Exercise 7.3** Prove theorem 7.2 (monotonicity in the single-dimensional case under CS^+ or CS^-) without making the differentiability assumptions on $y(\cdot)$. (Hint: Use the methodology introduced in chapter 13 to prove that reaction curves are monotonic in separable sequential games.)

Exercise 7.4** A buyer and a seller sign a contract for the delivery of one unit of a good. The buyer has valuation v and designs the contract (is the principal). The seller will receive an outside offer equal to $\theta + \varepsilon$ for the single unit he produces. A contract specifies an unconditional payment t by the buyer to the seller, and a liquidated damage ℓ to be paid by the seller to the buyer if the seller breaches the contract and accepts the outside offer instead. The seller knows θ when signing the contract and learns ε after signing the contract and before deciding whom to serve. The independent random variables θ and ε are drawn from distributions $P(\cdot)$ and $\tilde{P}(\cdot)$, respectively. The expectation of ε is equal to 0, and both parties are risk neutral. The buyer screens the seller's type θ by offering a menu of contracts $\{t(\hat{\theta}), \ell(\hat{\theta})\}$.

(a) What are the analogues of the variables t and x in the text for this model?

(b) Show that the seller's *net* utility (given that he will accept the outside offer if he does not sign the contract), $U_1(\theta)$, satisfies $\dot{U}_1(\theta) = -\tilde{P}(\ell(\theta) - \theta)$.

(c) Show that the buyer sets liquidated damages under the real damage v:

$$\ell(\theta) = v - \frac{P(\theta)}{p(\theta)}$$

(where $p(\theta) \equiv P'(\theta)$). Interpret. (For answers, see Stole 1990b.)

Exercise 7.5* Consider the following insurance model with adverse selection. The insuree can have a low probability of accident ($\underline{\theta}$) or a high one ($\bar{\theta} > \underline{\theta}$), with probabilities $\underline{p}$ and $\bar{p}$, respectively. The insuree knows his probability of accident, but the insurance company (which is a monopolist and which offers a menu of contracts) does not. The insuree has objective function

$$u_1(W_1, W_2, \theta) = (1 - \theta)U(W_1) + \theta\, U(W_2),$$

where W_1 and W_2 are his net incomes in states of nature 1 (no accident) and 2 (accident) and U is his von Neumann-Morgenstern utility function ($U' > 0$, $U'' < 0$). With W_0 denoting the insuree's initial wealth and D the (monetary) damage in case of accident, the (risk-neutral) insurer's expected utility is

$$u_0(W_1, W_2, \theta) = (1 - \theta)(W_0 - W_1) + \theta(W_0 - D - W_2).$$

(a) Give a diagrammatic description of the optimal (two-contract) menu for the insurance monopolist. In particular, draw the status-quo (no contract) point in the (W_1, W_2) space, and indifference curves corresponding to the two types for both the insuree and the insurer. Show that the binding IC constraint is that the high-risk insuree would not want to mimic the low-risk insuree, and that the high-risk insuree gets full insurance $(\bar{W}_1 = \bar{W}_2)$. Argue informally that the high-risk insuree may or may not get a rent (depending on the probability $\bar{p}$), and that he gets a rent if some insurance is given to the low-risk insuree.

(b) Perform the same analysis as in question a, but use algebra. Hints: Use question a to guess which constraints are binding (ignore the others and check them later); let Γ denote the inverse function of the insuree's von Neumann-Morgenstern utility function U. Describe a menu as $(\underline{U}_1, \underline{U}_2)$, $(\bar{U}_1, \bar{U}_2)$, where $\underline{U}_1$ is the low type's number of utils in state 1, etc. Notice that the monopoly's objective is concave in these utility levels and the constraints are linear. Solve the monopoly's program. (For the answer, see Stiglitz 1977.)

Exercise 7.6** A labor-managed firm under contract with the government makes profit $\pi = \theta f(x) - K + t$, where $\theta f(x)$ is output $(f' > 0, f'' < 0)$, K is a known fixed cost, t is a subsidy (possibly negative) from the government, and x is the number of workers. The goverment observes x and t, but not θ and π (which are private information to the firm). The firm's objective function is profit per worker: $u_1 = \pi/x$.

(a) Show that an increasing function $x(\theta)$ is implementable if and only if the marginal productivity of labor exceeds its average productivity $(f' > f/x)$.

(b) Suppose that the government has objective function $u_0 = \theta f(x) - K - wx$, where w is the opportunity wage of workers, and has prior density $p(\theta)$ over $[\underline{\theta}, \bar{\theta}]$. Using question a, show that if $f' < f/x$, the optimal policy for the government is to "bunch" all types at a single contract (t, x). (This exercise is from Guesnerie and Laffont 1984.)

Exercise 7.7** An entrepreneur has a project that yields revenue R with probability θ and 0 with probability $1 - \theta$. A debt contract specifies a reimbursement t to the lender if the project is successful, and an amount of collateral $C \geq 0$ to be paid to the lender if the project fails. The value of the collateral is βC for the lender, where $0 \leq \beta < 1$. The project involves a fixed nonmonetary cost b for the entrepreneur (the opportunity cost of his time). The entrepreneur's expected utility, u_1, is 0 if he does not borrow (the project is not realized) and $\theta(R - t) - (1 - \theta)C - b$ if he borrows. The amount borrowed is fixed and is equal to 1. Assume $\theta R > b + 1$ for any θ. The lender's utility is $u_0 = \theta t + (1 - \theta)\beta C - 1$ if he lends, and 0 otherwise.

The entrepreneur has private information about θ, which takes value $\underline{\theta}$ with probability $\underline{p}$ and $\bar{\theta}$ with probability $\bar{p}$ $(\underline{p} + \bar{p} = 1)$. Suppose first that there is a single creditor, who offers a debt contract to the entrepreneur.

(a) Show that if the lender knew θ he would not require any collateral.

(b) Suppose that the lender does not know θ. Proceeding by analogy with the price-discrimination example of section 7.1, what do you think are the binding IR and IC constraints? Assuming that the creditor wants to lend whatever θ, show that choosing $\underline{C} > 0$ *tightens* the IC constraint and that the lender offers the *pooling* contract $\{t = R - b/\underline{\theta}, C = 0\}$. Explain intuitively the difference with the price-discrimination example. (Hint: Think of the sorting condition and of which type's allocation ought to be distorted.)

(c) Suppose now that there is a *competitive* credit market (many lenders). Argue that the relevant IC constraint is not the same as in question b. Show that if a zero-profit, separating equilibrium exists, the levels of collateral for types θ and $\bar{\theta}$ are

$$C = 0$$

and

$$C = \frac{\bar{\theta} - \underline{\theta}}{\theta(1 - \theta) - \beta(1 - \bar{\theta})\underline{\theta}}$$

(assuming that the entrepreneur's initial wealth is at least $\bar{C}$; otherwise credit rationing may occur). (This exercise is from Besanko and Thakor 1987. See also Bester 1985.)

Exercise 7.8* A monopolist faces a single consumer. The consumer has utility $u_1 = \theta q - t$, where q is consumption and t is the transfer to the monopolist. The monopolist has cost $cq^2/2$ and offers a sales contract to the consumer. The consumer has reservation utility 0.

(a) Compute the transfer and the consumption under full information about θ.

(b) Suppose from now on that the monopolist has incomplete information about θ, which takes the value $\underline{\theta}$ with probability p and $\bar{\theta}$ with probability $\bar{p}$. Assume that $\underline{\theta} > p\bar{\theta}$. The monopolist's utility is

$$p\left(t - c\frac{q^2}{2}\right) + \bar{p}\left(\bar{t} - c\frac{\bar{q}^2}{2}\right).$$

Compute the optimal nonlinear tariff. Show that the equilibrium utility of type θ is $\bar{S} = (\bar{\theta} - \underline{\theta})(\theta - \bar{p}\bar{\theta})/c\underline{p}$.

(c) Suppose now that the consumer can purchase at the fixed cost f an alternative (bypass) technology that allows him to produce any amount q of the same good at cost $\tilde{c}q^2/2$. Suppose for simplicity that the consumer can consume only the monopolist's good or the alternative good (but not a mix of both), and that

$$\frac{\theta^2}{2\tilde{c}} - f > \bar{S} > 0 > \frac{\underline{\theta}^2}{2\tilde{c}} - f.$$

Is the tariff derived in question b still optimal for the monopolist? Discuss what may be optimal for the monopolist - in particular, why it may be optimal to have $cq > \bar{\theta}$. For example, consider what happens when f decreases from $\bar{\theta}^2/2\bar{c} - S$.

Exercise 7.9** A regulated firm has cost $C = (\theta - e)q + f$, where q is output, e is effort, and f is a known fixed cost. The regulator observes C and q. The firm's utility is $u_1 = t - \psi(e)$, where t is a net transfer from the regulator ($\psi(0) = 0, \psi' > 0, \psi'' > 0$). The firm has reservation utility 0. The technology parameter θ takes values $\underline{\theta}$ with probability $\underline{p}$ and $\bar{\theta}$ with probability $\bar{p}$. The social welfare function is

$$u_0 = S(q) - R(q) - (1 + \lambda)(t + C - R(q)) + u_1,$$

where $S(q)$ is gross consumer surplus, $R(q) \equiv P(q)q = S'(q)q$ is the firm's revenue from selling quanity q, and $\lambda > 0$ is the shadow cost of public funds.

(a) Determine optimal quantities and effort when the regulator has perfect information. Show that price is determined by a Ramsey-type formula. (The Lerner index—price minus marginal cost over price—is equal to a fraction of the inverse of the elasticity of demand.)

(b) Suppose the regulator does not observe the components of C. Argue intuitively that the regulator will base the allocation on marginal cost $c = (C - f)/q$. Infer from this that (for a given marginal cost) the price is given by the same Ramsey formula as in question a, but that the marginal cost changes. Show that the firm chooses effect $\underline{e}$ when $\theta = \underline{\theta}$ and $\bar{e}$ when $\theta = \bar{\theta}$, where

$$\psi'(\underline{e}) = \underline{q},$$

$$\psi'(\bar{e}) = \bar{q} - \frac{\lambda \underline{p}}{\bar{p}(1 + \lambda)} \Phi'(\bar{e}),$$

and

$$\Phi(e) \equiv \psi(e) - \psi(e - (\bar{\theta} - \underline{\theta})).$$

(This part of the exercise is from Laffont and Tirole 1986.)

(c) A regulator is responsible for two public utilities ($i = 1, 2$) located in separate geographic areas. Each utility produces a fixed amount of output (normalized at $q = 1$) and has a cost function $C_i = \alpha + \beta_i - e_i$, where α can be interpreted as some shock common to both firms, and β_i is an idiosyncratic shock with β_i independent of β_j (so $\theta_i = \alpha + \beta_i$). Social welfare is

$$u_0 = \sum_{i=1}^{2} [S - (1 + \lambda)(C_i + t_i) + u_i],$$

where t_i is the net transfer paid by the regulator to firm i, $u_i = t_i - \psi(e_i)$ is firm i's rent and S is the social surplus associated with a firm's production. Suppose first that $\beta_i = 0$ is common knowledge, but α is known only to the firms (common shock). Show that by offering the contract

$$t_i = -(C_i - C_j) + \psi(e^*),$$

where $\psi'(e^*) = 1$, to each firm, the regulator does as well as under full information. Explain. Suppose, second, that both α and β_i are random (common and idiosyncratic shock). Show that the regulator's lack of information about α has no welfare consequence as long as the two firms do not collude.

Exercise 7.10

(a)* A seller owns one unit of a good, which she values at c. (The value c can be thought of as the quality of the good.) A buyer may buy the unit from the seller. The seller's valuation is equal to $\underline{c}$ or $\bar{c}$ with equal probabilities (where $\underline{c} < \bar{c}$) and is private information to the seller. The buyer's valuation for the good is $\bar{v}$ if $c = \bar{c}$ and $\underline{v}$ if $c = \underline{c}$, where $\bar{v} > \bar{c}$ and $\underline{v} > \underline{c}$. The buyer thus has no private information. Assume that $(\bar{v} + \underline{v})/2 < \bar{c}$ (which implies that $\bar{c} > \underline{v}$). Show that efficiency is inconsistent with the seller's and the buyer's individual rationality and incentive compatibility. Give two reasons why the Myerson-Satterthwaite inefficiency result (theorem 7.5) cannot be applied here. With the quality interpretation in mind, suppose there are a continuum of sellers and a continuum of buyers. Buyers are homogeneous and have the same valuation for the good (which is either $\bar{v}$ or $\underline{v}$, depending on the quality of the seller's good). Each seller has probability $\frac{1}{2}$ of having a high-quality item (and therefore valuation $\bar{v}$). Qualities are "independent" across sellers. (Ignore technical subtleties concerning a continuum of independent variables.) Show that the inefficiency result carries over. (This is Akerlof's (1970) "lemons problem.")

(b)** Replicate the exchange economy of question a, but in such a way that each seller has private information that is relevant to a single buyer instead of to all buyers. Suppose that there are many duplexes (a continuum of them). In each duplex, there is one inhabitant on the first floor, who owns a smoke alarm, and one inhabitant on the second floor, who owns none. Second-floor residents have higher valuations (v) than first-floor residents (c) for the smoke alarm, but both valuations depend on whether the first-floor resident smokes ($\bar{v}$ and $\bar{c}$) or not ($\underline{v}$ and $\underline{c}$). Half of the first-floor residents smoke and half do not. Construct an efficient, individually rational, and incentive-compatible trading mechanism. (Hint: Construct a mechanism in which trade, if it occurs, occurs at a fixed price.) (Gul and Postlewaite (1988) give general limit efficiency results when an agent's private information is relevant only to a fixed number—one, in the above example—of agents, independent of the total number of agents in the economy.)

Exercise 7.11** A firm's profit is $x = \theta + e$, where e is the (single) manager's effort and θ is a productivity parameter known only to the manager.

θ takes value $\underline{\theta}$ with probability $\underline{p}$ and $\bar{\theta}$ with probability $\bar{p}$. The manager's objective is $u_1 = t - g(e)$, and the shareholders' utility function is $u_0 = x - t - Kq$, where q is the probability of audit and K the cost of auditing. The shareholders offer a contract $\{x(\hat{\theta}), t(\hat{\theta}), q(\hat{\theta})\}$, where $\hat{\theta}$ is the firm's announcement of its productivity parameter. If it announces $\hat{\theta}$, the firm is required to attain profit level, $x(\hat{\theta})$. After production takes place, the shareholders audit with probability $q(\hat{\theta})$. The audit yields a signal $\tilde{\theta} \in \{\underline{\theta}, \bar{\theta}\}$. The probability that the signal is truthful ($\tilde{\theta} = \theta$) is $r \in [\frac{1}{2}, 1]$. If $\tilde{\theta} = \hat{\theta}$, the manager receives $t(\hat{\theta})$. If $\tilde{\theta} \neq \hat{\theta}$, the manager, who is protected by limited liability, receives 0 (if you have time, show that this "maximal punishment" is indeed optimal).

Show that $q(\bar{\theta}) = 0$. Show that (for "K not too big") auditing always occurs for r close to 1 and $\hat{\theta} = \underline{\theta}$, and that when K varies there are three regimes (including one in which the first-best effort is attained). Indicate how $x(\underline{\theta})$ changes with r and K. Explain. (For more on auditing, see Baron and Besanko 1984b and Kofman and Lawarrée 1989.)

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IV DYNAMIC GAMES OF INCOMPLETE INFORMATION

8 Equilibrium Refinements: Perfect Bayesian Equilibrium, Sequential Equilibrium, and Trembling-Hand Perfection

8.1 Introduction†

The concept of subgame perfection, introduced in chapter 3, has no bite in games of incomplete information, even if the players observe one another's actions at the end of each period: Since the players do not know the others' types, the start of a period does not form a well-defined subgame until the players' posterior beliefs are specified, and so we cannot test whether the continuation strategies are a Nash equilibrium.[1]

The complications that incomplete information causes are easiest to see in "signaling games"—leader-follower games in which only the leader has private information. The leader moves first; the follower observes the leader's action, but not the leader's type, before choosing his own action. One example is Spence's (1974) famous model of the job market. In that model, the leader is a worker who knows her productivity and must choose a level of education; the follower, a firm (or a number of firms), observes the worker's education level but not her productivity and then decides what wage to offer her. The spirit of subgame perfection in this model is that, for any education level the worker chooses, the continuation play—that is, the offered wage—should be "reasonable" in the sense of being consistent with equilibrium play in the continuation game. Now, the reasonable wage to offer will typically depend on the firm's beliefs about the worker's productivity, which in turn can depend on the worker's observed level of education. If this level is one to which the equilibrium assigns positive probability, the posterior distribution of the worker's productivity can be computed using Bayes' rule. However, Bayes' rule does not determine the posterior distribution over productivity after the observation of an education level to which the equilibrium assigns probability 0, and the reasonable wage will depend on which posterior distribution is specified. Thus, in order to extend subgame perfection to these games, we will need to specify how players update their beliefs about their opponents' types after an observation that has prior probability 0.

This chapter starts by developing two solution concepts that extend subgame perfection to games of incomplete information: "perfect Bayesian equilibrium" and Kreps and Wilson's (1982a) "sequential equilibrium." Perfect Bayesian equilibrium results from combining the ideas of subgame perfection, Bayesian equilibrium, and Bayesian inference: Strategies are required to yield a Bayesian equilibrium in every "continuation game" given the posterior beliefs of the players, and the beliefs are required to be updated in accordance with Bayes' law whenever it is applicable. Sequential equilibrium is similar, but it imposes more restrictions on the way players update their beliefs. In the signaling games described above,

1. Formally the only proper subgame of a game of incomplete information is the whole game, so any Nash equilibrium is subgame perfect.

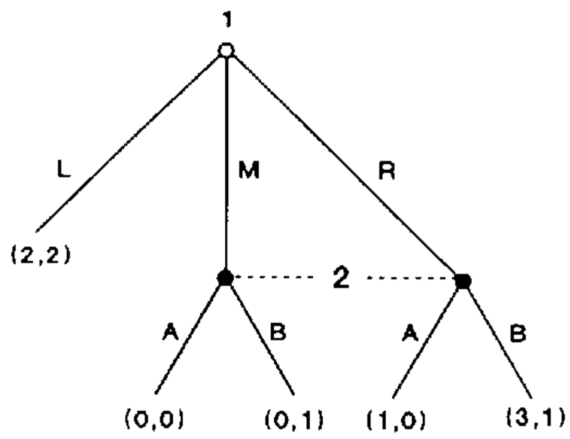


Figure 8.1

these two concepts are identical, and they place very weak restrictions on beliefs following probability-0 events: Any posterior beliefs that assign probability 1 to the support of the prior distribution of types are allowed. In more complex games, the two concepts can impose more restrictions on the allowed beliefs, and thus more restrictions on which equilibria are reasonable.

Since incomplete information is modeled as a kind of imperfect information (Harsanyi 1967–68), it should not be surprising that similar issues concerning out-of-equilibrium beliefs arise in games where the information is complete but imperfect. What matters is that a player's actions can convey information to the other players; this information can be anything that one player has observed and the others have not, including the player's own past actions. The simple example illustrated in figure 8.1 is due to Selten (1975). In this game, player 1 has three actions: L, M, and R. If he plays L, the game ends, with payoffs (2, 2). If he plays M or R, then player 2 must choose between A and B, and when making this choice player 2 does not know whether player 1 chose M or R. (This is why the dashed lines connect the nodes following M and R—in the terminology of section 3.3, they belong to the same information set for player 2.) If player 1 chooses M and player 2 chooses A, the payoffs are (0, 0); the payoffs to (M, B) are (0, 1), the payoffs to (R, A) are (1, 0), and the payoffs to (R, B) are (3, 1).

This game has two pure-strategy Nash equilibria, (L, A) and (R, B), both of which are subgame perfect. (To see that (L, A) is subgame perfect, note that since player 2 does not know player 1's action when choosing between A and B, we cannot test whether playing A is part of a "Nash equilibrium" from that point on. Any mixed-strategy profile where player 1 plays L and player 2 plays A with probability at least $\frac{1}{2}$ is a Nash equilibrium as well.) Note, though, that for any specification of player 2's "beliefs" about the relative probability of M and R when player 1 deviates and does not play L, player 2's optimal action is to play B, so that playing A in this game is analogous to offering the worker a wage that is not reasonable for any posterior beliefs about her productivity.

The simple version of perfect Bayesian equilibrium we develop in this chapter is limited to multi-stage games with observed actions and incomplete information, which we will simply call "multi-stage games" in this chapter. In contrast, sequential equilibrium is defined for general games and does rule out the equilibrium (L, A) in figure 8.1. This equilibrium is also ruled out by Selten's (1975) concept of "trembling-hand perfection," which historically preceded Kreps and Wilson's sequential equilibrium, and is quite closely related to it, as both papers develop refinements by considering perturbed games in which players "tremble" and play suboptimal actions with vanishingly small probability.

We reverse the historical order and discuss sequential equilibrium before trembling-hand perfection, because sequential equilibrium emphasizes the formation of the players' beliefs, and we find this approach easier to explain. Another way our development may be idiosyncratic is that, while most of the literature on refinements considers general extensive forms, we will begin by studying multi-stage games with observed actions, where the only relevant private information is each player's knowledge of his own type. This is the kind of private information most frequently encountered in the economics literature, and it is prominent in the applications of chapters 9 and 10.[2, 3]

Section 8.2 introduces the concept of perfect Bayesian equilibrium (PBE). Subsection 8.2.1 begins with the special case of PBE in signaling games. Subsection 8.2.2 gives applications to a game of predatory pricing (inspired by Kreps and Wilson (1982b) and Milgrom and Roberts (1982b)) and to Spence's model of job-market signaling; readers familiar with these or with similar examples will want to skip this subsection. Subsection 8.2.3 extends PBE to multi-stage games, and applies it to a repeated version of the public-good game of example 6.1.

PBE imposes more restrictions on beliefs than Bayes' rule alone, as it imposes some restrictions on beliefs after probability-0 events. Specifically, when initial beliefs are that types are independent, PBE requires that the posterior beliefs be that types are independent, that any two players have the same beliefs about the type of a third, and that if player i deviates and player j does not then beliefs about player j are updated in accordance with Bayes' rule.

2. This is not to say that no other games are economically relevant. For example, situations of moral hazard do not correspond to multi-stage games with observed actions.

3. Many early models of uncertainty and information were dynamic in nature and made some implicit use of perfect Bayesian equilibrium. Examples include the disarmament game of Aumann and Maschler (1966), the market games of Akerlof (1970) and Spence (1974), and Ortega-Reichert's (1967) analysis of repeated first-bid auctions. The first formal application of the idea was that of Milgrom and Roberts (1982a), which was followed by those of Kreps and Wilson (1982b) and Milgrom and Roberts (1982b).

Section 8.3 develops the concept of sequential equilibrium, which is defined for general extensive-form games and which places even more restrictions on beliefs after probability-0 events than PBE does. In a sequential equilibrium, players' beliefs are as if there were a small probability of a "tremble" or mistake at each information set, with the trembles at each information set being statistically independent of trembles at the others and with the probability of each tremble depending only on the information available at that information set. We discuss the desiderata that Kreps and Wilson used to motivate their concept, and compare it with PBE in the special case of multi-stage games. As we explain, sequential equilibrium is stronger than PBE unless the game has at most two periods or each player has at most two types. To illustrate the sequential-equilibrium definition, we then develop an "extended" version of PBE that is equivalent to sequential equilibrium in general multi-stage games with observed actions.

Section 8.4 describes related (and historically prior) refinements based on the strategic form. The focus on the strategic form does not imply a neglect of extensive-form considerations such as perfection. Selten's original (1975) idea was to introduce small trembles so that all pure strategies have positive probability, and to require that players optimize (subject to the constraint that they tremble with small probability) against their opponents' garbled strategies. Because all outcomes have positive probability, the issue of perfection—that in a Nash equilibrium a player can costlessly play a crazy strategy in some unreached subgame—does not arise. A "trembling-hand perfect equilibrium" is a limit of Nash equilibria with trembles as the trembles tend to 0. The sets of trembling-hand perfect and sequential equilibria coincide for almost all games. Section 8.4 then describes a refinement of trembling-hand perfect equilibrium due to Myerson (1978). A "proper equilibrium" requires that a player tremble less on strategies that are worse responses. Chapter 11 discusses stronger refinements related to the idea of "forward induction."

Finally, before introducing more equilibrium refinements, we should note that, since the concepts of this chapter strengthen subgame perfection, they are subject to the reservations we expressed in chapter 3, as well as to other reservations we will develop along the way. In particular, all these refinements suppose that all players expect an opponent to continue to play according to the equilibrium strategies even after that opponent deviates from the equilibrium path.

8.2 Perfect Bayesian Equilibrium in Multi-Stage Games of Incomplete Information[†]

8.2.1 The Basic Signaling Game

Signaling games are the simplest kind of game in which the issues of updating and perfection both arise. In these games, there are two players.

Player 1 is the leader (also called the sender, because he sends a signal), and player 2 is the follower (or receiver). Player 1 has private information about his type θ in Θ and chooses action a_1 in A_1. (We delete the subscript on player 1's type, as this will not lead to confusion.) Player 2, whose type is common knowledge for simplicity, observes a_1 and chooses a_2 in A_2. The spaces of mixed actions are $\mathscr{A}_1$ and $\mathscr{A}_2$ with elements α_1 and α_2. Player i's payoff is denoted $u_i(\alpha_1, \alpha_2, \theta)$. Before the game begins, it is common knowledge that player 2 has prior beliefs p about player 1's type. A strategy for player 1 prescribes a probability distribution $\sigma_1(\cdot|\theta)$ over actions a_1 for each type θ. A strategy for player 2 prescribes a probability distribution $\sigma_2(\cdot|a_1)$ over actions a_2 for each action a_1. Type θ's payoff to strategy $\sigma_1(\cdot|\theta)$ when player 2 plays $\sigma_2(\cdot|a_1)$ is

$$u_1(\sigma_1, \sigma_2, \theta) = \sum_{a_1} \sum_{a_2} \sigma_1(a_1|\theta)\sigma_2(a_2|a_1)u_1(a_1, a_2, \theta).$$

Players 2's (*ex ante*) payoff to strategy $\sigma_2(\cdot|a_1)$ when player 1 plays $\sigma_1(\cdot|\theta)$ is

$$\sum_{\theta} p(\theta)\left(\sum_{a_1} \sum_{a_2} \sigma_1(a_1|\theta)\sigma_2(a_2|a_1)u_2(a_1, a_2, \theta)\right).$$

Player 2, who observes player 1's move before choosing her own action, should update her beliefs about θ and base her choice of a_2 on the posterior distribution $\mu(\cdot|a_1)$ over Θ. How is this posterior formed? In a Bayesian equilibrium, player 1's action can depend on his type. Let $\sigma_1^*(\cdot|\theta)$ denote this strategy. Knowing σ_1^* and observing a_1, player 2 can use Bayes' rule to update $p(\cdot)$ into $\mu(\cdot|a_1)$. The natural extension of the subgame-perfect equilibrium to the signaling game is the perfect Bayesian equilibrium, which requires that player 2 maximize her payoff conditional on a_1 for each a_1, where the conditional payoff to strategy $\sigma_2(\cdot|a_1)$ is

$$\sum_{\theta} \mu(\theta|a_1)u_2(a_1, \sigma_2(\cdot|a_1), \theta) = \sum_{\theta} \sum_{a_2} \mu(\theta|a_1)\sigma_2(a_2|a_1)u_2(a_1, a_2, \theta).$$

Definition 8.1 A *perfect Bayesian equilibrium* (PBE) of a signaling game is a strategy profile σ^* and posterior beliefs $\mu(\cdot|a_1)$ such that:

$$(\mathrm{P}_1) \quad \forall\, \theta,\ \sigma_1^*(\cdot|\theta) \in \arg\max_{\alpha_1} u_1(\alpha_1, \sigma_2^*, \theta),$$

$$(\mathrm{P}_2) \quad \forall\, a_1,\ \sigma_2^*(\cdot|a_1) \in \arg\max_{\alpha_2} \sum_{\theta} \mu(\theta|a_1)u_2(a_1, \alpha_2, \theta),$$

and

$$(\mathrm{B}) \quad \mu(\theta|a_1) = p(\theta)\sigma_1^*(a_1|\theta) \Big/ \sum_{\theta' \in \Theta} p(\theta')\sigma_1^*(a_1|\theta')$$

$$\text{if } \sum_{\theta' \in \Theta} p(\theta')\sigma_1^*(a_1|\theta') > 0,$$

and $\mu(\cdot|a_1)$ is any probability distribution on Θ

if $\sum_{\theta' \in \Theta} p(\theta')\sigma_1^*(a_1|\theta') = 0.$

P_1 and P_2 are the perfection conditions. P_1 says that player 1 takes into account the effect of a_1 on player 2's action[4]; P_2 states that player 2 reacts optimally to player 1's action given her posterior beliefs about θ. B corresponds to the application of Bayes' rule. Note that if a_1 is not part of player 1's optimal strategy for some type, observing a_1 is a probability-0 event, and Bayes' rule does not pin down posterior beliefs. *Any* posterior beliefs $\mu(\cdot|a_1)$ are then admissible, and so any action a_2 can be played that is a best response for some beliefs. (This means that the only actions excluded are those which are dominated given that a_1 is played.) Indeed, the purpose of the refinements of the perfect Bayesian equilibrium concept is to put some restrictions on these posterior beliefs. As we will see in section 8.3, the concept of PBE defined here is equivalent to sequential equilibrium for the class of signaling games.

Thus, a PBE is simply a set of strategies and beliefs such that, at any stage of the game, strategies are optimal given the beliefs, and the beliefs are obtained from equilibrium strategies and observed actions using Bayes' rule.

Note the link between strategies and beliefs: The beliefs are consistent with the strategies, which are optimal given the beliefs. Because of this circularity, PBE cannot be determined by backward induction when there is incomplete information, even if players move one at a time. (Recall that, with perfect information, perfect equilibria can be determined by backward induction.)

8.2.2 Examples of Signaling Games

To help build intuition, we will analyze two examples of signaling games in a fair bit of detail. Readers already familiar with the ideas of separating and pooling equilibrium in, say, the Milgrom-Roberts limit-pricing model should probably skip to subsection 8.2.3.

Example 8.1: Two-Period Reputation Game

The following is a much-simplified version of the Kreps-Wilson (1982b)–Milgrom-Roberts (1982b) reputation model. There are two firms ($i = 1, 2$). In period 1, both firms are in the market. Only firm 1 (the "incumbent") takes an action a_1. The action space has two elements: "prey" and "accommodate." Firm 2 (the "entrant") has profit D_2 if firm 1 accommodates and

4. Recall that a mixed strategy is a best response if all actions in its support maximize the player's payoff, so condition P_1 is equivalent to

$$a_1 \in \text{support } \sigma_1^*(\cdot|\theta) \Leftrightarrow a_1 \in \arg\max_{\tilde{a}_1} u_1(\tilde{a}_1, \sigma_2^*(\cdot|\tilde{a}_1), \theta).$$

P_2 if firm 1 preys, with $D_2 > 0 > P_2$. Firm 1 has one of two potential types: "sane" and "crazy." A sane firm 1 makes D_1 when it accommodates and P_1 when it preys, where $D_1 > P_1$. Thus, a sane firm prefers to accommodate rather than to prey. However, it would prefer to be a monopoly, in which case it would make $M_1 > D_1$ per period. When crazy, firm 1 enjoys predation and thus preys (its utility function is such that it is always worth preying). Let p (respectively, $1 - p$) denote the prior probability that firm 1 is sane (respectively, crazy).

In period 2, only firm 2 chooses an action a_2. This action can take two values: "stay" and "exit." If firm 2 stays, it obtains payoff D_2 if firm 1 is actually sane and P_2 if it is crazy; if firm 2 exits, it obtains payoff 0. The idea is that, unless it is crazy, firm 1 will not prey in the second period, because there is no point to building or keeping a reputation at the end. (This assumption can be derived more formally from the description of the second-period competition.) The sane firm gets D_1 if firm 2 stays and M_1 if firm 2 exits. We let δ denote the discount factor between the two periods.

We presumed that the crazy type always preys. The interesting thing to study is thus the sane type's behavior. From a static point of view, it would want to accommodate in the first period; however, by preying it might convince firm 2 that it is of the crazy type, and thus induce exit (as $P_2 < 0$) and increase its second-period profit.

Let us first start with a taxonomy of potential perfect Bayesian equilibria. A *separating equilibrium* is an equilibrium in which the two types of firm 1 choose two different actions in the first period. Here, this means that the sane type chooses to accommodate. Note that in a separating equilibrium firm 2 has complete information in the second period:

$$\mu(\theta = \text{sane} \mid a_1 = \text{accommodate}) = 1$$

and

$$\mu(\theta = \text{crazy} \mid a_1 = \text{prey}) = 1.$$

A *pooling equilibrium* is an equilibrium in which firm 1's two types choose the same action in the first period. Here, this means that the sane type preys. In a pooling equilibrium firm 2 does not update its beliefs when observing the equilibrium action:

$$\mu(\theta = \text{sane} \mid a_1 = \text{prey}) = p.$$

There can also be *hybrid* or *semi-separating equilibria*. In the reputation game, the sane type may randomize between preying and accommodating, i.e., between pooling and separating. The posterior beliefs are then

$$\mu(\theta = \text{sane} \mid a_1 = \text{prey}) \in (0, p)$$

and

$\mu(\theta = \text{sane} \mid a_1 = \text{accommodate}) = 1.$

When do separating equilibria exist? In these equilibria, the sane type accommodates, thus revealing its type, and its payoff is $D_1(1 + \delta)$. (Firm 2 stays in because it expects $D_2 > 0$ in the second period.) If the sane type preyed, it would convince firm 2 that it was crazy and would obtain $P_1 + \delta M_1$. Thus, a necessary condition for the existence of a separating equilibrium is

$$\delta(M_1 - D_1) \leq (D_1 - P_1). \tag{8.1}$$

Conversely, suppose that equation 8.1 is satisfied, and consider the following strategies and beliefs: The sane incumbent accommodates, and the entrant (correctly) infers that the incumbent is sane when observing accommodation and therefore stays; the crazy incumbent preys and the entrant (correctly) infers that the incumbent is crazy when observing predation and therefore exits. Clearly, these strategies and beliefs form a PBE, so equation 8.1 is sufficient as well as necessary for the existence of a separating equilibrium.

In a pooling equilibrium, both types of incumbent prey, so the entrant's posterior beliefs are the same as its prior when predation is observed. Since predation is costly for the sane incumbent, it will prey only if doing so induces a positive probability of exit. Thus, a necessary condition for a pooling equilibrium is that the entrant's expected second-period payoff if it stays in is negative; that is,

$$pD_2 + (1 - p)P_2 \leq 0. \tag{8.2}$$

Conversely, assume that equation 8.2 holds, and consider the following strategies and beliefs: Both types prey; the entrant has posterior beliefs $\mu(\theta = \text{sane} \mid a_1 = \text{prey}) = p$ and $\mu(\theta = \text{sane} \mid a_1 = \text{accommodate}) = 1$, and stays in if and only if accommodation is observed. The sane type's equilibrium profit is $P_1 + \delta M_1$; it would receive $D_1(1 + \delta)$ from accommodation. Thus, if equation 8.1 is violated, the proposed strategies and beliefs form a pooling PBE. (Note that if equation 8.2 is satisfied with equality, there exists a *continuum* of such equilibria.[5])

We leave it to the reader to check that if both equation 8.1 and equation 8.2 are violated, the unique equilibrium is a hybrid PBE (with the entrant randomizing when observing predation and the sane incumbent randomizing between preying and accommodating[6]).

5. When $pD_2 + (1 - p)P_2 = 0$, any probability $x \geq \bar{x}$ that the entrant exits induces the sane incumbent to prey, where $\delta \bar{x}(M_1 - D_1) = D_1 - P_1$, so $0 < \bar{x} < 1$.

6. In this equilibrium, the entrant exits with the probability $\bar{x}$ defined in the previous footnote, and the sane incumbent preys with probability $\bar{y}$ such that $\tilde{p} = p\bar{y}/(p\bar{y} + 1 - p)$, where $\tilde{p}D_2 + (1 - \tilde{p})P_2 = 0$.

Remark The (generic) uniqueness of the PBE in this model is due to the fact that the "strong" type (the crazy incumbent) is assumed to always prey. Thus, predation is not a probability-0 event and, furthermore, if the sane type accommodates with positive probability, then accommodation reveals that player 1 is sane. The next example illustrates a more complex and a more common structure, for which refinements of the PBE concept are required if one insists on uniqueness of equilibrium.

Example 8.2: Spence's Education Game

Spence (1974) developed the following model of the choice of education level: Player 1 (a worker) chooses a level of education $a_1 \geq 0$. His private cost of investing a_1 units in education is a_1/θ, where θ is his type or "ability." The worker's productivity in a firm is equal to θ (to simplify, it is not affected by education). Player 2's (the firm's) objective is to minimize the quadratic difference of the wage a_2 offered to player 1 and player 1's productivity, so player 2 offers the expected productivity $a_2(a_1) = \mathrm{E}(\theta|a_1)$ in equilibrium. (Alternatively, we could suppose that there are several firms who make simultaneous wage offers.) Player 1's objective function is $a_2 - a_1/\theta$.

Player 1 has two possible types, θ' and θ'', with $0 < \theta' < \theta''$; the probabilities of these types are p' and p'', respectively. Player 1 knows θ, but player 2 does not.

Let σ_1' and σ_1'' denote the equilibrium strategies of types θ' and θ''. Note that if $a_1' \in \text{support } \sigma_1'$ and $a_1'' \in \text{support } \sigma_1''$, then $a_1' \leq a_1''$.[7] For, from equilibrium behavior,

$$a_2(a_1') - a_1'/\theta' \geq a_2(a_1'') - a_1''/\theta' \tag{8.3}$$

and

$$a_2(a_1'') - a_1''/\theta'' \geq a_2(a_1') - a_1'/\theta'' \tag{8.4}$$

Adding up these two inequalities yields $(1/\theta' - 1/\theta'')(a_1'' - a_1') \geq 0$, or $a_1' \leq a_1''$.

As in the reputation game of example 8.1, we can distinguish among separating, pooling, and hybrid equilibria.

In a *separating equilibrium*, the low-productivity worker reveals his type and therefore receives a wage equal to θ'. He therefore must choose $a_1' = 0$; if he did not, he would strictly gain by choosing $a_1' = 0$, because he would save on the education cost and would receive a wage which is necessarily a convex combination of θ' and θ'' and therefore is at least equal to θ'. Let $a_1'' > 0$ denote the equilibrium action of type θ'' (note that in a separating equilibrium type θ'' cannot play a mixed strategy, because all his equilibrium actions yield the same wage θ'' and therefore type θ'' prefers the

7. This monotonicity property is a special case of the general result in theorem 7.2.

one with the lowest education level). In order for $(a_1' = 0, a_1'')$ to be part of a separating equilibrium, it must be the case that type θ' does not prefer a_1'' to a_1':

$$\theta' \geq \theta'' - a_1''/\theta'$$

or

$$a_1'' \geq \theta'(\theta'' - \theta'). \tag{8.5}$$

Similarly, type θ'' cannot prefer a_1' to a_1'':

$$a_1'' \leq \theta''(\theta'' - \theta'). \tag{8.6}$$

Hence, $\theta'(\theta'' - \theta') \leq a_1'' \leq \theta''(\theta'' - \theta')$.

Conversely, suppose that a_1'' belongs to this interval. Consider the beliefs

$$\{\mu(\theta' | a_1) = 1 \text{ if } a_1 \neq a_1'', \mu(\theta' | a_1'') = 0\}.$$

Clearly, the two types prefer $a_1 = 0$ to any $a_1 \notin \{0, a_1''\}$, because any such a_1 yields the low wage θ' anyway. Because θ' prefers 0 to a_1'' (equation 8.5) and θ'' prefers a_1'' to 0 (equation 8.6), we have a continuum of separating equilibria. This continuum illustrates how the leeway in specifying off-the-equilibrium-path beliefs leads to a multiplicity of equilibria. We used the "pessimistic" beliefs under which any action other than a_1'' convinces player 2 that player 1 is the low type θ'. However, the separating equilibria can be supported by less extreme posterior beliefs. In particular, we can specify that $\mu(\theta' | a_1) = 0$ for all $a_1 \geq a_1''$, so that the posterior beliefs are monotonic in a_1, and we can use beliefs $\mu(\theta' | a_1)$ that are continuous in a_1.[8]

In a *pooling equilibrium*, both types choose the same action: $\tilde{a}_1 = a_1' = a_1''$. The wage is then $a_2(\tilde{a}_1) = p'\theta' + p''\theta''$. The easiest way to support $\tilde{a}_1$ as a pooling outcome is to assign pessimistic beliefs $\mu(\theta' | a_1) = 1$ to any action $a_1 \neq \tilde{a}_1$, as this minimizes both types' temptation to deviate. Therefore, $\tilde{a}_1$ is a pooling-equilibrium education level if and only if, for each θ,

$$\theta' \leq p'\theta' + p''\theta'' - \tilde{a}_1/\theta.$$

Since $\theta' < \theta''$, type θ' is the most tempted to deviate to $a_1 = 0$, to minimize education costs, and the binding constraint is

8. It is interesting to note that, of this continuum of separating equilibria, all but the "least-cost" one, where $a_1'' = \theta'(\theta'' - \theta') \equiv a_1^*$, can be eliminated by the following argument: No matter what education level player 1 chooses, player 2 should never choose any wage outside the interval $[\theta', \theta'']$. If player 1 realizes this, then type θ' will never choose any $a_1 > a_1^*$. If player 2 realizes that this is so, then she should respond to $a_1 > a_1^*$ with wage θ''; in that case, type θ'' will never choose $a_1 > a_1^*$. (This argument can be viewed either as an extension of the concept of iterated conditional dominance defined in chapter 4 or as an implication of iterated weak dominance.) However, with three types, θ', θ'', and θ''', $0 < \theta' < \theta'' < \theta'''$, this argument has little force. If as before we let a_1^* denote the education level where type θ' is indifferent between $(0, \theta')$ and (a_1^*, θ''), then even when the out-of-equilibrium wages are restricted to the interval $[\theta'', \theta''']$ type θ' may still be willing to choose $a_1 > a_1^*$.

$$\tilde{a}_1 \leq p''\theta'(\theta'' - \theta'), \tag{8.7}$$

so there is also a continuum of pooling equilibria. We leave it to the reader to derive the set of hybrid equilibria.

8.2.3 Multi-Stage Games with Observed Actions and Incomplete Information^{++}

We now consider a more general class of games we call "multi-stage games with observed actions and incomplete information." Each player i has a type θ_i in a finite set Θ_i. Letting $\theta \equiv (\theta_1, \ldots, \theta_I)$, we assume for the time being that types are independent, so that the prior distribution p is the product of marginals; that is,

$$p(\theta) = \prod_{i=1}^{I} p_i(\theta_i),$$

where $p_i(\theta_i)$ is the probability that player i's type is θ_i. At the beginning of the game, each player learns his type but is given no information about his opponents' types.

As in the multi-stage games of chapters 4, 5, and 13, these games are played in periods $t = 0, 1, 2, \ldots, T$, with the property that, at each period t, all players simultaneously choose an action that is revealed at the end of the period. (Recall that the set of feasible actions can be dependent on time and history, so that games with sequential moves such as the signaling game are included.) Players never receive additional observations of θ. For notational simplicity, we assume that each player's action set at each date is type-independent. Let $a_i^t \in A_i(h^t)$ denote player i's date-t action, $a^t = (a_1^t, \ldots, a_I^t)$ the vector of date-t actions, and let $h^t = (a^0, \ldots, a^{t-1})$ denote history at the beginning of date t. A behavior strategy σ_i maps the set of possible histories and types into the action spaces: $\sigma_i(a_i | h^t, \theta_i)$ is the probability of a_i given h^t and θ_i. Player i's payoff is $u_i(h^{T+1}, \theta)$.

To extend the spirit of subgame perfection to these games, we would like to require that the strategies yield a Bayesian Nash equilibrium, not only for the whole game, but also for the "continuation games" starting in each period t after every possible history h^t. Of course, these continuation games are not "proper subgames" because they do not stem from a singleton information set. Thus, to make the continuation games into true games we must specify the players' beliefs at the start of each continuation game. We denote player i's conditional probability that his opponents' types are θ_{-i} by $\mu_i(\theta_{-i} | \theta_i, h^t)$, and assume that it is defined for all players i, dates t, histories h^t, and types θ_i.

What restrictions should be imposed on player i's beliefs? Economic applications of incomplete-information games with independent types have typically made the following assumptions either explicitly or implicitly:

B(i) Posterior beliefs are independent, and all types of player i have the same beliefs: For all θ, t, and h^t,

$$\mu_i(\theta_{-i}|\theta_i, h^t) = \prod_{j \neq i} \mu_i(\theta_j | h^t).$$

B(i) requires that even unexpected observations do not make player i believe that his opponents' types are correlated.

B(ii) Bayes' rule is used to update beliefs from $\mu_i(\theta_j | h^t)$ to $\mu_i(\theta_j | h^{t+1})$ whenever possible: For all i, j, h^t, and $a_j^t \in A_j(h^t)$, if there exists $\hat{\theta}_j$ with $\mu_i(\hat{\theta}_j | h^t) > 0$ and $\sigma_j(a_j^t | h^t, \hat{\theta}_j) > 0$ (that is, player i assigns a_j^t positive probability given h^t), then, for all θ_j,

$$\mu_i(\theta_j | (h^t, a^t)) = \frac{\mu_i(\theta_j | h^t)\sigma_j(a_j^t | h^t, \theta_j)}{\sum_{\tilde{\theta}_j} \mu_i(\tilde{\theta}_j | h^t)\sigma_j(a_j^t | h^t, \tilde{\theta}_j)}.$$

B(ii) is stronger than simply using Bayes' rule in the usual fashion, as it applies to updating from period t to period $t + 1$ when the history h^t at period t has probability 0, and to beliefs about player j when h^t has positive probability and some player $k \neq j$ chooses a probability-0 action at date t. The motivation for this requirement is that if $\mu_i(\cdot | h^t)$ represents player i's beliefs given h^t, and nothing "surprising" occurs at t, then player i should use Bayes' rule to form his beliefs in period $t + 1$.

Note that B(ii) does not restrict the way beliefs about player j are updated if player j's period-t action had conditional probability 0.

The next condition says that even if player j does deviate at period t, the updating process should not be influenced by the actions of other players.

B(iii) For all h^t, i, j, θ_j, a^t, and $\hat{a}^t$,

$$\mu_i(\theta_j | (h^t, a^t)) = \mu_i(\theta_j | (h^t, \hat{a}^t)) \text{ if } a_j^t = \hat{a}_j^t.$$

This condition might be called "no signaling what you don't know," since players $k \neq j$ have no information about j's type not already known to player i.

Finally, most applications assume further that when types are independent players i and j should have the same beliefs about the type of a third player k. This restriction is defended as being in the spirit of equilibrium analysis, since equilibrium supposes that players have the same beliefs about each other's strategies.

B(iv) For all h^t, θ_k, and $i \neq j \neq k$,

$$\mu_i(\theta_k | h^t) = \mu_j(\theta_k | h^t) = \mu(\theta_k | h^t).$$

This condition implies that the posterior beliefs are consistent with a common joint distribution on Θ given h^t with

$$\mu(\theta_{-i} | h^t)\mu(\theta_i | h^t) = \mu(\theta | h^t).$$

Section 8.3 gives an example in which this restriction reduces the set of equilibrium outcomes. Although it is a standard assumption, we find it the least compelling of the four.

With a strategy σ and beliefs μ satisfying B(i)–B(iv), the natural way to extend subgame-perfect equilibrium is to require that for any t and h^t the strategies from h^t on are a Bayesian equilibrium of the continuation game. Formally, given probability distribution q and history h^t, let $u_i(\sigma | h^t, \theta_i, q)$ be type θ_i's expected payoff under profile σ conditional on reaching h^t. The relevant condition is then as follows:

(P) For each player i, type θ_i, player i's alternative strategy σ_i', and history h^t,

$$u_i(\sigma | h^t, \theta_i, \mu(\cdot | h^t)) \geq u_i((\sigma_i', \sigma_{-i}) | h^t, \theta_i, \mu(\cdot | h^t)).$$

Definition 8.2 A *perfect Bayesian equilibrium* is a (σ, μ) that satisfies P and B(i)–B(iv).

We now give an example of an application of the PBE concept. Other simple examples can be found in sections 9.1 and 10.1.

Example 8.3: The Repeated Public-Good Game

To illustrate the concept of PBE, we analyze the twice-repeated version of the public-good game studied in section 6.2. There are two players, $i = 1, 2$. In each period, $t = 0, 1$, players decide simultaneously whether to contribute to the period-t public good, and contributing is a 0-1 decision. In a given period, each player derives a benefit of 1 if at least one of them provides the public good and 0 if none does; player i's cost of contributing in a period is c_i and is the same in both periods. Per-period payoffs are depicted in figure 6.4. We assume that payoffs are discounted so that a player's objective function is the sum of his first-period payoff plus δ times his second-period payoff, where $0 < \delta < 1$. Though the benefits of the public good—1 each—are common knowledge, each player's cost is known only to that player. However, both players believe that the c_i are drawn independently from the same continuous and strictly increasing cumulative distribution function $P(\cdot)$ on $[0, \bar{c}]$, where $\bar{c} > 1$.

From chapter 6, we know that if there is a unique solution to the equation $c^* = 1 - P(1 - P(c^*))$ then the single-period version of the game has a unique Bayesian equilibrium, and c^* is (also) given by the equation $c^* = 1 - P(c^*)$ (the cost of contributing is equal to the probability that one's opponent won't contribute). Types $c_i \leq c^*$ contribute and the other types do not contribute.

In the repeated version of the game, the action space for each player is $\{0, 1\}$ in each period. A strategy for player i is a pair consisting of $\sigma_i^0(1 | c_i)$ (player i's probability of contributing in the first period when his cost is c_i) and $\sigma_i^1(1 | h^1, c_i)$ (the probability that player i contributes in the second period when his cost is c_i and when the history is $h^1 \in \{00, 01, 10, 11\}$).

Exercise 8.1 asks you to show that, in any PBE, there exists a cutoff cost $\hat{c}_i$ for each player i such that player i contributes in the first period if and only if $c_i \le \hat{c}_i$, and also to show that $0 < \hat{c}_i < 1$. We now look for a *symmetric* PBE, where $\hat{c}_1 = \hat{c}_2 = \hat{c}$. We start by solving the second-period Bayesian equilibrium given the posterior beliefs, which are determined by the equilibrium strategies and the first-period outcome.

Neither player contributed. Both players have learned that their opponent's cost exceeds $\hat{c}$. Posterior cumulative beliefs are thus the truncated beliefs

$$P(c_i|00) = \frac{P(c_i) - P(\hat{c})}{1 - P(\hat{c})}$$

for $c_i \in [\hat{c}, \bar{c}]$, and

$$P(c_i|00) = 0$$

for $c_i \le \hat{c}$. In a symmetric second-period equilibrium, each player i contributes if and only if $\hat{c} \le c_i \le \hat{\hat{c}}$ (from section 6.2, we know that a Bayesian equilibrium in period 2 involves a cutoff rule for each player). The cutoff cost $\hat{\hat{c}}$ is equal to the probability,

$$\frac{1 - P(\hat{\hat{c}})}{1 - P(\hat{c})},$$

that the opponent does not contribute. Note that $\hat{c} < \hat{\hat{c}} < 1$. We will later use the result that, because type $\hat{c}$ contributes in period 2 if no one has contributed in period 1, his second-period utility is $v^{00}(\hat{c}) = 1 - \hat{c}$.

Both players contributed. The posterior cumulative distribution is then

$$P(c_i|11) = \frac{P(c_i)}{P(\hat{c})}$$

for $c_i \in [0, \hat{c}]$, and

$$P(c_i|11) = 1$$

for $c_i \in [\hat{c}, \bar{c}]$. In a symmetric second-period equilibrium, each player i contributes if and only if $c_i \le \tilde{c}$, where $0 < \tilde{c} < \hat{c}$. Each player's cutoff cost is equal to the conditional probability that his opponent does not contribute:

$$\tilde{c} = \frac{P(\hat{c}) - P(\tilde{c})}{P(\hat{c})}. \tag{8.8}$$

Note in particular that type $\hat{c}$ does not contribute, so that his second-period utility is $v^{11}(\hat{c}) = P(\tilde{c})/P(\hat{c})$.

Only one player contributed. Suppose player i contributed in period 0 and player j did not. Hence, $c_i \leq \hat{c}$ and $c_j \geq \hat{c}$. One equilibrium in period 1 has player i contribute (recall that $\hat{c} < 1$) and player j not contribute, and this is the equilibrium we specify. (For some distributions—e.g., $P(\cdot)$ uniform on $[0, 2]$—this equilibrium is unique.[9]) The second-period utilities of type $\hat{c}$ are thus $v^{10}(\hat{c}) = 1 - \hat{c}$ and $v^{01}(\hat{c}) = 1$.

Let us now derive the first-period equilibrium. Type $\hat{c}$ must be indifferent between contributing and not contributing, or

$$\begin{aligned} &1 - \hat{c} + \delta\{P(\hat{c})v^{11}(\hat{c}) + [1 - P(\hat{c})]v^{10}(\hat{c})\} \\ &\quad = P(\hat{c}) + \delta\{P(\hat{c})v^{01}(\hat{c}) + [1 - P(\hat{c})]v^{00}(\hat{c})\}. \end{aligned} \tag{8.9}$$

Using the formulas of the second-period utilities and equation 8.8, we obtain

$$1 - P(\hat{c}) = \hat{c} + \delta P(\hat{c})\tilde{c}. \tag{8.10}$$

Equations 8.8 and 8.10 define $\hat{c}$. Equation 8.10 has a straightforward interpretation: By contributing in period 1, type $\hat{c}$ spends $\hat{c}$ but provides the public good when it would not have been provided otherwise (which has probability $1 - P(\hat{c})$). Moreover, he reveals that his type is at most $\hat{c}$ instead of signaling a type above $\hat{c}$ by not contributing. This makes no difference when the opponent's type is above $\hat{c}$, because type $\hat{c}$ will contribute in the second period whether or not he contributes today. Contributing does change type $\hat{c}$'s second-period payoff when the opponent has type under $\hat{c}$: Whereas not contributing in the first period would induce the opponent to contribute in the second period, contributing makes the opponent more reluctant to contribute in that he will contribute in the second period only if his cost is lower than $\tilde{c}$. Because a player's second-period payoff when not contributing is independent of his cost, and because type $\tilde{c}$ is indifferent between contributing and not contributing if both have contributed in the first period, type $\hat{c}$ gains $1 - (1 - \tilde{c}) = \tilde{c}$ by signaling a high cost in the first period when the opponent's cost is less than $\hat{c}$.

9. Note that it is a dominant strategy for type $c_j > 1$ not to contribute. It is therefore optimal for type $c_i = \varepsilon$ (very small) to contribute. Recalling that strategies in the second period are necessarily cutoff rules, let $\tilde{c}_i \leq \hat{c}$ denote the cutoff cost for player i in period 1, and $\mathring{c}_j \geq \hat{c}$ that for player j. They are given by

$$\tilde{c}_i = \frac{1 - P(\mathring{c}_j)}{1 - P(\hat{c})}$$

and

$$\mathring{c}_j = \frac{P(\hat{c}) - P(\tilde{c}_i)}{P(\hat{c})}.$$

For P uniform on $[0, 2]$, $\tilde{c}_i$ would thus be given by $\tilde{c}_i = -\hat{c}/(1 - \hat{c})^2$, which is impossible because $\tilde{c}_i$ must be a positive number.

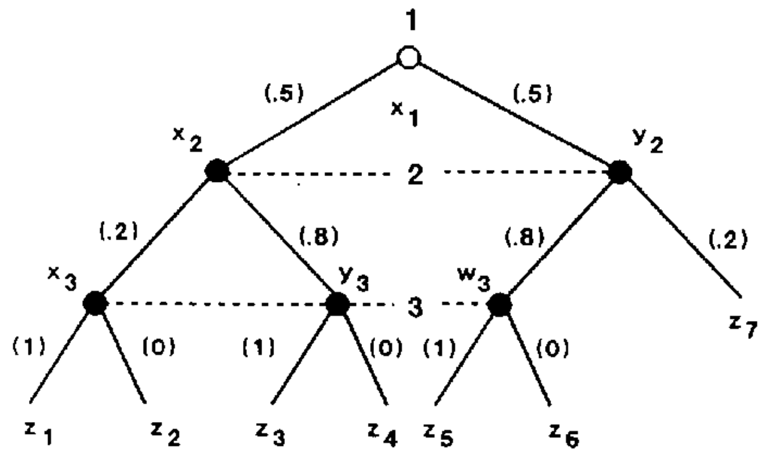
Because this occurs with probability $P(\hat{c})$, the expected second-period gain from not contributing in the first period is $P(\hat{c})\tilde{c}$.

Equation 8.10 implies that $\hat{c} < c^*$: In this equilibrium (which is the unique symmetric equilibrium under some assumptions), there is less contribution in the first period of the two-period game than in the one-period game. This follows from the fact that each player gains by developing a reputation for not being willing to supply the public good.

8.3 Extensive-Form Refinements††

8.3.1 Review of Game Trees

We defined extensive-form games in chapter 3. In the next two subsections and in section 8.4, we will consider games of perfect recall with a finite number of players ($i = 1, \dots, I$) and a finite number of decision nodes ($x \in X$). Let $h(x)$ denote the information set containing node x. (We follow standard notation; we hope that this will not create any confusion with the related notion of history.) The player playing at node x is denoted $i(x)$; terminal nodes are denoted by z. The mixed or behavior strategy of player $i = i(x)$ at node x is $\sigma_i(\cdot|x)$ or $\sigma_i(\cdot|h(x))$. (We will sometimes delete the conditioning of σ_i on the information set if player i moves at a single information set, as there cannot be any ambiguity.) Let Σ denote the set of



$p^\sigma(x_1)=1$
$p^\sigma(x_2)=p^\sigma(y_2)=.5$
$p^\sigma(x_3)=.1$; $p^\sigma(y_3)=p^\sigma(w_3)=.4$; $p^\sigma(h)=.9$ where $h=\{x_3,y_3,w_3\}$
$p^\sigma(z_1)=p^\sigma(z_7)=.1$; $p^\sigma(z_2)=p^\sigma(z_4)=p^\sigma(z_6)=0$;
$p^\sigma(z_3)=p^\sigma(z_5)=.4$
$\mu(x_3)=1/9$, $\mu(y_3)=\mu(w_3)=4/9$

Figure 8.2

all strategy profiles $\sigma = (\sigma_1, \ldots, \sigma_I)$, and let p denote the exogenous probability distribution over nature's moves. For example, in a game of incomplete information, nature's move is a choice of type for each player. As in our development so far, nature's moves are not considered to be given by a "strategy"; thus, when we perturb the game with "trembles" nature's moves will not be affected.

With σ given, $P^{\sigma}(x)$ and $P^{\sigma}(h)$ denote the respective probabilities that node x and information set h are reached. (These probabilities depend on the prior p, but we omit the superscript p because in a given extensive form the prior is fixed.) A system of beliefs μ specifies beliefs at each information set h: $\mu(x)$ denotes the probability player $i(x)$ assigns to node x conditional on reaching information set $h(x)$. In figure 8.2, which illustrates these concepts, the strategy profile σ is depicted on the tree.

Payoffs are determined by the terminal node of the game, and player i's payoff if z is reached is denoted $u_i(z)$. (Recall that z is a complete description of everything that happens before the terminal node is reached, including nature's choice of the players' private information.) Let $u_{i(h)}(\sigma \mid h, \mu(h))$ be the expected utility of player $i(h)$ given that information set h is reached, that the player's beliefs are given by $\mu(h)$, and that the strategies are σ.

An *assessment* (σ, μ) specifies a strategy profile σ and a system of beliefs μ. The set of all possible assessments is denoted by Ψ.

8.3.2 Sequential Equilibrium

We now describe how Kreps and Wilson (1982a) extend condition P into condition S (S for sequential rationality) and extend and refine condition B into condition C (C for consistency) for general finite games of perfect recall.

We noted in section 8.1 that the requirement that the players' strategies form a Nash equilibrium in each (proper) subgame is too weak, as there are few (proper) subgames in games of incomplete or imperfect information. We saw that in the imperfect-information game of figure 8.1 the only subgame is the whole game, and the Nash equilibrium (L, A) is subgame perfect. This equilibrium is nevertheless implausible, because whatever beliefs player 2 forms about which of M and R was player 1's move, he ought to play B if given the opportunity to move.

The appropriate generalization of condition P is that, given the system of beliefs, no player can gain by deviating at any information set:

(S) An assessment (σ, μ) is *sequentially rational* if, for any information set h and alternative strategy $\sigma'_{i(h)}$,

$$u_{i(h)}(\sigma \mid h, \mu(h)) \geq u_{i(h)}((\sigma'_{i(h)}, \sigma_{-i(h)}) \mid h, \mu(h)).$$

Note that players believe that their opponents will adhere to the equilibrium profile σ at every information set (including ones that should not be reached if all players adhere to σ). Condition S is equivalent to condition P for multi-stage games.

What conditions one should put on beliefs at an information set off the equilibrium path is a more difficult and controversial question. Kreps and Wilson introduce the notion of consistency. We first define consistency, and then discuss the desiderata that led Kreps and Wilson to offer this definition; we later explore what consistency implies for multi-stage games.

Let Σ^0 denote the set of all completely mixed (behavioral) strategies, i.e., profiles σ such that $\sigma_i(a_i|h) > 0$ for all h and $a_i \in A(h)$. If $\sigma \in \Sigma^0$, then $P^\sigma(x) > 0$ for all nodes x, so that Bayes' rule pins down beliefs at each information set: $\mu(x) = P^\sigma(x)/P^\sigma(h(x))$. Let Ψ^0 denote the set of all assessments (σ, μ) such that $\sigma \in \Sigma^0$ and μ is (uniquely) defined from σ by Bayes' rule.

(C) An assessment (σ, μ) is *consistent* if

$$(\sigma, \mu) = \lim_{n \to +\infty} (\sigma^n, \mu^n)$$

for some sequence (σ^n, μ^n) in Ψ^0.

Note that the strategies σ need not be totally mixed; however, they and the beliefs can be regarded as limits of totally mixed strategies and associated beliefs. Note also that condition C implies condition B for multi-stage games.

Since the probability distribution over nature's moves is not represented by a strategy, the definition of consistency does not apply "trembles" to nature's moves. Subsection 8.3.3 explains how allowing trembles by nature would change the properties of the equilibrium concept.

Definition 8.3 A *sequential equilibrium* is an assessment (σ, μ) that satisfies conditions S and C.

We now discuss the considerations that led Kreps and Wilson to propose the definition of consistency. Consider figure 8.3 (taken from their paper). Player 1 assigns probabilities $\frac{1}{3}$ and $\frac{2}{3}$ to nodes x and $x' \in h(x)$, respectively. His strategy is to play U. What should player 2 believe if player 1 deviates and plays D? Since player 1 cannot distinguish x and x', it seems natural to require that player 1 be "as likely" to deviate at both nodes. This idea leads to the requirement that player 2 put weights $\frac{1}{3}$ and $\frac{2}{3}$ on nodes y and y', respectively. However, any $\mu(y)$ is compatible with Bayes' rule, because event D has probability 0 in the equilibrium. Consistency yields the "right beliefs" in this game. Consider an arbitrary sequence ε^n converging to 0, and interpret ε^n as the probability that player 1 "trembles" and plays down. For this sequence,

$$\mu^n(y) = \frac{\mu^n(x)\varepsilon^n}{\mu^n(x)\varepsilon^n + \mu^n(x')\varepsilon^n} = \tfrac{1}{3}.$$

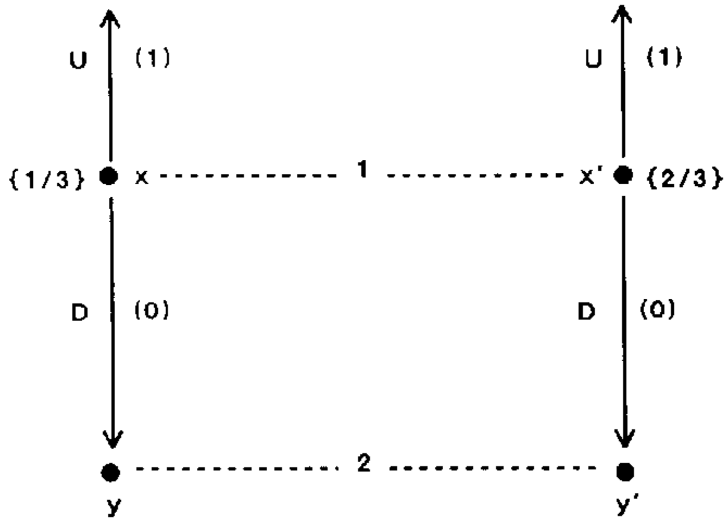


Figure 8.3

Thus, trembles ensure that the players' beliefs respect the information structure. This example also motivates Kreps and Wilson's definition of "structural consistency." (In subsection 8.3.4 we will exploit this example in a different way.)

An assessment (σ, μ) is *structurally consistent* if, for each information set h, there exists a strategy profile $\sigma_h \in \Sigma$ such that $P^{\sigma_h}(h) > 0$ and $\mu(x) = P^{\sigma_h}(x)/P^{\sigma_h}(h)$ for all x in h. That is, for each information set, the player on the move at the information set can find a strategy profile (not necessarily the same as σ) that would yield exactly the specified beliefs at the information set. The significance of structural consistency is the following: Suppose a player unexpectedly finds himself on the move at some information set h. What beliefs should he hold concerning the nodes in h? If he can find an alternative strategy profile σ_h that would reach h with positive probability, he could use this σ_h as a conjecture of the way the game had been played and then use Bayes' rule to form his beliefs at h. If the original equilibrium assessment (σ, μ) is structurally consistent, every player can, for every one of his off-the-equilibrium-path information sets, find such an alternative hypothesis to guide the formation of his beliefs.

Kreps and Wilson assert without proof that consistency implies structural consistency. Kreps and Ramey (1987) use figure 8.4 to show that this is incorrect. In figure 8.4, any assessment

$$\{\sigma_1(\mathrm{R}_1) = \sigma_2(\mathrm{R}_2) = 1,\ \sigma_3(\mathrm{R}_3) \in (0,1);\ \mu(x_2) = 0,\ \mu(y_2) = 1,\ \mu(x_3) = 0,$$

$$\mu(y_3) = \mu(w_3) = \tfrac{1}{2}\}$$

is consistent, because it is the limit of the assessment derived from the totally mixed assessments

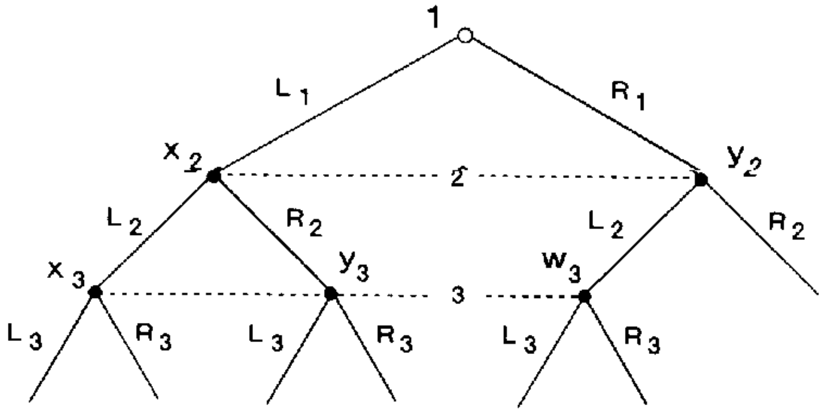


Figure 8.4

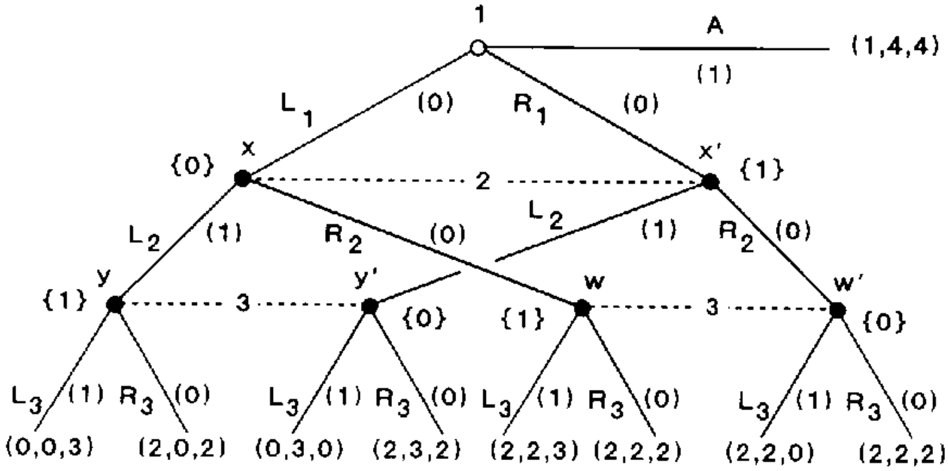


Figure 8.5

$$\{\sigma_1^n(R_1) = \sigma_2^n(R_2) = 1 - 1/n, \sigma_3^n(R_3) = \sigma_3(R_3); \mu^n(x_2) = 1/n,$$

$$\mu^n(y_2) = (n-1)/n, \mu^n(x_3) = 1/(2n-1),$$

$$\mu^n(y_3) = \mu^n(w_3) = (n-1)/(2n-1)\}.$$

This assessment is not structurally consistent, as no strategy giving positive weight to reaching nodes y_3 and w_3 gives weight 0 to reaching node x_3.[10,11]

Figure 8.5 shows how consistency reduces the set of equilibria by imposing common beliefs after deviations from equilibrium behavior. In this game, player 1 gets 2 by playing either L_1 or R_1 as long as at least one of

10. If y_3 is ever reached, then L_1 is sometimes played; if w_3 is ever reached, then L_2 is sometimes played. Therefore, the combination L_1 and L_2 is sometimes played, and therefore x_3 is reached with positive probability.

11. In response to this example, one might be tempted to add structural consistency to the definition of sequential equilibrium. However, Kreps and Ramey provide an example in which the unique sequential equilibrium does not satisfy structural consistency. They also show that any consistent assessment is the convex combination of structurally consistent assessments. In the example, the consistent beliefs are the convex combination of the structurally consistent beliefs $\mu(y_2) = \mu(w_3) = 1$ and $\mu(x_2) = \mu(y_3) = 1$.

the other players "cooperates" by playing right, so player 1 should play A only if both players 2 and 3 are likely to play left. Player 2's action does not affect player 3's payoff, and vice-versa. Consider the assessment (σ, μ), depicted in the figure, where $\sigma_1(A) = 1$, $\sigma_2(L_2) = 1$, $\sigma_3(L_3) = 1$ for each of player 3's two information sets, and $\mu(x') = \mu(y) = \mu(w) = 1$. This assessment is sequentially rational: If players 2 and 3 play left, player 1 should play A; if $\mu(x') = 1$, then player 2 gets 3 from L_2 and 2 from R_2; and if $\mu(y) = \mu(w) = 1$, then player 3 gets payoff 3 from L_3 and 2 from R_3. The assessment (σ, μ) is structurally consistent and obeys Bayes' rule where possible, but it is not consistent, as players 2 and 3 have different beliefs about the relative likelihood of player 1 playing L_1 and R_1, and this is not possible if the beliefs of player 2 and player 3 both are the limit of μ^n derived from the *same* sequence of totally mixed σ^n.

Moreover, there is no consistent assessment where player 1 plays A, since when players 2 and 3 have the same beliefs about player 1's move then at least one of them will play right: If $\mu(x) > \frac{1}{3}$ player 2 plays R_2, and if $\mu(y) < \frac{2}{3}$ player 3 plays R_3.

Although (σ, μ) is not consistent, it satisfies the weaker condition that for each player i there is a sequence $\sigma^n(i) \to \sigma$ of totally mixed strategy profiles such that, at each information set h, $\mu(x)$ is the limit of the beliefs $\mu^n(i)$ computed using Bayes' rule from $\sigma^n(i)$. Why should all players have the same theory to explain deviations that, after all, are either probability-0 events or very unlikely, depending on one's methodological point of view? The standard defense is that this requirement is in the spirit of equilibrium analysis, since equilibrium supposes that all players have common beliefs about the others' strategies. Although this restriction is usually imposed, we are not sure that we find it convincing.

8.3.3 Properties of Sequential Equilibrium (technical)

Existence

For any finite extensive-form game there exists at least one sequential equilibrium. Existence will be proved indirectly in section 8.4: Any trembling-hand perfect equilibrium is sequential, and because trembling-hand perfect equilibria exist in finite games, so do sequential equilibria.

Upper Hemi-Continuity

Like the Nash-equilibrium correspondence, the sequential-equilibrium correspondence is upper hemi-continuous with respect to payoffs. More precisely, fix an extensive form and prior beliefs p. For any sequence of utility functions u^n (defining a game) converging to some u, if the assessment (σ^n, μ^n) is a sequential equilibrium of game u^n for all n and converges to an assessment (σ, μ), then (σ, μ) is a sequential equilibrium of game u.

The proof of this is simple. We must show that (σ, μ) satisfies conditions S and C. The proof that it satisfies S follows the same lines as the proof that the Nash correspondence is upper hemi-continuous. That it satisfies C results from the fact that for each n there exists a sequence of assessments $(\sigma^{m,n}, \mu^{m,n})$ in Ψ^0 converging as m tends to infinity to (σ^n, μ^n), which in turn converges to (σ, μ). This upper-hemi-continuity property distinguishes sequential equilibrium from trembling-hand perfect equilibrium (see section 8.4).

What about upper hemi-continuity with respect to prior beliefs p? Consider a sequence p^n on a *fixed* set of initial nodes that converges to a distribution p. It is straightforward to check that the proof of upper hemi-continuity in the previous paragraph carries over as long as p assigns strictly positive probability to all of nature's moves.[12] However, if p assigns probability 0 to some of nature's moves, upper hemi-continuity with respect to beliefs may not hold. This lack of upper hemi-continuity can be illustrated in Spence's signaling model (example 8.2). In the (least-cost) separating equilibrium the high-productivity worker invests $\theta'(\theta'' - \theta')$ in education even if the probability of a low-productivity worker is very small. But when the latter probability is equal to 0, the high-productivity worker does not invest in education in the unique (subgame-) perfect equilibrium.

Note that if we modify the definition of consistency by requiring nature to tremble as well as the players, the separating equilibrium is still a sequential equilibrium when the prior probability of a low-productivity type is 0. More generally, with this definition the set of sequential equilibria is upper hemi-continuous with respect to prior beliefs on a fixed set of nature's moves. However, with the modified definition, the set of sequential equilibria can change when a probability-0 move by nature is added. That is, the set of sequential equilibria would depend not only on the prior beliefs but also on the set of nature's moves that are "conceivable." (A similar observation applies to the set of perfect Bayesian equilibria.)

Structure of Equilibria

Theorem 8.1 (Kreps and Wilson 1982a) For generic (i.e., generic end-point payoffs of) finite extensive-form games of perfect recall, the set of sequential-equilibrium probability distributions on terminal nodes is finite.

That is, for a fixed extensive form and fixed prior beliefs, the closure of the set of payoffs u such that the associated game has an infinite number of sequential-equilibrium outcomes has Lebesgue measure 0. The set of sequential-equilibrium assessments is in general infinite because of the leeway in specifying beliefs off the equilibrium path.

The set of sequential equilibrium strategies is in general infinite as well, because, when a player is indifferent between two actions at an off-path

12. To extend the proof, note that, although the sequence of beliefs $\mu^{m,n}$ is Bayes consistent with p^n and $\sigma^{m,n}$, it need not be consistent with p and $\sigma^{m,n}$. Thus, one replaces $\mu^{m,n}$ by $\tilde{\mu}^{m,n}$, which is Bayes consistent with p and $\sigma^{m,n}$.

information set, many different randomizing probabilities at that information set can be specified. This point is developed in detail in the appendix to the present chapter.

Addition of "Irrelevant Moves or Strategies"

Several authors have shown that the set of sequential equilibria can change when an apparently irrelevant move or strategy is added. We will return to this in more detail in chapter 11; we content ourselves with an example here.

Sequential equilibrium and the related concepts defined in section 8.4 have been criticized by Kohlberg and Mertens (1986) for allowing "strategically neutral" changes in the game tree to affect the equilibrium. Compare, for instance, figures 8.6a and 8.6b. Figure 8.6b is the same as figure 8.6a except that an apparently irrelevant move NA ("not across") has been added. Whereas A is a sequential-equilibrium outcome in figure 8.6a,[13] A is not a sequential-equilibrium outcome in figure 8.6b. In the "simultaneous-move" subgame following NA, the only Nash equilibrium

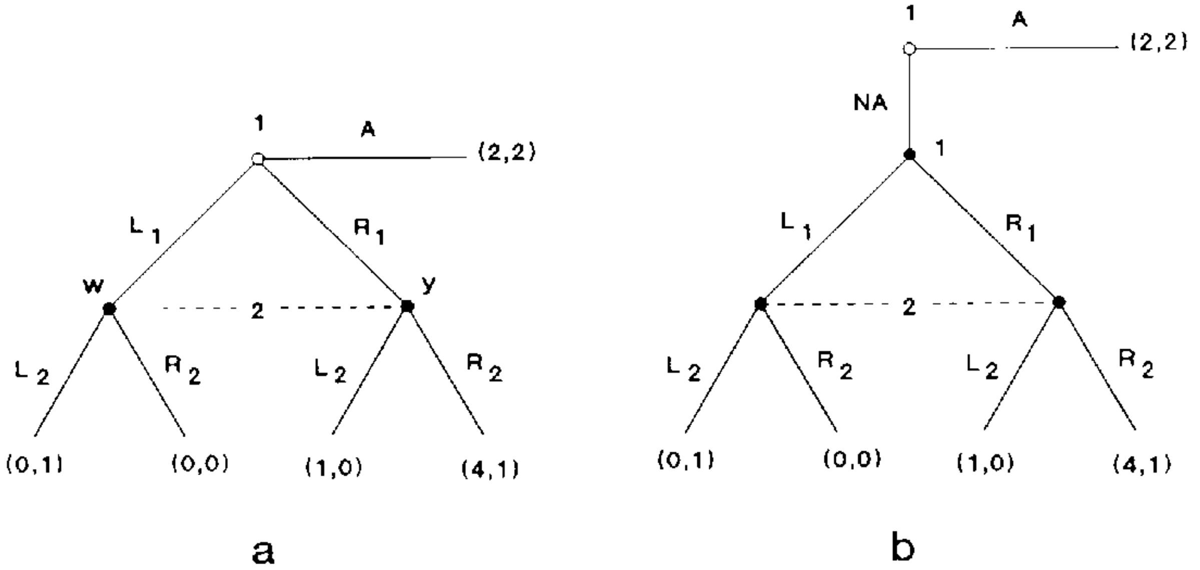


Figure 8.6

13. Consider the assessment $\{\sigma_1(A) = 1,\ \sigma_2(L_2) = 1;\ \mu(w) = 1\}$. This assessment satisfies condition S. To see that it satisfies condition C, consider the trembles

$$\sigma_1^n(A) = 1 - \frac{1}{n} - \frac{1}{n^2},$$

$$\sigma_1^n(L_1) = \frac{1}{n},$$

$$\sigma_2^n(L_2) = 1 - \frac{1}{n}.$$

Clearly, $\mu^n(w)$ converges to 1.

is (R_1, R_2), as L_1 is strictly dominated by R_1 for player 1. Hence, the only sequential-equilibrium payoffs are (4, 1). This example also illustrates that the deletion of a strictly dominated strategy affects the set of sequential-equilibrium payoffs: If L_1 is deleted in figure 8.6a, the unique sequential-equilibrium payoffs are (4, 1).

Chapter 11 has more discussion of when two similar trees should be expected to have the same solution. For now, let us note that if we take seriously the idea that players make "mistakes" at each information set, as the definition of sequential equilibrium might suggest, then it is not clear that the two figures are equivalent. In figure 8.6b, if player 1 makes the "mistake" of not playing A, he is still able to ensure that R_1 is more likely than L_1; in figure 8.6a he might play either action by "mistake" when intending to play A.

Correlated Sequential Equilibrium

Just as Nash equilibrium can be generalized to allow preplay observation of correlated signals, sequential equilibrium can be generalized to allow correlated strategies. There are three ways of doing so in multi-stage games (Forges 1986; Myerson 1986). First, one can allow players to receive information in the preplay phase ("at date -1") only. Second, one can allow the players to receive information slowly over time ("at each date"). Third, one can have players send private messages (inputs) at the beginning of each period to a "mediator" or a "machine," which then conveys private (but possibly correlated) messages (outputs) to the players. What differentiates the third possibility from the second is that the messages sent to the players can be contingent on their information. To show that each possibility allows more equilibria than the previous one (it clearly allows at least as many), consider the two examples shown in figures 8.7 and 8.8. Figure 8.7 illustrates the possibility that delaying the observation of "sunspots" may increase the equilibrium set. Payoffs (3, 3) can be obtained by having the players coordinate on (L_1, L_2) or (R_1, R_2) with equal probabilities after the occurrence of a sunspot at the beginning of period 1. If the realization of the sunspot leading to the coordination on (R_1, R_2) were known at date 0, player 1 would play ℓ_1 and payoffs (3, 3) could no longer be obtained.

In figure 8.8, player 2 would like to predict the state of nature. Suppose that player 1, a dummy player in figure 8.8, learns the state of nature and can communicate it before player 2 picks an action. Payoffs (0, 1) are now attainable, and thus communication increases the equilibrium payoff set.

These examples raise the question of the interpretation of an "extensive form." Should the complete rules of the game, including the "observation of sunspots" and "cheap talk," be explicitly described in the extensive form, or should any equilibrium concept allow for correlated moves and communi-

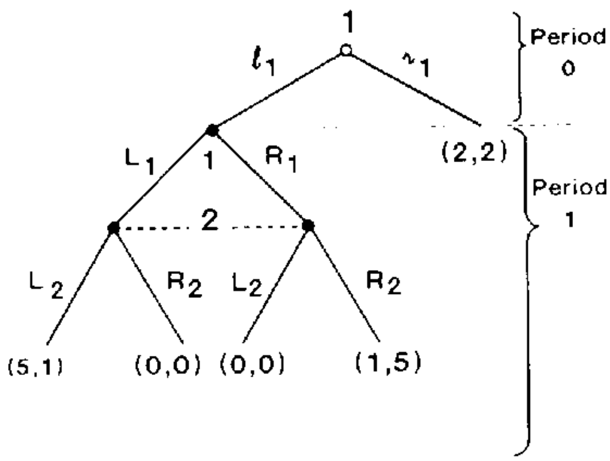


Figure 8.7

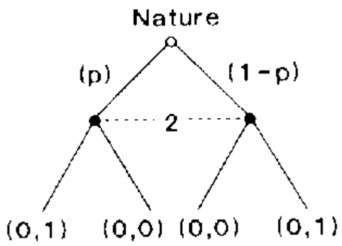


Figure 8.8

cation even if those possibilities are not explicitly described in the extensive form?

In the spirit of the revelation principle (see also sections 2.2 and 7.2), Forges and Myerson show that the set of equilibrium payoffs has a canonical representation. Any sequential equilibrium can be obtained by having, at each period, each player announce his information privately and truthfully to a mediator, who, having observed all messages, privately sends a recommended action or mixed strategy to each player, who then obeys the recommendation. For finite games, the sets of equilibrium strategies with *ex ante* correlation, correlation at each period, and correlation and communication at each period are convex polyhedrons, because in each case the equilibrium strategies are defined by a set of linear inequalities (the incentive-compatibility conditions).

8.3.4 Sequential Equilibrium Compared with Perfect Bayesian Equilibrium

The trembles underlying the definition of consistency give all paths positive probability so Bayes' rule pins down beliefs everywhere. There are two types of objections that can be made to the use of trembles. First, checking that an assessment in a finite game is consistent is a tedious process that is rarely carried out in applications. Furthermore, many applications involve an infinite number of actions or types; extending the formal definition of consistency to infinite games does not seem to require a conceptual innovation, but would face some technical difficulties. Second, and more important, one would like to know more about what consistency implies for behavior. In order to explain what the sequential-equilibrium restriction entails, we will compare it in some detail with the PBE concept of subsection 8.2.3.

Theorem 8.2 (Fudenberg and Tirole 1991) Consider a multi-stage game of incomplete information with independent types. If either each player has at most two possible types ($\#\Theta_i \leq 2$ for each i) or there are two periods, condition B is equivalent to condition C and therefore the sets of PBE and sequential equilibria coincide.

With more than two types per player and/or more than two periods, condition B is no longer sufficient to guarantee consistency, as figure 8.9 shows. This figure depicts a situation where player 1 has three possible types, θ_1', θ_1'', and θ_1^*, but where at time t Bayesian inference from the previous play has led to the conclusion that player 1 must be type θ_1^*. The equilibrium strategies at this point, which are given in parentheses in the figure, are for type θ_1' to play a_1', type θ_1'' to play a_1'', and type θ_1^* to play a_1^*. (For clarity, we do not depict type θ_1^*'s probability-0 actions.) Since the first two types have probability 0, player 2 expects to see player 1 play a_1^*. What should he believe if he sees one of the other two actions? The beliefs in figure 8.9 (given in braces) are that if player 2 sees a_1' he concludes that he is facing type θ_1'', while a_1'' is taken as a signal that player 1 is of type θ_1'. Since the definition of PBE places no constraints on the beliefs about a player who has just deviated (except that these beliefs are common to all players and that they do not depend on actions chosen by players other than the deviating one), the situation in figure 8.9 is compatible with PBE.

However, the situation of figure 8.9 cannot be part of a sequential equilibrium. To see this, imagine that there were trembles σ^n that converged to the given strategies σ and such that the associated beliefs μ^n converged to the given beliefs μ. Let the probability that μ^n assigns to type θ_1' at period t be ε'^n, and let the probability of type θ_1'' be ε''^n. Since μ^n converges to μ,

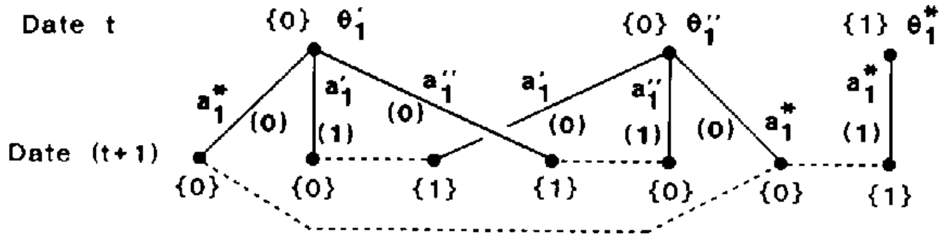


Figure 8.9

both ε'^n and ε''^n converge to 0, and $\sigma^n(a_1'\,|\,\theta_1'')$ and $\sigma^n(a_1''\,|\,\theta_1')$ converge to 0 as well. Since

$$\mu^n(\theta_1''\,|\,a_1') = \frac{\mu^n(\theta_1'')\sigma^n(a_1'\,|\,\theta_1'')}{\sum_{\theta_1} \mu^n(\theta_1)\sigma^n(a_1'\,|\,\theta_1)},$$

in order to have $\mu^n(\theta_1''|a_1')$ converge to 1 it must be that $\varepsilon'^n/\varepsilon''^n$ converges to 0: In order for the beliefs following a_1' to be concentrated on type θ_1'' when θ_1' plays the action with probability 1 while θ_1'' assigns it probability 0, the prior beliefs must be that θ_1'' is infinitely more likely than θ_1'. On its own, this requirement is compatible with sequential equilibrium. However, considering the beliefs that follow a_1'' leads to the conclusion that $\varepsilon''^n/\varepsilon'^n$ converges to 0, i.e., that θ_1' is infinitely more likely than θ_1'', and these two conditions are jointly incompatible with the beliefs being consistent. This restriction, though implied by sequential equilibrium, is stronger and different in spirit than the restrictions described by Kreps and Wilson to motivate the consistency requirement.

For PBE to imply consistency, the definition of beliefs must be extended to capture the relative probabilities of probability-0 types, and restrictions must be imposed on the way these relative probabilities are updated. Requiring that players assess relative probabilities of probability-0 states of nature is strong, but easy to formalize. Formally, one wants the posterior beliefs about each player to form a "system of relative beliefs," or a "conditional probability system."[14] That is, the players have beliefs $\mu^*(\theta_i\,|\,(\theta_i, \theta_i'), h^t)$ that player i is of type θ_i conditional on player i's being type θ_i or θ_i' and on history h^t, even if being type θ_i or θ_i' has probability 0 conditional on h^t. Note that a system of relative beliefs generates a system of absolute beliefs by the formula $\mu(\theta_i\,|\,h^t) \equiv \mu^*(\theta_i\,|\,\Theta_i, h^t)$.[15] A pair (σ, μ^*) is called a *generalized assessment*.

We now extend Bayesian conditions B(i)–B(iv) to require that *Bayes' rule and the no-signaling condition hold for relative beliefs and not only for absolute beliefs* (to simplify the statements, we assume right away that beliefs are common):

14. A conditional probability system (Myerson 1986) on a finite space Ω is a collection of functions $\nu(\cdot|\cdot)$ from $2^\Omega \times 2^\Omega$ to $[0, 1]$ such that for each $A \in 2^\Omega$, $\nu(\cdot|A)$ is a probability distribution on A, and such that for $A \subseteq B \subseteq C \in 2^\Omega$ with $B \neq \varnothing$, $\nu(A|B)\nu(B|C) = \nu(A|C)$.

15. More precisely,

$$\mu^*(\theta_i\,|\,\Theta_i, h^t) \equiv 1 \Big/ \sum_{\substack{\theta_i' \in \Theta_i \\ \theta_i' \neq \theta_i}} \left[\left(\frac{1}{\mu^*(\theta_i\,|\,(\theta_i, \theta_i'), h^t)} - 1\right) - 1\right],$$

if, for all $\theta_i' \neq \theta_i$, $\mu^*(\theta_i\,|\,(\theta_i, \theta_i'), h^t) > 0$,

$\equiv 0$ otherwise.

(B*) A generalized assessment (σ, μ^*) satisfies condition B* if
(i) Bayes' rule is used to update relative beliefs $\mu^*(\theta_i | (\theta_i, \theta_i'), h^t)$ into $\mu^*(\theta_i | (\theta_i, \theta_i'), (h^t, a^t))$ whenever possible: If a_i^t has positive probability conditional on (θ_i, θ_i') and h^t,

$$\mu^*(\theta_i | (\theta_i, \theta_i'), (h^t, a^t)) = \frac{\mu^*(\theta_i | (\theta_i, \theta_i'), h^t)\sigma_i(a_i^t | h^t, \theta_i)}{\sum_{\tilde{\theta}_i = \theta_i, \theta_i'} \mu^*(\tilde{\theta}_i | (\theta_i, \theta_i'), h^t)\sigma_i(a_i^t | h^t, \tilde{\theta}_i)};$$

(ii) the posterior beliefs are independent:

$$\mu(\theta | h^t) = \prod_i \mu(\theta_i | h^t);$$

(iii) the relative beliefs about player i at date $t+1$ depend only on h^t and on player i's period-t action:

$$\mu^*(\theta_i | (\theta_i, \theta_i'), (h^t, a^t)) = \mu^*(\theta_i | (\theta_i, \theta_i'), (h^t, \tilde{a}^t)) \text{ if } a_i^t = \tilde{a}_i^t.$$

Note that these conditions are the same as B(i)–B(iii) except that they apply to relative probabilities. Indeed, conditions B and B* coincide when $\mu(\cdot | h^t)$ has full support for all h^t. In particular, in a two-period game (such as the signaling game) all types have positive probability in period 0, so condition B* does not refine condition B (it does refine condition B for beliefs formed at the end of period 1, but those beliefs are irrelevant, as the game is over). In the case of at most two types per player, at most one type has probability 0 after any history and the issue of relative beliefs does not arise (absolute beliefs are also relative beliefs), so again condition B* coincides with condition B.

Condition B*(i) implies that if θ_i is infinitely more likely than θ_i' given h^t, and $\sigma_i(a_i^t | h^t, \theta_i) > 0$, then after observing a_i^t in period t, θ_i is still infinitely more likely than θ_i'. Similarly, if two types are "as likely" (in the sense that none is infinitely more likely than the other) given h^t, and both play action a_i^t with positive probability, the two types remain "as likely." Combined, these two implications rule out the beliefs in figure 8.9.

Definition 8.4 A *perfect extended Bayesian equilibrium* (PEBE) of a multi-stage game of incomplete information with independent types is a generalized assessment satisfying conditions P and B*.

Theorem 8.3 (Fudenberg and Tirole 1991) For multi-stage games of incomplete information with independent types, condition B* implies condition C, and any assessment satisfying C can be extended to a generalized assessment satisfying B*. Therefore, the sets of PEBE and sequential equilibria coincide.

The idea of the proof of these two results (that condition B for two types or two periods, or more generally that condition B* implies condition

C) is as follows: Suppose one has built trembles up to date t that yield strictly positive beliefs at the beginning of date t and converge to $\mu(\cdot|h^t)$. One then constructs trembles on probability-0 actions so as to obtain the posterior beliefs $\mu(\cdot|(h^t, a^t))$ in the limit. The no-signaling condition guarantees that these trembles can be built independently among players, and, with more than two types, condition B* guarantees that appropriate trembles exist that vindicate the relative beliefs. One then subtracts trembles on probability-0 actions from the (strictly positive) probabilities on equilibrium actions to ensure that, along the sequence of trembles, the probabilities of each player's actions add up to 1.

Correlated Types^{+++}

When types are correlated, it is convenient to transform the game into one with independent types and then map the resulting equilibrium strategies and beliefs to strategies and beliefs in the original game. Myerson (1985) shows that any Bayesian game can be transformed into one with independent types. Suppose that the prior distribution $\rho(\theta) = \rho(\theta_1, \ldots, \theta_I)$ has full support on Θ. And let $\hat{\rho}$ be the product of independent uniform marginal distributions $\hat{\rho}_i$ on Θ_i:

$$\hat{\rho}(\theta) \equiv 1 \Big/ \left(\prod_{i=1}^{I} (\#\Theta_i) \right) \text{ for all } \theta \text{ in } \Theta.$$

Define the fictitious von Neumann-Morgenstern payoff functions

$$\hat{u}_i(h^{T+1}, \theta_i, \theta_{-i}) \equiv \rho(\theta_{-i}|\theta_i) u_i(h^{T+1}, \theta_i, \theta_{-i}) \text{ for all } (h^{T+1}, \theta_i, \theta_{-i}).$$

Let (by the familiar abuse of notation) $u_i(\sigma, \theta)$ and $\hat{u}_i(\sigma, \theta)$ denote the utilities for strategy profile σ and types θ. With E_ρ and $\mathrm{E}_{\hat{\rho}}$ denoting the expectation operators with respect to distributions ρ and $\hat{\rho}$, $\mathrm{E}_\rho(u_i|\theta_i)$ and $\mathrm{E}_{\hat{\rho}}(\hat{u}_i|\theta_i)$ represent the same preferences for player i with type θ_i. The Bayesian equilibria of the game (u, ρ) with correlated types and the game $(\hat{u}, \hat{\rho})$ with independent types are therefore the same.

More generally, in a multi-stage game with incomplete information, it is straightforward to check that an assessment $(\hat{\sigma}, \hat{\mu})$ is a sequential equilibrium of the transformed game $(\hat{u}, \hat{\rho})$ if and only if the assessment (σ, μ) defined by $\sigma = \hat{\sigma}$ and

$$\mu(\theta_{-i}|\theta_i, h^t) \equiv \frac{\rho(\theta_{-i}|\theta_i)\hat{\mu}(\theta_{-i}|h^t)}{\sum_{\theta'_{-i}} \rho(\theta'_{-i}|\theta_i)\hat{\mu}(\theta'_{-i}|h^t)}$$

is a sequential equilibrium of the original game (u, ρ).

Imposing condition B or B* on the transformed game yields restrictions on beliefs for the original game. In particular, in a game with correlated types, a player's action conveys information about other players' types only to the extent that it conveys information about his own type. The date-

$(t+1)$ beliefs about θ_{-i} conditional on θ_i depend on the history h^t, the actions a^t_{-i}, and the conditional beliefs $\mu(\theta_{-i}|\theta_i, h^t)$ at date t, but not on player i's action a^t_i.

General Extensive-Form Games[†††]

Necessary conditions for sequential equilibrium in general extensive-form games can be given in terms of "no signaling what you don't know." The trembles associated with sequential equilibrium give rise to a conditional probability system on all terminal nodes. Conversely, assume that there is a conditional probability system on all terminal nodes that is compatible with the strategy profile σ. Let $\mu(x|x, y)$ denote the relative beliefs generated by the conditional probability system when nodes x and y belong to the same information set ($\mu(x|x, y)$ is the probability of terminal nodes that are successors of x conditional on the terminal node's being a successor of either x or y); similarly, let $\mu(\zeta(x, a)|x)$ denote the probability of the direct successor $\zeta(x, a)$ of node x through action $a \in A(h(x))$. The no-signaling conditions are simply (1) for any information set h and any node $x \in h$ and action $a \in A(h)$,

$$\mu(\zeta(x, a)|x) = \sigma(a|h)$$

and (2) for any information set h, nodes x and y in h and action a,

$$\mu(\zeta(x, a)|\zeta(x, a), \zeta(y, a)) = \mu(x|x, y).$$

That is, the player on move at h cannot distinguish among the nodes in h and therefore cannot signal information he does not have. Fudenberg and Tirole (1991) claimed that these conditions implied consistency, but Battigalli (1991) shows that this claim is false. The no-signaling conditions 1 and 2 are very weak, but the existence of a conditional probability system on all terminal nodes is quite strong: It allows the (common across players) comparison of probabilities of nodes at any two information sets. In contrast, for multi-stage games with incomplete information it suffices to be able to compare the likelihoods of types of a player in any given period.

8.4 Strategic-Form Refinements[††]

This section reviews two strategic-form refinements of Nash equilibrium. The concept of sequential equilibrium is closely related to that of trembling-hand perfect equilibrium (henceforth "perfect equilibrium") of Selten (1975). Perfect equilibrium requires that the strategies be the limit of totally mixed strategies and that, subject to the requirement that it must put at least a minimum weight (must tremble) on each pure strategy on the converging sequence, each player's strategy is (constrained) optimal against his opponents' (which include trembles themselves). The distinction with sequential equilibrium is thus that strategies must be in equilibrium along the converging subsequence and not only in the limit. This distinction turns out to make only a minor difference, as the sets of sequential and perfect equilibria coincide "for almost all games." We will also review Myerson's (1978) concept of proper equilibrium, which refines perfect equilibrium by requiring that, along the converging sequence of perturbed strategies, players are less likely to make "mistakes" that are more costly.

8.4.1 Trembling-Hand Perfect Equilibrium

Now we consider the concept of trembling-hand perfection in the *strategic form* and in the *agent-strategic form*. (Selten called the latter the "agent normal form.") As we will see, perfection in the strategic form does not imply subgame perfection. Selten introduced the agent-strategic form in order to rule out subgame-imperfect equilibria.

There are three equivalent definitions of trembling-hand perfection in the strategic form:

Definition 8.5A An "ε-constrained equilibrium" of a strategic-form game is a totally mixed strategy profile σ^ε such that, for each player i, σ_i^ε solves $\max_{\sigma_i} u_i(\sigma_i, \sigma_{-i}^\varepsilon)$ subject to $\sigma_i(s_i) \geq \varepsilon(s_i)$ for all s_i, for some $\{\varepsilon(s_i)\}_{s_i \in S_i, i \in \mathscr{I}}$ where $0 < \varepsilon(s_i) < \varepsilon$. A *perfect equilibrium* is any limit of ε-constrained equilibria σ^ε as ε tends to 0.

According to definition 8.5A, a perfect equilibrium is a limit of Nash equilibria of some sequence of constrained games. The standard sort of closed-graph argument shows that any perfect equilibrium is a Nash equilibrium of the game without the constraints.

For given $\{\varepsilon(s_i)\}$, a constrained equilibrium exists for the usual reasons. (The only difference with the proof of existence of a Nash equilibrium in mixed strategies (see section 1.3) is that each mixed strategy must belong to a subset of a simplex, as opposed to the simplex itself, but this difference is irrelevant because the subset is compact, convex and, for ε small, nonempty.) Thus, for any sequence of constraints $\{\varepsilon(s_i)\}$ there is a corresponding sequence of constrained equilibria. Because strategy spaces are compact, this sequence has a convergent subsequence, so a perfect equilibrium exists.

To see how trembles help refine the set of Nash equilibria, consider the game illustrated in figure 8.10, which Selten used to motivate subgame perfection. The Nash equilibrium $\{R_1, L_2\}$ is not the limit of constrained equilibria: If player 1 plays L_1 with positive probability, player 2 puts as much weight as possible on R_2.

The idea behind definition 8.5A is that players may tremble (make mistakes) and that their constrained strategies should be optimal when the trembles of their rivals are taken into account. Selten's second definition does not explicitly introduce minimum trembles, but requires that the

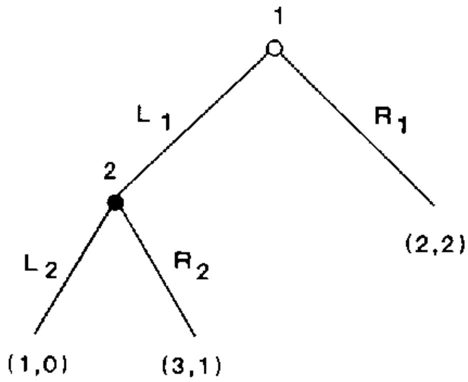


Figure 8.10

profile σ be a limit of a sequence of totally mixed profiles σ^n and that σ_i be a best response to the opponents' perturbed strategies σ_{-i}^n:

Definition 8.5B Strategy profile σ of a strategic form is a *perfect equilibrium* if there exists a sequence of totally mixed strategy profiles $\sigma^n \to \sigma$ such that, for all i, $u_i(\sigma_i, \sigma_{-i}^n) \geq u_i(s_i, \sigma_{-i}^n)$ for all $s_i \in S_i$.

Let us emphasize that in definition 8.5B strategy σ_i is a best response to *some* sequence σ_{-i}^n and not necessarily to all sequences converging to σ_{-i}. Likewise, in definition 8.5A, it suffices that σ is the limit of ε-constrained equilibria for *some* sequence of constraints, as opposed to all such sequences. The uniform versions of these definitions—requiring in definition 8.5B that σ_i be a best response to any sequence $\sigma_{-i}^n \to \sigma_{-i}$—yield the concept of "truly perfect equilibrium," which is much more demanding. For some games, truly perfect equilibria do not exist (see chapter 11).

The third definition of perfect equilibrium, due to Myerson (1978), does not quite refer to a conventional optimization:

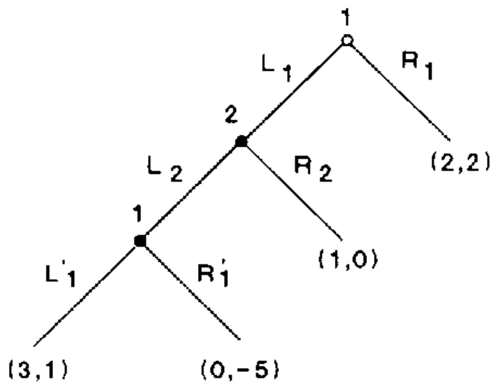
Definition 8.5C Strategy profile σ^ε of a strategic form is an ε-perfect equilibrium[16] if it is completely mixed, and, for all i and any s_i, if there exists s_i' with $u_i(s_i, \sigma_{-i}^\varepsilon) < u_i(s_i', \sigma_{-i}^\varepsilon)$, then $\sigma_i^\varepsilon(s_i) < \varepsilon$. A *perfect equilibrium* σ is any limit of ε-perfect strategy profiles σ^ε for some sequence ε of positive numbers that converges to 0.

That is, player i is not required to optimize against his rivals' strategies subject to an explicit constraint on minimum weights, but must put less than ε weight on strategies that are not best responses.

Theorem 8.4 The three definitions of perfect equilibrium (8.5A–8.5C) are equivalent.

Proof We show that definition A implies definition C, which implies definition B, which in turn implies definition A. First, by construction, the

16. Here the ε does *not* refer to ε-optimization, as in the ε-perfect equilibrium discussed in section 4.8.



a. Extensive Form

	L_2	R_2
R_1	2,2	2,2
L_1,L'_1	3,1	1,0
L_1,R'_1	0,−5	1,0

b. Strategic Form

Figure 8.11

sequence σ^ε defined in definition A is an ε-perfect equilibrium, so that σ^ε satisfies definition C if it satisfies definition A. Second, suppose that σ satisfies definition C. Then there is a sequence $\sigma^\varepsilon \to \sigma$ and a constant $d > 0$ with $\sigma_i^\varepsilon(s_i) > d$ for every s_i in the support of σ_i. Thus, every s_i in the support of σ_i must be a best response to σ_{-i}^ε, so that definition B is satisfied. Third, suppose that σ satisfies definition B, and let $\sigma^n \to \sigma$ be the hypothesized totally mixed strategy profiles. For s_i not in the support of σ_i define $\varepsilon^n(s_i) \equiv \sigma_i^n(s_i)$, and for s_i in the support of σ_i let $\varepsilon^n(s_i) \equiv 1/n$. Then consider the program $\{\mathrm{Max}_{\sigma_i} u_i(\sigma_i, \sigma_{-i}^\varepsilon)$ subject to $\sigma_i(s_i) \geq \varepsilon^n(s_i)$ for all $s_i \in S_i\}$. Because σ_i is a best response to σ_{-i}^n by assumption, one of the corresponding ε-constrained equilibria, σ^ε, has $\sigma_i^\varepsilon(s_i) = \varepsilon^n(s_i)$ for $s_i \notin$ support (σ_i), and $\sigma_i^\varepsilon(s_i) = \sigma_i(s_i)$ for $s_i \in$ support (σ_i).[17] $\varepsilon^n \equiv \max\{\varepsilon^n(s_i)\}$ tends to 0 as n tends to ∞. Hence, definition A is satisfied. ∎

As Selten noted, perfection in the strategic form is not totally satisfactory. Consider figure 8.11. The only subgame-perfect equilibrium is $\{L_1, L_2, L'_1\}$. But the subgame-imperfect Nash equilibrium $\{R_1, R_2, R'_1\}$ is the limit of equilibria with trembles. To see why, consider the corresponding (reduced) strategic form (displayed in figure 8.11b), and let player 1 play (L_1, L'_1) with

17. There can be other ε-constrained equilibria, because some $s_i \notin$ support (σ_i) could also be best responses to σ_{-i}^n.

probability ε^2 and (L_1, R_1') with probability ε. Then player 2 should put as much weight as possible on R_2, because player 1's probability of "playing" R_1' conditional on having "played" L_1 is $\varepsilon/(\varepsilon + \varepsilon^2) \simeq 1$ for ε small. The point is that strategic-form trembles allow correlation between a player's tremble and his play at subsequent information sets. In the above example, if a player "trembles" onto L_1 he is very likely to play R_1' and not L_1'.

One possible response to this is that since a player's trembles may indeed be correlated, subgame perfection is too strong. Recall that subgame perfection's premise is that reasonable play in a subgame depends only on that subgame, regardless of whether that subgame is in fact the whole tree or instead can be reached only if some player i deviates from the (perfect) equilibrium strategies in a longer game. If we take the trembles story literally this premise may or may not be compelling, depending on how and why mistakes occur, and subgame perfection loses some of its persuasiveness. (At this point the reader might want to reread the examples in section 3.6.)

Selten's view in his 1975 paper was rather that the trembles are a technical device, and that they are not intended to model actual "mistakes." In that spirit, he modified his trembling-hand concept to rule out correlation and thus rule out subgame-imperfect equilibria. The modification uses the concept of the *agent-strategic form*, which treats the two choices of player 1 in figure 8.11 as made by two different players whose trembles are independent.

More precisely, in the agent-strategic form each information set is "played" by a different "agent," and the agent on move at information h has the same payoffs over terminal nodes as the player $i(h)$ on move at h in the original game. A trembling-hand perfect equilibrium in the agent-strategic form of an extensive-form game is the trembling-hand perfect equilibrium of the corresponding extensive form.

It should be clear that the equivalence between the various definitions of perfect equilibrium carries over to perfection in the agent-strategic form, as does the proof of existence. From now on, by "perfect equilibrium" we will mean "trembling-hand perfect equilibrium in the agent-strategic form" (as opposed to "strategic-form perfect equilibrium," which allows correlated trembles across information sets of the same player). Figure 8.12 displays the agent strategic form associated with the extensive form in figure 8.11a. The "first incarnation" of player 1 chooses matrices and the "second incarnation" chooses columns. Because the two incarnations have the same payoffs, we merge these payoffs in the entries of the matrices.

Definition 8.5B makes it clear why a perfect equilibrium is a sequential equilibrium. The strategies σ are, by construction, limits of totally mixed strategies σ^n. To obtain a sequential equilibrium, one must construct beliefs μ such that (σ, μ) is consistent and σ is sequentially rational given μ. Because σ^n are totally mixed, associated beliefs μ^n at information sets of any exten-

	L'_1	R'_1
L_2	3,1	0,−5
R_2	1,0	1,0

L_1

	L'_1	R'_1
L_2	2,2	2,2
R_2	2,2	2,2

R_1

Figure 8.12

	L	R
U	1,1	0,0
D	0,0	0,0

Figure 8.13

sive form with this strategic form are uniquely defined by Bayes' rule. It then suffices to take the limit μ of a convergent subsequence μ^n. By construction, (σ, μ) is a consistent assessment. By the one-stage-deviation principle, σ_i is a best response for a "single player i" to σ_i^n, and since payoffs are continuous, (σ, μ) is sequentially rational.

A sequential equilibrium, however, need not be perfect, as is demonstrated in figure 8.13. In the simultaneous-move game with strategic form represented in this figure, the imperfect Nash equilibrium (D, R) is sequential. However, if one requires that strategies be optimal against some trembles, D and R cannot be chosen, because they are weakly dominated.

However, this game is nongeneric, because it relies on a player's (in this example, both players') being indifferent between an equilibrium strategy and a nonequilibrium strategy. Once indifference is broken by a small perturbation of payoffs, the sets of sequential and perfect equilibria coincide, as Kreps and Wilson (1982a) showed. The notion of genericity is the following: Fix an extensive form and prior beliefs and consider the family of games indexed by payoffs u at the ℓ terminal nodes. "Game u" is, by abuse of terminology, the game defined by the payoff vector u in $\mathbb{R}^{\ell \times I}$. A property is generic (satisfied for "almost all games") if the closure of the set of games that do not satisfy this property has Lebesgue measure 0 in $\mathbb{R}^{\ell \times I}$. We collect the results in theorem 8.5.

Theorem 8.5 In finite games, at least one perfect equilibrium exists (Selten 1975). A perfect equilibrium is sequential, but the converse is not true; however, for generic games the two concepts coincide (Kreps and Wilson 1982a).

The perfect-equilibrium correspondence need not be upper hemicontinuous in the payoffs. Figure 8.14 depicts a small perturbation of the

	L	R
U	1,1	0,0
D	0,0	$\frac{1}{n}, \frac{1}{n}$

Figure 8.14

game defined by figure 8.13. In figure 8.14, (D, R) is a perfect equilibrium: D is a best response to a σ_2^n that assigns probability $1 - 1/n^2$ to R, and R is a best response to a σ_1^n that assigns probability $1 - 1/n^2$ to D. However, the unique perfect equilibrium of the limit game is (U, L).

We have two final notes on the idea of trembles. First, observe that trembles can be interpreted as perturbations of the players' payoff functions. In the constrained game of definition 8.5A, player i must place probability at least $\varepsilon(s_i)$ on each $s_i \in S_i$; thus, strategy s_i is effectively replaced by the mixed strategy, which assigns probability $1 - \sum_{s_i' \neq s_i} \varepsilon(s_i')$ to s_i and $\varepsilon(s_i')$ to each s_i'. Equivalently, we could leave the strategies exactly as they were originally, and define new payoff functions

$$\hat{u}_i(s_i, \sigma_{-i}) = \left(1 - \sum_{s_i' \neq s_i} \varepsilon(s_i')\right) u_i(s_i, \sigma_{-i}) + \sum_{s_i' \neq s_i} \varepsilon(s_i') u_i(s_i', \sigma_{-i}).$$ [18]

Second, we should mention the work of Blume, Brandenburger, and Dekel (1990), which gives a characterization of perfect equilibrium in the strategic form in terms of "lexicographic beliefs" instead of trembles. This work stands in roughly the same relation to strategic-form perfect equilibrium as PBE does to sequential equilibrium.

8.4.2 Proper Equilibrium

Myerson (1978) considers perturbed games in which a player's second-best actions are assigned at most ε times the probability of the first-best actions, the third-best actions are assigned at most ε times the probability of the second-best actions, etc. The idea is that a player is "more likely to tremble" on an action that is not too detrimental to him, so that the probability of deviations from equilibrium behavior is inversely related to their costs.

Because a smaller set of trembles is considered, a proper equilibrium is clearly perfect in the strategic form. As we will see, proper equilibria are perfect in the agent-strategic form as well.[19]

18. We will see in chapter 12 that for generic *strategic-form* payoffs, any Nash equilibrium has a nearby Nash equilibrium in any game with nearby payoffs, so in generic strategic forms any Nash equilibrium is truly perfect. However, generic extensive-form payoffs do not generate generic strategic-form ones.

19. Although properness in the strategic form ensures backward induction, properness in the strategic form and properness in the agent-strategic form differ because two incarnations of the same player in the agent strategic form (associated with two different information sets) need not compare the payoffs of their probability-0 actions.

	L	M	R
U	1,1	0,0	−9,−9
M	0,0	0,0	−7,−7
D	−9,−9	−7,−7	−7,−7

Figure 8.15

To illustrate the notion of proper equilibrium, consider the game illustrated in figure 8.15 (due to Myerson), which adds weakly dominated strategies to the game defined in figure 8.12. This game has three pure-strategy Nash equilibria: (U, L), (M, M), and (D, R). D and R are weakly dominated strategies and therefore cannot be optimal when the other player trembles, so (D, R) is not perfect. (M, M) *is* perfect. To see this, consider the totally mixed strategy profile where each player plays M with probability $1 - 2\varepsilon$ and plays each of the other two strategies with probability ε. Deviating to U for player 1 (or to L for player 2) increases this player's payoff by $(\varepsilon - 9\varepsilon) - (-7\varepsilon) = -\varepsilon < 0$. However, (M, M) is not a proper equilibrium. Each player should put much more weight (tremble more) on his second-best strategy than on his third-best, which yields a lower payoff. But if player 1, say, puts weight ε on U and ε^2 on D, player 2 does better by playing L than by playing M, as $(\varepsilon - 9\varepsilon^2) - (-7\varepsilon^2) > 0$ for ε small. The only proper equilibrium in this game is (U, L).

Definition 8.6 An *ε-proper equilibrium* is a totally mixed strategy profile σ^ε such that, if $u_i(s_i, \sigma^\varepsilon_{-i}) < u_i(s_i', \sigma^\varepsilon_{-i})$, then $\sigma_i^\varepsilon(s_i) \le \varepsilon\, \sigma_i^\varepsilon(s_i')$. A *proper equilibrium* σ is any limit of ε-proper equilibria σ^ε as ε tends to 0.

Theorem 8.6 (Myerson 1978) All finite strategic-form games have proper equilibria.

Proof We first prove the existence of ε-proper equilibria. Let

$$\tilde{\Sigma}_i = \left\{ \sigma_i \in \Sigma_i^0 \,|\, \sigma_i(s_i) \ge \frac{\varepsilon^m}{m} \text{ for all } s_i \text{ in } S_i \right\},$$

where $m \equiv \max_i(\# S_i)$ and $0 < \varepsilon < 1$. Consider the constrained best-response correspondence of player i to strategies σ_{-i}:

$$\tilde{r}_i(\sigma_{-i}) = \{\sigma_i \in \tilde{\Sigma}_i \,|\, \text{if } u_i(s_i, \sigma_{-i}) < u_i(s_i', \sigma_{-i}) \text{ then } \sigma_i(s_i) \le \varepsilon\, \sigma_i(s_i')\ \forall (s_i, s_i') \in (S_i)^2\}.$$

Because $\tilde{r}_i$ is defined by a finite collection of linear weak inequalities, it is convex- and compact-valued; upper hemi-continuity of $\tilde{r}_i$ is straightforward. To prove that $\tilde{r}_i(\sigma_{-i})$ is nonempty, let $\rho(s_i)$ be the number of strategies s_i' such that $u_i(s_i, \sigma_{-i}) < u_i(s_i', \sigma_{-i})$. Then, if $\rho(s_i) > 0$, $\sigma_i \equiv \{\sigma_i(s_i)\}$, where

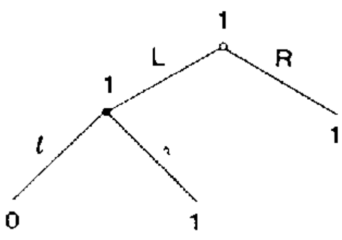
$$\sigma_i(s_i) = \varepsilon^{\rho(s_i)} \Big/ \left(\sum_{s_i' \in S_i} \varepsilon^{\rho(s_i')} \right) \geq \frac{\varepsilon^m}{m},$$

belongs to $\tilde{r}_i(\sigma_{-i})$. One then applies Kakutani's fixed-point theorem in the usual way to prove the existence of an ε-proper equilibrium in $\times_i \tilde{\Sigma}_i$. Letting $\tilde{\Sigma}_i$ tend to Σ_i and taking a convergent subsequence of the associated ε-proper equilibria completes the proof. ■

Let us conclude with two properties of proper equilibrium.[20]

First, proper equilibrium yields backward induction without the use of the agent strategic form, because the requirement on relative trembles ensures that the players play optimally off the equilibrium path. This is illustrated in figure 8.11. The strategy (L_1, R_1') is dominated by the strategy (L_1, L_1') as long as player 2 trembles. Hence, player 1 must put almost all the weight on L_1' if his second information set is reached.

Second, Kohlberg and Mertens (1986) have shown that every proper equilibrium of a strategic-form game is sequential in every extensive form with the given strategic form. Refer back to figure 8.6, which gave two allegedly equivalent descriptions of the "same game." Player 1 playing A is a sequential equilibrium outcome in figure 8.6a, but not in figure 8.6b. However, in either game, the only proper equilibrium is (R_1, R_2). In particular, (A, L_2) is not a proper equilibrium in figure 8.6a, since player 1 must give R_1 more weight than L_1 in any ε-proper equilibrium. Kohlberg and Mertens also observe that a proper equilibrium of a strategic form need not be a trembling-hand perfect equilibrium in (the agent strategic form of) every extensive form associated with this strategic form. Figure 8.16 considers a single-decision-maker problem with three pure strategies. (L, r) is proper in



R	1
(L, r)	1
(L, ℓ)	0

Strategic Form

Figure 8.16

20. See van Damme 1987 for a more extensive and very clear discussion of proper equilibrium and several of its variants. In particular, proper equilibria do not correspond to the limits of equilibria of games in which players optimize their error probabilities in the presence of "control costs." Intuitively, if strategy s_1 is almost as good a response as strategy s_1' but neither is a best response, then when error probabilities are optimized we would expect trembles on s_1 to be almost as likely as trembles on s_1'.

the strategic form, but not (agent-strategic-form) perfect in the tree: If the player's second incarnation trembles, his first incarnation prefers playing R.

Appendix: The Structure of Sequential Equilibria††

We remarked in subsection 8.3.3 that, although the set of sequential equilibrium *outcomes* is finite for generic extensive-form payoffs, the set of sequential equilibrium *assessments* is, in general, infinite. This appendix develops that remark in more detail.

Consider the game illustrated in figure 8.17, taken from Kreps and Wilson 1982a. This game has two sequential equilibrium outcomes: (L, ℓ) and A. There is a unique equilibrium assessment with outcome (L, ℓ), namely $\sigma_1(L) = 1 = \sigma_2(\ell)$, and $\mu(x) = 1$. In contrast, there are two one-parameter families of equilibrium assessments with outcome A. In the first family, $\sigma_1(A) = 1, \sigma_2(\ell) = 0$, and $\mu(x) < \frac{1}{2}$; in the second, $\sigma_1(A) = 1, \sigma_2(\ell) \in [0, \frac{3}{5}]$, and $\mu(x) = \frac{1}{2}$. Projecting the equilibrium assessments onto the pairs $(\mu(x), \sigma_2(\ell))$ gives the picture in figure 8.18.

As this example illustrates, for generic payoffs the set of sequential equilibrium assessments is the union of manifolds of varying dimensions;

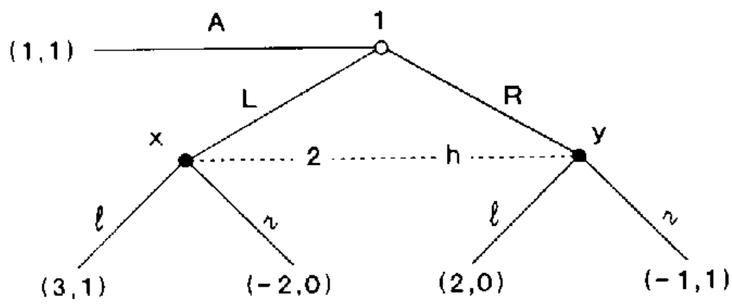


Figure 8.17

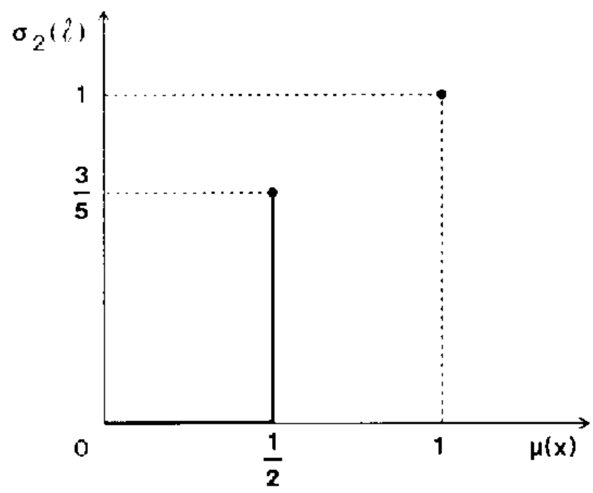


Figure 8.18

the dimensions of these manifolds is related to the number of "degrees of freedom" available in specifying the off-path strategies and beliefs. Since there are no off-path information sets in the equilibrium with outcome (L, ℓ), there are zero degrees of freedom, and the manifold associated with this outcome has dimension zero. (For the same reason, one-shot simultaneous-move games have a finite number of equilibria for generic payoffs, as we discuss in section 12.1.) In the equilibria with outcome A, player 2's information set is not reached. The horizontal segment of figure 8.18 reflects the one degree of freedom in specifying beliefs $\mu(x)$ that make r a better choice for player 2 than ℓ; the vertical segment corresponds to the degree of freedom in specifying mixed strategies for player 2 that make player 1 prefer A to L. Since player 2 must be indifferent between ℓ and r to randomize, the one degree of freedom in specifying beliefs must be lost to obtain the one degree of freedom in specifying mixed strategies.

Kreps and Wilson generalize these observations as follows. Let the *basis* b of an equilibrium assessment (σ, μ) be the collection of nodes and actions the assessment gives positive probability (i.e., a in $A(h)$ belongs to b iff $\sigma_{i(h)}(a|h) > 0$ and $x \in b$ iff $\mu(x) > 0$). Kreps and Wilson show that, for generic payoffs, the set of equilibria with a given basis is either empty, or is a manifold whose dimension is independent of the particular extensive-form payoffs specified.

Exercises

Exercise 8.1* Consider the public-good game of example 8.3.

(a) Show that in any PBE there exists $\hat{c}_i$ such that player i contributes if and only if $c_i \leq \hat{c}_i$. (Hint: Let z_j and z_j^{xy} denote player j's first-period probability of contributing and second-period probability of contributing conditional on whether player i contributed ($x = 1$ or 0) and player j contributed ($y = 1$ or 0) in the first period. Write the intertemporal expected payoffs for player i with type c_i from contributing and not contributing in the first period. Note that these payoffs involve second-period maximizations. What are their derivatives with respect to c_i?)

(b) Use question a to show that $\hat{c}_i < 1$ for each i. (Hint: Suppose, without loss of generality, that $\hat{c}_1 = \max\{\hat{c}_1, \hat{c}_2\} \geq 1$. Argue that player 1's not contributing in the first period induces "maximal contribution" by player 2 in the second period.)

(c) Use question b to show that $\hat{c}_i > 0$ for all i.

Exercise 8.2** In final-offer arbitration, the arbitrator is forced to choose one of the parties' offers as a settlement. Consider the following model of learning in final-offer arbitration, which is due to Gibbons (1988). In the first stage, an employer and a union simultaneously make wage offers a_e

and a_u in $\mathbb{R}$. The arbitrator then chooses $a_2 \in \{a_e, a_u\}$. The objective functions are $-\mathrm{E}a_2$ for the employer, $+\mathrm{E}a_2$ for the union, and $\mathrm{E}[-(a_2 - \omega)^2]$ for the arbitrator, where ω, the state of nature, is the arbitrator's bliss point. The information about ω is as follows: The employer and the union receive the same signal, $z_1 = \omega + \varepsilon_1$. The arbitrator receives signal $z_2 = \omega + \varepsilon_2$. The variables ω, ε_1, and ε_2 are independent and normally distributed with means m, 0, and 0 and precisions h, h_1, and h_2. (Recall that the precision is the inverse of the variance, and that the expectation of a variable given independent normally distributed signals—including the prior—is the weighted average of the signals, where the weights are the precisions.) Show that there exists an equilibrium in which $a_e + a_u$ perfectly reveals z_1 to the arbitrator, and in which each party $i = \mathrm{e}, \mathrm{u}$ offers $a_i = (hm + h_1 z_1)/(h + h_1) + k_i$, where k_i is a constant.

Exercise 8.3* Gilligan and Krehbiel (1988) depict the open rule in Congress as a cheap-talk game, that is, as a signaling game in which signals are costless. As a rough approximation, the committee proposes a policy, but the floor can introduce amendments and choose the policy it likes. The open rule is depicted as a two-player game, with a single member in the committee and a single representative on the floor (who stands for the median voter). The object of the decision is a policy a_2 in $\mathbb{R}$. The outcome given policy a_2 is $x = a_2 + \omega$, where ω is a random variable uniformly distributed between 0 and 1. The committee knows ω; the floor does not. The committee moves first and suggests a policy a_1 to the floor. The preferences of both are quadratic with bliss points $x = 0$ for the floor and $x = x_c \in (0, 1)$ for the committee: $u_1(x) = -(x - x_c)^2$ and $u_2(x) = -x^2$.

(a) Show that there always exists a "babbling" equilibrium in which a_1 is uninformative and $a_2 = -\frac{1}{2}$.

(b) Look for informative perfect Bayesian equilibria. In particular, find an equilibrium in which the committee "reports low" when $\omega \in [0, \omega^*]$ and "reports high" when $\omega \in (\omega^*, 1]$.

(c)** Analyze the closed rule, in which the committee proposes a policy a_1 and the floor chooses between a_1 and a reversion or *status quo* policy a_0. (Note that this is no longer a cheap-talk game.)

See Crawford and Sobel 1982 for the first example of a cheap-talk game.

Exercise 8.4** Consider the Chatterjee-Samuelson simultaneous-offer bargaining game developed in chapter 6. Assume that the buyer's valuation v and the seller's cost c are independently and uniformly distributed on $[0, 1]$. They make simultaneous offers. They trade if the seller's bid b_1 is less than the buyer's bid b_2 at a price $p = (b_1 + b_2)/2$. Add a preplay communication stage to this Chatterjee-Samuelson model. That is, before choosing their bids, the traders simultaneously send a message to each other. These messages are costless (are cheap talk). Show that the equilibria discussed in chapter 6 are still equilibria (the messages are simply ignored). But there exist other equilibria as well. For instance, show that the following is a perfect Bayesian equilibrium: Each trader announces either "keen" or "not keen" in the preplay communication stage. The buyer announces "keen" if and only if $v > v^* = (22 + 12\sqrt{2})/49$. The seller says "keen" if and only if $c < c^* = 1 - v^*$. If they both say "not keen," they "stop bargaining" (i.e., they play the continuation equilibrium in which they make nonserious offers, such as 0 for the buyer and 1 for the seller). If one of them says "keen" and the other "not keen," they play a Chatterjee-Samuelson linear equilibrium given posterior beliefs. If they both say "keen," both bids are equal to $\frac{1}{2}$. (See Farrell and Gibbons 1989 for the answer.)

Exercise 8.5*** Exercises 8.3 and 8.4 involve a player's transmission of information that is not verifiable by the other players. This exercise involves a two-stage game of transmission of verifiable information. There are I players, $i = 1, \ldots, I$. Player i's types, θ_i, belong to some finite ordered set Θ (for instance, of elements of the real line). Types are drawn independently from the prior distribution $p(\theta) = \prod_{i=1}^{I} p_i(\theta_i)$. In period 2, the players play some simultaneous-move game that results in (reduced-form) payoffs $v_i((\mu_i, \mu_{-i}), \theta_i)$ for player i, where μ_i is the posterior beliefs about θ_i and where $\mu_{-i} \equiv \prod_{j \neq i} \mu_j$ is the posterior beliefs about θ_{-i}. (Are we allowed to write posterior beliefs as a product if we look for sequential equilibrium?)

Beliefs μ_i (first-order) stochastically dominate beliefs μ_i' if for all θ_i

$$\sum_{\tilde{\theta}_i \leq \theta_i} \mu_i(\tilde{\theta}_i) \leq \sum_{\tilde{\theta}_i \leq \theta_i} \mu_i'(\tilde{\theta}_i),$$

with a strict inequality for at least some θ_i. Assume that each player prefers his opponents to believe he has "high types": For any μ_{-i} and θ_i, if μ_i stochastically dominates μ_i',

$$v_i((\mu_i, \mu_{-i}), \theta_i) > v_i((\mu_i', \mu_{-i}), \theta_i).$$

In the first period, players simultaneously announce messages $a_i^1 \in A_i^1(\theta_i)$. Messages do not enter the players' payoff function v_i. However, they affect beliefs for the second-stage game. Suppose that, for all i and θ_i, $A_i^1(\theta_i)$ contains a message that certifies that player i's type is at least equal to θ_i. Show that in a sequential equilibrium posterior beliefs are degenerate; that is, the first-period messages are fully revealing. (See Grossman 1980, Grossman and Hart 1980, and Milgrom 1981 for this result with one informed player; see Okuno-Fujiwara et al. 1990 for the many-informed-players version.)

Exercise 8.6* Introduce asymmetric information in the stag-hunt game of chapter 1. There are two players who must decide whether to hunt the stag or the rabbit. With probability p, each player has preferences that always make him hunt the stag (he does not like rabbit, or he is able to catch the stag by himself although he would prefer to hunt with the other player); with probability q, each player always hunts the rabbit (he does not like stag); with probability $1-p-q$, the player has the preferences described in chapter 1: He gets 1 if he hunts the rabbit, 2 if both hunt the stag, and 0 if he hunts the stag alone. Suppose that $2p > 1-q$ and $2q > 1-p$.

(a) Show that in the one-period version of the game there is a multiplicity of equilibria similar to the one in chapter 1 if $\max(p,q) < \frac{1}{2}$. Show that the equilibrium is unique if $p > \frac{1}{2}$ or $q > \frac{1}{2}$.

(b) Consider the two-period version of the above stag-hunt game with incomplete information. Show that for any first-period behavior the second-period behavior is uniquely determined.

(c) Assume that the discount factor between the periods is equal to 1, and that $\alpha \equiv (1+2p)/4 \in (p, 1-q)$ (which implies that $p < \frac{1}{2}$). Show the existence of a symmetric equilibrium, in which each player hunts the stag with probability α in the first period. Note that the type who does not have a dominant strategy sacrifices short-run utility to build a "reputation." (Reputational phenomena are studied in much detail in chapter 9.) Show that there are exactly two other equilibria.

Exercise 8.7** Consider the following two-player three-stage game with incomplete information. In each period, player 1 has three possible actions: S (hunt the stag), R (hunt the rabbit), and H (stay home), and player 2 has two possible actions: S (hunt the stag) and R (hunt the rabbit). Player 1 has one of three equally likely types: s, r, and h. In any given period, if player 1 has type s he gets 1 if both players play S, and 0 otherwise. Similarly, type r gets 1 if both players play R and 0 otherwise, and type h gets 1 if he plays H and 0 otherwise. Player 2 has no private information. In a given period, he gets 1 if both play S or both play R and 0 otherwise. Is there a sequential equilibrium in which the following observation has positive probability: player 1 plays H in period 0, both players play S in period 1, and player 2 plays R in period 2?

Exercise 8.8*** Consider a three-player two-stage game with incomplete information. Only players 1 and 2 have private information: Player i's type, θ_i, is equal to θ_i' or θ_i'' with equal probabilities, for $i = 1, 2$. Furthermore, the prior beliefs satisfy $\text{Prob}(\theta_1 = \theta_1' | \theta_2 = \theta_2') \equiv \text{Prob}(\theta_1 = \theta_1'' | \theta_2 = \theta_2'') = \frac{3}{4}$. Suppose that in a sequential equilibrium player i plays a_i^* in the first period whatever his type. Determine the set of player 3's joint probability distributions over (θ_1, θ_2) at the beginning of the second period that are compatible with sequential equilibrium when player 1 plays a_1^* but player 2 deviates in the first period.

Exercise 8.9*

(a) Show that (U_1, L_1) is the unique perfect equilibrium of the game illustrated in figure 8.19a.

	L_2	R_2
U_1	1,1	0,0
M_1	0,0	0,0

a

	L_2	M_2	R_2
U_1	1,1	0,0	−1,−2
M_1	0,0	0,0	0,−2
D_1	−2,−1	−2,0	−2,−2

b

Figure 8.19

(b) Use figure 8.19b to argue that adding dominated strategies may enlarge the set of perfect equilibria. (This exercise is from van Damme 1987.)

Exercise 8.10* Consider the following signaling game. There are two players, a plaintiff and a defendant in a civil suit. The plaintiff knows whether or not he will win the case if it goes to trial, but the defendant does not have this information. The defendant knows that the plaintiff knows who would win, and the defendant has prior beliefs that there is probability $\frac{1}{3}$ that the plaintiff will win; these prior beliefs are common knowledge. If the plaintiff wins, his payoff is 3 and the defendant's payoff is -4; if the plaintiff loses, his payoff is -1 and the defendant's is 0. (This corresponds to the defendant paying cash damages of 3 if the plaintiff wins, and the loser of the case paying court costs of 1.)

The plaintiff has two possible actions: He can ask for either a low settlement of $m = 1$ or a high settlement of $m = 2$. If the defendant accepts a settlement offer of m, the plaintiff's payoff is m and the defendant's is $-m$. If the defendant rejects the settlement offer, the case goes to court. List all the pure-strategy perfect Bayesian equilibria (PBE) strategy profiles. For each such profile, specify the posterior beliefs of the defendant as a function of m, and verify that the combination of these beliefs and the profile is in fact a PBE. Explain why the other profiles are not PBE.

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9 Reputation Effects

9.1 Introduction††

This chapter investigates the notion that a player who plays the same game repeatedly may try to develop a reputation for certain kinds of play. The idea is that if the player always plays in the same way, his opponents will come to expect him to play that way in the future and will adjust their own play accordingly. The question then is when and whether a player will be able to develop or maintain the reputation he desires. For example, if a central bank always implements the monetary policy it announces, will traders come to believe that it will do so in the future? That is, will reputation effects allow the central bank to effectively commit itself to implementing its announcements?

To model the possibility that players are concerned about their reputations, we suppose that there is incomplete information about each player's type, with different types expected to play in different ways. Each player's reputation is then summarized by his opponents' current beliefs about his type. For example, to model a central bank's reputation for sticking to its announced monetary policy, we assign positive prior probability to a type that always implements its announcements.[1] More generally, we may suppose that each player has several different types, each of which is associated with a different kind of play, and that no player's type enters directly as an argument into any other player's utility function.[2]

Intuitively, since reputations are like assets, a player is most likely to be willing to incur short-run costs to build up his reputation when he is patient and his planning horizon is long. A player with a short horizon will be less willing to make investments, so we should expect that investments in reputation will be more likely in long relationships than in short ones, and more likely at the beginning of a game than at its end. For this reason we will follow the literature and focus on reputations in long-run relationships, although reputations can also play an important role in short-run relationships.

1. An alternative approach is to identify reputations with equilibrium strategies in a repeated game of complete information. For example, in the repeated prisoner's dilemma the equilibrium in the "grim" strategies "cooperate until an opponent defects, and then defect thereafter" can be interpreted as describing a situation where each player has a "reputation" for cooperation that vanishes the first time he defects, and in a repeated quality-choice game the strategies "expect high quality until the firm produces low quality" can be interpreted as saying that the firm begins with a reputation for high quality that it can only maintain by making high-quality output. Of course, this reinterpretation does not change the set of equilibria, and so this version of reputation does not have predictive power. Also, modeling reputations as complete-information strategies cannot capture the idea that a player's reputation corresponds to something that his opponents have learned about him.

2. This is a narrower meaning of reputation than that suggested by common usage. For example, one might speak of a worker having a "reputation" for high productivity in the Spence signaling model, and of high-productivity workers investing in this reputation by choosing high levels of education.

Our main concern, then, is when and whether a long-lived player can take advantage of a small prior probability of a certain type or reputation to effectively commit himself to playing as if he were that type. For example, what prior distributions on types imply that in equilibrium a central bank's announcements will be credible?

A related question is whether models of reputation effects provide a way to pick and choose among the many equilibria of an infinitely repeated game, and in particular whether reputation effects can provide support for the intuitions that certain of these equilibria are particularly reasonable. For example, although many papers have used the "cooperative" equilibrium of the repeated prisoner's dilemma (chapter 4) to explain trust and cooperation in long-run relationships, there is also an equilibrium where players do not cooperate. Similarly, though a rough analog of the folk theorem holds in games where a single, long-run, patient player faces a sequence of short-run opponents (see subsection 5.3.1), economic applications typically examine only the equilibrium the long-run player most prefers. For example, a long-run firm that faces a sequence of short-run consumers may choose to produce high-quality output, even though doing so is more expensive in the short run, because switching to low quality would cost it sales in the future (Dybvig and Spatt 1980; Shapiro 1982). However, there is another equilibrium where the firm's quality is always low.

This case of a single long-run player is the one in which reputation effects have the strongest and most general implications. We discuss this case in section 9.2, beginning with the work on the chain-store paradox. Since there is only one player who has an incentive to maintain a reputation, it may not be surprising that reputation effects are quite powerful: In a simultaneous-move stage game, a weak full-support distribution on the prior distribution implies that a single patient player can use reputation effects to obtain the payoff he would obtain if he could publicly commit himself to whatever strategy he most prefers.

Another case where reputation effects might be thought to allow one player to commit himself is that of a single "large" player facing a great many long-lived but "small" opponents, since the large player has much more to gain from a successful commitment than his opponents. (One reason this case of small opponents is interesting is that it may be a better description of the situation facing a government entity such as the Internal Revenue Service or the Federal Reserve than the model in which the government entity is a long-run player facing a sequence of short-run private individuals.) Whether reputation effects allow the large player to commit himself turns out to depend on the fine structure of the game, as we discuss in section 9.4.

When all players are long-run, as in the repeated prisoner's dilemma, there is no single player whose interests might be expected to dominate play, and so it would seem unlikely that reputation effects could lead to strong general conclusions. It is true that strong results can be obtained for specific prior distributions over types. For example, in the repeated prisoner's dilemma, if player 2's payoffs are known to be as in the usual complete-information case, while player 1 is either a type who always plays the strategy "tit for tat" or a type with the usual payoffs, then with sufficiently patient players and a long finite horizon every sequential equilibrium has both players cooperate in almost every period. However, other outcomes can be obtained by varying the prior distribution; in fact, any feasible, individually rational payoffs of the complete-information game can be obtained as sequential equilibrium payoffs of an incomplete-information version of the game where the payoffs are the complete-information ones with probability close to 1. This confirms the intuition that reputation effects on their own have little power when all players are long-run. Reputation effects do pick out the unique Pareto-optimal payoffs in games of pure coordination when the prior distribution on types is restricted in a particular way. Section 9.3 presents these results.

9.2 Games with a Single Long-Run Player^{++}

9.2.1 The Chain-Store Game

We begin with a discussion of the work by Kreps and Wilson (1982) and Milgrom and Roberts (1982) on reputation effects in Selten's (1978) chain-store game. To set the stage for their work, we will first review a slight variant of Selten's model. A single long-run incumbent firm faces potential entry by a series of short-run firms, each of which plays only once but observes all previous play. Each period, a potential entrant decides whether to enter or stay out of a particular market. (Each entrant can enter only a single market, and the entrants' markets are distinct.) If the entrant stays out, the incumbent enjoys a monopoly in that market; if the entrant enters, the incumbent must choose whether to fight or to accommodate. The incumbent's payoffs are $a > 0$ if the entrant stays out, 0 if the entrant enters and the incumbent accommodates, and -1 if the entrant enters and the incumbent fights. The incumbent's objective is to maximize the discounted sum of its per-period payoffs; δ denotes the incumbent's discount factor. Each entrant has two possible types: tough and weak. Tough entrants always enter. A weak entrant has payoff 0 if it stays out, -1 if it enters and is fought, and $b > 0$ if it enters and the incumbent accommodates. Each entrant's type is private information, and each is tough with probability q^0 independent of the others. Thus, the incumbent has a short-run incentive to accommodate, whereas a weak entrant will enter only if it expects the probability of fighting to be less than $b/(b+1)$.

If the game has a finite horizon, there is a unique sequential equilibrium, as Selten (1978) observed: The incumbent accommodates in the last period, so the last entrant enters whatever his type and the history of the game; thus, the incumbent accommodates in the next-to-last period, and by backward induction the incumbent always accommodates and every entrant enters. Selten called this a "paradox" because when there are a large number of entrants the equilibrium seems counterintuitive: One suspects that the incumbent would be tempted to fight to try to deter entry. Of course, no matter how often the incumbent fights, he cannot deter the "tough" entrants, so a commitment to always fight is valuable only if the resulting per-period expected payoff of $a(1-q^0)-q^0$ exceeds the zero payoff from always accommodating. When this is the case, and when the incumbent's discount factor is close enough to 1, the infinite-horizon version of the model has an equilibrium where entry is deterred.[3]

Since there is also an infinite-horizon equilibrium where every entrant enters, this is only partial support for the intuition that entry deterrence is the reasonable outcome, and we are left with the puzzle of explaining why the entry-deterrence equilibrium is most plausible. In addition, we might believe that the outcome would be entry deterrence even with a fixed finite horizon. As we will see, allowing for reputation effects by introducing incomplete information responds to both of these points, and it does so in an intuitively appealing way: The incumbent fights to maintain its reputation for being a "tough" type that is likely to fight. After all, if the incumbent were to have fought in each of the preceding 100 periods, then it seems (to us) quite plausible that the next entrant should expect that it is likely to be fought!

To introduce reputation effects into the model, suppose that all players' payoffs are private information. With probability p^0, the incumbent is "tough," meaning that its payoffs are such that it will fight in every market along any equilibrium path.[4] The incumbent is "weak" (i.e., has the payoffs described above) with probability $1-p^0$. And each entrant is "tough" with probability q^0, independent of the others; tough entrants enter regardless of how they expect the incumbent to respond.[5]

3. This was observed by Milgrom and Roberts (1982). One such equilibrium is for the incumbent to fight all entrants so long as it has never accommodated and to accommodate entrants if it has accommodated at least once in the past, and for the entrants to stay out if the incumbent has never accommodated and enter if it ever does accommodate. This profile is an equilibrium if $a(1-q^0)-q^0>(1-\delta)/\delta$.

4. To construct such payoffs, it suffices that the tough type's payoff be equal to -1 times the number of times it fails to fight (or, more generally, the number of times it fails to follow the prescribed behavior). Alternatively, one can suppose that tough incumbents are simply unable to accommodate. Note also that the incumbent's type is chosen once and for all at the start of the game: The incumbent is either tough in all markets or tough in none of them.

5. Our presentation of the chain-store game is based on the summary by Fudenberg and Kreps (1987). Kreps and Wilson consider only the case $q^0=0$; Milgrom and Roberts consider a richer specification of payoffs.

To solve for the sequential equilibrium of the finite-horizon version of this game, we will first solve for the sequential equilibrium of the one-period game, then that of the two-period game, and proceed by induction to solve for the game with N periods. It is easy to determine the sequential equilibrium of a single play of this game: If there is entry, the incumbent accommodates if and only if it is weak, so that a weak entrant nets $(1 - p^0)b - p^0$ from entry. Thus, a weak entrant enters if $p^0 < b/(b+1) \equiv \bar{p}$ and stays out if the inequality is reversed. (We ignore the knife-edge case of equality.)

Now imagine that there are two periods remaining in the game the incumbent will play two different entrants in succession, in two different markets. Entrant 2 is faced first, and entrant 1 observes the outcome in market 2 before making its own entry decision.[6] The nature of the equilibrium depends on the prior probabilities and the parameters of the payoff functions:

(i) If $1 > a\delta(1 - q^0)$ or $q^0 > \bar{q} \equiv (a\delta - 1)/a\delta$, the maximum long-run benefit of fighting $(\delta a(1 - q^0))$ is less than its cost (which is 1), so a weak incumbent will not fight in market 2. Since the tough incumbent will fight, a weak entrant 2 enters if $p^0 < \bar{p}$ and stays out if $p^0 > \bar{p}$. A weak entrant 1 enters if the incumbent accommodates in market 2 and stays out if the incumbent fights.

(ii) If $q^0 < q$, the weak incumbent is willing to fight in market 2 if doing so deters entry, since accommodating reveals that the incumbent is weak and causes entry to occur. In this case, if entrant 2 enters, the weak incumbent must fight with positive probability: It cannot be a sequential equilibrium for the weak incumbent to accommodate with probability 1 in market 2, as then if the incumbent fights the entrants believe he is tough, and so fighting deters entry next period.

The exact nature of the equilibrium again depends on the prior p^0 that the incumbent is tough.

(iia) If $p^0 > p$, then, since the tough incumbent always fights, the posterior probability that the incumbent is tough given that he fights in market 2 is at least p^0, and so fighting in market 2 deters a weak entrant in market 1. Thus, the weak incumbent fights with probability 1 in market 2, the weak entrant stays out of market 2, and the weak incumbent's expected payoff is $[(1 - q^0)a - q^0] + \delta(1 - q^0)a$.

6. Example 8.1 considered a simplified version of this game in which entrant 2 has already entered, entrant 1 is assumed to always be "weak," the incumbent's decision in the final market as a function of its type has been solved out, and the discount factor δ equals 1. There we saw that if the cost of fighting today exceeds the gain from monopoly tomorrow—that is, if $a < 1$—then in the unique equilibrium the weak incumbent accommodates, whereas if $a > 1$ then in the unique equilibrium the weak incumbent fights with positive probability.

(iib) If $p^0 < \bar{p}$, it is not an equilibrium for the weak incumbent to fight with probability 1, as then the posterior probability of toughness after fighting would not deter entry, and the weak incumbent would prefer not to fight. Nor can it be an equilibrium for the weak incumbent to accommodate with probability 1, for then fighting would deter entry and the weak incumbent would prefer to fight. Thus, in equilibrium the weak incumbent must randomize, which requires that when the incumbent fights in market 2 the weak entrant 1 randomizes in a way that makes the weak incumbent indifferent in market 2. This, in turn, requires that the posterior probability that the incumbent is tough, conditional on fighting, be exactly the critical level $p = b/(b+1)$. If we let β be the conditional probability that a weak incumbent fights entry in market 2, and recall that the tough incumbent fights with probability 1, Bayes' rule gives

$$\text{Prob}(\text{tough}\,|\,\text{fight}) = p^0/[p^0 + \beta(1-p^0)],$$

and for this to equal $\bar{p}$, β must equal $p^0/(1-p^0)b$. The total probability that entry in market 2 is fought is

$$p^0 \cdot 1 + (1-p^0)\cdot[p^0/(1-p^0)b] = p^0(b+1)/b,$$

so the weak entrant will stay out of market 2 if $p^0 > [b/(b+1)]^2 = \bar{p}^2$. In this case the weak incumbent's expected average payoff is positive, whereas its payoff was 0 for the same parameters in the one-entrant game. If $p^0 < [b/(b+1)]^2$, the weak entrant enters in market 2, and the weak incumbent's payoff is 0.

Now we can see what happens with three periods remaining: If $p^0 > [b/(b+1)]^2$, the weak incumbent is certain to fight in market 3, and the weak entrant stays out. If p^0 is between $[b/(b+1)]^3$ and $[b/(b+1)]^2$, the weak incumbent randomizes and the weak entrant stays out; if $p^0 < [b/(b+1)]^3$, the weak incumbent randomizes and the weak entrant enters. More generally, for a fixed p^0 and N entrants, the weak entrant stays out until the first period k where $p^0 < [b/(b+1)]^k$, so that for the first $N-k$ periods the weak incumbent has expected payoff $a(1-q^0) - q^0$ per period.

The main point of the Kreps-Wilson and Milgrom-Roberts papers is that the size of the prior p^0 required to deter entry (when q^0 is sufficiently small) shrinks as the number of periods grows; indeed, it shrinks geometrically at the rate $b/(b+1)$. Thus, even a small amount of incomplete information can have a very large effect in long games. When $\delta = 1$, the unique equilibrium has the following form:

(a) If $q^0 > a/(a+1)$, then the weak incumbent accommodates at the first entry, which occurs (at the latest) the first time the entrant is tough. Hence, as the number of markets N tends to infinity, the incumbent's average payoff per period goes to 0.

(b) If $q^0 < a/(a + 1)$, then for every p^0 there is a number $n(p^0)$ so that if there are more than $n(p^0)$ markets remaining, the weak incumbent's strategy is to fight with probability 1. Thus, weak entrants stay out when there are more than $n(p^0)$ markets remaining, and the incumbent's average payoff approaches $(1 - q^0)a - q^0$ as $N \to \infty$.[7]

It is easy to explain the role played by the expression $a(1 - q^0) - q^0$ in the above. Imagine that the incumbent is given a choice at time 0 of making an observed and enforceable commitment either to always fight or to always accommodate. If the incumbent always fights, its expected payoff is $a(1 - q^0) - q^0$, as it must fight the tough entrants to deter the weak ones. The asymptotic nature of the equilibrium turns exactly on whether a commitment to always fight is better than a commitment to always accommodate, which yields payoff 0. Thus, one interpretation of the results is that reputation effects allow the incumbent to credibly make whichever of the two commitments it prefers.

Note, however, that neither of these commitments need be the one the incumbent would like most. If $a(1 - q^0) > q^0$, the incumbent is willing to fight the tough entrants to deter the weak ones, but it would do even better if it could commit itself to fight with the smallest probability that deters weak entrants, which is $b/(b + 1)$. This yields it an average payoff of $a(1 - q^0) - q^0 b/(b + 1)$, which is greater than the payoff $a(1 - q^0) - q^0$ from fighting with probability 1. Of course, when the prior distribution over the incumbent's types assigns positive probability only to the weak type and to a type that fights with probability 1, the incumbent cannot develop a reputation for fighting with a positive probability less than 1, as the first time that the incumbent accommodates it is revealed to be weak and its reputation is ruined. The next subsection discusses whether it is reasonable for the incumbent to be able to maintain a reputation for playing a mixed strategy, and shows how to change the model to make mixed-strategy reputations possible.

Although the commitment interpretation of reputation suggests that reputation effects are a "good thing" for the incumbent, this depends on the exact comparison that one has in mind. It is clear that the weak incumbent cannot lose from the fact that the entrants fear that it might be tough. An alternative comparison is to hold fixed the prior probabilities at p^0 and q^0 and to compare the game described above, where each entrant observes play in all previous markets, with the situation where each stage game is played in "informational isolation," meaning that the timing of play and the payoffs are as above but entrants do not observe play in other markets.

7. Note that we fix p^0 and take the limit as $N \to +\infty$. For fixed N and sufficiently small p^0, the weak incumbent must accommodate in each market in any sequential equilibrium. Exercise 9.1 asks you to extend this characterization to discount factors less than but close to 1.

Under informational isolation, the weak incumbent has no chance to build a reputation, and will accommodate in each market. Yet the weak incumbent's equilibrium payoff can be higher under informational isolation than in the "informational linkage" case where each entrant observes all past play. The reason is that informational linkage imposes a cost that Fudenberg and Kreps (1987) call a loss of "strategic flexibility": Under informational linkage the weak incumbent loses the ability to deter weak entrants while accommodating tough ones. Put differently, under linkage the incumbent must fight the tough entrants to deter the weak ones. When the cost of doing so is too high, the weak incumbent may choose not to develop a reputation for toughness (and hence get payoff 0).

Even when the weak incumbent does develop a reputation for toughness his payoff can be lower under informational linkage than under informational isolation. In the simple chain-store model, this is the case when $p^0 > p$, so that under informational isolation weak entrants do not enter and the weak incumbent has payoff $a(1 - q^0)$ per market. Under informational linkage, the weak incumbent does worse: His average payoff per market is $\max\{0, a(1 - q^0) - q^0\}$. Thus, although the incumbent may choose to develop a reputation given that markets are informationally linked, he might have been better off in a regime of informational isolation, where reputation building is not possible. More generally, informational linkage has both costs and benefits, and it is not obvious *a priori* when the benefits outweigh the costs.

9.2.2 Reputation Effects with a Single Long-Run Player: The General Case

If we view reputation effects as a way of supporting the intuition that the long-run player should be able to commit himself to any strategy he desires, the chain-store example raises several questions: Does the strong conclusion derived above depend on the fixed finite horizon, or do reputation effects have a similar impact in the infinitely repeated version of the game? Can the long-run player maintain a reputation for playing a mixed strategy when such a reputation would be desirable? How robust are the strong conclusions in the chain-store game to changes in the prior distribution to allow more possible types? And how does the commitment result extend to games with different payoffs and/or different extensive forms? What if the incumbent's action is not directly observed, as in a model of moral hazard?

To answer the first question—the role of the finite horizon—consider the infinite-horizon version of the game of the preceding subsection with $\delta > 1/(1 - q^0)(1 + a)$, so that even if the incumbent is known to be weak there is still an equilibrium where entry is deterred. If there is a prior probability $p^0 > 0$ that the incumbent is tough, entry deterrence is still an equilibrium. In this equilibrium, the weak incumbent fights all entrants, because the first time it fails to do so it is revealed to be weak and then all subsequent entrants enter and the weak incumbent accommodates from then on. However, this is not the only perfect Bayesian equilibrium of the infinite-horizon model. Here is another one: "The tough incumbent always fights. The weak incumbent accommodates the first entry, and then fights all subsequent entry if it has not accommodated two or more times in the past. Once the incumbent has accommodated twice, it accommodates all subsequent entry. Tough entrants always enter; weak entrants enter if there has been no previous entry or if the incumbent has already accommodated at least twice; weak entrants stay out otherwise." In this equilibrium, the weak incumbent reveals its type by accommodating in the first period; the incumbent is willing to do so because subsequent entrants stay out even after the incumbent's type is revealed.

These two equilibria (there are many more) show that reputation effects need not determine a unique equilibrium in an infinite-horizon model. At the same time, note that if the incumbent is patient it does almost as well here as in the equilibrium where all entry is deterred, so the second equilibrium does not show that reputation effects have no force. Finally, the multiplicity of equilibria suggests that it might be more convenient to try to characterize the set of equilibria without determining all of them explicitly.

This is the approach used by Fudenberg and Levine (1989, 1991). They extend the intuition developed in the chain-store example to general games where a single long-run player faces a sequence of short-run opponents. To generalize the introduction of a "tough type" in the chain-store game, they suppose that the short-run players assign positive prior probability to the long-run player's being one of several different "commitment types," each of which plays a particular fixed stage-game strategy in every period. The set of commitment types thus corresponds to the set of possible "reputations" that the long-run player might maintain. Instead of explicitly determining the set of equilibrium strategies, they obtain upper and lower bounds on the long-run player's payoff that hold in any *Nash* equilibrium of the game. (The 1991 paper allows the long-run player's actions to be imperfectly observed, as in the Cukierman-Meltzer (1986) model of the reputation of a central bank when the other players observe the realized inflation rate but not the bank's action.[8])

The upper bound on the long-run player's Nash-equilibrium payoff converges, as the number of periods grows and the discount factor goes to 1, to the long-run player's Stackelberg payoff, which is the most he could obtain by publicly committing himself to any of his stage-game strategies. If the short-run players' actions do not influence the information that is revealed about the long-run player's choice of stage-game strategy (as in a simultaneous-move game with observed actions), the lower bound on

8. Other models of reputation with imperfectly observed actions include those of Bénabou and Laroque (1989) and Diamond (1989).

payoffs converges to the most the long-run player could get by committing himself to any of the strategies for which the corresponding commitment type has positive prior probability. If moves in the stage game are not simultaneous, the lower bound must be modified, as we explain in subsection 9.2.3.

Consider a single long-run player 1 facing an infinite sequence of short-run player 2s in a "stage game" where players choose stage-game strategies a_i from finite sets A_i. Subsection 9.2.3 will allow the stage game to be a general, finite extensive form. This subsection treats the case where the stage game has simultaneous moves and the players' actions are revealed at the end of each period. Also, for the rest of this section, we consider infinite-horizon models; however, theorem 9.1 extends directly to the finite-horizon case. The history h^t at time t consists of past choices $(a_1^\tau, a_2^\tau)_{\tau=0,\dots,t-1}$. (Note that we now revert to counting time forward, instead of the backward counting we used in discussing the finite-horizon chain-store game. Note also that if the stage game has sequential moves it is not natural to suppose that the observed outcome at the end of period τ reveals the stage-game strategies a^τ the players used, as in a sequential-move game the a^τ prescribe play at information sets that may not be reached.) The long-run player's type, $\theta \in \Theta$, is private information; θ influences player 1's payoff but has no direct influence on player 2's payoff; θ has prior distribution p, which is common knowledge. Player 1's strategy is a sequence of maps σ_1^t from the set of possible histories H^t and the set of types Θ to the space of mixed stage-game actions $\mathscr{A}_1$; a strategy for the period-t player 2 is $\sigma_2^t : H^t \to \mathscr{A}_2$.

Since the short-run players are unconcerned about future payoffs, in any equilibrium each period's choice of mixed strategy α_2 will be a best response to the anticipated marginal distribution over player 1's actions. Let $r: \mathscr{A}_1 \rightrightarrows \mathscr{A}_2$ be the short-run player's best-response correspondence.

Two subsets of the set Θ of player 1's types are of particular interest. Types $\theta_0 \in \Theta_0$ are "sane types" whose preferences correspond to the expected discounted value of per-period payoffs $g_1(a_1, a_2, \theta_0)$. All sane types are assumed to use the same discount factor, δ, and to maximize their expected present discounted payoffs. (The chain-store papers had a single "sane type" whose probability was close to 1.) The "commitment types" are those who play the same stage-game strategy in every period; $\theta(\alpha_1)$ is the commitment type corresponding to α_1. The set of commitment strategies $C_1(p)$ are those for which the corresponding commitment strategies have positive prior probability under distribution p. We will present the case where Θ and thus C_1 are finite.

Define the *Stackelberg payoff* for $\theta_0 \in \Theta_0$ to be

$$g_1^s(\theta_0) = \max_{\alpha_1} \left[\max_{\alpha_2 \in r(\alpha_1)} g_1(\alpha_1, \alpha_2, \theta_0) \right],$$

and let the Stackelberg strategy be one that attains this maximum. This is the highest payoff type θ_0 could obtain if he could commit himself to always play any of his stage-game actions (including mixed actions). Note that the Stackelberg strategy need not be pure, as we saw in the chain-store game.

Note also that, since the long-run player's opponents are myopic, the long-run player could not do better than the Stackelberg payoff by committing himself to a strategy that varies over time with his opponents' past actions. If the opponents were themselves long-run players, player 1 might be able to do better than the Stackelberg payoff by using a strategy that induces the opponents not to play static best responses to avoid future punishment, as in the prisoner's-dilemma example we consider below. The support of p is allowed to include types who play such history-dependent strategies.

Given the set of possible (static) "reputations" $C_1(p)$, we ask which reputation from this set type θ_0 would most prefer, given that the short-run players may choose the best response that the long-run player likes least. This results in payoff

$$g_1^*(p,\theta_0) = \max_{\alpha_1 \in C_1(p)} \left[\min_{\alpha_2 \in r(\alpha_1)} g_1(\alpha_1, \alpha_2, \theta_0) \right].$$

The formal model allows for commitment types who play mixed strategies. Is this reasonable? Suppose that the incumbent has fought in 50 of the 100 periods to date where entry has occurred, and moreover that the distribution of "fight" versus "accommodate" looks consistent with the hypothesis of independent 50-50 randomization (i.e., tests based on run length do not reject independence). How then should the entrants predict that the incumbent will play? One can argue that at this point the entrants should assign a probability of about $\frac{1}{2}$ to the incumbent's fighting the next entrant, as opposed to their being certain that the incumbent will accommodate.[9]

Let $\underline{N}(\delta, p, \theta_0)$ and $\bar{N}(\delta, p, \theta_0)$ be the lowest and highest payoffs of type θ_0 in any Nash equilibrium of the game with discount factor δ and prior p.

Theorem 9.1 (Fudenberg and Levine 1991) Suppose that the long-run player's choice of a_1 is revealed at the end of each period. Then for all θ_0 with $p(\theta_0) > 0$, and all $\lambda > 0$, there is a $\underline{\delta} < 1$ such that, for all $\delta \in (\underline{\delta}, 1)$,

9. For those who are uncomfortable with the idea of types who "like" to play mixed strategies, an equivalent model identifies a countable set of types with each mixed strategy of the incumbent. Thus, one type always plays fight, the next accommodates in the first period and fights in all others, another fights at every other opportunity, and so on—one type for every sequence of fight and accommodate. Thus, every type plays a deterministic strategy, and by suitably choosing the relative probabilities of the types the aggregate distribution induced by all of the types will be the same as that of the given mixed strategy.

$$(1-\lambda)g_1^*(p,\theta_0)+\lambda \min_{\alpha} g_1(\alpha_1,\alpha_2,\theta_0)\leq \underline{N}(\delta,p,\theta_0) \tag{9.1a}$$

and

$$N(\delta,p,\theta_0)\leq(1-\lambda)g_1^s(\theta_0)+\lambda \max_{\alpha} g_1(\alpha_1,\alpha_2,\theta_0). \tag{9.1b}$$

Remarks

• The theorem says that if type θ_0 is patient he can obtain about his commitment payoff relative to the prior distribution, and that regardless of the prior probability distribution a patient type cannot obtain much more than his Stackelberg payoff. Note that the lower bound depends only on which feasible reputation type θ_0 wants to maintain, and is independent of the other types to which p assigns positive probability and of the relative likelihood of different types.

• Of course, the lower bound depends on the set of possible commitment types: If no commitment types have positive probability, reputation effects have no force! For a less trivial illustration, consider a modified version of the chain-store game presented in subsection 9.2.1, where each period's entrant, in addition to being tough or weak, is one of three "sizes" (large, medium, and small), and the entrant's size is public information. It is easy to specify payoffs so that the incumbent's best pure-strategy commitment is to fight the small and medium entrants and accommodate the large ones. The theorem shows that the sane incumbent can achieve the payoff associated with this strategy *if* the entrants assign it positive prior probability. However, if the entrants assign positive probability to only two types, one which is "weak" and another which fights all entrants regardless of size, then the incumbent cannot maintain a reputation for fighting only the small and medium entrants, for the first time it accommodates a large entrant it reveals that it is weak.

• For a fixed prior distribution p, the upper and lower bounds can have different limits as $\delta \to 1$ even if the Stackelberg type belongs to the prior distribution. Fudenberg and Levine (1991) show that in generic[10] simultaneous-move games, $g_1^*(p,\theta_0)=g_1^s(\theta_0)$ when the prior assigns a positive density to every commitment strategy.

• The Stackelberg payoff supposes that the short-run players correctly forecast the long-run player's stage-game action. The long-run player can obtain a higher payoff if his opponents mispredict his action. For this reason, for a fixed discount factor less than 1, some types of the long-run player can have an equilibrium payoff that strictly exceeds their Stackelberg level, as the short-run players may play a best response to the equilibrium actions of other types.

10. Genericity is needed to ensure that, by changing α_1 a bit, player 1 can always "break ties" in the right direction in the definition of $g_1^*(p,\theta_0)$, so that $g_1^*(p,\theta_0)=g_1^s(\theta_0)$.

For example, suppose that in the finite-horizon chain-store game

$$a(1-q^0) < q^0 b/(b+1),$$

so that the weak incumbent's Stackelberg payoff is 0, and suppose that the prior probability of the "tough" type is greater than $b/(b+1)$. Then the equilibrium is for the weak incumbent to always accommodate, and the weak entrants stay out until they have seen a tough entrant enter and the incumbent accommodate. Then the weak incumbent's equilibrium normalized payoff is

$$\frac{a(1-\delta)(1-q^0)}{1-\delta(1-q^0)} > 0.$$

For $\delta = 0$, the weak incumbent's payoff is $a(1-q^0)$ (which is higher than the Stackelberg payoff for any value of q^0): If the first entrant is weak it stays out, and if the first entrant is tough it enters and the incumbent accommodates. However, as $\delta \to 1$ the weak incumbent's payoff converges to its Stackelberg payoff of 0. Intuitively, the "supernormal" payoffs of the weak type are informational rents that come from the short-run players' not knowing its type. In the long run, the short-run players cannot be repeatedly "fooled" about the long-run player's play (unless $q^0 = 0$, in which case the long-run player's weakness is never tested), and the long-run player will have to bear the cost of fighting to maintain its reputation. This is why a patient long-run player cannot do better than its Stackelberg payoff. Reputation effects can serve to make commitments credible, but in the long run this is all they do.

• Although the theorem is stated for the limit $\delta \to 1$ in an infinite-horizon game, the same result covers the limit, as the horizon grows to infinity, of finite-horizon games with time-average payoffs.

• The key property of short-run players for the proof of theorem 9.1 is that they always play short-run best responses to the anticipated play of their opponents. Consider a single long-run "big" player facing a continuum of long-run "small" opponents in a repeated game. Suppose further that the various small players are anonymous, and that each player observes only the play of the big player and the play of subsets of small players of positive measure (see section 4.7 for a discussion of these assumptions). In this case the small players will play myopically, so the situation is equivalent to the case of short-run players and theorem 9.1 should be expected to apply. (At this writing no one has worked out a careful version of the argument, attending to the niceties of a continuum-of-players model.) It would be interesting to know if this observation extends to a limit result as the number of players grows. Section 9.4 discusses a game in which the small players are not anonymous and can try to maintain their own reputations; here the conclusions are much less sharp.

Sketch of Proof We will give an overview of the general argument and a detailed sketch for the case of commitment to a *pure strategy*. Fix a Nash equilibrium $(\hat{\sigma}_1, \hat{\sigma}_2)$ (Recall that σ denotes overall strategies.) This generates a joint probability distribution π over Θ and histories h^t for each t. The short-run players will use π to compute their posterior beliefs about θ at every history that π assigns positive probability. Now consider a type $\bar{\theta}$ with $p(\theta) > 0$, and imagine that player 1 chooses to play type $\bar{\theta}$'s equilibrium strategy, which we denote by $\bar{\sigma}_1$. This generates a sequence of actions with positive probability under π.

Since the short-run players are myopic, and best-response correspondences are upper hemi-continuous, Nash equilibrium requires that the short-run players' action be close to a best response to $\bar{\sigma}_1$ in any period where the observed history has positive probability and they expect the distribution over outcomes to be close to that generated by σ_1. Because the short-run players have a finite number of actions in the stage game, this conclusion can be sharpened: If the expected distribution over outcomes is close to that generated by σ_1, the short-run players must play a best response to $\bar{\sigma}_1$.

More precisely, for any h^t with $\pi(h^t) > 0$, let $\rho(h^t) = \pi[\alpha_1^t = \bar{\sigma}_1^t(\cdot|h^t)|h^t]$.

Claim For any $\bar{\theta}$, there is a $\bar{\rho} < 1$ such that $\hat{\sigma}_2^t \in r(\sigma_1^t(\cdot|h^t))$ whenever $\rho(h^t) > \bar{\rho}$. (Proving this is exercise 9.2.)

Conversely, in any period in which the short-run players do not play a best response to $\bar{\sigma}_1$, when player 1's action is observed there is a nonnegligible probability that they will be "surprised" and will increase the posterior probability that player 1 is type $\bar{\theta}$ by a nonnegligible amount. After sufficiently many of these surprises, the short-run players will attach a very high probability to player 1's playing σ_1 for the remainder of the game. In fact, one can show that for any ε there is a $K(\varepsilon)$ such that with probability $1 - \varepsilon$ the short-run players play best responses to $\bar{\sigma}_1$ in all but $K(\varepsilon)$ periods, and that this $K(\varepsilon)$ holds uniformly over all equilibria, all discount factors, and all priors p with the same prior probability of $\bar{\theta}$.

Once one obtains a $K(\varepsilon)$ that holds uniformly, one derives the lower bound on payoffs by considering $\bar{\theta}$ to be a commitment type that has positive prior probability and observing that type θ_0 gets at least the corresponding commitment payoff whenever the short-run players play a best response to $\bar{\sigma}_1$. To obtain the upper bound, let $\bar{\theta} = \theta_0$, so that type θ_0 plays his own equilibrium strategy. Whenever the short-run players are approximately correct in their expectations about the marginal distribution over actions, type θ_0 cannot obtain much more than his Stackelberg payoff.

In general, the stage-game strategies prescribed by $\bar{\sigma}_1$ may be mixed. Obtaining the bound $K(\varepsilon)$ on the number of "surprises" is particularly simple when $\bar{\sigma}_1$ prescribes the same pure strategy $\bar{a}_1$ in every period for every history. Fix an $\bar{a}_1$ such that the corresponding commitment type $\bar{\theta}$

has positive prior probability, and consider the strategy for player 1 of always playing $\bar{a}_1$. By upper hemi-continuity, there is a $\bar{\rho}$ such that, in any period where the player 2s do not play a best response to $\bar{a}_1$, $\rho(h^t) < \bar{\rho}$. We show that when player 1 plays $\bar{a}_1$ in every period there can be at most $\ln(p(\bar{\theta}))/\ln(\bar{\rho})$ periods where this inequality obtains. To see this, note that $\rho(h^t) \geq \mu(\bar{\theta}|h^t)$, because $\bar{\theta}$ always plays $\bar{a}_1$. Along any history with positive probability, Bayes' rule implies that

$$\mu(\bar{\theta}\,|\,h^{t+1}) = \mu(\bar{\theta}\,|\,(h^t, a^t)) = \frac{\pi(a^t\,|\,h^t, \bar{\theta})\mu(\bar{\theta}\,|\,h^t)}{\pi(a^t\,|\,h^t)}. \tag{9.2}$$

Then, since player 2's play is independent of θ, and the choices of the two players at time t are independent conditional on h^t,

$$\pi(a^t\,|\,h^t) = \pi(a_1^t\,|\,h^t)\cdot \pi(a_2^t\,|\,h^t)$$

and

$$\pi(a^t\,|\,\bar{\theta}, h^t) = \pi(a_1^t\,|\,\bar{\theta}, h^t)\cdot \pi(a_2^t\,|\,h^t).$$

If we now consider histories where $a_1^t = \bar{a}_1$ for all t,

$$\pi(a_1^t\,|\,\bar{\theta}, h^t) = 1,$$

and equation 9.2 simplifies to

$$\mu(\bar{\theta}\,|\,h^{t+1}) = \frac{\mu(\bar{\theta}\,|\,h^t)}{\pi(a_1^t\,|\,h^t)}. \tag{9.3}$$

Consequently $\mu(\bar{\theta}|h^{t+1})$ is nondecreasing, and increases by at least $1/\bar{\rho}$ whenever a best response to $\bar{a}_1$ is not played, as then $\pi(a_1^t\,|\,h^t) \leq \bar{\rho}$. Thus, there can be at most $\ln(p(\bar{\theta}))/\ln(\bar{\rho})$ periods where $\pi(a_1^t|h^t) \leq \bar{\rho}$, and the lower bound on payoffs follows. (The additional complication posed by types θ that play mixed strategies is that $\mu(\bar{\theta}|h^t)$ need not evolve deterministically when player 1 uses type θ's strategy.) ■

Note that the proof does not assert that $\mu(\bar{\theta}|h^t)$ converges to 1 when player 1 uses type $\bar{\theta}$'s strategy. This stronger assertion is not true. For example, in a pooling equilibrium where all types play the same strategy, $\mu(\bar{\theta}|h^t)$ is equal to the prior probability in every period. Rather, the proof shows that if player 1 always plays like type $\bar{\theta}$, eventually the short-run players become convinced that he will play like $\bar{\theta}$ in the future.

9.2.3 Extensive-Form Stage Games[†††]

Theorem 9.1 assumes that the long-run player's choice of stage-game strategy is revealed at the end of each period, as in a simultaneous-move game. The following example shows that the long-run player may do much less well than is predicted by theorem 9.1 if moves in the stage game are sequential. This may seem surprising, because the chain-store game con-

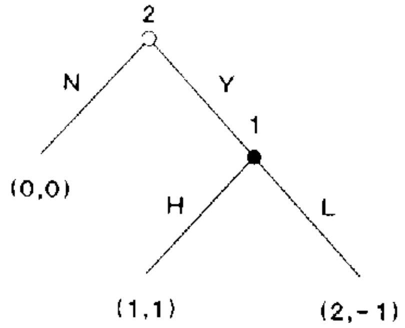

Figure 9.1

sidered by Kreps and Wilson (1982) and Milgrom and Roberts (1982) has sequential moves. Indeed, it has the same game tree as our example, but with different payoffs.

In figure 9.1, player 2 begins by choosing whether or not to purchase a good from player 1. If he does not buy, both players receive 0. If he buys, player 1 must decide whether to produce low or high quality. High quality gives each player a payoff of 1; low quality gives player 1 a payoff of 2 and gives -1 to player 2. If player 2 does not buy, player 1's (contingent) choice of quality is not revealed.

If player 1 could commit to high quality, all player 2s would purchase. Thus, if theorem 9.1 extended to this game it would say that if there is positive probability p^* that player 1 is a type who always produces high quality, then the Nash-equilibrium payoffs of a sane type θ_0 of player 1 (whose payoffs are as in figure 9.1) are bounded below by an amount that converges to 1 as the discount factor δ goes to 1.[11]

This extension is false, as the following infinite-horizon example shows. Take $p(\theta_0) = 0.99$ and $p^* = 0.01$, and consider the following strategies: The high-quality type always produces high quality. The "sane" type, θ_0, produces low quality if no more than one short-run player has ever made a purchase. Beginning with the second time a short-run player buys, type θ_0 produces high quality, and it continues to do so as long as its own past actions conform to this rule. If type θ_0 deviates and produces low quality more than once, it produces low quality forever afterward. The short-run players do not buy unless a previous short-run player has already bought, in which case they buy so long as all short-run purchasers but the first have received high quality. These strategies give type θ_0 a payoff of 0; exercise 9.3 asks you to verify not only that these strategies are a Nash equilibrium but also that they can be combined with consistent beliefs to form a sequential equilibrium.[12]

11. The Stackelberg strategy here is not "always H" but "H with probability $\frac{1}{2}$."
12. These strategies are not a sequential equilibrium if the horizon is finite. They thus do not form a counterexample to the sequential-equilibrium version of theorem 9.1 for finite-horizon games. (Theorem 9.1 is stated for an infinite horizon, but it holds equally well for large, finite horizons as long as δ is close to 1.) Kim (1990) has shown that, when this game is played with

The reason that reputation effects fail in this example is that when the short-run players do not buy, player 1 does not have an opportunity to signal his type. This problem did not arise in the chain-store game, for there the one action the entrant could take that "hid" the incumbent's action to stay out—was precisely the action the incumbent wished to be played. One response to the problem posed by the example is to assume that some consumers always purchase, so that there are no probability-0 information sets.

A second response is to weaken the theorem. Let the stage game be a finite extensive form of perfect recall without moves by nature. As in the example, the play of the stage game need not reveal player 1's choice of stage-game strategy a_1 (since the stage game need not be a simultaneous-move game, a_1 may be a contingent plan rather than an action). However, when both players use pure strategies the information revealed about player 1's play is deterministic. Let $0(a_1, a_2)$ be the subset of A_1 corresponding to strategies a_1' of player 1 such that (a_1', a_2) leads to the same terminal node as (a_1, a_2). We will say that these strategies are *observationally equivalent*. For each a_1, let $w(a_1)$ satisfy

$$w(a_1) = \{a_2 | \text{ for some } \alpha_1' \text{ with support in } 0(a_1, a_2), a_2 \in r(\alpha_1')\}. \tag{9.4}$$

In words, $w(a_1)$ is the set of pure-strategy best responses for player 2 to beliefs about player 1's strategy that are consistent with the true strategy being a_1 and with the information revealed when player 2's response is played. Then, if δ is near 1, player 1's equilibrium payoff should not be much less than

$$g_1^*(\theta_0) = \max_{a_1} \min_{a_2 \in w(a_1)} g_1(a_1, a_2, \theta_0). \tag{9.5}$$

This is verified in Fudenberg and Levine 1989.

This result, though not as strong as the assertion in theorem 9.1 that player 1 can pick out his preferred payoff in the graph of r, does suffice to prove that player 1 can develop a reputation for "toughness" in the sequential-move version of the chain-store game described in subsection 9.2.1, even if $q^0 = 0$ so that there are no "tough" entrants. In this game, $r(\text{fight}) = \{\text{stay out}\}$ and $r(\text{accommodate}) = \{\text{enter}\}$. Also, $0(\text{fight}, \text{stay out}) = 0(\text{accommodate}, \text{stay out}) = \{\text{accommodate}, \text{fight}\}$, whereas $0(\text{fight}, \text{enter}) = \{\text{fight}\}$ and $0(\text{accommodate}, \text{enter}) = \{\text{accommodate}\}$. First, we argue that $w(\text{fight}) = r(\text{fight})$. To see this, observe that $w(\text{fight})$ is at least as large as $r(\text{fight}) = \{\text{stay out}\}$. Moreover, "enter" is not a best response to "fight," and "accommodate" is not observationally equivalent to "fight"

a long but finite horizon, there is a unique sequential equilibrium, in which the firm does maintain a reputation for high quality. Kim is currently working on the question of the best lower bound for sequential-equilibrium payoffs in finite repetitions of general stage games with reputation effects.

when player 2 plays "enter." Consequently, no strategy placing positive weight on "enter" is in w(fight). Since player 1's Stackelberg action with observable strategies is fight, and $w(\text{fight}) = r(\text{fight})$, the generalized Stackelberg payoff and the usual one coincide in this game.

9.3 Games with Many Long-Run Players††

9.3.1 General Stage Games and General Reputations

Section 9.2 showed how reputation effects can allow a single, "long-run," or patient player to commit himself to his preferred strategy. Of course, there are also incentives to maintain reputations when all players are equally patient, but here it is difficult to draw general conclusions about how reputation effects influence play.

Kreps et al. (1982) analyze reputation effects in the finitely repeated prisoner's dilemma. They consider a game in which each player, if "sane," has payoffs corresponding to the expected average value of the per-period payoffs shown in figure 9.2. If both types are sane with probability 1, then the unique Nash equilibrium of the game is for both players to defect in every period, but intuition and experimental evidence suggest that even with a fixed finite horizon players may tend to cooperate. To explain this intuition, Kreps et al. introduced incomplete information about player 1's type, with player 1 either "sane" or a type who plays the strategy "tit for tat," which is "I play today whichever action you played yesterday." They showed that, for any fixed prior probability ε that player 1 is "tit for tat," there is a number K independent of the horizon length T such that, in any sequential equilibrium, both players must cooperate in almost all periods before date $T - K$, so that if T is sufficiently large the equilibrium payoffs will be close to those if the players always cooperated. The point is that a sane player 1 has an incentive to maintain a reputation for being "tit for tat," because if player 2 were convinced that player 1 plays "tit for tat" player 2 would cooperate until the last period of the game.

Just as in the chain-store game, adding a small amount of the right sort of incomplete information yields the "intuitive" outcome as the essentially unique prediction of the model with a long finite horizon. However, in contrast with games having a single long-run player, the resulting equi-

	Cooperate	Defect
Cooperate	2,2	−1,3
Defect	3,−1	0,0

Figure 9.2

librium is very sensitive to the exact nature of the incomplete information specified (Fudenberg and Maskin 1986).

Fix a two-player stage game g, and let V^* be the set of feasible, individually rational payoffs. Now consider repeated play of g with a fixed finite horizon T. Call player i "sane" if his payoff is the expected value of the sum of g_i. (Without loss of generality, we take $\delta = 1$ instead of "δ close to 1" because we consider a large but finite horizon.)

Theorem 9.2 (Fudenberg and Maskin 1986) For any $v = (v_1, v_2) \in V^*$ and any $\varepsilon > 0$, there exists a $\underline{T}$ such that, for all $T > \underline{T}$, there exists a T-period game such that each player i has probability $1 - \varepsilon$ of being sane, independent of the other, and there exists a sequential equilibrium of the game where player i's expected average payoff if sane is within ε of v_i.

Remark This theorem asserts the existence of a game and of an equilibrium; it does not say that all equilibria of the game have payoffs close to v. Note also that no restrictions are placed on the form of the payoffs that players have when they are not sane, i.e., on the support of the distribution of types: No possible types are excluded, and there is no requirement that certain types have positive prior probability. However, the theorem can be strengthened to assert the existence of a game with a strict equilibrium (subsection 1.2.1) where the sane types' payoffs are close to v; and a strict equilibrium of a game remains strict when additional types are added whose prior probability is sufficiently small. (Chapter 11 discusses this kind of robustness question.)

Partial Proof We will prove only the weaker theorem that any payoffs that Pareto dominate those of a static equilibrium can be approximated. Let e be a static-equilibrium profile with payoffs $y = (y_1, y_2)$, and let v be a payoff vector that Pareto dominates y. To avoid a discussion of public randomizations, assume that payoffs v can be attained with a pure-action profile a, i.e., $g(a) = v$.

Now consider a T-period game in which each player i has two possible types, "sane" and "crazy," and crazy types have payoffs that make the following strategy weakly dominant: "Play a_i as long as no deviations from a have occurred in the past; otherwise play e_i."

Let $\bar{g}_i = \max_a g_i(a)$ be player i's highest feasible stage-game payoff, and let $\underline{g}_i = \min_a g_i(a)$ be player i's lowest feasible stage-game payoff. Set

$$\underline{T} > \max_i \left(\frac{\bar{g}_i - (1-\varepsilon)\underline{g}_i - \varepsilon y_i}{\varepsilon(v_i - y_i)} \right). \tag{9.6}$$

Consider the extensive-form game corresponding to $T = \underline{T}$. This game has at least one sequential equilibrium for any specification of beliefs; pick one and call it the "endgame equilibrium."

Now consider $T > \underline{T}$. It will be convenient to number periods backwards, with period T the first one played and period 1 the last. Consider strategies that specify that profile a is played for all $t > \underline{T}$, and that if a deviation does occur at some $t > \underline{T}$ (i.e., "before $\underline{T}$") then e is played for the rest of the game, whereas if a is played in every period until $\underline{T}$ play follows the endgame equilibrium corresponding to prior beliefs. Let the beliefs prescribe that if any player deviates before $\underline{T}$ that player is believed to be sane with probability 1, and that if there are no such deviations before $\underline{T}$ then the beliefs are the same as the prior until $\underline{T}$ is reached.

We claim that these strategies form a sequential equilibrium. First, the beliefs are clearly consistent in the Kreps-Wilson sense.[13] They are sequentially rational by construction in the endgame equilibrium, and are also sequentially rational in all periods following a deviation before $\underline{T}$, where both types of both players play the static-equilibrium strategies.

It remains only to check that the strategies are sequentially rational along the path of play before $\underline{T}$. Pick a period $t > \underline{T}$ in which there have been no deviations to date. If player i plays anything but a_i, he receives at most $\bar{g}_i$ today and at most y_i thereafter, for a continuation payoff of

$$\bar{g}_i + (t-1)y_i. \tag{9.7}$$

If instead he follows the (not necessarily optimal) strategy of playing a_i each period until his opponent deviates and playing e_i thereafter, his expected payoff will be at least

$$\varepsilon t v_i + (1-\varepsilon)[\underline{g}_i + (t-1)y_i], \tag{9.8}$$

as this strategy yields tv_i if his opponent is crazy and at least $\underline{g}_i + (t-1)y_i$ if his opponent is sane. The definition of $\underline{T}$ has been chosen so that quantity 9.8 exceeds quantity 9.7 for $t > \underline{T}$, which shows that player i's best response to player j's strategy must involve playing a_i until $\underline{T}$. (A best response exists by standard arguments.) The key in the construction is that when players respond to deviations as we have specified, any deviation before $\underline{T}$ gives only a one-period gain (relative to y_i), whereas playing a_i until $\underline{T}$ gives probability ε of a gain $(v_i - y_i)$ that grows linearly in the time remaining, and risks only a one-period loss. This is why even a very small ε makes a difference when the horizon is sufficiently long.[14] ■

9.3.2 Common-Interest Games and Bounded-Recall Reputations†††

Aumann and Sorin (1989) consider reputation effects in the repeated play of two-player stage games of "common interests," which they define as stage

13. This is a two-types-per-player game of incomplete information. From chapter 8, we know that beliefs that are updated from one period to the next using Bayes' rule such that the updating about a player's type does not depend on the other player's action are consistent.
14. Note once again that as ε tends to 0 the $\underline{T}$ of the theorem tends to ∞, and that for a fixed horizon T a sufficiently small ε has no effect.

	L	R
U	9,9	0,8
D	8,0	7,7

Figure 9.3

games in which there is a payoff vector that strongly Pareto dominates all other feasible payoffs. In these games the Pareto-dominant payoff vector corresponds to a static Nash equilibrium; however, there can be other equilibria, as in the game illustrated in figure 9.3. This is the game we used in chapter 1 to show that even a unique Pareto-optimal payoff need not be the inevitable result of preplay negotiation: Player 1 should play D if he believes the probability that player 2 will play R is more than $\frac{1}{8}$. Also, player 1 would like player 2 to play L regardless of how player 1 intends to play. Thus, when the players meet, each will try to convince the other that he will play his first strategy, but these statements need not be compelling.

Aumann and Sorin show that when the possible reputations (i.e., crazy types) are all "pure strategies with bounded recall" (to be defined shortly) then reputation effects pick out the Pareto-dominant outcome so long as only pure-strategy equilibria are considered. A pure strategy for player i has recall k if it depends only on the last k choices of his opponent, that is, if all histories where i's opponent has played the same actions in the last k periods induce the same action by player i. (Note that when player i plays a pure strategy and does not contemplate deviations, conditioning on his own past moves is redundant.) When k is large this condition may seem innocuous, but it does rule out "grim" or "unrelenting" strategies that prescribe, e.g., reversion to the worst static Nash equilibrium for player i if player i ever deviates.

Aumann and Sorin consider perturbed games with independent types, where each player's type is private information, each player's payoff function depends only on his own type, and types are independently distributed. The prior p_i about player i's type is that player i is either the "sane" type θ_0, with the same payoffs as in the original game, or a type that plays a pure strategy with recall less than some bound ℓ. Moreover, p_i is required to assign positive probability to the types corresponding to each pure strategy of recall 0. These types play the same action in every period regardless of the history, just like the commitment types of Fudenberg and Levine. Such priors correspond to "admissible perturbations of recall ℓ," or "ℓ-perturbations" for short. Say that a sequence p^m of ℓ-perturbations supports a game G if $p^m(\theta_0^i) \to 1$ for all players i as $m \to \infty$ and if the conditional distribution $p^m(\theta^i | \theta^i \neq \theta_0^i)$ is constant.

Theorem 9.3 (Aumann and Sorin 1989) Let the stage game g be a game of common interests, and let z be its unique Pareto-optimal payoff vector. Fix a recall length ℓ, and let p^m be a sequence of ℓ-perturbations that support the associated discounted repeated game $G(\delta)$. Then the set of pure-strategy Nash equilibria of the games $G(\delta, p^m)$ is not empty, and the pure-strategy equilibrium payoffs converge to z for any sequence (δ, m) converging to $(1, \infty)$.

Idea of Proof We give a partial intuition for the convergence of equilibrium payoffs for the case in which δ goes to 1 much faster than m goes to ∞ (the theorem holds uniformly over sequences (δ, m)). Suppose more strongly that the game is symmetric and that a symmetric pure-strategy equilibrium exists. Fix $\varepsilon > 0$ and suppose further that, even when the probability of a sane type is very close to 1, a sane type's payoff is less than $(z - \varepsilon)$, where z is now the symmetric Pareto-optimal payoff. Since the equilibrium is pure, then, conditional on both types being sane, there must be some period in which the players fail to play the symmetric action $a(z)$ with payoff z. Then if player 1, say, adopts the strategy of always playing the action $a(z)$ corresponding to z, he will reveal that he is not sane. Suppose a pure-strategy equilibrium exists, and suppose its payoff is less than z. Consider the strategy for player 1 of always playing the action $a_1(z)$ corresponding to z. Since the equilibrium is pure, this strategy is certain to eventually reveal that player 1 is not of type θ_0. The commitment type $\theta_1(z)$ corresponding to $a_1(z)$ has positive probability by assumption, so if $\ell = 0$ player 2 will infer that player 1 is $\theta_1(z)$ and will play $a_2(z)$ from then on (because crazy types play constant strategies when $\ell = 0$). However, player 1 could be some other type with memory longer than 0, and to learn player 1's type will require player 2 to "experiment" to see how player 1 responds to different actions. Such experiments could be very costly if they provoked an unrelenting punishment by player 1; however, since player 1's crazy types all have recall at most ℓ, player 2's potential loss (in normalized payoff) from experimentation goes to 0 as δ goes to 1. Thus, if δ is sufficiently large we expect player 2 to eventually learn that player 1 has adopted the strategy "always play $a_1(z)$," and so when δ is close to 1 player 1 can obtain approximately z by always playing $a_1(z)$.

Remark Aumann and Sorin give counterexamples to show that the assumptions of bounded recall and full support on recall 0 are necessary, and to also show that there can be mixed-strategy equilibria whose payoffs are bounded away from z. They interpret the necessity of the bounded recall assumption with the remark that "in a culture in which irrational people have long memories, rational people are less likely to cooperate." Note that the theorem concerns the case where δ is large in comparison with the recall length ℓ, though one might expect that a more patient player would tend to have a longer memory. This is important for the proof: It is not clear that if ℓ grew with δ player 2 would try to learn player 1's strategy.

9.4 A Single "Big" Player against Many Simultaneous Long-Lived Opponents†††

Section 9.2 showed how reputation effects allow a single long-lived player to commit himself when facing a sequence of short-run opponents. An obvious question is whether a similar result obtains for a single "big" player who faces a large number of small but long-lived opponents. For example, one might ask if a large "government" or "employer" could maintain its desired reputation against small agents whose lifetimes are of the same order as the large player's. We will give an informal sketch of some of the issues involved based on the formal treatment of Fudenberg and Kreps (1987), who consider a special case where the large player plays each of the small ones in separate versions of the two-sided concession game studied by Kreps and Wilson (1982), which is essentially a continuous-time version of the chain-store game presented above.[15]

In the concession game, time is counted backward as in section 9.2. Thus, if $t \in [0, 1]$, time 0 is the final date. At each instant t, both players decide whether to "fight" or to "concede." The "tough" types always fight; the "weak" ones find fighting costly but are willing to fight to induce their opponent to concede in the future. More specifically, both weak types have a cost of 1 per unit time of fighting. If the entrant concedes first at t, the weak incumbent receives a flow of a per unit time until the end of the game, so the weak incumbent's payoff is $at - (1 - t)$ and the weak entrant's payoff is $-(1 - t)$. If the weak incumbent concedes first at t, the weak incumbent's payoff is $-(1 - t)$ and the weak entrant's payoff is $bt - (1 - t)$, where b is the entrant's flow payoff once the incumbent concedes. Thus, each weak player would like its opponent to concede, and each weak player will concede if it thinks its opponent is likely to fight until the end. The unique equilibrium involves the weak type of one player conceding with positive probability at date 0 (so the corresponding distribution of stopping times has an "atom" at 0); if there is no concession at date 0, both players concede according to smooth density functions thereafter.

Now suppose that a "large" incumbent is simultaneously involved in N such concession games against N different opponents, each of which plays only against the incumbent. The incumbent's type is perfectly correlated across games, in that the incumbent is tough in all the games with prior probability p^0 and weak in all of them with complementary probability $1 - p^0$. Each entrant is tough with probability q^0, independent of the

15. The concession game is also a variant of the incomplete-information war of attrition, studied in chapter 6.

others. Since the entrants are long-run, each has its own reputation to worry about.

The nature of the equilibrium depends on whether an entrant is allowed to reenter its market and resume fighting after it has dropped out. In the "captured contests" version of the game, if an entrant has ever conceded, it must concede from then on; the "reentry" version allows the entrant to revert to fighting after it has conceded. Note that when there is only one entrant, the "captured contests" and "reentry" versions have the same sequential equilibrium, as once the entrant chooses to concede it receives no subsequent information about the incumbent's type and thus will choose to concede from then on.[16]

One might guess that if there are enough entrants, the large incumbent can deter entry in either version of the game. This turns out not to be the case. Specifically, under captured contests, when each entrant has the same prior probability of being tough, no matter how many entrants the incumbent faces, equilibrium play in each market is exactly as if the incumbent played against only that entrant. To see why, suppose that there are N entrants, and that $N - k$ of them have conceded at time t, so that there are k entrants still fighting. If the equilibrium is symmetric (one can show that it must be), the incumbent then has the same posterior beliefs q^t about the type of each active entrant. Further, if the incumbent is randomizing at date t, it must be indifferent between conceding now (in which case it receives a continuation payoff of 0 in the remaining markets) and fighting on for a small interval dt and then conceding. The key is that, whatever happens in the active markets, the captured markets remain captured, so the incumbent does not consider them in making its current plans. If we denote the probability that each entrant concedes between t and $t - dt$ by σ^t, we have

$$0 = -k + k(1 - q^t)\sigma^t at. \tag{9.9}$$

Note that the number of active entrants, k, factors out of this equation, so that it is the same equation we have for the one-entrant case. This is why adding more entrants has no effect on equilibrium play.

In contrast, when reentry is allowed and there are many entrants, reputation effects can be shown to enable the incumbent to obtain approximately its commitment payoff. We say "can," rather than "will," because here the equilibrium is not unique; in one of the equilibria the incumbent can commit itself, but in another it cannot. The multiplicity comes from the fact

16. If there are several entrants and the incumbent plays them in succession, so that $t \in [0, 1]$ is against the first entrant, $t \in [1, 2]$ against the second, and so on, the first entrant might regret having conceded if it sees the incumbent concede to a subsequent entrant, but at that point the first entrant's contest is over, and once again the captured contests and reentry versions have the same equilibrium.

that in subgames where the weak incumbent has conceded and is thus revealed to be weak, the symmetric-information wars of attrition between the weak incumbent and the weak entrants who have previously conceded have multiple equilibria.

Fudenberg and Kreps focus on the equilibrium where, once the incumbent has conceded in any market, it concedes in all markets and all entrants who had previously conceded reenter. (This is the unique sequential equilibrium in the finite-horizon, discrete-time version of the game.) In this case, when the incumbent has captured a number of markets it has a great deal to lose by conceding. Here the incumbent's myopic incentive is to concede to entrants who have fought a long time and thus are likely to be tough, but the incumbent lacks the flexibility to concede to these active entrants without also conceding to the entrants who have already been revealed to be weak, and this lack of flexibility enables the incumbent to commit itself to tough play.

In contrast, if we specify that even if it is revealed to be weak the incumbent keeps control of any market in which an entrant has conceded, then, as in the case of captured contests, play in each market is exactly as if there were only one entrant, so that facing more entrants does not make the incumbent tougher. The reason is that since the incumbent keeps the flexibility to concede to the active entrants while threatening to fight the inactive ones, the presence of more entrants does not "stiffen the incumbent's backbone."

The moral of these observations is that the workings of reputation effects with one big player facing many small long-run opponents can depend on aspects of the game's structure that would be irrelevant if the small opponents were played sequentially. Thus, in applications of game theory one should be wary of blanket assertions that reputation effects will allow a large player to make its commitments credible.

An open question in this field is what happens when the incumbent's type need not be the same in each contest, so that the incumbent can be tough in some contests and weak in others.

Exercises

Exercise 9.1* Characterize the equilibria of the chain-store game of subsection 9.2.1 in the limit $N \to \infty$ for discount factors δ close to 1.

Exercise 9.2** Prove the claim in the proof of theorem 9.1.

Exercise 9.3** In subsection 9.2.3, check that the strategies yielding no purchase in the repeated quality game of figure 9.1 form a sequential equilibrium.

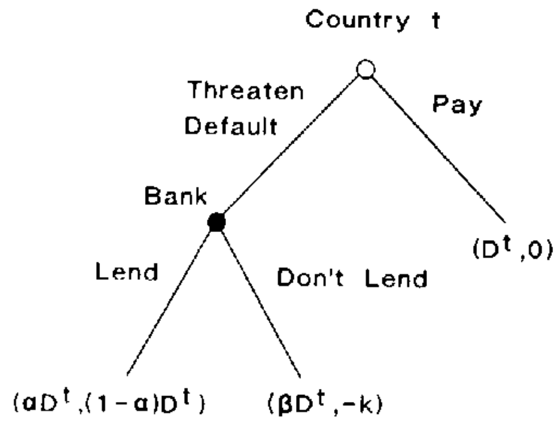

Figure 9.4

Exercise 9.4* Consider the chain-store game as described in subsection 9.2.1. Suppose that there is a single potential entrant, two markets (A and B), and two periods. The entrant can enter each market at most once and can enter at most one market per period, but he can choose which market to enter first. The incumbent is either tough in both markets or weak in both; the entrant is weak with probability 1. The tough incumbent always fights. Payoffs for the weak players in market A are as in subsection 9.2.1: The incumbent gets a if no entry, 0 if accommodate, -1 if fight; the entrant gets b if accommodate, 0 if no entry, -1 if fight. In market B, which is "big," all these payoffs are multiplied by 2. Which market should the entrant enter first? (Hint: Why might entering both markets at once, if feasible, be better than sequential entry?)

Exercise 9.5* Consider the following model of international debt repayment: A bank (representing the coalition of creditors) faces two countries sequentially. At date $t \in \{1, 2\}$, country t decides whether to pay its debt, D^t, or to threaten default. If it threatens default, the bank can either lend (or reschedule the debt) or not lend; the latter results in default. The stage game is illustrated in figure 9.4, where the first payoff is the bank's and the second is the country's. (See Armendariz de Aghion 1990 for more motivation.) Assume $1 > \alpha > 0$, $\alpha > \beta$, and $k > 0$. The bank can be "soft" (have payoffs as in figure 9.4) or "tough" (never lend, because of pessimism about future repayment, or because of costly acquisition of cash reserves). Only the bank knows whether it is soft or tough. Assume that the bank's discount factor is equal to 1, and that $(1 - p)(1 - \alpha)D^t - pk > 0$ for $t = 1, 2$, where p is the prior probability that the bank is tough.

Solve for the equilibrium of this two-period game. If the bank had the choice between facing the low-debt country or the high-debt country first, which one would it choose? (Compare your answer to that of exercise 9.4.)

Exercise 9.6** Consider the following two-period repeated game between a supervisor and an agent. In period $t = 1, 2$, the agent has type $\theta^t = 1$ with

		Supervisor	
		Denounce	Don't
Agent	Jam	$\theta^t - j, 0$	$\theta^t - j, 0$
	Don't	$0, w^t$	$\theta^t, 0$

Figure 9.5

probability α, and $\theta^t = 0$ with probability $1 - \alpha$. θ^t is learned by the agent at the beginning of the period, and θ^1 and θ^2 are independent. At cost $j > 0$ (for "jamming"), the agent can prevent the supervisor from observing θ^t. If the agent does not jam, the supervisor observes θ^t and chooses whether or not to report the agent's type to the manager. If the supervisor reports the agent's type, the manager adjusts the agent's contract to extract his rent, and the agent then gets payoff 0. If the supervisor does not report, the agent receives rent θ^t. (Think of θ^t as the agent's productivity.) The supervisor is sane (has payoffs as indicated in figure 9.5) with probability r, and "pro-agent" (never reports) with probability $1 - r$. Assume that $r > j$, so that the agent jams when he is of type 1 in the one-period version of the game. The discount factor is equal to 1.

(a) Assume that $w^1 > \alpha w^2$. Show that the agent "experiments" (i.e., does not jam) in period 1 when $\theta^1 = 1$ if and only if $r < j + \alpha(1 - r)j$. Interpret this condition.

(b) Assume that $0 < w^1 < \alpha w^2$. Show that the agent experiments in period 1 if and only if $1 - j < (1 - r)/(1 - j)$, and that the sane supervisor builds a reputation for trust with positive probability.

(Aghion and Caillaud (1988) develop a richer model of reputation and draw some inferences for organizational design.)

Exercise 9.7** This exercise (which concerns commitment in monetary policy) considers a central bank which chooses the level of the money supply as in the discussion of "time consistency" in chapter 3. The new wrinkles here are that the bank's preferences are private information and that the link between the money supply and inflation is stochastic. Specifically, suppose that the central bank's payoff in each period is $\theta N - \pi^2/2$, where N is the level of employment, π is the rate of inflation, and θ is a taste parameter.

The payoff functions of the "public" generate a link between employment and inflation given by a "Phillips curve,"

$$N = \alpha(\pi - \pi^e),$$

where π^e is the rate of inflation the public expects to occur. Thus, the

Phillips curve corresponds to the short-run reaction correspondence of the unmodelled economic agents.

The realized level of inflation depends on the central bank's action a and a random disturbance ε:

$$\pi = a + \varepsilon,$$

where ε has a normal distribution with mean 0 and variance v_ε.

(a) Show that if the game is played only once and θ is public information, the unique equilibrium is $a = \alpha\theta$. What is the central bank's equilibrium payoff?

(b) What is the Stackelberg action for the bank? What is the Stackelberg payoff?

(c) Still in the one-shot game, suppose that θ is private information for the bank, and that the prior distribution on θ is normal with mean $\bar{\theta}$ and variance v_θ. Show that the bank's equilibrium payoff as a function of θ is

$$[\theta^2\alpha^2 - v_\varepsilon]/2 - \alpha^2\bar{\theta}\theta.$$

Why do types with $\theta > 0$ prefer $\bar{\theta}$ to be very negative?

(d) Now consider a two-period version of this game, with prior beliefs as in question c. At the end of the first period, the public observes the realized inflation π^1 but not the bank's action a^1. Suppose the bank maximizes the discounted sum of its per-period payoffs with discount factor δ. Look for an equilibrium where the bank's first-period action a^1 is a linear function of θ: $a^1 = K\theta$. Show that K is defined implicitly by

$$K = \alpha[1 - \delta K v_\theta/(K^2 v_\theta + v_\varepsilon)].$$

Hint: If the bank uses a linear first-period strategy, the public's second-period beliefs about θ have a normal distribution, with a variance independent of first-period inflation π^1 and a mean equal to

$$\bar{\theta} + [K^2 v_\theta/(K^2 v_\theta + v_\varepsilon)](K\bar{\theta} - \pi^1)/K.$$

How does K depend on δ, v_ε, and v_θ? Explain. Is any type's equilibrium payoff above its Stackelberg level? Why?

(e) Suppose that the game is repeated infinitely often with discount factor near 1, and that θ has a finite support including $\theta = 0$. Suppose that type 0 always sets $a = 0$. Characterize each type's equilibrium payoff. (This exercise is based on Cukierman and Meltzer 1986. See Cukierman 1990 for more on central banks' reputations.)

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10 Sequential Bargaining under Incomplete Information

10.1 Introduction[††]

A bargaining situation involves players who must reach an agreement in order to realize gains from trade. The standard example is the problem of sharing a pie. No player can have any pie until they all agree about the shares each will receive. Negotiating about the shares is costly, and the pie may decay or disappear if the negotiations go on for very long.

At least since Edgeworth (1881) bargaining has been perceived as an important question in economics and political science. The first efforts to predict bargaining outcomes used the framework of "cooperative games." In this framework, axioms are developed on the result of the bargaining process, and in particular on how the result should vary with changes in the set of feasible utilities; these axioms are typically defended on normative grounds as well as positive ones. Cooperative game theory's use of axioms on outcomes distinguishes it from the noncooperative approach developed in this book, where outcomes depend explicitly on behavior and behavior is assumed to correspond to equilibrium play in exogenously determined games.

Nash (1950, 1953) used both the cooperative or axiomatic approach and the noncooperative one in his work on bargaining; he first characterized the unique outcome satisfying a set of axioms, and then proposed a noncooperative game whose equilibrium was precisely this outcome.[1] However, Nash's noncooperative model assumed that players had only one chance to reach an agreement, and that if they failed to do so they were unable to continue negotiating (see exercise 1.6). This game seemed too simple to capture the richness of bargaining, and (perhaps as a result) the noncooperative approach to bargaining received little attention until the 1970s.

The model of Ståhl (1972) and Rubinstein (1982), described in chapter 4, was the first bargaining model to reflect the fact that bargaining is a typically dynamic process involving offers and counteroffers. Ståhl and Rubinstein considered bargaining under complete information and found that sequential bargaining yields a unique, Pareto-efficient outcome in which the bargainers reach an efficient agreement without haggling. Ståhl and Rubinstein also gave some useful intuitions about what determines bargaining power; for instance, players who are more patient do better.

It is worth explaining the importance of the uniqueness and efficiency results. First, the interest of the uniqueness result derives in part from its contradicting the conventional wisdom that bargaining outcomes are arbitrary and that an outside observer is unable to predict which point of the

1. Actually, he considered a sequence of games whose equilibrium outcomes converged to this point in the limit.

Pareto frontier (if any) will prevail. Second, the Coase (1960) theorem makes efficiency a central issue. This theorem asserts that the distribution of ownership in an economy has no relevance for efficiency as long as the "transaction costs" are negligible, in the sense that bargaining results in efficient outcomes. Although neither efficiency nor equilibrium uniqueness obtains in all sequential-bargaining games with complete information (see for instance exercises 4.3 and 4.9), Ståhl and Rubinstein defined a class of games for which these obtain.

Since the early 1980s, a number of authors have developed models of sequential bargaining with incomplete information. It was clear from the start that the introduction of incomplete information would tend to introduce inefficiencies. As is noted in chapter 7, the simplest example of a bargaining process is monopoly pricing, in which a seller makes a "take it or leave it" offer to a buyer (or several buyers) who then decides whether to purchase the good. If the seller does not know the buyer's valuation for the good, suboptimal trade results. Because the seller charges a price above the marginal cost, trade does not take place when the buyer's valuation exceeds the marginal cost but is lower than the monopoly price, even though such trade would be efficient. Similar inefficiencies seem likely in more complex bargaining games where the buyer may have an incentive to reject a price that is below his valuation in the hope of obtaining a better price later on. Indeed, Myerson and Satterthwaite (1983) (discussed in subsection 7.4.4) give general sufficient conditions for all equilibria of a bargaining game to be inefficient when neither player knows the valuation of the other.[2] When bargaining can be inefficient, the choice of economic institutions—that is, the rules of the game—can influence the efficiency of the outcome. For instance, labor disputes may be explained as resulting from incomplete information about the firm's profitability, and arbitration clauses and labor law can influence the likelihood of strikes and lockouts.[3] Similarly, the distribution of ownership, by determining residual rights of control and therefore the *status quo* allocation in bargaining, has an effect on the efficiency of bargaining between two units.

Though we favor the noncooperative approach over the axiomatic approach, we should point out that the noncooperative approach has so far been unsuccessful in "solving the bargaining problem." There are two unresolved difficulties. The first is that, in both complete-information and incomplete-information models, the equilibrium outcomes are very sensitive to the choice of the extensive form. Even with complete information, any split of the pie can be obtained by changing the extensive form for

2. When only the buyer's valuation is private information, the game in which the buyer makes a "take it or leave it" offer to the seller leads to efficient trade.

3. Fudenberg, Levine, and Ruud (1985), Kennan and Wilson (1989, 1990), and Cramton and Tracy (1990) offer empirical analyses of strikes based on models of bargaining with incomplete information.

bargaining. This is worrisome because we, as outside observers of a bargaining process, usually have little information about which extensive form is being played, and furthermore the extensive form is likely to vary from one situation to the next. Of course, in any application of game theory the conclusions can vary with the extensive form chosen, but the issue seems more serious here than in other contexts, in which we may be able to limit attention to a smaller set of extensive forms.

The second difficulty with the noncooperative approach to bargaining is more specific to incomplete information. It was soon realized (Fudenberg and Tirole 1983; Cramton 1984; Rubinstein 1985) that games in which a bargainer with private information can propose agreements can have a great many perfect Bayesian equilibria. (This will not surprise the reader of the section on signaling games in chapter 8.) Thus, bargaining theory seems unlikely to offer unique predictions even if one knows the extensive form. Several authors have tried to select particular equilibria either on *a priori* grounds or by using stronger equilibrium refinements. (Chapter 11 discusses some, but far from all, of the refinements that have been used.) The theory of bargaining under incomplete information is currently more a series of examples than a coherent set of results. This is unfortunate because bargaining derives much of its interest from incomplete information.

Though many incomplete-information bargaining models do have many equilibria, strong results can be obtained in the special class of "one-sided-offer" bargaining games (section 10.2). A seller, who has one unit of a good and whose cost is common knowledge, makes sequential offers to a buyer, who has private information about his willingness to pay for the unit. Bargaining stops after the buyer has accepted an offer. With the buyer's strategy space restricted to "yes" and "no" at each stage, the issue of updating of beliefs about the buyer's valuation off the equilibrium path does not arise. Because this model avoids the multiplicity of equilibria associated with updating of beliefs, and because it illustrates most of the insights that have been obtained in bargaining theory, we devote a disproportionate amount of attention to it, but this extensive form is very special.

This "single-sale" model assumes that the seller sells the good once and for all to the buyer. (This does not mean that the good is consumed instantaneously by the buyer.) Section 10.3 takes up the case of repeated bargaining for a perishable good which the buyer must purchase anew each period. One interpretation of this model is that the perishable good is the current period's flow of service from a durable asset that belongs to the seller, so that each period's bargaining is over the current rental price.

Section 10.4 returns to the single-sale model and takes up the case of more complex bargaining processes, such as alternating-offer bargaining. It illustrates the difficulty in making predictions when an informed player makes offers. It also draws the link between static mechanism design (studied in chapter 7) and sequential bargaining. In particular, it discusses which incentive-compatible and individually rational outcomes can arise as *some* equilibrium of *some* sequential-bargaining game.

10.2 Intertemporal Price Discrimination: The Single-Sale Model††

10.2.1 The Framework

A seller and a buyer bargain over the trade of one unit of a good. The seller has known production (or opportunity) cost c incurred when transfer takes place. The buyer has valuation v for the good. In the single-sale model, v and c are stock variables; in particular, if the good is durable, v is the present discounted value of the buyer's per-period benefit from the date of purchase.

The seller makes offers at dates $t = 0, 1, \ldots, T$, where $T \leq +\infty$. In each period, the buyer says yes or no. In the single-sale model, an offer at date t is a purchase price m^t. A strategy for the seller is thus a sequence of prices m^t that are charged at date t conditional on the rejection of all previous offers. A strategy for the buyer is a choice of "accept" or "reject" in each period, and is conditional on the sequence of past and current offers. If $\delta \in (0, 1)$ denotes the (common) discount factor, the payoffs are $u_s = \delta^t(m^t - c)$ for the seller and $u_b = \delta^t(v - m^t)$ for the buyer if agreement is reached at date t at price m^t.

Two formulations of the asymmetry of information have been considered in the literature:

In the *two-type case*, v takes value $\bar{v}$ with probability $\bar{p}$ and value $\underline{v}$ with probability $\underline{p}$ (such that $\bar{p} + \underline{p} = 1$), where $\bar{v} > \underline{v} > c$.

In the *continuum-of-types case*, v takes a value in some interval $[\underline{v}, \bar{v}]$ with cumulative distribution function $P(\cdot)$ and continuous density $p(\cdot) > 0$ for all v, and $\bar{v} > c$. This case is divided into two subcases: the *gap case*, $\underline{v} > c$ (gains from trade bounded away from 0), and the *no-gap case*, $\underline{v} \leq c$ (there may not exist gains from trade).[4]

With any specification of the distribution of types, the model can be interpreted either as having a single buyer whose type is unknown (the "bargaining model") or as having a continuum of infinitesimal consumers, with the distribution of their willingness to pay given by $P(\cdot)$ (the "durable-good monopoly"). In the latter case, we suppose that the seller cannot tell the consumers apart, and that the seller observes only the measures of the sets who accept and reject.

In keeping with our focus on bargaining, we will assume in most of the chapter that there is a single buyer. However, because of the importance of

4. The assumption of a positive continuous density at $v = c$ is important. For instance, if there is no v in $[c - \varepsilon, c + \varepsilon]$, the no-gap case is equivalent to the gap case, because the seller never sells to a buyer with valuation under c.

the durable-good interpretation, we show how to switch from one interpretation to the other in our discussion of the example in subsection 10.2.3.

Whether the gap case or the no-gap case is more descriptive may depend on the context under consideration. Neither case is completely satisfactory, as both ignore the possibility that one party or the other may break off negotiations to bargain with a third party (in the single-buyer interpretation) and the possibility that there may be a steady influx of new potential buyers (in the continuum-of-buyers interpretation). We say more about these extensions, and about the relative merits of the gap and no-gap assumptions, in subsection 10.2.7.

From now on, we will simplify notation by assuming $c = 0$.

Our focus is on whether the equilibria display various properties implicitly and/or explicitly discussed in Coase's (1972) analysis of pricing by a durable-good monopolist.

The first set of properties relates to the dynamics of equilibrium behavior.

Coasian Dynamics

Skimming Property In a perfect Bayesian equilibrium, higher-valuation types of buyer buy earlier because they are more impatient to consume. (As we will see later, this property is a straightforward consequence of the sorting condition defined in chapter 7.)

Monotonicity of Prices The equilibrium path exhibits a weakly decreasing sequence of prices until one price is accepted. (As we will see, this property requires a stationarity assumption on strategies in the no-gap case).[5]

The second set of properties concerns the limit of the equilibrium outcomes as the time period between offers shrinks to 0, so that the per-period discount factor δ tends to 1. These properties were conjectured by Coase (1972).

Coase Conjecture

When offers take place very quickly ($\delta \to 1$),

Zero Profit The seller's profit tends to 0 and

Efficiency All potential gains from trade are realized almost instantaneously.

To study the Coase conjecture, we let r denote the rate of interest per unit of time and Δ be the length of time between offers. Hence, $\delta = e^{-r\Delta}$.

5. It may be useful here to distinguish between "offers" and "serious offers" (which are prices offered and accepted with positive probability). It can be seen that in any pure-strategy PBE the equilibrium sequence of serious offers is strictly decreasing even if $\underline{v} \leq c$ and no stationarity assumption is made (a buyer who rejects offer m^t at date t and accepts $m^{t+\tau} \geq m^t$ at date $t + \tau$, where $\tau > 0$, would be better off accepting m^t). The stationarity assumption implies that sales have a positive probability in every period (see note 19).

The focus of the Coase-conjecture analysis is thus the equilibrium behavior as Δ converges to 0.

We start with a two-period example that illustrates Coasian dynamics and an infinite-horizon example that satisfies the Coase conjecture. These examples may be skipped by a reader who has some familiarity with the topic. We then tackle the Coase conjecture in the two-type case, and more generally the gap case. We do the same in the no-gap case, and we conclude the section with some extensions of the sale model.

10.2.2 A Two-Period Introduction to Coasian Dynamics

Let $T = 1$, and let $v = \bar{v}$ with probability p and $\underline{v}$ with probability $\underline{p}$. Let m^0 denote a first-period price, let $\bar{\mu}(m^0)$ denote the posterior beliefs that $v = \bar{v}$ conditional on the rejection of offer m^0 in period 0, and define $\underline{\mu}(m^0) \equiv 1 - \bar{\mu}(m^0)$.

Because period 1 is the last period, the seller with beliefs $\bar{\mu}$ that $v = \bar{v}$ makes a "take it or leave it" offer m^1 so as to maximize that period's profit. The buyer will accept if and only if his valuation is at least m^1.[6] It is clear that the optimal offer is either $\bar{v}$ or $\underline{v}$. By charging $m^1 = \underline{v}$, the seller sells for sure and obtains $\underline{v}$; by charging $m^1 = \bar{v}$, the seller sells with probability $\bar{\mu}$ and has second-period profit $\bar{\mu}\bar{v}$. Therefore, the seller's optimal strategy at date $t = 1$ is

$$m^1 = \begin{cases} \underline{v} \text{ if } \bar{\mu} < \alpha \\ \bar{v} \text{ if } \bar{\mu} > \alpha \\ \text{any randomization between } \underline{v} \text{ and } \bar{v} \text{ if } \bar{\mu} = \alpha, \end{cases}$$

where $\alpha \equiv \underline{v}/\bar{v}$. We can rewrite this optimal strategy by introducing the probability x that the seller charges $\underline{v}$ in the second period:

$$x = \begin{cases} 1 \text{ if } \bar{\mu} < \alpha \\ 0 \text{ if } \bar{\mu} > \alpha \\ \in [0, 1] \text{ if } \bar{\mu} = \alpha. \end{cases}$$

Note that type $\underline{v}$ never obtains a surplus in the second period and, therefore, will behave myopically in the first period. Type $\bar{v}$ obtains a surplus only if the seller is sufficiently convinced that the type is $\underline{v}$. Consider now the buyer's behavior at $t = 0$ when offered price $m^0 \in [\underline{v}, \bar{v}]$ (it is straightforward to check that prices outside this interval are irrelevant). Price $m^0 = \underline{v}$ is accepted by both types, as they will not face a more favorable price at

6. Each type is actually indifferent between accepting and rejecting a price m^1 that exactly equals the type's valuation. However, if the supremum of the seller's payoff is attained in the limit of prices $m^1 = v - |\varepsilon|$ as $\varepsilon \to 0$, then existence of an equilibrium given the seller's beliefs requires that type v accept $m^1 = v$, and whether the other type accepts a price equal to its valuation is irrelevant.

$t = 1$.[7] Now consider $m^0 > \underline{v}$. The low-valuation type rejects this offer because buying would give him a negative surplus. The interesting part is type $\bar{v}$'s behavior.

Suppose, first, that rejection of m^0 generates "optimistic beliefs," meaning $\mu(m^0) > \alpha$. Then the seller charges $m^1 = \bar{v}$, and type $\bar{v}$ has no second-period surplus. Therefore, type $\bar{v}$ is better off accepting m^0. And since m^0 is rejected by type $\bar{v}$, Bayes' rule yields $\mu(m^0) = 0$, a contradiction.

Suppose, second, that rejection of m^0 generates "pessimistic beliefs": $\mu(m^0) < \alpha$. The seller then charges $m^1 = \underline{v}$ at date 1. Therefore, type $\bar{v}$ should accept m^0 only if

$$\bar{v} - m^0 \geq \delta(\bar{v} - \underline{v})$$

or

$$m^0 \leq \tilde{v} \equiv (1 - \delta)\bar{v} + \delta\underline{v}.$$

If $m^0 > \tilde{v}$, rejecting m^0 is optimal for type $\bar{v}$ (as it is for type $\underline{v}$), and therefore Bayes' rule yields $\bar{\mu}(m^0) = \bar{p}$ (the posterior beliefs coincide with the prior beliefs).

We are thus led to consider two cases:

$\bar{p} < \alpha$ In this case, for any $m^0 > \underline{v}$, $\bar{\mu}(m^0) \leq \bar{p} < \alpha$, and therefore the seller always charges $m^1 = \underline{v}$ at date 1. Type $\bar{v}$ accepts m^0 if and only if $m^0 \leq \tilde{v}$. The seller's optimal first-period strategy is either to charge $m^0 = \underline{v}$ and have payoff $U_s = \underline{v}$ or to charge $m^0 = \tilde{v}$ and have payoff $U_s = \bar{p}\tilde{v} + \delta\underline{p}\underline{v}$. If $\underline{v} > \bar{p}\tilde{v} + \delta\underline{p}\underline{v}$, no "price discrimination" occurs and agreement is reached instantaneously. On the other hand, $\underline{v} \leq \bar{p}\tilde{v} + \delta\underline{p}\underline{v}$ is ruled out by $\bar{p} < \alpha$, since

$$\bar{p}\tilde{v} + \delta\underline{p}\underline{v} = \overline{p}\overline{v} + \delta(\underline{v} - \overline{p}\overline{v}) < \underline{v}.$$

$\bar{p} > \alpha$ Then, when $m^0 \in (\tilde{v}, \bar{v}]$, in equilibrium type $\bar{v}$ cannot reject m^0 with probability 1, because in that case we would have $\bar{\mu}(m^0) = \bar{p} > \alpha$ and the seller charging $m^1 = \bar{v}$, so type $\bar{v}$ would be better off accepting m^0. But we already saw that type $\bar{v}$ cannot accept such an m^0 with probability 1 either. Hence, in equilibrium type $\bar{v}$ must randomize and the posterior probability must satisfy $\bar{\mu}(m^0) = \alpha$. Let $y(m^0)$ denote the probability that type $\bar{v}$ accepts m^0; $\mu(m^0) = \alpha$ is equivalent to

$$\frac{\bar{p}(1 - y(m^0))}{\bar{p}(1 - y(m^0)) + \underline{p}} = \alpha,$$

which defines a unique $y(m^0) = y$ in $[0, 1]$. Note that $y(m^0)$ is independent of m^0, a fact we will comment on later.

7. As in note 6, type $\underline{v}$ is actually indifferent between accepting and rejecting $m^0 = \underline{v}$, but our assumption involves no loss of generality.

Furthermore, in order for type $\overline{v}$ to be indifferent between accepting and rejecting m^0, it must be the case that $\overline{v} - m^0 = \delta x(m^0)(\overline{v} - \underline{v})$, which defines a unique probability $x(m^0)$ for $m^0 \in (\tilde{v}, \overline{v}]$.

Thus, when $\overline{p} > \alpha$, the seller's optimal price in the first period is one of the following:

$m^0 = \underline{v}$, which generates payoff $U_s = \underline{v}$,

$m^0 = \tilde{v}$, which generates payoff $U_s = \overline{p}\tilde{v} + \delta \underline{p}\underline{v}$,

$m^0 = v$, which generates payoff $U_s = pyv + \delta(\overline{p}(1 - y) + \underline{p})v$,

where the third payoff is computed using the fact that, for posterior beliefs α, $m^1 = v$ is an optimal price in period 1 for the seller. Any of these payoffs can be highest, depending on the parameters. Note that if the third payoff is highest, the seller never sells to the low-valuation type (as $x(\overline{v}) = 0$).

We thus conclude that for generic values of the parameters there exists a unique perfect Bayesian equilibrium, and that this equilibrium exhibits Coasian dynamics—that is, $\overline{\mu}(m^0) \leq \overline{p}$ for all m^0, so the seller becomes more pessimistic over time, and $m^1 \leq m^0$, so the seller's price decreases over time.

Fudenberg and Tirole (1983) characterize the set of equilibria of two-period bargaining games when the seller and the buyer each have two potential types (two-sided incomplete information), when the seller makes the two offers, and when the players alternate making offers. As is mentioned above, the fact that a player's offer can then signal his private information leads to a continuum of perfect Bayesian equilibria, as in the similar examples of chapter 8.

In contrast, when an uninformed seller makes offers to an informed buyer, the buyer has comparatively little scope to signal his type (he can do so only through his acceptance decision), and thus the leeway in specifying the beliefs after a probability-0 action has much less impact on the set of equilibria.

The two-period model raises the question of why the parties stop bargaining at the end of the second period if both offers have been rejected. The existence of unrealized gains from trade suggests that the parties would be better off if they continued to bargain. Thus, it would seem that a natural model of bargaining should have an infinite horizon unless one of the parties must quit for some reason. In practice, however, the players are likely to stop bargaining after some time even if they have not exhausted the gains from trade. It may be that they face a deadline for agreement (imposed by production constraints, for instance). Alternatively, the good may become obsolete at date 2 because of the introduction of a superior product at that date. The two-period model developed above applies directly to these two exogenous-horizon stituations, with the minor modification that if the good becomes obsolete the buyer's willingness to pay is lower in period 1 than in period 0, as the good is enjoyed only for one period instead of two. (This modification introduces a quantitative difference, but not a qualitative one.)

A more complex explanation for a finite horizon is that the players have a fixed bargaining cost per period or have outside opportunities, so that they may decide to stop bargaining or to bargain with someone else if they become sufficiently pessimistic about the gains from trade with their current partner. The bargaining horizon is then endogenously finite. The endogenous-horizon model is more complex than the model with an exogenously finite horizon. In particular, the endogenous-horizon model tends to have multiple equilibria, whereas equilibrium is unique in the exogenous-horizon model. To see this, note that if the seller expects the buyer to concede (buy) quickly, he becomes very pessimistic, so that if several offers are rejected he will break negotiation with the buyer and exercise his outside opportunity (which might be to sell to an alternative buyer). Also, if the seller is "switch-happy," the buyer is in a weak position and concedes quickly. Thus, fast concessions and fast switching are self-fulfilling; so are slow concessions and slow switching, which is why there are multiple equilibria (Fudenberg, Levine, and Tirole 1987).

10.2.3 An Infinite-Horizon Example of the Coase Conjecture

In this subsection we adopt the interpretation that a single seller faces a continuum of infinitesimal buyers. As is mentioned above, under this interpretation we suppose that the seller cannot distinguish between the buyers, and only observes the measures of the sets who accept and reject. We also explain how to interpret the model as having a single buyer.

Sobel and Takahashi (1983) study the following "linear demand curve" model.[8] The seller and the buyers are infinitely-lived, and the bargaining process has $T = +\infty$. The buyers' valuations are uniformly distributed on $[0, 1]$. (Sobel and Takahashi consider the more general distribution $P(v) = (v/\bar{v})^{\beta}$, where $\beta > 0$; the uniform distribution is $\beta = 1$.)

We look for an equilibrium with the following properties:

(i) If price m^t is offered at date t, types $v \geq w(m^t) = \lambda m^t$, where $\lambda > 1$, buy (if they have not purchased before) and types $v < w(m^t)$ do not.

(ii) If at some date t types greater than κ have purchased before and types less than κ have not (so that the seller's posterior beliefs are represented by the truncated uniform distribution on $[0, \kappa]$), the seller charges $m^t(\kappa) = \gamma\kappa$, where $0 < \gamma < 1$.

As mentioned above, the skimming property, which is proved in lemma 10.1 below, ensures that the buyers always use a cutoff rule of the form "accept the current offer if and only if the valuation exceeds some (possibly history-dependent) number." Hence, the real force of condition i is to

8. For early work on the Coase conjecture, see Bulow 1982 and Stokey 1981.

require that the cutoff valuation be stationary (it depends only on the current price and not on previous price offers) and linear (λ is independent of m^t). Condition ii requires that the seller's strategy also be stationary and linear. Note that because all players use a stationary strategy, each player loses nothing by using a stationary strategy himself.[9]

Let $U_s(\kappa)$ denote the seller's present discounted value of profits when the posterior beliefs are uniform on $[0, \kappa]$. From dynamic programming, $U_s(\cdot)$ must satisfy

$$U_s(\kappa) = \max_m \{(\kappa - \lambda m)m + \delta U_s(\lambda m)\}. \tag{10.1}$$

With a continuum of buyers, the term $(\kappa - \lambda m)$ in equation 10.1 is interpreted as the fraction of the population that will accept offer $m \leq \kappa/\lambda$, and $U_s(\lambda m)$ is the continuation present discounted value of profits. With a single buyer, equation 10.1 still holds as long as $U_s(\kappa)$ is interpreted as the product of the probability κ that the buyer has type below κ and the continuation expected present discounted value of profits. The term $(\kappa - \lambda m)$ in equation 10.1 is the probability that offer m is accepted by the buyer. Equations 10.5 and 10.6 are also valid in the single-buyer case.

If U_s is assumed to be differentiable, the maximization with respect to m yields

$$\kappa - 2\lambda m + \delta\lambda U_s'(\lambda m) = 0. \tag{10.2}$$

On the other hand, the envelope theorem can be applied to equation 10.1:

$$U_s'(\kappa) = m(\kappa) = \gamma\kappa. \tag{10.3}$$

Substituting equation 10.3 into equation 10.2 and eliminating κ yields

$$1 - 2\lambda\gamma + \delta\lambda^2\gamma^2 = 0. \tag{10.4}$$

We now look at the buyer's optimization. For type λm to be indifferent between accepting m and waiting one period and buying at price $\gamma\lambda m$, it must be the case that

$$\lambda m - m = \delta(\lambda m - \gamma\lambda m), \tag{10.5}$$

or

$$\lambda - 1 = \delta\lambda(1 - \gamma).^{10} \tag{10.6}$$

9. Condition ii is used only for convenience, as it is implied by condition i. To see this, note that, given condition i, *an* optimal policy for the seller is to use a stationary and linear strategy, as we show below. Furthermore, the valuation function $U_s(\cdot)$ is the quadratic function derived below (applying Blackwell's theorem—see, e.g., Stokey and Lucas 1989—to equation 10.1 shows that this valuation function is unique). The maximization in equation 10.1 then yields a unique optimal price, which therefore is a stationary and linear function of the cutoff type.

10. We can now check the second-order condition in the maximization in equation 10.1: $2\lambda + \delta\lambda^2\gamma \leq 0$, which is implied by equation 10.6.

Equations 10.4 and 10.6 yield

$$\lambda = \frac{1}{\sqrt{1-\delta}}$$

and

$$\gamma = \frac{\sqrt{1-\delta} - (1-\delta)}{\delta}.$$

This perfect Bayesian equilibrium exhibits Coasian dynamics. Furthermore, it satisfies the Coase conjecture. When offers take place very quickly, γ tends to 0. Hence, even the first offer m^0, which is the highest offer, converges to 0, and so does the seller's expected profit $U_s(1)$. To see that all potential gains from trade are realized almost instantaneously, consider a valuation v. By purchasing at or after *real* time $\tau > 0$, this type has a utility of at most $e^{-r\tau}v$, where r is the rate of interest. By buying in the first period, he gets $v - m^0(\delta)$, where $m^0(\delta) = \gamma(\delta) \to 0$. Hence, for any given τ, any type v buys before real time τ if δ is sufficiently close to 1.

10.2.4 The Skimming Property⁺⁺⁺

We now return to the single-buyer interpretation, and give a characterization of equilibrium.

The following lemma[11] considerably simplifies the study of buyer behavior:

Lemma 10.1 (skimming or cutoff-rule property) Suppose that the buyer accepts price m^t at date t when he has valuation v. Then he accepts price m^t with probability 1 when he has valuation $v' > v$.

Proof Let $h^t = (m^0, \ldots, m^{t-1})$ denote the history at date t (where the fact that the buyer has rejected all offers is implicit). Type v accepts m^t only if

$$v - m^t \geq \delta U_b(v,(h^t, m^t))$$

or

$$v - m^t \geq \mathrm{E} \sum_{\tau=1}^{T-t} [\delta^\tau (v - m^{t+\tau}(h^{t+\tau})) I^{t+\tau}(h^{t+\tau}, m^{t+\tau}, v) \mid (h^t, m^t)],$$

where $U_b(v,(h^t, m^t))$ is the continuation valuation of type v and where $I^{t+\tau}(h^{t+\tau}, m^{t+\tau}, v)$ is an indicator function indicating whether type v buys ($I = 1$) or not ($I = 0$) at price $m^{t+\tau}(h^{t+\tau})$ at date $t + \tau$. The random variables $m^{t+\tau}(h^{t+\tau})$ and $I^{t+\tau}(h^{t+\tau}, m^{t+\tau}, v)$ are determined by the history (h^t, m^t) at the end of the period if m^t is rejected and by the subsequent equilibrium strategies. Because the expected discounted volume of trade is always less than 1 and because type v' can mimic type v's bargaining strategy (and conversely),

11. Fudenberg et al. 1985.

$$|U_b(v',(h^t,m^t)) - U_b(v,(h^t,m^t))| \leq |v' - v|.$$

Therefore, for $v' > v$,

$$\begin{aligned} v' - m^t - \delta U_b(v',(h^t,m^t)) & \\ &\geq (v' - v) - \delta(U_b(v',(h^t,m^t)) - U_b(v,(h^t,m^t))) > 0. \quad \blacksquare \end{aligned}$$

Lemma 10.1 shows that because higher-valuation types are more eager to buy, they buy earlier. In particular, if v is drawn from a continuous distribution, the buyer's behavior is fully described by the cutoff rule $\kappa(\cdot)$: At date t, the buyer buys if $v > \kappa(h^t, m^t)$ and does not buy if $v < \kappa(h^t, m^t)$. (The type $v = \kappa(h^t, m^t)$ has probability 0, and the resolution of his indifference is irrelevant. Of course, with atoms in the distribution of types, as in the example of subsection 10.2.2, the cutoff rule still holds but the mixing behavior of the cutoff type becomes important.)

10.2.5 The Gap Case^{+++}

We now make the following assumptions on the distribution of types:

(G) $\underline{v} > 0$

(R) Either $P(\underline{v}) > 0$ or P admits a strictly positive and continuous density at $\underline{v}$.

Condition G asserts that there is a gap between the lowest valuation and the seller's cost. The regularity condition R allows either an atom or a strictly positive density at the lowest valuation.

Under assumption G, the Coase conjecture takes the following form: When $\delta \to 1$,

(c') the seller's profit tends to $\underline{v}$ and

(d) all gains from trade are realized almost instantaneously.

Next, we introduce a condition on the buyer's equilibrium strategy. This condition is satisfied by any equilibrium in the gap case; it will be imposed as an assumption in the no-gap case.

(S) The buyer's strategy satisfies property S if $\kappa(h^t, m^t) = \kappa(\tilde{h}^t, m^t)$ when m^t is lower than any price offered in histories h^t and $\tilde{h}^t$. That is, if the price is lower than in the past, the buyer's behavior is independent of previous prices.

Property S, which can be called "stationarity" or "the strong cutoff rule property," has a Markov flavor.[12]

Theorem 10.1 (Fudenberg, Levine, and Tirole 1985; Gul, Sonnenschein, and Wilson 1986) Suppose that the distribution of the buyer's type satisfies

12. In contrast with the Markov concept (see chapter 13 for the definition of Markov perfect equilibrium in games of complete information; see Maskin and Tirole 1989 for a definition of Markov perfect Bayesian equilibrium), property S cannot be required in states that are not reached in equilibrium.

conditions G and R. Then

(i) a perfect Bayesian equilibrium exists and is generically unique,

(ii) the equilibrium satisfies the Coase conjecture as $\delta \to 1$,

(iii) the equilibrium satisfies condition S, and

(iv) when $\underline{v} \to 0$, the equilibrium converges to a perfect Bayesian equilibrium of the no-gap case (that satisfies the Coase conjecture).[13]

Instead of proving this theorem, we give the flavor of the argument by analyzing the *two-type* case.[14] Suppose that $v = \bar{v}$ with probability $\bar{p}$, and $\underline{v}$ with probability $\underline{p}$. Let $\bar{\mu}^t$ denote the date-t posterior probability of $\bar{v}$, conditional on the history h^t of rejected prices. The first step in the proof is to show that the seller never makes an offer less than $\underline{v}$. Let $\underline{m}$ denote the infimum of equilibrium offers made by the seller for any period and history, and suppose that $\underline{m} < \underline{v}$.[15] Then we claim that $\underline{m}$ or any offer close to it is accepted with probability 1 by both types, as the most favorable offer made by the seller in the future is at best $\underline{m}$ (i.e., $v - m > \delta(v - \underline{m})$ for all v). Hence, the seller can raise his price by a discrete amount above $\underline{m}$ and still have his offer accepted with probability 1, which means that prices arbitrarily close to m cannot be optimal after all. So $\underline{m} \geq \underline{v}$. This implies that type $\underline{v}$ accepts all prices below $\underline{v}$, so that for any history h^t the seller can guarantee himself present payoff $\underline{v}$ by offering $\underline{v}$.

With these preliminary observations in hand, we turn to the heart of the proof, which uses an "upward induction on beliefs":

• If $\mu^t \leq \alpha \equiv \underline{v}/\bar{v}$, the seller's maximum profit from date t on is his "monopoly profit" $\underline{v}$. To see this, note that if the seller could commit himself to a single price, he would either choose $\underline{v}$ and get $\underline{v}$ or choose $\bar{v}$ and get $\bar{\mu}^t\bar{v}$, and because $\bar{\mu}^t\bar{v} \leq \underline{v}$, the optimal "commitment price" is $\underline{v}$. We saw in section 7.3 that commitment to a single price is an optimal mechanism for the seller; in particular, it weakly dominates the direct-revelation mechanism associated with the perfect Bayesian equilibrium of the bargaining game. Hence, $\underline{v}$ is an upper bound on the seller's profit from t on, and, furthermore, the seller can guarantee himself this upper bound by charging $\underline{v}$.

• Suppose now that $\bar{\mu}^t > \alpha$. Will the seller make an offer $m^t > \underline{v}$? If this offer results in posterior belief $\bar{\mu}^{t+1}(h^t, m^t) < \alpha$ (which implies that it is

13. Fudenberg et al. (1985) prove theorem 10.1 under assumption G and the following stronger assumption (R'): The distribution of types is smooth and has a density that is bounded and bounded away from zero ($0 < p_{\min} \leq p(v) \leq p_{\max}$ for all $v \in [\underline{v}, \bar{v}]$). Gul et al. (1986, theorem 1) generalize parts i–iii of their result by showing that a slightly weaker version of assumption R suffices; they also show that the seller does not randomize on the equilibrium path (although he does randomize off the path). Ausubel and Deneckere (1989a, theorem 4.2) obtain iv without assumptions on the distribution of types.

14. Hart 1989 provides more details of the two-type case.

15. That m is not $-\infty$ results from the fact that the buyer could guarantee himself a surplus close to $+\infty$ when such offers are made. Because aggregate gains from trade are finite, the seller would make a negative profit, which is impossible.

accepted by type $\bar{v}$ with positive probability), then, from our previous characterization, $m^{t+1}(h^t, m^t) = \underline{v}$. Therefore, $\bar{v} - m^t \geq \delta(\bar{v} - \underline{v})$ or $m^t \leq \tilde{v}_1 = \tilde{v} = v - \delta(\bar{v} - \underline{v})$. Conversely, any $m^t \leq \tilde{v}$ is accepted by type $\bar{v}$, as the best offer he will get in the future is $\underline{v}$. Furthermore, when $\bar{\mu}^t \geq \alpha$, the seller always prefers to charge $\tilde{v}$ rather than $\underline{v}$, as the payoff from doing so,

$$\mu^t(v - \delta(v - \underline{v})) + \delta(1 - \bar{\mu}^t)\underline{v},$$

exceeds $\underline{v}$ whenever $\bar{\mu}^t > \alpha$.

Now that we have established that for $\bar{\mu}^t > \alpha$ bargaining has at least two effective rounds, let us show that for $\bar{\mu}^t \in [\alpha, \alpha + \varepsilon]$, where ε is positive and small, the seller charges $\tilde{v}$ and then $\underline{v}$. Because $\bar{\mu}^{t+1} \geq \alpha$ if $m^t > \tilde{v}$, the seller's maximum profit from date t on, $U_s^{\sup}$, over all beliefs in $[\alpha, \alpha + \varepsilon]$ and all possible equilibria, satisfies

$$U_s^{\sup} \leq \max\left[\bar{\mu}^t\tilde{v} + \delta(1 - \bar{\mu}^t)\underline{v}, \frac{\bar{\mu}^t - \alpha}{1 - \alpha}\bar{v} + \delta\left(1 - \frac{\mu^t - \alpha}{1 - \alpha}\right)U_s^{\sup}\right]. \tag{10.7}$$

Clearly, for ε small, the max is obtained for the first term, and therefore it is optimal to charge $\tilde{v}$ at date t. Having pinned down equilibrium when $\mu^t \in [\alpha, \alpha + \varepsilon]$, one determines equilibrium for $\bar{\mu}^t \in [\alpha + \varepsilon, \alpha + 2\varepsilon]$, etc., until the first beliefs μ_2 such that the seller prefers to charge a price above $\tilde{v}$.[16]

Proceeding by upward induction on beliefs, one finds cutoff beliefs $\mu_0 = 0 < \mu_1 = \alpha < \mu_2 < \cdots$ such that, if $\mu^t \in [\mu_n, \mu_{n+1})$, there are $n + 1$ effective bargaining rounds (i.e., the seller charges strictly above $\underline{v}$ for n periods and then offers v). Posterior beliefs on the equilibrium path are decreasing (as the skimming property requires): $\bar{\mu}^{t+1} = \mu_{n-1}$, $\bar{\mu}^{t+2} = \mu_{n-2}, \ldots, \bar{\mu}^{t+n} = 0$. Prices are decreasing on the equilibrium path. They are given by type $\bar{v}$'s indifference between accepting in a given period and accepting in the following period: $m^{t+n} = \underline{v}$, $m^{t+n-1} = \tilde{v}$, and, more generally,

$$v - m^{t+k} = \delta(\bar{v} - m^{t+k+1}) \text{ or } m^{t+k} = v - \delta^{n-k}(\bar{v} - \underline{v}). \tag{10.8}$$

To check the Coase conjecture, it suffices to show that for any $\bar{p}$ there exists n such that $\mu_n > \bar{p}$ for any δ.[17] Equation 10.8 together with the last offer's being $\underline{v}$ then implies that, for δ close to 1, all offers in the effective length of bargaining are close to $\underline{v}$. Furthermore, because there are at most $n + 1$ offers, agreement is reached almost instantaneously.

16. It is easily checked that μ_2 satisfies

$$\frac{\mu_2 - \alpha}{1 - \alpha}[\bar{v} - \delta^2(\bar{v} - \underline{v})] + \left(1 - \frac{\mu_2 - \alpha}{1 - \alpha}\right)\delta v = \mu_2(\bar{v} - \delta(\bar{v} - \underline{v})) + (1 - \mu_2)\delta\underline{v}.$$

17. For instance, check that μ_2 does not converge to α when δ converges to 1.

10.2.6 The No-Gap Case†††

Assume now that

(NG) $\underline{v} = 0$.

Statement iv of theorem 10.1 shows that there exists an equilibrium satisfying condition S and the Coase conjecture. However, Gul, Sonnenschein, and Wilson (1986) show that there exist other equilibria.[18] They characterize the set of perfect Bayesian equilibria satisfying condition S as follows.

Theorem 10.2 (Gul, Sonnenschein, and Wilson 1986) Assume conditions NG and R. Then any perfect Bayesian equilibrium satisfying condition S satisfies the Coase conjecture.

We do not give the proof of theorem 10.2, which is complex. But it is worth sketching it, because the proof highlights the logic of the Coase conjecture better than the proof of theorem 10.1, which constructs the equilibrium and checks that it satisfies the conjecture. Under assumption S, the buyer's cutoff valuation $\kappa(\cdot)$ is independent of history. Therefore, the seller's expected profit from date t onward, given current beliefs $P(v)/P(\kappa^t)$ on $[0, \kappa^t]$ at the beginning of period t, depends only on the current cutoff valuation, κ^t, and not on history. Denote this payoff by $U_s(\kappa^t)$. Suppose for simplicity that the sequence of cutoffs κ^t is deterministic.

Fix a real time $\varepsilon > 0$, and let the time period Δ tend to 0 (so that there is a large number of offers between 0 and ε). For any $\eta > 0$, there exists a Δ sufficiently small and a $t \le \varepsilon/\Delta - 2$ such that

$$P(\kappa^t) - P(\kappa^{t+2}) < \eta,$$

that is, the seller sells with low probability between t and $t+2$. (Because of the large number of offers between 0 and ε, there must be some periods in which he sells with low probability.) The intuition for the Coase conjecture is that the seller would be eager to speed up the sale process at t if his continuation value at time ε were not negligible. To see this, note that the seller could offer m^{t+1} at t and thus create posterior beliefs κ^{t+2} at $t+1$, so

$$\begin{aligned}&[P(\kappa^t) - P(\kappa^{t+1})]m^t + \delta[P(\kappa^{t+1}) - P(\kappa^{t+2})]m^{t+1} + \delta^2 P(\kappa^{t+2})U_s(\kappa^{t+2})\\ &\quad > [P(\kappa^t) - P(\kappa^{t+2})]m^{t+1} + \delta P(\kappa^{t+2})U_s(\kappa^{t+2}).\end{aligned} \tag{10.9}$$

(This is where the stationarity assumption is crucial: The seller gets $U_s(\kappa^{t+2})$

18. Note that this multiplicity as well as that in theorem 10.3 depends on the set of possible valuations' being an interval. If the set of possible valuations is discrete and does not include the seller's cost, the no-gap model is equivalent to the gap model even if some types of buyers have valuation lower than cost. We know from theorem 10.1 that the equilibrium is then (generically) unique. (When the distribution is discrete and there is a type whose valuation is equal to cost, the uniqueness of equilibrium depends on what the seller is assumed to do when he has sold to all types with valuation above his cost.)