables—for example, prices or advertising expenditures. Moreover, we will construct best-response *curves* to help us find Nash equilibria, and we will see that such games are less likely—by virtue of the continuity of strategy choices—to have no equilibrium in pure strategies.

B. Ordinal Payoffs

Best-response analysis depends (only) on how players rank different outcomes. Each player’s ranking of the possible outcomes is referred to as her ordinal payoff. Even if two game tables show different payoff numbers (*cardinal payoffs*) in each cell, the players’ rankings over the cells may be the same in each table. For instance, compare the new version of the prisoners’ dilemma shown in Figure 4.10a with that shown earlier in Figure 4.4. The numbers of years that Husband and Wife would spend in jail in each possible outcome have been changed in Figure 4.10a. However, the way in which the players rank the four possible outcomes is the same in both games. These ordinal payoffs for the prisoners’ dilemma, shown in Figure 4.10b, are identical for *any* prisoners’ dilemma regardless of the cardinal payoff numbers involved. In every prisoners’ dilemma, “only I confess” is best (ordinal payoff of 4), “both deny” is second-best (ordinal payoff of 3), “both confess” is third-best (ordinal payoff of 2), and “only I deny” is worst (ordinal payoff of 1).

A prisoners’ dilemma has three essential defining features. First, each player has two strategies: Cooperate (with the other player—deny any involvement with the crime, in our example) or Defect (from cooperation with the other player—here, confess to the crime). Second, each player has a dominant strategy: Defect. Third, both players are worse off when they both play their dominant strategy, Defect, than when they both play Cooperate. These defining features of the prisoners’ dilemma imply that the players’ ordinal payoffs must be as shown in Figure 4.10b. To see why, consider the generic row player, Rowan. (The same logic applies to the generic column player, Colette.) Because Defect is Rowan’s dominant strategy, (Defect, Cooperate) must be better than

(Cooperate, Cooperate) and (Defect, Defect) must be better than (Cooperate, Defect). Since both players are worse off when they both confess than when they both deny, Rowan’s ranking of the possible outcomes must therefore be (Defect, Cooperate) > (Cooperate, Cooperate) > (Defect, Defect) > (Cooperate, Defect), as shown in Figure 4.10b.

(a) Cardinal payoffs for new prisoners’ dilemma

		WIFE	
		Confess (Defect)	Deny (Cooperate)
HUSBAND	Confess (Defect)	5 yr, 5 yr	0 yr, 50 yr
	Deny (Cooperate)	50 yr, 0 yr	1 yr, 1 yr

You may need to scroll left and right to see the full figure.

(b) Ordinal payoffs for all prisoners’ dilemmas

		COLLETTE	
		Confess (Defect)	Deny (Cooperate)
ROWAN	Confess (Defect)	2, 2	4, 1

You may need to scroll left and right to see the full figure.

		COLLETTE	
		Confess (Defect)	Deny (Cooperate)
	Deny (Cooperate)	1, 4	3, 3

You may need to scroll left and right to see the full figure.

FIGURE 4.10 Cardinal and Ordinal Payoffs in the Prisoners' Dilemma

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More generally, the best responses among the pure strategies available in a game depend only on the way players rank the outcomes associated with those strategies. Therefore, Nash equilibria in pure strategies also depend only on ordinal payoffs, not on the actual payoff numbers. The same will not be true in games with Nash equilibria in mixed strategies. When we consider mixed strategies in Chapter 7, we will need to take averages of payoff numbers weighted by the probabilities of choosing those strategies, so the actual numbers will matter.

Endnotes

- Alternatively and equivalently, one could mark in some way the choices that are *not* made. For example, in Figure 4.3, Row will not choose Top, High, or Bottom as responses to Column's Right; one could show this by drawing slashes through Row's payoffs in these cases: 10, 6, and 9, respectively. When this is done for all strategies of both players, (Low, Middle) has both of its payoffs unslashed; it is thus the Nash equilibrium of the game. The alternatives of circling choices that are made and slashing choices that are not made stand in a relation to each other that is conceptually similar to that between the alternatives of showing chosen branches with arrows and pruning unchosen branches for sequential-move games. We prefer the first alternative in each case because the resulting picture looks cleaner and tells the story better. Return to reference 11

Glossary

best-response analysis
: Finding the Nash equilibria of a game by calculating the best-response functions or curves of each player and solving them simultaneously for the strategies of all the players.

ordinal payoffs
: Each player’s ranking of the possible outcomes in a game.

6 THREE PLAYERS

So far, we have analyzed only games between two players. All the methods of analysis that have been discussed, however, can be used to find pure-strategy Nash equilibria in any simultaneous-move game among any number of players. When a game is played by more than two players, each of whom has a relatively small number of pure strategies available, we can use a game table for our analysis, as we did in the first five sections of this chapter.

In Chapter 3, we described a game among three players, each of whom had two pure strategies. The three players, Emily, Nina, and Talia, had to choose whether to contribute toward the creation of a flower garden for their small street. We assumed that the garden would be no better when all three contributed than when only two contributed, and that a garden with just one contributor would be so sparse that it was as bad as no garden at all. Now let us suppose instead that the three players make their choices simultaneously and that there is a somewhat richer variety of possible outcomes and payoffs. In particular, the size and splendor of the garden will now differ according to the exact number of contributors: Three contributors will produce the largest and best garden, two contributors will produce a medium-sized garden, and one contributor will produce a small garden.

Suppose Emily is contemplating the possible outcomes of the street-garden game. There are six possible choices for her to consider. Emily can choose either to contribute or not to contribute when both Nina and Talia contribute, or when neither of them contributes, or when just one of them contributes. From her perspective, the best possible outcome, with a rank of 6, would be to take advantage of her good-hearted neighbors and to have both Nina and Talia contribute while she does not. Emily could then enjoy a medium-sized garden without putting up her own hard-earned cash. If both of the others contribute and Emily also contributes, she gets to enjoy a large, splendid garden, but at the cost of her own contribution; she ranks this outcome second best, or 5.

At the other end of the spectrum are the outcomes that arise when neither Nina nor Talia contributes to the garden. If that is the case, Emily would again prefer not to contribute, because she would foot the entire bill for a small public garden that everyone could enjoy; she would rather have the flowers in her own yard. Thus, when neither of the other players is contributing, Emily ranks the outcome in which she contributes as a 1 and the outcome in which she does not as a 2.

In between these cases are the situations in which either Nina or Talia contributes to the flower garden, but not both of them. When one of them contributes, Emily knows that she can enjoy a small garden without contributing; she also feels that the cost of her contribution outweighs the increase in benefit that she would get by increasing the size of the garden. Thus, she ranks the outcome in which she does not contribute but still enjoys the small garden as a 4 and the outcome in which she does contribute, thereby providing a medium-sized garden, as a 3. Because Nina and Talia have the same views as Emily on the costs and benefits of contributions and garden size, each of them ranks the different outcomes in the same way—the worst outcome being the one in which each contributes and the other two do not, and so on.

If all three women decide whether to contribute to the garden without knowing what their neighbors will do, we have a three-person simultaneous-move game. To find the Nash equilibrium of this game, we then need a game table. For a three-player game, the table must be three-dimensional, and the third player’s strategies must correspond to the new dimension. The easiest way to add a third dimension to a two-dimensional game table is to add pages. The first page of the table shows payoffs for the third player’s first strategy, the second page shows payoffs for the third player’s second strategy, and so on.

TALIA chooses:

Contribute

		NINA	
		Contribute	**Don’t**
EMILY	Contribute	5, 5, 5	3, 6, 3
	Don’t	6, 3, 3	4, 4, 1

Don’t Contribute

		NINA	
		Contribute	**Don’t**
EMILY	Contribute	3, 3, 6	1, 4, 4
	Don’t	4, 1, 4	2, 2, 2

You may need to scroll left and right to see the full figure.

Figure 4.11 Street-Garden Game

We show the three-dimensional table for the street-garden game in Figure 4.11. It has two rows for Emily’s two strategies, two columns for Nina’s two strategies, and two pages for Talia’s two strategies. We show the pages side by side here so that you can see everything at the same time. In each cell, payoffs are listed for the row player first, the column player second, and the page player third; in this case, the order is Emily, Nina, Talia.

Our first test should be to determine whether there are dominant strategies for any of the players. In one-page game tables, we found this test to be simple: we just compared the outcomes associated with one of a player’s strategies with the outcomes associated with another of her strategies. In practice, this comparison required, for the row player, a simple check within the columns of the single page of the table, and vice versa for the column player. Here, we must check both pages of the table to determine whether any player has a dominant strategy.

For Emily, we compare the two rows of both pages of the table and note that when Talia contributes, Emily has a dominant strategy, Don’t Contribute, and when Talia does not contribute, Emily also has a dominant strategy, Don’t Contribute. Thus, the best thing for Emily to do, regardless of what either of the other players does, is not to contribute. Similarly, we see that Nina’s dominant strategy—in both pages of the table—is Don’t Contribute. When we check for a dominant strategy for Talia, we have to be a bit more careful. We must compare outcomes that keep Emily’s and Nina’s behavior constant, checking Talia’s payoffs from choosing Contribute versus Don’t Contribute. That is, we must compare cells across pages of the table—the top-left cell in the first page (on the left) with the top-left cell in the second page (on the right), and so on. As for the first two players, this process indicates that Talia also has a dominant strategy, Don’t Contribute.

Each player in this game has a dominant strategy, which must therefore be her equilibrium strategy. The pure-strategy Nash equilibrium of the street-garden game entails all three players choosing not to contribute to the street garden and getting their second-worst payoffs: The garden is not planted, but no one has to contribute to it, either.

TALIA chooses:

Contribute

	NINA	
	Contribute	Don’t

		NINA	
		Contribute	Don’t
EMILY	Contribute	5, 5, 5	3, 6, 3
	Don’t	6, 3, 3	4, 4, 1

Don’t Contribute

		NINA	
		Contribute	Don’t
EMILY	Contribute	3, 3, 6	1, 4, 4
	Don’t	4, 1, 4	2, 2, 2
You may need to scroll left and right to see the full figure.			

Figure 4.12 Best-Response Analysis in the Street-Garden Game

Notice that this game is yet another example of a prisoners’ dilemma. It has a unique Nash equilibrium in which all players receive a payoff of 2. Yet there is another outcome in the game—in which all three neighbors contribute to the garden—that yields higher payoffs of 5 for all three players. Even though it would be beneficial to each of them for all of them to pitch in to build the garden, none of them has the individual incentive to do so. As a result, gardens of this type are often not planted at all, or they are paid for with tax dollars because the town government can require its citizens to pay such taxes. In Chapter 11, we will encounter more such dilemmas of collective action and study some methods for resolving them.

The Nash equilibrium of this game can also be found using best-response analysis, as shown in Figure 4.12. Because each player has Don’t Contribute as her dominant strategy, all of

Emily's best responses are on her Don't Contribute row, all of Nina's best responses are on her Don't Contribute column, and all of Talia's best responses are on her Don't Contribute page. The bottom-right cell on the right page contains all three best responses; therefore, it gives us the Nash equilibrium.

7 MULTIPLE EQUILIBRIA IN PURE STRATEGIES

Most of the games considered in the preceding sections have had a unique pure-strategy Nash equilibrium. In general, however, games need not have unique Nash equilibria. We illustrate this possibility using a class of games that has many applications. As a group, they may be labeled coordination games. The players in such games have some (but not always completely) common interests, but because they act independently (by virtue of the nature of noncooperative games), the coordination of actions needed to achieve a jointly preferred outcome is problematic.

A. Pure Coordination: Will Holmes Meet Watson?

To illustrate the idea behind coordination games, we can imagine a situation involving Sherlock Holmes and Dr. Watson early in 1881, the first year of their partnership. They are hurriedly leaving their Baker Street apartment, each with a specific task to accomplish regarding a new case. As they rush off in different directions, Holmes shouts back to Watson that they should rendezvous at the end of the day to compare notes, at “four o’clock at our meeting place.” Later, while collecting evidence for the case, each realizes that in their rush that morning they failed to be precise about the meaning of “our meeting place.” Had Holmes meant to imply “the place where we first met each other” in January—St. Bartholomew’s Hospital? Or had he meant to imply “the place where we met recently for a meal”—Simpson’s in the Strand? Unfortunately, these two locations are sufficiently far apart across the crowded city of London that it will be impossible to try both. Having no other way to contact the other, each is left with a quandary: Which meeting place should he choose?

The game table in Figure 4.13 illustrates this situation. Each player has two choices: St. Bart’s and Simpson’s. The payoffs for each are 1 if they meet and 0 if they do not. Best-response analysis quickly reveals that the game has two Nash equilibria, one where both choose St. Bart’s and the other where both choose Simpson’s. It is important for both that they achieve one of the equilibria, but which one is immaterial because the two yield equal payoffs. All that matters is that they coordinate on the same action; it does not matter which action. That is why the game is said to be one of pure coordination.

But will they coordinate successfully? Or will they end up in different locations, each thinking that the other has left him or succumbed to some disastrous fate? Alas, that risk exists. Holmes might think that Watson will go to St. Bart’s because he said something about following up a lead near the hospital that morning. But Watson might have the opposite belief about what Holmes will do. When there are multiple Nash equilibria, if the players are to select one successfully, they need some way to coordinate their beliefs or expectations about each other’s actions.

Their situation is similar to that of the heroes of the “Which tire?” game in Chapter 1, where we labeled the coordination device a focal point. In the present context, one of the two locations might be closer to where each of the partners was heading that day. But it is not enough that Holmes knows this to be the case. He must know that Watson knows, and that he knows that Holmes knows, and so on. In other words, their expectations must *converge* on the focal point. Otherwise, Holmes may be doubtful about where Watson will go because he does not know what Watson is thinking about where Holmes will go, and similar doubts may arise at the third or fourth or higher levels of thinking about thinking.[12]

		WATSON	
		St. Bart’s	Simpson’s
HOLMES	St. Bart’s	1, 1	0, 0
	Simpson’s	0, 0	1, 1

FIGURE 4.13 Pure Coordination

Over time, the two players here might come to understand “our meeting place” to mean exactly one of the possible interpretations. (As, in fact, they did. Simpson’s became “our Strand restaurant” for Holmes and Watson, but not until much later in their relationship.) But without the benefit of a lengthy partnership, they have no obvious choice.

In general, whether players in coordination games can find a focal point depends on whether they have some commonly known point of contact, whether historical, cultural, or linguistic.

B. Assurance: Will Holmes Meet Watson? And Where?

Now let’s change the game’s payoffs a little. It may be that our pair is not quite indifferent about which location they choose. Meeting at the hospital might be safer, as they will be protected from the prying eyes of other Londoners at dinner. Or they may both want to choose the location at which they can acquire a spot of tea in nice surroundings. Suppose they both prefer to meet at Simpson’s; then the payoff to each is 2 when they meet there versus 1 when they meet at St. Bart’s. The new payoff matrix is shown in Figure 4.14.

Again, there are two Nash equilibria. But in this version of the game, each prefers the equilibrium where both choose Simpson’s. Unfortunately, their mere liking of that outcome is not guaranteed to bring it about. First of all (and as always in our analysis), the payoffs have to be common knowledge—both have to know the entire payoff matrix, both have to know that both know, and so on. Such detailed knowledge about the game could arise if the two had discussed and agreed on the relative merits of the locations, but Holmes simply forgot to clarify that they should meet at Simpson’s. Even then, Holmes might think that Watson has some other reason for choosing St. Bart’s, or he might think that Watson thinks that he does, and so on. Without genuine convergence of expectations about actions, the players may choose the worse equilibrium, or worse still, they may fail to coordinate actions and get payoffs of 0 each.

		WATSON	
		St. Bart’s	Simpson’s
HOLMES	St. Bart’s	1, 1	0, 0

		WATSON	
		St. Bart's	Simpson's
	Simpson's	0, 0	2, 2

FIGURE 4.14 Assurance

Thus, players in the game illustrated in Figure 4.14 can get the preferred equilibrium outcome only if each has enough certainty or assurance that the other is choosing the appropriate action. For this reason, such games are called assurance games.[13]

In many real-life situations of this kind, such assurance is easily obtained, given even a small amount of communication between the players. Their interests are perfectly aligned, so if one of them says to the other, "I will go to Simpson's," the other has no reason to doubt the truth of this statement and will go to Simpson's to get the mutually preferred outcome. That is why we had to construct the story with the two friends dashing off in different directions with no later means of communication. If the players' interests conflict, truthful communication becomes more problematic. We examine this problem further when we consider strategic manipulation of information in games in Chapter 9.

In larger groups, communication can be achieved by scheduling meetings or by making announcements. These devices work only if everyone knows that everyone else is paying attention to them, because successful coordination requires the desired equilibrium to be a focal point. The players' expectations must converge on it; everyone should know that everyone knows that . . . everyone is choosing it. Many social institutions and arrangements play this role. Meetings where the participants sit in a circle facing inward ensure that everyone sees everyone else paying attention. Advertisements during the Super Bowl, especially when they are proclaimed in advance as major attractions, assure each viewer that many others are viewing them also. That makes such ads especially attractive to companies making products that are more desirable for any one buyer when many others are buying them, too; such products include those produced by the computer, telecommunication, and Internet industries.[14]

		WATSON	
		St. Bart's	Simpson's
HOLMES	St. Bart's	2, 1	0, 0
	Simpson's	0, 0	1, 2

FIGURE 4.15 Battle of the Sexes

C. Battle of the Sexes: Will Holmes Meet Watson? And Where?

Now let’s introduce another complication to the coordination game: Both players want to meet, but prefer different locations. So Holmes might get a payoff of 2 and Watson a payoff of 1 from meeting at St. Bart’s, and the other way around from meeting at Simpson’s. This payoff matrix is shown in Figure 4.15.

This game is called the battle of the sexes. The name derives from the story concocted for this payoff structure by game theorists in the sexist 1950s. A husband and wife were supposed to choose between going to a boxing match and going to a ballet, and (presumably for evolutionary genetic reasons) the husband was supposed to prefer the boxing match and the wife the ballet. The name has stuck, and we will keep it, but our example should make it clear that it does not necessarily have sexist connotations.

What will happen in this game? There are still two Nash equilibria. If Holmes believes that Watson will choose St. Bart’s, it is best for him to do likewise. For similar reasons, Simpson’s is also a Nash equilibrium. To achieve either of these equilibria and avoid the outcomes where the two go to different locations, the players need a focal point, or convergence of expectations, as in the pure-coordination and assurance games. But the risk of coordination failure is greater in the battle of the sexes. The players are initially in quite symmetric situations, but each of the two Nash equilibria gives them asymmetric payoffs, and their preferences between the two outcomes are in conflict. Holmes prefers the outcome where they meet at St. Bart’s, and Watson prefers to meet at Simpson’s. They must find some way of breaking the symmetry.

In an attempt to achieve his preferred equilibrium, each player may try to “act tough” and follow the strategy leading to that equilibrium. In Chapter 8, we will consider in detail such advance devices, called *strategic moves*, that players in such games can adopt to try to achieve their preferred outcomes. Or each may try to be nice, leading to the unfortunate situation where Holmes goes to Simpson’s because he wants to please Watson, only to find that Watson has chosen to please him and has gone to St. Bart’s, like the couple choosing Christmas presents for each other in O. Henry’s short story titled “The Gift of the Magi.” Or if the game is repeated, successful coordination may be negotiated and maintained as an equilibrium. For example, Holmes and Watson may arrange to alternate between meeting locations on various days or during different cases. In Chapter 10, we will examine such tacit cooperation in repeated games in the context of a prisoners’ dilemma.

D. Chicken: Will James Meet Dean?

Our final example in this section is a slightly different kind of coordination game. In this game, the players want to avoid, not choose, the same strategies. Further, the consequences of coordination failure in this kind of game are far more drastic than in the other kinds.

The story comes from a game that was supposedly played by American teenagers in the 1950s. Two teenagers take their cars to opposite ends of Main Street, Middle-of-Nowhere, USA, at midnight and start to drive toward each other. The one who swerves to prevent a collision is the "chicken," and the one who keeps going straight is the winner. If both maintain a straight course, there is a collision in which both cars are damaged and both players injured.[15]

The payoffs for games of chicken depend on how negatively one rates the "bad" outcome—being hurt and damaging your car in this case—against being labeled chicken. As long as words hurt less than crunching metal, a reasonable payoff table for the 1950s version of chicken is the one found in Figure 4.16. Each player's highest-ranked outcome is to win, having the other be chicken, and each player's lowest-ranked outcome is the crash of the two cars. In between these two extremes, it is better to have your rival be chicken (to save face) than to be chicken yourself.

This story has four essential features that define any game of chicken. First, each player has one strategy that is the "tough" strategy and one that is the "weak" strategy. Second, there are two pure-strategy Nash equilibria. These are the outcomes in which exactly one of the players is chicken, or weak. Third, each player strictly prefers that equilibrium in which the other player chooses chicken.

Fourth, the payoffs when both players are tough are very bad for both players. In games such as this one, the real game becomes a test of how to achieve one's preferred equilibrium.

		DEAN	
		Swerve (Chicken)	Straight (Tough)
JAMES	Swerve (Chicken)	0, 0	−1, 1
	Straight (Tough)	1, −1	−2, −2

FIGURE 4.16 Chicken

We are now back in a situation similar to the battle of the sexes. One expects most real-life games of chicken to be even worse as battles than most battles of the sexes—the benefit of winning is larger, as is the cost of a crash, so all the problems of conflict of interest and asymmetry between the players are aggravated. Each player will want to influence the outcome. One player may try to create an aura of toughness that everyone recognizes so as to intimidate all rivals.[16] Another possibility is to come up with some other way to convince your rival that you will not be chicken by making a visible and irreversible commitment to going straight. (In Chapter 8, we consider just how to make such commitment moves.) In addition, both players also want to try to prevent the worst outcome (a crash) if at all possible.

As with the battle of the sexes, if the game is repeated, tacit coordination is a better route to a solution. That is, if two teenagers played chicken every Saturday night at midnight, they would have the benefit of knowing that the game had both a history and a future when deciding on their strategies. In such a situation, they might logically choose to alternate between the two equilibria, taking turns being the winner every other week. (But if the others found out about this deal, both players would lose face.)

There is one final point arising from coordination games that must be addressed: The concept of Nash equilibrium requires each player to have the correct belief about the other's choice of strategy. When we look for Nash equilibria in pure strategies, the concept requires each player to be confident about the other's choice. But our analysis of coordination games shows that thinking about the other's choice in such games is fraught with strategic uncertainty. How can we incorporate such uncertainty in our analysis? In Chapter 7, we introduce the concept of a mixed strategy, where actual choices are made randomly among the available actions. This approach generalizes the concept of Nash equilibrium to situations where the players may be unsure about each other's actions.

Endnotes

- Thomas Schelling presented the classic treatment of coordination games and developed the concept of a focal point in his book *The Strategy of Conflict* (Cambridge, Mass.: Harvard University Press, 1960); see pp. 54 – 58, 89 – 118. His explanation of focal points included the results garnered when he posed several questions to his students and colleagues. The best remembered of these is, “Suppose you have arranged to meet someone in New York City on a particular day, but have failed to arrange a specific place or time, and have no way of communicating with the other person. Where will you go and at what time?” Fifty years ago, when the question was first posed, the clock at Grand Central Station was the usual focal place; now it might be the stairs at TKTS in Times Square. The focal time remains 12:00 noon. Return to reference 12
- The classic example of an assurance game usually offered is the stag hunt described by the eighteenth-century French philosopher Jean-Jacques Rousseau. Several people can successfully hunt a stag, thereby getting a large quantity of meat, if they collaborate. If any one of them is sure that all the others will collaborate, he also stands to benefit by joining the group. But if he is unsure whether the group will be large enough, he will do better to hunt for a smaller animal, a hare, on his own. However, it can be argued that Rousseau believed that each hunter would prefer to go after a hare regardless of what the others were doing, which would make the stag hunt a multiplayer prisoners’ dilemma, not an assurance game. We discuss this example in the context of collective action Chapter 11. Return to reference 13
- Michael Chwe develops this theme in *Rational Ritual: Culture, Coordination, and Common Knowledge* (Princeton,

N.J.: Princeton University Press, 2001). Return to reference 14

- A slight variant was made famous by the 1955 James Dean movie *Rebel without a Cause*. There, two players drove their cars in parallel, very fast, toward a cliff. The first to jump out of his car before it went over the cliff was the chicken. The other, if he left too late, risked going over the cliff in his car to his death. The characters in the film referred to this as a "chicky game." In the mid-1960s, the British philosopher Bertrand Russell and other peace activists used the game of chicken as an analogy for the nuclear arms race between the United States and the USSR, and the game theorist Anatole Rapoport gave a formal game-theoretic statement to that effect. Other game theorists have chosen to interpret the arms race as a prisoners' dilemma or as an assurance game. For a review and interesting discussion, see Barry O'Neill, "Game Theory Models of Peace and War," in *The Handbook of Game Theory*, vol. 2, ed. Robert J. Aumann and Sergiu Hart (Amsterdam: North Holland, 1994), pp. 995–1053. Return to reference 15
- Why would a potential rival play chicken against someone with a reputation for never giving in? The problem is that participation in chicken, as in lawsuits, is not really voluntary. Put another way, choosing whether to play chicken is itself a game of chicken. As Thomas Schelling says, "If you are publicly invited to play chicken and say you would rather not, then you have just played [and lost]." *Arms and Influence* (New Haven, Conn.: Yale University Press, 1965), p. 118. Return to reference 16

Glossary

coordination game
: A game with multiple Nash equilibria, where the players are unanimous about the relative merits of the equilibria, and prefer any equilibrium to any of the nonequilibrium possibilities. Their actions must somehow be coordinated to achieve the preferred equilibrium as the outcome.

pure coordination game
: A coordination game where the payoffs of each player are the same in all the Nash equilibria. Thus, all players are indifferent among all the Nash equilibria, and coordination is needed only to ensure avoidance of a nonequilibrium outcome.

focal point
: A configuration of strategies for the players in a game, which emerges as the outcome because of the convergence of the players' expectations on it.

convergence of expectations
: A situation where the players in a noncooperative game can develop a common understanding of the strategies they expect will be chosen.

assurance game
: A game where each player has two strategies, say, Cooperate and Not, such that the best response of each is to Cooperate if the other cooperates, Not if not, and the outcome from (Cooperate, Cooperate) is better for both than the outcome of (Not, Not).

battle of the sexes
: A game where each player has two strategies, say, Hard and Soft, such that [1] (Hard, Soft) and (Soft, Hard) are both Nash equilibria, [2] of the two Nash equilibria, each player prefers the outcome where he is Hard and the other is Soft, and [3] both prefer the Nash equilibria to

the other two possibilities, (Hard, Hard) and (Soft, Soft).

chicken

A game where each player has two strategies, say Tough and Weak, such that [1] both (Tough, Weak) and (Weak, Tough) are Nash equilibria, [2] of the two, each prefers the outcome where she plays Tough and the other plays Weak, and [3] the outcome (Tough, Tough) is worst for both.

8 NO EQUILIBRIUM IN PURE STRATEGIES

Each of the games considered so far has had at least one Nash equilibrium in pure strategies. Some of these games, such as the coordination games in Section 7, had more than one Nash equilibrium, whereas games in earlier sections had exactly one. Unfortunately, not all games that we come across in the study of strategy and game theory will have easily definable outcomes in which players always choose one particular action as an equilibrium strategy. In this section, we look at games in which there is not even one pure-strategy Nash equilibrium—games in which none of the players would consistently choose one strategy as his equilibrium action.

A simple example of a game with no equilibrium in pure strategies is a single point in a tennis match. Recall the tennis match we first introduced in Chapter 1, Section 2.A, between the two all-time best women players—Martina Navratilova and Chris Evert.[17] Navratilova, at the net, has just volleyed a ball to Evert on the baseline, and Evert is about to attempt a passing shot. She can try to send the ball either down the line (DL; a hard, straight shot) or crosscourt (CC; a softer, diagonal shot). Navratilova must likewise prepare to cover one side or the other. Each player is aware that she must not give any indication of her planned action to her opponent, knowing that such information will be used against her: Navratilova will move to cover the side to which Evert is planning to hit, or Evert will hit to the side that Navratilova is not planning to cover. Both must act in a fraction of a second, and both are equally good at concealing their intentions until the last possible moment; therefore, their actions are effectively simultaneous, and we can analyze the point as a two-player simultaneous-move game.

Payoffs in this tennis-point game are given by the fraction of times a player wins the point in any particular combination of passing shot and covering play. Given that a down-the-line passing shot is stronger than a crosscourt shot and that Evert is more likely to win the point when Navratilova moves to cover the wrong side of the court, we can work out a reasonable set of payoffs. Suppose Evert is successful with a down-the-line passing shot 80% of the time if Navratilova covers crosscourt, but only 50% of the time if Navratilova covers down the line. Similarly, Evert is successful with her crosscourt passing shot 90% of the time if Navratilova covers down the line. This success rate is higher than when Navratilova covers crosscourt, in which case Evert wins only 20% of the time.

		NAVRATILOVA	
		DL	CC
EVERT	DL	50, 50	80, 20
	CC	90, 10	20, 80

FIGURE 4.17 No Equilibrium in Pure Strategies

Clearly, the percentage of times that Navratilova wins this tennis point is simply the difference between 100% and the percentage of times that Evert wins. Thus, the game is zero-sum (even though the two payoffs technically sum to 100), and we can represent all the necessary information in the payoff table with just the payoff to Evert in each cell. Figure 4.17 shows that payoff table and the percentage of times that Evert wins the point against Navratilova in each of the four possible combinations of their strategy choices.

The rules for solving simultaneous-move games tell us to look first for dominant or dominated strategies and then to use best-response analysis to find a Nash equilibrium. It is a useful exercise to verify that no dominant strategies exist here. Going on to best-response analysis, we find that Evert's best response to DL is CC, and that her best response to CC is DL. By contrast, Navratilova's best response to DL is DL, and her best response to CC is CC. None of the cells in the game table is a Nash equilibrium, because someone always prefers to change her strategy. For example, if we start in the upper-left cell of the table, we find that Evert prefers to deviate from DL to CC, increasing her own payoff from 50% to 90%. But in the lower-left cell of the table, we find that Navratilova, too, prefers to switch from DL to CC, raising her payoff from 10% to 80%. As you can verify, Evert similarly prefers to deviate from the lower-right cell, and Navratilova prefers to deviate from the upper-right cell. In every cell, one player always wants to change her play, and we cycle through the table endlessly without finding an equilibrium.

An important message is contained in the absence of a Nash equilibrium in this game and similar ones. What is important in games of this type is not what players should do, but what players should *not* do. In our example, each tennis player should neither always nor systematically pick the same shot when faced with the same situation. If either player engages in any determinate behavior of that type, the other can take advantage of it. (So if Evert consistently went crosscourt with her passing shot, Navratilova would learn to cover crosscourt every time and would thereby reduce Evert's chances of success with her crosscourt shot.) The most reasonable thing for players to do here is to act somewhat unsystematically, hoping for the element of surprise. An unsystematic approach entails choosing each strategy part of the time. (Evert should be using her weaker shot with enough frequency to guarantee that Navratilova cannot predict which shot will come her way. She should not, however, use the two shots in any set pattern, because that, too, would cause her to lose the element of surprise.) This approach, in which players randomize their actions, is known as mixing strategies, and it is the focus of Chapter 7. The game illustrated in Figure 4.17 may not have an equilibrium in pure strategies, but it can still be solved by looking for an equilibrium in mixed strategies, as we will do in Chapter 7, Section 2.

Endnotes

- For those among you who remember only the latest phenom who shines for a couple of years and then burns out, here are some amazing facts about these two women, who were at the top levels of the game for almost two decades and ran a memorable rivalry all that time. Navratilova was a left-handed serve-and-volley player. In grand-slam tournaments, she won 18 singles titles, 31 doubles, and 7 mixed doubles. In all tournaments, she won 167, a record. Evert, a right-handed baseliner, had a record win-loss percentage (90% wins) in her career and 150 titles, of which 18 were for singles in grand-slam tournaments. She probably invented (and certainly popularized) the two-handed backhand that is now so common. From 1973 to 1988, the two played each other 80 times, and Navratilova ended up with a slight edge, 43–37. Return to reference 17

SUMMARY

In simultaneous-move games, players make their strategy choices without knowledge of the choices being made by other players. Such games are illustrated by *game tables*, in which cells show the payoff of each choice to each player, and the dimensionality of the table equals the number of players. Two-person *zero-sum games* may be illustrated in shorthand with only one player's payoff in each cell of the game table.

Nash equilibrium is the concept used to solve simultaneous-move games, It consists of a list of strategies, one for each player, such that each player has chosen her best response to the other's choice. Nash equilibrium can also be defined as a list of strategies in which each player has correct *beliefs* about the others' strategies and chooses the best strategy for herself given those beliefs. Nash equilibria can be found by searching for *dominant strategies*, by *successive elimination of dominated strategies*, or with *best-response analysis.*

There are many classes of simultaneous-move games. *Prisoners' dilemma* games appear in many contexts. Coordination games, such as *assurance*, *chicken*, and *battle of the sexes*, have multiple equilibria, and the solution of such games requires players to achieve coordination by some means. If a game has no equilibrium in *pure strategies*, we must look for an equilibrium in *mixed strategies*, the analysis of which is presented in Chapter 7.

KEY TERMS

Glossary

assurance game
: A game where each player has two strategies, say, Cooperate and Not, such that the best response of each is to Cooperate if the other cooperates, Not if not, and the outcome from (Cooperate, Cooperate) is better for both than the outcome of (Not, Not).

battle of the sexes
: A game where each player has two strategies, say, Hard and Soft, such that [1] (Hard, Soft) and (Soft, Hard) are both Nash equilibria, [2] of the two Nash equilibria, each player prefers the outcome where he is Hard and the other is Soft, and [3] both prefer the Nash equilibria to the other two possibilities, (Hard, Hard) and (Soft, Soft).

belief
: The notion held by one player about the strategy choices of the other players and used when choosing his own optimal strategy.

best response
: The strategy that is optimal for one player, given the strategies actually played by the other players, or the belief of this player about the other players' strategy choices.

best-response analysis
: Finding the Nash equilibria of a game by calculating the best-response functions or curves of each player and solving them simultaneously for the strategies of all the players.

chicken
: A game where each player has two strategies, say Tough and Weak, such that [1] both (Tough, Weak) and (Weak, Tough) are Nash equilibria, [2] of the two, each prefers the outcome where she plays Tough and the other plays

Weak, and [3] the outcome (Tough, Tough) is worst for both.

convergence of expectations

A situation where the players in a noncooperative game can develop a common understanding of the strategies they expect will be chosen.

coordination game

A game with multiple Nash equilibria, where the players are unanimous about the relative merits of the equilibria, and prefer any equilibrium to any of the nonequilibrium possibilities. Their actions must somehow be coordinated to achieve the preferred equilibrium as the outcome.

dominance solvable

A game where iterated elimination of dominated strategies leaves a unique outcome, or just one strategy for each player.

dominant strategy

A strategy X is dominant for a player if the outcome when playing X is always better than the outcome when playing any other strategy, no matter what strategies other players adopt.

dominated strategy

A strategy X is dominated by another strategy Y for a player if the outcome when playing X is always worse than the outcome when playing Y, no matter what strategies other players adopt.

focal point

A configuration of strategies for the players in a game, which emerges as the outcome because of the convergence of the players' expectations on it.

game table

A spreadsheetlike table whose dimension equals the number of players in the game; the strategies available to each player are arrayed along one of the dimensions (row, column, page, . . .); and each cell shows the payoffs of all the players in a specified order, corresponding to the configuration of strategies that yield that cell. Also called payoff table.

[iterated elimination of dominated strategies]

Considering the players in turns and repeating the process in rotation, eliminating all strategies that are dominated for one at a time, and continuing doing so until no such further elimination is possible. Also called successive elimination of dominated strategies.

[mixed strategy]

A mixed strategy for a player consists of a random choice, to be made with specified probabilities, from his originally specified pure strategies.

[Nash equilibrium]

A configuration of strategies (one for each player) such that each player's strategy is best for him, given those of the other players. (Can be in pure or mixed strategies.)

[normal form]

Representation of a game in a game matrix, showing the strategies (which may be numerous and complicated if the game has several moves) available to each player along a separate dimension (row, column, etc.) of the matrix and the outcomes and payoffs in the multidimensional cells. Also called strategic form.

[ordinal payoffs]

Each player's ranking of the possible outcomes in a game.

[payoff matrix]

Same as payoff table and game table.

[payoff table]

Same as game table.

[prisoners' dilemma]

A game where each player has two strategies, say Cooperate and Defect, such that [1] for each player, Defect dominates Cooperate, and [2] the outcome (Defect, Defect) is worse for both than the outcome (Cooperate, Cooperate).

pure coordination game
A coordination game where the payoffs of each player are the same in all the Nash equilibria. Thus, all players are indifferent among all the Nash equilibria, and coordination is needed only to ensure avoidance of a nonequilibrium outcome.

pure strategy
A rule or plan of action for a player that specifies without any ambiguity or randomness the action to take in each contingency or at each node where it is that player’s turn to act.

strategic form
Same as normal form.

successive elimination of dominated strategies
Same as iterated elimination of dominated strategies.

superdominant
A strategy is superdominant for a player if the worst possible outcome when playing that strategy is better than the best possible outcome when playing any other strategy.

weakly dominant
A strategy is weakly dominant for a player if the outcome when playing that strategy is never worse than the outcome when playing any other strategy, no matter what strategies other players adopt.

SOLVED EXERCISES

1. Find all Nash equilibria in pure strategies for the following games. First, check for dominant strategies. If there are none, solve the games using iterated elimination of dominated strategies. Explain your reasoning.

1.

		COLIN	
		Left	Right
ROWENA	Up	4, 0	3, 1
	Down	2, 2	1, 3

You may need to scroll left and right to see the full figure.

2.

		COLIN	
		Left	Right
ROWENA	Up	2, 4	1, 0
	Down	6, 5	4, 2

You may need to scroll left and right to see the full figure.

3.

		COLIN		
		Left	Middle	Right
ROWENA	Up	1, 5	2, 4	5, 1
	Straight	2, 4	4, 2	3, 3

You may need to scroll left and right to see the full figure.

	COLIN		
	Left	Middle	Right
Down	1, 5	3, 3	3, 3

You may need to scroll left and right to see the full figure.

4.

		COLIN		
		Left	Middle	Right
ROWENA	Up	5, 2	1, 6	3, 4
	Straight	6, 1	1, 6	2, 5
	Down	1, 6	0, 7	0, 7

You may need to scroll left and right to see the full figure.

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2. For each of the four games in Exercise S1, identify whether the game is zero-sum or non-zero-sum. Explain your reasoning.
3. For each of the four games in Exercise S1, identify which players (if any) have a superdominant strategy. Explain your reasoning.
4. Another method for solving zero-sum games, important because it was developed long before Nash developed his concept of equilibrium for non-zero-sum games, is the *minimax* method. To use this method, assume that no matter which strategy a player chooses, her rival will choose to give her the worst possible payoff from that strategy. For each zero-sum game identified in Exercise S2, use the minimax method to find the game's equilibrium strategies by doing the following:
    1. For each of Rowena's strategies, write down the minimum possible payoff to her (the worst that Colin can do to her in each case). For each of Colin's strategies, write down the minimum possible payoff to

him (the worst that Rowena can do to him in each case).
2. Determine the strategy (or strategies) that gives each player the best of these worst payoffs. This strategy is called a *minimax strategy*.

(Because we are considering zero-sum games, players' best responses do indeed involve minimizing each other's payoffs, so the minimax strategies are the same as the Nash equilibrium strategies. John von Neumann proved the existence of a minimax equilibrium in zero-sum games in 1928, more than 20 years before Nash generalized the theory to include zero-sum games.)

5. Find all Nash equilibria in pure strategies in the following non-zero-sum games. Describe the steps that you used in finding the equilibria.

1.

		COLIN	
		Left	Right
ROWENA	Up	3, 2	2, 3
	Down	4, 1	1, 4

You may need to scroll left and right to see the full figure.

2.

		COLIN	
		Left	Right
ROWENA	Up	1, 1	0, 1
	Down	1, 0	1, 1

You may need to scroll left and right to see the full figure.

3.

COLIN				
		Left	Middle	Right
ROWENA	Up	0, 1	9, 0	2, 3
	Straight	5, 9	7, 3	1, 7
	Down	7, 5	10, 10	3, 5

You may need to scroll left and right to see the full figure.

4.

COLIN				
		West	Center	East
ROWENA	North	2, 3	8, 2	7, 4
	Up	3, 0	4, 5	6, 4
	Down	10, 4	6, 1	3, 9
	South	4, 5	2, 3	5, 2

You may need to scroll left and right to see the full figure.

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6. Consider the following game table:

COLIN					
		North	South	East	West
ROWENA	Earth	1, 3	3, 1	0, 2	1, 1
	Water	1, 2	1, 2	2, 3	1, 1
	Wind	3, 2	2, 1	1, 3	0, 3
	Fire	2, 0	3, 0	1, 1	2, 2

You may need to scroll left and right to see the full figure.

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1. Does either Rowena or Colin have a dominant strategy? Explain why or why not.
2. Use iterated elimination of dominated strategies to reduce the game as much as possible. Give the order in which the eliminations occur and give the reduced form of the game.
3. Is this game dominance solvable? Explain why or why not.
4. State the Nash equilibrium (or equilibria) of this game.

7. "If a player has a dominant strategy in a simultaneous-move game, then she is sure to get her best possible outcome." True or false? Explain and give an example of a game that illustrates your answer.
8. An old lady is looking for help crossing the street. Only one person is needed to help her; if more people help her, this is no better. You and I are the two people in the vicinity who can help; we have to choose simultaneously whether to do so. Each of us will get pleasure worth a payoff of 3 from her success (no matter who helps her). But each one of us who helps will bear a cost of 1, this being the value of our time taken up in helping her. If neither player helps, the payoff for each player is 0. Set up this situation as a game. Draw the game table and find all pure-strategy Nash equilibria.
9. A university is contemplating whether to build a new laboratory or a new theater on campus. The science faculty would rather see a new lab built, and the humanities faculty would prefer a new theater. However, the funding for the project (whichever it may turn out to be) is contingent on unanimous support from the faculty. If there is disagreement, neither project will go forward, leaving each group with no new building and their worst payoff. Meetings of the two separate faculty groups to discuss which proposal to support occur simultaneously. Payoffs are given in the following table.

		HUMANITIES FACULTY	
		Lab	Theater
SCIENCE FACULTY	Lab	4, 2	0, 0
	Theater	0, 0	1, 5

---

1. What are the pure-strategy Nash equilibria of this game?
2. Which game described in this chapter is most similar to this game? Explain your reasoning.

10. Suppose two game-show contestants, Alex and Bob, each separately select one of three doors, numbered 1, 2, and 3. Both players get dollar prizes if their choices match, as indicated in the following payoff table:

		BOB		
		1	2	3
ALEX	1	10, 10	0, 0	0, 0
	2	0, 0	15, 15	0, 0
	3	0, 0	0, 0	15, 15
You may need to scroll left and right to see the full figure.				

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1. What are the Nash equilibria of this game? Which, if any, is likely to emerge as the focal point? Explain.
2. Consider a slightly changed game in which the choices are again doors 1, 2, and 3, but the payoffs in the two cells with (15, 15) in the table become (25, 25). What is the expected (average) payoff to each player if each flips a coin to decide whether to play 2 or 3? Is this outcome better than the outcome of both of them choosing 1 as a focal point? How should you account for the risk that Alex might do one thing while Bob does the other?

11. At the very end of the British game show Golden Balls, two players (Rowena and Colin) simultaneously decide whether to split or steal a large cash prize. If both choose Split, they each walk away with half of the prize. If one player chooses Split and the other chooses Steal, the stealer gets all the money. Finally, if both choose Steal, both get nothing.
    1. Draw the game table for this game, assuming for concreteness that the cash prize is worth $10,000 and that the players care only about how much money they wind up winning.
    2. In this game, both players view Steal as a ________ strategy. Fill in the blank with one of the following answers: "superdominant," "strictly dominant (but not superdominant)," "weakly dominant (but not strictly dominant)," or "not dominant." Explain your answer.
    3. Find all pure-strategy Nash equilibria of this game.
    4. Suppose that, in addition to wanting to win money, each player wants to avoid looking foolish. In particular, suppose that each player views the outcome in which they Split and the other player Steals as the very worst possibility. How does this extra consideration change your answer to part (b), and how does it change the set of pure-strategy Nash equilibria relative to part (c)? Explain your answers.
    5. Now, suppose that Rowena has preferences as described in part (d), but Colin is a kinder-hearted soul who would rather Rowena get all the money than for both of them to get nothing.[18] How does this change your answer to part (b), and how does it change the set of pure-strategy Nash equilibria relative to part (d)? Explain your answers.
12. Marta has three sons: Arturo, Bernardo, and Carlos. She discovers a broken lamp in her living room and knows that one of her sons must have broken it at play. Carlos was

actually the culprit, but Marta doesn't know this. She cares more about finding out the truth than she does about punishing the child who broke the lamp, so Marta announces that her sons are to play the following game.

Each child will write down his name on a piece of paper and write down either "Yes, I broke the lamp," or "No, I didn't break the lamp." If at least one child claims to have broken the lamp, she will give the normal allowance of $2 to each child who claims to have broken the lamp, and $5 to each child who claims not to have broken the lamp. If all three children claim not to have broken the lamp, none of them receives any allowance (each receives $0).

1. Draw the game table. Make Arturo the row player, Bernardo the column player, and Carlos the page player.
2. Find all Nash equilibria of this game.
3. This game has multiple Nash equilibria. Which one would you consider to be a focal point?

13. Consider a game in which there is a prize worth $30. There are three contestants, Larry, Curly, and Moe. Each can buy a ticket worth $15 or $30 or not buy a ticket at all. They make these choices simultaneously and independently. Then, knowing the ticket-purchase decisions, the game organizer awards the prize. If no one has bought a ticket, the prize is not awarded. Otherwise, the prize is awarded to the buyer of the highest-cost ticket if there is only one such player or is split equally between two or three if there are ties among the highest-cost ticket buyers. Show this game in strategic form, using Larry as the row player, Curly as the column player, and Moe as the page player. Find all pure-strategy Nash equilibria.
14. Anne and Bruce would like to rent a movie, but they can't decide what kind of movie to choose. Anne wants to

rent a comedy, and Bruce wants to rent a drama. They decide to choose randomly by playing "Evens or Odds." On the count of three, each of them shows one or two fingers. If the sum of all the fingers is even, Anne wins, and they rent the comedy; if the sum is odd, Bruce wins, and they rent the drama. Each of them earns a payoff of 1 for winning and 0 for losing Evens or Odds.

1. Draw the game table for Evens or Odds.
2. Demonstrate that this game has no Nash equilibrium in pure strategies.

15. In the film *A Beautiful Mind*, John Nash and three of his graduate-school colleagues find themselves faced with a dilemma while at a bar. There are five young women at the bar, four brunettes and one blonde. Each young man wants to win the attention of one young woman, most of all the blonde, but can approach only one of them. The catch is that if two or more of the young men approach the blonde, she will reject all of them and then the brunettes will also reject the men because they don't like being approached second. Each young man gets a payoff of 10 if he gains the blonde's attention, a payoff of 5 if he gains a brunette's attention, and a payoff of 0 if he is rejected by all of the women. (Thus, the only way that one of the men gets a payoff of 10 is if he is the sole person to approach the blonde.)
    1. First, consider a simpler situation in which there are only two young men instead of four. (And there are only two brunettes and one blonde, but these women merely respond in the manner just described and are not active players in the game.) Show the payoff table for the game and find all pure-strategy Nash equilibria of the game.
    2. Now show the (three-dimensional) payoff table for the case in which there are three young men (and three brunettes and one blonde who are not active players). Again, find all Nash equilibria of the game.

3. Without the use of a table, give all Nash equilibria for the case in which there are four young men (as well as four brunettes and a blonde).
4. (**Optional**) Use your results in parts (a), (b), and (c) to generalize your analysis to the case in which there are n young men. Do not attempt to draw an n-dimensional payoff table; merely find the payoff to one player when k of the others approach Blonde and $(n - k - 1)$ approach Brunette, for $k = 0, 1, \ldots (n - 1)$. Can the outcome specified in the movie as the Nash equilibrium of the game—that all the young men approach brunettes—ever really be a Nash equilibrium of the game?

UNSOLVED EXERCISES

1. Find all Nash equilibria in pure strategies for the following games. First, check for dominated strategies. If there are none, solve the games using successive elimination of dominated strategies.

    1.

		COLIN	
		Left	Right
ROWENA	Up	3, 1	4, 2
	Down	5, 2	2, 3

You may need to scroll left and right to see the full figure.

    2.

		COLIN		
		Left	Middle	Right
ROWENA	Up	2, 9	5, 5	6, 2
	Straight	6, 4	9, 2	5, 3
	Down	4, 3	2, 7	7, 1

You may need to scroll left and right to see the full figure.

    3.

		COLIN		
		Left	Middle	Right
ROWENA	Up	5, 3	3, 5	2, 6
	Straight	6, 2	4, 4	3, 5

You may need to scroll left and right to see the full figure.

	COLIN			
		Left	Middle	Right
	Down	1, 7	6, 2	2, 6
You may need to scroll left and right to see the full figure.				

4.

		COLIN			
		North	South	East	West
ROWENA	Up	6, 4	7, 3	5, 5	6, 4
	High	7, 3	3, 7	4, 6	5, 5
	Low	8, 2	6, 4	3, 7	2, 8
	Down	3, 7	5, 5	4, 6	5, 5
You may need to scroll left and right to see the full figure.					

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2. For each of the four games in Exercise U1, identify whether the game is zero-sum or non-zero-sum. Explain your reasoning.
3. For each of the four games in Exercise U1, identify which players (if any) have a superdominant strategy. Explain your reasoning.
4. As in Exercise S4 above, use the minimax method to find the Nash equilibria for the zero-sum games identified in Exercise U2.
5. Find all Nash equilibria in pure strategies for the following games. Describe the steps that you used in finding the equilibria.

1.

	COLIN	
	Left	Right
You may need to scroll left and right to see the full figure.		

		COLIN	
		Left	Right
ROWENA	Up	1, −1	4, −4
	Down	2, −2	3, −3

You may need to scroll left and right to see the full figure.

2.

		COLIN	
		Left	Right
ROWENA	Up	0, 0	0, 0
	Down	0, 0	1, 1

You may need to scroll left and right to see the full figure.

3.

		COLIN	
		Left	Right
ROWENA	Up	1, 3	2, 2
	Down	4, 0	3, 1

You may need to scroll left and right to see the full figure.

4.

		COLIN		
		Left	Middle	Right
ROWENA	Up	5, 3	7, 2	2, 1
	Straight	1, 2	6, 3	1, 4

You may need to scroll left and right to see the full figure.

	COLIN		
	Left	Middle	Right
Down	4, 2	6, 4	3, 5

You may need to scroll left and right to see the full figure.

6. Use successive elimination of dominated strategies to solve the following game. Explain the steps you followed. Show that your solution is a Nash equilibrium.

		COLIN		
		Left	Middle	Right
ROWENA	Up	4, 3	2, 7	0, 4
	Down	5, 0	5, −1	−4, −2

You may need to scroll left and right to see the full figure.

7. Find all pure-strategy Nash equilibria for the following game. Describe the process that you used to find the equilibria. Use this game to explain why it is important to describe an equilibrium by using the strategies employed by the players, not merely the payoffs received in equilibrium.

		COLIN		
		Left	Center	Right
ROWENA	Up	1, 2	2, 1	1, 0
	Level	0, 5	1, 2	7, 4
	Down	−1, 1	3, 0	5, 2

You may need to scroll left and right to see the full figure.

8. Consider the following game table:

		COLIN		
		Left	Center	Right
ROWENA	Top	4, __	__, 2	3, 1
	Middle	3, 5	2, __	2, 3
	Down	__, 3	3, 4	4, 2

You may need to scroll left and right to see the full figure.

1. Complete the payoffs in the game table so that Colin has a dominant strategy. State which strategy is dominant and explain why. (Note: There are many equally correct answers.)
2. Complete the payoffs in the game table so that neither player has a dominant strategy, but also so that each player does have a dominated strategy. State which strategies are dominated and explain why. (Again, there are many equally correct answers.)

9. The *Battle of the Bismarck Sea* (named for that part of the southwestern Pacific Ocean separating the Bismarck Archipelago from Papua New Guinea) was a naval engagement between the United States and Japan during World War II. In 1943, a Japanese admiral was ordered to move a convoy of ships to New Guinea; he had to choose between a rainy northern route and a sunnier southern route, both of which required three days' sailing time. The Americans knew that the convoy would sail and wanted to send bombers after it, but they did not know which route it would take. The Americans had to send reconnaissance planes to scout for the convoy, but they had only enough reconnaissance planes to explore one route at a time. Both the Japanese and the Americans had to make their

decisions with no knowledge of the plans being made by the other side.

If the convoy was on the route that the Americans explored first, they could send bombers right away; if not, they lost a day of bombing. Poor weather on the northern route would also hamper bombing. If the Americans explored the northern route and found the Japanese right away, they could expect only two (of three) good bombing days; if they explored the northern route and found that the Japanese had gone south, they could also expect two days of bombing. If the Americans chose to explore the southern route first, they could expect three full days of bombing if they found the Japanese right away, but only one day of bombing if they found that the Japanese had gone north.

1. Illustrate this game in a game table.
2. Identify any dominant strategies in the game and solve for the Nash equilibrium.

10. Two players, Jack and Jill, are put in separate rooms, and each is told the rules of the game. Each is to pick one of six letters: G, K, L, Q, R, or W. If the two happen to choose the same letter, both get prizes as follows:

Letter	G	K	L	Q	R	W
Jack's Prize	3	2	6	3	4	5
Jill's Prize	6	5	4	3	2	1

If they choose different letters, each gets 0. This whole schedule is revealed to both players, and both are told that both know the schedules, and so on.

1. Draw the game table for this game. What are the Nash equilibria in pure strategies?
2. Can one of the equilibria be a focal point? Which one? Why?

11. The widget market is currently monopolized by Widgets R Us (or simply Widgets), but another firm (Wadgets) is deciding whether to enter that market. If Wadgets stays out of the market, Widgets will earn $100 million profit. However, if Wadgets enters, Widgets can *either* share the market, in which case the two companies enjoy a total $20 million in profit, *or* wage a ruinous price war, in which case both companies lose big and go bankrupt (call this $0 profit for concreteness). The only sane choice for Widgets is to share the market, but before Wadgets chooses whether to enter, the Widgets Board of Directors has the opportunity to hire a new CEO—and this new CEO might just be crazy enough to wage a price war!
    1. Draw the game table for this game, where the relevant players are Wadgets, which decides whether to enter the market, and the Widgets Board, which decides whether to hire a crazy CEO who will wage a price war if Wadgets enters or hire a sane CEO who will share the market if Wadgets enters. (Assume that Wadgets has no way of knowing if the newly hired Widgets CEO is crazy or sane, making this a simultaneous-move game.)
    2. In this game, the Widgets Board views Hire a Sane CEO as a _______ strategy. Fill in the blank with one of the following answers: “superdominant,” “strictly dominant (but not superdominant),” “weakly dominant (but not strictly dominant),” or “not dominant.” Explain your answer.
    3. Find all pure-strategy Nash equilibria of this game.

4. Suppose that, in addition to wanting to maximize profits and avoid bankruptcy, the Widgets Board would prefer not to have a crazy CEO. How does this extra consideration change your answer to part (b), and how does it change the set of pure-strategy Nash equilibria relative to part (c)? Explain your answers.

12. Three friends (Julie, Kristin, and Larissa) independently go shopping for dresses for their high-school prom. On reaching the store, each girl sees only three dresses worth considering: one black, one lavender, and one yellow. Each girl, furthermore, knows that her two friends would consider the same set of three dresses because all three have somewhat similar tastes.

    Each girl would prefer to have a unique dress, so each girl's payoff is 0 if she ends up purchasing the same dress as at least one of her friends. All three know that Julie strongly prefers black to both lavender and yellow, so she would get a payoff of 3 if she were the only one wearing the black dress, and a payoff of 1 if she were either the only one wearing the lavender dress or the only one wearing the yellow dress. Similarly, all know that Kristin prefers lavender and secondarily prefers yellow, so her payoff would be 3 for uniquely wearing lavender, 2 for uniquely wearing yellow, and 1 for uniquely wearing black. Finally, all know that Larissa prefers yellow and secondarily prefers black, so she would get 3 for uniquely wearing yellow, 2 for uniquely wearing black, and 1 for uniquely wearing lavender.

    1. Provide the game table for this three-player game. Make Julie the row player, Kristin the column player, and Larissa the page player.
    2. Identify any dominated strategies in this game, or explain why there are none.
    3. What are the pure-strategy Nash equilibria in this game?

13. Bruce, Colleen, and David are all getting together at Bruce's house on Friday evening to play their favorite game, Monopoly. They all love to eat sushi while they play. They all know from previous experience that two orders of sushi are just the right amount to satisfy their hunger. If they wind up with fewer than two orders, they all end up going hungry and don't enjoy the evening. More than two orders would be a waste, however, because they can't manage to eat a third order, and the extra sushi just goes bad. Their favorite restaurant, Fishes in the Raw, packages its sushi in such large containers that each individual person can feasibly purchase at most one order of sushi. Fishes in the Raw offers takeout, but unfortunately doesn't deliver.

    Suppose that each player enjoys $20 worth of utility from having enough sushi to eat on Friday evening, and $0 from not having enough to eat. The cost to each player of picking up an order of sushi is $10.

    Unfortunately, the players have forgotten to communicate about who should be buying sushi this Friday, and none of the players has a cell phone, so they must each make an independent decision about whether to buy (B) or not buy (N) an order of sushi.

    1. Write down this game in strategic form.
    2. Find all the Nash equilibria in pure strategies.
    3. Which equilibrium would you consider to be a focal point? Explain your reasoning.

14. Roxanne, Sara, and Ted all love to eat cookies, but there's only one left in the package. No one wants to split the cookie, so Sara proposes the following extension of "Evens or Odds" (see Exercise S14) to determine who gets to eat it. On the count of three, each of them will show one or two fingers, they'll add them up, and then divide the sum by 3. If the remainder is 0,

Roxanne gets the cookie, if the remainder is 1, Sara gets it, and if it is 2, Ted gets it. Each of them receives a payoff of 1 for winning (and eating the cookie) and 0 otherwise.

   1. Represent this three-player game in normal form, with Roxanne as the row player, Sara as the column player, and Ted as the page player.
   2. Find all the pure-strategy Nash equilibria of this game. Is this game a fair mechanism for allocating the cookie? Explain why or why not.

15. (**Optional**) Construct the payoff matrix for your own two-player game that satisfies the following requirements: First, each player should have three strategies. Second, the game should not have any dominant strategies. Third, the game should not be solvable using the minimax method. Fourth, the game should have exactly two pure-strategy Nash equilibria. Provide your payoff matrix, then demonstrate that all of the above conditions are true.

Endnotes

- In an episode of *Golden Balls* that WNYC Studio’s *Radiolab* called “one of the strangest moments in game show history,” one player (Nick Corrigan) made a brilliant move to ensure that the other player (Ibrahim Hussein) strictly preferred to Split even while believing that Nick was certain to Steal. Learn more and watch the episode at https://www.wnycstudios.org/story/golden-rule. Return to reference 18

5 ■ Simultaneous-Move Games: Continuous Strategies, Discussion, and Evidence

THE DISCUSSION OF SIMULTANEOUS-MOVE GAMES in Chapter 4 focused on games in which each player had a discrete set of actions from which to choose. Discrete-strategy games of this type include sporting contests in which a small number of well-defined plays can be used in a given situation—such as soccer penalty kicks, in which the kicker can choose to go high, low, to a corner, or to the center. Other examples include coordination and prisoners' dilemma games in which players have only two or three available strategies. Such games are amenable to analysis with the use of a game table, at least for situations with a reasonable number of players and available actions.

Many simultaneous-move games differ from those considered so far in that they entail players choosing strategies from a wide range of possibilities. Games in which manufacturers choose prices for their products, philanthropists choose charitable contribution amounts, or contractors choose project bid levels are examples in which players have a virtually infinite set of choices. Technically, prices and other amounts of money do have a minimum unit, such as a cent, so there is actually a finite set of discrete price strategies available. But in practice, the minimum unit is very small, and allowing discreteness in our analyses would require us to give each player too many distinct strategies and make the game table too large; therefore, it is simpler and better to regard such choices as continuously variable real numbers. When players have such a large range of actions available, game tables become virtually useless as analytical tools; they become too unwieldy to be of practical use. For these games, we need a different solution technique. We present the analytical tools for handling games with such continuous strategies in the first part of this chapter.

This chapter also takes up some broader matters relevant to player behavior in simultaneous-move games and to the concept of Nash equilibrium. We review the empirical evidence on Nash equilibrium play that has been collected both from the laboratory and from real-life situations. We also present some theoretical criticisms of the Nash equilibrium concept and rebuttals of these criticisms. You will see that game-theoretic predictions are often a reasonable starting point for understanding actual behavior, with some caveats.

Glossary

continuous strategy
: A choice over a continuous range of real numbers available to a player.

1 PURE STRATEGIES THAT ARE CONTINUOUS VARIABLES

In Chapter 4, we developed the method of best-response analysis for finding all pure-strategy Nash equilibria of simultaneous-move games. Now we extend that method to games in which each player—for example, a firm setting the price of a product—has available a continuous range of choices. To calculate best responses in this type of game, we find, for each possible value of one firm's price, the value of the other firm's price that is best for it (maximizes its payoff). The continuity of the sets of strategies allows us to use algebraic formulas to show how strategies generate payoffs and to show the best responses as curves in a graph, with each player's price (or any other continuous strategy) on one of the axes. In such an illustration, the Nash equilibrium of the game occurs where the two curves meet. We develop this idea and technique by using two stories.

A. Price Competition

Our first story is set in a small town, Eten, which has two restaurants, Xavier’s Tapas Bar and Yvonne’s Bistro. To keep the story simple, we assume that each place has a set menu, and that Xavier and Yvonne must set the prices of the meals on their respective menus. Prices are their strategic choices in the game of competing with each other; each restaurant’s goal is to set its prices to maximize profit, the payoff in this game. We assume that they get their menus printed separately without knowing each other’s prices, so the game has simultaneous moves.[1] Because prices can take any value within an (almost) infinite range, we start with general or algebraic symbols for them. We then find best-response rules that we can use to solve the game and to determine equilibrium prices. Let us call Xavier’s price P_x and Yvonne’s price P_y.

In setting its price, each restaurant has to calculate the consequences for its profit. To keep things relatively simple, we put the two restaurants in a very symmetric relationship, but readers with a little more mathematical skill can do a similar analysis by using much more general numbers or even algebraic symbols. Suppose the cost of serving each customer is $8 for each restaurateur. Suppose further that experience or market surveys have shown that when Xavier’s price is P_x and Yvonne’s price is P_y, the numbers of customers they serve, Q_x and Q_y, respectively (measured in hundreds per month), are given by the demand equations[2]

$$Q_x = 44 - 2P_x + P_y,$$

$$Q_y = 44 - 2P_y + P_x.$$

The key idea in these equations is that, if one restaurant raises its price by $1 (say, if Yvonne increases P_y by $1), its sales will go down by 200 per month (Q_y changes by -2) and those of the other restaurant will go up by 100 per month (Q_x changes by 1).

(Presumably, 100 of Yvonne's customers switch to Xavier's and another 100 stay at home.)

Xavier's profit per week (in hundreds of dollars per week), which we call Π_x—the Greek letter Π (pi) is the traditional economic symbol for profit—is given by the product of the net revenue per customer (price less cost, or $P_x - 8$) and the number of customers served:

$$\Pi_x = (P_x - 8)\ Q_x = (P_x - 8)\ (44 - 2P_x + P_y).$$

By multiplying out and rearranging the terms on the right-hand side of this expression, we can write profit as a function of increasing powers of P_x:

$$\Pi_x = -8(44 + P_y) + (16 + 44 + P_y)P_x - 2(P_x)^2$$

$$= -8(44 + P_y) + (60 + P_y)P_x - 2(P_x)^2.$$

Xavier sets his price P_x to maximize this payoff. Doing so for each possible level of Yvonne's price P_y gives us Xavier's best-response rule, and we can then graph it.

Many simple illustrative examples where one real number (such as a price) is chosen to maximize another real number that depends on it (such as a profit or payoff) have a similar form. (In mathematical jargon, we would describe the second number as a function of the first.) In the appendix to this chapter, we develop a simple general technique for performing such maximization; you will find many occasions to use it. Here we simply state the formula.

The function we want to maximize takes the general form

$$Y = A + BX - CX^2,$$

where we have used the descriptor Y for the number we want to maximize and X for the number we want to choose so as to maximize that Y. In our specific example, profit, Π_x, would be represented by Y, and the price, P_x, by X. Similarly, although in any specific problem the terms A, B, and C in the equation above would be known numbers, we have denoted them with general algebraic symbols here so that our formula can be applied across a wide variety of similar problems. (The technical term for the terms A, B, and C is *parameters*, or *algebraic constants*.) Because most of our applications involve nonnegative X entities, such as prices, and the maximization of the Y entity, we require that $B > 0$ and $C > 0$. Then the formula giving the choice of X to maximize Y in terms of the known parameters A, B, and C is simply $X = B/(2C)$. Observe that A does not appear in the formula, although it will of course affect the value of Y that results.

Comparing the general function in the preceeding equation and the specific example of the profit function in the restaurant pricing game, we have[3]

$B = 60 + P_y$ and $C = 2$.

Therefore, Xavier's choice of price to maximize his profit will satisfy the formula $B/(2C)$ and will be

$P_x = 15 + 0.25P_y$.

This equation determines the value of P_x that maximizes Xavier's profit, given a particular value of Yvonne's price, P_y. In other words, it is exactly what we want: the rule for Xavier's best response.

Yvonne's best-response rule can be found similarly. Because the costs and sales of the two restaurants are entirely symmetric, that equation is obviously going to be

$P_y = 15 + 0.25P_x$.

Both rules are used in the same way to develop best-response graphs. If Xavier sets a price of 16, for example, then Yvonne plugs this value into her best-response rule to find $P_y = 15 + 0.25(16) = 19$; similarly, Xavier's best response to Yvonne's P_y

= 16 is P_x = 19, and each restaurant's best response to the other's price of 4 is 16, that to 8 is 17, and so on.

Figure 5.1 shows the graphs of these two rules, called the best-response curves. Owing to the special features of our example—namely, the linear relationship between quantity sold and prices charged, and the constant cost of producing each meal—each of the two best-response curves is a straight line. For other specifications of demands and costs, the curves can be other than straight, but the method of obtaining them is the same—namely, first holding one restaurant's price (say, P_y) fixed and finding the value of the other's price (say, P_x) that maximizes that other restaurant's profit, and then the other way around.

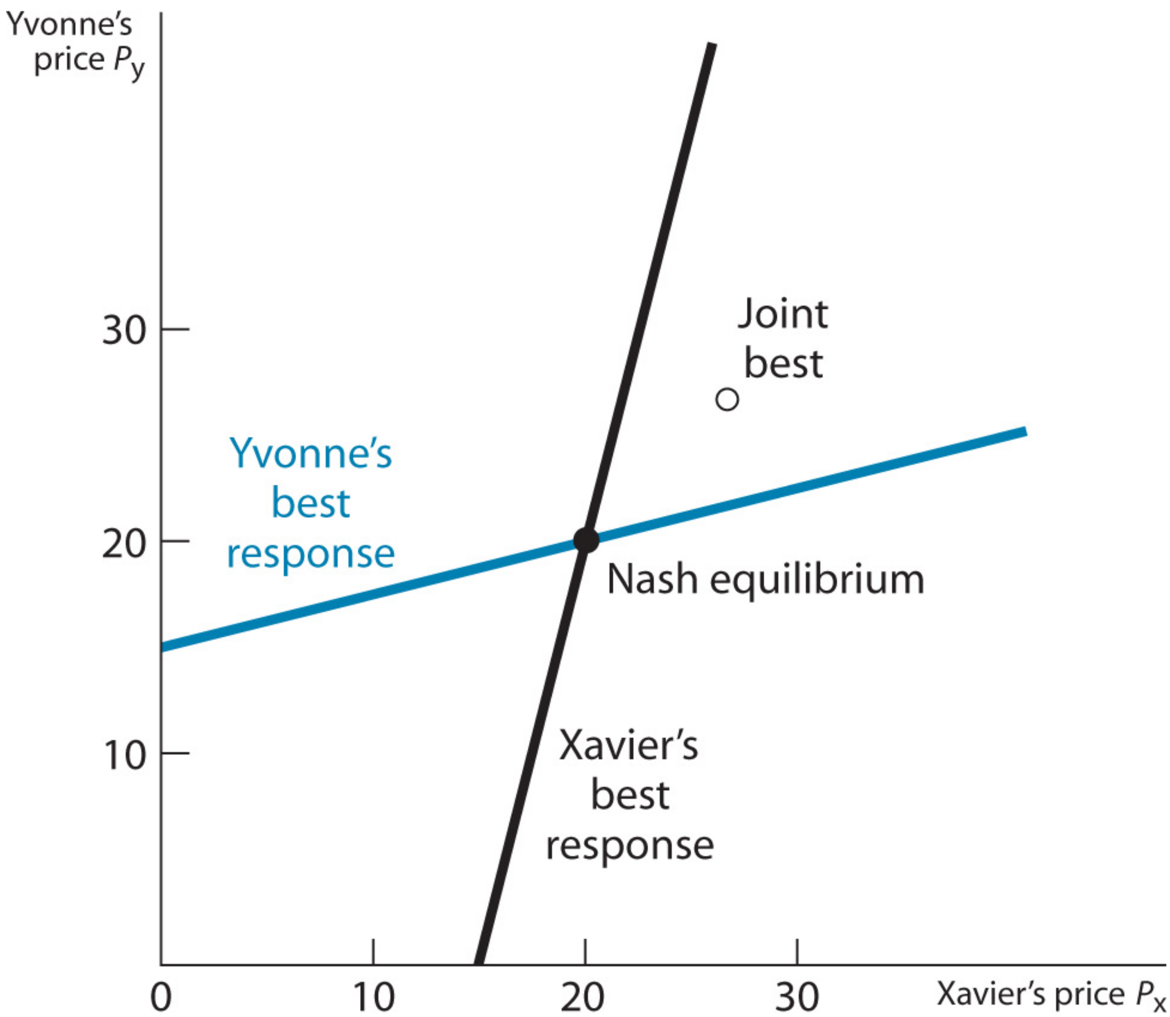

Figure 5.1 Best-Response Curves and Nash Equilibrium in the Restaurant Pricing Game

The point of intersection of the two best-response curves is the Nash equilibrium of the pricing game between the two restaurants. That point represents the pair of prices, one for each restaurant, that are best responses to each other. The specific values for each restaurant's pricing strategy in equilibrium can be found algebraically by solving the two best-response rules jointly for P_x and P_y. We deliberately chose our example to make the equations linear, and the solution is easy. In this case, we simply substitute the expression for P_x into the expression for P_y to find

$$P_y = 15 + 0.25P_x = 15 + 0.25(15 + 0.25P_y) = 18.75 + 0.0625\ P_y.$$

This last equation simplifies to $P_y = 20$. Given the symmetry of the problem, it is simple to determine that $P_x = 20$ also.[4] Thus, in equilibrium, each restaurant charges $20 for its menu and makes a profit of $12 on each of the 2,400 customers [2,400 = (44−(2 × 20) + 20) hundred] that it serves each month, for a total profit of $28,800 per month.

B. Some Economics of Oligopoly

Our main purpose in presenting the restaurant pricing example was to illustrate how the Nash equilibrium can be found in a game where the strategies are continuous variables, such as prices. But it is interesting to take a further look into some of the economics behind pricing strategies and profits when a small number of firms (here just two) compete. In the jargon of economics, such competition is referred to as *oligopoly*, from the Greek words for "a small number of sellers."

Begin by observing that each firm's best-response curve slopes upward. Specifically, when one restaurant raises its price by $1, the other's best response is to raise its own price by 0.25, or 25 cents. When one restaurant raises its price, some of its customers switch to the rival restaurant, which can then profit from these new customers by raising its price part of the way. Thus, a restaurant that raises its price is also helping to increase its rival's profit. In Nash equilibrium, where each restaurant chooses its price independently and out of concern for its own profit, it does not take into account this benefit that it conveys to the other. Could the two restaurants get together and cooperatively agree to raise their prices, thereby raising profits for both? Yes. Suppose the two restaurants charged $24 each. Then each would make a profit of $16 on each of the 2,000 customers [2,000 = (44 − (2 × 24) + 24) hundred] that it would serve each month, for a total profit of $32,000 per month.

This pricing game is exactly like the prisoners' dilemma game presented in Chapter 4, but now the strategies are continuous variables. In the story in Chapter 4, Husband and Wife were each tempted to cheat the other and confess to the police, but when they both did so, both ended up with longer prison sentences (worse outcomes). In the same way, the more profitable price of $24 is not a Nash equilibrium. The separate calculations of the two restaurants will lead them to undercut such a price. Suppose that Yvonne somehow starts charging $24. Using the best-response formula, we see that Xavier will then charge 15 + 0.25 × 24 =

21. Then Yvonne will come back with her best response to that price: 15 + 0.25 × 21 = 20.25. Continuing this process, the prices of both will converge toward the Nash equilibrium price of $20.

But what price is jointly best for the two restaurants? Given the symmetry, suppose both charge the same price P. Then the profit of each will be

$$\Pi_x = \Pi_y = (P - 8)(44 - 2P + P) = (P - 8)(44 - P) = -352 + 52P - P^2.$$

The two can choose P to maximize this expression. Using the formula provided in Section 1.A, we see that the solution is $P = 52/2 = 26$. The resulting profit for each restaurant is $32,400 per month.

In the jargon of economics, a group of firms that collude to raise prices to the jointly optimal level is called a *cartel*. The high prices hurt consumers, and regulatory agencies of the U.S. government often try to prevent the formation of cartels and to make firms compete with one another. Explicit collusion over price is illegal, but it may be possible to maintain tacit collusion in a repeated prisoners' dilemma; we examine such repeated games in Chapter 10.[5]

But collusion need not always lead to higher prices. In the preceding example, if one restaurant lowers its price, its sales increase, in part because it draws some customers away from its rival because the products (meals) of the two restaurants are *substitutes* for each other. In other contexts, however, two firms may be selling products that are *complements* to each other—for example, hardware and software. In that case, if one firm lowers its price, the sales of both firms increase. In a Nash equilibrium, where the firms act independently, they do not take into account the benefit that would accrue to each of them if they both lowered their prices. Therefore, they keep prices higher than they would if they were able to coordinate their actions. Allowing them to cooperate would lead to lower prices and thus be beneficial to the consumers as well. We examine such collective-action problems in more detail in Chapter 11.

Competition need not always involve the use of prices as the strategic variables. For example, fishing fleets may compete to bring a larger catch to market; this is quantity competition as opposed to the price competition considered in this section. We will consider quantity competition later in this chapter and in several of the end-of-chapter exercises.

C. Political Campaign Advertising

Our second example is one drawn from politics. It requires just a little more mathematics than we normally use, but we explain the intuition behind the calculations in words and with a graph.

Consider an election contested by two parties or candidates. Each is trying to win votes away from the other by advertising—either with positive ads that highlight the good things about themselves or with negative ads that emphasize the bad things about their opponent. To keep matters simple, suppose the voters start out entirely ignorant and unconcerned and form opinions solely as a result of the ads. (Many people would claim that this is an accurate description of U.S. politics, but more advanced analyses in political science do recognize that there are informed and strategic voters. We address the behavior of such voters in detail in Chapter 16.) Even more simply, suppose the vote share of a party equals its share of the total campaign advertising that is done. Call the parties or candidates L and R; when L spends \$$x$ million on advertising and R spends \$$y$ million, L will get a share $x/(x+y)$ of the votes and R will get $y/(x+y)$. (Readers who get interested in this application can find more general treatments in specialized political science writings.)

Raising money to pay for these ads includes a cost: money to send letters and make phone calls; the time and effort of the candidates, party leaders, and activists; and future political payoffs to large contributors, along with possible future political costs if these payoffs are exposed and lead to scandals. For simplicity of analysis, let us suppose that all these costs are proportional to the direct campaign expenditures for advertising, x and y. Specifically, let us suppose that party L’s payoff is measured by its vote percentage minus its advertising expenditure, $100x/(x+y) - x$. Similarly, party R’s payoff is $100y/(x+y) - y$.

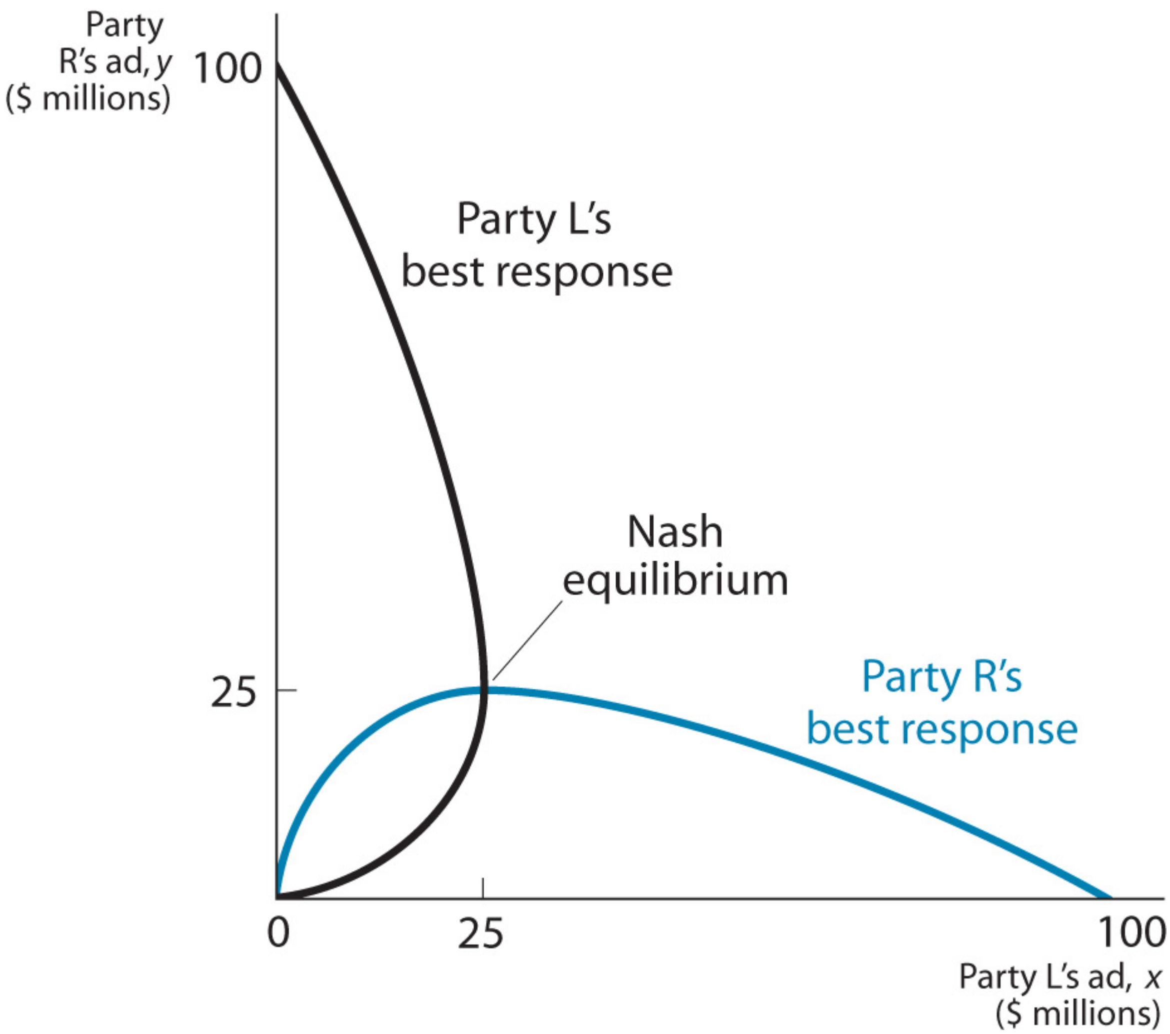

Figure 5.2 Best Responses and Nash Equilibrium in the Campaign Advertising Game

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Now we can find the best responses. Because we cannot do so without calculus, we derive the formula mathematically and then explain in words its general meaning intuitively. For a given strategy x of party L, party R chooses y to maximize its payoff. The calculus first-order condition is found by holding x fixed and setting the derivative of $100y/(x+y) - y$ with respect to y equal to 0. It is $100x/(x+y)^2 - 1 = 0$, or

$$y = 10\sqrt{x} - x.$$

Figure 5.2 shows its graph and that of

the analogous best-response function of party L—namely,

$$x = 10\sqrt{y} - y.$$

Look at the best-response curve of party R. As the value of party L's x increases, party R's y increases for a while and then decreases. If the other party is advertising very little, then one's own ads have a high reward in the form of votes, and it pays to respond to a small increase in the other party's expenditures by spending more oneself to compete harder. But if the other party already spends a great deal on ads, then one's own ads get only a small return in relation to their cost, so it is better to respond to the other party's increase in spending by scaling back.

As it happens, the two parties' best-response curves intersect at their peak points. Again, some algebraic manipulation of the equations for the two curves yields us the equilibrium values of x and y. You should verify that here x and y are each equal to 25, or $25 million. (This is presumably a congressional election; Senate and presidential elections cost much more these days.)

As in the pricing game, we have a prisoners' dilemma. If both parties were to cut back on their ads in equal proportions, their vote shares would be entirely unaffected, but both would save on their expenditures, and so both would have a larger payoff. Unlike a producers' cartel for substitute products (which keeps prices high and hurts consumers), a politicians' cartel to advertise less would probably benefit voters and society, just as a producers' cartel for complementary products would lead to lower prices and benefit consumers. We could all benefit from finding ways to resolve this particular prisoners' dilemma. The United States currently does not limit spending on political advertising by candidates, but some European countries do impose such limits, and the United Kingdom prohibits paid advertising of any kind on the part of political candidates.

What if the parties are not symmetrically situated? Two kinds of asymmetries can arise. One party (say, R) may be able to advertise at a lower cost because it has favored access to the media. Or R’s advertising dollars may be more effective than L’s—for example, L’s vote share may be $x/(x + 2y)$, while R’s vote share is $2y/(x + 2y)$.

In the first of these cases, R exploits its cheaper access to advertising by choosing a higher level of expenditures y for any given x for party L—that is, R’s best-response curve in Figure 5.2 shifts upward. The Nash equilibrium shifts to the northwest along L’s unchanged best-response curve. Thus, R ends up advertising more and L ends up advertising less than before. It is as if the advantaged party uses its muscle and the disadvantaged party gives up to some extent in the face of this adversity.

In the second case, both parties’ best-response curves shift in more complex ways. The outcome is that both spend equal amounts, but less than the $25 million that they spent in the symmetric case. In our example where R’s dollars are twice as effective as L’s, it turns out that their common expenditure level is $200/9 = 22.2 < 25$. (Thus the symmetric case is the one of most intense competition.) When R’s spending is more effective, it is also true that the best-response curves are asymmetric in such a way that the new Nash equilibrium, rather than being at the peak points of the two best-response curves, is on the downward part of L’s best-response curve and on the upward part of R’s best-response curve. That is to say, although both parties spend the same dollar amount, the favored party, R, spends more than the amount that would bring forth the maximum response from party L, and the underdog party, L, spends less than the amount that would bring forth the maximum response from party R. We include an optional exercise (Exercise U12) in this chapter that lets mathematically advanced students derive these results.

D. General Method for Finding Nash Equilibria

Although the strategies (prices or campaign expenditures) and payoffs (profits or vote shares) in the two previous examples are specific to the context of competition between firms or political parties, the method for finding the Nash equilibrium of a game with continuous strategies is perfectly general. Here we state its steps so that you can use it as a recipe for solving other games of this kind.

Suppose the players are numbered 1, 2, 3, ... Label their strategies x, y, z, . . . in the same order, and their payoffs with the corresponding uppercase letters X, Y, Z, ... The payoff of each player is, in general, a function of the choices of all; label the respective functions F, G, H, ... Construct payoffs from the information about the game, and write them as

$X = F(x, y, z, . . .), Y = G(x, y, z, . . .), Z = H(x, y, z, . . .)$.

Using this general format to describe our example of price competition between two players (firms) makes the strategies x and y become the prices P_x and P_y. The payoffs X and Y are the profits Π_x and Π_y. The functions F and G are the quadratic formulas

$\Pi_x = -8(44 + P_y) + (16 + 44 + P_y)P_x - 2(P_x)$,

and similarly for Π_y.

In the general approach, player 1 regards the strategies of players 2, 3, . . . as outside his control, and chooses his own strategy to maximize his own payoff. Therefore, for each given set of values of y, z, . . . , player 1's choice of x maximizes $X = F(x, y, z, . . .)$. If you use calculus, the condition for this maximization is that the derivative of X with respect to x

holding y, z, . . . constant (the partial derivative) equals 0. For special functions, simple formulas are available, such as the one we stated and used above for the quadratic. And even if an algebra or calculus formulation is too difficult, computer programs can tabulate or graph best-response functions for you. Whatever method you use, you can find an equation for the choice of x for given y, z, . . . that is player 1's best-response function. Similarly, you can find the best-response functions for each of the other players.

The best-response functions are equal in number to the number of strategies in the game and can be solved simultaneously while regarding the strategy variables as the unknowns. The solution is the Nash equilibrium we seek. Some games may have multiple solutions, yielding multiple Nash equilibria. Other games may have no solution, requiring further analysis, such as inclusion of mixed strategies.

Endnotes

- In reality, the competition extends over time, so each can observe the other's past choices. This repetition of the game introduces new considerations, which we cover in Chapter 10. Return to reference 1
- Readers who know some economics will recognize that the equations linking quantities to prices are demand functions for the two products *X* and *Y*. The quantity demanded of each product is decreasing with its own price (demands are downward sloping) and increasing with the price of the other product (the two products are substitutes). Return to reference 2
- Although P_y, chosen by Yvonne, is a variable in the full game, here we are considering only a part of the game—namely, Xavier's best response, where he regards Yvonne's choice as outside his control and therefore like a constant. Return to reference 3
- Without this symmetry, the two best-response equations will be different, but given our other specifications, still linear. So it is not much harder to solve the asymmetric case. You will have a chance to do so in Exercise S2 at the end of this chapter. Return to reference 4
- Firms do try to achieve explicit collusion when they think they can get away with it. An entertaining and instructive story of one such episode is in *The Informant*, by Kurt Eichenwald (New York: Broadway Books, 2000). Return to reference 5

Glossary

best-response rule
: A function expressing the strategy that is optimal for one player, for each of the strategy combinations actually played by the other players, or the belief of this player about the other players' strategy choices.

best-response curve
: A graph showing the best strategy of one player as a function of the strategies of the other player(s) over the entire range of those strategies.

2 CRITICAL DISCUSSION OF THE NASH EQUILIBRIUM CONCEPT

Although Nash equilibrium is the primary solution concept for simultaneous-move games, it has been subject to several theoretical criticisms. In this section, we briefly review some of these criticisms and some rebuttals, in each case by using an example.[6] Some of the criticisms are mutually contradictory, and some can be countered by thinking of the games themselves in a better way. Others tell us that the Nash equilibrium concept by itself is not enough and suggest some augmentations or relaxations of it that have better properties. We develop one such alternative here and point to some others that will appear in later chapters. We believe our presentation will leave you with renewed but cautious confidence in using the Nash equilibrium concept. But some serious doubts remain unresolved, indicating that game theory is not yet a settled science. Even this should give encouragement to budding game theorists because it shows that there is a lot of room for new thinking and new research in the subject. A totally settled science would be a dead science.

We begin by considering the basic appeal of the Nash equilibrium concept. Most of the games in this book are noncooperative, in the sense that every player takes her action independently. Therefore, it seems natural to suppose that if her action is not the best according to her own value system (payoff scale) given what everyone else does, she will change it. In other words, it is appealing to suppose that every player's action will be the best response to the actions of all the others. Nash equilibrium has just this property of "simultaneous best responses"; indeed, that is its very definition. In any purported final outcome that is not a Nash equilibrium, at least one player could have done better by switching to a different action.

This consideration led Nobel laureate Roger Myerson to rebut those criticisms of the Nash equilibrium that were based on the intuitive appeal of playing a different strategy. His rebuttal simply shifted the burden of proof onto the critic. “When asked why players in a game should behave as in some Nash equilibrium,” he said, “my favorite response is to ask ‘Why not?’ and to let the challenger specify what he thinks the players should do. If this specification is not a Nash equilibrium, then . . . we can show that it would destroy its own validity if the players believed it to be an accurate description of each other’s behavior.”[7]

A. The Treatment of Risk in Nash Equilibrium

Some critics argue that the Nash equilibrium concept does not pay due attention to risk. In some games, people might find strategies different from their Nash equilibrium strategies to be safer and might therefore choose those strategies. We offer two examples of this kind. The first comes from John Morgan, an economics professor at the University of California, Berkeley; Figure 5.3 shows the game table for this example.

Best-response analysis quickly reveals that this game has a unique Nash equilibrium—namely, (A, A), yielding the payoffs (2, 2). But you may think, as did several participants in an experiment conducted by Morgan, that playing C has a lot of appeal, for the following reasons: First, it *guarantees* you the same payoff as you would get in the Nash equilibrium—namely, 2—whereas if you play your Nash equilibrium strategy A, you will get a 2 only if the other player also plays A. Why take that chance? What is more, if you think the other player might use this rationale for playing C, then you would be making a serious mistake by playing A; you would get only a 0 when you could have gotten a 2 by playing C.

Myerson would respond to Morgan, "Not so fast. If you really believe that the other player would think this way and play C, then you should play B to get the payoff 3. And if you think the other person would think this way and play B, then your best response to B should be A. And if you think the other person would figure this out, too, you should be playing your best response to A—namely, A. Back to the Nash equilibrium!" As you can see, criticizing Nash equilibrium and rebutting the criticisms is itself something of an intellectual game, and quite a fascinating one.

		COLUMN		
		A	B	C
ROW	A	2, 2	3, 1	0, 2
	B	1, 3	2, 2	3, 2
	C	2, 0	2, 3	2, 2

You may need to scroll left and right to see the full figure.

FIGURE 5.3 A Game with a Questionable Nash Equilibrium

		B	
		Left	Right
A	Up	9, 10	8, 9.9
	Down	10, 10	−1000, 9.9

FIGURE 5.4 Disastrous Nash Equilibrium?

The second example, which comes from David Kreps, an economist at Stanford Business School, is even more dramatic. The payoff matrix is shown in Figure 5.4. Before doing any theoretical analysis of this game, pretend that you are actually playing the game and that you are player A. Which of the two actions would you choose?

Keep in mind your answer to the preceding question as we proceed to analyze the game. If we start by looking for dominant strategies, we see that player A does not have a dominant strategy, but player B does. Playing Left guarantees B a payoff of 10, no matter what A does, versus the payoff of 9.9 earned from playing Right (also no matter what A does). Thus, player B should play Left. Given that player B is going to go Left, player A does better to go Down. The unique pure-strategy Nash equilibrium of this game is therefore (Down, Left); each player achieves a payoff of 10 with this outcome.

The problem that arises here is that many people assigned to be Player A would not choose to play Down. (What did you choose?) This is true for those who have been students of game theory for years as well as for those who have never heard of the subject. If A has *any* doubts about *either* B's payoffs *or* B's rationality, then it is a lot safer for A to play Up than to play her Nash equilibrium strategy of Down. What if A thought the payoffs were as illustrated in Figure 5.4, but in reality B's payoffs were the reverse—the 9.9 payoff went with Left and the 10 payoff went with Right? What if the 9.9 payoff were only an approximation and the exact payoff was actually 10.1? What if B was a player with a value system substantially different from A's, or was not a truly rational player and might choose the "wrong" action just for fun? Obviously, our assumptions of perfect information and rationality can be crucial to the analysis that we use in the study of strategy. Doubts about players can alter equilibria from those that we would normally predict and can call the reasonableness of the Nash equilibrium concept into question.

However, the real problem with many such examples is not that the Nash equilibrium concept is inappropriate, but that the examples illustrate it in an inappropriately simplistic way. In this example, if there are any doubts about B's payoffs, then this fact should be made an integral part of the analysis. If A does not know B's payoffs, the game is one of asymmetric information (which we won't have the tools to discuss fully until Chapter 9). But this particular example is a relatively simple game of that kind, and we can figure out its equilibrium very easily.

Suppose A thinks there is a probability p that B's payoffs from Left and Right are the reverse of those shown in Figure 5.4; $(1 - p)$ is then the probability that B's payoffs are as stated in that figure. Because A must take her action without knowing what B's actual payoffs are, she must choose her strategy to be "best on average." In this game, the calculation is simple because in each case B has a dominant strategy; the only problem for A is that in the two different cases, different strategies are dominant for B. With probability $(1 - p)$, B's dominant strategy is Left (the case shown in the figure), and with probability p, it is Right (the opposite case). Therefore, if A chooses Up, then with probability $(1 - p)$, he will meet B playing Left and so get a payoff of 9; with probability p, he will meet B playing Right and so get a payoff of 8. Thus, A's statistical, or probability-weighted, average payoff from playing Up is $9(1 - p) + 8p$. Similarly, A's statistical average payoff from playing Down is $10(1 - p) - 1{,}000p$. Therefore, it is better for A to choose Up if

$$9(1 - p) + 8p > 10(1 - p) - 1{,}000p, \text{ or } p > 1/1{,}009.$$

Thus, even if there is only a very slight chance that B's payoffs are the opposite of those in Figure 5.4, it is optimal for A to play Up. In this case, analysis based on rational behavior, when done correctly, contradicts neither the intuitive suspicion nor the experimental evidence after all.

B. Multiplicity of Nash Equilibria

Another criticism of the Nash equilibrium concept is based on the observation that many games have multiple Nash equilibria. Thus, the argument goes, the concept fails to pin down outcomes of games sufficiently precisely to give unique predictions. This argument does not automatically require us to abandon the Nash equilibrium concept. Rather, it suggests that if we want a unique prediction from our theory, we must add some criterion for deciding which one of the multiple Nash equilibria we want to select.

In Chapter 4, we studied many games of coordination with multiple equilibria. From among these equilibria, the players may be able to select one as a focal point if they have some common social, cultural, or historical knowledge. Consider the following coordination game played by students at Stanford University. One player was assigned the city of Boston and the other was assigned San Francisco. Each was then given a list of nine other U.S. cities—Atlanta, Chicago, Dallas, Denver, Houston, Los Angeles, New York, Philadelphia, and Seattle—and asked to choose a subset of those cities. The two chose simultaneously and independently. If and only if their choices divided up the nine cities completely and without any overlap between them, both got a prize. Despite the existence of 512 different Nash equilibria, when both players were Americans or long-time U.S. residents, more than 80% of the time they chose a unique equilibrium based on geography. The student assigned Boston chose all the cities east of the Mississippi, and the student assigned San Francisco chose all the cities west of the Mississippi. Such coordination was much less likely when one or both students were non-U.S. residents. In such pairs, the choices were sometimes made alphabetically, but with much less likelihood of achieving a non-overlapping split of the full list.[8]

The features of the game itself, combined with shared cultural background, can help players' expectations to converge. As another example of multiplicity of equilibria, consider a game where two players write down, simultaneously and independently, the share that each wants from a total prize of $100. If the amounts that they write down add up to $100 or less, each player receives the amount she wrote down. If the two add up to more than $100, neither gets anything. For any x, one player writing x and the other writing $(100 - x)$ is a Nash equilibrium. Thus, the game has an (almost) infinite range of Nash equilibria. But, in practice, 50:50 emerges as a focal point. This social norm of equality or fairness seems so deeply ingrained as to be almost an instinct; players who choose 50 say that it is the obvious answer. To be a true focal point, not only should that answer be obvious to each, but everyone should know that it is obvious to each, and everyone should know that everyone knows, and so on; in other words, its obviousness should be common knowledge. That is not always the case, as we see when we consider a situation in which one player is a woman from an enlightened and egalitarian society who believes that 50:50 is the obvious choice and the other is a man from a patriarchal society who believes it is obvious that, in any matter of division, a man should get three times as much as a woman. Then each will do what is obvious to her or him, and they will end up with nothing, because neither's obvious solution is obvious as common knowledge to both.

The existence of focal points is often a matter of coincidence, and creating them where none exist is basically an art that requires a lot of attention to the historical and cultural context of a game and not merely its mathematical description. This bothers many game theorists, who would prefer that the outcome depend only on an abstract specification of a game—that players and their strategies be identified by numbers without any external associations. We disagree. We think that historical and cultural contexts are just as important to a game as is its purely mathematical description, and if such context helps us select a unique outcome from multiple Nash equilibria, that is all to the good.

In Chapter 6, we will see that sequential-move games can have multiple Nash equilibria. There, we will introduce the requirement of *credibility* that enables us to select a particular equilibrium; it turns out that this equilibrium is in fact the rollback equilibrium of Chapter 3. For more complex games with information asymmetries or additional complications, other restrictions, called refinements, have been developed to identify and rule out Nash equilibria that are unreasonable in some way. In Chapter 9, we will consider one such refinement process that selects an outcome called a *perfect Bayesian equilibrium*. The motivation for a refinement is often specific to a particular type of game. A refinement stipulates how players update their information when they observe what moves other players made or failed to make. Each such stipulation is often perfectly reasonable in its context, and in many games it is not difficult to eliminate most of the Nash equilibria and therefore to reduce the ambiguity in predictions.

The opposite of the criticism that some games may have too many Nash equilibria is that some games may have none at all. In Section 8 of Chapter 4, where we presented an example, we argued that by extending the concept of strategy to random mixtures, Nash equilibrium could be restored. In Chapter 7, we will explain and consider Nash equilibria in mixed strategies. In higher reaches of game theory, there are esoteric examples of games that have no Nash equilibrium in mixed strategies either. However, this added complication is not relevant for the types of analyses and applications that we deal with in this book, so we do not attempt to address it here.

C. Requirements of Rationality for Nash Equilibrium

Remember that Nash equilibrium can be regarded as a system of the strategy choices of each player and the belief that each player holds about the other players' choices. In equilibrium, (1) the choice of each should give her the best payoff given her belief about the others' choices, and (2) the belief of each player should be correct—that is, the other players' actual choices should be the same as what this player believes them to be. These features seem to be natural expressions of the requirements of the mutual consistency of individual rationality. If all players have common knowledge that they are all rational, how can any one of them rationally believe something about others' choices that would be inconsistent with a rational response to her own actions?

To begin to address this question, we consider the three-by-three game in Figure 5.5. Best-response analysis quickly reveals that it has only one Nash equilibrium—namely, (R2, C2), leading to payoffs (3, 3). In this equilibrium, Row plays R2 because she believes that Column is playing C2. Why does she believe this? Because she knows Column to be rational, Row must simultaneously believe that Column believes that Row is choosing R2, because C2 would not be Column's best choice if she believed Row would be playing either R1 or R3. Thus, the claim goes, in any rational process of formation of beliefs and responses, beliefs would have to be correct.

	COLUMN		
	C1	C2	C3
You may need to scroll left and right to see the full figure.			

		COLUMN		
		C1	C2	C3
ROW	R1	0, 7	2, 5	7, 0
	R2	5, 2	3, 3	5, 2
	R3	7, 0	2, 5	0, 7

You may need to scroll left and right to see the full figure.

FIGURE 5.5 Justifying Choices by Chains of Beliefs and Responses

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The trouble with this argument is that it stops after one round of thinking about beliefs. If we allow it to go far enough, we can justify other choice combinations. We can, for example, rationally justify Row's choice of R1. To do so, we note that R1 is Row's best choice if she believes Column is choosing C3. Why does she believe this? Because she believes that Column believes that Row is playing R3. Row justifies this belief by thinking that Column believes that Row believes that Column is playing C1, believing that Row is playing R1, believing in turn . . . . This is a chain of beliefs, each link of which is perfectly rational.

Thus, rationality alone does not justify Nash equilibrium. There are more sophisticated arguments of this kind that do justify a special form of Nash equilibrium in which players can condition their strategies on a publicly observable randomization device. But we leave that to more advanced treatments. In the next section, we develop a simpler concept that captures what is logically implied by the players' common knowledge of their rationality alone.

Endnotes

- David M. Kreps, *Game Theory and Economic Modelling* (Oxford: Clarendon Press, 1990), gives an excellent in-depth discussion. Return to reference 6
- Roger Myerson, *Game Theory* (Cambridge, Mass.: Harvard University Press, 1991), p. 106. Return to reference 7
- See David Kreps, *A Course in Microeconomic Theory* (Princeton, N.J.: Princeton University Press, 1990), pp. 392–93, 414–15. Return to reference 8

Glossary

refinement
: A restriction that narrows down possible outcomes when multiple Nash equilibria exist.

3 RATIONALIZABILITY

What strategy choices in games can be justified on the basis of rationality alone? For the two-player game shown in Figure 5.5, we can justify any pair of strategies, one for each player, by using the same type of logic that we used in Section 2.C. In other words, we can justify any one of the nine logically conceivable combinations. Thus, rationality alone does not give us any power to narrow down or predict outcomes of this game. Is this a general feature of all games? No. For example, if a strategy is dominated, rationality alone can rule it out of consideration. And when players recognize that other players, being rational, will not play dominated strategies, iterated elimination of dominated strategies can be performed on the basis of common knowledge of rationality. Is this the best that can be done? No. Some more ruling out of strategies can be done by using a property slightly stronger than dominance in pure strategies. This property identifies strategies that are never a best response. The set of strategies that survive elimination on this ground are called rationalizable, and the concept itself is known as rationalizability.

Why introduce this additional concept, and what does it do for us? As for why, it is useful to know how far we can narrow down the possible outcomes of a game on the basis of the players' rationality alone, without invoking correctness of beliefs about the other player's actual choice. It is sometimes possible to figure out that the other player *will not* choose some available action or actions, even when it is not possible to pin down the single action that she *will* choose. As for what it achieves, that depends on the context. In some cases, rationalizability may not narrow down the outcomes at all. This was so in the three-by-three example game shown in Figure 5.5. In other cases, it narrows the possibilities to some extent, but not all the way down to the Nash equilibrium, if the game has a unique one, or to the set of Nash equilibria, if there are several. We will consider an example of such a situation in Section 3.A. In still other cases, the narrowing goes all the way down to the Nash equilibrium; in these cases, we have a more powerful justification for the Nash equilibrium that relies on rationality alone, without assuming correctness of beliefs. In Section 3.B below, we will present an example of quantity competition in which the rationalizability argument takes us all the way to the game’s unique Nash equilibrium.

A. Applying the Concept of Rationalizability

Consider the game in Figure 5.6, which is the same as Figure 5.5 but with an additional strategy for each player.[9] We indicated in Section 2.C that the original nine strategy combinations in Figure 5.5 can all be justified by a chain of the players' beliefs about each other's beliefs. That remains true in this enlarged matrix. But can R4 and C4 be justified in this way?

Could Row ever believe that Column would play C4? Such a belief would have to be justified by Column's beliefs about Row's choice. What might Column believe about Row's choice that would make C4 Column's best response? Nothing. If Column believes that Row will play R1, then Column's best choice is C1. If Column believes that Row will play R2, then Column's best choice is C2. If Column believes that Row will play R3, then C3 is Column's best choice. And, if Column believes that Row will play R4, then C1 and C3 are tied for her best choice. Thus, C4 is never a best response for Column.[10] This means that Row, knowing Column to be rational, can never attribute to Column any belief about Row's choice that would justify Column's choice of C4. Therefore, Row should never believe that Column would choose C4.

		COLUMN			
		C1	C2	C3	C4
ROW	R1	0, 7	2, 5	7, 0	0, 1
	R2	5, 2	3, 3	5, 2	0, 1
	R3	7, 0	2, 5	0, 7	0, 1
	R4	0, 0	0, −2	0, 0	10, −1

You may need to scroll left and right to see the full figure.

FIGURE 5.6 Rationalizable Strategies

Note that, although C4 is never a best response, it is not dominated by any of Column’s other strategies, C1, C2, or C3. For Column, C4 does better than C1 against Row’s R3, better than C2 against Row’s R4, and better than C3 against Row’s R1. If a strategy *is* dominated, it also can never be a best response. Thus, “never a best response” is a more general concept than “dominated.” Eliminating strategies that are never a best response may be possible even when eliminating dominated strategies is not. So eliminating strategies that are never a best response can narrow down the set of possible outcomes more than can elimination of dominated strategies.[11]

The elimination of never-a-best-response strategies can also be carried out iteratively. Because a rational Row can never believe that a rational Column will play C4, a rational Column should foresee this. Because R4 is Row’s best response only against C4, Column should never believe that Row will play R4. Thus, R4 and C4 can never figure in the set of rationalizable strategies. The concept of rationalizability allows us to narrow down the set of possible outcomes of this game to this extent.

If a game has a Nash equilibrium, it is rationalizable and, in fact, can be sustained by a simple one-round system of beliefs, as we saw in Section 2.C above. But, more generally, even if a game does not have a Nash equilibrium, it may have rationalizable outcomes. Consider the two-by-two game we can obtain from Figure 5.5 or Figure 5.6 by retaining just the strategies R1 and R3 for Row and C1 and C3 for Column. It is easy to see that this game has no Nash equilibrium in pure strategies. But all four outcomes are rationalizable with the use of exactly the chain of beliefs, constructed earlier, that went around and around these strategies.

Thus, the concept of rationalizability provides a possible way of solving games that do not have a Nash equilibrium. And more importantly, it tells us how far we can narrow down the possibilities in a game on the basis of rationality alone.

B. Rationalizability Can Take Us All the Way to Nash Equilibrium

In some games, iterated elimination of never-a-best-response strategies can narrow things down all the way to Nash equilibrium. Note that we said *can*, not *must*. But if it does, that is useful, because in these games we can strengthen the case for Nash equilibrium by arguing that it follows purely from the players' rational thinking about each other's thinking. Interestingly, one class of games that can be solved in this way is very important in economics. This class consists of games of competition between firms that choose the quantities that they produce, knowing that the total quantity that is put on the market will determine the price.

We illustrate a game of this type in the context of a small coastal town. It has two fishing boats that go out every evening and return the following morning to put their night's catch on the market. The game is played in an era before modern refrigeration, so all the fish has to be sold and eaten the same day it is caught. Fish are plentiful in the ocean near the town, so the owner of each boat can decide how much to catch each night. But each knows that if the total brought to the market is too large, the glut of fish will mean a low price and low profits.

Specifically, we suppose that, if one boat brings R barrels and the other brings S barrels of fish to the market, the price P (measured in ducats per barrel) will be $P = 60 - (R + S)$. We also suppose that the two boats and their crews are somewhat different in their fishing efficiency. Fishing costs the first boat 30 ducats per barrel and the second boat 36 ducats per barrel.

Now we can write down the profits U and V of the two boat owners in terms of their strategies R and S:

$U = [(60 - R - S) - 30]R = (30 - S)R - R^2,$

$V = [(60 - R - S) - 36]S = (24 - R)S - S^2.$

With these payoff expressions, we can construct best-response curves and find the Nash equilibrium. As in our price competition example from Section 1, each player's payoff is a quadratic function of his own strategy, holding the strategy of the other player constant. Therefore, the same mathematical methods we develop there and in the appendix to this chapter can be applied.

The first boat's best response R should maximize U for each given value of the other boat's S. With the use of calculus, this means that we should differentiate U with respect to R, holding S fixed, and set the derivative equal to 0, which gives

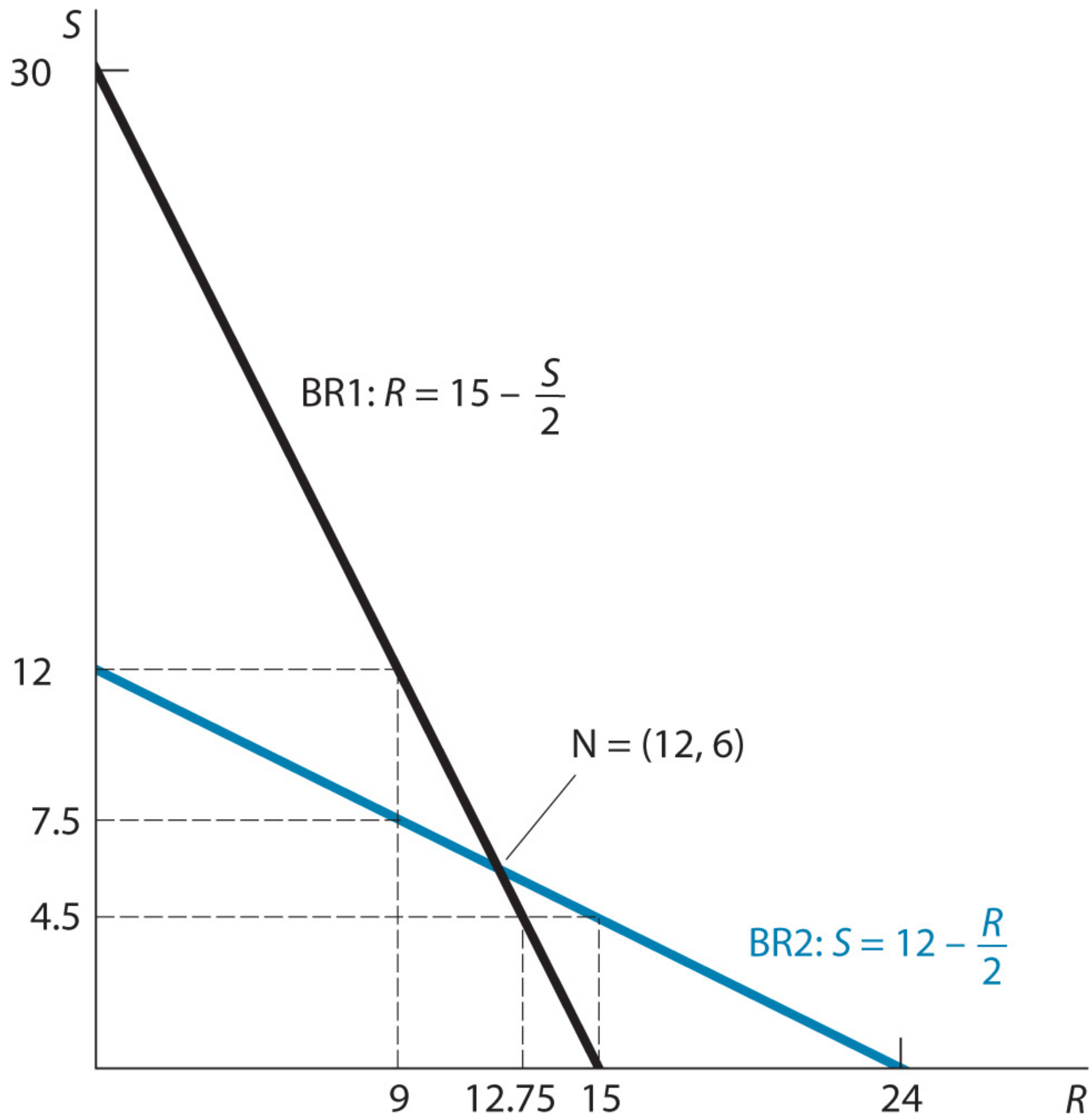

Figure 5.7 Nash Equilibrium through Rationalizability

$$(30-R)-2R=0;\text{ so } R=15-\frac{S}{2}.$$

The noncalculus approach uses the result that the U-maximizing value of $R = B/(2C)$, where in this case $B = 30 - S$ and $C = 1$.

This gives $R = (30 - S)/2$, or $R = 15 - S/2$.

Similarly, the best-response equation for the second boat is found by choosing S to maximize V for each fixed R, yielding

$$S = \frac{24 - R}{2};\ \text{so}\ S = 12 - \frac{R}{2}.$$

The Nash equilibrium is found by solving the two best-response equations jointly for R and S, which is easy to do.[12] So we state only the results here: quantities are $R = 12$ and $S = 6$; price is $P = 42$; and profits are $U = 144$ and $V = 36$.

Figure 5.7 shows the two fishermen's best-response curves (labeled BR1 and BR2 with the equations displayed) and the Nash equilibrium (labeled N with its coordinates displayed) at the intersection of the two curves. Figure 5.7 also shows how the players' beliefs about each other's choices can be narrowed down by iteratively eliminating strategies that are never best responses.

What values of S can the first boat owner rationally believe the second owner will choose? That depends on what the second owner thinks the first owner will produce. But no matter what that quantity might be, the whole range of the second owner's best responses is between 0 and 12. So the first owner cannot rationally believe that the second owner will choose anything else; all negative choices of S (obviously) and all choices of S greater than 12 (less obviously) are eliminated. Similarly, the second owner cannot rationally think that the first owner will produce anything less than 0 or greater than 15.

Now take this process to the second round. When the first owner has restricted the second owner's choices of S to the range between 0 and 12, her own choices of R are restricted to the range of best responses to S's range. The best response to $S = 0$ is $R = 15$, and the best response to $S = 12$ is $R = 15 - 12/2 = 9$.

Because BR1 has a negative slope throughout, the whole range of R allowed at this round of thinking is between 9 and 15. Similarly, the second owner's choice of S is restricted to the range of best responses to R between 0 and 15—namely, values between S = 12 and S = 12 − 15/2 = 4.5. Figure 5.7 shows these restricted ranges on the axes.

The third round of thinking narrows the ranges still further. Because R must be at least 9 and BR2 has a negative slope, S can be at most the best response to 9—namely, S = 12 − 9/2 = 7.5. In the second round, S was already shown to be at least 4.5. Thus, S is now restricted to the range between 4.5 and 7.5. Similarly, because S must be at least 4.5, R can be at most 15 − 4.5/2 = 12.75. In the second round, R was shown to be at least 9, so now it is restricted to the range between 9 and 12.75.

This succession of rounds can be carried on as far as you like, but it is already evident that the successive narrowing of the two ranges is converging on the Nash equilibrium, R = 12 and S = 6. Thus, the Nash equilibrium is the only outcome that survives the iterated elimination of strategies that are never best responses.[13] We know that in general, rationalizability need not narrow down the outcomes of a game to its Nash equilibria, so that is a special feature of this example. Actually, the iterative process works for an entire class of games: those games that have a unique Nash equilibrium at the intersection of downward-sloping best-response curves.[14]

This argument should be carefully distinguished from an older one based on a succession of best responses. The old reasoning proceeded as follows: Start at any strategy for one of the players—say, R = 18. Then, the best response of the other is S = 12 − 18/2 = 3. The best response of R to S = 3 is then R = 15 − 3/2 = 13.5. In turn, the best response of S to R = 13.5 is 12 − 13.5/2 = 5.25. Then, in its turn, the best R against this S is R = 15 − 5.25/2 = 12.375. And so on.

The chain of best responses in the old argument also converges to the Nash equilibrium. But the argument is flawed. The game is played once with simultaneous moves. Therefore, it is not possible for one player to respond to what the other player has chosen, then have the first player respond in turn, and so on. If such dynamics of actual play were allowed, would each player not foresee how the other was going to respond and so do something different in the first place?

The rationalizability argument is different. It clearly incorporates the fact that the game is played only once and with simultaneous moves. All the thinking regarding the chain of best responses is done in advance, and all the successive rounds of thinking and responding are purely conceptual. Players are not responding to actual choices, but are merely calculating those choices that will never be made. The dynamics are purely in the minds of the players.

Endnotes

- This example comes from Douglas Bernheim, “Rationalizable Strategic Behavior,” *Econometrica*, vol. 52, no. 4 (July 1984), pp. 1007–28, an article that originally developed the concept of rationalizability. See also Andreu Mas-Colell, Michael Whinston, and Jerry Green, *Microeconomic Theory* (New York: Oxford University Press, 1995), pp. 242–45. Return to reference 9
- Note that in each case, the best choice is strictly better than C4 for Column. Thus, C4 is never even tied for best response. We can distinguish between weak and strong senses of never being a best response just as we distinguished between weak and strict dominance. Here, we have the strong sense. Return to reference 10
- When one allows for mixed strategies, as we will do in Chapter 7, there arises the possibility of a pure strategy being dominated by a mixture of other pure strategies. With such an expanded definition of a dominated strategy, iterated elimination of strictly dominated strategies turns out to be equivalent to rationalizability. The details are best left for a more advanced course in game theory. Return to reference 11
- Although they are incidental to our purpose, some interesting properties of the solution are worth pointing out. The quantities differ because the costs differ; the more efficient (lower-cost) boat gets to sell more. The cost and quantity differences together imply even bigger differences in the resulting profits. The cost advantage of the first boat over the second is only 20%, but it makes four times as much profit as the second boat. Return to reference 12
- This example can also be solved by iteratively eliminating dominated strategies, but proving dominance is harder and needs more calculus, whereas the never-a-best-response property is obvious from Figure 5.7, so we use the simpler argument. Return to reference 13
- A similar argument works with upward-sloping best-response curves, such as those in the pricing game of Figure 5.1, for

narrowing the range of best responses starting at low prices. Narrowing from the higher end is possible only if there is some obvious starting point. This starting point might be a very high price that can never be exceeded for some externally enforced reason—if, for example, people simply do not have the money to pay prices beyond a certain level.
Return to reference 14

Glossary

never a best response
: A strategy is never a best response for a player if, for each list of strategies that the other players choose (or for each list of strategies that this player believes the others are choosing), some other strategy is this player’s best response. (The other strategy can be different for different lists of strategies of the other players.)

rationalizable
: A strategy is called rationalizable for a player if it is his optimal choice given some belief about what (pure or mixed strategy) the other player(s) would choose, provided this belief is formed recognizing that the other players are making similar calculations and forming beliefs in the same way. (This concept is more general than that of the Nash equilibrium and yields outcomes that can be justified on the basis only of the players’ common knowledge of rationality.)

rationalizability
: A solution concept for a game. A list of strategies, one for each player, is a rationalizable outcome of the game if each strategy in the list is rationalizable for the player choosing it.

4 EMPIRICAL EVIDENCE CONCERNING NASH EQUILIBRIUM

In Chapter 3, when we considered empirical evidence on sequential-move games and rollback, we presented empirical evidence from observations on games played in the real world as well as games deliberately constructed for testing the theory in the laboratory. There, we pointed out the different merits and drawbacks of the two methods for assessing the validity of rollback equilibrium predictions. Similar issues arise in securing and interpreting evidence on Nash equilibrium in simultaneous-move games.

Real-world games are played for substantial stakes, often by experienced players who have the knowledge and the incentives to employ good strategies. But these situations include many factors beyond those considered in the theory. In particular, in real-life situations, it is difficult to observe the quantitative payoffs that players would have earned for all possible combinations of strategies. Therefore, if their behavior does not bear out the predictions of the theory, we cannot tell whether the theory is wrong or whether some other factors overwhelm the strategic considerations.

Laboratory experiments attempt to control for other factors to provide cleaner tests of the theory. But they bring in inexperienced players and provide them with little time and relatively weak incentives to learn the game and play it well. Confronted with a new game, most of us would initially flounder and try things out at random. Thus, the first several plays of the game in an experimental setting may represent this learning phase and not the equilibrium strategies that experienced players would learn to play. Experiments often control for inexperience and learning by discarding several initial plays from their data, but the learning phase may last longer than the one morning or one afternoon that is the typical limit of laboratory sessions.

A. Laboratory Experiments

Researchers have conducted numerous laboratory experiments in the past three decades to test discover how people act when placed in certain interactive strategic situations. In particular, such research asks, Do participants play their Nash equilibrium strategies? Reviewing this work, Douglas Davis and Charles Holt conclude that “in the laboratory, the Nash equilibrium appears to explain behavior fairly well when it is unique (but [with] some important exceptions).”[15] But the theory’s success is more mixed in more complex situations, such as when multiple Nash equilibria exist, when emotional factors modify payoffs beyond the stated cash amounts, when the calculations for finding a Nash equilibrium are more complex, or when the game is repeated with the same partners. We briefly consider the predictive performance of Nash equilibrium in several of these circumstances.

I. CHOOSING AMONG MULTIPLE EQUILIBRIA In Section 2.B above, we presented examples demonstrating that focal points sometimes emerge to help players choose among multiple Nash equilibria. Players may not manage to coordinate 100% of the time, but circumstances often enable players to achieve much more coordination than would result from random choices across possible equilibrium strategies. Here we present a coordination game designed with an interesting trade-off: The equilibrium with the highest payoff to all players also happens to be the riskiest one to play, in the sense of Section 2.A above.

John Van Huyck, Raymond Battalio, and Richard Beil describe a 16-player game in which each player simultaneously chooses an “effort” level between 1 and 7. Individual payoffs depend on group “output,” a function of the minimum effort level chosen by any player in the group, minus the cost of one’s individual effort. The game has exactly seven Nash equilibria in pure strategies; any outcome in which all players choose the same effort level is an equilibrium. The highest possible payoff ($1.30 per player) occurs when all players choose an effort level of 7, while the lowest equilibrium payoff ($0.70 per player) occurs when all players choose an effort level of 1. The highest-payoff equilibrium is a natural candidate for a focal point, but in this case there is a risk to choosing the highest effort level: If just one other player chooses a lower effort level than you, then your extra effort is wasted. For example, if you play 7 and at least one other player chooses 1, you get a payoff of just $0.10, far worse than the worst equilibrium payoff of $0.70. This makes players nervous about whether others will choose the maximum effort level, and as a result, large groups typically fail to coordinate on the best equilibrium. A few players inevitably choose lower than the maximum effort, and in repeated rounds, play converges toward the lowest-effort equilibrium.[16]

II. EMOTIONS AND SOCIAL NORMS In Chapter 3, we saw several examples in sequential-move games where players were more generous to each other than Nash equilibrium would predict. Similar observations occur in simultaneous-move games such as the prisoners' dilemma. One reason may be that the players' payoffs are different from those assumed by the experimenter: In addition to cash, their payoffs may include the experience of emotions such as empathy, anger, or guilt. In other words, the players' value systems may have internalized some social norms of niceness and fairness that have proved useful in the larger social context and that therefore carry over to their behavior in the experimental game.[17] Seen through this lens, these observations do not show any deficiency of the Nash equilibrium concept itself, but they do warn us against using the concept under naive or mistaken assumptions about people's payoffs. It might be a mistake, for example, to assume that players are always driven by the selfish pursuit of money.

III. COGNITIVE ERRORS As we saw in the experimental evidence on rollback equilibrium in Chapter 3, players do not always fully think through the entire game before playing, nor do they always expect other players to do so. Behavior in a game known as the travelers' dilemma illustrates a similar limitation of Nash equilibrium in simultaneous-move games. In this game, two travelers purchase collections of souvenirs of identical value while on vacation, and the airline loses both of their bags on the return trip. The airline announces to the two players that it intends to reimburse them for their losses, but it does not know the exact amount to reimburse. It knows that the correct amount is between $80 and $200 per traveler, so it designs a game as follows. Each player may submit a claim between $80 and $200. The airline will reimburse both players at an amount equal to the lower of the two claims submitted. In addition, if the two claims differ, the airline will pay a reward of $5 to the person making the smaller claim and deduct a penalty of $5 from the reimbursement of the person making the larger claim.

With these rules, irrespective of the actual value of the lost luggage, each player has an incentive to undercut the other's claim. In fact, it turns out that the only Nash equilibrium—and indeed, the only rationalizable outcome—is for both players to report the minimum value of $80. However, in the laboratory, players rarely claim $80; instead, they claim amounts much closer to $200. (Real payoff amounts in the laboratory are typically in cents rather than in dollars.) Interestingly, if the penalty/reward parameter is increased by a factor of 10, from $5 to $50, player behavior conforms much more closely to the Nash equilibrium, with reported amounts generally near $80. Thus, behavior in this experiment varies tremendously with a parameter that does not affect the Nash equilibrium at all; the unique Nash equilibrium is $80, regardless of the penalty/reward amount.

To explain these results from their laboratory, Monica Capra and her coauthors employed a theoretical model called quantal-response equilibrium (QRE), originally proposed by Richard McKelvey and Thomas Palfrey. This model's mathematics are beyond the scope of this text, but its main contribution is allowing for the possibility that players make errors, with the probability of a given error being much smaller for costly mistakes than for mistakes that reduce one's payoff by very little. Furthermore, the model assumes that players expect each other to make errors in this way. It turns out that QRE can explain the data quite well. Reporting a high claim is not very costly when the penalty is only $5, so players are more willing to report values near $200—especially knowing that their rivals are likely to behave similarly, so that the payoff for reporting a high number could be large. However, with a penalty/reward of $50 instead of $5, reporting a high claim becomes quite costly, so players are less likely to expect each other to make such a mistake. This expectation pushes behavior toward the Nash equilibrium claim of $80. Building on this success, QRE has become a very active area of game-theoretic research.[18]

IV. COMMON KNOWLEDGE OF RATIONALITY We have just seen that to better explain experimental results, QRE allows for the possibility that players do not believe that others are perfectly rational. Another way to explain these results is to allow for the possibility that different players engage in different levels of reasoning. A strategic guessing game that is often used in classrooms or laboratories asks each participant to choose a number between 0 and 100. Typically, the players are handed cards on which to write their name and their choice, so this game is a simultaneous-move game. When the cards are collected, the average of the numbers is calculated. The person whose choice is closest to a specified fraction—say, two-thirds—of the average is the winner. The rules of the game (this whole procedure) are announced in advance.

The Nash equilibrium of this game is for everyone to choose 0. In fact, the game is dominance solvable. Even if everyone chooses 100, two-thirds of the average can never exceed 67, so for each player, any choice above 67 is dominated by 67.[19] But all players should rationally figure this out, so the average can never exceed 67, and two-thirds of it can never exceed 44, and so any choice above 44 is dominated by 44. The iterated elimination of dominated strategies goes on in this way until only 0 is left.

However, when a group actually plays this game for the first time, the winner is not a person who plays 0. Typically, the winning number is somewhere around 15 or 20. The most commonly observed choices are 33 and 22, suggesting that a large number of players perform one or two rounds of iterated elimination of dominated strategies without going further. That is, “level-1” players imagine that all other players will choose numbers randomly, with an average of 50, so they best-respond with a choice of two-thirds of this amount, or 33. Similarly, “level-2” players imagine that everyone else will be a “level-1” player, so they best-respond by playing two-thirds of 33, or 22. Note that all of these choices are far from the Nash equilibrium of 0. It appears that many players follow a limited number of steps of iterated elimination of dominated strategies, in some cases because they expect others to be limited in their number of rounds of thinking as well.[20]

V. LEARNING AND MOVING TOWARD EQUILIBRIUMWhat happens when the strategic guessing game is repeated with the same group of players? In classroom experiments, we find that the winning number can easily drop 50% in each subsequent round, as the students expect all their classmates to play numbers as low as the previous round’s winning number or lower. By the third round of play, winning numbers tend to be as low as 5 or less.

How should one interpret this result? Critics would say that unless the exact Nash equilibrium is reached, the theory is refuted. Indeed, they would argue, if you have good reason to believe that other players will not play their Nash equilibrium strategies, then your best response is not your Nash equilibrium strategy either. If you can figure out how others will deviate from their Nash equilibrium strategies, then you should play your best response to what you believe they are choosing. Others would argue that theories in social science can never hope for the kind of precise prediction that we expect in sciences such as physics and chemistry. If observed outcomes are close to the Nash equilibrium, that is a vindication of the theory. In this case, the experiment not only produces such a vindication, but illustrates the process by which people gather experience and learn to play strategies close to Nash equilibrium. We tend to agree with this latter viewpoint.

Interestingly, we have found that people learn a game somewhat faster by observing others playing it than by playing it themselves. This may be because, as observers, they are free to focus on the game as a whole and think about it analytically. Players' brains are occupied with the task of making their own choices, and they are less able to take the broader perspective.

We should clarify the concept of gaining experience by playing a game. The quotation from Davis and Holt at the start of this section spoke of "repetitions with different partners." In other words, experience should be gained by playing the game frequently, but with different opponents each time. However, for any learning process to generate outcomes increasingly closer to the Nash equilibrium, the whole *population* of learners needs to be stable. If novices keep appearing on the scene and trying new experimental strategies, then the original group members may unlearn what they had learned by playing against one another.

If a game is played repeatedly between two players, or even among the same small group of known players, then any pair is likely to play each other repeatedly. In such a situation, the whole set of repetitions becomes a game in its own right. And that game can have Nash equilibria very different from those that simply repeat the Nash equilibrium of a single play. For example, tacit cooperation may emerge in repeated prisoners’ dilemmas, owing to the expectation that any temporary gain from cheating will be more than offset by the subsequent loss of trust. If games are repeated in this way, then learning about them must come from playing whole sets of repetitions frequently, against different partners each time.

B. Real-World Evidence

While the field environment does not allow for as much direct observation as the laboratory does, observations outside the laboratory can also provide valuable evidence about the relevance of Nash equilibrium. Conversely, Nash equilibrium often provides a valuable starting point for social scientists seeking to make sense of the real world.

I. APPLICATIONS OF NASH EQUILIBRIUMOne of the earliest applications of the Nash equilibrium concept to real-world behavior was in the area of international relations. Thomas Schelling pioneered the use of game theory to explain phenomena such as the escalation of arms races, even between countries that have no intention of attacking each other, and to evaluate the credibility of deterrent threats. Subsequent applications in this area have included the questions of when and how a country can credibly signal its resolve in diplomatic negotiations or in the face of a potential war. Game theory began to be used systematically in economics and business in the mid-1970s, and such applications continue to proliferate.[21]

As we saw earlier in this chapter, price competition is one important application of Nash equilibrium in business. Other strategic choices by firms include product quality, investment, and R&D. The theory has also helped us to understand when and how the established firms in an industry can make credible commitments to deter new competition—for example, to wage a destructive price war against any new entrant to the market. Game-theoretic models based on the Nash equilibrium concept and its dynamic generalizations fit the data for many major industries, such as automobile manufacturers, reasonably well. They also give us a better understanding of the determinants of competition than older models, which assumed perfect competition and estimated supply and demand curves.[22]

Pankaj Ghemawat, a professor at the Stern School of Business, New York University, has developed a number of case studies of individual firms or industries, supported by statistical analysis of the data. His game-theoretic models are remarkably successful in improving our understanding of several initially puzzling observed business decisions on pricing, capacity, innovation, and so on. For example, DuPont constructed an enormous amount of manufacturing capacity for titanium dioxide in the 1970s. It added capacity in excess of the projected growth in worldwide demand over the next decade. At first glance, this choice looked like a terrible strategy because the excess capacity could lead to lower market prices for this commodity. However, DuPont successfully foresaw that by having excess capacity in reserve, it could punish competitors that cut prices by increasing its production and driving prices even lower. This ability made it a price leader in the industry, and it enjoyed high profit margins. The strategy worked quite well, and DuPont continued to be a worldwide leader in titanium dioxide 50 years later.[23]

More recently, game theory has become the tool of choice for the study of political systems and institutions. As we will see in Chapter 16, game theory has shown how agenda setting and voting in committees and elections can be strategically manipulated in pursuit of one's ultimate objectives. Part Four of this book will develop other applications of Nash equilibrium to auctions, voting, and bargaining. We will also develop our own case study of the Cuban missile crisis in Chapter 13.

Some critics remain unpersuaded of the value of Nash equilibrium, claiming that the same understanding of these phenomena can be obtained using previously known general principles of economics, political science, and so on. In one sense they are right: A few of these analyses existed before Nash equilibrium came along. For example, the equilibrium of the interaction between two price-setting firms, which we developed in Section 1 of this chapter, has been known in economics for more than 100 years. One can think of Nash equilibrium as just a general formulation of that equilibrium concept for all games. Some theories of strategic voting date to the eighteenth century, and some notions of credibility can be found in history as far back as Thucydides's *Peloponnesian War*. Nash equilibrium, however, unifies all these applications and thereby facilitates the development of new ones.

Furthermore, the development of game theory has also led directly to a wealth of new ideas and applications and thus to new discoveries—for example, how the existence of a second-strike capability reduces the fear of surprise attack, how different auction rules affect bidding behavior and seller revenues, how governments can successfully manipulate fiscal and monetary policies to achieve reelection even when voters are sophisticated and aware of such attempts, and so on. If these examples had all been amenable to previously known approaches, they would have been discovered long ago.

II. REAL-WORLD EXAMPLES OF LEARNINGWe conclude by offering a beautiful example of equilibrium and the learning process in the real-world game of major-league baseball, discovered by Stephen Jay Gould.[24] In this game, the stakes are high, and players play more than 100 games per year, so players have strong motivation and good opportunities to learn. The best batting average recorded in a baseball season declined over most of the twentieth century. In particular, instances of a player averaging .400 or better used to be much more frequent than they are now. Devotees of baseball history often explain this decline by invoking nostalgia: "There were giants in those days." A moment's thought should make one wonder why there were no corresponding pitching giants who would have kept batting averages low. But Gould demolishes such arguments in a more systematic way: He points out that we should look at all batting averages, not just the top ones. The worst batting averages are not as bad as they used to be; there are also many fewer .150 hitters in the major leagues than there used to be. Gould argues that this overall decrease in *variation* is a standardization or stabilization effect:

> When baseball was very young, styles of play had not become sufficiently regular to foil the antics of the very best. Wee Willie Keeler could “hit ’em where they ain’t” (and compile an average of .432 in 1897) because fielders didn’t yet know where they should be. Slowly, players moved toward *optimal* methods of positioning, fielding, pitching, and batting—and variation inevitably declined. The best [players] now met an opposition too finely honed to its own perfection to permit the extremes of achievement that characterized a more casual age [emphasis added].

In other words, through a succession of adjustments of strategies to counter one another, the system settled down into its (Nash) equilibrium.

Gould marshals decades of hitting statistics to demonstrate that such a decrease in variation actually occurred, except for occasional “blips.” And indeed, the blips confirm his thesis, because they occur soon after an equilibrium is disturbed by an externally imposed change. Whenever the rules of baseball are altered (the strike zone is enlarged or reduced, the pitching mound is lowered, or new teams and many new players enter when an expansion takes place) or its technology changes (a livelier ball is used or, perhaps in the future, aluminum bats are allowed), the preceding system of mutual best responses is thrown out of equilibrium.

Variation increases for a while as players experiment, and some strategies succeed while others fail. Finally, a new equilibrium is attained, and variation goes down again. That is exactly what we should expect in the framework of learning and adjustment leading to a Nash equilibrium.

Michael Lewis's 2003 book *Moneyball* (later made into a movie starring Brad Pitt) describes a related example of movement toward equilibrium in baseball. Instead of focusing on the strategies of individual players, it focuses on the teams' back-office strategies for choosing players. The book documents Oakland A's general manager Billy Beane's decision to use "sabermetrics" in hiring decisions—that is, paying close attention to baseball statistics with the aim of maximizing runs scored and minimizing runs given up to opponents. His hiring decisions involved paying more attention to attributes undervalued by the market, such as a player's documented ability to earn walks. Such decisions arguably led to the A's becoming a very strong team, going to the playoffs in five out of seven seasons despite having less than half the payroll of larger-market teams such as the New York Yankees. Beane's innovative hiring strategies have subsequently been adopted by other teams, such as the Boston Red Sox. Using Beane's methods, Boston, under general manager Theo Epstein, managed to break the "curse of the Bambino" in 2004 to win its first World Series in 86 years, and it has remained a winning or strongly contending team in the 15 years since then. In that same era, over the course of a decade, nearly a dozen teams decided to hire full-time sabermetricians, with Beane noting in September 2011 that he was once again "fighting uphill" against larger-market teams that had learned to best-respond to his strategies. Real-world games often involve innovation followed by gradual convergence to equilibrium; the two examples from baseball both give evidence of movement toward equilibrium, although full convergence may sometimes take years or even decades to complete.[25]

We take up additional evidence about other game-theoretic predictions at appropriate points in later chapters. For now, the experimental and empirical evidence that we have presented should make you cautiously optimistic about using Nash equilibrium, especially as a first approach. On the whole, we believe you should have considerable confidence in using the Nash equilibrium concept when the game you are interested in is played frequently by players from a reasonably stable population and under relatively unchanging rules and conditions. When the game is new, or is played just once, and the players are inexperienced, you should use the equilibrium concept more cautiously and should not be surprised if the outcome that you observe is not the equilibrium that you calculate. But even then, your first step in the analysis should be to look for a Nash equilibrium; then you can judge whether it seems a plausible outcome and, if not, proceed to the further step of asking why not.[26] Often the reason will be your misunderstanding of the players' objectives, not the players' failure to play the game correctly giving their true objectives.

Endnotes

- Douglas D. Davis and Charles A. Holt, *Experimental Economics* (Princeton, N.J.: Princeton University Press, 1993), pp. 101–102. Return to reference 15
- See John B. Van Huyck, Raymond C. Battalio, and Richard O. Beil, "Tacit Coordination Games, Strategic Uncertainty, and Coordination Failure," *American Economic Review*, vol. 80, no. 1 (March 1990), pp. 234–48. Subsequent research has suggested methods that can promote coordination on the best equilibrium. Subhasish Dugar, "Non-monetary Sanction and Behavior in an Experimental Coordination Game," *Journal of Economic Behavior & Organization*, vol. 73, no. 3 (March 2010), pp. 377–86, shows that players gradually manage to coordinate on the highest-payoff outcome merely by allowing players, between rounds, to express the numeric strength of their disapproval for each other player's decision. Roberto A. Weber, "Managing Growth to Achieve Efficient Coordination in Large Groups," *American Economic Review*, vol. 96, no. 1 (March 2006), pp. 114–26, shows that starting with a small group and slowly adding additional players can sustain the highest-payoff equilibrium, suggesting that a firm may do well to expand slowly and make sure that employees understand the corporate culture of cooperation. Return to reference 16
- The distinguished game theorist Jörgen Weibull argues this position in detail in "Testing Game Theory," in *Advances in Understanding Strategic Behaviour: Game Theory, Experiments and Bounded Rationality: Essays in Honour of Werner Güth*, ed. Steffen Huck (Basingstoke, UK: Palgrave MacMillan, 2004), pp. 85–104. Return to reference 17
- See Kaushik Basu, "The Traveler's Dilemma," *Scientific American*, vol. 296, no. 6 (June 2007), pp. 90–95. The

experiments and modeling can be found in C. Monica Capra, Jacob K. Goeree, Rosario Gomez, and Charles A. Holt, "Anomalous Behavior in a Traveler's Dilemma?" *American Economic Review*, vol. 89, no. 3 (June 1999), pp. 678–90. Quantal-response equilibrium (QRE) was first proposed by Richard D. McKelvey and Thomas R. Palfrey, "Quantal Response Equilibria for Normal Form Games," *Games and Economic Behavior*, vol. 10, no. 1 (July 1995), pp. 6–38. Return to reference 18

- If you factor in your own choice, the calculation is strengthened. Suppose there are N players. In the "worst-case scenario," where all the other $(N - 1)$ players choose 100 and you choose x, the average is $[x + (N - 1)100]/N$. Then your best choice is two-thirds of this, so $x = (2/3)[x + (N - 1)100]/N$, or $x = 100(2N - 2)/(3N - 2)$. If $N = 10$, then $x = (18/28)/100 = 64$ (approximately). So any choice above 64 is dominated by 64. The same reasoning applies to the successive rounds. Return to reference 19
- You will analyze similar games in Exercises S12 and U11. For a summary of results from large-scale experiments run in European newspapers with thousands of players, see Rosemarie Nagel, Antoni Bosch-Domènech, Albert Satorra, and Juan Garcia-Montalvo, "One, Two, (Three), Infinity: Newspaper and Lab Beauty-Contest Experiments," *American Economic Review*, vol. 92, no. 5 (December 2002), pp. 1687–1701. Return to reference 20
- For those who would like to see more applications, here are some suggested sources. Thomas Schelling's *The Strategy of Conflict* (Cambridge, Mass.: Harvard University Press, 1960) and *Arms and Influence* (New Haven, Conn.: Yale University Press, 1966) are still required reading for all students of game theory. The classic textbook on game-theoretic treatment of industries is Jean Tirole, *The Theory of Industrial Organization* (Cambridge, Mass.: MIT Press, 1988). In political science, an early classic is William H. Riker,

Liberalism against Populism (San Francisco: W. H. Freeman, 1982). For advanced-level surveys of research, see several articles in *The Handbook of Game Theory with Economic Applications*, ed. Robert J. Aumann and Sergiu Hart (Amsterdam: North-Holland/Elsevier, 1992, 1994, 2002), particularly Barry O'Neill, "Game Theory Models of Peace and War," in volume 2, and Kyle Bagwell and Asher Wolinsky, "Game Theory and Industrial Organization," and Jeffrey Banks, "Strategic Aspects of Political Systems," both of which are in volume 3. Return to reference 21

- For simultaneous-move models of price competition, see Timothy F. Bresnahan, "Empirical Studies of Industries with Market Power," in *Handbook of Industrial Organization*, vol. 2, ed. Richard L. Schmalensee and Robert D. Willig (Amsterdam: North-Holland/Elsevier, 1989), pp. 1011–57. For models of entry, see Steven Berry and Peter Reiss, "Empirical Models of Entry and Market Structure," in *Handbook of Industrial Organization*, vol. 3, ed. Mark Armstrong and Robert Porter (Amsterdam: North-Holland/Elsevier, 2007), pp. 1845–86. Return to reference 22
- Pankaj Ghemawat, "Capacity Expansion in the Titanium Dioxide Industry," *Journal of Industrial Economics*, vol. 33, no. 2 (December 1984), pp. 145–63. For more examples, see Pankaj Ghemawat, *Games Businesses Play: Cases and Models* (Cambridge, Mass.: MIT Press, 1997). Return to reference 23
- Stephen Jay Gould, "Losing the Edge," in *The Flamingo's Smile: Reflections in Natural History* (New York: W. W. Norton, 1985), pp. 215–29. Return to reference 24
- Susan Slusser, "Michael Lewis on A's 'Moneyball' Legacy," *San Francisco Chronicle*, September 18, 2011, p. B-1. The original book is Michael Lewis, *Moneyball: The Art of Winning an Unfair Game* (New York: W. W. Norton, 2003). Return to reference 25

- In an article probing the weaknesses of Nash equilibrium in experimental data and proposing QRE-style alternative models for dealing with them, two prominent researchers write, "We will be the first to admit that we begin the analysis of a new strategic problem by considering the equilibria derived from standard game theory before considering" other possibilities. Jacob K. Goeree and Charles A. Holt, "Ten Little Treasures of Game Theory and Ten Intuitive Contradictions," *American Economic Review*, vol. 91, no. 5 (December 2001), pp. 1402–22. Return to reference 26

Glossary

quantal-response equilibrium (QRE)
: Solution concept that allows for the possibility that players make errors, with the probability of a given error smaller for more costly mistakes.

SUMMARY

When players in a simultaneous-move game have a continuous range of actions to choose from, best-response analysis yields mathematical *best-response rules* that can be solved simultaneously to obtain Nash equilibrium. The best-response rules can be shown as *best-response curves* on a graph, where the intersection of the two curves represents the Nash equilibrium. Games played among firms choosing price or quantity from a large range of possible values and among political parties choosing campaign advertising expenditure levels are examples of games with *continuous strategies*.

Theoretical criticisms of the Nash equilibrium concept have argued that the concept does not adequately account for risk, that it is of limited use because many games have multiple equilibria, and that it cannot be justified on the basis of rationality alone. In many cases, a better description of the game and its payoff structure or a *refinement* of the Nash equilibrium concept can lead to better predictions or fewer potential equilibria. *Quantal response equilibrium* is one such alternative. Another is *rationalizability*, which relies on the iterated elimination of strategies that are *never a best response* to obtain a set of *rationalizable* outcomes. When a game has a Nash equilibrium, that outcome will be rationalizable, but rationalizability also allows one to predict equilibrium outcomes in games that have no Nash equilibria.

The results of laboratory tests of the Nash equilibrium concept show that a common cultural background is essential for coordination in games with multiple equilibria. Repeated play of some games shows that players can learn from experience and begin to choose strategies that approach their Nash equilibrium choices. Further, predictions of equilibria

are accurate only when the experimenters’ assumptions match the true preferences of players. Real-world applications of game theory have helped economists and political scientists, in particular, to understand important consumer, firm, voter, legislative, and government behaviors.

KEY TERMS

best-response curve (133)

best-response rule (132)

continuous strategy (131)

never a best response (146)

quantal-response equilibrium (QRE) (154)

rationalizability (146)

rationalizable (146)

refinement (144)

Glossary

best-response curve
: A graph showing the best strategy of one player as a function of the strategies of the other player(s) over the entire range of those strategies.

best-response rule
: A function expressing the strategy that is optimal for one player, for each of the strategy combinations actually played by the other players, or the belief of this player about the other players' strategy choices.

continuous strategy
: A choice over a continuous range of real numbers available to a player.

never a best response
: A strategy is never a best response for a player if, for each list of strategies that the other players choose (or for each list of strategies that this player believes the others are choosing), some other strategy is this player's best response. (The other strategy can be different for different lists of strategies of the other players.)

quantal-response equilibrium (QRE)
: Solution concept that allows for the possibility that players make errors, with the probability of a given error smaller for more costly mistakes.

rationalizability
: A solution concept for a game. A list of strategies, one for each player, is a rationalizable outcome of the game if each strategy in the list is rationalizable for the player choosing it.

rationalizable
: A strategy is called rationalizable for a player if it is his optimal choice given some belief about what (pure or mixed strategy) the other player(s) would choose,

provided this belief is formed recognizing that the other players are making similar calculations and forming beliefs in the same way. (This concept is more general than that of the Nash equilibrium and yields outcomes that can be justified on the basis only of the players' common knowledge of rationality.)

refinement

A restriction that narrows down possible outcomes when multiple Nash equilibria exist.

SOLVED EXERCISES

1. In the political campaign advertising game in Section 1.C, party L chooses an advertising budget x (millions of dollars), and party R similarly chooses an advertising budget y (millions of dollars). We showed there that the best-response rules in that game are $y = 10\sqrt{x} - x$ for party R and $x = 10\sqrt{y} - y$ for party L.
    1. What is party R's best response if party L spends $16 million?
    2. Use the specified best-response rules to verify that the Nash equilibrium advertising budgets are $x = y = 25$, or $25 million.
2. The restaurant pricing game illustrated in Figure 5.1 defines customer demand equations for meals at Xavier's (Q_x) and Yvonne's (Q_y) as $Q_x = 44 - 2P_x + P_y$, and $Q_y = 44 - 2P_y + P_x$. Profits for each firm also depend on the costs of serving each customer. Suppose that Yvonne's is able to reduce its costs to a mere $2 per customer by completely eliminating the waitstaff (customers pick up their orders at a counter, and a few remaining employees bus the tables). Xavier's continues to incur a cost of $8 per customer.
    1. Recalculate the best-response rules and the Nash equilibrium prices for the two restaurants given this change in one restaurant's costs.
    2. Graph the two best-response curves and describe the differences between your graph and Figure 5.1. In particular, which curve has moved, and by how much? Explain why these changes occurred in the graph.
3. The small town of Eten has two food stores, La Boulangerie, which sells bread, and La Fromagerie, which sells cheese. It costs $1 to make a loaf of bread and $2 to make a pound of cheese. If La Boulangerie's price is P_1 dollars per loaf of bread and La Fromagerie's price is P_2 dollars per pound of cheese, then their

respective weekly sales, Q_1 thousand loaves of bread and Q_2 thousand pounds of cheese, are given by the following equations:

$Q_1 = 14 - P_1 - 0.5\ P_2$

$Q_2 = 19 - 0.5P_1 - P_2$.

1. For each of the stores, write the profit as a function of P_1 and P_2 (in the exercises that follow, we will call this "the profit function" for brevity). Then find their respective best-response rules. Graph the best-response curves, and find the Nash equilibrium prices in this game.
2. Suppose that the two stores collude and set prices jointly to maximize the sum of their profits. Find the joint profit-maximizing prices for the stores.
3. Provide a short intuitive explanation for the differences between the Nash equilibrium prices and those that maximize joint profit. Why is joint profit maximization not a Nash equilibrium?
4. In this problem, bread and cheese are *complements* to each other. They are often consumed together, so a drop in the price of one increases the sales of the other. The products in our restaurant example in Section 1.A are *substitutes* for each other. How does this difference explain the differences between your findings for the best-response rules, the Nash equilibrium prices, and the joint profit-maximizing prices in this question and the corresponding entities in the restaurant example in the text?

4. The game illustrated in Figure 5.3 has a unique Nash equilibrium in pure strategies. However, all nine outcomes in that game are rationalizable. Confirm this assertion, explaining your reasoning for each outcome.
5. For the game presented in Exercise S6 in Chapter 4, what are the rationalizable strategies for each player? Explain your reasoning.
6. Section 3.B of this chapter describes a fishing-boat game played in a small coastal town. When the best-response rules for the two boats have been derived, rationalizability can be used to justify the Nash equilibrium in the game. In the description in the text, we take the process of narrowing down strategies that can never be best responses through three rounds. By the third round, we know that R (the number of barrels of fish brought home by boat

1) must be at least 9, and that S (the number of barrels of fish brought home by boat 2) must be at least 4.5. The iterative elimination process in that round restricted R to the range between 9 and 12.75 while restricting S to the range between 4.5 and 7.5. Take this process of narrowing through one additional (fourth) round and show the reduced ranges of R and S that are obtained at the end of that round.

7. Two carts selling coconut milk (from the coconut) are located at points 0 and 1, 1 mile apart on the beach in Rio de Janeiro. (They are the only two coconut-milk carts on the beach.) The carts—Cart 0 and Cart 1—charge prices p_0 and p_1, respectively, for each coconut. One thousand beachgoers buy coconut milk, and these customers are uniformly distributed along the beach between carts 0 and 1. Each beachgoer will purchase one coconut milk in the course of her day at the beach, and in addition to the price, each will incur a transport cost of $0.5 \times d^2$, where d is the distance (in miles) from her beach blanket to the coconut-milk cart. In this system, Cart 0 sells to all the beachgoers located between 0 and x, and Cart 1 sells to all the beachgoers located between x and 1, where x is the location of the beachgoer who pays the same total price whether she goes to Cart 0 or Cart 1. Location x is then defined by the expression

   $$p_0 + 0.5\, x^2 = p_1 + 0.5(1 - x)^2.$$

   Each cart will set its prices to maximize its profit, Π, which is determined by revenue (the cart's price times its number of customers) and cost (each cart incurs a cost of \$0.25 per coconut times the number of coconuts sold).

   1. For each cart, determine the expression for the number of customers served as a function of p_0 and p_1. [Recall that Cart 0 gets the customers between 0 and x, or just x, while Cart 1 gets the customers between x and 1, or $1 - x$. That is, Cart 0 sells to x customers, where x is measured in thousands, and Cart 1 sells to $(1 - x)$.]
   2. Write the profit functions for the two carts. Find the best-response rule for each cart as a function of its rival's price.
   3. Graph the best-response rules, and then calculate (and show on your graph) the Nash equilibrium price for coconut milk on the beach.

8. Crude oil is transported across the globe in enormous tanker ships called Very Large Crude Carriers (VLCCs). The vast majority of all new VLCCs are built in South Korea and Japan. Assume that the price of new VLCCs (in millions of dollars) is determined by the function $P = 180 - Q$, where $Q = q_{\text{Korea}} + q_{\text{Japan}}$, the sum of the quantities produced in each of South Korea and Japan (assuming that only these two countries produce VLCCs, so they are a duopoly.) Assume that the cost of building each ship is \$30 million in both Korea and Japan. That is, $c_{\text{Korea}} = c_{\text{Japan}} = 30$, where the cost per ship is measured in millions of dollars.
   1. Write the profit function for each country in terms of q_{Korea} and q_{Japan} and in terms of either c_{Korea} or c_{Japan}. Find each country's best-response function.
   2. Using the best-response functions found in part (a), solve for the Nash equilibrium quantity of VLCCs produced by each country per year. What is the price of a VLCC? How much profit is made in each country?
   3. Labor costs in Korean shipyards are actually much lower than in their Japanese counterparts. Assume now that the cost per ship in Japan is \$40 million and that in Korea it is only \$20 million. Given $c_{\text{Korea}} = 20$ and $c_{\text{Japan}} = 40$, what is the market share of each country (that is, the percentage of ships that each country sells relative to the total number of ships sold)? What are the profits for each country?
9. Extending the previous problem, suppose China decides to enter the VLCC construction market. The duopoly now becomes a triopoly, so that although price is still $P = 180 - Q$, quantity is now given by $Q = q_{\text{Korea}} + q_{\text{Japan}} + q_{\text{China}}$. Assume that all three countries have a per-ship cost of \$30 million: $c_{\text{Korea}} = c_{\text{Japan}} = c_{\text{China}} = 30$.
   1. Write the profit function for each of the three countries in terms of q_{Korea}, q_{Japan}, and q_{China}, and in terms of c_{Korea}, c_{Japan}, or c_{China}. Find each country's best-response rule.
   2. Using your answer to part (a), find the quantity produced, the market share captured [see Exercise S8, part (c)], and the profits earned by each country. This will require the solution of three equations in three unknowns.
   3. What happens to the price of a VLCC in the new triopoly relative to that in the duopoly situation in Exercise S8, part (b)? Why?

10. Monica and Nancy have formed a business partnership to provide consulting services in the golf industry. They each have to decide how much effort to put into the business. Let m be the amount of effort put into the business by Monica, and n be the amount of effort put in by Nancy.

    The joint profits of the partnership are given by $4m + 4n + mn$, in tens of thousands of dollars, and the two partners split these profits equally. However, each partner's payoff also includes the cost of her own effort; the cost to Monica of her effort is m^2, while the cost to Nancy of her effort is n^2 (both measured in tens of thousands of dollars). Each partner must make her effort decision without knowing what effort decision the other player has made.

    1. If Monica and Nancy each put in effort of $m = n = 1$, then what are their payoffs?
    2. If Monica puts in effort of $m = 1$, then what is Nancy's best response?
    3. What is the Nash equilibrium of this game?

11. Nash equilibrium can be achieved through rationalizability in games with upward-sloping best-response curves if the rounds of eliminating strategies that are never a best response begin with the smallest possible values. Consider the pricing game between Xavier's Tapas Bar and Yvonne's Bistro that is illustrated in Figure 5.1. Use Figure 5.1 and the best-response rules from which it is derived to begin rationalizing the Nash equilibrium in that game. Start with the lowest possible prices for the two restaurants and describe (at least) two rounds of narrowing the set of rationalizable prices toward the Nash equilibrium.
12. A professor presents the following game to Elsa and her 49 classmates. Each of the students simultaneously and privately writes down a number between 0 and 100 on a piece of paper, and they all hand in their numbers. The professor then computes the mean of these numbers and defines X to be the mean of the students' numbers. The student who submits the number closest to one-half of X wins \$50. If multiple students tie, they split the prize equally.
    1. Show that choosing the number 80 is a dominated strategy.
    2. What would the set of best responses be for Elsa if she knew that all of her classmates would submit the number 40? That

is, what is the range of numbers for which each number in the range is closer to the winning number than 40?

3. What would the set of best responses be for Elsa if she knew that all of her classmates would submit the number 10?
4. Find a symmetric Nash equilibrium for this game. That is, what number is a best response to everyone else submitting that same number?
5. Which strategies are rationalizable in this game?

UNSOLVED EXERCISES

1. Diamond Trading Company (DTC), a subsidiary of De Beers, is the dominant supplier of high-quality diamonds for the wholesale market. For simplicity, assume that DTC has a monopoly on wholesale diamonds. The quantity that DTC chooses to sell thus has a direct impact on the wholesale price of diamonds. Let the wholesale price of diamonds (in hundreds of dollars) be given by the following (inverse) demand equation: $P = 120 - Q_{\text{DTC}}$. Assume that DTC has a cost of 12 (hundred dollars) per high-quality diamond.
    1. Write DTC’ s profit function in terms of Q_{DTC}, and solve for DTC’ s profit-maximizing quantity. What will be the wholesale price of diamonds at that quantity? What will DTC’ s profit be?

    Frustrated with DTC’ s monopoly, several diamond mining interests and large retailers collectively set up a joint venture called Adamantia to act as a competitor to DTC in the wholesale market for diamonds. The wholesale price is now given by $P = 120 - Q_{\text{DTC}} - Q_{\text{ADA}}$. Assume that Adamantia has a cost of 12 (hundred dollars) per high-quality diamond.

    1. (b) Write the best-response functions for both DTC and Adamantia. What quantity does each wholesaler supply to the market in equilibrium? What wholesale price do these quantities imply? What will the profit of each supplier be in this duopoly situation?
    2. (c) Describe the differences between the market for wholesale diamonds under the duopoly of DTC and Adamantia and under the monopoly of DTC. What happens to the quantity supplied in the market and the market price when Adamantia enters? What happens to the collective profit of DTC and Adamantia?
2. There are two movie theaters in the town of Harkinsville: Modern Multiplex, which shows first-run movies, and Sticky Shoe, which shows movies that have been out for a while at a cheaper price. The demand for movies at Modern Multiplex is given by $Q_{\text{MM}} = 14 - P_{\text{MM}} + P_{\text{SS}}$, while the demand for movies at Sticky Shoe is $Q_{\text{SS}} = 8 - 2P_{\text{SS}} + P_{\text{MM}}$, where prices are in dollars and quantities are measured in hundreds of moviegoers. Modern Multiplex has a per-

customer cost of \$4, while Sticky Shoe has a per-customer cost of only \$2.

1. In the demand equations alone, what indicates whether Modern Multiplex and Sticky Shoe offer services that are substitutes or complements?
2. Write the profit function for each theater in terms of P_{SS} and P_{MM}. Find each theater’ s best-response rule.
3. Find the Nash equilibrium price, quantity, and profit for each theater.
4. What would each theater’ s price, quantity, and profit be if the two decided to collude to maximize their joint profits in this market? Why isn’ t the outcome of this collusion a Nash equilibrium?

3. Fast forward a decade beyond the situation in Exercise S3. Eten’ s demand for bread and cheese has decreased, and the town’ s two food stores, La Boulangerie and La Fromagerie, have been bought out by a third company: L’ Épicerie. It still costs \$1 to make a loaf of bread and \$2 to make a pound of cheese, but the quantities of bread and cheese sold (Q_1 and Q_2 respectively, measured in thousands) are now given by the equations

   $Q_1 = 8 - P_1 - 0.5P_2, \; Q_2 = 16 - 0.5P_1 - P_2.$

   Again, P_1 is the price in dollars of a loaf of bread, and P_2 is the price in dollars of a pound of cheese.

   1. Initially, L’ Épicerie runs La Boulangerie and La Fromagerie as if they were separate firms, with independent managers who each try to maximize their own profit. What are the Nash equilibrium quantities, prices, and profits for the two divisions of L’ Épicerie, given the new quantity equations?
   2. The owners of L’ Épicerie think that they can make more total profit by coordinating the pricing strategies of the two Eten divisions of their company. What are the joint profit-maximizing prices for bread and cheese under this arrangement? What quantities of each good do La Boulangerie and La Fromagerie sell, and what is the profit that each division earns separately?
   3. In general, why might companies sell some of their goods at prices below cost? That is, explain the rationale of such *loss leaders*, using your answer from part (b) as an illustration.

4. The coconut-milk carts from Exercise S7 set up again the next day. Nearly everything is exactly the same as in Exercise S7: The carts are in the same locations, the number and distribution of beachgoers is identical, and the demand of the beachgoers for exactly one coconut milk each is unchanged. The only difference is that it is a particularly hot day, so that now each beachgoer incurs a higher transport cost of $0.6d^2$. Again, Cart 0 sells to all the beachgoers located between 0 and x, and Cart 1 sells to all the beachgoers located between x and 1, where x is the location of the beachgoer who pays the same total price whether she goes to Cart 0 or Cart 1. However, location x is now defined by the expression

   $$p_0 + 0.6x^2 = p_1 + 0.6(1 - x)^2.$$

   Again, each cart has a cost of $0.25 per coconut sold.

   1. For each cart, determine the expression for the number of customers served as a function of p_0 and p_1. [Recall that Cart 0 gets the customers between 0 and x, or just x, while Cart 1 gets the customers between x and 1, or $1 - x$. That is, Cart 0 sells to x customers, where x is measured in thousands, and Cart 1 sells to $(1 - x)$.]
   2. Write out profit functions for the two carts and find the two best-response rules.
   3. Calculate the Nash equilibrium price for coconuts on the beach. How does this price compare with the price found in Exercise S7? Why?

5. The game illustrated in Figure 5.4 has a unique Nash equilibrium in pure strategies. Find that Nash equilibrium, and then show that it is also the unique rationalizable outcome in that game.
6. What are the rationalizable strategies of the game of “Evens or Odds” from Exercise S14 in Chapter 4?
7. In the fishing-boat game of Section 3.B, we showed that it is possible for a game to have a uniquely rationalizable outcome in continuous strategies that is also a Nash equilibrium. However, this is not always the case; there may be many rationalizable strategies, and not all of them will necessarily be part of a Nash equilibrium.

Returning to the political advertising game of Exercise S1, find the set of rationalizable strategies for party L. (Due to their symmetric payoffs, the set of rationalizable strategies will be the same for party R.) Explain your reasoning.

8. Intel and AMD, the primary producers of computer central processing units (CPUs), compete with each other in the mid-range chip category (among other categories). Assume that global demand for mid-range chips depends on the quantity that the two firms make, so that the price (in dollars) for mid-range chips is given by $P = 210 - Q$, where $Q = q_{\text{Intel}} + q_{\text{AMD}}$ and where the quantities are measured in millions. Each mid-range chip costs Intel \$60 to produce. AMD's production process is more streamlined; each chip costs it only \$48 to produce.
   1. Write the profit function for each firm in terms of q_{Intel} and q_{AMD}. Find each firm's best-response rule.
   2. Find the Nash equilibrium price, quantity, and profit for each firm.
   3. (**Optional**) Suppose Intel acquires AMD, so that it now has two separate divisions with two different production costs. The merged firm wishes to maximize total profits from the two divisions. How many chips should each division produce? (Hint: You may need to think carefully about this problem, rather than blindly applying mathematical techniques.) What is the market price and the total profit for the firm?
9. Return to the VLCC triopoly game of Exercise S9. In reality, the three countries do not have identical production costs. China has been gradually entering the VLCC construction market for several years, and its production costs started out rather high due to its lack of experience.
   1. Solve for the triopoly quantities, market shares, prices, and profits for the case where the per-ship costs are \$20 million for Korea, \$40 million for Japan, and \$60 million for China ($c_{\text{Korea}} = 20$, $c_{\text{Japan}} = 40$, and $c_{\text{China}} = 60$).

   After it gains experience and adds production capacity, China's per-ship cost will decrease dramatically. Because labor is even cheaper in China than in Korea, eventually the per-ship cost will be even lower in China than it is in Korea.

   1. (b) Repeat part (a) with the adjustment that China's per-ship cost is \$16 million ($c_{\text{Korea}} = 20$, $c_{\text{Japan}} = 40$, and $c_{\text{China}} =$

16).

10. Return to the story of Monica and Nancy from Exercise S10. After some additional professional training, Monica is more productive on the job, so that the joint profits of their company are now given by $5m + 4n + mn$, in tens of thousands of dollars. Again, m is the amount of effort put into the business by Monica, n is the amount of effort put in by Nancy, and the costs of their efforts are m^2 and n^2 to Monica and Nancy respectively (in tens of thousands of dollars).

    The terms of their partnership still require that the joint profits be split equally, despite the fact that Monica is more productive. Assume that their effort decisions are made simultaneously.

    1. What is Monica's best response if she expects Nancy to put in an effort of $n = 4/3$?
    2. What is the Nash equilibrium of this game?
    3. Compared with the old Nash equilibrium found in Exercise S10, part (c), does Monica now put in more, less, or the same amount of effort? What about Nancy?
    4. What are the final payoffs to Monica and Nancy in the new Nash equilibrium (after splitting the joint profits and accounting for the costs of their efforts)? How do these payoffs compare to the payoffs to each of them under the old Nash equilibrium? In the end, who receives more benefit from Monica's additional training?
11. A professor presents a new game to Elsa and her 49 classmates (similar to the situation in Exercise S12). As before, each of the students simultaneously and privately writes down a number between 0 and 100 on a piece of paper, and the professor computes the mean of these numbers and calls it X. This time, the student who submits the number closest to $2/3 \times (X + 9)$ wins \$50. Again, if multiple students tie, they split the prize equally.
    1. Find a symmetric Nash equilibrium for this game. That is, what number is a best response to everyone else submitting the same number?
    2. Show that choosing the number 5 is a dominated strategy. (Hint: What would class average X have to be for the target number to be 5?)
    3. Show that choosing the number 90 is a dominated strategy.
    4. What are all of the dominated strategies?

5. Suppose Elsa believes that none of her classmates will play the dominated strategies you found in part (d). Given her beliefs, what strategies are never a best response for Elsa?
6. Which strategies do you think are rationalizable in this game? Explain your reasoning.

12. (**Optional**—requires calculus) Recall the political campaign advertising example from Section 1.C concerning parties L and R. In that example, when L spends $\$x$ million on advertising and R spends $\$y$ million, L gets a share $x/(x + y)$ of the votes and R gets a share $y/(x + y)$. We also mentioned that two types of asymmetries can arise between the parties in that model: One party—say, R—may be able to advertise at a lower cost, or R's advertising dollars may be more effective in generating votes than L's. To allow for both possibilities, we can write the payoff functions of the two parties as

$$V_L = \frac{x}{x+ky} - x \text{ and } V_n = \frac{ky}{x+ky} - cy, \text{ where } k>0 \text{ and } c>0.$$

These payoff functions show that R has an advantage in the relative effectiveness of its ads when k is high and that R has an advantage in the cost of its ads when c is low.

1. Use the payoff functions to derive the best-response functions for R (which chooses y) and L (which chooses x).
2. Use your calculator or your computer to graph these best-response functions when $k = 1$ and $c = 1$. Compare the graph with the one for the case in which $k = 1$ and $c = 0.8$. What is the effect of having an advantage in the cost of advertising?
3. Compare the graph from part (b), when $k = 1$ and $c = 1$, with the one for the case in which $k = 2$ and $c = 1$. What is the effect of having an advantage in the effectiveness of advertising dollars?
4. Solve the best-response functions that you found in part (a), jointly for x and y, to show that the campaign advertising expenditures in Nash equilibrium are

$$x = \frac{ck}{(c+k)^2} - x \text{ and } y = \frac{k}{(c+k)^2}.$$

5. Let $k = 1$ in the equilibrium equations in part (d) and show how the two equilibrium spending levels vary with changes in c (that is, interpret the signs of dx/dc and dy/dc). Then let $c = 1$ and show how the two equilibrium spending levels vary with changes in k (that is, interpret the signs of dx/dk and dy/dk). Do your answers support the effects that you observed in parts (b) and (c) of this exercise?

■ Appendix: Finding a Value to Maximize a Function

Here we develop in a simple way the method for choosing a variable X to obtain the maximum value of a variable that is a function of it, say $Y = F(X)$. Our applications will mostly be to cases where the function is quadratic, such as $Y = A + BX - CX^2$. For such functions we derive the formula $X = B/(2C)$ that was stated and used in the chapter text. We develop the general idea using calculus, and then offer an alternative approach that does not use calculus but applies only to the quadratic function.[27]

The calculus method tests a value of X for optimality by seeing what happens to the value of the function for other values on either side of X. If X does indeed maximize $Y = F(X)$, then the effect of increasing or decreasing X should be a drop in the value of Y. Calculus gives us a quick way to perform such a test.

Figure 5A.1 illustrates the basic idea. It shows the graph of a function $Y = F(X)$, where we have used a function of the type that fits our application, even though the idea is perfectly general. Start at any point P with coordinates (X, Y) on the graph. Consider a slightly different value of X, say $(X + h)$. Let k be the resulting change in $Y = F(X)$, so the point Q with coordinates $(X + h, Y + k)$ is also on the graph. The slope of the chord joining P to Q is the ratio k/h. If this ratio is positive, then h and k have the same sign; as X increases, so does Y. If the ratio is negative, then h and k have opposite signs; as X increases, Y decreases.

If we now consider smaller and smaller changes h in X, and the corresponding smaller and smaller changes k in Y, the chord PQ will approach the tangent to the graph at P. The slope of this tangent is the limiting value of the ratio k/h. It is called the derivative of the function $Y = F(X)$ at the point X. Symbolically, it is written as $F'(X)$ or dY/dX. Its sign tells us whether the function is increasing or decreasing at precisely the point X.

For the quadratic function in our application, $Y = A + BX - CX^2$ and

$$Y + k = A + B(X + h) - C(X + h)^2.$$

Therefore, we can find an expression for k as follows:

$$k = [A + B(X + h) - C(X + h)^2] - (A + BX - CX^2)$$

$$= Bh - C[(X + h)^2 - X^2]$$

$$= Bh - C(X^2 + 2Xh + h^2 - X^2)$$

$$= (B - 2CX)h - Ch^2.$$

Then $k/h = (B - 2CX) - Ch$. In the limit as h goes to zero, $k/h = (B - 2CX)$. This last expression is then the derivative of our function.

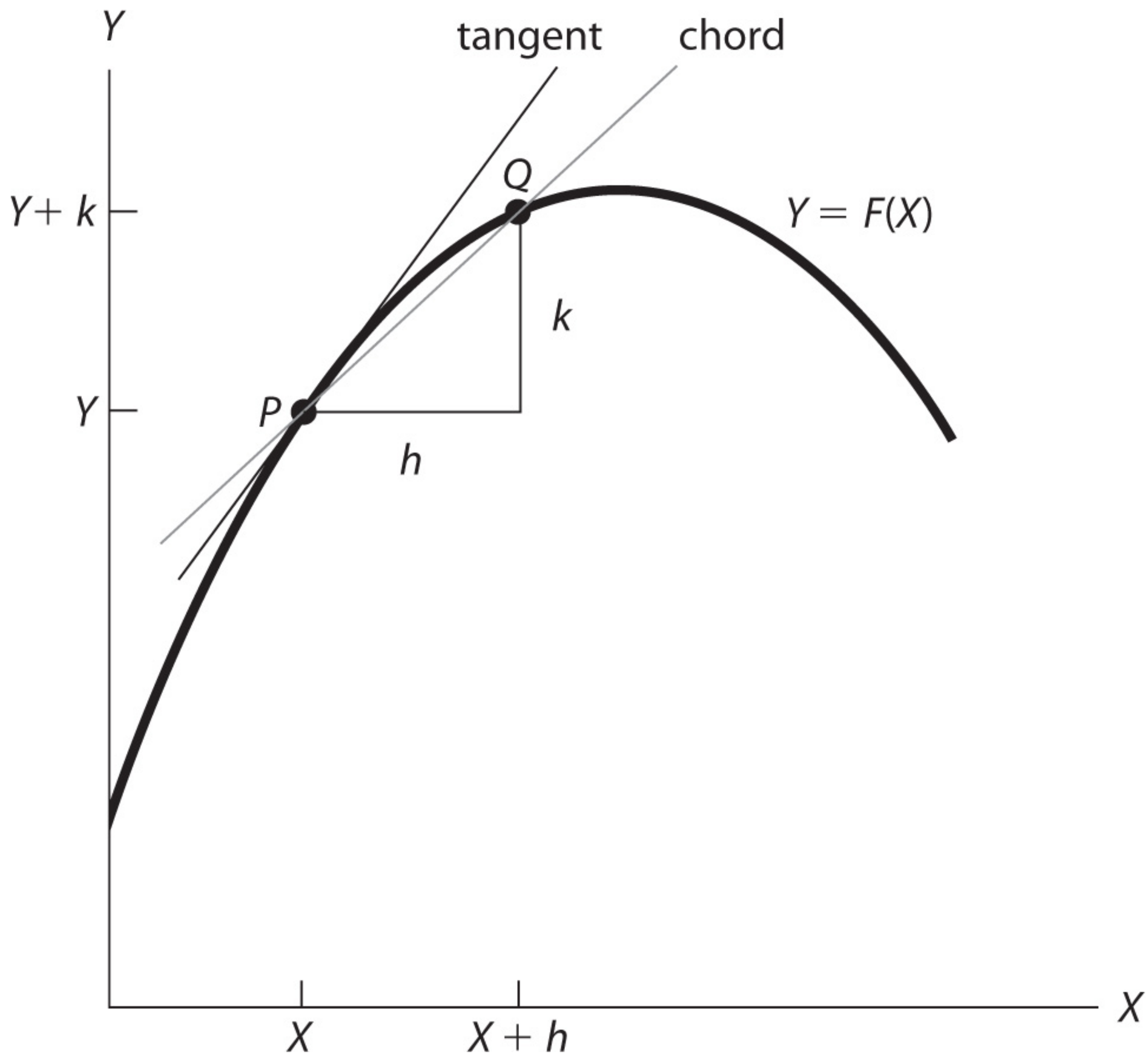

Figure 5A.1 Derivative of a Function Illustrated

Now we use the derivative to find a test for optimality. Figure 5A.2 illustrates the idea. The point M yields the highest value of $Y = F(X)$. The function increases as we approach the point M from the left and decreases after we have passed to the right of M. Therefore, the derivative $F'(X)$ should be positive for values of X smaller than M and negative for values of X larger than M. By continuity, the derivative precisely at M should be 0. In ordinary language, the graph of the function should be flat where it peaks.

In our quadratic example, the derivative is $F'(X) = B - 2CX$. Our optimality test implies that the function is optimized when this is 0, or at $X = B/(2C)$. This is exactly the formula given in the chapter text.

One additional check needs to be performed. If we turn the whole figure upside down, M is the minimum value of the upside-down function, and at this trough the graph will also be flat. So, for a general function $F(X)$, setting $F'(X) = 0$ might yield an X that gives its minimum rather than its maximum. How do we distinguish the two possibilities?

At a maximum, the function will be increasing to its left and decreasing to its right. Therefore, the derivative will be positive for values of X smaller than the purported maximum, and negative for larger values. In other words, the derivative, itself regarded as a function of X, will be decreasing at this point. A decreasing function has a negative derivative. Therefore, the derivative of the derivative, what is called the second derivative of the original function, written as $F'(X)$ or d^2Y/dX^2, should be negative at a maximum. Similar logic shows that the second derivative should be positive at a minimum; that is what distinguishes the two cases.

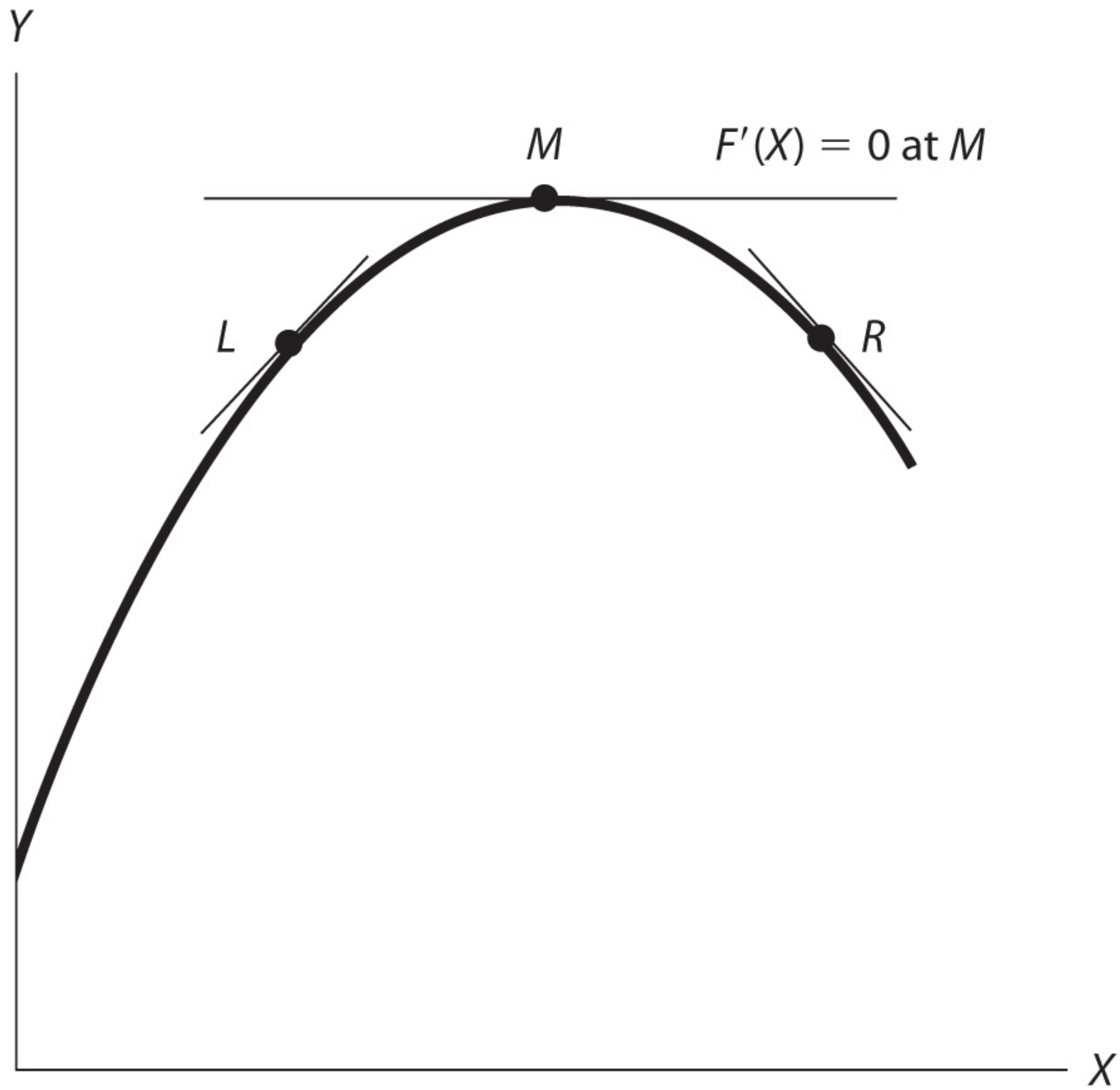

Figure 5A.2 Optimum of a Function

For the derivative $F'(X) = B - 2CX$ of our quadratic example, applying the same h, k procedure to $F'(X)$ as we did to $F(X)$ shows $F'(X) = -2C$. This result is negative so long as C is positive, which we assumed when stating the problem in the chapter text. The test $F'(X) = 0$ is called the first-order condition for maximization of $F(X)$, and $F'(X) < 0$ is the second-order condition.

To fix the idea further, let us apply it to the specific example of Xavier's best response that we considered in the chapter text. We had the expression

$\Pi_x = -8(44 + P_y) + (16 + 44 + P_y)P_x - 2(P_x)^2.$

This is a quadratic function of P_x (holding the other restaurant's price, P_y, fixed). Our method gives its derivative:

$$\frac{d\Pi_x}{dP_x} = (60 + P_y) - 4P_x.$$

The first-order condition for P_x to maximize Π_x is that this derivative should be 0. Setting it equal to 0 and solving for P_x gives the same equation as derived in Section 1.A. (The second-order condition is $d^2\Pi_x/dP_x^2 < 0$, which is satisfied because the second-order derivative is just −4.)

We hope you will regard the calculus method as simple enough and that you will have occasion to use it again in a few places later —for example, in Chapter 11 on collective action. But if you find it too difficult, here is a noncalculus alternative method that works for quadratic functions. Rearrange terms to write the function as

$$
\begin{aligned}
Y &= A + BX - CX^2 \\
&= A + \frac{B^2}{4C} - \frac{B^2}{4C} + BX - CX^2 \\
&= A + \frac{B^2}{4C} - C\left(\frac{B^2}{4C^2} - 2\frac{B}{C} + X^2 \right) \\
&= A + \frac{B^2}{4C} - C\left(\frac{B}{2C} - X \right)^2 .
\end{aligned}
$$

In the final form of the expression, X appears only in the last term, where a square involving it is being subtracted (remember $C > 0$). The whole expression is maximized when this subtracted term is made as small as possible, which happens when $X = B/(2C)$. Voila!

This method of "completing the square" works for quadratic functions and therefore will suffice for most of our uses. It also avoids calculus. But we must admit that it smacks of magic. Calculus is more general and more methodical. It repays a little study many times over.

Endnotes

- Needless to say, we give only the briefest, quickest treatment, leaving out all issues of functions that don't have derivatives, functions that are maximized at an extreme point of the interval over which they are defined, and so on. Some readers will know all we say here; some will know much more. Others who want to find out more should refer to any introductory calculus textbook. Return to reference 27

6 ■ Combining Sequential and Simultaneous Moves

IN CHAPTER 3, we considered games with purely sequential moves; Chapters 4 and 5 dealt with games with purely simultaneous moves. We developed concepts and techniques of analysis appropriate to these pure game types: game trees and rollback equilibrium for sequential-move games, game tables and Nash equilibrium for simultaneous-move games. In reality, however, many strategic situations contain elements of both types of interactions. Furthermore, although we used game trees (extensive forms) as the sole method of illustrating sequential-move games and game tables (strategic forms) as the sole method of illustrating simultaneous-move games, we can use either form for any type of game.

In this chapter, we examine many of these possibilities. We begin by showing how games that combine sequential and simultaneous moves can be solved by combining game trees and game tables, and by combining rollback and Nash equilibrium analysis, in appropriate ways. Then we consider the effects of changing the nature of the interactions in a particular game. Specifically, we look at the effects of changing the rules of a game to convert sequential play into simultaneous play, and vice versa, and of changing the order of moves in sequential play. This topic gives us an opportunity to compare the equilibria found by using the concept of rollback, in a sequential-move game, with those found by using the Nash equilibrium concept, in the simultaneous-move version of the same game. From this comparison, we extend the concept of Nash equilibria to sequential-move games. It turns out that the rollback equilibrium is a special case, usually called a refinement, of these Nash equilibria.

1 GAMES WITH BOTH SIMULTANEOUS AND SEQUENTIAL MOVES

As mentioned several times thus far, most real games that you will encounter will be made up of numerous smaller components. Each of these components may entail simultaneous play or sequential play, so that the full game requires you to be familiar with both. The most obvious examples of strategic interactions containing both sequential and simultaneous components are those between two (or more) players over an extended period of time. You may play a number of different simultaneous-move games against your roommate during your year together: Your action in any one of these games is influenced by the history of your interactions up to then and by your expectations about the interactions to come. Many sporting events, interactions between competing firms in an industry, and political relationships are also sequentially linked series of simultaneous-move games. Such games can be analyzed by combining the tools presented in Chapter 3 (game trees and rollback) and in Chapters 4 and 5 (game tables and Nash equilibria).[1] The only difference is that the actual analysis becomes more complicated as the number of moves and interactions increases.

A. Two-Stage Games and Subgames

Our main illustrative example of a game with both simultaneous and sequential components tells a story of two would-be telecom giants, CrossTalk and GlobalDialog. Each firm can choose whether to invest $10 billion in the purchase of a fiber-optic network. The two firms make their investment decisions simultaneously. If neither chooses to make the investment, that is the end of the game. If one invests and the other does not, then the investor has to make a pricing decision for its telecom services. It can choose either a high price, which will attract 60 million customers, from each of whom it will make an operating profit (gross profit not including the cost of its investment) of $400, or a low price, which will attract 80 million customers, from each of whom it will make an operating profit of $200. If both firms acquire fiber-optic networks and enter the market, then their pricing choices become a second simultaneous-move game. Each firm can choose either the high or the low price. If both choose the high price, they will split the total market equally, each will get 30 million customers and an operating profit of $400 from each customer. If both choose the low price, again they will split the total market equally, so each will get 40 million customers and an operating profit of $200 from each customer. If one firm chooses the high price and the other the low price, then the low-price firm will get all the 80 million customers at that price, and the high-price firm will get nothing.

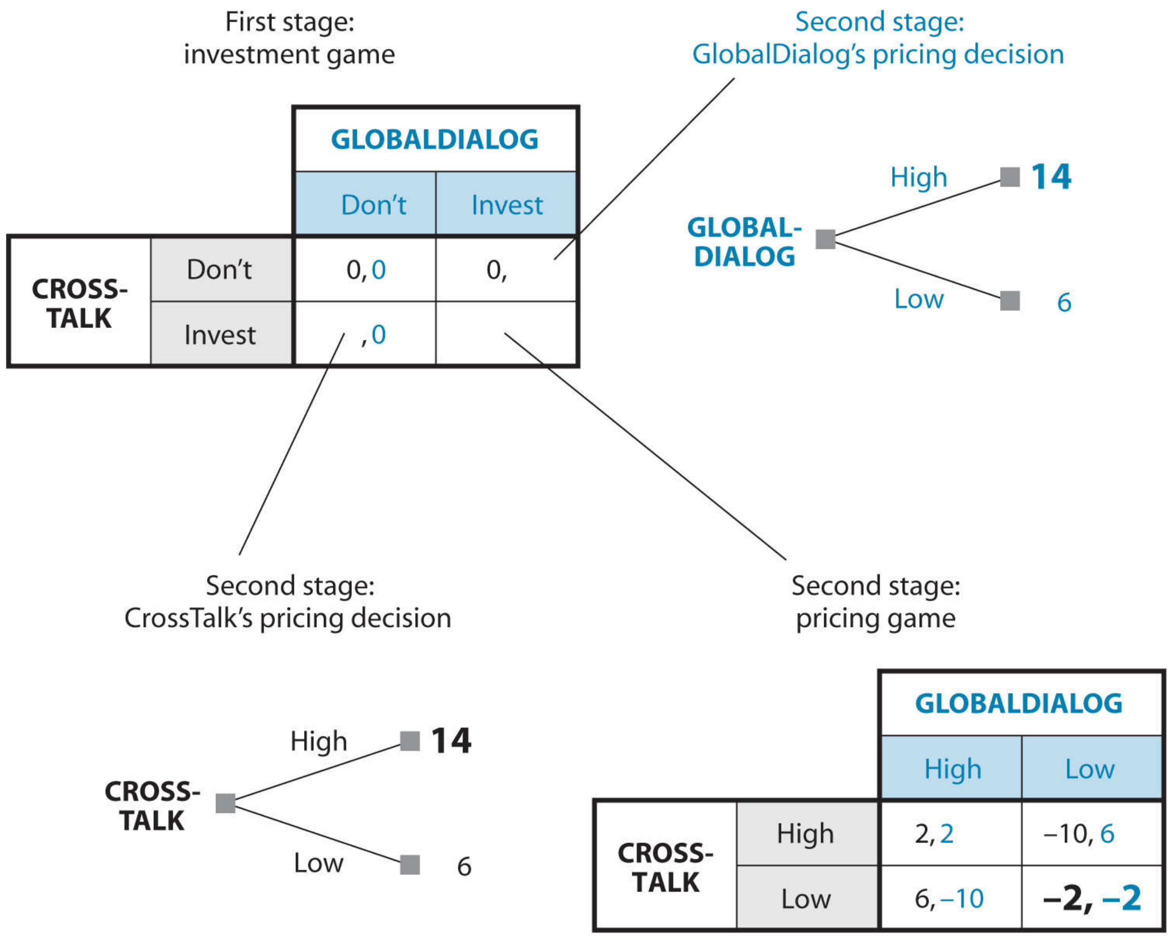

Figure 6.1 Two-Stage Game Combining Sequential and Simultaneous Moves

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The interaction between CrossTalk and GlobalDialog forms a two-stage game. Of the four combinations of simultaneous-move choices at the first (investment) stage, one ends the game, two lead to a second-stage (pricing) decision by just one player, and the last leads to a simultaneous-move (pricing) game at the second stage. We show this game pictorially in Figure 6.1.

Regarded as a whole, Figure 6.1 illustrates a game tree, but one that is more complex than the trees in Chapter 3. You can think of it as an elaborate “tree house” with multiple levels. The levels are shown in different parts of the same two-dimensional figure, as if you are looking down at the tree from a helicopter positioned directly above it.

The first-stage game is represented by the payoff table in the top-left quadrant of Figure 6.1. You can think of it as the first floor of the tree house. It has four “rooms.” The room in the northwest corner corresponds to “Don’t invest” first-stage moves by both firms. If the firms’ decisions take the game to this room, there are no further choices to be made, so we can think of this room as equivalent to a terminal node of a game tree, and we can show the payoffs in that cell of the table: Both firms get 0. However, all the other combinations of actions for the two firms lead to rooms that lead to further choices, so we cannot yet show all the payoffs for those cells. Instead, we show branches leading to the second floor of the tree house. The northeast and southwest rooms show only the payoff to the firm that has not invested; the branches leading from each of these rooms take us to single-firm pricing decisions in the second stage. The branch from the southeast room leads to a multiroom second-floor structure within the tree house, which represents the second-stage pricing game that is played if both firms have invested in the first stage. This second-floor structure has four rooms corresponding to the four combinations of the two firms’ pricing moves. All the second-floor rooms are like terminal nodes of a game tree, so we can show the payoffs in each case. Those payoffs consist of each firm’s net profit—operating profit minus its previous investment costs; payoff values are written in billions of dollars.

To see how the payoff values are calculated, consider first the second-stage pricing decision illustrated in the southwest corner of Figure 6.1. The game arrives in that corner if CrossTalk is the only firm that has invested. Then, if it chooses the high price, its operating profit is $400 × 60 million = $24 billion; after subtracting the $10 billion investment cost, its payoff (net profit) is $14 billion, which we write as 14. In the same corner, if CrossTalk chooses the low price, then its operating profit is $200 × 80 million = $16 billion, yielding the payoff 6 after accounting for its original investment. In this situation, GlobalDialog’s payoff is 0, as shown in the southwest room of the first floor of our tree house. Similar calculations for the case in which GlobalDialog is the only firm to invest give us the payoffs shown in the northeast corner of Figure 6.1; again, the payoff of 0 for CrossTalk is shown in the northeast room of the first-stage game table.

If both firms invest, both play the second-stage pricing game illustrated in the southeast corner of the figure. When both choose the high price in the second stage, each gets operating profit of $400 × 30 million (half of the market), or $12 billion; after subtracting the $10 billion investment cost, each is left with a net profit of $2 billion, or a payoff of 2. If both firms choose the low price in the second stage, each gets operating profit of $200 × 40 million = $8 billion, and after subtracting the $10 billion investment cost, each is left with a net loss of $2 billion, or a payoff of −2. Finally, if one firm charges the high price and the other firm the low price, then the low-price firm has operating profit of $200 × 80 million = $16 billion, leading to the payoff 6, while the high-price firm gets no operating profit and simply loses its $10 billion investment, for a payoff of −10.

As with any multistage game in Chapter 3, we must solve this game backward, starting with the second-stage game. In the case of the two single-firm pricing decisions, we see at once that the high-price decision yields the higher payoff. We highlight this by showing that payoff in larger type.

The second-stage pricing game has to be solved by using methods developed in Chapter 4. It is immediately evident, however, that this game is a prisoners' dilemma. Low price is the dominant strategy for each firm, so the outcome is the room in the southeast corner of the second-stage game table: Each firm gets a payoff of −2.[2] Again, we show these payoffs in large type to highlight the fact that they are the payoffs obtained in the second-stage equilibrium.

		GLOBALDIALOG	
		Don't	Invest
CROSSTALK	Don't	0, 0	0, 14
	Invest	14, 0	−2, −2

FIGURE 6.2 First-Stage Investment Game (After Substituting Rolled-Back Payoffs from the Equilibrium of the Second Stage)

Rollback analysis now tells us that each set of first-stage moves should be evaluated by looking ahead to the equilibrium of the second-stage game (or the optimal second-stage decision in cases where only one firm has an action to take) and the resulting payoffs. We can therefore substitute the second-stage payoffs that we have just calculated into the previously empty or partly empty rooms on the first floor of our tree house. This substitution gives us a first floor with known payoffs, shown in Figure 6.2.

Now we can use the methods of Chapter 4 to solve this simultaneous-move game. You should immediately recognize the game in Figure 6.2 as a game of chicken. It has two pure-strategy Nash equilibria, each of which entails one firm choosing Invest and the other choosing Don't. The firm that invests makes a huge profit, so each firm prefers the equilibrium in which it is the investor while the other firm stays out of the market. In Chapter 4, we briefly discussed the ways in which one of the two equilibria might get selected. We also pointed out the possibility that each firm might try to get its preferred outcome, with the result, in this case, that both of them invest and both lose money. In Chapter 7, we will investigate this type of game further, showing that it has a third Nash equilibrium, in mixed strategies.

Analysis of Figure 6.2 shows that the first-stage game in our example does not have a unique Nash equilibrium. This problem is not too serious, because we can leave the solution ambiguous to the extent we did in the preceding paragraph. Matters would be more difficult if the second-stage game did not have a unique equilibrium. Then it would be essential to specify the precise process by which an outcome gets selected so that we could figure out the second-stage payoffs and use them to roll back to the first stage.

The second-stage pricing game shown in the payoff table in the bottom-right quadrant of Figure 6.1 is one part of the complete two-stage game. However, it is also a full-fledged game in its own right, with a fully specified structure of players, strategies, and payoffs. To bring out this dual nature more explicitly, it is called a subgame of the full game.

More generally, a subgame is the part of a multimove game that begins at a particular node of the original game. The tree for a subgame is then just that part of the tree for the full game that takes this node as its root, or initial node. A multimove game has as many subgames as it has decision nodes.

B. Configurations of Multistage Games

In the multistage game illustrated in Figure 6.1, each stage consists of a simultaneous-move game. However, that may not always be the case. Simultaneous and sequential components may be mixed and matched in any way. In this section, we give two more examples to clarify this point and to reinforce the ideas introduced in the preceding section.

The first example is a slight variation of the CrossTalk–GlobalDialog game. Suppose one of the firms—say, GlobalDialog—has already made the $10 billion investment in the fiber-optic network. CrossTalk knows of this investment and now has to decide whether to make its own investment. If CrossTalk does not invest, then GlobalDialog will have a simple pricing decision to make. If CrossTalk invests, then the two firms will play the second-stage pricing game already described. The tree for this multistage game has conventional branches starting at the initial node and has a simultaneous-move subgame starting at one of the nodes to which these initial branches lead. The complete tree is shown in Figure 6.3.

When the tree has been set up, it is easy to analyze the game. We show the rollback analysis in Figure 6.3 by using large type for the equilibrium payoffs that result from the second-stage game or decision and a thicker branch for CrossTalk's first-stage choice. In words, CrossTalk figures out that if it invests, the ensuing prisoners' dilemma of pricing will leave it with payoff −2, whereas staying out will get it 0. Thus, it prefers the latter. GlobalDialog gets 14 instead of the −2 that it would have gotten if CrossTalk had invested, but CrossTalk's concern is to maximize its own payoff and not to ruin GlobalDialog deliberately.

This analysis does raise the possibility, though, that GlobalDialog may try to get its investment done quickly, before CrossTalk makes its decision, so as to ensure its most preferred outcome from the full game. And CrossTalk may try to beat

GlobalDialog to the punch in the same way. In Chapter 8, we study some methods, called strategic moves, that may enable players to secure such advantages.

Our second example comes from football. Before each play, the coach for the offense chooses the play that his team will run; simultaneously, the coach for the defense sends his team out with instructions on how they should align themselves to counter the offense. Thus, these moves are simultaneous. Suppose the offense has just two alternatives, a safe play and a risky play, and the defense can align itself to counter either of them. If the offense has planned to run the risky play and the quarterback sees the defensive alignment that will counter it, he can call for a change in the play at the line of scrimmage. And the defense, hearing the change, can respond by changing its own alignment. Thus, we have a simultaneous-move game at the first stage, and one of the combinations of moves at this stage leads to a sequential-move subgame. Figure 6.4 shows the complete tree.

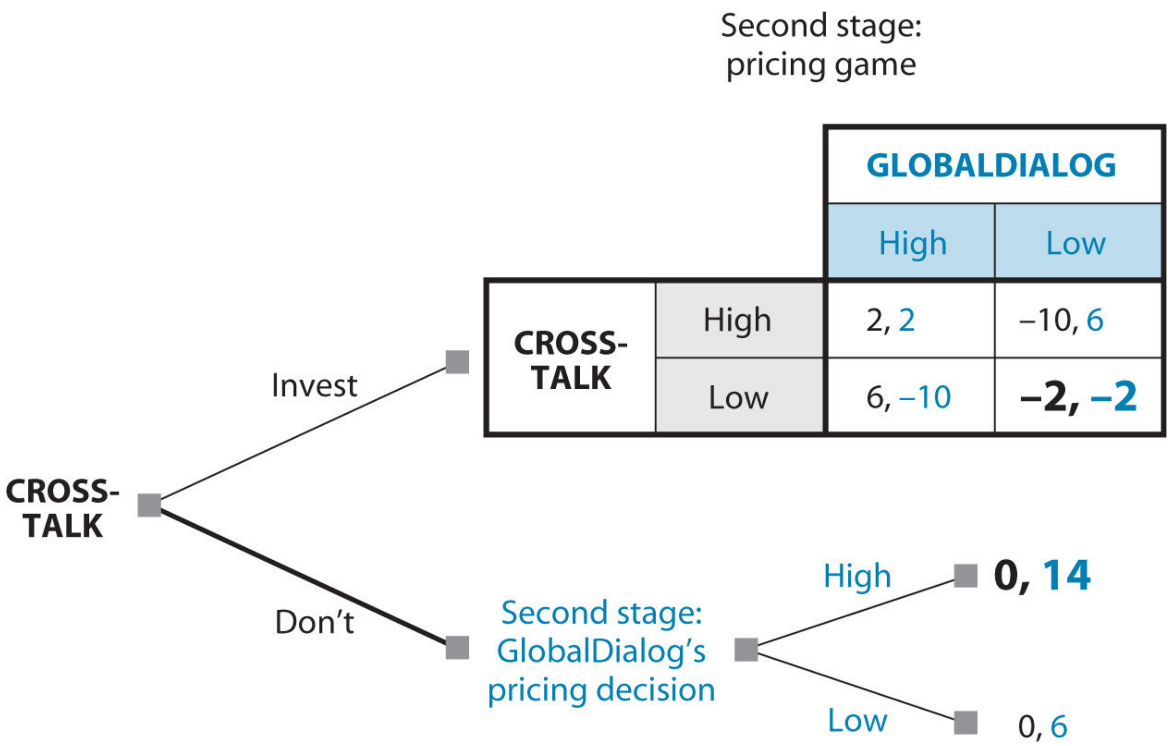


Figure 6.3 Two-Stage Game When One Firm Has Already Invested

This is a zero-sum game in which the offense's payoffs are measured in the number of yards that it expects to gain, and the defense's payoffs are exactly the opposite, measured in the number of yards it expects to give up. The safe play for the offense gets it 2 yards, even if the defense is ready for it; if the defense is not ready for it, the safe play does not do much better, gaining 6 yards. The risky play, if it catches the defense unready to cover it, gains 30 yards. But if the defense is ready for the risky play, the offense loses 10 yards. We show this set of payoffs of −10 for the offense and 10 for the defense at the terminal node in the subgame where the offense does not change the play. If the offense does change the play (back to safe), the payoffs are (2, −2) if the defense responds and (6, −6) if it does not; these payoffs are the same as those that arise when the offense plans the safe play from the start.

We show the chosen branches in the sequential subgame as thick lines in Figure 6.4. It is easy to see that if the offense changes its play, the defense will respond to keep its payoff at −2 rather than −6, and that the offense will still prefer to change its play to get 2 rather than −10. Rolling back, we put the resulting set of payoffs, (2, −2), in the bottom-right cell of the simultaneous-move game of the first stage. Then we see that this game has no Nash equilibrium in pure strategies. The reason is the same as that in the tennis-point game of Chapter 4, Section 8: One player (defense) wants to match its opponent's moves (align to counter the play that the offense is choosing) while the other (offense) wants to unmatch its opponent's moves (catch the defense in the wrong alignment). In Chapter 7, we show how to calculate the mixed-strategy equilibrium of such a game. It turns out that the offense should choose the risky play with probability ⅛, or 12.5%.

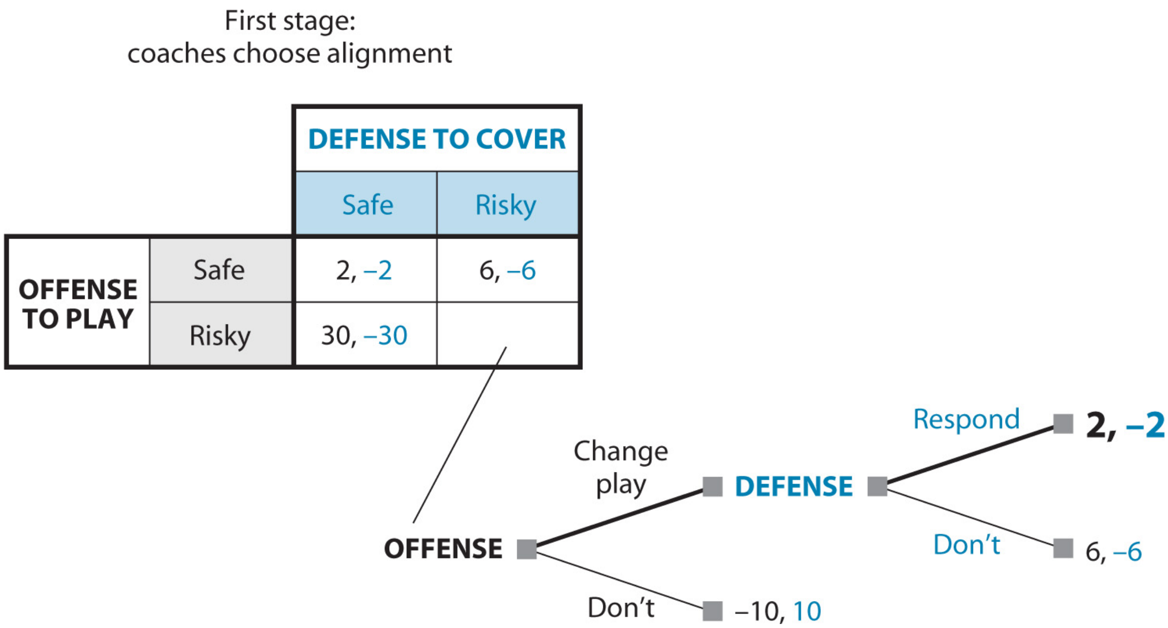


Figure 6.4 Simultaneous-Move First Stage Followed by Sequential Moves

Endnotes

- Sometimes the simultaneous part of the game will have equilibria in mixed strategies; then, the tools we develop in Chapter 7 will be required. We mention this possibility in this chapter where relevant and give you an opportunity to use such methods in exercises for the later chapters. Return to reference 1
- As is usual in a prisoners' dilemma, if the firms could successfully collude and charge high prices, both could get the higher payoff of 2. But this outcome is not an equilibrium because each firm is tempted to cheat to try to get the much higher payoff of 6. Return to reference 2

Glossary

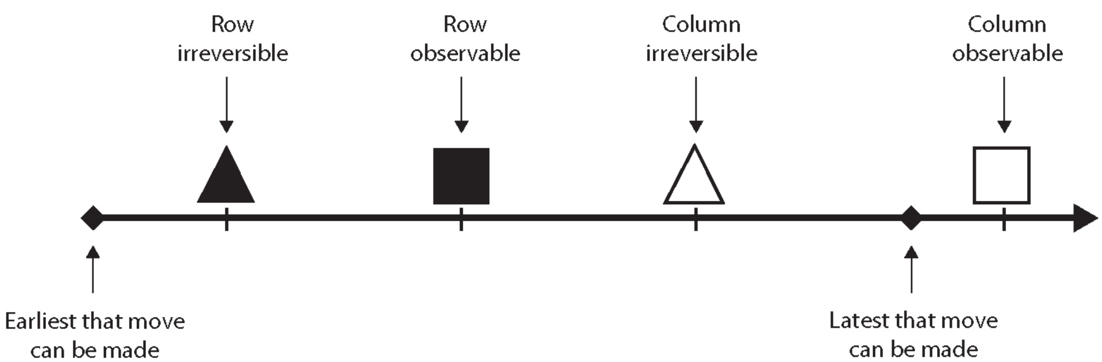
subgame
: A game comprising a portion or remnant of a larger game, starting at a noninitial node of the larger game.

2 CHANGING THE ORDER OF MOVES IN A GAME

Thus far, we have presented every game as having a fixed order of moves that is out of the players' control. This approach has allowed us to develop the appropriate concepts and methods to analyze different types of games: backward induction to find the rollback equilibrium in sequential-move games (Chapter 3), best-response analysis to find Nash equilibria in simultaneous-move games (Chapters 4 and 5), or a blend of both methods for games with elements of both sequential and simultaneous play (Section 1 of this chapter). In this section, we focus in more detail on what determines the strategic order of moves (or simply "order of moves") in a game, and we consider a player's ability and incentive to change the order of moves.

A. What Determines the Order of Moves?

The strategic order of moves in a two-player game depends on two key factors: when each move becomes irreversible and when it becomes observable. To determine the strategic order of moves, we first ask, When does each player’ s move become irreversible? In other words, When does each player “make her move” ? If both players make their moves at the same exact moment in time, then the game obviously has simultaneous moves. But what if one player makes her move before the other player in chronological time? Whether the game has sequential or simultaneous moves depends on when this earlier move becomes observable.



(a) Example: Row moves first


(b) Example: Column moves first