Also show that an alternative way to write this test is $p^0 \cdot (x^1 - x^0) < 0$, and depict the test for the case where $L = 2$ in (x_1, x_2) space. [*Hint*: Locate the point x^0 on the set $\{x \in \mathbb{R}^L_+: u(x) = u^0\}$.]

3.I.12$^{\mathbf{B}}$ Extend the compensating and equivalent variation measures of welfare change to the case of changes in both prices and wealth, so that we change from (p^0, w^0) to (p^1, w^1). Also extend the "partial information" test developed in Section 3.I to this case.

3.J.1$^{\mathbf{C}}$ Show that when $L = 2$, $x(p, w)$ satisfies the strong axiom if and only if it satisfies the weak axiom.

3.AA.1$^{\mathbf{B}}$ Suppose that the consumption set is $X = \{x \in \mathbb{R}^2_+: x_1 + x_2 \geq 1\}$ and the utility function is $u(x) = x_2$. Represent graphically, and show (a) that the locally cheaper consumption test fails at $(p, w) = (1, 1, 1)$ and (b) that market demand is not continuous at this point. Interpet economically.

3.AA.2$^{\mathbf{C}}$ Under the conditions of Proposition 3.AA.1, show that $h(p, u)$ is upper hemicontinuous and that $e(p, u)$ is continuous (even if we replace minimum by infimum and allow $p \geq 0$). Also, assuming that $h(p, u)$ is a function, give conditions for its differentiability.

4 Aggregate Demand

4.A Introduction

For most questions in economics, the aggregate behavior of consumers is more important than the behavior of any single consumer. In this chapter, we investigate the extent to which the theory presented in Chapters 1 to 3 can be applied to *aggregate demand*, a suitably defined sum of the demands arising from all the economy's consumers. There are, in fact, a number of different properties of individual demand that we might hope would also hold in the aggregate. Which ones we are interested in at any given moment depend on the particular application at hand.

In this chapter, we ask three questions about aggregate demand:

(i) Individual demand can be expressed as a function of prices and the individual's wealth level. *When can aggregate demand be expressed as a function of prices and aggregate wealth?*

(ii) Individual demand derived from rational preferences necessarily satisfies the weak axiom of revealed preference. *When does aggregate demand satisfy the weak axiom?* More generally, when can we apply in the aggregate the demand theory developed in Chapter 2 (especially Section 2.F)?

(iii) Individual demand has welfare significance; from it, we can derive measures of welfare change for the consumer, as discussed in Section 3.I. *When does aggregate demand have welfare significance?* In particular, when do the welfare measures discussed in Section 3.I have meaning when they are computed from the aggregate demand function?

These three questions could, with a grain of salt, be called the *aggregation theories of*, respectively, *the econometrician*, *the positive theorist*, and *the welfare theorist*.

The econometrician is interested in the degree to which he can impose a simple structure on aggregate demand functions in estimation procedures. One aspect of these concerns, which we address here, is the extent to which aggregate demand can be accurately modeled as a function of only *aggregate* variables, such as aggregate (or, equivalently, average) consumer wealth. This question is important because the econometrician's data may be available only in an aggregate form.

The positive (behavioral) theorist, on the other hand, is interested in the degree to which the positive restrictions of individual demand theory apply in the aggregate. This can be significant for deriving predictions from models of market equilibrium in which aggregate demand plays a central role.[1]

The welfare theorist is interested in the normative implications of aggregate demand. He wants to use the measures of welfare change derived in Section 3.I to evaluate the welfare significance of changes in the economic environment. Ideally, he would like to treat aggregate demand as if it were generated by a "representative consumer" and use the changes in this fictional individual's welfare as a measure of aggregate welfare.

Although the conditions we identify as important for each of these aggregation questions are closely related, the questions being asked in the three cases are conceptually quite distinct. Overall, we shall see that, in all three cases, very strong restrictions will need to hold for the desired aggregation properties to obtain. We discuss these three questions, in turn, in Sections 4.B to 4.D.

Finally, Appendix A discusses the regularizing (i.e., "smoothing") effects arising from aggregation over a large number of consumers.

4.B Aggregate Demand and Aggregate Wealth

Suppose that there are I consumers with rational preference relations $\succsim_i$ and corresponding Walrasian demand functions $x_i(p, w_i)$. In general, given prices $p \in \mathbb{R}^L$ and wealth levels $(w_1, \ldots, w_I)$ for the I consumers, aggregate demand can be written as

$$x(p, w_1, \ldots, w_I) = \sum_{i=1}^{I} x_i(p, w_i).$$

Thus, aggregate demand depends not only on prices but also on the specific wealth levels of the various consumers. In this section, we ask when we are justified in writing aggregate demand in the simpler form $x(p, \sum_i w_i)$, where aggregate demand depends only on aggregate wealth $\sum_i w_i$.

For this property to hold in all generality, aggregate demand must be identical for any two distributions of the same total amount of wealth across consumers. That is, for any $(w_1, \ldots, w_I)$ and $(w'_1, \ldots, w'_I)$ such that $\sum_i w_i = \sum_i w'_i$, we must have $\sum_i x_i(p, w_i) = \sum_i x_i(p, w'_i)$.

To examine when this condition is satisfied, consider, starting from some initial distribution $(w_1, \ldots, w_I)$, a differential change in wealth $(dw_1, \ldots, dw_I) \in \mathbb{R}^I$ satisfying $\sum_i dw_i = 0$. If aggregate demand can be written as a function of aggregate wealth, then assuming differentiability of the demand functions, we must have

$$\sum_i \frac{\partial x_{\ell i}(p, w_i)}{\partial w_i} dw_i = 0 \quad \text{for every } \ell.$$

This can be true for all redistributions $(dw_1, \ldots, dw_I)$ satisfying $\sum_i dw_i = 0$ and from any initial wealth distribution $(w_1, \ldots, w_I)$ if and only if the coefficients of the different

1. The econometrician may also be interested in these questions because a priori restrictions on the properties of aggregate demand can be incorporated into his estimation procedures.

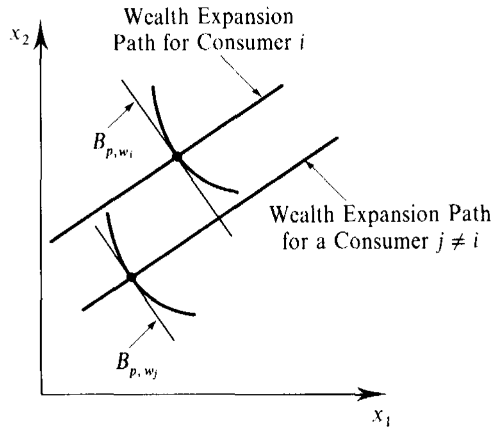


Figure 4.B.1
Invariance of aggregate demand to redistribution of wealth implies wealth expansion paths that are straight and parallel across consumers.

dw_i are equal; that is,

$$\frac{\partial x_{\ell i}(p, w_i)}{\partial w_i} = \frac{\partial x_{\ell j}(p, w_j)}{\partial w_j} \tag{4.B.1}$$

for every ℓ, any two individuals i and j, and all $(w_1, \dots, w_I)$.[2]

In short, for any fixed price vector p, and any commodity ℓ, the wealth effect at p must be the same whatever consumer we look at and whatever his level of wealth.[3] It is indeed fairly intuitive that in this case, the individual demand changes arising from any wealth redistribution across consumers will cancel out. Geometrically, the condition is equivalent to the statement that all consumers' wealth expansion paths are parallel, straight lines. Figure 4.B.1 depicts parallel, straight wealth expansion paths.

One special case in which this property holds arises when all consumers have identical preferences that are homothetic. Another is when all consumers have preferences that are quasilinear with respect to the same good. Both cases are examples of a more general result shown in Proposition 4.B.1.

Proposition 4.B.1: A necessary and sufficient condition for the set of consumers to exhibit parallel, straight wealth expansion paths at any price vector p is that preferences admit indirect utility functions of the Gorman form with the coefficients on w_i the same for every consumer i. That is:

$$v_i(p, w_i) = a_i(p) + b(p)w_i.$$

Proof: You are asked to establish sufficiency in Exercise 4.B.1 (this is not too difficult; use Roy's identity). Keep in mind that we are neglecting boundaries (alternatively, the significance of a result such as this is only local). You should not attempt to prove necessity. For a discussion of this result, see Deaton and Muellbauer (1980). ■

2. As usual, we are neglecting boundary constraints; hence, strictly speaking, the validity of our claims in this section is only local.

3. Note that $\partial x_{\ell i}(p, w_i)/\partial w_i = \partial x_{\ell i}(p, w_i')/\partial w_i$ for all $w_i \neq w_i'$ because for any values of $w_j, j \neq i$, (4.B.1) must hold for the wealth distributions $(w_1, \dots, w_{i-1}, w_i, w_{i+1}, \dots, w_I)$ and $(w_1, \dots, w_{i-1}, w_i', w_{i+1}, \dots, w_I)$. Hence, $\partial x_{\ell i}(p, w_i)/\partial w_i = \partial x_{\ell j}(p, w_j)/\partial w_j = \partial x_{\ell i}(p, w_i')/\partial w_i$ for any $j \neq i$.

Thus, aggregate demand can be written as a function of aggregate wealth if and only if all consumers have preferences that admit indirect utility functions of the Gorman form with equal wealth coefficients $b(p)$. Needless to say, this is a very restrictive condition on preferences.[4]

Given this conclusion, we might ask whether less restrictive conditions can be obtained if we consider aggregate demand functions that depend on a wider set of aggregate variables than just the total (or, equivalently, the mean) wealth level. For example, aggregate demand might be allowed to depend on both the mean and the variance of the statistical distribution of wealth or even on the whole statistical distribution itself. Note that the latter condition is still restrictive. It implies that aggregate demand depends only on how many rich and poor there are, not on who in particular is rich or poor.

These more general forms of dependence on the distribution of wealth are indeed valid under weaker conditions than those required for aggregate demand to depend only on aggregate wealth. For a trivial example, note that aggregate demand depends only on the statistical distribution of wealth whenever all consumers possess identical but otherwise arbitrary preferences and differ only in their wealth levels. We shall not pursue this topic further here; good references are Deaton and Muellbauer (1980), Lau (1982) and Jorgenson (1990).

There is another way in which we might be able to get a more positive answer to our question. So far, the test that we have applied is whether the aggregate demand function can be written as a function of aggregate wealth for *any* distribution of wealth across consumers. The requirement that this be true for every conceivable wealth distribution is a strong one. Indeed, in many situations, individual wealth levels may be generated by some underlying process that restricts the set of individual wealth levels which can arise. If so, it may still be possible to write aggregate demand as a function of prices and aggregate wealth.

For example, when we consider general equilibrium models in Part IV, individual wealth is generated by individuals' shareholdings of firms and by their ownership of given, fixed stocks of commodities. Thus, the individual levels of real wealth are determined as a function of the prevailing price vector.

Alternatively, individual wealth levels may be determined in part by various government programs that redistribute wealth across consumers (see Section 4.D). Again, these programs may limit the set of possible wealth distributions that may arise.

To see how this can help, consider an extreme case. Suppose that individual i's wealth level is generated by some process that can be described as a function of prices p and aggregate wealth w, $w_i(p, w)$. This was true, for example, in the general equilibrium illustration above. Similarly, the government program may base an individual's taxes (and hence his final wealth position) on his wage rate and the total (real) wealth of the society. We call a family of functions $(w_1(p, w), \dots, w_I(p, w))$ with $\sum_i w_i(p, w) = w$ for all (p, w) a *wealth distribution rule*. When individual wealth levels

4. Recall, however, that it includes some interesting and important classes of preferences. For example, if preferences are quasilinear with respect to good ℓ, then there is an indirect utility of the form $a_i(p) + w_i/p_\ell$, which, letting $b(p) = 1/p_\ell$, we can see is of the Gorman type with identical $b(p)$.

are generated by a wealth distribution rule, we can indeed *always* write aggregate demand as a function $x(p, w) = \sum_i x_i(p, w_i(p, w))$, and so aggregate demand depends only on prices and aggregate wealth.

4.C Aggregate Demand and the Weak Axiom

To what extent do the positive properties of individual demand carry over to the aggregate demand function $x(p, w_1, \dots, w_I) = \sum_i x_i(p, w_i)$? We can note immediately three properties that do: continuity, homogeneity of degree zero, and Walras' law [that is, $p \cdot x(p, w_1, \dots, w_I) = \sum_I w_i$ for all $(p, w_1, \dots, w_I)$]. In this section, we focus on the conditions under which aggregate demand also satisfies the weak axiom, arguably the most central positive property of the individual Walrasian demand function.

To study this question, we would like to operate on an aggregate demand written in the form $x(p, w)$, where w is aggregate wealth. This is the form for which we gave the definition of the weak axiom in Chapter 2. We accomplish this by supposing that there is a wealth distribution rule $(w_1(p, w), \dots, w_I(p, w))$ determining individual wealths from the price vector and total wealth. We refer to the end of Section 4.B for a discussion of wealth distribution rules.[5] With the wealth distribution rule at our disposal, aggregate demand can automatically be written as

$$x(p, w) = \sum_i x_i(p, w_i(p, w)).$$

Formally, therefore, the aggregate demand function $x(p, w)$ depends then only on aggregate wealth and is therefore a market demand function in the sense discussed in Chapter 2.[6] We now investigate the fulfillment of the weak axiom by $x(\cdot, \cdot)$.

In point of fact, and merely for the sake of concreteness, we shall be even more specific and focus on a particularly simple example of a distribution rule. Namely, we restrict ourselves to the case in which relative wealths of the consumers remain fixed, that is, are independent of prices. Thus, we assume that we are given wealth shares $\alpha_i \geq 0$, $\sum_i \alpha_i = 1$, so that $w_i(p, w) = \alpha_i w$ for every level $w \in \mathbb{R}$ of aggregate wealth.[7] We have then

$$x(p, w) = \sum_i x_i(p, \alpha_i w).$$

We begin by recalling from Chapter 2 the definition of the weak axiom.

Definition 4.C.1: The aggregate demand function $x(p, w)$ satisfies the weak axiom (WA) if $p \cdot x(p', w') \leq w$ and $x(p, w) \neq x(p', w')$ imply $p' \cdot x(p, w) > w'$ for any (p, w) and (p', w').

5. There is also a methodological advantage to assuming the presence of a wealth distribution rule. It avoids confounding different aggregation issues because the aggregation problem studied in Section 4.B (invariance of demand to redistributions) is then entirely assumed away.

6. Note that it assigns commodity bundles to price wealth combinations, and, provided every $w_i(\cdot, \cdot)$ is continuous and homogeneous of degree one, that it is continuous, homogeneous of degree zero, and satisfies Walras's law.

7. Observe that this distribution rule amounts to leaving the wealth levels $(w_1, \dots, w_I)$ unaltered and considering only changes in the price vector p. This is because the homogeneity of degree zero of $x(p, w_1, \dots, w_I)$ implies that any proportional change in wealths can also be captured by a proportional change in prices. The description by means of shares is, however, analytically more convenient.

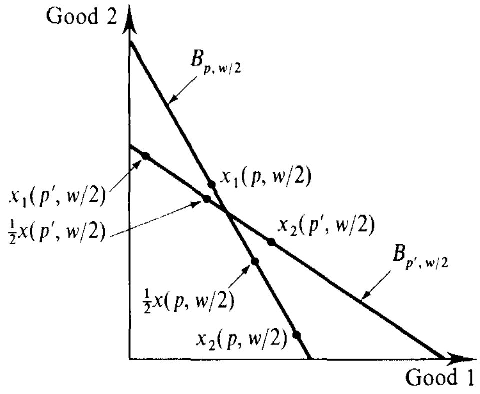


Figure 4.C.1
Failure of aggregate demand to satisfy the weak axiom.

We next provide an example illustrating that aggregate demand may not satisfy the weak axiom.

Example 4.C.1: *Failure of Aggregate Demand to Satisfy the WA.* Suppose that there are two commodities and two consumers. Wealth is distributed equally so that $w_1 = w_2 = w/2$, where w is aggregate wealth. Two price vectors p and p' with corresponding individual demands $x_1(p, w/2)$ and $x_2(p, w/2)$ under p, and $x_1(p', w/2)$ and $x_2(p', w/2)$ under p', are depicted in Figure 4.C.1.

These individual demands satisfy the weak axiom, but the aggregate demands do not. Figure 4.C.1 shows the vectors $\frac{1}{2}x(p, w)$ and $\frac{1}{2}x(p', w)$, which are equal to the average of the two consumers' demands; (and so for each price vector, they must lie at the midpoint of the line segment connecting the two individuals' consumption vectors). As illustrated in the figure, we have

$$\tfrac{1}{2}p\cdot x(p', w) < w/2 \qquad \text{and} \qquad \tfrac{1}{2}p'\cdot x(p, w) < w/2,$$

which (multiply both sides by 2) constitutes a violation of the weak axiom at the price-wealth pairs considered. ■

The reason for the failure illustrated in Example 4.C.1 can be traced to wealth effects. Recall from Chapter 2 (Proposition 2.F.1) that $x(p, w)$ satisfies the weak axiom if and only if it satisfies the law of demand for *compensated* price changes. Precisely, if and only if for any (p, w) and any price change p' that is compensated [so that $w' = p'\cdot x(p, w)$], we have

$$(p' - p)\cdot[x(p', w') - x(p, w)] \le 0, \tag{4.C.1}$$

with strict inequality if $x(p, w) \neq x(p', w')$.[8]

If the price-wealth change under consideration, say from (p, w) to (p', w'), happened to be a compensated price change for *every* consumer i—that is, if $\alpha_i w' = p'\cdot x_i(p, \alpha_i w)$ for all i—then because individual demand satisfies the weak axiom, we would know (again by Proposition 2.F.1) that for all $i = 1, \dots, I$:

$$(p' - p)\cdot[x_i(p', \alpha_i w') - x_i(p, \alpha_i w)] \le 0, \tag{4.C.2}$$

8. Note that if $p\cdot x(p', w') \le w$ and $x(p', w') \neq x(p, w)$, then we must have $p'\cdot x(p, w) > w'$, in agreement with the weak axiom.

with strict inequality if $x_i(p', \alpha_i w) \neq x_i(p, \alpha_i w')$. Adding (4.C.2) over i gives us precisely (4.C.1). Thus, we conclude that aggregate demand must satisfy the WA for any price wealth change that is compensated for every consumer.

The difficulty arises because a price–wealth change that is compensated in the aggregate, so that $w' = p' \cdot x(p, w)$, need not be compensated for each individual; we may well have $\alpha_i w' \neq p' \cdot x_i(p, \alpha_i w)$ for some or all i. If so, the individual wealth effects [which, except for the condition $p \cdot D_{w_i} x(p, \alpha_i w) = 1$, are essentially unrestricted] can play havoc with the well-behaved but possibly small individual substitution effects. The result may be that (4.C.2) fails to hold for some i, thus making possible the failure of the similar expression (4.C.1) in the aggregate.

Given that a property of individual demand as basic as the WA cannot be expected to hold generally for aggregate demand, we might wish to know whether there are any restrictions on individual preferences under which it must be satisfied. The preceding discussion suggests that it may be worth exploring the implications of assuming that the law of demand, expression (4.C.2), holds at the individual level for price changes that are left uncompensated. Suppose, indeed, that given an initial position (p, w_i), we consider a price change p' that is not compensated, namely, we leave $w_i' = w_i$. If (4.C.2) nonetheless holds, then by addition so does (4.C.1). More formally, we begin with a definition.

Definition 4.C.2: The individual demand function $x_i(p, w_i)$ satisfies the *uncompensated law of demand* (*ULD*) property if

$$(p' - p) \cdot [x_i(p', w_i) - x_i(p, w_i)] \leq 0 \tag{4.C.3}$$

for any p, p', and w_i, with strict inequality if $x_i(p', w_i) \neq x_i(p, w_i)$.
The analogous definition applies to the aggregate demand function $x(p, w)$.

In view of our discussion of the weak axiom in Section 2.F, the following differential version of the ULD property should come as no surprise (you are asked to prove it in Exercise 4.C.1):

> If $x_i(p, w_i)$ satisfies the ULD property, then $D_p x_i(p, w_i)$ is negative semidefinite; that is, $dp \cdot D_p x_i(p, w_i)\, dp \leq 0$ for all dp.

As with the weak axiom, there is a converse to this:

> If $D_p x_i(p, w_i)$ is negative definite for all p, then $x_i(p, w_i)$ satisfies the ULD property.

The analogous differential version holds for the aggregate demand function $x(p, w)$.

The great virtue of the ULD property is that, in contrast with the WA, it does, in fact, aggregate. Adding the individual condition (4.C.3) for $w_i = \alpha_i w$ gives us $(p' - p) \cdot [x(p', w) - x(p, w)] \leq 0$, with strict inequality if $x(p, w) \neq x'(p, w)$. This leads us to Proposition 4.C.1.

Proposition 4.C.1: If every consumer's Walrasian demand function $x_i(p, w_i)$ satisfies the uncompensated law of demand (ULD) property, so does the aggregate demand $x(p, w) = \sum_i x_i(p, \alpha_i w)$. As a consequence, the aggregate demand $x(p, w)$ satisfies the weak axiom.

Proof: Consider any (p, w), (p', w) with $x(p, w) \neq x(p', w)$. We must have

$$x_i(p, \alpha_i w) \neq x_i(p', \alpha_i w)$$

for some i. Therefore, adding (4.C.3) over i, we get

$$(p' - p)\cdot[x(p, w) - x(p', w)] < 0.$$

This holds for all p, p', and w.

To verify the WA, take any (p, w), (p', w') with $x(p, w) \neq x(p', w')$ and $p \cdot x(p', w') \leq w$.[9] Define $p'' = (w/w')p'$. By homogeneity of degree zero, we have $x(p'', w) = x(p', w')$. From $(p'' - p)\cdot[x(p'', w) - x(p, w)] < 0$, $p \cdot x(p'', w) \leq w$, and Walras' law, it follows that $p'' \cdot x(p, w) > w$. That is, $p' \cdot x(p, w) > w'$. ■

How restrictive is the ULD property as an axiom of individual behavior? It is clearly not implied by preference maximization (see Exercise 4.C.3). Propositions 4.C.2 and 4.C.3 provide sufficient conditions for individual demands to satisfy the ULD property.

Proposition 4.C.2: If $\succsim_i$ is homothetic, then $x_i(p, w_i)$ satisfies the uncompensated law of demand (ULD) property.

Proof: We consider the differentiable case [i.e., we assume that $x_i(p, w_i)$ is differentiable and that $\succsim_i$ is representable by a differentiable utility function]. The matrix $D_p x_i(p, w_i)$ is

$$D_p x_i(p, w_i) = S_i(p, w_i) - \frac{1}{w_i} x_i(p, w_i)\, x_i(p, w_i)^T,$$

where $S_i(p, w_i)$ is consumer i's Slutsky matrix. Because $[dp \cdot x_i(p, w_i)]^2 > 0$ except when $dp \cdot x_i(p, w_i) = 0$ and $dp \cdot S_i(p, w_i)\, dp < 0$ except when dp is proportional to p, we can conclude that $D_p x_i(p, w_i)$ is negative definite, and so the ULD condition holds. ■

In Proposition 4.C.2, the conclusion is obtained with minimal help from the substitution effects. Those could all be arbitrarily small. The wealth effects by themselves turn out to be sufficiently well behaved. Unfortunately, the homothetic case is the only one in which this is so (see Exercise 4.C.4). More generally, for the ULD property to hold, the substitution effects (which are always well behaved) must be large enough to overcome possible "perversities" coming from the wealth effects. The intriguing result in Proposition 4.3.C [due to Mitiushin and Polterovich (1978) and Milleron (1974); see Mas-Colell (1991) for an account and discussion of this result] gives a concrete expression to this relative dominance of the substitution effects.

Proposition 4.C.3: Suppose that $\succsim_i$ is defined on the consumption set $X = \mathbb{R}^L_+$ and is representable by a twice continuously differentiable concave function $u_i(\cdot)$. If

$$-\frac{x_i \cdot D^2 u_i(x_i) x_i}{x_i \cdot \nabla u_i(x_i)} < 4 \quad \text{for all } x_i,$$

then $x_i(p, w_i)$ satisfies the unrestricted law of demand (ULD) property.

9. Strictly speaking, this proof is required because although we know that the WA is equivalent to the law of demand for compensated price changes, we are now dealing with uncompensated price changes.

The proof of Proposition 4.C.3 will not be given. The courageous reader can attempt it in Exercise 4.C.5.

The condition in Proposition 4.C.3 is not an extremely stringent one. In particular, notice how amply the homothetic case fits into it (Exercise 4.C.6). So, to the question "How restrictive is the ULD property as an axiom of individual behavior?" perhaps we can answer: "restrictive, but not extremely so."[10]

Note, in addition, that for the ULD property to hold for aggregate demand, it is not necessary that the ULD be satisfied at the individual level. It may arise out of aggregation itself. The example in Proposition 4.C.4, due to Hildenbrand (1983), is not very realistic, but it is nonetheless highly suggestive.

Proposition 4.C.4: Suppose that all consumers have identical preferences $\succsim$ defined on $\mathbb{R}^L_+$ [with individual demand functions denoted $\tilde{x}(p, w)$] and that individual wealth is uniformly distributed on an interval $[0, \bar{w}]$ (strictly speaking, this requires a continuum of consumers). Then the aggregate (rigorously, the average) demand function

$$x(p) = \int_0^{\bar{w}} \tilde{x}(p, w)\, dw$$

satisfies the unrestricted law of demand (ULD) property.

Proof: Consider the differentiable case. Take $v \neq 0$. Then

$$v \cdot Dx(p)v = \int_0^{\bar{w}} v \cdot D_p\tilde{x}(p, w)v\, dw.$$

Also

$$D_p\tilde{x}(p, w) = S(p, w) - D_w\tilde{x}(p, w)\tilde{x}(p, w)^T,$$

where $S(p, w)$ is the Slutsky matrix of the individual demand function $x(\cdot, \cdot)$ at (p, w). Hence,

$$v \cdot Dx(p)v = \int_0^{\bar{w}} v \cdot S(p, w)v\, dw - \int_0^{\bar{w}} (v \cdot D_w\tilde{x}(p, w))(v \cdot \tilde{x}(p, w))\, dw.$$

The first term of this sum is negative, unless v is proportional to p. For the second, note that

$$2(v \cdot D_w\tilde{x}(p, w))(v \cdot \tilde{x}(p, w)) = \frac{d(v \cdot \tilde{x}(p, w))^2}{dw}.$$

So

$$-\int_0^{\bar{w}} (v \cdot D_w\tilde{x}(p, w))(v \cdot \tilde{x}(p, w))\, dw = -\frac{1}{2}\int_0^{\bar{w}} \frac{d(v \cdot \tilde{x}(p, w))^2}{dw}\, dw = -\frac{1}{2}(v \cdot \tilde{x}(p, \bar{w}))^2 \leq 0,$$

where we have used $\tilde{x}(p, 0) = 0$. Observe that the sign is negative when v is proportional to p. ■

Recall that the ULD property is additive across groups of consumers. Therefore, what we need in order to apply Proposition 4.C.4 is, not that preferences be identical, but that for every preference relation, the distribution of wealth conditional on that preference be uniform over

10. Not to misrepresent the import of this claim, we should emphasize that Proposition 4.C.1, which asserts that the ULD property is preserved under addition, holds for the price-independent distribution rules that we are considering in this section. When the distribution of real wealth may depend on prices (as it typically will in the general equilibrium applications of Part IV), then aggregate demand may violate the WA even if individual demand satisfies the ULD property (see Exercise 4.C.13). We discuss this point further in Section 17.F.

some interval that includes the level 0 (in fact, a nonincreasing density function is enough; see Exercise 4.C.7).

One lesson of Proposition 4.C.4 is that the properties of aggregate demand will depend on how preferences and wealth are distributed. We could therefore pose the problem quite generally and ask which distributional conditions on preferences and wealth will lead to satisfaction of the weak axiom by aggregate demand.[11]

As mentioned in Section 2.F, a market demand function $x(p, w)$ can be shown to satisfy the WA if for all (p, w), the Slutsky matrix $S(p, w)$ derived from the function $x(p, w)$ satisfies $dp \cdot S(p, w)\, dp < 0$ for every $dp \neq 0$ not proportional to p. We now examine when this property might hold for the aggregate demand function.

The Slutsky equation for the aggregate demand function is

$$S(p, w) = D_p x(p, w) + D_w x(p, w)\, x(p, w)^{\mathrm{T}}. \tag{4.C.4}$$

Or, since $x(p, w) = \sum_i x_i(p, \alpha_i w)$,

$$S(p, w) = D_p x(p, w) + [\textstyle\sum_i \alpha_i D_{w_i} x_i(p, \alpha_i w)]\, x(p, w)^{\mathrm{T}} \tag{4.C.5}$$

Next, let $S_i(p, w_i)$ denote the individual Slutsky matrices. Adding the individual Slutsky equations gives

$$\textstyle\sum_i S_i(p, \alpha_i w) = \sum_i D_p x_i(p, \alpha_i w) + \sum_i D_{w_i} x_i(p, \alpha_i w)\, x_i(p, \alpha_i w)^{\mathrm{T}} \tag{4.C.6}$$

Since $D_p x(p, w) = \sum_i D_p x_i(p, \alpha_i w)$, we can substitute (4.C.6) into (4.C.5) to get

$$S(p, w) = \textstyle\sum_i S_i(p, w_i) - \sum_i \alpha_i [D_{w_i} x_i(p, \alpha_i w) - D_w x(p, w)] \left[\dfrac{1}{\alpha_i} x_i(p, \alpha_i w) - x(p, w)\right]^{\mathrm{T}}. \tag{4.C.7}$$

Note that because of wealth effects, the Slutsky matrix of aggregate demand is *not* the sum of the individual Slutsky matrices. The difference

$$\begin{aligned} C(p, w) &= \textstyle\sum_i S_i(p, \alpha w) - S(p, w) \\ &= \textstyle\sum_i \alpha_i [D_{w_i} x_i(p, \alpha_i w) - D_w x(p, w)] \left[\dfrac{1}{\alpha_i} x_i(p, \alpha_i w) - x(p, w)\right]^{\mathrm{T}} \end{aligned} \tag{4.C.8}$$

is a covariance matrix between wealth effect vectors $D_{w_i} x_i(p, \alpha_i w)$ and proportionately adjusted consumption vectors $(1/\alpha_i) x_i(p, \alpha_i w)$. The former measures how the marginal dollar is spent across commodities; the latter measures the same thing for the average dollar [e.g., $(1/\alpha_i w) x_{\ell i}(p, \alpha_i w)$ is the per-unit-of-wealth consumption of good ℓ by consumer i]. Every "observation" receives weight α_i. Note also that, as it should be, we have

$$\textstyle\sum_i \alpha_i [D_{w_i} x_i(p, \alpha_i w) - D_w x(p, w)] = 0 \quad \text{and} \quad \sum_i \alpha_i [(1/\alpha_i) x_i(p, \alpha_i w) - x(p, w)] = 0.$$

For an individual Slutsky matrix $S_i(\cdot, \cdot)$ we always have $dp \cdot S_i(p, \alpha_i w)\, dp < 0$ for $dp \neq 0$ not proportional to p. Hence, a *sufficient* condition for the Slutsky matrix of aggregate demand to have the desired property is that $C(p, w)$ be positive semidefinite. Speaking loosely, this will be the case if, on average, there is a *positive association* across consumers between consumption (per unit of wealth) in one commodity and the wealth effect for that commodity.

Figure 4.C.2(a) depicts a case for $L = 2$ in which, assuming a uniform distribution of wealth across consumers, this association is positive: Consumers with higher-than-average

11. In the next few paragraphs, we follow Jerison (1982) and Freixas and Mas-Colell (1987).

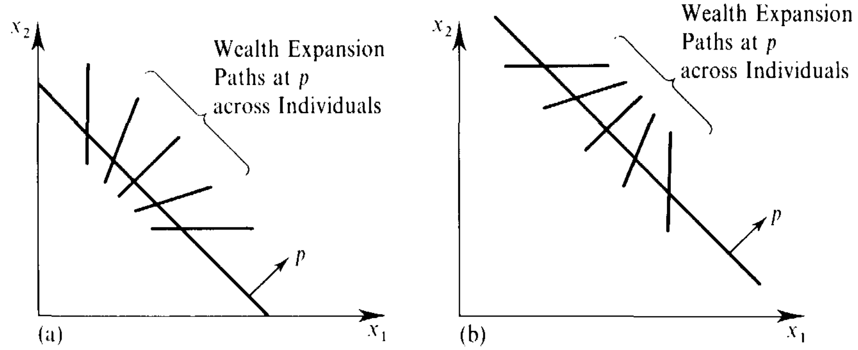


Figure 4.C.2
The relation across consumers between expenditure per unit of wealth on a commodity and its wealth effect when all consumers have the same wealth.
(a) Positive relation.
(b) Negative relation.

consumption of one good spend a higher-than-average fraction of their last unit of wealth on that good. The association is negative in Figure 4.C.2(b).[12,13]

From the preceding derivation, we can see that aggregate demand satisfies the WA in two cases of interest: (i) All the $D_{w_i}x_i(p, \alpha_i w)$ are equal (there are equal wealth effects), and (ii) all the $(1/\alpha_i)x_i(p, \alpha_i w)$ are equal (there is proportional consumption). In both cases, we have $C(p, w) = 0$, and so $dp \cdot S(p, w)\, dp < 0$ whenever $dp \neq 0$ is not proportional to p.

Case (i) has important implications. In particular, if every consumer has indirect utility functions of the Gorman form $v_i(p, w_i) = a_i(p) + b(p)w$, with the coefficient $b(p)$ identical across consumers, then (as we saw in Section 4.B) the wealth effects are the same for all consumers and we can therefore conclude that the WA is satisfied. We know from Section 4.B that one is led to this family of indirect utility functions by the requirement that aggregate demand be invariant to redistribution of wealth. Thus, aggregate demand satisfying the weak axiom for a fixed distribution of wealth is a less demanding property than the invariance to redistribution property considered in Section 4.B. In particular, if the second property holds, then the first also holds, but aggregate demand (for a fixed distribution of wealth) may satisfy the weak axiom even though aggregate demand may not be invariant to redistribution of wealth (e.g., individual preferences may be homothetic but not identical).

Having spent all this time investigating the weak axiom (WA), you might ask: "What about the strong axiom (SA)?" We have not focused on the Strong Axiom for three reasons.

First, the WA is a robust property, whereas the SA (which, remember, yields the symmetry of the Slutsky matrix) is not; a priori, the chances of it being satisfied by a real economy are essentially zero. For example, if we start with a group of consumers with identical preferences and wealth, then aggregate demand obviously satisfies the SA. However, if we now perturb every preference slightly and independently across consumers, the negative semidefiniteness of the Slutsky matrices (and therefore the WA) may well be preserved but the symmetry (and therefore the SA) will almost certainly not be.

12. You may want to verify that the wealth expansion paths of Example 4.C.1 must indeed look like Figure 4.C.2(b).

13. A priori, we cannot say which form is more likely. Because the demand at zero wealth is zero, it is true that for a consumer, *some* dollar must be spent among the two goods according to shares similar to the shares of the average dollar. But if the levels of wealth are not close to zero, it does not follow that this is the case for the *marginal* dollar. It may even happen that because of incipient satiation, the shares of the marginal dollar display consumption propensities that are the reverse of the ones exhibited by the average dollar. See Hildenbrand (1994) for an account of empirical research on this matter.

Second, many of the strong positive results of general equilibrium (to be reviewed in Part IV, especially Chapters 15 and 17) to which one wishes to apply the aggregation theory discussed in this chapter depend on the weak axiom, not on the strong axiom, holding in the aggregate.

Third, while one might initially think that the existence of a preference relation explaining aggregate behavior (which is what we get from the SA) would be the condition required to use aggregate demand measures (such as aggregate consumer surplus) as welfare indicators, we will see in Section 4.D that, in fact, more than this condition is required anyway.

4.D Aggregate Demand and the Existence of a Representative Consumer

The aggregation question we pose in this section is: When can we compute meaningful measures of aggregate welfare using the aggregate demand function and the welfare measurement techniques discussed in Section 3.I for individual consumers? More specifically, when can we treat the aggregate demand function as if it were generated by a fictional *representative consumer* whose preferences can be used as a measure of aggregate societal (or *social*) welfare?

We take as our starting point a distribution rule $(w_1(p, w), \dots, w_I(p, w))$ that to every level of aggregate wealth $w \in \mathbb{R}$ assigns individual wealths. We assume that $\sum_i w_i(p, w) = w$ for all (p, w) and that every $w_i(\cdot, \cdot)$ is continuous and homogeneous of degree one. As discussed in Sections 4.B and 4.C, aggregate demand then takes the form of a conventional market demand function $x(p, w) = \sum_i x_i(p, w_i(p, w))$. In particular, $x(p, w)$ is continuous, is homogeneous of degree zero, and satisfies Walras' law. It is important to keep in mind that the aggregate demand function $x(p, w)$ depends on the wealth distribution rule (except under the special conditions identified in Section 4.B).

It is useful to begin by distinguishing two senses in which we could say that there is a representative consumer. The first is a positive, or behavioral, sense.

Definition 4.D.1: A *positive representative consumer* exists if there is a rational preference relation $\succsim$ on $\mathbb{R}^L_+$ such that the aggregate demand function $x(p, w)$ is precisely the Walrasian demand function generated by this preference relation. That is, $x(p, w) \succ x$ whenever $x \neq x(p, w)$ and $p \cdot x \leq w$.

A positive representative consumer can thus be thought of as a fictional individual whose utility maximization problem when facing society's budget set $\{x \in \mathbb{R}^L_+ : p \cdot x \leq w\}$ would generate the economy's aggregate demand function.

For it to be correct to treat aggregate demand as we did individual demand functions in Section 3.I, there must be a positive representative consumer.[14] However, although this is a necessary condition for the property of aggregate demand that we seek, it is not sufficient. We also need to be able to assign welfare significance to this

14. Note that if there is a positive representative consumer, then aggregate demand satisfies the positive properties sought in Section 4.C. Indeed, not only will aggregate demand satisfy the weak axiom, but it will also satisfy the strong axiom. Thus, the aggregation property we are after in this section is stronger than the one discussed in Section 4.C.

fictional individual's demand function. This will lead to the definition of a *normative* representative consumer. To do so, however, we first have to be more specific about what we mean by the term *social welfare*. We accomplish this by introducing the concept of a *social welfare function*, a function that provides a summary (social) utility index for any collection of individual utilities.

Definition 4.D.2: A *(Bergson-Samuelson) social welfare function* is a function $W: \mathbb{R}^I \to \mathbb{R}$ that assigns a utility value to each possible vector $(u_1, \ldots, u_I) \in \mathbb{R}^I$ of utility levels for the I consumers in the economy.

The idea behind a social welfare function $W(u_1, \ldots, u_I)$ is that it accurately expresses society's judgments on how individual utilities have to be compared to produce an ordering of possible social outcomes. (We do not discuss in this section the issue of where this social preference ranking comes from. Chapters 21 and 22 cover this point in much more detail.) We also assume that social welfare functions are increasing, concave, and whenever convenient, differentiable.

Let us now hypothesize that there is a process, a benevolent central authority perhaps, that, for any given prices p and aggregate wealth level w, redistributes wealth in order to maximize social welfare. That is, for any (p, w), the wealth distribution $(w_1(p, w), \ldots, w_I(p, w))$ solves

$$\begin{aligned} \underset{w_1, \ldots, w_I}{\text{Max}} \quad & W(v_1(p, w_1), \ldots, v_I(p, w_I)) \\ \text{s.t.} \quad & \textstyle\sum_{i=1}^{I} w_i \leq w, \end{aligned} \tag{4.D.1}$$

where $v_i(p, w)$ is consumer i's indirect utility function.[15,16] The optimum value of problem (4.D.1) defines a social indirect utility function $v(p, w)$. Proposition 4.D.1 shows that this indirect utility function provides a positive representative consumer for the aggregate demand function $x(p, w) = \sum_i x_i(p, w_i(p, w))$.

Proposition 4.D.1: Suppose that for each level of prices p and aggregate wealth w, the wealth distribution $(w_1(p, w), \ldots, w_I(p, w))$ solves problem (4.D.1). Then the value function $v(p, w)$ of problem (4.D.1) is an indirect utility function of a positive representative consumer for the aggregate demand function $x(p, w) = \sum_i x_i(p, w_i(p, w))$.

Proof: In Exercise 4.D.2, you are asked to establish that $v(p, w)$ does indeed have the properties of an indirect utility function. The argument for the proof then consists of using Roy's identity to derive a Walrasian demand function from $v(p, w)$, which we denote by $x_R(p, w)$, and then establishing that it actually equals $x(p, w)$.

We begin by recording the first-order conditions of problem (4.D.1) for a

15. We assume in this section that our direct utility functions $u_i(\cdot)$ are concave. This is a weak hypothesis (once quasiconcavity has been assumed) which makes sure that in all the optimization problems to be considered, the first-order conditions are sufficient for the determination of global optima. In particular, $v_i(p, \cdot)$ is then a concave function of w_i.

16. In Exercise 4.D.1, you are asked to show that if so desired, problem (4.D.1) can be equivalently formulated as one where social utility is maximized, not by distributing wealth, but by distributing bundles of goods with aggregate value at prices p not larger than w. The fact that in optimally redistributing goods, we can also restrict ourselves to redistributing wealth is, in essence, a version of the second fundamental theorem of welfare economics, which will be covered extensively in Chapter 16.

given value of (p, w). Neglecting boundary solutions, these require that for some $\lambda \geq 0$, we have

$$\lambda = \frac{\partial W}{\partial v_1}\frac{\partial v_1}{\partial w_1} = \cdots = \frac{\partial W}{\partial v_I}\frac{\partial v_I}{\partial w_I} \tag{4.D.2}$$

(For notational convenience, we have omitted the points at which the derivatives are evaluated.) Condition (4.D.2) simply says that at a socially optimal wealth distribution, the social utility of an extra unit of wealth is the same irrespective of who gets it.

By Roy's identity, we have $x_R(p, w) = -[1/(\partial v(p, w)/\partial w)]\, \nabla_p v(p, w)$. Since $v(p, w)$ is the value function of problem (4.D.1), we know that $\partial v/\partial w = \lambda$. (See Section M.K of the Mathematical Appendix) In addition, for any commodity ℓ, the chain rule and (4.D.2)—or, equivalently, the envelope theorem—give us

$$\frac{\partial v}{\partial p_\ell} = \sum_i \frac{\partial W}{\partial v_i}\frac{\partial v_i}{\partial p_\ell} + \lambda \sum_i \frac{\partial w_i}{\partial p_\ell} = \sum_i \frac{\partial W}{\partial v_i}\frac{\partial v_i}{\partial p_\ell},$$

where the second equality follows because $\sum_i w_i(p, w) = w$ for all (p, w) implies that $\sum_i (\partial w_i/\partial p_\ell) = 0$. Hence, in matrix notation, we have

$$\nabla_p v(p, w) = \textstyle\sum_i (\partial W/\partial v_i)\, \nabla_p v_i(p, w_i(p, w)).$$

Finally, using Roy's identity and the first-order condition (4.D.2), we get

$$\begin{aligned} x_R(p, w) &= -\frac{1}{\lambda}\textstyle\sum_i \left[\frac{\lambda}{\partial v_i/\partial w_i}\right] \nabla_p v_i(p, w_i(p, w)) \\ &= -\textstyle\sum_i \left[\frac{1}{\partial v_i/\partial w_i}\right] \nabla_p v_i(p, w_i(p, w)) \\ &= \textstyle\sum_i x_i(p, w_i(p, w)) = x(p, w), \end{aligned}$$

as we wanted to show. ■

Equipped with Proposition 4.D.1, we can now define a *normative representative consumer*.

Definition 4.D.3: The positive representative consumer $\succsim$ for the aggregate demand $x(p, w) = \sum_i x_i(p, w_i(p, w))$ is a *normative representative consumer* relative to the social welfare function $W(\cdot)$ if for every (p, w), the distribution of wealth $(w_1(p, w), \ldots, w_I(p, w))$ solves problems (4.D.1) and, therefore, the value function of problem (4.D.1) is an indirect utility function for $\succsim$.

If there is a normative representative consumer, the preferences of this consumer have welfare significance and the aggregate demand function $x(p, w)$ can be used to make welfare judgments by means of the techniques described in Section 3.I. In doing so, however, it should never be forgotten that a given wealth distribution rule [the one that solves (4.D.1) for the given social welfare function] is being adhered to and that the "level of wealth" should always be understood as the "optimally distributed level of wealth." For further discussion, see Samuelson (1956) and Chipman and Moore (1979).

Example 4.D.1: Suppose that consumers all have homothetic preferences represented by utility functions homogeneous of degree one. Consider now the social welfare function $W(u_1, \ldots, u_I) = \sum_i \alpha_i \ln u_i$ with $\alpha_i > 0$ and $\sum_i \alpha_i = 1$. Then the optimal wealth distribution function [for problem (4.D.1)] is the price-independent rule that we adopted in Section 4.C: $w_i(p, w) = \alpha_i w$. (You are asked to demonstrate this fact in Exercise 4.D.6.) Therefore, in the homothetic case, the aggregate demand $x(p, w) = \sum_i x_i(p, \alpha_i w)$ can be viewed as originating from the normative representative consumer generated by this social welfare function. ■

Example 4.D.2: Suppose that all consumers' preferences have indirect utilities of the Gorman form $v_i(p, w_i) = a_i(p) + b(p)w_i$. Note that $b(p)$ does not depend on i, and recall that this includes as a particular case the situation in which preferences are quasilinear with respect to a common numeraire. From Section 4.B, we also know that aggregate demand $x(p, w)$ is independent of the distribution of wealth.[17]

Consider now the *utilitarian* social welfare function $\sum_i u_i$. Then *any* wealth distribution rule $(w_1(p, w), \dots, w_I(p, w))$ solves the optimization problem (4.D.1), and the indirect utility function that this problem generates is simply $v(p, w) = \sum_i a_i(p) + b(p)w$. (You are asked to show these facts in Exercise 4.D.7.) One conclusion is, therefore, that when indirect utility functions have the Gorman form [with common $b(p)$] and the social welfare function is utilitarian, then aggregate demand can *always* by viewed as being generated by a normative representative consumer.

When consumers have Gorman-form indirect utility functions [with common $b(p)$], the theory of the normative representative consumer admits an important strengthening. In general, the preferences of the representative consumer depend on the form of the social welfare function. *But not in this case.* We now verify that if the indirect utility functions of the consumers have the Gorman form [with common $b(p)$], then the preferences of the representative consumer are independent of the particular social welfare function used.[18] In fact, we show that $v(p, w) = \sum_i a_i(p) + b(p)w$ is an admissible indirect utility function for the normative representative consumer relative to *any* social welfare function $W(u_1, \dots, u_I)$.

To verify this claim, consider a particular social welfare function $W(\cdot)$, and denote the value function of problem (4.D.1), relative to $W(\cdot)$, by $v^*(p, w)$. We must show that the ordering induced by $v(\cdot)$ and $v^*(\cdot)$ is the same, that is, that for any pair (p, w) and (p', w') with $v(p, w) < v(p', w')$, we have $v^*(p, w) < v^*(p', w')$. Take the vectors of individual wealths $(w_1, \dots, w_I)$ and $(w'_1, \dots, w'_I)$ reached as optima of (4.D.1), relative to $W(\cdot)$, for (p, w) and (p', w'), respectively. Denote $u_i = a_i(p) + b(p)w_i$, $u'_i = a_i(p) + b(p)w'_i$, $u = (u_1, \dots, u_I)$, and $u' = (u'_1, \dots, u'_I)$. Then $v^*(p, w) = W(u)$ and $v^*(p', w') = W(u')$. Also $v(p, w) = \sum_i a_i(p) + b(p)w = \sum_i u_i$, and similarly, $v(p', w') = \sum_i u'_i$. Therefore, $v(p, w) < v(p', w')$ implies $\sum_i u_i < \sum_i u'_i$. We argue that $\nabla W(u') \cdot (u - u') < 0$, which, $W(\cdot)$ being concave, implies the desired result, namely $W(u) < W(u')$.[19] By expression (4.D.2), at an optimum we have $(\partial W/\partial v_i)(\partial v_i/\partial w_i) = \lambda$ for all i. But in our case, $\partial v_i/\partial w_i = b(p)$ for all i. Therefore, $\partial W/\partial v_i = \partial W/\partial v_j > 0$ for any i, j. Hence, $\sum_i u_i < \sum_i u'_i$ implies $\nabla W(u') \cdot (u - u') < 0$.

The previous point can perhaps be better understood if we observe that when

17. As usual, we neglect the nonnegativity constraints on consumption.

18. But, of course, the optimal distribution rules will typically depend on the social welfare function. Only for the utilitarian social welfare function will it not matter how wealth is distributed.

19. Indeed, concavity of $W(\cdot)$ implies $W(u') + \nabla W(u') \cdot (u - u') \geq W(u)$; see Section M.C of the Mathematical Appendix.

preferences have the Gorman form [with common $b(p)$], then (p', w') is socially better than (p, w) for the utilitarian social welfare function $\sum_i u_i$ *if and only if* when compared with (p, w), (p', w') passes the following *potential compensation test*: For any distribution $(w_1, \ldots, w_I)$ of w, there is a distribution $(w'_1, \ldots, w'_I)$ of w' such that $v_i(p', w'_i) > v_i(p, w_i)$ for all i. To verify this is straightforward. Suppose that

$$(\textstyle\sum_i a_i(p') + b(p')w') - (\textstyle\sum_i a_i(p) + b(p)w) = c > 0.$$

Then the wealth levels w'_i implicitly defined by $a_i(p') + b(p')w'_i = a_i(p) + b(p)w_i + c/I$ will be as desired.[20] Once we know that (p', w') when compared with (p, w) passes the potential compensation test, it follows merely from the definition of the optimization problem (4.D.1) that (p', w') is better than (p, w) for any normative consumer, that is, for any social welfare function that we may wish to employ (see Exercise 4.D.8).

The two properties just presented—independence of the representative consumer's preferences from the social welfare function and the potential compensation criterion will be discussed further in Sections 10.F and 22.C. For the moment, we simply emphasize that they are not general properties of normative representative consumers. By choosing the distribution rules that solve (4.D.1), we can generate a normative representative consumer for any set of individual utilities and any social welfare function. For the properties just reviewed to hold, the individual preferences have been required to have the Gorman form [with common $b(p)$]. ■

It is important to stress the distinction between the concepts of a positive and a normative representative consumer. It is *not* true that whenever aggregate demand can be generated by a positive representative consumer, this representative consumer's preferences have normative content. It may even be the case that a positive representative consumer exists but that there is *no* social welfare function that leads to a normative representative consumer. We expand on this point in the next few paragraphs [see also Dow and Werlang (1988) and Jerison (1994)].

We are given a distribution rule $(w_1(p, w), \ldots, w_I(p, w))$ and assume that a positive representative consumer with utility function $u(x)$ exists for the aggregate demand $x(p, w) = \sum_i x_i(p, w_i(p, w))$. In principle, using the integrability techniques presented in Section 3.H, it should be possible to determine the preferences of the representative consumer from the knowledge of $x(p, w)$. Now fix any $(\bar{p}, \bar{w})$, and let $\bar{x} = x(\bar{p}, \bar{w})$. Relative to the aggregate consumption vector $\bar{x}$, we can define an at-least-as-good-as set for the representative consumer:

$$B = \{x \in \mathbb{R}^L_+ : u(x) \geq (\bar{x})\} \subset \mathbb{R}^L_+.$$

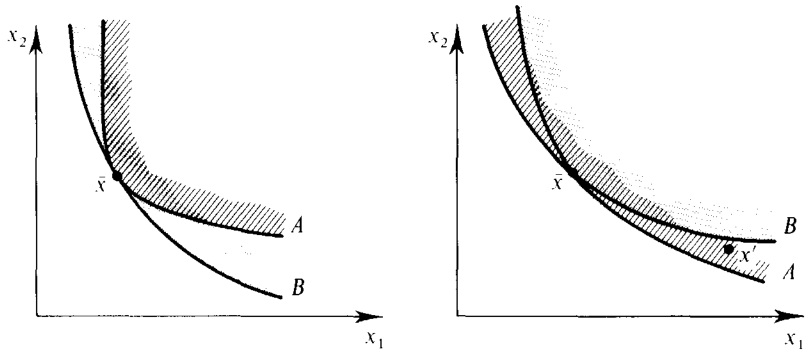
Next, let $\bar{w}_i = w_i(\bar{p}, \bar{w})$ and $\bar{x}_i = x_i(\bar{p}, \bar{w}_i)$, and consider the set

$$A = \{x = \textstyle\sum_i x_i : x_i \succsim_i \bar{x}_i \text{ for all } i\} \subset \mathbb{R}^L_+.$$

In words, A is the set of aggregate consumption vectors for which there is a distribution of commodities among consumers that makes every consumer as well off as under $(\bar{x}_1, \ldots, \bar{x}_I)$. The boundary of this set is sometimes called a *Scitovsky contour*. Note that both set A and set B are supported by the price vector $\bar{p}$ at $\bar{x}$ (see Figure 4.D.1).

If the given wealth distribution comes from the solution to a social welfare optimization problem of the type (4.D.1) (i.e., if the positive representative consumer is in fact a normative

20. We continue to neglect nonnegativity constraints on wealth.



(a) (b)

$A = \{x \in \Sigma_i x_i: u_i(x_i) \geq u_i(\bar{x}_i) \text{ for all } i\}$
$B = \{x \in \mathbb{R}^2_+: u(x) \geq u(\bar{x})\}$

Figure 4.D.1
Comparing the at-least-as-good-as set of the positive representative consumer with the sum of the at-least-as-good-as sets of the individual consumers.
(a) The positive representative consumer could be a normative representative consumer.
(b) The positive representative consumer cannot be a normative representative consumer.

representative consumer), then this places an important restriction on how sets A and B relate to each other: Every element of set A must be an element of set B. This is so because the social welfare function underlying the normative representative consumer is increasing in the utility level of every consumer (and thus any aggregate consumption bundle that could be distributed in a manner that guarantees to every consumer a level of utility as high as the levels corresponding to the optimal distribution of $\bar{x}$ must receive a social utility higher than the latter; see Exercise 4.D.4). That is, a *necessary* condition for the existence of a normative representative consumer is that $A \subset B$. A case that satisfies this necessary condition is depicted in Figure 4.D.1(a).

However, there is nothing to prevent the existence, in a particular setting, of a positive representative consumer with a utility function $u(x)$ that fails to satisfy this condition, as in Figure 4.D.1(b). To provide some further understanding of this point, Exercise 4.D.9 asks you to show that $A \subset B$ implies that $\sum_i S_i(\bar{p}, \bar{w}_i) - S(\bar{p}, \bar{w})$ is positive semidefinite, where $S(p, w)$ and $S_i(p, w_i)$ are the Slutsky matrices of aggregate and individual demand, respectively. Informally, we could say that the substitution effects of aggregate demand must be larger in absolute value than the sum of individual substitution effects (geometrically, this corresponds to the boundary of B being flatter at $\bar{x}$ than the boundary of A). This observation allows us to generate in a simple manner examples in which aggregate demand can be rationalized by preferences but, nonetheless, there is no normative representative consumer.

Suppose, for example, that the wealth distribution rule is of the form $w_i(p, w) = \alpha_i w$. Suppose also that $S(p, w)$ happens to be symmetric for all (p, w); if $L = 2$, this is automatically satisfied. Then, from integrability theory (see Section 3.H), we know that a sufficient condition for the existence of underlying preferences is that, for all (p, w), we have $dp \cdot S(p, w)\, dp < 0$ for all $dp \neq 0$ not proportional to p (we abbreviate this as the *n.d. property*). On the other hand, as we have just seen, a necessary condition for the existence of a normative representative consumer is that $C(\bar{p}, \bar{w}) = \sum_i S_i(p, w_i) - S(\bar{p}, \bar{w})$ be positive semidefinite [this is the same matrix discussed in Section 4.C; see expression (4.C.8)]. Thus, if $S(p, w)$ has the n.d. property for all (p, w) but $C(\bar{p}, \bar{w})$ is not positive semidefinite [i.e., wealth effects are such that $S(\bar{p}, \bar{w})$ is "less negative" than $\sum_i S_i(\bar{p}, \bar{w})$], then a positive representative consumer exists that, nonetheless, cannot be made normative for any social welfare function. (Exercise 4.D.10 provides an instance where this is indeed the case.) In any example of this nature we have moves in aggregate consumption that would pass a potential compensation test (each consumer's welfare could be made better off by an appropriate distribution of the move) but are regarded as socially inferior under the utility function that rationalizes aggregate demand. [In Figure 4.D.1(b), this could be the move from $\bar{x}$ to x'.]

The moral of all this is clear: The existence of preferences that explain behavior is not enough to attach to them any welfare significance. For the latter, it is also necessary that these preferences exist for the right reasons. ■

APPENDIX A: REGULARIZING EFFECTS OF AGGREGATION

This appendix is devoted to making the point that although aggregation can be deleterious to the preservation of the good properties of individual demand, it can also have helpful *regularizing* effects. By regularizing, we mean that the average (per-consumer) demand will tend to be more continuous or smooth, as a function of prices, than the individual components of the sum.

Recall that if preferences are strictly convex, individual demand functions are continuous. As we noted, aggregate demand will then be continuous as well. But average demand can be (nearly) continuous even when individual demands are not. The key requirement is one of *dispersion* of individual preferences.

Example 4.AA.1: Suppose that there are two commodities. Consumers have quasilinear preferences with the second good as numeraire. The first good, on the other hand, is available only in integer amounts, and consumers have no wish for more than one unit of it. Thus, normalizing the utility of zero units of the first good to be zero, the preferences of consumer i are completely described by a number v_{1i}, the utility in terms of numeraire of holding one unit of the first good. It is then clear that the demand for the first good by consumer i is given by the correspondence

$$\begin{aligned} x_{1i}(p_1) &= 1 && \text{if } p_1 < v_{1i}, \\ &= \{0, 1\} && \text{if } p_1 = v_{1i}, \\ &= 0 && \text{if } p_1 > v_{1i}, \end{aligned}$$

which is depicted in Figure 4.AA.1(a). Thus, individual demand exhibits a sudden, discontinuous jump in demand from 0 to 1 as the price crosses the value $p_1 = v_{1i}$.

Suppose now that there are many consumers. In fact, consider the limit situation where there is an actual continuum of consumers. We could then say that individual preferences are *dispersed* if there is no concentrated group of consumers having any particular value of v_1 or, more precisely, if the statistical distribution function of the v_1's, $G(v_1)$, is continuous. Then, denoting by $x_1(p_1)$ the average demand for the first good, we have $x_1(p_1) =$ "mass of consumers with $v_1 > p_1$" $= 1 - G(p_1)$.

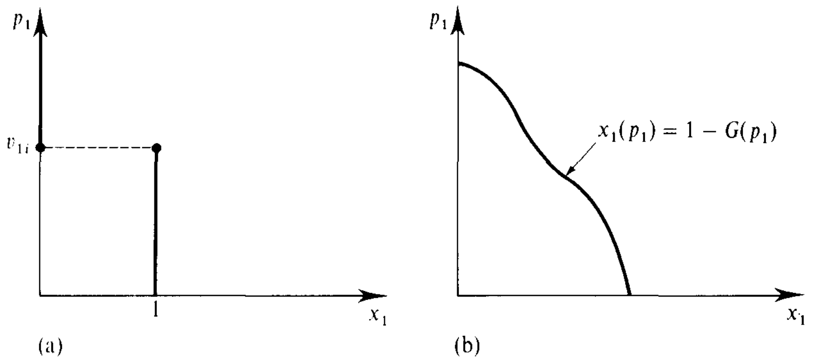


Figure 4.AA.1
The regularizing effect of aggregation.
(a) Individual demand.
(b) Aggregate demand when the distribution of the v_1's is $G(\cdot)$.

Hence, the aggregate demand $x_1(\cdot)$, shown in Figure 4.AA.1(b), is a nice continuous function even though none of the individual demand correspondences are so. Note that with only a finite number of consumers, the distribution function $G(\cdot)$ cannot quite be a continuous function; but if the consumers are many, then it can be nearly continuous. ∎

The regularizing effects of aggregation are studied again in Section 17.I. We show there that in general (i.e., without dispersedness requirements), the aggregation of numerous individual demand correspondences will generate a (nearly) *convex-valued* average demand correspondence.

REFERENCES

Chipman, J. S., and J. Moore. (1979). On social welfare functions and the aggregation of preferences. *Journal of Economic Theory* **21**: 111–39.

Deaton, A., and J. Muellbauer. (1980). *Economics and Consumer Behavior*. Cambridge, UK: Cambridge University Press.

Dow, J., and S. Werlang. (1988). The consistency of welfare judgments with a representative consumer. *Journal of Economic Theory* **44**: 265–80.

Freixas, X., and A. Mas-Colell. (1987). Engel curves leading to the weak axiom in the aggregate. *Econometrica* **21**: 63–80.

Hildenbrand, W. (1983). On the "law of demand." *Econometrica* **51**: 997–1020.

Hildenbrand, W. (1994). *Market Demand: Theory and Empirical Evidence*. Princeton: Princeton University Press.

Jerison, M. (1982). The representative consumer and the weak axiom when the distribution of income is fixed. Working paper, Department of Economics, SUNY Albany.

Jerison, M. (1994). Optimal income distribution rules and representative consumers. *Review of Economic Studies* **61**: 739–71.

Jorgenson, D. (1990). Aggregate consumer behavior and the measurement of social welfare *Econometrica* **58**: 1007–1030.

Lau, L. (1982). A note on the fundamental theorem of exact aggregation, *Economic letters* **9**: 119–26.

Mas-Colell, A. (1991). On the uniqueness of equilibrium once again. Chap. 12 in *Equilibrium Theory and Applications*, edited by W. Barnett, B. Cornet, C. d'Aspremont, J. Gabszewicz, A. Mas-Colell. Cambridge, U.K.: Cambridge University Press.

Milleron, J. C. (1974). Unicité et stabilité de l'équilibre en économie de distribution. Seminaire d'Econométrie Roy-Malinvaud, preprint.

Mitiushin, L. G., and W. M. Polterovich (1978). Criteria for monotonicity of demand functions [in Russian]. *Ekonomika i Matematichskie Metody* **14**: 122–28.

Samuelson, P. (1956). Social indifference curves. *Quarterly Journal of Economics* **70**: 1–22.

EXERCISES

4.B.1[B] Prove the sufficiency part of Proposition 4.B.1. Show also that if preferences admit Gorman-form indirect utility functions with the same $b(p)$, then preferences admit expenditure functions of the form $e_i(p, u_i) = c(p)u_i + d_i(p)$.

4.B.2[B] Suppose that there are I consumers and L commodities. Consumers differ only by their wealth levels w_i and by a taste parameter s_i, which we could call *family size*. Thus, denote the indirect utility function of consumer i by $v(p, w_i, s_i)$. The corresponding Walrasian demand function for consumer i is $x(p, w_i, s_i)$.

(a) Fix $(s_1, \ldots, s_I)$. Show that if for any $(w_1, \ldots, w_I)$ aggregate demand can be written as a function of only p and aggregate wealth $w = \sum_i w_i$ (or, equivalently, average wealth), and if every consumer's preference relationship $\succsim_i$ is homothetic, then all these preferences must be identical [and so $x(p, w_i, s_i)$ must be independent of s_i].

(b) Give a sufficient condition for aggregate demand to depend only on aggregate wealth w and $\sum_i s_i$ (or, equivalently, average wealth and average family size).

4.C.1^{C} Prove that if $x_i(p, w_i)$ satisfies the ULD, then $D_p x_i(p, w_i)$ is negative semidefinite [i.e., $dp \cdot D_p x_i(p, w_i)\, dp \leq 0$ for all dp]. Also show that if $D_p x_i(p, w_i)$ is negative definite for all p, then $x_i(p, w_i)$ satisfies the ULD (this second part is harder).

4.C.2^{A} Prove a version of Proposition 4.C.1 by using the (sufficient) differential versions of the ULD and the WA. (Recall from the small type part of Section 2.F that a sufficient condition for the WA is that $v \cdot S(p, w)v < 0$ whenever v is not proportional to p.)

4.C.3^{A} Give a graphical two-commodity example of a preference relation generating a Walrasian demand that does not satisfy the ULD property. Interpret.

4.C.4^{C} Show that if the preference relation $\succsim_i$ on $\mathbb{R}^2_+$ has L-shaped indifference curves and the demand function $x_i(p, w_i)$ has the ULD property, then $\succsim_i$ must be homothetic. [*Hint*: The L shape of indifference curves implies $S_i(p, w_i) = 0$ for all (p, w_i); show that if $D_{w_i} x_i(\bar{p}, \bar{w}_i) \neq (1/\bar{w}_i) x_i(\bar{p}, \bar{w}_i)$, then there is $v \in \mathbb{R}^L$ such that $v \cdot D_p x_i(\bar{p}, \bar{w}_i) v > 0$.]

4.C.5^{C} Prove Proposition 4.C.3. To that effect, you can fix $w = 1$. The proof is best done in terms of the indirect demand function $g_i(x) = (1/x \cdot \nabla u_i(x))\, \nabla u_i(x)$ [note that $x = x_i(g_i(x), 1)$]. For an individual consumer, the ULD is self-dual; that is, it is equivalent to $(g_i(x) - g_i(y)) \cdot (x - y) < 0$ for all $x \neq y$. In turn, this property is implied by the negative definiteness of $Dg_i(x)$ for all x. Hence, concentrate on proving this last property. More specifically, let $v \neq 0$, and denote $q = \nabla u_i(x)$ and $C = D^2 u_i(x)$. You want to prove $v \cdot Dg_i(x) v < 0$. [*Hint*: You can first assume $q \cdot v = q \cdot x$; then differentiate $g_i(x)$, and make use of the equality $v \cdot Cv - x \cdot Cv = (v - \frac{1}{2}x) \cdot C(v - \frac{1}{2}x) - \frac{1}{4} x \cdot Cx$.]

4.C.6^{A} Show that if $u_i(x_i)$ is homogeneous of degree one, so that $\succsim_i$ is homothetic, then $\sigma_i(x_i) = 0$ for all x_i [$\sigma_i(x_i)$ is the quotient defined in Proposition 4.C.3].

4.C.7^{B} Show that Proposition 4.C.4 still holds if the distribution of wealth has a nonincreasing density function on $[0, \bar{w}]$. A more realistic distribution of wealth would be *unimodal* (i.e., an increasing and then decreasing density function with a single peak). Argue that there are unimodal distributions for which the conclusions of the proposition do not hold.

4.C.8^{A} Derive expression (4.C.7), the aggregate version of the Slutsky matrix.

4.C.9^{A} Verify that if individual preferences $\succsim_i$ are homothetic, then the matrix $C(p, w)$ defined in expression (4.C.8) is positive semidefinite.

4.C.10^{C} Argue that for the Hildenbrand example studied in Proposition 4.C.4, $C(p, w)$ is positive semidefinite. Conclude that aggregate demand satisfies the WA for that wealth distribution. [Note: You must first adapt the definition of $C(p, w)$ to the continuum-of-consumers situation of the example.]

4.C.11^{B} Suppose there are two consumers, 1 and 2, with utility functions over two goods, 1 and 2, of $u_1(x_{11}, x_{21}) = x_{11} + 4\sqrt{x_{21}}$ and $u_2(x_{12}, x_{22}) = 4\sqrt{x_{12}} + x_{22}$. The two consumers have identical wealth levels $w_1 = w_2 = w/2$.

(a) Calculate the individual demand functions and the aggregate demand function.

(b) Compute the individual Slutsky matrices $S_i(p, w/2)$ (for $i = 1, 2$) and the aggregate

Slutsky matrix $S(p, w)$. [*Hint*: Note that for this two-good example, only one element of each matrix must be computed to determine the entire matrix.] Show that $dp \cdot S(p, w)\, dp < 0$ for all $dp \neq 0$ not proportional to p. Conclude that aggregate demand satisfies the *WA*.

(c) Compute the matrix $C(p, w) = \sum_i S_i(p, w/2) - S(p, w)$ for prices $p_1 = p_2 = 1$. Show that it is positive semidefinite if $w > 16$ and that it is negative semidefinite if $8 < w < 16$. In fact, argue that in the latter case, $dp \cdot C(p, w)\, dp < 0$ for some dp [so that $C(p, w)$ is *not* positive semidefinite]. Conclude that $C(p, w)$ positive semidefinite is not necessary for the WA to be satisfied.

(d) For each of the two cases $w > 16$ and $8 < w < 16$, draw a picture in the (x_1, x_2) plane depicting each consumer's consumption bundle and his wealth expansion path for the prices $p_1 = p_2 = 1$. Compare your picture with Figure 4.C.2.

4.C.12[B] The results presented in Sections 4.B and 4.C indicate that if for any $(w_1, \ldots, w_I)$ aggregate demand can be written as a function of only aggregate wealth [i.e., as $x(p, \sum_i w_i)$], then aggregate demand must satisfy the WA. The *distribution function* $F\colon [0, \infty) \to [0, 1]$ of $(w_1, \ldots, w_I)$ is defined as $F(w) = (1/I)(\textit{number of } i\textit{'s with } w_i \leq w)$ for any w. Suppose now that for any $(w_1, \ldots, w_I)$, aggregate demand can be written as a function of the corresponding aggregate *distribution* $F(\cdot)$ of wealth. Show that aggregate demand does not necessarily satisfy the WA. [*Hint*: It suffices to give a two-commodity, two-consumer example where preferences are identical, wealths are $w_1 = 1$ and $w_2 = 3$, and the WA fails. Try to construct the example graphically. It is a matter of making sure that four suitably positioned indifference curves can be fitted together without crossing.]

4.C.13[C] Consider a two-good environment with two consumers. Let the wealth distribution rule be $w_1(p, w) = wp_1/(p_1 + p_2)$, $w_2(p, w) = wp_2/(p_1 + p_2)$. Exhibit an example in which the two consumers have homothetic preferences but, nonetheless, the aggregate demand fails to satisfy the weak axiom. A good picture will suffice. Why does not Proposition 4.C.1 apply?

4.D.1[B] In this question we are concerned with a normative representative consumer. Denote by $v(p, w)$ the optimal value of problem (4.D.1), and by $(w_1(p, w), \ldots, w_I(p, w))$ the corresponding optimal wealth distribution rules. Verify that $v(p, w)$ is also the optimal value of

$$\begin{aligned} \operatorname*{Max}_{x_1, \ldots, x_I} \quad & W(u_1(x_1), \ldots, u_I(x_I)) \\ \text{s.t.} \quad & p \cdot \left(\textstyle\sum_i x_i\right) \leq w \end{aligned}$$

and that $[x_1(p, w_1(p, w)), \ldots, x_I(p, w_I(p, w))]$ is a solution to this latter problem. Note the implication that to maximize social welfare given prices p and wealth w, the planner need not control consumption directly, but rather need only distribute wealth optimally and allow consumers to make consumption decisions independently given prices p.

4.D.2[B] Verify that $v(p, w)$, defined as the optimal value of problem (4.D.1), has the properties of an indirect utility function (i.e., that it is homogeneous of degree zero, increasing in w and decreasing in p, and quasiconvex).

4.D.3[B] It is good to train one's hand in the use of inequalities and the Kuhn–Tucker conditions. Prove Proposition 4.D.1 again, this time allowing for corner solutions.

4.D.4[C] Suppose that there is a normative representative consumer with wealth distribution rule $(w_1(p, w), \ldots, w_I(p, w))$. For any $x \in \mathbb{R}^L_+$, define

$$\begin{aligned} u(x) = \operatorname*{Max}_{(x_1, \ldots, x_I)} \quad & W(u_1(x_1), \ldots, u_I(x_I)) \\ \text{s.t.} \quad & \textstyle\sum_i x_i \leq x. \end{aligned}$$

(a) Give conditions implying that $u(\cdot)$ has the properties of a utility function; that is, it is monotone, continuous, and quasiconcave (and even concave).

(b) Show that for any (p, w), the Walrasian demand generated from the problem $\text{Max}_x\ u(x)$ s.t. $p \cdot x \leq w$ is equal to the aggregate demand function.

4.D.5^{A} Suppose that there are I consumers and that consumer i's utility function is $u_i(x_i)$, with demand function $x_i(p, w_i)$. Consumer i's wealth w_i is generated according to a wealth distribution rule $w_i = \alpha_i w$, where $\alpha_i \geq 0$ and $\sum_i \alpha_i = 1$. Provide an example (i.e., a set of utility functions) in which this economy does *not* admit a positive representative consumer.

4.D.6^{B} Establish the claims made in Example 4.D.1.

4.D.7^{B} Establish the claims made in the second paragraph of Example 4.D.2.

4.D.8^{A} Say that (p', w') passes the *potential compensation test* over (p, w) if for any distribution $(w_1, \ldots, w_I)$ of w there is a distribution $(w'_1, \ldots, w'_I)$ of w' such that $v_i(p', w'_i) > v_i(p, w_i)$ for all i. Show that if $(p'w')$ passes the potential compensation test over (p, w), any normative representative consumer must prefer (p', w') over (p, w).

4.D.9^{B} Show that $A \subset B$ (notation as in Section 4.D) implies that $S(\bar{p}, \bar{w}) - \sum_i S_i(\bar{p}, \bar{w}_i)$ is negative semidefinite. [*Hint*: Consider $g(p) = e(p, u(\bar{x})) - \sum_i e_i(p, u_i(\bar{x}_i))$, where $e(\cdot)$ is the expenditure function for $u(\cdot)$ and $e_i(\cdot)$ is the expenditure function for $u_i(\cdot)$. Note that $A = \sum_i \{x_i : u_i(x_i) \geq u_i(\bar{x}_i)\}$ implies that $\sum_i e_i(p, u_i(\bar{x}_i))$ is the optimal value of the problem $\text{Min}_{x \in A}\ p \cdot x$. From this and $A \subset B$, you get $g(p) \leq 0$ for all p and $g(\bar{p}) = 0$. Therefore, $D^2 g(\bar{p})$ is negative semidefinite. Show then that $D^2 g(\bar{p}) = S(\bar{p}, \bar{w}) - \sum_i S_i(\bar{p}, \bar{w}_i)$.]

4.D.10^{A} Argue that in the example considered in Exercise 4.C.11, there is a positive representative consumer rationalizing aggregate demand but that there cannot be a normative representative consumer.

4.D.11^{C} Argue that for $L > 2$, the Hildenbrand case of Proposition 4.C.4 need not admit a positive representative consumer. [*Hint*: Argue that the Slutsky matrix may fail to be symmetric.]

CHAPTER

5 Production

5.A Introduction

In this chapter, we move to the supply side of the economy, studying the process by which the goods and services consumed by individuals are produced. We view the supply side as composed of a number of productive units, or, as we shall call them, "firms." Firms may be corporations or other legally recognized businesses. But they must also represent the productive possibilities of individuals or households. Moreover, the set of all firms may include some potential productive units that are never actually organized. Thus, the theory will be able to accommodate both active production processes and potential but inactive ones.

Many aspects enter a full description of a firm: Who owns it? Who manages it? How is it managed? How is it organized? What can it do? Of all these questions, we concentrate on the last one. Our justification is not that the other questions are not interesting (indeed, they are), but that we want to arrive as quickly as possible at a minimal conceptual apparatus that allows us to analyze market behavior. Thus, our model of production possibilities is going to be very parsimonious: The firm is viewed merely as a "black box", able to transform inputs into outputs.

In Section 5.B, we begin by introducing the firm's *production set*, a set that represents the production *activities*, or *production plans*, that are technologically feasible for the firm. We then enumerate and discuss some commonly assumed properties of production sets, introducing concepts such as *returns to scale*, *free disposal*, and *free entry*.

After studying the firm's technological possibilities in Section 5.B, we introduce its objective, the goal of *profit maximization*, in Section 5.C. We then formulate and study the firm's profit maximization problem and two associated objects, the firm's *profit function* and its *supply correspondence*. These are, respectively, the value function and the optimizing vectors of the firm's profit maximization problem. Related to the firm's goal of profit maximization is the task of achieving cost-minimizing production. We also study the firm's cost minimization problem and two objects associated with it: The firm's *cost function* and its *conditional factor demand correspondence*. As with the utility maximization and expenditure minimization problems in the theory of demand, there is a rich duality theory associated with the profit maximization and cost minimization problems.

Section 5.D analyzes in detail the geometry associated with cost and production relationships for the special but theoretically important case of a technology that produces a single output.

Aggregation theory is studied in Section 5.E. We show that aggregation on the supply side is simpler and more powerful than the corresponding theory for demand covered in Chapter 4.

Section 5.F constitutes an excursion into welfare economics. We define the concept of *efficient production* and study its relation to profit maximization. With some minor qualifications, we see that profit-maximizing production plans are efficient and that when suitable convexity properties hold, the converse is also true: An efficient plan is profit maximizing for an appropriately chosen vector of prices. This constitutes our first look at the important ideas of the *fundamental theorems of welfare economics.*

In Section 5.G, we point out that profit maximization does not have the same primitive status as preference maximization. Rigorously, it should be derived from the latter. We discuss this point and related issues.

In Appendix A, we study in more detail a particular, important case of production technologies: Those describable by means of linear constraints. It is known as the *linear activity model.*

5.B Production Sets

As in the previous chapters, we consider an economy with L commodities. A *production vector* (also known as an *input–output*, or *netput*, vector, or as a *production plan*) is a vector $y = (y_1, \dots, y_L) \in \mathbb{R}^L$ that describes the (net) outputs of the L commodities from a production process. We adopt the convention that positive numbers denote outputs and negative numbers denote inputs. Some elements of a production vector may be zero; this just means that the process has no net output of that commodity.

Example 5.B.1: Suppose that $L = 5$. Then $y = (-5, 2, -6, 3, 0)$ means that 2 and 3 units of goods 2 and 4, respectively, are produced, while 5 and 6 units of goods 1 and 3, respectively, are used. Good 5 is neither produced nor used as an input in this production vector. ■

To analyze the behavior of the firm, we need to start by identifying those production vectors that are technologically possible. The set of all production vectors that constitute feasible plans for the firm is known as the *production set* and is denoted by $Y \subset \mathbb{R}^L$. Any $y \in Y$ is possible; any $y \notin Y$ is not. The production set is taken as a primitive datum of the theory.

The set of feasible production plans is limited first and foremost by technological constraints. However, in any particular model, legal restrictions or prior contractual commitments may also contribute to the determination of the production set.

It is sometimes convenient to describe the production set Y using a function $F(\cdot)$, called the *transformation function.* The transformation function $F(\cdot)$ has the property that $Y = \{y \in \mathbb{R}^L\colon F(y) \le 0\}$ and $F(y) = 0$ if and only if y is an element of the boundary of Y. The set of boundary points of Y, $\{y \in \mathbb{R}^L\colon F(y) = 0\}$, is known as the *transformation frontier.* Figure 5.B.1 presents a two-good example.

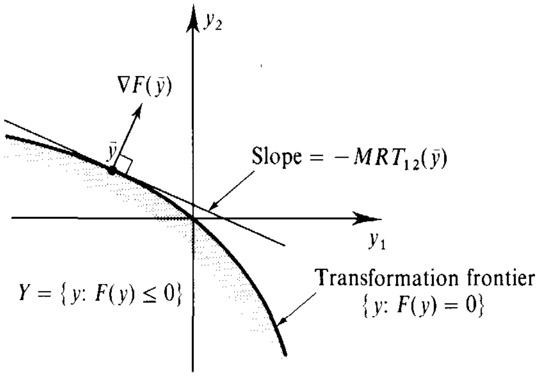


Figure 5.B.1
The production set and transformation frontier.

If $F(\cdot)$ is differentiable, and if the production vector $\bar{y}$ satisfies $F(\bar{y}) = 0$, then for any commodities ℓ and k, the ratio

$$MRT_{\ell k}(\bar{y}) = \frac{\partial F(\bar{y})/\partial y_\ell}{\partial F(\bar{y})/\partial y_k}$$

is called the *marginal rate of transformation* (MRT) *of good ℓ for good k at $\bar{y}$*.[1] The marginal rate of transformation is a measure of how much the (net) output of good k can increase if the firm decreases the (net) output of good ℓ by one marginal unit. Indeed, from $F(\bar{y}) = 0$, we get

$$\frac{\partial F(\bar{y})}{\partial y_k} dy_k + \frac{\partial F(\bar{y})}{\partial y_\ell} dy_\ell = 0,$$

and therefore the slope of the transformation frontier at $\bar{y}$ in Figure 5.B.1 is precisely $-MRT_{12}(\bar{y})$.

Technologies with Distinct Inputs and Outputs

In many actual production processes, the set of goods that can be outputs is distinct from the set that can be inputs. In this case, it is sometimes convenient to notationally distinguish the firm's inputs and outputs. We could, for example, let $q = (q_1, \ldots, q_M) \ge 0$ denote the production levels of the firm's M outputs and $z = (z_1, \ldots, z_{L-M}) \ge 0$ denote the amounts of the firm's $L - M$ inputs, with the convention that the amount of input z_ℓ used is now measured as a *nonnegative* number (as a matter of notation, we count all goods not actually used in the process as inputs).

One of the most frequently encountered production models is that in which there is a single output. A single-output technology is commonly described by means of a *production function* $f(z)$ that gives the maximum amount q of output that can be produced using input amounts $(z_1, \ldots, z_{L-1}) \ge 0$. For example, if the output is good L, then (assuming that output can be disposed of at no cost) the production function $f(\cdot)$ gives rise to the production set:

$$Y = \{(-z_1, \ldots, -z_{L-1}, q)\colon q - f(z_1, \ldots, z_{L-1}) \le 0 \quad \text{and} \quad (z_1, \ldots, z_{L-1}) \ge 0\}.$$

Holding the level of output fixed, we can define the *marginal rate of technical*

1. As in Chapter 3, in computing ratios such as this, we always assume that $\partial F(\bar{y})/\partial y_k \neq 0$.

substitution (MRTS) of input ℓ *for input* k *at* $\bar{z}$ as

$$MRTS_{\ell k}(\bar{z}) = \frac{\partial f(\bar{z})/\partial z_\ell}{\partial f(\bar{z})/\partial z_k}.$$

The number $MRTS_{\ell k}(\bar{z})$ measures the additional amount of input k that must be used to keep output at level $\bar{q} = f(\bar{z})$ when the amount of input ℓ is decreased marginally. It is the production theory analog to the consumer's marginal rate of substitution. In consumer theory, we look at the trade-off between commodities that keeps utility constant, here, we examine the trade-off between inputs that keeps the amount of output constant. Note that $MRTS_{\ell k}$ is simply a renaming of the marginal rate of transformation of input ℓ for input k in the special case of a single-output, many-input technology.

Example 5.B.2: *The Cobb–Douglas Production Function* The *Cobb–Douglas* production function with two inputs is given by $f(z_1, z_2) = z_1^\alpha z_2^\beta$, where $\alpha \geq 0$ and $\beta \geq 0$. The marginal rate of technical substitution between the two inputs at $z = (z_1, z_2)$ is $MRTS_{12}(z) = \alpha z_2/\beta z_1$. ■

Properties of Production Sets

We now introduce and discuss a fairly exhaustive list of commonly assumed properties of production sets. The appropriateness of each of these assumptions depends on the particular circumstances (indeed, some of them are mutually exclusive).[2]

(i) *Y is nonempty.* This assumption simply says that the firm has something it can plan to do. Otherwise, there is no need to study the behavior of the firm in question.

(ii) *Y is closed.* The set Y includes its boundary. Thus, the limit of a sequence of technologically feasible input–output vectors is also feasible; in symbols, $y^n \to y$ and $y^n \in Y$ imply $y \in Y$. This condition should be thought of as primarily technical.[3]

(iii) *No free lunch.* Suppose that $y \in Y$ and $y \geq 0$, so that the vector y does not use any inputs. The no-free-lunch property is satisfied if this production vector cannot produce output either. That is, whenever $y \in Y$ and $y \geq 0$, then $y = 0$; it is not possible to produce something from nothing. Geometrically, $Y \cap \mathbb{R}^L_+ \subset \{0\}$. For $L = 2$, Figure 5.B.2(a) depicts a set that violates the no-free-lunch property, the set in Figure 5.B.2(b) satisfies it.

(iv) *Possibility of inaction* This property says that $0 \in Y$: Complete shutdown is possible. Both sets in Figure 5.B.2, for example, satisfy this property. The point in time at which production possibilities are being analyzed is often important for the validity of this assumption. If we are contemplating a firm that could access a set of technological possibilities but that has not yet been organized, then inaction is clearly

2. For further discussion of these properties, see Koopmans (1957) and Chapter 3 of Debreu (1959).

3. Nonetheless, we show in Exercise 5.B.4 that there is an important case of economic interest when it raises difficulties.

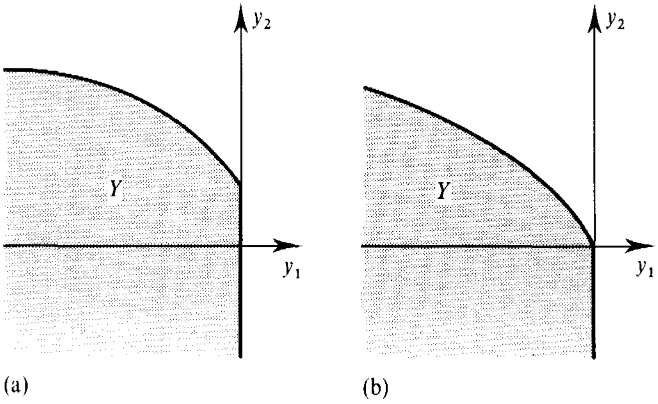


Figure 5.B.2
The no free lunch property.
(a) Violates no free lunch.
(b) Satisfies no free lunch.

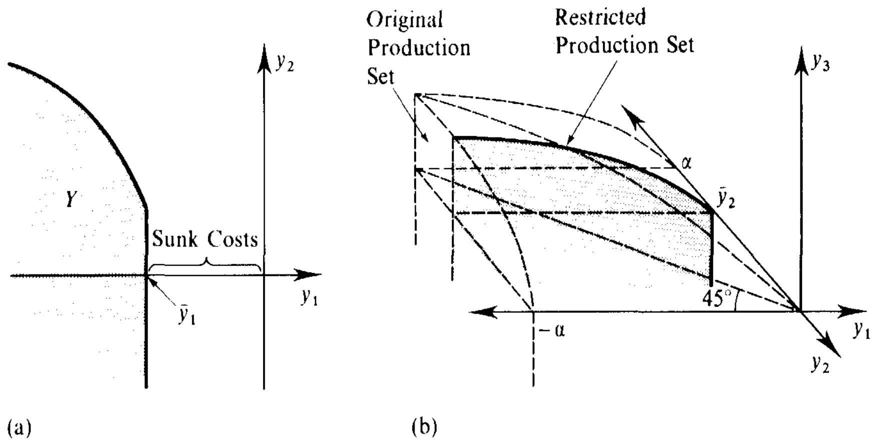


Figure 5.B.3
Two production sets with sunk costs.
(a) A minimal level of expenditure committed.
(b) One kind of input fixed.

possible. But if some production decisions have already been made, or if irrevocable contracts for the delivery of some inputs have been signed, inaction is not possible. In that case, we say that some costs are *sunk*. Figure 5.B.3 depicts two examples. The production set in Figure 5.B.3(a) represents the *interim* production possibilities arising when the firm is already committed to use at least $-\bar{y}_1$ units of good 1 (perhaps because it has already signed a contract for the purchase of this amount); that is, the set is a *restricted production set* that reflects the firm's remaining choices from some original production set Y like the ones in Figure 5.B.2. In Figure 5.B.3(b), we have a second example of sunk costs. For a case with one output (good 3) and two inputs (goods 1 and 2), the figure illustrates the restricted production set arising when the level of the second input has been irrevocably set at $\bar{y}_2 < 0$ [here, in contrast with Figure 5.B.3(a), increases in the use of the input are impossible].

(v) *Free disposal.* The property of free disposal holds if the absorption of any additional amounts of inputs without any reduction in output is always possible. That is, if $y \in Y$ and $y' \leq y$ (so that y' produces at most the same amount of outputs using at least the same amount of inputs), then $y' \in Y$. More succinctly, $Y - \mathbb{R}^L_+ \subset Y$ (see Figure 5.B.4). The interpretation is that the extra amount of inputs (or outputs) can be disposed of or eliminated at no cost.

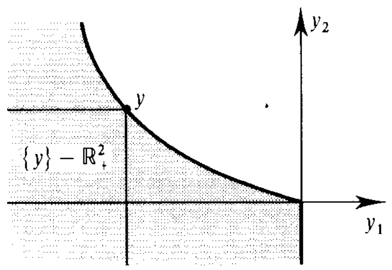
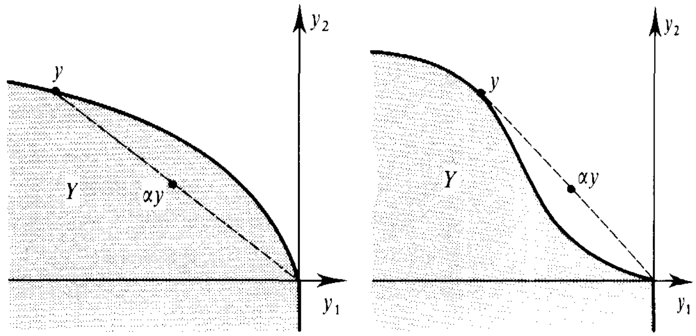


Figure 5.B.4 (left) The free disposal property.



(a) (b)

Figure 5.B.5 (right) The nonincreasing returns to scale property. (a) Nonincreasing returns satisfied. (b) Nonincreasing returns violated.

(vi) *Irreversibility*. Suppose that $y \in Y$ and $y \neq 0$. Then irreversiblity says that $-y \notin Y$. In words, it is impossible to reverse a technologically possible production vector to transform an amount of output into the same amount of input that was used to generate it. If, for example, the description of a commodity includes the time of its availability, then irreversibility follows from the requirement that inputs be used before outputs emerge.

Exercise 5.B.1: Draw two production sets: one that violates irreversibility and one that satisfies this property.

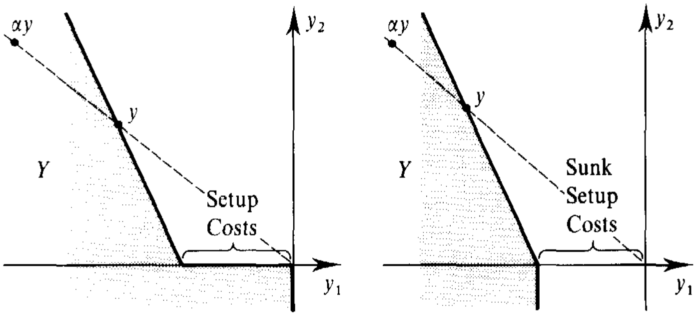
(vii) *Nonincreasing returns to scale*. The production technology Y exhibits nonincreasing returns to scale if for any $y \in Y$, we have $\alpha y \in Y$ for all scalars $\alpha \in [0, 1]$. In words, any feasible input–output vector can be scaled down (see Figure 5.B.5). Note that nonincreasing returns to scale imply that inaction is possible [property (iv)].

(viii) *Nondecreasing returns to scale*. In contrast with the previous case, the production process exhibits nondecreasing returns to scale if for any $y \in Y$, we have $\alpha y \in Y$ for any scale $\alpha \geq 1$. In words, any feasible input–output vector can be scaled up. Figure 5.B.6(a) presents a typical example; in the figure, units of output (good 2) can be produced at a constant cost of input (good 1) except that in order to produce at all, a fixed setup cost is required. It does not matter for the existence of nondecreasing returns if this fixed cost is sunk [as in Figure 5.B.6(b)] or not [as in Figure 5.B.6(a), where inaction is possible].

(ix) *Constant returns to scale*. This property is the conjunction of properties (vii) and (viii). The production set Y exhibits constant returns to scale if $y \in Y$ implies $\alpha y \in Y$ for any scalar $\alpha \geq 0$. Geometrically, Y is a *cone* (see Figure 5.B.7).

For single-output technologies, properties of the production set translate readily into properties of the production function $f(\cdot)$. Consider Exercise 5.B.2 and Example 5.B.3.

Exercise 5.B.2: Suppose that $f(\cdot)$ is the production function associated with a single-output technology, and let Y be the production set of this technology. Show that Y satisfies constant returns to scale if and only if $f(\cdot)$ is homogeneous of degree one.



(a) (b)

Figure 5.B.6 (left)
The nondecreasing returns to scale property.

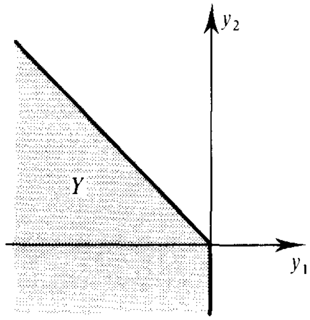


Figure 5.B.7 (right)
A technology satisfying the constant returns to scale property.

Example 5.B.3: *Returns to Scale with the Cobb–Douglas Production Function*: For the Cobb–Douglas production function introduced in Example 5.B.2, $f(2z_1, 2z_2) = 2^{\alpha+\beta}z_1^\alpha z_2^\beta = 2^{\alpha+\beta}f(z_1, z_2)$. Thus, when $\alpha + \beta = 1$, we have constant returns to scale; when $\alpha + \beta < 1$, we have decreasing returns to scale; and when $\alpha + \beta > 1$, we have increasing returns to scale. ■

(x) *Additivity* (*or free entry*). Suppose that $y \in Y$ and $y' \in Y$. The additivity property requires that $y + y' \in Y$. More succinctly, $Y + Y \subset Y$. This implies, for example, that $ky \in Y$ for any positive *integer* k. In Figure 5.B.8, we see an example where Y is additive. Note that in this example, output is available only in integer amounts (perhaps because of indivisibilities). The economic interpretation of the additivity condition is that if y and y' are both possible, then one can set up two plants that do not interfere with each other and carry out production plans y and y' independently. The result is then the production vector $y + y'$.

Additivity is also related to the idea of entry. If $y \in Y$ is being produced by a firm and another firm enters and produces $y' \in Y$, then the net result is the vector $y + y'$. Hence, the *aggregate production set* (the production set describing feasible production plans for the economy as a whole) must satisfy additivity whenever unrestricted entry, or (as it is called in the literature) *free entry*, is possible.

(xi) *Convexity*. This is one of the fundamental assumptions of microeconomics. It postulates that the production set Y is convex. That is, if $y, y' \in Y$ and $\alpha \in [0, 1]$, then $\alpha y + (1 - \alpha)y' \in Y$. For example, Y is convex in Figure 5.B.5(a) but is not convex in Figure 5.B.5(b).

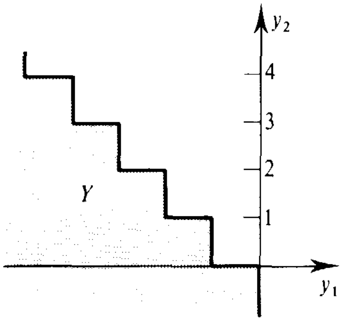


Figure 5.B.8
A production set satisfying the additivity property.

The convexity assumption can be interpreted as incorporating two ideas about production possibilities. The first is nonincreasing returns. In particular, if inaction is possible (i.e., if $0 \in Y$), then convexity implies that Y has nonincreasing returns to scale. To see this, note that for any $\alpha \in [0, 1]$, we can write $\alpha y = \alpha y + (1 - \alpha)0$. Hence, if $y \in Y$ and $0 \in Y$, convexity implies that $\alpha y \in Y$. Second, convexity captures the idea that "unbalanced" input combinations are not more productive than balanced ones (or, symmetrically, that "unbalanced" output combinations are not least costly to produce than balanced ones). In particular, if production plans y and y' produce exactly the same amount of output but use different input combinations, then a production vector that uses a level of each input that is the average of the levels used in these two plans can do at least as well as either y or y'.

Exercise 5.B.3 illustrates these two ideas for the case of a single-output technology.

Exercise 5.B.3: Show that for a single-output technology, Y is convex if and only if the production function $f(z)$ is concave.

(xii) *Y is a convex cone.* This is the conjunction of the convexity (xi) and constant returns to scale (ix) properties. Formally, Y is a convex cone if for any production vector $y, y' \in Y$ and constants $\alpha \geq 0$ and $\beta \geq 0$, we have $\alpha y + \beta y' \in Y$. The production set depicted in Figure 5.B.7 is a convex cone.

An important fact is given in Proposition 5.B.1.

Proposition 5.B.1: The production set Y is additive and satisfies the nonincreasing returns condition if and only if it is a convex cone.

Proof: The definition of a convex cone directly implies the nonincreasing returns and additivity properties. Conversely, we want to show that if nonincreasing returns and additivity hold, then for any $y, y' \in Y$ and any $\alpha > 0$, and $\beta > 0$, we have $\alpha y + \beta y' \in Y$. To this effect, let k be any integer such that $k > \text{Max}\,\{\alpha, \beta\}$. By additivity, $ky \in Y$ and $ky' \in Y$. Since $(\alpha/k) < 1$ and $\alpha y = (\alpha/k)ky$, the nonincreasing returns condition implies that $\alpha y \in Y$. Similarly, $\beta y' \in Y$. Finally, again by additivity, $\alpha y + \beta y' \in Y$. ■

Proposition 5.B.1 provides a justification for the convexity assumption in production. Informally, we could say that if feasible input–output combinations can always be scaled down, and if the simultaneous operation of several technologies without mutual interference is always possible, then, in particular, convexity obtains. (See Appendix A of Chapter 11 for several examples in which there is mutual interference and, as a consequence, convexity does not arise.)

It is important not to lose sight of the fact that the production set describes technology, not limits on resources. It can be argued that if all inputs (including, say, entrepreneurial inputs) are explicitly accounted for, then it should always be possible to replicate production. After all, we are not saying that doubling output is actually feasible, only that in principle it would be possible if *all* inputs (however esoteric, be they marketed or not) were doubled. In this view, which originated with Marshall and has been much emphasized by McKenzie (1959), decreasing returns must reflect the scarcity of an underlying, unlisted input of production. For this reason, some economists believe that among models with convex technologies the constant returns model is the most fundamental. Proposition 5.B.2 makes this idea precise.

Proposition 5.B.2: For any convex production set $Y \subset \mathbb{R}^L$ with $0 \in Y$, there is a constant returns, convex production set $Y' \subset \mathbb{R}^{L+1}$ such that $Y = \{y \in \mathbb{R}^L : (y, -1) \in Y'\}$.

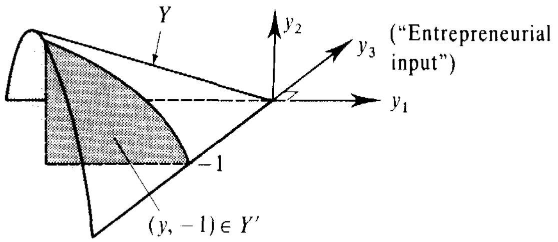


Figure 5.B.9
A constant returns production set with an "entrepreneurial factor."

Proof: Simply let $Y' = \{y' \in \mathbb{R}^{L+1}: y' = \alpha(y, -1) \text{ for some } y \in Y \text{ and } \alpha \geq 0\}$. (See Figure 5.B.9.) ■

The additional input included in the extended production set (good $L + 1$) can be called the "entrepreneurial factor." (The justification for this can be seen in Exercise 5.C.12; in a competitive environment, the return to this entrepreneurial factor is precisely the firm's profit.) In essence, the implication of Proposition 5.B.2 is that in a competitive, convex setting, there may be little loss of conceptual generality in limiting ourselves to constant returns technologies.

5.C Profit Maximization and Cost Minimization

In this section, we begin our study of the market behavior of the firm. In parallel to our study of consumer demand, we assume that there is a vector of prices quoted for the L goods, denoted by $p = (p_1, \ldots, p_L) \gg 0$, and that these prices are independent of the production plans of the firm (the *price-taking assumption*).

We assume throughout this chapter that the firm's objective is to maximize its profit. (It is quite legitimate to ask why this should be so, and we will offer a brief discussion of the issue in Section 5.G.) Moreover, we always assume that the firm's production set Y satisfies the properties of *nonemptiness*, *closedness*, and *free disposal* (see Section 5.B).

The Profit Maximization Problem

Given a price vector $p \gg 0$ and a production vector $y \in \mathbb{R}^L$, the profit generated by implementing y is $p \cdot y = \sum_{\ell=1}^{L} p_\ell y_\ell$. By the sign convention, this is precisely the total revenue minus the total cost. Given the technological constraints represented by its production set Y, the firm's *profit maximization problem* (*PMP*) is then

$$\begin{aligned} \operatorname*{Max}_{y} \quad & p \cdot y \\ \text{s.t.} \quad & y \in Y. \end{aligned} \tag{PMP}$$

Using a transformation function to describe Y, $F(\cdot)$, we can equivalently state the PMP as

$$\begin{aligned} \operatorname*{Max}_{y} \quad & p \cdot y \\ \text{s.t.} \quad & F(y) \leq 0. \end{aligned}$$

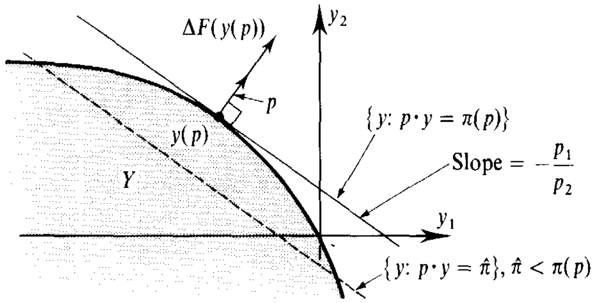


Figure 5.C.1
The profit maximization problem.

Given a production set Y, the firm's *profit function* $\pi(p)$ associates to every p the amount $\pi(p) = \text{Max}\ \{p \cdot y: y \in Y\}$, the value of the solution to the PMP. Correspondingly, we define the firm's *supply correspondence* at p, denoted $y(p)$, as the set of profit-maximizing vectors $y(p) = \{y \in Y: p \cdot y = \pi(p)\}$.[4] Figure 5.C.1 depicts the supply to the PMP for a strictly convex production set Y. The optimizing vector $y(p)$ lies at the point in Y associated with the highest level of profit. In the figure, $y(p)$ therefore lies on the *iso-profit line* (a line in $\mathbb{R}^2$ along which all points generate equal profits) that intersects the production set farthest to the northeast and is, therefore, tangent to the boundary of Y at $y(p)$.

In general, $y(p)$ may be a set rather than a single vector. Also, it is possible that no profit-maximizing production plan exists. For example, the price system may be such that there is no bound on how high profits may be. In this case, we say that $\pi(p) = +\infty$.[5] To take a concrete example, suppose that $L = 2$ and that a firm with a constant returns technology produces one unit of good 2 for every unit of good 1 used as an input. Then $\pi(p) = 0$ whenever $p_2 \leq p_1$. But if $p_2 > p_1$, then the firm's profit is $(p_2 - p_1)y_2$, where y_2 is the production of good 2. Clearly, by choosing y_2 appropriately, we can make profits arbitrarily large. Hence, $\pi(p) = +\infty$ if $p_2 > p_1$.

Exercise 5.C.1: Prove that, in general, if the production set Y exhibits nondecreasing returns to scale, then either $\pi(p) \leq 0$ or $\pi(p) = +\infty$.

If the transformation function $F(\cdot)$ is differentiable, then first-order conditions can be used to characterize the solution to the PMP. If $y^* \in y(p)$, then, for some $\lambda \geq 0$, y^* must satisfy the first-order conditions

$$p_\ell = \lambda \frac{\partial F(y^*)}{\partial y_\ell} \quad \text{for } \ell = 1, \ldots, L$$

or, equivalently, in matrix notation,

$$p = \lambda \nabla F(y^*). \tag{5.C.1}$$

4. We use the term *supply correspondence* to keep the parallel with the *demand* terminology of the consumption side. Recall however that $y(p)$ is more properly thought of as the firm's *net* supply to the market. In particular, the negative entries of a supply vector should be interpreted as demand for inputs.

5. Rigorously, to allow for the possibility that $\pi(p) = +\infty$ (as well as for other cases where no profit-maximizing production plan exists), the profit function should be defined by $\pi(p) = \text{Sup}\ \{p \cdot y: y \in Y\}$. We will be somewhat loose, however, and continue to use *Max* while allowing for this possiblity.

In words, the *price vector p and the gradient* $\nabla F(y^*)$ *are proportional* (Figure 5.C.1 depicts this fact). Condition (5.C.1) also yields the following ratio equality: $p_\ell/p_k = MRT_{\ell k}(y^*)$ for all ℓ, k. For $L = 2$, this says that the slope of the transformation frontier at the profit-maximizing production plan must be equal to the negative of the price ratio, as shown in Figure 5.C.1. Were this not so, a small change in the firm's production plan could be found that increases the firm's profits.

When Y corresponds to a single-output technology with differentiable production function $f(z)$, we can view the firm's decision as simply a choice over its input levels z. In this special case, we shall let the scalar $p > 0$ denote the price of the firm's output and the vector $w \gg 0$ denote its input prices.[6] The input vector z^* maximizes profit given (p, w) if it solves

$$\underset{z \geq 0}{\text{Max}}\ pf(z) - w \cdot z.$$

If z^* is optimal, then the following first-order conditions must be satisfied for $\ell = 1, \ldots, L-1$:

$$p \frac{\partial f(z^*)}{\partial z_\ell} \leq w_\ell, \quad \text{with equality if } z_\ell^* > 0,$$

or, in matrix notation,

$$p\,\nabla f(z^*) \leq w \qquad \text{and} \qquad [p\,\nabla f(z^*) - w]\cdot z^* = 0.^7 \tag{5.C.2}$$

Thus, the marginal product of every input ℓ actually used (i.e., with $z_\ell^* > 0$) must equal its price in terms of output, w_ℓ/p. Note also that for any two inputs ℓ and k with $(z_\ell^*, z_k^*) \gg 0$, condition (5.C.2) implies that $MRTS_{\ell k} = w_\ell/w_k$; that is, the marginal rate of technical substitution between the two inputs is equal to their price ratio, the economic rate of substitution between them. This ratio condition is merely a special case of the more general condition derived in (5.C.1).

If the production set Y is convex, then the first-order conditions in (5.C.1) and (5.C.2) are not only necessary but also sufficient for the determination of a solution to the PMP.

Proposition 5.C.1, which lists the properties of the profit function and supply correspondence, can be established using methods similar to those we employed in Chapter 3 when studying consumer demand. Observe, for example, that mathematically the concept of the profit function should be familiar from the discussion of duality in Chapter 3. In fact, $\pi(p) = -\mu_{-Y}(p)$, where $\mu_{-Y}(p) = \text{Min}\,\{p\cdot(-y)\colon y \in Y\}$ is the support function of the set $-Y$. Thus, the list of important properties in Proposition 5.C.1 can be seen to follow from the general properties of support functions discussed in Section 3.F.

6. Up to now, we have always used the symbol p for an overall vector of prices; here we use it only for the output price and we denote the vector of input prices by w. This notation is fairly standard. As a rule of thumb, unless we are in a context of explicit classification of commodities as inputs or outputs (as in the single-output case), we will continue to use p to denote an overall vector of prices $p = (p_1, \ldots, p_L)$.

7. The concern over boundary conditions arises here, but not in condition (5.C.1), because the assumption of distinct inputs and outputs requires that $z \geq 0$, whereas the formulation leading to (5.C.1) allows the net output of every good to be either positive or negative. Nonetheless, when using the first-order conditions (5.C.2), we will typically assume that $z^* \gg 0$.

Proposition 5.C.1: Suppose that $\pi(\cdot)$ is the profit function of the production set Y and that $y(\cdot)$ is the associated supply correspondence. Assume also that Y is closed and satisfies the free disposal property. Then

(i) $\pi(\cdot)$ is homogeneous of degree one.

(ii) $\pi(\cdot)$ is convex.

(iii) If Y is convex, then $Y = \{y \in \mathbb{R}^L: p\cdot y \leq \pi(p) \text{ for all } p \gg 0\}$.

(iv) $y(\cdot)$ is homogeneous of degree zero.

(v) If Y is convex, then $y(p)$ is a convex set for all p. Moreover, if Y is strictly convex, then $y(p)$ is single-valued (if nonempty).

(vi) (*Hotelling's lemma*) If $y(\bar{p})$ consists of a single point, then $\pi(\cdot)$ is differentiable at $\bar{p}$ and $\nabla\pi(\bar{p}) = y(\bar{p})$.

(vii) If $y(\cdot)$ is a function differentiable at $\bar{p}$, then $Dy(\bar{p}) = D^2\pi(\bar{p})$ is a symmetric and positive semidefinite matrix with $Dy(\bar{p})\bar{p} = 0$.

Properties (ii), (iii), (vi), and (vii) are the nontrivial ones.

Exercise 5.C.2: Prove that $\pi(\cdot)$ is a convex function [Property (ii) of Proposition 5.C.1]. [*Hint*: Suppose that $y \in y(\alpha p + (1-\alpha)p')$. Then

$$\pi(\alpha p + (1-\alpha)p') = \alpha p\cdot y + (1-\alpha)p'\cdot y \leq \alpha\pi(p) + (1-\alpha)\pi(p').]$$

Property (iii) tells us that if Y is closed, convex, and satisfies free disposal, then $\pi(p)$ provides an alternative ("dual") description of the technology. As for the indirect utility function's (or expenditure function's) representation of preferences (discussed in Chapter 3), it is a less primitive description than Y itself because it depends on the notions of prices and of price-taking behavior. But thanks to property (vi), it has the great virtue in applications of often allowing for an immediate computation of supply.

Property (vi) relates supply behavior to the derivatives of the profit function. It is a direct consequence of the duality theorem (Proposition 3.F.1). As in Proposition 3.G.1, the fact that $\nabla\pi(\bar{p}) = y(\bar{p})$ can also be established by the related arguments of the envelope theorem and of first-order conditions.

The positive semidefiniteness of the matrix $Dy(p)$ in property (vii), which in view of property (vi) is a consequence of the convexity of $\pi(\cdot)$, is the general mathematical expression of the *law of supply*: *Quantities respond in the same direction as price changes*. By the sign convention, this means that *if the price of an output increases* (all other prices remaining the same), then *the supply of the output increases*; and *if the price of an input increases*, then *the demand for the input decreases*.

Note that the law of supply holds for *any* price change. Because, in contrast with demand theory, there is no budget constraint, there is no compensation requirement of any sort. In essence, we have no wealth effects here, only substitution effects.

In nondifferentiable terms, the law of supply can be expressed as

$$(p - p')\cdot(y - y') \geq 0 \tag{5.C.3}$$

for all p, p', $y \in y(p)$, and $y' \in y(p')$. In this form, it can also be established by a straightforward revealed preference argument. In particular,

$$(p - p')\cdot(y - y') = (p\cdot y - p\cdot y') + (p'\cdot y' - p'\cdot y) \geq 0,$$

where the inequality follows from the fact that $y \in y(p)$ and $y' \in y(p')$ (i.e., from the fact that y is profit maximizing given prices p and y' is profit maximizing for prices p').

Property (vii) of Proposition 5.C.1 implies that the matrix $Dy(p)$, the *supply substitution matrix*, has properties that parallel (although with the reverse sign) those for the substitution matrix of demand theory. Thus, own-substitution effects are nonnegative as noted above $[\partial y_\ell(p)/\partial p_\ell \geq 0$ for all $\ell]$, and substitution effects are symmetric $[\partial y_\ell(p)/\partial p_k = \partial y_k(p)/\partial p_\ell$ for all $\ell, k]$. The fact that $Dy(p)p = 0$ follows from the homogeneity of $y(\cdot)$ [property (iv)] in a manner similar to the parallel property of the demand substitution matrix discussed in Chapter 3.

Cost Minimization

An important implication of the firm choosing a profit-maximizing production plan is that there is no way to produce the same amounts of outputs at a lower total input cost. Thus, cost minimization is a necessary condition for profit maximization. This observation motivates us to an independent study of the firm's *cost minimization problem*. The problem is of interest for several reasons. First, it leads us to a number of results and constructions that are technically very useful. Second, as we shall see in Chapter 12, when a firm is not a price taker in its output market, we can no longer use the profit function for analysis. Nevertheless, as long as the firm is a price taker in its input market, the results flowing from the cost minimization problem continue to be valid. Third, when the production set exhibits nondecreasing returns to scale, the value function and optimizing vectors of the cost minimization problem, which keep the levels of outputs fixed, are better behaved than the profit function and supply correspondence of the PMP (e.g., recall from Exercise 5.C.1 that the profit function can take only the values 0 and $+\infty$).

To be concrete, we focus our analysis on the single-output case. As usual, we let z be a nonnegative vector of inputs, $f(z)$ the production function, q the amounts of output, and $w \gg 0$ the vector of input prices. The *cost minimization problem* (CMP) can then be stated as follows (we assume free disposal of output):

$$\begin{aligned} \underset{z \geq 0}{\text{Min}} \quad & w \cdot z \\ \text{s.t. } & f(z) \geq q. \end{aligned} \qquad \text{(CMP)}$$

The optimized value of the CMP is given by the *cost function* $c(w, q)$. The corresponding optimizing set of input (or factor) choices, denoted by $z(w, q)$, is known as the *conditional factor demand correspondence* (or *function* if it is always single-valued). The term *conditional* arises because these factor demands are conditional on the requirement that the output level q be produced.

The solution to the CMP is depicted in Figure 5.C.2(a) for a case with two inputs. The shaded region represents the set of input vectors z that can produce at least the amount q of output. It is the projection (into the positive orthant of the input space) of the part of the production set Y than generates output of at least q, as shown in Figure 5.C.2(b). In Figure 5.C.2(a), the solution $z(w, q)$ lies on the iso-cost line (a line in $\mathbb{R}^2$ on which all input combinations generate equal cost) that intersects the set $\{z \in \mathbb{R}^L_+ : f(z) \geq q\}$ closest to the origin.

If z^* is optimal in the CMP, and if the production function $f(\cdot)$ is differentiable,

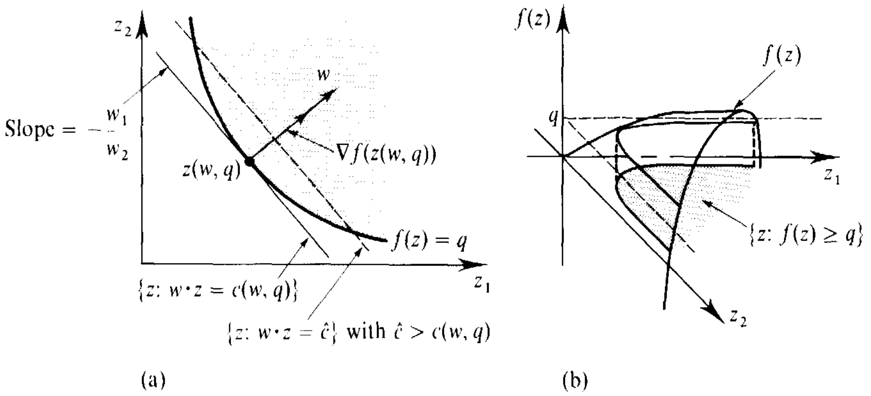


Figure 5.C.2 The cost minimization problem. (a) Two inputs. (b) The isoquant as a section of the production set.

then for some $\lambda \geq 0$, the following first-order conditions must hold for every input $\ell = 1, \ldots, L-1$:

$$w_\ell \geq \lambda \frac{\partial f(z^*)}{\partial z_\ell}, \quad \text{with equality if } z_\ell^* > 0,$$

or, in matrix notation,

$$w \geq \lambda \nabla f(z^*) \quad \text{and} \quad [w - \lambda \nabla f(z^*)] \cdot z^* = 0. \tag{5.C.4}$$

As with the PMP, if the production set Y is convex [i.e., if $f(\cdot)$ is concave], then condition (5.C.4) is not only necessary but also sufficient for z^* to be an optimum in the CMP.[8]

Condition (5.C.4), like condition (5.C.2) of the PMP, implies that for any two inputs ℓ and k with $(z_\ell, z_k) \gg 0$, we have $MRTS_{\ell k} = w_\ell / w_k$. This correspondence is to be expected because, as we have noted, profit maximization implies that input choices are cost minimizing for the chosen output level q. For $L = 2$, condition (5.C.4) entails that the slope at z^* of the *isoquant* associated with production level q is exactly equal to the negative of the ratio of the input prices $-w_1/w_2$. Figure 5.C.2(a) depicts this fact as well.

As usual, the Lagrange multiplier λ can be interpreted as the marginal value of relaxing the constraint $f(z^*) \geq q$. Thus, λ equals $\partial c(w, q)/\partial q$, the *marginal cost of production*.

Note the close formal analogy with consumption theory here. Replace $f(\cdot)$ by $u(\cdot)$, q by u, and z by x (i.e., interpret the production function as a utility function), and the CMP becomes the expenditure minimization problem (EMP) discussed in Section 3.E. Therefore, in Proposition 5.C.2, properties (i) to (vii) of the cost function and conditional factor demand correspondence follow from the analysis in Sections 3.E to 3.G by this reinterpretation. [You are asked to prove properties (viii) and (ix) in Exercise 5.C.3.]

Proposition 5.C.2: Suppose that $c(w, q)$ is the cost function of a single-output technology Y with production function $f(\cdot)$ and that $z(w, q)$ is the associated

8. Note, however, that the first-order conditions are sufficient for a solution to the CMP as long as the set $\{z: f(z) \geq q\}$ is convex. Thus, the key condition for the sufficiency of the first-order conditions of the CMP is the *quasiconcavity* of $f(\cdot)$. This is an important fact because the quasiconcavity of $f(\cdot)$ is compatible with increasing returns to scale (see Example 5.C.1).

conditional factor demand correspondence. Assume also that Y is closed and satisfies the free disposal property. Then

(i) $c(\cdot)$ is homogeneous of degree one in w and nondecreasing in q.

(ii) $c(\cdot)$ is a concave function of w.

(iii) If the sets $\{z \geq 0 : f(z) \geq q\}$ are convex for every q, then $Y = \{(-z, q) : w \cdot z \geq c(w, q) \text{ for all } w \gg 0\}$.

(iv) $z(\cdot)$ is homogeneous of degree zero in w.

(v) If the set $\{z \geq 0 : f(z) \geq q\}$ is convex, then $z(w, q)$ is a convex set. Moreover, if $\{z \geq 0 : f(z) \geq q\}$ is a strictly convex set, then $z(w, q)$ is single-valued.

(vi) (*Shepard's lemma*) If $z(\bar{w}, q)$ consists of a single point, then $c(\cdot)$ is differentiable with respect to w at $\bar{w}$ and $\nabla_w c(\bar{w}, q) = z(\bar{w}, q)$.

(vii) If $z(\cdot)$ is differentiable at $\bar{w}$, then $D_w z(\bar{w}, q) = D_w^2 c(\bar{w}, q)$ is a symmetric and negative semidefinite matrix with $D_w z(\bar{w}, q)\bar{w} = 0$.

(viii) If $f(\cdot)$ is homogeneous of degree one (i.e., exhibits constant returns to scale), then $c(\cdot)$ and $z(\cdot)$ are homogeneous of degree one in q.

(ix) If $f(\cdot)$ is concave, then $c(\cdot)$ is a convex function of q (in particular, marginal costs are nondecreasing in q).

In Exercise 5.C.4 we are asked to show that properties (i) to (vii) of Proposition 5.C.2 also hold for technologies with multiple outputs.

The cost function can be particularly useful when the production set is of the constant returns type. In this case, $y(\cdot)$ is not single-valued at any price vector allowing for nonzero production, making Hotelling's lemma [Proposition 5.C.1(vi)] inapplicable at these prices. Yet, the conditional input demand $z(w, q)$ may nevertheless be single-valued, allowing us to use Shepard's lemma. Keep in mind, however, that the cost function does not contain more information than the profit function. In fact, we know from property (iii) of Propositions 5.C.1 and 5.C.2 that under convexity restrictions there is a one-to-one correspondence between profit and cost functions; that is, from either function, the production set can be recovered, and the other function can then be derived.

Using the cost function, we can restate the firm's problem of determining its profit-maximizing production level as

$$\underset{q \geq 0}{\text{Max}}\ pq - c(w, q). \tag{5.C.5}$$

The necessary first-order condition for q^* to be profit maximizing is then

$$p - \frac{\partial c(w, q^*)}{\partial q} \leq 0, \quad \text{with equality if } q^* > 0. \tag{5.C.6}$$

In words, at an interior optimum (i.e., if $q^* > 0$), *price equals marginal cost.*[9] If $c(w, q)$ is convex in q, then the first-order condition (5.C.6) is also sufficient for q^* to be the firm's optimal output level. (We study the relationship between the firm's supply behavior and the properties of its technology and cost function in detail in Section 5.D.)

9. This can also be seen by noting that the first-order condition (5.C.4) of the CMP coincides with first-order condition (5.C.2) of the PMP if and only if $\lambda = p$. Recall that λ, the multiplier on the constraint in the CMP, is equal to $\partial c(w, q)/\partial q$.

We could go on for many pages analyzing profit and cost functions. Some examples and further properties are contained in the exercises. See McFadden (1978) for an extensive treatment of this topic.

Example 5.C.1: *Profit and Cost Functions for the Cobb–Douglas Production Function.* Here we derive the profit and cost functions for the Cobb–Douglas production function of Example 5.B.2, $f(z_1, z_2) = z_1^\alpha z_2^\beta$. Recall from Example 5.B.3 that $\alpha + \beta = 1$ corresponds to the case of constant returns to scale, $\alpha + \beta < 1$ corresponds to decreasing returns, and $\alpha + \beta > 1$ corresponds to increasing returns.

The conditional factor demand equations and cost function have exactly the same form, and are derived in exactly the same way, as the expenditure function in Section 3.E (see Example 3.E.1; the only difference in the computations is that we now do not impose $\alpha + \beta = 1$):

$$z_1(w_1, w_2, q) = q^{1/(\alpha+\beta)}(\alpha w_2/\beta w_1)^{\beta/(\alpha+\beta)},$$
$$z_2(w_1, w_2, q) = q^{1/(\alpha+\beta)}(\beta w_1/\alpha w_2)^{\alpha/(\alpha+\beta)},$$

and

$$c(w_1, w_2, q) = q^{1/(\alpha+\beta)}[(a/\beta)^{\beta/(\alpha+\beta)} + (\alpha/\beta)^{-\alpha/(\alpha+\beta)}]w_1^{\alpha/(\alpha+\beta)}w_2^{\beta/(\alpha+\beta)}.$$

This cost function has the form $c(w_1, w_2, q) = q^{1/(\alpha+\beta)}\theta\phi(w_1, w_2)$, where

$$\theta = [(\alpha/\beta)^{\beta/(\alpha+\beta)} + (\alpha/\beta)^{-\alpha/(\alpha+\beta)}]$$

is a constant and $\phi(w_1, w_2) = w_1^{\alpha/(\alpha+\beta)}w_2^{\beta/(\alpha+\beta)}$ is a function that does not depend on the output level q. When we have constant returns, $\theta\phi(w_1, w_2)$ is the per-unit cost of production.

One way to derive the firm's supply function and profit function is to use this cost function and solve problem (5.C.5). Applying (5.C.6), the first-order condition for this problem is

$$p \leq \theta\phi(w_1, w_2)\left(\frac{1}{\alpha+\beta}\right)q^{(1/(\alpha+\beta))-1}, \quad \text{with equality if } q > 0 \tag{5.C.7}$$

The first-order condition (5.C.7) is sufficient for a maximum when $\alpha + \beta \leq 1$ because the firm's cost function is then convex in q.

When $\alpha + \beta < 1$, (5.C.7) can be solved for a unique optimal output level:

$$q(w_1, w_2, p) = (\alpha + \beta)[p/\theta\phi(w_1, w_2)]^{(\alpha+\beta)/(1-\alpha-\beta)}.$$

The factor demands can then be obtained through substitution,

$$z_\ell(w_1, w_2, p) = z_\ell(w_1, w_2, q(w_1, w_2, p)) \quad \text{for } \ell = 1, 2,$$

as can the profit function,

$$\pi(w_1, w_2, p) = pq(w_1, w_2, p) - w \cdot z(w_1, w_2, q(w_1, w_2, p)).$$

When $\alpha + \beta = 1$, the right-hand side of the first-order condition (5.C.7) becomes $\theta\phi(w_1, w_2)$, the unit cost of production (which is independent of q). If $\theta\phi(w_1, w_2)$ is greater than p, then $q = 0$ is optimal; if it is smaller than p, then no solution exists (again, unbounded profits can be obtained by increasing q); and when $\theta\phi(w_1, w_2) = p$, any non-negative output level is a solution to the PMP and generates zero profits.

Finally, when $\alpha + \beta > 1$ (so that we have increasing returns to scale), a quantity q satisfying the first-order condition (5.C.7) does not yield a profit-maximizing production. [Actually, in this case, the cost function is strictly concave in q, so that any solution to the first-order condition (5.C.7) yields a local *minimum* of profits, subject to output being always produced at minimum cost]. Indeed, since $p > 0$, a doubling of the output level starting from any q doubles the firm's revenue but increases input costs only by a factor of $2^{1/(\alpha+\beta)} > 2$. With enough doublings, the firm's profits can therefore be made arbitrarily large. Hence, with increasing returns to scale, there is no solution to the PMP. ■

5.D The Geometry of Cost and Supply in the Single-Output Case

In this section, we continue our analysis of the relationships among a firm's technology, its cost function, and its supply behavior for the special but commonly used case in which there is a single output. A significant advantage of considering the single-output case is that it lends itself to extensive graphical illustration.

Throughout, we denote the amount of output by q and hold the vector of factor prices constant at $\bar{w} \gg 0$. For notational convenience, we write the firm's cost function as $C(q) = c(\bar{w}, q)$. For $q > 0$, we can denote the firm's average cost by $AC(q) = C(q)/q$ and assuming that the derivative exists, we denote its *marginal cost* by $C'(q) = dC(q)/dq$.

Recall from expression (5.C.6) that for a given output price p, all profit-maximizing output levels $q \in q(p)$ must satisfy the first-order condition [assuming that $C'(q)$ exists]:

$$p \leq C'(q) \quad \text{with equality if } q > 0. \tag{5.D.1}$$

If the production set Y is convex, $C(\cdot)$ is a convex function [see property (ix) of Proposition 5.C.2], and therefore marginal cost is nondecreasing. In this case, as we noted in Section 5.C, satisfaction of this first-order condition is also sufficient to establish that q is a profit-maximizing output level at price p.

Two examples of convex production sets are given in Figures 5.D.1 and 5.D.2. In the figures, we assume that there is only one input, and we normalize its price to equal 1 (you can think of this input as the total expense of factor use).[10] Figure 5.D.1 depicts the production set (a), cost function (b), and average and marginal cost functions (c) for a case with decreasing returns to scale. Observe that the cost function is obtained from the production set by a 90-degree rotation. The determination of average cost and marginal cost from the cost function is shown in Figure 5.D.1(b) (for an output level $\hat{q}$). Figure 5.D.2 depicts the same objects for a case with constant returns to scale.

In Figures 5.D.1(c) and 5.D.2(c), we use a heavier trace to indicate the firm's profit-maximizing supply locus, the graph of $q(\cdot)$. (*Note*: *In this and subsequent figures, the supply locus is always indicated by a heavier trace.*) Because the technologies in these two examples are convex, the supply locus in each case coincides exactly with the (q, p) combinations that satisfy the first-order condition (5.D.1).

If the technology is not convex, perhaps because of the presence of some underlying indivisibility, then satisfaction of the first-order necessary condition

10. Thus, the single input can be thought of as a Hicksian composite commodity in a sense analogous to that in Exercise 3.G.5.

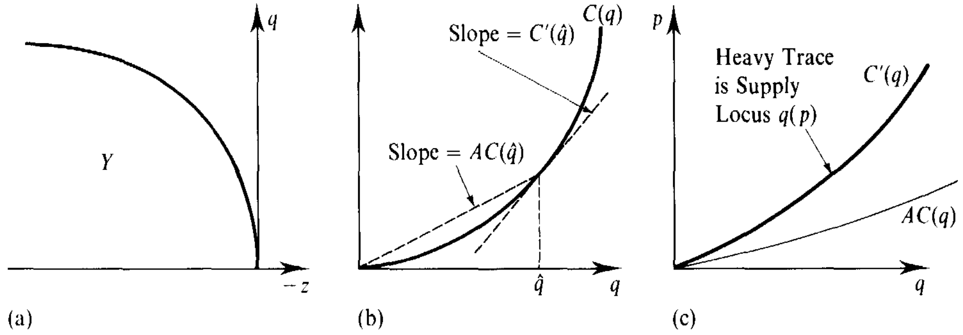


Figure 5.D.1
A strictly convex technology (strictly decreasing returns to scale).
(a) Production set.
(b) Cost function.
(c) Average cost, marginal cost, and supply.

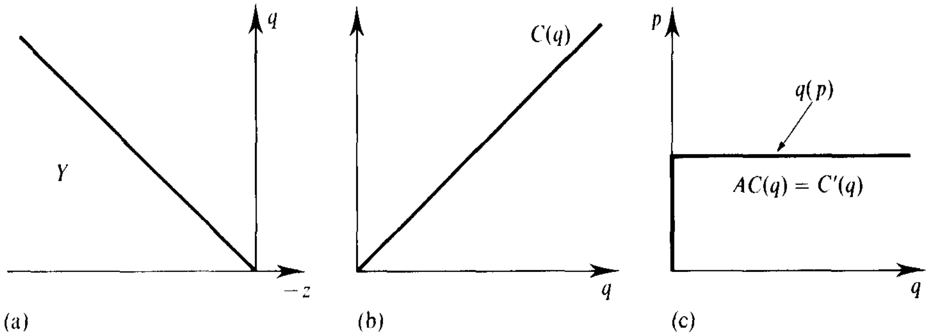


Figure 5.D.2
A constant returns to scale technology.
(a) Production set.
(b) Cost function.
(c) Average cost, marginal cost, and supply.

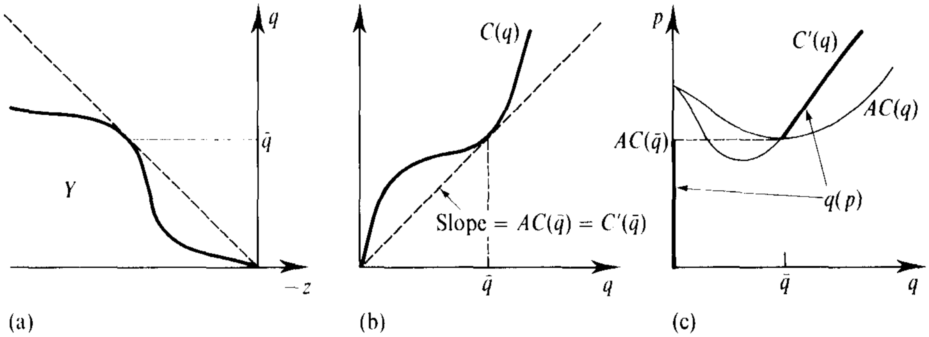


Figure 5.D.3
A nonconvex technology.
(a) Production set.
(b) Cost function.
(c) Average cost, marginal cost, and supply.

(5.D.1) no longer implies that q is profit maximizing. The supply locus will then be only a subset of the set of (q, p) combinations that satisfy (5.D.1).

Figure 5.D.3 depicts a situation with a nonconvex technology. In the figure, we have an initial segment of increasing returns over which the average cost decreases and then a region of decreasing returns over which the average cost increases. The level (or levels) of production corresponding to the minimum average cost is called the *efficient scale*, which, if unique, we denote by $\bar{q}$. Looking at the cost functions in Figure 5.D.3(a) and (b), we see that at $\bar{q}$ we have $AC(\bar{q}) = C'(\bar{q})$. In Exercise 5.D.1, you are asked to establish this fact as a general result.

Exercise 5.D.1: Show that $AC(\bar{q}) = C'(\bar{q})$ at any $\bar{q}$ satisfying $AC(\bar{q}) \leq AC(q)$ for all q. Does this result depend on the differentiability of $C(\cdot)$ everywhere?

The supply locus for this nonconvex example is depicted by the heavy trace in

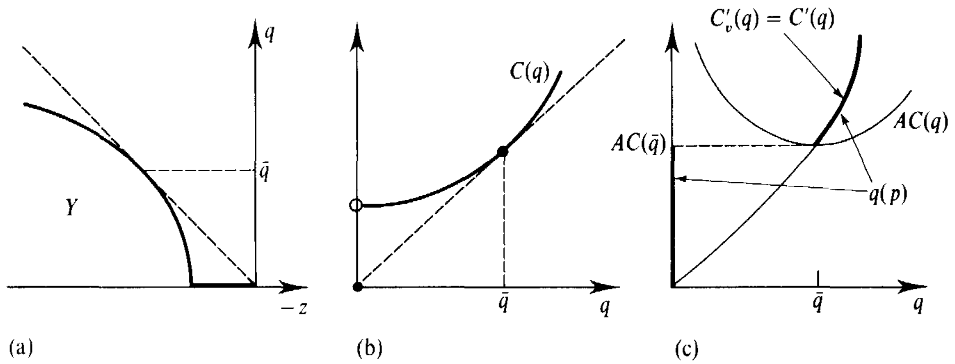


Figure 5.D.4
Strictly convex variable costs with a nonsunk setup cost. (a) Production set. (b) Cost function. (c) Average cost, marginal cost, and supply.

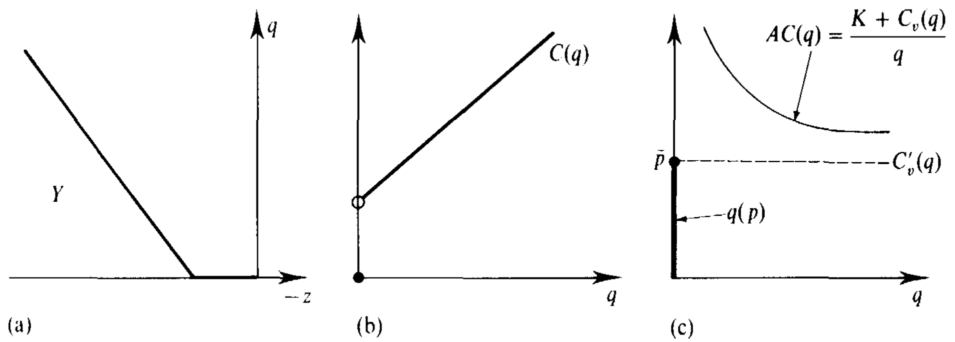


Figure 5.D.5
Constant returns variable costs with a nonsunk setup cost. (a) Production set. (b) Cost function. (c) Average cost, marginal cost, and supply.

Figure 5.D.3(c). When $p > AC(\bar{q})$, the firm maximizes its profit by producing at the unique level of q satisfying $p = C'(q) > AC(q)$. [Note that the firm earns strictly positive profits doing so, exceeding the zero profits earned by setting $q = 0$, which in turn exceed the strictly negative profits earned by choosing any $q > 0$ with $p = C'(q) < AC(q)$.] On the other hand, when $p < AC(\bar{q})$, any $q > 0$ earns strictly negative profits, and so the firm's optimal supply is $q = 0$ [note that $q = 0$ satisfies the necessary first-order condition (5.D.1) because $p < C'(0)$]. When $p = AC(\bar{q})$, the profit-maximizing set of output levels is $\{0, \bar{q}\}$. The supply locus is therefore as shown in Figure 5.D.3(c).

An important source of nonconvexities is fixed setup costs. These may or may not be sunk. Figures 5.D.4 and 5.D.5 (which parallel 5.D.1 and 5.D.2) depict two cases with nonsunk fixed setup costs (so inaction is possible). In these figures, we consider a case in which the firm incurs a fixed cost K if and only if it produces a positive amount of output and otherwise has convex costs. In particular, total cost is of the form $C(0) = 0$, and $C(q) = C_v(q) + K$ for $q > 0$, where $K > 0$ and $C_v(q)$, the *variable cost function*, is convex [and has $C_v(0) = 0$]. Figure 5.D.4 depicts the case in which $C_v(\cdot)$ is strictly convex, whereas $C_v(\cdot)$ is linear in Figure 5.D.5. The supply loci are indicated in the figures. In both illustrations, the firm will produce a positive amount of output only if its profit is sufficient to cover not only its variable costs but also the fixed cost K. You should read the supply locus in Figure 5.D.5(c) as saying that for $p > \bar{p}$, the supply is "infinite," and that $q = 0$ is optimal for $p \leq \bar{p}$.

In Figure 5.D.6, we alter the case studied in Figure 5.D.4 by making the fixed costs sunk, so that $C(0) > 0$. In particular, we now have $C(q) = C_v(q) + K$ for all $q \geq 0$; therefore, the firm must pay K whether or not it produces a positive quantity.

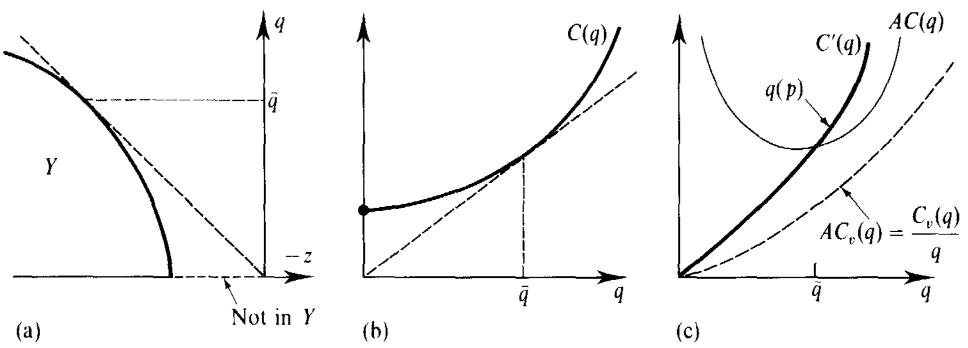


Figure 5.D.6 Strictly convex variable costs with sunk costs. (a) Production set. (b) Cost function. (c) Average cost, marginal cost, and supply.

Although inaction is not possible here, the firm's cost function is convex, and so we are back to the case in which the first-order condition (5.D.1) is sufficient. Because the firm must pay K regardless of whether it produces a positive output level, it will not shut down simply because profits are negative. Note that because $C_v(\cdot)$ is convex and $C_v(0) = 0$, $p = C_v'(q)$ implies that $pq > C_v(q)$; hence, the firm covers its variable costs when it sets output to satisfy its first-order condition. The firm's supply locus is therefore that depicted in Figure 5.D.6(c). Note that its supply behavior is exactly the same as if it did not have to pay the sunk cost K at all [compare with Figure 5.D.1(c)].

Exercise 5.D.2: Depict the supply locus for a case with partially sunk costs, that is, where $C(q) = K + C_v(q)$ if $q > 0$ and $0 < C(0) < K$.

As we noted in Section 5.B, one source of sunk costs, at least in the short run, is input choices irrevocably set by prior decisions. Suppose, for example, that we have two inputs and a production function $f(z_1, z_2)$. Recall that we keep the prices of the two inputs fixed at $(\bar{w}_1, \bar{w}_2)$. In Figure 5.D.7(a), the cost function excluding any prior input commitments is depicted by $C(\cdot)$. We call it the *long-run cost function*. If one input, say z_2, is fixed at level $\bar{z}_2$ in the short-run, then the *short-run cost function* of the firm becomes $C(q \mid \bar{z}_2) = \bar{w}_1 z_1 + \bar{w}_2 \bar{z}_2$, where z_1 is chosen so that $f(z_1, \bar{z}_2) = q$. Several such short-run cost functions corresponding to different levels of z_2 are illustrated in Figure 5.D.7(a). Because restrictions on the firm's input decisions can only increase its costs of production, $C(q \mid \bar{z}_2)$ lies above $C(q)$ at all q except the q for

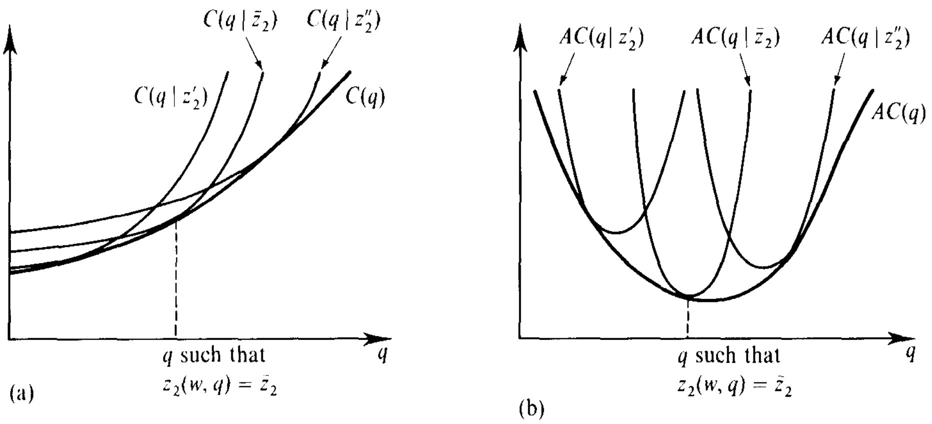


Figure 5.D.7 Costs when an input level is fixed in the short run but is free to vary in the long run. (a) Long-run and short-run cost functions. (b) Long-run and short-run average cost.

which $\bar{z}_2$ is the optimal long-run input level [i.e., the q such that $z_2(\bar{w}, q) = \bar{z}_2$]. Thus, $C(q|z_2(\bar{w},q)) = C(q)$ for all q. It follows from this and from the fact that $C(q'|z_2(\bar{w}, q)) \geq C(q')$ for all q', that $C'(q) = C'(q|z_2(\bar{w}, q))$ for all q; that is, if the level of z_2 is at its long-run value, then the short-run marginal cost equals the long-run marginal cost. Geometrically, $C(\cdot)$ is the lower envelope of the family of short-run functions $C(q|z_2)$ generated by letting z_2 take all possible values.

Observe finally that given the long-run and short-run cost functions, the long-run and short-run average cost functions and long-run and short-run supply functions of the firm can be derived in the manner discussed earlier in the section. The average-cost version of Figure 5.D.7(a) is given in Figure 5.D.7(b). (Exercise 5.D.3 asks you to investigate the short-run and long-run supply behavior of the firm in more detail.)

5.E Aggregation

In this section, we study the theory of aggregate (net) supply. As we saw in Section 5.C, the absence of a budget constraint implies that individual supply is not subject to wealth effects. As prices change, there are only substitution effects along the production frontier. In contrast with the theory of aggregate demand, this fact makes for an aggregation theory that is simple and powerful.[11]

Suppose there are J production units (firms or, perhaps, plants) in the economy, each specified by a production set $Y_1, \ldots, Y_J$. We assume that each Y_j is nonempty, closed, and satisfies the free disposal property. Denote the profit function and supply correspondences of Y_j by $\pi_j(p)$ and $y_j(p)$, respectively. The *aggregate supply correspondence* is the sum of the individual supply correspondences:

$$y(p) = \sum_{j=1}^{J} y_j(p) = \{y \in \mathbb{R}^L \colon y = \textstyle\sum_j y_j \text{ for some } y_j \in y_j(p), j = 1, \ldots, J\}.$$

Assume, for a moment, that every $y_j(\cdot)$ is a single-valued, differentiable function at a price vector p. From Proposition 5.C.1, we know that every $Dy_j(p)$ is a symmetric, positive semidefinite matrix. Because these two properties are preserved under addition, we can conclude that the matrix $Dy(p)$ is *symmetric and positive semidefinite.*

As in the theory of individual production, the positive semidefiniteness of $Dy(p)$ implies the *law of supply* in the aggregate: If a price increases, then so does the corresponding *aggregate* supply. As with the law of supply at the firm level, this property of aggregate supply holds for *all* price changes. We can also prove this aggregate law of supply directly because we know from (5.C.3) that $(p - p')\cdot[y_j(p) - y_j(p')] \geq 0$ for every j; therefore, adding over j, we get

$$(p - p')\cdot[y(p) - y(p')] \geq 0.$$

The symmetry of $Dy(p)$ suggests that underlying $y(p)$ there is a "representative producer." As we now show, this is true in a particularly strong manner.

Given $Y_1, \ldots, Y_J$, we can define the *aggregate production set* by

$$Y = Y_1 + \cdots + Y_J = \{y \in \mathbb{R}^L \colon y = \textstyle\sum_j y_j \text{ for some } y_j \in Y_J, j = 1, \ldots, J\}.$$

11. A classical and very readable account for the material in this section and in Section 5.F is Koopmans (1957).

The aggregate production set Y describes the production vectors that are feasible in the aggregate if all the production sets are used together. Let $\pi^*(p)$ and $y^*(p)$ be the profit function and the supply correspondence of the aggregate production set Y. They are the profit function and supply correspondence that would arise if a single price-taking firm were to operate, under the same management so to speak, all the individual production sets.

Proposition 5.E.1 establishes a strong aggregation result for the supply side: *The aggregate profit obtained by each production unit maximizing profit separately taking prices as given is the same as that which would be obtained if they were to coordinate their actions* (i.e., *their* y_js) *in a joint profit maximizing decision.*

Proposition 5.E.1: For all $p \gg 0$, we have

(i) $\pi^*(p) = \sum_j \pi_j(p)$

(ii) $y^*(p) = \sum_j y_j(p)$ $(= \{\sum_j y_j\colon y_j \in y_j(p) \text{ for every } j\})$.

Proof: (i) For the first equality, note that if we take any collection of production plans $y_j \in Y_j$, $j = 1, \ldots, J$, then $\sum_j y_j \in Y$. Because $\pi^*(\cdot)$ is the profit function associated with Y, we therefore have $\pi^*(p) \geq p \cdot (\sum_j y_j) = \sum_j p \cdot y_j$. Hence, it follows that $\pi^*(p) \geq \sum_j \pi_j(p)$. In the other direction, consider any $y \in Y$. By the definition of the set Y, there are $y_j \in Y_j, j = 1, \ldots, J$, such that $\sum_j y_j = y$. So $p \cdot y = p \cdot (\sum_j y_j) = \sum_j p \cdot y_j \leq \sum_j \pi_j(p)$ for all $y \in Y$. Thus, $\pi^*(p) \leq \sum_j \pi_j(p)$. Together, these two inequalities imply that $\pi^*(p) = \sum_j \pi(p)$.

(ii) For the second equality, we must show that $\sum_j y_j(p) \subset y^*(p)$ and that $y^*(p) \subset \sum_j y_j(p)$. For the former relation, consider any set of individual production plans $y_j \in y_j(p), j = 1, \ldots, J$. Then $p \cdot (\sum_j y_j) = \sum_j p \cdot y_j = \sum_j \pi_j(p) = \pi^*(p)$, where the last equality follows from part (i) of the proposition. Hence, $\sum_j y_j \in y^*(p)$, and therefore, $\sum_j y_j(p) \subset y^*(p)$. In the other direction, take any $y \in y^*(p)$. Then $y = \sum_j y_j$ for some $y_j \in Y_j, j = 1, \ldots, J$. Since $p \cdot (\sum_j y_j) = \pi^*(p) = \sum_j \pi_j(p)$ and, for every j, we have $p \cdot y_j \leq \pi_j(p)$, it must be that $p \cdot y_j = \pi_j(p)$ for every j. Thus, $y_j \in y_j(p)$ for all j, and so $y \in \sum_j y_j(p)$. Thus, we have shown that $y^*(p) \subset \sum_j y_j(p)$. ■

The content of Proposition 5.E.1 is illustrated in Figure 5.E.1. The proposition can be interpreted as a decentralization result: To find the solution of the aggregate profit maximization problem for given prices p, it is enough to add the solutions of the corresponding individual problems.

Simple as this result may seem, it nevertheless has many important implications. Consider, for example, the single-output case. The result tells us that if firms are maximizing profit facing output price p and factor prices w, then their supply behavior maximizes aggregate profits. But this must mean that if $q = \sum_j q_j$ is the aggregate output produced by the firms, then the total cost of production is exactly equal to $c(w, q)$, the value of the *aggregate cost function* (the cost function corresponding to the aggregate production set Y). *Thus, the allocation of the production of output level q among the firms is cost minimizing.* In addition, this allows us to relate the firms' aggregate supply function for output $q(p)$ to the aggregate cost function in the same manner as done in Section 5.D for an individual firm. (This fact will prove useful when we study partial equilibrium models of competitive markets in Chapter 10.)

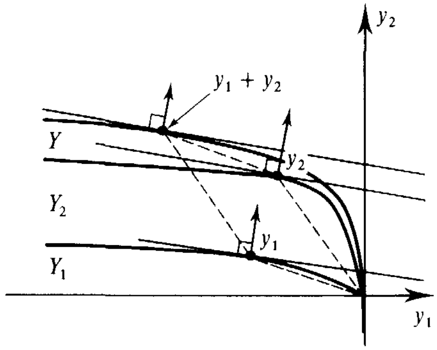


Figure 5.E.1
Joint profit maximization as a result of individual profit maximization.

In summary: If firms maximize profits taking prices as given, then the production side of the economy aggregates beautifully.

As in the consumption case (see Appendix A of Chapter 4), aggregation can also have helpful regularizing effects in the production context. An interesting and important fact is that the existence of many firms or plants with technologies that are not too dissimilar can make the *average* production set almost convex, even if the individual production sets are not so. This is illustrated in Figure 5.E.2, where there are J firms with identical production sets equal to

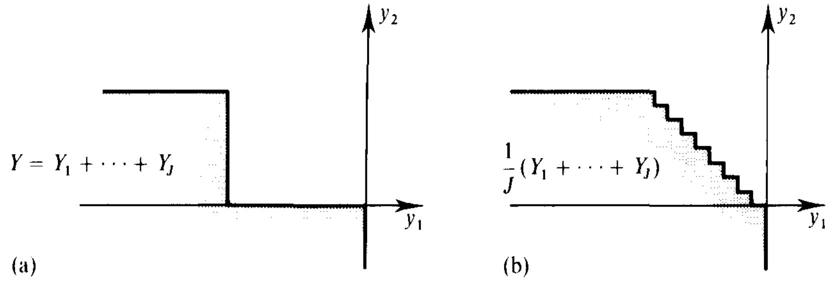


Figure 5.E.2
An example of the convexifying effects of aggregation.
(a) The individual production set.
(b) The average production set.

that displayed in 5.E.2(a). Defining the average production set as $(1/J)(Y_1 + \cdots + Y_J) = \{y: y = (1/J)(y_1 + \cdots + y_J)$ for some $y_j \in Y_j, j = 1, \ldots, J\}$, we see that for large J, this set is nearly convex, as depicted in Figure 5.E.2(b).[12]

5.F Efficient Production

Because much of welfare economics focuses on efficiency (see, for example, Chapters 10 and 16), it is useful to have algebraic and geometric characterizations of productions plans that can unambiguously be regarded as nonwasteful. This motivates Definition 5.F.1.

12. Note that this production set is bounded above. This is important because it insures that the individual nonconvexity is of finite size. If the individual production set was like that shown in, say, Figure 5.B.4, where neither the set nor the nonconvexity is bounded, then the average set would display a large nonconvexity (for any J). In Figure 5.B.5, we have a case of an unbounded production set but with a bounded nonconvexity; as for Figure 5.E.2, the average set will in this case be almost convex.

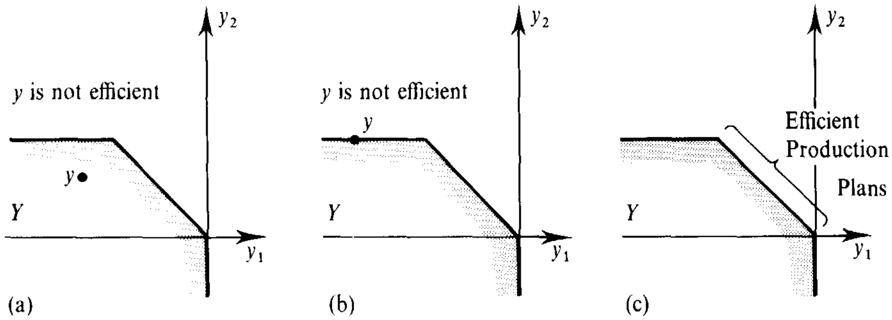


Figure 5.F.1
An efficient production plan must be on the boundary of Y, but not all points on the boundary of Y are efficient.
(a) An inefficient production plan in the interior of Y.
(b) An inefficient production plan at the boundary of Y.
(c) The set of efficient production plans.

Definition 5.F.1: A production vector $y \in Y$ is *efficient* if there is no $y' \in Y$ such that $y' \geq y$ and $y' \neq y$.

In words, a production vector is efficient if there is no other feasible production vector that generates as much output as y using no additional inputs, and that actually produces more of some output or uses less of some input.

As we see in Figure 5.F.1, every efficient y must be on the boundary of Y, but the converse is not necessarily the case: There may be boundary points of Y that are not efficient.

We now show that the concept of efficiency is intimately related to that of supportability by profit maximization. This constitutes our first look at a topic that we explore in much more depth in Chapter 10 and especially in Chapter 16

Proposition 5.F.1 provides an elementary but important result. It is a version of the *first fundamental theorem of welfare economics.*

Proposition 5.F.1: If $y \in Y$ is profit maximizing for some $p \gg 0$, then y is efficient.

Proof: Suppose otherwise: That there is a $y' \in Y$ such that $y' \neq y$ and $y' \geq y$. Because $p \gg 0$, this implies that $p \cdot y' > p \cdot y$, contradicting the assumption that y is profit maximizing. ■

It is worth emphasizing that Proposition 5.F.1 is valid even if the production set is nonconvex. This is illustrated in Figure 5.F.2.

When combined with the aggregation results discussed in Section 5.E, Proposition 5.F.1 tells us that *if a collection of firms each independently maximizes profits with respect to the same fixed price vector $p \gg 0$, then the aggregate production is*

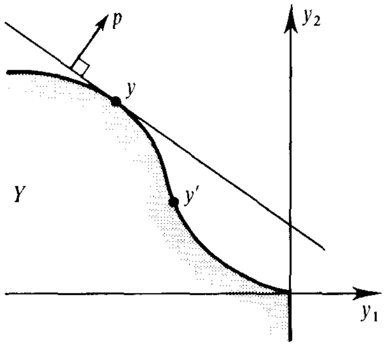


Figure 5.F.2
A profit-maximizing production plan (for $p \gg 0$) is efficient.

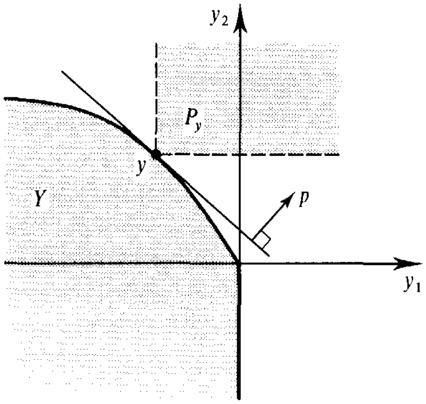


Figure 5.F.3
The use of the separating hyperplane theorem to prove Proposition 5.F.2: If Y is convex, every efficient $y \in Y$ is profit maximizing for some $p \geq 0$.

socially efficient. That is, there is no other production plan for the economy as a whole that could produce more output using no additional inputs. This is in line with our conclusion in Section 5.E that, in the single-output case, the aggregate output level is produced at the lowest-possible cost when all firms maximize profits facing the same prices.

The need for strictly positive prices in Proposition 5.F.1 is unpleasant, but it cannot be dispensed with, as Exercise 5.F.1 asks you to demonstrate.

Exercise 5.F.1: Give an example of a $y \in Y$ that is profit maximizing for some $p \geq 0$ with $p \neq 0$ but that is also inefficient (i.e. not efficient).

A converse of Proposition 5.F.1 would assert that any efficient production vector is profit maximizing for *some* price system. However, a glance at the efficient production y' in Figure 5.F.2 shows that this cannot be true in general. Nevertheless, this converse does hold with the added assumption of convexity. Proposition 5.F.2, which is less elementary than Proposition 5.F.1, is a version of the so-called *second fundamental theorem of welfare economics*.

Proposition 5.F.2: Suppose that Y is convex. Then every efficient production $y \in Y$ is a profit-maximizing production for some nonzero price vector $p \geq 0$.[13]

Proof: This proof is an application of the separating hyperplane theorem for convex sets (see Section M.G of the Mathematical Appendix). Suppose that $y \in Y$ is efficient, and define the set $P_y = \{y' \in \mathbb{R}^L: y' \gg y\}$. The set P_y is depicted in Figure 5.F.3. It is convex, and because y is efficient, we have $Y \cap P_y = \varnothing$. We can therefore invoke the separating hyperplane theorem to establish that there is *some* $p \neq 0$ such that $p \cdot y' \geq p \cdot y''$ for every $y' \in P_y$ and $y'' \in Y$ (see Figure 5.F.3). Note, in particular, that this implies $p \cdot y' \geq p \cdot y$ for every $y' \gg y$. Therefore, we must have $p \geq 0$ because if $p_\ell < 0$ for some ℓ, then we would have $p \cdot y' < p \cdot y$ for some $y' \gg y$ with $y'_\ell - y_\ell$ sufficiently large.

Now take any $y'' \in Y$. Then $p \cdot y' \geq p \cdot y''$ for every $y' \in P_y$. Because y' can be chosen to be arbitrarily close to y, we conclude that $p \cdot y \geq p \cdot y''$ for any $y'' \in Y$; that is, y is profit maximizing for p. ■

13. As the proof makes clear, the result also applies to *weakly efficient* productions, that is, to productions such as y in Figure 5.F.1(b) where there is no $y' \in Y$ such that $y' \gg y$.

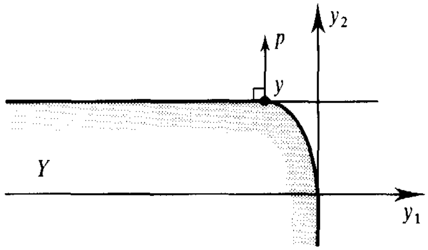


Figure 5.F.4
Proposition 5.C.2 cannot be extended to require $p \gg 0$.

The second part of Proposition 5.F.2 cannot be strengthened to read "$p \gg 0$." In Figure 5.F.4, for example, the production vector y is efficient, but it cannot be supported by any strictly positive price vector.

As an illustration of Proposition 5.F.2, consider a single-output, concave production function $f(z)$. Fix an input vector $\bar{z}$, and suppose that $f(\cdot)$ is differentiable at $\bar{z}$ and $\nabla f(\bar{z}) \gg 0$. Then the production plan that uses input vector $\bar{z}$ to produce output level $f(\bar{z})$ is efficient. Letting the price of output be 1, condition (5.C.2) tells us that the input price vector that makes this efficient production profit maximizing is precisely $w = \nabla f(\bar{z})$, the vector of marginal productivities.

5.G Remarks on the Objectives of the Firm

Although it is logical to take the assumption of preference maximization as a primitive concept for the theory of the consumer, the same cannot be said for the assumption of profit maximization by the firm. Why this objective rather than, say, the maximization of sales revenues or the size of the firm's labor force? The objectives of the firm assumed in our economic analysis should emerge from the objectives of those individuals who control it. Firms in the type of economies we consider are owned by individuals who, wearing another hat, are also consumers. A firm owned by a single individual has well-defined objectives: those of the owner. In this case, the only issue is whether this objective coincides with profit maximization. Whenever there is more than one owner, however, we have an added level of complexity. Indeed, we must either reconcile any conflicting objectives the owners may have or show that no conflict exists.

Fortunately, it is possible to resolve these issues and give a sound theoretical grounding to the objective of profit maximization. We shall now show that under reasonable assumptions this is the goal that all owners would agree upon.

Suppose that a firm with production set Y is owned by consumers. Ownership here simply means that each consumer $i = 1, \ldots, I$ is entitled to a share $\theta_i \geq 0$ of profits, where $\sum_i \theta_i = 1$ (some of the θ_i's may equal zero). Thus, if the production decision is $y \in Y$, then a consumer i with utility function $u_i(\cdot)$ achieves the utility level

$$\begin{aligned} \underset{x_i \geq 0}{\text{Max}} \quad & u_i(x_i) \\ \text{s.t. } & p \cdot x_i \leq w_i + \theta_i p \cdot y, \end{aligned}$$

where w_i is consumer i's nonprofit wealth. Hence at fixed prices, higher profit increases consumer-owner i's overall wealth and expands her budget set, a desirable outcome. It follows that at any fixed price vector p, the consumer–owners *unanimously* prefer that the firm implement a production plan $y' \in Y$ instead of $y \in Y$ whenever $p \cdot y' > p \cdot y$. Hence, we conclude that if we maintain the assumption of price-taking behavior, all owners would agree, whatever their utility functions, to instruct the manager of the firm to maximize profits.[14]

It is worth emphasizing three of the implicit assumptions in the previous reasoning: (i) prices are fixed and do not depend on the actions of the firm, (ii) profits are not uncertain, and (iii) managers can be controlled by owners. We comment on these assumptions very informally.

(i) If prices may depend on the production of the firm, the objective of the owners may depend on their tastes as consumers. Suppose, for example, that each consumer has no wealth from sources other than the firm ($w_i = 0$), that $L = 2$, and that the firm produces good 1 from good 2 with production function $f(\cdot)$. Also, normalize the price of good 2 to be 1, and suppose that the price of good 1, in terms of good 2, is $p(q)$ if output is q. If, for example, the preferences of the owners are such that they care only about the consumption of good 2, then they will unanimously want to solve $\text{Max}_{z \geq 0}\, p(f(z))f(z) - z$. This maximizes the amount of good 2 that they get to consume. On the other hand, if they want to consume only good 1, then they will wish to solve $\text{Max}_{z \geq 0}\, f(z) - [z/p(f(z))]$ because if they earn $p(f(z))f(z) - z$ units of good 2, then end up with $[p(f(z))f(z) - z]/p(f(z))$ units of good 1. But these two problems have different solutions. (Check the first-order conditions.) Moreover, as this suggests, if the owners differ in their tastes as consumers, then they will not agree about what they want the firm to do (Exercise 5.G.1 elaborates on this point.)

(ii) If the output of the firm is random, then it is crucial to distinguish whether the output is sold before or after the uncertainty is resolved. If the output is sold after the uncertainty is resolved (as in the case of agricultural products sold in spot markets after harvesting), then the argument for a unanimous desire for profit maximization breaks down. Because profit, and therefore derived wealth, are now uncertain, the risk attitudes and expectations of owners will influence their preferences with regard to production plans. For example, strong risk averters will prefer relatively less risky production plans than moderate risk averters.

On the other hand, if the output is sold before uncertainty is resolved (as in the case of agricultural products sold in futures markets before harvesting), then the risk is fully carried by the buyer. The profit of the firm is not uncertain, and the argument for unanimity in favor of profit maximization still holds. In effect, the firm can be thought of as producing a commodity that is sold before uncertainty is resolved in a market of the usual kind. (Further analysis of this issue would take us too far afield. We come back to it in Section 19.G after covering the foundations of decision theory under uncertainty in Chapter 6.)

(iii) It is plain that shareholders cannot usually exercise control directly. They need managers, who, naturally enough, have their own objectives. Especially if ownership is very diffuse, it is an important theoretical challenge to understand how and to what extent managers are, or can be, controlled by owners. Some relevant considerations are factors such as the degree of observability of managerial actions

14. In actuality, there are public firms and quasipublic organizations such as universities that do not have *owners* in the sense that private firms have shareholders. Their objectives may be different, and the current discussion does not apply to them.

and the stake of individual owners. [These issues will be touched on in Section 14.C (agency contracts as a mechanism of internal control) and in Section 19.G (stock markets as a mechanism of external control).]

APPENDIX A: THE LINEAR ACTIVITY MODEL

The saliency of the model of production with convexity and constant returns to scale technologies recommends that we examine it in some further detail.

Given a constant returns to scale technology Y, the *ray* generated (or spanned) by a vector $\bar{y} \in Y$ is the set $\{y \in Y: y = \alpha\bar{y}$ for some scalar $\alpha \geq 0\}$. We can think of a ray as representing a production *activity* that can be run at any *scale of operation.* That is, the production plan $\bar{y}$ can be scaled up or down by any factor $\alpha \geq 0$, generating, in this way, other possible production plans.

We focus here on a particular case of constant returns to scale technologies that lends itself to explicit computation and is therefore very important in applications. We assume that we are given as a primitive of our theory a list of *finitely many activities* (say M), each of which can be run at any scale of operation and any number of which can be run simultaneously. Denote the M activities, to be called the *elementary activities*, by $a_1 \in \mathbb{R}^L, \dots, a_M \in \mathbb{R}^L$. Then, the production set is

$$Y = \{y \in \mathbb{R}^L: y = \sum_{m=1}^{M} \alpha_m a_m \text{ for some scalars } (\alpha_1, \dots, \alpha_M) \geq 0\}.$$

The scalar α_m is called the *level of elementary activity m*; it measures the scale of operation of the mth activity. Geometrically, Y is a *polyhedral cone*, a set generated as the convex hull of a finite number of rays.

An activity of the form $(0, \dots, 0, -1, 0, \dots, 0)$, where -1 is in the ℓth place, is known as the *disposal activity* for good ℓ. Henceforth, we shall always assume that, in addition to the M listed elementary activities, the L disposal activities are also available. Figure 5.AA.1 illustrates a production set arising in the case where $L = 2$ and $M = 2$.

Given a price vector $p \in \mathbb{R}^L_+$, a profit-maximizing plan exists in Y if and only if $p \cdot a_m \leq 0$ for every m. To see this, note that if $p \cdot a_m < 0$, then the profit-maximizing level of activity m is $\alpha_m = 0$. If $p \cdot a_m = 0$, then any level of activity m generates zero profits. Finally, if $p \cdot a_m > 0$ for some m, then by making α_m arbitrarily large, we could generate arbitrarily large profits. Note that the presence of the disposal activities implies that we must have $p \in \mathbb{R}^L_+$ for a profit-maximizing plan to exist. If $p_\ell < 0$, then the ℓth disposal activity would generate strictly positive (hence, arbitrarily large) profits.

For any price vector p generating zero profits, let $A(p)$ denote the set of activities that generate exactly zero profits: $A(p) = \{a_m: p \cdot a_m = 0\}$. If $a_m \notin A(p)$, then $p \cdot a_m < 0$, and so activity m is not used at prices p. The profit-maximizing supply set $y(p)$ is therefore the convex cone generated by the activities in $A(p)$; that is, $y(p) = \{\sum_{a_m \in A(p)} \alpha_m a_m: \alpha_m \geq 0\}$. The set $y(p)$ is also illustrated in Figure 5.AA.1. In the figure, at price vector p, activity a_1 makes exactly zero profits, and activity a_2

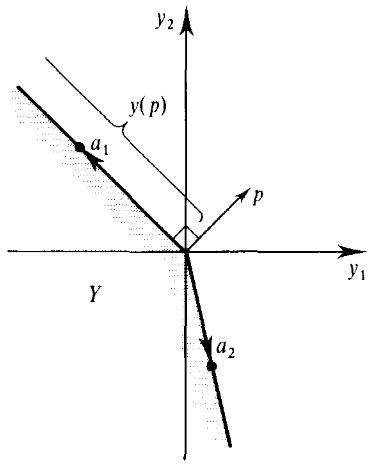


Figure 5.AA.1
A production set generated by two activities.

incurs a loss (if operated at all). Therefore, $A(p) = \{a_1\}$ and $y(p) = \{y: y = \alpha_1 a_1$ for any scalar $\alpha_1 \geq 0\}$, the ray spanned by activity a_1.

A significant result that we shall not prove is that for the linear activity model the converse of the efficiency Proposition 5.F.1 holds exactly; that is, we can strengthen Proposition 5.F.2 to say: *Every efficient $y \in Y$ is a profit-maximizing production for some $p \gg 0$.*

An important special case of the linear activity model is *Leontief's input–output model*. It is characterized by two additional features:

(i) There is one commodity, say the Lth, which is not produced by any activity. For this reason, we will call it the *primary factor*. In most applications of the Leontief model, the primary factor is labor.
(ii) Every elementary activity has at most a single positive entry. This is called the assumption of *no joint production*. Thus, it is as if every good except the primary factor is produced from a certain type of constant returns production function using the other goods and the primary factor as inputs.

The Leontief Input–Output Model with No Substitution Possibilities

The simplest Leontief model is one in which each producible good is produced by only one activity. In this case, it is natural to label the activity that produces good $\ell = 1, \dots, L-1$ as $a_\ell = (a_{1\ell}, \dots, a_{L\ell}) \in \mathbb{R}^L$. So the number of elementary activities M is equal to $L-1$. As an example, in Figure 5.AA.2, for a case where $L = 3$, we represent the unit production isoquant [the set $\{(z_2, z_3): f(z_2, z_3) = 1\}$] for the implied production function of good 1. In the figure, the disposal activities for goods 2 and 3 are used to get rid of any excess of inputs. Because inputs must be used in fixed proportions (disposal aside), this special case is called a *Leontief model with no substitution possibilities*.

If we normalize the activity vectors so that $a_{\ell\ell} = 1$ for all $\ell = 1, \dots, L-1$, then the vector $\alpha = (\alpha_1, \dots, \alpha_{L-1}) \in \mathbb{R}^{L-1}$ of activity levels equals the vector of *gross* production of goods 1 through $L-1$. To determine the levels of *net* production, it is convenient to denote by A the $(L-1) \times (L-1)$ matrix in which the ℓth column is

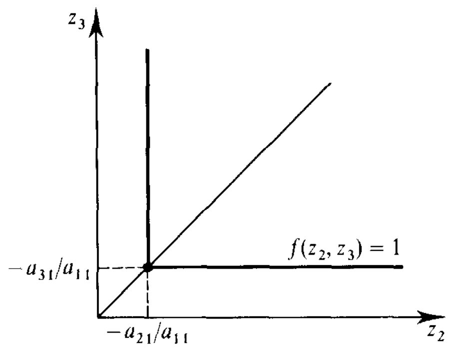


Figure 5.AA.2 Unit isoquant of production function for good 1 in the Leontief model with no substitution.

the negative of the activity vector a_ℓ except that its last entry has been deleted and entry $a_{\ell\ell}$ has been replaced by a zero (recall that entries $a_{k\ell}$ with $k \neq \ell$ are nonpositive):

$$A = \begin{bmatrix} 0 & -a_{12} & \cdots & -a_{1,L-1} \\ -a_{21} & 0 & \cdots & -a_{2,L-1} \\ \vdots & & \ddots & \\ -a_{L-1,1} & -a_{L-1,2} & \cdots & 0 \end{bmatrix}.$$

The matrix A is known as the *Leontief input–output matrix*. Its $k\ell$th entry, $-a_{k\ell} \geq 0$, measures how much of good k is needed to produce one unit of good ℓ. We also denote by $b \in \mathbb{R}^{L-1}$ the vector of primary factor requirements, $b = (-a_{L1}, \ldots, -a_{L,L-1})$. The vector $(I - A)\alpha$ then gives the *net* production levels of the $L - 1$ outputs when the activities are run at levels $\alpha = (\alpha_1, \ldots, \alpha_{L-1})$. To see this, recall that the activities are normalized so that the gross production levels of the $L - 1$ produced goods are exactly $\alpha = (\alpha_1, \ldots, \alpha_{L-1})$. On the other hand, $A\alpha$ gives the amounts of each of these goods that are used as inputs for other produced goods. The difference, $(I - A)\alpha$, is therefore the net production of goods $1, \ldots, L - 1$. In addition, the scalar $b \cdot \alpha$ gives the total use of the primary factor. In summary, with this notation, we can write the set of technologically feasible production vectors (assuming free disposal) as

$$Y = \left\{ y\colon y \leq \begin{bmatrix} I - A \\ -b \end{bmatrix} \alpha \text{ for some } \alpha \in \mathbb{R}^L_+ \right\}.$$

If $(I - A)\bar{\alpha} \gg 0$ for some $\bar{\alpha} \geq 0$, the input–output matrix A is said to be *productive*. That is, the input–output matrix A is productive if there is *some* production plan that can produce positive net amounts of the $L - 1$ outputs, provided only that there is a sufficient amount of primary input available.

A remarkable fact of Leontief input–output theory is the all-or-nothing property stated in Proposition 5.AA.1.

Proposition 5.AA.1: If A is productive, then for any nonnegative amounts of the $L - 1$ producible commodities $c \in \mathbb{R}^{L-1}_+$, there is a vector of activity levels $\alpha \geq 0$ such that $(I - A)\alpha = c$. That is, if A is productive, then it is possible to produce *any* nonnegative net amount of outputs (perhaps for purposes of final consumption), provided only that there is enough primary factor available.

Proof: We will show that if A is productive, then the inverse of the matrix $(I - A)$ exists and is nonnegative. This will give the result because we can then achieve net output levels $c \in \mathbb{R}_+^{L-1}$ by setting the (nonnegative) activity levels $\alpha = (I - A)^{-1}c$.

To prove the claim, we begin by establishing a matrix-algebra fact. We show that if A is productive, then the matrix $\sum_{n=0}^{N} A^n$, where A^n is the nth power of A, approaches a limit as $N \to \infty$. Because A has only nonnegative entries, every entry of $\sum_{n=0}^{N} A^n$ is *nondecreasing* with N. Therefore, to establish that $\sum_{n=0}^{N} A^n$ has a limit, it suffices to show that there is an upper bound for its entries. Since A is productive, there is an $\bar{\alpha}$ and $\bar{c} \gg 0$ such that $\bar{c} = (I - A)\bar{\alpha}$. If we premultiply both sides of this equality by $\sum_{n=0}^{N} A^n$, we get $(\sum_{n=0}^{N} A^n)\bar{c} = (I - A^{N+1})\bar{\alpha}$ (recall that $A^0 = I$). But $(I - A^{N+1})\bar{\alpha} \le \bar{\alpha}$ because all elements of the matrix A^{N+1} are nonnegative. Therefore, $(\sum_{n=0}^{N} A^n)\bar{c} \le \bar{\alpha}$. With $\bar{c} \gg 0$, this implies that no entry of $\sum_{n=0}^{N} A^n$ can exceed $\{\text{Max}\,\{\bar{\alpha}_1, \ldots, \bar{\alpha}_{L-1}\}/\text{Min}\,\{\bar{c}_1, \ldots, \bar{c}_{L-1}\}]$, and so we have established the desired upper bound. We conclude, therefore, that $\sum_{n=0}^{\infty} A^n$ exists.

The fact that $\sum_{n=0}^{\infty} A^n$ exists must imply that $\lim_{N\to\infty} A^N = 0$. Thus, since $(\sum_{n=0}^{N} A^n)(I - A) = (I - A^{N+1})$ and $\lim_{N\to\infty} (I - A^{N+1}) = I$, it must be that $\sum_{n=0}^{\infty} A^n = (I - A)^{-1}$. (If A is a single number, this is precisely the high-school formula for adding up the terms of a geometric series.) The conclusion is that $(I - A)^{-1}$ exists and that all its entries are nonnegative. This establishes the result. ■

The focus on $\sum_{n=0}^{N} A^n$ in the proof of Proposition 5.AA.1 makes economic sense. Suppose we want to produce the vector of final consumptions $c \in \mathbb{R}_+^{L-1}$. How much total production will be needed? To produce final outputs $c = A^0 c$, we need to use as inputs the amounts $A(A^0 c) = Ac$ of produced goods. In turn, to produce these amounts requires that $A(Ac) = A^2 c$ of additional produced goods be used, and so on ad infinitum. The *total* amounts of goods required to be produced is therefore the limit of $(\sum_{n=0}^{N} A^n)c$ as $N \to \infty$. Thus, we can conclude that the vector $c \ge 0$ will be producible if and only if $\sum_{n=0}^{\infty} A^n$ is well defined (i.e., all its entries are finite).

Example 5.AA.1: Suppose that $L = 3$, and let $a_1 = (1, -1, -2)$ and $a_2 = (-\beta, 1, -4)$ for some constant $\beta \ge 0$. Activity levels $\alpha = (\alpha_1, \alpha_2)$ generate a positive net output of good 2 if $\alpha_2 > \alpha_1$; they generate a positive net output of good 1 if $\alpha_1 - \beta\alpha_2 > 0$. The input-output matrix A and the matrix $(I - A)^{-1}$ are

$$A = \begin{bmatrix} 0 & \beta \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad (I - A)^{-1} = \frac{1}{1-\beta}\begin{bmatrix} 1 & \beta \\ 1 & 1 \end{bmatrix}.$$

Hence, matrix A is productive if and only if $\beta < 1$. Figure 5.AA.3(a) depicts a case where A is productive. The shaded region represents the vectors of net outputs that can be generated using the two activity vectors; note how the two activity vectors can span all of $\mathbb{R}_+^2$. In contrast, in Figure 5.AA.3(b), the matrix A is not productive: No strictly positive vector of net outputs can be achieved by running the two activities at nonnegative scales. [Again, the shaded region represents those vectors that can be generated using the two activity vectors, here a set whose only intersection with $\mathbb{R}_+^2$ is the point $(0, 0)$]. Note also that the closer β is to the value 1, the larger the levels of activity required to produce any final vector of consumptions. ■

The Leontief Model with Substitution Possibilities

We now move to the consideration of the general Leontief model in which each good may have more than one activity capable of producing it. We shall see that the

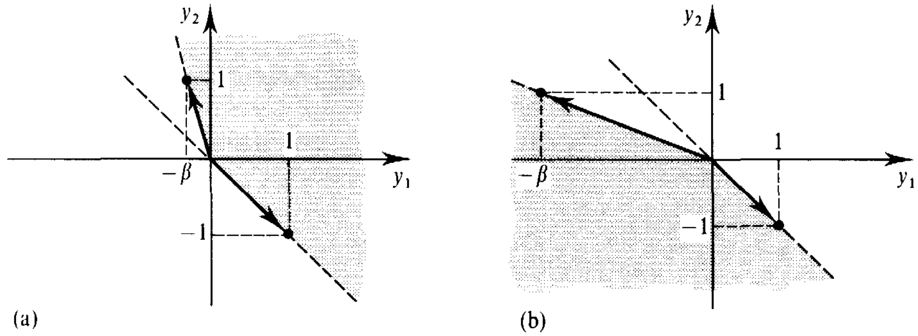


Figure 5.AA.3 Leontief model of Example 5.AA.1. (a) Productive ($\beta < 1$). (b) Unproductive ($\beta \geq 1$).

properties of the nonsubstitution model remain very relevant for the more general case where substitution is possible.

The first thing to observe is that the computation of the production function of a good, say good 1, now becomes a linear programming problem (see Section M.M of the Mathematical Appendix). Indeed, suppose that $a_1 \in \mathbb{R}^L, \ldots, a_{M_1} \in \mathbb{R}^L$ is a list of M_1 elementary activities capable of producing good 1 and that we are given initial levels of goods $2, \ldots, L$ equal to $z_2, \ldots, z_L$. Then the maximal possible production of good 1 given these available inputs $f(z_2, \ldots, z_L)$ is the solution to the problem

$$\begin{aligned} \underset{\alpha_1 \geq 0, \ldots, \alpha_{M_1} \geq 0}{\text{Max}} \quad & \alpha_1 a_{11} + \cdots + \alpha_{M_1} a_{1M_1} \\ \text{s.t.} \quad & \textstyle\sum_{m=1}^{M_1} \alpha_m a_{\ell m} \geq -z_\ell \quad \text{for all } \ell = 2, \ldots, L. \end{aligned}$$

We also know from linear programming theory that the $L-1$ dual variables $(\lambda_2, \ldots, \lambda_L)$ of this problem (i.e., the multipliers associated with the $L-1$ constraints) can be interpreted as the marginal productivities of the $L-1$ inputs. More precisely, for any $\ell = 2, \ldots, L$, we have $(\partial f/\partial z_\ell)^+ \leq \lambda_\ell \leq (\partial f/\partial z_\ell)^-$, where $(\partial f/\partial z_\ell)^+$ and $(\partial f/\partial z_\ell)^-$ are, respectively, the left-hand and right-hand ℓth partial derivatives of $f(\cdot)$ at $(z_2, \ldots, z_L)$.

Figure 5.AA.4 illustrates the unit isoquant for the case in which good 1 can be produced using two other goods (goods 2 and 3) as inputs with two possible activities $a_1 = (1, -2, -1)$ and $a_2 = (1, -1, -2)$. If the ratio of inputs is either higher than 2 or lower than $\frac{1}{2}$, one of the disposal activities is used to eliminate any excess inputs.

For any vector $y \in \mathbb{R}^L$, it will be convenient to write $y = (y_{-L}, y_L)$, where $y_{-L} = (y_1, \ldots, y_{L-1})$. We shall assume that our Leontief model is *productive* in the sense that there is a technologically feasible vector $y \in Y$ such that $y_{-L} \gg 0$.

A striking implication of the Leontief structure (constant returns, no joint products, single primary factor) is that we can associate with each good a *single optimal technique* (which could be a mixture of several of the elementary techniques corresponding to that good). What this means is that optimal techniques (one for each output) supporting efficient production vectors can be chosen independently of the particular output vector that is being produced (as long as the net output of every producible good is positive). Thus, although substitution is possible in principle, efficient production requires no substitution of techniques as desired final consumption levels change. This is the content of the celebrated *non-substitution theorem* (due to Samuelson [1951]).

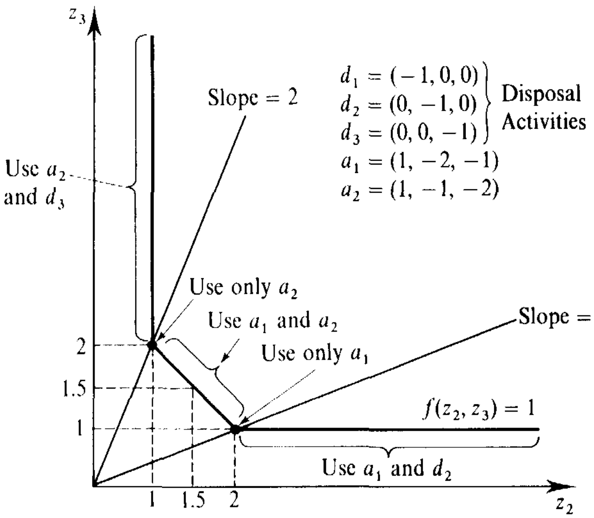


Figure 5.AA.4
Unit isoquant of production function of good 1, in the Leontief model with substitution.

Proposition 5.AA.2: (*The Nonsubstitution Theorem*) Consider a productive Leontief input output model with $L - 1$ producible goods and $M_\ell \geq 1$ elementary activities for the producible good $\ell = 1, \ldots, L - 1$. Then there exist $L - 1$ activities $(a_1, \ldots, a_{L-1})$, with a_ℓ possibly a nonnegative linear combination of the M_ℓ elementary activities for producing good ℓ, such that *all* efficient production vectors with $y_{-L} \gg 0$ can be generated with these $L - 1$ activities.

Proof: Let $y \in Y$ be an efficient production vector with $y_{-L} \gg 0$. As a general matter, the vector y must be generated by a collection of $L - 1$ activities $(a_1, \ldots, a_{L-1})$ (some of these may be "mixtures" of the original activities) run at activity levels $(\alpha_1, \ldots, \alpha_{L-1}) \gg 0$; that is, $y = \sum_{\ell=1}^{L-1} \alpha_\ell a_\ell$. We show that any efficient production plan y' with $y'_{-L} \gg 0$ can be achieved using the activities $(a_1, \ldots, a_{L-1})$.

Since $y \in Y$ is efficient, there exists a $p \gg 0$ such that y is profit maximizing with respect to p (this is from Proposition 5.F.2, as strengthened for the linear activity model). From $p \cdot a_\ell \leq 0$ for all $\ell = 1, \ldots, L - 1$, $\alpha_\ell > 0$, and

$$0 = p \cdot y = p \cdot \left(\sum_{\ell=1}^{L-1} \alpha_\ell a_\ell \right) = \sum_{\ell=1}^{L-1} \alpha_\ell p \cdot a_\ell,$$

it follows that $p \cdot a_\ell = 0$ for all $\ell = 1, \ldots, L - 1$.

Consider now any other efficient production $y' \in Y$ with $y'_{-L} \gg 0$. We want to show that y' can be generated from the activities $(a_1, \ldots, a_{L-1})$. Denote by A the input output matrix associated with $(a_1, \ldots, a_{L-1})$. Because $y_{-L} \gg 0$, it follows by definition that A is productive. Therefore, by Proposition 5.AA.1, we know that there are activity levels $(\alpha''_1, \ldots, \alpha''_{L-1})$ such that the production vector $y'' = \sum_{\ell=1}^{L-1} \alpha''_\ell a_\ell$ has $y''_{-L} = y'_{-L}$. Note that since $p \cdot a_\ell = 0$ for all $\ell = 1, \ldots, L - 1$, we must have $p \cdot y'' = 0$. Thus, y'' is profit maximizing for $p \gg 0$ (recall that the maximum profits for p are zero), and so it follows that y'' is efficient by Proposition 5.F.1. But then we have two production vectors, y' and y'', with $y'_{-L} = y''_{-L}$, and both are efficient. It must therefore be that $y''_L = y'_L$. Hence, we conclude that y' can be produced using only the activities $(a_1, \ldots, a_{L-1})$, which is the desired result. ■

The nonsubstitution theorem depends critically on the presence of only one primary factor. This makes sense. With more than one primary factor, the optimal choice of techniques should depend on the relative prices of these factors. In turn, it is logical to expect that these relative prices will not be independent of the composition of final demand (e.g., if demand moves from land-intensive goods toward labour-intensive goods, we would expect the price of labor relative to the price of land to increase). Nonetheless, it is worth mentioning that the nonsubstitution result remains valid as long as the prices of the primary factors do not change.

For further reading on the material discussed in this appendix see Gale (1960).

REFERENCES

Champsaur, P., and J.-C. Milleron (1983). *Advanced Exercises in Microeconomics.* Cambridge, Mass.: Harvard University Press.

Debreu, G. (1959). *Theory of Value.* New York: Wiley.

Gale, D. (1960). *The Theory of Linear Economic Models.* New York: McGraw-Hill.

Koopmans, T. (1957). *Three Essays on the State of Economic Science.* Essay 1. New York: McGraw-Hill.

McFadden, D. (1978). Cost, revenue and profit functions. In *Production economics: A dual approach to theory and applications*, edited by M. Fuss and D. McFadden. Amsterdam: North-Holland.

McKenzie, L. (1959). On the existence of a general equilibrium for a competitive market. *Econometrica* **27**: 54–71.

Samuelson, P. (1951). Abstract of a theorem concerning substitutability in open Leontief models. In *Activity analysis of production and allocation*, edited by T. Koopmans. New York: Wiley.

EXERCISES

5.B.1^A In text

5.B.2^A In text.

5.B.3^A In text.

5.B.4^B Suppose that Y is a production set, interpreted now as the technology of a single production unit. Denote by Y^+ the additive closure of Y, that is, the smallest production set that is additive and contains Y (in other words, Y^+ is the total production set if technology Y can be replicated an arbitrary number of times). Represent Y^+ for each of the examples of production sets depicted graphically in Section 5.B. In particular, note that for the typical decreasing returns technology of Figure 5.B.5(a), the additive closure Y^+ violates the closedness condition (ii). Discuss and compare with the case corresponding to Figure 5.B.5(b), where Y^+ is closed.

5.B.5^C Show that if Y is closed and convex, and $-\mathbb{R}^L_+ \subset Y$, then free disposal holds.

5.B.6^B There are three goods. Goods 1 and 2 are inputs. The third, with amounts denoted by q, is an output. Output can be produced by two techniques that can be operated simultaneously or separately. The techniques are not necessarily linear. The first (respectively, the second) technique uses only the first (respectively, the second) input. Thus, the first (respectively, the second) technique is completely specified by $\phi_1(q_1)$ [respectively, $\phi_2(q_2)$], the minimal amount of input one (respectively, two) *sufficient* to produce the amount of output q_1 (respectively, q_2). The two functions $\phi_1(\cdot)$ and $\phi_2(\cdot)$ are increasing and $\phi_1(0) = \phi_2(0) = 0$.

(a) Describe the three-dimensional production set associated with these two techniques. Assume free disposal.

(b) Give sufficient conditions on $\phi_1(\cdot), \phi_2(\cdot)$ for the production set to display additivity.

(c) Suppose that the input prices are w_1 and w_2. Write the first-order necessary conditions for profit maximization and interpret. Under which conditions on $\phi_1(\cdot), \phi_2(\cdot)$ will the necessary conditions be sufficient?

(d) Show that if $\phi_1(\cdot)$ and $\phi_2(\cdot)$ are strictly concave, then a cost-minimizing plan cannot involve the simultaneous use of the two techniques. Interpret the meaning of the concavity requirement, and draw isoquants in the two-dimensional space of input uses.

5.C.1^{A} In text.

5.C.2^{A} In text.

5.C.3^{B} Establish properties (viii) and (ix) of Proposition 5.C.2. [*Hint*: Property (viii) is easy; (ix) is more difficult. Try the one-input case first.]

5.C.4^{A} Establish properties (i) to (vii) of Proposition 5.C.2 for the case in which there are multiple outputs.

5.C.5^{A} Argue that for property (iii) of Proposition 5.C.2 to hold, it suffices that $f(\cdot)$ be quasiconcave. Show that quasiconcavity of $f(\cdot)$ is compatible with increasing returns.

5.C.6^{C} Suppose $f(z)$ is a concave production function with $L-1$ inputs $(z_1, \ldots, z_{L-1})$. Suppose also that $\partial f(z)/\partial z_\ell \geq 0$ for all ℓ and $z \geq 0$ and that the matrix $D^2 f(z)$ is negative definite at all z. Use the firm's first-order conditions and the implicit function theorem to prove the following statements:

(a) An increase in the output price always increases the profit-maximizing level of output.

(b) An increase in output price increases the demand for *some* input.

(c) An increase in the price of an input leads to a reduction in the demand for the input.

5.C.7^{C} A price-taking firm producing a single product according to the technology $q = f(z_1, \ldots, z_{L-1})$ faces prices p for its output and $w_1, \ldots, w_{L-1}$ for each of its inputs. Assume that $f(\cdot)$ is strictly concave and increasing, and that $\partial^2 f(z)/\partial z_\ell\, \partial z_k < 0$ for all $\ell \neq k$. Show that for all $\ell = 1, \ldots, L-1$, the factor demand functions $z_\ell(p, w)$ satisfy $\partial z_\ell(p, w)/\partial p > 0$ and $\partial z_\ell(p, w)/\partial w_k < 0$ for all $k \neq \ell$.

5.C.8^{B} Alpha Incorporated (AI) produces a single output q from two inputs z_1 and z_2. You are assigned to determine AI's technology. You are given 100 monthly observations. Two of these monthly observations are shown in the following table:

	Input prices		*Input levels*		*Output price*	*Output level*
Month	w_1	w_2	z_1	z_2	p	q
3	3	1	40	50	4	60
95	2	2	55	40	4	60

In light of these two monthly observations, what problem will you encounter in trying to accomplish your task?

5.C.9^{A} Derive the profit function $\pi(p)$ and supply function (or correspondence) $y(p)$ for the single-output technologies whose production functions $f(z)$ are given by

(a) $f(z) = \sqrt{z_1} + z_2$.

(b) $f(z) = \sqrt{\text{Min}\{z_1, z_2\}}$.

(c) $f(z) = (z_1^\rho + z_2^\rho)^{1/\rho}$, for $\rho \leq 1$.

5.C.10$^{\text{A}}$ Derive the cost function $c(w, q)$ and conditional factor demand functions (or correspondences) $z(w, q)$ for each of the following single-output constant return technologies with production functions given by

(a) $f(z) = z_1 + z_2$ (perfect substitutable inputs)

(b) $f(z) = \text{Min}\{z_1, z_2\}$ (Leontief technology)

(c) $f(z) = (z_1^\rho + z_2^\rho)^{1/\rho}$, $\rho \leq 1$ (constant elasticity of substitution technology)

5.C.11$^{\text{A}}$ Show that $\partial z_\ell(w, q)/\partial q > 0$ if and only if marginal cost at q is increasing in w_ℓ.

5.C.12$^{\text{A}}$ We saw at the end of Section 5.B that any convex Y can be viewed as the section of a constant returns technology $Y' \subset \mathbb{R}^{L+1}$, where the $L + 1$ coordinate is fixed at the level -1. Show that if $y \in Y$ is profit maximizing at prices p then $(y, -1) \in Y'$ is profit maximizing at $(p, \pi(p))$, that is, profits emerge as the price of the implicit fixed input. The converse is also true: If $(y, -1) \in Y'$ is profit maximizing at prices (p, p_{L+1}), then $y \in Y$ is profit maximizing at p and the profit is p_{L+1}.

5.C.13$^{\text{B}}$ A price-taking firm produces output q from inputs z_1 and z_2 according to a differentiable concave production function $f(z_1, z_2)$. The price of its output is $p > 0$, and the prices of its inputs are $(w_1, w_2) \gg 0$. However, there are two unusual things about this firm. First, rather than maximizing profit, the firm maximizes revenue (the manager wants her firm to have bigger dollar sales than any other). Second, the firm is cash constrained. In particular, it has only C dollars on hand before production and, as a result, its total expenditures on inputs cannot exceed C.

Suppose one of your econometrician friends tells you that she has used repeated observations of the firm's revenues under various output prices, input prices, and levels of the financial constraint and has determined that the firm's revenue level R can be expressed as the following function of the variables (p, w_1, w_2, C):

$$R(p, w_1, w_2, C) = p[\gamma + \ln C - \alpha \ln w_1 - (1 - \alpha) \ln w_2].$$

(γ and α are scalars whose values she tells you.) What is the firm's use of input z_1 when prices are (p, w_1, w_2) and it has C dollars of cash on hand?

5.D.1$^{\text{A}}$ In text.

5.D.2$^{\text{A}}$ In text.

5.D.3$^{\text{B}}$ Suppose that a firm can produce good L from $L - 1$ factor inputs ($L > 2$). Factor prices are $w \in \mathbb{R}^{L-1}$ and the price of output is p. The firm's differentiable cost function is $c(w, q)$. Assume that this function is strictly convex in q. However, although $c(w, q)$ is the cost function when all factors can be freely adjusted, factor 1 cannot be adjusted in the short run.

Suppose that the firm is initially at a point where it is producing its long-run profit-maximizing output level of good L given prices w and p, $q(w, p)$ [i.e., the level that is optimal under the long-run cost conditions described by $c(w, q)$], and that all inputs are optimally adjusted [i.e., $z_\ell = z_\ell(w, q(w, p))$ for all $\ell = 1, \ldots, L - 1$, where $z_\ell(\cdot,\cdot)$ is the long-run input demand function]. Show that the firm's profit-maximizing output response to a marginal increase in the price of good L is larger in the long run than in the short run. [*Hint*: Define a short-run cost function $c_s(w, q \mid z_1)$ that gives the minimized costs of producing output level q given that input 1 is fixed at level z_1.]

5.D.4[B] Consider a firm that has a distinct set of inputs and outputs. The firm produces M outputs; let $q = (q_1, \ldots, q_M)$ denote a vector of its output levels. Holding factor prices fixed, $C(q_1, \ldots, q_M)$ is the firm's cost function. We say that $C(\cdot)$ is *subadditive* if for all $(q_1, \ldots, q_M)$, there is no way to break up the production of amounts $(q_1, \ldots, q_M)$ among several firms, each with cost function $C(\cdot)$, and lower the costs of production. That is, there is no set of, say, J firms and collection of production vectors $\{q_j = (q_{1j}, \ldots, q_{Mj})\}_{j=1}^{J}$ such that $\sum_j q_j = q$ and $\sum_j C(q_j) < C(q)$. When $C(\cdot)$ is subadditive, it is usual to say that the industry is a *natural monopoly* because production is cheapest when it is done by only one firm.

(a) Consider the single-output case, $M = 1$. Show that if $C(\cdot)$ exhibits decreasing average costs, then $C(\cdot)$ is subadditive.

(b) Now consider the multiple-output case, $M > 1$. Show by example that the following multiple-output extension of the decreasing average cost assumption is *not* sufficient for $C(\cdot)$ to be subadditive:

$C(\cdot)$ exhibits *decreasing ray average cost* if for any $q \in \mathbb{R}_+^M$,

$$C(q) > C(kq)/k \text{ for all } k > 1.$$

(c) (Harder) Prove that, if $C(\cdot)$ exhibits decreasing ray average cost *and* is quasiconvex, then $C(\cdot)$ is subadditive. [Assume that $C(\cdot)$ is continuous, increasing, and satisfies $C(0) = 0$.]

5.D.5[B] Suppose there are two goods: an input z and an output q. The production function is $q = f(z)$. We assume that $f(\cdot)$ exhibits increasing returns to scale.

(a) Assume that $f(\cdot)$ is differentiable. Do the increasing returns of $f(\cdot)$ imply that the average product is necessarily nondecreasing in input? What about the marginal product?

(b) Suppose there is a representative consumer with the utility function $u(q) - z$ (the negative sign indicates that the input is taken away from the consumer). Suppose that $\bar{q} = f(\bar{z})$ is a production plan that maximizes the representative consumer utility. Argue, either mathematically or economically (disregard boundary solutions), that the equality of marginal utility and marginal cost is a necessary condition for this maximization problem.

(c) Assume the existence of a representative consumer as in (b). "The equality of marginal cost and marginal utility is a sufficient condition for the optimality of a production plan." Right or wrong? Discuss.

5.E.1[A] Assuming that every $\pi_j(\cdot)$ is differentiable and that you already know that $\pi^*(p) = \sum_{j=1}^{J} \pi_j(p)$, give a proof of $y^*(p) = \sum_{j=1}^{J} y_j(p)$ using differentiability techniques.

5.E.2[A] Verify that Proposition 5.E.1 and its interpretation do not depend on any convexity hypothesis on the sets $Y_1, \ldots, Y_J$.

5.E.3[B] Assuming that the sets $Y_1, \ldots, Y_J$ are convex and satisfy the free disposal property, and that $\sum_{j=1}^{J} Y_j$ is closed, show that the latter set equals $\{y: p \cdot y \leq \sum_{j=1}^{J} \pi_j(p) \text{ for all } p \gg 0\}$.

5.E.4[B] One output is produced from two inputs. There are many technologies. Every technology can produce up to one unit of output (but no more) with fixed and proportional input requirements z_1 and z_2. So a technology is characterized by $z = (z_1, z_2)$, and we can describe the population of technologies by a density function $g(z_1, z_2)$. Take this density to be uniform on the square $[0, 10] \times [0, 10]$.

(a) Given the input prices $w = (w_1, w_2)$, solve the profit maximization problem of a firm with characteristics z. The output price is 1.

(b) More generally, find the profit function $\pi(w_1, w_2, 1)$ for

$$w_1 \geq \frac{1}{10} \quad \text{and} \quad w_2 \geq \frac{1}{10}.$$

(c) Compute the aggregate input demand function. Ideally, do that directly, and check that the answer is correct by using your finding in (b); this way you also verify (b).

(d) What can you say about the aggregate production function? If you were to assume that the profit function derived in (b) is valid for $w_1 \geq 0$ and $w_2 \geq 0$, what would the underlying aggregate production function be?

5.E.5^A (M. Weitzman) Suppose that there are J single-output plants. Plant j's average cost is $AC_j(q_j) = \alpha + \beta_j q_j$ for $q_j \geq 0$. Note that the coefficient α is the same for all plants but that the coefficient β_j may differ from plant to plant. Consider the problem of determining the cost-minimizing aggregate production plan for producing a total output of q, where $q < (\alpha/\mathrm{Max}_j |\beta_j|)$.

(a) If $\beta_j > 0$ for all j, how should output be allocated among the J plants?

(b) If $\beta_j < 0$ for all j, how should output be allocated among the J plants?

(c) What if $\beta_j > 0$ for some plants and $\beta_j < 0$ for others?

5.F.1^A In text.

5.G.1^B Let $f(z)$ be a single-input, single-output production function. Suppose that owners have quasilinear utilities with the firm's input as the numeraire.

(a) Show that a necessary condition for consumer–owners to unanimously agree to a production plan z is that consumption shares among owners at prices $p(z)$ coincide with ownership shares.

(b) Suppose that ownership shares are identical. Comment on the conflicting instructions to managers and how they depend on the consumer–owners' tastes for output.

(c) With identical preferences and ownership shares, argue that owners will unanimously agree to maximize profits in terms of input. (Recall that we are assuming preferences are quasilinear with respect to input; hence, the numeraire is intrinsically determined.)

5.AA.1^A Compute the cost function $c(w, 1)$ and the input demand $z(w, 1)$ for the production function in Figure 5.AA.4. Verify that whenever $z(w, 1)$ is single-valued, we have $z(w, 1) = \nabla_w c(w, 1)$.

5.AA.2^B Consider a Leontief input–output model with no substitution. Assume that the input matrix A is productive and that the vector of primary factor requirements b is strictly positive.

(a) Show that for any $\alpha \geq 0$, the production plan

$$y = \begin{bmatrix} I - A \\ -b \end{bmatrix} \alpha.$$

is efficient.

(b) Fixing the price of the primary factor to equal 1, show that any production plan with $\alpha \gg 0$ is profit maximizing at a unique vector of prices.

(c) Show that the prices obtained in (b) have the interpretation of amounts of the primary factor directly or indirectly embodied in the production of one unit of the different goods.

(d) (Harder) Suppose that A corresponds to the techniques singled out by the nonsubstitution theorem for a model that, in principle, admits substitution. Show that every component of the price vector obtained from A in (c) is less than or equal to the corresponding component of the price vector obtained from any other selection of techniques.

5.AA.3[B] There are two produced goods and labor. The input–output matrix is

$$A = \begin{bmatrix} 0 & 1 \\ \alpha & 0 \end{bmatrix}.$$

Here $a_{\ell k}$ is the amount of good ℓ required to produce one unit of good k.

(a) Let $\alpha = \frac{1}{2}$, and suppose that the labor coefficients vector is

$$b = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

where b_1 (respectively, b_2) is the amount of labor required to produce one unit of good 1 (respectively, good 2). Represent graphically the production possibility set (i.e., the locus of possible productions) for the two goods if the total availability of labor is 10.

(b) For the values of α and b in (a), compute equilibrium prices p_1, p_2 (normalize the wage to equal 1) from the profit maximization conditions (assume positive production of the two goods).

(c) For the values of α and b in (a), compute the amount of labor directly or indirectly incorporated into the production of one net (i.e., available for consumption) unit of good 1. How does this amount relate to your answer in **(b)**?

(d) Suppose there is a second technique to produce good 2. To

$$\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad b_2 = 2$$

we now add

$$\begin{bmatrix} a'_{12} \\ a'_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}, \qquad b'_2 = \beta.$$

Taking the two techniques into account, represent graphically the locus of amounts of good 1 and of labor necessary to produce one unit of good 2. (Assume free disposal.)

(e) In the context of **(d)**, what does the nonsubstitution theorem say? Determine the value of β at which there is a switch of optimal techniques.

5.AA.4[B] Consider the following linear activity model:

$$\begin{aligned} a_1 &= (1, -1, \;\; 0, \;\; 0) \\ a_2 &= (0, -1, \;\; 1, \;\; 0) \\ a_3 &= (0, \;\; 0, -1, \;\; 1) \\ a_4 &= (2, \;\; 0, \;\; 0, -1) \end{aligned}$$

(a) For each of the following input–output vectors, check whether they belong or do not belong to the aggregate production set. Justify your answers:

$$\begin{aligned} y_1 &= (6, \;\; 0, \;\; 0, -2) \\ y_2 &= (5, -3, \;\; 0, -1) \\ y_3 &= (6, -3, \;\; 0, \;\; 0) \\ y_4 &= (0, -4, \;\; 0, \;\; 4) \\ y_5 &= (0, -3, \;\; 4, \;\; 0) \end{aligned}$$

(b) The input–output vector $y = (0, -5, 5, 0)$ is efficient. Prove this by finding a $p \gg 0$ for which y is profit-maximizing.

(c) The input–output vector $y = (1, -1, 0, 0)$ is feasible, but it is not efficient. Why?

5.AA.5[B] [This exercise was inspired by an exercise of Champsaur and Milleron (1983).] There are four commodities indexed by $\ell = 1, 2, 3, 4$. The technology of a firm is described by eight elementary activities a_m, $m = 1, \ldots, 8$. With the usual sign convention, the numerical values of these activities are

$$\begin{aligned}
a_1 &= (\ -3,\ \ -6,\ \ \ 4,\ \ 0) \\
a_2 &= (\ -7,\ \ -9,\ \ \ 3,\ \ 2) \\
a_3 &= (\ -1,\ \ -2,\ \ \ 3,\ -1) \\
a_4 &= (\ -8,\ -13,\ \ \ 3,\ \ 1) \\
a_5 &= (-11,\ -19,\ 12,\ \ 0) \\
a_6 &= (\ -4,\ \ -3,\ -2,\ \ 5) \\
a_7 &= (\ -8,\ \ -5,\ \ \ 0,\ 10) \\
a_8 &= (\ -2,\ \ -4,\ \ \ 5,\ \ 2)
\end{aligned}$$

It is assumed that any activity can be operated at any nonnegative level $\alpha_m \geq 0$ and that all activities can operate simultaneously at any scale (i.e., for any $\alpha_m \geq 0$, $m = 1, \ldots, 8$, the production $\sum_m \alpha_m a_m$ is feasible).

(a) Define the corresponding production set Y, and show that it is convex.

(b) Verify the no-free-lunch property.

(c) Verify that Y does *not* satisfy the free-disposal property. The free-disposal property would be satisfied if we added new elementary activities to our list. How would you choose them (given specific numerical values)?

(d) Show by direct comparison of a_1 with a_5, a_2 with a_4, a_3 with a_8, and a_6 with a_7 that four of the elementary activities are not efficient.

(e) Show that a_1 and a_2 are inefficient be exhibiting two positive linear combinations of a_3 and a_7 that dominate a_1 and a_2, respectively.

(f) Could you venture a complete description of the set of efficient production vectors?

(g) Suppose that the amounts of the four goods available as initial resources to the firm are

$$s_1 = 480, \qquad s_2 = 300, \qquad s_3 = 0, \qquad s_4 = 0.$$

Subject to those limitations on the net use of resources, the firm is interested in maximizing the net production of the third good. How would you set up the problem as a linear program?

(h) By using all the insights you have gained on the set of efficient production vectors, can you solve the optimization problem in **(g)**? [*Hint*: It can be done graphically.]

6 Choice Under Uncertainty

6.A Introduction

In previous chapters, we studied choices that result in perfectly certain outcomes. In reality, however, many important economic decisions involve an element of risk. Although it is formally possible to analyze these situations using the general theory of choice developed in Chapter 1, there is good reason to develop a more specialized theory: Uncertain alternatives have a structure that we can use to restrict the preferences that "rational" individuals may hold. Taking advantage of this structure allows us to derive stronger implications than those based solely on the framework of Chapter 1.

In Section 6.B, we begin our study of choice under uncertainty by considering a setting in which alternatives with uncertain outcomes are describable by means of objectively known probabilities defined on an abstract set of possible outcomes. These representations of risky alternatives are called *lotteries*. In the spirit of Chapter 1, we assume that the decision maker has a rational preference relation over these lotteries. We then proceed to derive the *expected utility theorem*, a result of central importance. This theorem says that under certain conditions, we can represent preferences by an extremely convenient type of utility function, one that possesses what is called the *expected utility form*. The key assumption leading to this result is the *independence axiom*, which we discuss extensively.

In the remaining sections, we focus on the special case in which the outcome of a risky choice is an amount of money (or any other one-dimensional measure of consumption). This case underlies much of finance and portfolio theory, as well as substantial areas of applied economics.

In Section 6.C, we present the concept of *risk aversion* and discuss its measurement. We then study the comparison of risk aversions both across different individuals and across different levels of an individual's wealth.

Section 6.D is concerned with the comparison of alternative distributions of monetary returns. We ask when one distribution of monetary returns can unambiguously be said to be "better" than another, and also when one distribution can be said to be "more risky than" another. These comparisons lead, respectively, to the concepts of *first-order* and *second-order stochastic dominance*.

In Section 6.E, we extend the basic theory by allowing utility to depend on *states of nature* underlying the uncertainty as well as on the monetary payoffs. In the process, we develop a framework for modeling uncertainty in terms of these underlying states. This framework is often of great analytical convenience, and we use it extensively later in this book.

In Section 6.F, we consider briefly the theory of *subjective probability*. The assumption that uncertain prospects are offered to us with known objective probabilities, which we use in Section 6.B to derive the expected utility theorem, is rarely descriptive of reality. The subjective probability framework offers a way of modeling choice under uncertainty in which the probabilities of different risky alternatives are not given to the decision maker in any objective fashion. Yet, as we shall see, the theory of subjective probability offers something of a rescue for our earlier objective probability approach.

For further reading on these topics, see Kreps (1988) and Machina (1987). Diamond and Rothschild (1978) is an excellent sourcebook for original articles.

6.B Expected Utility Theory

We begin this section by developing a formal apparatus for modeling risk. We then apply this framework to the study of preferences over risky alternatives and to establish the important expected utility theorem.

Description of Risky Alternatives

Let us imagine that a decision maker faces a choice among a number of risky alternatives. Each risky alternative may result in one of a number of possible *outcomes*, but which outcome will actually occur is uncertain at the time that he must make his choice.

Formally, we denote the set of all possible outcomes by C.[1] These outcomes could take many forms. They could, for example, be consumption bundles. In this case, $C = X$, the decision maker's consumption set. Alternatively, the outcomes might take the simpler form of monetary payoffs. This case will, in fact, be our leading example later in this chapter. Here, however, we treat C as an abstract set and therefore allow for very general outcomes.

To avoid some technicalities, we assume in this section that the number of possible outcomes in C is finite, and we index these outcomes by $n = 1, \dots, N$.

Throughout this and the next several sections, we assume that the probabilities of the various outcomes arising from any chosen alternative are *objectively known*. For example, the risky alternatives might be monetary gambles on the spin of an unbiased roulette wheel.

The basic building block of the theory is the concept of a *lottery*, a formal device that is used to represent risky alternatives.

Definition 6.B.1: A *simple lottery* L is a list $L = (p_1, \dots, p_N)$ with $p_n \geq 0$ for all n and $\sum_n p_n = 1$, where p_n is interpreted as the probability of outcome n occurring.

1. It is also common, following Savage (1954), to refer to the elements of C as *consequences*.

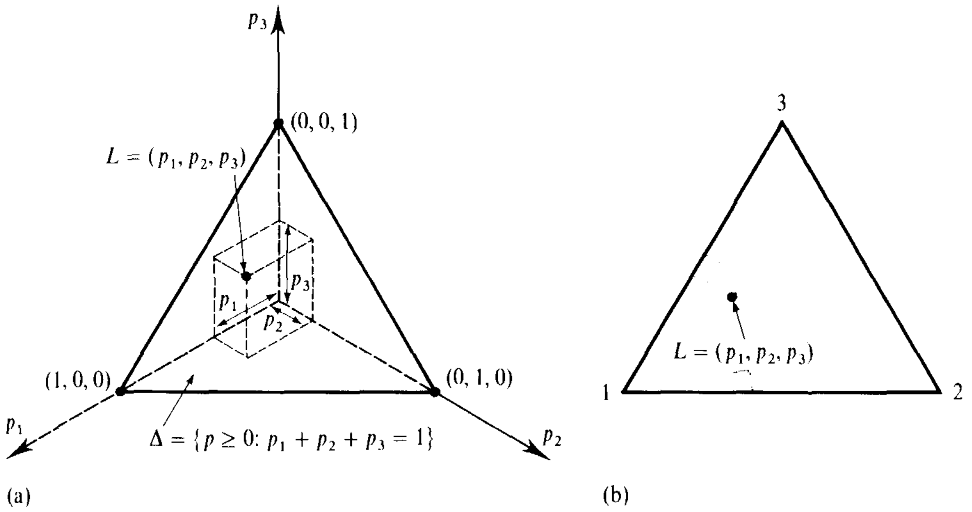


Figure 6.B.1
Representations of the simplex when $N = 3$.
(a) Three-dimensional representation.
(b) Two-dimensional representation.

A simple lottery can be represented geometrically as a point in the $(N - 1)$ *dimensional simplex*, $\Delta = \{p \in \mathbb{R}^N_+ \colon p_1 + \cdots + p_N = 1\}$. Figure 6.B.1(a) depicts this simplex for the case in which $N = 3$. Each vertex of the simplex stands for the degenerate lottery where one outcome is certain and the other two outcomes have probability zero. Each point in the simplex represents a lottery over the three outcomes. When $N = 3$, it is convenient to depict the simplex in two dimensions, as in Figure 6.B.1(b), where it takes the form of an equilateral triangle.[2]

In a simple lottery, the outcomes that may result are certain. A more general variant of a lottery, known as a *compound lottery*, allows the outcomes of a lottery themselves to be simple lotteries.[3]

Definition 6.B.2: Given K simple lotteries $L_k = (p_1^k, \ldots, p_N^k)$, $k = 1, \ldots, K$, and probabilities $\alpha_k \geq 0$ with $\sum_k \alpha_k = 1$, the *compound lottery* $(L_1, \ldots, L_K; \alpha_1, \ldots, \alpha_K)$ is the risky alternative that yields the simple lottery L_k with probability α_k for $k = 1, \ldots, K$.

For any compound lottery $(L_1, \ldots, L_K; \alpha_1, \ldots, \alpha_K)$, we can calculate a corresponding *reduced lottery* as the simple lottery $L = (p_1, \ldots, p_N)$ that generates the same ultimate distribution over outcomes. The value of each p_n is obtained by multiplying the probability that each lottery L_k arises, α_k, by the probability p_n^k that outcome n arises in lottery L_k, and then adding over k. That is, the probability of outcome n in the reduced lottery is

$$p_n = \alpha_1 p_n^1 + \cdots + \alpha_K p_n^K$$

2. Recall that equilateral triangles have the property that the sum of the perpendiculars from any point to the three sides is equal to the altitude of the triangle. It is therefore common to depict the simplex when $N = 3$ as an equilateral triangle with altitude equal to 1 because by doing so, we have the convenient geometric property that the probability p_n of outcome n in the lottery associated with some point in this simplex is equal to the length of the perpendicular from this point to the side opposite the vertex labeled n.

3. We could also define compound lotteries with more than two stages. We do not do so because we will not need them in this chapter. The principles involved, however, are the same.

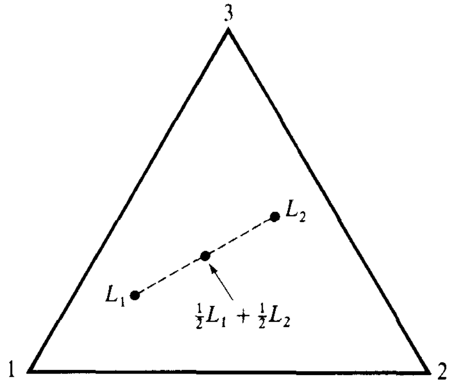


Figure 6.B.2
The reduced lottery of a compound lottery.

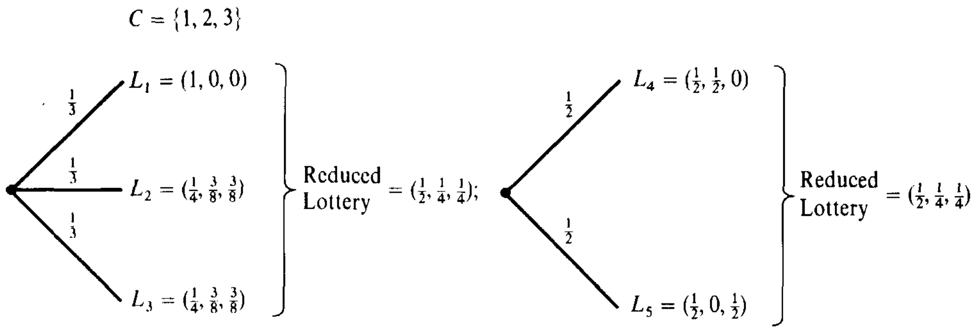


Figure 6.B.3
Two compound lotteries with the same reduced lottery.

for $n = 1, \ldots, N$.[4] Therefore, the reduced lottery L of any compound lottery $(L_1, \ldots, L_K; \alpha_1, \ldots, \alpha_K)$ can be obtained by vector addition:

$$L = \alpha_1 L_1 + \cdots + \alpha_K L_K \in \Delta.$$

In Figure 6.B.2, two simple lotteries L_1 and L_2 are depicted in the simplex Δ. Also depicted is the reduced lottery $\frac{1}{2}L_1 + \frac{1}{2}L_2$ for the compound lottery $(L_1, L_2; \frac{1}{2}, \frac{1}{2})$ that yields either L_1 or L_2 with a probability of $\frac{1}{2}$ each. This reduced lottery lies at the midpoint of the line segment connecting L_1 and L_2. The linear structure of the space of lotteries is central to the theory of choice under uncertainty, and we exploit it extensively in what follows.

Preferences over Lotteries

Having developed a way to model risky alternatives, we now study the decision maker's preferences over them. The theoretical analysis to follow rest on a basic *consequentialist* premise: We assume that for any risky alternative, only the reduced lottery over final outcomes is of relevance to the decision maker. Whether the probabilities of various outcomes arise as a result of a simple lottery or of a more complex compound lottery has no significance. Figure 6.B.3 exhibits two different compound lotteries that yield the same reduced lottery. Our consequentialist hypothesis requires that the decision maker view these two lotteries as equivalent.

4. Note that $\sum_n p_n = \sum_k \alpha_k (\sum_n p_n^k) = \sum_k \alpha_k = 1$.

We now pose the decision maker's choice problem in the general framework developed in Chapter 1 (see Section 1.B). In accordance with our consequentialist premise, we take the set of alternatives, denoted here by $\mathscr{L}$, to be *the set of all simple lotteries over the set of outcomes C*. We next assume that the decision maker has a rational preference relation $\succsim$ on $\mathscr{L}$, a complete and transitive relation allowing comparison of any pair of simple lotteries. It should be emphasized that, if anything, the rationality assumption is stronger here than in the theory of choice under certainty discussed in Chapter 1. The more complex the alternatives, the heavier the burden carried by the rationality postulates. In fact, their realism in an uncertainty context has been much debated. However, because we want to concentrate on the properties that are specific to uncertainty, we do not question the rationality assumption further here.

We next introduce two additional assumptions about the decision maker's preferences over lotteries. The most important and controversial is the *independence axiom*. The first, however, is a continuity axiom similar to the one discussed in Section 3.C.

Definition 6.B.3: The preference relation $\succsim$ on the space of simple lotteries $\mathscr{L}$ is *continuous* if for any $L, L', L'' \in \mathscr{L}$, the sets

$$\{\alpha \in [0, 1]: \alpha L + (1-\alpha)L' \succsim L''\} \subset [0, 1]$$

and

$$\{\alpha \in [0, 1]: L'' \succsim \alpha L + (1-\alpha)L'\} \subset [0, 1]$$

are closed.

In words, continuity means that small changes in probabilities do not change the nature of the ordering between two lotteries. For example, if a "beautiful and uneventful trip by car" is preferred to "staying home," then a mixture of the outcome "beautiful and uneventful trip by car" with a sufficiently small but positive probability of "death by car accident" is still preferred to "staying home." Continuity therefore rules out the case where the decision maker has lexicographic ("safety first") preferences for alternatives with a zero probability of some outcome (in this case, "death by car accident").

As in Chapter 3, the continuity axiom implies the existence of a utility function representing $\succsim$, a function $U: \mathscr{L} \to \mathbb{R}$ such that $L \succsim L'$ if and only if $U(L) \geq U(L')$. Our second assumption, the independence axiom, will allow us to impose considerably more structure on $U(\cdot)$.[5]

Definition 6.B.4: The preference relation $\succsim$ on the space of simple lotteries $\mathscr{L}$ satisfies the *independence axiom* if for all $L, L', L'' \in \mathscr{L}$ and $\alpha \in (0, 1)$ we have

$$L \succsim L' \quad \text{if and only if} \quad \alpha L + (1-\alpha)L'' \succsim \alpha L' + (1-\alpha)L''.$$

In other words, if we mix each of two lotteries with a third one, then the preference ordering of the two resulting mixtures does not depend on (is *independent* of) the particular third lottery used.

5. The independence axiom was first proposed by von Neumann and Morgenstern (1944) as an incidental result in the theory of games.

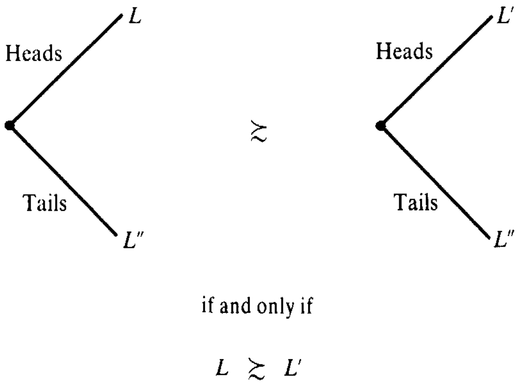


Figure 6.B.4
The independence axiom.

Suppose, for example, that $L \succsim L'$ and $\alpha = \frac{1}{2}$. Then $\frac{1}{2}L + \frac{1}{2}L''$ can be thought of as the compound lottery arising from a coin toss in which the decision maker gets L if heads comes up and L'' if tails does. Similarly, $\frac{1}{2}L' + \frac{1}{2}L''$ would be the coin toss where heads results in L' and tails results in L'' (see Figure 6.B.4). Note that conditional on heads, lottery $\frac{1}{2}L + \frac{1}{2}L''$ is at least as good as lottery $\frac{1}{2}L' + \frac{1}{2}L''$; but conditional on tails, the two compound lotteries give identical results. The independence axiom requires the sensible conclusion that $\frac{1}{2}L + \frac{1}{2}L''$ be at least as good as $\frac{1}{2}L' + \frac{1}{2}L''$.

The independence axiom is at the heart of the theory of choice under uncertainty. It is unlike anything encountered in the formal theory of preference-based choice discussed in Chapter 1 or its applications in Chapters 3 to 5. This is so precisely because it exploits, in a fundamental manner, the structure of uncertainty present in the model. In the theory of consumer demand, for example, there is no reason to believe that a consumer's preferences over various bundles of goods 1 and 2 should be independent of the quantities of the other goods that he will consume. In the present context, however, it *is* natural to think that a decision maker's preference between two lotteries, say L and L', should determine which of the two he prefers to have as part of a compound lottery *regardless* of the other possible outcome of this compound lottery, say L''. This other outcome L'' should be irrelevant to his choice because, in contrast with the consumer context, he does not consume L or L' together with L'' but, rather, only *instead* of it (if L or L' is the realized outcome).

Exercise 6.B.1: Show that if the preferences $\succsim$ over $\mathscr{L}$ satisfy the independence axiom, then for all $\alpha \in (0, 1)$ and $L, L', L'' \in \mathscr{L}$ we have

$$L \succ L' \quad \text{if and only if} \quad \alpha L + (1-\alpha)L'' \succ \alpha L' + (1-\alpha)L''$$

and

$$L \sim L' \quad \text{if and only if} \quad \alpha L + (1-\alpha)L'' \sim \alpha L' + (1-\alpha)L''.$$

Show also that if $L \succ L'$ and $L'' \succ L'''$, then $\alpha L + (1-\alpha)L'' \succ \alpha L' + (1-\alpha)L'''$.

As we will see shortly, the independence axiom is intimately linked to the representability of preferences over lotteries by a utility function that has an *expected utility form*. Before obtaining that result, we define this property and study some of its features.

Definition 6.B.5: The utility function $U: \mathscr{L} \to \mathbb{R}$ has an *expected utility form* if there is an assignment of numbers $(u_1, \ldots, u_N)$ to the N outcomes such that for every simple lottery $L = (p_1, \ldots, p_N) \in \mathscr{L}$ we have

$$U(L) = u_1 p_1 + \cdots + u_N p_N.$$

A utility function $U: \mathscr{L} \to \mathbb{R}$ with the expected utility form is called a *von Neumann–Morgenstern (v.N–M) expected utility function*.

Observe that if we let L^n denote the lottery that yields outcome n with probability one, then $U(L^n) = u_n$. Thus, the term *expected utility* is appropriate because with the v.N–M expected utility form, the utility of a lottery can be thought of as the expected value of the utilities u_n of the N outcomes.

The expression $U(L) = \sum_n u_n p_n$ is a general form for a *linear function in the probabilities* $(p_1, \ldots, p_N)$. This linearity property suggests a useful way to think about the expected utility form.

Proposition 6.B.1: A utility function $U: \mathscr{L} \to \mathbb{R}$ has an expected utility form if and only if it is *linear*, that is, if and only if it satisfies the property that

$$U\left(\sum_{k=1}^{K} \alpha_k L_k\right) = \sum_{k=1}^{K} \alpha_k U(L_k) \tag{6.B.1}$$

for any K lotteries $L_k \in \mathscr{L}$, $k = 1, \ldots, K$, and probabilities $(\alpha_1, \ldots, \alpha_K) \geq 0$, $\sum_k \alpha_k = 1$.

Proof: Suppose that $U(\cdot)$ satisfies property (6.B.1). We can write any $L = (p_1, \ldots, p_N)$ as a convex combination of the degenerate lotteries $(L^1, \ldots, L^N)$, that is, $L = \sum_n p_n L^n$. We have then $U(L) = U(\sum_n p_n L^n) = \sum_n p_n U(L^n) = \sum_n p_n u_n$. Thus, $U(\cdot)$ has the expected utility form.

In the other direction, suppose that $U(\cdot)$ has the expected utility form, and consider any compound lottery $(L_1, \ldots, L_K; \alpha_1, \ldots, \alpha_K)$, where $L_k = (p_1^k, \ldots, p_N^k)$. Its reduced lottery is $L' = \sum_k \alpha_k L_k$. Hence,

$$U\left(\sum_k \alpha_k L_k\right) = \sum_n u_n \left(\sum_k \alpha_k p_n^k\right) = \sum_k \alpha_k \left(\sum_n u_n p_n^k\right) = \sum_k \alpha_k U(L_k).$$

Thus, property (6.B.1) is satisfied. ■

The expected utility property is a *cardinal* property of utility functions defined on the space of lotteries. In particular, the result in Proposition 6.B.2 shows that the expected utility form is preserved only by increasing *linear* transformations.

Proposition 6.B.2: Suppose that $U: \mathscr{L} \to \mathbb{R}$ is a v.N–M expected utility function for the preference relation $\succsim$ on $\mathscr{L}$. Then $\tilde{U}: \mathscr{L} \to \mathbb{R}$ is another v.N–M utility function for $\succsim$ if and only if there are scalars $\beta > 0$ and γ such that $\tilde{U}(L) = \beta U(L) + \gamma$ for every $L \in \mathscr{L}$.

Proof: Begin by choosing two lotteries $\bar{L}$ and $\underline{L}$ with the property that $\bar{L} \succsim L \succsim \underline{L}$ for all $L \in \mathscr{L}$.[6] If $\bar{L} \sim \underline{L}$, then every utility function is a constant and the result follows immediately. Therefore, we assume from now on that $\bar{L} \succ \underline{L}$.

6. These best and worst lotteries can be shown to exist. We could, for example, choose a maximizer and a minimizer of the linear, hence continuous, function $U(\cdot)$ on the simplex of probabilities, a compact set.

Note first that if $U(\cdot)$ is a v.N–M expected utility function and $\tilde{U}(L) = \beta U(L) + \gamma$, then

$$\begin{aligned}\tilde{U}\left(\sum_{k=1}^{K} \alpha_k L_k\right) &= \beta U\left(\sum_{k=1}^{K} \alpha_k L_k\right) + \gamma \\ &= \beta\left[\sum_{k=1}^{K} \alpha_k U(L_k)\right] + \gamma \\ &= \sum_{k=1}^{K} \alpha_k[\beta U(L_k) + \gamma] \\ &= \sum_{k=1}^{K} \alpha_k \tilde{U}(L_k).\end{aligned}$$

Since $\tilde{U}(\cdot)$ satisfies property (6.B.1), it has the expected utility form.

For the reverse direction, we want to show that if both $\tilde{U}(\cdot)$ and $U(\cdot)$ have the expected utility form, then constants $\beta > 0$ and γ exist such that $\tilde{U}(L) = \beta U(L) + \gamma$ for all $L \in \mathscr{L}$. To do so, consider any lottery $L \in \mathscr{L}$, and define $\lambda_L \in [0, 1]$ by

$$U(L) = \lambda_L U(\bar{L}) + (1 - \lambda_L)U(\underline{L}).$$

Thus

$$\lambda_L = \frac{U(L) - U(\underline{L})}{U(\bar{L}) - U(\underline{L})}. \tag{6.B.2}$$

Since $\lambda_L U(\bar{L}) + (1 - \lambda_L)U(\underline{L}) = U(\lambda_L \bar{L} + (1 - \lambda_L)\underline{L})$ and $U(\cdot)$ represents the preferences $\succsim$, it must be that $L \sim \lambda_L \bar{L} + (1 - \lambda_L)\underline{L}$. But if so, then since $\tilde{U}(\cdot)$ is also linear and represents these same preferences, we have

$$\begin{aligned}\tilde{U}(L) &= \tilde{U}(\lambda_L \bar{L} + (1 - \lambda_L)\underline{L}) \\ &= \lambda_L \tilde{U}(\bar{L}) + (1 - \lambda_L)\tilde{U}(\underline{L}) \\ &= \lambda_L(\tilde{U}(\bar{L}) - \tilde{U}(\underline{L})) + \tilde{U}(\underline{L}).\end{aligned}$$

Substituting for λ_L from (6.B.2) and rearranging terms yields the conclusion that $\tilde{U}(L) = \beta U(L) + \gamma$, where

$$\beta = \frac{\tilde{U}(\bar{L}) - \tilde{U}(\underline{L})}{U(\bar{L}) - U(\underline{L})}$$

and

$$\gamma = \tilde{U}(\underline{L}) - U(\underline{L})\frac{\tilde{U}(\bar{L}) - \tilde{U}(\underline{L})}{U(\bar{L}) - U(\underline{L})}.$$

This completes the proof ■

A consequence of Proposition 6.B.2 is that for a utility function with the expected utility form, differences of utilities have meaning. For example, if there are four outcomes, the statement "the difference in utility between outcomes 1 and 2 is greater than the difference between outcomes 3 and 4," $u_1 - u_2 > u_3 - u_4$, is equivalent to

$$\tfrac{1}{2}u_1 + \tfrac{1}{2}u_4 > \tfrac{1}{2}u_2 + \tfrac{1}{2}u_3.$$

Therefore, the statement means that the lottery $L = (\frac{1}{2}, 0, 0, \frac{1}{2})$ is preferred to the lottery $L' = (0, \frac{1}{2}, \frac{1}{2}, 0)$. This ranking of utility differences is preserved by all linear transformations of the v.N–M expected utility function.

Figu
utili

Note that if a preference relation $\succsim$ on $\mathscr{L}$ is representable by a utility function $U(\cdot)$ that has the expected utility form, then since a linear utility function is continuous, it follows that $\succsim$ is continuous on $\mathscr{L}$. More importantly, the preference relation $\succsim$ must also satisfy the independence axiom. You are asked to show this in Exercise 6.B.2.

Exercise 6.B.2: Show that if the preference relation $\succsim$ on $\mathscr{L}$ is represented by a utility function $U(\cdot)$ that has the expected utility form, then $\succsim$ satisfies the independence axiom.

The expected utility theorem, the central result of this section, tells us that the converse is also true.

The Expected Utility Theorem

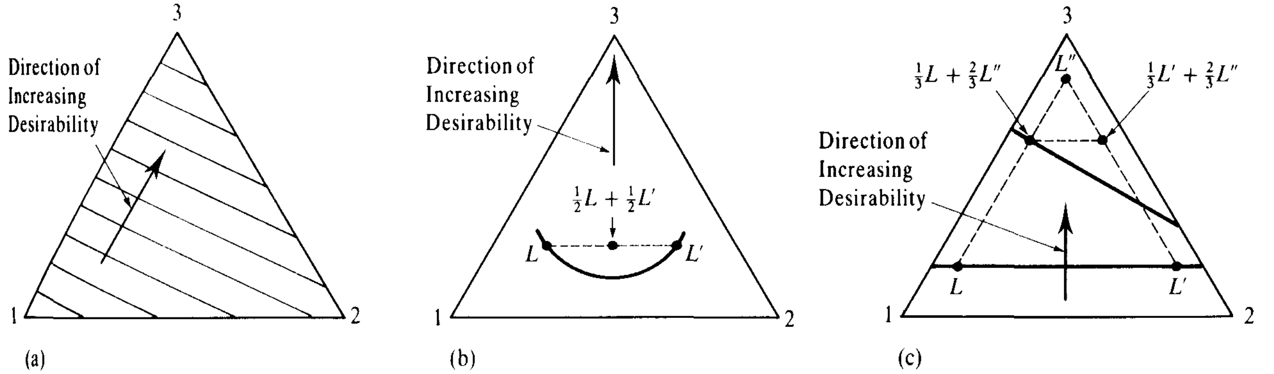
The *expected utility theorem* says that if the decision maker's preferences over lotteries satisfy the continuity and independence axioms, then his preferences are representable by a utility function with the expected utility form. It is the most important result in the theory of choice under uncertainty, and the rest of the book bears witness to its usefulness.

Before stating and proving the result formally, however, it may be helpful to attempt an intuitive understanding of why it is true.

Consider the case where there are only three outcomes. As we have already observed, the continuity axiom insures that preferences on lotteries can be represented by some utility function. Suppose that we represent the indifference map in the simplex, as in Figure 6.B.5. Assume, for simplicity, that we have a conventional map with one-dimensional indifference curves. Because the expected utility form is linear in the probabilities, representability by the expected utility form is equivalent to these indifference curves being straight, parallel lines (you should check this). Figure 6.B.5(a) exhibits an indifference map satisfying these properties. We now argue that these properties are, in fact, consequences of the independence axiom.

Indifference curves are straight lines if, for every pair of lotteries L, L', we have that $L \sim L'$ implies $\alpha L + (1-\alpha)L' \sim L$ for all $\alpha \in [0,1]$. Figure 6.B.5(b) depicts a situation where the indifference curve is not a straight line; we have $L' \sim L$ but

Figure 6.B.5 Geometric explanation of the expected utility theorem. (a) $\succsim$ is representable by a utility function with the expected utility form. (b) Contradiction of the independence axiom. (c) Contradiction of the independence axiom.



$\frac{1}{2}L' + \frac{1}{2}L \succ L$. This is equivalent to saying that

$$\tfrac{1}{2}L' + \tfrac{1}{2}L \succ \tfrac{1}{2}L + \tfrac{1}{2}L. \tag{6.B.3}$$

But since $L \sim L'$, the independence axiom implies that we must have $\frac{1}{2}L' + \frac{1}{2}L \sim \frac{1}{2}L + \frac{1}{2}L$ (see Exercise 6.B.1). This contradicts (6.B.3), and so we must conclude that indifference curves are straight lines.

Figure 6.B.5(c) depicts two straight but nonparallel indifference lines. A violation of the independence axiom can be constructed in this case, as indicated in the figure. There we have $L \succsim L'$ (in fact, $L \sim L'$), but $\frac{1}{3}L + \frac{2}{3}L'' \succsim \frac{1}{3}L' + \frac{2}{3}L''$ does not hold for the lottery L'' shown in the figure. Thus, indifference curves must be parallel, straight lines if preferences satisfy the independence axiom.

In Proposition 6.B.3, we formally state and prove the expected utility theorem.

Proposition 6.B.3: (*Expected Utility Theorem*) Suppose that the rational preference relation $\succsim$ on the space of lotteries $\mathscr{L}$ satisfies the continuity and independence axioms. Then $\succsim$ admits a utility representation of the expected utility form. That is, we can assign a number u_n to each outcome $n = 1, \ldots, N$ in such a manner that for any two lotteries $L = (p_1, \ldots, p_N)$ and $L' = (p'_1, \ldots, p'_N)$, we have

$$L \succsim L' \quad \text{if and only if} \quad \sum_{n=1}^{N} u_n p_n \geq \sum_{n=1}^{N} u_n p'_n. \tag{6.B.4}$$

Proof: We organize the proof in a succession of steps. For simplicity, we assume that there are best and worst lotteries in $\mathscr{L}$, $\bar{L}$ and $\underline{L}$ (so, $\bar{L} \succsim L \succsim \underline{L}$ for any $L \in \mathscr{L}$).[7] If $\bar{L} \sim \underline{L}$, then all lotteries in $\mathscr{L}$ are indifferent and the conclusion of the proposition holds trivially. Hence, from now on, we assume that $\bar{L} \succ \underline{L}$.

Step 1. If $L \succ L'$ and $\alpha \in (0, 1)$, then $L \succ \alpha L + (1 - \alpha)L' \succ L'$.

This claim makes sense. A nondegenerate mixture of two lotteries will hold a preference position strictly intermediate between the positions of the two lotteries. Formally, the claim follows from the independence axiom. In particular, since $L \succ L'$, the independence axiom implies that (recall Exercise 6.B.1)

$$L = \alpha L + (1 - \alpha)L \succ \alpha L + (1 - \alpha)L' \succ \alpha L' + (1 - \alpha)L' = L'.$$

Step 2. Let $\alpha, \beta \in [0, 1]$. Then $\beta\bar{L} + (1 - \beta)\underline{L} \succ \alpha\bar{L} + (1 - \alpha)\underline{L}$ if and only if $\beta > \alpha$.

Suppose that $\beta > \alpha$. Note first that we can write

$$\beta\bar{L} + (1 - \beta)\underline{L} = \gamma\bar{L} + (1 - \gamma)[\alpha\bar{L} + (1 - \alpha)\underline{L}],$$

where $\gamma = [(\beta - \alpha)/(1 - \alpha)] \in (0, 1]$. By Step 1, we know that $\bar{L} \succ \alpha\bar{L} + (1 - \alpha)\underline{L}$. Applying Step 1 again, this implies that $\gamma\bar{L} + (1 - \gamma)(\alpha\bar{L} + (1 - \alpha)\underline{L}) \succ \alpha\bar{L} + (1 - \alpha)\underline{L}$, and so we conclude that $\beta\bar{L} + (1 - \beta)\underline{L} \succ \alpha\bar{L} + (1 - \alpha)\underline{L}$.

For the converse, suppose that $\beta \leq \alpha$. If $\beta = \alpha$, we must have $\beta\bar{L} + (1 - \beta)\underline{L} \sim \alpha\bar{L} + (1 - \alpha)\underline{L}$. So suppose that $\beta < \alpha$. By the argument proved in the previous

7. In fact, with our assumption of a finite set of outcomes, this can be established as a consequence of the independence axiom (see Exercise 6.B.3).

paragraph (reversing the roles of α and β), we must then have $\alpha\bar{L} + (1-\alpha)\underline{L} \succ \beta\bar{L} + (1-\beta)\underline{L}$.

Step 3. For any $L \in \mathscr{L}$, there is a unique α_L such that $[\alpha_L\bar{L} + (1-\alpha_L)\underline{L}] \sim L$.

Existence of such an α_L is implied by the continuity of $\succsim$ and the fact that $\bar{L}$ and $\underline{L}$ are, respectively, the best and the worst lottery. Uniqueness follows from the result of Step 2.

The existence of α_L is established in a manner similar to that used in the proof of Proposition 3.C.1. Specifically, define the sets

$$\{\alpha \in [0,1]: \alpha\bar{L} + (1-\alpha)\underline{L} \succsim L\} \quad \text{and} \quad \{\alpha \in [0,1]: L \succsim \alpha\bar{L} + (1-\alpha)\underline{L}\}.$$

By the continuity and completeness of $\succsim$, both sets are closed, and any $\alpha \in [0,1]$ belongs to at least one of the two sets. Since both sets are nonempty and $[0,1]$ is connected, it follows that there is some α belonging to both. This establishes the existence of an α_L such that $\alpha_L\bar{L} + (1-\alpha_L)\underline{L} \sim L$.

Step 4. The function $U: \mathscr{L} \to \mathbb{R}$ that assigns $U(L) = \alpha_L$ for all $L \in \mathscr{L}$ represents the preference relation $\succsim$.

Observe that, by Step 3, for any two lotteries $L, L' \in \mathscr{L}$, we have

$$L \succsim L' \quad \text{if and only if} \quad \alpha_L\bar{L} + (1-\alpha_L)\underline{L} \succsim \alpha_{L'}\bar{L} + (1-\alpha_{L'})\underline{L}.$$

Thus, by Step 2, $L \succsim L'$ if and only if $\alpha_L \geq \alpha_{L'}$.

Step 5. The utility function $U(\cdot)$ that assigns $U(L) = \alpha_L$ for all $L \in \mathscr{L}$ is linear and therefore has the expected utility form.

We want to show that for any $L, L' \in \mathscr{L}$, and $\beta \in [0,1]$, we have $U(\beta L + (1-\beta)L') = \beta U(L) + (1-\beta)U(L')$. By definition, we have

$$L \sim U(L)\bar{L} + (1-U(L))\underline{L}$$

and

$$L' \sim U(L')\bar{L} + (1-U(L'))\underline{L}.$$

Therefore, by the independence axiom (applied twice),

$$\begin{aligned}\beta L + (1-\beta)L' &\sim \beta[U(L)\bar{L} + (1-U(L))\underline{L}] + (1-\beta)L' \\ &\sim \beta[U(L)\bar{L} + (1-U(L))\underline{L}] + (1-\beta)[U(L')\bar{L} + (1-U(L'))\underline{L}].\end{aligned}$$

Rearranging terms, we see that the last lottery is algebraically identical to the lottery

$$[\beta U(L) + (1-\beta)U(L')]\bar{L} + [1 - \beta U(L) - (1-\beta)U(L')]\underline{L}.$$

In other words, the compound lottery that gives lottery $[U(L)\bar{L} + (1-U(L))\underline{L}]$ with probability β and lottery $[U(L')\bar{L} + (1-U(L'))\underline{L}]$ with probability $(1-\beta)$ has the same reduced lottery as the compound lottery that gives lottery $\bar{L}$ with probability $[\beta U(L) + (1-\beta)U(L')]$ and lottery $\underline{L}$ with probability $[1 - \beta U(L) - (1-\beta)U(L')]$. Thus

$$\beta L + (1-\beta)L' \sim [\beta U(L) + (1-\beta)U(L')]\bar{L} + [1 - \beta U(L) - (1-\beta)U(L')]\underline{L}.$$

By the construction of $U(\cdot)$ in Step 4, we therefore have

$$U(\beta L + (1-\beta)L') = \beta U(L) + (1-\beta)U(L'),$$

as we wanted.

Together, Steps 1 to 5 establish the existence of a utility function representing $\succsim$ that has the expected utility form. ■

Discussion of the Theory of Expected Utility

A first advantage of the expected utility theorem is technical: It is extremely convenient analytically. This, more than anything else, probably accounts for its pervasive use in economics. It is very easy to work with expected utility and very difficult to do without it. As we have already noted, the rest of the book attests to the importance of the result. Later in this chapter, we will explore some of the analytical uses of expected utility.

A second advantage of the theorem is normative: Expected utility may provide a valuable guide to action. People often find it hard to think systematically about risky alternatives. But if an individual believes that his choices should satisfy the axioms on which the theorem is based (notably, the independence axiom), then the theorem can be used as a guide in his decision process. This point is illustrated in Example 6.B.1.

Example 6.B.1: *Expected Utility as a Guide to Introspection.* A decision maker may not be able to assess his preference ordering between the lotteries L and L' depicted in Figure 6.B.6. The lotteries are too close together, and the differences in the probabilities involved are too small to be understood. Yet, if the decision maker believes that his preferences should satisfy the assumptions of the expected utility theorem, then he may consider L'' instead, which is on the straight line spanned by L and L' but at a significant distance from L. The lottery L'' may not be a feasible choice, but if he determines that $L'' \succ L$, then he can conclude that $L' \succ L$. Indeed, if $L'' \succ L$, then there is an indifference curve separating these two lotteries, as shown in the figure, and it follows from the fact that indifference curves are a family of parallel straight lines that there is also an indifference curve separating L' and L, so that $L' \succ L$. Note that this type of inference is not possible using only the general

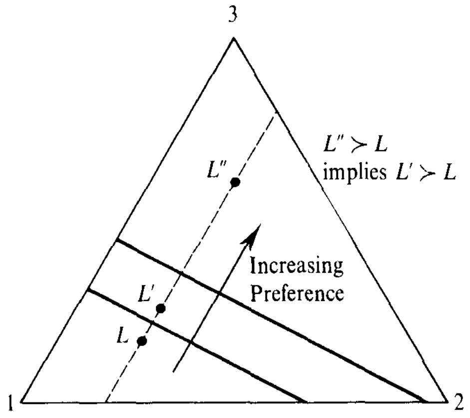


Figure 6.B.6
Expected utility as a guide to introspection.

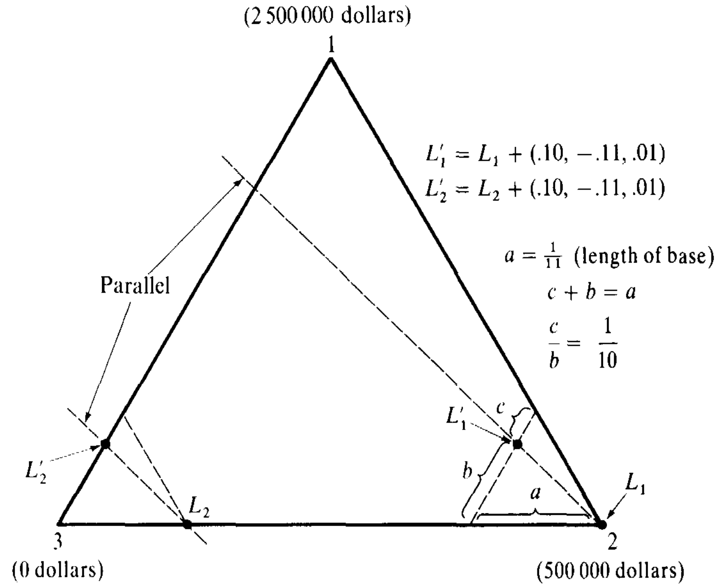


Figure 6.B.7
Depiction of the Allais paradox in the simplex.

choice theory of Chapter 1 because, without the hypotheses of the expected utility theorem, the indifference curves need not be straight lines (with a general indifference map, we could perfectly well have $L'' \succ L$ and $L \succ L'$).

A concrete example of this use of the expected utility theorem is developed in Exercise 6.B.4. ■

As a descriptive theory, however, the expected utility theorem (and, by implication, its central assumption, the independence axiom), is not without difficulties. Examples 6.B.2 and 6.B.3 are designed to test its plausibility.

Example 6.B.2: *The Allais Paradox.* This example, known as the Allais paradox [from Allais (1953)], constitutes the oldest and most famous challenge to the expected utility theorem. It is a thought experiment. There are three possible monetary prizes (so the number of outcomes is $N = 3$):

First Prize	*Second Prize*	*Third Prize*
2 500 000 dollars	500 000 dollars	0 dollars

The decision maker is subjected to two choice tests. The first consists of a choice between the lotteries L_1 and L_1':

$$L_1 = (0, 1, 0) \qquad L_1' = (.10, .89, .01).$$

The second consists of a choice between the lotteries L_2 and L_2':

$$L_2 = (0, .11, .89) \qquad L_2' = (.10, 0, .90).$$

The four lotteries involved are represented in the simplex diagram of Figure 6.B.7.

It is common for individuals to express the preferences $L_1 \succ L_1'$ and $L_2' \succ L_2$.[8]

8. In our classroom experience, roughly half the students choose this way.

The first choice means that one prefers the certainty of receiving 500 000 dollars over a lottery offering a 1/10 probability of getting five times more but bringing with it a tiny risk of getting nothing. The second choice means that, all things considered, a 1/10 probability of getting 2 500 000 dollars is preferred to getting only 500 000 dollars with the slightly better odds of 11/100.

However, these choices are not consistent with expected utility. This can be seen in Figure 6.B.7: The straight lines connecting L_1 to L'_1 and L_2 to L'_2 are parallel. Therefore, if an individual has a linear indifference curve that lies in such a way that L_1 is preferred to L'_1, then a parallel linear indifference curve must make L_2 preferred to L'_2, and vice versa. Hence, choosing L_1 and L'_2 is inconsistent with preferences satisfying the assumptions of the expected utility theorem.

More formally, suppose that there was a v.N–M expected utility function. Denote by u_{25}, u_{05}, and u_0 the utility values of the three outcomes. Then the choice $L_1 \succ L'_1$ implies

$$u_{05} > (.10)u_{25} + (.89)u_{05} + (.01)u_0.$$

Adding $(.89)u_0 - (.89)u_{05}$ to both sides, we get

$$(.11)u_{05} + (.89)u_0 > (.10)u_{25} + (.90)u_0,$$

and therefore any individual with a v.N–M utility function must have $L_2 \succ L'_2$. ■

There are four common reactions to the Allais paradox. The first, propounded by J. Marshack and L. Savage, goes back to the normative interpretation of the theory. It argues that choosing under uncertainty is a reflective activity in which one should be ready to correct mistakes if they are proven inconsistent with the basic principles of choice embodied in the independence axiom (much as one corrects arithmetic mistakes).

The second reaction maintains that the Allais paradox is of limited significance for economics as a whole because it involves payoffs that are out of the ordinary and probabilities close to 0 and 1.

A third reaction seeks to accommodate the paradox with a theory that defines preferences over somewhat larger and more complex objects than simply the ultimate lottery over outcomes. For example, the decision maker may value not only what he receives but also what he receives compared with what he might have received by choosing differently. This leads to *regret theory*. In the example, we could have $L_1 \succ L'_1$ because the expected regret caused by the possibility of getting zero in lottery L'_1, when choosing L_1 would have assured 500 000 dollars, is too great. On the other hand, with the choice between L_2 and L'_2, no such clear-cut regret potential exists; the decision maker was very likely to get nothing anyway.

The fourth reaction is to stick with the original choice domain of lotteries but to give up the independence axiom in favor of something weaker. Exercise 6.B.5 develops this point further.

Example 6.B.3: *Machina's paradox.* Consider the following three outcomes: "a trip to Venice," "watching an excellent movie about Venice," and "staying home." Suppose that you prefer the first to the second and the second to the third.

Now you are given the opportunity to choose between two lotteries. The first lottery gives "a trip to Venice" with probability 99.9% and "watching an excellent movie about Venice" with probability 0.1%. The second lottery gives "a trip to

Venice," again with probability 99.9% and "staying home" with probability 0.1%. The independence axiom forces you to prefer the first lottery to the second. Yet, it would be understandable if you did otherwise. Choosing the second lottery is the rational thing to do if you anticipate that in the event of not getting the trip to Venice, your tastes over the other two outcomes will change: You will be severely *disappointed* and will feel miserable watching a movie about Venice.

The idea of disappointment has parallels with the idea of regret that we discussed in connection with the Allais paradox, but it is not quite the same. Both ideas refer to the influence of "what might have been" on the level of well-being experienced, and it is because of this that they are in conflict with the independence axiom. But disappointment is more directly concerned with what might have been if another outcome of a given lottery had come up, whereas regret should be thought of as regret over a choice not made. ■

Because of the phenomena illustrated in the previous two examples, the search for a useful theory of choice under uncertainty that does not rely on the independence axiom has been an active area of research [see Machina (1987) and also Hey and Orme (1994)]. Nevertheless, the use of the expected utility theorem is pervasive in economics.

An argument sometimes made against the practical significance of violations of the independence axiom is that individuals with such preferences would be weeded out of the marketplace because they would be open to the acceptance of so-called "Dutch books," that is, deals leading to a sure loss of money. Suppose, for example, that there are three lotteries such that $L \succ L'$ and $L \succ L''$ but, in violation of the independence axiom, $\alpha L' + (1-\alpha)L'' \succ L$ for some $\alpha \in (0, 1)$. Then, when the decision maker is in the initial position of owning the right to lottery L, he would be willing to pay a small fee to trade L for a compound lottery yielding lottery L' with probability α and lottery L'' with probability $(1-\alpha)$. But as soon as the first stage of this lottery is over, giving him either L' or L'' we could get him to pay a fee to trade this lottery for L. Hence, at that point, he would have paid the two fees but would otherwise be back to his original position.

This may well be a good argument for convexity of the not-better-than sets of $\succsim$, that is, for it to be the case that $L \succsim \alpha L' + (1-\alpha)L''$ whenever $L \succsim L'$ and $L \succsim L''$. This property is implied by the independence axiom but is weaker than it. Dutch book arguments for the full independence axiom are possible, but they are more contrived [see Green (1987)].

Finally, one must use some caution in applying the expected utility theorem because in many practical situations the final outcomes of uncertainty are influenced by actions taken by individuals. Often, these actions should be explicitly modeled but are not. Example 6.B.4 illustrates the difficulty involved.

Example 6.B.4: *Induced preferences.* You are invited to a dinner where you may be offered fish (F) or meat (M). You would like to do the proper thing by showing up with white wine if F is served and red wine if M is served. The action of buying the wine must be taken *before* the uncertainty is resolved.

Suppose now that the cost of the bottle of red or white wine is the same and that you are also indifferent between F and M. If you think of the possible outcomes as F and M, then you are apparently indifferent between the lottery that gives F with certainty and the lottery that gives M with certainty. The independence axiom would then seem to require that you also be indifferent to a lottery that gives F or M with probability $\frac{1}{2}$ each. But you would clearly not be indifferent, since knowing that either F or M will be served with certainty allows you to buy the right wine, whereas, if you are not certain, you will either have to buy both wines or else bring the wrong wine with probability $\frac{1}{2}$.

Yet this example does not contradict the independence axiom. To appeal to the axiom, the decision framework must be set up so that the satisfaction derived from an outcome does not depend on any action taken by the decision maker before the uncertainty is resolved. *Thus, preferences should not be induced or derived from ex ante actions.*[9] Here, the action "acquisition of a bottle of wine" is taken before the uncertainty about the meal is resolved.

To put this situation into the framework required, we must include the ex ante action as part of the description of outcomes. For example, here there would be four outcomes: "bringing red wine when served M," "bringing white wine when served M," "bringing red wine when served F," and "bringing white wine when served F." For any underlying uncertainty about what will be served, you induce a lottery over these outcomes by your choice of action. In this setup, it is quite plausible to be indifferent among "having meat and bringing red wine," "having fish and bringing white wine," or any lottery between these two outcomes, as the independence axiom 6
requires. ■

Although it is not a contradiction to the postulates of expected utility theory, and therefore it is not a serious conceptual difficulty, the induced preferences example nonetheless raises a practical difficulty in the use of the theory. The example illustrates the fact that, in applications, many economic situations do not fit the pure framework of expected utility theory. Preferences are almost always, to some extent, induced.[10]

The expected utility theorem does impose some structure on induced preferences. For example, suppose the complete set of outcomes is $B \times A$, where $B = \{b_1, \dots, b_N\}$ is the set of possible realizations of an exogenous randomness and A is the decision maker's set of possible (ex ante) actions. Under the conditions of the expected utility theorem, for every $a \in A$ and $b_n \in B$, we can assign some utility value $u_n(a)$ to the outcome (b_n, a). Then, for every exogenous lottery $L = (p_1, \dots, p_N)$ on B, we can define a derived utility function by maximizing expected utility:

$$U(L) = \operatorname*{Max}_{a \in A} \sum_n p_n u_n(a).$$

In Exercise 6.B.6, you are asked to show that while $U(L)$, a function on $\mathscr{L}$, need not be linear,

9. Actions taken ex post do not create problems. For example, suppose that $u_n(a_n)$ is the utility derived from outcome n when action a_n is taken after the realization of uncertainty. The decision maker therefore chooses a_n to solve $\text{Max}_{a_n \in A_n} u_n(a_n)$, where A_n is the set of possible actions when outcome n occurs. We can then let $u_n = \text{Max}_{a_n \in A_n} u_n(a_n)$ and evaluate lotteries over the N outcomes as in expected utility theory.

10. Consider, for example, preferences for lotteries over amounts of money available tomorrow. Unless the individual's preferences over consumption today and tomorrow are additively separable, his decision of how much to consume today—a decision that must be made before the resolution of the uncertainty concerning tomorrow's wealth—affects his preferences over these lotteries in a manner that conflicts with the fulfillment of the independence axiom.

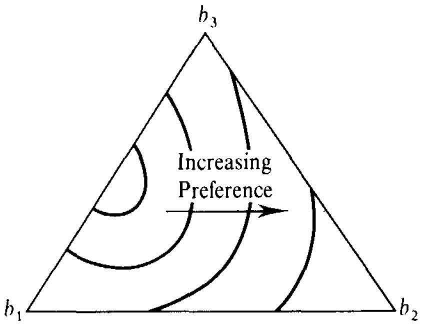


Figure 6.B.8
An indifference map for induced preferences over lotteries on $B = \{b_1, b_2, b_3\}$.

it is nonetheless always *convex*; that is,

$$U(\alpha L + (1-\alpha)L') \le \alpha U(L) + (1-\alpha)U(L').$$

Figure 6.B.8 represents an indifference map for induced preferences in the probability simplex for a case where $N = 3$.

6.C Money Lotteries and Risk Aversion

In many economic settings, individuals seem to display aversion to risk. In this section, we formalize the notion of *risk aversion* and study some of its properties.

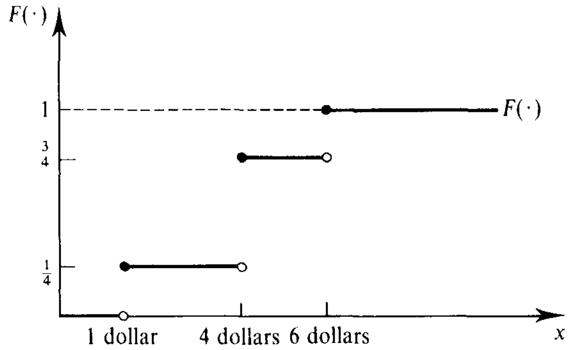
From this section through the end of the chapter, we concentrate on risky alternatives whose outcomes are amounts of money. It is convenient, however, when dealing with monetary outcomes, to treat money as a continuous variable. Strictly speaking, the derivation of the expected utility representation given in Section 6.B assumed a finite number of outcomes. However, the theory can be extended, with some minor technical complications, to the case of an infinite domain. We begin by briefly discussing this extension.

Lotteries over Monetary Outcomes and the Expected Utility Framework

Suppose that we denote amounts of money by the continuous variable x. We can describe a monetary lottery by means of a *cumulative distribution function* $F\colon \mathbb{R} \to [0,1]$. That is, for any x, $F(x)$ is the probability that the realized payoff is less than or equal to x. Note that if the distribution function of a lottery has a density function $f(\cdot)$ associated with it, then $F(x) = \int_{-\infty}^{x} f(t)\,dt$ for all x. The advantage of a formalism based on distribution functions over one based on density functions, however, is that the first is completely general. It does not exclude a priori the possibility of a discrete set of outcomes. For example, the distribution function of a lottery with only three monetary outcomes receiving positive probability is illustrated in Figure 6.C.1.

Note that distribution functions preserve the linear structure of lotteries (as do density functions). For example, the final distribution of money, $F(\cdot)$, induced by a compound lottery $(L_1, \ldots, L_K; \alpha_1, \ldots, \alpha_K)$ is just the weighted average of the distributions induced by each of the lotteries that constitute it: $F(x) = \sum_k \alpha_k F_k(x)$, where $F_k(\cdot)$ is the distribution of the payoff under lottery L_k.

From this point on, we shall work with distribution functions to describe lotteries over monetary outcomes. We therefore take the lottery space $\mathscr{L}$ to be the *set of all*



$$\begin{Bmatrix} \text{Prob(1 dollar)} = \frac{1}{4} \\ \text{Prob(4 dollars)} = \frac{1}{2} \\ \text{Prob(6 dollars)} = \frac{1}{4} \end{Bmatrix} \rightarrow F(x) = \begin{cases} 0 & \text{if } x < 1 \\ \frac{1}{4} & \text{if } x \in [1, 4) \\ \frac{3}{4} & \text{if } x \in [4, 6) \\ 1 & \text{if } x \geq 6 \end{cases}$$

Figure 6.C.1
A distribution function.

distribution functions over nonnegative amounts of money, or, more generally, over an interval $[a, +\infty)$.

As in Section 6.B, we begin with a decision maker who has rational preferences $\succsim$ defined over $\mathscr{L}$. The application of the expected utility theorem to outcomes defined by a continuous variable tells us that under the assumptions of the theorem, there is an assignment of utility values $u(x)$ to nonnegative amounts of money with the property that any $F(\cdot)$ can be evaluated by a utility function $U(\cdot)$ of the form

$$U(F) = \int u(x)\, dF(x). \tag{6.C.1}$$

Expression (6.C.1) is the exact extension of the expected utility form to the current setting. The v.N–M utility function $U(\cdot)$ is the mathematical expectation, over the realizations of x, of the values $u(x)$. The latter takes the place of the values $(u_1, \ldots, u_N)$ used in the discrete treatment of Section 6.B.[11] Note that, as before, $U(\cdot)$ is linear in $F(\cdot)$.

The strength of the expected utility representation is that it preserves the very useful expectation form while making the utility of monetary lotteries sensitive not only to the mean but also to the higher moments of the distribution of the monetary payoffs. (See Exercise 6.C.2 for an illuminating quadratic example.)

It is important to distinguish between the utility function $U(\cdot)$, defined on lotteries, and the utility function $u(\cdot)$ defined on sure amounts of money. For this reason, we call $U(\cdot)$ the *von-Neumann–Morgenstern (v.N–M) expected utility function* and $u(\cdot)$ the *Bernoulli utility function.*[12]

11. Given a distribution function $F(x)$, the expected value of a function $\phi(x)$ is given by $\int \phi(x)\, dF(x)$. When $F(\cdot)$ has an associated density function $f(x)$, this expression is exactly equal to $\int \phi(x) f(x)\, dx$. Note also that for notational simplicity, we do not explicitly write the limits of integration when the integral is over the full range of possible realizations of x.

12. The terminology is not standardized. It is common to call $u(\cdot)$ the v.N–M utility function or the expected utility function. We prefer to have a name that is specific to the $u(\cdot)$ function, and so we call it the Bernoulli function for Daniel Bernoulli, who first used an instance of it.

Although the general axioms of Section 6.B yield the expected utility representation, they place no restrictions whatsoever on the Bernoulli utility function $u(\cdot)$. In large part, the analytical power of the expected utility formulation hinges on specifying the Bernoulli utility function $u(\cdot)$ in such a manner that it captures interesting economic attributes of choice behavior. At the simplest level, it makes sense in the current monetary context to postulate that $u(\cdot)$ is *increasing* and *continuous*.[13] We maintain both of these assumptions from now on.

Another restriction, based on a subtler argument, is the *boundedness* (above and below) of $u(\cdot)$. To argue the plausibility of boundedness above (a similar argument applies for boundedness below), we refer to the famous *St. Petersburg–Menger paradox*. Suppose that $u(\cdot)$ is unbounded, so that for every integer m there is an amount of money x_m with $u(x_m) > 2^m$. Consider the following lottery: we toss a coin repeatedly until tails comes up. If this happens in the mth toss, the lottery gives a monetary payoff of x_m. Since the probability of this outcome is $1/2^m$, the expected utility of this lottery is $\sum_{m=1}^{\infty} u(x_m)(1/2^m) \geq \sum_{m=1}^{\infty} (2^m)(1/2^m) = +\infty$. But this means that an individual should be willing to give up all his wealth for the opportunity to play this lottery, a patently absurd conclusion (how much would you pay?).[14]

The rest of this section concentrates on the important property of *risk aversion*, its formulation in terms of the Bernoulli utility function $u(\cdot)$, and its measurement.[15]

Risk Aversion and Its Measurement

The concept of risk aversion provides one of the central analytical techniques of economic analysis, and it is assumed in this book whenever we handle uncertain situations. We begin our discussion of risk aversion with a general definition that does not presume an expected utility formulation.

Definition 6.C.1: A decision maker is a *risk averter* (or exhibits *risk aversion*) if for any lottery $F(\cdot)$, the degenerate lottery that yields the amount $\int x\, dF(x)$ with certainty is at least as good as the lottery $F(\cdot)$ itself. If the decision maker is always [i.e., for any $F(\cdot)$] indifferent between these two lotteries, we say that he is *risk neutral*. Finally, we say that he is *strictly risk averse* if indifference holds only when the two lotteries are the same [i.e., when $F(\cdot)$ is degenerate].

If preferences admit an expected utility representation with Bernoulli utility function $u(x)$, it follows directly from the definition of risk aversion that the decision maker is risk averse if and only if

$$\int u(x)\, dF(x) \leq u\left(\int x\, dF(x)\right) \quad \text{for all } F(\cdot). \tag{6.C.2}$$

Inequality (6.C.2) is called *Jensen's inequality*, and it is the defining property of a concave function (see Section M.C of the Mathematical Appendix). Hence, in the

13. In applications, an exception to continuity is sometimes made at $x = 0$ by setting $u(0) = -\infty$.

14. In practice, most utility functions commonly used are not bounded. Paradoxes are avoided because the class of distributions allowed by the modeler in each particular application is a limited one. Note also that if we insisted on $u(\cdot)$ being defined on $(-\infty, \infty)$ then any nonconstant $u(\cdot)$ could not be both concave and bounded (above and below).

15. Arrow (1971) and Pratt (1964) are the classical references in this area.

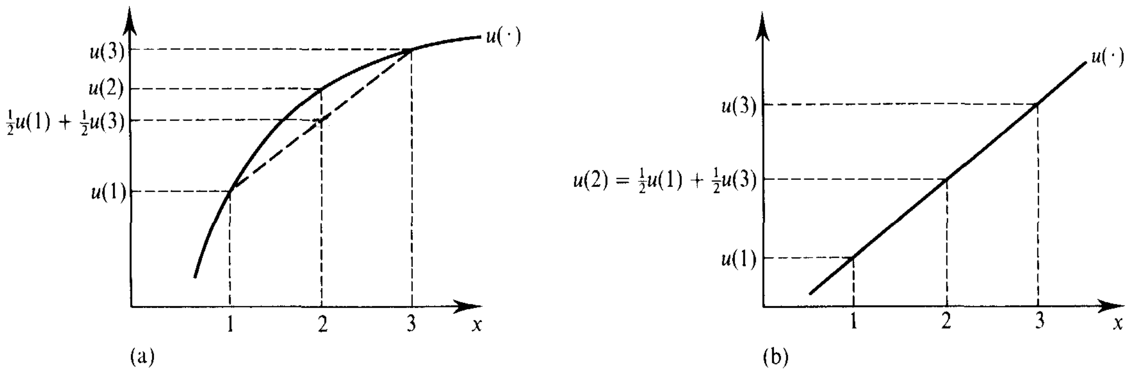


Figure 6.C.2 Risk aversion (a) and risk neutrality (b).

context of expected utility theory, we see that *risk aversion is equivalent to the concavity of* $u(\cdot)$ and that strict risk aversion is equivalent to the strict concavity of $u(\cdot)$. This makes sense. Strict concavity means that the marginal utility of money is decreasing. Hence, at any level of wealth x, the utility gain from an extra dollar is smaller than (the absolute value of) the utility loss of having a dollar less. It follows that a risk of gaining or losing a dollar with even probability is not worth taking. This is illustrated in Figure 6.C.2(a); in the figure we consider a gamble involving the gain or loss of 1 dollar from an initial position of 2 dollars. The (v.N–M)utility of this gamble, $\frac{1}{2}u(1) + \frac{1}{2}u(3)$, is strictly less than that of the initial certain position $u(2)$.

For a risk-neutral expected utility maximizer, (6.C.2) must hold with *equality* for all $F(\cdot)$. Hence, the decision maker is risk neutral if and only if the Bernoulli utility function of money $u(\cdot)$ is linear. Figure 6.C.2(b) depicts the (v.N–M) utility associated with the previous gamble for a risk neutral individual. Here the individual is indifferent between the gambles that yield a mean wealth level of 2 dollars and a certain wealth of 2 dollars. Definition 6.C.2 introduces two useful concepts for the analysis of risk aversion.

Definition 6.C.2: Given a Bernoulli utility function $u(\cdot)$ we define the following concepts:

(i) The *certainty equivalent of* $F(\cdot)$, denoted $c(F, u)$, is the amount of money for which the individual is indifferent between the gamble $F(\cdot)$ and the certain amount $c(F, u)$; that is,

$$u(c(F, u)) = \int u(x)\, dF(x). \tag{6.C.3}$$

(ii) For any fixed amount of money x and positive number ε, the *probability premium* denoted by $\pi(x, \varepsilon, u)$, is the excess in winning probability over fair odds that makes the individual indifferent between the certain outcome x and a gamble between the two outcomes $x + \varepsilon$ and $x - \varepsilon$. That is

$$u(x) = (\tfrac{1}{2} + \pi(x, \varepsilon, u))\,u(x + \varepsilon) + (\tfrac{1}{2} - \pi(x, \varepsilon, u))\,u(x - \varepsilon). \tag{6.C.4}$$

These two concepts are illustrated in Figure 6.C.3. In Figure 6.C.3(a), we exhibit the geometric construction of $c(F, u)$ for an even probability gamble between 1 and 3 dollars. Note that $c(F, u) < 2$, implying that some expected return is traded for certainty. The satisfaction of the inequality $c(F, u) \leq \int x\, dF(x)$ for all $F(\cdot)$ is, in fact,

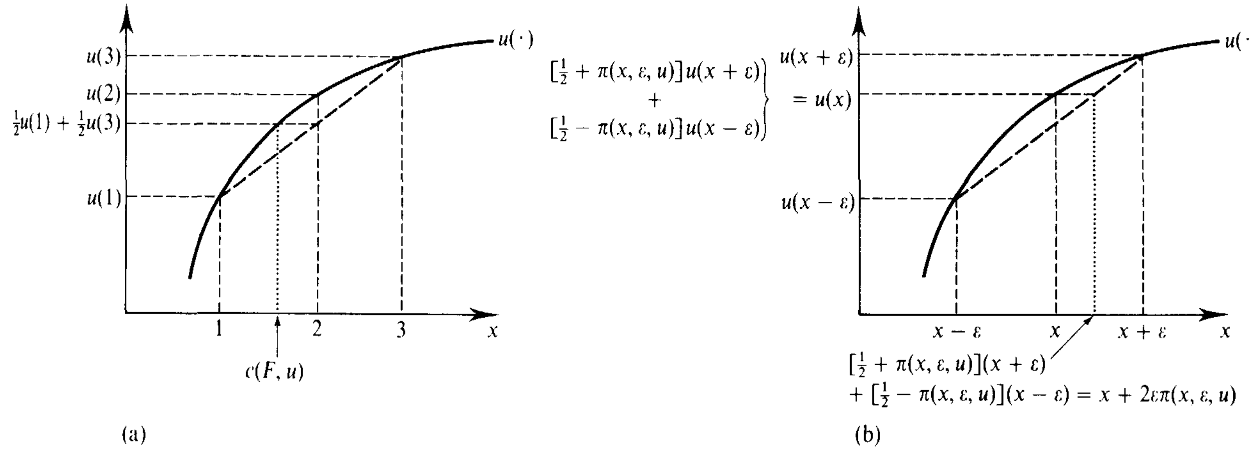


Figure 6.C.3 The certainty equivalent (a) and the probability premium (b).

equivalent to the decision maker being a risk averter. To see this, observe that since $u(\cdot)$ is nondecreasing, we have

$$c(F, u) \le \int x\, dF(x) \Leftrightarrow u(c(F, u)) \le u\left(\int x\, dF(x)\right) \Leftrightarrow \int u(x)\, dF(x) \le u\left(\int x\, dF(x)\right),$$

where the last $\Leftrightarrow$ follows from the definition of $c(F, u)$.

In Figure 6.C.3(b), we exhibit the geometric construction of $\pi(x, \varepsilon, u)$. We see that $\pi(x, \varepsilon, u) > 0$; that is, better than fair odds must be given for the individual to accept the risk. In fact, the satisfaction of the inequality $\pi(x, \varepsilon, u) \ge 0$ for all x and $\varepsilon > 0$ is also equivalent to risk aversion (see Exercise 6.C.3).

These points are formally summarized in Proposition 6.C.1.

Proposition 6.C.1: Suppose a decision maker is an expected utility maximizer with a Bernoulli utility function $u(\cdot)$ on amounts of money. Then the following properties are equivalent:

(i) The decision maker is risk averse.
(ii) $u(\cdot)$ is concave.[16]
(iii) $c(F, u) \le \int x\, dF(x)$ for all $F(\cdot)$.
(iv) $\pi(x, \varepsilon, u) \ge 0$ for all x, ε.

Examples 6.C.1 to 6.C.3 illustrate the use of the risk aversion concept.

Example 6.C.1: *Insurance.* Consider a strictly risk-averse decision maker who has an initial wealth of w but who runs a risk of a loss of D dollars. The probability of the loss is π. It is possible, however, for the decision maker to buy insurance. One unit of insurance costs q dollars and pays 1 dollar if the loss occurs. Thus, if α units of insurance are bought, the wealth of the individual will be $w - \alpha q$ if there is no loss and $w - \alpha q - D + \alpha$ if the loss occurs. Note, for purposes of later discussion, that the decision maker's expected wealth is then $w - \pi D + \alpha(\pi - q)$. The decision maker's problem is to choose the optimal level of α. His utility maximization problem is

pay

16. Recall that if $u(\cdot)$ is twice differentiable then concavity is equivalent to $u''(x) \le 0$ for all x.

therefore

$$\operatorname*{Max}_{\alpha \geq 0} \ (1-\pi)u(w-\alpha q) + \pi u(w-\alpha q - D + \alpha).$$

If α^* is an optimum, it must satisfy the first-order condition:

$$-q(1-\pi)u'(w-\alpha^* q) + \pi(1-q)u'(w-D+\alpha^*(1-q)) \leq 0,$$

with equality if $\alpha^* > 0$.

Suppose now that the price q of one unit of insurance is *actuarially fair* in the sense of it being equal to the expected cost of insurance. That is, $q = \pi$. Then the first-order condition requires that

$$u'(w-D+\alpha^*(1-\pi)) - u'(w-\alpha^*\pi) \leq 0,$$

with equality if $\alpha^* > 0$.

Since $u'(w-D) > u'(w)$, we must have $\alpha^* > 0$, and therefore

$$u'(w-D+\alpha^*(1-\pi)) = u'(w-\alpha^*\pi).$$

Because $u'(\cdot)$ is strictly decreasing, this implies

$$w - D + \alpha^*(1-\pi) = w - \alpha^*\pi,$$

or, equivalently,

$$\alpha^* = D.$$

Thus, *if insurance is actuarially fair, the decision maker insures completely.* The individual's final wealth is then $w - \pi D$, regardless of the occurrence of the loss.

This proof of the complete insurance result uses first-order conditions, which is instructive but not really necessary. Note that if $q = \pi$, then the decision maker's expected wealth is $w - \pi D$ for any α. Since setting $\alpha = D$ allows him to reach $w - \pi D$ with certainty, the definition of risk aversion directly implies that this is the optimal level of α. ■

Example 6.C.2: *Demand for a Risky Asset.* An *asset* is a divisible claim to a financial return in the future. Suppose that there are two assets, a safe asset with a return of 1 dollar per dollar invested and a risky asset with a random return of z dollars per dollar invested. The random return z has a distribution function $F(z)$ that we assume satisfies $\int z\, dF(z) > 1$; that is, its mean return exceeds that of the safe asset.

An individual has initial wealth w to invest, which can be divided in any way between the two assets. Let α and β denote the amounts of wealth invested in the risky and the safe asset, respectively. Thus, for any realization z of the random return, the individual's *portfolio* (α, β) pays $\alpha z + \beta$. Of course, we must also have $\alpha + \beta = w$.

The question is how to choose α and β. The answer will depend on $F(\cdot)$, w, and the Bernoulli utility function $u(\cdot)$. The utility maximization problem of the individual is

$$\operatorname*{Max}_{\alpha, \beta \geq 0} \int u(\alpha z + \beta)\, dF(z)$$
$$\text{s.t. } \alpha + \beta = w.$$

Equivalently, we want to maximize $\int u(w + \alpha(z-1))\, dF(z)$ subject to $0 \leq \alpha \leq w$. If

α^* is optimal, it must satisfy the Kuhn–Tucker first-order conditions:[17]

$$\phi(\alpha^*) = \int u'(w + \alpha^*[z - 1])(z - 1)\, dF(z) \begin{cases} \leq 0 & \text{if } \alpha^* < w, \\ \geq 0 & \text{if } \alpha^* > 0. \end{cases}$$

Note that $\int z\, dF(z) > 1$ implies $\phi(0) > 0$. Hence, $\alpha^* = 0$ cannot satisfy this first-order condition. We conclude that the optimal portfolio has $\alpha^* > 0$. The general principle illustrated in this example, is that *if a risk is actuarially favorable, then a risk averter will always accept at least a small amount of it.*

This same principle emerges in Example 6.C.1 if insurance is not actuarially fair. In Exercise 6.C.1, you are asked to show that if $q > \pi$, then the decision maker will not fully insure (i.e., will accept some risk). ■

Example 6.C.3: *General Asset Problem.* In the previous example, we could define the utility $U(\alpha,\beta)$ of the portfolio (α,β) as $U(\alpha,\beta) = \int u(\alpha z + \beta)\, dF(z)$. Note that $U(\cdot)$ is then an increasing, continuous, and concave utility function. We now discuss an important generalization. We assume that we have N assets (one of which may be the safe asset) with asset n giving a return of z_n per unit of money invested. These returns are jointly distributed according to a distribution function $F(z_1, \ldots, z_N)$. The utility of holding a *portfolio* of assets $(\alpha_1, \ldots, \alpha_N)$ is then

$$U(\alpha_1, \ldots, \alpha_N) = \int u(\alpha_1 z_1 + \cdots + \alpha_N z_N)\, dF(z_1, \ldots, z_N).$$

This utility function for portfolios, defined on $\mathbb{R}^N_+$, is also increasing, continuous, and concave (see Exercise 6.C.4). This means that, formally, we can treat assets as the usual type of commodities and apply to them the demand theory developed in Chapters 2 and 3. Observe, in particular, how risk aversion leads to a convex indifference map for portfolios. ■

Suppose that the lotteries pay in vectors of physical goods rather than in money. Formally, the space of outcomes is then the consumption set $\mathbb{R}^L_+$ (all the previous discussion can be viewed as the special case in which there is a single good). In this more general setting, the concept of risk aversion given by Definition 6.C.1 is perfectly well defined. Furthermore, if there is a Bernoulli utility function $u\colon \mathbb{R}^L_+ \to \mathbb{R}$, *then risk aversion is still equivalent to the* concavity of $u(\cdot)$. Hence, we have here another justification for the convexity assumption of Chapter 3: Under the assumptions of the expected utility theorem, the convexity of preferences for perfectly certain amounts of the physical commodities must hold if for any lottery with commodity payoffs the individual always prefers the certainty of the mean commodity bundle to the lottery itself.

In Exercise 6.C.5, you are asked to show that if preferences over lotteries with commodity payoffs exhibit risk aversion, then, at given commodity prices, the induced preferences on money lotteries (where consumption decisions are made after the realization of wealth) are also risk averse. Thus, in principle, it is possible to build the theory of risk aversion on the more primitive notion of lotteries over the final consumption of goods.

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17. The objective function is concave in α because the concavity of $u(\cdot)$ implies that $\int u''(w + \alpha(z - 1))(z - 1)^2\, dF(x) \leq 0$.

The Measurement of Risk Aversion

Now that we know what it means to be risk averse, we can try to measure the extent of risk aversion. We begin by defining one particularly useful measure and discussing some of its properties.

Definition 6.C.3: Given a (twice-differentiable) Bernoulli utility function $u(\cdot)$ for money, the *Arrow–Pratt coefficient of absolute risk aversion* at x is defined as $r_A(x) = -u''(x)/u'(x)$.

The Arrow–Pratt measure can be motivated as follows: We know that risk neutrality is equivalent to the linearity of $u(\cdot)$, that is, to $u''(x) = 0$ for all x. Therefore, it seems logical that the degree of risk aversion be related to the *curvature* of $u(\cdot)$. In Figure 6.C.4, for example, we represent two Bernoulli utility functions $u_1(\cdot)$ and $u_2(\cdot)$ normalized (by choice of origin and units) to have the same utility and marginal utility values at wealth level x. The certainty equivalent for a small risk with mean x is smaller for $u_2(\cdot)$ than for $u_1(\cdot)$, suggesting that risk aversion increases with the curvature of the Bernoulli utility function at x. One possible measure of curvature of the Bernoulli utility function $u(\cdot)$ at x is $u''(x)$. However, this is not an adequate measure because it is not invariant to positive linear transformations of the utility function. To make it invariant, the simplest modification is to use $u''(x)/u'(x)$. If we change sign so as to have a positive number for an increasing and concave $u(\cdot)$, we get the Arrow–Pratt measure.

A more precise motivation for $r_A(x)$ as a measure of the degree of risk aversion can be obtained by considering a fixed wealth x and studying the behavior of the probability premium $\pi(x, \varepsilon, u)$ as $\varepsilon \to 0$ [for simplicity, we write it as $\pi(\varepsilon)$]. Differentiating the identity (6.C.4) that defines $\pi(\cdot)$ twice with respect to ε (assume that $\pi(\cdot)$ is differentiable), and evaluating at $\varepsilon = 0$, we get $4\pi'(0)u'(x) + u''(x) = 0$. Hence

$$r_A(x) = 4\pi'(0).$$

Thus, $r_A(x)$ measures the rate at which the probability premium increases at certainty with the small risk measured by ε.[18] As we go along, we will find additional related interpretations of the Arrow–Pratt measure.

18. For a similar derivation relating $r_A(\cdot)$ to the rate of change of the certainty equivalent with respect to a small increase in a small risk around certainty, see Exercise 6.C.20.

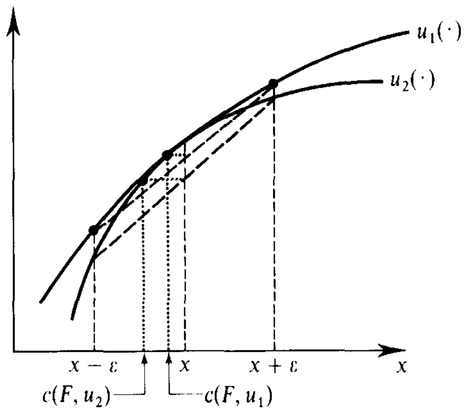


Figure 6.C.4
Differing degrees of risk aversion.

Note that, up to two integration constants, the utility function $u(\cdot)$ can be recovered from $r_A(\cdot)$ by integrating twice. The integration constants are irrelevant because the Bernoulli utility is identified only up to two constants (origin and units). Thus, the Arrow–Pratt risk aversion measure $r_A(\cdot)$ fully characterizes behavior under uncertainty.

Example 6.C.4: Consider the utility function $u(x) = -e^{-ax}$ for $a > 0$. Then $u'(x) = ae^{-ax}$ and $u''(x) = -a^2e^{-ax}$. Therefore, $r_A(x, u) = a$ for all x. It follows from the observation just made that the general form of a Bernoulli utility function with an Arrow–Pratt measure of absolute risk aversion equal to the constant $a > 0$ at all x is $u(x) = -\alpha e^{-ax} + \beta$ for some $\alpha > 0$ and β. ■

Once we are equipped with a measure of risk aversion, we can put it to use in comparative statics exercises. Two common situations are the comparisons of risk attitudes across individuals with different utility functions and the comparison of risk attitudes for one individual at different levels of wealth.

Comparisons across individuals

Given two Bernoulli utility functions $u_1(\cdot)$ and $u_2(\cdot)$, when can we say that $u_2(\cdot)$ is unambiguously *more risk averse than* $u_1(\cdot)$? Several possible approaches to a definition seem plausible:

(i) $r_A(x, u_2) \geq r_A(x, u_1)$ for every x.

(ii) There exists an increasing concave function $\psi(\cdot)$ such that $u_2(x) = \psi(u_1(x))$ at all x; that is, $u_2(\cdot)$ is a concave transformation of $u_1(\cdot)$. [In other words, $u_2(\cdot)$ is "more concave" than $u_1(\cdot)$.]

(iii) $c(F, u_2) \leq c(F, u_1)$ for any $F(\cdot)$.

(iv) $\pi(x, \varepsilon, u_2) \geq \pi(x, \varepsilon, u_1)$ for any x and ε.

(v) Whenever $u_2(\cdot)$ finds a lottery $F(\cdot)$ at least as good as a riskless outcome $\bar{x}$, then $u_1(\cdot)$ also finds $F(\cdot)$ at least as good as $\bar{x}$. That is, $\int u_2(x)\, dF(x) \geq u_2(\bar{x})$ implies $\int u_1(x)\, dF(x) \geq u_1(\bar{x})$ for any $F(\cdot)$ and $\bar{x}$.[19]

In fact, these five definitions are equivalent.

Proposition 6.C.2: Definitions (i) to (v) of the *more-risk-averse-than* relation are equivalent.

Proof: We will not give a complete proof. (You are asked to establish some of the implications in Exercises 6.C.6 and 6.C.7.) Here we will show the equivalence of (i) and (ii) under differentiability assumptions.

Note, first that we always have $u_2(x) = \psi(u_1(x))$ for some increasing function $\psi(\cdot)$; this is true simply because $u_1(\cdot)$ and $u_2(\cdot)$ are ordinally identical (more money is preferred to less). Differentiating, we get

$$u_2'(x) = \psi'(u_1(x))u_1'(x)$$

and

$$u_2''(x) = \psi'(u_1(x))u_1''(x) + \psi''(u_1(x))(u_1'(x))^2.$$

Dividing both sides of the second expression by $u_2'(x) > 0$, and using the first

19. In other words, any risk that $u_2(\cdot)$ would accept starting from a position of certainty would also be accepted by $u_1(\cdot)$.

expression, we get

$$r_A(x, u_2) = r_A(x, u_1) - \frac{\psi''(u_1(x))}{\psi'(u_1(x))} u_1'(x).$$

Thus, $r_A(x, u_2) \geq r_A(x, u_1)$ for all x if and only if $\psi''(u_1) \leq 0$ for all u_1 in the range of $u_1(\cdot)$. ■

The more-risk-averse-than relation is a *partial ordering* of Bernoulli utility functions; it is transitive but far from complete. Typically, two Bernoulli utility functions $u_1(\cdot)$ and $u_2(\cdot)$ will not be comparable; that is, we will have $r_A(x, u_1) > r_A(x, u_2)$ at some x but $r_A(x', u_1) < r_A(x', u_2)$ at some other $x' \neq x$.

Example 6.C.2 continued: We take up again the asset portfolio problem between a safe and a risky asset discussed in Example 6.C.2. Suppose that we now have two individuals with Bernoulli utility functions $u_1(\cdot)$ and $u_2(\cdot)$, and denote by α_1^* and α_2^* their respective optimal investments in the risky asset. We will show that *if* $u_2(\cdot)$ *is more risk averse than* $u_1(\cdot)$, *then* $\alpha_2^* < \alpha_1^*$; that is, the second decision maker invests less in the risky asset than the first.

To repeat from our earlier discussion, the asset allocation problem for $u_1(\cdot)$ is

$$\underset{0 \leq \alpha \leq w}{\text{Max}} \int u_1(w - \alpha + \alpha z)\, dF(z).$$

Assuming an interior solution, the first-order condition is

$$\int (z - 1) u_1'(w + \alpha_1^*[z - 1])\, dF(z) = 0. \qquad (6.C.5)$$

The analogous expression for the utility function $u_2(\cdot)$ is

$$\phi_2(\alpha_2^*) = \int (z - 1) u_2'(w + \alpha_2^*[z - 1])\, dF(z) = 0. \qquad (6.C.6)$$

As we know, the concavity of $u_2(\cdot)$ implies that $\phi_2(\cdot)$ is decreasing. Therefore, if we show that $\phi_2(\alpha_1^*) < 0$, it must follow that $\alpha_2^* < \alpha_1^*$, which is the result we want. Now, $u_2(x) = \psi(u_1(x))$ allows us to write

$$\phi_2(\alpha_1^*) = \int (z - 1)\psi'(u_1(w + \alpha_1^*[z - 1]))u_1'(w + \alpha_1^*[z - 1])\, dF(z) < 0. \qquad (6.C.7)$$

To understand the final inequality, note that the integrand of expression (6.C.7) is the same as that in (6.C.5) except that it is multiplied by $\psi'(\cdot)$, a positive decreasing function of z [recall that $u_2(\cdot)$ more risk averse than $u_1(\cdot)$ means that the increasing function $\psi(\cdot)$ is concave; that is, $\psi'(\cdot)$ is positive and decreasing]. Hence, the integral (6.C.7) underweights the positive values of $(z - 1)u_1'(w + \alpha_1^*[z - 1])$, which obtain for $z > 1$, relative to the negative values, which obtain for $z < 1$. Since, in (6.C.5), the integral of the positive and the negative parts of the integrand added to zero, they now must add to a negative number. This establishes the desired inequality. ■

Comparisons across wealth levels

It is a common contention that wealthier people are willing to bear more risk than poorer people. Although this might be due to differences in utility functions across people, it is more likely that the source of the difference lies in the possibility that richer people "can afford to take a chance." Hence, we shall explore the implications of the condition stated in Definition 6.C.4.

Definition 6.C.4: The Bernoulli utility function $u(\cdot)$ for money exhibits *decreasing absolute risk aversion* if $r_A(x, u)$ is a decreasing function of x.

Individuals whose preferences satisfy the decreasing absolute risk aversion property take more risk as they become wealthier. Consider two levels of initial wealth $x_1 > x_2$. Denote the increments or decrements to wealth by z. Then the individual evaluates risk at x_1 and x_2 by, respectively, the induced Bernoulli utility functions $u_1(z) = u(x_1 + z)$ and $u_2(z) = u(x_2 + z)$. Comparing an individual's attitudes toward risk as his level of wealth changes is like comparing the utility functions $u_1(\cdot)$ and $u_2(\cdot)$, a problem we have just studied. If $u(\cdot)$ displays decreasing absolute risk aversion, then $r_A(z, u_2) \geq r_A(z, u_1)$ for all z. This is condition (i) of Proposition 6.C.2. Hence, the result in Proposition 6.C.3 follows directly from Proposition 6.C.2.

Proposition 6.C.3: The following properties are equivalent:

(i) The Bernoulli utility function $(\cdot)$ exhibits decreasing absolute risk aversion.

(ii) Whenever $x_2 < x_1$, $u_2(z) = u(x_2 + z)$ is a concave transformation of $u_1(z) = u(x_1 + z)$.

(iii) For any risk $F(z)$, the certainty equivalent of the lottery formed by adding risk z to wealth level x, given by the amount c_x at which $u(c_x) = \int u(x + z)\, dF(z)$, is such that $(x - c_x)$ is decreasing in x. That is, the higher x is, the less is the individual willing to pay to get rid of the risk.

(iv) The probability premium $\pi(x, \varepsilon, u)$ is decreasing in x.

(v) For any $F(z)$, if $\int u(x_2 + z)\, dF(z) \geq u(x_2)$ and $x_2 < x_1$, then $\int u(x_1 + z)\, dF(z) \geq u(x_1)$.

Exercise 6.C.8: Assume that the Bernoulli utility function $u(\cdot)$ exhibits decreasing absolute risk aversion. Show that for the asset demand model of Example 6.C.2 (and Example 6.C.2 continued), the optimal allocation between the safe and the risky assets places an increasing amount of wealth in the risky asset as w rises (i.e., the risky asset is a normal good).

The assumption of decreasing absolute risk aversion yields many other economically reasonable results concerning risk-bearing behavior. However, in applications, it is often too weak and, because of its analytical convenience, it is sometimes complemented by a stronger assumption: *nonincreasing relative risk aversion.*

To understand the concept of relative risk aversion, note that the concept of absolute risk aversion is suited to the comparison of attitudes toward risky projects whose outcomes are *absolute gains or losses* from current wealth. But it is also of interest to evaluate risky projects whose outcomes are *percentage* gains or losses of current wealth. The concept of relative risk aversion does just this.

Let $t > 0$ stand for *proportional* increments or decrements of wealth. Then, an individual with Bernoulli utility function $u(\cdot)$ and initial wealth x can evaluate a random percentage risk by means of the utility function $\tilde{u}(t) = u(tx)$. The initial wealth position corresponds to $t = 1$. We already know that for a small risk around $t = 1$, the degree of risk aversion is well captured by $\tilde{u}''(1)/\tilde{u}'(1)$. Noting that $\tilde{u}''(1)/\tilde{u}'(1) = xu''(x)/u'(x)$, we are led to the concept stated in Definition 6.C.5.

Definition 6.C.5: Given a Bernoulli utility function $u(\cdot)$, the *coefficient of relative risk aversion at* x is $r_R(x, u) = -xu''(x)/u'(x)$.

Consider now how this measure varies with wealth. The property of *nonincreasing relative risk aversion* says that the individual becomes less risk averse with regard to gambles that are proportional to his wealth as his wealth increases. This is a stronger assumption than decreasing absolute risk aversion: Since $r_R(x, u) = xr_A(x, u)$, a risk-averse individual with decreasing relative risk aversion will exhibit decreasing absolute risk aversion, but the converse is not necessarily the case.

As before, we can examine various implications of this concept. Proposition 6.C.4 is an abbreviated parallel to Proposition 6.C.3.

Proposition 6.C.4: The following conditions for a Bernoulli utility function $u(\cdot)$ on amounts of money are equivalent:

(i) $r_R(x, u)$ is decreasing in x.

(ii) Whenever $x_2 < x_1$, $\tilde{u}_2(t) = u(tx_2)$ is a concave transformation of $\tilde{u}_1(t) = u(tx_1)$.

(iii) Given any risk $F(t)$ on $t > 0$, the certainty equivalent $\bar{c}_x$ defined by $u(\bar{c}_x) = \int u(tx)\, dF(t)$ is such that $x/\bar{c}_x$ is decreasing in x.

Proof: Here we show only that (i) implies (iii). To this effect, fix a distribution $F(t)$ on $t > 0$, and, for any x, define $u_x(t) = u(tx)$. Let $c(x)$ be the usual certainty equivalent (from Definition 6.C.2): $u_x(c(x)) = \int u_x(t)\, dF(t)$. Note that $-u_x''(t)/u_x'(t) = -(1/t)tx[u''(tx)/u'(tx)]$ for any x. Hence if (i) holds, then $u_{x'}(\cdot)$ is less risk averse than $u_x(\cdot)$ whenever $x' > x$. Therefore, by Proposition 6.C.2, $c(x') > c(x)$ and we conclude that $c(\cdot)$ is increasing. Now, by the definition of $u_x(\cdot)$, $u_x(c(x)) = u(xc(x))$. Also

$$u_x(c(x)) = \int u_x(t)\, dF(t) = \int u(tx)\, dF(t) = u(\bar{c}_x).$$

Hence, $\bar{c}_x/x = c(x)$, and so $x/\bar{c}_x$ is decreasing. This concludes the proof. ∎

Example 6.C.2 continued: In Exercise 6.C.11, you are asked to show that if $r_R(x, u)$ is decreasing in x, then the proportion of wealth invested in the risky asset $\gamma = \alpha/w$ is increasing with the individual's wealth level w. The opposite conclusion holds if $r_R(x, u)$ is increasing in x. If $r_R(x, u)$ is a constant independent of x, then the fraction of wealth invested in the risky asset is independent of w [see Exercise 6.C.12 for the specific analytical form that $u(\cdot)$ must have]. Models with constant relative risk aversion are encountered often in finance theory, where they lead to considerable analytical simplicity. Under this assumption, no matter how the wealth of the economy and its distribution across individuals evolves over time, the portfolio decisions of individuals in terms of budget shares do not vary (as long as the safe return and the distribution of random returns remain unchanged). ∎

6.D Comparison of Payoff Distributions in Terms of Return and Risk

In this section, we continue our study of lotteries with monetary payoffs. In contrast with Section 6.C, where we compared utility functions, our aim here is to compare payoff distributions. There are two natural ways that random outcomes can be compared: according to the level of returns and according to the dispersion of returns. We will therefore attempt to give meaning to two ideas: that of a distribution $F(\cdot)$ yielding unambiguously higher returns than $G(\cdot)$ and that of $F(\cdot)$ being unambiguously less risky than $G(\cdot)$. These ideas are known, respectively, by the technical terms of *first-order stochastic dominance* and *second-order stochastic dominance*.[20]

In all subsequent developments, we restrict ourselves to distributions $F(\cdot)$ such that $F(0) = 0$ and $F(x) = 1$ for some x.

First-Order Stochastic Dominance

We want to attach meaning to the expression: "The distribution $F(\cdot)$ yields unambiguously higher returns than the distribution $G(\cdot)$." At least two sensible criteria suggest themselves. First, we could test whether every expected utility maximizer who values more over less prefers $F(\cdot)$ to $G(\cdot)$. Alternatively, we could verify whether, for every amount of money x, the probability of getting at least x is higher under $F(\cdot)$ than under $G(\cdot)$. Fortunately, these two criteria lead to the same concept.

Definition 6.D.1: The distribution $F(\cdot)$ *first-order stochastically dominates* $G(\cdot)$ if, for every nondecreasing function $u: \mathbb{R} \to \mathbb{R}$, we have

$$\int u(x)\, dF(x) \geq \int u(x)\, dG(x).$$

Proposition 6.D.1: The distribution of monetary payoffs $F(\cdot)$ first-order stochastically dominates the distribution $G(\cdot)$ if and only if $F(x) \leq G(x)$ for every x.

Proof: Given $F(\cdot)$ and $G(\cdot)$ denote $H(x) = F(x) - G(x)$. Suppose that $H(\bar{x}) > 0$ for some $\bar{x}$. Then we can define a nondecreasing function $u(\cdot)$ by $u(x) = 1$ for $x > \bar{x}$ and $u(x) = 0$ for $x \leq \bar{x}$. This function has the property that $\int u(x)\, dH(x) = -H(\bar{x}) < 0$, and so the "only if" part of the proposition follows.

For the "if" part of the proposition we first put on record, without proof, that it suffices to establish the equivalence for differentiable utility functions $u(\cdot)$. Given $F(\cdot)$ and $G(\cdot)$, denote $H(x) = F(x) - G(x)$. Integrating by parts, we have

$$\int u(x)\, dH(x) = [u(x)H(x)]_0^\infty - \int u'(x)H(x)\, dx.$$

Since $H(0) = 0$ and $H(x) = 0$ for large x, the first term of this expression is zero. It follows that $\int u(x)\, dH(x) \geq 0$ [or, equivalently, $\int u(x)\, dF(x) - \int u(x)\, dG(x) \geq 0$] if and only if $\int u'(x)H(x)\, dx \leq 0$. Thus, if $H(x) \leq 0$ for all x and $u(\cdot)$ is increasing, then $\int u'(x)H(x)\, dx \leq 0$ and the "if" part of the proposition follows. ■

In Exercise 6.D.1 you are asked to verify Proposition 6.D.1 for the case of lotteries over three possible outcomes. In Figure 6.D.1, we represent two distributions $F(\cdot)$ and $G(\cdot)$. Distribution $F(\cdot)$ first-order stochastically dominates $G(\cdot)$ because the graph of $F(\cdot)$ is uniformly below the graph of $G(\cdot)$. Note two important points: First, first-order stochastic dominance does *not* imply that every possible return of the

20. They were introduced into economics in Rothschild and Stiglitz (1970).

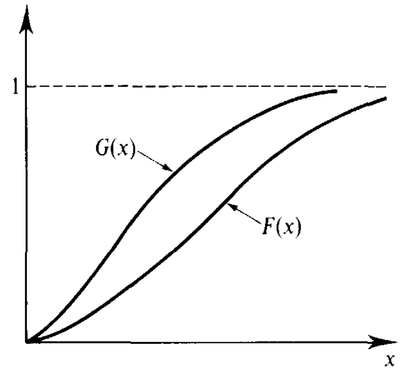


Figure 6.D.1
$F(\cdot)$ first-order stochastically dominates $G(\cdot)$.

superior distribution is larger than every possible return of the inferior one. In the figure, the set of possible outcomes is the same for the two distributions. Second, although $F(\cdot)$ first-order stochastically dominating $G(\cdot)$ implies that the mean of x under $F(\cdot)$, $\int x\,dF(x)$, is greater than its mean under $G(\cdot)$, a ranking of the means of two distributions does *not* imply that one first-order stochastically dominates the other; rather, the entire distribution matters (see Exercise 6.D.3).

Example 6.D.1: Consider a compound lottery that has as its first stage a realization of x distributed according to $G(\cdot)$ and in its second stage applies to the outcome x of the first stage an "upward probabilistic shift." That is, if outcome x is realized in the first stage, then the second stage pays a final amount of money $x + z$, where z is distributed according to a distribution $H_x(z)$ with $H_x(0) = 0$. Thus, $H_x(\cdot)$ generates a *final* return of at least x with probability one. (Note that the distributions applied to different x's may differ.)

Denote the resulting reduced distribution by $F(\cdot)$. Then for any nondecreasing function $u\colon \mathbb{R} \to \mathbb{R}$, we have

$$\int u(x)\,dF(x) = \int\left[\int u(x+z)\,dH_x(z)\right]dG(x) \geq \int u(x)\,dG(x).$$

So $F(\cdot)$ first-order stochastically dominates $G(\cdot)$.

A specific example is illustrated in Figure 6.D.2. As Figure 6.D.2(a) shows, $G(\cdot)$ is an even randomization between 1 and 4 dollars. The outcome "1 dollar" is then

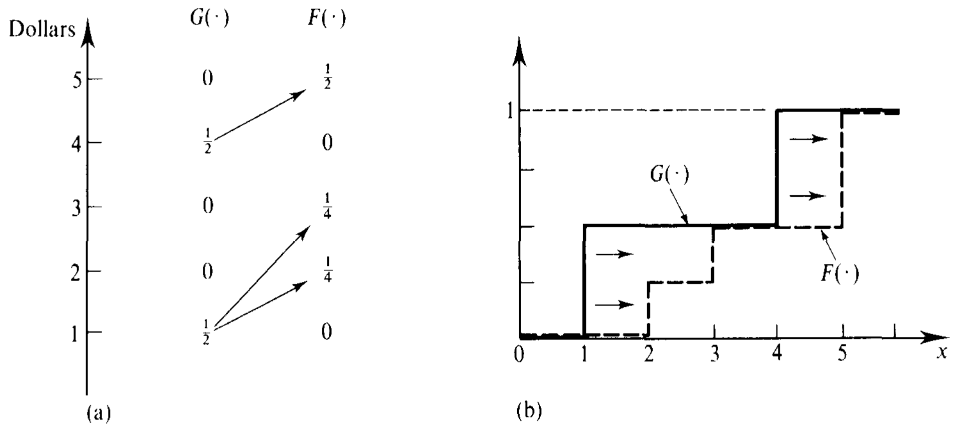


Figure 6.D.2
$F(\cdot)$ first-order stochastically dominates $G(\cdot)$.

shifted up to an even probability between 2 and 3 dollars, and the outcome "4 dollars" is shifted up to 5 dollars with probability one. Figure 6.D.2(b) shows that $F(x) \leq G(x)$ at all x.

It can be shown that the reverse direction also holds. Whenever $F(\cdot)$ first-order stochastically dominates $G(\cdot)$, it is possible to generate $F(\cdot)$ from $G(\cdot)$ in the manner suggested in this example. Thus, this provides yet another approach to the characterization of the first-order stochastic dominance relation. ■

Second-Order Stochastic Dominance

First-order stochastic dominance involves the idea of "higher/better" vs. "lower/worse." We want next to introduce a comparison based on relative *riskiness* or *dispersion*. To avoid confusing this issue with the trade-off between returns and risk, *we will restrict ourselves for the rest of this section to comparing distributions with the same mean.*

Once again, a definition suggests itself: Given two distributions $F(\cdot)$ and $G(\cdot)$ with the same mean [that is, with $\int x\, dF(x) = \int x\, dG(x)$], we say that $G(\cdot)$ is riskier than $F(\cdot)$ if every risk averter prefers $F(\cdot)$ and $G(\cdot)$. This is stated formally in Definition 6.D.2.

Definition 6.D.2: For any two distributions $F(x)$ and $G(\cdot)$ with the same mean, $F(\cdot)$ *second-order stochastically dominates* (or *is less risky than*) $G(\cdot)$ if for every nondecreasing concave function $u: \mathbb{R}_+ \to \mathbb{R}$, we have

$$\int u(x)\, dF(x) \geq \int u(x)\, dG(x).$$

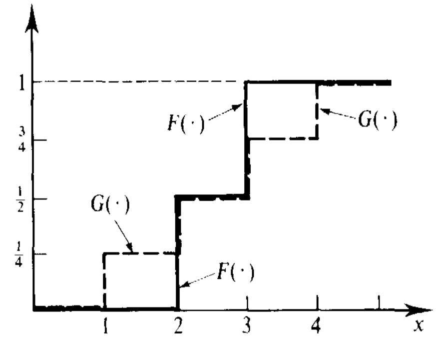
Example 6.D.2 introduces an alternative way to characterize the second-order stochastic dominance relation.

Example 6.D.2: *Mean-Preserving Spreads.* Consider the following compound lottery: In the first stage, we have a lottery over x distributed according to $F(\cdot)$. In the second stage, we randomize each possible outcome x further so that the final payoff is $x + z$, where z has a distribution function $H_x(z)$ with a mean of zero [i.e., $\int z\, dH_x(z) = 0$]. Thus, the mean of $x + z$ is x. Let the resulting reduced lottery be denoted by $G(\cdot)$. When lottery $G(\cdot)$ can be obtained from lottery $F(\cdot)$ in this manner for some distribution $H_x(\cdot)$, we say that $G(\cdot)$ is a *mean-preserving spread* of $F(\cdot)$.

For example, $F(\cdot)$ may be an even probability distribution between 2 and 3 dollars. In the second step we may spread the 2 dollars outcome to an even probability between 1 and 3 dollars, and the 3 dollars outcome to an even probability between 2 and 4 dollars. Then $G(\cdot)$ is the distribution that assigns probability $\frac{1}{4}$ to the four outcomes: 1, 2, 3, 4 dollars. These two distributions $F(\cdot)$ and $G(\cdot)$ are depicted in Figure 6.D.3.

The type of two-stage operation just described keeps the mean of $G(\cdot)$ equal to that of $F(\cdot)$. In addition, if $u(\cdot)$ is concave, we can conclude that

$$\int u(x)\, dG(x) = \int \left(\int u(x + z)\, dH_x(z) \right) dF(x) \leq \int u\left(\int (x + z)\, dH_x(z) \right) dF(x)$$
$$= \int u(x)\, dF(x),$$



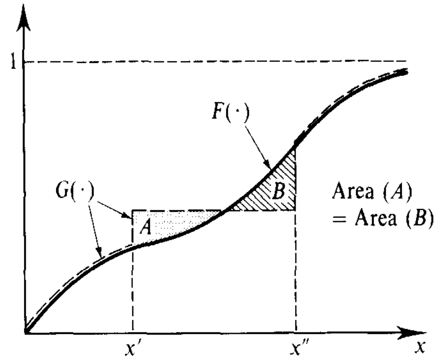


Figure 6.D.3 (left) $G(\cdot)$ is a mean-preserving spread of $F(\cdot)$.

Figure 6.D.4 (right) $G(\cdot)$ is an elementary increase in risk from $F(\cdot)$.

and so $F(\cdot)$ second-order stochastically dominates $G(\cdot)$. It turns out that the converse is also true: If $F(\cdot)$ second-order stochastically dominates $G(\cdot)$, then $G(\cdot)$ is a mean-preserving spread of $F(\cdot)$. Hence, *saying that $G(\cdot)$ is a mean-preserving spread of $F(\cdot)$ is equivalent to saying that $F(\cdot)$ second-order stochastically dominates $G(\cdot)$.* ■

Example 6.D.3 provides another illustration of a mean-preserving spread.

Example 6.D.3: *An Elementary Increase in Risk.* We say that $G(\cdot)$ constitutes an *elementary increase in risk* from $F(\cdot)$ if $G(\cdot)$ is generated from $F(\cdot)$ by taking all the mass that $F(\cdot)$ assigns to an interval $[x', x'']$ and transferring it to the endpoints x' and x'' in such a manner that the mean is preserved. This is illustrated in Figure 6.D.4. An elementary increase in risk is a mean-preserving spread. [In Exercise 6.D.3, you are asked to verify directly that if $G(\cdot)$ is an elementary increase in risk from $F(\cdot)$, then $F(\cdot)$ second-order stochastically dominates $G(\cdot)$.] ■

We can develop still another way to capture the second-order stochastic dominance idea. Suppose that we have two distributions $F(\cdot)$ and $G(\cdot)$ with the same mean. Recall that, for simplicity, we assume that $F(\bar{x}) = G(\bar{x}) = 1$ for some $\bar{x}$. Integrating by parts (and recalling the equality of the means) yields

$$\int_0^{\bar{x}} (F(x) - G(x))\,dx = -\int_0^{\bar{x}} x\,d(F(x) - G(x)) + (F(\bar{x}) - G(\bar{x}))\bar{x} = 0. \qquad (6.D.1)$$

That is, the areas below the two distribution functions are the same over the interval $[0, \bar{x}]$. Because of this fact, the regions marked A and B in Figure 6.D.4 must have the same area. Note that for the two distributions in the figure, this implies that

$$\int_0^x G(t)\,dt \geq \int_0^x F(t)\,dt \quad \text{for all } x. \qquad (6.D.2)$$

It turns out that property (6.D.2) is equivalent to $F(\cdot)$ second-order stochastically dominating $G(\cdot)$.[21] As an application, suppose that $F(\cdot)$ and $G(\cdot)$ have the same mean and that the graph of $G(\cdot)$ is initially above the graph of $F(\cdot)$ and then moves

21. We will not prove this. The claim can be established along the same lines used to prove Proposition 6.D.1 except that we must integrate by parts twice and take into account expression (6.D.1).

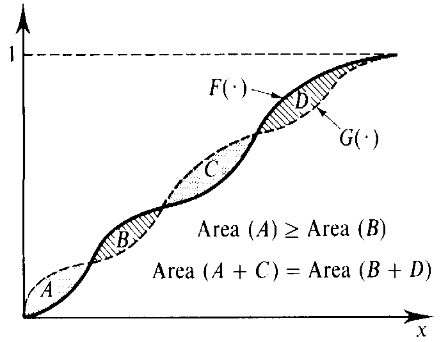


Figure 6.D.5
$F(\cdot)$ second-order stochastically dominates $G(\cdot)$.

permanently below it (as in Figures 6.D.3 and 6.D.4). Then because of (6.D.1), condition (6.D.2) must be satisfied, and we can conclude that $G(\cdot)$ is riskier than $F(\cdot)$. As a more elaborate example, consider Figure 6.D.5, which shows two distributions having the same mean and satisfying (6.D.2). To verify that (6.D.2) is satisfied, note that area A has been drawn to be at least as large as area B and that the equality of the means [i.e., (6.D.1)] implies that the areas $B + D$ and $A + C$ must be equal.

We state Proposition 6.D.2 without proof.

Proposition 6.D.2: Consider two distributions $F(\cdot)$ and $G(\cdot)$ with the same mean. Then the following statements are equivalent:

(i) $F(\cdot)$ second-order stochastically dominates $G(\cdot)$.
(ii) $G(\cdot)$ is a mean-preserving spread of $F(\cdot)$.
(iii) Property (6.D.2) holds.

In Exercise 6.D.4, you are asked to verify the equivalence of these three properties in the probability simplex diagram.

6.E State-dependent Utility

In this section, we consider an extension of the analysis presented in the preceding two sections. In Sections 6.C and 6.D, we assumed that the decision maker cares solely about the distribution of monetary payoffs he receives. This says, in essence, that the underlying cause of the payoff is of no importance. If the cause is one's state of health, however, this assumption is unlikely to be fulfilled.[22] The distribution function of monetary payoffs is then not the appropriate object of individual choice. Here we consider the possibility that the decision maker may care not only about his monetary returns but also about the underlying events, or *states of nature*, that cause them.

We begin by discussing a convenient framework for modeling uncertain alternatives that, in contrast to the lottery apparatus, recognizes underlying states of nature. (We will encounter it repeatedly throughout the book, especially in Chapter 19.)

22. On the other hand, if it is an event such as the price of some security in a portfolio, the assumption is more likely to be a good representation of reality.

State-of-Nature Representations of Uncertainty

In Sections 6.C and 6.D, we modeled a risky alternative by means of a distribution function over monetary outcomes. Often, however, we know that the random outcome is generated by some underlying causes. A more detailed description of uncertain alternatives is then possible. For example, the monetary payoff of an insurance policy might depend on whether or not a certain accident has happened, the payoff on a corporate stock on whether the economy is in a recession, and the payoff of a casino gamble on the number selected by the roulette wheel.

We call these underlying causes *states*, or *states of nature.* We denote the set of states by S and an individual state by $s \in S$. For simplicity, we assume here that the set of states is finite and that each state s has a well-defined, objective probability $\pi_s > 0$ that it occurs. We abuse notation slightly by also denoting the total number of states by S.

An uncertain alternative with (nonnegative) monetary returns can then be described as a function that maps realizations of the underlying state of nature into the set of possible money payoffs $\mathbb{R}_+$. Formally, such a function is known as a *random variable.*

Definition 6.E.1: A *random variable* is a function $g\colon S \to \mathbb{R}_+$ that maps states into monetary outcomes.[23]

Every random variable $g(\cdot)$ gives rise to a money lottery describable by the distribution function $F(\cdot)$ with $F(x) = \sum_{\{s: g(s) \leq x\}} \pi_s$ for all x. Note that there is a loss in information in going from the random variable representation of uncertainty to the lottery representation; we do not keep track of which states give rise to a given monetary outcome, and only the aggregate probability of every monetary outcome is retained.

Because we take S to be finite, we can represent a random variable with monetary payoffs by the vector $(x_1, \ldots, x_S)$, where x_s is the nonnegative monetary payoff in state s. The set of all nonnegative random variables is then $\mathbb{R}^S_+$.

State-Dependent Preferences and the Extended Expected Utility Representation

The primitive datum of our theory is now a rational preference relation $\succsim$ on the set $\mathbb{R}^S_+$ of nonnegative random variables. Note that this formal setting is parallel to the one developed in Chapters 2 to 4 for consumer choice. The similarity is not merely superficial. If we define commodity s as the random variable that pays one dollar if and only if state s occurs (this is called a *contingent commodity* in Chapter 19), then the set of nonnegative random variables $\mathbb{R}^S_+$ is precisely the set of nonnegative bundles of these S contingent commodities.

As we shall see, it is very convenient if, in the spirit of the previous sections of this chapter, we can represent the individual's preferences over monetary outcomes by a utility function that possesses an *extended expected utility form.*

23. For concreteness, we restrict the outcomes to be nonnegative amounts of money. As we did in Section 6.B, we could equally well use an abstract outcome set C instead of $\mathbb{R}_+$.

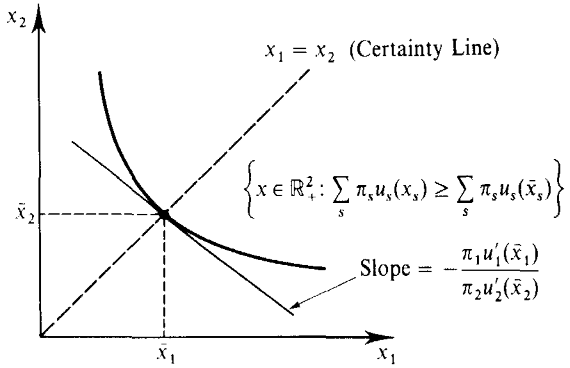


Figure 6.E.1
State-dependent preferences.

Definition 6.E.2: The preference relation $\succsim$ has an *extended expected utility representation* if for every $s \in S$, there is a function $u_s\colon \mathbb{R}_+ \to \mathbb{R}$ such that for any $(x_1, \ldots, x_S) \in \mathbb{R}^S_+$ and $(x'_1, \ldots, x'_S) \in \mathbb{R}^S_+$,

$$(x_1, \ldots, x_S) \succsim (x'_1, \ldots, x'_S) \quad \text{if and only if} \quad \sum_s \pi_s u_s(x_s) \geq \sum_s \pi_s u_s(x'_s).$$

To understand Definition 6.E.2, recall the analysis in Section 6.B. If only the distribution of money payoffs mattered, and if preferences on money distributions satisfied the expected utility axioms, then the expected utility theorem leads to a *state-independent* (we will also say *state-uniform*) expected utility representation $\sum_s \pi_s u(x_s)$, where $u(\cdot)$ is the Bernoulli utility function on amounts of money.[24] The generalization in Definition 6.E.2 allows for a different function $u_s(\cdot)$ in every state.

Before discussing the conditions under which an extended utility representation exists, we comment on its usefulness as a tool in the analysis of choice under uncertainty. This usefulness is primarily a result of the behavior of the indifference sets around the *money certainty line*, the set of random variables that pay the same amount in every state. Figure 6.E.1 depicts state-dependent preferences in the space $\mathbb{R}^S_+$ for a case where $S = 2$ and the $u_s(\cdot)$ functions are concave (as we shall see later, concavity of these functions follows from risk aversion considerations). The certainty line in Figure 6.E.1 is the set of points with $x_1 = x_2$. The marginal rate of substitution at a point $(\bar{x}, \bar{x})$ is $\pi_1 u_1'(\bar{x})/\pi_2 u_2'(\bar{x})$. Thus, the slope of the indifference curves on the certainty line reflects the nature of state dependence as well as the probabilities of the different states. In contrast, with state-uniform (i.e., identical across states) utility functions, the marginal rate of substitution at any point on the certainty line equals the ratio of the probabilities of the states (implying that this slope is the same at all points on the certainty line).

Example 6.E.1: *Insurance with State-dependent Utility.* One interesting implication of state dependency arises when actuarially fair insurance is available. Suppose there are two states: State 1 is the state where no loss occurs, and state 2 is the state where a loss occurs. (This economic situation parallels that in Example 6.C.1.) The individual's initial situation (i.e., in the absence of any insurance purchase) is a

24. Note that the random variable $(x_1, \ldots, x_S)$ induces a money lottery that pays x_s with probability π_s. Hence, $\sum_s \pi_s u(x_s)$ is its expected utility.

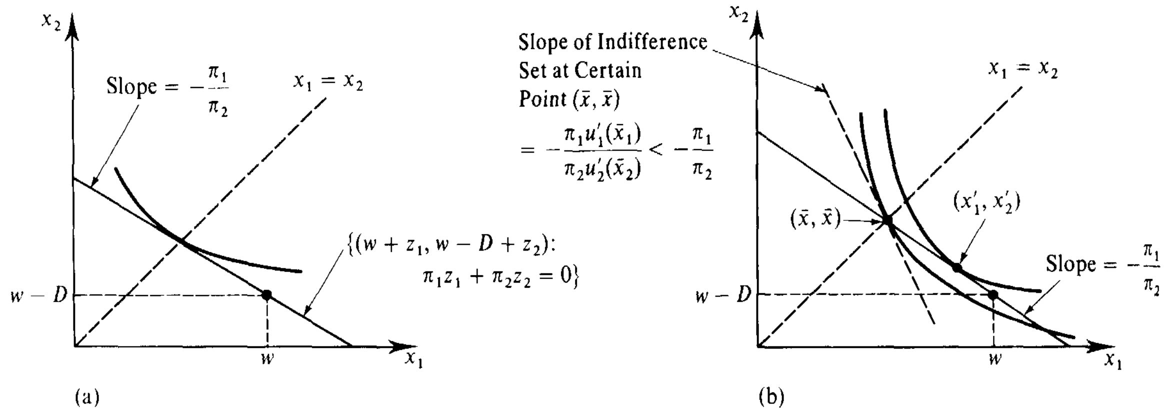


Figure 6.E.2 Insurance purchase with state-dependent utility. (a) State-uniform utility. (b) State-dependent utility.

random variable $(w, w - D)$ that gives the individual's wealth in the two states. This is depicted in Figure 6.E.2(a). We can represent an insurance contract by a random variable $(z_1, z_2) \in \mathbb{R}^2$ specifying the net change in wealth in the two states (the insurance payoff in the state less any premiums paid). Thus, if the individual purchases insurance contract (z_1, z_2), his final wealth position will be $(w + z_1, w - D + z_2)$. The insurance policy (z_1, z_2) is actuarially fair if its expected payoff is zero, that is, if $\pi_1 z_1 + \pi_2 z_2 = 0$.

Figure 6.E.2(a) shows the optimal insurance purchase when a risk-averse expected utility maximizer with state-uniform preferences can purchase any actuarially fair insurance policy he desires. His budget set is the straight line drawn in the figure. We saw in Example 6.C.2 that under these conditions, a decision maker with state-uniform utility would insure completely. This is confirmed here because if there is no state dependency, the budget line is tangent to an indifference curve at the certainty line.

Figure 6.E.2(b) depicts the situation with state-dependent preferences. The decision maker will now prefer a point such as (x_1', x_2') to the certain outcome $(\bar{x}, \bar{x})$. This creates a desire to have a higher payoff in state 1, where $u_1'(\cdot)$ is relatively higher, in exchange for a lower payoff in state 2. ■

Existence of an Extended Expected Utility Representation

We now investigate conditions for the existence of an extended expected utility representation.

Observe first that since $\pi_s > 0$ for every s, we can formally include π_s in the definition of the utility function at state s. That is, to find an extended expected utility representation, it suffices that there be functions $u_s(\cdot)$ such that

$$(x_1, \ldots, x_S) \succsim (x_1', \ldots, x_S') \quad \text{if and only if} \quad \sum_s u_s(x_s) \geq \sum_s u_s(x_s').$$

This is because if such functions $u_s(\cdot)$ exist, then we can define $\tilde{u}_s(\cdot) = (1/\pi_s)u_s(\cdot)$ for each $s \in S$, and we will have $\sum_s u_s(x_s) \geq \sum_s u_s(x_s')$ if and only if $\sum_s \pi_s \tilde{u}_s(x_s) \geq \sum_s \pi_s \tilde{u}_s(x_s')$. Thus, from now on, we focus on the existence of an additively separable form $\sum_s u_s(\cdot)$, and the π_s's cease to play any role in the analysis.

It turns out that the extended expected utility representation can be derived *in exactly the same way* as the expected utility representation of Section 6.B if we appropriately enlarge the domain over which preferences are defined.[25] Accordingly, we now allow for the possibility that within each state s, the monetary payoff is not a certain amount of money x_s but a random amount with distribution function $F_s(\cdot)$. We denote these uncertain alternatives by $L = (F_1, \ldots, F_S)$. Thus, L is a kind of compound lottery that assigns well-defined monetary gambles as prizes contingent on the realization of the state of the world s. We denote by $\mathscr{L}$ the set of all such possible lotteries.

Our starting point is now a rational preference relation $\succsim$ on $\mathscr{L}$. Note that $\alpha L + (1-\alpha)L' = (\alpha F_1 + (1-\alpha)F'_1, \ldots, \alpha F_S + (1-\alpha)F'_S)$ has the usual interpretation as the reduced lottery arising from a randomization between L and L', although here we are dealing with a reduced lottery within each state s. Hence, we can appeal to the same logic as in Section 6.B and impose an independence axiom on preferences.

Definition 6.E.3: The preference relation $\succsim$ on $\mathscr{L}$ satisfies the *extended independence axiom* if for all $L, L', L'' \in \mathscr{L}$ and $\alpha \in (0, 1)$, we have

$$L \succsim L' \quad \text{if and only if} \quad \alpha L + (1-\alpha)L'' \succsim \alpha L' + (1-\alpha)L''.$$

We also make a continuity assumption: Except for the reinterpretation of $\mathscr{L}$, this *continuity axiom* is exactly the same as that in Section 6.B; we refer to Definition 6.B.3 for its statement.

Proposition 6.E.1: (*Extended Expected Utility Theorem*) Suppose that the preference relation $\succsim$ on the space of lotteries $\mathscr{L}$ satisfies the continuity and extended independence axioms. Then we can assign a utility function $u_s(\cdot)$ for money in every state s such that for any $L = (F_1, \ldots, F_S)$ and $L' = (F'_1, \ldots, F'_S)$, we have

$$L \succsim L' \quad \text{if and only if} \quad \sum_s \left(\int u_s(x_s)\, dF_s(x_s) \right) \geq \sum_s \left(\int u_s(x_s)\, dF'_s(x_s) \right).$$

Proof: The proof is identical, almost word for word, to the proof of the expected utility theorem (Proposition 6.B.2).

Suppose, for simplicity, that we restrict ourselves to a finite number $\{x_1, \ldots, x_N\}$ of monetary outcomes. Then we can identify the set $\mathscr{L}$ with Δ^S, where Δ is the $(N-1)$-dimensional simplex. Our aim is to show that $\succsim$ can be represented by a linear utility function $U(L)$ on Δ^S. To see this, note that, up to an additive constant that can be neglected, $U(p_1^1, \ldots, p_N^1, \ldots, p_1^S, \ldots, p_N^S)$ is a linear function of its arguments if it can be written as $U(L) = \sum_{n,s} u_{n,s} p_n^s$ for some values $u_{n,s}$. In this case, we can write $U(L) = \sum_s (\sum_n u_{n,s} p_n^s)$, which, letting $u_s(x_n) = u_{n,s}$, is precisely the form of a utility function on $\mathscr{L}$ that we want.

Choose $\bar{L}$ and $\underline{L}$ such that $\bar{L} \succsim L \succsim \underline{L}$ for all $L \in \mathscr{L}$. As in the proof of Proposition 6.B.2, we can then define $U(L)$ by the condition

$$L \sim U(L)\bar{L} + (1 - U(L))\underline{L}.$$

Applying the extended independence axiom in exactly the same way as we applied the independence axiom in the proof of Proposition 6.B.2 yields the result that $U(L)$ is indeed a linear utility function on $\mathscr{L}$. ■

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25. By pushing the enlargement further than we do here, it would even be possible to view the existence of an extended utility representation as a corollary of the expected utility theorem.