14.B.5[B] Suppose that in the hidden action model explored in Section 14.B the manager can not only choose how much effort to exert but can also, after observing the realization of the firm's profits π, unobservably reduce them in a way that is of no direct benefit to him (e.g., he can voluntarily offer to pay more for his inputs). Show that in this case there is always an optimal incentive scheme that is nondecreasing in observed profits.

14.B.6[B] Amend the two-effort-level model studied in Section 14.B as follows: Suppose now that effort has distinct effects on revenues R and costs C, where $\pi = R - C$. Let $f_R(R \mid e)$ and $f_C(C \mid e)$ denote the density functions of R and C conditional on e, and assume that, conditional on e, R and C are independently distributed. Assume $R \in [\underline{R}, \bar{R}]$, $C \in [\underline{C}, \bar{C}]$, and that for all e, $f_R(R \mid e) > 0$ for all $R \in [\underline{R}, \bar{R}]$ and $f_C(C \mid e) > 0$ for all $C \in [\underline{C}, \bar{C}]$.

The two effort choices are now $\{e_R, e_C\}$, where e_R is an effort choice that devotes more time to revenue enhancement and less to cost reduction, and the opposite is true for e_C. In particular, assume that $F_R(R \mid e_R) < F_R(R \mid e_C)$ for all $R \in (\underline{R}, \bar{R})$ and that $F_C(C \mid e_C) > F_C(C \mid e_R)$ for all $C \in (\underline{C}, \bar{C})$. Moreover, assume that the monotone likelihood ratio property holds for each of these variables in the following form: $[f_R(R \mid e_R)/f_R(R \mid e_C)]$ is increasing in R, and $[f_C(C \mid e_R)/f_C(C \mid e_C)]$ is increasing in C. Finally, the manager prefers revenue enhancement over cost reduction: that is, $g(e_C) > g(e_R)$.

(a) Suppose that the owner wants to implement effort choice e_C and that both R and C are observable. Derive the first-order condition for the optimal compensation scheme $w(R, C)$. How does it depend on R and C?

(b) How would your answer to **(a)** change if the manager could always unobservably reduce the revenues of the firm (in a way that is of no direct benefit to him)?

(c) What if, in addition, costs are now unobservable by a court (so that compensation can be made contingent only on revenues)?

14.B.7[C] Consider a two-period model that involves two repetitions of the two-effort-level hidden action model studied in Section 14.B. There is no discounting by either the firm or the manager. The manager's expected utility over the two periods is the sum of his two single-period expected utilities $E[v(w) - g(e)]$, where $v'(\cdot) > 0$ and $v''(\cdot) < 0$.

Suppose that a contract can be signed ex ante that gives payoffs in each period as a function of performance up until then. Will period 2 wages depend on period 1 profits in the optimal contract?

14.B.8[C] Amend the two-effort-choice hidden action model discussed in Section 14.B as follows: Suppose the principal can, for a cost of c, observe an extra signal $\tilde{y}$ of the agent's effort. Profits π and the signal y have a joint distribution $f(\pi, y \mid e)$ conditional on e. The decision to investigate the value of y can be made after observing π.

A contract now specifies a wage schedule $w(\pi)$ in the event of no investigation, a wage schedule $w(\pi, y)$ if an investigation occurs, and a probability $p(\pi)$ of investigation conditional on π. Characterize the optimal contract for implementing effort level e_H.

14.C.1[C] Analyze the extension of the hidden information model discussed in Section 14.C where there are an arbitrary finite number of states $(\theta_1, \ldots, \theta_N)$ where $\theta_{i+1} > \theta_i$ for all i.

14.C.2[B] Consider the hidden information model in Section 14.C, but now let the manager be risk neutral with utility function $v(w) = w$. Show that the owner can do as well when θ is unobservable as when it is observable. In particular, show that he can accomplish this with a contract that offers the manager a compensation scheme of the form $w(\pi) = \pi - \alpha$ and allows him to choose any effort level he wants. Graph this function and the manager's choices in (w, e)-space. What revelation mechanism would give this same outcome?

14.C.3$^{\mathbf{B}}$ Suppose that in the two-state hidden information model examined in Section 14.C, $u(w, e, \theta) = v(w) - g(e, \theta)$.

(a) Characterize the optimal contract under full observability.

(b) Is this contract feasible when the state θ is not observable?

14.C.4$^{\mathbf{C}}$ Characterize the solution to the two-state principal-agent model with hidden information when the manager is risk averse, but not infinitely so.

14.C.5$^{\mathbf{B}}$ Confirm that the analysis in Section 14.C could not change if the owner's profits depended on the state and were not publicly observable and if, letting $\pi_i(e)$ denote the profits in state θ_i for $i = L, H$, $\pi'_H(e) \geq \pi'_L(e) > 0$ for all $e \geq 0$. What happens if this condition is not satisfied?

14.C.6$^{\mathbf{C}}$ Reconsider the labor market screening model in Exercise 13.D.1, but now suppose that there is a single employer. Characterize the solution to this firm's screening problem (assume that both types of workers have a reservation utility level of 0). Compare the task levels in this solution with those in the equilibrium of the competitive screening model (assuming an equilibrium exists) that you derived in Exercise 13.D.1.

14.C.7$^{\mathbf{B}}$ (J. Tirole) Assume that there are two types of consumers for a firm's product, θ_H and θ_L. The proportion of type θ_L consumers is λ. A type θ's utility when consuming amount x of the good and paying a total of T for it is $u(x, T) = \theta v(x) - T$, where

$$v(x) = \frac{1 - (1 - x)^2}{2}.$$

The firm is the sole producer of this good, and its cost of production per unit is $c > 0$.

(a) Consider a nondiscriminating monopolist. Derive his optimal pricing policy. Show that he serves both classes of consumers if either θ_L or λ is "large enough."

(b) Consider a monopolist who can distinguish the two types (by some characteristic) but can only charge a simple price p_i to each type θ_i. Characterize his optimal prices.

(c) Suppose the monopolist cannot distinguish the types. Derive the optimal two-part tariff (a pricing policy consisting of a lump-sum charge F plus a linear price per unit purchased of p) under the assumption that the monopolist serves both types. Interpret. When will the monopolist serve both types?

(d) Compute the fully optimal nonlinear tariff. How do the quantities purchased by the two types compare with the levels in **(a)** to **(c)**?

14.C.8$^{\mathbf{B}}$ Air Shangri-la is the only airline allowed to fly between the islands of Shangri-la and Nirvana. There are two types of passengers, tourist and business. Business travelers are willing to pay more than tourists. The airline, however, cannot tell directly whether a ticket purchaser is a tourist or a business traveler. The two types do differ, though, in how much they are willing to pay to avoid having to purchase their tickets in advance. (Passengers do not like to commit themselves in advance to traveling at a particular time.)

More specifically, the utility levels of each of the two types net of the price of the ticket, P, for any given amount of time W prior to the flight that the ticket is purchased are given by

$$\begin{aligned} &\textit{Business:} \quad v - \theta_B P - W, \\ &\textit{Tourist:} \quad v - \theta_T P - W, \end{aligned}$$

where $0 < \theta_B < \theta_T$. (Note that for any given level of W, the business traveler is willing to pay more for his ticket. Also, the business traveler is willing to pay more for any given reduction in W.)

The proportion of travelers who are tourists is λ. Assume that the cost of transporting a passenger is c.

Assume in **(a)** to **(d)** that Air Shangri-la wants to carry both types of passengers.

(a) Draw the indifference curves of the two types in (P, W)-space. Draw the airline's isoprofit curves. Now formulate the optimal (profit-maximizing) price discrimination problem mathematically that Air Shangri-la would want to solve. [*Hint*: Impose nonnegativity of prices as a constraint since, if it charged a negative price, it would sell an infinite number of tickets at this price.]

(b) Show that in the optimal solution, tourists are indifferent between buying a ticket and not going at all.

(c) Show that in the optimal solution, business travelers never buy their ticket prior to the flight and are just indifferent between doing this and buying when tourists buy.

(d) Describe fully the optimal price discrimination scheme under the assumption that they sell to both types. How does it depend on the underlying parameters λ, θ_B, θ_T, and c?

(e) Under what circumstances will Air Shangri-la choose to serve only business travelers?

14.C.9$^{\text{C}}$ Consider a risk-averse individual who is an expected utility maximizer with a Bernoulli utility function over wealth $u(\cdot)$. The individual has initial wealth W and faces a probability θ of suffering a loss of size L, where $W > L > 0$.

An insurance contract may be described by a pair (c_1, c_2), where c_1 is the amount of wealth the individual has in the event of no loss and c_2 is the amount the individual has if a loss is suffered. That is, in the event no loss occurs the individual pays the insurance company an amount $(W - c_1)$, whereas if a loss occurs the individual receives a payment $[c_2 - (W - L)]$ from the company.

(a) Suppose that the individual's only source of insurance is a risk-neutral monopolist (i.e., the monopolist seeks to maximize its expected profits). Characterize the contract the monopolist will offer the individual in the case in which the individual's probability of loss, θ, is observable.

(b) Suppose, instead, that θ is not observable by the insurance company (the individual knows θ). The parameter θ can take one of two values $\{\theta_L, \theta_H\}$, where $\theta_H > \theta_L > 0$ and Prob $(\theta_L) = \lambda$. Characterize the optimal contract offers of the monopolist. Can one speak of one type of insured individual being "rationed" in his purchases of insurance (i.e., he would want to purchase more insurance if allowed to at fair odds)? Intuitively, why does this rationing occur? [*Hint*: It might be helpful to draw a picture in (c_1, c_2)-space. To do so, start by locating the individual's endowment point, that is, what he gets if he does not purchase any insurance.]

(c) Compare your solution in **(b)** with your answer to Exercise 13.D.2.

14.AA.1$^{\text{B}}$ Show that $[f_e(\pi \mid e)/f(\pi \mid e)]$ is increasing in π for all $e \in [a, b] \subset \mathbb{R}$ if and only if for any $e', e'' \in [a, b]$, with $e'' > e'$, $[f(\pi \mid e'')/f(\pi \mid e')]$ is increasing in π.

14.AA.2$^{\text{B}}$ Consider a hidden action model with $e \in [0, \bar{e}]$ and two outcomes π_H and π_L, with $\pi_H > \pi_L$. The probability of π_H given effort level e is $f(\pi_H \mid e)$. Give sufficient conditions for the first-order approach to be valid. Characterize the optimal contract when these conditions are satisfied.

14.BB.1$^{\text{B}}$ Try solving problem (14.BB.1) by first solving it while ignoring constraint (iv) and then arguing that the solution you derive to this "relaxed" problem is actually the solution to problem (14.BB.1).

P A R T   F O U R

General Equilibrium

Part IV is devoted to an examination of competitive market economies from a *general equilibrium* perspective. Our use of the term "general equilibrium" refers both to a methodological point of view and to a substantive theory.

Methodologically, the general equilibrium approach has two central features. First, it views the economy as a *closed* and *interrelated* system in which we must simultaneously determine the equilibrium values of all variables of interest. Thus, when we evaluate the effects of a perturbation in the economic environment, the equilibrium levels of the entire set of endogenous variables in the economy needs to be recomputed. This stands in contrast to the *partial equilibrium* approach, where the impact on endogenous variables not directly related to the problem at hand is explicitly or implicitly disregarded.

A second central feature of the general equilibrium approach is that it aims at reducing the set of variables taken as exogenous to a small number of physical realities (e.g., the set of economic agents, the available technologies, the preferences and physical endowments of goods of various agents).

From a substantive viewpoint, general equilibrium theory has a more specific meaning: It is a theory of the determination of equilibrium prices and quantities in a system of perfectly competitive markets. This theory is often referred to as the *Walrasian theory* of markets [from L. Walras (1874)], and it is the object of our study in Part IV. The Walrasian theory of markets is very ambitious. It attempts no less than to predict the complete vector of final consumptions and productions using only the fundamentals of the economy (the list of commodities, the state of technology, preferences and endowments), the institutional assumption that a price is quoted for every commodity (including those that will not be traded at equilibrium), and the behavioral assumption of price taking by consumers and firms.

Strictly speaking, we introduced a particular case of the general equilibrium model in Chapter 10. There, we carried out an equilibrium and welfare analysis of perfectly competitive markets under the assumption that consumers had quasilinear preferences. In that setting, consumer demand functions do not display wealth effects (except for a single commodity, called the *numeraire*); as a consequence, the analysis of a single market (or small group of markets) could be pursued in a manner understandable as traditional partial equilibrium analysis. A good deal of what we do in Part IV can be viewed as an attempt to extend the ideas of Chapter 10 to a world in which wealth effects are significant. The primary motivation for this is the increase in realism it brings. To make practical use of equilibrium analysis for studying the performance of an entire economy, or for evaluating policy interventions that affect large numbers of markets simultaneously, wealth effects, a primary source of linkages across markets, cannot be neglected, and therefore the general equilibrium approach is essential.

Although knowledge of the material discussed in Chapter 10 is not a strict prerequisite for Part IV, we nonetheless strongly recommend that you study it, especially Sections 10.B to 10.D. It constitutes an introduction to the main issues and provides a simple and analytically very useful example. We will see in the different chapters of Part IV that quite a number of the important results established in Chapter 10 for the quasilinear situation carry over to the case of general preferences. But many others do not. To understand why this may be so, recall from Chapters 4 and 10 that a group of consumers with quasilinear preferences (with respect to the same numeraire) admits the existence of a (normative) representative consumer. This is a powerful restriction on the behavior of aggregate demand that will not be available to us in the more general settings that we study here.

It is important to note that, relative to the analysis carried out in Part III, we incur a cost for accomplishing the task that general equilibrium sets itself to do: the assumptions of price-taking behavior and universal price quoting—that is, the existence of markets for every relevant commodity (with the implication of symmetric information)—are present in nearly all the theory studied in Part IV. Thus, in many respects, we are not going as deep as we did in Part III in the microanalysis of markets, of market failure, and of the strategic interdependence of market actors. The trade-off in conceptual structure between Parts III and IV reflects, in a sense, the current state of the frontier of microeconomic research.

The content of Part IV is organized into six chapters.

Chapter 15 presents a preliminary discussion. Its main purpose is to illustrate the issues that concern general equilibrium theory by means of three simple examples: the *two-consumer Edgeworth box economy*; the *one-consumer, one-firm economy*, and the *small open economy* model.

Chapters 16 and 17 constitute the heart of the formal analysis in Part IV. Chapter 16 presents the formal structure of the general equilibrium model and introduces two central concepts of the theory: the notions of *Pareto optimality* and *price-taking equilibrium* (and, in particular, *Walrasian equilibrium*). The chapter is devoted to the examination of the relationship between these two concepts. The emphasis is therefore *normative*, focusing on the welfare properties of price-taking equilibria. The core of the chapter is concerned with the formulation and proof of the two *fundamental theorems of welfare economics*.

In Chapter 17, the emphasis is, instead, on *positive* (or *descriptive*) properties of Walrasian equilibria. We study a number of questions pertaining to the predictive power of the Walrasian theory, including the existence, local and global uniqueness, and comparative statics behavior of Walrasian equilibria.

Chapters 18 to 20 explore extensions of the basic analysis presented in Chapters 16 and 17. Chapter 18 covers a number of topics whose origins lie in normative theory or the cooperative theory of games; these topics share the feature that they provide a deeper look at the foundations of price-taking equilibria by exploiting properties derived from the mass nature of markets. We study the important *core equivalence theorem*, examine further the idea of Walrasian equilibria as the limit of noncooperative equilibria as markets grow large (a subject already broached in Section 12.F), and present two normative characterizations of Walrasian equilibria: one in terms of *envy-freeness* (or *anonymity*) and the other in terms of a *marginal productivity principle*. Appendix A of Chapter 18 offers a brief introduction to the cooperative theory of games.

Chapter 19 covers the modeling of uncertainty in a general equilibrium context. The ability to do this in a theoretically satisfying way has been one of the success stories of general equilibrium theory. The concepts of *contingent commodities*, *Arrow–Debreu equilibrium*, *sequential trade* (in a two-period setting), *Radner equilibrium*, *arbitrage*, *rational expectations equilibrium*, and *incomplete markets* are all introduced and studied here. The chapter provides a natural link to the modern theory of finance.

Chapter 20 considers the application of the general theory to dynamic competitive economies (but with no uncertainty) and also studies a number of issues specific to this environment. Notions such as *impatience*, *dynamic efficiency*, and *myopic* versus *overall utility maximization* are introduced. The chapter first analyzes dynamic representative consumer economies (including the *Ramsey–Solow model*), then generalizes to the case of a finite number of consumers, and concludes with a brief presentation of the *overlapping generations model*. In the process, we explore a wide range of dynamic behaviors. The chapter provides a natural link to macroeconomic theory.

The modern classics of general equilibrium theory are Debreu (1959) and Arrow and Hahn (1971). These texts provide further discussion of topics treated here. For extensions, we recommend the encyclopedic coverage of Arrow and Intriligator (1981, 1982, 1986) and Hildenbrand and Sonnenschein (1991). See also the more recent textbook account of Ellickson (1993). General equilibrium analysis has a very important applied dimension that we do not touch on in this book but that accounts in good part for the importance of the theory. For a review, we recommend Shoven and Whalley (1992).

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CHAPTER

15

General Equilibrium Theory: Some Examples

15.A Introduction

The purpose of this chapter is to present a preliminary discussion. In it, we describe and analyze three simple examples of general equilibrium models. These examples introduce some of the questions, concepts, and common techniques that will occupy us for the rest of Part IV.

In most economies, three basic economic activities occur: production, exchange, and consumption. In Section 15.B, we restrict our focus to exchange and consumption. We analyze the case of a *pure exchange economy*, in which no production is possible and the commodities that are ultimately consumed are those that individuals possess as *endowments*. Individuals trade these endowments among themselves in the market-place for mutual advantage. The model we present is the simplest-possible exchange problem: two consumers trading two goods between each other. In this connection, we introduce an extremely handy graphical device, the *Edgeworth box*.

In Section 15.C, we introduce production by studying an economy formed by one firm and one consumer. Using this simple model, we explore how the production and consumption sides of the economy fit together.

In Section 15.D, we examine the production side of the economy in greater detail by discussing the allocation of resources among several firms. To analyze this issue in isolation, we study the case of a small open economy that takes the world prices of its outputs as fixed, a central model in international trade literature.

Section 15.E illustrates, by means of an example, some of the potential dangers of adopting a partial equilibrium perspective when a general equilibrium approach is called for.

As we noted in the introduction of Part IV, Chapter 10 contains another simple example of a general equilibrium model: that of an economy in which consumers have preferences admitting a quasilinear representation.

15.B Pure Exchange: The Edgeworth Box

A *pure exchange economy* (or, simply, an *exchange economy*) is an economy in which there are no production opportunities. The economic agents of such an economy are consumers who possess initial stocks, or *endowments*, of commodities. Economic activity consists of trading and consumption.

The simplest economy with the possibility of profitable exchange is one with two commodities and two consumers. As it turns out, this case is amenable to analysis by a graphical device known as the *Edgeworth box*, which we use extensively in this section. Throughout, we assume that the two consumers act as price takers. Although this may not seem reasonable with only two traders, our aim here is to illustrate some of the features of general equilibrium models in the simplest-possible way.[1]

To begin, assume that there are two consumers, denoted by $i = 1, 2$, and two commodities, denoted by $\ell = 1, 2$. Consumer i's consumption vector is $x_i = (x_{1i}, x_{2i})$; that is, consumer i's consumption of commodity ℓ is $x_{\ell i}$. We assume that consumer i's consumption set is $\mathbb{R}^2_+$ and that he has a preference relation $\succsim_i$ over consumption vectors in this set. Each consumer i is initially endowed with an amount $\omega_{\ell i} \geq 0$ of good ℓ. Thus, consumer i's *endowment vector* is $\omega_i = (\omega_{1i}, \omega_{2i})$. The *total endowment* of good ℓ in the economy is denoted by $\bar{\omega}_\ell = \omega_{\ell 1} + \omega_{\ell 2}$; we assume that this quantity is strictly positive for both goods.

An *allocation* $x \in \mathbb{R}^4_+$ in this economy is an assignment of a nonnegative consumption vector to each consumer: $x = (x_1, x_2) = ((x_{11}, x_{21}), (x_{12}, x_{22}))$. We say that an allocation is *feasible* for the economy if

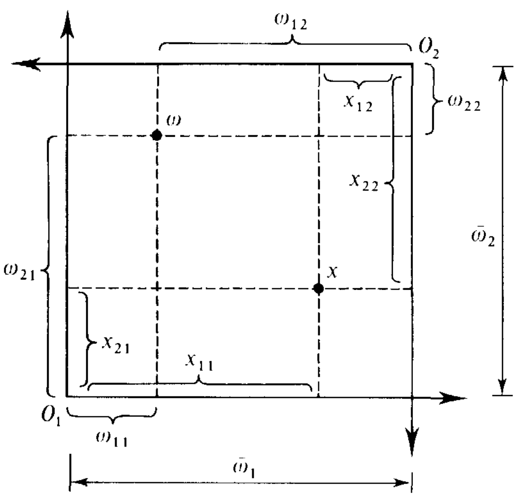
$$x_{\ell 1} + x_{\ell 2} \leq \bar{\omega}_\ell \qquad \text{for } \ell = 1, 2, \tag{15.B.1}$$

that is, if the total consumption of each commodity is no more than the economy's aggregate endowment of it (note that in this notion of feasibility, we are implicitly assuming that there is free disposal of commodities).

The feasible allocations for which equality holds in (15.B.1) could be called *nonwasteful*. Nonwasteful feasible allocations can be depicted by means of an *Edgeworth box*, shown in Figure 15.B.1. In the Edgeworth box, consumer 1's quantities are measured in the usual way, with the southwest corner as the origin. In contrast, consumer 2's quantities are measured using the northeast corner as the origin. For both consumers, the vertical dimension measures quantities of good 2, and the horizontal dimension measures quantities of good 1. The length of the box is $\bar{\omega}_1$, the economy's total endowment of good 1; its height is $\bar{\omega}_2$, the economy's total endowment of good 2. Any point in the box represents a (nonwasteful) division of the economy's total endowment between consumers 1 and 2. For example, Figure 15.B.1 depicts the endowment vector $\omega = ((\omega_{11}, \omega_{21}), (\omega_{12}, \omega_{22}))$ of the two consumers. Also depicted is another possible nonwasteful allocation, $x = ((x_{11}, x_{21}), (x_{12}, x_{22}))$; the fact that it is nonwasteful means that $(x_{12}, x_{22}) = (\bar{\omega}_1 - x_{11}, \bar{\omega}_2 - x_{21})$.

As is characteristic of general equilibrium theory, the wealth of a consumer is not given exogenously. Rather, for any prices $p = (p_1, p_2)$, consumer i's wealth equals the market value of his endowments of commodities, $p \cdot \omega_i = p_1 \omega_{1i} + p_2 \omega_{2i}$. Wealth levels are therefore determined by the values of prices. Hence, given the consumer's endowment vector ω_i, his budget set can be viewed solely as a

1. Alternatively, we could assume that each consumer (perhaps better called a *consumer type*) stands, not for an individual, but for a large number of identical consumers. This would make the price-taking assumption more plausible; and with equal numbers of the two types of consumers, the analysis in this section would be otherwise unaffected.



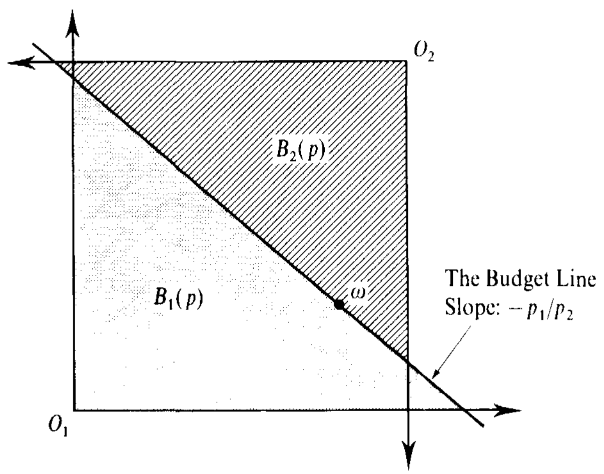


Figure 15.B.1 (left)
An Edgeworth box.

Figure 15.B.2 (right)
Budget sets.

function of prices:

$$B_i(p) = \{x_i \in \mathbb{R}^2_+ : p \cdot x_i \le p \cdot \omega_i\}.$$

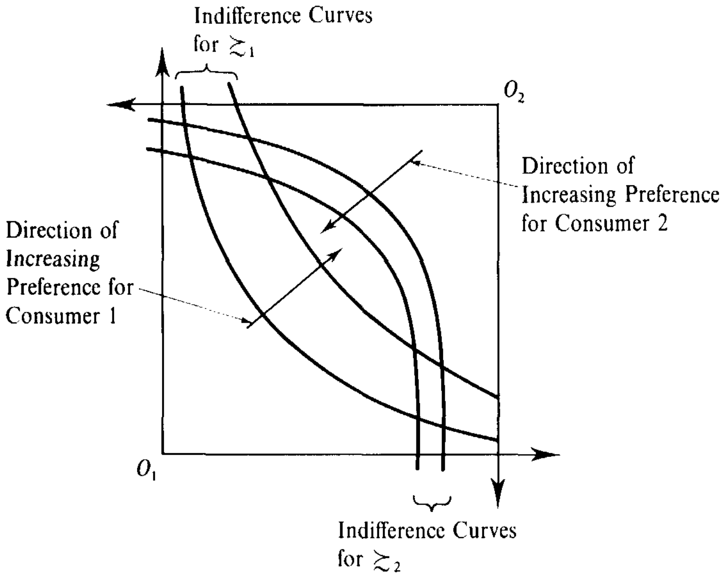
The budget sets of the two consumers can be represented in the Edgeworth box in a simple manner. To do so, we draw a line, known as the *budget line*, through the endowment point ω with slope $-(p_1/p_2)$, as shown in Figure 15.B.2. Consumer 1's budget set consists of all the nonnegative vectors below and to the left of this line (the shaded set). Consumer 2's budget set, on the other hand, consists of all the vectors above and to the right of this same line which give consumer 2 nonnegative consumption levels (the hatched set).[2] Observe that only allocations on the budget line are affordable to both consumers simultaneously at prices (p_1, p_2).[3]

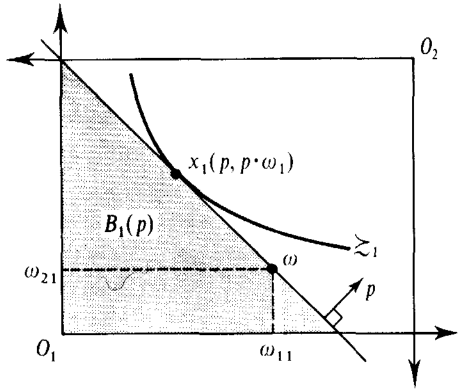
We can also depict the preferences $\succsim_i$ of each consumer i in the Edgeworth box, as in Figure 15.B.3. Except where otherwise noted, we assume that $\succsim_i$ is strictly convex, continuous, and strongly monotone (see Sections 3.B and 3.C for discussion of these conditions).

Figure 15.B.4 illustrates how the consumption vector demanded by consumer 1 can be determined for any price vector p. Given p, the consumer demands his most preferred point in $B_1(p)$, which can be expressed using his demand function as $x_1(p, p \cdot \omega_1)$ (this is the same demand function studied in Chapters 2 to 4; here wealth is $w_1 = p \cdot \omega_1$). In Figure 15.B.5, we see that as the price vector p varies, the budget line pivots around the endowment point ω, and the demanded consumptions trace out a curve, denoted by OC_1, that is called the *offer curve* of consumer 1. Note that this curve passes through the endowment point. Because at every p the endowment vector $\omega_1 = (\omega_{11}, \omega_{21})$ is affordable to consumer 1, it follows that this consumer must find every point on his offer curve at least as good as his endowment point.

2. Note, in particular, that the budget sets of the consumers may well extend outside the box.

3. There are other feasible allocations that are simultaneously affordable; but in these allocations some resources are not consumed by either consumer, and thus they cannot be depicted in an Edgeworth box. Because of the nonsatiation assumption to be made on preferences, we will not have to worry about such allocations.





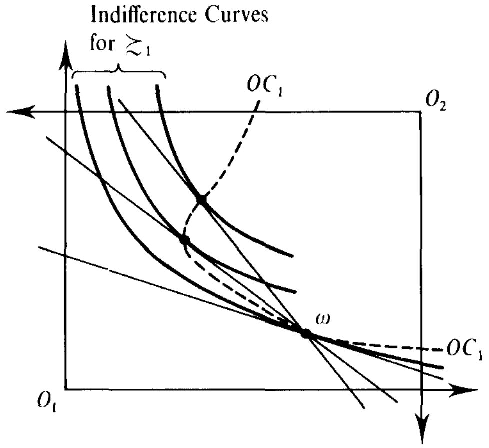


Figure 15.B.3 (top left) Preferences in the Edgeworth box.

Figure 15.B.4 (top right) Optimal consumption for consumer 1 at prices p.

Figure 15.B.5 (bottom) Consumer 1's offer curve.

This implies that the consumer's offer curve lies within the upper contour set of ω_1 and that, if indifference curves are smooth, the offer curve must be tangent to the consumer's indifference curve at the endowment point.

Figure 15.B.6 represents the demanded bundles of the two consumers at some arbitrary price vector p. Note that the demands expressed by the two consumers are not compatible. The total demand for good 2 exceeds its total supply in the economy $\bar{\omega}_2$, whereas the total demand for good 1 is strictly less than its endowment $\bar{\omega}_1$. Put somewhat differently, consumer 1 is a *net demander* of good 2 in the sense that he wants to consume more than his endowment of that commodity. Although consumer 2 is willing to be a *net supplier* of that good (he wants to consume less than his endowment), he is not willing to supply enough to satisfy consumer 1's needs. Good 2 is therefore in *excess demand* in the situation depicted in the figure. In contrast, good 1 is in *excess supply*.

At a market equilibrium where consumers take prices as given, markets should clear. That is, the consumers should be able to fulfill their desired purchases and

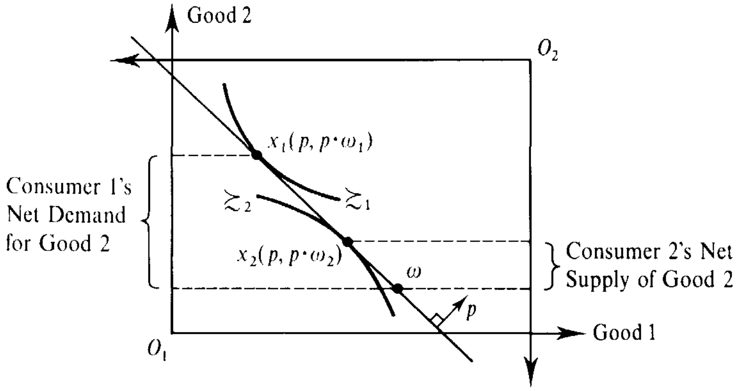


Figure 15.B.6
A price vector with excess demand for good 2 and excess supply for good 1.

sales of commodities at the going market prices. Thus, if one consumer wishes to be *a net demander* of some good, the other must be a *net supplier* of this good in exactly the same amount; that is, demand should equal supply. This gives us the notion of equilibrium presented in Definition 15.B.1.

Definition 15.B.1: A *Walrasian* (or *competitive*) *equilibrium* for an Edgeworth box economy is a price vector p^* and an allocation $x^* = (x_1^*, x_2^*)$ in the Edgeworth box such that for $i = 1, 2$,

$$x_i^* \succsim_i x_i' \quad \text{for all} \quad x_i' \in B_i(p^*).$$

A Walrasian equilibrium is depicted in Figure 15.B.7. In Figure 15.B.7(a), we represent the equilibrium price vector p^* and the equilibrium allocation $x^* = (x_1^*, x_2^*)$. Each consumer i's demanded bundle at price vector p^* is x_i^*, and one consumer's net demand for a good is exactly matched by the other's net supply. Figure 15.B.7(b) adds to the depiction the consumers' offer curves and their indifference curves through ω. Note that at any equilibrium, the offer curves of the two consumers intersect. In fact, *any* intersection of the consumers' offer curves at an allocation different from the endowment point ω corresponds to an equilibrium because if $x^* = (x_1^*, x_2^*)$ is any such point of intersection, then x_i^* is the optimal consumption bundle for each consumer i for the budget line that goes through the two points ω and x^*.

In Figure 15.B.8, we show a Walrasian equilibrium where the equilibrium allocation lies on the boundary of the Edgeworth box. Once again, at price vector p^*, the two consumers' demands are compatible.

Note that each consumer's demand is homogeneous of degree zero in the price vector $p = (p_1, p_2)$; that is, if prices double, then the consumer's wealth also doubles and his budget set remains unchanged. Thus, from Definition 15.B.1, we see that if $p^* = (p_1^*, p_2^*)$ is a Walrasian equilibrium price vector, then so is $\alpha p^* = (\alpha p_1^*, \alpha p_2^*)$ for any $\alpha > 0$. In short, only the *relative* prices p_1^*/p_2^* are determined in an equilibrium.

Example 15.B.1: Suppose that each consumer i has the Cobb–Douglas utility function $u_i(x_{1i}, x_{2i}) = x_{1i}^{\alpha} x_{2i}^{1-\alpha}$. In addition, endowments are $\omega_1 = (1, 2)$ and $\omega_2 = (2, 1)$. At prices $p = (p_1, p_2)$, consumer 1's wealth is $(p_1 + 2p_2)$ and therefore his demands lie on the offer curve (recall the derivation in Example 3.D.1):

$$OC_1(p) = \left(\frac{\alpha(p_1 + 2p_2)}{p_1}, \frac{(1-\alpha)(p_1 + 2p_2)}{p_2}\right).$$

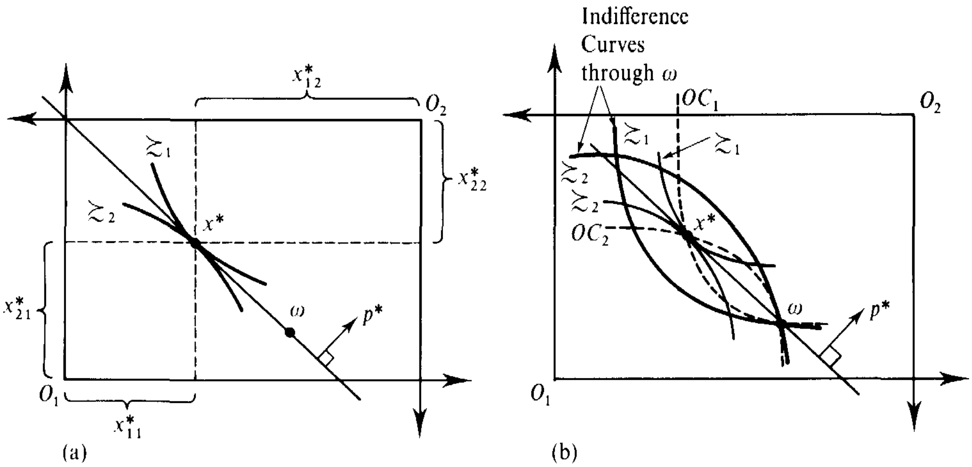


Figure 15.B.7 (top)

(a) A Walrasian equilibrium.
(b) The consumer's offer curves intersect at the Walrasian equilibrium allocation.

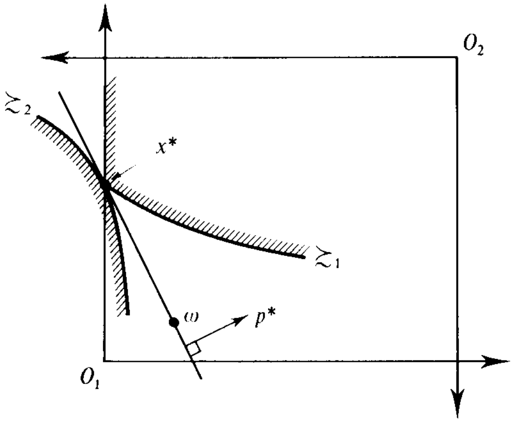


Figure 15.B.8 (bottom)

A Walrasian equilibrium allocation on the boundary of the Edgeworth box.

Observe that the demands for the first and the second good are, respectively, decreasing and increasing with p_1. This is how we have drawn OC_1 in Figure 15.B.7(b). Similarly, $OC_2(p) = (\alpha(2p_1 + p_2)/p_1, (1-\alpha)(2p_1 + p_2)/p_2)$. To determine the Walrasian equilibrium prices, note that at these prices the total amount of good 1 consumed by the two consumers must equal 3 $(=\omega_{11} + \omega_{12})$. Thus,

$$\frac{\alpha(p_1^* + 2p_2^*)}{p_1^*} + \frac{\alpha(2p_1^* + p_2^*)}{p_1^*} = 3.$$

Solving this equation yields

$$\frac{p_1^*}{p_2^*} = \frac{\alpha}{1-\alpha}. \tag{15.B.2}$$

Observe that at any prices (p_1^*, p_2^*) satisfying condition (15.B.2), the market for good 2 clears as well (you should verify this). This is a general feature of an Edgeworth box economy: To determine equilibrium prices we need only determine prices at which one of the markets clears; the other market will necessarily clear at these prices. This point can be seen graphically in the Edgeworth box: Because both consumers' demanded bundles lie on the same budget line, if the amounts of commodity 1 demanded are compatible, then so must be those for commodity 2. (See also Exercise 15.B.1.) ■

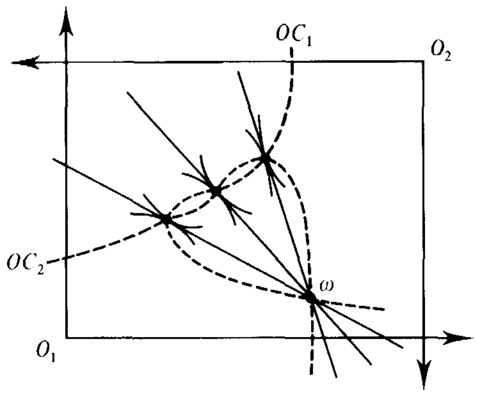


Figure 15.B.9
Multiple Walrasian equilibria.

The Edgeworth box, simple as it is, is remarkably powerful. There are virtually no phenomena or properties of general equilibrium exchange economies that cannot be depicted in it. Consider, for example, the issue of the uniqueness of Walrasian equilibrium. In Chapter 10, we saw that if there is a numeraire commodity relative to which preferences admit a quasilinear representation, then (with strict convexity of preferences) the equilibrium consumption allocation and relative prices are unique. In Figure 15.B.7, we also have uniqueness (see Exercise 15.B.2 for a more explicit discussion). Yet, as the Edgeworth box in Figure 15.B.9 shows, this property does not generalize. In that figure, preferences (which are entirely nonpathological) are such that the offer curves change curvature and interlace several times. In particular, they intersect for prices such that p_1/p_2 is equal to $\frac{1}{2}$, 1, and 2. For the sake of completeness, we present an analytical example with the features of the figure.

Example 15.B.2: Let the two consumers have utility functions

$$u_1(x_{11}, x_{21}) = x_{11} - \tfrac{1}{8}x_{21}^{-8} \text{ and } u_2(x_{12}, x_{22}) = -\tfrac{1}{8}x_{12}^{-8} + x_{22}.$$

Note that the utility functions are quasilinear (which, in particular, facilitates the computation of demand), but with respect to *different* numeraires. The endowments are $\omega_1 = (2, r)$ and $\omega_2 = (r, 2)$, where r is chosen to guarantee that the equilibrium prices turn out to be round numbers. Precisely, $r = 2^{8/9} - 2^{1/9} > 0$. In Exercise 15.B.5, you are asked to compute the offer curves of the two consumers. They are:

$$OC_1(p_1, p_2) = \left(2 + r\left(\frac{p_2}{p_1}\right) - \left(\frac{p_2}{p_1}\right)^{8/9}, \left(\frac{p_2}{p_1}\right)^{-1/9}\right) \gg 0$$

and

$$OC_2(p_1, p_2) = \left(\left(\frac{p_1}{p_2}\right)^{-1/9}, 2 + r\left(\frac{p_1}{p_2}\right) - \left(\frac{p_1}{p_2}\right)^{8/9}\right) \gg 0.$$

Note that, as illustrated in Figure 15.B.9, and in contrast with Example 15.B.1, consumer 1's demand for good 1 (and symmetrically for consumer 2) may be increasing in p_1.

To compute the equilibria it is sufficient to solve the equation that equates the total demand of the second good to its total supply, or

$$\left(\frac{p_2}{p_1}\right)^{-1/9} + 2 + r\left(\frac{p_1}{p_2}\right) - \left(\frac{p_1}{p_2}\right)^{8/9} = 2 + r.$$

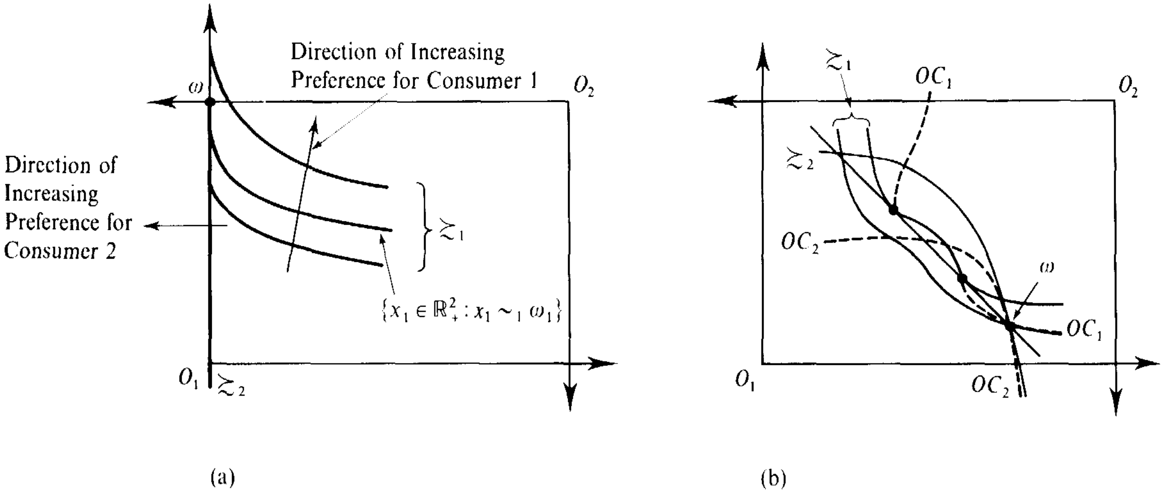


Figure 15.B.10 (a) and (b): Two examples of nonexistence of Walrasian equilibrium.

Recalling the value of r, this equation has three solutions for p_1/p_2: 2, 1, and $\frac{1}{2}$ (you should check this). ■

It may also happen that a pure exchange economy does not have *any* Walrasian equilibria. For example, Figure 15.B.10(a) depicts a situation in which the endowment lies on the boundary of the Edgeworth box (in the northwest corner). Consumer 2 has all the endowment of good 1 and desires only good 1. Consumer 1 has all the endowment of good 2 and his indifference set containing ω_1, $\{x_1 \in \mathbb{R}^2_+ : x_1 \sim_1 \omega_1\}$, has an infinite slope at ω_1 (note, however, that at ω_1, consumer 1 would strictly prefer receiving more of good 1). In this situation, there is no price vector p^* at which the consumers' demands are compatible. If $p_2/p_1 > 0$ then consumer 2 optimal demand is to keep his initial bundle ω_2, whereas the initial bundle ω_1 is never consumer 1's optimal demand (no matter how large the relative price of the first good, consumer 1 always wishes to buy a strictly positive amount of it). On the other hand, consumer 1's demand for good 2 is infinite when $p_2/p_1 = 0$. Note for future reference that consumer 2's preferences in this example are not strongly monotone.

Figure 15.B.10(b) depicts a second example of nonexistence. There, consumer 1's preferences are nonconvex. As a result, consumer 1's offer curve is disconnected, and there is no crossing point of the two consumers' offer curves (other than the endowment point, which is not an equilibrium allocation here).

In Chapter 17, we will study the conditions under which the existence of a Walrasian equilibrium is assured.

Welfare Properties of Walrasian Equilibria

A central question in economic theory concerns the welfare properties of equilibria. Here we shall focus on the notion of Pareto optimality, which we have already encountered in Chapter 10 (see, in particular, Section 10.B). An economic outcome is *Pareto optimal* (or *Pareto efficient*) if there is no alternative feasible outcome at which every individual in the economy is at least as well off and some individual is

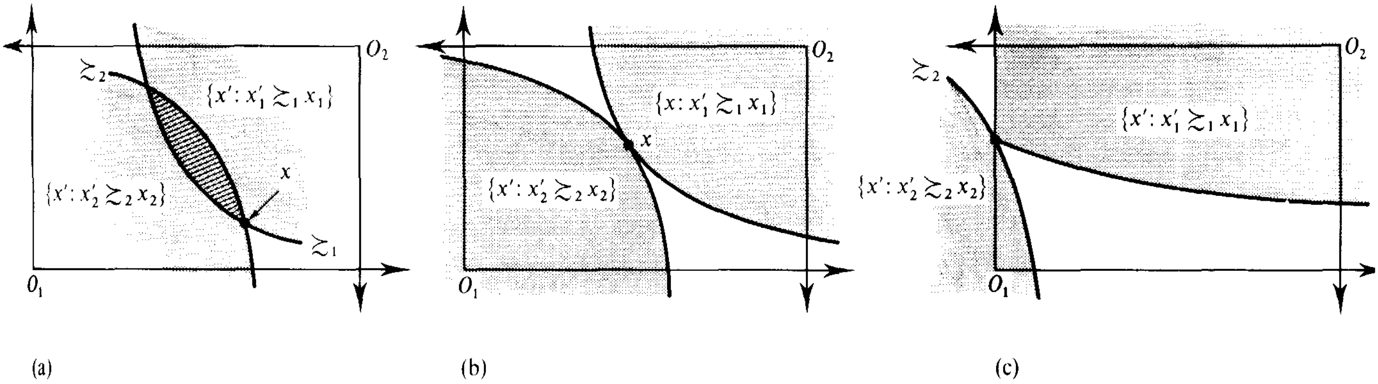


Figure 15.B.11 (a) Allocation x is not Pareto optimal. (b) Allocation x is Pareto optimal. (c) Allocation x is Pareto optimal.

strictly better off. Definition 15.B.2 expresses this idea in the setting of our two-consumer, pure exchange economy.

Definition 15.B.2: An allocation x in the Edgeworth box is *Pareto optimal* (or *Pareto efficient*) if there is no other allocaton x' in the Edgeworth box with $x'_i \succsim_i x_i$ for $i = 1, 2$ and $x'_i \succ_i x_i$ for some i.

Figure 15.B.11(a) depicts an allocation x that is not Pareto optimal. Any allocation in the interior of the crosshatched region of the figure, the intersection of the sets $\{x'_1 \in \mathbb{R}^2_+ : x'_1 \succsim_1 x_1\}$ and $\{x'_2 \in \mathbb{R}^2_+ : x'_2 \succsim_2 x_2\}$ within the Edgeworth box, is a feasible allocation that makes both consumers strictly better off than at x. The allocation x depicted in Figure 15.B.11(b), on the other hand, is Pareto optimal because the intersection of the sets $\{x'_i \in \mathbb{R}^2_+ : x'_i \succsim_i x_i\}$ for $i = 1, 2$ consists only of the point x. Note that if a Pareto optimal allocation x is an interior point of the Edgeworth box, then the consumers' indifference curves through x must be tangent (assuming that they are smooth). Figure 15.B.11(c) depicts a Pareto optimal allocation x that is not interior; at such a point, tangency need not hold.

The set of all Pareto optimal allocations is known as the *Pareto set*. An example is illustrated in Figure 15.B.12. The figure also displays the *contract curve*, the part of the

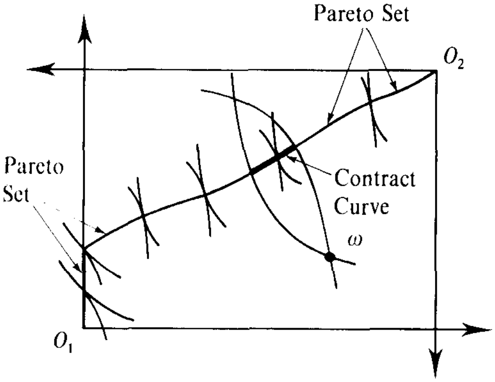


Figure 15.B.12
The Pareto set and the contract curve.

Pareto set where both consumers do at least as well as at their initial endowments. The reason for this term is that we might expect any bargaining between the two consumers to result in an agreement to trade to some point on the contract curve; these are the only points at which both of them do as well as at their initial endowments and for which there is no alternative trade that can make both consumers better off.

We can now verify a simple but important fact: *Any Walrasian equilibrium allocation x^* necessarily belongs to the Pareto set.* To see this, note that by the definition of a Walrasian equilibrium the budget line separates the two at-least-as-good-as sets associated with the equilibrium allocation, as seen in Figures 15.B.7(a) and 15.B.8. The only point in common between these two sets is x^* itself. Thus, at any competitive allocation x^*, there is no alternative feasible allocation that can benefit one consumer without hurting the other. The conclusion that Walrasian allocations yield Pareto optimal allocations is an expression of the *first fundamental theorem of welfare economics*, a result that, as we shall see in Chapter 16, holds with great generality. Note, moreover, that since each consumer must be at least as well off in a Walrasian equilibrium as by simply consuming his endowment, any Walrasian equilibrium lies in the contract curve portion of the Pareto set.

The first fundamental welfare theorem provides, for competitive market economies, a formal expression of Adam Smith's "invisible hand." Under perfectly competitive conditions, any equilibrium allocation is a Pareto optimum, and the only possible welfare justification for intervention in the economy is the fulfillment of distributional objectives.

The *second fundamental theorem of welfare economics*, which we also discuss extensively in Chapter 16, offers a (partial) converse result. Roughly put, it says that *under convexity assumptions (not required for the first welfare theorem), a planner can achieve any desired Pareto optimal allocation by appropriately redistributing wealth in a lump-sum fashion and then "letting the market work."* Thus, the second welfare theorem provides a theoretical affirmation for the use of competitive markets in pursuing distributional objectives.

Definition 15.B.3 is a more formal statement of the concept of an equilibrium with lump-sum wealth redistribution.

Definition 15.B.3: An allocation x^* in the Edgeworth box is supportable as an *equilibrium with transfers* if there is a price system p^* and wealth transfers T_1 and T_2 satisfying $T_1 + T_2 = 0$, such that for each consumer i we have

$$x_i^* \succsim_i x_i' \quad \text{for all } x_i' \in \mathbb{R}^2_+ \quad \text{such that} \quad p^* \cdot x_i' \leq p^* \cdot \omega_i + T_i.$$

Note that the transfers sum to zero in Definition 15.B.3; the planner runs a balanced budget, merely redistributing wealth between the consumers.

Equipped with Definition 15.B.3, we can state more formally a version of the second welfare theorem as follows: if the preferences of the two consumers in the Edgeworth box are continuous, convex, and strongly monotone, then *any Pareto optimal allocation is supportable as an equilibrium with transfers.* This result is illustrated in Figure 15.B.13(a), where the consumer's endowments are at point ω. Suppose that for distributional reasons, the socially desired allocation is the Pareto optimal allocation x^*. Then if a tax authority constructs a transfer of wealth between the two consumers that shifts the budget line to the location

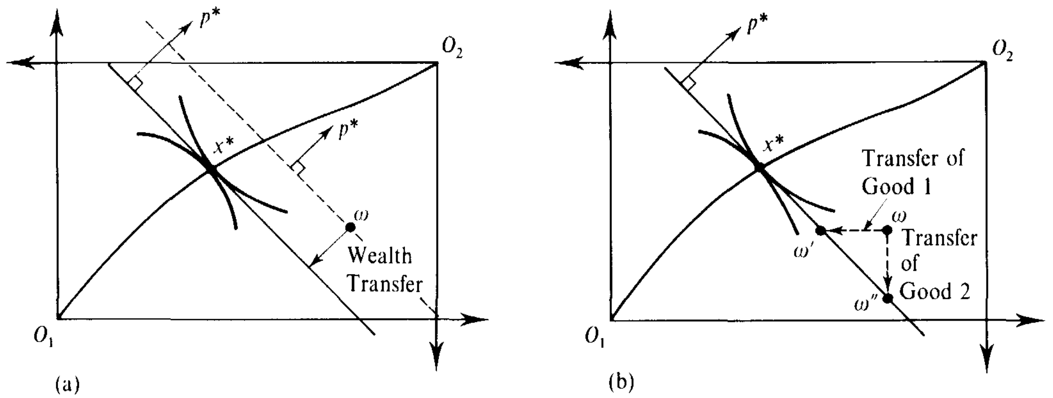


Figure 15.B.13 The second fundamental welfare theorem. (a) Using wealth transfers. (b) Using transfers of endowments.

indicated in the figure, the price vector p^* clears the markets for the two goods, and allocation x^* results.

Note that this wealth transfer may also be accomplished by directly transferring endowments. As Figure 15.B.13(b) illustrates, a transfer of good 1 that moves the endowment vector to ω' will have the price vector p^* and allocation x^* as a Walrasian equilibrium. A transfer of good 2 that changes endowments to ω'' does so as well. In fact, if all commodities can be easily transferred, then we could equally well move the endowment vector directly to allocation x^*. From this new endowment point, the Walrasian equilibrium involves no trade.[4]

Figure 15.B.14 shows that the second welfare theorem may fail to hold when preferences are not convex. In the figure, $x^* = (x_1^*, x_2^*)$ is a Pareto optimal allocation that is not supportable as an equilibrium with transfers. At the budget line with the property that consumer 2 would demand x_2^*, consumer 1 would prefer a point other than x_1^* (such as x'_1). Convexity, as it turns out, is a critical assumption for the second welfare theorem.

A failure of the second welfare theorem of a different kind is illustrated in Figure 15.B.10(a). There, the initial endowment allocation ω is a Pareto optimal allocation, but it cannot be supported as an equilibrium with transfers (you should check this). In this case, it is the assumption that consumers' preferences are strongly monotone that is violated.

For further illustrations of Edgeworth box economies see, for example, Newman (1965).

15.C The One-Consumer, One-Producer Economy

We now introduce the possibility of production. To do so in the simplest-possible setting, we suppose that there are two price-taking economic agents, a single

4. In practice, endowments may be difficult to transfer (e.g., human capital), and so the ability to use wealth transfers (or transfers of only a limited number of commodities) may be important. It is worth observing that there is one attractive feature of transferring endowments directly to the desired Pareto optimal allocation: we can be assured that x^* is the *unique* Walrasian equilibrium allocation after the transfers (strictly speaking this requires a strict convexity assumption on preferences).

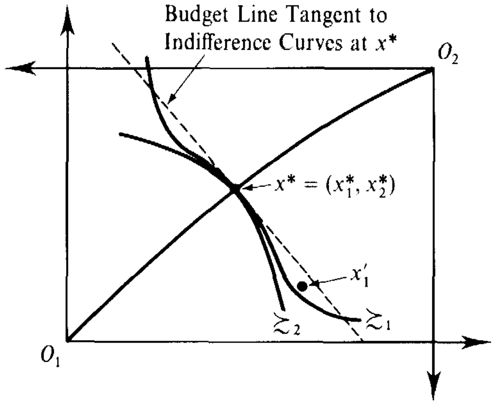


Figure 15.B.14
Failure of the second welfare theorem with nonconvex preferences.

consumer and a single firm, and two goods, the labor (or leisure) of the consumer and a consumption good produced by the firm.[5]

The consumer has continuous, convex, and strongly monotone preferences $\succsim$ defined over his consumption of leisure x_1 and the consumption good x_2. He has an endowment of $\bar{L}$ units of leisure (e.g., 24 hours in a day) and no endowment of the consumption good.

The firm uses labor to produce the consumption good according to the increasing and strictly concave production function $f(z)$, where z is the firm's labor input. Thus, to produce output, the firm must hire the consumer, effectively purchasing some of the consumer's leisure from him. We assume that the firm seeks to maximize its profits taking market prices as given. Letting p be the price of its output and w be the price of labor, the firm solves

$$\operatorname*{Max}_{z \geq 0} \quad pf(z) - wz. \tag{15.C.1}$$

Given prices (p, w), the firm's optimal labor demand is $z(p, w)$, its output is $q(p, w)$, and its profits are $\pi(p, w)$.

As we noted in Chapter 5, firms are owned by consumers. Thus, we assume that the consumer is the sole owner of the firm and receives the profits earned by the firm $\pi(p, w)$. As with the price-taking assumption, the idea of the consumer being hired by his own firm through an anonymous labor market may appear strange in this model with only two agents. Nevertheless, bear with us; our aim is to illustrate the workings of more complicated many-consumer general equilibrium models in the simplest-possible way.[6]

Letting $u(x_1, x_2)$ be a utility function representing $\succsim$, the consumer's problem given prices (p, w) is

$$\begin{aligned} \operatorname*{Max}_{(x_1, x_2) \in \mathbb{R}^2_+} \quad & u(x_1, x_2) \\ \text{s.t.} \quad & px_2 \leq w(\bar{L} - x_1) + \pi(p, w). \end{aligned} \tag{15.C.2}$$

5. One-consumer economies are sometimes referred to as *Robinson Crusoe economies*.

6. The point made in footnote 1 can be repeated here: we could imagine that the firm and the consumer stand for a large number of identical firms and consumers. We comment a bit more on this interpretation at the end of this Section.

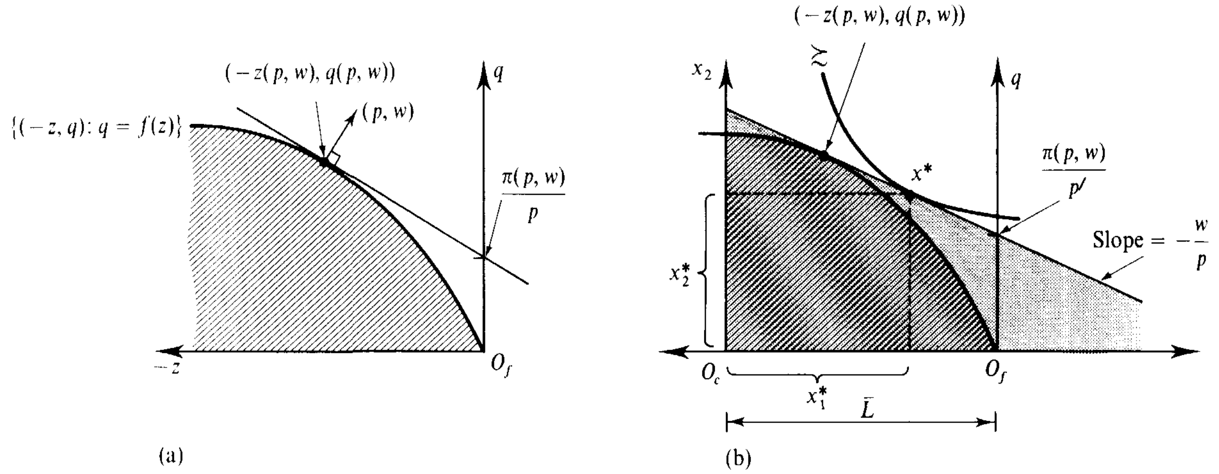


Figure 15.C.1 (a) The firm's problem. (b) The consumer's problem.

The budget constraint in (15.C.2) reflects the two sources of the consumer's purchasing power: If the consumer supplies an amount $(\bar{L} - x_1)$ of labor when prices are (p, w), then the total amount he can spend on the consumption good is his labor earnings $w(\bar{L} - x_1)$ plus the profit distribution from the firm $\pi(p, w)$. The consumer's optimal demands in problem (15.C.2) for prices (p, w) are denoted by $(x_1(p, w), x_2(p, w))$.

A Walrasian equilibrium in this economy involves a price vector (p^*, w^*) at which the consumption and labor markets clear; that is, at which

$$x_2(p^*, w^*) = q(p^*, w^*) \tag{15.C.3}$$

and

$$z(p^*, w^*) = \bar{L} - x_1(p^*, w^*) \tag{15.C.4}$$

Figure 15.C.1 illustrates the working of this one-consumer, one-firm economy. Figure 15.C.1(a) depicts the firm's problem. As in Chapter 5, we measure the firm's use of labor input on the horizontal axis as a negative quantity. Its output is depicted on the vertical axis. The production set associated with the production function $f(z)$ is also shown, as are the profit-maximizing input and output levels at prices (p, w), $z(p, w)$ and $q(p, w)$, respectively.

Figure 15.C.1(b) adapts this diagram to represent the consumer's problem. Leisure and consumption levels are measured from the origin denoted O_c at the lower-left-hand corner of the diagram, which is determined by letting the length of the segment $[O_c, O_f]$ be equal to $\bar{L}$, the total labor endowment. The figure depicts the consumer's (shaded) budget set given prices (p, w) and profits $\pi(p, w)$. Note that if the consumer consumes $\bar{L}$ units of leisure then since he sells no labor, he can purchase $\pi(p, w)/p$ units of the consumption good. Thus, the budget line must cut the vertical q-axis at height $\pi(p, w)/p$. In addition, for each unit of labor he sells, the consumer earns w and can therefore afford to purchase w/p units of x_2. Hence, the budget line has slope $-(w/p)$. Observe that the consumer's budget line is exactly the isoprofit line associated with the solution to the firm's profit-maximization problem in Figure 15.C.1(a), that is, the set of points $\{(-z, q): pq - wz = \pi(p, w)\}$ that yield profits of $\pi(p, w)$.

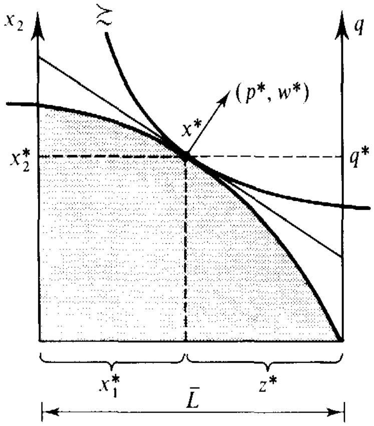


Figure 15.C.2
A Walrasian equilibrium.

The prices depicted in Figure 15.C.1(b) are not equilibrium prices; at these prices, there is an excess demand for labor (the firm wants more labor than the consumer is willing to supply) and an excess supply of the produced good. An equilibrium price vector (p^*, w^*) that clears the markets for the two goods is depicted in Figure 15.C.2.

There is a very important fact to notice from Figure 15.C.2: *A particular consumption leisure combination can arise in a competitive equilibrium if and only if it maximizes the consumer's utility subject to the economy's technological and endowment constraints.* That is, the Walrasian equilibrium allocation is the same allocation that would be obtained if a planner ran the economy in a manner that maximized the consumer's well-being. Thus, we see here an expression of the fundamental theorems of welfare economics: Any Walrasian equilibrium is Pareto optimal, and the Pareto optimal allocation is supportable as a Walrasian equilibrium.[7]

The indispensability of convexity for the second welfare theorem can again be observed in Figure 15.C.3(a). There, the allocation x^* maximizes the welfare of the consumer, but for the only value of relative prices that could support x^* as a utility-maximizing bundle, the firm does not maximize profits even locally (i.e., at the relative prices w/p, there are productions arbitrarily close to x^* yielding higher profits). In contrast, the first welfare theorem remains applicable even in the presence of nonconvexities. As Figure 15.C.3(b) suggests, any Walrasian equilibrium maximizes the well-being of the consumer in the feasible production set.

Under certain circumstances, the model studied in this section can be rigorously justified as representing the outcome of a more general economy by interpreting the "firm" as a representative producer (see Section 5.E) and the "consumer" as a representative consumer (see Section 4.D). The former is always possible, but the latter—that is, the existence of a (normative) representative consumer—requires strong conditions. If, however, the economy

7. In a single-consumer economy, the test for Pareto optimality reduces to the question of whether the well-being of the single consumer is being maximized (subject to feasibility constraints). Note that given the convexity of preferences and the strict convexity of the aggregate production set assumed here, there is a unique Pareto optimal consumption vector (and therefore a unique equilibrium).

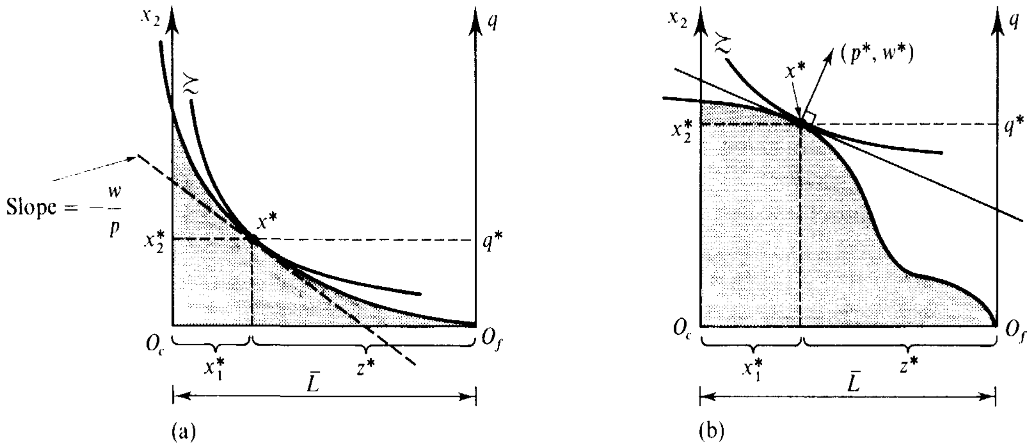


Figure 15.C.3 (a) Failure of the second welfare theorem with a nonconvex technology. (b) The first welfare theorem applies even with a nonconvex technology.

is composed of many consumers with identical concave utility functions and identical initial endowments, and if society has a strictly concave social welfare function in which these consumers are treated symmetrically, then a (normative) representative consumer exists who has the same utility function as the consumers over levels of per capita consumption.[8] (We can also think of the representative firm's input and output choices as being on a per capita basis). For more general conditions under which a representative consumer exists, see Section 4.D.

15.D The 2 × 2 Production Model

In this section, we discuss an example that concentrates on general equilibrium effects in production.

To begin, consider an economy in which the production sector consists of J firms. Each firm j produces a consumer good q_j directly from a vector of L primary (i.e., nonproduced) inputs, or *factors*, $z_j = (z_{1j}, \ldots, z_{Lj}) \geq 0$.[9] Firm j's production takes place by means of a concave, strictly increasing, and differentiable production function $f_j(z_j)$. Note that there are no intermediate goods (i.e., produced goods that are used as inputs). The economy has total endowments of the L factor inputs, $(\bar{z}_1, \ldots, \bar{z}_L) \gg 0$. These endowments are initially owned by consumers and have a use only as production inputs (i.e., consumers do not wish to consume them).

To concentrate on the factor markets of the economy, we suppose that the prices of the J produced consumption goods are fixed at $p = (p_1, \ldots, p_J)$, The leading example for this assumption is that of a small open economy whose trading decisions in the world markets for consumption goods have little effect on the world prices of

8. To see this, note that an equal distribution of wealth (which is what occurs here in the absence of any wealth transfers given the identical endowments of the consumers) maximizes social welfare for any price vector and aggregate wealth level.

9. Some of these outputs may be the same good; that is, firms j and j' may produce the same commodity.

these goods.[10] Output is sold in world markets. Factors, on the other hand, are immobile and must be used for production within the country.

The central question for our analysis concerns the equilibrium in the factor markets; that is, we wish to determine the equilibrium factor prices $w = (w_1, \dots, w_L)$ and the allocation of the economy's factor endowments among the J firms.[11]

Given output prices $p = (p_1, \dots, p_J)$ and input prices $w = (w_1, \dots, w_L)$, a profit-maximizing production plan for firm j solves

$$\underset{z_j \geq 0}{\text{Max}} \quad p_j f_j(z_j) - w \cdot z_j.$$

We denote firm j's set of optimal input demands given prices (p, w) by $z(p, w) \subset \mathbb{R}^L_+$. Because consumers have no direct use for their factor endowments, the total factor supply will be $(\bar{z}_1, \dots, \bar{z}_L)$ as long as the input prices w_ℓ are strictly positive (the only case that will concern us here). An equilibrium for the factor markets of this economy given the fixed output prices p therefore consists of an input price vector $w^* = (w_1^*, \dots, w_L^*) \gg 0$ and a factor allocation

$$(z_1^*, \dots, z_J^*) = ((z_{11}^*, \dots, z_{L1}^*), \dots, (z_{1J}^*, \dots, z_{LJ}^*)),$$

such that firms receive their desired factor demands under prices (p, w^*) and all the factor markets clear, that is, such that

$$z_j^* \in z_j(p, w) \qquad \text{for all } j = 1, \dots, J$$

and

$$\sum_j z_{\ell j}^* = \bar{z}_\ell \qquad \text{for all } \ell = 1, \dots, L.$$

Because of the concavity of firms' production functions, first-order conditions are both necessary and sufficient for the characterization of optimal factor demands. Therefore, the $L(J + 1)$ variables formed by the factor allocation $(z_1^*, \dots, z_J^*) \in \mathbb{R}^{LJ}_+$ and the factor prices $w^* = (w_1^*, \dots, w_L^*)$ constitute an equilibrium if and only if they satisfy the following $L(J + 1)$ equations (we assume an interior solution here):

$$p_j \frac{\partial f_j(z_j^*)}{\partial z_{\ell j}} = w_\ell^* \qquad \text{for } j = 1, \dots, J \text{ and } \ell = 1, \dots, L \tag{15.D.1}$$

and

$$\sum_j z_{\ell j}^* = \bar{z}_\ell \qquad \text{for } \ell = 1, \dots, L. \tag{15.D.2}$$

The equilibrium output levels are then $q_j^* = f_j(z_j^*)$ for every j.

Equilibrium conditions for *outputs* and factor prices can alternatively be stated using the firms' cost functions $c_j(w, q_j)$ for $j = 1, \dots, J$. Output levels $(q_1^*, \dots, q_J^*) \gg 0$ and factor prices $w^* \gg 0$ constitute an equilibrium if and only if the following

10. See Exercise 15.D.4 for an endogenous determination (up to a scalar multiple) of the prices $p = (p_1, \dots, p_J)$.

11. Note that once the factor prices and allocations are determined, each consumer's demands can be readily determined from his demand function given the exogenous prices $(p_1, \dots, p_J)$ and the wealth derived from factor input sales and profit distributions. Recall that the current model is completed by assuming that this demand is met in the world markets.

conditions hold:

$$p_j = \frac{\partial c_j(w^*, q_j^*)}{\partial q_j} \qquad \text{for } j = 1, \ldots, J, \tag{15.D.3}$$

$$\sum_j \frac{\partial c_j(w^*, q_j^*)}{\partial w_\ell} = \bar{z}_\ell \qquad \text{for } \ell = 1, \ldots, L. \tag{15.D.4}$$

Conditions (15.D.3) and (15.D.4) constitute a system of $L + J$ equations in the $L + J$ endogenous variables $(w_1, \ldots, w_L)$ and $(q_1, \ldots, q_J)$. Condition (15.D.3) states that each firm must be at a profit-maximizing output level given prices p and w^*. If so, firm j's optimal demand for the ℓth input is $z_{\ell j}^* = \partial c_j(w, q_j^*)/\partial w_\ell$ (this is Shepard's lemma; see Proposition 5.C.2). Condition (15.D.4) is therefore the factor market-clearing condition.

Before examining the determinants of the equilibrium factor allocation in greater detail, we note that the equilibrium factor allocation $(z_1^*, \ldots, z_J^*)$ in this model is exactly the factor allocation that would be chosen by a revenue-maximizing planner, thus providing us with yet another expression of the welfare-maximizing property of competitive allocations (the first welfare theorem).[12] To see this, consider the problem faced by a planning authority who is charged with coordinating factor allocations for the economy in order to maximize the gross revenues from the economy's production activities:

$$\underset{(z_1, \ldots, z_J) \geq 0}{\text{Max}} \quad \sum_j p_j f_j(z_j) \tag{15.D.5}$$

$$\text{s.t.} \quad \textstyle\sum_j z_j = \bar{z}.$$

How does the equilibrium factor allocation $(z_1^*, \ldots, z_J^*)$ compare with what this planner does? Recall from Section 5.E that whenever we have a collection of J price-taking firms, their profit-maximizing behavior is compatible with the behavior we would observe if the firms were to maximize their profits jointly taking the prices of outputs and factors as given. That is, the factor demands $(z_1^*, \ldots, z_J^*)$ solve

$$\underset{(z_1, \ldots, z_J) \geq 0}{\text{Max}} \quad \sum_j (p_j f_j(z_j) - w^* \cdot z_j). \tag{15.D.6}$$

Since $\sum_j z_j^* = \bar{z}$ (by the equilibrium property of market clearing), the factor demands $(z_1^*, \ldots, z_J^*)$ must also solve problem (15.D.6) subject to the further constraint that $\sum_j z_j = \bar{z}$. But this implies that the factor demands $(z_1^*, \ldots, z_J^*)$ in fact solve problem (15.D.5): if we must have $\sum_j z_j = \bar{z}$, then the total cost $w^* \cdot (\sum_j z_j)$ is given, and so the joint profit-maximizing problem (15.D.6) reduces to the revenue-maximizing problem (15.D.5).

One benefit of the property just established is that it can be used to obtain the equilibrium factor allocation without a previous explicit computation of the equilibrium factor prices; we simply need to solve problem (15.D.5) directly. It also provides a useful way of viewing the equilibrium factor prices. To see this, consider again the joint profit-maximization problem (15.D.6). We can approach this problem in an equivalent manner by first deriving an aggregate

12. Note that maximization of economy-wide revenue from production would be the goal of any planner who wanted to maximize consumer welfare: it allows for the maximal purchases of consumption goods, at the fixed world prices.

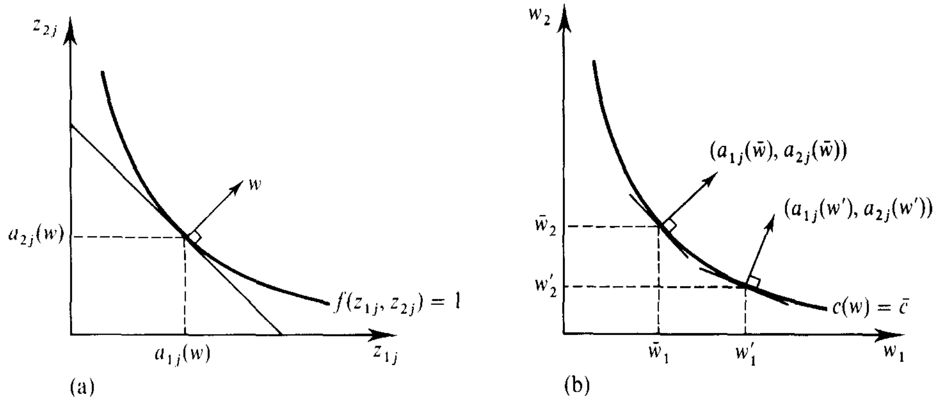


Figure 15.D.1
(a) A unit isoquant.
(b) The unit cost function.

production function for dollars:

$$f(z) = \underset{(z_1,\ldots,z_J)\geq 0}{\text{Max}} \quad p_1 f_1(z_1) + \cdots + p_J f_J(z_J)$$
$$\text{s.t. } \textstyle\sum_j z_j = z.$$

The aggregate factor demands must then solve $\text{Max}_{z \geq 0}(f(z) - w \cdot z)$. For every ℓ, the first-order condition for this problem is $w_\ell = \partial f(z)/\partial z_\ell$. Moreover, at an equilibrium, the aggregate usage of factor ℓ must be exactly $\bar{z}_\ell$. Hence, the equilibrium factor price of factor ℓ must be $w_\ell = \partial f(\bar{z})/\partial z_\ell$; that is, *the price of factor ℓ must be exactly equal to its aggregate marginal productivity (in terms of revenue)*. Since $f(\cdot)$ is concave, this observation by itself generates some interesting comparative statics. For example, a change in the endowment of a single input must change the equilibrium price of the input in the opposite direction.

Let us now be more specific and take $J = L = 2$, so that the economy under study produces two outputs from two primary factors. We also assume that the production functions $f_1(z_{11}, z_{21})$, $f_2(z_{12}, z_{22})$ are homogeneous of degree one (so the technologies exhibit constant returns to scale; see Section 5.B). This model is known as the *2 × 2 production model*. In applications, factor 1 is often thought of as labor and factor 2 as capital.

For every vector of factor prices $w = (w_1, w_2)$, we denote by $c_j(w)$ the minimum cost of producing one unit of good j and by $a_j(w) = (a_{1j}(w), a_{2j}(w))$ the input combination (assumed unique) at which this minimum cost is reached. Recall again from Proposition 5.C.2 that $\nabla c_j(w) = (a_{1j}(w), a_{2j}(w))$.

Figure 15.D.1(a) depicts the unit isoquant of firm j,

$$\{(z_{1j}, z_{2j}) \in \mathbb{R}^2_+ : f_j(z_{1j}, z_{2j}) = 1\},$$

along with the cost-minimizing input combination $(a_{1j}(w), a_{2j}(w))$. In Figure 15.D.1(b), we draw a level curve of the unit cost function, $\{(w_1, w_2)\colon c_j(w_1, w_2) = \bar{c}\}$. This curve is downward sloping because as w_1 increases, w_2 must fall in order to keep the minimized costs of producing one unit of good j unchanged. Moreover, the set $\{(w_1, w_2)\colon c_j(w_1, w_2) \geq \bar{c}\}$ is convex because of the concavity of the cost function $c_j(w)$ in w. Note that the vector $\nabla c_j(\bar{w})$, which is normal to the level curve at $\bar{w} = (\bar{w}_1, \bar{w}_2)$, is exactly $(a_{1j}(\bar{w}), a_{2j}(\bar{w}))$. As we move along the curve toward higher w_1 and lower w_2, the ratio $a_{1j}(w)/a_{2j}(w)$ falls.

Consider, first, the efficient factor allocations for this model. In Figure 15.D.2, we

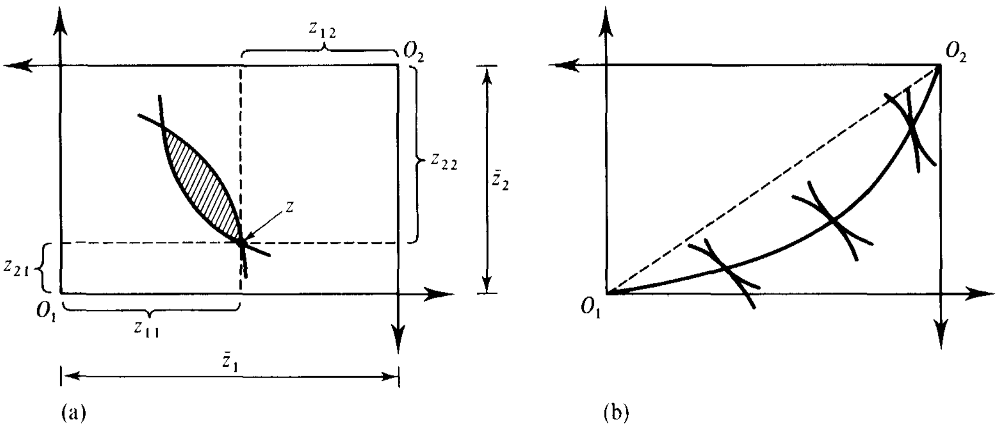


Figure 15.D.2 (a) An inefficient factor allocation. (b) The Pareto set of factor allocations.

represent the possible allocations of the factor endowments between the two firms in an Edgeworth box of size $\bar{z}_1$ by $\bar{z}_2$. The factors used by firm 1 are measured from the southwest corner; those used by firm 2 are measured from the northeast corner. We also represent the isoquants of the two firms in this Edgeworth box. Figure 15.D.2(a) depicts an inefficient allocation z of the inputs between the two firms: Any allocation in the interior of the hatched region generates more output of *both* goods than does z. Figure 15.D.2(b), on the other hand, depicts the Pareto set of factor allocations, that is, the set of factor allocations at which it is not possible, with the given total factor endowments, to produce more of one good without producing less of the other.

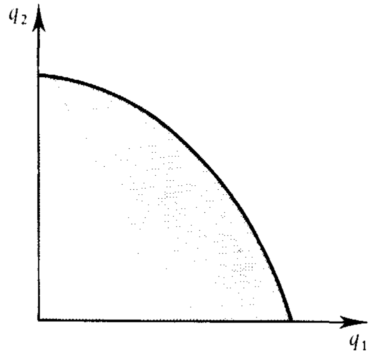
The Pareto set (endpoints excluded) must lie all above or all below or be coincident with the diagonal of the Edgeworth box. If it ever cuts the diagonal then because of constant returns, the isoquants of the two firms must in fact be tangent all along the diagonal, and so the diagonal must be the Pareto set (see also Exercise 15.B.7). Moreover, you should convince yourself of the correctness of the following claims.

Exercise 15.D.1: Suppose that the Pareto set of the 2 × 2 production model does not coincide with the diagonal of the Edgeworth box.

(a) Show that in this case, the factor intensity (the ratio of a firm's use of factor 1 relative to factor 2) of one of the firms exceeds that of the other at every point along the Pareto set.

(b) Show that in this case, any ray from the origin of either of the firms can intersect the Pareto set at most once. Conclude that the factor intensities of the two firms and the supporting relative factor prices change monotonically as we move along the Pareto set from one origin to the other.

In Figure 15.D.3, we depict the set of nonnegative output pairs (q_1, q_2) that can be produced using the economy's available factor inputs. This set is known as the *production possibility set*. Output pairs on the frontier of this set arise from factor allocations lying in the Pareto set of Figure 15.D.2(b). (Exercise 15.D.2 asks you to prove that the production possibility set is convex, as shown in Figure 15.D.3.)



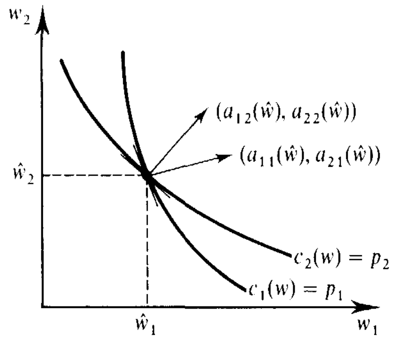


Figure 15.D.3 (left)
The production possibility set.

Figure 15.D.4 (right)
The equilibrium factor prices and factor intensities in an interior equilibrium.

With the purpose of examining more closely the determinants of the equilibrium factor allocation (z_1^*, z_2^*) and the corresponding equilibrium factor prices $w^* = (w_1^*, w_2^*)$, we now assume that the *factor intensities* of the two firms bear a systematic relation to one another. In particular, we assume that in the production of good 1, there is, relative to good 2, a greater need for the first factor. In Definition 15.D.1 we make precise the meaning of "greater need".

Definition 15.D.1: The production of good 1 is *relatively more intensive in factor 1* than is the production of good 2 if

$$\frac{a_{11}(w)}{a_{21}(w)} > \frac{a_{12}(w)}{a_{22}(w)}$$

at *all* factor prices $w = (w_1, w_2)$.

To determine the equilibrium factor prices, suppose that we have an *interior* equilibrium in which the production levels of the two goods are strictly positive (otherwise, we say that the equilibrium is *specialized*). Given our constant returns assumption, a necessary condition for (w_1^*, w_2^*) to be the factor prices in an interior equilibrium is that it satisfies the system of equations

$$c_1(w_1, w_2) = p_1 \qquad \text{and} \qquad c_2(w_1, w_2) = p_2. \tag{15.D.7}$$

That is, at an interior equilibrium, prices must be equal to unit cost. This gives us two equations for the two unknown factor prices w_1 and w_2.[13]

Figure 15.D.4 depicts the two unit cost functions in (15.D.7). By expression (15.D.7), a necessary condition for $(\hat{w}_1, \hat{w}_2)$ to be the factor prices of an interior equilibrium is that these curves cross at $(\hat{w}_1, \hat{w}_2)$. Moreover, the factor intensity assumption implies that whenever the two curves cross, the curve for firm 2 must be flatter (less negatively sloped) than that for firm 1 [recall that $\nabla c_j(w) = (a_{1j}(w), a_{2j}(w))$]. From this, it follows that the two curves can cross at most once.[14] Hence, under the

13. Expression (15.D.7) is the constant returns version of (15.D.3). Note that the effect of the constant returns to scale assumption is to make (15.D.3) independent of the output levels $(q_1, \ldots, q_J)$ (for interior equilibria).

14. If they crossed several times, then the curve for firm 2 must cross the curve for firm 1 at least once from above. At this crossing point, the curve for firm 2 would be steeper than the curve for firm 1, contradicting the factor intensity condition.

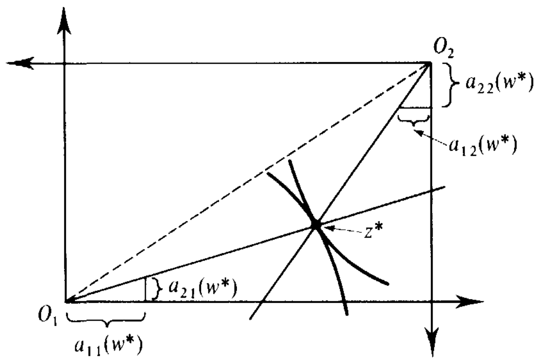


Figure 15.D.5
The equilibrium factor allocation.

factor intensity condition, there is at most a single pair of factor prices that can arise as the equilibrium factor prices of an interior equilibrium.[15]

Once the equilibrium factor prices w^* are known, the equilibrium output levels can be found graphically by determining the unique point (z_1^*, z_2^*) in the Edgeworth box of factor allocations at which both firms have the factor intensities associated with factor prices w^*, that is,

$$\frac{z_{11}^*}{z_{21}^*} = \frac{a_{11}(w^*)}{a_{21}(w^*)} \quad \text{and} \quad \frac{z_{12}^*}{z_{22}^*} = \frac{a_{12}(w^*)}{a_{22}(w^*)}.$$

The construction is depicted in Figure 15.D.5.

An important consequence of this discussion is that in the 2 × 2 production model, if the factor intensity condition holds, then as long as the economy does not specialize in the production of a single good [and therefore (15.D.7) holds], the equilibrium factor prices depend *only on the technologies of the two firms and on the output prices p*. Thus, the levels of the endowments matter only to the extent that they determine whether the economy specializes. This result is known in the international trade literature as the *factor price equalization theorem*. The theorem provides conditions (which include the presence of tradable consumption goods, identical production technologies in each country, and price-taking behavior) under which the prices of nontradable factors are equalized across nonspecialized countries.

We now present two comparative statics exercises. We first ask: How does a change in the price of one of the outputs, say p_1, affect the equilibrium factor prices and factor allocations? Figure 15.D.6(a), which depicts the induced change in Figure 15.D.4, identifies the change in factor prices. The increase in p_1 shifts firm 1's curve

15. Note, however, that although $(\hat{w}_1, \hat{w}_2)$ may solve (15.D.7), this is not sufficient to ensure that $(\hat{w}_1, \hat{w}_2)$ are equilibrium factor prices. In particular, even though $(\hat{w}_1, \hat{w}_2)$ solve (15.D.7), no interior equilibrium may exist. In Exercise 15.D.6, you are asked to show that under the factor intensity condition, the equilibrium will involve positive production of the two goods if and only if

$$\frac{a_{11}(\hat{w})}{a_{21}(\hat{w})} > \frac{\bar{z}_1}{\bar{z}_2} > \frac{a_{12}(\hat{w})}{a_{22}(\hat{w})},$$

where $\hat{w} = (\hat{w}_1, \hat{w}_2)$ is the unique solution to (15.D.7). In words, the factor intensity of the overall economy must be intermediate between the factor intensities of the two firms computed at the sole vector of factor prices at which diversification can conceivably occur.

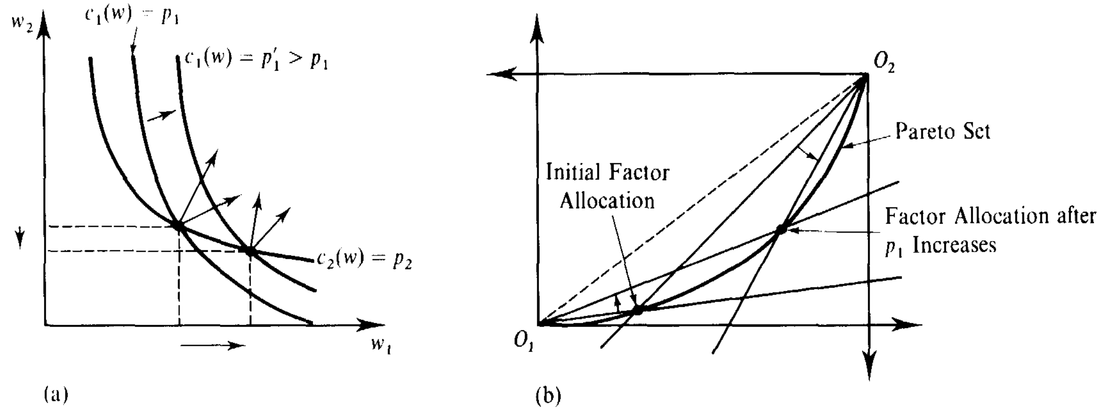


Figure 15.D.6 The Stolper–Samuelson theorem. (a) The change in equilibrium factor prices. (b) The change in the equilibrium factor allocation.

[the set $\{(w_1, w_2): c_1(w_1, w_2) = p_1\}$] outward toward higher factor price levels; the point of intersection of the two curves moves out along firm 2's curve to a higher level of w_1 and a lower level of w_2.

Formally, this gives us the result presented in Proposition 15.D.1.

Proposition 15.D.1: (*Stolper Samuelson Theorem*) In the 2×2 production model with the factor intensity assumption, if p_j increases, then the equilibrium price of the factor more intensively used in the production of good j increases, while the price of the other factor decreases (assuming interior equilibria both before and after the price change).[16]

Proof: For illustrative purposes, we provide a formal proof to go along with the graphical analysis of Figure 15.D.6 presented above. Note that it suffices to prove the result for an infinitesimal change $dp = (1, 0)$.

Differentiating conditions (15.D.7), we have

$$dp_1 = \nabla c_1(w^*)\cdot dw = a_{11}(w^*)\, dw_1 + a_{21}(w^*)\, dw_2,$$
$$dp_2 = \nabla c_2(w^*)\cdot dw = a_{12}(w^*)\, dw_1 + a_{22}(w^*)\, dw_2,$$

or in matrix notation,

$$dp = \begin{bmatrix} a_{11}(w^*) & a_{21}(w^*) \\ a_{12}(w^*) & a_{22}(w^*) \end{bmatrix} dw.$$

16. See Exercise 15.D.3 for a strengthening of this conclusion. We also note that, strictly speaking, the factor inensity condition is not required for this result. The reason is that, as we saw in Exercise 15.D.1, the firm that uses one factor, say factor 1, more intensely is the same for any point in the Pareto set of factor allocations. Suppose, for example, that we are as in Figure 15.D.2(b), where firm 1 uses factor 1 more intensively. Then, when p_1 rises, we can see from Figure 15.D.3, and the overall revenue-maximizing property of equilibrium discussed earlier in this section, that the output of good 1 increases and that of good 2 decreases. This implies that we move along the Pareto set in Figure 15.D.2(b) toward firm 2's origin. Therefore, recalling Exercise 15.D.1, both firms' intensity of use of factor 1 decreases. Hence, the equilibrium factor price ratio w_1^*/w_2^* must increase. Finally, since firm 2 is still breaking even and its output price has not changed, this implies that w_1^* increases and w_2^* decreases.

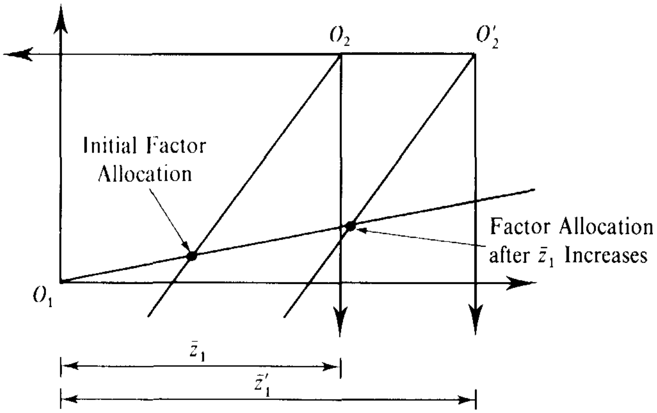


Figure 15.D.7
The Rybcszynski theorem.

Denote this 2×2 matrix by A. The factor intensity assumption implies that $|A| = a_{11}(w^*)a_{22}(w^*) - a_{12}(w^*)a_{21}(w^*) > 0$. Therefore A^{-1} exists and we can compute it to be

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} a_{22}(w^*) & -a_{21}(w^*) \\ -a_{12}(w^*) & a_{11}(w^*) \end{bmatrix}.$$

Hence, the entries of A^{-1} are positive at the diagonal and negative off the diagonal. Since $dw = A^{-1}\,dp$, this implies that for $dp = (1, 0)$ we have $dw_1 > 0$ and $dw_2 < 0$, as we wanted. ■

We have just seen that if p_1 increases, then w_1^*/w_2^* increases. Therefore, both firms must move to a less intensive use of factor 1. Figure 15.D.6(b) depicts the resulting change in the equilibrium allocation of factors. As can be seen, the factor allocation moves to a new point in the Pareto set at which the output of good 1 has risen and that of good 2 has fallen.

For the second comparative statics exercise, suppose that the total availability of factor 1 increases from $\bar{z}_1$ to $\bar{z}_1'$. What is the effect of this on equilibrium factor prices and output levels? Because neither the output prices nor the technologies have changed, the factor input prices remain unaltered (as long as the economy does not specialize). As a result, factor intensities also do not change. The new input allocation is then easily determined in the superimposed Edgeworth boxes of Figure 15.D.7; we merely find the new intersection of the two rays associated with the unaltered factor intensity levels.

Thus, examination of Figure 15.D.7 gives us the result presented in Proposition 15.D.2.

Proposition 15.D.2: (*Rybcszynski Theorem*) In the 2×2 production model with the factor intensity assumption, if the endowment of a factor increases, then the production of the good that uses this factor relatively more intensively increases and the production of the other good decreases (assuming interior equilibria both before and after the change of endowment).

For further discussion of the 2×2 production model see, for example, Johnson (1971).

Consider the general case of an arbitrary number of factors L and outputs J. For given output prices, the zero-profit conditions [i.e., the general analog of expression (15.D.7)] constitute a (nonlinear) system of J equations in L unknowns. If $L > J$, then there are too many unknowns and we cannot hope that the zero-profit conditions alone will determine the factor prices. The total factor endowments will play a role. If $J > L$, then there are too many equations and, for typical world prices, they cannot all be satisfied simultaneously. What this means is that the economy will specialize in the production of a number of goods equal to the number of factors L. The set of goods chosen may well depend on the endowments of factors. Beyond the 2×2 situation (the analysis of which, as we have seen, is quite instructive), the case $L = J$ seems too coincidental to be of interest. Nevertheless, we point out that in this case the zero-profit conditions are nonlinear and that in order to guarantee a unique solution (and versions of the Stolper–Samuelson and the Rybcszynski theorems), we need a generalization of the factor intensity condition. These generalizations exist, but they cannot be interpreted economically in as simple a manner as can the factor intensity condition of the 2×2 model.

15.E General Versus Partial Equilibrium Theory

There are some problems that are inherently general equilibrium problems. It would be hard to envision convincing analyses of economic growth, demographic change, international economic relations, or monetary policy that were restricted to only a subset of commodities and did not consider economy-wide feedback effects.

Partial equilibrium models of markets, or of systems of related markets, determine prices, profits, productions, and the other variables of interest adhering to the assumption that there are no feedback effects from these endogenous magnitudes to the underlying demand or cost curves that are specified in advance. Individuals' wealth is another variable that general equilibrium theory regards as endogenously determined but that is often treated as exogenous in partial equilibrium theory.

If general equilibrium analysis did not change any of the predictions or conclusions of partial equilibrium analysis, it would be of limited significance when applied to problems amenable to partial equilibrium treatment. It might be of comfort because we would then know that our partial equilibrium conclusions are valid, but it would not change our view of how markets work. However, things are not that simple. The choice of methodology may be far from innocuous. We now present an example [due to Bradford (1978)] in which a naive application of partial equilibrium analysis leads us seriously astray. See Sections 3.I and 10.G for some discussion of when partial equilibrium theory is (approximately) justified.

A Tax Incidence Example

Consider an economy with a large number of towns, N. Each town has a single price-taking firm that produces a consumption good by means of the strictly concave production function $f(z)$ (once again, we could reinterpret the model as having many identical firms in each town to make the price-taking hypothesis more palatable). This consumption good, which is identical across towns, is traded in a national market. The overall economy has M units of labor, inelastically supplied by workers who derive utility only from the output of the firms. Workers are free to move from town to town and do so to seek the highest wage. We normalize the price of the consumption good to be 1, and we denote the wage rate in town n's labor market by w_n.

Given that workers can move freely in search of the highest wage, at an equilibrium the wage rates across towns must be equal; that is, we must have $w_1 = \cdots = w_N = \bar{w}$. From the symmetry of the problem, it must be that each firm hires exactly M/N units of labor in an equilibrium. As a result, the equilibrium wage rate must be $\bar{w} = f'(M/N)$. The equilibrium profits of an individual firm are therefore $f(M/N) - f'(M/N)(M/N)$.

Now suppose that town 1 levies a tax on the labor used by the firm located there. We investigate the "incidence" of the tax on workers and firms (or, more properly, on the firms' owners); that is, we examine the extent to which each group bears the burden of the tax. If the tax rate is t and the wage in town 1 is w_1, the labor demand of the firm in town 1 will be the amount z_1 such that $f'(z_1) = t + w_1$. At this point, we may be tempted to argue that, since N is large, we can approximate and take the wage rates elsewhere, $\bar{w}$, to be unaffected by this change in town 1. Moreover, since labor moves freely, the supply correspondence of workers in town 1 should then be 0 at $w_1 < \bar{w}$, ∞ at $w_1 > \bar{w}$, and $[0, \infty]$ at $w_1 = \bar{w}$. Thus, taking a partial equilibrium view, the equilibrium wage rate in the town 1 labor market remains equal to $\bar{w}$, and the labor employed in town 1 falls to the level z_1 such that $f'(z_1) = t + \bar{w}$ (hence, some labor will shift to the other towns). By adopting this sort of partial equilibrium view of the labor market of town 1, we are therefore led to conclude that the income of workers remains the same, as does the profit of every firm not located in town 1. Only the profit of the firm in town 1 decreases. The qualitative conclusion is that firms (actually, firms' owners) "bear" all of the tax burden. Labor, because it is free to move and because the number of untaxed firms is large, "escapes."

Alas, this conclusion constitutes an egregious mistake, and it will be overturned by a general equilibrium view of the same model.

We now look at the general equilibrium across the labor markets of all the towns. We know that the equilibrium wage rate must be such that $w_1 = \cdots = w_N$ and that all M units of labor are employed. Let $w(t)$ be this common equilibrium wage when the tax rate in town 1 is t. By symmetry, the firms in towns $2, \ldots, N$ will each employ the same amount of labor, $z(t)$. Let $z_1(t)$ be the equilibrium labor demand of the firm in town 1 when town 1's tax rate is t. Then the equilibrium conditions are

$$(N - 1)z(t) + z_1(t) = M. \tag{15.E.1}$$

$$f'(z(t)) = w(t). \tag{15.E.2}$$

$$f'(z_1(t)) = w(t) + t. \tag{15.E.3}$$

Consider the impact on wages of the introduction of a small tax dt. Substituting from (15.E.1) for $z_1(t)$ in (15.E.3), differentiating with respect to t, and evaluating at $t = 0$ [at which point $z_1(0) = z(0) = (M/N)$], we get

$$-f''(M/N)(N - 1)z'(0) = w'(0) + 1. \tag{15.E.4}$$

But from (15.E.2), we get

$$f''(M/N)z'(0) = w'(0). \tag{15.E.5}$$

Substituting from (15.E.5) into (15.E.4) yields

$$w'(0) = -\frac{1}{N}.$$

Therefore, once the general equilibrium effects are taken into account, we see that the wage rate in all towns falls with the imposition of the tax in town 1. However, we see that this fall in the wage rate approaches zero as N grows large. Thus, at this point, it may still seem that our partial equilibrium approximation will have given us the correct answers for large N. But this is not so. Consider the effect of the tax on total profits. The partial equilibrium approach told us that workers escaped the tax; all the tax fell as a burden on firms. But letting $\pi(w)$ be the profit function of a representative firm, the change in aggregate profits from the imposition of this tax is[17]

$$(N-1)\pi'(\bar{w})w'(0) + \pi'(\bar{w})(w'(0)+1) = \pi'(\bar{w})\left(-\frac{N-1}{N} + \frac{N-1}{N}\right) = 0.$$

Aggregate profits stay constant! Thus, all of the burden of a small tax falls on laborers, not on the owners of firms. Although the partial equilibrium approximation is correct as far as getting prices and wages about right, it errs by just enough, and in just such a direction, that the conclusions of the tax incidence analysis based on it are completely reversed.[18]

17. Recall that the profits of the firm in town 1 are $\pi(w(t)+t)$.

18. We note that the justifications of partial equilibrium analysis in terms of small individual budget shares that we informally described in Sections 3.I and 10.G do not apply here because the "consumption" goods in this example (jobs in different towns) are perfect substitutes and therefore individual budget shares are not guaranteed to be small at all prices.

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EXERCISES

15.B.1^{A} Consider an Edgeworth box economy in which the two consumers have locally nonsatiated preferences. Let $x_{\ell i}(p)$ be consumer i's demand for good ℓ at prices $p = (p_1, p_2)$.

(a) Show that $p_1(\sum_i x_{1i}(p) - \bar{\omega}_1) + p_2(\sum_i x_{2i}(p) - \bar{\omega}_2) = 0$ for all prices p.

(b) Argue that if the market for good 1 clears at prices $p^* \gg 0$, then so does the market for good 2; hence, p^* is a Walrasian equilibrium price vector.

15.B.2^{A} Consider an Edgeworth box economy in which the consumers have the Cobb–Douglas utility functions $u_1(x_{11}, x_{21}) = x_{11}^{\alpha} x_{21}^{1-\alpha}$ and $u_2(x_{12}, x_{22}) = x_{12}^{\beta} x_{22}^{1-\beta}$. Consumer i's endowments are $(\omega_{1i}, \omega_{2i}) \gg 0$, for $i = 1, 2$. Solve for the equilibrium price ratio and allocation. How do these change with a differential change in ω_{11}?

15.B.3^{B} Argue (graphically) that in an Edgeworth box economy with locally nonsatiated preferences, a Walrasian equilibrium is Pareto optimal.

15.B.4[C] Consider an Edgeworth box economy. An offer curve has the *gross substitute* property if an increase in the price of one commodity decreases the demand for that commodity and increases the demand for the other one.

(a) Represent in an Edgeworth box the shape of an offer curve with the gross substitute property.

(b) Assume that the offer curves of the two consumers have the gross substitute property. Show then that the offer curves can intersect only once (not counting the intersection at the initial endowments).

Let us denote an offer curve as *normal* if an increase in the price of one commodity leads to an increase in the demand for that commodity only if the demands of the two commodities both increase.

(c) Represent in the Edgeworth box the shape of a normal offer curve that does not satisfy the gross substitute property.

(d) Show that there are preferences giving rise to offer curves that are not normal. Show that the demand function for such preferences is not normal (i.e., at some prices some good is inferior).

(e) Show in the Edgeworth box that if the offer curve of one consumer is normal and that of the other satisfies the gross substitute property, then the offer curves can intersect at most once (not counting the intersection at the initial endowments).

(f) Show that two normal offer curves can intersect several times.

15.B.5[A] Verify that the offer curves of Example 15.B.2 are as claimed. Solve also for the claimed values of relative prices.

15.B.6[B] (D. Blair) Compute the equilibria of the following Edgeworth box economy (there is more than one):

$$u_1(x_{11}, x_{21}) = (x_{11}^{-2} + (12/37)^3 x_{21}^{-2})^{-1/2}, \qquad \omega_1 = (1, 0),$$
$$u_2(x_{12}, x_{22}) = ((12/37)^3 x_{12}^{-2} + x_{22}^{-2})^{-1/2}, \qquad \omega_2 = (0, 1).$$

15.B.7[C] Show that if both consumers in an Edgeworth box economy have continuous, strongly monotone, and strictly convex preferences, then the Pareto set has no "holes": precisely, it is a connected set. Show that if, in addition, the preferences of both consumers are homothetic, then the Pareto set lies entirely on one side of the diagonal of the box.

15.B.8[B] Suppose that both consumers in an Edgeworth box have continuous and strictly convex preferences that admit a quasilinear utility representation with the first good as numeraire. Show that any two Pareto optimal allocations in the interior of the Edgeworth box then involve the same consumptions of the second good. Connect this with the discussion of Chapter 10.

15.B.9[B] Suppose that in a pure exchange economy (i.e., an economy without production), we have two consumers, Alphanse and Betatrix, and two goods, Perrier and Brie. Alphanse and Betatrix have the utility functions:

$$u_\alpha = \text{Min}\{x_{p\alpha}, x_{b\alpha}\} \qquad \text{and} \qquad u_\beta = \text{Min}\{x_{p\beta}, (x_{b\beta})^{1/2}\}$$

(where $x_{p\alpha}$ is Alphanse's consumption of Perrier, and so on). Alphanse starts with an endowment of 30 units of Perrier (and none of Brie); Betatrix starts with 20 units of Brie (and none of Perrier). Neither can consume negative amounts of a good. If the two consumers behave as price takers, what is the equilibrium?

Suppose instead that Alphanse begins with only 5 units of Perrier while Betatrix's initial endowment remains 20 units of Brie, 0 units of Perrier. What happens now?

15.B.10^{C} *(The Transfer Paradox)* In a two-consumer, two-commodity pure exchange economy with continuous, strictly convex and strongly monotone preferences, consider the comparative statics of the welfare of consumer 1 with changes in the initial endowments $\omega_1 = (\omega_{11}, \omega_{21})$ and $\omega_2 = (\omega_{12}, \omega_{22})$.

(a) Suppose first that the preferences of the two consumers are quasilinear with respect to the same numeraire. Show that if the endowments of consumer 1 are increased to $\omega_1' \gg \omega_1$ while ω_2 remains the same, then at equilibrium the utility of consumer 1 *may* decrease. Interpret this observation and relate it to the theory of a quantity-setting monopoly.

(b) Suppose now that the increase in resources of consumer 1 constitute a transfer from consumer 2, that is, $\omega_1' = \omega_1 + z$ and $\omega_2' = \omega_2 - z$ with $z \geq 0$. Under the same assumption as in **(a)**, show that the utility of consumer 1 cannot decrease.

(c) Consider again a transfer as in **(b)**, but this time preferences may not be quasilinear. Suppose that the transfer z is small and that similarly the change in the equilibrium (relative) price is restricted to be small. Show that it is possible for the utility of consumer 1 to decrease (this is called the *transfer paradox*). A graphical illustration in the Edgeworth box suffices to make the point. Interpret in terms of the interplay between substitution and wealth effects.

(d) Show that in this Edgeworth box example (but, be warned, not more generally) the transfer paradox can happen only if there is a multiplicity of equilibria. [*Hint:* Argue graphically in the Edgeworth box. Show that if a transfer to consumer 1 leads to a decrease of the utility of consumer 1, then there must be an equilibrium at the no-transfer situation where consumer 1 gets an even lower level of utility.]

15.C.1^{B} This exercise refers to the one-consumer, one-firm economy discussed in Section 15.C.

(a) Prove that in an economy with one firm, one consumer, and strictly convex preferences and technology, the equilibrium level of production is unique.

(b) Fix the price of output to be 1. Define the excess demand function for labor as

$$z_1(w) = x_1(w, w\bar{L} + \pi(w)) + y_1(w) - \bar{L},$$

where w is the wage rate, $\pi(\cdot)$ is the profit function, and $x_1(\cdot, \cdot)$, $y_1(\cdot)$ are, respectively, the consumer's demand function for leisure and the firm's demand function for labor. Show that the slope of the excess demand function is not necessarily of one sign throughout the range of prices but that it is necessarily negative in a neighborhood of the equilibrium.

(c) Give an example to show that there can be multiple equilibria in a strictly convex economy with one firm and two individuals, each of whom is endowed with labor alone. (Assume that profits are split equally between the two consumers.) Can this happen if the firm operates under constant rather than strictly decreasing returns to scale?

15.C.2^{A} Consider the one-consumer, one-producer economy discussed in Section 15.C. Compute the equilibrium prices, profits, and consumptions when the production function is $f(z) = z^{1/2}$, the utility function is $u(x_1, x_2) = \ln x_1 + \ln x_2$, and the total endowment of labor is $\bar{L} = 1$.

15.D.1^{B} In text.

15.D.2^{A} Show that in the 2×2 production model the production possibility set is convex (assume free disposal).

15.D.3^{B} Show that the Stolper–Samuelson theorem (Proposition 15.D.1) can be strengthened to assert that the increase in the price of the intensive factor is proportionally larger than the increase in the price of the good (and therefore the well-being of a consumer who owns only the intensive factor must increase).

15.D.4^{C} Consider a general equilibrium problem with two consumer–workers ($i = 1, 2$), two constant returns firms ($j = 1, 2$) with concave technologies, two factors of production ($\ell = 1, 2$), and two consumption goods ($j = 1, 2$) produced, respectively, by the two firms. Assume that the production of consumption good 1 is relatively more intensive in factor 1. Neither consumer consumes either of the factors. Consumer 1 owns one unit of factor 1 while consumer 2 owns one unit of factor 2.

(a) Set up the equilibrium problem as one of clearing the factor and goods markets (in a closed economy context) under the assumption that prices are taken as given and productions are profit maximizing.

(b) Suppose that consumer 1 has a taste only for the second consumption good and that consumer 2 cares only for the first good. Argue that there is at most one equilibrium.

(c) Suppose now that consumer 1 has a taste only for the first good and that consumer 2 cares only for the second good. Argue that a multiplicity of equilibria is possible.

[*Hint:* Parts **(b)** and **(c)** can be answered by graphical analysis in the Edgeworth box of factors of production.]

15.D.5^{B} Show that the Rybcszynski theorem (Proposition 15.D.2) can be strengthened to assert that the proportional increase in the production of the good that uses the increased factor relatively more intensively is greater than the proportional increase in the endowment of the factor.

15.D.6^{C} Suppose you are in the 2×2 production model with output prices (p_1, p_2) given (the economy could be a small open economy). The factor intensity condition is satisfied (production of consumption good 1 uses factor 1 more intensely). The total endowment vector is $z \in \mathbb{R}^2$.

(a) Set up the equilibrium conditions for factor prices (w_1^*, w_2^*) and outputs (q_1^*, q_2^*) allowing for the possibility of specialization.

(b) Suppose that $\hat{w} = (\hat{w}_1, \hat{w}_2)$ are factor prices with the property that for each of the two goods the unit cost equals the price. Show that the necessary and sufficient condition for the equilibrium determined in **(a)** to have $(q_1^*, q_2^*) \gg 0$ is that $\bar{z}$ belongs to the set

$$\{(z_1, z_2) \in \mathbb{R}^2_+ : a_{11}(\hat{w})/a_{21}(\hat{w}) > z_1/z_2 > a_{12}(\hat{w})/a_{22}(\hat{w})\},$$

where $a_{\ell j}(\hat{w})$ is the optimal usage (at factor prices $\hat{w}$) of the input ℓ in the production of one unit of good j. This set is called the *diversification cone*.

(c) The unit-dollar isoquant of good j is the set of factor combinations that produce an amount of good j of 1 dollar value. Show that under the factor intensity condition the unit-dollar isoquants of the two goods can intersect at most once. Use the unit-dollar isoquants to construct graphically the diversification cone. [*Hint:* If they intersect twice then there are two points (one in each isoquant) proportional to each other and such that the slopes of the isoquants at these points are identical.]

(d) When the total factor endowment is not in the diversification cone, the equilibrium is specialized. Can you determine, as a function of total factor endowments, in which good the economy will specialize and what the factor prices will be? Be sure to verify the inequality conditions in **(a)**. To answer this question you can make use of the graphical apparatus developed in **(c)**.

15.D.7[B] Suppose there are two output goods and two factors. The production functions for the two outputs are

$$f_1(z_{11}, z_{21}) = 2(z_{11})^{1/2} + (z_{21})^{1/2} \quad \text{and} \quad f_2(z_{12}, z_{22}) = (z_{12})^{1/2} + 2(z_{22})^{1/2}.$$

The international prices for these goods are $p = (1, 1)$. Firms are price takers and maximize profits. The total factor endowments are $\bar{z} = (\bar{z}_1, \bar{z}_2)$. Consumers have no taste for the consumption of factors of production. Derive the equilibrium factor allocation $((z_{11}^*, z_{21}^*), (z_{12}^*, z_{22}^*))$ and the equilibrium factor prices (w_1^*, w_2^*) as a function of $(\bar{z}_1, \bar{z}_2)$. Verify that you get the same result whether you proceed through equations (15.D.1) and (15.D.2) or by solving (15.D.5).

15.D.8[B] The setting is as in the 2×2 production model. The production functions for the two outputs are of the Cobb–Douglas type:

$$f_1(z_{11}, z_{21}) = (z_{11})^{2/3}(z_{21})^{1/3} \quad \text{and} \quad f_2(z_{12}, z_{22}) = (z_{12})^{1/3}(z_{22})^{2/3}.$$

The international output price vector is $p = (1, 1)$ and the total factor endowments vector is $\bar{z} = (\bar{z}_1, \bar{z}_2) \gg 0$. Compute the equilibrium factor allocations and factor prices for all possible values of $\bar{z}$. Be careful in specifying the region of total endowment vectors where the economy will specialize in the production of a single good.

15.D.9[C] (*The Heckscher–Ohlin Theorem*) Suppose there are two consumption goods, two factors, and two countries A and B. Each country has technologies as in the 2×2 production model. The technologies for the production of each consumption good are the same in the two countries. The technology for the production of the first consumption good is relatively more intensive in factor 1. The endowments of the two factors are $\bar{z}_A \in \mathbb{R}^2_+$ and $\bar{z}_B \in \mathbb{R}^2_+$ for countries A and B, respectively. We assume that country A is relatively better endowed with factor 1, that is, $\bar{z}_{1A}/\bar{z}_{2A} > \bar{z}_{1B}/\bar{z}_{2B}$. Consumers are identical within and between countries. Their preferences are representable by increasing, concave, and homogeneous utility functions that depend only on the amount consumed of the two consumption goods.

Suppose that factors are not mobile and that each country is a price taker with respect to the international prices for consumption goods. Suppose then that at the international prices $p = (p_1, p_2)$ we have that, first, neither of the two countries specializes and, second, the international markets for consumption goods clear. Prove that country A must be exporting good 1, the good whose production is relatively more intensive in the factor that is relatively more abundant in country A.

16 Equilibrium and Its Basic Welfare Properties

16.A Introduction

With this chapter, we begin our systematic study of equilibrium in economies where agents act as price takers. We consider a world with L commodities in which consumers and firms interact through a market system. In this market system, a price is quoted for every commodity, and economic agents take these prices as independent of their individual actions.

We concentrate in this chapter on a presentation of the basic welfare properties of equilibria. Some more advanced topics in welfare economics are discussed in Chapter 18 and in Part V.

We begin, in Section 16.B, by specifying the formal model of an economy to be studied here and for the rest of Part IV. Its essential ingredients—commodities, consumers, and firms—we have already encountered in Part I. The remainder of Section 16.B introduces the main concepts that will concern us throughout the chapter. We define first the normative notion of a *Pareto optimal allocation*, an allocation with the property that it is impossible to make any consumer better off without making some other consumer worse off. Then, we present two notions of price-taking equilibrium: *Walrasian* (or *competitive*) *equilibrium*, and its generalization, a *price equilibrium with transfers*. The Walrasian equilibrium concept applies to the case of a *private ownership economy*, in which a consumer's wealth is derived from her ownership of endowments and from claims to profit shares of firms. The more general notion of a price equilibrium with transfers allows instead for an arbitrary distribution of wealth among consumers.

The remaining sections of the chapter are devoted to exploring the relationships between these equilibrium concepts and Pareto optimality.

Section 16.C focuses on the statement of the (very weak) conditions implying that every price equilibrium with transfers (and, hence, every Walrasian equilibrium) results in a Pareto optimal allocation. This is the *first fundamental theorem of welfare economics*, a formal expression for competitive market economies of Adam Smith's claimed "invisible hand" property of markets.

In Section 16.D, we study the converse issue. We state conditions (convexity assumptions are the crucial ones) under which every Pareto optimal allocation can be supported as a price equilibrium with transfers. This result is known as the *second fundamental theorem of welfare economics.* It tells us that if its assumptions are satisfied, then through the use of appropriate lump-sum wealth transfers, a welfare authority can, in principle, implement any desired Pareto optimal allocation as a price-taking equilibrium. We also discuss the practical limitations of this result.

In Section 16.E, we introduce the problem of maximizing a *social welfare function* and relate it to the Pareto optimality concept. We uncover a close formal relationship between these two notions of welfare optimality.

Section 16.F reexamines the Pareto optimality concept and associated results by making differentiability assumptions and analyzing first-order conditions. There we see how equilibrium prices can be interpreted as the Lagrange multipliers, or shadow prices, that arise in the associated Pareto optimality problem.

Section 16.G discusses several applications of the concepts and results previously developed. We first present some examples that rely on particular interpretations of the L abstract commodities; one of them concerns the case of *public goods.* We then consider an application of our results to a world with nonconvex production sets, which leads to a brief exposition of the theory of *marginal cost pricing.*

Appendix A deals with some technical issues concerning the boundedness of the set of feasible allocations and the existence of Pareto optima.

Classical accounts of the material at the heart of this chapter are given by Koopmans (1957), Debreu (1959), and Arrow and Hahn (1971).

16.B The Basic Model and Definitions

In this chapter, we study an economy composed of $I > 0$ consumers and $J > 0$ firms in which there are L commodities. These L commodities can be given many possible interpretations; we discuss some examples in Section 16.G.

Each consumer $i = 1, \ldots, I$ is characterized by a consumption set $X_i \subset \mathbb{R}^L$ and a preference relation $\succsim_i$ defined on X_i. We assume that these preferences are rational (i.e., complete and transitive). Chapters 1 to 3 provide an extensive discussion of these concepts.

Each firm $j = 1, \ldots, J$ is characterized by a technology, or production set, $Y_j \subset \mathbb{R}^L$. We assume that every Y_j is nonempty and closed. See Chapter 5 for a discussion of production sets and their properties.

The initial resources of commodities in the economy—that is, the economy's *endowments*—are given to us by a vector $\bar{\omega} = (\bar{\omega}_1, \ldots, \bar{\omega}_L) \in \mathbb{R}^L$.

Thus, the basic data on preferences, technologies, and resources for this economy are summarized by $(\{(X_i, \succsim_i)\}_{i=1}^I, \{Y_j\}_{j=1}^J, \bar{\omega})$.

The Edgeworth box pure exchange economy discussed in Section 15.B, for example, corresponds to the case in which $L = 2$, $I = 2$, $X_1 = X_2 = \mathbb{R}^L_+$, $J = 1$, and $Y_1 = -\mathbb{R}^2_+$ (the disposal technology). More generally, we say that an economy is a *pure exchange economy* if its only technological possibility is that of free disposal, that is, if $Y_j = -\mathbb{R}^L_+$ for all $j = 1, \ldots, J$.

Definition 16.B.1: An *allocation* $(x, y) = (x_1, \ldots, x_I, y_1, \ldots, y_J)$ is a specification of a consumption vector $x_i \in X_i$ for each consumer $i = 1, \ldots, I$ and a production vector $y_j \in Y_j$ for each firm $j = 1, \ldots, J$. An allocation (x, y) is *feasible* if

$\sum_i x_{\ell i} = \bar{\omega}_\ell + \sum_j y_{\ell j}$ for every commodity ℓ. That is, if

$$\sum_i x_i = \bar{\omega} + \sum_j y_j. \tag{16.B.1}$$

We denote the set of feasible allocations by

$$A = \{(x, y) \in X_1 \times \cdots \times X_I \times Y_1 \times \cdots \times Y_J : \sum_i x_i = \bar{\omega} + \sum_j y_j\} \subset \mathbb{R}^{L(I+J)}.$$

The notion of a socially desirable outcome that we focus on is that of a *Pareto optimal* allocation.

Definition 16.B.2: A feasible allocation (x, y) is *Pareto optimal* (or *Pareto efficient*) if there is no other allocation $(x', y') \in A$ that *Pareto dominates* it, that is, if there is no feasible allocation (x', y') such that $x'_i \succsim_i x_i$ for all i and $x'_i \succ_i x_i$ for some i.

An allocation is Pareto optimal if there is no waste: It is impossible to make any consumer strictly better off without making some other consumer worse off. Note that the Pareto optimality concept does not concern itself with distributional issues. For example, in a pure exchange economy, an allocation that gives all of society's endowments to one consumer who has strongly monotone preferences is necessarily Pareto optimal.

In Appendix A, we provide conditions on the primitives of the economy implying that the set of feasible allocations A is nonempty, closed, and bounded and that Pareto optimal allocations exist.

Private Ownership Economies

Throughout Part IV, we study the properties of competitive *private ownership economies*. In such economies, every good is traded in a market at publicly known prices that consumers and firms take as unaffected by their own actions. Consumers trade in the marketplace to maximize their well-being, and firms produce and trade to maximize profits. The wealth of consumers is derived from individual endowments of commodities and from ownership claims (*shares*) to the profits of the firms, which are therefore thought of as being owned by consumers.[1]

Formally, consumer i has an initial endowment vector of commodities $\omega_i \in \mathbb{R}^L$ and a claim to a share $\theta_{ij} \in [0, 1]$ of the profits of firm j (where $\bar{\omega} = \sum_i \omega_i$ and $\sum_i \theta_{ij} = 1$ for every firm j). Thus, the basic preference, technological, resource, and ownership data of a private ownership economy are summarized by $(\{(X_i, \succsim_i)\}_{i=1}^I, \{Y_j\}_{j=1}^J, \{(\omega_i, \theta_{i1}, \ldots, \theta_{iJ})\}_{i=1}^I)$.

The notion of a price-taking equilibrium for a competitive private ownership economy is that of a *Walrasian equilibrium*.

Definition 16.B.3: Given a private ownership economy specified by $(\{(X_i, \succsim_i)\}_{i=1}^I, \{Y_j\}_{j=1}^J, \{(\omega_i, \theta_{i1}, \ldots, \theta_{iJ})\}_{i=1}^I)$, an allocation (x^*, y^*) and a price vector $p = (p_1, \ldots, p_L)$ constitute a *Walrasian* (or *competitive*) *equilibrium* if:

(i) For every j, y_j^* maximizes profits in Y_j; that is,

$$p \cdot y_j \leq p \cdot y_j^* \quad \text{for all } y_j \in Y_j.$$

1. Recall from Section 5.G that, under our present assumptions, the consumer owners of a firm are unanimously in favor of the objective of profit maximization.

(ii) For every i, x_i^* is maximal for $\succsim_i$ in the budget set

$$\{x_i \in X_i \colon p \cdot x_i \le p \cdot \omega_i + \sum_j \theta_{ij} p \cdot y_j^*\}.^2$$

(iii) $\sum_i x_i^* = \bar{\omega} + \sum_j y_j^*$.

Condition (i) of Definition 16.B.3 says that at a Walrasian equilibrium, firms are maximizing their profits given the equilibrium prices p. The logic of profit maximization is examined extensively in Chapter 5. Condition (ii) says that consumers are maximizing their well-being given, first, the equilibrium prices and, second, the wealth derived from their holdings of commodities and from their shares of profits. See Chapter 3 for extensive discussion of preference maximization. Finally, condition (iii) says that markets must clear at an equilibrium; that is, all consumers and firms must be able to achieve their desired trades at the going market prices.

Price Equilibria with Transfers

The aim of this chapter is to relate the idea of Pareto optimality to supportability by means of price-taking behavior. To this end, it is useful to introduce a notion of equilibrium that allows for a more general determination of consumers' wealth levels than that in a private ownership economy. By way of motivation, we can imagine a situation where a social planner is able to carry out (lump-sum) redistributions of wealth, and where society's aggregate wealth can therefore be redistributed among consumers in any desired manner.

Definition 16.B.4: Given an economy specified by $(\{(X_i, \succsim_i)\}_{i=1}^I, \{Y_j\}_{j=1}^J, \bar{\omega})$ an allocation (x^*, y^*) and a price vector $p = (p_1, \ldots, p_L)$ constitute a *price equilibrium with transfers* if there is an assignment of wealth levels $(w_1, \ldots, w_I)$ with $\sum_i w_i = p \cdot \bar{\omega} + \sum_j p \cdot y_j^*$ such that

(i) For every j, y_j^* maximizes profits in Y_j; that is,

$$p \cdot y_j \le p \cdot y_j^* \qquad \text{for all } y_j \in Y_j.$$

(ii) For every i, x_i^* is maximal for $\succsim_i$ in the budget set

$$\{x_i \in X_i \colon p \cdot x_i \le w_i\}.$$

(iii) $\sum_i x_i^* = \bar{\omega} + \sum_j y_j^*$.

The concept of a price equilibrium with transfers requires only that there be *some* wealth distribution such that allocation (x^*, y^*) and price vector $p \in \mathbb{R}^L$ constitute an equilibrium. It captures the idea of price-taking market behavior without any supposition about the determination of consumers' wealth levels. Note that a Walrasian equilibrium is a special case of an equilibrium with transfers. It amounts to the case in which, for every i, consumer i's wealth level is determined by the initial endowment vector ω_i and by the profit shares $(\theta_{i1}, \ldots, \theta_{iJ})$ without any further wealth transfers, that is, where $w_i = p \cdot \omega_i + \sum_j \theta_{ij} p \cdot y_j^*$ for all $i = 1, \ldots, I$.

2. The terminology "x_i is maximal for $\succsim_i$ in set B" means that x_i is a preference-maximizing choice for consumer i in the set B; that is, $x_i \in B$ and $x_i \succsim_i x_i'$ for all $x_i' \in B$.

16.C The First Fundamental Theorem of Welfare Economics

The first fundamental theorem of welfare economics states conditions under which any price equilibrium with transfers, and in particular any Walrasian equilibrium, is a Pareto optimum. For competitive market economies, it provides a formal and very general confirmation of Adam Smith's asserted "invisible hand" property of the market. A single, very weak assumption, the *local nonsatiation of preferences* (see Section 3.B), is all that is required for the result. Notably, we need not appeal to any convexity assumption whatsoever.

Recall the definition of locally nonsatiated preferences from Section 3.B (Definition 3.B.3).

Definition 16.C.1: The preference relation $\succsim_i$ on the consumption set X_i is *locally nonsatiated* if for every $x_i \in X_i$ and every $\varepsilon > 0$, there is an $x'_i \in X_i$ such that $\|x'_i - x_i\| \leq \varepsilon$ and $x'_i \succ_i x_i$.

Intuitively, the local nonsatiation condition will be satisfied if there are some desirable commodities. Note also a significant implication of the condition: if $\succsim_i$ is continuous and locally nonsatiated, then any closed consumption set X_i must be unbounded. Otherwise, there would by necessity exist a global (hence, local) satiation point (see Exercise 16.C.1).

Proposition 16.C.1: (*First Fundamental Theorem of Welfare Economics*) If preferences are locally nonsatiated, and if (x^*, y^*, p) is a price equilibrium with transfers, then the allocation (x^*, y^*) is Pareto optimal. In particular, any Walrasian equilibrium allocation is Pareto optimal.

Proof: Suppose that (x^*, y^*, p) is a price equilibrium with transfers and that the associated wealth levels are $(w_1, \dots, w_I)$. Recall that $\sum_i w_i = p \cdot \bar{\omega} + \sum_j p \cdot y_j^*$.

The preference maximization part of the definition of a price equilibrium with transfers [i.e., part (ii) of Definition 16.B.4] implies that

$$\text{If } x_i \succ_i x_i^* \qquad \text{then } p \cdot x_i > w_i. \tag{16.C.1}$$

That is, anything that is strictly preferred by consumer i to x_i^* must be unaffordable to her. The significance of the local nonsatiation condition for the purpose at hand is that with it (16.C.1) implies an additional property:

$$\text{If } x_i \succsim_i x_i^* \qquad \text{then } p \cdot x_i \geq w_i. \tag{16.C.2}$$

That is, anything that is at least as good as x_i^* is at best just affordable. This property is easily verified (you are asked to do so in Exercise 16.C.2).

Now consider an allocation (x, y) that Pareto dominates (x^*, y^*). That is, $x_i \succsim_i x_i^*$ for all i and $x_i \succ_i x_i^*$ for some i. By (16.C.2), we must have $p \cdot x_i \geq w_i$ for all i, and by (16.C.1) $p \cdot x_i > w_i$ for some i. Hence,

$$\textstyle\sum_i p \cdot x_i > \sum_i w_i = p \cdot \bar{\omega} + \sum_j p \cdot y_j^*.$$

Moreover, because y_j^* is profit maximizing for firm j at price vector p,

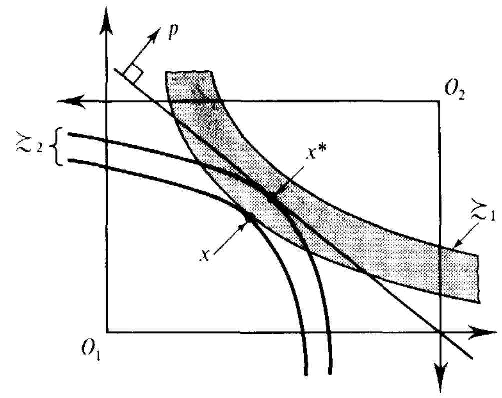


Figure 16.C.1
A price equilibrium with transfers that is not a Pareto optimum.

we have $p \cdot \bar{\omega} + \sum_j p \cdot y_j^* \geq p \cdot \bar{\omega} + \sum_j p \cdot y_j$. Thus,

$$\sum_i p \cdot x_i > p \cdot \bar{\omega} + \sum_j p \cdot y_j. \tag{16.C.3}$$

But then (x, y) cannot be feasible. Indeed, $\sum_i x_i = \bar{\omega} + \sum_j y_j$ implies $\sum_i p \cdot x_i = p \cdot \bar{\omega} + \sum_j p \cdot y_j$, which contradicts (16.C.3). We conclude that the equilibrium allocation (x^*, y^*) must be Pareto optimal. ■

The central idea in the proof of Proposition 16.C.1 can be put as follows: At any feasible allocation (x, y), the total cost of the consumption bundles $(x_1, \ldots, x_I)$, evaluated at prices p, must be equal to the social wealth at those prices, $p \cdot \bar{\omega} + \sum_j p \cdot y_j$. Moreover, because preferences are locally nonsatiated, if (x, y) Pareto dominates (x^*, y^*) then the total cost of consumption bundles $(x_1, \ldots, x_I)$ at prices p, and therefore the social wealth at those prices, must exceed the total cost of the equilibrium consumption allocation $p \cdot (\sum_i x_i^*) = p \cdot \bar{\omega} + \sum_j p \cdot y_j^*$. But by the profit-maximization of Definition 16.B.4, there are no technologically feasible production levels that attain a value of social wealth at prices p in excess of $p \cdot \bar{\omega} + \sum_j p \cdot y_j^*$.

The importance of the nonsatiation assumption for the result can be seen in Figure 16.C.1, which depicts an Edgeworth box where local nonsatiation fails for consumer 1 (note that consumer 1's indifference "curve" is thick) and where the allocation x^*, a price equilibrium for the price vector $p = (p_1, p_2)$ (you should verify this), is not Pareto optimal. Consumer 1 is indifferent about a move to allocation x, and consumer 2, having strongly monotone preferences, is strictly better off. (See Exercise 16.C.3 for a first welfare theorem compatible with satiation.)

Two points about Proposition 16.C.1 should be noted. First, although the result may appear to follow from very weak hypotheses, our theoretical structure already incorporates two strong assumptions: *universal price quoting of commodities* (market completeness) and *price taking* by economic agents. In Part III, we studied a number of circumstances (externalities, market power, and asymmetric information) in which these conditions are not satisfied and market equilibria fail to be Pareto optimal. Second, the first welfare theorem is entirely silent about the desirability of the equilibrium allocation from a distributional standpoint. In Section 16.D, we study the second fundamental theorem of welfare economics. That result, a partial converse to the first welfare theorem, gives us conditions under which any desired distributional aims can be achieved through the use of competitive (price-taking) markets.

16.D The Second Fundamental Theorem of Welfare Economics

The second fundamental welfare theorem gives conditions under which a Pareto optimum allocation can be supported as a price equilibrium with transfers. It is a converse of the first welfare theorem in the sense that it tells us that, under its assumptions, we can achieve any desired Pareto optimal allocation as a market-based equilibrium using an appropriate lump-sum wealth distribution scheme.

The second welfare theorem is more delicate than the first, and its validity requires additional assumptions. To see this, reconsider some of the examples discussed in Chapter 15. In Figure 15.C.3 (a) we saw that in a one-consumer, one-firm economy a Pareto optimal allocation may not be supportable as an equilibrium if the firm's technology is not convex. Figure 15.B.14 depicted a similar failure in a two-consumer Edgeworth box economy where the preferences of one consumer were not convex. Both figures suggest that the assumption of convexity will have to play a central role in establishing the second welfare theorem. Notice that convexity was not appealed to in any way for the first welfare theorem in Section 16.C.

The Edgeworth box of Figure 15.B.10(a) illustrates a different type of failure of supportability by means of prices. In that figure, both consumers have convex preferences, but the Pareto optimal allocation (ω_1, ω_2) cannot be supported as a price equilibrium with transfers; ω_2 is an optimal demand for consumer 2 for any price vector $p = (p_1, p_2) \geq 0$ when her wealth is $w_2 = p \cdot \omega_2$, but ω_1 is an optimal demand for consumer 1 for *no* price vector $p \geq 0$ and wealth level w_1.

It is convenient to tackle the problems raised by these two types of examples in two steps. The first step consists of establishing a version of the second fundamental theorem in which the sort of failure arising in Figure 15.B.10(a) is allowed. This is accomplished by defining the concept of a *price quasiequilibrium with transfers*, a weakening of the notion of a price equilibrium with transfers. We prove that if all preferences and technologies are convex, any Pareto optimal allocation can be achieved as a price quasiequilibrium with transfers. The second step consists of giving sufficient conditions for a price quasiequilibrium to be a full-fledged equilibrium. This division of labor is convenient because the first step is very general and isolates the central role of convexity, whereas the assumptions for the second step tend to be more special, often being tailored to the particulars of the model under consideration.

The definition of a quasiequilibrium with transfers, Definition 16.D.1, is identical to Definition 16.B.4 except that the preference maximization condition that anything preferred to x_i^* must cost more than w_i (i.e., "if $x_i \succ_i x_i^*$, then $p \cdot x_i > w_i$") is replaced by the weaker requirement that anything preferred to x_i^* cannot cost less than w_i (i.e., "if $x_i \succ_i x_i^*$, then $p \cdot x_i \geq w_i$").

Definition 16.D.1: Given an economy specified by $(\{(X_i, \succsim_i)\}_{i=1}^I, \{Y_j\}_{j=1}^J, \bar{\omega})$ an allocation (x^*, y^*) and a price vector $p = (p_1, \ldots, p_L) \neq 0$ constitute a *price quasi-equilibrium with transfers* if there is an assignment of wealth levels $(w_1, \ldots, w_I)$ with $\sum_i w_i = p \cdot \bar{\omega} + \sum_j p \cdot y_j^*$ such that

(i) For every j, y_j^* maximizes profits in Y_j; that is,

$$p \cdot y_j \leq p \cdot y_j^* \qquad \text{for all } y_j \in Y_j.$$

(ii) For every i, if $x_i \succ_i x_i^*$ then $p \cdot x_i \geq w_i$.

(iii) $\sum_i x_i^* = \bar{\omega} + \sum_j y_j^*$.

Part (ii) of Definition 16.D.1 is implied by the preference maximization condition of the definition of a price equilibrium with transfers [part (ii) of Definition 16.B.4]: If x_i^* is preference maximizing in the set $\{x_i \in X_i: p \cdot x_i \leq w_i\}$, then no $x_i \succ_i x_i^*$ with $p \cdot x_i < w_i$ can exist. Hence, any price equilibrium with transfers is a price quasi-equilibrium with transfers. However, as we discuss later in this section, the converse is not true.

Note also that when consumers' preferences are locally nonsatiated, part (ii) of Definition 16.D.1 implies $p \cdot x_i^* \geq w_i$ for every i.[3] In addition, from part (iii), we get $\sum_i p \cdot x_i^* = p \cdot \bar{\omega} + \sum_j p \cdot y_j^* = \sum_i w_i$. Therefore, *under the assumption of locally nonsatiated preferences*, which we always make, *we must have $p \cdot x_i^* = w_i$ for every i.* This means that we could just as well not mention the w_i's explicitly and replace part (ii) of Definition 16.D.1 by

(ii′) If $x_i \succ_i x_i^*$ then $p \cdot x_i \geq p \cdot x_i^*$.

That is, allocation (x^*, y^*) and price vector p constitute a price quasiequilibrium with transfers if and only if conditions (i), (ii′), and (iii) hold.[4] Moreover, with locally nonsatiated preferences, condition (ii′) is equivalent to saying that x_i^* is expenditure minimizing on the set $\{x_i \in X: x_i \succsim_i x_i^*\}$ (see Exercise 16.D.1). Thus, our discussion later in this section of the conditions under which a price quasiequilibrium with transfers is a price equilibrium with transfers can be interpreted in the locally nonsatiated case as providing conditions under which expenditure minimization on the set $\{x_i \in X_i: x_i \succsim_i x_i^*\}$ implies preference maximization on the set $\{x_i \in X_i: p \cdot x_i \leq p \cdot x_i^*\} = \{x_i \in X_i: p \cdot x_i \leq w_i\}$. ■

Proposition 16.D.1 states a version of the second fundamental welfare theorem.

Proposition 16.D.1: (*Second Fundamental Theorem of Welfare Economics*) Consider an economy specified by $(\{(X_i, \succsim_i)\}_{i=1}^I, \{Y_j\}_{j=1}^J, \bar{\omega})$, and suppose that every Y_j is convex and every preference relation $\succsim_i$ is convex [i.e., the set $\{x_i' \in X_i: x_i' \succsim_i x_i\}$ is convex for every $x_i \in X_i$] and locally nonsatiated. Then, for every Pareto optimal allocation (x^*, y^*), there is a price vector $p = (p_1, \ldots, p_L) \neq 0$ such that (x^*, y^*, p) is a price quasiequilibrium with transfers.

Proof: In its essence, the proof is just an application of the separating hyperplane theorem for convex sets (see Section M.G. of the Mathematical Appendix). To facilitate comprehension, we organize the proof into a number of small steps.

3. To see this, observe that if preferences are locally nonsatiated and $p \cdot x_i^* < w_i$, then close to x_i^* there is an x_i with $x_i \succ_i x_i^*$ and $p \cdot x_i < w_i$, contradicting condition (ii) of Definition 16.D.1.

4. A similar observation applies, incidentally, to the definition of price equilibrium with transfers (Definition 16.B.4). If preferences are locally nonsatiated, we get an equivalent definition by not referring explicitly to the w_i's and replacing part (ii) of the definition by (ii″): If $x_i \succ_i x_i^*$ then $p \cdot x_i > p \cdot x_i^*$. Thus, in this locally nonsatiated case, condition (ii″) says that x_i^* is preference maximizing on $\{x_i \in X_i: p \cdot x_i \leq p \cdot x_i^*\}$.

We begin by defining, for every i, the set V_i of consumptions preferred to x_i^*, that is, $V_i = \{x_i \in X_i : x_i \succ_i x_i^*\} \subset \mathbb{R}^L$. Then define

$$V = \sum_i V_i = \left\{\sum_i x_i \in \mathbb{R}^L : x_1 \in V_1, \ldots, x_I \in V_I\right\}$$

and

$$Y = \sum_j Y_j = \left\{\sum_j y_j \in \mathbb{R}^L : y_1 \in Y_1, \ldots, y_J \in Y_J\right\}.$$

Thus, V is the set of aggregate consumption bundles that could be split into I individual consumptions, each preferred by its corresponding consumer to x_i^*. The set Y is simply the aggregate production set. Note that the set $Y + \{\bar{\omega}\}$, which geometrically is the aggregate production set with its origin shifted to $\bar{\omega}$, is the set of aggregate bundles producible with the given technology and endowments and usable, in principle, for consumption.

Step 1: Every set V_i is convex. Suppose that $x_i \succ_i x_i^*$ and $x_i' \succ_i x_i^*$. Take $0 \leq \alpha \leq 1$. We want to prove that $\alpha x_i + (1 - \alpha)x_i' \succsim_i x_i^*$. Because preferences are complete, we can assume without loss of generality that $x_i \succsim_i x_i'$. Therefore, by convexity of preferences, we have $\alpha x_i + (1 - \alpha)x_i' \succsim_i x_i'$, which by transitivity yields the desired conclusion: $\alpha x_i + (1 - \alpha)x_i' \succ_i x_i^*$ [recall part (iii) of Proposition 1.B.1].

Step 2: The sets V and $Y + \{\bar{\omega}\}$ are convex. This is just a general, and easy-to-prove, mathematical fact: The sum of any two (and therefore any number of) convex sets is convex.

Step 3: $V \cap (Y + \{\bar{\omega}\}) = \varnothing$. This is a consequence of the Pareto optimality of (x^*, y^*). If there were a vector both in V and in $Y + \{\bar{\omega}\}$, then this would mean that with the given endowments and technologies it would be possible to produce an aggregate vector that could be used to give every consumer i a consumption bundle that is preferred to x_i^*.

Step 4: There is $p = (p_1, \ldots, p_L) \neq 0$ and a number r such that $p \cdot z \geq r$ for every $z \in V$ and $p \cdot z \leq r$ for every $z \in Y + \{\bar{\omega}\}$. This follows directly from the separating hyperplane theorem (see Section M. G. the Mathematical Appendix). It is illustrated in Figure 16.D.1.

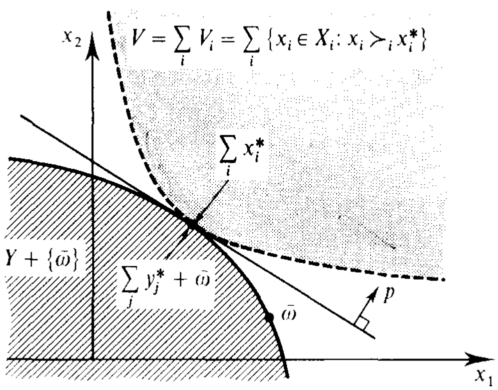


Figure 16.D.1
The separation argument in the proof of the second welfare theorem.

Step 5: If $x_i \succsim_i x_i^*$ for every i then $p\cdot(\sum_i x_i) \geq r$. Suppose that $x_i \succsim_i x_i^*$ for every i. By local nonsatiation, for each consumer i there is a consumption bundle $\hat{x}_i$ arbitrarily close to x_i such that $\hat{x}_i \succ_i x_i$, and therefore $\hat{x}_i \in V_i$. Hence, $\sum_i \hat{x}_i \in V$, and so $p\cdot(\sum_i \hat{x}_i) \geq r$, which, taking the limit as $\hat{x}_i \to x_i$, gives $p\cdot(\sum_i x_i) \geq r$.[5]

Step 6: $p\cdot(\sum_i x_i^*) = p\cdot(\bar{\omega} + \sum_j y_j^*) = r$. Because of step 5, we have $p\cdot(\sum_i x_i^*) \geq r$. On the other hand, $\sum_i x_i^* = \sum_j y_j^* + \bar{\omega} \in Y + \{\bar{\omega}\}$, and therefore $p\cdot(\sum_i x_i^*) \leq r$. Thus, $p\cdot(\sum_i x_i^*) = r$. Since $\sum_i x_i^* = \bar{\omega} + \sum_j y_j^*$, we also have $p\cdot(\bar{\omega} + \sum_j y_j^*) = r$.

Step 7: For every j, we have $p\cdot y_j \leq p\cdot y_j^*$ for all $y_j \in Y_j$. For any firm j and $y_j \in Y_j$, we have $y_j + \sum_{h\neq j} y_h^* \in Y$. Therefore,

$$p\cdot\left(\bar{\omega} + y_j + \sum_{h\neq j} y_h^*\right) \leq r = p\cdot\left(\bar{\omega} + y_j^* + \sum_{h\neq j} y_h^*\right).$$

Hence, $p\cdot y_j \leq p\cdot y_j^*$.

Step 8: For every i, if $x_i \succ_i x_i^*$, then $p\cdot x_i \geq p\cdot x_i^*$. Consider any $x_i \succ_i x_i^*$. Because of steps 5 and 6, we have

$$p\cdot\left(x_i + \sum_{k\neq i} x_k^*\right) \geq r = p\cdot\left(x_i^* + \sum_{k\neq i} x_k^*\right).$$

Hence, $p\cdot x_i \geq p\cdot x_i^*$.

Step 9: The wealth levels $w_i = p\cdot x_i^*$ for $i = 1, \ldots, I$ support (x^*, y^*, p) as a price quasiequilibrium with transfers. Conditions (i) and (ii) of Definition 16.D.1 follow from steps 7 and 8; condition (iii) follows from the feasibility of the Pareto optimal allocation (x^*, y^*). ■

In Exercise 16.D.2, you are asked to show that the local nonsatiation condition is required in Proposition 16.D.1.

When will a price quasiequilibrium with transfers be a price equilibrium with transfers? The example in Figure 15.B.10(a), reproduced in Figure 16.D.2, indicates that there is indeed a problem. Figure 16.D.2 depicts the quasiequilibrium associated with the Pareto optimal allocation labeled x^*. The unique price vector (normalizing $p_1 = 1$) that supports x^* as a quasiequilibrium allocation is $p = (1, 0)$; the associated wealth levels are $w_1 = p\cdot x_1^* = (1, 0)\cdot(0, x_{21}^*) = 0$ and $w_2 = p\cdot x_2^*$. However, although the consumption bundle x_1^* satisfies part (ii) of Definition 16.D.1 (indeed, $p\cdot x_1 \geq 0 = w_1$ for *any* $x_1 \geq 0$), it is *not* consumer 1's preference-maximizing bundle in her budget set $\{(x_{11}, x_{21}) \in \mathbb{R}^2_+ : (1, 0)\cdot(x_{11}, x_{21}) \leq 0\} = \{(x_{11}, x_{21}) \in \mathbb{R}^2_+ : x_{11} = 0\}$.

An important feature of the example just discussed, however, is that consumer 1's wealth level at the quasiequilibrium is zero. As we shall see, this is key to the failure of the quasiequilibrium to be an equilibrium. Our next result provides a sufficient condition under which the condition "$x_i \succ_i x_i^*$ implies $p\cdot x_i \geq w_i$" is equivalent to the preference maximization condition "$x_i \succ_i x_i^*$ implies $p\cdot x_i > w_i$."

5. Geometrically, what we have done here is show that the set $\sum_i \{x_i \in X_i : x_i \succsim_i x_i^*\}$ is contained in the closure of V (see Section M.F of the Mathematical Appendix for this concept), which, in turn, is contained in the half-space $\{v \in \mathbb{R}^L : p\cdot v \geq r\}$.

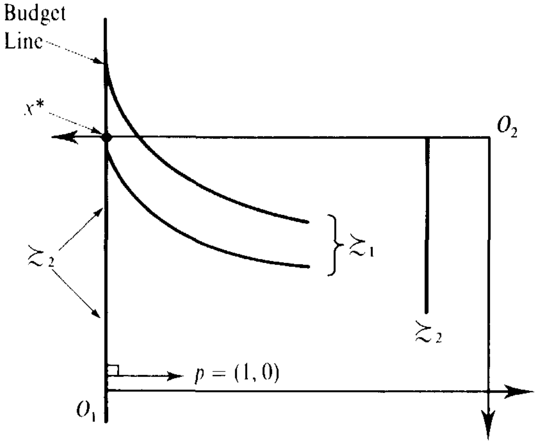


Figure 16.D.2 (left)
A price quasi-equilibrium that is not a price equilibrium.

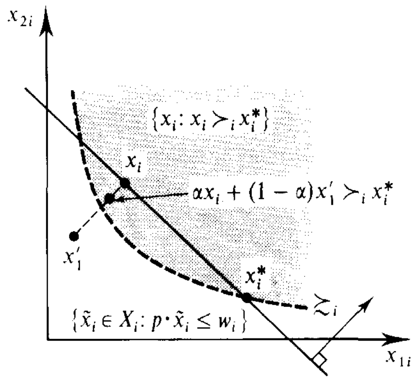


Figure 16.D.3 (right)
Suppose there exists a "cheaper consumption" (an $x'_i \in X_i$ such that $p\cdot x'_i < w_i$). Then if the preferred set does intersect the budget set ($p\cdot x_i \le w_i$ for some $x_i \succ_i x_i^*$), it follows that the preferred set does intersect the interior of the budget set ($p\cdot x_i < w_i$ for some $x_i \succ_i x_i^*$).

Proposition 16.D.2: Assume that X_i is convex and $\succsim_i$ is continuous. Suppose also that the consumption vector $x_i^* \in X_i$, the price vector p, and the wealth level w_i are such that $x_i \succ_i x_i^*$ implies $p\cdot x_i \ge w_i$. Then, if there is a consumption vector $x'_i \in X_i$ such that $p\cdot x'_i < w_i$ [a *cheaper consumption* for (p, w_i)], it follows that $x_i \succ_i x_i^*$ implies $p\cdot x_i > w_i$.[6]

Proof: The idea of the proof is indicated in Figure 16.D.3 (where we take $p\cdot x_i^* = w_i$ only because this is the leading case; the fact plays no role in the proof). Suppose that, contrary to the assertion of the proposition, there is an $x_i \succ_i x_i^*$ with $p\cdot x_i = w_i$. By the cheaper consumption assumption, there exists an $x'_i \in X_i$ such that $p\cdot x'_i < w_i$. Then for all $\alpha \in [0, 1)$, we have $\alpha x_i + (1-\alpha)x'_i \in X_i$ and $p\cdot(\alpha x_i + (1-\alpha)x'_i) < w_i$.[7] But if α is close enough to 1, the continuity of $\succsim_i$ implies that $\alpha x_i + (1-\alpha)x'_i \succ_i x_i^*$, which constitutes a contradiction because we have then found a consumption bundle that is preferred to x_i^* and costs less than w_i. ■

Note that in the example of Figure 16.D.2, we have $w_1 = 0$ in the price quasiequilibrium supporting allocation x^*, and so there is no cheaper consumption for (p, w_1).[8]

As a consequence of Proposition 16.D.2, we have Proposition 16.D.3.

Proposition 16.D.3: Suppose that for every i, X_i is convex, $0 \in X_i$, and $\succsim_i$ is continuous. Then any price quasiequilibrium with transfers that has $(w_1, \ldots, w_I) \gg 0$ is a price equilibrium with transfers.

6. If, as in all our applications, $\succsim_i$ is locally nonsatiated and $w_i = p\cdot x_i^*$, then Proposition 16.D.2 offers sufficient conditions for the equivalence of the statements "x_i^* minimizes expenditure relative to p in the set $\{x_i \in X_i: x_i \succsim_i x_i^*\}$" and "$x_i^*$ is maximal for $\succsim_i$ in the budget set $\{x_i \in X_i: p\cdot x_i \le p\cdot x_i^*\}$."

7. A similar argument can be used to show that if X_i is convex and the Walrasian demand function $x_i(p, w_i)$ is well defined, then there is a cheaper consumption for (p, w_i) if and only if there is an x'_i arbitrarily close to $x_i(p, w_i)$ with $p\cdot x'_i < w_i$. In the Appendix A, of Chapter 3 the latter concept was called the *locally cheaper consumption condition.*

8. Note also that Proposition 16.D.2 generalizes the result in Proposition 3.E.1(ii), which assumed local nonsatiation, $w_i = p\cdot x_i^* > 0$, and $X_i = \mathbb{R}^L_+$.

Consider the implications of Proposition 16.D.3 for a pure exchange economy in which $\bar{\omega} \gg 0$ and every consumer has $X_i = \mathbb{R}^L_+$ and continuous, locally nonsatiated preferences. In such an economy, by free disposal and profit maximization, we must have $p \geq 0$ and $p \neq 0$ at any price quasiequilibrium.[9] Thus, under these assumptions, any price quasiequilibrium with transfers in which $x_i^* \gg 0$ for all i is a price equilibrium with transfers (since then $w_i = p \cdot x_i^* > 0$ for all i). But there is more. Suppose that, in addition, preferences are strongly monotone. Then we must have $p \gg 0$ in any price quasiequilibrium with transfers. To see this, note that $p \geq 0$, $p \neq 0$, and $\bar{\omega} \gg 0$ imply that $\sum_i w_i = p \cdot \bar{\omega} > 0$ and therefore that $w_i > 0$ for *some* i. But by Proposition 16.D.2, this consumer must then be maximizing her preferences in her budget set $\{x_i \in \mathbb{R}^L_+ : p \cdot x_i \leq w_i\}$, which, by strong monotonicity of preferences, cannot occur if prices are not strictly positive. Once we know that we must have $p \gg 0$, we can conclude that *any* price quasiequilibrium with transfers in this economy is a price equilibrium with transfers: if consumer i's allocation satisfies $x_i^* \neq 0$, then $p \cdot x_i^* = w_i > 0$ and Proposition 16.D.2 applies. On the other hand, if $x_i^* = 0$, then $w_i = 0$ and the result follows from the fact that $x_i^* = 0$ is the only vector in the set $\{x_i \in \mathbb{R}^L_+ : p \cdot x_i \leq 0\}$. (Exercise 16.D.3 asks you to extend the arguments presented in this paragraph to the case of an economy with production.)

The second welfare theorem (combined with Propositions 16.D.2 and 16.D.3) identifies conditions under which any Pareto optimal allocation can be implemented through competitive markets and offers a strong conceptual affirmation of the use of competitive markets, even for dealing with distributional concerns. Yet, it is important to discuss some of the practical limitations on the use of this theoretical result.

The first observation to make is that a planning authority wishing to implement a particular Pareto optimal allocation must be able to insure that the supporting prices $(p_1, \ldots, p_L)$ will be taken as given by consumers and firms. If the market structure is such that price-taking behavior would not automatically hold (say, because economic agents are not all of negligible size), then the planning authority must somehow enforce these prices, either by monitoring all transactions or, perhaps, by credibly offering to buy or sell any amount of any good ℓ at price p_ℓ.

A second observation is that the information of a planning authority that wants to use the second welfare theorem must be very good indeed. To begin with, it must have sufficiently good information to identify the Pareto optimal allocation to be implemented and to compute the right supporting price vector. For this purpose, the authority must know, at least, the statistical joint distribution of preferences, endowments, and other relevant characteristics of the agents that actually exist in the economy. However, to implement the correct transfer levels for each consumer, the planning authority must know *more*: it must have the ability to tell who is who by observing each individual's private characteristics (e.g., preferences and endowments) perfectly. Such information is extremely unlikely to be available in practice; as a result, most common transfer schemes fail to be lump-sum schemes. For example, if the planning authority wants to transfer wealth from those who have a great deal of a highly valuable labor skill to those who do not, the only way it may have to tell which consumers are which may be by observing their *actual*

9. Indeed, if we had $p_\ell < 0$, then unboundedly large profits could be generated through disposal of the ℓth good.

earnings. But if transfers are based on observed earnings, they will cease to be lump-sum in nature. Individuals will recognize that by altering their earnings, they will change their tax burden.

Finally, even if the planning authority observes all the required information, it must actually have the power to enforce the necessary wealth transfers through some tax-and-transfer mechanism that individuals cannot evade.

Because of these informational and enforceability limitations, it is in practice unlikely that extensive lump-sum taxation will be possible.[10] We shall see in Section 18.D that if these types of transfers are not possible, then the second welfare theorem collapses in the sense that, for a typical economy, only a limited range of Pareto optima are supportable by means of prices supplemented by the usual sort of taxation systems. For the typical economy, redistribution schemes are *distortionary*; that is, they trade off distributional aims against Pareto optimality. The analysis of this trade-off is the subject of *second-best* welfare economics, some elements of which are presented in Chapter 22. (Chapter 23 discusses in much more depth what is and what is not implementable by a planning authority who faces informational and enforceability constraints.)

In summary, the second welfare theorem is a very useful *theoretical* reference point. But it is far from a direct prescription for policy practice. On the contrary, by pointing out what is necessary to achieve any desired Pareto optimal allocation, it serves a cautionary purpose.

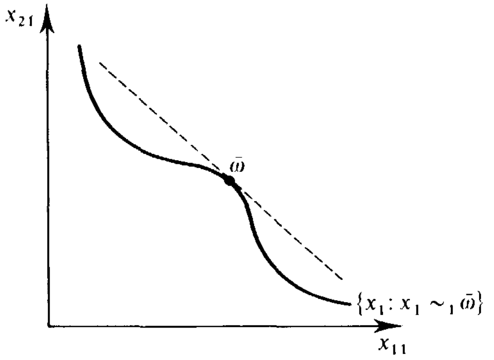
It is clear from our discussion that convexity plays a central role in the second welfare theorem. But it is a role that deserves a very important qualification. The interpretation of the second welfare theorem is at its strongest when the number of economic agents is large. This is so because the price-taking assumption is then enforced by the market itself (otherwise, it is almost inescapable that there must be some sort of centralized mechanism guaranteeing the fixity of prices). It turns out, however, that if consumers are numerous (in the limit, a continuum), and if the nonconvexities of production sets are bounded in a certain sense, then the assumptions of convex preferences and production sets are *not* required for the second welfare theorem.

To see the idea behind this, it is useful to consider the one-consumer economy depicted in Figure 16.D.4 where, because of nonconvexities, the (trivially Pareto optimal) allocation $x_1 = \bar{\omega}$ cannot be price supported. Suppose, however, that we replicate the economy so that we have *two* consumers and the total endowments are *doubled* to $2\bar{\omega}$. Again, the allocation $x_1 = x_2 = \bar{\omega}$ cannot be price supported, but now *this symmetric allocation is no longer a Pareto optimum.* In Figure 16.D.5, we can see that the asymmetric allocation

$$x'_1 = \bar{\omega} + (1, -1) \qquad \text{and} \qquad x'_2 = \bar{\omega} + (-1, 1)$$

Pareto dominates $x_1 = x_2 = \bar{\omega}$. It certainly does not follow from this that with just two replicas any allocation that cannot be price supported is not a Pareto optimum. (In Figure 16.D.6, we represent the Edgeworth box associated with Figure 16.D.5. We have drawn it so that the allocation x' is actually Pareto optimal and yet cannot be price supported). However, what

10. Note that the extent of the required lump-sum taxation depends not only on the final wealth levels $(w_1, \dots, w_I)$ but also on the initial situation. For example, if we are in private ownership economy, then the net transfer to consumer i is $w_i - p \cdot \omega_i - \sum_j \theta_{ij} p \cdot y_j^*$ and so depends on the consumer's initial endowments. The Walrasian equilibria correspond to the no-taxation situations; and the farther from the Walrasian wealth levels that we try to go, the larger the required transfer.



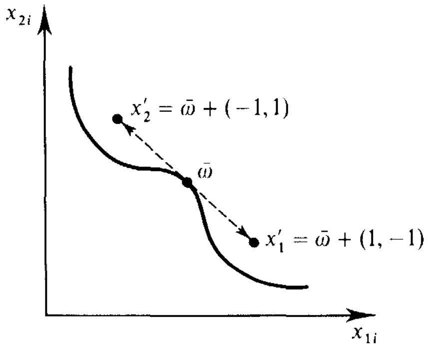


Figure 16.D.4 (top left)
A one-consumer economy (without production) where the initial endowment is not supportable by prices.

Figure 16.D.5 (top right)
Allocation (x_1', x_2') Pareto dominates allocation $(\bar{\omega}, \bar{\omega})$ in a two-consumer replica of the economy in Figure 16.D.4.

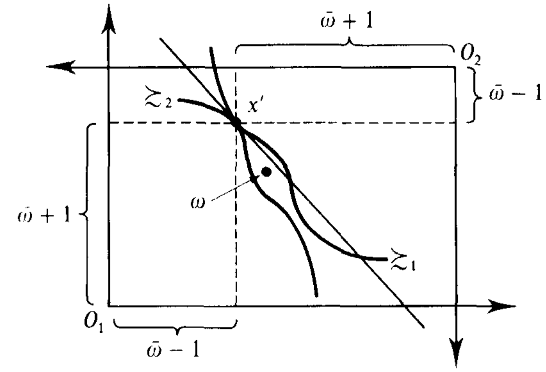


Figure 16.D.6 (bottom)
Allocation $x' = (x_1', x_2')$ is Pareto optimal but not price supportable.

can be shown is that if the number of replicas is large enough, then any feasible allocation that fails (significantly) to be price supportable can be Pareto dominated, and therefore any Pareto optimum must be (almost) price supportable. (See Exercise 16.D.4 for more on this.)[11]

16.E Pareto Optimality and Social Welfare Optima

In this section, we discuss the relationship between the Pareto optimality concept and the maximization of a social welfare function (see Section 4.D and Chapter 22 for more on this concept).

Given a family $u_i(\cdot)$ of (continuous) utility functions representing the preferences $\succsim_i$ of the I consumers, we can capture the attainable vectors of utility levels for an economy specified by $(\{(X_i, \succsim_i)\}_{i=1}^I, \{Y_j\}_{j=1}^J, \bar{\omega})$ by means of the *utility possibility set*:

$$U = \{(u_1, \ldots, u_I) \in \mathbb{R}^I\text{: there is a feasible allocation } (x, y) \text{ such that}$$
$$u_i \leq u_i(x_i) \text{ for } i = 1, \ldots, I\}.$$

11. Two facts established in Chapter 17 lend plausibility to this claim. First, in Section 17.I, we show that convexity is not required for the (approximate) existence of a Walrasian equilibrium in a large economy. Second, in Section 17.C, we argue that the second welfare theorem can be rephrased as an assertion of the existence of a Walrasian equilibrium for economies in which endowments are distributed in a particular manner, and it can therefore be seen as implied by the conditions guaranteeing the general existence of Walrasian equilibria.

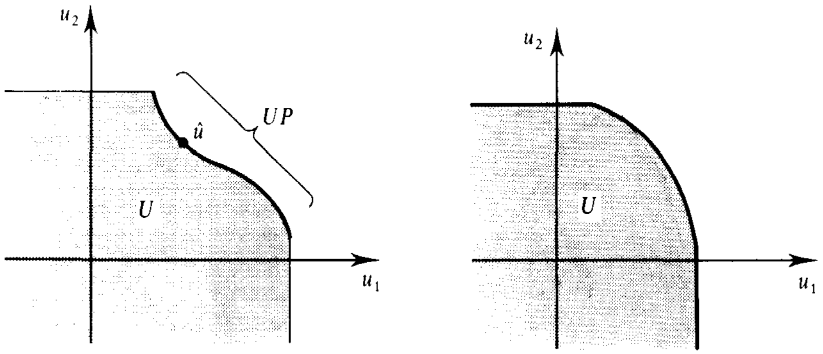


Figure 16.E.1 (left)
The utility possibility set.

Figure 16.E.2 (right)
A convex utility possibility set.

Figure 16.E.1 depicts this set for a two-consumer economy. (Note that we show $U \subset \mathbb{R}^I$ as a closed set; Appendix A discusses sufficient conditions to guarantee that the set is indeed closed.)

By the definition of Pareto optimality, the utility values of a Pareto optimal allocation must belong to the boundary of the utility possibility set.[12] More precisely, we define the *Pareto frontier* UP, also shown in Figure 16.E.1, by

$$UP = \{(u_1, \ldots, u_I) \in U\colon \text{there is no } (u'_1, \ldots, u'_I) \in U \text{ such that } u'_i \geq u_i \text{ for all } i \text{ and } u'_i > u_i \text{ for some } i\}.$$

Proposition 16.E.1 is then intuitive.

Proposition 16.E.1: A feasible allocation $(x, y) = (x_1, \ldots, x_I, y_1, \ldots, y_J)$ is a Pareto optimum if and only if $(u_1(x_1), \ldots, u_I(x_I)) \in UP$.

Proof: If $(u_1(x_1), \ldots, u_I(x_I)) \notin UP$, then there is $(u'_1, \ldots, u'_I) \in U$ such that $u'_i \geq u_i(x_i)$ for all i and $u'_i > u_i(x_i)$ for some i. But $(u'_1, \ldots, u'_I) \in U$ only if there is a feasible allocation (x', y') such that $u_i(x'_i) \geq u'_i$ for all i. It follows then that (x', y') Pareto dominates (x, y). Conversely, if (x, y) is not a Pareto optimum, then it is Pareto dominated by some feasible (x', y'), which means that $u_i(x'_i) \geq u_i(x_i)$ for all i and $u_i(x'_i) > u_i(x_i)$ for some i. Hence, $(u_1(x_1), \ldots, u_I(x_I)) \notin UP$. ■

We also note that if every X_i and every Y_j is convex, and if the utility functions $u_i(\cdot)$ are concave, then the utility possibility set U is convex (see Exercise 16.E.2).[13] One such utility possibility set is represented in Figure 16.E.2.

Suppose now that society's distributional principles can be summarized in a *social welfare function* $W(u_1, \ldots, u_I)$ assigning social utility values to the various possible vectors of utilities for the I consumers. We concentrate here on a particularly simple class of social welfare functions: those that take the *linear* form

$$W(u_1, \ldots, u_I) = \sum_i \lambda_i u_i$$

12. However, not every point in the boundary must be Pareto optimal. Go back, for example, to Figure 16.C.1: The utility values associated with x^* belong to the boundary of the utility possibility set because it is impossible to make *both* consumers better off. Yet, x^* is not a Pareto optimum.

13. It can be shown that under a mild technical strengthening of the strict convexity assumption on preferences (essentially the same condition used to guarantee differentiability of the Walrasian demand function in Appendix A of Chapter 3), there are in the family of utility functions $u_i(\cdot)$ that represent $\succsim_i$ some utility functions that are not only quasiconcave but also concave.

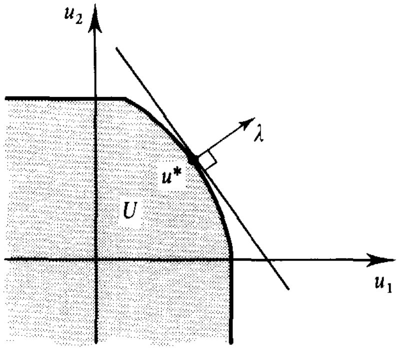


Figure 16.E.3
Maximizing a linear social welfare function.

for some constants $\lambda = (\lambda_1, \ldots, \lambda_I)$.[14] Letting $u = (u_1, \ldots, u_I)$, we can also write $W(u) = \lambda \cdot u$. Because social welfare should be nondecreasing in the consumer's utility levels, we assume that $\lambda \geq 0$.

Armed with a linear social welfare function, we can select points in the utility possibility set U that maximize our measure of social welfare by solving

$$\underset{u \in U}{\text{Max}} \quad \lambda \cdot u. \tag{16.E.1}$$

Figure 16.E.3 depicts the solution to problem (16.E.1). As the figure suggests, we have the result presented in Proposition 16.E.2.

Proposition 16.E.2: If $u^* = (u_1^*, \ldots, u_I^*)$ is a solution to the social welfare maximization problem (16.E.1) with $\lambda \gg 0$, then $u^* \in UP$; that is, u^* is the utility vector of a Pareto optimal allocation. Moreover, if the utility possibility set U is convex, then for any $\tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_I) \in UP$, there is a vector of welfare weights $\lambda = (\lambda_1, \ldots, \lambda_I) \geq 0$, $\lambda \neq 0$, such that $\lambda \cdot \tilde{u} \geq \lambda \cdot u$ for all $u \in U$, that is, such that $\tilde{u}$ is a solution to the social welfare maximization problem (16.E.1).

Proof: The first part is immediate: if u^* were not Pareto optimal, then there would exist a $u \in U$ with $u \geq u^*$ and $u \neq u^*$; and so because $\lambda \gg 0$, we would have $\lambda \cdot u > \lambda \cdot u^*$.

For the second part, note that if $\tilde{u} \in UP$, then $\tilde{u}$ is in the boundary of U. By the supporting hyperplane theorem (see Section M.G of the Mathematical Appendix), there exists a $\lambda \neq 0$ such that $\lambda \cdot \tilde{u} \geq \lambda \cdot u$ for all $u \in U$. Moreover, since the set U has been constructed so that $U - \mathbb{R}^I_+ \subset U$, we must have $\lambda \geq 0$ (indeed, if $\lambda_i < 0$, then by choosing a $u \in U$ with $u_i < 0$ large enough in absolute value, we would have $\lambda \cdot u > \lambda \cdot \tilde{u}$). ■ 1

Proposition 16.E.2 tells us that for economies with convex utility possibility sets, there is a close relation between Pareto optima and linear social welfare optima: Every linear social welfare optimum with weights $\lambda \gg 0$ is Pareto optimal, and every Pareto optimal allocation (and hence, every Walrasian equilibrium) is a social welfare optimum for *some* welfare weights $(\lambda_1, \ldots, \lambda_I) \geq 0$.[15]

14. See Chapter 22 for a discussion of more general types of social welfare functions.

15. The necessity of allowing for some λ_i to equal zero in the second part of this statement parallels the similar feature encountered in the characterization of efficient production vectors in Proposition 5.F.2.

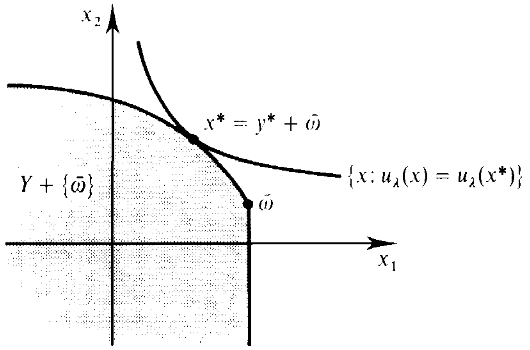


Figure 16.E.4 Maximizing the utility of a representative consumer.

As usual, in the absence of convexity of the set U, we cannot be assured that a Pareto optimum can be supported as a maximum of a linear social welfare function. The point $\hat{u}$ in Figure 16.E.1 provides an example where it cannot.

By using the social welfare weights associated with a particular Pareto optimal allocation (perhaps a Walrasian equilibrium), we can view the latter as the welfare optimum in a certain single-consumer, single-firm economy. To see this, let (x^*, y^*) be a Pareto optimal allocation and suppose that $\lambda = (\lambda_1, \ldots, \lambda_I) \gg 0$ is a vector of welfare weights supporting U at $(u_1(x_1^*), \ldots, u_I(x_I^*))$. Define then a utility function $u_\lambda(\bar{x})$ on aggregate consumption vectors in $X = \sum_i X_i \subset \mathbb{R}^L$ by

$$u_\lambda(x) = \underset{(x_1,\ldots,x_I)}{\text{Max}} \quad \sum_i \lambda_i u_i(x_i) \tag{16.E.2}$$
$$\text{s.t. } x_i \in X_i \text{ for all } i \text{ and } \textstyle\sum_i x_i = \bar{x}.$$

The utility function $u_\lambda(\cdot)$ is the (direct) utility function of a normative representative consumer in the sense discussed in Section 4.D (see, in particular, Exercise 4.D.4). Letting $Y = \sum_j Y_j$ be the aggregate production set, the pair $(\sum_i x_i^*, \sum_j y_j^*)$ is then a solution to the problem

$$\text{Max} \quad u_\lambda(\bar{x})$$
$$\text{s.t. } \bar{x} = \bar{\omega} + \bar{y},\ \bar{x} \in X,\ \bar{y} \in Y.$$

This solution is depicted in Figure 16.E.4.

It is important to emphasize, however, that the particular utility function chosen for the representative consumer depends on the weights $(\lambda_1, \ldots, \lambda_I)$ and therefore on the Pareto optimal allocation under consideration.

16.F First-Order Conditions for Pareto Optimality

This section intends to be pedagogical. The emphasis is not on minimal assumptions, and its pace is deliberately slow. Making differentiability assumptions, we show how prices and the optimality properties of price-taking behavior emerge naturally from an examination of the first-order conditions associated with Pareto optimality problems. Along the way we redo, and in some aspects also generalize, the analysis in Sections 16.C and 16.D on the two fundamental welfare theorems.[16]

16. Early contributions along the line of the discussion in this section are Allais (1953), Lange (1942), and Samuelson (1947).

To begin with, we assume that the consumption set of every consumer is $\mathbb{R}^L_+$ and that preferences are represented by utility functions $u_i(x_i)$ that are twice continuously differentiable and satisfy $\nabla u_i(x_i) \gg 0$ at all x_i (hence, preferences are strongly monotone). We also normalize so that $u_i(0) = 0$.

The production set of firm j takes the form $Y_j = \{y \in \mathbb{R}^L: F_j(y) \leq 0\}$, where $F_j(y) = 0$ defines firm j's transformation frontier. We assume that $F_j: \mathbb{R}^L \rightarrow \mathbb{R}$ is twice continuously differentiable, that $F_j(0) \leq 0$, and that $\nabla F_j(y_j) = (\partial F_j(y_j)/\partial y_{1j}, \ldots, \partial F_j(y_j)/\partial y_{Lj}) \gg 0$ for all $y_j \in \mathbb{R}^L$. The meaning of the last condition is that if $F_j(y_j) = 0$, so that y_j is in the transformation frontier of Y_j, then any attempt to produce more of some output or use less of some input makes the value of $F_j(\cdot)$ positive and pushes us out of Y_j (in other words, y_j is production efficient, in the sense discussed in Section 5.F, in the production set Y_j).[17]

Note that, for the moment, no convexity assumptions have been made on preferences or production sets.

The problem of identifying the Pareto optimal allocations for this economy can be reduced to the selection of allocations

$$(x, y) = (x_1, \ldots, x_I, y_1, \ldots, y_J) \in \mathbb{R}^{LI}_+ \times \mathbb{R}^{LJ}$$

that solve the following problem:

$$\begin{aligned} \text{Max} \quad & u_1(x_{11}, \ldots, x_{L1}) && && (16.F.1) \\ \text{s.t.} \ (1) \quad & u_i(x_{1i}, \ldots, x_{Li}) \geq \bar{u}_i && i = 2, \ldots, I \\ (2) \quad & \textstyle\sum_i x_{\ell i} \leq \bar{\omega}_\ell + \sum_j y_{\ell j} && \ell = 1, \ldots, L \\ (3) \quad & F_j(y_{1j}, \ldots, y_{Lj}) \leq 0 && j = 1, \ldots, J. \end{aligned}$$

Problem (16.F.1) states the Pareto optimality problem as one of trying to maximize the well-being of consumer 1 subject to meeting certain required utility levels for the other consumers in the economy [constraints (1)] and the resource and technological limitations on what is feasible [constraints (2) and (3), respectively]. By solving problem (16.F.1) for varying required levels of utility for these other consumers $(\bar{u}_2, \ldots, \bar{u}_I)$, we can identify all the Pareto optimal allocations for this economy. Indeed, you should pause to convince yourself of this by solving Exercise 16.F.1.

Exercise 16.F.1: Show that any allocation that is a solution to problem (16.F.1) is Pareto optimal and that any Pareto optimal allocation for this economy must be a solution to problem (16.F.1) for *some* choice of utility levels $(\bar{u}_2, \ldots, \bar{u}_I)$. [*Hint:* Use the fact that preferences are strongly monotone.]

Because utility functions are normalized to take nonnegative values, from now on we consider only required utility levels that satisfy $\bar{u}_i \geq 0$ for all i.

The point of Exercise 16.F.1 can be seen by examining the utility possibility set U in Figure 16.F.1. If we fix a required nonnegative utility level for consumer 2, we can locate a point on the frontier of the utility possibility set U by maximizing

17. For expositional convenience, we have taken every $F_j(\cdot)$ to be defined on the entire $\mathbb{R}^L$. A consequence of this (and the assumption that $\nabla F_j(y_j) \gg 0$ for all y_j) is that every commodity is both an input and an output of the production process. Because this is unrealistic, we emphasize that no more than expositional ease is involved here.

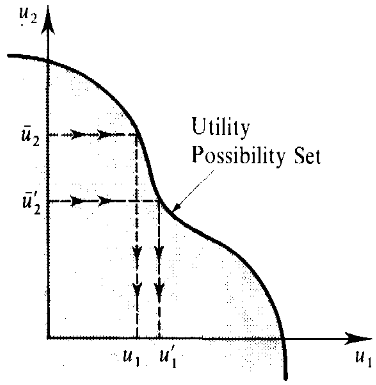


Figure 16.F.1
Parameterizing the frontier of the utility possibility set when $I = 2$ by the required utility level of consumer 2.

consumer 1's utility subject to the required utility constraint for consumer 2. By varying 2's required utility level, we trace out the set of Pareto optimal points.

Under our assumptions (recall that $\bar{u}_i \geq 0$ for $i \geq 2$), all the constraints of problem (16.F.1) will be binding at a solution. Denote by $(\delta_2, \ldots, \delta_I) \geq 0$, $(\mu_1, \ldots, \mu_L) \geq 0$, and $(\gamma_1, \ldots, \gamma_J) \geq 0$ the multipliers associated with the constraints (1), (2), and (3) of problem (16.F.1), respectively, and define $\delta_1 = 1$. In Exercise 16.F.2, you are asked to verify that the first-order (Kuhn–Tucker) necessary conditions for problem (16.F.1) can be written as follows (all the derivatives, here and elsewhere in this section, are evaluated at the solution):[18]

$$x_{\ell i}: \quad \delta_i \frac{\partial u_i}{\partial x_{\ell i}} - \mu_\ell \begin{cases} \leq 0 \\ = 0 \text{ if } x_{\ell i} > 0 \end{cases} \qquad \text{for all } i, \ell, \tag{16.F.2}$$

$$y_{\ell j}: \quad \mu_\ell - \gamma_j \frac{\partial F_j}{\partial y_{\ell j}} = 0 \qquad \text{for all } j, \ell. \tag{16.F.3}$$

As is well known from Kuhn–Tucker theory (see Section M.K of the Mathematical Appendix), the value of the multiplier μ_ℓ at an optimal solution is exactly equal to the increase in consumer 1's utility derived from a relaxation of the corresponding constraint, that is, from a marginal increase in the available social endowment $\bar{\omega}_\ell$ of good ℓ. Thus, the multiplier μ_ℓ can be interpreted as the marginal value or "shadow price" (in terms of consumer 1's utility) of good ℓ. The multiplier δ_i, on the other hand, equals the marginal change in consumer 1's utility if we decrease the utility requirement $\bar{u}_i$ that must be met for consumer $i \neq 1$. Condition (16.F.2) therefore says that, at an optimal interior allocation, the increase in the utility of any consumer i from receiving an additional unit of good ℓ, weighted (if $i \neq 1$) by the amount that relaxing consumer i's utility constraint is worth in terms of raising consumer 1's utility, should be equal to the marginal value μ_ℓ of good ℓ.

Similarly, the multiplier γ_j can be interpreted as the marginal benefit from relaxing the jth production constraint or, equivalently, the marginal cost from tightening it.

18. Recall that for expositional ease we are not imposing any boundary constraints on the vectors y_j. We note also that the assumption of strictly positive gradients of the functions $u_i(\cdot)$ and $F_j(\cdot)$ implies that the constraint qualification for the necessity of the Kuhn–Tucker conditions is satisfied. (See Section M.K of the Mathematical Appendix for the specifics of first-order conditions for optimization problems under constraints.)

Hence, $\gamma_j(\partial F_j/\partial y_{\ell j})$ is the marginal cost of increasing $y_{\ell j}$ and thereby effectively tightening the constraint on the net outputs of the other goods. Condition (16.F.3) says, then, that at an optimum this marginal cost is equated, for every j, to the marginal benefit μ_ℓ of good ℓ.

If we suppose that we have an interior solution (i.e., $x_i \gg 0$ for all i), then conditions (16.F.2) and (16.F.3) imply that three types of ratio conditions must hold (see Exercise 16.F.3):

$$\frac{\partial u_i/\partial x_{\ell i}}{\partial u_i/\partial x_{\ell' i}} = \frac{\partial u_{i'}/\partial x_{\ell i'}}{\partial u_{i'}/\partial x_{\ell' i'}} \quad \text{for all } i, i', \ell, \ell'. \tag{16.F.4}$$

$$\frac{\partial F_j/\partial y_{\ell j}}{\partial F_j/\partial y_{\ell' j}} = \frac{\partial F_{j'}/\partial y_{\ell j'}}{\partial F_{j'}/\partial y_{\ell' j'}} \quad \text{for all } j, j', \ell, \ell'. \tag{16.F.5}$$

$$\frac{\partial u_i/\partial x_{\ell i}}{\partial u_i/\partial x_{\ell' i}} = \frac{\partial F_j/\partial y_{\ell j}}{\partial F_j/\partial y_{\ell' j}} \quad \text{for all } i, j, \ell, \ell'. \tag{16.F.6}$$

Condition (16.F.4) says that in any Pareto optimal allocation, all consumers' marginal rates of substitution between every pair of goods must be equalized [see Figures 15.B.11(b) and 15.B.12 for an illustration in the two-good, two-consumer case]; condition (16.F.5) says that all firms' marginal rates of transformation between every pair of goods must be equalized [see Figure 15.D.2(b) for an illustration in the two-good, two-firm case]; and condition (16.D.6) says that every consumer's marginal rate of substitution must equal every firm's marginal rate of transformation for all pairs of goods [see Figure 15.C.2 for an illustration in the case of the one-consumer, one-firm model with two goods].

Conditions (16.F.4) to (16.F.6) correspond to three types of efficiency embodied in a Pareto optimal allocation (see Exercise 16.F.4).

(i) *Optimal allocation of available goods across consumers.* Given some aggregate amounts $(\bar{x}_1, \ldots, \bar{x}_L)$ of goods available for consumption purposes, we want to distribute them to maximize consumer 1's well-being while meeting the utility requirements $(\bar{u}_2, \ldots, \bar{u}_I)$ for consumers $2, \ldots, I$. That is, we want to solve

$$\begin{aligned} \underset{(x_1,\ldots,x_I)}{\text{Max}} \quad & u_1(x_{11}, \ldots, x_{L1}) \\ \text{s.t. } (1) \quad & u_i(x_{1i}, \ldots, x_{Li}) \geq \bar{u}_i \qquad i = 2, \ldots, I \\ (2) \quad & \textstyle\sum_i x_{\ell i} \leq \bar{x}_\ell \qquad \ell = 1, \ldots, L. \end{aligned} \tag{16.F.7}$$

The first-order conditions for this problem lead to condition (16.F.4).

(ii) *Efficient production across technologies.* The aggregate production vector should be *efficient* in the sense discussed in Section 5.F. That is, it should be impossible to reassign production plans across individual production sets so as to produce, in the aggregate, more of a particular output (or use less of it as an input) without producing less of another. Focusing, in particular, on the first good, this means that given required total productions $(\bar{y}_2, \ldots, \bar{y}_L)$ of the other goods, we want to solve

$$\begin{aligned} \underset{(y_1,\ldots,y_J)}{\text{Max}} \quad & \textstyle\sum_j y_{1j} \\ \text{s.t. } (1) \quad & \textstyle\sum_j y_{\ell j} \geq \bar{y}_\ell \qquad \ell = 2, \ldots, L \\ (2) \quad & F_j(y_j) \leq 0 \qquad j = 1, \ldots, J. \end{aligned} \tag{16.F.8}$$

The first-order conditions for this problem lead to condition (16.F.5).

(iii) *Optimal aggregate production levels.* We also must have picked aggregate production levels that generate a desirable assortment of commodities available for consumption. Keeping the utility requirements $(\bar{u}_2, \dots, \bar{u}_I)$ fixed, let $u(\bar{x}_1, \dots, \bar{x}_L)$ and $f(\bar{y}_2, \dots, \bar{y}_L)$ denote, respectively, the value functions for problems (16.F.7) and (16.F.8). Then we want to solve

$$\begin{aligned} \underset{(\bar{y}_1,\dots,\bar{y}_L)}{\text{Max}} \quad & u(\bar{\omega}_1 + \bar{y}_1, \dots, \bar{\omega}_L + \bar{y}_L) \\ \text{s.t. } & \bar{y}_1 \le f(\bar{y}_2, \dots, \bar{y}_L). \end{aligned} \tag{16.F.9}$$

The first-order conditions of this problem lead to condition (16.F.6).

To explore the relationship of the first-order conditions (16.F.2) and (16.F.3) to the first and second welfare theorems, we make the further, and substantive, assumption that every $u_i(\cdot)$ is a quasiconcave function (hence, preferences are convex) and that every $F_j(\cdot)$ is a convex function (hence, production sets are convex). The virtue of this assumption is that with it we do not have to worry about second-order conditions; in all the maximization problems to be considered, the first-order necessary conditions are automatically sufficient.

In this differentiable, convex framework, conditions (16.F.2) and (16.F.3) can be used to establish a version of the two welfare theorems. To see this, note first that (x^*, y^*, p) is a price equilibrium with transfers (with associated wealth levels $w_i = p \cdot x_i^*$ for $i = 1, \dots, I$) if and only if the first-order conditions for the budget-constrained utility maximization problems

$$\begin{aligned} \underset{x_i \ge 0}{\text{Max}} \quad & u_i(x_i) \\ \text{s.t. } & p \cdot x_i \le w_i \end{aligned}$$

and the profit maximization problems

$$\begin{aligned} \underset{y_j}{\text{Max}} \quad & p \cdot y_j \\ \text{s.t. } & F_j(y_j) \le 0 \end{aligned}$$

are satisfied. Denoting by α_i and β_j the respective multipliers for the constraints of these problems, the first-order conditions [evaluated at (x^*, y^*)] can be written as follows:

$$x_{\ell i}: \quad \frac{\partial u_i}{\partial x_{\ell i}} - \alpha_i p_\ell \begin{cases} \le 0 \\ = 0 \text{ if } x_{\ell i} > 0 \end{cases} \qquad \text{for all } i, \ell, \tag{16.F.10}$$

$$y_{\ell j}: \quad p_\ell - \beta_j \frac{\partial F_j}{\partial y_{\ell j}} = 0 \qquad \text{for all } i, \ell. \tag{16.F.11}$$

Letting $\mu_\ell = p_\ell$, $\delta_i = 1/\alpha_i$, and $\gamma_j = \beta_j$, we see that there is an exact correspondence between conditions (16.F.2)–(16.F.3) and (16.F.10)–(16.F.11). Since both sets of conditions are necessary and sufficient for their respective problems, this implies that the allocation (x^*, y^*) is Pareto optimal if and only if it is a price equilibrium with transfers with respect to some price vector $p = (p_1, \dots, p_L)$. Note, moreover, that the equilibrium price p_ℓ exactly equals μ_ℓ, the marginal value of good ℓ in the Pareto optimality problem.

Suppose that, in addition, every $u_i(\cdot)$ is concave. Then it is also instructive to examine the marginal conditions for the maximization of a linear social welfare function (see Section 16.E). Consider the problem

$$\begin{aligned} \underset{x,y}{\text{Max}} \quad & \sum_i \lambda_i u_i(x_{1i}, \ldots, x_{Li}) & & \quad (16.F.12) \\ \text{s.t. } (1) \quad & \textstyle\sum_i x_{\ell i} \le \bar{\omega}_\ell + \sum_j y_{\ell j} & \ell = 1, \ldots, L & \\ (2) \quad & F_j(y_{1j}, \ldots, y_{Lj}) \le 0 & j = 1, \ldots, J. & \end{aligned}$$

where $\lambda_i > 0$ for all i. Letting $(\psi_1, \ldots, \psi_L)$ and $(\eta_1, \ldots, \eta_J)$ denote the multipliers on constraints (1) and (2), respectively, the necessary and sufficient first-order conditions for this problem can be written as follows:

$$x_{\ell i}: \quad \lambda_i \frac{\partial u_i}{\partial x_{\ell i}} - \psi_\ell \begin{cases} \le 0 \\ = 0 \text{ if } x_{\ell i} > 0 \end{cases} \qquad \text{for all } i, \ell, \qquad (16.F.13)$$

$$y_{\ell j}: \quad \psi_\ell - \eta_j \frac{\partial F_j}{\partial y_{\ell j}} = 0 \qquad \text{for all } j, \ell. \qquad (16.F.14)$$

Note that by letting $\delta_1 = \lambda_i / / \lambda_1$, $\mu_\ell = \psi_\ell / \lambda_1$, and $\gamma_j = \eta_j / \lambda_1$, we have an exact correspondence between (16.F.2)–(16.F.3) and (16.F.13)–(16.F.14). Therefore, any solution to (16.F.13) and (16.F.14) is a solution to (16.F.2) and (16.F.3) and, hence, a Pareto optimum.[19] Conversely, any Pareto optimum that for some multipliers satisfies (16.F.2) and (16.F.3) is also a solution of (16.F.13) and (16.F.14), and consequently of problem (16.F.12), for an appropriate choice of $\lambda = (\lambda_1, \ldots, \lambda_I)$.

It is also enlightening to compare (16.F.13) and (16.F.14) with the first-order conditions (16.F.10) and (16.F.11) for the optimization problems associated with a price equilibrium with transfers. We get an exact correspondence between them by letting $p_\ell = \psi_\ell$, $\alpha_i = 1/\lambda_i$, and $\beta_j = \eta_j$. Once again, the price p_ℓ represents the marginal social value of good ℓ. In addition, note that α_i, which is the marginal utility of wealth for consumer i at prices p and wealth level $w_i = p \cdot x_i^*$, equals the reciprocal of λ_i. Hence, we can draw the conclusion presented in Proposition 16.F.1.

Proposition 16.F.1: Under the assumptions made about the economy [in particular, the concavity of every $u_i(\cdot)$ and the convexity of every $F_j(\cdot)$], every Pareto optimal allocation (and, hence, every price equilibrium with transfers) maximizes a weighted sum of utilities subject to the resource and technological constraints. Moreover, the weight λ_i of the utility of the ith consumer equals the reciprocal of consumer i's marginal utility of wealth evaluated at the supporting prices and imputed wealth.

16.G Some Applications

In this section, we present some applications of the ideas covered in the previous sections of the chapter. We first discuss three examples that introduce particular interpretations of the commodity space. We then present an extension of the second welfare theorem that relies on a concept of *marginal cost pricing.*

19. Recall that by the concavity–convexity assumptions, (16.F.2)–(16.F.3) and (16.F.13)–(16.F.14) are necessary and sufficient conditions for their respective problems.

Interpretations of the Commodity Space

Up to now we have treated our commodities as abstractly defined objects. This has not been for formalism's sake, but to facilitate a wide applicability of the theory.

It is easy to think of the case in which commodities are distinct, physically tradeable real objects. But there are many other interesting possibilities. The theory presented in the previous sections has proven to be remarkably flexible and subtle in the interpretations that can be given to the commodities, consumption sets, preferences, and production sets.

Two important examples, *commodities contingent on the state of the world* and *dated commodities*, are discussed extensively in Chapters 19 and 20, respectively. For the sake of completeness, we devote a few words to contingent commodities in this section. We then briefly discuss two other examples: *occupational choice* and *public goods*.

Example 16.G.1: *Contingent Commodities.* An interesting use of artificial commodities appears in the area of general equilibrium under uncertainty. A full formal description is presented in Chapter 19, but the basic idea can be conveyed here. The usefulness of a commodity may depend on uncertain, external circumstances. For example, medical care is much more important if one is ill than if one is healthy. To ensure an efficient allocation of resources, we have to make sure not only that the right commodities are delivered to the right people but also that they are allocated under the right circumstances, that is, according to the realization of the uncertain external states. To model this type of resource allocation problem, we can use the concept of a *contingent commodity*. A commodity such as medical care can be subdivided into many different "artificial commodities," each of which has the interpretation "medical care is provided under circumstance s." For example, suppose that there are I consumers in the economy, each of whom may turn out ex post to be either "sick" or "healthy." A consumer's need for medical care depends, of course, on her state of health. From an economy-wide perspective, there are then 2^I different states of nature, each corresponding to a different configuration of ill health across the population. We can therefore imagine 2^I different commodities called "medical care," one for each of these configurations. A consumer buying "medical care in state s" receives care when state s occurs.[20]

One of the strengths of general equilibrium theory is its ability to deal easily with an arbitrary number of commodities. There are very few results that depend on the number of commodities, and none of them is of general interest. Therefore, even though it seems difficult to conceive of a very large number of markets for a very large number of contingent commodities, all the welfare propositions that we have developed turn out to be easily applicable to this uncertainty setting (to be sure, we are taking a theoretical, rather than a practical perspective here). In Chapter 19, we discuss these points in more detail. ■

20. Thus, to purchase medical care when she is sick, the consumer actually buys a large number of different "contingent medical care commodities" (in fact, 2^{I-1} of them).

Example 16.G.2: *Occupational Choice* Suppose that every individual could, in principle, work either as a classics scholar or as an economics professor. But not all individuals are equally good at both things. A way to capture the different comparative advantage is to assume that for every individual i, there is an $\alpha_i \geq 0$ measuring how many "effective hours of economics professorial services" it takes to produce "an effective hour of classical scholarship." A relatively low α_i indicates comparative advantage in classical scholarship. Suppose also that every individual i has an amount $\bar{T}_i$ of professorial hours that she can supply; we assume that 1 professorial hour can produce 1 effective hour of economics professorial services or $1/\alpha_i$ effective hours of classical scholarship by individual i. There is a single consumption good on which the individual i can spend her earnings.

It is important to be able to imbed this problem in our formal structure because we certainly want to be able to analyze how, for example, competitive labor markets will perform when individuals have *occupational choices* as well as choices about *how much* labor to supply.

This is how it can be done (it is not the only possible way): suppose we list consumption and effective hours supplied as a three-dimensional vector (c_i, t_{ci}, t_{ei}), where c_i is individual i's consumption and $t_{ci} \leq 0$ and $t_{ei} \leq 0$ are the effective hours spent working as a classics scholar and as an economics professor, respectively. Because the latter two quantities are supplies—that is, services offered by the individual to the market we follow the convention of measuring them as negative numbers. We can then define the consumption set of individual i as

$$X_i = \{(c_i, t_{ci}, t_{ei}): c_i \geq 0, t_{ci} \leq 0, t_{ei} \leq 0, \bar{T}_i + t_{ei} + \alpha_i t_{ci} \geq 0\}.$$

One should interpret the nonpositivity constraints as the inability to consume labor services. The amount $\bar{T}_i + t_{ei} + \alpha_i t_{ci}$ is the time available for leisure activities. Preferences are defined on X_i. Because the consumption good is desirable, the local nonsatiation condition is satisfied. The assumption that preferences are continuous and convex is also natural. We can complete the model by having a concave production function $f(z_c, z_e)$ that transforms input combinations (z_c, z_e) of effective hours of classics and economics scholarship, respectively, into the consumption good.

We now have a complete general equilibrium system to which we can pose a number of interesting questions: If the occupational choice is directed by a competitive (i.e., price-taking) market system, will the outcome result in an efficient exploitation of comparative advantage? Conversely, can every efficient arrangement of occupations be sustained by a market system (supplemented perhaps by lump-sum transfers)? The results of Sections 16.C and 16.D tell us that the answer to both questions is in the affirmative. ■

Example 16.G.3: *Public Goods.* The notion of a "public good" and the more general concept of an "externality" were discussed in Chapter 11, where we also introduced the key idea of "personalized" prices.[21] Consequently, we can be rather brief here. (The basic references on public goods were also given in Chapter 11.)

21. In fact, the current discussion is more general than the one in Chapter 11 because there we restricted ourselves to a quasilinear setting.

Suppose there are I consumers and two commodities; a "private" good, say labor, and a public good (the theory presented extends without essential modification to any number of private or public goods). A *private good* is like the goods discussed up to now: A unit of the good can be consumed only once, so if consumer i consumes it then it is unavailable for consumption by others. We let x_{1i} denote consumer i's consumption of the private good. But in the case of a (pure) *public good*, its consumption by consumer i does not prevent its availability to other consumers. Thus, if x_2 is the total amount of public good produced, then x_2 can be made available at no extra cost to *every* consumer.

We assume that every consumer i has the consumption set $\mathbb{R}^2_+$ and continuous, convex, locally nonsatiated preferences $\succsim_i$ defined on pairs (x_{1i}, x_2). The model is completed by having some amount $\bar{\omega}_1$ of the private good as the initial total endowment (there is no endowment of the public good) and a firm that transforms amounts $z \in \mathbb{R}_+$ of the private good into the public good by means of an increasing, concave production function $f(z)$.

An allocation $((x_{11}, \ldots, x_{1I}, x_2), (q, z)) \geq 0$ is feasible if

$$q \leq f(z), \quad \sum_i x_{1i} + z = \bar{\omega}_1, \quad \text{and} \quad q = x_2.$$

It is Pareto optimal if there is no other feasible allocation $((x'_{11}, \ldots, x'_{1I}, x'_2), (q', z'))$ such that $(x'_{1i}, x'_2) \succsim_i (x'_{1i}, x'_2)$ for all i and $(x'_{1i}, x'_2) \succ_i (x'_{1i}, x'_2)$ for some i.

We now describe this model in an artificial but equivalent way, with the advantage, that, formally speaking, it reduces the public commodities to private ones and therefore makes the results of Sections 16.C and 16.D applicable. The "trick" is to define a personalized commodity x_{2i} for every consumer i, to be interpreted as "commodity 2 as received by the ith consumer." Formally, consumer i cares only about good 1 and the ith personalized commodity. We therefore denote her consumption bundle by $x_i = (x_{1i}, x_{2i})$. The single firm is now viewed as producing a *joint* bundle of personalized commodities with a technology that produces these commodities in fixed proportions. Formally, its (convex) production set is

$$Y = \{(-z, q_1, \ldots, q_I) \in \mathbb{R}^{I+1}_+ : z \geq 0 \text{ and } q_1 = \cdots = q_I = q \leq f(z)\}.$$

With this reinterpretation, the model fits into the structure analyzed throughout the chapter. A price equilibrium with transfers for this artificial economy is known as a *Lindahl equilibrium*.[22]

Definition 16.G.1: A *Lindahl equilibrium* for the public goods economy is a price equilibrium with transfers for the artificial economy with personalized commodities. That is, an allocation $(x_1^*, \ldots, x_I^*, q^*, z^*) \in \mathbb{R}^{2I} \times \mathbb{R} \times \mathbb{R}$ and a price system $(p_1, p_{21}, \ldots, p_{2I}) \in \mathbb{R}^{I+1}$ constitute a Lindahl equilibrium if there is a set of wealth levels $(w_1, \ldots, w_I)$ satisfying $\sum_i w_i = \sum_i p_1 x_{1i}^* + (\sum_i p_{2i}) q^* - p_1 z^*$ and such that:

(i) $q^* \leq f(z^*)$ and $(\sum_i p_{2i}) q^* - p_1 z^* \geq (\sum_i p_{2i}) q - p_1 z$ for all (q, z) with $z \geq 0$ and $q \leq f(z)$.

22. More properly, we should say a *Lindahl equilibrium with transfers*.

(ii) For every i, $x_i^* = (x_{1i}^*, x_{2i}^*)$ is maximal for $\succsim_i$ in the set $\{(x_{1i}, x_{2i}) \in X_i: p_1 x_{1i} + p_{2i} x_{2i} \leq w_i\}$.
(iii) $\sum_i x_{1i}^* + z^* = \bar{\omega}_1$ and $x_{2i}^* = q^*$ for every i.

The first and second fundamental welfare theorems tell us that every Lindahl equilibrium is Pareto optimal and every Pareto optimal allocation can be implemented using a Lindahl equilibrium (with appropriate wealth transfers and, perhaps, with the usual quasiequilibrium qualification).[23] There is an important caveat, however: unlike economies where with large numbers of agents each agent becomes small relative to the size of the market, in markets for personalized goods, each consumer is necessarily large with respect to the market in her personalized good. As we multiply the number of consumers, we also multiply, as a matter of definition, the number of commodities. As a result, it is very unlikely that the critical assumption of price taking will be satisfied. Thus, the descriptive content of this equilibrium concept is low.

Nevertheless, the second welfare theorem may still be of some interest. In particular, it tells us that *if* the planning authority has a means to enforce the prices, then we have a mechanism involving voluntary purchases of the public good that achieves the desired Pareto optimal allocation. Even for this purpose, however, further difficulties arise that are inherent to the public goods setting: First, to calculate the personalized supporting prices, statistical information (e.g. information on the distribution of preferences across the economy) will not do; the fact that prices are personalized means that personal, private information is required. This information may be difficult to get because individuals will often not have incentives to reveal this information truthfully (see Chapters 11 and 23 for more on this issue). Second, for a personalized market voluntary mechanism to work, individuals must expect to receive precisely the amount of public good they purchase. This requires that the public good be *excludable*; that is, there must be some procedure to deny total or partial use of the public good to anyone who does not pay for it. In many cases, such exclusion is difficult, if not impossible (consider, for example, national defense). ■

Nonconvex Production Technologies and Marginal Cost Pricing

The second welfare theorem runs into difficulties in the presence of nonconvex production sets (in this section, we do not question the assumption of convexity on the consumption side). In the first place, large nonconvexities caused by the presence of fixed costs or extensive increasing returns lead to a world of a small number of large firms (in the limit, production efficiency may require a single firm, a so-called "natural monopoly"), making the assumption of price taking less plausible. Yet, even

23. Suppose that the production function $f(\cdot)$ is differentiable and the indifference curves are smooth, and consider a Lindahl equilibrium that is interior. Then preference maximization implies that $p_{2i}/p_1 = -MRS_{21}^i$, where MRS_{21}^i is consumer i's marginal rate of substitution of good 2 for good 1. On the other hand, profit maximization entails $\sum_i p_{2i}/p_1 = -MRT_{21}$, where MRT_{21} is the firm's marginal rate of transformation of good 2 for good 1 (the marginal cost of output in terms of input). Hence, in any Lindahl equilibrium, we must have $\sum_i MRS_{21}^i = MRT_{21}$, which is exactly the Samuelson optimality condition for a public good (see Section 11.G for its derivation in the case of quasilinear preferences).

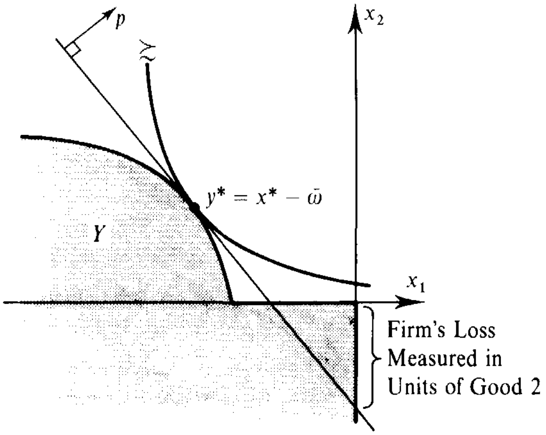


Figure 16.G.1
The firm incurs a loss at the prices that locally support the Pareto optimal allocation.

if price taking can somehow be relied on (perhaps because a planning authority can enforce prices), it may still be impossible to support a given Pareto optimal allocation. Examples are provided by Figures 15.C.3(a) and 16.G.1. In Figure 16.G.1, at the only relative prices that could support the production y^* locally, the firm sustains a loss and would rather avoid it by shutting down. In Figure 15.C.3(a), on the other hand, not even local profit maximization can be guaranteed (see the discussion in Section 15.C).

Although nonconvexities may prevent us from supporting the Pareto optimal production allocation as a profit-maximizing choice, under the differentiability assumptions of Section 16.F we can use the first-order necessary conditions derived there to formulate a weaker result that parallels the second welfare theorem (see Exercise 16.G.1).

Proposition 16.G.1: Suppose that the basic assumptions of Section 16.F hold[24] and that, in addition, all consumers have convex preferences (so utility functions are quasiconcave). If (x^*, y^*) is Pareto optimal, then there exists a price vector $p = (p_1, \dots, p_L)$ and wealth levels $(w_1, \dots, w_I)$ with $\sum_i w_i = p \cdot \bar{\omega} + \sum_j p \cdot y_j^*$ such that:

(i) For any firm j, we have

$$p = \gamma_j \nabla F_j(y_j^*) \quad \text{for some } \gamma_j > 0. \tag{16.G.1}$$

(ii) For any i, x_i^* is maximal for $\succsim_i$ in the budget set

$$\{x_i \in X_i \colon p \cdot x_i \le w_i\}.$$

(iii) $\sum_i x_i^* = \bar{\omega} + \sum_j y_j^*$.

The type of equilibrium represented by conditions (i) to (iii) of Proposition 16.G.1 is called a *marginal cost price equilibrium with transfers*. The motivation for this terminology comes from the one-output, one-input case.[25]

24. That is, the assumptions leading up to conditions (16.F.2) and (16.F.3).

25. We point out that for the general case, the term *marginal cost price equilibrium* is, strictly speaking, inappropriate. Exercise 16.G.3 explains why. However, the terminology is by now standard, and we retain it.

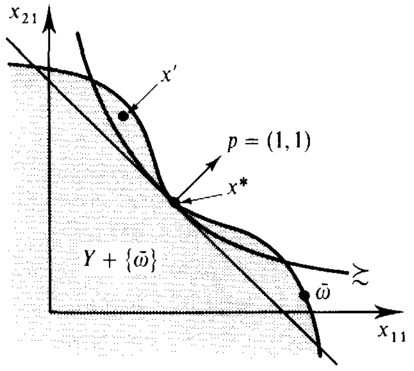


Figure 16.G.2
A marginal cost price equilibrium need not be Pareto optimal.

Exercise 16.G.2: Suppose there are only two goods: an input and an output. Show that in this case, condition (16.G.1) simply says that the price of the input must equal the price of the output multiplied by the marginal productivity of the input or, equivalently, that the price of the output equals its marginal cost.

As we have noted, condition (16.G.1) does *not* imply that the $(y_1^*, \ldots, y_J^*)$ are profit-maximizing production plans for price-taking firms. The condition says only that small changes in production plans have no first-order effect on profit. But small changes may *still* have positive second-order effects (as in Figure 15.C.3, where at a marginal cost price equilibrium the firm actually chooses the production that *minimizes* profits among the efficient productions) and, at any rate, large changes may increase profits (as in Figure 16.G.1). Thus, to achieve allocation (x^*, y^*) may require that a regulatory agency prevent the managers of nonconvex firms from attempting to maximize profits at the given prices p.[26]

See Quinzii (1992) for extensive background and discussion on the material presented in this section.

It should be noted that the converse result to Proposition 16.G.1, which would assert that every marginal cost price equilibrium is Pareto optimal, is *not true*. In Figure 16.G.2, for example, we show a one-consumer economy with a nonconvex production set. In the figure, x^* is a marginal cost price equilibrium with transfers for the price system $p = (1, 1)$. Yet, allocation x' yields the consumer a higher utility. Informally, this occurs because marginal cost pricing neglects second-order conditions and it may therefore happen that, as at allocation x^*, the second-order conditions for the social utility maximization problem are not fulfilled. As a result, satisfaction of the first-order marginal optimality conditions (which in the case of Figure 16.G.2 amounts simply to the tangency of the indifference curve and the production surface) does not ensure that the allocaton is Pareto optimal. (See Exercise 16.G.4 for more on this topic.)

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26. In the context of Figure 16.G.1, the regulator could reach the desired outcome by merely prohibiting the firm from shutting down and otherwise letting it maximize profits at the "supporting" prices (assuming that the firm will act as a price taker; otherwise, the regulator may also need to enforce those prices).

APPENDIX A: TECHNICAL PROPERTIES OF THE SET OF FEASIBLE ALLOCATIONS

The set of feasible allocations is

$$A = \{(x_1, \ldots, x_I, y_1, \ldots, y_J) \in X_1 \times \cdots \times X_I \times Y_1 \times \cdots \times Y_J : \textstyle\sum_i x_i = \bar{\omega} + \sum_j y_j\}$$
$$\subset \mathbb{R}^{LI} \times \mathbb{R}^{LJ}.$$

Our economic problem would not be very interesting if there were no feasible allocations for the economy or if we could give every consumer an unboundedly large consumption vector. We might therefore simply assume that A is nonempty, bounded, and, for good measure, closed (i.e., nonempty and compact). In Chapter 17, where this technical point becomes important for the study of the existence of Walrasian equilibria, we assume exactly this. Nonetheless, it is useful to give, once and for all, a set of sufficient conditions for these very basic properties to hold.

Proposition 16.AA.1: Suppose that

(i) Every X_i:
- (i.1) is closed;
- (i.2) is bounded below (i.e., there is $r > 0$ such that $x_{\ell i} > -r$ for every ℓ and i; in words, no consumer can supply to the market an arbitrarily large amount of any good).

(ii) Every Y_j is closed. Moreover, the aggregate production set $Y = \sum_j Y_j$:[27]
- (ii.1) is convex:
- (ii.2) admits the possibility of inaction (i.e., $0 \in Y$):
- (ii.3) satisfies the no-free-lunch property (i.e., $y \geq 0$ and $y \in Y$ implies $y = 0$):
- (ii.4) is irreversible ($y \in Y$ and $-y \in Y$ implies $y = 0$).

Then the set of feasible allocations A is closed and bounded [i.e., there is $r > 0$ such that $|x_{\ell i}| < r$ and $|y_{\ell j}| < r$ for all i, j, ℓ and any $(x, y) \in A$]. If, moreover, $-\mathbb{R}^L_+ \subset Y$ and we can choose $\hat{x}_i \in X_i$ for every i in such a manner that $\sum_i \hat{x}_i \leq \bar{\omega}$, then A is nonempty.

Proof: The proof of this proposition is rather technical, and we shall not give it. Nonetheless, we shall say a few words regarding the logic of the result.

The nonemptiness part is clear enough because we have $\hat{x}_i \in X_i$ for every i and $\sum_i \hat{x}_i - \bar{\omega} \in -\mathbb{R}^L_+ \subset Y$. Thus, an allocation with individual consumptions $(\hat{x}_1, \ldots, \hat{x}_I)$ and aggregate production vector $\sum_i \hat{x}_i - \bar{\omega}$ is feasible.

Similarly, the closedness of A is a direct consequence of the closedness assumptions on the consumption and production sets (see Exercise 16.AA.1).

What remains is to show that A is bounded. To gain some understanding, suppose that $J = 1$ and $X_i = \mathbb{R}^L_+$ for every i (as long as every X_i is bounded below, the argument for general consumption sets is similar). In Figure 16.AA.1, we represent the set of feasible aggregate consumption bundles $(Y + \{\bar{\omega}\}) \cap \mathbb{R}^L_+$, that is, the set of nonnegative vectors obtained when the origin of Y is shifted to $\bar{\omega} \geq 0$. It is intuitive from the figure that this set can be unbounded above only if Y contains nonnegative, nonzero vectors and so violates the no-free-lunch condition.

27. See Section 5.B for a discussion of conditions (*ii*.1) to (*ii*.4).

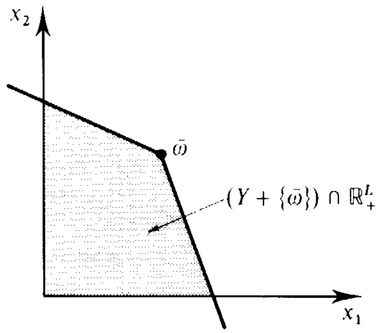

Figure 16.AA.1 The set of feasible aggregate consumption vectors is compact.

You should verify, however, that this intuition also depends on three facts: $0 \in Y$, Y is closed, and Y is convex (see Exercise 16.AA.2). Note now that if the set of feasible aggregate consumptions is bounded, then, a fortiori, so are the sets of feasible individual consumptions and the set of feasible productions (because $J = 1$, we get this set by subtracting $\bar{\omega}$ from each feasible aggregate consumption vector). Hence, A is bounded.

The case with several production sets is more delicate, and it is here that the irreversibility assumption comes to the rescue. Very informally, we can derive, as in the preceding paragraph, the boundedness of feasible *aggregate* productions and feasible individual consumptions. Now, the only way that unboundedness would be possible at the individual production level while remaining bounded in the aggregate is if, so to speak, the unboundedness in one individual production plan was to be canceled by the unboundedness of another. However, this would imply that the collection of all technologies in the economy (i.e., the aggregate production set) allows the reversal of some technologies (see Exercise 16.AA.3 for more details). Incidentally, it can also be shown that irreversibility, with the other assumptions, yields the closedness of Y, so we do not actually need to assume this separately. ■

Proposition 16.AA.2 gives an important implication of the compactness of the set of feasible allocations for the form of the utility possibility set.

Proposition 16.AA.2: Suppose that the set of feasible allocations A is nonempty, closed, and bounded and that utility functions $u_i(\cdot)$ are continuous. Then the utility possibility set U is closed and bounded above.

Proof: Note that $U = U' - \mathbb{R}^L_+$ where

$$U' = \{(u_1(x_1), \ldots, u_I(x_I)) \colon (x, y) \in A\} \subset \mathbb{R}^L.$$

Thus, U' is the image of the compact set A under a continuous function and is therefore itself a compact set (see Section M.F of the Mathematical Appendix). From this, the closedness and the boundedness above of $U = U' - \mathbb{R}^L_+$ follow directly. ■

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EXERCISES

16.C.1$^{\mathrm{B}}$ Show that if a consumption set $X_i \subset \mathbb{R}^L$ is nonempty, closed, and bounded and the preference relation $\succsim_i$ on X_i is continuous, then $\succsim_i$ cannot be locally nonsatiated. [*Hint*: Show that the continuous utility function representing $\succsim_i$ must have a maximum of X_i; see Section M.F of the Mathematical Appendix on this.]

16.C.2$^{\mathrm{A}}$ Suppose that the preference relation $\succsim_i$ is locally nonsatiated and that x_i^* is maximal for $\succsim_i$ in set $\{x_i \in X_i \colon p \cdot x_i \leq w_i\}$. Prove that the following property holds: "If $x_i \succsim_i x_i^*$ then $p \cdot x_i \geq w_i$."

16.C.3$^{\mathrm{B}}$ In this exercise you are asked to establish the first welfare theorem under a set of assumptions compatible with satiation. Suppose that every X_i is nonempty and convex and that every $\succsim_i$ is strictly convex (i.e., if $x_i' \succsim_i x_i$ and $x_i' \neq x_i$ then $\alpha x_i' + (1 - \alpha)x_i \succ_i x_i$ whenever $0 < \alpha < 1$). Prove the following:

(a) Every i can have at most one global satiation point and preferences are locally nonsatiated at any consumption bundle different from the single global satiation point.

(b) Any price equilibrium with transfers is a Pareto optimum.

16.C.4$^{\mathrm{A}}$ Suppose that for each individual there is a "pleasure function" depending on her own consumption only, given by $u_i(x_i)$. Every individual's utility depends positively on her own and everyone else's "pleasure" according to the utility function

$$U_i(x_1, \ldots, x_I) = U_i(u_1(x_1), u_2(x_2), \ldots, u_I(x_I)).$$

Show that if $x = (x_1, \ldots, x_I)$ is Pareto optimal relative to the $U_i(\cdot)$'s, then $(x_1, \ldots, x_I)$ is also a Pareto optimum relative to the u_i's. Does this mean that a community of altruists can use competitive markets to attain Pareto optima? Does your argument depend on the concavity of the u_i's, or the U_i's?

16.D.1$^{\mathrm{A}}$ Prove that if preferences are locally nonsatiated then the condition: "if $x_i \succ_i x_i^*$ then $p \cdot x_i \geq p \cdot x_i^*$" is equivalent to the condition: "x_i^* is expenditure minimizing for the price vector p in the set $\{x_i \in X_i \colon x_i \succsim_i x_i^*\}$."

16.D.2$^{\mathrm{B}}$ Exhibit a one-firm, one-consumer economy in which the production set is convex, the preference relation is continuous and convex, and there is nevertheless a Pareto optimal allocation that can be supported neither as a price equilibrium with transfers nor as a price quasiequilibrium with transfers. Which condition of Proposition 16.D.1 fails?

16.D.3$^{\mathrm{B}}$ Suppose that we have an economy with continuous and strongly monotone preferences (consumption sets are $X_i = \mathbb{R}^L_+$). Suppose also that a strictly positive production is possible; that is, there are $y_j \in Y_j$ such that $\sum_j y_j + \bar{\omega} \gg 0$. Prove that any price quasiequilibrium with transfers must be a price equilibrium with transfers. [*Hint*: Show first that $w_i > 0$ for some i and then argue that $p \gg 0$.]

16.D.4$^{\mathrm{C}}$ Consider a two-good exchange economy with r identical consumers. The consumption set is $\mathbb{R}^2_+$, the individual endowments are $\omega \in \mathbb{R}^2_{++}$, and the preferences are continuous and strongly monotone but not necessarily convex. Argue that the symmetric allocation in which every consumer gets her initial allocation is either a Walrasian equilibrium (for some price vector p) or, if it is not, then for r large enough it is not a Pareto optimum. [*Hint*: the differentiable case is simpler.]

16.E.1$^{\text{B}}$ Given a utility possibility set U, denote by $U' \subset U$ the subset actually achieved by feasible allocations:

$$U' = \{(u_1(x_1), \ldots, u_I(x_I)) \colon \textstyle\sum_i x_i = \sum_j y_j + \bar{\omega} \text{ for some } y_j \in Y_j\}.$$

(Relative to U', the set U allows for free disposal of utility).

(a) Give a two-consumer, two-commodity exchange example showing that it is possible for a point of U' to belong to the boundary of U and *not* be a Pareto optimum.

(b) Suppose that every Y_j satisfies free disposal and $0 \in Y_j$. Also, assume that for every i, $X_i = \mathbb{R}^L_+$ and $\succsim_i$ is continuous and strongly monotone. Show that any boundary point of U that belongs to U' is then a Pareto optimum. [*Hint*: Let $u_i(0) = 0$ for all i and show first that $U' = U \cap \mathbb{R}^L_+$. Next argue that if $u \in U$ is a Pareto optimum and $0 \leq u' \leq u$, $u' \neq u$, then we must be able to reach u' with a surplus of goods relative to u.]

(c) Consider an exchange economy with consumption sets equal to $\mathbb{R}^L_+$, continuous, locally nonsatiated preferences, and a strictly positive total endowment vector. Show that if $u = (u_1, \ldots, u_I)$ is the utility vector corresponding to a price quasiequilibrium with transfers then u cannot be in the interior of U; that is, there is no feasible allocation yielding higher utility to *every* consumer. [*Hint*: Show that $w_i > 0$ for some i and then apply Proposition 16.D.2.]

16.E.2$^{\text{B}}$ Show that the utility possibility set U of an economy with convex production and consumption sets and with concave utility functions is convex.

16.F.1$^{\text{B}}$ In text.

16.F.2$^{\text{A}}$ Derive the first-order conditions (16.F.2) and (16.F.3) of the maximization problem (16.F.1).

16.F.3$^{\text{A}}$ Derive conditions (16.F.4), (16.F.5), and (16.F.6) from the first-order conditions (16.F.2) and (16.F.3).

16.F.4$^{\text{A}}$ Derive the first-order conditions (16.F.4), (16.F.5), and (16.F.6) from problems (16.F.7), (16.F.8), and (16.F.9), respectively.

16.G.1$^{\text{A}}$ Prove Proposition 16.G.1 using the first-order conditions (16.F.2) and (16.F.3).

16.G.2$^{\text{A}}$ In text.

16.G.3$^{\text{B}}$ Exhibit graphically a one-consumer, one-firm economy with two inputs and one output where at the (unique) marginal cost price equilibrium, cost is *not* minimized. [*Hint*: Choose the production function to violate quasiconcavity.]

16.G.4$^{\text{B}}$ Show that under the general conditions of Section 16.G if there is a single consumer (perhaps a normative representative consumer) with convex preferences, then there exists at least one marginal cost price equilibrium that is an optimum.

16.G.5$^{\text{B}}$ In a certain economy there are two commodities, education (e) and food (f), produced by using labor (L) and land (T) according to the production functions

$$e = (\text{Min}\,\{L, T\})^2 \qquad \text{and} \qquad f = (LT)^{1/2}$$

There is a single consumer with the utility function

$$u(e, f) = e^{\alpha} f^{(1-\alpha)},$$

and endowment (ω_L, ω_T). To ease the calculations, take $\omega_L = \omega_T = 1$ and $\alpha = \frac{1}{2}$.

(a) Find the optimal allocation of the endowments to their productive uses.

(b) Recognizing that the production of education entails increasing returns to scale, the government of this economy decides to control the education industry and finance its operation with a lump-sum tax on the consumer. The consumption of education is competitive in the sense that the consumer can choose any amount of education and food desired at the going prices. The food industry remains competitive in both its production and consumption aspects. Assuming that the education industry minimizes cost, find the marginal cost of education at the optimum. Show that if this price of education were announced, together with a lump-sum tax to finance the deficit incurred when the education sector produces the optimal amount at this price for its product, then the consumer's choice of education will be at the optimal level.

(c) What is the level of the lump-sum tax necessary to decentralize this optimum in the manner described in **(b)**?

Now suppose there are two consumers and that their preferences are identical to those above. One owns all of the land and the other owns all of the labor. In this society, arbitrary lump-sum taxes are not possible. It is the law that any deficit incurred by a public enterprise must be covered by a tax on the value of land.

(d) In appropriate notation, write the transfer from the landowner as a function of the government's planned production of education.

(e) Find a marginal cost price equilibrium for this economy where transfers have to be compatible with the transfer function specified in **(d)**. Is it Pareto optimal?

16.AA.1^{A} Show that if every X_i and every Y_j is closed, then the set A of feasible allocations is closed.

16.AA.2^{B} Show that $(Y + \{\bar{\omega}\}) \cap \mathbb{R}^L_+$ is compact if the following four assumptions are satisfied: (i) Y is closed, (ii) Y is convex, (iii) $0 \in Y$, and (iv) if $v \in Y \cap \mathbb{R}^L_+$ then $v = 0$. Exhibit graphically four examples showing that each of the four assumptions is indispensable.

16.AA.3^{B} Suppose that $Y = Y_1 + Y_2 \subset \mathbb{R}^L_+$ satisfies the assumptions given in Exercise 16.AA.2 and that $0 \in Y_1$, $0 \in Y_2$. Argue that if the irreversibility assumption holds for Y then $\{y_1 \in Y_1 : y_1 + y_2 + \bar{\omega} \geq 0 \text{ for some } y_2 \in Y_2\}$ is bounded.

17 The Positive Theory of Equilibrium

17.A Introduction

In this chapter, we study the theoretical predictive power of the Walrasian equilibrium model. Thus, in contrast with Chapter 16, our outlook here is positive rather than normative.

We begin in Section 17.B by laying the foundations for our analysis. We recall the basic model of a *private ownership economy* and the definition of a *Walrasian equilibrium* presented in Section 16.B. We then introduce the notion of an *aggregate excess demand function* and, in a framework of strong assumptions, we characterize Walrasian equilibria as solutions to a system of aggregate excess demand equations. The analysis of this system of equations serves throughout the chapter as our primary method for the study of Walrasian equilibria. (In Appendix A, we discuss another useful equation system based instead on the welfare properties of Walrasian equilibria.)

In Section 17.C (and with much more generality in Appendix B), we present conditions guaranteeing the existence of a Walrasian equilibrium. The identification of an interesting set of conditions ensuring existence assures us that the object of our study in this chapter is not vacuous. A key condition turns out to be the *convexity* of the decision problems of individual economic agents.

Sections 17.D to 17.H all deal with properties of the set of equilibria. Section 17.D reaches a general conclusion: typically (or in the usual terminology, *generically*), there is a finite number of equilibria, and each equilibrium is therefore "locally isolated." Even more, this number is odd, and the equilibria fall naturally into two categories according to the sign of their "index." Section 17.E brings bad news: without further assumptions on the nature of preferences, endowments, or technologies, we cannot say anything more; the behavior of excess demand functions and hence the properties of Walrasian equilibria are not restricted in any manner that goes beyond the facts established in Section 17.D. This negative message reverberates in Section 17.F on uniqueness, Section 17.G on comparative statics, and Section 17.H on (tâtonnement) stability. The purpose of these three sections is precisely to find interesting sufficient conditions for, respectively, the uniqueness of equilibria, good comparative statics properties, and the stability of equilibria. A common theme of the three sections is the role of two sufficient conditions: the *weak axiom of revealed preference in the aggregate* (a way of saying that wealth effects do not cancel in the aggregate the positive influence of the substitution effects), and the property of *gross substitution* (a way of saying that there are not strong complementarities among the goods in the economy).

In Section 17.I, we return to the role of convexity in guaranteeing the existence of Walrasian equilibrium. We qualify this role by showing that nonconvexities that are "small" relative to the aggregate economy (e.g., the indivisibility represented by a car) are not an obstacle to the (near) existence of equilibria, even if they are "large" from the standpoint of an individual agent.

This chapter is of interest from both methodological and substantive points of view. From a substantive standpoint, it deals with an important theory: that of Walrasian equilibrium. Methodologically, the questions that we ask (e.g., does an equilibrium exist? Are the equilibria typically isolated? Is the equilibrium unique? Is it stable? What are the effects of shocks?) and the techniques that we use are questions and techniques that are of relevance to any theory of equilibrium.

17.B Equilibrium: Definitions and Basic Equations

The concept of a private ownership economy was described in Section 16.B. In such an economy, there are I consumers and J firms. Every consumer i is specified by a consumption set $X_i \subset \mathbb{R}^L$, a preference relation $\succsim_i$ on X_i, an initial endowment vector $\omega_i \in \mathbb{R}^L$, and an ownership share $\theta_{ij} \geq 0$ of each firm $j = 1, \ldots, J$ (where $\sum_i \theta_{ij} = 1$). Each firm j is characterized by a production set $Y_j \subset \mathbb{R}^L$. An *allocation* for such an economy is a collection of consumption and production vectors:

$$(x, y) = (x_1, \ldots, x_I, y_1, \ldots, y_J) \in X_1 \times \cdots \times X_I \times Y_1 \times \cdots \times Y_J.$$

The object of investigation in this chapter is the notion of *Walrasian equilibrium*, which we take as a positive prediction for the outcome of a system of markets in which consumers and firms are price takers and the wealth of consumers derives from their initial endowments and profit shares. The formal notion of a Walrasian equilibrium was already introduced in Definition 16.B.3. Definition 17.B.1 repeats it.

Definition 17.B.1: Given a private ownership economy specified by

$$(\{(X_i, \succsim_i)\}_{i=1}^I, \{Y_j\}_{j=1}^J, \{(\omega_i, \theta_{i1}, \ldots, \theta_{iJ})\}_{i=1}^I),$$

an allocation (x^*, y^*) and a price vector $p = (p_1, \ldots, p_L)$ constitute a *Walrasian* (or *competitive*, or *market*, or *price-taking*) equilibrium if

(i) For every j, $y_j^* \in Y_j$ maximizes profits in Y_j; that is,

$$p \cdot y_j \leq p \cdot y_j^* \quad \text{for all } y_j \in Y_j.$$

(ii) For every i, $x_i^* \in X_i$ is maximal for $\succsim_i$ in the budget set

$$\{x_i \in X_i : p \cdot x_i \leq p \cdot \omega_i + \sum_j \theta_{ij} p \cdot y_j^*\}.[1]$$

(iii) $\sum_i x_i^* = \sum_i \omega_i + \sum_j y_j^*$.

1. The terminology "x_i is maximal for $\succsim_i$ in set B" means that x_i is consumer i's preference-maximizing choice in the set B; that is, $x_i \in B$ and $x_i \succsim_i x_i'$ for all $x_i' \in B$.

For purposes of formal analysis, it is extremely helpful to be able to express equilibria as the solutions of a system of equations. We devote the remainder of this section to the study of how this may be done. In what follows, we aim at being very concrete and impose strong assumptions to simplify the analysis.

Exchange Economies and Excess Demand Functions

We begin our derivation of equilibrium equations by studying the case of a pure exchange economy. Recall that a pure exchange economy is one in which the only possible production activities are those of free disposal. Formally, we let $J = 1$ and $Y_1 = -\mathbb{R}^L_+$. We take $X_i = \mathbb{R}^L_+$ and we assume at the outset that each consumer's preferences are *continuous*, *strictly convex*, and *locally nonsatiated* (shortly we shall strengthen local nonsatiation to strong monotonicity). We also assume that $\sum_i \omega_i \gg 0$.

For a pure exchange economy satisfying the above assumptions, the three conditions of Definition 17.B.1 can be equivalently restated as: $(x^*, y^*) = (x_1^*, \ldots, x_I^*, y_1^*)$ and $p \in \mathbb{R}^L$ constitute a Walrasian equilibrium if and only if

(i′) $y_1^* \leq 0$, $p \cdot y_1^* = 0$, and $p \geq 0$.

(ii′) $x_i^* = x_i(p, p \cdot \omega_i)$ for all i [where $x_i(\cdot)$ is consumer i's Walrasian demand function].

(iii′) $\sum_i x_i^* - \sum_i \omega_i^* = y_1^*$.

Condition (i′) is the only one that is not immediate. In Exercise 17.B.1, you are asked to show that it is equivalent to condition (i) of Definition 17.B.1.

Conditions (i′) to (iii′) yield the simple result of Proposition 17.B.1.

Proposition 17.B.1: In a pure exchange economy in which consumer preferences are continuous, strictly convex, and locally nonsatiated, $p \geq 0$ is a Walrasian equilibrium price vector if and only if:

$$\sum_i (x_i(p, p \cdot \omega_i) - \omega_i) \leq 0. \tag{17.B.1}$$

Proof: That (17.B.1) must hold in any Walrasian equilibrium of such an economy follows immediately from conditions (i′) to (iii′). In the other direction, suppose that (17.B.1) holds. If we let $y_1^* = \sum_i (x_i(p, p \cdot \omega_i) - \omega_i)$ and $x_i^* = x_i(p, p \cdot \omega_i)$, then $(x_1^*, \ldots, x_I^*, y_1^*)$ and p satisfy conditions (i′) to (iii′). In particular, note that $p \cdot y_1^* = p \cdot \sum_i (x_i(p, p \cdot \omega_i) - \omega_i) = \sum_i (p \cdot x_i(p, p \cdot \omega_i) - p \cdot \omega_i) = 0$, where the last equality follows because with local nonsatiation we have $p \cdot x_i(p, p \cdot \omega_i) = p \cdot \omega_i$ for all i. ■

The vector $x_i(p, p \cdot \omega_i) - \omega_i \in \mathbb{R}^L$ lists consumer i's net, or *excess*, demand for each good over and above the amount that he possesses in his endowment vector ω_i. Condition (17.B.1) suggests that it may be useful to have a formal representation of this excess demand vector, and of its sum over the I consumers, as a function of prices. This is given in Definition 17.B.2.

Definition 17.B.2: The *excess demand function of consumer i* is

$$z_i(p) = x_i(p, p \cdot \omega_i) - \omega_i,$$

where $x_i(p, p \cdot \omega_i)$ is consumer i's Walrasian demand function. The (*aggregate*) *excess demand function* of the economy is

$$z(p) = \sum_i z_i(p).$$

The domain of this function is a set of nonnegative price vectors that includes all strictly positive price vectors.

Using the economy's excess demand function $z(p)$, condition (17.B.1) can now be expressed more succinctly as follows:

$$\text{“}p \in \mathbb{R}^L_+ \text{ is an equilibrium price vector if and only if } z(p) \leq 0.\text{”} \qquad (17.B.1')$$

Note that if p is a Walrasian equilibrium price vector in a pure exchange economy with locally nonsatiated preferences, then $p \geq 0$, $z(p) \leq 0$, and $p \cdot z(p) = \sum_i p \cdot z_i(p) = \sum_i (p \cdot x_i(p, p \cdot \omega_i) - p \cdot \omega_i) = 0$ (the last equality follows once again from local nonsatiation). Therefore, for every ℓ, we not only have $z_\ell(p) \leq 0$, but also $z_\ell(p) = 0$ if $p_\ell > 0$. Thus, we see that at an equilibrium a good ℓ can be in excess supply (i.e., have $z_\ell(p) < 0$), but only if it is free (i.e., only if $p_\ell = 0$).[2]

To simplify matters even more, we go one step further by assuming that consumer preferences are *strongly monotone*. Thus, for the rest of this section (and, in fact, for all sections of this chapter except Section 17.I and Appendix B), we let $X_i = \mathbb{R}^L_+$ for all i and assume that *all preference relations $\succsim_i$ are continuous, strictly convex, and strongly monotone.*

With strongly monotone preferences, any Walrasian equilibrium must involve a *strictly positive* price vector $p \gg 0$; otherwise consumers would demand an unboundedly large quantity of all the free goods. As a result, we conclude that with strong monotonicity of preferences, *a price vector $p = (p_1, \ldots, p_L)$ is a Walrasian equilibrium price vector if and only if it "clears all markets"; that is, if and only if it solves the system of L equations in L unknowns:*

$$z_\ell(p) = 0 \quad \text{for every } \ell = 1, \ldots, L, \qquad (17.B.2)$$

or, in more compact notation, $z(p) = 0$.

Throughout this chapter, we study the properties of Walrasian equilibria largely by examining the properties of the system of equilibrium equations (17.B.2). We should point out, however, that this is not the only system of equations that we could use to characterize Walrasian equilibria. In Appendix A, for example, we discuss an important alternative system that exploits the welfare properties of Walrasian equilibria identified in Chapter 16.

Proposition 17.B.2 enumerates the properties of the aggregate excess demand function, in pure exchange economies with strongly monotone preferences, that are essential to the developments of this chapter.

Proposition 17.B.2: Suppose that, for every consumer i, $X_i = \mathbb{R}^L_+$ and $\succsim_i$ is continuous, strictly convex, and strongly monotone. Suppose also that $\sum_i \omega_i \gg 0$. Then the aggregate excess demand function $z(p)$, defined for all price vectors $p \gg 0$,

2. As a simple example, good ℓ might be a "bad." Then, we would expect that good ℓ's price would be zero, consumer demand for the good would be zero, and the excess supply $z_\ell(p) = \omega_\ell > 0$ would be eliminated using the disposal technology.

satisfies the properties:

(i) $z(\cdot)$ is continuous.
(ii) $z(\cdot)$ is homogeneous of degree zero.
(iii) $p \cdot z(p) = 0$ for all p (*Walras' law*).
(iv) There is an $s > 0$ such that $z_\ell(p) > -s$ for every commodity ℓ and all p.
(v) If $p^n \to p$, where $p \neq 0$ and $p_\ell = 0$ for some ℓ, then

$$\text{Max}\,\{z_1(p^n), \ldots, z_L(p^n)\} \to \infty.$$

Proof: With the exception of property (v), all these properties are direct consequences of the definition and the parallel properties of demand functions.[3] The bound in (iv) follows from the nonnegativity of demand (i.e., the fact that $X_i = \mathbb{R}^L_+$), which implies that a consumer's total net supply to the market of any good ℓ can be no greater than his initial endowment. You are asked to prove property (v) in Exercise 17.B.2. The intuition for it is this: As some prices go to zero, a consumer whose wealth tends to a strictly positive limit [note that, because $p \cdot (\sum_i \omega_i) > 0$, there must be at least one such consumer] and with strongly monotone preferences will demand an increasingly large amount of some of the commodities whose prices go to zero (but perhaps not of all such commodities: relative prices still matter). ∎

Finally, note that because of Walras' law, to verify that a price vector $p \gg 0$ clears all markets [i.e., has $z_\ell(p) = 0$ for all ℓ] it suffices to check that it clears *all markets but one*. In particular, if $p \gg 0$ and $z_1(p) = \cdots = z_{L-1}(p) = 0$, then because $p \cdot z(p) = \sum_\ell p_\ell z_\ell(p) = 0$ and $p_L > 0$, we must also have $z_L(p) = 0$. Hence, if we denote the vector of $L-1$ excess demands for goods 1 through $L-1$ by

$$\hat{z}(p) = (z_1(p), \ldots, z_{L-1}(p)),$$

we see that a strictly positive price vector p is a Walrasian equilibrium if and only if $\hat{z}(p) = 0$.

Production Economies

It is possible to extend the methodology based on excess demand equations to the general production case. Assume, to begin with, that production sets are closed, strictly convex, and bounded above. Then, for any price vector $p \gg 0$, we can let $\pi_j(p)$ and $y_j(p)$ be the maximum profits and the profit-maximizing production vector, respectively, for firm j. Defining

$$\tilde{z}(p) = \sum_i x_i\Big(p, p \cdot \omega_i + \sum_j \theta_{ij}\pi_j(p)\Big) - \sum_i \omega_i - \sum_j y_j(p) \qquad \text{(17.B.3)}$$

as the *production inclusive excess demand function*, we see that p is a Walrasian equilibrium price vector if and only if it solves the system of equations $\tilde{z}(p) = 0$. In Exercise 17.B.4, you are asked to show that under a weak hypothesis (that a strictly positive aggregate consumption bundle is producible using the initial endowments), the function $\tilde{z}(\cdot)$ satisfies properties (i) to (v) of Proposition 17.B.2.

Note that if the production sets are not bounded above, then $\tilde{z}(p)$ may fail to be

3. Note, incidentally, that properties (i) to (iv) continue to hold even if preferences are only locally nonsatiated.

defined for some $p \gg 0$ [because we may have $\pi_j(p) = \infty$ for some j]. Nevertheless, an equilibrium price vector is still characterized by $\tilde{z}(p) = 0$.

When production sets are not strictly convex, matters become more complicated because the correspondences $y_j(p)$ may no longer be single-valued. Indeed, a production situation of considerable theoretical and practical importance—and one that we certainly do not want to rule out by assumption—is the case of constant returns to scale. With constant returns, however, production sets are neither strictly convex nor bounded above (except for the trivial case in which no positive amount of any good can be produced). In principle, we could still view the equilibria as the zeros of a "production inclusive excess demand *correspondence*," defined as in (17.B.3) for a subset of strictly positive prices.[4] Correspondences, however, do not make good equational systems (e.g., they cannot be differentiated). It is therefore usually much more convenient in such cases to capture the equilibria as the solutions of an extended system of equations involving the production and the consumption sides of the economy. We illustrate this idea in the small type discussion that follows.

To see how an extended system of equations can be constructed, consider the case in which production is of the linear activity type (this case is reviewed in Appendix A of Chapter 5). Say that, in addition to the disposal technologies, we have J basic activities $a_1, \ldots, a_J \in \mathbb{R}^L$. That is, the aggregate production set is

$$Y = \left\{ y \in \mathbb{R}^L\colon y \leq \sum_j \alpha_j a_j \text{ for some } (\alpha_1, \ldots, \alpha_J) \geq 0 \right\}.$$

Because preferences are strongly monotone, there can be no free goods at an equilibrium (i.e., we must have $p \gg 0$). Also, productions should be profit maximizing, and because of constant returns, these maximum profits must be zero. Therefore, a pair (p, α) formed by a price vector $p \in \mathbb{R}^L_+$ and a vector of activity levels $\alpha \in \mathbb{R}^J_+$ constitute an equilibrium if and only if they solve

$$z(p) - \sum_j \alpha_j a_j = 0 \qquad (17.B.4)$$

and

$$p \cdot a_j \leq 0, \qquad \alpha_j(p \cdot a_j) = 0 \qquad \text{for all } j, \qquad (17.B.5)$$

where $z(\cdot)$ is the consumers' aggregate excess demand function of Definition 17.B.2. Note that, if so desired, condition (17.B.5) can be expressed as a system of equations: just replace "$p \cdot a_j \leq 0$, $\alpha_j(p \cdot a_j) = 0$" by "$\alpha_j p \cdot a_j + \text{Max}\,\{0, p \cdot a_j\} = 0$." Exercise 17.B.5 presents an extension of the current discussion to a more general production case allowing for continuous substitution of activities.

It is worth emphasizing that, at least for the case of convex technologies, there would not be much loss of conceptual generality if we assumed that the production sector of the economy was composed of a single firm endowed with a constant returns production technology. To see this, recall from Proposition 5.B.2 that by creating for each firm j an extra, firm-specific, factor of production, we can always assume that every Y_j exhibits constant returns (when we do so, we transform each consumer's ownership shares of the profits of the jth firm into endowments of the jth new physical resource). Because profits at an equilibrium must then be zero, we see that once this is done there is no need to keep the identity of firms separate in order to compute the wealth of consumers. Moreover, from the point of view of production decisions, again no such need arises. As we saw in Section 5.E, we could as well work with the aggregate "representative firm" $Y = \sum_j Y_j$.

4. That is, p would be an equilibrium price vector if and only if $0 \in \tilde{z}(p)$.

A single constant returns Y can be interpreted as a description of a long-run state of knowledge that is freely available to every agent in the economy (i.e., to every consumer), for the purpose of setting up firms or simply for household production. In fact, we could go one step further and formally dispense with the separate consideration of firms and of the profit-maximization condition. In Exercise 17.B.6, you are asked to show that a Walrasian equilibrium can be redefined in terms of the following two-stage process: first, consumers choose a vector $v_i \in \mathbb{R}^L$ subject to the budget constraint $p \cdot v_i \leq p \cdot \omega_i$ (the equilibrium market-clearing condition is $\sum_i v_i = \sum_i \omega_i$); second, every i uses v_i and the technology Y for household production of a most preferred consumption bundle.[5]

17.C Existence of Walrasian Equilibrium

When studying a positive theory, the first question to ask is: under what conditions does the formal model possess a solution? That is, is it capable of predicting a definite outcome? This is known as the *existence* problem. Conceptually, the assurance of existence of an equilibrium means that our equilibrium notion passes the logical test of consistency. It tells us that the mathematical model is well suited to the purposes it has been designed for. Although an existence theorem can hardly be the end of the story, it is, in a sense, the door that opens into the house of analysis.[6]

The existence of a Walrasian equilibrium can be established in considerable generality. To maintain the natural flow of exposition, in this section we offer a detailed examination of the existence problem for the particular case that will be our primary focus throughout the chapter: pure exchange economies modeled by means of excess demand functions. In Appendix B, we discuss the existence problem in the general case.

We have seen in Section 17.B that the excess demand function $z(\cdot)$ of an exchange economy with $\sum_i \omega_i \gg 0$ and continuous, strictly convex, and strongly monotone preferences satisfies properties (i) to (v) of Proposition 17.B.2. We now argue that *any* function $z(\cdot)$ satisfying these five conditions admits a solution, that is, a price vector p such that $z(p) = 0$. By doing so, we establish that a Walrasian equilibrium exists under the conditions of Proposition 17.B.2.

To start simply, suppose that there are only two commodities (i.e., $L = 2$). For this case, it is an easy matter to establish the existence of an equilibrium. First, by the homogeneity of degree zero of $z(\cdot)$ (condition (ii) of Proposition 17.B.2), we can normalize $p_2 = 1$ and look for equilibrium price vectors of the form $(p_1, 1)$. Then, by Walras' law (condition (iii) of Proposition 17.B.2), an equilibrium can be obtained as a solution to the single equation $z_1(p_1, 1) = 0$. This one-variable problem is

5. This process formally reduces the production economy to an economy where only exchange takes place. But we do not mean to suggest that the induced exchange economy satisfies all of the strong assumptions that we have imposed in this chapter.

6. It should be emphasized that finding a class of conditions that guarantee the existence of a Walrasian equilibrium does not say that this is the outcome that will occur whenever preferences, endowments, and technologies satisfy the assumptions of the existence theorem: the behavioral assumption of price taking and the institutional assumptions of complete markets must also hold. However, when the conditions required for existence are *not* satisfied by preferences, endowments, or technologies, it does suggest that the type of equilibrium under consideration may not be the right one to look for.

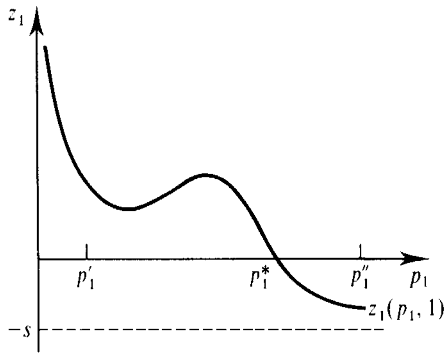

Figure 17.C.1
Proof of existence of an equilibrium for the case $L = 2$.

represented in Figure 17.C.1.[7] When p'_1 is very small, we must have $z_1(p'_1, 1) > 0$; if p''_1 is very large, we have $z_1(p''_1, 1) < 0$. These two boundary restrictions follow by using conditions (iv) and (v) of Proposition 17.B.2 to identify the commodity with positive excess demand as the one whose relative price is very low.[8] Because the function $z_1(p_1, 1)$ is continuous [condition (i) of Proposition 17.B.2], there must be an intermediate value $p^*_1 \in [p'_1, p''_1]$ with $z_1(p^*_1, 1) = 0$ and, hence, an equilibrium price vector must exist.

In the general case of more than two commodities, the proof that a solution exists is more delicate, and involves the use of some powerful mathematical tools. In Proposition 17.C.1, we follow a traditional approach that invokes Kakutani's fixed-point theorem (see Section M.I of the Mathematical Appendix). We should point out that the proof of Proposition 17.C.1 has to deal with the technical complication that excess demand is not defined when the prices of some commodities are zero. The reader may actually gain a more direct insight into the nature of the fixed-point argument from Proposition 17.C.2, which contains a very simple proof for the case of excess demand functions defined for all nonzero, nonnegative prices.

Proposition 17.C.1: Suppose that $z(p)$ is a function defined for all strictly positive price vectors $p \in \mathbb{R}^L_{++}$ and satisfying conditions (i) to (v) of Proposition 17.B.2. Then the system of equations $z(p) = 0$ has a solution. Hence, a Walrasian equilibrium exists in any pure exchange economy in which $\sum_i \omega_i \gg 0$ and every consumer has continuous, strictly convex, and strongly monotone preferences.

7. Note that we revert to the usual mathematical convention of representing the independent variable p_1 on the horizontal axis. The partial equilibrium convention of putting prices on the vertical axis, to which we have adhered throughout Part III, is a vestige of the origins of the theory in Marshall (1920) where, in contrast with Walras (1874), prices are, in fact, dependent variables.

8. In particular, property (iv) implies that the value of intended sales is bounded. By Walras' law, the value of intended purchases must therefore also be bounded. Because, by property (v), intended purchases become unbounded in physical terms for *some* good as $p_1 \to 0$, it follows that it must be good 1 whose demand becomes unbounded as $p_1 \to 0$. Hence, $z_1(p'_1, 1) > 0$ for p'_1 sufficiently small. By symmetry, as $p_1 \to \infty$ [which, by the homogeneity of degree zero of $z(\cdot)$, is equivalent to $p_2 \to 0$ holding p_1 fixed], for p''_1 large enough we must have $z_2(p''_1, 1) > 0$, and therefore $z_1(p''_1, 1) < 0$.

Proof: We begin by normalizing prices in a convenient way. Denote by

$$\Delta = \left\{p \in \mathbb{R}^L_+ : \sum_\ell p_\ell = 1\right\}$$

the unit simplex in $\mathbb{R}^L$. Because the function $z(\cdot)$ is homogeneous of degree zero, we can restrict ourselves, in our search for an equilibrium, to price vectors in Δ. Note, however, that the function $z(\cdot)$ is well defined only for price vectors in the set

$$\text{Interior } \Delta = \{p \in \Delta : p_\ell > 0 \text{ for all } \ell\}.$$

We shall proceed in five steps. In the first two, we construct a certain correspondence $f(\cdot)$ from Δ to Δ. In the third, we argue that any fixed point of $f(\cdot)$, that is, any p^* with $p^* \in f(p^*)$, has $z(p^*) = 0$. The fourth step proves that $f(\cdot)$ is convex valued and upper hemicontinuous (or, equivalently, that it has a closed graph). Finally, the fifth step applies Kakutani's fixed-point theorem to show that a p^* with $p^* \in f(p^*)$ necessarily exists.

For notational clarity, in defining the set $f(p) \subset \Delta$, we denote the vectors that are elements of $f(p)$ by the symbol q.

Step 1: Construction of the fixed-point correspondence for $p \in$ Interior Δ. Whenever $p \gg 0$, we let

$$f(p) = \{q \in \Delta : z(p) \cdot q \geq z(p) \cdot q' \quad \text{for all } q' \in \Delta\}.$$

In words: Given the current "proposal" p, the "counterproposal" assigned by the correspondence $f(\cdot)$ is any price vector q that, among the permissible price vectors (i.e., among the members of Δ), maximizes the value of the excess demand vector $z(p)$. This makes economic sense; thinking of $f(\cdot)$ as a rule that adjusts current prices in a direction that eliminates any excess demand, the correspondence $f(\cdot)$ as defined above assigns the highest prices to the commodities that are most in excess demand. In particular, we have

$$f(p) = \{q \in \Delta : q_\ell = 0 \text{ if } z_\ell(p) < \text{Max}\{z_1(p), \ldots, z_L(p)\}\}.$$

Observe that if $z(p) \neq 0$ for $p \gg 0$, then because of Walras' law we have $z_\ell(p) < 0$ for some ℓ and $z_{\ell'}(p) > 0$ for some $\ell' \neq \ell$. Thus, for such a p, any $q \in f(p)$ has $q_\ell = 0$ for some ℓ. Therefore, if $z(p) \neq 0$ then $f(p) \subset$ Boundary $\Delta = \Delta \backslash$Interior Δ. In contrast, if $z(p) = 0$ then $f(p) = \Delta$.

Step 2: Construction of the fixed-point correspondence for $p \in$ Boundary Δ. If $p \in$ Boundary Δ, we let

$$f(p) = \{q \in \Delta : p \cdot q = 0\} = \{q \in \Delta : q_\ell = 0 \text{ if } p_\ell > 0\}.$$

Because $p_\ell = 0$ for some ℓ, we have $f(p) \neq \varnothing$. Note also that with this construction, no price from Boundary Δ can be a fixed point; that is, $p \in$ Boundary Δ and $p \in f(p)$ cannot occur because $p \cdot p > 0$ while $p \cdot q = 0$ for all $q \in f(p)$.

Step 3: A fixed point of $f(\cdot)$ is an equilibrium. Suppose that $p^* \in f(p^*)$. As we pointed out in step 2, we must have $p^* \notin$ Boundary Δ. Therefore $p^* \gg 0$. If $z(p^*) \neq 0$, then we saw in step 1 that $f(p^*) \subset$ Boundary Δ, which is incompatible with $p^* \in f(p^*)$ and $p^* \gg 0$. Hence, if $p^* \in f(p^*)$ we must have $z(p^*) = 0$.

Step 4: The fixed-point correspondence is convex-valued and upper hemicontinuous. To establish convex-valuedness, note that, both when $p \in$ Interior Δ and when $p \in$ Boundary Δ, $f(p)$ equals a level set of a linear function defined on the convex set Δ [that is, a set of the form $\{q \in \Delta: \lambda \cdot q = k\}$ for some scalar k and vector $\lambda \in \mathbb{R}^L$], and so it is convex. (Exercise 17.C.1 asks for a more explicit verification.)[9]

To establish upper hemicontinuity (see Section M.M of the Mathematical Appendix for definitions), consider sequences $p^n \to p$, $q^n \to q$ with $q^n \in f(p^n)$ for all n. We have to show that $q \in f(p)$. There are two cases: $p \in$ Interior Δ and $p \in$ Boundary Δ.

If $p \in$ Interior Δ, then $p^n \gg 0$ for n sufficiently large. From $q^n \cdot z(p^n) \geq q' \cdot z(p^n)$ for all $q' \in \Delta$ and the continuity of $z(\cdot)$, we get $q \cdot z(p) \geq q' \cdot z(p)$ for all q'; that is, $q \in f(p)$.

Now suppose that $p \in$ Boundary Δ. Take any ℓ with $p_\ell > 0$. We shall argue that for n sufficiently large we have $q_\ell^n = 0$ and therefore it must be that $q_\ell = 0$; from this, $q \in f(p)$ follows. Because $p_\ell > 0$, there is an $\varepsilon > 0$ such that $p_\ell^n > \varepsilon$ for n sufficiently large. If, in addition, $p^n \in$ Boundary Δ then $q_\ell^n = 0$ by the definition of $f(p^n)$. If, instead, $p^n \gg 0$ then the boundary conditions (iv) and (v) of Proposition 17.B.2 come into play. They imply that, for n sufficiently large, we must have

$$z_\ell(p^n) < \text{Max}\,\{z_1(p^n), \ldots, z_L(p^n)\}$$

and therefore that, again, $q_\ell^n = 0$. To prove the above inequality, note that by condition (v) the right-hand side of the above expression goes to infinity with n (because $p \in$ Boundary Δ, some prices go to zero as $n \to \infty$). But the left-hand side is bounded above because if it is positive then

$$z_\ell(p^n) \leq \frac{1}{\varepsilon} p_\ell^n z_\ell(p^n) = -\frac{1}{\varepsilon} \sum_{\ell' \neq \ell} p_{\ell'}^n z_{\ell'}(p^n) < \frac{s}{\varepsilon} \sum_{\ell' \neq \ell} p_{\ell'}^n < \frac{s}{\varepsilon},$$

where s is the bound in excess supply given by condition (iv).[10] In summary, for p^n close enough to Boundary Δ, the maximal demand corresponds to some of the commodities whose price is close to zero. Therefore, we conclude that, for large n, any $q^n \in f(p^n)$ will put nonzero weight only on commodities whose prices approach zero. But this guarantees $p \cdot q = 0$, and so $q \in f(p)$.

Step 5: A fixed point exists. Kakutani's fixed-point theorem (see Section M.I of the Mathematical Appendix) says that a convex-valued, upper hemicontinuous correspondence from a nonempty, compact, convex set into itself has a fixed point. Since Δ is a nonempty, convex, and compact set, and since $f(\cdot)$ is a convex-valued upper hemicontinuous correspondence from Δ to Δ, we conclude that there is a $p^* \in \Delta$ with $p^* \in f(p^*)$. ■

It is instructive to examine which of properties (i) to (v) of Proposition 17.B.2 fail to hold for the excess demand functions corresponding to the Edgeworth boxes of Figures 15.B.10(a) and (b), where, as we saw, no Walrasian equilibrium existed. In

9. Note also that for any $p \in \Delta$, the set $f(p)$ is always a face of the simplex Δ; that is, it is one of the subsets of Δ spanned by a finite subset of unit coordinates. For $p \in$ Boundary Δ, $f(p)$ is the face of Δ spanned by the zero coordinates of p. For $p \in$ Interior Δ, $f(p)$ is the face spanned by the coordinates corresponding to commodities with maximal excess demand.

10. In words, the last chain of inequalities says that the expenditure on commodity ℓ is bounded because it has to be financed by, and therefore cannot be larger than, the bounded value of excess supplies.

the case of Figure 15.B.10(b), preferences are not convex and therefore $z(\cdot)$ is not a function, let alone a continuous one (condition (i)).[11] For Figure 15.B(10(a), it is property (v) that fails: For any sequence of prices $(p_1^n, p_2^n) \to (1, 0)$, excess demand stays bounded. Note that in the limit there is a single consumer with positive wealth, but the preferences for this consumer, while monotone, are not strongly monotone.

To facilitate a clear understanding of the nature of the fixed-point argument it is helpful to consider Proposition 17.C.2, in which boundary complications are eliminated by studying continuous, homogeneous of degree zero functions $z(p)$ satisfying Walras' law and defined for all nonnegative, nonzero price vectors. Within a framework of continuous and strictly convex preferences, this type of excess demand function is not compatible with monotone preferences but can arise with preferences that are locally nonsatiated. Recall also that the equilibrium condition when zero prices are allowed is $z(p) \leq 0$; see expression (17.B.1').

Proposition 17.C.2: Suppose that $z(p)$ is a function defined for all nonzero, nonnegative price vectors $p \in \mathbb{R}_+^L$ and satisfying conditions (i) to (iii) of Proposition 17.B.2 (i.e. continuity, homogeneity of degree zero and Walras' law). Then there is a price vector p^* such that $z(p^*) \leq 0$.

Proof: Because of homogeneity of degree zero we can restrict our search for an equilibrium to the unit simplex $\Delta = \{p \in \mathbb{R}_+^L : \sum_\ell p_\ell = 1\}$.

Define on Δ the function $z^+(\cdot)$ by $z_\ell^+(p) = \text{Max}\,\{z_\ell(p), 0\}$. Note that $z^+(\cdot)$ is continuous and that $z^+(p)\cdot z(p) = 0$ implies $z(p) \leq 0$.

Denote $\alpha(p) = \sum_\ell [p_\ell + z_\ell^+(p)]$. We have $\alpha(p) \geq 1$ for all p.

Define a continuous function $f(\cdot)$ from the closed, convex set Δ into itself by

$$f(p) = [1/\alpha(p)](p + z^+(p)).$$

Note that, corresponding to intuition, this fixed-point function tends to increase the price of commodities in excess demand. The construction of the function is illustrated in Figure 17.C.2 for the case $L = 2$.

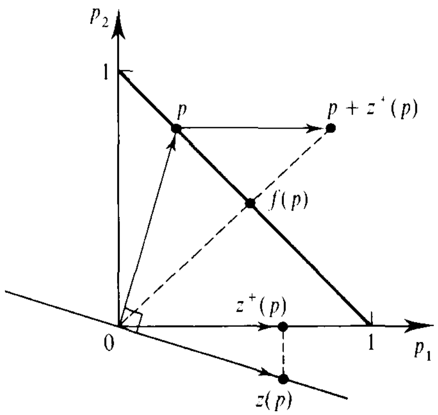

Figure 17.C.2
The fixed-point function for Proposition 17.C.2.

11. As we shall mention shortly in this section, existence would still obtain if $z(\cdot)$ was a convex-valued and upper hemicontinuous correspondence. In Figure 15.B.10(b), however, excess demand fails to satisfy the convex-valuedness property.

By Brouwer's fixed-point theorem (see Section M.I of the Mathematical Appendix) there is a $p^* \in \Delta$ such that $p^* = f(p^*)$. We show that $z(p^*) \leq 0$.

By Walras' law:

$$0 = p^* \cdot z(p^*) = f(p^*) \cdot z(p^*) = [1/\alpha(p^*)] z^+(p^*) \cdot z(p^*).$$

Therefore, $z^+(p^*) \cdot z(p^*) = 0$. But, as we have already pointed out, this implies $z(p^*) \leq 0$. ■

The applicability of Proposition 17.C.1 is not limited to exchange economies. We saw, for example, in Section 17.B (and Exercise 17.B.4), that if we allow for production sets that are closed, strictly convex, and bounded above (and if a positive aggregate consumption bundle is producible from the initial aggregate endowments), then the production inclusive excess demand function $\tilde{z}(\cdot)$ satisfies conditions (i) to (v) of Proposition 17.B.2. Hence, Proposition 17.C.1 also implies that a Walrasian equilibrium necessarily exists in this case.

We also note for later reference that Proposition 17.C.1 holds as well for a convex-valued and upper hemicontinuous *correspondence* $z(p)$ that satisfies conditions (ii) to (v) (properly adapted) of Proposition 17.B.2. In this case, there exists a p such that $0 \in z(p)$. (See Exercise 17.C.2 for more on this point.)

Although Proposition 17.C.1 tells us that an equilibrium exists, it does not give us the equilibrium price vectors or allocations in an explicit manner. The issue of how to actually find equilibria was first considered by Scarf (1973). By now, a variety of useful techniques are available. They are very important for applied work, where the ability to compute solutions is key. See Shoven and Whaley (1992) for an account of applied general equilibrium.

The second welfare theorem of Section 16.D can be seen as a particular case of the current existence result. To see this, suppose that $x = (x_1, \ldots, x_I)$ is a Pareto optimal allocation of a pure exchange economy satisfying the assumptions leading to Proposition 17.C.1. Then, by Proposition 17.C.1, a Walrasian equilibrium price vector p and allocation $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_I)$ exist for the economy in which endowments are $\omega_i = x_i$ for all i. Since x_i is affordable at prices p for every consumer i, we must have $\hat{x}_i \succsim_i x_i$ for all i. Hence, it follows from the Pareto optimality of $x = (x_1, \ldots, x_I)$ that $\hat{x}_i \sim_i x_i$ for all i. But since $\hat{x}_i$ is consumer i's optimal demand given prices p and wealth $w_i = p \cdot \omega_i = p \cdot x_i$, x_i must also be an optimal demand for consumer i for price wealth pair $(p, p \cdot x_i)$. Hence, we see that the price vector p and the wealth levels $w_i = p \cdot x_i$ support the allocation x as a price equilibrium with transfers in the sense of Definition 16.B.4.[12]

17.D Local Uniqueness and the Index Theorem

Having established in Section 17.C (and Appendix B) conditions under which a Walrasian equilibrium is guaranteed to exist, we now begin a study of some issues related to its uniqueness or multiplicity.

12. The fact that the second welfare theorem can be viewed as a corollary of theorems asserting the existence of Walrasian equilibrium is valid much beyond the economies studied in this section.

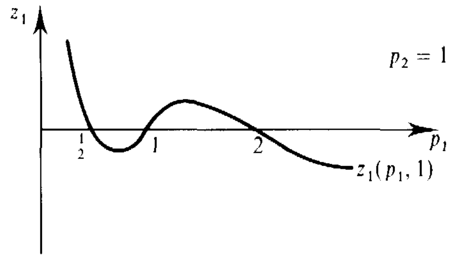

Figure 17.D.1
The excess demand function for an economy with multiple equilibria.

For a theorist, the best of the all possible worlds is one in which the social situation being analyzed can be formalized in a manner that, on the one hand, is very parsimonious (i.e., uses as inputs only the most indisputable and sturdy traits of the reality being modeled) and, on the other, manages to predict a unique outcome.

The Walrasian model of perfect competition is indeed very parsimonious. Essentially, it attempts to give a complete theoretical account of an economy by using as fundamentals only the list of commodities, the state of the technology, and the preferences and endowments of consumers. However, the other side of the coin is that the theory is not completely deterministic. We shall see in Section 17.F that the uniqueness of equilibria is assured only under special conditions. The Edgeworth box of Figure 15.B.9 and Example 15.B.2 provides a simple illustration that, under the assumptions we have made, multiplicity of equilibria is possible. Figure 17.D.1 represents the excess demand function for good 1 of Example 15.B.2 as a function of p_1 (normalizing to $p_2 = 1$). For another example of multiplicity, see Exercise 17.D.1.

From the theoretical point of view, if uniqueness is not achievable, the next-best property is local uniqueness. We say that an equilibrium price vector is *locally unique*, or *locally isolated*, if we cannot find another (normalized) price vector arbitrarily close to it. If every equilibrium of an economy is locally unique, we say that the *local uniqueness property* holds for the economy. The local uniqueness property is of interest because, if it prevails, then it may not be difficult to complete the theory in any particular application. For example, history may have determined the region where equilibrium lies (it could be the region where equilibrium used to be before a small unanticipated shock to the economy), and in that region we may have a unique equilibrium. In this case, the theory retains its predictive power, albeit only locally. We say that a theory that guarantees the local uniqueness of equilibria is *locally* (as opposed to globally) *determinate*.

The question is then: Is the Walrasian theory locally determinate? The example of Figure 17.D.1 suggests that it is: Every solution to the excess demand equation is locally isolated. But Figures 17.D.2 and 17.D.3 provide a counterexample. The figures depict the offer curves and the excess demand function of an exchange economy with a continuum of Walrasian equilibria. Nonetheless, we should not despair. The situation displayed in Figures 17.D.2 and 17.D.3 has an obvious pathological feel about it; it looks like a coincidence. And indeed, it was shown by Debreu (1970) that such an occurrence is not robust: it can happen only by accident.

We now turn to a formal discussion of these issues. For the sake of concreteness we restrict ourselves, as usual, to the analysis of exchange economies formed by I consumers. Every consumer i is specified by $(\succsim_i, \omega_i)$, where $\succsim_i$ is a continuous,

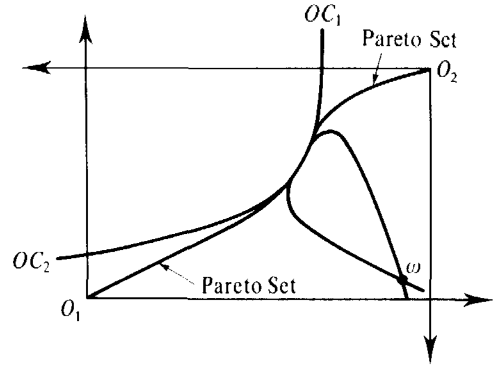

Figure 17.D.2 (left)
A continuum of equilibria is possible: Edgeworth box.

Figure 17.D.3 (right)
A continuum of equilibria is possible: excess demand.

strictly convex, and strongly monotone preference relation on $\mathbb{R}^L_+$ and $\omega_i \gg 0$. As we know, the aggregate excess demand function $z(\cdot)$ then satisfies conditions (i) to (v) of Proposition 17.B.2. We further assume that $z(\cdot)$ is continuously differentiable.[13]

Because we can only hope to determine *relative* prices, we normalize $p_L = 1$ and, as we did in Section 17.B, denote by

$$\hat{z}(p) = (z_1(p), \ldots, z_{L-1}(p))$$

the vector of excess demands for the first $L - 1$ goods.[14] A normalized price vector $p = (p_1, \ldots, p_{L-1}, 1)$ constitutes a Walrasian equilibrium if and only if it solves the system of $L - 1$ equations in $L - 1$ unknowns:

$$\hat{z}(p) = 0.$$

Regular Economies

It is useful to begin by introducing the concept stated in Definition 17.D.1.

Definition 17.D.1: An equilibrium price vector $p = (p_1, \ldots, p_{L-1})$ is *regular* if the $(L - 1) \times (L - 1)$ matrix of price effects $D\hat{z}(p)$ is nonsingular, that is, has rank $L - 1$. If every normalized equilibrium price vector is regular, we say that the *economy is regular*.

In Figure 17.D.1, every equilibrium is regular because the slope of excess demand, $\partial z_1(p_1, 1)/\partial p_1$, is nonzero at every solution. In contrast, none of the equilibria of Figure 17.D.3 are regular because at any equilibrium price vector the slope of the excess demand function is zero. Later in this section we shall argue that, in a sense that we will make precise, "almost every" economy is regular.

The significance of the technical concept of regularity derives from the fact that a regular (normalized) equilibrium price vector is isolated, and a regular economy can only have a finite number of (normalized) price equilibria. This is formally stated in Proposition 17.D.1.

13. In Appendix A to Chapter 3, we discussed conditions for the differentiability of demand functions and therefore of excess demand functions.

14. Nothing in what follows depends on the particular normalization. It can be shown, for example, that if $z(p) = 0$ and the $L \times L$ matrix $Dz(p)$ has rank $L - 1$, then the $(L - 1) \times (L - 1)$ matrix $D\hat{z}(p)$ has rank $L - 1$ whichever good we choose to normalize. Even more, the sign of its determinant is independent of the normalization (see Section M.D of the Mathematical Appendix).

Proposition 17.D.1: Any regular (normalized) equilibrium price vector

$$p = (p_1, \ldots, p_{L-1}, 1)$$

is *locally isolated* (or *locally unique*). That is, there is an $\varepsilon > 0$ such that if $p' \neq p$, $p'_L = p_L = 1$, and $\|p' - p\| < \varepsilon$, then $z(p') \neq 0$. Moreover, if the economy is regular, then the number of normalized equilibrium price vectors is finite.

Proof: The local uniqueness of a regular solution is a direct consequence of the inverse function theorem (see Section M.E of the Mathematical Appendix). Intuitively, this is clear enough. For any infinitesimal change in normalized prices, $dp = (dp_1, \ldots, dp_{L-1}, 0) \neq 0$, the nonsingularity of $D\hat{z}(p)$ implies that $D\hat{z}(p)\,dp \neq 0$. Hence, we cannot remain at equilibrium.

Once we know that every equilibrium is locally isolated, the finiteness of the number of equilibria is a consequence of the boundary condition (v) of Proposition 17.B.2 on the excess demand function. Because of this condition (which, recall, follows from the strong monotonicity of preferences), equilibrium is not compatible with relative prices that are arbitrarily close to zero. That is, there is an $r > 0$ such that if $\hat{z}(p) = 0$ and $p_L = 1$, then $1/r < p_\ell < r$ for every ℓ. The continuity of $\hat{z}(\cdot)$ adds to this the fact that the set of equilibrium price vectors is a closed subset of $\mathbb{R}^{L-1}$. But a set that is closed and bounded (i.e., compact) in $\mathbb{R}^{L-1}$ and discrete (i.e., with all its points locally isolated) must necessarily be finite (see Section M.F of the Mathematical Appendix). ■

Our next aim is suggested by reexamining Figure 17.D.1; we see that for a regular economy with two commodities, we can assert more than the finiteness of the number of equilibria. Indeed, the boundary conditions on the excess demand function $z_1(\cdot)$ (excess demand is positive if p_1 is very low and negative if it is very high) necessarily imply that, for a regular economy, first, there is an odd number of equilibria and, second, the slopes of the excess demand function at the equilibrium must alternate between being negative and being positive, starting with negative. If we say that an equilibrium with an associated negative slope of excess demand has an *index* of $+1$ and that one with a positive slope has an *index* of -1, then, no matter how many equilibria there are, the sum of the indices of the equilibria of a regular economy is always $+1$. With appropriate definitions, it turns out that this invariance of index property also holds in the general case with any number of commodities, where it has some important implications for comparative statics and uniqueness questions.

Let us generalize the definition of the index of a regular equilibrium that we have just suggested for the case $L = 2$ to the case of many commodities.

Definition 17.D.2: Suppose that $p = (p_1, \ldots, p_{L-1}, 1)$ is a regular equilibrium of the economy. Then we denote

$$\text{index } p = (-1)^{L-1} \text{ sign } |D\hat{z}(p)|,$$

where $|D\hat{z}(p)|$ is the determinant of the $(L-1) \times (L-1)$ matrix $D\hat{z}(p)$.[15]

If $L = 2$, then $|D\hat{z}(p)|$ is merely the slope of $z_1(\cdot)$ at p. Hence, we see that for this case the index is $+1$ or -1 according to whether the slope is negative or positive.

15. For any number $\alpha \neq 0$, sign $\alpha = +1$ or -1 according to whether $\alpha > 0$ or $\alpha < 0$.

A regular economy has a finite number of equilibria (Proposition 17.D.1). Therefore, for a regular economy, the expression

$$\sum_{\{p:\, z(p)=0,\, p_L=1\}} \text{index } p$$

makes sense. The next proposition (the *index theorem*) says that the value of this expression is always equal to $+1$.

Proposition 17.D.2: (*The Index Theorem*) For any regular economy, we have

$$\sum_{\{p:\, z(p)=0,\, p_L=1\}} \text{index } p = +1.$$

A brief discussion of why this result is true is given at the end of this section. Here we point out some of its implications and why it is useful and significant. Note, first, that it implies that the number of equilibria of a regular economy is odd.[16] In particular, this number cannot be zero; so the existence of at least one equilibrium is a particular case of the proposition. Second, the index concept provides a classification of equilibria into two types. In a sense, Proposition 17.D.2 tells us that the type with positive index is more fundamental because the presence of at least one equilibrium of positive type is unavoidable. In fact, it is typically the case that any search for well-behaved equilibria (what this means depends on the particular application) can be confined to the positive index equilibria. Third, as we shall see in Section 17.F, the index result has implications for the uniqueness and the multiplicity of equilibria. Fourth, as we shall discuss in Section 17.E, part of the importance of the index theorem is that this is all we can hope to derive without imposing (strong) additional assumptions.

We next proceed to argue that typically (or, in the usual jargon, *generically*) economies are regular. Hence, generically, the solutions to the excess demand equations are locally isolated and finite in number, and the index formula holds.[17]

Genericity Analysis

To emphasize the wide scope of the methodology to be presented, we discuss it first in terms of a general system of equations. We then specialize our discussion to the economic problem at hand and spell out its consequences for the excess demand equations.

The essence of genericity analysis rests on counting equations and unknowns. Suppose we have a system of M equations in N unknowns:

$$\begin{aligned} f_1(v_1, \ldots, v_N) &= 0, \\ &\vdots \\ f_M(v_1, \ldots, v_N) &= 0, \end{aligned} \tag{17.D.1}$$

or, more compactly, $f(v) = 0$. The normal situation should be one in which, with N unknowns and M equations, we have $N - M$ degrees of freedom available for the

16. This result was first shown by Dierker (1972).

17. For advanced treatments on the topic of this section, refer to Balasko (1988) or Mas-Colell (1985).

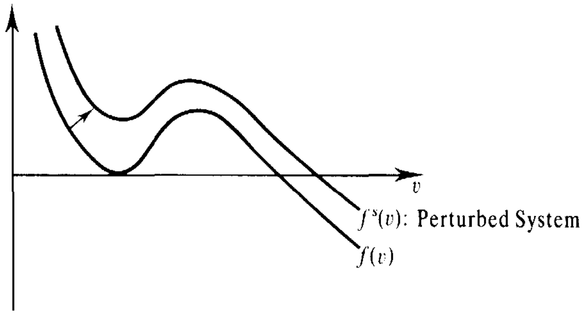

Figure 17.D.4
The regular case is typical.

description of the solution set. In particular, if $M > N$, the system should be *over-determined* and have no solution; if $M = N$, the system should be *exactly determined* with the solutions locally isolated; and if $M < N$, the system should be *under-determined* and the solutions not locally isolated. Clearly, all these statements are not always true (you can see this just by considering examples with linear equations). So, what does it mean to be in the "normal case"? The implicit function theorem provides an answer: one needs the equations (which we assume are differentiable) to be independent (that is, truly distinct) at the solutions. Definition 17.D.3 captures this notion.

Definition 17.D.3: The system of M equations in N unknowns $f(v) = 0$ is *regular* if rank $Df(v) = M$ whenever $f(v) = 0$.

For a regular system, the implicit function theorem (see Section M.E of the Mathematical Appendix) yields the existence of the right number of degrees of freedom. If $M < N$, we can choose M variables corresponding to M linearly independent columns of $Df(v)$ and we can express the values of these M variables that solve the M equations $f(v) = 0$ as a function of the $N - M$ remaining variables (see Exercise 17.D.2). If $M = N$, equilibria must be locally isolated for the same reasons as discussed earlier in this section for the system $\hat{z}(p) = 0$. And if $M > N$, then rank $Df(v) \le N < M$ for all v; in this case, Definition 17.D.3 simply says that, as a matter of definition, the equation system $f(v) = 0$ is regular if and only if the system admits no solution.

It remains to be argued that the regular case is the "normal" one. Figure 17.D.4 suggests how this can be approached. In the figure, the one-equation, one-unknown system $f(v) = 0$ is not regular [because of the tangency point of the graph of $f(\cdot)$ and the horizontal axis]. But clearly this phenomenon is not robust: if we slightly perturb the equation in an arbitrary manner [say that the shocked system is $f^s(\cdot)$], we get a regular system. On the other hand, the regularity of a system that is already regular is preserved for any small perturbation.[18]

This intuitive idea of a perturbation can be formalized as follows. Suppose there are some parameters $q = (q_1, \dots, q_S)$ such that, for every q, we have a system of equations $f(v; q) = 0$, as above. The set of possible parameter values is $\mathbb{R}^S$ (or an open region of $\mathbb{R}^S$). We can then justifiably say that $f(\cdot\,; q')$ is a perturbation of

18. The perturbation should control the values *and* the derivatives of the function. In technical language, it should be a C^1 perturbation.

$f(\cdot; q)$ if q' is close to q. Hence, the notion that the regularity of a system $f(\cdot; q) = 0$ is typical, or generic, could be captured by demanding that *for almost every* q, $f(\cdot; q) = 0$ be regular; in other words, that nonregular systems have probability zero of occurring (with respect to say, a nondegenerate normal distribution on $\mathbb{R}^S$).[19] It stands to reason that some condition will be required on the dependence of $f(\cdot; q)$ on q for this to hold. At the very least, $f(\cdot; q)$ has to actually depend on q. The important mathematical theorem to be presented next tells us that little beyond this is needed.[20]

Proposition 17.D.3: (*The Transversality Theorem*) If the $M \times (N + S)$ matrix $Df(v; q)$ has rank M whenever $f(v; q) = 0$ then for almost every q, the $M \times N$ matrix $D_v f(v; q)$ has rank M whenever $f(v; q) = 0$.

Heuristically, the assumption of the transversality theorem requires that there be enough variation in our universe. If $Df(v; q)$ has rank M whenever $f(v; q) = 0$, then from any solution it is always possible to (differentially) alter the values of the function f in any prescribed direction by adjusting the v and q variables. The conclusion of the theorem is that, if this can always be done, then whenever we are initially at a nonregular situation an arbitrary random displacement in q breaks us away from nonregularity. In fanciful language, if our universe is nondegenerate, then so will be almost every world in it. Note one of the strengths of the theorem: the matrix $Df(v; q)$ has M rows and $N + S$ columns. Hence, if S is large, so that there are many perturbation parameters, then the assumption of the theorem is likely to be satisfied; after all, we only need to find M linearly independent columns. On the other hand, $D_v f(v; q)$ has M rows but only N columns. It is thus harder to guarantee in advance that at a solution $D_v f(v; q)$ has M linearly independent columns. But the theorem tells us that this is so for almost every q. Observe that if $M > N$ (more equations than unknowns), then the $M \times N$ matrix $D_v f(v; q)$ cannot have rank M. Hence, the theorem tells us that in this case, generically (i.e., for almost every q), $f(v; q) = 0$ has no solution.

Let us now specialize our discussion to the case of a system of $L - 1$ excess demand equations in $L - 1$ unknowns, $\hat{z}(p) = 0$. We have seen by example that nonregular economies are possible. We wish to argue that they are not typical. To

19. More formally, we could say that in a system defined by finitely many parameters (taking values in, say, an open set) a property is *generic in the first sense* if it holds for a set of parameters of full measure (i.e., the complement of the set for which it holds has measure zero). The property is *generic in the second sense* if it holds in an open set of full measure. A full measure set is dense but it need not be open. Hence, the second sense is stronger than the first. Yet in many applications (all of ours in fact), the property under consideration holds in an open set, and so genericity in the first sense automatically yields genericity in the second sense. In some applications there is no finite number of parameters and no notion of measure to appeal to. In those cases we could say that a property is *generic in the third sense* if the property holds in an open and dense set. When no measure is available, this still provides a sensible way to capture the idea that the property is typical; but it should be noted that with finitely many parameters a set may be open, dense and have arbitrarily small (positive) measure. In this entire section we deal with genericity in the first sense, and we simply call it *genericity*.

20. For this theorem, we assume that $f(v; q)$ is as many times differentiable in its two arguments as is necessary.

do so, we could resort to a wide variety of perturbation parameters influencing preferences or endowments (or, in a more general setting, technologies). A natural set of parameters are the initial endowments themselves:

$$\omega = (\omega_{11}, \ldots, \omega_{L1}, \ldots, \omega_{1I}, \ldots, \omega_{LI}) \in \mathbb{R}^{LI}_{++}.$$

We can write the dependence of the economy's excess demand function on endowments explicitly as $\hat{z}(p; \omega)$. We then have Proposition 17.D.4.

Proposition 17.D.4: For any p and ω, rank $D_\omega \hat{z}(p; \omega) = L - 1$.

Proof: It suffices to consider the endowments of a single consumer, say consumer 1, and to show that the $(L-1) \times L$ matrix $D_{\omega_1}\hat{z}(p; \omega)$ has rank $L - 1$ [this implies that rank $D_\omega \hat{z}(p; \omega) = L - 1$]. To show this, we can either compute $D_{\omega_1}\hat{z}(p; \omega)$ explicitly (Exercise 17.D.3) or simply note that any perturbation of ω_1, say $d\omega_1$, that leaves the wealth of consumer 1 at prices p unaltered will not change demand and therefore will change excess demand by exactly $-d\omega_1$. Specifically, if $p \cdot d\omega_1 = 0$ then, denoting $d\hat{\omega}_1 = (d\omega_{11}, \ldots, d\omega_{L-1,1})$, we have $D_{\omega_1}\hat{z}(p; \omega)\, d\omega_1 = D_{\omega_1}\hat{z}_1(p; \omega)\, d\hat{\omega}_1 = -d\hat{\omega}_1$. Because the condition $p \cdot d\omega_1 = 0$ on $d\omega_1$ places no restrictions on $d\hat{\omega}_1$, it follows that by changing consumer 1's endowments we can move $\hat{z}(p; \omega)$ in any desired direction in $\mathbb{R}^{L-1}$, and so rank $D_{\omega_1}\hat{z}(p; \omega) = L - 1$. ■

We are now ready to state the main result [due to Debreu (1970)].

Proposition 17.D.5: For almost every vector of initial endowments $(\omega_1, \ldots, \omega_I) \in \mathbb{R}^{LI}_{++}$, the economy defined by $\{(\succsim_i, \omega_i)\}_{i=1}^{i=I}$ is regular.[21]

Proof: Because of Proposition 17.D.4, the result follows from the transversality theorem (Proposition 17.D.3). ■

See Exercises 17.D.4 to 17.D.6 for variations on the theme of Proposition 17.D.4.

In Figure 17.D.5, we represent the equilibrium set $E = \{(\omega_1, \omega_2, p_1)\colon \hat{z}(p_1, 1; \omega) = 0\}$ of an Edgeworth box economy with total endowment $\bar{\omega} = \omega_1 + \omega_2$. The set E is the graph of the correspondence that assigns equilibrium prices to economies $\omega = (\omega_1, \omega_2)$.

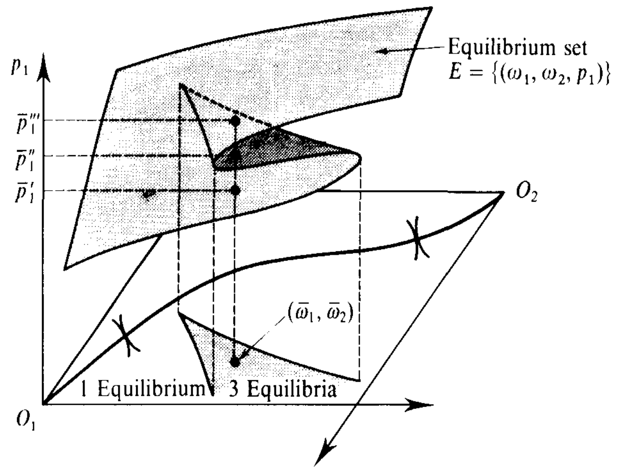

Figure 17.D.5
The equilibrium set.

21. To be quite explicit, this means that the set of endowments that yield nonregular economies is a subset of $\mathbb{R}^{LI}$ that has (LI-dimensional) Lebesgue measure zero, or, equivalently, probability zero for, say, a nondegenerate LI-dimensional normal distribution.

Because of the index theorem, this picture, in which the number of equilibria changes discontinuously from 3 to 1 at some points in the space of endowments is typical of the multiple-equilibrium case. A very extensive analysis of this equilibrium set has been carried out by Balasko (1988).

We conclude the discussion of genericity with two observations: First, the generic local determinateness of the theory extends to cases with externalities, taxes, or other "imperfections" leading to the failure of the first welfare theorem. (See Exercise 17.D.6.) This should be clear from the generality of the mathematical techniques which, in essence, rely only on the ability to express the equilibria of the theory as the zeros of a natural system of equations with the same number of equations and unknowns. Second, "finiteness of the number of equilibria" is a blunt conclusion. It is not the same if the "finite" stands for three or for a few million. Unfortunately, short of going all the way to uniqueness conditions (as we do in Section 17.F), we have no technique that allows us to refine our conclusions. We want to put on record, however, that it should not be presumed that in all generality "finite" means "small." In this respect, we mention, tentatively, that there seems to be a distinction between market equilibrium situations for which the first welfare theorem holds (in which, indeed, examples with "many" equilibria seem contrived) and situations with a variety of market failures (where examples are easy to produce). See Exercise 17.D.7 and the discussion on "sunspots" in Section 19.F.

On the Index Theorem

The index result (Proposition 17.D.2) is, in its essence, a purely mathematical fact. An attempt at a rigorous proof would take us too far afield. Nonetheless, it is instructive to give an argument for its validity. It is an argument, we note incidentally, that can be made into a rigorous proof.

Denote our given, normalized, excess demand function by $\hat{z}(p)$. We begin by availing ourselves of some other excess demand function $\hat{z}^0(p)$ with the properties that (i) there is a unique $\bar{p}$ such that $\hat{z}^0(\bar{p}) = 0$ and (ii) $\text{sign}|D\hat{z}^0(\bar{p})| = (-1)^{L-1}$. For example, $\hat{z}^0(p)$ could be generated from a single-consumer Cobb–Douglas economy (Exercise 17.D.8). The idea is that $\hat{z}^0(p)$ is both simple and familiar to us and that, as a consequence, we can use it to learn about the properties of the unfamiliar $\hat{z}(p)$.

Consider the following one-parameter family (in technical language, a *homotopy*) of excess demand functions:

$$\hat{z}(p, t) = t\hat{z}(p) + (1 - t)\hat{z}^0(p) \qquad \text{for } 0 \le t \le 1.$$

The system $\hat{z}(p, t) = 0$ has $L - 1$ equations and L unknowns: $(p_1, \ldots, p_{L-1}, t)$. Typically, therefore, the solution set $E = \{(p, t): \hat{z}(p, t) = 0\}$ has *one and only one* degree of freedom at any of its points (that is, it looks locally like a segment). Moreover, since this solution set cannot escape to infinite or zero prices (because of the boundary conditions on excess demand) and is closed [because of the continuity of $\hat{z}(p, t)$], it follows that the general situation is well represented in Figure 17.D.6.

In Figure 17.D.6, we depict E as formed, so to speak, by a finite number of circle-like and segment-like components, with the endpoints of the segments at the $t = 0$ and $t = 1$ boundaries. Since there are two endpoints per segment, there is an even number of such endpoints. By construction, $\bar{p}$ is the only endpoint at the $t = 0$ boundary.[22] Therefore, there must be an odd number of endpoints at the $t = 1$ boundary; that is, there is an odd number of solutions to $\hat{z}(p) = \hat{z}(p, 1) = 0$. Suppose now that we follow a segment from end to end. What

22. More generally, if $\hat{z}(p; t)$ is an arbitrary homotopy then the typical situation is well represented by any of the Figures 17.G.1(a), (b), or (c).

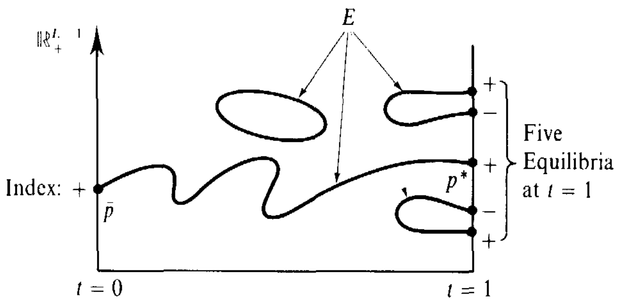

Figure 17.D.6
The equilibrium set under a homotopy.

is the relation between the indices at the two ends? A moment's reflection (keeping the implicit function theorem in mind) reveals that as long as we move in a given direction relative to t (i.e., forward or backward), the index, $(-1)^{L-1}$ sign $|D_p\hat{z}(p, t)|$, does not change, and that the index changes sign precisely when we reverse direction.[23] Now, a segment that begins and ends at the same boundary must reverse direction an odd number of times; hence, the indices at the two endpoints have opposite signs. You can verify this in Figure 17.D.6. Therefore, the sum of the indices at $t = 1$ equals the index of the lone equilibrium of $\hat{z}(\cdot)$ connected by a segment to the equilibrium $\bar{p}$ of $\hat{z}^0(\cdot)$ at the boundary $t = 0$. It is represented by p^* in Figure 17.D.6. The segment that connects $\bar{p}$ to p^* in E reverses directions an even number of times (possibly none); therefore, we conclude that the index of this equilibrium at $t = 1$ equals the index of $\bar{p}$ for $\hat{z}^0(\cdot)$, which, by construction, is $+1$. Hence, the sum of the indices at $t = 1$ is $+1$, as Proposition 17.D.2 asserts to be true in complete generality.

17.E Anything Goes: The Sonnenschein–Mantel–Debreu Theorem

We have seen that under a number of general assumptions (of which the most substantial concerns convexity), an equilibrium exists and the number of equilibria is typically finite. Those are important properties, but we would like to know if we could say more, especially for predictive or comparative-statics purposes (see Section 17.G). We may well suspect by now (especially if the message of Chapter 4 on the difficulties of demand aggregation has been well understood) that the answer is likely to be negative; that is, that, in general, we will not be able to impose further restrictions on excess demand than those in Proposition 17.B.2, and therefore that no further general restrictions on the nature of Walrasian equilibria than those already studied can be hoped for. Special assumptions will have to be made to derive stronger implications (such as uniqueness; see Section 17.F).

In this section, we confirm this and bring home the negative message in a particularly strong manner. The theme, culminating in Propositions 17.E.3 and 17.E.4, is: *Anything satisfying the few properties that we have already shown must hold, can actually occur.*

23. To see this, think of the case where $L = 2$. Applying the implicit function theorem to $\hat{z}_1(p_1, t) = 0$, verify then that a reversal of direction occurs precisely where $\partial\hat{z}_1(p_1, t)/\partial p_1 = 0$.

The analysis that follows develops the logic of this conclusion through a series of intermediate results that have independent interest. Some readers may wish, in a first reading of this section, to skip these results and examine directly the statements of Propositions 17.E.3 and 17.E.4 and the accompanying discussion of their interpretations.

To be specific, we concentrate the analysis, as usual, on exchange economies formalized by means of excess demand equations. Focusing on exchange economies makes sense because, as we know from Chapter 5, aggregation effects are unproblematic in production. The source of the aggregation problem rests squarely with the wealth effects of the consumption side.

We begin by posing a relatively simple but nonetheless quite important question: To what extent can we derive restrictions on the behavior of excess demand at a given price p. In particular, we ask for possible restrictions on the $L \times L$ matrix of price effects $Dz(p)$.[24]

Suppose that $z(p)$ is a differentiable aggregate excess demand function. In Exercise 17.E.1, you are asked to show that

$$\sum_k \frac{\partial z_\ell(p)}{\partial p_k} p_k = 0 \qquad \text{for all } \ell \text{ and } p \text{ [or } Dz(p)p = 0] \tag{17.E.1}$$

$$\sum_k p_k \frac{\partial z_k(p)}{\partial p_\ell} = -z_\ell(p) \qquad \text{for all } \ell \text{ and } p \text{ [or } p \cdot Dz(p) = -z(p)] \tag{17.E.2}$$

These are the excess demand counterparts of expressions (2.E.1) and (2.E.4) for demand functions. They follow, respectively, from the homogeneity of degree zero and the Walras' law properties of excess demand. More interestingly, from $z(p) = \sum_i (x_i(p, p \cdot \omega_i) - \omega_i)$ we also get

$$Dz(p) = \sum_i [S_i(p, p \cdot \omega_i) - D_{w_i} x_i(p, p \cdot \omega_i) z_i(p)^{\mathrm{T}}] \tag{17.E.3}$$

where, as usual, $S_i(p, p \cdot \omega_i)$ is the substitution matrix (see Exercise 17.E.2).

Expression (17.E.3) is very instructive. It tells us that if it were not for the wealth effects, $Dz(p)$ would inherit the negative semidefiniteness (n.s.d.) property of the substitution matrices. How much havoc can the wealth effects cause? Notice that the matrix

$$D_{w_i} x_i(p, p \cdot \omega_i) z_i(p)^{\mathrm{T}} = \begin{bmatrix} \dfrac{\partial x_{1i}(p, p \cdot \omega_i)}{\partial w_i} z_{1i}(p) & \cdots & \dfrac{\partial x_{1i}(p, p \cdot \omega_i)}{\partial w_i} z_{Li}(p) \\ & \ddots & \\ \dfrac{\partial x_{Li}(p, p \cdot \omega_i)}{\partial w_i} z_{1i}(p) & \cdots & \dfrac{\partial x_{Li}(p, p \cdot \omega_i)}{\partial w_i} z_{Li}(p) \end{bmatrix}$$

is of rank 1 (any two columns, or rows, are proportional). Therefore, we could informally surmise that the wealth effect of consumer i can hurt in at most one

24. Note that $z(p)$ can take any value. You need only specify a consumer with an endowment vector ω such that $\omega + z(p) \gg 0$ and then choose a utility function that has $\omega + z(p)$ as the demanded point.

direction of price change.[25] Thus, we should expect that if $I < L$ then there are some negative semidefiniteness restrictions left on $Dz(p)$. That this is the case is formalized in Proposition 17.E.1.

Proposition 17.E.1: Suppose that $I < L$. Then for any equilibrium price vector p there is some direction of price change $dp \neq 0$ such that $p \cdot dp = 0$ (hence, dp is not proportional to p) and $dp \cdot Dz(p)\, dp \leq 0$.

Proof: Because $z(p) = \sum_i z_i(p) = 0$, at most I of the $I + 1$ vectors,

$$\{p, z_1(p), \ldots, z_I(p)\} \subset \mathbb{R}^L$$

can be linearly independent. Since $I < L$, it follows that we can find a nonzero vector $dp \in \mathbb{R}^L$ such that $p \cdot dp = 0$ and $z_i(p) \cdot dp = 0$ for all i. In words: dp is a nonproportional price change that is compensated (i.e., there is no change in real wealth) for *every* consumer. But then from (17.E.3) we obtain

$$dp \cdot Dz(p)\, dp = \sum_i dp \cdot S_i(p, p \cdot \omega_i)\, dp \leq 0. \qquad \blacksquare$$

Parallel reasoning should make us expect that if $I \geq L$ (i.e., if there are at least as many consumers as commodities), then there may not be any restriction left on $Dz(p)$ beyond (17.E.1) and (17.E.2). After all, the direction of an individual wealth effect vector at a given price is quite arbitrary (and can be chosen independently of the substitution effects of the corresponding individual); and with $I \geq L$ wealth effect vectors to be specified, there is considerable room to maneuver. Proposition 17.E.2 confirms this suspicion.

Proposition 17.E.2: Given a price vector p, let $z \in \mathbb{R}^L$ be an arbitrary vector and A an arbitrary $L \times L$ matrix satisfying $p \cdot z = 0$, $Ap = 0$ and $p \cdot A = -z$. Then there is a collection of L consumers generating an aggregate excess demand function $z(\cdot)$ such that $z(p) = z$ and $Dz(p) = A$.

Proof: To keep the argument simple, we restrict ourselves to a search for consumers that at their demanded vectors have a null substitution matrix, $S_i(p, p \cdot \omega_i) = 0$, that is, whose indifference sets exhibit a vertex at the chosen point.[26]

We can always formally rewrite the given $L \times L$ matrix A as

$$A = \sum_\ell e^\ell a^\ell,$$

where e^ℓ is the ℓth unit column vector (i.e., all the entries of e^ℓ are 0 except the ℓth entry, which equals 1) and a^ℓ is the ℓth row of A [i.e., $a^\ell = (a_{\ell 1}, \ldots, a_{\ell L})$].

Suppose now that we could specify L consumers, $i = 1, \ldots, L$, with the property that, for every i, consumer i has, at the price vector p, an excess demand vector $z_i(p) = -p_i(a^i)^{\mathrm{T}}$, a wealth effect vector $D_{w_i} x_i(p, p \cdot \omega_i) = (1/p_i)e^i$, and a substitution matrix $S_i(p, p \cdot \omega_i) = 0$ (where $a^1, \ldots, a^L$ and $e^1, \ldots, e^L$ are as defined above). Then we would have both

$$z(p) = \sum_i z_i(p) = -\sum_\ell p_\ell (a^\ell)^{\mathrm{T}} = -A^{\mathrm{T}} p = -p \cdot A = z$$

25. For example, it cannot hurt in any direction of price change that is orthogonal to the wealth effects vector $D_{w_i} x_i(p, p \cdot \omega_i)$ or to the excess demand vector $z_i(p)$. A more precise argument is given in Proposition 17.E.1.

26. The term "vertex" refers to what is usually called a "kink" in the case $L = 2$.

and

$$Dz(p) = -\sum_i D_{w_i}x_i(p, p\cdot\omega_i)z_i(p)^{\mathrm{T}}$$

$$= \sum_{\ell}(1/p_\ell)e^\ell(p_\ell a^\ell) = \sum_{\ell} e^\ell a^\ell = A,$$

and so we would have accomplished our objective.

Can we find these L consumers? The answer is "yes." Begin by choosing a collection of endowments $(\omega_1, \ldots, \omega_I)$ yielding strictly positive consumptions when excess demands are $z_i(p) = -p_i(a^i)^{\mathrm{T}}$; that is, $x_i = \omega_i - p_i(a^i)^{\mathrm{T}} \gg 0$ for every i. Observe then that, for every $i = 1, \ldots, L$, the candidate individual excess demand satisfies Walras' law

$$p\cdot z_i(p) = -p_i p\cdot a^i = 0 \quad \text{(because } Ap = 0\text{)},$$

and, also, that the candidate wealth effect vector satisfies the necessary condition of Proposition 2.E.3

$$p\cdot D_{w_i}x_i(p, p\cdot\omega_i) = (1/p_i)p\cdot e^i = 1.$$

Figure 17.E.1 should then be persuasive enough in convincing us that we can assign preferences to $i = 1, \ldots, L$ in such a way that the chosen consumption at p is x_i, the wealth effect vector at p is proportional to e^i (and therefore must equal $(1/p_i)e^i$),[27] and the indifference map has a kink at x_i. The figure illustrates the complete construction for the case $L = 2$.[28] In Exercise 17.E.3, you are asked to write an explicit utility function. ■

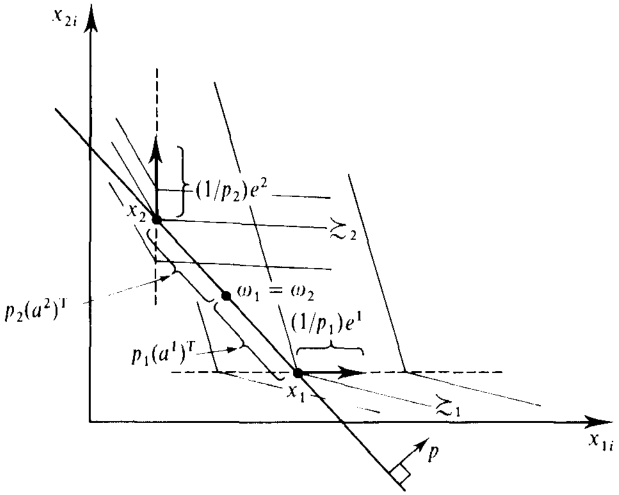

Figure 17.E.1
Decomposition of excess demand and price effects at a price vector p (for $L = 2$).

27. Indeed, if $Dx_i(p, p\cdot\omega_i) = \alpha_i e^i$, then $1 = p\cdot Dx_i(p, p\cdot\omega_i) = \alpha_i p\cdot e^i = \alpha_i p_i$. Hence, $\alpha_i = 1/p_i$.

28. At no extra cost, we could actually accomplish a bit more. We could also require the substitution matrices of the consumers $i = 1, \ldots, L$ to be any arbitrary collection of $L \times L$ matrices S_i satisfying the properties: S_i is symmetric, negative semidefinite, $p\cdot S_i = 0$, and $S_i p = 0$. The specification of consumers generating excess demand $z(p)$ and excess demand effects $Dz(p)$ at p would proceed in a manner similar to the proof just given except that the argument would now be applied to $A - \sum_i S_i$. By using matrices S_i of maximal rank (i.e., of rank $L - 1$), we could insure that the resulting L consumers display smooth indifference sets at their chosen consumptions.

Up to now, we have studied the possibility of restrictions on the behavior of excess demand at a single price vector. Although the results of Propositions 17.E.1 and 17.E.2 are already quite useful, we can go further. The essence of the negative point being made is, unfortunately, much more general. Consider an arbitrary function $z(p)$, and let us for the moment sidestep boundary issues by having $z(p)$ be defined on a domain where relative prices are bounded away from zero; that is, for a small constant $\varepsilon > 0$, we consider only price vectors p with $p_\ell/p_{\ell'} \geq \varepsilon$ for every ℓ and ℓ'. We could then ask: "Can $z(\cdot)$ coincide with the excess demand function of an economy *for every p in its domain*?" Of course, in its domain, $z(\cdot)$ must fulfill three obvious necessary conditions: it must be continuous, it must be homogeneous of degree zero, and it must satisfy Walras' law. But for any $z(\cdot)$ satisfying these three conditions, it turns out that the answer is, again, "yes."[29]

Proposition 17.E.3: Suppose that $z(\cdot)$ is a continuous function defined on

$$P_\varepsilon = \{p \in \mathbb{R}^L_+ : p_\ell/p_{\ell'} \geq \varepsilon \text{ for every } \ell \text{ and } \ell'\}$$

and with values in $\mathbb{R}^L$. Assume that, in addition, $z(\cdot)$ is homogeneous of degree zero and satisfies Walras' law. Then there is an economy of L consumers whose aggregate excess demand function coincides with $z(p)$ in the domain P_ε.[30]

Proof: At the end of this section, we offer (in small-type) a brief discussion of the general proof of this result. Here, we limit ourselves to the comparatively simple case where $L = 2$.

Suppose then that $L = 2$ and that an $\varepsilon > 0$ and a function $z(\cdot)$ satisfying the assumption of the proposition are given to us. The continuity and homogeneity of degree zero of $z(\cdot)$ imply the existence of a number $r > 0$ such that $|z_1(p)| < r$ for every $p \in P_\varepsilon$. We now specify two functions $z^1(\cdot)$ and $z^2(\cdot)$ with domain P_ε and values in $\mathbb{R}^2$, which are also continuous and homogeneous of degree zero, and satisfy Walras' law. In particular, we let

$$z_1^1(p) = \tfrac{1}{2}z_1(p) + r \qquad [\text{accordingly, } z_2^1(p) = -(p_1/p_2)z_1^1(p)]$$

and

$$z_1^2(p) = \tfrac{1}{2}z_1(p) - r \qquad [\text{accordingly, } z_2^2(p) = -(p_1/p_2)z_1^2(p)].$$

Note that $z(p) = z^1(p) + z^2(p)$ for every $p \in P_\varepsilon$. We shall show that for $i = 1, 2$ the function $z^i(\cdot)$ coincides in the domain P_ε with the excess demand function of a consumer. To this effect, we use the following properties of $z^i(\cdot)$: continuity, homogeneity of degree zero, satisfaction of Walras law, *and* the fact that there is no $p \in P_\varepsilon$ such that $z^i(p) = 0$. In Exercise 17.E.4, you are asked to show by example that this last requirement is needed.

Choose a $\omega_i \gg 0$ such that $\omega_i + z^i(p) \gg 0$ for every $p \in P_\varepsilon$. In Figure 17.E.2, we represent the offer curve OC_i associated with $z^i(\cdot)$ in the domain P_ε. In the figure, for every $p \in P_\varepsilon$,

29. The question was posed by Sonnenschein (1973). He conjectured that the answer was that, indeed, on the domain where $p_\ell \geq \varepsilon$ for all ℓ, the three properties were not only necessary but also sufficient; that is, we could always find such an economy. He also proved that this is so for the two-commodity case. The problem was then solved by Mantel (1974) for any number of commodities. Mantel made use of $2L$ consumers. Shortly afterwards, Debreu (1974) gave a different and very simple proof requiring the indispensable minimum of L consumers. This was topped by Mantel (1976), who refined his earlier proof to show that L homothetic consumers (with no restrictions in their initial endowments) would do.

30. Note, in particular, that this result implies that for any $I \geq L$, there is an economy of I consumers that generates $z(\cdot)$ on P_ε. We need only add to the L consumers identified by the proposition $I - L$ consumers who have no endowments (or, alternatively, whose most preferred consumption bundle at all price vectors in P_ε is their endowment vector).

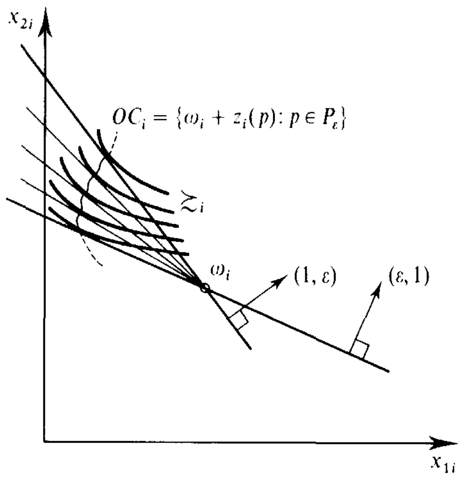

Figure 17.E.2 Construction of preferences (in the case $L = 2$) for the offer curve of an excess demand function $z_i(\cdot)$ such that $z_i(p) = 0$ has no solution with $1/\varepsilon < p_1/p_2 < \varepsilon$.

$\omega_i + z^i(p)$ is the intersection point of the offer curve with the budget line perpendicular to p. The offer curve is continuous and, because $z^i(p) = 0$ has no solution in P_ε, it does not touch the initial endowment point. We then see in the figure that no matter how complicated the offer curve may otherwise be, we can always fit an indifference map so that for any $p \in P_\varepsilon$ we generate precisely the demands $\omega_i + z^i(p)$. ■

Strictly speaking, Proposition 17.E.3 does not yet settle our original question, "Can we assert anything more about the equilibria of an economy than what we have derived in Sections 17.C and 17.D?" The problem is that Proposition 17.E.3 characterizes the behavior of excess demand away from the boundary, whereas it is the power of the boundary conditions that yields some of the restrictions we have already established: existence, (generic) finiteness, oddness, the index formula.[31] To argue that we cannot hope for more restrictions than these on the equilibrium set, we need to guarantee that if a candidate equilibrium set satisfies them, then the construction of the "explaining" economy will not add new equilibria. The result presented in Proposition 17.E.4, whose proof we omit, provides therefore the final answer to our question.[32]

Proposition 17.E.4: For any $N \geq 1$, suppose that we assign to each $n = 1, \ldots, N$ a price vector p^n, normalized to $\|p^n\| = 1$, and an $L \times L$ matrix A_n of rank $L - 1$, satisfying $A_n p^n = 0$ and $p^n \cdot A_n = 0$. Suppose that, in addition, the index formula $\sum_n (-1)^{L-1} \text{sign } |\hat{A}_n| = +1$ holds.[33] If $L = 2$, assume also that positive and negative index equilibria alternate.

Then there is an economy with L consumers such that the aggregate excess demand $z(\cdot)$ has the properties:

(i) $z(p) = 0$ for $\|p\| = 1$ if and only if $p = p^n$ for some n.
(ii) $Dz(p^n) = A_n$ for every n.

31. Note, for example, that although a candidate function $z(\cdot)$ defined on P_ε may not have any solution, we can still successfully generate it from an economy. What happens, of course, is that the equilibria of the economy (which must exist) are all outside of P_ε.

32. For this and more general results, see Mas-Colell (1977).

33. Here, $\hat{A}_n$ is the $L - 1 \times L - 1$ matrix obtained by deleting one row and corresponding column from A.

Proposition 17.E.4 tells us that for any finite collection of price vectors $\{p^1, \ldots, p^N\}$ and matrices of price effects $\{Dz(p^1), \ldots, Dz(p^N)\}$, we can find an economy with L consumers for which these price vectors are equilibrium price vectors and $\{Dz(p^1), \ldots, Dz(p^N)\}$, are the corresponding price effects at these equilibria. The result implies that to derive further restrictions on Walrasian equilibria we will need to make additional (and, as we shall see, strong) assumptions. This is the subject of the next three sections. An excellent survey for further reading on the topic of this section is Shafer and Sonnenschein (1982).

We should point out that the initial endowments of the consumers obtained by means of Propositions 17.E.2, 17.E.3 or 17.E.4 are not a priori limited in any way. If there are constraints on permissible initial endowments, the nonnegativity conditions on consumption come into play and there may, in fact, be other restrictions on the function $z(\cdot)$. For example, you are asked in Exercise 17.E.5 to verify that the excess demand vectors $z(p)$ and $z(p')$ represented in Figure 17.E.3 cannot be decomposed into individual excess demand functions generated by rational preferences if the amount of any commodity that any consumer may possess as an initial endowment is prescribed to be at most 1 and if consumptions must be nonnegative.

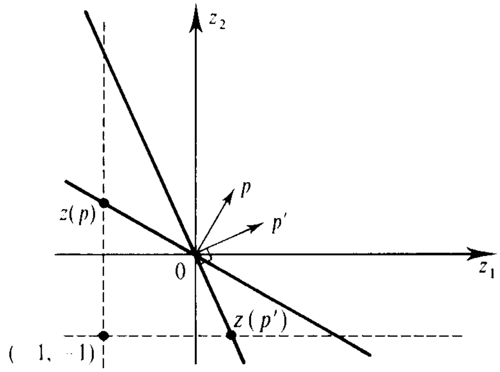

Figure 17.E.3
Excess demand choices that cannot be decomposed due to boundary constraints.

Proof of Proposition 17.E.3 continued: Although a complete proof of the proposition for the case of any number of commodities would take us too far afield, the essentials of the proof by Debreu (1974) are actually not too difficult to convey. We shall attempt to do so. We note that, when carefully examined, the proof can be seen as a generalization of the argument for the $L = 2$ case presented earlier.

In Section 3.J, we saw that the strong axiom of revealed preference (SA) for demand functions is equivalent to the existence of rationalizing preferences. The same is true for excess demand functions: If an excess demand function $z^i(\cdot)$ satisfies the SA (we will give a precise definition in a moment), then $z^i(\cdot)$ can be generated from rational preferences.[34] It is thus reasonable to redefine our problem as: Given a function $z(\cdot)$ that, on the domain P_ε, is continuous, homogeneous of degree zero, and satisfies Walras' law (for short, we refer to these functions as *excess demand functions*), can we find L excess demand functions $z^i(\cdot)$, each satisfying the SA, such that $\sum_i z^i(p) = z(p)$ for every $p \in P_\varepsilon$?

Before proceeding, let us define the SA for an excess demand function $z^i(\cdot)$. The definition is just a natural adaptation of the definition for demand functions. We say that *p is directly revealed preferred to p'* if

34. We refer to the proof of Proposition 3.J.1 for the justification of this claim.

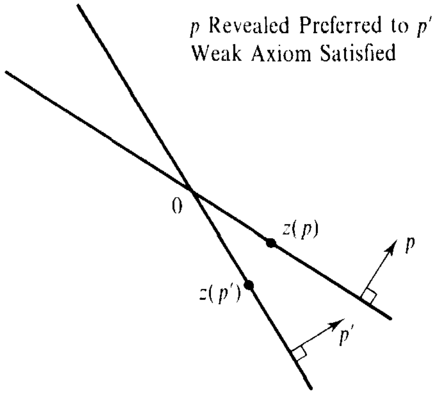

Figure 17.E.4
Revealed preference for excess demand.

$$z^i(p) \neq z^i(p') \qquad \text{and} \qquad p \cdot z^i(p') \leq 0 \tag{17.E.4}$$

(see Figure 17.E.4). We say that p *is indirectly revealed preferred to* p' if there is a finite chain $p^1, \ldots, p^N$ such that $p^1 = p$, $p^N = p'$, and p^n is directly revealed preferred to p^{n+1} for all $n \leq N - 1$. The SA then says:

For every p and p', if p is (directly or indirectly) revealed preferred to p', then p' cannot be (directly) revealed preferred to p.

From now on, we let prices be normalized. A convenient normalization here is $\|p\|^2 = p \cdot p = \sum_\ell (p_\ell)^2 = 1$.

We say that an excess demand function $z^i(\cdot)$ is *proportionally one-to-one* if $p \neq p'$ implies that $z^i(p)$ is not proportional to $z^i(p')$; in particular, we have $z^i(p) \neq z^i(p')$. For proportionally one-to-one excess demand functions (and normalized prices), we can restate the "directly revealed preferred" definition (17.E.4) as

$$p \neq p' \qquad \text{and} \qquad p \cdot z^i(p') \leq 0. \tag{17.E.4'}$$

Suppose that $\alpha_i(\cdot)$ is an arbitrary real-valued function of p such that $\alpha_i(p) > 0$ for all $p \in P_\varepsilon$. The basic observation of the proof is then the following: if $z^i(\cdot)$ *is a proportionally one-to-one excess demand function that satisfies the SA, then the same properties are true of the function* $\alpha_i(\cdot)z^i(\cdot)$. Indeed, for any p and p' the revealed preference inequalities (17.E.4') hold for $z^i(\cdot)$ *if and only if* they hold for $\alpha_i(\cdot)z^i(\cdot)$, and if $z^i(\cdot)$ is proportionally one-to-one, then so is $\alpha_i(\cdot)z^i(\cdot)$. This observation suggests a way to proceed. We could look for L proportionally one-to-one excess demand functions $z^i(\cdot)$ satisfying the SA and such that, at every $p \in P_\varepsilon$, the vectors $\{z^1(p), \ldots, z^L(p)\}$ constitute a basis capable of spanning $z(p)$ by means of a strictly positive linear combination, that is, such that for every $p \in P_\varepsilon$ we can write $z(p) = \sum_i \alpha_i(p) z^i(p)$ for some numbers $\alpha_i(p) > 0$. This is precisely what we will now do.

For every normalized $p \in P_\varepsilon$, denote $T_p = \{z \in \mathbb{R}^L : p \cdot z = 0\}$ and for every $i = 1, \ldots, L$, let $z^i(p) \in T_p$ be the point that minimizes the Euclidean distance $\|z - e^i\|$ (or, equivalently, maximizes the concave "utility function" $-\|z - e^i\|$) for $z \in T_p$, where e^i is the ith unit vector (the column vector whose ith entry is 1 with zeros elsewhere). Geometrically, $z^i(p)$ is the perpendicular projection of e^i on the budget hyperplane T_p; that is, $z^i(p) = e^i - p_i p$, where p_i is the ith component of the vector p (recall that $i \leq L$). Then $z^i(\cdot)$ is proportionally one-to-one (see Exercise 17.E.6) and satisfies the SA (since it is derived from utility maximization; see also Exercise 17.E.7).

Now let $r > 0$ be a large-enough number for us to have $z(p) + rp \gg 0$ for every normalized $p \in P_\varepsilon$ [such an r exists by the continuity of $z(\cdot)$ and the fact that the set of normalized price vectors in P_ε is compact and includes only strictly positive price vectors]. For every $i = 1, \ldots, L$ and every normalized $p \in P_\varepsilon$, define $\alpha_i(p) = z_i(p) + rp_i > 0$, where $z_i(p)$ is the ith component

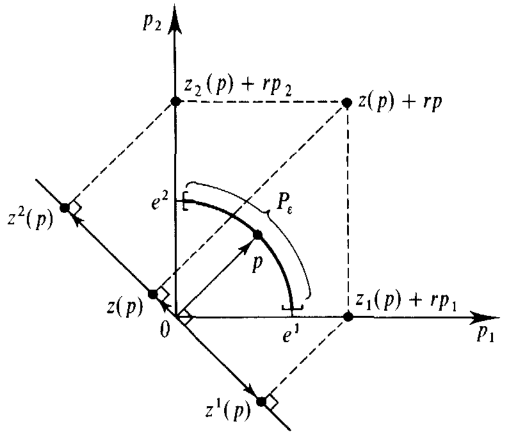

Figure 17.E.5 Illustration of the construction of individual excess demand in the proof of Proposition 17.E.3.

of the vector $z(p)$. We claim, that $\sum_i \alpha_i(p)z^i(p) = z(p)$, and this concludes the proof. Indeed,

$$\begin{aligned}\sum_i (z_i(p) + rp_i)(e^i - p_i p) &= \sum_i z_i(p)e^i + \sum_i rp_i e^i - \left(\sum_i p_i z_i(p)\right)p - r\left(\sum_i p_i^2\right)p \\ &= z(p) \qquad + rp \qquad - 0 \qquad\qquad - rp \\ &= z(p).\end{aligned}$$

Geometrically, what we are doing is projecting every $\alpha_i(p)e^i$ on the hyperplane $\{z \in \mathbb{R}^L\colon p \cdot z = 0\}$. By the definition of $\alpha_i(p)$, we have $\sum_i \alpha_i(p)e^i = z(p) + rp$. Therefore, when we project both sides, we get $\sum_i \alpha_i(p)z^i(p) = z(p)$. The construction is illustrated in Figure 17.E.5. ■

17.F Uniqueness of Equilibria

Up to this point, we have concentrated on the determination of the general properties of the Walrasian equilibrium model. We now take a different tack. We focus on a particular, important property—the uniqueness of equilibrium—and we investigate conditions, necessarily special, under which it obtains.[35]

The presentation is organized into four headings. The first contemplates a general setting with production and discusses conditions on the demand side of the economy that, by themselves (i.e., without the help of further restrictions on the production side), guarantee the uniqueness of equilibrium. The second discusses the gross substitution property, an important class of conditions with uniqueness implications for exchange economies. The third presents a limited result that relies on the Pareto optimality property of equilibrium. The fourth analyzes the role of the index formula as a source of uniqueness and nonuniqueness results.

Throughout Section 17.F, we assume that individual preferences are continuous, strictly convex, and strongly monotone.

35. Reviews for this topic are Kehoe (1985) and (1991), and Mas-Colell (1991).

The Weak Axiom for Aggregate Excess Demand

Suppose that the production side of the economy is given to us by an arbitrary technology $Y \subset \mathbb{R}^L$ of the constant returns, convex type (i.e., Y is a convex cone). What conditions involving only the demand side of the economy guarantee the uniqueness of equilibrium allocations?[36] From the analysis presented in Section 4.D, we already know one answer: if a welfare authority makes sure that wealth is always distributed so as to maximize a (strictly concave) social welfare function, then the economy admits a (strictly concave) normative representative consumer, and the equilibrium necessarily corresponds to the unique Pareto optimum of this one-consumer economy (as in Section 15.C). In our current framework, however, this is not a promising approach because wealth is derived from initial endowments and only by coincidence can we expect that the induced distribution of wealth maximizes a social welfare function. We will therefore concentrate on a weaker and, for the purpose at hand, more interesting condition: that the *weak axiom of revealed preference* holds for the aggregate excess demand of the consumers.

To begin, suppose that $z(p) = \sum_i (x_i(p, p \cdot \omega_i) - \omega_i)$ is the aggregate excess demand function of the consumers. For this economy with production, Proposition 17.F.1 provides a useful restatement of the definition of Walrasian equilibrium (Definition 17.B.1) in terms of $z(\cdot)$.

Proposition 17.F.1: Given an economy specified by the constant returns technology Y and the aggregate excess demand function of the consumers $z(\cdot)$, a price vector p is a Walrasian equilibrium price vector if and only if

(i) $p \cdot y \leq 0$ for every $y \in Y$, and
(ii) $z(p)$ is a feasible production; that is, $z(p) \in Y$.

Proof: If p is a Walrasian equilibrium price vector, then (ii) follows from market clearing and (i) is a necessary condition for profit maximization with a constant returns technology. In the other direction, if (i) and (ii) hold, then consumptions $x_i^* = x_i(p, p \cdot \omega_i)$ for $i = 1, \dots, I$, production vector $y^* = z(p) \in Y$, and price vector p constitute a Walrasian equilibrium. To verify this, the only condition that is not immediate is profit maximization. However, $y^* = z(p) \in Y$ is profit maximizing because $p^* \cdot y \leq 0$ for all $y \in Y$ [since (i) holds] and $p \cdot y^* = p \cdot z(p) = 0$ (from Walras' law). ■

We next define the weak axiom for excess demand functions.

Definition 17.F.1: (*The Weak Axiom for Excess Demand Functions*) The excess demand function $z(\cdot)$ satisfies the weak axiom of revealed preference (WA) if for any pair of price vectors p and p', we have

$$z(p) \neq z(p') \text{ and } p \cdot z(p') \leq 0 \text{ implies } p' \cdot z(p) > 0.$$

36. The set Y can be thought as an aggregate production set. The restriction that Y be of constant returns is made merely for convenience of exposition. It allows us, for example, not to worry about the distribution of profits to consumers (since profits are zero in any equilibrium). Note also that the constant returns model includes pure exchange as a special case (where $Y = -\mathbb{R}^L_+$).

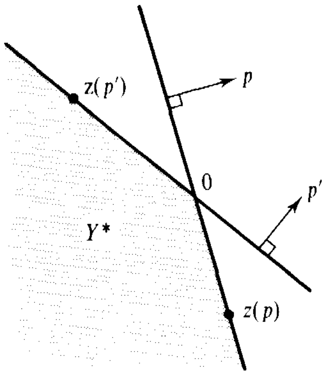

Figure 17.F.1
A violation of the weak axiom implies multiplicity of equilibria for some Y.

In words, the definition says that if p is revealed preferred to p', then p' cannot be revealed preferred to p [i.e., $z(p)$ cannot be affordable under p']. It is the same definition used in Sections 1.G and 2.F, but now applied to excess demand functions.[37] The axiom is always satisfied by the excess demand function of a single individual, but it is a strong condition for *aggregate* excess demand (see Section 4.C for a discussion of this point).

We first note that, given $z(\cdot)$, the WA is a necessary condition for us to be assured of a unique equilibrium for every possible convex, constant returns technology Y that $z(\cdot)$ is coupled with. To see this, suppose that the WA was violated; that is, suppose that for some p and p' we have $z(p) \neq z(p')$, $p \cdot z(p') \leq 0$, and $p' \cdot z(p) \leq 0$. Then we claim that both p and p' are equilibrium prices for the convex, constant returns production set given by

$$Y^* = \{y \in \mathbb{R}^L: p \cdot y \leq 0 \text{ and } p' \cdot y \leq 0\}.$$

Figure 17.F.1 depicts this production set for the case $L = 2$. Note that we have $z(p) \in Y^*$ and $p \cdot y \leq 0$ for every $y \in Y^*$. Thus, by Proposition 17.F.1, p is an equilibrium price vector. The same is true for p'. Since $z(p) \neq z(p')$, we conclude that the equilibrium is not unique for the economy formed by $z(\cdot)$ and the production set Y^*.

What about sufficiency? The weak axiom is not quite a sufficient condition for uniqueness, but Proposition 17.F.2 shows that it does guarantee that for any convex, constant returns Y, *the set of equilibrium price vectors is convex.* Although this convexity property is certainly not the same as uniqueness, it has an immediate uniqueness implication: if an economy has only a *finite* number of (normalized) price equilibria (a generic situation according to Section 17.D),[38] the equilibrium must be unique.

37. A formal, and inessential, difference is that we now define the revealed preference relation on the budget sets (i.e., on price vectors) directly rather than on the choices (i.e., on commodity vectors).

38. Although our discussion in Section 17.D focused on the case of exchange economies, its conclusions regarding generic local uniqueness and finiteness of the equilibrium set can be extended to the present production context.