4.3.1 The Cholesky Factorization

We would like to generate N joint values of ϵ that display the correlation structure $V(\epsilon) = E(\epsilon\epsilon') = R$. Because the matrix R is a symmetric real matrix, it can be decomposed into its so-called Cholesky factors:

$$R = TT' \tag{4.16}$$

where T is a lower triangular matrix with zeros on the upper right corners (above the diagonal). This is known as the **Cholesky factorization.**

As in the previous section, we first generate a vector of independent η. Thus, their covariance matrix is $V(\eta) = I$, where I is the identity matrix with zeros everywhere except for the diagonal.

We then construct the transformed variable $\epsilon = T\eta$. The covariance matrix is now $V(\epsilon) = E(\epsilon\epsilon') = E((T\eta)(T\eta)') = E(T\eta\eta'T') = TE(\eta\eta')T' = TV(\eta)T' = TIT' = TT' = R$. This transformation therefore generates ϵ variables with the desired correlations.

To illustrate, let us go back to our two-variable case. The correlation matrix can be decomposed into its Cholesky factors:

$$\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} \\ 0 & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11}^2 & a_{11}a_{21} \\ a_{21}a_{11} & a_{21}^2 + a_{22}^2 \end{bmatrix}$$

To find the entries a_{11}, a_{21}, a_{22}, we solve each of the three equations

$$a_{11}^2 = 1$$
$$a_{11}a_{21} = \rho$$
$$a_{21}^2 + a_{22}^2 = 1$$

This gives $a_{11} = 1, a_{21} = \rho$, and $a_{22} = (1 - \rho^2)^{1/2}$. The Cholesky factorization is then

$$\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \rho & (1-\rho^2)^{1/2} \end{bmatrix} \begin{bmatrix} 1 & \rho \\ 0 & (1-\rho^2)^{1/2} \end{bmatrix}$$

Note that this conforms to Equation (4.15):

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \rho & (1-\rho^2)^{1/2} \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$$

In practice, this decomposition yields a number of useful insights. The decomposition will fail if the number of independent factors implied in the correlation matrix is less than N. For instance, if $\rho = 1$, the two assets are perfectly correlated, meaning that we have twice the same factor. This could be for instance the case of two currencies fixed to each other. The decomposition gives $a_{11} = 1, a_{21} = 1, a_{22} = 0$. The new variables are then $\epsilon_1 = \eta_1$ and $\epsilon_2 = \eta_1$. In this case, the second variable η_2 is totally superfluous and simulations can be performed with one variable only.

4.3.2 The Curse of Dimensionality

Modern risk management is about measuring the risk of large portfolios, typically exposed to a large number of risk factors. The problem is that the number of computations increases geometrically with the number of factors N. The covariance matrix, for instance, has dimensions $N(N+1)/2$. A portfolio with 500 variables requires a matrix with 125,250 entries.

In practice, the risk manager should simplify the number of risk factors, discarding those that do not contribute significantly to the risk of the portfolio. Simulations based on the full set of variables would be inordinately time-consuming. The art of simulation is to design parsimonious experiments that represent the breadth of movements in risk factors.

This can be done by an economic analysis of the risk factors and portfolio strategies, as done in Part Three of this handbook. Alternatively, the risk manager can perform a statistical decomposition of the covariance matrix. A widely-used method for this is the **principal-component analysis** (PCA), which finds linear combinations of the risk factors that have maximal explanatory power. This type of analysis, which is as much an art as it is a science, can be used to reduce the dimensionality of the risk space.

EXAMPLE 4.10: FRM EXAM 2007—QUESTION 28

Let N be an $n \times 1$ vector of independent draws from a standard normal distribution, and let V be a covariance matrix of market time-series data. Then, if L is a diagonal matrix of the eigenvalues of V, E is a matrix of the eigenvectors of V, and $C'C$ is the Cholesky factorization of V, which of the following would generate a normally distributed random vector with mean zero and covariance matrix V to be used in a Monte Carlo simulation?

a. $NC'CN'$
b. NC'
c. $E'LE$
d. Cannot be determined from data given

EXAMPLE 4.11: FRM EXAM 2006—QUESTION 82

Consider a stock that pays no dividends, has a volatility of 25% pa and an expected return of 13% pa. The current stock price is $S_0 = \$30$. This implies the model $S_{t+1} = S_t(1 + 0.13\Delta t + 0.25\sqrt{\Delta t}\,\epsilon)$, where ϵ is a standard normal random variable. To implement this simulation, you generate a path of the stock price by starting at $t = 0$, generating a sample for ϵ, updating the stock price according to the model, incrementing t by 1 and repeating this process until the end of the horizon is reached. Which of the following strategies for generating a sample for ϵ will implement this simulation properly?

- **a.** Generate a sample for ϵ by using the inverse of the standard normal cumulative distribution of a sample value drawn from a uniform distribution between 0 and 1.
- **b.** Generate a sample for ϵ by sampling from a normal distribution with mean 0.13 and standard deviation 0.25.
- **c.** Generate a sample for ϵ by using the inverse of the standard normal cumulative distribution of a sample value drawn from a uniform distribution between 0 and 1. Use Cholesky decomposition to correlate this sample with the sample from the previous time interval.
- **d.** Generate a sample for ϵ by sampling from a normal distribution with mean 0.13 and standard deviation 0.25. Use Cholesky decomposition to correlate this sample with the sample from the previous time interval.

EXAMPLE 4.12: FRM EXAM 2006—QUESTION 83

Continuing with the previous question, you have implemented the simulation process discussed above using a time interval $\Delta t = 0.001$, and you are analyzing the following stock price path generated by your implementation.

t	S_{t-1}	ϵ	ΔS
0	30.00	0.0930	0.03
1	30.03	0.8493	0.21
2	30.23	0.9617	0.23
3	30.47	0.2460	0.06
4	30.53	0.4769	0.12
5	30.65	0.7141	0.18

Given this sample, which of the following simulation steps most likely contains an error.

- **a.** Calculation to update the stock price
- **b.** Generation of random sample value for ϵ
- **c.** Calculation of the change in stock price during each period
- **d.** None of the above

4.4 IMPORTANT FORMULAS

The Wiener process: $\Delta z \sim N(0, \Delta t)$

The generalized Wiener process: $\Delta x = a\Delta t + b\Delta z$

The Ito process: $\Delta x = a(x, t)\Delta t + b(x, t)\Delta z$

The geometric Brownian motion: $\Delta S = \mu S \Delta t + \sigma S \Delta z$

One-factor equilibrium model for yields: $\Delta r_t = \kappa(\theta - r_t)\Delta t + \sigma r_t^{\gamma} \Delta z_t$

Vasicek model, $\gamma = 0$
Lognormal model, $\gamma = 1$
CIR model, $\gamma = 0.5$

No-arbitrage models:

Ho and Lee model, $\Delta r_t = \theta(t)\Delta t + \sigma \Delta z_t$
Hull and White model, $\Delta r_t = [\theta(t) - ar_t]\Delta t + \sigma \Delta z_t$
Heath, Jarrow, and Morton model, $\Delta r_t = [\theta(t) - ar_t]\Delta t + \sigma(t)\Delta z_t$

Binomial trees: $u = e^{\sigma\sqrt{\Delta t}}, \quad d = (1/u), \ p = \frac{e^{\mu\Delta t} - d}{u - d}$

Cholesky factorization: $R = TT', \epsilon = T\eta$

4.5 ANSWERS TO CHAPTER EXAMPLES

Example 4.1: FRM Exam 2003—Question 40

b. Both dS/S or $d\ln(S)$ are normally distributed. As a result, S is lognormally distributed. The only incorrect answer is I.

Example 4.2: FRM Exam 2002—Question 126

a. All the statements are correct except a., which is too strong. The expected price is higher than today's price but certainly not the price in all states of the world.

Example 4.3: FRM Exam 1999—Question 25

b. This model postulates only one source of risk in the fixed-income market. This is a single-factor term-structure model.

Example 4.4: FRM Exam 1999—Question 26

c. This represents the expected return with mean reversion.

Example 4.5: FRM Exam 2000—Question 118

a. These are no-arbitrage models of the term structure, implemented as either one-factor or two-factor models.

Example 4.6: FRM Exam 2000—Question 119

b. Both the Vasicek and CIR models are one-factor equilibrium models with mean reversion. The Hull-White model is a no-arbitrage model with mean reversion. The Ho and Lee model is an early no-arbitrage model without mean-reversion.

Example 4.7: FRM Exam 2005—Question 67

b. MC simulations do account for options. The first step is to simulate the process of the risk factor. The second step prices the option, which properly accounts for non-linearity.

Example 4.8: FRM Exam 2007—Question 66

d. Short option positions have long left tails, which makes it more difficult to estimate a left-tailed quantile precisely. Accuracy with independent draws increases with the square root of K. Thus increasing the number of replications should shrink the standard error, so answer b. is incorrect.

Example 4.9: Sampling Variation

b. Sampling variability (or imprecision) increases with (1) fewer observations and (2) greater confidence levels. To show (1), we can refer to the formula for the precision of the sample mean, which varies inversely with the square root of the number of data points. A similar reasoning applies to (2). A greater confidence level involves fewer observations in the left tails, from which VAR is computed.

Example 4.10: FRM Exam 2007—Question 28

b. In the notation of the text, N is the vector of i.i.d. random variables η and $C'C = TT'$. The transformed variable is $T\eta$, or $C'N$, or its transpose.

Example 4.11: FRM Exam 2006—Question 82

a. The variable ϵ should have a standard normal distribution, i.e., with mean zero and unit standard deviation. Answer b. is incorrect because ϵ is transformed afterward to the desired mean and standard deviation. The Cholesky decomposition is not applied here because the sequence of random variables has no serial correlation.

Example 4.12: FRM Exam 2006—Question 83

b. The random variable ϵ should have a standard normal distribution, which means that it should have negative as well as positive values, which should average close to zero. This is not the case here. This is probably a uniform variable instead.

PART Two

Capital Markets

CHAPTER 5

Introduction to Derivatives

This chapter provides an overview of derivative instruments. Derivatives are financial contracts traded in private **over-the-counter** (OTC) markets, or on **organized exchanges**. As the term implies, derivatives derive their value from some underlying index, typically the price of an asset. Depending on the type of relationship, they can be broadly classified into two categories: linear and nonlinear instruments.

To the first category belong forward contracts, futures, and swaps. Their value is a linear function of the underlying index. These are *obligations* to exchange payments according to a specified schedule. Forward contracts are relatively simple to evaluate and price. So are futures, which are traded on exchanges. Swaps are more complex but generally can be reduced to portfolios of forward contracts. To the second category belong options, which are traded both OTC and on exchanges. Their value is a nonlinear function of the underlying index. These will be covered in the next chapter.

This chapter describes the general characteristics as well as the pricing of linear derivatives. Pricing is the first step toward risk measurement. The second step consists of combining the valuation formula with the distribution of underlying risk factors to derive the distribution of contract values. This will be done later, in the market risk section.

Section 5.1 provides an overview of the size of the derivatives markets. Section 5.2 then presents the valuation and pricing of forwards. Sections 5.3 and 5.4 introduce futures and swap contracts, respectively.

5.1 OVERVIEW OF DERIVATIVES MARKETS

A **derivative instrument** can be generally defined as a private contract whose value derives from some underlying asset price, reference rate or index—such as a stock, bond, currency, or a commodity. In addition, the contract must also specify a principal, or **notional** amount, which is defined in terms of currency, shares, bushels, or some other unit. Movements in the value of the derivative depend on the notional and the underlying price or index.

In contrast with **securities**, such as stocks and bonds, which are issued to raise capital, derivatives are **contracts**, or private agreements between two parties. Thus the sum of gains and losses on derivatives contracts must be zero. For any gain made by one party, the other party must have suffered a loss of equal magnitude.

At the broadest level, derivatives markets can be classified by the underlying instrument, as well as by the type of trading. Table 5.1 describes the size and growth of the global derivatives markets. As of 2007, the total notional amounts add up to $677 trillion, of which $596 trillion is on OTC markets and $81 trillion on organized exchanges. These markets have grown exponentially, from $56 trillion in 1995.

The table shows that interest rate contracts are the most widespread type of derivatives, especially swaps. On the OTC market, currency contracts are also widely used, especially outright forwards and **forex swaps**, which are a combination of spot and short-term forward transactions. Among exchange-traded instruments, interest rate futures and options are the most common.

The magnitude of the notional amount of $677 trillion is difficult to grasp. This number is several times the world **gross domestic product (GDP)**, which amounted to approximately $48 trillion in 2006. It is also greater than the total

TABLE 5.1 Global Derivatives Markets, 1995–2007 (Billions of U.S. Dollars)

	Notional Amounts	
	March 1995	Dec. 2007
OTC Instruments	47,530	596,004
Interest rate contracts	26,645	393,138
Forwards (FRAs)	4,597	26,599
Swaps	18,283	309,588
Options	3,548	56,951
Foreign exchange contracts	13,095	56,238
Forwards and forex swaps	8,699	29,144
Swaps	1,957	14,347
Options	2,379	12,748
Equity-linked contracts	579	8,509
Forwards and swaps	52	2,233
Options	527	6,276
Commodity contracts	318	9,000
Credit default swaps	0	57,894
Others	6,893	71,225
Exchange-Traded Instruments	8,838	80,578
Interest rate contracts	8,380	71,052
Futures	5,757	26,770
Options	2,623	44,282
Foreign exchange contracts	88	292
Futures	33	159
Options	55	133
Stock-index contracts	370	9,234
Futures	128	1,132
Options	242	8,102
Total	55,910	676,582

Source: Bank for International Settlements outstanding value of stocks, which was \$54 trillion and of debt securities, which was \$69 trillion at that time.

Notional amounts give an indication of equivalent positions in cash markets. For example, a long futures contract on a stock index with a notional of \$1 million is equivalent to a cash position in the stock market of the same magnitude.

Notional amounts, however, do not give much information about the risks of the positions. The current (positive) market value of OTC derivatives contracts, for instance, is estimated at \$15 trillion. This is less than 3% of the notional. More generally, the risk of these derivatives is best measured by the potential change in mark-to-market values over the horizon. In other words, by a value-at-risk measure.

5.2 FORWARD CONTRACTS

5.2.1 Definition

The most common transactions in financial instruments are **spot transactions**, that is, for physical delivery as soon as practical (perhaps in two business days or in a week). Historically, grain farmers went to a centralized location to meet buyers for their product. As markets developed, the farmers realized that it would be beneficial to trade for delivery at some future date. This allowed them to hedge out price fluctuations for the sale of their anticipated production.

This gave rise to **forward contracts**, which are private agreements to exchange a given asset against cash (or sometimes another asset) at a fixed point in the future. The terms of the contract are the quantity (number of units or shares), date, and price at which the exchange will be done.

A position which implies buying the asset is said to be **long**. A position to sell is said to be **short**. Any gain to one party must be a loss to the other.

These instruments represent contractual obligations, as the exchange must occur whatever happens to the intervening price, unless default occurs. Unlike an option contract, there is no choice in taking delivery or not.

To avoid the possibility of losses, the farmer could enter a forward sale of grain for dollars. By so doing, he locks up a price now for delivery in the future. We then say that the farmer is **hedged** against movements in the price.

We use the notations

$$
\begin{aligned}
t &= \text{current time} \\
T &= \text{time of delivery} \\
\tau = T - t &= \text{time to maturity} \\
S_t &= \text{current spot price of the asset in dollars} \\
F_t(T) &= \text{current forward price of the asset for delivery at } T \\
&\quad\ \text{(also written as } F_t \text{ or } F \text{ to avoid clutter)} \\
V_t &= \text{current value of contract} \\
r &= \text{current domestic risk-free rate for delivery at } T \\
n &= \text{quantity, or number of units in contract}
\end{aligned}
$$

The **face amount**, or **principal value** of the contract is defined as the amount nF to pay at maturity, like a bond. This is also called the **notional amount.** We will assume that interest rates are continuously compounded so that the present value of a dollar paid at expiration is $\text{PV}(\$1) = e^{-r\tau}$.

Say that the initial forward price is $F_t = \$100$. A speculator agrees to buy $n = 500$ units for F_t at T. At expiration, the payoff on the forward contract is determined as follows:

1. The speculator pays $nF = \$50{,}000$ in cash and receives 500 units of the underlying.
2. The speculator could then sell the underlying at the prevailing spot price S_T, for a profit of $n(S_T - F)$. For example, if the spot price is at $S_T = \$120$, the profit is $500 \times (\$120 - \$100) = \$10{,}000$. This is also the mark-to-market value of the contract at expiration.

In summary, the value of the forward contract at expiration, for one unit of the underlying asset is

$$V_T = S_T - F \tag{5.1}$$

Here, the value of the contract at expiration is derived from the purchase and **physical delivery** of the underlying asset. There is a payment of cash in exchange for the actual asset.

Another mode of settlement is **cash settlement**. This involves simply measuring the market value of the asset upon maturity, S_T, and agreeing for the "long" to receive $nV_T = n(S_T - F)$. This amount can be positive or negative, involving a profit or loss.

Figures 5.1 and 5.2 present the payoff patterns on long and short positions in a forward contract, respectively. It is important to note that the payoffs are *linear*

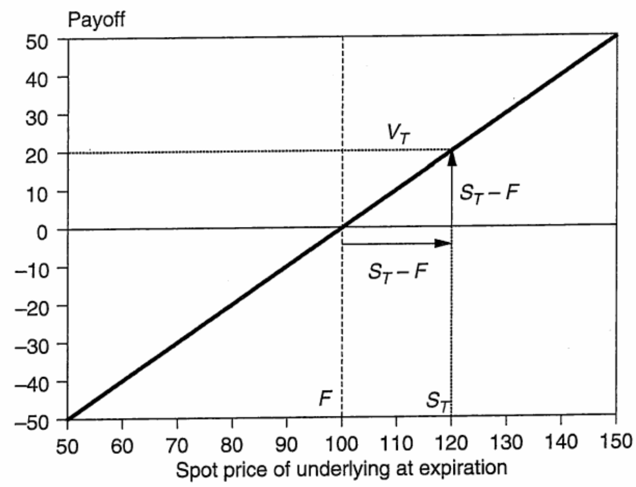


FIGURE 5.1 Payoff of Profits on Long Forward Contract

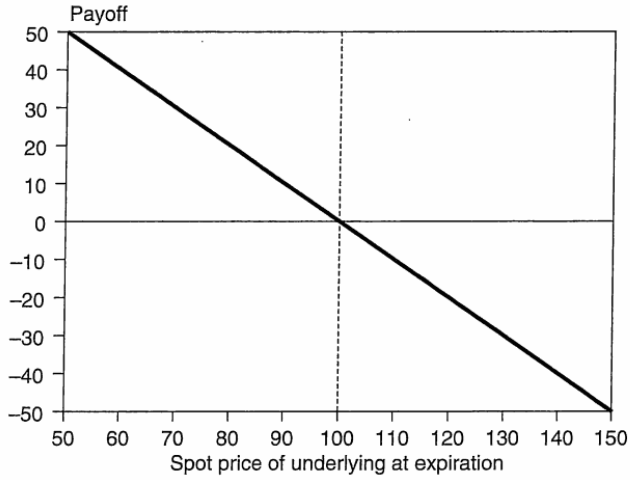


FIGURE 5.2 Payoff of Profits on Short Forward Contract

in the underlying spot price. Also, the positions in the two figures are symmetrical around the horizontal axis. For a given spot price, the sum of the profit or loss for the long and the short is zero, because these are private contracts.

5.2.2 Valuing Forward Contracts

When evaluating forward contracts, two important questions arise. First, how is the current forward price F_t determined? Second, what is the current value V_t of an outstanding forward contract?

Initially, we assume that the underlying asset pays no income. This will be generalized in the next section. We also assume no transaction costs, that is, zero bid–ask spread on spot and forward quotations as well as the ability to lend and borrow at the same risk-free rate.

Generally, forward contracts are established so that their initial value is zero. This is achieved by setting the forward price F_t appropriately by a **no-arbitrage relationship** between the cash and forward markets. No-arbitrage is a situation where positions with the same payoffs have the same price. This rules out situations where **arbitrage profits** can exist. Arbitrage is a zero-risk, zero-net investment strategy that still generates profits.

Consider these strategies:

- Buy one share/unit of the underlying asset at the spot price S_t and hold to time T.
- Enter a forward contract to buy one share/unit of same underlying asset at the forward price F_t. In order to have sufficient funds at maturity to pay F_t, we invest the present value of F_t in an interest-bearing account. This is the present value $F_t e^{-r\tau}$. The forward price F_t is set so that the initial cost of the forward contract, V_t, is zero.

The two portfolios are economically equivalent because they will be identical at maturity. Each will contain one share of the asset. Hence their up-front cost must be the same. To avoid arbitrage, we must have:

$$S_t = F_t e^{-r\tau} \tag{5.2}$$

This equation defines the fair forward price F_t such that the initial value of the contract is zero. More generally, the term multiplying F_t is the present value factor for maturity τ, or PV(\$1). For instance, assuming $S_t = \$100$, $r = 5\%$, $\tau = 1$, we have $F_t = S_t e^{r\tau} = \$100 \times \exp(0.05 \times 1) = \$105.13$.

We see that the forward rate is higher than the spot rate. This reflects the fact that there is no down payment to enter the forward contract, unlike for the cash position. As a result, the forward price must be higher than the spot price to reflect the time value of money.

Abstracting from transaction costs, any deviation creates an arbitrage opportunity. This can be taken advantage of by buying the cheap asset and selling the expensive one. Assume for instance that $F = \$110$. We determined that the fair value is $S_t e^{r\tau} = \$105.13$, based on the cash price. We apply the principle of buying low at \$105.13 and selling high at \$110. We can lock in a sure profit by:

1. Buying now the asset spot at \$100
2. Selling now the asset forward at \$110

This can be done by borrowing the \$100 to buy the asset now. At expiration, we will owe principal plus interest, or \$105.13 but receive \$110, for a profit of \$4.87. This would be a blatant arbitrage opportunity, or "money machine."

Now consider a mispricing where $F = \$102$. We apply the principle of buying low at \$102 and selling high at \$105.13. We can lock in a sure profit by:

1. Short-selling now the asset spot at \$100
2. Buying now the asset forward at \$102

From the short sale, we invest the cash, which will grow to \$105.13. At expiration, we will have to deliver the stock but this will be acquired through the forward purchase. We pay \$102 for this and are left with a profit of \$3.13.

This transaction involves the **short-sale** of the asset, which is more involved than an outright purchase. When purchasing, we pay \$100 and receive one share of the asset. When short-selling, we borrow one share of the asset and promise to give it back at a future date; in the meantime, we sell it at \$100.[1]

[1] In practice, we may not get full access to the proceeds of the sale when it involves individual stocks. The broker will typically only allow us to withdraw 50% of the cash. The rest is kept as a performance bond should the transaction lose money.

5.2.3 Valuing an Off-Market Forward Contract

We can use the same reasoning to evaluate an outstanding forward contract, with a locked-in delivery price of K. In general, such a contract will have non zero value because K differs from the prevailing forward rate. Such a contract is said to be **off-market**.

Consider these strategies:

- Buy one share/unit of the underlying asset at the spot price S_t and hold it until time T.
- Enter a forward contract to buy one share/unit of the same underlying asset at the price K; in order to have sufficient funds at maturity to pay K, we invest the present value of K in an interest-bearing account. This present value is also $Ke^{-r\tau}$. In addition, we have to pay the market value of the forward contract, or V_t.

The up-front cost of the two portfolios must be identical. Hence, we must have $V_t + Ke^{-r\tau} = S_t$, or

$$V_t = S_t - Ke^{-r\tau} \tag{5.3}$$

which defines the market value of an outstanding long position.[2] This gains value when the underlying S increases in value. A short position would have the reverse sign. Later, we will extend this relationship to the measurement of risk by considering the distribution of the underlying risk factors, S_t and r.

For instance, assume we still hold the previous forward contract with $F_t = \$105.13$ and after one month the spot price moves to $S_t = \$110$. The fixed rate is $K = \$105.13$ throughout the life of the contract. The interest has not changed at $r = 5\%$, but the maturity is now shorter by one month, $\tau = 11/12$. The new value of the contract is $V_t = S_t - Ke^{-r\tau} = \$110 - \$105.13\exp(-0.05 \times 11/12) = \$110 - \$100.42 = \$9.58$. The contract is now more valuable than before because the spot price has moved up.

5.2.4 Valuing Forward Contracts with Income Payments

We previously considered a situation where the asset produces no income payment. In practice, the asset may be

- A stock that pays a regular dividend
- A bond that pays a regular coupon
- A stock index that pays a dividend stream approximated by a continuous yield
- A foreign currency that pays a foreign-currency denominated interest rate

Whichever income is paid on the asset, we can usefully classify the payment into **discrete**, that is, fixed dollar amounts at regular points in time, or on a **continuous**

[2] Note that V_t is not the same as the forward price F_t. The former is the value of the contract; the latter refers to a specification of the contract.

basis, that is, accrued in proportion to the time the asset is held. We must assume that the income payment is fixed or is certain. More generally, a storage cost is equivalent to a negative dividend.

We use these definitions:

$$\begin{aligned} D &= \text{discrete (dollar) dividend or coupon payment} \\ r_t^*(T) &= \text{foreign risk-free rate for delivery at } T \\ q_t(T) &= \text{dividend yield} \end{aligned}$$

Whether the payment is a dividend or a foreign interest rate, the principle is the same. We can afford to invest less in the asset up-front to get one unit at expiration. This is because the income payment can be reinvested into the asset. Alternatively, we can borrow against the value of the income payment to increase our holding of the asset.

It is also important to note that all prices (S, F) are measured in the domestic currency. For example S could be expressed in terms of the U.S. dollar price of the euro, in which case r is the U.S. interest rate and r^* is the euro interest rate. Conversely, if S is the Japanese yen price of the U.S. dollar, r will represent the Japanese interest rate, and r^* the U.S. interest rate.

Continuing our example, consider a stock priced at \$100 that pays a dividend of $D =$ \$1 in three months. The present value of this payment discounted over three months is $De^{-r\tau} = \$1 \exp(-0.05 \times 3/12) = \$0.99$. We only need to put up $S_t - \text{PV}(D) = \$100.00 - 0.99 = \$99.01$ to get one share in one year. Put differently, we buy 0.9901 fractional shares now and borrow against the (sure) dividend payment of \$1 to buy an additional 0.0099 fractional share, for a total of one share.

The pricing formula in Equation (5.2) is extended to

$$F_t e^{-r\tau} = S_t - \text{PV}(D) \tag{5.4}$$

where PV(D) is the present value of the dividend/coupon payments. If there is more than one payment, PV(D) represents the sum of the present values of each individual payment, discounted at the appropriate risk-free rate. With storage costs, we need to *add* the present value of storage costs PV(C) to the right side of Equation (5.4).

The approach is similar for an asset that pays a continuous income, defined per unit time instead of discrete amounts. Holding a foreign currency, for instance, should be done through an interest-bearing account paying interest that accrues with time. Over the horizon τ, we can afford to invest less up-front, $S_t e^{-r^*\tau}$ in order to receive one unit at maturity. The right-hand side of Equation (5.4) is now

$$F_t e^{-r\tau} = S_t e^{-r^*\tau} \tag{5.5}$$

Hence, the forward price should be

$$F_t = S_t e^{-r^*\tau} / e^{-r\tau} \tag{5.6}$$

If instead interest rates are annually compounded, this gives

$$F_t = S_t (1 + r)^\tau / (1 + r^*)^\tau \tag{5.7}$$

Equation (5.6) can be also written in terms of the forward premium or discount, which is

$$\frac{(F_t - S_t)}{S_t} = e^{-r^*\tau}/e^{-r\tau} = \exp(r - r^*)\tau \approx (r - r^*)\tau \tag{5.8}$$

If $r^* < r$, we have $F_t > S_t$ and the asset trades at a **forward premium.** Conversely, if $r^* > r$, $F_t < S_t$ and the asset trades at a **forward discount.** Thus, the forward price is higher or lower than the spot price, depending on whether the yield on the asset is lower than or higher than the domestic risk-free interest rate.

Equation (5.6) is also known as **interest rate parity** when dealing with currencies. Also note that both the spot and forward prices must be expressed in dollars per unit of the foreign currency when the domestic currency interest rate is r. This is the case, for example, for the dollar euro or dollar/pound exchange rate. If, on the other hand, the exchange rate is expressed in foreign currency per dollar, then r must be the rate on the foreign currency. For the yen/dollar rate, for example, S is in yen per dollar, r is the yen interest rate, and r^* is the dollar interest rate.

KEY CONCEPT

The forward price differs from the spot price to reflect the time value of money and the income yield on the underlying asset. It is higher than the spot price if the yield on the asset is lower than the domestic risk-free interest rate, and vice versa.

With income payments, the value of an outstanding forward contract is

$$V_t = S_t e^{-r^*\tau} - Ke^{-r\tau} \tag{5.9}$$

If F_t is the new, current forward price, we can also write

$$V_t = F_t e^{-r\tau} - Ke^{-r\tau} = (F_t - K)e^{-r\tau} \tag{5.10}$$

This provides a useful alternative formula for the valuation of a forward contract. The intuition here is that we could liquidate the outstanding forward contract by entering a reverse position at the current forward rate. The payoff at expiration is $(F - K)$, which, discounted back to the present, gives Equation (5.10).

KEY CONCEPT

The current value of an outstanding forward contract can be found by entering an offsetting forward position and discounting the net cash flow at expiration.

EXAMPLE 5.1: FRM EXAM 2005—QUESTION 2

What is the no-arbitrage price of a forward contract if the time to expiration is three months, the underlying asset is worth $1,000, the continuously compounded annualized risk-free rate is 6%, and storage costs are expressed in terms of a continuous annualized yield of 3%?

a. USD 1,008
b. USD 972
c. USD 1,023
d. USD 1,039

EXAMPLE 5.2: FRM EXAM 2005—QUESTION 16

Suppose that U.S. interest rates rise from 3% to 4% this year. The spot exchange rate quotes at 112.5 JPY/USD and the forward rate for a one-year contract is at 110.5. What is the Japanese interest rate?

a. 1.81%
b. 2.15%
c. 3.84%
d. 5.88%

EXAMPLE 5.3: FRM EXAM 2002—QUESTION 56

Consider a forward contract on a stock market index. Identify the *false* statement. Everything else being constant,

a. The forward price depends directly upon the level of the stock market index.
b. The forward price will fall if underlying stocks increase the level of dividend payments over the life of the contract.
c. The forward price will rise if time to maturity is increased.
d. The forward price will fall if the interest rate is raised.

EXAMPLE 5.4: FRM EXAM 2007—QUESTION 119

A three-month futures contract on an equity index is currently priced at USD 1,000. The underlying index stocks are valued at USD 990 and pay dividends at a continuously compounded rate of 2%. The current continuously compounded risk-free rate is 4%. The potential arbitrage profit per contract, given this set of data, is closest to

a. USD 10.00
b. USD 7.50
c. USD 5.00
d. USD 1.50

5.3 FUTURES CONTRACTS

5.3.1 Definitions of Futures

Forward contracts allow users to take positions that are economically equivalent to those in the underlying cash markets. Unlike cash markets, however, they do not involve substantial up-front payments. Thus, forward contracts can be interpreted as having *leverage*. Leverage is efficient as it makes our money work harder.

Leverage creates credit risk for the counterparty, however. For a cash trade, there is no leverage. When a speculator buys a stock at the price of $100, the counterparty receives the cash and has no credit risk. Instead, when a speculator enters a forward contract to buy an asset at the price of $105, there is no up-front payment. In effect, the speculator borrows from the counterparty to invest in the asset. There is a risk of default should the value of the contract to the speculator fall sufficiently. In response, futures contracts have been structured so as to minimize credit risk for all counterparties. Otherwise, from a market risk standpoint, futures contracts are basically identical to forward contracts.

Futures contracts are standardized, negotiable, and exchange-traded contracts to buy or sell an underlying asset. They differ from forward contracts as follows.

- **Trading on organized exchanges.** In contrast to forwards, which are OTC contracts tailored to customers' needs, futures are traded on organized exchanges (either with a physical location or electronic).
- **Standardization.** Futures contracts are offered with a limited choice of expiration dates. They trade in fixed contract sizes. This standardization ensures an active secondary market for many futures contracts, which can be easily traded, purchased, or resold. In other words, most futures contracts have good liquidity. The trade-off is that futures are less precisely suited to the need of some hedgers, which creates basis risk (to be defined later).

- **Clearinghouse.** Futures contracts are also standardized in terms of the counterparty. After each transaction is confirmed, the clearinghouse basically interposes itself between the buyer and the seller, ensuring the performance of the contract. Thus, unlike forward contracts, counterparties do not have to worry about the credit risk of the other side of the trade. Instead, the credit risk is that of the clearinghouse (or the broker), which is generally excellent.
- **Marking-to-market.** As the clearinghouse now has to deal with the credit risk of the two original counterparties, it has to monitor credit risk closely. This is achieved by daily marking-to-market, which involves settlement of the gains and losses on the contract every day. This will avoid the accumulation of large losses over time, potentially leading to an expensive default.
- **Margins.** Although daily settlement accounts for past losses, it does not provide a buffer against future losses. This is the goal of **margins**, which represent upfront posting of collateral that can be seized should the other party default. If the equity in the account falls below the **maintenance margin**, the customer is required to provide additional funds to cover the initial margin. The level of margin depends on the instrument and the type of position; in general, less volatile instruments or hedged positions require lower margins.

Example: Margins for a Futures Contract

Consider a futures contract on 1,000 units of an asset worth \$100. A long futures position is economically equivalent to holding \$100,000 worth of the asset directly. To enter the futures position, a speculator has to post only \$5,000 in margin, for example. This amount is placed in an equity account with the broker.

The next day, the futures price moves down by \$3, leading to a loss of \$3,000 for the speculator. The profit or loss is added to the equity account, bringing it down to \$5,000–\$3,000 = \$2,000. The speculator would then receive a **margin call** from the broker, asking to have an additional \$3,000 of capital posted to the account. If he or she fails to meet the margin call, the broker has the right to liquidate the position.

Since futures trading is centralized on an exchange, it is easy to collect and report aggregate trading data. **Volume** is the number of contracts traded during the day, which is a flow item. **Open interest** represents the outstanding number of contracts at the close of the day, which is a stock item.

5.3.2 Valuing Futures Contracts

Valuation principles for futures contracts are very similar to those for forward contracts. The main difference between the two types of contracts is that any profit or loss accrues *during* the life of the futures contract instead of all at once, at expiration.

When interest rates are assumed constant or deterministic, forward and futures prices must be equal. With stochastic interest rates, there may be a small difference, depending on the correlation between the value of the asset and interest rates.

If the correlation is zero, then it makes no difference whether payments are received earlier or later. The futures price must be the same as the forward price. In contrast, consider a contract whose price is positively correlated with the interest rate. If the value of the contract goes up, it is more likely that interest rates will go up as well. This implies that profits can be withdrawn and reinvested at a higher rate. Relative to forward contracts, this marking-to-market feature is beneficial to a long futures position. As a result, the futures price must be higher in equilibrium.

In practice, this effect is only observable for interest-rate futures contracts, whose value is *negatively* correlated with interest rates. Because this feature is unattractive for the long position, the futures price must be *lower* than the forward price. Chapter 8 will explain how to compute the adjustment, called the **convexity effect.**

EXAMPLE 5.5: FRM EXAM 2004—QUESTION 38

An investor enters into a short position in a gold futures contract at USD 294.20. Each futures contract controls 100 troy ounces. The initial margin is USD 3,200, and the maintenance margin is USD 2,900. At the end of the first day, the futures price drops to USD 286.6. Which of the following is the amount of the variation margin at the end of the first day?

a. 0
b. USD 34
c. USD 334
d. USD 760

EXAMPLE 5.6: FRM EXAM 2004—QUESTION 66

Which one of the following statements is *incorrect* regarding the margining of exchange-traded futures contracts?

a. Day trades and spread transactions require lower margin levels.
b. If an investor fails to deposit variation margin in a timely manner the positions may be liquidated by the carrying broker.
c. Initial margin is the amount of money that must be deposited when a futures contract is opened.
d. A margin call will be issued only if the investor's margin account balance becomes negative.

5.4 SWAP CONTRACTS

Swap contracts are OTC agreements to exchange a *series* of cash flows according to prespecified terms. The underlying asset can be an interest rate, an exchange rate, an equity, a commodity price, or any other index. Typically, swaps are established for longer periods than forwards and futures.

For example, a 10-year currency swap could involve an agreement to exchange every year 5 million dollars against 3 million pounds over the next 10 years, in addition to a principal amount of 100 million dollars against 50 million pounds at expiration. The principal is also called **notional principal.**

Another example is that of a five-year interest rate swap in which one party pays 8% of the principal amount of 100 million dollars in exchange for receiving an interest payment indexed to a floating interest rate. In this case, since both payments are the same amount in the same currency, there is no need to exchange principal at maturity.

Swaps can be viewed as a portfolio of forward contracts. They can be priced using valuation formulas for forwards. Our currency swap, for instance, can be viewed as a combination of 10 forward contracts with various face values, maturity dates, and rates of exchange. We will give detailed examples in later chapters.

5.5 IMPORTANT FORMULAS

Forward price, no income on the asset: $F_t e^{-r\tau} = S_t$

Forward price, income on the asset:

discrete dividend, $F_t e^{-r\tau} = S_t - \text{PV}(D)$,

continuous dividend, $F_t e^{-r\tau} = S_t e^{-r^*\tau}$

Forward premium or discount: $\frac{(F_t - S_t)}{S_t} \approx (r - r^*)\tau$

Valuation of outstanding forward contract:

$$V_t = S_t e^{-r^*\tau} - K e^{-r\tau} = F_t e^{-r\tau} - K e^{-r\tau} = (F_t - K)e^{-r\tau}$$

5.6 ANSWERS TO CHAPTER EXAMPLES

Example 5.1: FRM Exam 2005—Question 2

c. The asset has a carrying cost, which is the equivalent to a negative dividend yield. Using Equation (5.6), with $S = 1000$, $r = 0.06$, $r^* = -0.03$, and $T = 3/12$, we have $F = 1000 \times \exp(+0.03 \times 1/12)/\exp(-0.06 \times 1/12) = 1{,}022.8$.

Example 5.2: FRM Exam 2005—Question 16

b. As is the convention in the currency markets, the exchange rate is defined as the yen price of the dollar, which is the foreign currency. The foreign currency interest rate is the latest U.S. dollar rate, or 4%. Assuming discrete compounding, the pricing formula for forward contracts is $F(JPY/USD)/(1+rT) = S(JPY/USD)/(1+r^*T)$. Therefore, $(1+rT) = (F/S)(1+r^*T) = (110.5/112.5)(1.04) = 1.0215$, and $r = 2.15\%$. Using continuous compounding gives a similar result. Another approach would consider the forward discount on the dollar, which is $(F - S)/S = -1.8\%$. Thus, the dollar is 1.8% cheaper forward than spot, which must mean that the Japanese interest rate must be approximately 1.8% lower than the U.S. interest rate.

Example 5.3: FRM Exam 2002—Question 56

d. Defining the dividend yield as q, the forward price depends on the cash price according to $F\exp(-rT) = S\exp(-qT)$. This can also be written as $F = S\exp[+(r-q)T]$. Generally, $r > q$. Statement a. is correct: F depends directly on S. Statement b. is also correct, as higher q decreases the term between brackets and hence F. Statement c. is correct because the term $r - q$ is positive, leading to a larger term in brackets as the time to maturity T increases. Statement d. is false, as increasing r makes the forward contract more attractive, or increases F.

Example 5.4: FRM Exam 2007—Question 119

c. The fair value of the futures contract is given by $F = S\exp(-r^{*T})/\exp(-rT) = 990\exp(-0.02 \times 3/12)\exp(-0.04 \times 3/12) = 994.96$. Hence, the actual futures price is too high by $(1{,}000 - 995) = 5$.

Example 5.5: FRM Exam 2004—Question 38

a. This is a tricky question. Because the investor is short and the price fell, the position creates a profit and there is no variation margin.

On the other hand, for the long the loss is \$760, which would bring the equity to $3{,}200 - 760 = 2{,}440$. Because this is below the maintenance margin of 2,900, an additional payment of \$760 is required to bring back the equity to the initial margin.

Example 5.6: FRM Exam 2004—Question 66

d. All the statements are correct, except d. If the margin account balance falls below the maintenance margin (not zero), a margin call will be issued.

CHAPTER 6

Options

This chapter now turns to nonlinear derivatives, or options. As described in Table 5.1, options account for a large part of the derivatives markets. On organized exchanges, options represent \$52 trillion in derivatives outstanding. Over-the-counter options add up to about \$76 trillion in notional amounts.

Although the concept behind these instruments are not new, option markets have blossomed since the early 1970s, because of a breakthrough in pricing options, the Black–Scholes formula, and advances in computing power. We start with plain, **vanilla** options, calls and puts. These are the basic building blocks of many financial instruments. They are also more common than complicated, **exotic** options.

The purpose of this chapter is to present a compact overview of important concepts for options, including their pricing. We will cover option sensitivities (the "Greeks") in a future chapter. Section 6.1 presents the payoff functions on basic options and combinations thereof. We then discuss option premiums in Section 6.2. The Black–Scholes pricing approach is presented in Section 6.3. Next, Section 6.4 briefly summarizes more complex options. Finally, Section 6.5 shows how to value options using a numerical, binomial tree model.

6.1 OPTION PAYOFFS

6.1.1 Basic Options

Options are instruments that give their holder the *right* to buy or sell an asset at a specified price until a specified expiration date. The specified delivery price is known as the **delivery price**, or **exercise price**, or **strike price**, and is denoted by K.

Options to buy are **call options**. Options to sell are **put options**. As options confer a right to the purchaser of the option, but not an obligation, they will be exercised only if they generate profits. In contrast, forwards involve an obligation to either buy or sell and can generate profits or losses. Like forward contracts, options can be purchased or sold. In the latter case, the seller is said to **write** the option.

Depending on the timing of exercise, options can be classified into European or American options. **European options** can be exercised at maturity only. **American options** can be exercised at any time, before or at maturity. Because American options include the right to exercise at maturity, they must be at least as valuable as European options. In practice, however, the value of this early exercise feature is small, as an investor can generally receive better value by reselling the option on the open market instead of exercising it.

We use these notations, in addition to those in the previous chapter:

$$\begin{aligned} K &= \text{exercise price} \\ c &= \text{value of European call option} \\ C &= \text{value of American call option} \\ p &= \text{value of European put option} \\ P &= \text{value of American put option} \end{aligned}$$

To illustrate, take an option on an asset that currently trades at \$85 with a delivery price of \$100 in one year. If the spot price stays at \$85 at expiration, the holder of the call will not **exercise** the option, because the option is not profitable with a stock price less than \$100. In contrast, if the price goes to \$120, the holder will exercise the right to buy at \$100, will acquire the stock now worth \$120, and will enjoy a "paper" profit of \$20. This profit can be realized by selling the stock. For put options, a profit accrues if the spot price falls below the exercise price $K = \$100$.

Thus, the payoff profile of a long position in the call option at expiration is

$$C_T = \mathrm{Max}(S_T - K, 0) \tag{6.1}$$

The payoff profile of a long position in a put option is

$$P_T = \mathrm{Max}(K - S_T, 0) \tag{6.2}$$

If the current asset price S_t is close to the strike price K, the option is said to be **at-the-money**. If the current asset price S_t is such that the option could be exercised now at a profit, the option is said to be **in-the-money**. If the remaining situation, the option is said to be **out-of-the-money**. A call will be in-the-money if $S_t > K$. A put will be in-the-money if $S_t < K$.

As in the case of forward contracts, the payoff at expiration can be cash settled. Instead of actually buying the asset, the contract could simply pay \$20 if the price of the asset is \$120.

Because buying options can generate only profits (at worst zero) at expiration, an option contract must be a valuable asset (or at worst have zero value). This means that a payment is needed to acquire the contract. This up-front payment, which is much like an insurance premium, is called the option "premium." This premium cannot be negative. An option becomes more expensive as it moves in-the-money.

Thus, the payoffs on options must take into account this cost (for long positions) or benefit (for short positions). To compute the total payoff, we should translate all option payoffs by the *future* value of the premium, that is, $ce^{r\tau}$, for European call options.

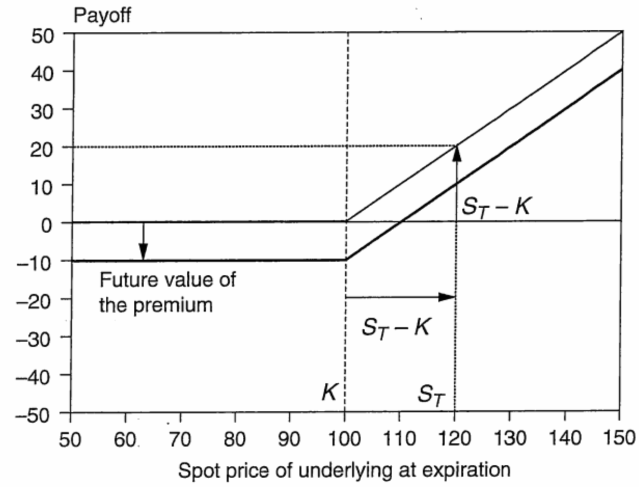


FIGURE 6.1 Profit Payoffs on Long Call

Figure 6.1 displays the total profit payoff on a call option as a function of the asset price at expiration. Assuming that $S_T = \$120$, the proceeds from exercise are $\$120 - \$100 = \$20$, from which we have to subtract the future value of the premium, say \$10. In the graphs that follow, we always take into account the cost of the option.

Figure 6.2 summarizes the payoff patterns on long and short positions in a call and a put contract. Unlike those of forwards, these payoffs are **nonlinear** in the underlying spot price. Sometimes they are referred to as the "hockey stick" diagrams. This is because forwards are obligations, whereas options are rights. Note that the positions for the same contract are symmetrical around the horizontal axis. For a given spot price, the sum of the profit or loss for the long and for the short is zero.

In the market-risk section (Part Three) of this handbook, we will combine these payoffs with the distribution of the risk factors. Even so, it is immediately

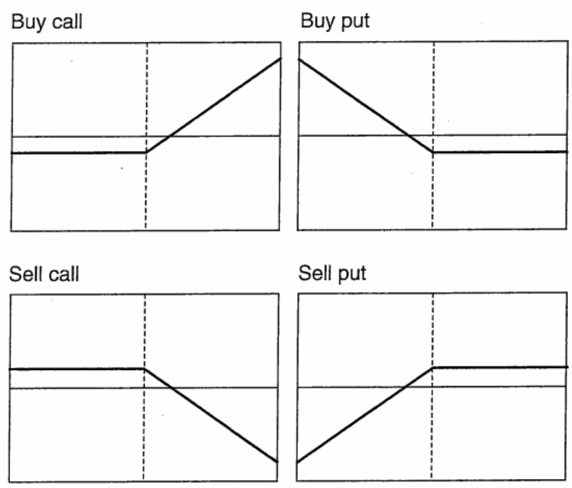


FIGURE 6.2 Profit Payoffs on Long and Short Calls and Puts

obvious that long option positions have limited downside risk, which is the loss of the premium. Short call option positions have unlimited downside risk because there is no upper limit on S. The worst loss on short put positions occurs if S goes to zero.

So far, we have covered options on cash instruments. Options can also be struck on futures. When exercising a call, the investor becomes long the futures contract. Conversely, exercising a put creates a short position in the futures contract. Because positions in futures are equivalent to leveraged positions in the underlying cash instrument, options on cash instruments and on futures are equivalent.

6.1.2 Put–Call Parity

These option payoffs can be used as the basic building blocks for more complex positions. A long position in the underlying asset can be decomposed into a long call plus a short put, as shown in Figure 6.3.

The figure shows that the long call provides the equivalent of the upside while the short put generates the same downside risk as holding the asset. This link creates a relationship between the value of the call and that of the put, also known as **put–call parity**. The relationship is illustrated in Table 6.1, which examines the payoff at initiation and at expiration under the two possible states of the world. We only consider European options with the same maturity and exercise price. Also, we assume that there is no income payment on the underlying asset.

The portfolio consists of a long position in the call, a short position in the put, and an investment to ensure that we will be able to pay the exercise price at maturity. Long positions are represented by negative values, as they are outflows.

The table shows that the final payoffs to portfolio (1) add up to S_T in the two states of the world, which is the same as a long position in the asset itself. Hence, to avoid arbitrage, the initial payoff must be equal to the current cost of the asset,

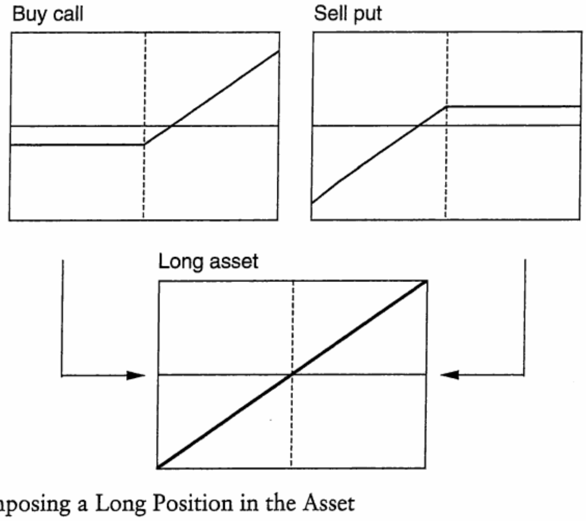


FIGURE 6.3 Decomposing a Long Position in the Asset

TABLE 6.1 Put–Call Parity

			Final Payoff	
Portfolio	Position	Initial Payoff	$S_T < K$	$S_T \geq K$
(1)	Buy call	$-c$	0	$S_T - K$
	Sell put	$+p$	$-(K - S_T)$	0
	Invest	$-Ke^{-r\tau}$	K	K
	Total	$-c + p - Ke^{-r\tau}$	S_T	S_T
(2)	Buy asset	$-S$	S_T	S_T

which is $S_t = S$. So, we must have $-c + p - Ke^{-r\tau} = -S$. More generally, with income paid at the rate of r^*, put-call parity can be written as

$$c - p = Se^{-r^*\tau} - Ke^{-r\tau} = (F - K)e^{-r\tau} \tag{6.3}$$

Because $c \geq 0$ and $p \geq 0$, this relationship can be also used to determine lower bounds for European calls and puts. Note that the relationship does not hold exactly for American options since there is a likelihood of early exercise, which could lead to mismatched payoffs.

Finally, this relationship can be used to determine the **implied dividend yield** from market prices. We observe c, p, S, and r and can solve for y or r^*. This yield is used for determining the forward rate in **dividend swaps,** which are contracts where the payoff is indexed to the actual dividends paid over the horizon, minus the implied dividends.

KEY CONCEPT

A long position in an asset is equivalent to a long position in a European call with a short position in an otherwise identical put, combined with a risk-free position.

EXAMPLE 6.1: FRM EXAM 2007—QUESTION 84

According to put–call parity, buying a put option on a stock is equivalent to

a. Buying a call option and buying the stock with funds borrowed at the risk-free rate
b. Selling a call option and buying the stock with funds borrowed at the risk-free rate
c. Buying a call option, selling the stock, and investing the proceeds at the risk-free rate
d. Selling a call option, selling the stock, and investing the proceeds at the risk-free rate

EXAMPLE 6.2: FRM EXAM 2005—QUESTION 72

A one-year European put option on a non-dividend-paying stock with strike at EUR 25 currently trades at EUR 3.19. The current stock price is EUR 23 and its annual volatility is 30%. The annual risk-free interest rate is 5%. What is the price of a European call option on the same stock with the same parameters as that of the above put option? Assume continuous compounding.

a. EUR 1.19
b. EUR 3.97
c. EUR 2.41
d. Cannot be determined with the data provided

EXAMPLE 6.3: FRM EXAM 2002—QUESTION 25

The price of a non-dividend-paying stock is \$20. A six-month European call option with a strike price of \$18 sells for \$4. A European put option on the same stock, with the same strike price and maturity, sells for \$1.47. The continuously compounded risk-free interest rate is 6% per annum. Are these three securities (the stock and the two options) consistently priced?

a. No, there is an arbitrage opportunity worth \$2.00.
b. No, there is an arbitrage opportunity worth \$2.53.
c. No, there is an arbitrage opportunity worth \$14.00.
d. Yes.

EXAMPLE 6.4: FRM EXAM 2006—QUESTION 74

Jeff is an arbitrage trader, who wants to calculate the implied dividend yield on a stock while looking at the over-the-counter price of a five-year European put and call on that stock. He has the following data: $S = \$85$, $K = \$90$, $r = 5\%$, $c = \$10$, $p = \$15$. What is the continuous implied dividend yield of that stock?

a. 2.48%
b. 4.69%
c. 5.34%
d. 7.71%

6.1.3 Combination of Options

Options can be combined in different ways, either with each other or with the underlying asset. Consider first combinations of the underlying asset and an option. A long position in the stock can be accompanied by a short sale of a call to collect the option premium. This operation, called a **covered call**, is described in Figure 6.4. Likewise, a long position in the stock can be accompanied by a purchase of a put to protect the downside. This operation is called a **protective put**.

Options can also be combined with an underlying position to limit the range of potential gains and losses. Suppose an investor is long a stock, currently trading at $10. The investor can buy a put with a low strike price (e.g., $7), partially financed by the sale of a call with a high strike (e.g., $12). Ignoring the net premium, the highest potential gain is $2 and the worst loss is $3. Such strategy is called a **collar**. If the strike prices were the same, this would be equivalent to a short stock position, which creates a net payoff of exactly zero.

We can also combine a call and a put with the same or different strike prices and maturities. When the strike prices of the call and the put, and their maturities, are the same, the combination is referred to as a **straddle**. Figure 6.5 shows how to construct a long straddle, i.e., buying a call and a put with the same maturity and strike price. This position is expected to benefit from a large price move, whether up or down. The reverse position is a short straddle. When the strike prices are different, the combination is referred to as a **strangle**. Since strangles are out-of-the-money, they are cheaper to buy than straddles.

Thus far, we have concentrated on positions involving two classes of options. One can, however, establish positions with one class of options, called **spreads**. Calendar, or **horizontal spreads** correspond to different maturities. **Vertical spreads** correspond to different strike prices. The names of the spreads are derived from the manner in which they are listed in newspapers: time is listed horizontally and strike prices are listed vertically. **Diagonal spreads** move across maturities and strike prices.

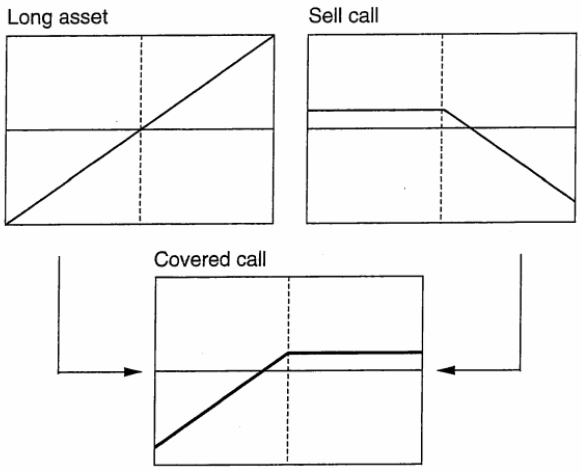


FIGURE 6.4 Creating a Covered Call

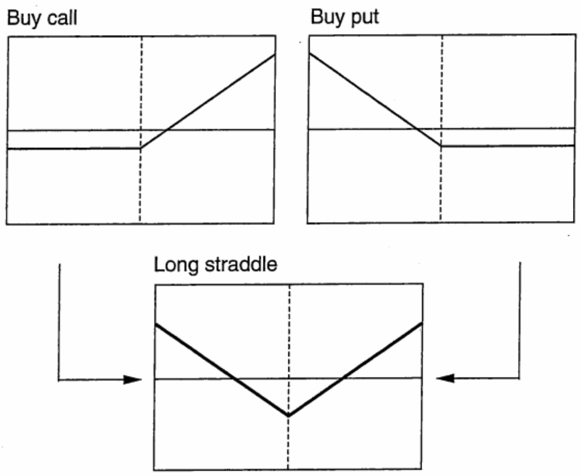


FIGURE 6.5 Creating a Long Straddle

For instance, a **bull spread** is positioned to take advantage of an increase in the price of the underlying asset. Conversely, a **bear spread** represents a bet on a falling price. Figure 6.6 shows how to construct a bull(ish) vertical spread with two calls with the same maturity. This could also be constructed with puts, however. Here, the spread is formed by buying a call option with a low exercise price K_1 and selling another call with a higher exercise price K_2. Note that the cost of the first call $c(S, K_1)$ must exceed the cost of the second call $c(S, K_2)$, because the first option is more in-the-money than the second. Hence, the sum of the two premiums represents a net cost. At expiration, when $S_T > K_2$, the payoff is $\text{Max}(S_T - K_1, 0) - \text{Max}(S_T - K_2, 0) = (S_T - K_1) - (S_T - K_2) = K_2 - K_1$, which is positive. Thus, this position is expected to benefit from an upmove, while incurring only limited downside risk.

Spreads involving more than two positions are referred to as butterfly or sandwich spreads. A **butterfly spread** involves three types of options with the same maturity: for example, a long call at a strike price K_1, two short calls at a

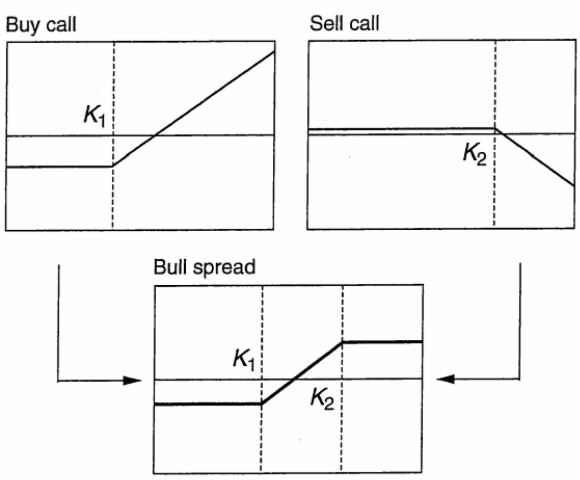


FIGURE 6.6 Creating a Bull Spread

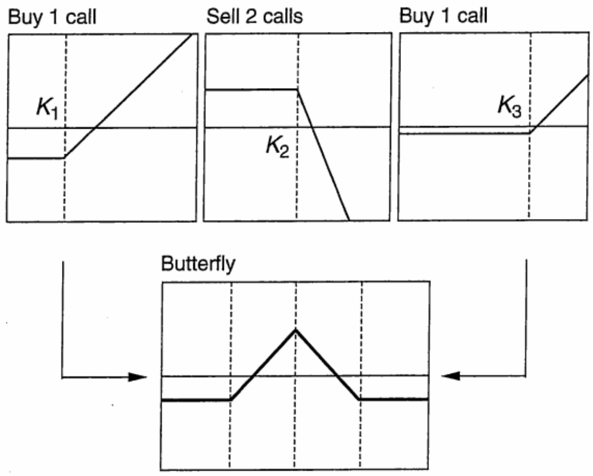


FIGURE 6.7 Creating a Butterfly Spread

higher strike price K_2, and a long call position at a higher strike price K_3, with the same spacing. Figure 6.7 shows that this position is expected to benefit when the underlying asset price stays stable, close to K_2. The double position in the middle is called the body, and the others the wings. A sandwich spread is the opposite of a butterfly spread.

EXAMPLE 6.5: FRM EXAM 2001—QUESTION 90

Which of the following is the riskiest form of speculation using option contracts?

a. Setting up a spread using call options
b. Buying put options
c. Writing naked call options
d. Writing naked put options

EXAMPLE 6.6: FRM EXAM 2007—QUESTION 103

An investor sells a June 2008 call of ABC Limited with a strike price of USD 45 for USD 3 and buys a June 2008 call of ABC Limited with a strike price of USD 40 for USD 5. What is the name of this strategy and the maximum profit and loss the investor could incur?

a. Bear spread, maximum loss USD 2, maximum profit USD 3
b. Bull spread, maximum loss Unlimited, maximum profit USD 3
c. Bear spread, maximum loss USD 2, maximum profit unlimited
d. Bull spread, maximum loss USD 2, maximum profit USD 3

EXAMPLE 6.7: FRM EXAM 2006—QUESTION 45

A portfolio manager wants to hedge his bond portfolio against changes in interest rates. He intends to buy a put option with a strike price below the portfolio's current price in order to protect against rising interest rates. He also wants to sell a call option with a strike price above the portfolio's current price in order to reduce the cost of buying the put option. What strategy is the manager using?

a. Bear spread
b. Strangle
c. Collar
d. Straddle

EXAMPLE 6.8: FRM EXAM 2002—QUESTION 42

Consider a bearish option strategy of buying one $50 strike put for $7, selling two $42 strike puts for $4 each, and buying one $37 put for $2. All options have the same maturity. Calculate the final profit (P/L) per share of the strategy if the underlying is trading at $33 at expiration.

a. $1 per share
b. $2 per share
c. $3 per share
d. $4 per share

EXAMPLE 6.9: FRM EXAM 2003—QUESTION 72

Which of the following regarding option strategies is/are *not* correct?

I. A long strangle involves buying a call and a put with equal strike prices.
II. A *short* bull spread involves selling a call at lower strike price and buying another call at higher strike price.
III. Vertical spreads are formed by options with different maturities.
IV. A long butterfly spread is formed by buying two options at two different strike prices and selling another two options at the same strike price.

a. I only
b. I and III only
c. I and II only
d. III and IV only

6.2 OPTION PREMIUMS

6.2.1 General Relationships

So far, we have examined the payoffs at expiration only. Also important is the instantaneous relationship between the option value and the current price S, which is displayed in Figures 6.8 and 6.9.

For a call, a higher price S increases the current value of the option, but in a nonlinear, convex fashion. For a put, lower values for S increase the value of the option, also in a convex fashion. As time goes by, the curved line approaches the hockey stick line.

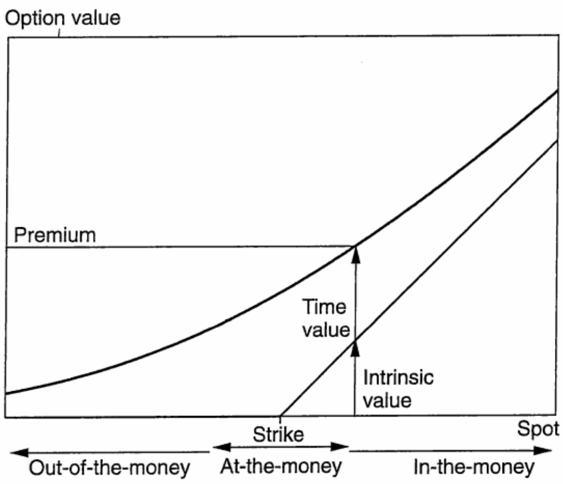


FIGURE 6.8 Relationship between Call Value and Spot Price

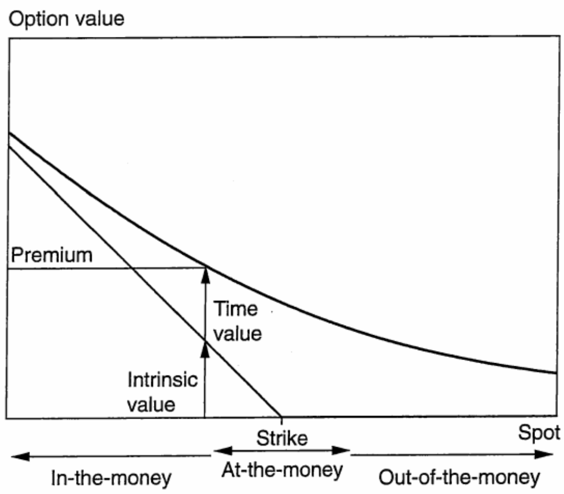


FIGURE 6.9 Relationship between Put Value and Spot Price

Figures 6.8 and 6.9 decompose the current premium into

- An **intrinsic value**, which basically consists of the value of the option if exercised today, or $\text{Max}(S_t - K, 0)$ for a call, and $\text{Max}(K - S_t, 0)$ for a put
- A **time value**, which consists of the remainder, reflecting the possibility that the option will create further gains in the future

Consider for example a one-year call with strike $K = \$100$. The current price is $S = \$120$ and interest rate $r = 5\%$. The asset pays no dividend. Say the call premium is \$26.17. This can be decomposed into an intrinsic value of $\$120 - \$100 = \$20$ and time value of \$6.17. The time value increases with the volatility of the underlying asset. It also generally increases with the maturity of the option.

As shown in Figures 6.8 and 6.9, options can be classified into:

- **At-the-money**, when the current spot price is close to the strike price
- **In-the-money**, when the intrinsic value is large
- **Out-of-the-money**, when the spot price is much below the strike price for calls and conversely for puts (out-of-the-money options have zero intrinsic value)

We can also identify some general bounds for European options that should always be satisfied; otherwise there would be an arbitrage opportunity (a money machine). For simplicity, assume there is no dividend. We know that a European option is worth less than an American option. First, the current value of a call must be less than, or equal to, the asset price:

$$c_t \leq C_t \leq S_t \tag{6.4}$$

This is because, in the limit, an option with zero exercise price is equivalent to holding the stock. We are sure to exercise the option.

Second, the value of a European call must be greater than, or equal to, the price of the asset minus the present value of the strike price:

$$c_t \geq S_t - Ke^{-r\tau} \tag{6.5}$$

To prove this, we could simply use put-call parity, or Equation (6.3) with $r^* = 0$, imposing the condition that $p \geq 0$. Note that, since $e^{-r\tau} < 1$, we must have $S_t - Ke^{-r\tau} > S_t - K$ before expiration. Thus, $S_t - Ke^{-r\tau}$ is a more informative lower bound than $S_t - K$. As an example, continue with our call option. The lower bound is $S_t - Ke^{-r\tau} = \$120 - \$100 \exp(-5\% \times 1) = \$24.88$. This is more informative than $S - K = \$20$.

We can also describe upper and lower bounds for put options. The value of a put cannot be worth more than K

$$p_t \leq P_t \leq K \tag{6.6}$$

which is the upper bound if the price falls to zero. Using put–call parity, we can show that the value of a European put must satisfy the following lower bound

$$p_t \geq Ke^{-r\tau} - S_t \tag{6.7}$$

6.2.2 Early Exercise of Options

These relationships can be used to assess the value of early exercise for American options. The basic trade-off arises between the value of the American option **dead**, that is, exercised, or **alive**, that is, nonexercised. Thus, the choice is between exercising the option and selling it on the open market.

Consider an American call on a non-dividend-paying stock. By exercising early, the holder gets exactly $S_t - K$. The value of the option alive, however, must be worth more than that of the equivalent European call. From Equation (6.5), this must satisfy $c_t \geq S_t - Ke^{-r\tau}$, which is strictly greater than $S_t - K$. Hence, an American call on a non-dividend-paying stock *should never* be exercised early.

In our example, the lower bound on the European call is \$24.88. If we exercise the American call, we only get $S - K = \$120 - \$100 = \$20$. Because this is less than the minimum value of the European call, the American call should not be exercised. As a result, the value of the American feature is zero and we always have $c_t = C_t$.

The only reason to exercise early a call is to capture a dividend payment. Intuitively, a high income payment makes holding the asset more attractive than holding the option. Thus, American options on income-paying assets may be exercised early. Note that this applies also to options on futures, since the implied income stream on the underlying is the risk-free rate.

KEY CONCEPT

An American call option on a non-dividend-paying stock (or asset with no income) should never be exercised early. If the asset pays income, early exercise may occur, with a probability that increases with the size of the income payment.

For an American put, we must have

$$P_t \geq K - S_t \tag{6.8}$$

because it could be exercised now. Unlike the relationship for calls, this lower bound $K - S_t$ is strictly greater than the lower bound for European puts $Ke^{-r\tau} - S_t$. So, we could have early exercise.

To decide whether to exercise early, the holder of the option has to balance the benefit of exercising, which is to receive K now instead of later, against the loss of killing the time value of the option. Because it is better to receive money now than later, it may be worth exercising the put option early.

Thus, American puts on nonincome paying assets *could* be exercised early, unlike calls. The probability of early exercise decreases for lower interest rates and with higher income payments on the asset. In each case, it becomes less attractive to sell the asset.

KEY CONCEPT

An American put option on a non-dividend-paying stock (or asset with no income) may be exercised early. If the asset pays income, the possibility of early exercise decreases with the size of the income payments.

EXAMPLE 6.10: FRM EXAM 2002—QUESTION 50

Given strictly positive interest rates, the best way to close out a long American call option position early (option written on a stock that pays no dividends) would be to

a. Exercise the call
b. Sell the call
c. Deliver the call
d. Do none of the above

EXAMPLE 6.11: FRM EXAM 2005—QUESTION 15

You have been asked to verify the pricing of a two-year European call option with a strike price of USD 45. You know that the initial stock price is USD 50, and the continuous risk-free rate is 3%. To verify the possible price range of this call, you consider using price bounds. What is the difference between the upper and lower bounds for that European call?

a. 0.00
b. 7.62
c. 42.38
d. 45.00

6.3 VALUING OPTIONS

6.3.1 Pricing by Replication

We now turn to the pricing of options. The philosophy of pricing models consists of replicating the payoff on the instrument. To avoid arbitrage, the price of the instrument must then equal the price of the replicating portfolio.

Consider a call option on a stock whose price is represented by a binomial process. The initial price of $S_0 = \$100$ can only move up or down, to two values (hence the name "binomial"), $S_1 = \$150$ or $S_2 = \$50$. The option is a call with $K = \$100$, and therefore can only take values of $c_1 = \$50$ or $c_2 = \$0$. We assume that the rate of interest is $r = 25\%$, so that a dollar invested now grows to \$1.25 at maturity.

$$\boxed{\begin{array}{lcll} & & S_1 = \$150 & c_1 = \$50 \\ & \nearrow & & \\ S_0 = \$100 & & & \\ & \searrow & & \\ & & S_2 = \$50 & c_2 = \$0 \end{array}}$$

The key idea of derivatives pricing is that of **replication**. In other words, we replicate the payoff on the option by a suitable portfolio of the underlying asset plus a position, long or short, in a risk-free bill. This is feasible in this simple setup because we have two states of the world and two instruments, the stock and the bond. To prevent arbitrage, the current value of the derivative must be the same as that of the portfolio.

The portfolio consists of n shares and a risk-free investment currently valued at B (a negative value implies borrowing). We set $c_1 = nS_1 + B$, or $\$50 = n\$150 + B$ and $c_2 = nS_2 + B$, or $\$0 = n\$50 + B$ and solve the 2 by 2 system, which gives $n = 0.5$ and $B = -\$25$. At time $t = 0$, the value of the loan is $B_0 = \$25/1.25 = \$20$. The current value of the portfolio is $nS_0 + B_0 = 0.5 \times \$100 - \$20 = \$30$. Hence, the current value of the option must be $c_0 = \$30$. This derivation shows the essence of option pricing methods.

Note that we did not need the actual probability of an upmove. Define this as p. To see how this can be derived, we write the current value of the stock as the discounted expected payoff assuming investors were risk-neutral:

$$S_0 = [p \times S_1 + (1 - p) \times S_2]/(1 + r) \tag{6.9}$$

where the term between brackets is the expectation of the future spot price, given by the probability times its value for each state. Solving for $100 = [p \times 150 + (1 - p) \times 50]/1.25$, we find a risk-neutral probability of $p = 0.75$. We now value the option in the same fashion:

$$c_0 = [p \times c_1 + (1 - p) \times c_2]/(1 + r) \tag{6.10}$$

which gives

$$c_0 = [0.75 \times \$50 + 0.25 \times \$0]/1.25 = \$30$$

This simple example illustrates a very important concept, which is that of **risk-neutral pricing.**

6.3.2 Black–Scholes Valuation

The Black–Scholes (BS) model is an application of these ideas that provides an elegant closed-form solution to the pricing of European calls. The derivation of the model is based on four assumptions:

Black–Scholes Model Assumptions

- *The price of the underlying asset moves in a continuous fashion.*
- *Interest rates are known and constant.*
- *The variance of underlying asset returns is constant.*
- *Capital markets are perfect* (i.e., short-sales are allowed, there are no transaction costs or taxes, and markets operate continuously).

The most important assumption behind the model is that prices are continuous. This rules out discontinuities in the sample path, such as jumps, which cannot be hedged in this model.

The statistical process for the asset price is modeled by a geometric Brownian motion: Over a very short time interval, dt, the logarithmic return has a normal distribution with mean $= \mu dt$ and variance $= \sigma^2 dt$. The total return can be modeled as

$$dS/S = \mu dt + \sigma dz \tag{6.11}$$

where the first term represents the drift component, and the second is the stochastic component, with dz distributed normally with mean zero and variance dt.

This process implies that the logarithm of the ending price is distributed as

$$\ln(S_T) = \ln(S_0) + (\mu - \sigma^2/2)\tau + \sigma\sqrt{\tau}\,\epsilon \tag{6.12}$$

where ϵ is a $N(0, 1)$ random variable. Hence, the price is lognormally distributed.

Based on these assumptions, Black and Scholes (1972) derived a closed-form formula for European options on a non-dividend-paying stock, called the **Black–Scholes model.** The key point of the analysis is that a position in the option can be replicated by a "delta" position in the underlying asset. Hence, a portfolio combining the asset and the option in appropriate proportions is "locally" risk-free, that is, for small movements in prices. To avoid arbitrage, this portfolio must return the risk-free rate.

As a result, we can directly compute the present value of the derivative as the discounted expected payoff

$$f_t = E_{\mathrm{RN}}[e^{-r\tau}F(S_T)] \tag{6.13}$$

where the underlying asset is assumed to grow at the risk-free rate, and the discounting is also done at the risk-free rate. Here, the subscript RN refers to the fact that the analysis assumes **risk neutrality**. In a risk-neutral world, the expected return on all securities must be the risk-free rate of interest, r. The reason is that risk-neutral investors do not require a risk premium to induce them to take risks. The BS model value can be computed assuming that all payoffs grow at the risk-free rate and are discounted at the same risk-free rate.

This risk-neutral valuation approach is without a doubt the most important tool in derivatives pricing. Before the Black–Scholes breakthrough, Samuelson had derived a very similar model in 1965, but with the asset growing at the rate μ and discounting as some other rate μ^*.[1] Because μ and μ^* are unknown, the Samuelson model was not practical. The risk-neutral valuation is merely an artificial method to obtain the correct solution, however. It does not imply that investors are risk-neutral.

Furthermore, this approach has limited uses for risk management. The BS model can be used to derive the **risk-neutral probability** of exercising the option. For risk management, however, what matters is the actual probability of exercise, also called **physical probability**. This can differ from the RN probability.

We now turn to the formulation of the BS model. In the case of a European call, the final payoff is $F(S_T) = \mathrm{Max}(S_T - K, 0)$. Initially, we assume no dividend payment on the asset. The current value of the call is given by:

$$c = SN(d_1) - Ke^{-r\tau}N(d_2) \tag{6.14}$$

where $N(d)$ is the cumulative distribution function for the standard normal distribution:

$$N(d) = \int_{-\infty}^{d} \Phi(x)dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{1}{2}x^2}dx$$

with Φ defined as the standard normal density function. $N(d)$ is also the area to the left of a standard normal variable with value equal to d, as shown in Figure 6.10. Note that, since the normal density is symmetrical, $N(d) = 1 - N(-d)$, or the area to the left of d is the same as the area to the right of $-d$.

[1] Samuelson, Paul (1965), Rational Theory of Warrant Price, *Industrial Management Review* 6, 13–39.

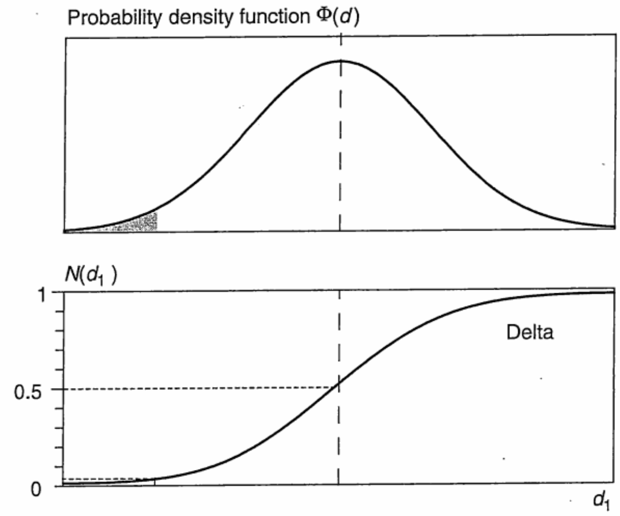


FIGURE 6.10 Cumulative Distribution Function

The values of d_1 and d_2 are:

$$d_1 = \frac{ln\,(S/Ke^{-r\tau})}{\sigma\sqrt{\tau}} + \frac{\sigma\sqrt{\tau}}{2}, \quad d_2 = d_1 - \sigma\sqrt{\tau}$$

By put-call parity, the European put option value is

$$p = S[N(d_1) - 1] - Ke^{-r\tau}[N(d_2) - 1] \tag{6.15}$$

Example: Computing the Black–Scholes Value

Consider an at-the-money call on a stock worth $S = \$100$, with a strike price of $K = \$100$ and maturity of six months. The stock has annual volatility of $\sigma = 20\%$ and pays no dividend. The risk-free rate is $r = 5\%$.

First, we compute the present value factor, which is $e^{-r\tau} = \exp(-0.05 \times 6/12) = 0.9753$. We then compute the value of $d_1 = \ln[S/Ke^{-r\tau}]/\sigma\sqrt{\tau} + \sigma\sqrt{\tau}/2 = 0.2475$ and $d_2 = d_1 - \sigma\sqrt{\tau} = 0.1061$. Using standard normal tables or the NORMSDIST Excel function, we find $N(d_1) = 0.5977$ and $N(d_2) = 0.5422$. Note that both values are greater than 0.5 since d_1 and d_2 are both positive. The option is at-the-money. As S is close to K, d_1 is close to zero and $N(d_1)$ close to 0.5.

The value of the call is $c = SN(d_1) - Ke^{-r\tau}N(d_2) = \$6.89$.

The value of the call can also be viewed as an equivalent position of $N(d_1) = 59.77\%$ in the stock and some borrowing: $c = \$59.77 - \$52.88 = \$6.89$. Thus this is a leveraged position in the stock.

The value of the put is \$4.42. Buying the call and selling the put costs $\$6.89 - \$4.42 = \$2.47$. This indeed equals $S - Ke^{-r\tau} = \$100 - \$97.53 = \$2.47$, which confirms put–call parity.

We should note that Equation (6.14) can be reinterpreted in view of the discounting formula in a risk-neutral world, Equation (6.13)

$$c = E_{\text{RN}}[e^{-r\tau}\text{Max}(S_T - K, 0)] = e^{-r\tau}[\int_K^{\infty} Sf(S)dS] - Ke^{-r\tau}[\int_K^{\infty} f(S)dS] \quad (6.16)$$

We see that the integral term multiplying K is the risk-neutral probability of exercising the call, or that the option will end up in-the-money $S > K$. Matching this up with (6.14), this gives

$$\text{Risk} - \text{neutral probability of exercise} = \left[\int_K^{\infty} f(S)dS\right] = N(d_2) \quad (6.17)$$

6.3.3 Extensions

Merton (1973) expanded the BS model to the case of a stock paying a continuous dividend yield q. Garman and Kohlhagen (1983) extended the formula to foreign currencies, reinterpreting the yield as the foreign rate of interest $q = r^*$, in what is called the **Garman-Kohlhagen model.**

The Merton model then replaces all occurrences of S by $Se^{-r^*\tau}$. The call is worth

$$c = Se^{-r^*\tau}N(d_1) - Ke^{-r\tau}N(d_2) \quad (6.18)$$

It is interesting to take the limit of Equation (6.14) as the option moves more in-the-money, that is, when the spot price S is much greater than K. In this case, d_1 and d_2 become very large and the functions $N(d_1)$ and $N(d_2)$ tend to unity. The value of the call then tends to

$$c(S >> K) = Se^{-r^*\tau} - Ke^{-r\tau} \quad (6.19)$$

which is the valuation formula for a forward contract. A call that is deep in-the-money is equivalent to a long forward contract, because we are almost certain to exercise.

The **Black model** (1976) applies the same formula to options on futures. The only conceptual difference lies in the income payment to the underlying instrument. With an option on cash, the income is the dividend or interest on the cash instrument. In contrast, with a futures contract, the economically equivalent stream of income is the riskless interest rate. The intuition is that a futures can be viewed as equivalent to a position in the underlying asset with the investor setting aside an amount of cash equivalent to the present value of F.

KEY CONCEPT

With an option on futures, the implicit income is the risk-free rate of interest.

For the Black model, we simply replace S by F, the current futures quote, and replace r^* by r, the domestic risk-free rate. The Black model for the valuation of options on futures is:

$$c = [F\,N(d_1) - K\,N(d_2)]e^{-r\tau} \tag{6.20}$$

Finally, we should note that standard options involve a choice to exchange cash for the asset. This is a special case of an **exchange option**, which involves the surrender of an asset (call it B) in exchange for acquiring another (call it A). The payoff on such a call is

$$c_T = \text{Max}(S_T^A - S_T^B, 0) \tag{6.21}$$

where S^A and S^B are the respective spot prices. Some financial instruments involve the maximum of the value of two assets, which is equivalent to a position in one asset plus an exchange option:

$$\text{Max}(S_t^A, S_t^B) = S_T^B + \text{Max}(S_T^A - S_T^B, 0) \tag{6.22}$$

Margrabe (1978)[2] has shown that the valuation formula is similar to the usual model, except that K is replaced by the price of asset B (S_B), and the risk-free rate by the yield on asset B (q_B).[2] The volatility σ is now that of the difference between the two assets, which is

$$\sigma_{AB}^2 = \sigma_A^2 + \sigma_B^2 - 2\rho_{AB}\sigma_A\sigma_B \tag{6.23}$$

These options also involve the correlation coefficient. So, if we have a triplet of options, involving A, B, and the option to exchange B into A, we can compute σ_A, σ_B, and σ_{AB}. This allows us to infer the correlation coefficient. The pricing formula is called the **Margrabe model.**

[2] Margrabe, W. (1978), The Value of an Option to Exchange One Asset for Another, *Journal of Finance* 33, 177–186. See also Stulz, R. (1982), Options on the Minimum or the Maximum of Two Risky Assets: Analysis and Applications, *Journal of Financial Economics* 10, 161–185.

6.3.4 Market versus Model Prices

In practice, the BS model is widely used to price options. All of the parameters are observable, except for the volatility. If we observe a market price, however, we can solve for the volatility parameter that sets the model price equal to the market price. This is called the **implied standard deviation** (ISD).

If the model were correct, the ISD should be constant across strike prices. In fact, this is not what we observe. Plots of the ISD against the strike price generally display what is called a **volatility smile** pattern, meaning that ISDs increase for low and high values of K. This effect has been observed in a variety of markets, such as foreign currency options. For equity index options, the effect is more asymmetrical, with very high ISDs for low strike prices. Because of the negative slope, this is called a **volatility skew**. Before the stock market crash of October 1987, this effect was minor. Since then, it has become more pronounced.

A related concept is the term **structure of volatility**, which refers to the observation that the ISD differs across maturities. This arises because the option market incorporates many types of events over the life of the option. For instance, the realized volatility of a stock tends to increase on the day of an earnings announcement.

These observations can be usefully summarized by an **implied volatility surface**, which is a three-dimensional plot of ISD across maturities and strike prices. Option traders typically observe the shape of this volatility surface to identify sectors where the ISDs seem out of line. To predict returns on options, however, traders also need to forecast the evolution of the implied volatility surface.

Figure 6.11 gives an example of the evolution of a volatility skew. The initial curve has an ISD of 18% for ATM options, with a strike price of $K = 100$. The

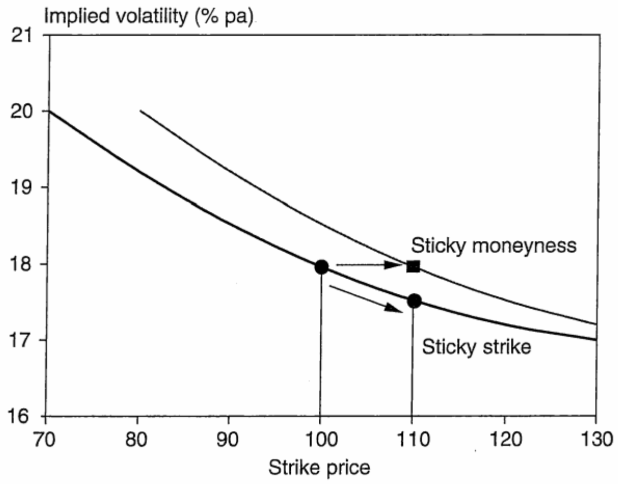


FIGURE 6.11 Evolution of Volatility Skew

question is what the curve will look like if the spot price moves from $S = 100$ to $S = 110$. Traders typically use heuristic approaches to the extrapolation of the curve over the investment horizon.

In a first scenario, called **sticky strike**, the curve does not change and the ISD drops from 18 to 17.5. This assumes that there is no structural change in the volatility curve and that the price movement is largely temporary. In a second scenario, called **sticky moneyness**, the curve shifts to the right and the ISD stays at 18 (the moneyness is not changed and is still 100%). This assumes a permanent shift in the volatility curve. Based on these assumptions, the trader can then examine the return on different option trading strategies and choose the most appropriate one. More generally, this demonstrates that the implied volatility is a major risk factor when trading options.

Finally, the ISD of a portfolio of assets can be related to the ISD of its components through the **implied correlation.** Normally, the portfolio variance is related to the individual volatilities using

$$\sigma_p^2 = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + 2 \sum_{i=1}^{N} \sum_{j<i}^{N} w_i w_j \rho_{ij} \sigma_i \sigma_j \tag{6.24}$$

Assume now that there is a constant correlation ρ across all pairs of assets that maintains the portfolio variance

$$\sigma_p^2 = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + 2 \sum_{i=1}^{N} \sum_{j<i}^{N} w_i w_j (\rho) \sigma_i \sigma_j \tag{6.25}$$

This correlation is a weighted average of the pairwise correlations ρ_{ij}. With option ISDs measured for the portfolio and all the constituents, we can use Equation (6.25) to infer the portfolio implied correlation. This implied correlation is a summary measure of diversification benefits across the portfolio. All else equal, an increasing correlation increases the total portfolio risk.

EXAMPLE 6.12: FRM EXAM 2005—QUESTION 40

Which of the following statements is *wrong*?

a. The Black–Scholes formula holds only in a risk-neutral world.
b. The futures price of a stock depends on the risk-free rate.
c. An American put option is generally priced higher than a similar European put option.
d. Binomial trees can be used to price American options.

EXAMPLE 6.13: FRM EXAM 2001—QUESTION 91

Using the Black–Scholes model, calculate the value of a European call option given the following information: Spot rate = 100; Strike price = 110; Risk-free rate = 10%; Time to expiry = 0.5 years; $N(d_1) = 0.457185$; $N(d_2) = 0.374163$.

a. \$10.90
b. \$9.51
c. \$6.57
d. \$4.92

EXAMPLE 6.14: PROBABILITY OF EXERCISE

In the Black–Scholes expression for a European call option the term used to compute option probability of exercise is

a. d_1
b. d_2
c. $N(d_1)$
d. $N(d_2)$

6.4 OTHER OPTION CONTRACTS

The options described so far are standard, plain-vanilla options. Many other types of options, however, have been developed.

Binary options, also called **digital options** pay a fixed amount, say Q, if the asset price ends up above the strike price

$$c_T = Q \times I(S_T - K) \tag{6.26}$$

where $I(x)$ is an indicator variable that takes the value of 1 if $x \geq 0$ and 0 otherwise. The payoff function is illustrated in Figure 6.12 when $K = \$100$.

Because the probability of ending in-the-money in a risk-neutral world is $N(d_2)$, the initial value of this option is simply

$$c = Qe^{-r\tau} N(d_2) \tag{6.27}$$

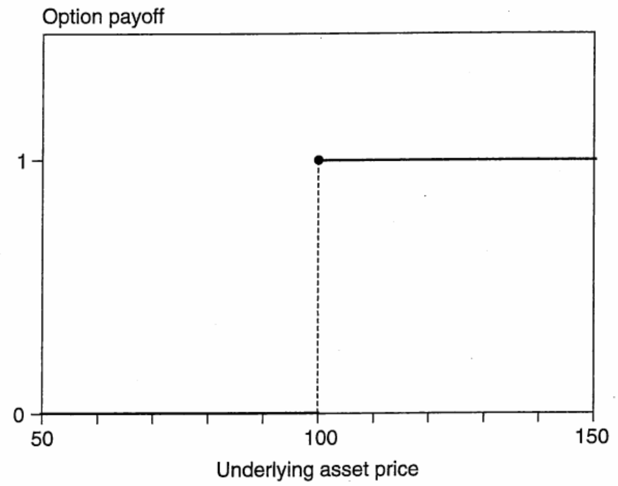


FIGURE 6.12 Payoff on a Binary Option

These options involve a sharp discontinuity around the strike price. Just below K, their value is zero. Just above, the value is the notional Q. Due to this discontinuity, these options are very difficult to hedge.

Another important class of options are barrier options. **Barrier options** are options where the payoff depends on the value of the asset hitting a barrier during a certain period of time. A **knock-out option** disappears if the price hits a certain barrier. A **knock-in option** comes into existence when the price hits a certain barrier.

An example of a knock-out option is the **down-and-out call**. This disappears if S hits a specified level H during its life. In this case, the knock-out price H must be lower than the initial price S_0. The option that appears at H is the **down-and-in call**. With identical parameters, the two options are perfectly complementary. When one disappears, the other appears. As a result, these two options must add up to a regular call option. Similarly, an **up-and-out call** ceases to exist when S reaches $H > S_0$. The complementary option is the **up-and-in call**.

Figure 6.13 compares price paths for the four possible combinations of calls. In all figures, the dark line describes the relevant price path, during which the option is alive; the grey line describes the remaining path.

The graphs illustrates that the down-and-out call and down-and-in call add up to the regular price path of a regular European call option. Thus at initiation, the value of these two options must add up to

$$c = c_{DO} + c_{DI} \tag{6.28}$$

Because all these values are positive (or at worst zero), the value of each premium c_{DO} and c_{DI} must be no greater than that of c. A similar reasoning applies to the two options in the right panels. Sometimes the option offers a **rebate** if it is knocked out.

Similar combinations exist for put options. An **up-and-out put** ceases to exist when S reaches $H > S_0$. A **down-and-out put** ceases to exist when S reaches $H < S_0$. The only difference with Figure 6.13 is that the option is exercised at maturity if $S < K$.

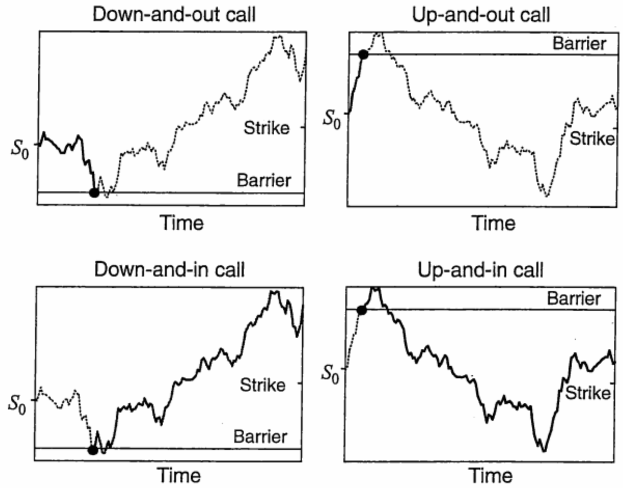


FIGURE 6.13 Paths for Knock-Out and Knock-In Call Options

Barrier options are attractive because they are "cheaper" than the equivalent European option. This, of course, reflects the fact that they are less likely to be exercised than other options.

In addition, these options are difficult to hedge because a discontinuity arises as the spot price get closer to the barrier. Just above the barrier, the option has positive value. For a very small movement in the asset price, going below the barrier, this value disappears.

Another widely used class of options are Asian options. **Asian options**, or **average rate options**, generate payoffs that depend on the average value of the underlying spot price during the life of the option, instead of the ending value. Define this as $S_{\mathrm{AVE}}(t, T)$. The final payoff for a call is

$$c_T = \mathrm{Max}(S_{\mathrm{AVE}}(t, T) - K, 0) \tag{6.29}$$

Because an average is less variable than the final value at the end of the same period, such options are "cheaper" than regular options due to lower volatility. In fact, the price of the option can be treated like that of an ordinary option with the volatility set equal to $\sigma/\sqrt{3}$ and an adjustment to the dividend yield.[3] As a result of the averaging process, such options are easier to hedge than ordinary options.

Chooser options allow the holder to choose whether the option is a call or a put. At that point in time, the value of the option is

$$f_t = \mathrm{Max}(c_t, p_t) \tag{6.30}$$

Thus, it is a package of two options, a regular call plus an option to convert to a put. As a result, these options are more expensive than plain-vanilla options.

[3] This is only strictly true when the averaging is a geometric average. In practice, average options involve an arithmetic average, for which there is no analytic solution; the lower volatility adjustment is just an approximation.

Finally, **lookback options** have payoffs that depend on the extreme values of S over the option's life. Define S_{MAX} as the maximum and S_{MIN} as the minimum. A fixed strike lookback call option pays $\text{Max}(S_{MAX} - K, 0)$. A floating strike lookback call option pays $\text{Max}(S_T - S_{MIN}, 0)$. These options are even more expensive than regular options.

EXAMPLE 6.15: FRM EXAM 2003—QUESTION 34

Which of the following options is strongly path-dependent?

a. An Asian option
b. A binary option
c. An American option
d. A European call option

EXAMPLE 6.16: FRM EXAM 2006—QUESTION 59

All else being equal, which of the following options would cost more than plain-vanilla options that are currently at-the-money?

I. Lookback options
II. Barrier options
III. Asian options
IV. Chooser option

a. I only
b. I and IV
c. II and III
d. I, III, and IV

EXAMPLE 6.17: FRM EXAM 2002—QUESTION 19

Of the following options, which one does *not* benefit from an increase in the stock price when the current stock price is \$100 and the barrier has not yet been crossed:

a. A down-and-out call with out barrier at \$90 and strike at \$110
b. A down-and-in call with in barrier at \$90 and strike at \$110
c. An up-and-in put with barrier at \$110 and strike at \$100
d. An up-and-in call with barrier at \$110 and strike at \$100

6.5 VALUING OPTIONS BY NUMERICAL METHODS

Some options have analytical solutions, such as the Black–Scholes models for European vanilla options. For more general options, however, we need to use numerical methods.

The basic valuation formula for derivatives is Equation (6.13), which states that the current value is the discounted present value of expected cash flows, where all assets grow at the risk-free rate and are discounted at the same risk-free rate.

We can use the Monte Carlo simulation methods presented in Chapter 4 to generate sample paths, final option values, and discount them into the present. Such simulation methods can be used for European or even path-dependent options, such as Asian options.

Table 6.2 gives an example. Suppose we need to price a European call with parameters $S = 100, K = 100, T = 1, r = 5\%, r^* = 0, \sigma = 20\%$. We set up the simulation with, for instance, $n = 100$ steps over the horizon of one year. For each step, the trend is $r/n = 0.05/100$; the volatility is $\sigma/\sqrt{(n)} = 0.20/\sqrt{100}$. Each replication starts from a price of \$100 until the horizon. For instance, the first replication gives a final price of $S_T = \$114.06$. The option is in-the-money and is worth $c_T = \$14.06$. We then discount this number into the present and get \$13.37. In the second replication, $S_T = \$75.83$ and the option expires worthless $c_T = 0$. Averaging across the K replications gives an average of \$10.33 in this case. The result is close to the actual Black–Scholes model price of \$10.45, obtained with Equation (6.14). The simulation, however, is much more general. The payoff at expiration could be a complicated function of the final price or even its intermediate values.

Simulation methods, however, cannot account for the possibility of early exercise. Instead, binomial trees must be used to value American options. As explained previously, the method consists of chopping up the time horizon into n intervals Δt and setting up the tree so that the characteristics of price movements fit the lognormal distribution.

At each node, the initial price S can go up to uS with probability p or down to dS with probability $(1 - p)$. The parameters u, d, p are chosen so that, for a small time interval, the expected return and variance equal those of the continuous

TABLE 6.2 Example of Simulation for a European Call Option

Replication	Final Payoff S_T	Final Payoff c_T	Discounted Value
1	114.06	14.06	13.37
2	75.83	0.00	0.00
3	108.76	8.76	8.33
...			
Average			10.33

process. One could choose, for instance,

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = (1/u), \quad p = \frac{e^{\mu\Delta t} - d}{u - d} \tag{6.31}$$

Since this is a risk-neutral process, the total expected return must be equal to the risk-free rate r. Allowing for an income payment of r^*, this gives $\mu = r - r^*$.

The tree is built starting from the current time to maturity, from the left to the right. Next, the derivative is valued by starting at the end of the tree, and working backward to the initial time, from the right to the left.

Consider first a European call option. At time T (maturity) and node j, the call option is worth $\text{Max}(S_{Tj} - K, 0)$. At time $T - 1$ and node j, the call option is the discounted expected value of the option at time T and nodes j and $j + 1$:

$$c_{T-1,j} = e^{-r\Delta t}[pc_{T,j+1} + (1 - p)c_{T,j}] \tag{6.32}$$

We then work backward through the tree until the current time.

For American options, the procedure is slightly different. At each point in time, the holder compares the value of the option *alive* and *dead* (i.e., exercised). The American call option value at node $T - 1$, j is

$$C_{T-1,j} = \text{Max}[(S_{T-1,j} - K), c_{T-1,j}] \tag{6.33}$$

Example: Computing an American Option Value

Consider an at-the-money call on a foreign currency with a spot price of \$100, a strike price of $K = \$100$, and a maturity of six months. The annualized volatility is $\sigma = 20\%$. The domestic interest rate is $r = 5\%$; the foreign rate is $r^* = 8\%$. Note that we require an income payment for the American feature to be valuable. If $r^* = 0$, we know that the American option is worth the same as a European option, which can be priced with the Black–Scholes model. There would be no point in using a numerical method.

First, we divide the period into four intervals, for instance, so that $\Delta t = 0.50/4 = 0.125$. The discounting factor over one interval is $e^{-r\Delta t} = 0.9938$. We then compute:

$$u = e^{\sigma\sqrt{\Delta t}} = e^{0.20\sqrt{0.125}} = 1.0733$$

$$d = (1/u) = 0.9317$$

$$a = e^{(r-r^*)\Delta t} = e^{(-0.03)0.125} = 0.9963$$

$$p = \frac{a - d}{u - d} = (0.9963 - 0.9317)/(1.0733 - 0.9317) = 0.4559$$

The procedure for pricing the option is detailed in Table 6.3. First, we lay out the tree for the spot price, starting with $S = 100$ at time $t = 0$, then $uS = 107.33$ and $dS = 93.17$ at time $t = 1$, and so on.

This allows us to value the European call. We start from the end, at time $t = 4$, and set the call price to $c = S - K = 132.69 - 100.00 = 32.69$ for the highest spot price, 15.19 for the next price and so on, down to $c = 0$ if the spot price is below $K = 100.00$. At the previous step and highest node, the value of the call is

$$c = 0.9938[0.4559 \times 32.69 + (1 - 0.4559) \times 15.19] = 23.02$$

Continuing through the tree to time 0 yields a European call value of \$4.43. The Black–Scholes formula gives an exact value of \$4.76. Note how close the binomial approximation is, with just four steps. A finer partition would quickly improve the approximation.

Next, we examine the American call. At time $t = 4$, the values are the same as above since the call expires. At time $t = 3$ and node $j = 4$, the option holder can either keep the call, in which case the value is still \$23.02, or exercise. When exercised, the option payoff is $S - K = 123.63 - 100.00 = 23.63$. Since this is greater than the value of the option alive, the holder should optimally exercise the option. We replace the European option value by \$23.63 at that node. Continuing through the tree in the same fashion, we find a starting value of \$4.74. The value of the American call is slightly greater than the European call price, as expected.

TABLE 6.3 Computation of American Option Value

	0	1	2	3	4
Spot Price S_t	→	→	→	→	→
					132.69
				123.63	115.19
			115.19	107.33	100.00
		107.33	100.00	93.17	86.81
	100.00	93.17	86.81	80.89	75.36
European Call c_t	←	←	←	←	←
					32.69
				23.02	15.19
			14.15	6.88	0.00
		8.10	3.12	0.00	0.00
	4.43	1.41	0.00	0.00	0.00
Exercised Call $S_t - K$					32.69
				23.63	15.19
			15.19	7.33	0.00
		7.33	0.00	0.00	0.00
	0.00	0.00	0.00	0.00	0.00
American Call C_t	←	←	←	←	←
					32.69
				23.63	15.19
			15.19	7.33	0.00
		8.68	3.32	0.00	0.00
	4.74	1.50	0.00	0.00	0.00

EXAMPLE 6.18: FRM EXAM 2006—QUESTION 86

Which of the following statements about American options is *incorrect*?

a. American options can be exercised at any time until maturity.
b. American options are always worth at least as much as European options.
c. American options can easily be valued with Monte Carlo simulation.
d. American options can be valued with binomial trees.

6.6 IMPORTANT FORMULAS

Payoff on a long call and put: $C_T = \text{Max}(S_T - K, 0)$, $P_T = \text{Max}(K - S_T, 0)$

Put–call parity: $c - p = Se^{-r^*\tau} - Ke^{-r\tau} = (F - K)e^{-r\tau}$

Bounds on call value (no dividends): $c_t \leq C_t \leq S_t$, $c_t \geq S_t - Ke^{-r\tau}$

Bounds on put value (no dividends): $p_t \leq P_t \leq K$, $p_t \geq Ke^{-r\tau} - S_t$

Geometric Brownian motion: $\ln(S_T) = \ln(S_0) + (\mu - \sigma^2/2)\tau + \sigma\sqrt{\tau}\,\epsilon$

Risk-neutral discounting formula: $f_t = E_{\text{RN}}[e^{-r\tau}F(S_T)]$

Black–Scholes call option pricing: $c = SN(d_1) - Ke^{-r\tau}N(d_2)$,

$$d_1 = \frac{\ln(S/Ke^{-r\tau})}{\sigma\sqrt{\tau}} + \frac{\sigma\sqrt{\tau}}{2},\quad d_2 = d_1 - \sigma\sqrt{\tau}$$

Black–Scholes put option pricing: $p = S[N(d_1) - 1] - Ke^{-r\tau}[N(d_2) - 1]$

Black–Scholes pricing with dividend, Garman-Kohlhagen model:

$$c = Se^{-r^*\tau}N(d_1) - Ke^{-r\tau}N(d_2)$$

Black model, option on futures: $c = [FN(d_1) - KN(d_2)]e^{-r\tau}$

Margrabe model: Replace S by S_A, K by S_B, set $\sigma^2_{AB} = \sigma^2_A + \sigma^2_B - 2\rho_{AB}\sigma_A\sigma_B$

Pricing of binary option: $c = Qe^{-r\tau}N(d_2)$

Asian option: $c_T = \text{Max}(S_{\text{AVE}}(t, T) - K, 0)$

Binomial process: $u = e^{\sigma\sqrt{\Delta t}}$, $d = (1/u)$, $p = \frac{e^{\mu\Delta t} - d}{u - d}$

6.7 ANSWERS TO CHAPTER EXAMPLES

Example 6.1: FRM Exam 2007—Question 84

c. Buying a put creates a gain if the stock price falls, which is similar to selling the stock on the downside. On the upside, the loss is capped by buying a call.

Example 6.2: FRM Exam 2005—Question 72

c. By put–call parity, $c = p + Se^{-r^*\tau} - Ke^{-r\tau} = 3.19 + 23 - 25\exp(-0.05 \times 1) = 3.19 - 0.78 = 2.409$. Note that the volatility information is not useful.

Example 6.3: FRM Exam 2002—Question 25

d. Put–call parity applies to these European options. With no dividend, the relationship is $c - p = S - K\exp(-r\tau)$. The first term is $c - p = \$4 - \$1.47 = \$2.53$. The second term is $S - K\ exp(-r\tau) = \$20 - \$18\exp[-6\%(6/12)] = \$2.53$. Because the two numbers are the same, there is no arbitrage opportunity.

Example 6.4: FRM Exam 2006—Question 74

c. By put–call parity, $c - p = Se^{-r^*\tau} - Ke^{-r\tau}$. Therefore, $Se^{-r^*\tau} = (c - p + Ke^{-r\tau}) = (10 - 15 + 90\exp(0.05 \times 5) = 65.09$. The dividend yield is then $y = -(1/T)\ln(65.09/85) = 5.337\%$.

Example 6.5: FRM Exam 2001—Question 90

c. Long positions in options can lose at worst the premium, so b. is wrong. Spreads involve long and short positions in options and have limited downside loss, so a. is wrong. Writing options exposes the seller to very large losses. In the case of puts, the worst loss is the strike price K, if the asset price goes to zero. In the case of calls, however, the worst loss is in theory unlimited because there is a small probability of a huge increase in S. Between c. and d., c. is the better answer.

Example 6.6: FRM Exam 2007—Question 103

d. This position is graphed in Figure 6.6. It benefits from an increase in the price between 40 and 45, so it is a bull spread. The worst loss occurs below $K_1 = 40$, when none of the options is exercised and the net lost premium is $5 - 3 = 2$. The maximum profit occurs above $K_2 = 45$, when the two options are exercised, for a net profit of \$5 minus the lost premium, which gives \$3.

Example 6.7: FRM Exam 2006—Question 45

c. The manager is long a portfolio, which is protected by buying a put with low strike price and selling a call with higher strike price. This locks in a range of profits and losses and is a collar. If the strike prices were the same, the hedge would be perfect.

Example 6.8: FRM Exam 2002—Question 42

b. Because the final price is below the lowest of the three strike prices, all the puts will be exercised. The final payoff is $(\$50 - \$33) - 2(\$42 - \$33) + (\$37 - \$33) = \$17 - \$18 + \$4 = \$3$. From this, we have to deduct the up-front cost, which is $-\$7 + 2(\$4) - \$2 = -\$1$. The total profit is then, ignoring the time value of money, $\$3 - \$1 = \$2$ per share.

Example 6.9: FRM Exam 2003—Question 72

b. A strangle involves two different strike prices, so I. is incorrect. A long bull spread involves buying a call and selling a call with $K_1 < K_2$; the short position is inverted, so that II. is correct. Options with different maturities are called horizontal spreads, so answer III. is incorrect. A long butterfly spread indeed involves options with three strike prices, so IV. is correct. Hence, I. and III. are incorrect, and answer b. is the (correct) solution.

Example 6.10: FRM Exam 2002—Question 50

b. When there is no dividend, there is never any reason to exercise an American call early. Instead, the option should be sold to another party.

Example 6.11: FRM Exam 2005—Question 15

d. The upper bound is $S = 50$. The lower bound is $c \geq S - Ke^{-r\tau} = 50 - 45\exp(-0.03 \times 2) = 42.379$.

Example 6.12: FRM Exam 2005—Question 40

a. The BS formula relies on a method called risk-neutral valuation but does apply to the real world. Otherwise, it would be useless.

Example 6.13: FRM Exam 2001—Question 91

c. We use Equation (6.14) assuming there is no income payment on the asset. This gives $c = SN(d_1) - K\exp(-r\tau)N(d_2) = 100 \times 0.457185 - 110\exp(-0.1 \times 0.5) \times 0.374163 = \$6.568$.

Example 6.14: Probability of Exercise

d. This is the term multiplying the present value of the strike price, by Equation (6.17).

Example 6.15: FRM Exam 2003—Question 34

a. The payoff of an Asian option depends on the average value of S and therefore is path-dependent.

Example 6.16: FRM Exam 2006—Question 59

b. Lookback options use the maximum stock price over the period, which must be more than the value at the end. Hence, they must be more valuable than regular European options. Chooser options involve an additional choice during the life of the option, and as a result are more valuable than regular options. Asian options involve the average, which is less volatile than the final price, so must be less expensive than regular options. Finally, barrier options can be structured so that the sum of two barrier options is equal to a regular option. Because each premium is positive, a barrier option must be less valuable than regular options.

Example 6.17: FRM Exam 2002—Question 19

b. A down-and-out call where the barrier has not been touched is still alive and hence benefits from an increase in S, so a. is incorrect. A down-and-in call only comes alive when the barrier is touched, so an increase in S brings it away from the barrier. This is not favorable, so b. is correct. An up-and-in put would benefit from an increase in S as this brings it closer to the barrier of \$110, so c. is not correct. Finally, an up-and-in call would also benefit if S gets closer to the barrier.

Example 6.18: FRM Exam 2006—Question 86

c. This statement is incorrect because Monte Carlo simulations are strictly backward-looking, and cannot take into account optimal future exercise, which a binomial tree can do.

CHAPTER 7

Fixed-Income Securities

The next two chapters provide an overview of fixed-income markets, securities, and their derivatives. Originally, **fixed-income securities** referred to bonds that promise to make fixed coupon payments. Over time, this narrow definition has evolved to include any security that obligates the borrower to make specific payments to the bondholder on specified dates. Thus, a **bond** is a security that is issued in connection with a borrowing arrangement. In exchange for receiving cash, the borrower becomes obligated to make a series of payments to the bondholder.

Fixed-income derivatives are instruments whose value derives from some bond price, interest rate, or other bond market variable. Due to their complexity, these instruments are analyzed in the next chapter.

Section 7.1 provides an overview of the different segments of the bond market. Section 7.2 then introduces the various types of fixed-income securities. Section 7.3 reviews the basic tools for analyzing fixed-income securities, including the determination of cash flows, the measurement of duration, and the term structure of interest rates and forward rates. Because of their importance, mortgage-backed securities (MBSs) are analyzed in great detail. MBSs are an example of **securitization**, which is the process by which assets are pooled and securities representing interests in the pool are issued. This topic is covered in Section 7.4. Section 7.5 then discusses MBSs and collateralized mortgage obligations (CMOs). These new structures illustrate the creativity of financial engineering.

7.1 OVERVIEW OF DEBT MARKETS

Fixed-Income markets are truly global. To help sort the various categories of the bond markets, Table 7.1 provides a broad classification of bonds by borrower and currency types. Bonds issued by resident entities and denominated in the domestic currency are called **domestic bonds.** In contrast, **foreign bonds** are those floated by a foreign issuer in the domestic currency and subject to domestic country regulations (e.g., by the government of Sweden in dollars in the United States). **Eurobonds** are mainly placed outside the country of the currency in which they are denominated and are sold by an international syndicate of financial institutions (e.g., a dollar-denominated bond issued by IBM and marketed in London).[1]

[1] These should not be confused with bonds denominated in the euro, which can be of any type.

TABLE 7.1 Classification of Bond Markets

	By resident	By non-resident
In domestic currency	Domestic Bond	Foreign Bond
In foreign currency	Eurobond	Eurobond

Foreign bonds and Eurobonds constitute the **international bond market**. Finally, **global bonds** are placed at the same time in the Eurobond and one or more domestic markets with securities fungible between these markets.

The domestic bond market can be further decomposed into these categories:

- **Government bonds**, issued by central governments, or also called **sovereign bonds** (e.g., by the United States in dollars)
- **Government agency and guaranteed bonds**, issued by agencies or guaranteed by the central government (e.g., by Fannie Mae, a U.S. government agency), which are public financial institutions
- **State and local bonds**, issued by local governments, other than the central government, also known as **municipal bonds** (e.g., by the state of California)
- Bonds issued by private **financial institutions**, including banks, insurance companies, or issuers of asset-backed securities (e.g., by Citibank in the U.S. market)
- **Corporate bonds**, issued by private nonfinancial corporations, including industrials and utilities (e.g., by IBM in the U.S. market)

Table 7.2 breaks down the world debt securities market, which was worth \$80 trillion at the end of 2007. This includes the **bond markets**, traditionally defined as fixed-income securities with remaining maturities beyond one year, and the shorter-term **money markets**, with maturities below one year. The table includes all publicly tradable debt securities sorted by country of issuer and issuer type.

The table shows that U.S. entities have issued a total of \$24.4 trillion in domestic debt and \$4.8 trillion in international debt, for a total amount of \$29.2 trillion, by far the biggest debt securities market. Next comes the Eurozone market, with a size of \$20.0 trillion, and the Japanese market, with \$9.0 trillion.

As Table 7.2 shows, the largest sector is for domestic government debt. This sector includes sovereign debt issued by emerging countries in their own currencies, e.g., Mexican peso-denominated debt issued by the Mexican government. Few of these markets have long-term issues, because of their history of high inflation, which renders long-term bonds very risky. In Mexico, for instance, the market consists mainly of **Cetes**, which are peso-denominated, short-term Treasury bills.

Among international debt, the emerging market sector also includes debt denominated in United States dollars, such as **Brady bonds**, which are sovereign bonds issued in exchange for bank loans, and the **Tesebonos**, which are dollar-denominated bills issued by the Mexican government. Brady bonds are hybrid

TABLE 7.2 Global Debt Securities Markets - 2007 (Billions of U.S. dollars)

Country of Issuer	Domestic	Type: Gov't	Type: Financials	Type: Corporates	Int'l	Total
United States	24,430	6,592	14,907	2,931	4,770	29,200
Japan	8,856	7,145	983	728	165	9,021
Germany	2,631	1,393	1,048	190	2,193	4,824
Italy	3,044	1,772	934	338	929	3,973
France	2,849	1,405	1,123	321	1,414	4,263
United Kingdom	1,359	903	433	23	3,160	4,519
Canada	1,144	734	277	133	433	1,577
Spain	1,641	496	600	545	1,224	2,865
Netherlands	993	397	519	77	1,633	2,626
South Korea	1,077	466	380	231	106	1,183
Belgium	550	388	113	49	207	757
China	1,687	1,137	446	104	21	1,708
Denmark	586	84	469	33	111	697
Australia	695	117	534	44	483	1,178
Brazil	953	694	251	8	82	1,035
Sweden	389	145	207	37	248	637
Switzerland	243	116	111	16	26	269
Austria	252	120	102	30	352	604
Eurozone	11,960	5,971	4,439	1,550	7,952	19,912
Subtotal	53,379	24,104	23,437	5,838	17,557	70,936
Others	3,794	2,662	650	483	5,157	8,951
Total	57,173	26,766	24,087	6,321	22,714	79,887

Source: Bank for International Settlements

securities whose principal is collateralized by U.S. Treasury zero-coupon bonds. As a result, there is no risk of default on the principal, unlike on coupon payments.

The domestic financial market is also important, especially for mortgage-backed securities. **Mortgage-backed securities** (MBSs), are securities issued in conjunction with mortgage loans, which are loans secured by the collateral of a specific real estate property. Payments on MBSs are repackaged cash flows supported by mortgage payments made by property owners. MBSs can be issued by government agencies as well as by private financial corporations. More generally, **asset-backed securities** (ABSs) are securities whose cash flows are supported by assets such as credit card receivables or car loan payments.

Finally, the remainder of the domestic market represents bonds raised by private, nonfinancial corporations. This sector, by far the largest in the United States, is growing rather quickly as corporations recognize that bond issuances are a lower-cost source of funds than bank debt. The advent of the common currency, the euro, is also leading to a growing, more liquid and efficient, corporate bond market in Europe.

7.2 FIXED-INCOME SECURITIES

7.2.1 Instrument Types

Bonds pay interest on a regular basis, semiannual for U.S. Treasury and corporate bonds, annual for others such as Eurobonds, or quarterly for others. The most common types of bonds are

- **Fixed-coupon bonds**, which pay a fixed percentage of the principal every period and the principal as a **balloon**, one-time payment at maturity
- **Zero-coupon bonds**, which pay no coupons but only the principal; their return is derived from price appreciation only
- **Annuities**, which pay a constant amount over time which includes interest plus amortization, or gradual repayment, of the principal
- **Perpetual bonds** or **consols**, which have no set redemption date and whose value derives from interest payments only
- **Floating-coupon bonds**, which pay interest equal to a reference rate plus a margin, reset on a regular basis; these are usually called **floating-rate notes** (FRNs)
- **Structured notes**, which have more complex coupon patterns to satisfy the investor's needs
- **Inflation-protected notes**, whose principal is indexed to the **Consumer Price Index** (CPI), hence providing protection against an increasing rate of inflation[2]

There are many variations on these themes. For instance, **step-up bonds** have fixed coupons that start at a low rate and increase over time.

It is useful to consider floating-rate notes in more detail. Take for instance a 10-year \$100 million FRN paying semiannually six-month LIBOR in arrears.[3] Here, **LIBOR** is the London Interbank Offer Rate, a benchmark cost of borrowing for highly-rated (AA) credits. Every semester, on the **reset date**, the value of six-month LIBOR is recorded. Say LIBOR is initially at 6%. At the next coupon date, the payment will be $(\frac{1}{2}) \times \$100 \times 6\% = \$3$ million. Simultaneously, we record a new value for LIBOR, say 8%. The next payment will then increase to \$4 million, and so on. At maturity, the issuer pays the last coupon plus the principal. Like a cork at the end of a fishing line, the coupon payment "floats" with the current interest rate.

[2] In the United States, these government bonds are called **Treasury inflation-protected securities** (TIPS). The coupon payment is fixed in real terms, say 3%. If after six months, the cumulative inflation is 2%, the principal value of the bond increases from \$100 to $\$100 \times (1+2\%) = \$102$. The first semiannual coupon payment is then $(3\%/2) \times \$102 = \$1.53$.

[3] Note that the index could be defined differently. The floating payment could be tied to a Treasury rate, or LIBOR with a different maturity—say 3-month LIBOR. The pricing of the FRN will depend on the index. Also, the coupon will typically be set to LIBOR plus some spread that depends on the creditworthiness of the issuer.

APPLICATION: LIBOR AND OTHER BENCHMARK INTEREST RATES

LIBOR, the London Interbank Offer Rate, is a reference rate based on interest rates at which banks borrow unsecured funds from each other in the London interbank market.

LIBOR is published daily by the **British Bankers' Association** (BBA) around 11:45A.M., London Time, and is computed from an average of the distribution of rates provided by reporting banks. LIBID, the London Interbank Bid Rate, represents the average deposit rate.

LIBOR is calculated for 10 different currencies and various maturities, from overnight to one year. LIBOR rates are the most widely used reference rates for short-term futures contracts, such as the Eurodollar futures.

For the euro, however, **EURIBOR**, or Euro Interbank Offer Rate, is most often used. It is sponsored by the European Banking Federations (EBF) and published by Reuters at 11A.M., Central European Time (CET). In addition, **EONIA** (Euro Overnight Index Average) is an overnight unsecured lending rate that is published every day before 7P.M., CET. The same panel of banks contributes to EURIBOR and EONIA. The equivalent for sterling is **SONIA** (Sterling Overnight Index Average).

Among structured notes, we should mention **inverse floaters**, also known as reverse floaters, which have coupon payments that vary inversely with the level of interest rates. A typical formula for the coupon is $c = 12\% - \text{LIBOR}$, if positive, payable semiannually. Assume the principal is \$100 million. If LIBOR starts at 6%, the first coupon will be $(1/2) \times \$100 \times (12\% - 6\%) = \$3$ million. If after six months LIBOR moves to 8%, the second coupon will be $(1/2) \times \$100 \times (12\% - 8\%) = \$2$ million. The coupon will go to zero if LIBOR moves above 12%. Conversely, the coupon will increase if LIBOR drops. Hence, inverse floaters do best in a falling interest rate environment.

Bonds can also be issued with option features. The most important are

- **Callable bonds,** where the issuer has the right to "call" back the bond at fixed prices on fixed dates, the purpose being to call back the bond when the cost of issuing new debt is lower than the current coupon paid on the bond
- **Puttable bonds,** where the investor has the right to "put" the bond back to the issuer at fixed prices on fixed dates, the purpose being to dispose of the bond should its price deteriorate
- **Convertible bonds,** where the bond can be converted into the common stock of the issuing company at a fixed price on a fixed date, the purpose being to partake in the good fortunes of the company (these will be covered in the chapter on equities)

The key to analyzing these bonds is to identify and price the option feature. For instance, a callable bond can be decomposed into a long position in a straight bond minus a call option on the bond price. The call feature is unfavorable for investors who require a lower price to purchase the bond, thereby increasing its yield. Conversely, a put feature will make the bond more attractive, increasing its price and lowering its yield. Similarly, the convertible feature allows companies to issue bonds at a lower yield than otherwise.

EXAMPLE 7.1: FRM EXAM 2003—QUESTION 95

With any other factors remaining unchanged, which of the following statements regarding bonds is *not* valid?

- **a.** The price of a callable bond increases when interest rates increase.
- **b.** Issuance of a callable bond is equivalent to a short position in a straight bond plus a long call option on the bond price.
- **c.** The put feature in a puttable bond lowers its yield compared with the yield of an equivalent straight bond.
- **d.** The price of an inverse floater decreases as interest rates increase.

7.2.2 Methods of Quotation

Most bonds are quoted on a **clean price** basis, that is, without accounting for the accrued income from the last coupon. For U.S. bonds, this clean price is expressed as a percent of the face value of the bond with fractions in thirty-seconds, for instance as 104-12, which means $104 + 12/32$, for the 6.25% May 2030 Treasury bond. Transactions are expressed in number of units, e.g., \$20 million face value.

Actual payments, however, must account for the accrual of interest. This is factored into the **gross price**, also known as the **dirty price**, which is equal to the clean price plus accrued interest. In the U.S. Treasury market, accrued interest (AI) is computed on an *actual/actual* basis:

$$AI = \text{Coupon} \times \frac{\text{Actual number of days since last coupon}}{\text{Actual number of days between last and next coupon}} \tag{7.1}$$

The fraction involves the actual number of days in both the numerator and denominator. For instance, say the 6.25% of May 2030 paid the last coupon on November 15 and will pay the next coupon on May 15. The denominator is, counting the number of days in each month, $15 + 31 + 31 + 29 + 31 + 30 + 15 = 182$. If the trade settles on April 26, there are $15 + 31 + 31 + 29 + 31 + 26 = 163$

days into the period. The accrued is computed from the \$3.125 coupon as

$$\$3.125 \times \frac{163}{182} = \$2.798763$$

The total, gross price for this transaction is:

$$(\$20,000,000/100) \times [(104 + 12/32) + 2.798763] = \$21,434,753$$

Different markets have different day count conventions. A 30/360 convention, for example, considers that all months count for 30 days exactly. The computation of the accrued interest is tedious but must be performed precisely to settle the trades.

We should note that the accrued interest in the **LIBOR** market is based on *actual/360*. For instance, the interest accrued on a 6% \$1 million loan over 92 days is

$$\$1,000,000 \times 0.06 \times \frac{92}{360} = \$15,333.33$$

Another notable pricing convention is the discount basis for Treasury bills. These bills are quoted in terms of an annualized discount rate (DR) to the face value, defined as

$$\mathrm{DR} = (\mathrm{Face} - \mathrm{P})/\mathrm{Face} \times (360/t) \tag{7.2}$$

where P is the price and t is the actual number of days. The dollar price can be recovered from

$$P = \mathrm{Face} \times [1 - \mathrm{DR} \times (t/360)] \tag{7.3}$$

For instance, a bill quoted at a 5.19% discount with 91 days to maturity could be purchased for

$$\$100 \times [1 - 5.19\% \times (91/360)] = \$98.6881$$

This price can be transformed into a conventional yield to maturity, using

$$F/P = (1 + y \times t/365) \tag{7.4}$$

which gives 5.33% in this case. Note that the yield is greater than the discount rate because it is a rate of return based on the initial price. Because the price is lower than the face value, the yield must be greater than the discount rate.

7.3 ANALYSIS OF FIXED-INCOME SECURITIES

7.3.1 The NPV Approach

Fixed-income securities can be valued by, first, laying out their cash flows and, second, computing their net present value (NPV) using the appropriate discount rate. Let us write the market value of a bond P as the present value of future cash flows:

$$P = \sum_{t=1}^{T} \frac{C_t}{(1+y)^t} \tag{7.5}$$

where: C_t = the cash flow (coupon and/or principal repayment) in period t
t = the number of periods (e.g., half-years) to each payment
T = the number of periods to final maturity
y = the yield to maturity for this particular bond
P = the price of the bond, including accrued interest

For a fixed-rate bond with face value F, the cash flow C_t is cF each period, where c is the coupon rate, plus F upon maturity. Other cash flow patterns are possible, however. Figure 7.1 illustrates the time profile of the cash flows C_t for three bonds with initial market value of \$100, 10-year maturity and 6% annual interest. The figure describes a straight coupon-paying bond, an annuity, and a zero-coupon bond. As long as the cash flows are predetermined, the valuation is straightforward.

Given the market price, solving for y gives the yield to maturity. This yield is another way to express the price of the bond and is more convenient to compare various bonds. The yield is also the *expected* rate of return on the bond, provided all coupons are reinvested at the same rate. This interpretation fails, however, when the cash flows are random or when the life of the bond can change due to optionlike features.

7.3.2 Pricing

We can also use information from the fixed-income market to assess the fair value of the bond. Say we observe that the yield to maturity for comparable bonds is y_T. We can then discount the cash flows using the same, market-determined yield. This gives a fair value for the bond:

$$\hat{P} = \sum_{t=1}^{T} \frac{C_t}{(1+y_T)^t} \tag{7.6}$$

Note that the discount rate y_T does not depend on t, but is fixed for all payments for this bond.

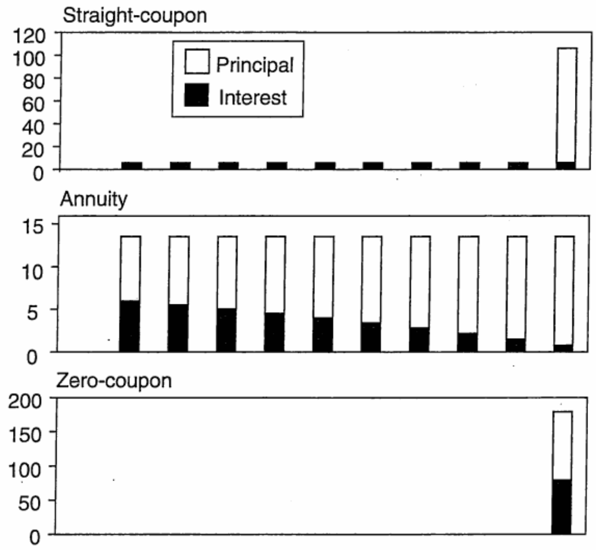


FIGURE 7.1 Time Profile of Cash Flows

This approach, however, ignores the shape of the term structure of interest rates. Short maturities, for example, could have much lower rates in which case it is inappropriate to use the same yield. We should really be discounting each cash flow at the zero-coupon rate that corresponds to each time period. Define R_t as the **spot interest rate** for maturity t and this risk class (i.e., same currency and credit risk). The fair value of the bond is then:

$$\hat{P} = \sum_{t=1}^{T} \frac{C_t}{(1 + R_t)^t} \tag{7.7}$$

We can then check whether the market price is greater or lower. If the term structure is flat, the two approaches will be identical.

Alternatively, to assess whether a bond is rich or cheap, we can add a fixed amount SS, called the **static spread** to the spot rates so that the NPV equals the current price:

$$P = \sum_{t=1}^{T} \frac{C_t}{(1 + R_t + \mathrm{SS})^t} \tag{7.8}$$

All else equal, a bond with a large static spread is preferable to another with a lower spread. It means the bond is cheaper, or has a higher expected rate of return.

It is simpler, but less accurate to compute a **yield spread**, YS, using yield to maturity, such that

$$P = \sum_{t=1}^{T} \frac{C_t}{(1 + y_T + \mathrm{YS})^t} \tag{7.9}$$

Table 7.3 gives an example of a 7% coupon, two-year bond. The term structure environment, consisting of spot rates and par yields, is described on the left side.

TABLE 7.3 Bond Price and Term Structure

Maturity (Year) i	Term Structure Spot Rate R_i	Par Yield y_i	7% Bond PVCF Discounted at Spot SS = 0	Yield+YS $\Delta y = 0.2386$	Spot + SS $SSs = 0.2482$
1	4.0000	4.0000	6.7308	6.5926	6.7147
2	6.0000	5.9412	95.2296	94.9074	94.7853
Sum			101.9604	101.5000	101.5000
Price			101.5000	101.5000	101.5000

The right side lays out the present value of the cash flows (PVCF). Discounting the two cash flows at the spot rates gives a fair value of $\hat{P} = \$101.9604$. In fact, the bond is selling at a price of $P = \$101.5000$. So, the bond is cheap.

We can convert the difference in prices to annual yields. The yield to maturity on this bond is 6.1798%, which implies a yield spread of YS = 6.1798 − 5.9412 = 0.2386%. Using the static spread approach, we find that adding SS = 0.2482% to the spot rates gives the current price. The second measure is more accurate than the first.

Cash flows with different credit risks need to be discounted with different rates. For example, the principal on Brady bonds is collateralized by U.S. Treasury securities and carries no default risk, in contrast to the coupons. As a result, it has become common to separate the discounting of the principal from that of the coupons. Valuation is done in two steps. First, the principal is discounted into a present value using the appropriate Treasury yield. The present value of the principal is subtracted from the market value. Next, the coupons are discounted at what is called the **stripped yield**, which accounts for the credit risk of the issuer.

7.3.3 Duration

Armed with a cash flow profile, we can proceed to compute duration. As we have seen in Chapter 1, **duration** is a measure of the exposure, or sensitivity, of the bond price to movements in yields. When cash flows are fixed, duration is measured as the weighted maturity of each payment, where the weights are proportional to the present value of the cash flows. Using the same notations as in Equation (7.5), recall that **Macaulay duration** is

$$D = \sum_{t=1}^{T} t \times w_t = \sum_{t=1}^{T} t \times \frac{C_t/(1+y)^t}{\sum C_t/(1+y)^t} \tag{7.10}$$

KEY CONCEPT

Duration can be viewed as the weighted average time to wait for each payment, but only when the bond's cash flows are predetermined.

More generally, duration is a measure of interest-rate exposure:

$$\frac{dP}{dy} = -\frac{D}{(1+y)}P = -D^*P \tag{7.11}$$

where D^* is **modified duration.** The second term D^*P is also known as the **dollar duration.** Sometimes this sensitivity is expressed in **dollar value of a basis point** (also known as DV01), defined as

$$\frac{dP}{0.01\%} = \text{DVBP} \tag{7.12}$$

For fixed cash flows, duration can be computed using Equation (7.10). Otherwise, we can infer duration from an economic analysis of the security. Consider a **floating-rate note** (FRN) with no credit risk. Just before the reset date, we know that the coupon will be set to the prevailing interest rate. The FRN is then similar to cash, or a money market instrument, which has no interest rate risk and hence is selling at par with zero duration. Just after the reset date, the investor is locked into a fixed coupon over the accrual period. The FRN is then economically equivalent to a zero-coupon bond with maturity equal to the time to the next reset date.

KEY CONCEPT

The duration of a floating-rate note is the time to wait until the next reset period, at which time the FRN should be at par.

EXAMPLE 7.2: CALLABLE BOND DURATION

A 10-year zero-coupon bond is callable annually at par (its face value) starting at the beginning of year six. Assume a flat yield curve of 10%. What is the bond duration?

- **a.** 5 years
- **b.** 7.5 years
- **c.** 10 years
- **d.** Cannot be determined based on the data given

EXAMPLE 7.3: DURATION OF FLOATERS

A money markets desk holds a floating-rate note with an eight-year maturity. The interest rate is floating at three-month LIBOR rate, reset quarterly. The next reset is in one week. What is the approximate duration of the floating-rate note?

a. 8 years
b. 4 years
c. 3 months
d. 1 week

7.4 SPOT AND FORWARD RATES

In addition to the cash flows, we also need detailed information on the term structure of interest rates to value fixed-income securities and their derivatives. This information is provided by **spot rates,** which are zero-coupon investment rates that start at the current time. From spot rates, we can infer **forward rates,** which are rates that start at a future date. Both are essential building blocks for the pricing of bonds.

Consider for instance a one-year rate that starts in one year. This forward rate is defined as $F_{1,2}$ and can be inferred from the one-year and two-year spot rates, R_1 and R_2. The forward rate is the break-even future rate that equalizes the return on investments of different maturities. An investor has the choice to lock in a two-year investment at the two-year rate, or to invest for a term of one year and roll over at the one-to-two year forward rate. The two portfolios will have the same payoff when the future rate $F_{1,2}$ is such that

$$(1 + R_2)^2 = (1 + R_1)(1 + F_{1,2}) \tag{7.13}$$

For instance, if $R_1 = 4.00\%$ and $R_2 = 4.62\%$, we have $F_{1,2} = 5.24\%$.

More generally, the T-period spot rate can be written as a geometric average of the spot and consecutive one-year forward rates

$$(1 + R_T)^T = (1 + R_1)(1 + F_{1,2}) \ldots (1 + F_{T-1,T}) \tag{7.14}$$

where $F_{i,i+1}$ is the forward rate of interest prevailing now (at time t) over a horizon of i to $i + 1$. This sequence is shown in Figure 7.2. Table 7.4 displays a sequence of spot rates, forward rates, and par yields, using annual compounding. The last column is the **discount function,** which is simply the current price of a dollar paid at t.

Alternatively, one could infer a series of forward rates for various maturities, all starting in one year

$$(1 + R_3)^3 = (1 + R_1)(1 + F_{1,3})^2, \ldots, \ (1 + R_T)^T = (1 + R_1)(1 + F_{1,T})^{T-1} \tag{7.15}$$

This defines a term structure in one year, $F_{1,2}, F_{1,3}, \ldots, F_{1,T}$.

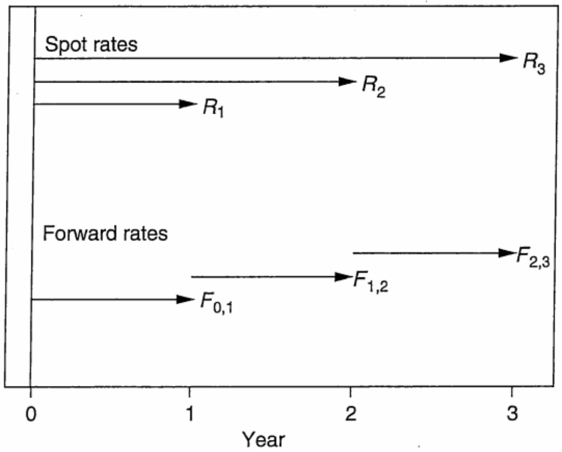


FIGURE 7.2 Spot and Forward Rates

The forward rate can be interpreted as a measure of the slope of the term structure. We can, for instance, expand both sides of Equation (7.13). After neglecting cross-product terms, we have

$$F_{1,2} \approx R_2 + (R_2 - R_1) \tag{7.16}$$

Thus, with an upward-sloping term structure, R_2 is above R_1, and $F_{1,2}$ will also be above R_2.

We can also show that in this situation, the spot rate curve is above the par yield curve. Consider a bond with two payments. The two-year par yield y_2 is implicitly defined from:

$$P = \frac{cF}{(1+y_2)} + \frac{(cF+F)}{(1+y_2)^2} = \frac{cF}{(1+R_1)} + \frac{(cF+F)}{(1+R_2)^2}$$

where P is set to par $P = F$. The par yield can be viewed as a weighted average of spot rates. In an upward-sloping environment, par yield curves involve coupons

TABLE 7.4 Spot Rates, Forward Rates, and Par Yields

Maturity (Year) i	Spot Rate R_i	Forward Rate $F_{i-1,i}$	Par Yield y_i	Discount Function $D(t_i)$
1	4.000	4.000	4.000	0.9615
2	4.618	5.240	4.604	0.9136
3	5.192	6.350	5.153	0.8591
4	5.716	7.303	5.640	0.8006
5	6.112	7.712	6.000	0.7433
6	6.396	7.830	6.254	0.6893
7	6.621	7.980	6.451	0.6383
8	6.808	8.130	6.611	0.5903
9	6.970	8.270	6.745	0.5452
10	7.112	8.400	6.860	0.5030

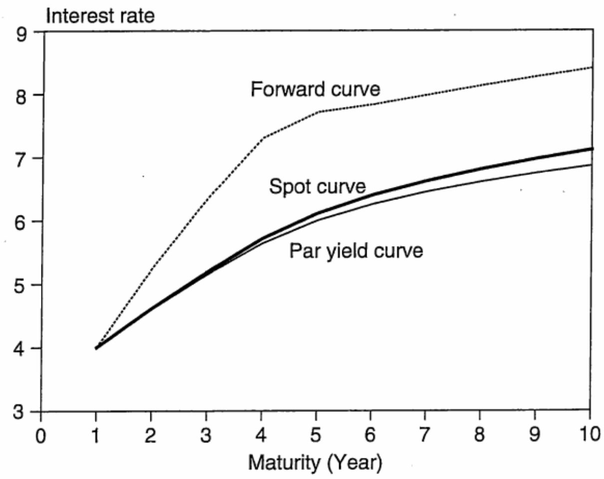


FIGURE 7.3 Upward-Sloping Term Structure

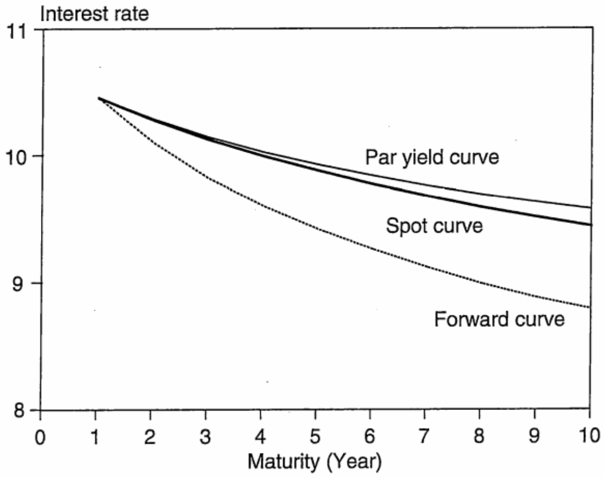


FIGURE 7.4 Downward-Sloping Term Structure

that are discounted at shorter and thus lower rates than the final payment. As a result, the par yield curve lies below the spot rate curve.[4] When the spot rate curve is flat, the spot curve is identical to the par yield curve and to the forward curve. In general, the curves differ. Figure 7.3 displays the case of an upward-sloping term structure. It shows the yield curve is below the spot curve while the forward curve is above the spot curve. With a downward-sloping term structure, as shown in Figure 7.4, the yield curve is above the spot curve, which is above the forward curve.

[4] For a formal proof, consider a two-period par bond with a face value of \$1 and coupon of y_2. We can write the price of this bond as $1 = y_2/(1+R_1) + (1+y_2)/(1+R_2)^2$. After simplification, this gives $y_2 = R_2(2+R_2)/(2+F_{1,2})$. In an upward-sloping environment, $F_{1,2} > R_2$ and thus $y_2 < R_2$.

Note that, because interest rates have to be positive, forward rates have to be positive, otherwise there would be an arbitrage opportunity.[5]

Forward rates allow us to project future cash flows that depend on future rates. The $F_{1,2}$ forward rate, for example, can be taken as the market's expectation of the second coupon payment on an FRN with annual payments and resets. We will also show later that positions in forward rates can be taken easily with derivative instruments.

As a result, the forward rate can be viewed as an expectation of the future spot rate. According to the **expectations hypothesis**

$$F_{1,2}^{t} = E(R_1^{t+1}) \tag{7.17}$$

This assumes that there is no risk premium embedded in forward rates. An upward-sloping term structure implies that short-term rates will rise in the future. In Figure 7.3, the forward curve traces out the path of future one-year spot rates.

If this hypothesis is correct, then it does not matter which maturity should be selected for investment purposes. Longer maturities benefit from higher coupons but will suffer a capital loss, due to the increase in rates, that will offset this benefit exactly.

KEY CONCEPT

In an upward-sloping term structure environment, the forward curve is above the spot curve, which is above the par yield curve. According to the expectations hypothesis, this implies a forecast for rising interest rates.

EXAMPLE 7.4: FRM EXAM 2007—QUESTION 32

The price of a three-year zero-coupon government bond is 85.16. The price of a similar four-year bond is 79.81. What is the one-year implied forward rate from year 3 to year 4?

a. 5.4%
b. 5.5%
c. 5.8%
d. 6.7%

[5] We abstract from transaction costs and assume we can invest and borrow at the same rate. For instance, $R_1 = 11.00\%$ and $R_2 = 4.62\%$ gives $F_{1,2} = -1.4\%$. This means that $(1 + R_1) = 1.11$ is greater than $(1 + R_2)^2 = 1.094534$. To take advantage of this discrepancy, we borrow \$1 million for two years and invest it for one year. After the first year, the proceeds are kept in cash, or under the proverbial mattress, for the second period. The investment gives \$1,110,000 and we have to pay back \$1,094,534 only. This would create a profit of \$15,466 out of thin air, which is highly unlikely in practice.

EXAMPLE 7.5: FRM EXAM 1999—QUESTION 1

Suppose that the yield curve is upward sloping. Which of the following statements is *true*?

a. The forward rate yield curve is above the zero-coupon yield curve, which is above the coupon-bearing bond yield curve.
b. The forward rate yield curve is above the coupon-bearing bond yield curve, which is above the zero-coupon yield curve.
c. The coupon-bearing bond yield curve is above the zero-coupon yield curve, which is above the forward rate yield curve.
d. The coupon-bearing bond yield curve is above the forward rate yield curve, which is above the zero-coupon yield curve.

EXAMPLE 7.6: FRM EXAM 2004—QUESTION 61

According to the pure expectations hypothesis, which of the following statements is *correct* concerning the expectations of market participants in an upward-sloping yield curve environment?

a. Interest rates will increase and the yield curve will flatten.
b. Interest rates will increase and the yield curve will steepen.
c. Interest rates will decrease and the yield curve will flatten.
d. Interest rates will decrease and the yield curve will steepen.

7.5 PREPAYMENT

7.5.1 Describing Prepayment Speed

So far, we considered fixed-income securities with fixed cash flows. In practice, many instruments have uncertain cash flows. Consider an investment in a traditional fixed-rate mortgage. The homeowner has the possibility of making early payments of principal. For the borrower, this represents a long position in an option. For the lender, this is a short position.

In some cases, these prepayments are random, for instance when the homeowner sells the home due to changing job or family conditions. In other cases, these prepayments are more predictable. When interest rates fall, prepayments increase as homeowners can refinance at a lower cost. This also applies to callable bonds, where the borrower has the option to call back its bonds at a fixed prices at fixed points in time. Generally, these factors affect mortgage refinancing patterns:

- *Spread between the mortgage rate and current rates*: Increases in the spread increase prepayments. Like a callable bond, there is a greater benefit to refinancing if it achieves a significant cost saving.
- *Age of the loan*: Prepayment rates are generally low just after the mortgage loan has been issued and gradually increase over time until they reach a stable, or "seasoned," level. This effect is known as **seasoning**.
- *Refinancing incentives*: The smaller the costs of refinancing, the more likely home-owners will refinance often.
- *Previous path of interest rates*: Refinancing is more likely to occur if rates have been high in the past but recently dropped. In this scenario, past prepayments have been low but should rise sharply. In contrast, if rates are low but have been so for a while, most of the principal will already have been prepaid. This path dependence is usually referred to as **burnout**.
- *Level of mortgage rates*: Lower rates increase affordability and turnover.
- *Economic activity*: An economic environment where more workers change job location creates greater job turnover, which is more likely to lead to prepayments.
- *Seasonal effects*: There is typically more home-buying in the spring, leading to increased prepayments in early fall.

The prepayment rate is summarized into what is called the **conditional prepayment rate (CPR)**, which is expressed in annual terms. This number can be translated into a monthly number, known as the **single monthly mortality (SMM) rate** using the adjustment:

$$(1 - \text{SMM})^{12} = (1 - \text{CPR}) \tag{7.18}$$

For instance, if CPR $= 6\%$ annually, the monthly proportion of principal paid early will be SMM $= 1 - (1 - 0.06)^{1/12} = 0.005143$, or 0.514% monthly. For a loan with a beginning monthly balance (BMB) of BMB = \$50,525 and a scheduled principal payment of SP = \$67, the prepayment will be $0.005143 \times (\$50{,}525 - \$67) = \$260$.

To price the mortgage, the portfolio manager should describe the schedule of projected prepayments during the remaining life of the bond. This depends on many factors, including the age of the loan.

Prepayments can be described using an industry standard, known as the **Public Securities Association (PSA)** prepayment model. The PSA model assumes a CPR of 0.2% for the first month, going up by 0.2% per month for the next 30 months, until 6% thereafter. Formally, this is:

$$\text{CPR} = \text{Min}[6\% \times (t/30), 6\%] \tag{7.19}$$

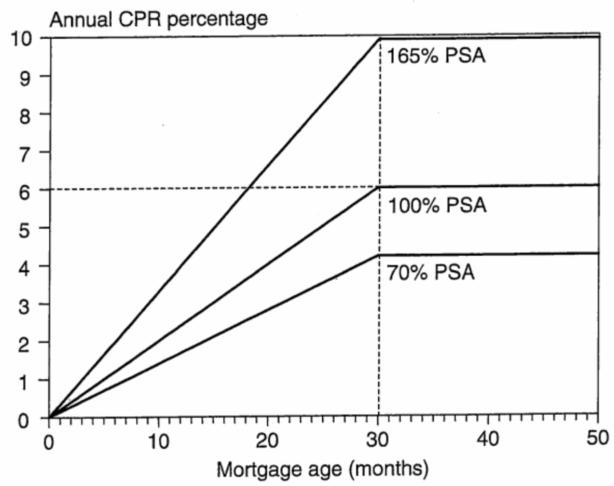


FIGURE 7.5 Prepayment Pattern

This pattern is described in Figure 7.5 as the 100% PSA speed. By convention, prepayment patterns are expressed as a percentage of the PSA speed, for example 165% for a faster pattern and 70% PSA for a slower pattern.

Example: Computing the CPR

Consider a mortgage issued 20 months ago with a speed of 150% PSA. What are the CPR and SMM?

The PSA speed is $\text{Min}[6\% \times (20/30), 6\%] = 0.04$. Applying the 150 factor, we have $\text{CPR} = 150\% \times 0.04 = 0.06$. This implies $\text{SMM} = 0.514\%$.

The next step is to project cash flows based on the prepayment speed pattern. Figure 7.6 displays cash-flow patterns for a 30-year loan with a face amount of $1 million and 6% interest rate. The horizontal, "No prepayment" line, describes the fixed annuity payment of $6,000 without any prepayment. The "100% PSA" line describes an increasing pattern of cash flows, peaking in 30 months and decreasing thereafter. This point corresponds to the stabilization of the CPR at 6%. This pattern is more marked for the "165% PSA" line, which assumes a faster prepayment speed.

Early prepayments create less payments later, which explains why the 100% PSA line is initially higher than the 0% line, then lower as the principal has been paid off more quickly.

7.5.2 Prepayment Risk

Like other fixed-income instruments, mortgages are subject to market risk, due to fluctuations in interest rates, and to credit risk, due to homeowner default. They are also, however, subject to **prepayment risk**, which is the risk that the principal will be repaid early.

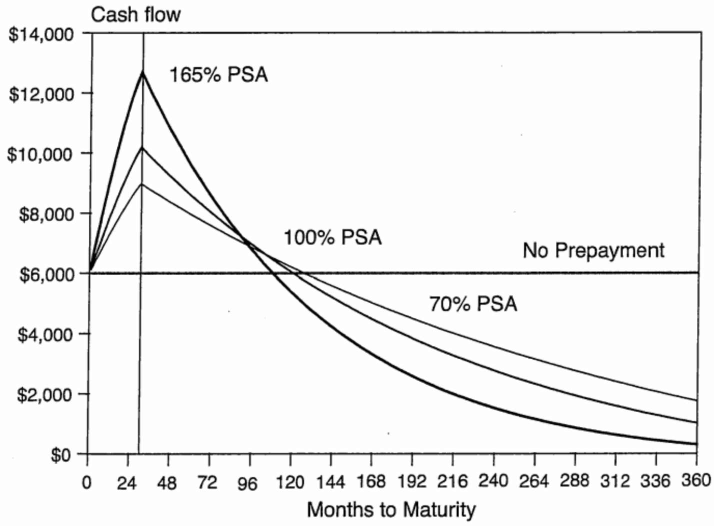


FIGURE 7.6 Cash Flows for Various PSA Speeds

Consider for instance an 8% mortgage, which is illustrated in Figure 7.7. If rates drop to 6%, homeowners will rationally prepay early to refinance the loan. Because the average life of the loan is shortened, this is called **contraction risk.** Conversely, if rates increase to 10%, homeowners will be less likely to refinance early, and prepayments will slow down. Because the average life of the loan is extended, this is called **extension risk.**

As shown in Figure 7.7, this feature creates "negative convexity" at point A. This reflects the fact that the investor in a mortgage is short an option. At point B, interest rates are very high and it is unlikely that the homeowner will refinance early. The option is nearly worthless and the mortgage behaves like a regular bond, with positive convexity.

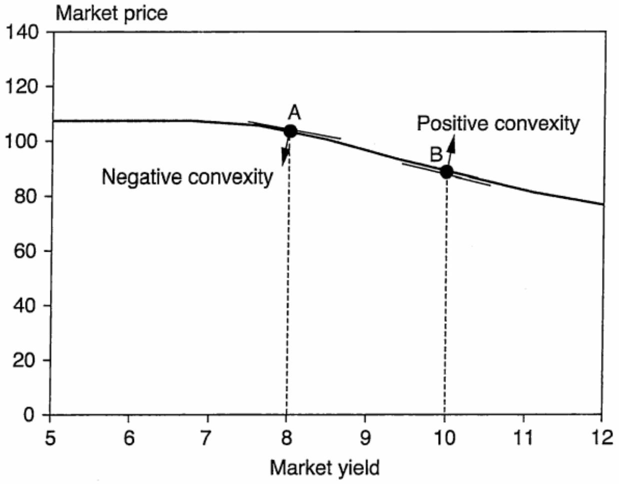


FIGURE 7.7 Negative Convexity of Mortgages

TABLE 7.5 Computing Effective Duration and Convexity

	Initial	Unchanged PSA		Changed PSA	
Yield	7.50%	+25bp	−25bp	+25bp	−25bp
PSA		165PSA	165PSA	150PSA	200PSA
Price	100.125	98.75	101.50	98.7188	101.3438
Duration		5.49y		5.24y	
Convexity		0		−299	

This changing cash-flow pattern makes standard duration measures unreliable. Instead, sensitivity measures are computed using **effective duration** and **effective convexity**, as explained in Chapter 1. The measures are based on the estimated price of the mortgage for three yield values, making suitable assumptions about how changes in rates should affect prepayments.

Table 7.5 shows an example. In each case, we consider an upmove and downmove of 25bp. In the first, "unchanged" panel, the PSA speed is assumed to be constant at 165 PSA. In the second, "changed" panel, we assume a higher PSA speed if rates drop and lower speed if rates increase. When rates drop, the mortgage value goes up but slightly less than with a constant PSA speed. This reflects contraction risk. When rates increase, the mortgage value drops by more than if the prepayment speed had not changed. This reflects extension risk.

As we have seen in Chapter 1, **effective duration** is measured as

$$D^E = \frac{P(y_0 - \Delta y) - P(y_0 + \Delta y)}{(2P_0 \Delta y)} \tag{7.20}$$

Effective convexity is measured as

$$C^E = \left[\frac{P(y_0 - \Delta y) - P_0}{(P_0 \Delta y)} - \frac{P_0 - P(y_0 + \Delta y)}{(P_0 \Delta y)}\right] \Big/ \Delta y \tag{7.21}$$

In Table 7.5, in the "unchanged" panel, the effective duration is 5.49 years and the convexity is close to zero. In the "changed" panel, the effective duration is 5.24 years and the convexity is negative, as expected, and quite large.

KEY CONCEPT

Mortgage investments have negative convexity, which reflects the short position in an option granted to the homeowner to repay early. This creates extension risk when rates increase or contraction risk when rates fall.

The option feature in mortgages increases their yield. To ascertain whether the securities represent good value, portfolio managers need to model the option component. The approach most commonly used is the **option-adjusted spread** (OAS).

Starting from the **static spread**, the OAS method involves running simulations of various interest rate scenarios and prepayments to establish the option cost. The OAS is then

$$\text{OAS} = \text{Static spread} - \text{Option cost} \tag{7.22}$$

which represents the net richness or cheapness of the instrument. Within the same risk class, a security trading at a high OAS is preferable to others.

The OAS is more stable over time than the spread, because the latter is affected by the option component. This explains why during market rallies, i.e., when long-term Treasury yields fall, yield spreads on current coupon mortgages often widen. These mortgages are more likely to be prepaid early, which makes them less attractive. Their option cost increases, pushing up the yield spread.

EXAMPLE 7.7: FRM EXAM 1999—QUESTION 51

Suppose the annual prepayment rate CPR for a mortgage-backed security is 6%. What is the corresponding single-monthly mortality rate SMM?

a. 0.514%
b. 0.334%
c. 0.5%
d. 1.355%

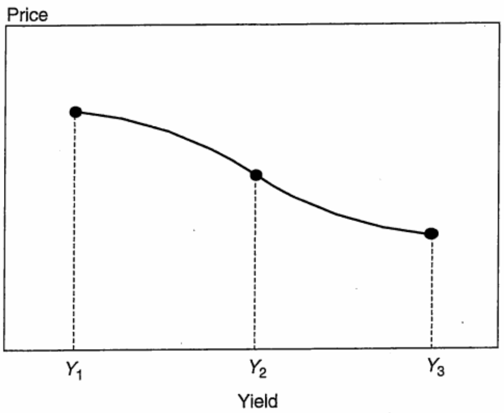
EXAMPLE 7.8: FRM EXAM 2000—QUESTION 3

How would you describe the typical price behavior of a low premium mortgage pass-through security?

a. It is similar to a U.S. Treasury bond.
b. It is similar to a plain-vanilla corporate bond.
c. When interest rates fall, its price increase would exceed that of a comparable duration U.S. Treasury bond.
d. When interest rates fall, its price increase would lag that of a comparable duration U.S. Treasury bond.

EXAMPLE 7.9: FRM EXAM 2003—QUESTION 52

What bond type does the following price-yield curve represent and at which yield level is convexity equal to zero?



a. Puttable bond with convexity close to zero at Y_2.
b. Puttable bond with convexity close to zero at Y_1 and Y_3.
c. Callable bond with convexity close to zero at Y_2.
d. Callable bond with convexity close to zero at Y_1 and Y_3.

EXAMPLE 7.10: FRM EXAM 2006—QUESTION 93

You are analyzing two comparable (same credit rating, maturity, liquidity, rate) U.S. callable corporate bonds. The following data is available for the nominal spread over the U.S. Treasury yield curve and Z spread and option-adjusted spread (OAS) relative to the U.S. Treasury spot curve.

	X	Y
Nominal spread	145	130
Z spread	120	115
OAS	100	105

The nominal spread on the comparable option-free bonds in the market is 100 basis points. Which of the following statements is correct?

a. X only is undervalued.
b. Y only is undervalued.
c. X and Y both are undervalued.
d. Neither X nor Y is undervalued.

7.6 SECURITIZATION

7.6.1 Principles of Securitization

One problem with mortgage loans is that they are not tradable. In the past, they were originated and held by financial institutions such as savings and loans. This arrangement, however, concentrates risk in an industry that may not be able to hedge it efficiently. Also, it limits the amount of capital that can flow into mortgages. **Mortgage-backed securities** (MBSs) were created to solve this problem. MBSs are tradable securities that represent claims on pools of mortgage loans.[6]

This is an example of **securitization**, which is the process by which assets are pooled and securities representing interests in the pool are issued. These assets are created by an **originator**, or **issuer**.

The first step of the process is to create a new legal entity, called a **special-purpose vehicle** (**SPV**), or **special-purpose entity** (**SPE**). The originator then pools a group of assets and sells them to the SPV. In the next step, the SPV issues tradable claims, or securities, that are backed by the financial assets. Figure 7.8 describes a basic securitization structure.

A major advantage of this structure is that it shields the ABS investor from the credit risk of the originator. This requires, however, a clean sale of the assets to the SPV. Otherwise, the creditors of the originators might try to seize the SPV's assets in a bankruptcy proceeding. Other advantages are that pooling offers ready-made diversification across many assets.

The growth of securitization is being fueled by the **disintermediation** of banks as main providers of capital to everyone. When banks act as financial intermediaries, they raise funds (recorded as liabilities on the balance sheet) that are used for making loans (recorded as assets). With securitization, both assets and liabilities are removed from the balance sheet, requiring less equity capital to operate. Securitization provides regulatory capital relief if it enables the originator to hold proportionately less equity capital than otherwise. For instance, if the capital requirements for mortgages are too high, the bank will benefit from spinning off mortgage loans into securities because its required capital will drop sharply.

For the originator, securitization creates an additional source of funding. Securitization can also be used to manage the bank's risk profile. If the securitized assets have the same risk as the rest of the bank's assets, the relative risk of the bank is not changed, even though its size shrinks. In contrast, if the collateral is much riskier than the rest of the assets, the bank will have lowered its risk profile with securitization.

All sorts of assets can be included in ABSs, including mortgage loans, auto loans, student loans, credit card receivables, accounts receivables, and debt obligations. These assets are called **collateral**. In general, collecting payments on the collateral requires ongoing servicing activities. This is done by the **servicing agent**. Usually, the originator also performs the servicing, in exchange for a servicing fee.

[6] The MBS market was developed largely by Salomon Brothers in the early 1980s. This is described in a very entertaining book by Michael Lewis (1989), *Liar's Poker*, New York: Norton.

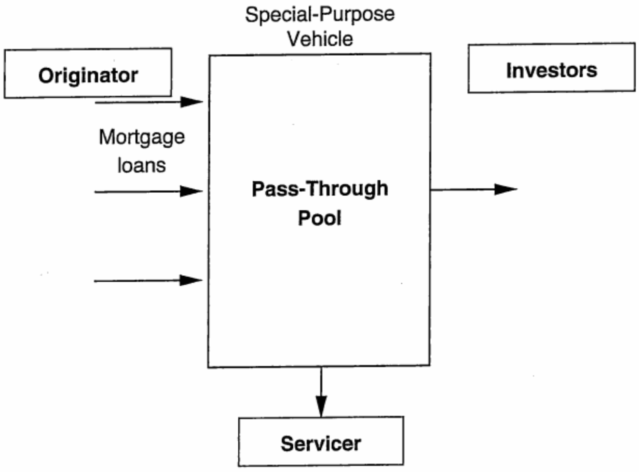


FIGURE 7.8 Securitization

The cash flows from the assets, minus the servicing fees, flow through the SPV to securities holders. When the securitization is structured as a **pass-through**, there is one class of bonds, and all investor receive the same proportional interests in the cash flows. When the SPV issues different classes of securities, the bonds are called **tranches**.[7] In addition, derivative instruments can be created to exchange claims on the ABS tranches, as we shall see in Chapter 22.

So far, we examined off–balance sheet securitizations. Another group is on–balance sheet securitizations, called **covered bonds** or **Pfandbriefe** in Germany. In these structures, the bank originates the loans and issues securities secured by these loans, which are kept on its books. Such structures are similar to secured corporate bonds, but have stronger legal protection in many European civil-law countries. Another difference is that investors have recourse against the bank in the case of defaults on the mortgages. Effectively, the bank provides a guarantee against credit risk.

In the case of MBS securitizations, the collateral consists of residential or commercial mortgage loans. These are called RMBS and CMBS, respectively. Their basic cash-flow patterns start from an annuity, where the homeowner makes a monthly fixed payment that covers principal and interest. As a result, the net present value of these cash flows is subject to interest rate risk, prepayment risk, and default risk.

In practice, however, most MBSs have third-party guarantees against credit risk. For instance, MBSs issued by Fannie Mae, an agency that is sponsored by the U.S. government, carry a guarantee of full interest and principal payment, even if the original borrower defaults. In this case, the **government-sponsored enterprise** (GSE) is the **mortgage insurer**. Such mortgage pass-throughs are sometimes called

[7] This is the French word for *slice*, as in a cake.

participation certificates. In contrast, private-label MBSs are exposed to credit risk, and may receive a credit rating.[8]

7.6.2 Issues with Securitization

The financial crisis that started in 2007 has highlighted serious deficiencies with the securitization process. Securitization allows banks to move assets off their balance sheets, freeing up capital and spreading the risk among many different investors. In theory, this provides real benefits.

In practice, failures in the **originate-to-distribute** model contributed to the credit crisis. The model failed in a number of key places, including underwriting, credit rating, and investor due diligence.

In the traditional banking model, banks or savings institutions underwrite mortgage loans and keep them on their balance sheet. This creates an incentive to screen loans carefully and to monitor their quality closely. In contrast, when a loan is securitized, the originator has less incentives to worry about the quality of the loans because its revenues depend on the volume of issuance. After all, another party bears the losses. This has led to a large increase in poor quality loans, in particular within the subprime category, which were packaged in securities and went bad rather quickly. This is a **moral hazard** problem, where an institution behaves differently than if it was fully exposed to the risk of the activity.

An additional problem arises when a bank decides to securitize a loan because of negative information about the borrower, rather than for diversification or capital relief reasons. This creates a systematic bias toward lower quality loans among securitized loans. To protect themselves, investors in asset-backed securities can require the issuing bank to retain some of the risk of the loans but this is not always sufficient. This is an **adverse selection** problem, due to **asymmetric information**. Indeed, Berndt and Gupta (2008) find that borrowers whose loans are sold in the secondary market have stock prices that underperform other firms.[9]

In addition, most of these loans ended up in complex securitization structures. Investors had relied blindly on credit ratings that turned out to be inaccurate and had not performed sufficient analysis of the risk of the underlying assets.

Finally, because it was so easy to securitize assets, the total amount of lending in the residential sector has gone up, creating additional demand for housing and perhaps pushing housing prices even further away from their fundamental value.

The securitization process also came back to haunt originating banks, many of which had created separate entities that invested in asset-backed securities. One such example are **structured investment vehicles** (SIVs), which are basically virtual banks, investing in asset-backed securities and funding themselves using

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[8] Within this class, "Alt-A" loans contain nonstandard features but have borrowers of "A" creditworthiness. At the low end of the credit scale, securities that are backed by subprime mortgage loans are classified as "home equity ABS" rather than MBS.

[9] An alternative explanation is that borrowers whose loans are sold are not subject to the discipline of bank monitoring, and undertake suboptimal investment. See Berndt, A. and A. Gupta (2008), Moral Hazard and Adverse Selection in the Originate-to-Distribute Model of Bank Credit, Working Paper, Carnegie Mellon University.

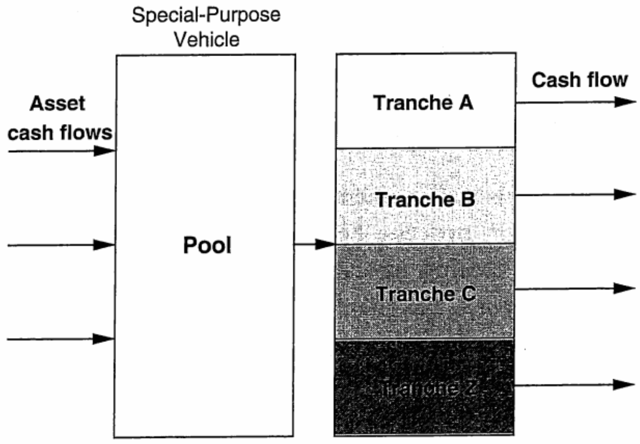

FIGURE 7.9 Tranching

short-term debt. During the credit crisis, investors in this debt became worried about the solvability of these SIVs and refused to roll over their investments. This forced many banks to absorb the assets of the failing SIVs they had sponsored back on their balance sheet, requiring them to raise additional equity capital in a particularly unfavorable environment. Thus securitization failed to provide the expected capital relief.

7.6.3 Tranching

In the case of MBSs, however, a major remaining risk is prepayment risk, which induces negative convexity. This feature is unattractive to investors who want fixed-income securities with predictable payments, so that they can match their liabilities.

In response, the industry has developed new classes of securities based on MBSs with more appealing characteristics. These are the **collateralized mortgage obligations (CMOs)**, which are new securities that redirect the cash flows of an MBS pool to various segments.

Figure 7.9 illustrates this tranching process. The cash flows from the MBS pool go into the SPV, which issues different claims, or tranches with various characteristics. These are structured so that the cash flow from the first tranche, for instance, is more predictable than the original cash flows. The uncertainty is then pushed into the other tranches.

Starting from an MBS pool, financial engineering creates securities that are better tailored to investors' needs. It is important to realize, however, that the cash flows and risks are fully preserved. They are only redistributed across tranches. Whatever transformation is brought about, the resulting package must obey basic laws of conservation for the underlying securities and package of resulting securities.[10]

[10]As Lavoisier, the French chemist who was executed during the French revolution said, *Rien ne se perd, rien ne se crée (nothing is lost, nothing is created).*

At every single point in time, we must have the same cash flows going into and coming out of the SPV. As a result, we must have the same market value and the same risk profile. In particular, the weighted duration and convexity of the portfolio of tranches must add up to the original duration and convexity. If Tranche A has less convexity than the underlying securities, the other tranches must have more convexity.

Similar structures apply to **collateralized bond obligations** (CBOs), **collateralized loan obligations** (CLOs), and **collateralized debt obligations** (CDOs), which are a set of tradable bonds backed by bonds, loans, or debt (bonds and loans), respectively. These structures rearrange credit risk and will be explained in more detail in a later chapter.

KEY CONCEPT

Tranching rearranges the total cash flows, total value, and total risk of the underlying securities. At all times, the total cash flows, value, and risk of the tranches must equal those of the collateral. If some tranches are less risky than the collateral, others must be more risky.

EXAMPLE 7.11: FRM EXAM 2000—QUESTION 13

A CLO is generally

a. A set of loans that can be traded individually in the market
b. A pass-through
c. A set of bonds backed by a loan portfolio
d. None of the above

EXAMPLE 7.12: FRM EXAM 2004—QUESTION 57

When evaluating asset-backed securitization issues, which of the following would be *least* important during the investor's analysis process?

a. The liability concentration levels of the asset originator
b. The structure of the underlying securitization transaction
c. The quality of the loan servicer for the underlying assets in the transaction
d. The quality of the underlying assets within the securitization structure

7.6.4 Tranching: Inverse Floaters

To illustrate the concept of tranching, we consider a simple example. with a two-tranche structure. The collateral consists of a regular five-year, 6% coupon \$100 million note. This can be split up into a floater, that pays LIBOR on a notional of \$50 million, and an **inverse floater**, that pays 12% − LIBOR on a notional of \$50 million. Because the coupon C_{IF} on the inverse floater cannot go below zero, this imposes another condition on the floater coupon C_F. The exact formulas are:

$$\text{Coupon}_F = \text{Min}(\text{LIBOR}, 12\%) \quad \text{Coupon}_{\text{IF}} = \text{Max}(12\% - \text{LIBOR}, 0)$$

We verify that the outgoing cash flows exactly add up to the incoming flows. For each coupon payment, we have, in millions

$$\$50 \times \text{LIBOR} + \$50 \times (12\% - \text{LIBOR}) = \$100 \times 6\% = \$6$$

so this is a perfect match. At maturity, the total payments of twice \$50 million add up to \$100 million, so this matches as well.

We can also decompose the risk of the original structure into that of the two components. Assume a flat-term structure and say the duration of the original five-year note is $D = 4.5$ years. The portfolio dollar duration is:

$$\$50,000,000 \times D_F + \$50,000,000 \times D_{\text{IF}} = \$100,000,000 \times D$$

Just before a reset, the duration of the floater is close to zero $D_F = 0$. Hence, the duration of the inverse floater must be $D_{\text{IF}} = (\$100,000,000/\$50,000,000) \times D = 2 \times D$, or nine years, which is twice that of the original note. Note that the duration of the inverse floater is much greater than its maturity. This illustrates the point that duration is an interest rate sensitivity measure. When cash flows are uncertain, duration is not necessarily related to maturity. Intuitively, the first tranche, the floater, has zero risk so that all of the risk must be absorbed into the second tranche. The total risk of the portfolio is conserved.

This analysis can be easily extended to inverse floaters with greater leverage. Suppose the coupon is tied to twice LIBOR, for example 18% − 2 × LIBOR. The principal must be allocated in the amount x, in millions, for the floater and $100 - x$ for the inverse floater so that the coupon payment is preserved. We set

$$x \times \text{LIBOR} + (100 - x) \times (18\% - 2 \times \text{LIBOR}) = \$6$$

$$[x - 2(100 - x)] \times \text{LIBOR} + (100 - x) \times 18\% = \$6$$

Because LIBOR will change over time, this can only be satisfied if the term between brackets is always zero. This implies $3x - 200 = 0$, or $x = \$66.67$ million. Thus, two-thirds of the notional must be allocated to the floater, and one-third to the inverse floater. The inverse floater now has three times the duration of the original note.

EXAMPLE 7.13: FRM EXAM 1999—QUESTION 79

Suppose that the coupon and the modified duration of a 10-year bond priced to par are 6.0% and 7.5, respectively. What is the approximate modified duration of a 10-year inverse floater priced to par with a coupon of 18% − 2 × LIBOR?

a. 7.5
b. 15.0
c. 22.5
d. 0.0

EXAMPLE 7.14: FRM EXAM 2004—QUESTION 69

With LIBOR at 4%, a manager wants to increase the duration of his portfolio. Which of the following securities should he acquire to increase the duration of his portfolio the most?

a. A 10-year reverse floater that pays 8%− LIBOR, payable annually
b. A 10-year reverse floater that pays 12% − 2×LIBOR, payable annually
c. A 10-year floater that pays LIBOR, payable annually
d. A 10-year fixed rate bond carrying a coupon of 4% payable annually

EXAMPLE 7.15: FRM EXAM 2003—QUESTION 91

Which of the following statements most accurately reflects characteristics of a reverse floater (with no options attached)?

a. A portfolio of reverse floaters carries a marginally higher duration risk than a portfolio of similar maturity normal floaters.
b. A holder of a reverse floater can synthetically convert his position into a fixed rate bond by receiving floating and paying fixed on an interest rate swap.
c. A reverse floater hedges against rising benchmark yields.
d. A reverse floater's price changes by as much as that in a similar maturity fixed rate bond for a given change in yield.

7.6.5 Tranching: CMOs

When the collateral consists of mortgages, CMOs can be defined by prioritizing the payment of principal into different tranches. This is defined as **sequential-pay tranches**. Tranche A, for instance, will receive the principal payment on the whole underlying mortgages first. This creates more certainty in the cash flows accruing to Tranche A, which makes it more appealing to some investors. Of course, this is to the detriment of others. After principal payments to Tranche A are exhausted, Tranche B then receives all principal payments on the underlying MBS, and so on for other tranches.

Prepayment risk can be minimized further with a **planned amortization class** (PAC). All prepayment risk is then transferred to other bonds in the CMO structure, called **support** bonds. PAC bond[5] offer a fixed redemption schedule as long as prepayments on the collateral stay within a specific PSA range, say 100 to 250 PSA, called the **PAC collar**. When the structure is set up, the principal payment is set at the minimum payment of these two extreme values for every month of its life. Over time, this ensures a more stable pattern of payments.

Another popular construction is the IO/PO structure. This strips the MBS into two components. The **interest-only (IO)** tranche receives only the interest payments on the underlying MBS. The **principal-only (PO)** tranche then receives only the principal payments. As before, the market value of the IO and PO must exactly add to that of the MBS. Figure 7.10 describes the price behavior of the IO and PO. Note that the vertical addition of the two components always equals the value of the MBS.

To analyze the PO, it is useful to note that the sum of all principal payments is constant (because we have no default risk). Only the timing is uncertain. In contrast, the sum of all interest payments depends on the timing of principal payments. Later principal payments create greater total interest payments.

If interest rates fall, principal payments will come early, which reflects contraction risk. Because the principal is paid earlier and the discount rate decreases, the

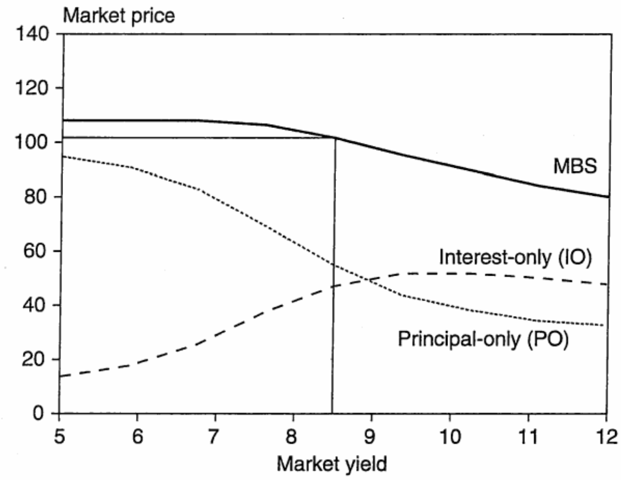

FIGURE 7.10 Creating an IO and PO from an MBS

PO should appreciate sharply in value. On the other hand, the faster prepayments mean less interest payments over the life of the MBS, which is unfavorable to the IO. The IO should depreciate.

Conversely, if interest rates rise, prepayments will slow down, which reflects extension risk. Because the principal is paid later and the discount rate increases, the PO should lose value. On the other hand, the slower prepayments mean more interest payments over the life of the MBS, which is favorable to the IO. The IO appreciates in value, up to the point where the higher discount rate effect dominates. Thus, IOs are bullish securities with negative duration, as shown in Figure 7.10.

EXAMPLE 7.16: FRM EXAM 2006—QUESTION 43

Which of the following mortgage-backed securities has a negative duration?

a. Interest-only strips (IO)
b. Inverse floater
c. Mortgage pass-through
d. Principal-only strips (PO)

EXAMPLE 7.17: FRM EXAM 2004—QUESTION 45

As the CRO of a firm specializing in MBSs, you have been asked to explain how interest-only (IO) strips and principal-only (PO) strips would react if interest rates change. Which of the following is *true*?

a. When interest rates fall, both PO and IO strips will increase in value.
b. When interest rates fall, POs will increase in value, IOs decrease in value.
c. When interest rates rise, POs will increase in value, IOs decrease in value.
d. When interest rates rise, both PO and IO strips will increase in value.

7.7 IMPORTANT FORMULAS

Quotation of Treasury bill as discount rate: $DR = (Face - P)/Face \times (360/t)$
Pricing using spot rate: $\hat{P} = \sum_{t=1}^{T} \frac{C_t}{(1+R_t)^t}$
Spot and forward rate: $(1 + R_2)^2 = (1 + R_1)(1 + F_{1,2})$, $F_{1,2} \approx R_2 + (R_2 - R_1)$

Conditional prepayment rate (CPR), single monthly mortality (SMM) rate:

$$(1 - \text{SMM})^{12} = (1 - \text{CPR})$$

Public Securities Association (PSA) model: $\text{CPR} = \text{Min}[6\% \times (t/30), 6\%]$

7.8 ANSWERS TO CHAPTER EXAMPLES

Example 7.1: FRM Exam 2003—Question 95

a. Answer b. is valid because a short position in a callable bond is the same as a short position in a straight bond plus a long position in a call. (The issuer can call the bond back.) Answer c. is valid because a put is favorable for the investor, so lowers the yield. Answer d. is valid because an inverse floater has high duration.

Example 7.2: Callable Bond Duration

c. Because this is a zero-coupon bond, it will always trade below par, and the call should never be exercised. Hence, its duration is the maturity, 10 years.

Example 7.3: Duration of Floaters

d. Duration is not related to maturity when coupons are not fixed over the life of the investment. We know that at the next reset, the coupon on the FRN will be set at the prevailing rate. Hence, the market value of the note will be equal to par at that time. The duration or price risk is only related to the time to the next reset, which is one week here.

Example 7.4: FRM Exam 2007—Question 32

d. The forward rate can be inferred from $P_4 = P_3/(1 + F_{3,4})$, or $(1 + R_4)^4 = (1 + R_3)^3(1 + F_{3,4})$. Solving, this gives $F_{3,4} = (85.16/79.81) - 1 = 0.067$.

Example 7.5: FRM Exam 1999—Question 1

a. See Figures 7.3 and 7.4. The coupon yield curve is an average of the spot, zero-coupon curve, hence has to lie below the spot curve when it is upward-sloping. The forward curve can be interpreted as the spot curve plus the slope of the spot curve. If the latter is upward-sloping, the forward curve has to be above the spot curve.

Example 7.6: FRM Exam 2004—Question 61

a. An upward-sloping term structure implies forward rates higher than spot rates, or that short-term rates will increase. Because short-term rates increase more than long-term rates, this implies a flattening of the yield curve.

Example 7.7: FRM Exam 1999—Question 51

a. Using $(1 - 6\%) = (1 - \text{SMM})^{12}$, we find SMM = 0.51%.

Example 7.8: FRM Exam 2000—Question 3

d. MBSs are unlike regular bonds, Treasuries, or corporates, because of their negative convexity. When rates fall, homeowners prepay early, which means that the price appreciation is less than that of comparable duration regular bonds.

Example 7.9: FRM Exam 2003—Question 52

c. This has to be a callable bond because the price is capped if rates fall, reflecting the fact that the borrower would call back the bond. At Y_1, convexity is negative, at Y_2, close to zero.

Example 7.10: FRM Exam 2006—Question 93

b. The nominal spreads and Z spreads do not take into account the call option, Instead, comparisons should focus on the OAS, which is higher for Y, and also higher than the 100bp for option-free bonds.

Example 7.11: FRM Exam 2000—Question 13

c. Like a CMO, a CLO represents a set of tradable securities backed by some collateral, in this case a loan portfolio.

Example 7.12: FRM Exam 2004—Question 57

a. Bankruptcy by the originator would not affect the SPV, so the financial condition of the originator is the least important factor. All of the other factors would be important in evaluating the securitization.

Example 7.13: FRM Exam 1999—Question 79

c. Following the same reasoning as above, we must divide the fixed-rate bonds into 2/3 FRN and 1/3 inverse floater. This will ensure that the inverse floater payment is related to twice LIBOR. As a result, the duration of the inverse floater must be 3 times that of the bond.

Example 7.14: FRM Exam 2004—Question 69

b. The duration of a floater is about zero. The duration of a 10-year regular bond is about nine years. The first reverse floater has a duration of about $2 \times 9 = 18$ years, the second, $3 \times 9 = 27$ years.

Example 7.15: FRM Exam 2003—Question 91

b. The duration of a reverse floater is higher than that of a FRN, which is close to zero, or even than that of a fixed-date bond with the same maturity. So, answers a. and d. are wrong. It loses money when yields rise, so c. is wrong. A reverse floater is equivalent as a long position in a fixed-rate bond plus a receive-fixed/pay-floating swap. Hence, b. is correct.

Example 7.16: FRM Exam 2006—Question 43

a. IOs increase in value as interest rates increase because in this scenario, there will be less prepayment of mortgages. Less early payment means more total interest payments, which increases the value of the IO.

Example 7.17: FRM Exam 2004—Question 45

b. POs have positive duration, IOs negative. Hence, they react in opposite directions to falls in interest rates.

CHAPTER 8

Fixed-Income Derivatives

This chapter turns to the analysis of fixed-income derivatives. These are instruments whose value derives from a bond price, interest rate, or other bond market variable. As discussed in Chapter 5, fixed-income derivatives account for the largest proportion of the global derivatives markets. Understanding fixed-income derivatives is also important because many fixed-income securities have derivative-like characteristics.

This chapter focuses on the use of fixed-income derivatives, as well as their pricing. Pricing involves finding the fair market value of the contract. For risk management purposes, however, we also need to assess the range of possible movements in contract values. This will be further examined in the chapters on market risk (Chapters 10–15) and in Chapter 21 on credit exposure.

This chapter presents the most important interest rate derivatives and discusses fundamentals of pricing. Section 8.1 discusses interest rate forward contracts, also known as forward rate agreements. Section 8.2 then turns to the discussion of interest rate futures, covering Eurodollar and Treasury bond futures. Although these products are dollar-based, similar products exist on other capital markets. Swaps are analyzed in Section 8.3. Swaps are very important instruments due to their widespread use. Finally, interest rate options are covered in Section 8.4, including caps and floors, swaptions, and exchange-traded options.

8.1 FORWARD CONTRACTS

Forward Rate Agreements (FRAs) are over-the-counter financial contracts that allow counterparties to lock in an interest rate starting at a future time. The buyer of an FRA locks in a borrowing rate, the seller locks in a lending rate. In other words, the "long" benefits from an increase in rates and the "short" benefits from a fall in rates.

As an example, consider an FRA that settles in one month on three-month LIBOR. Such an FRA is called 1×4. The first number corresponds to the first settlement date, the second to the time to final maturity. Call τ the period to which LIBOR applies, three months in this case. On the settlement date, in one month, the payment to the long involves the net value of the difference between the spot

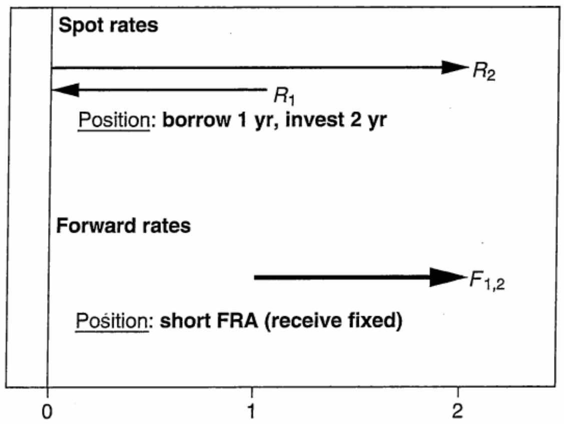

FIGURE 8.1 Decomposition of a Short FRA Position

rate S_T (the prevailing three-month LIBOR rate) and of the locked-in forward rate F. The payoff is $S_T - F$, as with other forward contracts, present valued to the first settlement date. This gives

$$V_T = (S_T - F) \times \tau \times \text{Notional} \times \text{PV}(\$1) \tag{8.1}$$

where $\text{PV}(\$1) = \$1/(1 + S_T\tau)$. The amount is settled in cash.

Figure 8.1 shows that a *short* position in an FRA is equivalent to borrowing short-term to finance a long-term investment. In both cases, there is no up-front investment. The duration is equal to the difference between the durations of the two legs, and can be inferred from the derivative of Equation (8.1). The duration of a short FRA is τ. Its dollar duration is $\text{DD} = \tau \times \text{Notional} \times \text{PV}(\$1)$.

Example: Using an FRA

A company will receive \$100 million in six months to be invested for a six-month period. The Treasurer is afraid rates will fall, in which case the investment return will be lower. The company needs to take a position that will offset this loss by generating a gain when rates fall. Because a short FRA gains when rates fall, the Treasurer needs to *sell* a 6×12 FRA on \$100 million at the rate of, say, $F = 5\%$. This locks in an investment rate of 5% starting in six months.

When the FRA expires in six months, assume that the prevailing six-month spot rate is $S_T = 3\%$. This will lower the investment return on the cash received, which is the scenario the Treasurer feared. Using Equation (8.1), the FRA has a payoff of $V_T = -(3\% - 5\%) \times (6/12) \times \$100 \text{ million} = \$1{,}000{,}000$, which multiplied by the 4% present value factor gives \$980,392. In effect, this payment offsets the lower return that the company received on a floating investment, guaranteeing a return equal to the forward rate. This contract is also equivalent to borrowing the present value of \$100 million for six months and investing the proceeds for 12 months.

KEY CONCEPT

A long FRA position benefits from an increase in rates. A short FRA position is similar to a long position in a bond. Its duration is positive and equal to the difference between the two maturities.

EXAMPLE 8.1: FRM EXAM 2002—QUESTION 27

A long position in a FRA 2 × 5 is equivalent to the following positions in the spot market:

a. Borrowing in two months to finance a five-month investment
b. Borrowing in five months to finance a two-month investment
c. Borrowing half a loan amount at two months and the remainder at five months
d. Borrowing in two months to finance a three-month investment

EXAMPLE 8.2: FRM EXAM 2005—QUESTION 57

ABC, Inc., entered a forward rate agreement (FRA) to receive a rate of 3.75% with continuous compounding on a principal of USD 1 million between the end of year 1 and the end of year 2. The zero rates are 3.25% and 3.50% for one and two years. What is the value of the FRA when the deal is just entered?

a. USD 35,629
b. USD 34,965
c. USD 664
d. USD 0

EXAMPLE 8.3: FRM EXAM 2001—QUESTION 70

Consider the buyer of a 6 × 9 FRA. The contract rate is 6.35% on a notional amount of $10 million. Calculate the settlement amount of the *seller* if the settlement rate is 6.85%. Assume a 30/360 day count basis.

a. −12,500
b. −12,290
c. +12,500
d. +12,290

8.2 FUTURES

Whereas FRAs are over-the-counter contracts, futures are traded on organized exchanges. We will cover the most important types of futures contracts, Eurodollar and T-bond futures.

8.2.1 Eurodollar Futures

Eurodollar futures are futures contracts tied to a forward LIBOR rate. Since their creation on the Chicago Mercantile Exchange, Eurodollar futures have spread to equivalent contracts such as Euribor futures (denominated in euros),[1] Euroyen futures (denominated in Japanese yen), and so on. These contracts are akin to FRAs involving three-month forward rates starting on a wide range of dates, up to 10 years into the future.

The formula for calculating the value of one contract is

$$P_t = 10{,}000 \times [100 - 0.25(100 - \mathrm{FQ}_t)] = 10{,}000 \times [100 - 0.25F_t] \qquad (8.2)$$

where FQ_t is the quoted Eurodollar futures price. This is quoted as 100.00 minus the interest rate F_t, expressed in percent, that is, $\mathrm{FQ}_t = 100 - F_t$. The 0.25 factor represents the three-month maturity, or 0.25 years. For instance, if the market quotes $\mathrm{FQ}_t = 94.47$, we have $F_t = 100 - 94.47 = 5.53$, and the contract value is $P = 10{,}000[100 - 0.25 \times 5.53] = \$986{,}175$. At expiration, the contract value settles to

$$P_T = 10{,}000 \times [100 - 0.25S_T] \qquad (8.3)$$

where S_T is the three-month Eurodollar spot rate prevailing at T. Payments are cash settled.

As a result, F_t can be viewed as a three-month forward rate that starts at the maturity of the futures contract. The formula for the contract price may look complicated but in fact is structured so that an increase in the interest rate leads to a decrease in the price of the contract, as is usual for fixed-income instruments. Also, because the change in the price is related to the interest rate by a factor of 0.25, this contract has a constant duration of three months. It is useful to remember that the DV01 is $\$10{,}000 \times 0.25 \times 0.01 = \$25$.

Chapter 5 has explained that the pricing of forwards is similar to those of futures, except when the value of the futures contract is strongly correlated with the reinvestment rate. This is the case with Eurodollar futures.

Interest rate futures contracts are designed to move like a bond, that is, to lose value when interest rates increase. The correlation is negative. This implies that when interest rates rise, the futures contract loses value and in addition funds have to be provided precisely when the borrowing cost or reinvestment rate is higher.

[1] Euribor futures are based on the European Bankers Federations' Euribor Offered Rate (EBF Euribor).

Example: Using Eurodollar Futures

As in the previous section, the Treasurer wants to hedge a future investment of \$100 million in six months for a six-month period. The company needs to take a position that will offset the earnings loss by generating a gain when rates fall. Because a long Eurodollar futures position gains when rates fall, the Treasurer should *buy* Eurodollar futures.

If the futures contract trades at $FQ_t = 95.00$, the dollar value of one contract is $P = 10{,}000 \times [100 - 0.25(100 - 95)] = \$987{,}500$. The Treasurer needs to buy a suitable number of contracts that will provide the best hedge against the loss of earnings. The computation of this number will be detailed in a future chapter.

Conversely when rates drop, the contract gains value and the profits can be withdrawn but are now reinvested at a lower rate. Relative to forward contracts, this marking-to-market feature is *disadvantageous* to long futures positions. This has to be offset by a *lower* futures contract value. Given that the value is negatively related to the futures rate, by $P_t = 10{,}000 \times [100 - 0.25 \times F_t]$, this implies a *higher* Eurodollar futures rate F_t.

The difference is called the **convexity adjustment** and can be described as[2]

$$\textbf{Futures Rate} = \textbf{Forward Rate} + (1/2)\sigma^2 t_1 t_2 \tag{8.4}$$

where σ is the volatility of the change in the short-term rate, t_1 is the time to maturity of the futures contract, and t_2 is the maturity of the rate underlying the futures contract.

Example: Convexity Adjustment

Consider a 10-year Eurodollar contract, for which $t_1 = 10$, $t_2 = 10.25$. The maturity of the futures contract itself is 10 years and that of the underlying rate is 10 years plus three months.

Typically, $\sigma = 1\%$, so that the adjustment is $(1/2)0.01^2 \times 10 \times 10.25 = 0.51\%$. So, if the forward price is 6%, the equivalent futures rate would be 6.51%. Note that the effect is significant for long maturities only. Changing t_1 to one year and t_2 to 1.25, for instance, reduces the adjustment to 0.006%, which is negligible.

8.2.2 T-Bond Futures

T-bond futures are futures contracts tied to a pool of Treasury bonds that consists of all bonds with a remaining maturity greater than 15 years (and noncallable within 15 years). Similar contracts exist on shorter rates, including 2-, 5-, and

[2] This formula is derived from the Ho-Lee model. See for instance Hull (2000), *Options, Futures, and Other Derivatives*, Upper Saddle River, NJ: Prentice-Hall.

10-year Treasury notes. Government bond futures also exist in other markets, including Canada, the United Kingdom, the Eurozone, and Japan.

Futures contracts are quoted like T-bonds, for example 97-02, in percent plus thirty seconds, with a notional of \$100,000. Thus the price of the contract is $P = \$100{,}000 \times (97 + 2/32)/100 = \$97{,}062.50$. The next day, if yields go up and the quoted price falls to 96-0, the new value is \$96,000, and the loss on the long position is $P_2 - P_1 = -\$1{,}062.50$.

It is important to note that the T-bond futures contract is settled by physical delivery. To ensure interchangeability between the deliverable bonds, the futures contract uses a **conversion factor** (CF) for delivery. This factor multiplies the futures price for payment to the short. The goal of the CF is to attempt to equalize the net cost of delivering the various eligible bonds.

The conversion factor is needed because bonds trade at widely different prices. High coupon bonds trade at a premium, low coupon bonds at a discount. Without this adjustment, the party with the short position (the "short") would always deliver the same, cheap bond and there would be little exchangeability between bonds. Exchangeability is an important feature, however, as it minimizes the possibility of market squeezes. A **squeeze** occurs when holders of the short position cannot acquire or borrow the securities required for delivery under the terms of the contract.

So, the "short" buys a bond, delivers it, and receives the quoted futures price times a conversion factor that is specific to the delivered bond (plus accrued interest). The short should rationally pick the bond that minimizes the net cost,

$$\text{Cost} = \text{Price} - \text{Futures Quote} \times \text{CF} \tag{8.5}$$

The bond with the lowest net cost is called **cheapest to deliver** (CTD).

In practice, the CF is set by the exchange at initiation of the contract for each bond. It is computed by discounting the bond's cash flows at a notional 6% rate, assuming a flat term structure. Take for instance the 7 5/8% of 2025. The CF is computed as

$$\text{CF} = \frac{(7.625\%/2)}{(1 + 6\%/2)^1} + \cdots + \frac{(1 + 7.625\%/2)}{(1 + 6\%/2)^T} \tag{8.6}$$

which gives CF = 1.1717. High coupon bonds have higher CFs. Also, because the coupon is greater than 6%, the CF is greater than 1.

The net cost calculations are illustrated in Table 8.1 for three bonds. The net cost for the first bond in the table is $\$104.375 - 110.8438 \times 0.9116 = \$3.330$. For the 6% coupon bond, the CF is exactly unity. The net cost for the third bond in the table is \$1.874. Because this is the lowest entry, this bond is the CTD for this group. Note how the CF adjustment brings the cost of all bonds much closer to each other than their original prices.

The adjustment is not perfect when current yields are far from 6%, or when the term structure is not flat, or when bonds do not trade at their theoretical prices. Assume for instance that we operate in an environment where yields are flat at

TABLE 8.1 Calculation of CTD

Bond	Price	Futures	CF	Cost
5 1/4% Nov 2028	104.3750	110.8438	0.9116	3.330
6% Feb 2026	112.9063	110.8438	1.0000	2.063
7 5/8% Feb 2025	131.7500	110.8438	1.1717	1.874

5% and all bonds are priced at par. Discounting at 6% will create CF factors that are lower than 1; the longer the maturity of the bond, the greater the difference. The net cost $P - F \times \text{CF}$ will then be greater for longer-term bonds. This tends to favor short-term bonds for delivery. When the term structure is upward sloping, the opposite occurs, and there is a tendency for long-term bonds to be delivered.

As a first approximation, this CTD bond drives the characteristics of the futures contract. As before, the equilibrium futures price is given by

$$F_t e^{-r\tau} = S_t - \text{PV}(D) \tag{8.7}$$

where S_t is the gross price of the CTD and PV(D) is the present value of the coupon payments. This has to be further divided by the conversion factor for this bond. The duration of the futures contract is also given by that of the CTD. In fact, this relationship is only approximate because the short has an *option* to deliver the cheapest of a group of bonds. The value of this delivery option should depress the futures price because the party who is long the futures is also short the option. As a result, he requires a lower acquisition price. Unfortunately, this complex option is not easy to evaluate.

EXAMPLE 8.4: FRM EXAM 2005—QUESTION 49

John H., a portfolio manager, is shorting a U.S. Treasury bond futures contract and has decided to deliver. The quoted futures price is USD 95.5.

Among the four deliverable bonds, which is the cheapest-to-deliver?

Bond	A	B	C	D
Quote	125.69	90.31	87.6	128.56
Conversion Factor	1.1979	0.8109	0.8352	1.2249

a. Bond A
b. Bond B
c. Bond C
d. Bond D

EXAMPLE 8.5: FRM EXAM 2007—QUESTION 80

Consider an FRA (forward rate agreement) with the same maturity and compounding frequency as a Eurodollar futures contract. The FRA has a LIBOR underlying. Which of the following statements are true about the relationship between the forward rate and the futures rate?

a. The forward rate is normally higher than the futures rate.
b. They have no fixed relationship.
c. The forward rate is normally lower than the futures rate.
d. They should be exactly the same.

8.3 SWAPS

Swaps are agreements by two parties to exchange cash flows in the future according to a prearranged formula. Interest rate swaps have payments tied to an interest rate. The most common type of swap is the **fixed-for-floating** swap, where one party commits to pay a fixed percentage of notional against a receipt that is indexed to a floating rate, typically LIBOR. The risk is that of a change in the level of rates.

Other types of swaps are **basis swaps**, where both payments are indexed to a floating rate. For instance, the swap can involve exchanging payments tied to three-month LIBOR against a three-month Treasury bill rate. The risk is that of a change in the spread between the reference rates.

8.3.1 Instruments

Consider two counterparties, A and B, that can raise funds either at fixed or floating rates, $100 million over 10 years. A wants to raise floating, and B wants to raise fixed.

Table 8.2a displays capital costs. Company A has an **absolute advantage** in the two markets as it can raise funds at rates systematically lower than B. Company

TABLE 8.2a Cost of Capital Comparison

Company	Fixed	Floating
A	10.00%	LIBOR + 0.30%
B	11.20%	LIBOR + 1.00%