1 The Rules of the Game

1.1: Definitions

Game theory is concerned with the actions of decision makers who are conscious that their actions affect each other. When the only two publishers in a city choose prices for their newspapers, aware that their sales are determined jointly, they are players in a game with each other. They are not in a game with the readers who buy the newspapers, because each reader ignores his effect on the publisher. Game theory is not useful when decisionmakers ignore the reactions of others or treat them as impersonal market forces.

The best way to understand which situations can be modelled as games and which cannot is to think about examples like the following:

1. OPEC members choosing their annual output;
2. General Motors purchasing steel from USX;
3. two manufacturers, one of nuts and one of bolts, deciding whether to use metric or American standards;
4. a board of directors setting up a stock option plan for the chief executive officer;
5. the US Air Force hiring jet fighter pilots;
6. an electric company deciding whether to order a new power plant given its estimate of demand for electricity in ten years.

The first four examples are games. In (1), OPEC members are playing a game because Saudi Arabia knows that Kuwait's oil output is based on Kuwait's forecast of Saudi output, and the output from both countries matters to the world price. In (2), a significant portion of American trade in steel is between General Motors and USX, companies which realize that the quantities traded by each of them affect the price. One wants the price low, the other high, so this is a game with conflict between the two players. In (3), the nut and bolt manufacturers are not in conflict, but the actions of one do affect the desired actions of the other, so the situation is a game none the less. In (4), the board of directors chooses a stock option plan anticipating the effect on the actions of the CEO.

Game theory is inappropriate for modelling the final two examples. In (5), each individual pilot affects the US Air Force insignificantly, and each pilot makes his employment decision without regard for the impact on the Air Force's policies. In (6), the electric company faces a complicated decision, but it does not face another rational agent. These situations are more appropriate for the use of **decision theory** than game theory, decision theory being the careful analysis of how one person makes a decision when he may be faced with uncertainty, or an entire sequence of decisions that interact with each other, but when he is not faced with having to interact strategically with other single decision makers. Changes in the important economic variables could,however, turn examples (5) and (6) into games. The appropriate model changes if the Air Force faces a pilots' union or if the public utility commission pressures the utility to change its generating capacity.

Game theory as it will be presented in this book is a modelling tool, not an axiomatic system. The presentation in this chapter is unconventional. Rather than starting with mathematical definitions or simple little games of the kind used later in the chapter, we will start with a situation to be modelled, and build a game from it step by step.

Describing a Game

The essential elements of a game are **players**, **actions**, **payoffs**, and **information**– PAPI, for short. These are collectively known as the **rules of the game**, and the modeller's objective is to describe a situation in terms of the rules of a game so as to explain what will happen in that situation. Trying to maximize their payoffs, the players will devise plans known as **strategies** that pick actions depending on the information that has arrived at each moment. The combination of strategies chosen by each player is known as the **equilibrium**. Given an equilibrium, the modeller can see what actions come out of the conjunction of all the players' plans, and this tells him the **outcome** of the game.

This kind of standard description helps both the modeller and his readers. For the modeller, the names are useful because they help ensure that the important details of the game have been fully specified. For his readers, they make the game easier to understand, especially if, as with most technical papers, the paper is first skimmed quickly to see if it is worth reading. The less clear a writer's style, the more closely he should adhere to the standard names, which means that most of us ought to adhere very closely indeed.

Think of writing a paper as a game between author and reader, rather than as a single-player production process. The author, knowing that he has valuable information but imperfect means of communication, is trying to convey the information to the reader. The reader does not know whether the information is valuable, and he must choose whether to read the paper closely enough to find out.[1]

To define the terms used above and to show the difference between game theory and decision theory, let us use the example of an entrepreneur trying to decide whether to start a dry cleaning store in a town already served by one dry cleaner. We will call the two firms "NewCleaner" and "OldCleaner." NewCleaner is uncertain about whether the economy will be in a recession or not, which will affect how much consumers pay for dry cleaning, and must also worry about whether OldCleaner will respond to entry with a price war or by keeping its initial high prices. OldCleaner is a well-established firm, and it would survive any price war, though its profits would fall. NewCleaner must itself decide whether to

[1]Once you have read to the end of this chapter: What are the possible equilibria of this game?

initiate a price war or to charge high prices, and must also decide what kind of equipment to buy, how many workers to hire, and so forth.

Players *are the individuals who make decisions. Each player's goal is to maximize his utility by choice of actions.*

In the Dry Cleaners Game, let us specify the players to be NewCleaner and OldCleaner. Passive individuals like the customers, who react predictably to price changes without any thought of trying to change anyone's behavior, are not players, but environmental parameters. Simplicity is the goal in modelling, and the ideal is to keep the number of players down to the minimum that captures the essence of the situation.

Sometimes it is useful to explicitly include individuals in the model called **pseudo-players** whose actions are taken in a purely mechanical way.

Nature *is a pseudo-player who takes random actions at specified points in the game with specified probabilities.*

In the Dry Cleaners Game, we will model the possibility of recession as a move by Nature. With probability 0.3, Nature decides that there will be a recession, and with probability 0.7 there will not. Even if the players always took the same actions, this random move means that the model would yield more than just one prediction. We say that there are different **realizations** of a game depending on the results of random moves.

An **action** *or* **move** *by player i, denoted a_i, is a choice he can make.*

Player i's **action set**, $A_i = \{a_i\}$, *is the entire set of actions available to him.*

An **action combination** *is an ordered set $a = \{a_i\}$, $(i = 1, \ldots, n)$ of one action for each of the n players in the game.*

Again, simplicity is our goal. We are trying to determine whether Newcleaner will enter or not, and for this it is not important for us to go into the technicalities of dry cleaning equipment and labor practices. Also, it will not be in Newcleaner's interest to start a price war, since it cannot possibly drive out Oldcleaners, so we can exclude that decision from our model. Newcleaner's action set can be modelled very simply as $\{Enter, Stay\ Out\}$. We will also specify Oldcleaner's action set to be simple: it is to choose price from $\{Low, High\}$.

By player i's **payoff** $\pi_i(s_1, \ldots, s_n)$, *we mean either:*
(1) The utility player i receives after all players and Nature have picked their strategies and the game has been played out; or
(2) The expected utility he receives as a function of the strategies chosen by himself and the other players.

For the moment, think of "strategy" as a synonym for "action". Definitions (1) and (2) are distinct and different, but in the literature and this book the term "payoff" is used for both the actual payoff and the expected payoff. The context will make clear which is meant. If one is modelling a particular real-world situation, figuring out the payoffs is often the hardest part of constructing a model. For this pair of dry cleaners, we will pretend we have looked over all the data and figured out that the payoffs are as given by Table 1a if the economy is normal, and that if there is a recession the payoff of each player who operates in the market is 60 thousand dollars lower, as shown in Table 1b.

Table 1a: The Dry Cleaners Game: Normal Economy

		OldCleaner	
		Low price	*High price*
NewCleaner	*Enter*	-100, -50	100, 100
	Stay Out	0,50	0,300

Payoffs to: (NewCleaner, OldCleaner) in thousands of dollars

Table 1b: The Dry Cleaners Game: Recession

		OldCleaner	
		Low price	*High price*
NewCleaner	*Enter*	-160, -110	40, 40
	Stay Out	0,-10	0,240

Payoffs to: (NewCleaner, OldCleaner) in thousands of dollars

Information is modelled using the concept of the **information set**, a concept which will be defined more precisely in Section 2.2. For now, think of a player's information set as his knowledge at a particular time of the values of different variables. The elements of the information set are the different values that the player thinks are possible. If the information set has many elements, there are many values the player cannot rule out; if it has one element, he knows the value precisely. A player's information set includes not only distinctions between the values of variables such as the strength of oil demand, but also knowledge of what actions have previously been taken, so his information set changes over the course of the game.

Here, at the time that it chooses its price, OldCleaner will know NewCleaner's decision about entry. But what do the firms know about the recession? If both firms know about the recession we model that as Nature moving before NewCleaner; if only OldCleaner knows, we put Nature's move after NewCleaner; if neither firm knows whether there is a recession at the time they must make their decisions, we put Nature's move at the end of the game. Let us do this last.

It is convenient to lay out information and actions together in an **order of play**. Here is the order of play we have specified for the Dry Cleaners Game:

1 Newcleaner chooses its entry decision from $\{Enter, Stay\ Out\}$.
2 Oldcleaner chooses its price from $\{Low, High\}$.
3 Nature picks demand, D, to be $Recession$ with probability 0.3 or $Normal$ with probability 0.7.

The purpose of modelling is to explain how a given set of circumstances leads to a particular result. The result of interest is known as the outcome.

The **outcome** *of the game is a set of interesting elements that the modeller picks from the values of actions, payoffs, and other variables after the game is played out.*

The definition of the outcome for any particular model depends on what variables the modeller finds interesting. One way to define the outcome of the Dry Cleaners Game would be as either *Enter* or *Stay Out*. Another way, appropriate if the model is being constructed to help plan NewCleaner's finances, is as the payoff that NewCleaner realizes, which is, from Tables 1a and 1b, one element of the set {*0, 100, -100, 40, -160*}.

Having laid out the assumptions of the model, let us return to what is special about the way game theory models a situation. Decision theory sets up the rules of the game in much the same way as game theory, but its outlook is fundamentally different in one important way: there is only one player. Return to NewCleaner's decision about entry. In decision theory, the standard method is to construct a **decision tree** from the rules of the game, which is just a graphical way to depict the order of play.

Figure 1 shows a decision tree for the Dry Cleaners Game. It shows all the moves available to NewCleaner, the probabilities of states of nature (actions that NewCleaner cannot control), and the payoffs to NewCleaner depending on its choices and what the environment is like. Note that although we already specified the probabilities of Nature's move to be 0.7 for $Normal$, we also need to specify a probability for OldCleaner's move, which is set at probability 0.5 of *Low price* and probability 0.5 of *High price*.

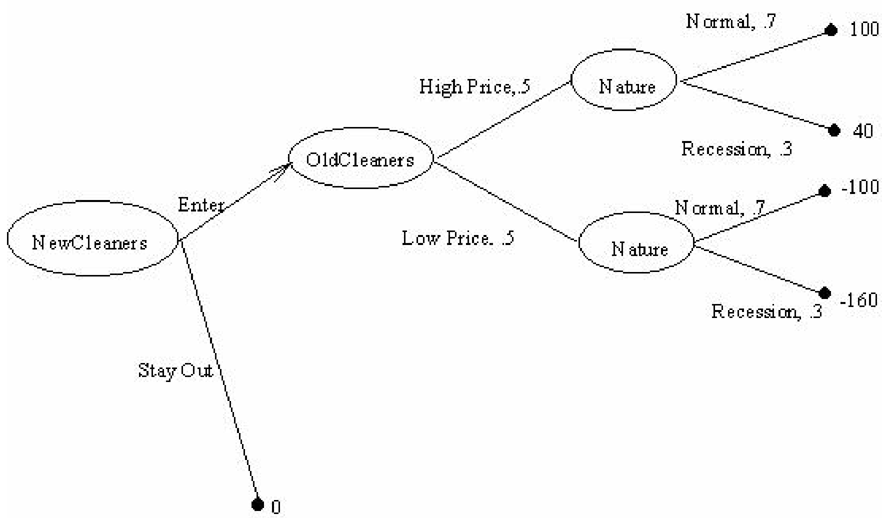

Figure 1: The Dry Cleaners Game as a Decision Tree

Once a decision tree is set up, we can solve for the optimal decision which maximizes the expected payoff. Suppose NewCleaner has entered. If OldCleaner chooses a high price, then NewCleaner's expected payoff is 82, which is 0.7(100) + 0.3(40). If OldCleaner chooses a low price, then NewCleaner's expected payoff is -118, which is 0.7(-100) + 0.3(-160). Since there is a 50-50 chance of each move by OldCleaner, NewCleaner's overall expected payoff from *Enter* is -18. That is worse than the 0 which NewCleaner could get by choosing *stay out*, so the prediction is that NewCleaner will stay out.

That, however, is wrong. This is a game, not just a decision problem. The flaw in the reasoning I just went through is the assumption that OldCleaner will choose *High price* with probability 0.5. If we use information about OldCleaner' payoffs and figure out what moves OldCleaner will take in solving its own profit maximization problem, we will come to a different conclusion.

First, let us depict the order of play as a **game tree** instead of a decision tree. Figure 2 shows our model as a game tree, with all of OldCleaner's moves and payoffs.

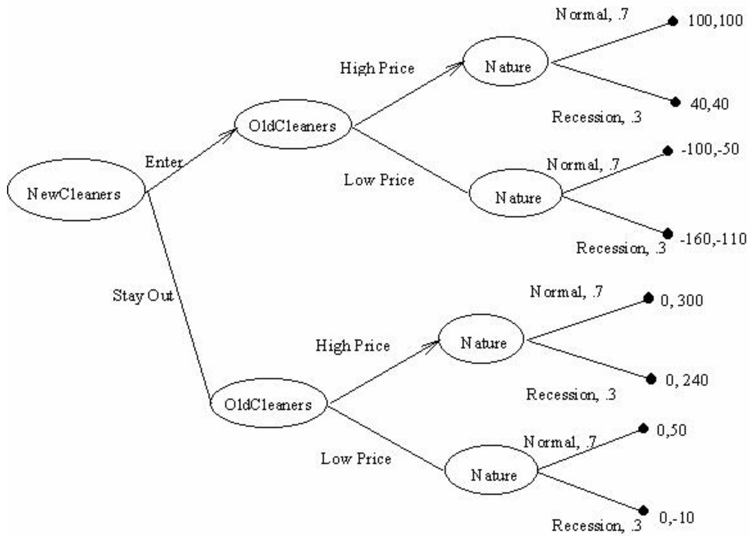

Figure 2: The Dry Cleaners Game as a Game Tree

Viewing the situation as a game, we must think about both players' decision making. Suppose NewCleaner has entered. If OldCleaner chooses *High price*, OldCleaner's expected profit is 82, which is 0.7(100) + 0.3(40). If OldCleaner chooses *Low price*, OldCleaner's expected profit is -68, which is 0.7(-50) + 0.3(-110). Thus, OldCleaner will choose *High price*, and with probability 1.0, not 0.5. The arrow on the game tree for *High price* shows this conclusion of our reasoning. This means, in turn, that NewCleaner can predict an expected payoff of 82, which is 0.7(100) + 0.3(40), from *Enter*.

Suppose NewCleaner has not entered. If OldCleaner chooses *High price*, OldCleaner' expected profit is 282, which is 0.7(300) + 0.3(240). If OldCleaner chooses *Low price*, OldCleaner's expected profit is 32, which is 0.7(50) + 0.3(-10). Thus, OldCleaner will choose *High price*, as shown by the arrow on *High price*. If NewCleaner chooses *Stay out*, NewCleaner will have a payoff of 0, and since that is worse than the 82 which NewCleaner can predict from *Enter*, NewCleaner will in fact enter the market.

This switching back from the point of view of one player to the point of view of another is characteristic of game theory. The game theorist must practice putting himself in *everybody* else's shoes. (Does that mean we become kinder, gentler people? – Or do we just get trickier?)

Since so much depends on the interaction between the plans and predictions of different players, it is useful to go a step beyond simply setting out actions in a game. Instead, the modeller goes on to think about **strategies**, which are action plans.

Player i's **strategy** s_i *is a rule that tells him which action to choose at each instant of the game, given his information set.*

Player i's **strategy set** *or* **strategy space** $S_i = \{s_i\}$ *is the set of strategies available to him.*

A **strategy profile** $s = (s_1, \ldots, s_n)$ *is an ordered set consisting of one strategy for each of the* n *players in the game.*[2]

Since the information set includes whatever the player knows about the previous actions of other players, the strategy tells him how to react to their actions. In the Dry Cleaners Game, the strategy set for NewCleaner is just { *Enter, Stay Out* } , since NewCleaner moves first and is not reacting to any new information. The strategy set for OldCleaner, though, is

$$\left\{ \begin{array}{l} \text{High Price if NewCleaner Entered, Low Price if NewCleaner Stayed Out} \\ \text{Low Price if NewCleaner Entered, High Price if NewCleaner Stayed Out} \\ \text{High Price No Matter What} \\ \text{Low Price No Matter What} \end{array} \right\}$$

The concept of the strategy is useful because the action a player wishes to pick often depends on the past actions of Nature and the other players. Only rarely can we predict a player's actions unconditionally, but often we can predict how he will respond to the outside world.

Keep in mind that a player's strategy is a complete set of instructions for him, which tells him what actions to pick in every conceivable situation, even if he does not expect to reach that situation. Strictly speaking, even if a player's strategy instructs him to commit suicide in 1989, it ought also to specify what actions he takes if he is still alive in 1990. This kind of care will be crucial in Chapter 4's discussion of "subgame perfect" equilibrium. The completeness of the description also means that strategies, unlike actions, are unobservable. An action is physical, but a strategy is only mental.

Equilibrium

To predict the outcome of a game, the modeller focusses on the possible strategy profiles, since it is the interaction of the different players' strategies that determines what happens. The distinction between strategy profiles, which are sets of strategies, and outcomes, which are sets of values of whichever variables are considered interesting, is a common source of confusion. Often different strategy profiles lead to the same outcome. In the Dry Cleaners Game, the single outcome of *NewCleaner Enters* would result from either of the following two strategy profiles:

[2] I used "strategy combination" instead of "strategy profile" in the third edition, but "profile" seems well enough established that I'm switching to it.

$$\left\{\begin{array}{l} \textit{High Price if NewCleaner Enters, Low Price if NewCleaner Stays Out} \\ \textit{Enter} \end{array}\right\}$$

$$\left\{\begin{array}{l} \textit{Low Price if NewCleaner Enters, High Price if NewCleaner Stays Out} \\ \textit{Enter} \end{array}\right\}$$

Predicting what happens consists of selecting one or more strategy profiles as being the most rational behavior by the players acting to maximize their payoffs.

An **equilibrium** $s^* = (s_1^*, \ldots, s_n^*)$ *is a strategy profile consisting of a best strategy for each of the* n *players in the game.*

The **equilibrium strategies** are the strategies players pick in trying to maximize their individual payoffs, as distinct from the many possible strategy profiles obtainable by arbitrarily choosing one strategy per player. Equilibrium is used differently in game theory than in other areas of economics. In a general equilibrium model, for example, an equilibrium is a set of prices resulting from optimal behavior by the individuals in the economy. In game theory, that set of prices would be the **equilibrium outcome**, but the equilibrium itself would be the strategy profile— the individuals' rules for buying and selling— that generated the outcome.

People often carelessly say "equilibrium" when they mean "equilibrium outcome," and "strategy" when they mean "action." The difference is not very important in most of the games that will appear in this chapter, but it is absolutely fundamental to thinking like a game theorist. Consider Germany's decision on whether to remilitarize the Rhineland in 1936. France adopted the strategy: *Do not fight*, and Germany responded by remilitarizing, leading to World War II a few years later. If France had adopted the strategy: *Fight if Germany remilitarizes; otherwise do not fight*, the outcome would still have been that France would not have fought. No war would have ensued,however, because Germany would not remilitarized. Perhaps it was because he thought along these lines that John von Neumann was such a hawk in the Cold War, as MacRae describes in his biography (MacRae [1992]). This difference between actions and strategies, outcomes and equilibria, is one of the hardest ideas to teach in a game theory class, even though it is trivial to state.

To find the equilibrium, it is not enough to specify the players, strategies, and payoffs, because the modeller must also decide what "best strategy" means. He does this by defining an equilibrium concept.

An **equilibrium concept** *or* **solution concept** $F : \{S_1, \ldots, S_n, \pi_1, \ldots, \pi_n\} \rightarrow s^*$ *is a rule that defines an equilibrium based on the possible strategy profiles and the payoff functions.*

We have implicitly already used an equilibrium concept in the analysis above, which picked one strategy for each of the two players as our prediction for the game (what we implicitly used is the concept of **subgame perfectness** which will reappear in chapter 4). Only a few equilibrium concepts are generally accepted, and the remaining sections of this chapter are devoted to finding the equilibrium using the two best-known of them: dominant strategy and Nash equilibrium.

Uniqueness

Accepted solution concepts do not guarantee uniqueness, and lack of a unique equilibrium is a major problem in game theory. Often the solution concept employed leads us to believe that the players will pick one of the two strategy profiles A or B, not C or D, but we cannot say whether A or B is more likely. Sometimes we have the opposite problem and the game has no equilibrium at all. By this is meant either that the modeller sees no good reason why one strategy profile is more likely than another, or that some player wants to pick an infinite value for one of his actions.

A model with no equilibrium or multiple equilibria is underspecified. The modeller has failed to provide a full and precise prediction for what will happen. One option is to admit that his theory is incomplete. This is not a shameful thing to do; an admission of incompleteness like Section 5.2's Folk Theorem is a valuable negative result. Or perhaps the situation being modelled really is unpredictable. Another option is to renew the attack by changing the game's description or the solution concept. Preferably it is the description that is changed, since economists look to the rules of the game for the differences between models, and not to the solution concept. If an important part of the game is concealed under the definition of equilibrium, in fact, the reader is likely to feel tricked and to charge the modeller with intellectual dishonesty.

1.2 Dominated and Dominant Strategies: The Prisoner's Dilemma

In discussing equilibrium concepts, it is useful to have shorthand for "all the other players' strategies."

For any vector $y = (y_1, \ldots, y_n)$, denote by y_{-i} the vector $(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n)$, which is the portion of y not associated with player i.

Using this notation, s_{-Smith}, for instance, is the profile of strategies of every player except player *Smith*. That profile is of great interest to Smith, because he uses it to help choose his own strategy, and the new notation helps define his best response.

Player i's **best response** *or* **best reply** *to the strategies s_{-i} chosen by the other players is the strategy s_i^* that yields him the greatest payoff; that is,*

$$\pi_i(s_i^*, s_{-i}) \geq \pi_i(s_i', s_{-i}) \quad \forall s_i' \neq s_i^*. \tag{1}$$

The best response is strongly best if no other strategies are equally good, and weakly best otherwise.

The first important equilibrium concept is based on the idea of **dominance**.

The strategy s_i^d *is a* **dominated strategy** *if it is strictly inferior to some other strategy no matter what strategies the other players choose, in the sense that whatever strategies they pick, his payoff is lower with* s_i^d. *Mathematically,* s_i^d *is dominated if there exists a single* s_i' *such that*

$$\pi_i(s_i^d, s_{-i}) < \pi_i(s_i', s_{-i}) \;\; \forall s_{-i}. \tag{2}$$

Note that s_i^d is not a dominated strategy if there is no s_{-i} to which it is the best response, but sometimes the better strategy is s_i' and sometimes it is s_i''. In that case, s_i^d could have the redeeming feature of being a good compromise strategy for a player who cannot predict what the other players are going to do. A dominated strategy is unambiguously inferior to some single other strategy.

There is usually no special name for the superior strategy that beats a dominated strategy. In unusual games, however, there is some strategy that beats *every* other strategy. We call that a "dominant strategy".

The strategy s_i^* *is a* **dominant strategy** *if it is a player's strictly best response to* any *strategies the other players might pick, in the sense that whatever strategies they pick, his payoff is highest with* s_i^*. *Mathematically,*

$$\pi_i(s_i^*, s_{-i}) > \pi_i(s_i', s_{-i}) \;\; \forall s_{-i}, \;\; \forall s_i' \neq s_i^*. \tag{3}$$

A **dominant strategy equilibrium** *is a strategy profile consisting of each player's dominant strategy.*

A player's dominant strategy is his strictly best response even to wildly irrational actions by the other players. Most games do not have dominant strategies, and the players must try to figure out each others' actions to choose their own.

The Dry Cleaners Game incorporated considerable complexity in the rules of the game to illustrate such things as information sets and the time sequence of actions. To illustrate equilibrium concepts, we will use simpler games, such as the Prisoner's Dilemma. In the Prisoner's Dilemma, two prisoners, Messrs Row and Column, are being interrogated separately. If both confess, each is sentenced to eight years in prison; if both deny their involvement, each is sentenced to one year.[3] If just one confesses, he is released but the other prisoner is sentenced to ten years. The Prisoner's Dilemma is an example of a **2-by-2 game**, because each of the two players— Row and Column— has two possible actions in his action set: $Confess$ and *Deny*. Table 2 gives the payoffs (The arrows represent a player's preference between actions, as will be explained in Section 1.4).

Table 2: The Prisoner's Dilemma

[3] Another way to tell the story is to say that if both deny, then with probability 0.1 they are convicted anyway and serve ten years, for an expected payoff of $(-1, -1)$.

		Column		
		Deny		*Confess*
	Deny	-1,-1	→	-10, 0
Row		↓		↓
	Confess	0,-10	→	**- 8,-8**

Payoffs to: (Row,Column)

Each player has a dominant strategy. Consider Row. Row does not know which action Column is choosing, but if Column chooses *Deny*, Row faces a *Deny* payoff of -1 and a *Confess* payoff of 0, whereas if Column chooses *Confess*, Row faces a *Deny* payoff of -10 and a *Confess* payoff of -8. In either case Row does better with *Confess*. Since the game is symmetric, Column's incentives are the same. The dominant strategy equilibrium is (*Confess*, *Confess*), and the equilibrium payoffs are $(-8, -8)$, which is worse for both players than $(-1, -1)$. Sixteen, in fact, is the greatest possible combined total of years in prison.

The result is even stronger than it seems, because it is robust to substantial changes in the model. Because the equilibrium is a dominant strategy equilibrium, the information structure of the game does not matter. If Column is allowed to know Row's move before taking his own, the equilibrium is unchanged. Row still chooses *Confess*, knowing that Column will surely choose *Confess* afterwards.

The Prisoner's Dilemma crops up in many different situations, including oligopoly pricing, auction bidding, salesman effort, political bargaining, and arms races. Whenever you observe individuals in a conflict that hurts them all, your first thought should be of the Prisoner's Dilemma.

The game seems perverse and unrealistic to many people who have never encountered it before (although friends who are prosecutors assure me that it is a standard crime-fighting tool). If the outcome does not seem right to you, you should realize that very often the chief usefulness of a model is to induce discomfort. Discomfort is a sign that your model is not what you think it is— that you left out something essential to the result you expected and didn't get. Either your original thought or your model is mistaken; and finding such mistakes is a real if painful benefit of model building. To refuse to accept surprising conclusions is to reject logic.

Cooperative and Noncooperative Games

What difference would it make if the two prisoners could talk to each other before making their decisions? It depends on the strength of promises. If promises are not binding, then although the two prisoners might agree to *Deny*, they would $Confess$ anyway when the time came to choose actions.

A **cooperative game** *is a game in which the players can make binding commitments, as opposed to a* **noncooperative game***, in which they cannot.*

This definition draws the usual distinction between the two theories of games, but the real difference lies in the modelling approach. Both theories start off with the rules of the game, but they differ in the kinds of solution concepts employed. Cooperative game theory is axiomatic, frequently appealing to pareto-optimality,[4] fairness, and equity. Noncooperative game theory is economic in flavor, with solution concepts based on players maximizing their own utility functions subject to stated constraints. Or, from a different angle: cooperative game theory is a reduced-form theory, which focusses on properties of the outcome rather than on the strategies that achieve the outcome, a method which is appropriate if modelling the process is too complicated. Except for Section 12.2 in the chapter on bargaining, this book is concerned exclusively with noncooperative games. For a good defense of the importance of cooperative game theory, see the essay by Aumann (1996).

In applied economics, the most commonly encountered use of cooperative games is to model bargaining. The Prisoner's Dilemma is a noncooperative game, but it could be modelled as cooperative by allowing the two players not only to communicate but to make binding commitments. Cooperative games often allow players to split the gains from cooperation by making **side-payments**— transfers between themselves that change the prescribed payoffs. Cooperative game theory generally incorporates commitments and side-payments via the solution concept, which can become very elaborate, while noncooperative game theory incorporates them by adding extra actions. The distinction between cooperative and noncooperative games does *not* lie in conflict or absence of conflict, as is shown by the following examples of situations commonly modelled one way or the other:

A cooperative game without conflict. Members of a workforce choose which of equally arduous tasks to undertake to best coordinate with each other.

A cooperative game with conflict. Bargaining over price between a monopolist and a monopsonist.

A noncooperative game with conflict. The Prisoner's Dilemma.

A noncooperative game without conflict. Two companies set a product standard without communication.

1.3 Iterated Dominance: The Battle of the Bismarck Sea

[4]If outcome X **strongly pareto-dominates** outcome Y, then all players have higher utility under outcome X. If outcome X **weakly pareto-dominates** outcome Y, some player has higher utility under X, and no player has lower utility. A zero-sum game does not have outcomes that even weakly pareto-dominate other outcomes. All of its equilibria are pareto-efficient, because no player gains without another player losing.

It is often said that strategy profile x "pareto dominates" or "dominates" strategy profile y. Taken literally, this is meaningless, since strategies do not necessarily have any ordering at all— one could define *Deny* as being bigger than $Confess$, but that would be arbitrary. The statement is really shorthand for "The payoff profile resulting from strategy profile x pareto-dominates the payoff profile resulting from strategy y."

Very few games have a dominant strategy equilibrium, but sometimes dominance can still be useful even when it does not resolve things quite so neatly as in the Prisoner's Dilemma. The Battle of the Bismarck Sea, a game I found in Haywood (1954), is set in the South Pacific in 1943. General Imamura has been ordered to transport Japanese troops across the Bismarck Sea to New Guinea, and General Kenney wants to bomb the troop transports. Imamura must choose between a shorter northern route or a longer southern route to New Guinea, and Kenney must decide where to send his planes to look for the Japanese. If Kenney sends his planes to the wrong route he can recall them, but the number of days of bombing is reduced.

The players are Kenney and Imamura, and they each have the same action set, $\{North, South\}$, but their payoffs, given by Table 3, are never the same. Imamura loses exactly what Kenney gains. Because of this special feature, the payoffs could be represented using just four numbers instead of eight, but listing all eight payoffs in Table 3 saves the reader a little thinking. The 2-by-2 form with just four entries is a **matrix game**, while the equivalent table with eight entries is a **bimatrix game.** Games can be represented as matrix or bimatrix games even if they have more than two moves, as long as the number of moves is finite.

Table 3: The Battle of the Bismarck Sea

			Imamura	
		North		South
	North	**2,-2**	$\leftrightarrow$	$2, -2$
Kenney		$\uparrow$		$\downarrow$
	South	$1, -1$	$\leftarrow$	$3, -3$

Payoffs to: (Kenney, Imamura)

Strictly speaking, neither player has a dominant strategy. Kenney would choose *North* if he thought Imamura would choose *North*, but *South* if he thought Imamura would choose *South.* Imamura would choose *North* if he thought Kenney would choose *South*, and he would be indifferent between actions if he thought Kenney would choose *North.* This is what the arrows are showing. But we can still find a plausible equilibrium, using the concept of "weak dominance".

Strategy s_i' is **weakly dominated** *if there exists some other strategy s_i'' for player i which is possibly better and never worse, yielding a higher payoff in some strategy profile and never yielding a lower payoff. Mathematically, s_i' is weakly dominated if there exists s_i'' such that*

$$\begin{array}{ll} \pi_i(s_i'', s_{-i}) \geq \pi_i(s_i', s_{-i}) & \forall s_{-i}, \text{ and} \\ \pi_i(s_i'', s_{-i}) > \pi_i(s_i', s_{-i}) & \text{for some } s_{-i}. \end{array} \tag{4}$$

One might define a **weak dominance equilibrium** as the strategy profile found by deleting all the weakly dominated strategies of each player. Eliminating weakly dominated strategies does not help much in the Battle of the Bismarck Sea, however. Imamura's strategy of *South* is weakly dominated by the strategy *North* because his payoff from *North* is never smaller than his payoff from *South*, and it is greater if Kenney picks *South*. For Kenney, however, neither strategy is even weakly dominated. The modeller must therefore go a step further, to the idea of the iterated dominance equilibrium.

An **iterated dominance equilibrium** *is a strategy profile found by deleting a weakly dominated strategy from the strategy set of one of the players, recalculating to find which remaining strategies are weakly dominated, deleting one of them, and continuing the process until only one strategy remains for each player.*

Applied to the Battle of the Bismarck Sea, this equilibrium concept implies that Kenney decides that Imamura will pick *North* because it is weakly dominant, so Kenney eliminates "Imamura chooses *South*" from consideration. Having deleted one column of Table 3, Kenney has a strongly dominant strategy: he chooses *North*, which achieves payoffs strictly greater than *South*. The strategy profile (*North, North*) is an iterated dominance equilibrium, and indeed (*North*, *North*) was the outcome in 1943.

It is interesting to consider modifying the order of play or the information structure in the Battle of the Bismarck Sea. If Kenney moved first, rather than simultaneously with Imamura, (*North*, *North*) would remain an equilibrium, but (*North*, *South*) would also become one. The payoffs would be the same for both equilibria, but the outcomes would be different.

If Imamura moved first, (*North*, *North*) would be the only equilibrium. What is important about a player moving first is that it gives the other player more information before he acts, not the literal timing of the moves. If Kenney has cracked the Japanese code and knows Imamura's plan, then it does not matter that the two players move literally simultaneously; it is better modelled as a sequential game. Whether Imamura literally moves first or whether his code is cracked, Kenney's information set becomes either {Imamura moved *North*} or {Imamura moved *South*} after Imamura's decision, so Kenney's equilibrium strategy is specified as (*North* if Imamura moved *North*, *South* if Imamura moved *South*).

Game theorists often differ in their terminology, and the terminology applied to the idea of eliminating dominated strategies is particularly diverse. The equilibrium concept used in the Battle of the Bismarck Sea might be called **iterated dominance equilibrium** or **iterated dominant strategy equilibrium**, or one might say that the game is **dominance solvable**, that it can be **solved by iterated dominance**, or that the equilibrium strategy profile is **serially undominated**. Sometimes the terms are used to mean deletion of strictly dominated strategies and sometimes to mean deletion of weakly dominated strategies.

The significant difference is between strong and weak dominance. Everyone agrees that no rational player would use a strictly dominated strategy, but it is harder to argue against weakly dominated strategies. In economic models, firms and individuals are often indifferent about their behavior in equilibrium. In standard models of perfect competition, firms earn zero profits but it is crucial that some firms be active in the market and some stay out and produce nothing. If a monopolist knows that customer Smith is willing to pay up to ten dollars for a widget, the monopolist will charge exactly ten dollars to Smith in equilibrium, which makes Smith indifferent about buying and not buying, yet there is no equilibrium unless Smith buys. It is impractical, therefore, to rule out equilibria in which a player is indifferent about his actions. This should be kept in mind later when we discuss the "open-set problem" in Section 4.3.

Another difficulty is multiple equilibria. The dominant strategy equilibrium of any game is unique if it exists. Each player has at most one strategy whose payoff in any strategy profile is strictly higher than the payoff from any other strategy, so only one strategy profile can be formed out of dominant strategies. A strong iterated dominance equilibrium is unique if it exists. A weak iterated dominance equilibrium may not be, because the order in which strategies are deleted can matter to the final solution. If all the weakly dominated strategies are eliminated simultaneously at each round of elimination, the resulting equilibrium is unique, if it exists, but possibly no strategy profile will remain.

Consider Table 4's Iteration Path Game. The strategy profile (r_1, c_1) and (r_1, c_3) are both iterated dominance equilibria, because each of those strategy profile can be found by iterated deletion. The deletion can proceed in the order (r_3, c_3, c_2, r_2) or in the order (r_2, c_2, c_1, r_3).

Table 4: The Iteration Path Game

		Column		
		c_1	c_2	c_3
	r_1	**2,12**	1,10	**1,12**
Row	r_2	0,12	0,10	0,11
	r_3	0,12	1,10	0,13

Payoffs to: (Row, Column)

Despite these problems, deletion of weakly dominated strategies is a useful tool, and it is part of more complicated equilibrium concepts such as Section 4.1's "subgame perfectness".

If we may return to the Battle of the Bismarck Sea, that game is special because the payoffs of the players always sum to zero. This feature is important enough to deserve a name.

A **zero-sum game** *is a game in which the sum of the payoffs of all the players is zero whatever strategies they choose. A game which is not zero-sum is* **nonzero-sum game** *or* **variable- sum.**

In a zero-sum game, what one player gains, another player must lose. The Battle of the Bismarck Sea is a zero-sum game, but the Prisoner's Dilemma and the Dry Cleaners Game are not, and there is no way that the payoffs in those games can be rescaled to make them zero-sum without changing the essential character of the games.

If a game is zero-sum the utilities of the players can be represented so as to sum to zero under any outcome. Since utility functions are to some extent arbitrary, the sum can also be represented to be non-zero even if the game is zero-sum. Often modellers will refer to a game as zero-sum even when the payoffs do not add up to zero, so long as the payoffs add up to some constant amount. The difference is a trivial normalization.

Although zero-sum games have fascinated game theorists for many years, they are uncommon in economics. One of the few examples is the bargaining game between two players who divide a surplus, but even this is often modelled nowadays as a nonzero-sum game in which the surplus shrinks as the players spend more time deciding how to divide it. In reality, even simple division of property can result in loss— just think of how much the lawyers take out when a divorcing couple bargain over dividing their possessions.

Although the 2-by-2 games in this chapter may seem facetious, they are simple enough for use in modelling economic situations. *The Battle of the Bismarck Sea*, for example, can be turned into a game of corporate strategy. Two firms, Kenney Company and Imamura Incorporated, are trying to maximize their shares of a market of constant size by choosing between the two product designs *North* and *South*. Kenney has a marketing advantage, and would like to compete head-to-head, while Imamura would rather carve out its own niche. The equilibrium is (*North*, *North*).

1.4 Nash Equilibrium: *Boxed Pigs*, the *Battle of the Sexes*, and *Ranked Coordination*

For the vast majority of games, which lack even iterated dominance equilibria, modellers use Nash equilibrium, the most important and widespread equilibrium concept. To introduce Nash equilibrium we will use the game Boxed Pigs from Baldwin & Meese (1979). Two pigs are put in a box with a special control panel at one end and a food dispenser at the other end. When a pig presses the panel, at a utility cost of 2 units, 10 units of food are dispensed at the dispenser. One pig is "dominant" (let us assume he is bigger), and if he gets to the dispenser first, the other pig will only get his leavings, worth 1 unit. If, instead, the small pig is at the dispenser first, he eats 4 units, and even if they arrive at the same time the small pig gets 3 units. Table 5 summarizes the payoffs for the strategies *Press* the panel and $Wait$ by the dispenser at the other end.

Table 5: Boxed Pigs

		Small Pig		
		Press		Wait
Big Pig	Press	$5, 1$	$\rightarrow$	$\boxed{\mathbf{4}}, \boxed{\mathbf{4}}$
		$\downarrow$		$\uparrow$
	Wait	$\boxed{9}, -1$	$\rightarrow$	$0, \boxed{0}$

Payoffs to: (Big Pig, Small Pig)

Boxed Pigs has no dominant strategy equilibrium, because what the big pig chooses depends on what he thinks the small pig will choose. If he believed that the small pig would press the panel, the big pig would wait by the dispenser, but if he believed that the small pig would wait, the big pig would press the panel. There does exist an iterated dominance equilibrium, $(Press, Wait)$, but we will use a different line of reasoning to justify that outcome: Nash equilibrium.

Nash equilibrium is the standard equilibrium concept in economics. It is less obviously correct than dominant strategy equilibrium but more often applicable. Nash equilibrium is so widely accepted that the reader can assume that if a model does not specify which equilibrium concept is being used it is Nash or some refinement of Nash.

The strategy profile s^* is a **Nash equilibrium** *if no player has incentive to deviate from his strategy given that the other players do not deviate. Formally,*

$$\forall i, \;\; \pi_i(s_i^*, s_{-i}^*) \geq \pi_i(s_i', s_{-i}^*), \;\; \forall s_i'. \tag{5}$$

The strategy profile $(Press, Wait)$ is a Nash equilibrium. The way to approach Nash equilibrium is to propose a strategy profile and test whether each player's strategy is a best response to the others' strategies. If the big pig picks $Press$, the small pig, who faces a choice between a payoff of 1 from pressing and 4 from waiting, is willing to wait. If the small pig picks $Wait$, the big pig, who has a choice between a payoff of 4 from pressing and 0 from waiting, is willing to press. This confirms that $(Press, Wait)$ is a Nash equilibrium, and in fact it is the unique Nash equilibrium.[5]

It is useful to draw arrows in the tables when trying to solve for the equilibrium, since the number of calculations is great enough to soak up quite a bit of mental RAM. Another solution tip, illustrated in Boxed Pigs, is to circle payoffs that dominate other payoffs (or

[5] This game, too, has its economic analog. If Bigpig, Inc. introduces granola bars, at considerable marketing expense in educating the public, then Smallpig Ltd. can imitate profitably without ruining Bigpig's sales completely. If Smallpig introduces them at the same expense, however, an imitating Bigpig would hog the market.

box, them, as is especially suitable here). Double arrows or dotted circles indicate weakly dominant payoffs. Any payoff profile in which every payoff is circled, or which has arrows pointing towards it from every direction, is a Nash equilibrium. I like using arrows better in 2-by-2 games, but circles are better for bigger games, since arrows become confusing when payoffs are not lined up in order of magnitude in the table (see Chapter 2's Table 2).

The pigs in this game have to be smarter than the players in the Prisoner's Dilemma. They have to realize that the only set of strategies supported by self-consistent beliefs is $(Press, Wait)$. The definition of Nash equilibrium lacks the "$\forall s_{-i}$" of dominant strategy equilibrium, so a Nash strategy need only be a best response to the other Nash strategies, not to all possible strategies. And although we talk of "best responses," the moves are actually simultaneous, so the players are predicting each others' moves. If the game were repeated or the players communicated, Nash equilibrium would be especially attractive, because it is even more compelling that beliefs should be consistent.

Like a dominant strategy equilibrium, a Nash equilibrium can be either weak or strong. The definition above is for a weak Nash equilibrium. To define strong Nash equilibrium, make the inequality strict; that is, require that no player be indifferent between his equilibrium strategy and some other strategy.

Every dominant strategy equilibrium is a Nash equilibrium, but not every Nash equilibrium is a dominant strategy equilibrium. If a strategy is dominant it is a best response to *any* strategies the other players pick, including their equilibrium strategies. If a strategy is part of a Nash equilibrium, it need only be a best response to the other players' *equilibrium* strategies.

The Modeller's Dilemma of Table 6 illustrates this feature of Nash equilibrium. The situation it models is the same as the Prisoner's Dilemma, with one major exception: although the police have enough evidence to arrest the prisoner's as the "probable cause" of the crime, they will not have enough evidence to convict them of even a minor offense if neither prisoner confesses. The northwest payoff profile becomes (0,0) instead of $(-1, -1)$.

Table 6: The Modeller's Dilemma

			Column	
		Deny		*Confess*
Row	*Deny*	**0**, **0**	↔	-10, 0
		↕		↓
	Confess	0,-10	→	$\boxed{\textbf{-8}}$,$\boxed{\textbf{-8}}$

Payoffs to: (Row, Column)

The Modeller's Dilemma does not have a dominant strategy equilibrium. It does have what might be called a weak dominant strategy equilibrium, because *Confess* is still a weakly dominant strategy for each player. Moreover, using this fact, it can be seen that *(Confess, Confess)* is an iterated dominance equilibrium, and it is a strong Nash equilibrium as well. So the case for *(Confess, Confess)* still being the equilibrium outcome seems very strong.

There is, however, another Nash equilibrium in the Modeller's Dilemma: *(Deny, Deny)*, which is a weak Nash equilibrium. This equilibrium is weak and the other Nash equilibrium is strong, but *(Deny, Deny)* has the advantage that its outcome is pareto-superior: (0, 0) is uniformly greater than $(-8, -8)$. This makes it difficult to know which behavior to predict.

The Modeller's Dilemma illustrates a common difficulty for modellers: what to predict when two Nash equilibria exist. The modeller could add more details to the rules of the game, or he could use an **equilibrium refinement**, adding conditions to the basic equilibrium concept until only one strategy profile satisfies the refined equilibrium concept. There is no single way to refine Nash equilibrium. The modeller might insist on a strong equilibrium, or rule out weakly dominated strategies, or use iterated dominance. All of these lead to *(Confess, Confess)* in the Modeller's Dilemma. Or he might rule out Nash equilibria that are pareto-dominated by other Nash equilibria, and end up with *(Deny, Deny)*. Neither approach is completely satisfactory. In particular, do not be misled into thinking that weak Nash equilibria are to be despised. Often, no Nash equilibrium at all will exist unless the players have the expectation that player B chooses X when he is indifferent between X and Y. It is not that we are picking the equilibrium in which it is assumed B does X when he is indifferent. Rather, we are finding the *only* set of consistent expectations about behavior. (You will read more about this in connection with the "open-set problem" of Section 4.2.)

The Battle of the Sexes

The third game we will use to illustrate Nash equilibrium is the Battle of the Sexes, a conflict between a man who wants to go to a prize fight and a woman who wants to go to a ballet. While selfish, they are deeply in love, and would, if necessary, sacrifice their preferences to be with each other. Less romantically, their payoffs are given by Table 7.

Table 7: The Battle of the Sexes [6]

		Woman		
		Prize Fight		*Ballet*
	Prize Fight	**2,1**	←	0, 0
Man		↑		↓
	Ballet	0, 0	→	**1,2**

Payoffs to: (Man, Woman)

[6] Political correctness has led to bowdlerized versions of this game being presented in many game theory books. This is the original, unexpurgated game.

The Battle of the Sexes does not have an iterated dominance equilibrium. It has two Nash equilibria, one of which is the strategy profile (*Prize Fight, Prize Fight*). Given that the man chooses *Prize Fight*, so does the woman; given that the woman chooses *Prize Fight*, so does the man. The strategy profile (*Ballet*, *Ballet*) is another Nash equilibrium by the same line of reasoning.

How do the players know which Nash equilibrium to choose? Going to the fight and going to the ballet are both Nash strategies, but for different equilibria. Nash equilibrium assumes correct and consistent beliefs. If they do not talk beforehand, the man might go to the ballet and the woman to the fight, each mistaken about the other's beliefs. But even if the players do not communicate, Nash equilibrium is sometimes justified by repetition of the game. If the couple do not talk, but repeat the game night after night, one may suppose that eventually they settle on one of the Nash equilibria.

Each of the Nash equilibria in the Battle of the Sexes is pareto-efficient; no other strategy profile increases the payoff of one player without decreasing that of the other. In many games the Nash equilibrium is not pareto-efficient: (*Confess*, *Confess*), for example, is the unique Nash equilibrium of the Prisoner's Dilemma, although its payoffs of $(-8, -8)$ are pareto- inferior to the $(-1, -1)$ generated by (*Deny*, *Deny*).

Who moves first is important in the Battle of the Sexes, unlike any of the three previous games we have looked at. If the man could buy the fight ticket in advance, his commitment would induce the woman to go to the fight. In many games, but not all, the player who moves first (which is equivalent to commitment) has a **first-mover advantage.**

The Battle of the Sexes has many economic applications. One is the choice of an industrywide standard when two firms have different preferences but both want a common standard to encourage consumers to buy the product. A second is to the choice of language used in a contract when two firms want to formalize a sales agreement but they prefer different terms. Both sides might, for example, want to add a "liquidated damages" clause which specifies damages for breach, rather than trust to the courts to estimate a number later, but one firm wants the value to be $10,000 and the other firm wants $12,000.

Coordination Games

Sometimes one can use the size of the payoffs to choose between Nash equilibria. In the following game, players Smith and Jones are trying to decide whether to design the computers they sell to use large or small floppy disks. Both players will sell more computers if their disk drives are compatible, as shown in Table 8.

Table 8: Ranked Coordination

$$
\begin{array}{llccc}
 & & & \textbf{Jones} & \\
 & & \textit{Large} & & \textit{Small} \\
 & \textit{Large} & \mathbf{2,2} & \leftarrow & -1,-1 \\
\textbf{Smith} & & \uparrow & & \downarrow \\
 & \textit{Small} & -1,-1 & \rightarrow & \mathbf{1,1}
\end{array}
$$

Payoffs to: (Smith, Jones)

The strategy profiles (*Large*, *Large*) and (*Small*, *Small*) are both Nash equilibria, but (*Large*, *Large*) pareto-dominates (*Small*, *Small*). Both players prefer (*Large*, *Large*), and most modellers would use the pareto- efficient equilibrium to predict the actual outcome. We could imagine that it arises from pre-game communication between Smith and Jones taking place outside of the specification of the model, but the interesting question is what happens if communication is impossible. Is the pareto-efficient equilibrium still more plausible? The question is really one of psychology rather than economics.

Ranked Coordination is one of a large class of games called **coordination games**, which share the common feature that the players need to coordinate on one of multiple Nash equilibria. Ranked Coordination has the additional feature that the equilibria can be pareto ranked. Section 3.2 will return to problems of coordination to discuss the concepts of "correlated strategies" and "cheap talk." These games are of obvious relevance to analyzing the setting of standards; see, e.g., Michael Katz & Carl Shapiro (1985) and Joseph Farrell & Garth Saloner (1985). They can be of great importance to the wealth of economies– just think of the advantages of standard weights and measures (or read Charles Kindleberger (1983) on their history). Note, however, that not all apparent situations of coordination on pareto-inferior equilibria turn out to be so. One oft-cited coordination problem is that of the QWERTY typewriter keyboard, developed in the 1870s when typing had to proceed slowly to avoid jamming. QWERTY became the standard, although it has been claimed that the faster speed possible with the Dvorak keyboard would amortize the cost of retraining full-time typists within ten days (David [1985]). Why large companies would not retrain their typists is difficult to explain under this story, and Liebowitz & Margolis (1990) show that economists have been too quick to accept claims that QWERTY is inefficient. English language spelling is a better example.

Table 9 shows another coordination game, Dangerous Coordination, which has the same equilibria as Ranked Coordination, but differs in the off-equilibrium payoffs. If an experiment were conducted in which students played Dangerous Coordination against each other, I would not be surprised if *(Small,Small)*, the pareto-dominated equilibrium, were the one that was played out. This is true even though *(Large, Large)* is still a Nash equilibrium; if Smith thinks that Jones will pick *Large*, Smith is quite willing to pick *Large* himself. The problem is that if the assumptions of the model are weakened, and Smith cannot trust Jones to be rational, well-informed about the payoffs of the game, and unconfused, then Smith will be reluctant to pick *Large* because his payoff if Jones picks *Small* is then -1,000. He would play it safe instead, picking *Small* and ensuring a payoff of at least -1. In reality, people do make mistakes, and with such an extreme difference in payoffs, even a small probability of a mistake is important, so *(Large, Large)* would be a bad prediction.

Table 9: Dangerous Coordination

		Jones		
		Large		*Small*
	Large	**2,2**	←	$-1000, -1$
Smith		↑		↓
	Small	$-1, -1$	→	**1,1**

Payoffs to: (Smith, Jones)

Games like Dangerous Coordination are a major concern in the 1988 book by Harsanyi and Selten, two of the giants in the field of game theory. I will not try to describe their approach here, except to say that it is different from my own. I do not consider the fact that one of the Nash equilibria of Dangerous Coordination is a bad prediction as a heavy blow against Nash equilibrium. The bad prediction is based on two things: using the Nash equilibrium concept, and using the game Dangerous Coordination. If Jones might be confused about the payoffs of the game, then the game actually being played out is not Dangerous Coordination, so it is not surprising that it gives poor predictions. The rules of the game ought to describe the probabilities that the players are confused, as well as the payoffs if they take particular actions. If confusion is an important feature of the situation, then the two-by-two game of Table 9 is the wrong model to use, and a more complicated game of incomplete information of the kind described in Chapter 2 is more appropriate. Again, as with the Prisoner's Dilemma, the modeller's first thought on finding that the model predicts an odd result should not be "Game theory is bunk," but the more modest "Maybe I'm not describing the situation correctly" (or even "Maybe I should not trust my 'common sense' about what will happen").

Nash equilibrium is more complicated but also more useful than it looks. Jumping ahead a bit, consider a game slightly more complex than the ones we have seen so far. Two firms are choosing outputs Q_1 and Q_2 simultaneously. The Nash equilibrium is a pair of numbers (Q_1^*, Q_2^*) such that neither firm would deviate unilaterally. This troubles the beginner, who says to himself,

> "Sure, Firm 1 will pick Q_1^* if it thinks Firm 2 will pick Q_2^*. But Firm 1 will realize that if it makes Q_1 bigger, then Firm 2 will react by making Q_2 smaller. So the situation is much more complicated, and (Q_1^*, Q_2^*) is not a Nash equilibrium. Or, if it is, Nash equilibrium is a bad equilibrium concept."

If there is a problem in this model, it is not Nash equilibrium but the model itself. Nash equilibrium makes perfect sense as a stable outcome in this model. The beginner's hypothetical is false because if Firm 1 chooses something other than Q_1^*, Firm 2 would not observe the deviation till it was too late to change Q_2– remember, this is a simultaneous move game. The beginner's worry is really about the rules of the game, not the equilibrium concept. He seems to prefer a game in which the firms move sequentially, or maybe a repeated version of the game. If Firm 1 moved first, and then Firm 2, then Firm 1's strategy would still be a single number, Q_1, but Firm 2's strategy– its action rule– would have to be a function, $Q_2(Q_1)$. A Nash equilibrium would then consist of an equilibrium number, Q_1^{**}, and an equilibrium function, $Q_2^{**}(Q_1)$. The two outputs actually chosen, Q_1^{**} and $Q_2^{**}(Q_1^{**})$, will be different from the Q_1^* and Q_2^* in the original game. And they should be different– the new model represents a very different real-world situation. Look ahead, and you will see that these are the Cournot and Stackelberg models of Chapter 3.

One lesson to draw from this is that it is essential to figure out the mathematical form the strategies take before trying to figure out the equilibrium. In the simultaneous move game, the strategy profile is a pair of non-negative numbers. In the sequential game, the strategy profile is one nonnegative number and one function defined over the nonnegative numbers. Students invariably make the mistake of specifying Firm 2's strategy as a number, not a function. This is a far more important point than any beginner realizes. Trust me– you're going to make this mistake sooner or later, so it's worth worrying about.

1.5 Focal Points

Schelling's book, *The Strategy of Conflict* (1960) is a classic in game theory, even though it contains no equations or Greek letters. Although it was published more than 40 years ago, it is surprisingly modern in spirit. Schelling is not a mathematician but a strategist, and he examines such things as threats, commitments, hostages, and delegation that we will examine in a more formal way in the remainder of this book. He is perhaps best known for his coordination games. Take a moment to decide on a strategy in each of the following games, adapted from Schelling, which you win by matching your response to those of as many of the other players as possible.

1 Circle one of the following numbers: 100, 14, 15, 16, 17, 18.

2 Circle one of the following numbers 7, 100, 13, 261, 99, 666.

3 Name Heads or Tails.

4 Name Tails or Heads.

5 You are to split a pie, and get nothing if your proportions add to more than 100 percent.

6 You are to meet somebody in New York City. When? Where?

Each of the games above has many Nash equilibria. In example (1), if each player thinks every other player will pick 14, he will too, and this is self-confirming; but the same is true if each player thinks every other player will pick 15. But to a greater or lesser extent they also have Nash equilibria that seem more likely. Certain of the strategy profiles are **focal points:** Nash equilibria which for psychological reasons are particularly compelling.

Formalizing what makes a strategy profile a focal point is hard and depends on the context. In example (1), 100 is a focal point, because it is a number clearly different from all the others, it is biggest, and it is first in the listing. In example (2), Schelling found 7 to be the most common strategy, but in a group of Satanists, 666 might be the focal point. In repeated games, focal points are often provided by past history. Examples (3) and (4) are identical except for the ordering of the choices, but that ordering might make a difference. In (5), if we split a pie once, we are likely to agree on 50:50. But if last year we split a pie in the ratio 60:40, that provides a focal point for this year. Example (6) is the most interesting of all. Schelling found surprising agreement in independent choices, but the place chosen depended on whether the players knew New York well or were unfamiliar with the city.

The **boundary** is a particular kind of focal point. If player Russia chooses the action of putting his troops anywhere from one inch to 100 miles away from the Chinese border, player China does not react. If he chooses to put troops from one inch to 100 miles *beyond* the border, China declares war. There is an arbitrary discontinuity in behavior at the boundary. Another example, quite vivid in its arbitrariness, is the rallying cry, "Fifty-Four Forty or Fight!," which refers to the geographic parallel claimed as the boundary by jingoist Americans in the Oregon dispute between Britain and the United States in the 1840s.[7]

Once the boundary is established it takes on additional significance because behavior with respect to the boundary conveys information. When Russia crosses an established boundary, that tells China that Russia intends to make a serious incursion further into China. Boundaries must be sharp and well known if they are not to be violated, and a large part of both law and diplomacy is devoted to clarifying them. Boundaries can also arise in business: two companies producing an unhealthful product might agree not to mention relative healthfulness in their advertising, but a boundary rule like "Mention unhealthfulness if you like, but don't stress it," would not work.

Mediation and **communication** are both important in the absence of a clear focal point. If players can communicate, they can tell each other what actions they will take, and sometimes, as in Ranked Coordination, this works, because they have no motive to lie. If the players cannot communicate, a mediator may be able to help by suggesting an equilibrium to all of them. They have no reason not to take the suggestion, and they would use the mediator even if his services were costly. Mediation in cases like this is as effective as arbitration, in which an outside party imposes a solution.

One disadvantage of focal points is that they lead to inflexibility. Suppose the pareto-superior equilibrium (*Large*, *Large*) were chosen as a focal point in Ranked Coordination, but the game was repeated over a long interval of time. The numbers in the payoff matrix

[7]The threat was not credible: that parallel is now deep in British Columbia.

might slowly change until (*Small, Small*) and (*Large, Large*) both had payoffs of, say, 1.6, and (*Small, Small*) started to dominate. When, if ever, would the equilibrium switch?

In Ranked Coordination, we would expect that after some time one firm would switch and the other would follow. If there were communication, the switch point would be at the payoff of 1.6. But what if the first firm to switch is penalized more? Such is the problem in oligopoly pricing. If costs rise, so should the monopoly price, but whichever firm raises its price first suffers a loss of market share.

NOTES

N1.2 Dominant Strategies: The Prisoner's Dilemma

- Many economists are reluctant to use the concept of cardinal utility (see Starmer [2000]), and even more reluctant to compare utility across individuals (see Cooter & Rappoport [1984]). Noncooperative game theory never requires interpersonal utility comparisons, and only ordinal utility is needed to find the equilibrium in the Prisoner's Dilemma. So long as each player's rank ordering of payoffs in different outcomes is preserved, the payoffs can be altered without changing the equilibrium. In general, the dominant strategy and pure strategy Nash equilibria of games depend only on the ordinal ranking of the payoffs, but the mixed strategy equilibria depend on the cardinal values. Compare Section 3.2's Chicken game with Sectio 5.6's Hawk-Dove.

- If we consider only the ordinal ranking of the payoffs in 2-by-2 games, there are 78 distinct games in which each player has strict preference ordering over the four outcomes and 726 distinct games if we allow ties in the payoffs. Rapoport, Guyer & Gordon's 1976 book, *The 2x2 Game*, contains an exhaustive description of the possible games.

- The Prisoner's Dilemma was so named by Albert Tucker in an unpublished paper, although the particular 2-by-2 matrix, discovered by Dresher and Flood, was already well known. Tucker was asked to give a talk on game theory to the psychology department at Stanford, and invented a story to go with the matrix, as recounted in Straffin (1980), pp. 101-18 of Poundstone (1992), and pp. 171-3 of Raiffa (1992).

- In the Prisoner's Dilemma the notation *cooperate* and *defect* is often used for the moves. This is bad notation, because it is easy to confuse with *cooperative* games and with *deviations*. It is also often called the Prisoners' Dilemma (rs', not r's) ; whether one looks at from the point of the individual or the group, the prisoners have a problem.

- The Prisoner's Dilemma is not always defined the same way. If we consider just ordinal payoffs, then the game in Table 10 is a Prisoner's Dilemma if $T(temptation) > R(revolt) > P(punishment) > S(Sucker)$, where the terms in parentheses are mnemonics. This is standard notation; see, for example, Rapoport, Guyer & Gordon (1976), p. 400. If the game is repeated, the cardinal values of the payoffs can be important. The requirement $2R > T + S > 2P$ should be added if the game is to be a standard Prisoner's Dilemma, in which $(Deny, Deny)$ and $(Confess, Confess)$ are the best and worst possible outcomes in terms of the sum of payoffs. Section 5.3 will show that an asymmetric game called the One-Sided Prisoner's Dilemma has properties similar to the standard Prisoner's Dilemma, but does not fit this definition.

  Sometimes the game in which $2R < T + S$ is also called a prisoner's dilemma, but in it the sum of the players' payoffs is maximized when one confesses and the other denies. If the game were repeated or the prisoners could use the correlated equilibria defined in Section 3.2, they would prefer taking turns being confessed against, which would make the game a coordination game similar to the Battle of the Sexes. David Shimko has suggested the name "Battle of the Prisoners" for this (or, perhaps, the "Sex Prisoners' Dilemma").

Table 10: A General Prisoner's Dilemma

		Column		
		Deny		*Confess*
	Deny	R,R	$\rightarrow$	S,T
Row		$\downarrow$		$\downarrow$
	Confess	T,S	$\rightarrow$	**P,P**

Payoffs to: (Row, Column)

- Herodotus (429 B.C., III-71) describes an early example of the reasoning in the Prisoner's Dilemma in a conspiracy against the Persian emperor. A group of nobles met and decided to overthrow the emperor, and it was proposed to adjourn till another meeting. One of them named Darius then spoke up and said that if they adjourned, he knew that one of them would go straight to the emperor and reveal the conspiracy, because if nobody else did, he would himself. Darius also suggested a solution— that they immediately go to the palace and kill the emperor.

  The conspiracy also illustrates a way out of coordination games. After killing the emperor, the nobles wished to select one of themselves as the new emperor. Rather than fight, they agreed to go to a certain hill at dawn, and whoever's horse neighed first would become emperor. Herodotus tells how Darius's groom manipulated this randomization scheme to make him the new emperor.

- Philosophers are intrigued by the Prisoner's Dilemma: see Campbell & Sowden (1985), a collection of articles on the Prisoner's Dilemma and the related Newcombe's paradox. Game theory has even been applied to theology: if one player is omniscient or omnipotent, what kind of equilibrium behavior can we expect? See Brams (1983).

N1.4 Nash Equilibrium: Boxed Pigs, the Battle of the Sexes, and Ranked Coordination

- I invented the payoffs for Boxed Pigs from the description of one of the experiments in Baldwin & Meese (1979). They do *not* think of this as an experiment in game theory, and they describe the result in terms of "reinforcement." The Battle of the Sexes is taken from p. 90 of Luce & Raiffa (1957). I have changed their payoffs of $(-1,-1)$ to $(-5,-5)$ to fit the story.

- Some people prefer the term "equilibrium point" to "Nash equilibrium," but the latter is more euphonious, since the discoverer's name is "Nash" and not "Mazurkiewicz."

- Bernheim (1984a) and Pearce (1984) use the idea of mutually consistent beliefs to arrive at a different equilibrium concept than Nash. They define a **rationalizable strategy** to be a strategy which is a best response for some set of rational beliefs in which a player believes that the other players choose their best responses. The difference from Nash is that not all players need have the same beliefs concerning which strategies will be chosen, nor need their beliefs be consistent.

  This idea is attractive in the context of Bertrand games (see Section 3.6). The Nash equilibrium in the Bertrand game is weakly dominated— by picking any other price above marginal cost, which yields the same profit of zero as does the equilibrium. Rationalizability rules that out.

- Jack Hirshleifer (1982) uses the name "the Tender Trap" for a game essentially the same as Ranked Coordination, and the name "the Assurance Game" has also been used for it.

- O. Henry's story,"The Gift of the Magi" is about a coordination game noteworthy for the reason communication is ruled out. A husband sells his watch to buy his wife combs for Christmas, while she sells her hair to buy him a watch fob. Communication would spoil the surprise, a worse outcome than discoordination.

- Macroeconomics has more game theory in it than is readily apparent. The macroeconomic concept of *rational expectations* faces the same problems of multiple equilibria and consistency of expectations as Nash equilibrium. Game theory is now often explicitly used in macroeconomics; see the books by Canzoneri & Henderson (1991) and Cooper (1999).

N1.5 Focal Points

- Besides his 1960 book, Schelling has written books on diplomacy (1966) and the oddities of aggregation (1978). Political scientists are now looking at the same issues more technically; see Brams & Kilgour (1988) and Ordeshook (1986). Riker (1986) and Muzzio's 1982 book, *Watergate Games* are absorbing examples of how game theory can be used to analyze specific historical episodes.

- In Chapter 12 of *The General Theory*, Keynes (1936) suggests that the stock market is a game with multiple equilibria, like a contest in which a newspaper publishes the faces of 20 girls, and contestants submit the name of the one they think most people would submit as the prettiest. When the focal point changes, big swings in predictions about beauty and value result.

- Not all of what we call boundaries have an arbitrary basis. If the Chinese cannot defend themselves as easily once the Russians cross the boundary at the Amur River, they have a clear reason to fight there.

- Crawford & Haller (1990) take a careful look at focalness in repeated coordination games by asking which equilibria are objectively different from other equilibria, and how a player can learn through repetition which equilibrium the other players intend to play. If on the first repetition the players choose strategies that are Nash with respect to each other, it seems focal for them to continue playing those strategies, but what happens if they begin in disagreement?

Problems

1.1. Nash and Iterated Dominance

(a) Show that every iterated dominance equilibrium s^* is Nash.

(b) Show by counterexample that not every Nash equilibrium can be generated by iterated dominance.

(c) Is every iterated dominance equilibrium made up of strategies that are not weakly dominated?

1.2. 2-by-2 Games
Find examples of 2-by-2 games with the following properties:

(a) No Nash equilibrium (you can ignore mixed strategies).

(b) No weakly pareto-dominant strategy profile.

(c) At least two Nash equilibria, including one equilibrium that pareto-dominates all other strategy profiles.

(d) At least three Nash equilibria.

1.3. Pareto Dominance (from notes by Jong-Shin Wei)

(a) If a strategy profile s^* is a dominant strategy equilibrium, does that mean it weakly pareto-dominates all other strategy profiles?

(b If a strategy profile s strongly pareto-dominates all other strategy profiles, does that mean it is a dominant strategy equilibrium?

(c) If s weakly pareto-dominates all other strategy profiles, then must it be a Nash equilibrium?

1.4. Discoordination
Suppose that a man and a woman each choose whether to go to a prize fight or a ballet. The man would rather go to the prize fight, and the woman to the ballet. What is more important to them, however, is that the man wants to show up to the same event as the woman, but the woman wants to avoid him.

(a) Construct a game matrix to illustrate this game, choosing numbers to fit the preferences described verbally.

(b) If the woman moves first, what will happen?

(c) Does the game have a first-mover advantage?

(d) Show that there is no Nash equilibrium if the players move simultaneously.

1.5. Drawing Outcome Matrices
It can be surprisingly difficult to look at a game using new notation. In this exercise, redraw the outcome matrix in a different form than in the main text. In each case, read the description of the game and draw the outcome matrix as instructed. You will learn more if you do this from the description, without looking at the conventional outcome matrix.

(a) The Battle of the Sexes (Table 7). Put (*Prize Fight, Prize Fight*) in the northwest corner, but make the woman the row player.

(b) The Prisoner's Dilemma (Table 2). Put (*Confess, Confess*) in the northwest corner.

(c) The Battle of the Sexes (Table 7). Make the man the row player, but put (*Ballet, Prize Fight*) in the northwest corner.

1.6. Finding Nash Equilibria
Find the Nash equilibria of the game illustrated in Table 11. Can any of them be reached by iterated dominance?

Table 11: An Abstract Game

		Column		
		Left	*Middle*	*Right*
	Up	10,10	$0,0$	$-1,15$
Row:	*Sideways*	$-12,1$	8, 8	$-1,-1$
	Down	15,1	8,-1	$0,0$

Payoffs to: (Row, Column).

1.7. Finding More Nash Equilibria
Find the Nash equilibria of the game illustrated in Table 12. Can any of them be reached by iterated dominance?

Table 12: Flavor and Texture

		Brydox	
		Flavor	*Texture*
Apex:	*Flavor*	-2,0	0,1
	Texture	-1,-1	0,-2

Payoffs to: (Apex, Brydox).

1.8. Which Game?
Table 13 is like the payoff matrix for what game that we have seen?
(a) a version of the Battle of the Sexes.

(b) a version of the Prisoner's Dilemma.
(c) a version of Pure Coordination.
(d) a version of the Legal Settlement Game.
(e) none of the above.

Table 13: Which Game?

		COL	
		A	B
ROW	A	3,3	0,1
	B	5,0	-1,-1

1.9. Choosing Computers
The problem of deciding whether to adopt IBM or HP computers by two offices in a company is most like which game that we have seen?

1.10. Finding Equilibria
Find the pure-strategy Nash equilibria of the game in Table 14.

1.11. Campaign Contributions
The large Wall Street investment banks have recently agreed not to make campaign contributions to state treasurers, which up till now has been a common practice. What was the game in the past, and why can the banks expect this agreement to hold fast?

1.12. Three-by-Three Equilibria
Identify any dominated strategies and any Nash equilibria in pure strategies in the game of Table 15.

Table 15: A Three-By-Three Game

			Column	
		Left	**Middle**	*Right*
	Up	**1,4**	$5,-1$	$0,1$
Row:	**Sideways**	$-1,0$	-2,-2	$-3,4$
	Down	$0,3$	$9,-1$	$5,0$

Payoffs to: (Row, Column).

1.13. A Sequential Prisoner's Dilemma
Suppose Row moves first, then Column, in the Prisoner's Dilemma. What are the possible actions? What are the possible strategies? Construct a normal form, showing the relationship between strategy profiles and payoffs.

Hint: The normal form is *not* a two-by-two matrix here.

2 Information

6 September 1999. . December 7, 2003. January 1, 2005. 25 March 2005. xxx Footnotes starting with xxx are the author's notes to himself. Comments are welcomed.

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2.1 The Strategic and Extensive Forms of a Game

If half of strategic thinking is predicting what the other player will do, the other half is figuring out what he knows. Most of the games in Chapter 1 assumed that the moves were simultaneous, so the players did not have a chance to learn each other's private information by observing each other. Information becomes central as soon as players move in sequence. The important difference, in fact, between simultaneous-move games and sequential-move games is that in sequential-move games the second player acquires the information on how the first player moved before he must make his own decision.

Section 2.1 shows how to use the strategic form and the extensive form to describe games with sequential moves. Section 2.2 shows how the extensive form, or game tree, can be used to describe the information available to a player at each point in the game. Section 2.3 classifies games based on the information structure. Section 2.4 shows how to redraw games with incomplete information so that they can be analyzed using the Harsanyi transformation, and derives Bayes's Rule for combining a player's prior beliefs with information which he acquires in the course of the game. Section 2.5 concludes the chapter with the Png Settlement Game, an example of a moderately complex sequential-move game.

The Strategic Form and the Outcome Matrix

Games with moves in sequence require more care in presentation than single-move games. In Section 1.4 we used the 2-by-2 form, which for the game Ranked Coordination is shown in Table 1.

Table 1: Ranked Coordination

		Jones		
		Large		*Small*
	Large	**2,2**	←	−1, −1
Smith		↑		↓
	Small	−1, −1	→	**1,1**

Payoffs to: (Smith, Jones)

Because strategies are the same as actions in Ranked Coordination and the outcomes are simple, the 2-by-2 form in Table 1 accomplishes two things: it relates strategy profiles to payoffs, and action profiles to outcomes. These two mappings are called the strategic form and the outcome matrix, and in more complicated games they are distinct from each other. The strategic form shows what payoffs result from each possible strategy profile, while the outcome matrix shows what outcome results from each possible action profile. The definitions below use n to denote the number of players, k the number of variables in the outcome vector, p the number of strategy profiles, and q the number of action profiles.

The **strategic form** *(or* **normal form***) consists of*
1 All possible strategy profiles $s^1, s^2, \ldots, s^p$.
2 Payoff functions mapping s^i *onto the payoff* n*-vector* π^i, $(i = 1, 2, \ldots, p)$.

The **outcome matrix** *consists of*
1 All possible action profiles $a^1, a^2, \ldots, a^q$.
2 Outcome functions mapping a^i *onto the outcome* k*-vector* z^i, $(i = 1, 2, \ldots, q)$.

Consider the following game based on Ranked Coordination, which we will call Follow-the-Leader I since we will create several variants of the game. The difference from Ranked Coordination is that Smith moves first, committing himself to a certain disk size no matter what size Jones chooses. The new game has an outcome matrix identical to Ranked Coordination, but its strategic form is different because Jones's strategies are no longer single actions. Jones's strategy set has four elements,

$$\left\{ \begin{array}{l} \text{(If Smith chose } \textit{Large}\text{, choose } \textit{Large}\text{; if Smith chose } \textit{Small}\text{, choose } \textit{Large}\text{),} \\ \text{(If Smith chose } \textit{Large}\text{, choose } \textit{Large}\text{; if Smith chose } \textit{Small}\text{, choose } \textit{Small}\text{),} \\ \text{(If Smith chose } \textit{Large}\text{, choose } \textit{Small}\text{; if Smith chose } \textit{Small}\text{, choose } \textit{Large}\text{),} \\ \text{(If Smith chose } \textit{Large}\text{, choose } \textit{Small}\text{; if Smith chose } \textit{Small}\text{, choose } \textit{Small}\text{)} \end{array} \right\}$$

which we will abbreviate as

$$\left\{ \begin{array}{l} (L|L,\ L|S), \\ (L|L,\ S|S), \\ (S|L,\ L|S), \\ (S|L,\ S|S) \end{array} \right\}$$

Follow-the-Leader I illustrates how adding a little complexity can make the strategic form too obscure to be very useful. The strategic form is shown in Table 2, with equilibria boldfaced and labelled E_1, E_2, and E_3.

		Jones			
		J_1	J_2	J_3	J_4
		$L\|L,\ L\|S$	$L\|L,\ S\|S$	$S\|L,\ L\|S$	$S\|L,\ S\|S$
Smith	S_1 : *Large*	$\boxed{\mathbf{2}}$, $\mathbf{2}$ (E_1)	$\boxed{\mathbf{2}}$, $\mathbf{2}$ (E_2)	$-1, -1$	$-1, -1$
	S_2 : *Small*	$-1, -1$	$1, 1$	$-1, -1$	$\boxed{\mathbf{1}}$, $\mathbf{1}$ (E_3)

Payoffs to: (Smith, Jones)

Table 2: Follow-the-Leader I

Equilibrium	**Strategies**	**Outcome**
E_1	{*Large, (L*|*L, L*|*S)*}	Both pick *Large*
E_2	{*Large, (L*|*L, S*|*S)*}	Both pick *Large*
E_3	{*Small,(S*|*L, S*|*S)*}	Both pick *Small*

Consider why E_1, E_2, and E_3 are Nash equilibria. In Equilibrium E_1, Jones will respond with *Large* regardless of what Smith does, so Smith quite happily chooses *Large.* Jones would be irrational to choose *Large* if Smith chose *Small* first, but that event never happens in equilibrium. In Equilibrium E_2, Jones will choose whatever Smith chose, so Smith chooses *Large* to make the payoff 2 instead of 1. In Equilibrium E_3, Smith chooses *Small* because he knows that Jones will respond with *Small* whatever he does, and Jones is willing to respond with *Small* because Smith chooses *Small* in equilibrium. Equilibria E_1 and E_3 are not completely sensible, because the choices $Large|Ssmall$ (as specified in E_1) and $Small|Large$ (as specified in E_3) would reduce Jones's payoff if the game ever reached a point where he had to actually play them. Except for a little discussion in connection with Figure 1, however, we will defer to Chapter 4 the discussion of how to redefine the equilibrium concept to rule them out.

The Order of Play

The "normal form" is rarely used in modelling games of any complexity. Already, in Section 1.1, we have seen an easier way to model a sequential game: the *order of play.* For it Follow the Leader I, this would be:

1 Smith chooses his disk size to be either $Large$ or $Small$.
2 Jones chooses his disk size to be either $Large$ or $Small$.

The reason I have retained the concept of the normal form in this edition is that it reinforces the idea of laying out all the possible strategies and comparing their payoffs. The order of play, however, gives us a better way to describe games, as I will explain next.

The Extensive Form and the Game Tree

Two other ways to describe a game are the extensive form and the game tree. First we need to define their building blocks. As you read the definitions, you may wish to refer to Figure 1 as an example.

A **node** *is a point in the game at which some player or Nature takes an action, or the game ends.*

A **successor** *to node X is a node that may occur later in the game if X has been reached.*

A **predecessor** *to node X is a node that must be reached before X can be reached.*

A **starting node** *is a node with no predecessors.*

An **end node** *or* **end point** *is a node with no successors.*

A **branch** *is one action in a player's action set at a particular node.*

A **path** *is a sequence of nodes and branches leading from the starting node to an end node.*

These concepts can be used to define the extensive form and the game tree.

The **extensive form** *is a description of a game consisting of*
1 A configuration of nodes and branches running without any closed loops from a single starting node to its end nodes.
2 An indication of which node belongs to which player.
3 The probabilities that Nature uses to choose different branches at its nodes.
4 The information sets into which each player's nodes are divided.
5 The payoffs for each player at each end node.

The **game tree** *is the same as the extensive form except that (5) is replaced with*
5′ The outcomes at each end node.

"Game tree" is a looser term than "extensive form." If the outcome is defined as the payoff profile, one payoff for each player, then the extensive form is the same as the game tree.

The extensive form for Follow-the-Leader I is shown in Figure 1. We can see why Equilibria E_1 and E_3 of Table 2 are unsatisfactory even though they are Nash equilibria. If the game actually reached nodes J_1 or J_2, Jones would have dominant actions, *Small* at J_1 and *Large* at J_2, but E_1 and E_3 specify other actions at those nodes. In Chapter 4 we will return to this game and show how the Nash concept can be refined to make E_2 the only equilibrium.

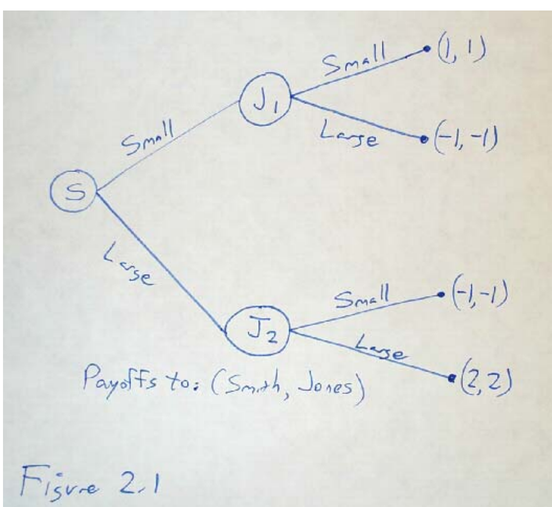


Figure 1: Follow-the-Leader I in Extensive Form

The extensive form for *Ranked Coordination*, shown in Figure 2, adds dotted lines to the extensive form for *Follow-the-Leader I*. Each player makes a single decision between two actions. The moves are simultaneous, which we show by letting Smith move first, but not letting Jones know how he moved. The dotted line shows that Jones's knowledge stays the same after Smith moves. All Jones knows is that the game has reached some node within the information set defined by the dotted line; he does not know the exact node reached.

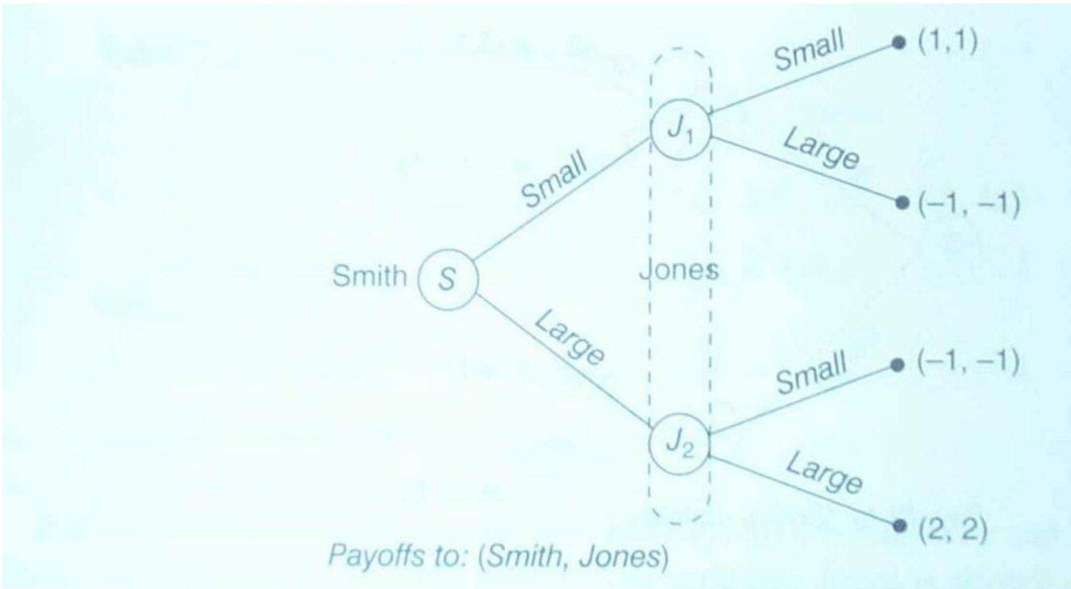


Figure 2: Ranked Coordination in Extensive Form

The Time Line

The **time line**, a line showing the order of events, is another way to describe games. Time lines are particularly useful for games with continuous strategies, exogenous arrival of information, and multiple periods, games that are frequently used in the accounting and finance literature. A typical time line is shown in Figure 3a, which represents a game that will be described in Section 11.5.

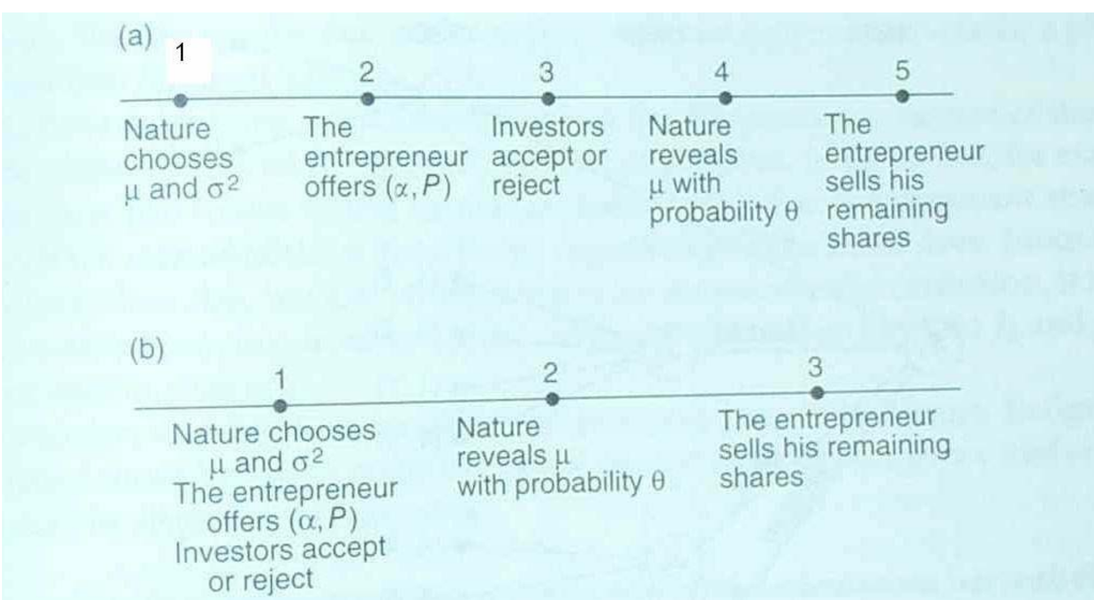


Figure 3: The Time Line for Stock Underpricing: (a) A Good Time Line; (b) A Bad Time Line

The time line illustrates the order of actions and events, not necessarily the passage of time. Certain events occur in an instant, others over an interval. In Figure 3a, events 2 and 3 occur immediately after event 1, but events 4 and 5 might occur ten years later. We sometimes refer to the sequence in which decisions are made as **decision time** and the interval over which physical actions are taken as **real time.** A major difference is that players put higher value on payments received earlier in real time because of time preference (on which see the appendix).

A common and bad modelling habit is to restrict the use of the dates on the time line to separating events in real time. Events 1 and 2 in Figure 2.3a are not separated by real time: as soon as the entrepreneur learns the project's value, he offers to sell stock. The modeller might foolishly decide to depict his model by a picture like Figure 3b in which both events happen at date 1. Figure 3b is badly drawn, because readers might wonder which event occurs first or whether they occur simultaneously. In more than one seminar, 20 minutes of heated and confusing debate could have been avoided by 10 seconds care to delineate the order of events.

2.2: Information Sets

A game's information structure, like the order of its moves, is often obscured in the strategic form. During the Watergate affair, Senator Baker became famous for the question "How much did the President know, and when did he know it?". In games, as in scandals, these are the big questions. To make this precise, however, requires technical definitions so that one can describe who knows what, and when. This is done using the "information set," the set of nodes a player thinks the game might have reached, as the basic unit of knowledge.

Player i's **information set** ω_i *at any particular point of the game is the set of different nodes in the game tree that he knows might be the actual node, but between which he cannot distinguish by direct observation.*

As defined here, the information set for player i is a set of nodes belonging to one player but on different paths. This captures the idea that player i knows whose turn it is to move, but not the exact location the game has reached in the game tree. Historically, player i's information set has been defined to include only nodes at which player i moves, which is appropriate for single-person decision theory, but leaves a player's knowledge undefined for most of any game with two or more players. The broader definition allows comparison of information across players, which under the older definition is a comparison of apples and oranges.

In the game in Figure 4, Smith moves at node S_1 in 1984 and Jones moves at nodes J_1, J_2, J_3, and J_4 in 1985 or 1986. Smith knows his own move, but Jones can tell only whether Smith has chosen the moves which lead to J_1, J_2, or "other"; he cannot distinguish between J_3 and J_4. If Smith has chosen the move leading to J_3, his own information set is simply $\{J_3\}$, but Jones's information set is $\{J_3, J_4\}$.

One way to show information sets on a diagram is to put dashed lines around or between nodes in the same information set. The resulting diagrams can be very cluttered, so it is often more convenient to draw dashed lines around the information set of just the player making the move at a node. The dashed lines in Figure 4 show that J_3 and J_4 are in the same information set for Jones, even though they are in different information sets for Smith. An expressive synonym for information set which is based on the appearance of these diagrams is "**cloud**": one would say that nodes J_3 and J_4 are in the same cloud, so that while Jones can tell that the game has reached that cloud, he cannot pierce the fog to tell exactly which node has been reached.

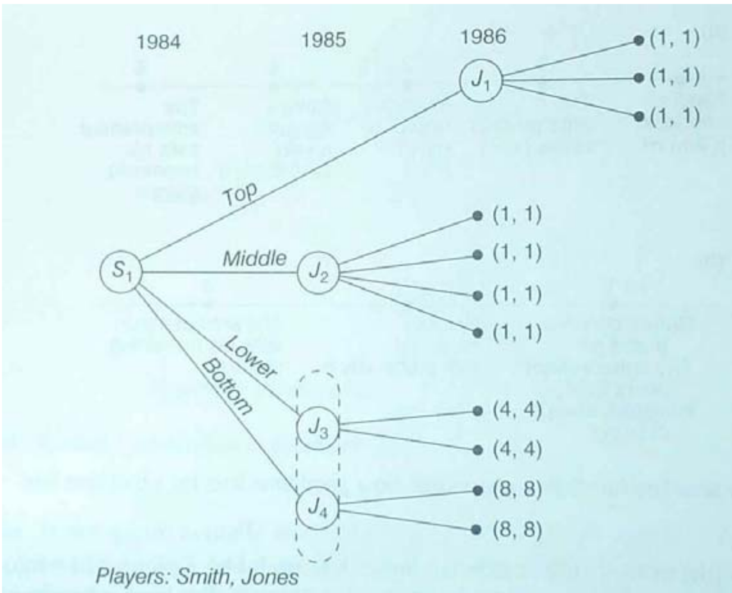


Figure 4: Information Sets and Information Partitions.

One node cannot belong to two different information sets of a single player. If node J_3 belonged to information sets $\{J_2,J_3\}$ and $\{J_3,J_4\}$ (unlike in Figure 4), then if the game reached J_3, Jones would not know whether he was at a node in $\{J_2, J_3\}$ or a node in $\{J_3, J_4\}$— which would imply that they were really the same information set.

If the nodes in one of Jones's information sets are nodes at which he moves, his action set must be the same at each node, because he knows his own action set (though his actions might differ later on in the game depending on whether he advances from J_3 or J_4). Jones has the same action sets at nodes J_3 and J_4, because if he had some different action available at J_3 he would know he was there and his information set would reduce to just $\{J_3\}$. For the same reason, nodes J_1 and J_2 could not be put in the same information set; Jones must know whether he has three or four moves in his action set. We also require end nodes to be in different information sets for a player if they yield him different payoffs.

With these exceptions, we do not include in the information structure of the game any information acquired by a player's rational deductions. In Figure 4, for example, it seems clear that Smith would choose *Bottom*, because that is a dominant strategy — his payoff is 8 instead of the 4 from *Lower*, regardless of what Jones does. Jones should be able to deduce this, but even though this is an uncontroversial deduction, it is none the less a deduction, not an observation, so the game tree does not split J_3 and J_4 into separate information sets.

Information sets also show the effects of unobserved moves by Nature. In Figure 4, if the initial move had been made by Nature instead of by Smith, Jones's information sets would be depicted the same way.

Player i's **information partition** *is a collection of his information sets such that*
1 Each path is represented by one node in a single information set in the partition, and
2 The predecessors of all nodes in a single information set are in one information set.

The information partition represents the different positions that the player knows he will be able to distinguish from each other at a given stage of the game, carving up the set of all possible nodes into the subsets called information sets. One of Smith's information partitions is $(\{J_1\},\{J_2\},\{J_3\},\{J_4\})$. The definition rules out information set $\{S_1\}$ being in that partition, because the path going through S_1 and J_1 would be represented by two nodes. Instead, $\{S_1\}$ is a separate information partition, all by itself. The information partition refers to a stage of the game, not chronological time. The information partition $(\{J_1\},\{J_2\},\{J_3,J_4\})$ includes nodes in both 1985 and 1986, but they are all immediate successors of node S_1.

Jones has the information partition $(\{J_1\},\{J_2\},\{J_3,J_4\})$. There are two ways to see that his information is worse than Smith's. First is the fact that one of his information *sets*, $\{J_3,J_4\}$, contains *more* elements than Smith's, and second, that one of his information *partitions*, $(\{J_1\},\{J_2\},\{J_3,J_4\})$, contains *fewer* elements.

Table 3 shows a number of different information partitions for this game. Partition I is Smith's partition and partition II is Jones's partition. We say that partition II is **coarser**, and partition I is **finer**. A profile of two or more of the information sets in a partition, which reduces the number of information sets and increases the numbers of nodes in one or more of them is a **coarsening**. A splitting of one or more of the information sets in a partition, which increases the number of information sets and reduces the number of nodes in one or more of them, is a **refinement**. Partition II is thus a coarsening of partition I, and partition I is a refinement of partition II. The ultimate refinement is for each information set to be a **singleton**, containing one node, as in the case of partition I. As in bridge, having a singleton can either help or hurt a player. The ultimate coarsening is for a player not to be able to distinguish between any of the nodes, which is partition III in Table 3.[1]

A finer information partition is the formal definition for "better information." Not all information partitions are refinements or coarsenings of each other, however, so not all information partitions can be ranked by the quality of their information. In particular, just because one information partition contains more information sets does not mean it is a refinement of another information partition. Consider partitions II and IV in Figure 3. Partition II separates the nodes into three information sets, while partition IV separates them into just two information sets. Partition IV is not a coarsening of partition II,

[1]Note, however, that partitions III and IV are not really allowed in this game, because Jones could tell the node from the actions available to him, as explained earlier.

however, because it cannot be reached by combining information sets from partition II, and one cannot say that a player with partition IV has worse information. If the node reached is J_1, partition II gives more precise information, but if the node reached is J_4, partition IV gives more precise information.

Nodes	I	II	III	IV
J_1	$\{J_1\}$	$\{J_1\}$	$\{J_1, J_2, J_3, J_4\}$	$\{J_1, J_2, J_3\}$
J_2	$\{J_2\}$	$\{J_2\}$		
J_3	$\{J_3\}$	$\{J_3, J_4\}$		
J_4	$\{J_4\}$			$\{J_4\}$

Table 3: Information Partitions

Information quality is defined independently of its utility to the player: it is possible for a player's information to improve and for his equilibrium payoff to fall as a result. Game theory has many paradoxical models in which a player prefers having worse information, not a result of wishful thinking, escapism, or blissful ignorance, but of cold rationality. Coarse information can have a number of advantages. (a) It may permit a player to engage in trade because other players do not fear his superior information. (b) It may give a player a stronger strategic position because he usually has a strong position and is better off not knowing that in a particular realization of the game his position is weak. Or, (c) as in the more traditional economics of uncertainty, poor information may permit players to insure each other.

I will wait till later chapters to discuss points (a) and (b), the strategic advantages of poor information (go to Section 6.3 on entry deterrence and Chapter 9 on used cars if you feel impatient), but it is worth pausing here to think about point (c), the insurance advantage. Consider the following example, which will illustrate that even when information is symmetric and behavior is nonstrategic, better information in the sense of a finer information partition, can actually reduce everybody's utility.

Suppose Smith and Jones, both risk averse, work for the same employer, and both know that one of them chosen randomly will be fired at the end of the year while the other will be promoted. The one who is fired will end with a wealth of 0 and the one who is promoted will end with 100. The two workers will agree to insure each other by pooling their wealth: they will agree that whoever is promoted will pay 50 to whoever is fired. Each would then end up with a guaranteed utility of U(50). If a helpful outsider offers to tell them who will be fired before they make their insurance agreement, they should cover their ears and refuse to listen. Such a refinement of their information would make both worse off, in expectation, because it would wreck the possibility of the two of them agreeing on an insurance arrangement. It would wreck the possibility because if they knew who would be promoted, the lucky worker would refuse to pool with the unlucky one. Each worker's expected utility with no insurance but with someone telling them what will happen is .5 *U(0) + .5* U(100), which is less than 1.0*U(50) if they are risk averse. They would prefer not to know, because better information would reduce the expected utility of both of them.

Common Knowledge

We have been implicitly assuming that the players know what the game tree looks like. In fact, we have assumed that the players also know that the other players know what the game tree looks like. The term "common knowledge" is used to avoid spelling out the infinite recursion to which this leads.

Information is **common knowledge** *if it is known to all the players, if each player knows that all the players know it, if each player knows that all the players know that all the players know it, and so forth ad infinitum.*

Because of this recursion (the importance of which will be seen in Section 6.3), the assumption of common knowledge is stronger than the assumption that players have the same beliefs about where they are in the game tree. Hirshleifer & Riley (1992, p. 169) use the term **concordant beliefs** to describe a situation where players share the same belief about the probabilities that Nature has chosen different states of the world, but where they do not necessarily know they share the same beliefs. (Brandenburger [1992] uses the term **mutual knowledge** for the same idea.)

For clarity, models are set up so that information partitions are common knowledge. Every player knows how precise the other players' information is, however ignorant he himself may be as to which node the game has reached. Modelled this way, the information partitions are independent of the equilibrium concept. Making the information partitions common knowledge is important for clear modelling, and restricts the kinds of games that can be modelled less than one might think. This will be illustrated in Section 2.4 when the assumption will be imposed on a situation in which one player does not even know which of three games he is playing.

2.3 Perfect, Certain, Symmetric, and Complete Information

We categorize the information structure of a game in four different ways, so a particular game might have perfect, complete, certain, and symmetric information. The categories are summarized in Table 4.

Information category	Meaning
Perfect	Each information set is a singleton
Certain	Nature does not move after any player moves
Symmetric	No player has information different from other players when he moves, or at the end nodes
Complete	Nature does not move first, or her initial move is observed by every player

Table 4: Information Categories

The first category divides games into those with perfect and those with imperfect information.

In a game of **perfect information** *each information set is a singleton. Otherwise the game is one of* **imperfect information**.

The strongest informational requirements are met by a game of perfect information, because in such a game each player always knows exactly where he is in the game tree. No moves are simultaneous, and all players observe Nature's moves. Ranked Coordination is a game of imperfect information because of its simultaneous moves, but Follow-the-Leader I is a game of perfect information. Any game of incomplete or asymmetric information is also a game of imperfect information.

A game of **certainty** *has no moves by Nature after any player moves. Otherwise the game is one of* **uncertainty.**

The moves by Nature in a game of uncertainty may or may not be revealed to the players immediately. A game of certainty can be a game of perfect information if it has no simultaneous moves. The notion "game of uncertainty" is new with this book, but I doubt it would surprise anyone. The only quirk in the definition is that it allows an initial move by Nature in a game of certainty, because in a game of incomplete information Nature moves first to select a player's "type." Most modellers do not think of this situation as uncertainty.

We have already talked about information in Ranked Coordination, a game of imperfect, complete, and symmetric information with certainty. The Prisoner's Dilemma falls into the same categories. Follow-the-Leader I, which does not have simultaneous moves, is a game of perfect, complete, and symmetric information with certainty.

We can easily modify Follow-the-Leader I to add uncertainty, creating the game Follow-the-Leader II (Figure 5). Imagine that if both players pick *Large* for their disks, the market yields either zero profits or very high profits, depending on the state of demand, but demand would not affect the payoffs in any other strategy profile. We can quantify this by saying that if (*Large*, *Large*) is picked, the payoffs are (10,10) with probability 0.2, and (0,0) with probability 0.8, as shown in Figure 5.

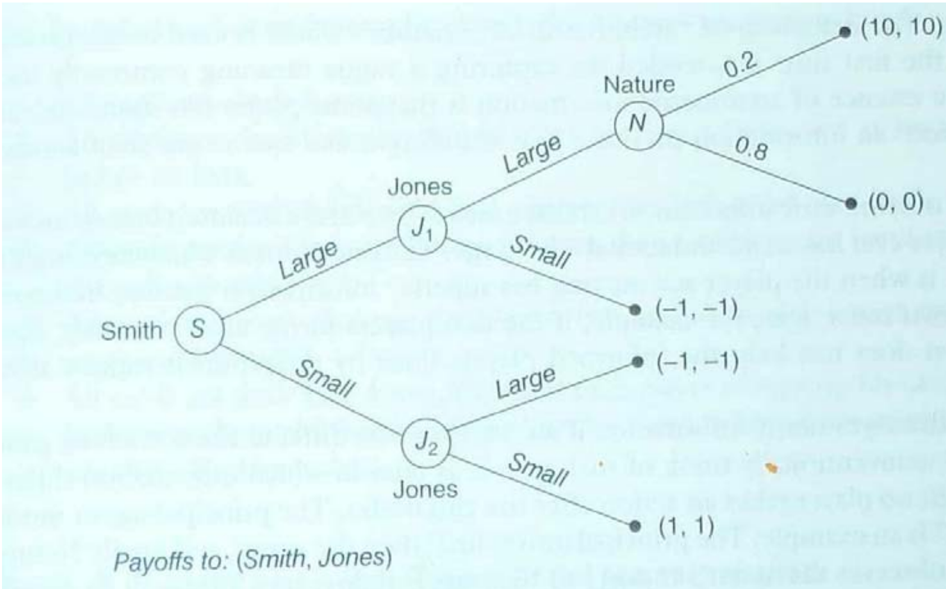


Figure 5: Follow-the-Leader II

When players face uncertainty, we need to specify how they evaluate their uncertain future payoffs. The obvious way to model their behavior is to say that the players maximize the expected values of their utilities. Players who behave in this way are said to have **von Neumann-Morgenstern utility functions,** a name chosen to underscore von Neumann & Morgenstern's (1944) development of a rigorous justification of such behavior.

Maximizing their expected utilities, the players would behave exactly the same as in Follow-the-Leader I. Often, a game of uncertainty can be transformed into a game of certainty without changing the equilibrium, by eliminating Nature's moves and changing the payoffs to their expected values based on the probabilities of Nature's moves. Here we could eliminate Nature's move and replace the payoffs 10 and 0 with the single payoff 2 ($= 0.2[10] + 0.8[0]$). This cannot be done, however, if the actions available to a player depend on Nature's moves, or if information about Nature's move is asymmetric.

The players in Figure 5 might be either risk averse or risk neutral. Risk aversion is implicitly incorporated in the payoffs because they are in units of utility, not dollars. When players maximize their expected utility, they are not necessarily maximizing their expected dollars. Moreover, the players can differ in how they map money to utility. It could be that (0,0) represents ($0, $5,000), (10,10) represents ($100,000, $100,000), and (2,2), the expected utility, could here represent a non-risky ($3,000, $7,000).

In a game of **symmetric information**, *a player's information set at*
1 any node where he chooses an action, or
2 an end node
contains at least the same elements as the information sets of every other player. Otherwise the game is one of **asymmetric information.**

In a game of asymmetric information, the information sets of players differ in ways relevant to their behavior, or differ at the end of the game. Such games have imperfect information, since information sets which differ across players cannot be singletons. The definition of "asymmetric information" which is used in the present book for the first time is intended for capturing a vague meaning commonly used today. The essence of asymmetric information is that some player has useful **private information**: an information partition that is different and not worse than another player's.

A game of symmetric information can have moves by Nature or simultaneous moves, but no player ever has an informational advantage. The one point at which information may differ is when the player *not* moving has superior information because he knows what his own move *was*; for example, if the two players move simultaneously. Such information does not help the informed player, since by definition it cannot affect his move.

A game has asymmetric information if information sets differ at the end of the game because we conventionally think of such games as ones in which information differs, even though no player takes an action after the end nodes. The principal-agent model of Chapter 7 is an example. The principal moves first, then the agent, and finally Nature. The agent observes the agent's move, but the principal does not, although he may be able to deduce it. This would be a game of symmetric information except for the fact that information continues to differ at the end nodes.

In a game of **incomplete information***, Nature moves first and is unobserved by at least one of the players. Otherwise the game is one of* **complete information.**

A game with incomplete information also has imperfect information, because some player's information set includes more than one node. Two kinds of games have complete but imperfect information: games with simultaneous moves, and games where, late in the game, Nature makes moves not immediately revealed to all players.

Many games of incomplete information are games of asymmetric information, but the two concepts are not equivalent. If there is no initial move by Nature, but Smith takes a move unobserved by Jones, and Smith moves again later in the game, the game has asymmetric but complete information. The principal-agent games of Chapter 7 are again examples: the agent knows how hard he worked, but his principal never learns, not even at the end nodes. A game can also have incomplete but symmetric information: let Nature, unobserved by either player, move first and choose the payoffs for $(Confess, Confess)$ in the Prisoner's Dilemma to be either $(-6, -6)$ or $(-100, -100)$.

Harris & Holmstrom (1982) have a more interesting example of incomplete but symmetric information: Nature assigns different abilities to workers, but when workers are young their ability is known neither to employers nor to themselves. As time passes, the abilities become common knowledge, and if workers are risk averse and employers are risk neutral, the model shows that equilibrium wages are constant or rising over time.

Poker Examples of Information Classification

In the game of poker, the players make bets on who will have the best hand of cards at the end, where a ranking of hands has been pre-established. How would the following rules for behavior before betting be classified? (Answers are in note N2.3)

1. All cards are dealt face up.

2. All cards are dealt face down, and a player cannot look even at his own cards before he bets.

3. All cards are dealt face down, and a player can look at his own cards.

4. All cards are dealt face up, but each player then scoops up his hand and secretly discards one card.

5. All cards are dealt face up, the players bet, and then each player receives one more card face up.

6. All cards are dealt face down, but then each player scoops up his cards without looking at them and holds them against his forehead so all the *other* players can see them (Indian poker).

2.4 The Harsanyi Transformation and Bayesian Games

The Harsanyi Transformation: Follow-the-Leader III

The term "incomplete information" is used in two quite different senses in the literature, usually without explicit definition. The definition in Section 2.3 is what economists commonly *use*, but if asked to *define* the term, they might come up with the following, older, definition.

Old definition
In a game of **complete information***, all players know the rules of the game. Otherwise the game is one of* **incomplete information.**

The old definition is not meaningful, since the game itself is ill defined if it does not specify exactly what the players' information sets are. Until 1967, game theorists spoke of games of incomplete information to say that they could not be analyzed. Then John Harsanyi pointed out that any game that had incomplete information under the old definition could be remodelled as a game of complete but imperfect information without changing its essentials, simply by adding an initial move in which Nature chooses between different sets of rules. In the transformed game, all players know the new meta-rules, including the fact that Nature has made an initial move unobserved by them. Harsanyi's suggestion trivialized the definition of incomplete information, and people began using the term to refer to the transformed game instead. Under the old definition, a game of incomplete information was transformed into a game of complete information. Under the new definition, the original game is ill defined, and the transformed version is a game of incomplete information.

Follow-the-Leader III serves to illustrate the Harsanyi transformation. Suppose that Jones does not know the game's payoffs precisely. He does have some idea of the payoffs, and we represent his beliefs by a subjective probability distribution. He places a 70 percent probability on the game being game (A) in Figure 6 (which is the same as Follow-the-Leader I), a 10 percent chance on game (B), and a 20 percent on game (C). In reality the game has a particular set of payoffs, and Smith knows what they are. This is a game of incomplete information (Jones does not know the payoffs), asymmetric information (when Smith moves, Smith knows something Jones does not), and certainty (Nature does not move after the players do.)

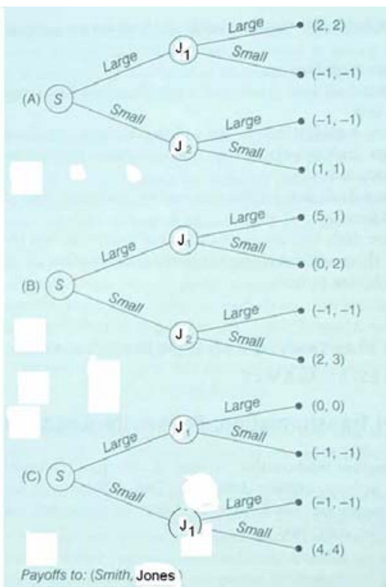


Figure 6: Follow-the-Leader III: Original

The game cannot be analyzed in the form shown in Figure 6. The natural way to approach such a game is to use the Harsanyi transformation. We can remodel the game to look like Figure 7, in which Nature makes the first move and chooses the payoffs of game (A), (B), or (C), in accordance with Jones's subjective probabilities. Smith observes Nature's move, but Jones does not. Figure 7 depicts the same game as Figure 6, but now we can analyze it. Both Smith and Jones know the rules of the game, and the difference between them is that Smith has observed Nature's move. Whether Nature actually makes the moves with the indicated probabilities or Jones just imagines them is irrelevant, so long as Jones's initial beliefs or fantasies are common knowledge.

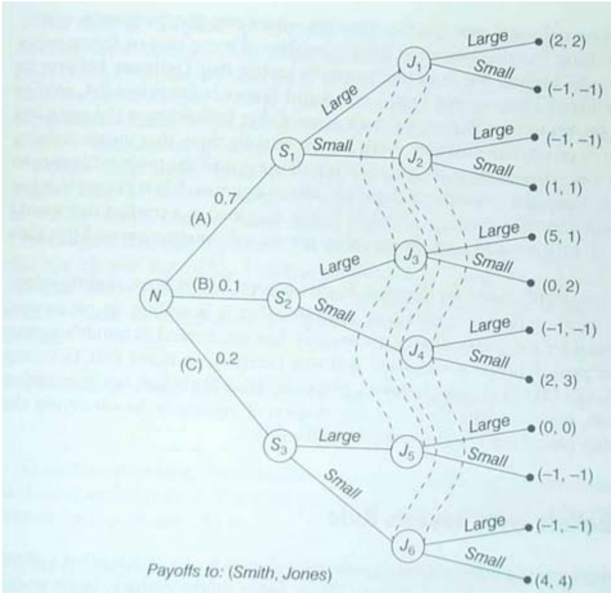


Figure 7: Follow-the-Leader III: After the Harsanyi Transformation

Often what Nature chooses at the start of a game is the strategy set, information partition, and payoff function of one of the players. We say that the player can be any of several "types," a term to which we will return in later chapters. When Nature moves, especially if she affects the strategy sets and payoffs of both players, it is often said that Nature has chosen a particular "state of the world." In Figure 7 Nature chooses the state of the world to be (A), (B), or (C).

A player's **type** *is the strategy set, information partition, and payoff function which Nature chooses for him at the start of a game of incomplete information.*

A **state of the world** *is a move by Nature.*

As I have already said, it is good modelling practice to assume that the structure of the game is common knowledge, so that though Nature's choice of Smith's type may really just represent Jones's opinions about Smith's possible type, Smith knows what Jones's possible opinions are and Jones knows that they are just opinions. The players may have different beliefs, but that is modelled as the effect of their observing different moves by Nature. All players begin the game with the same beliefs about the probabilities of the moves Nature will make— the same priors, to use a term that will shortly be introduced. This modelling assumption is known as the **Harsanyi doctrine**. If the modeller is following it, his model can never reach a situation where two players possess exactly the same information but disagree as to the probability of some past or future move of Nature. A model cannot, for example, begin by saying that Germany believes its probability of winning a war against France is 0.8 and France believes it is 0.4, so they are both willing to go to war. Rather, he must assume that beliefs begin the same but diverge because of private information. Both players initially think that the probability of a German victory is 0.4 but that if General Schmidt is a genius the probability rises to 0.8, and then Germany discovers that Schmidt is indeed a genius. If it is France that has the initiative to declare war, France's mistaken beliefs may lead to a conflict that would be avoidable if Germany could credibly reveal its private information about Schmidt's genius.

An implication of the Harsanyi doctrine is that players are at least slightly open-minded about their opinions. If Germany indicates that it is willing to go to war, France must consider the possibility that Germany has discovered Schmidt's genius and update the probability that Germany will win (keeping in mind that Germany might be bluffing). Our next topic is how a player updates his beliefs upon receiving new information, whether it be by direct observation of Nature or by observing the moves of another player who might be better informed.

Updating Beliefs with Bayes's Rule

When we classify a game's information structure we do not try to decide what a player can deduce from the other players' moves. Player Jones might deduce, upon seeing Smith choose *Large,* that Nature has chosen state (A), but we do not draw Jones's information set in Figure 7 to take this into account. In drawing the game tree we want to illustrate only the exogenous elements of the game, uncontaminated by the equilibrium concept. But to find the equilibrium we do need to think about how beliefs change over the course of the game.

One part of the rules of the game is the collection of **prior beliefs** (or **priors**) held by the different players, beliefs that they update in the course of the game. A player holds prior beliefs concerning the types of the other players, and as he sees them take actions he updates his beliefs under the assumption that they are following equilibrium behavior.

The term **Bayesian equilibrium** is used to refer to a Nash equilibrium in which players update their beliefs according to Bayes's Rule. Since Bayes's Rule is the natural and standard way to handle imperfect information, the adjective, "Bayesian," is really optional. But the two-step procedure of checking a Nash equilibrium has now become a three-step procedure:

1 Propose a strategy profile.
2 See what beliefs the strategy profile generates when players update their beliefs in response to each others' moves.
3 Check that given those beliefs together with the strategies of the other players each player is choosing a best response for himself.

The rules of the game specify each player's initial beliefs, and Bayes's Rule is the rational way to update beliefs. Suppose, for example, that Jones starts with a particular prior belief, $Prob(Nature\ chose\ (A))$. In Follow-the-Leader III, this equals 0.7. He then observes Smith's move — *Large*, perhaps. Seeing *Large* should make Jones update to the **posterior** belief, $Prob(Nature\ chose\ (A))|Smith\ chose\ Large)$, where the symbol "|" denotes "conditional upon" or "given that."

Bayes's Rule shows how to revise the prior belief in the light of new information such as Smith's move. It uses two pieces of information, the likelihood of seeing Smith choose *Large* given that Nature chose state of the world (A), $Prob(Large|(A))$, and the likelihood of seeing Smith choose *Large* given that Nature did not choose state (A), $Prob(Large|(B)\ or\ (C))$. From these numbers, Jones can calculate $Prob(Smith\ chooses\ Large)$, the **marginal likelihood** of seeing *Large* as the result of one or another of the possible states of the world that Nature might choose.

$$\begin{aligned} Prob(Smith\ chooses\ Large) &= Prob(Large|A)Prob(A) + Prob(Large|B)Prob(B) \\ &\quad + Prob(Large|C)Prob(C). \end{aligned} \tag{1}$$

To find his posterior, $Prob(Nature\ chose\ (A))|Smith\ chose\ Large)$, Jones uses the likelihood and his priors. The joint probability of both seeing Smith choose *Large* and Nature having chosen (A) is

$$Prob(Large, A) = Prob(A|Large)Prob(Large) = Prob(Large|A)Prob(A). \tag{2}$$

Since what Jones is trying to calculate is $Prob(A|Large)$, rewrite the last part of (2) as follows:

$$Prob(A|Large) = \frac{Prob(Large|A)Prob(A)}{Prob(Large)}. \tag{3}$$

Jones needs to calculate his new belief — his posterior — using $Prob(Large)$, which he calculates from his original knowledge using (1). Substituting the expression for $Prob(Large)$ from (1) into equation (3) gives the final result, a version of Bayes's Rule.

$$Prob(A|Large) = \frac{Prob(Large|A)Prob(A)}{Prob(Large|A)Prob(A) + Prob(Large|B)Prob(B) + Prob(Large|C)Prob(C)}. \tag{4}$$

More generally, for Nature's move x and the observed data,

$$Prob(x|data) = \frac{Prob(data|x)Prob(x)}{Prob(data)} \tag{5}$$

Equation (6) is a verbal form of Bayes's Rule, which is useful for remembering the terminology, summarized in Table 5.

$$(Posterior\ for\ Nature's\ \ Move) = \frac{(Likelihood\ of\ Player's\ \ Move) \cdot (Prior\ for\ Nature's\ \ Move)}{(Marginal\ likelihood\ of\ Player's\ \ Move)}. \tag{6}$$

Bayes's Rule is not purely mechanical. It is the only way to rationally update beliefs. The derivation is worth understanding, because Bayes's Rule is hard to memorize but easy to rederive.

Table 5: Bayesian terminology

Name	**Meaning**
Likelihood	$Prob(data|event)$
Marginal likelihood	$Prob(data)$
Conditional Likelihood	$Prob(data\ \mathrm{X}|data\ \mathrm{Y},\ event)$
Prior	$Prob(event)$
Posterior	$Prob(event|data)$

Updating Beliefs in Follow-the-Leader III

Let us now return to the numbers in Follow-the-Leader III to use the belief- updating rule that was just derived. Jones has a prior belief that the probability of event "Nature picks state (A)" is 0.7 and he needs to update that belief on seeing the data "Smith picks *Large*". His prior is $Prob(A) = 0.7$, and we wish to calculate $Prob(A|Large)$.

To use Bayes's Rule from equation (4), we need the values of $Prob(Large|A)$, $Prob(Large|B)$, and $Prob(Large|C)$. These values depend on what Smith does in equilibrium, so Jones's beliefs cannot be calculated independently of the equilibrium. This is the reason for the three-step procedure suggested above, for what the modeller must do is propose an equilibrium and then use it to calculate the beliefs. Afterwards, he must check that the equilibrium strategies are indeed the best responses given the beliefs they generate.

A candidate for equilibrium in Follow-the-Leader III is for Smith to choose *Large* if the state is (A) or (B) and *Small* if it is (C), and for Jones to respond to *Large* with *Large* and to *Small* with *Small*. This can be abbreviated as $(L|A, L|B, S|C; L|L, S|S)$. Let us test that this is an equilibrium, starting with the calculation of $Prob(A|Large)$.

If Jones observes *Large*, he can rule out state (C), but he does not know whether the state is (A) or (B). Bayes's Rule tells him that the posterior probability of state (A) is

$$\begin{aligned} Prob(A|Large) &= \frac{(1)(0.7)}{(1)(0.7)+(1)(0.1)+(0)(0.2)} \\ &= 0.875. \end{aligned} \tag{7}$$

The posterior probability of state (B) must then be $1 - 0.875 = 0.125$, which could also be calculated from Bayes's Rule, as follows:

$$\begin{aligned} Prob(B|Large) &= \frac{(1)(0.1)}{(1)(0.7)+(1)(0.1)+(0)(0.2)} \\ &= 0.125. \end{aligned} \tag{8}$$

Figure 8 shows a graphic intuition for Bayes's Rule. The first line shows the total probability, 1, which is the sum of the prior probabilities of states (A), (B), and (C). The second line shows the probabilities, summing to 0.8, which remain after *Large* is observed and state (C) is ruled out. The third line shows that state (A) represents an amount 0.7 of that probability, a fraction of 0.875. The fourth line shows that state (B) represents an amount 0.1 of that probability, a fraction of 0.125.

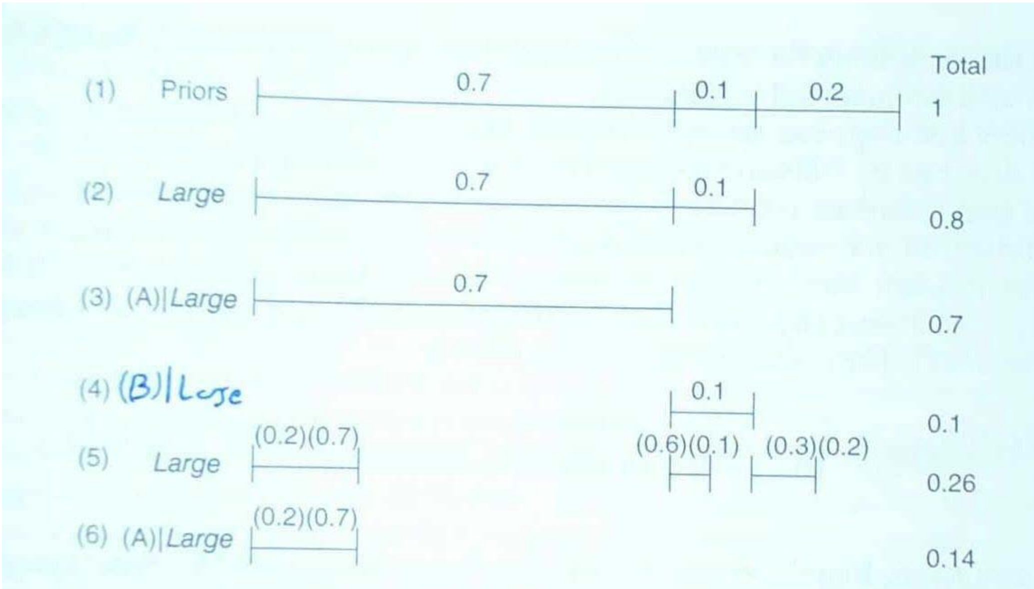


Figure 8: Bayes's Rule

Jones must use Smith's strategy in the proposed equilibrium to find numbers for $Prob(Large|A)$, $Prob(Large|B)$, and $Prob(Large|C)$. As always in Nash equilibrium, the modeller assumes that the players know which equilibrium strategies are being played out, even though they do not know which particular actions are being chosen.

Given that Jones believes that the state is (A) with probability 0.875 and state (B) with probability 0.125, his best response is *Large*, even though he knows that if the state were actually (B) the better response would be *Small*. Given that he observes *Large*, Jones's expected payoff from *Small* is -0.625 ($= 0.875[-1] + 0.125[2]$), but from *Large* it is 1.875 ($= 0.875[2]+0.125[1]$). The strategy profile $(L|A, L|B, S|C; L|L, S|S)$ is a Bayesian equilibrium.

A similar calculation can be done for $Prob(A|Small)$. Using Bayes's Rule, equation (4) becomes

$$Prob(A|Small) = \frac{(0)(0.7)}{(0)(0.7) + (0)(0.1) + (1)(0.2)} = 0. \tag{9}$$

Given that he believes the state is (C), Jones's best response to *Small* is *Small*, which agrees with our proposed equilibrium.

Smith's best responses are much simpler. Given that Jones will imitate his action, Smith does best by following his equilibrium strategy of $(L|A, L|B, S|C)$.

The calculations are relatively simple because Smith uses a nonrandom strategy in equilibrium, so, for instance, $Prob(Small|A) = 0$ in equation (9). Consider what happens if Smith uses a random strategy of picking *Large* with probability 0.2 in state (A), 0.6 in state (B), and 0.3 in state (C) (we will analyze such "mixed" strategies in Chapter 3). The equivalent of equation (7) is

$$Prob(A|Large) = \frac{(0.2)(0.7)}{(0.2)(0.7) + (0.6)(0.1) + (0.3)(0.2)} = 0.54 \ \ (rounded). \tag{10}$$

If he sees *Large*, Jones's best guess is still that Nature chose state (A), even though in state (A) Smith has the smallest probability of choosing *Large*, but Jones's subjective posterior probability, $Pr(A|Large)$, has fallen to 0.54 from his prior of $Pr(A) = 0.7$.

The last two lines of Figure 8 illustrate this case. The second-to- last line shows the total probability of *Large*, which is formed from the probabilities in all three states and sums to 0.26 ($=0.14 + 0.06 + 0.06$). The last line shows the component of that probability arising from state (A), which is the amount 0.14 and fraction 0.54 (rounded).

Regression to the Mean

Regression to the mean is an old statistical idea that has a Bayesian interpretation. Suppose that each student's performance on a test results partly from his ability and partly from random error because of his mood the day of the test. The teacher does not know the individual student's ability, but does know that the average student will score 70 out of 100. If a student scores 40, what should the teacher's estimate of his ability be?

It should not be 40. A score of 30 points below the average score could be the result of two things: (1) the student's ability is below average, or (2) the student was in a bad mood the day of the test. Only if mood is completely unimportant should the teacher use 40 as his estimate. More likely, both ability and luck matter to some extent, so the teacher's best guess is that the student has an ability below average but was also unlucky. The best estimate lies somewhere between 40 and 70, reflecting the influence of both ability and luck. Of the students who score 40 on the test, more than half can be expected to score above 40 on the next test. Since the scores of these poorly performing students tend to float up towards the mean of 70, this phenomenon is called "regression to the mean." Similarly, students who score 90 on the first test will tend to score less well on the second test.

This is "regression to the mean" ("towards" would be more precise) not "regression beyond the mean." A low score does indicate low ability, on average, so the predicted score on the second test is still below average. Regression to the mean merely recognizes that both luck and ability are at work.

In Bayesian terms, the teacher in this example has a prior mean of 70, and is trying to form a posterior estimate using the prior and one piece of data, the score on the first test. For typical distributions, the posterior mean will lie between the prior mean and the data point, so the posterior mean will be between 40 and 70.

In a business context, regression to the mean can be used to explain business conservatism. It is sometimes claimed that businesses pass up profitable investments because they have an excessive fear of risk. Let us suppose that the business is risk neutral, because the risk associated with the project and the uncertainty over its value are nonsystematic — that is, they are risks that a widely held corporation can distribute in such a way that each shareholder's risk is trivial. Suppose that the firm will not spend $100,000 on an investment with a present value of $105,000. This is easily explained if the $105,000 is an estimate and the $100,000 is cash. If the average value of a new project of this kind is less than $100,000 — as is likely to be the case since profitable projects are not easy to find — the best estimate of the value will lie between the measured value of $105,000 and that average value, unless the staffer who came up with the \$105,000 figure has already adjusted his estimate. Regressing the \$105,000 to the mean may regress it past \$100,000. Put a bit differently, if the prior mean is, let us say, \$80,000, and the data point is \$105,000, the posterior may well be less than \$100,000.

It is important to keep regression to the mean in mind as an alternative to strategic behavior in explaining odd phenomena. In analyzing test scores, one might try to explain the rise in the scores of poor students by changes in their effort level in an attempt to achieve a target grade in the course with minimum work. In analyzing business decisions, one might try to explain why apparently profitable projects are rejected because of managers' dislike for innovations that would require them to work harder. These explanations might well be valid, but models based on Bayesian updating or regression to the mean might explain the situation just as well and with fewer hard-to- verify assumptions about the utility functions of the individuals involved.

2.5: An Example: The Png Settlement Game

The Png (1983) model of out-of-court settlement is an example of a game with a fairly complicated extensive form.[2] The plaintiff alleges that the defendant was negligent in providing safety equipment at a chemical plant, a charge which is true with probability q. The plaintiff files suit, but the case is not decided immediately. In the meantime, the defendant and the plaintiff can settle out of court.

What are the moves in this game? It is really made up of two games: the one in which the defendant is liable for damages, and the one in which he is blameless. We therefore start the game tree with a move by Nature, who makes the defendant either liable or blameless. At the next node, the plaintiff takes an action: *Sue* or *Grumble.* If he decides on *Grumble* the game ends with zero payoffs for both players. If he decides to *Sue*, we go to the next node. The defendant then decides whether to *Resist* or *Offer* to settle. If the defendant chooses *Offer*, then the plaintiff can *Settle* or *Refuse*; if the defendant chooses to *Resist*, the plaintiff can *Try* the case or *Drop* it. The following description adds payoffs to this model.

The Png Settlement Game

Players
The plaintiff and the defendant.

The Order of Play
0 Nature chooses the defendant to be Liable for injury to the plaintiff with probability $q = 0.13$ and Blameless otherwise. The defendant observes this but the plaintiff does not.
1 The plaintiff decides to Sue or just to Grumble.
2 The defendant Offers a settlement amount of $S = 0.15$ to the plaintiff, or Resist, setting $S = 0$.
3a If the defendant offered $S = 0.15$, the plaintiff agrees to Settle or he Refuses and goes to trial.

[2] "Png," by the way, is pronounced the same way it is spelt.

3b If the defendant offered $S = 0$, the plaintiff Drops the case, for legal costs of $P = 0$ and $D = 0$ for himself and the defendant, or chooses to Try it, creating legal costs of $P = 0.1$ and $D = 0.2$
4 If the case goes to trial, the plaintiff wins damages of $W = 1$ if the defendant is Liable and $W = 0$ if the defendant is Blameless. If the case is dropped, $W = 0$.

Payoffs
The plaintiff's payoff is $S + W - P$. The defendant's payoff is $-S - W - D$.

We can also depict this on a game tree, as in Figure 9.

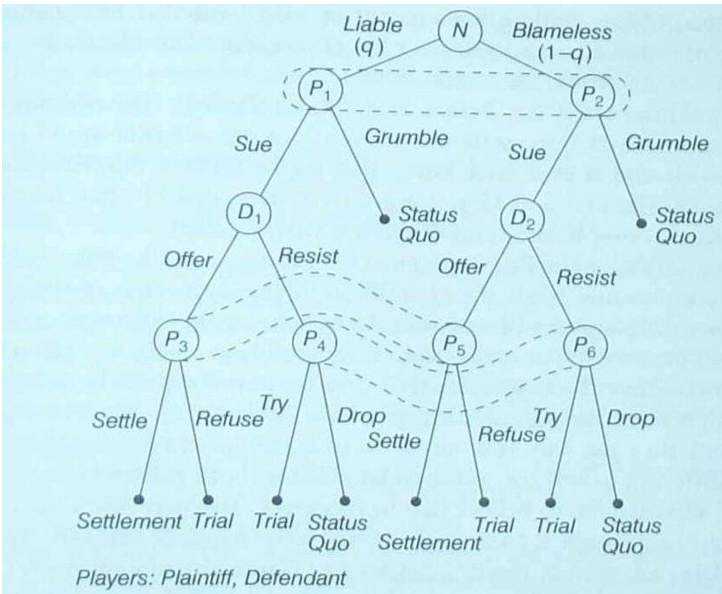


Figure 9: The Game Tree for the Png Settlement Game

This model assumes that the settlement amount, $S = 0.15$, and the amounts spent on legal fees are exogenous. Except in the infinitely long games without end nodes that will appear in Chapter 5, an extensive form should incorporate all costs and benefits into the payoffs at the end nodes, even if costs are incurred along the way. If the court required a \$100 filing fee (which it does not in this game, although a fee will be required in the similar game of Nuisance Suits in Section 4.3), it would be subtracted from the plaintiff's payoffs at every end node except those resulting from his choice of *Grumble*. Such consolidation makes it easier to analyze the game and would not affect the equilibrium strategies unless payments along the way revealed information, in which case what matters is the information, not the fact that payoffs change.

We assume that if the case reaches the court, justice is done. In addition to his legal fees D, the defendant pays damages $W = 1$ only if he is liable. We also assume that the players are risk neutral, so they only care about the expected dollars they will receive, not the variance. Without this assumption we would have to translate the dollar payoffs into utility, but the game tree would be unaffected.

This is a game of certain, asymmetric, imperfect, and incomplete information. We have assumed that the defendant knows whether he is liable, but we could modify the game by assuming that he has no better idea than the plaintiff of whether the evidence is sufficient to prove him so. The game would become one of symmetric information and we could reasonably simplify the extensive form by eliminating the initial move by Nature and setting the payoffs equal to the expected values. We cannot perform this simplification in the original game, because the fact that the defendant, and only the defendant, knows whether he is liable strongly affects the behavior of both players.

Let us now find the equilibrium. Using dominance we can rule out one of the plaintiff's strategies immediately — *Grumble* — which is dominated by (*Sue, Settle, Drop*).

Whether a strategy profile is a Nash equilibrium depends on the parameters of the model—S, W, P, D and q, which are the settlement amount, the damages, the court costs for the plaintiff and defendant, and the probability the defendant is liable. Depending on the parameter values, three outcomes are possible: settlement (if the settlement amount is low), trial (if expected damages are high and the plaintiff's court costs are low), and the plaintiff dropping the action (if expected damages minus court costs are negative). Here, I have inserted the parameter values $S = 0.15, D = 0.2, W = 1, q = 0.13$, and $P = 0.1$. Two Nash equilibria exist for this set of parameter values, both weak.

One equilibrium is the strategy profile $\{$(*Sue, Settle, Try*), (*Offer, Offer*)$\}$. The plaintiff sues, the defendant offers to settle (whether liable or not), and the plaintiff agrees to settle. Both players know that if the defendant did not offer to settle, the plaintiff would go to court and try the case. Such **out-of-equilibrium** behavior is specified by the equilibrium, because the threat of trial is what induces the defendant to offer to settle, even though trials never occur in equilibrium. This is a Nash equilibrium because given that the plaintiff chooses (*Sue, Settle, Try*), the defendant can do no better than (*Offer, Offer*), settling for a payoff of -0.15 whether he is liable or not; and, given that the defendant chooses (*Offer, Offer*), the plaintiff can do no better than the payoff of 0.15 from (*Sue, Settle, Try*).

The other equilibrium is $\{$(*Sue, Refuse, Try*), (*Resist, Resist*)$\}$. The plaintiff sues, the defendant resists and makes no settlement offer, the plaintiff would refuse any offer that was made, and goes to trial. Given that the he foresees that the plaintiff will refuse a settlement offer of $S = 0.15$ and will go to trial no matter what, the defendant is willing to resist because it makes no difference what he does.

One final observation on the Png Settlement Game: the game illustrates the Harsanyi doctrine in action, because while the plaintiff and defendant differ in their beliefs as to the probability the plaintiff will win, they do so because the defendant has different information, not because the modeller assigns them different beliefs at the start of the game. This seems awkward compared to the everday way of approaching this problem in which we simply note that potential litigants have different beliefs, and will go to trial if they both think they can win. It is very hard to make the story consistent, however, because if the differing beliefs are common knowledge, both players know that one of them is wrong, and each has to believe that he is correct. This may be fine as a "reduced form," in which the attempt is to simply describe what happens without explaining it in any depth. After all, even in The Png Settlement Game, if a trial occurs it is because the players differ in their beliefs, so one could simply chop off the first part of the game tree. But that is also the problem with violating the Harsanyi doctrine: one cannot analyze how the players react to each other's moves if the modeller simply assigns them inflexible beliefs. In the Png Settlement Game, a settlement is rejected and a trial can occur under certain parameters because the plaintiff weighs the probability that the defendant knows he will win versus the probablility that he is bluffing, and sometimes decides to risk a trial. Without the Harsanyi doctrine it is very hard to evaluate such an explanation for trials.

NOTES

N2.1 The strategic and extensive forms of a game

- The term "outcome matrix" is used in Shubik (1982, p. 70), but never formally defined there.
- The term "node" is sometimes defined to include only points at which a player or Nature makes a decision, which excludes the end points.

N2.2 Information Sets

- If you wish to depict a situation in which a player does not know whether the game has reached node A_1 or A_2 and he has different action sets at the two nodes, restructure the game. If you wish to say that he has action set (X,Y,Z) at A_1 and (X,Y) at A_2, first add action Z to the information set at A_2. Then specify that at A_2, action Z simply leads to a new node, A_3, at which the choice is between X and Y.
- The term "common knowledge" comes from Lewis (1969). Discussions include Brandenburger (1992) and Geanakoplos (1992). For rigorous but nonintuitive definitions of common knowledge, see Aumann (1976) and Milgrom (1981a). Following Milgrom, let (Ω, p) be a probability space, let P and Q be partitions of Ω representing the information of two agents, and let R be the finest common coarsening of P and Q. Let ω be an event (an item of information) and $R(\omega)$ be that element of R which contains ω.

  An event A is **common knowledge** *at ω if $R(\omega) \subset A$.*

N2.3 Perfect, Certain, Symmetric, and Complete Information

- Tirole (1988, p. 431) (and more precisely Fudenberg & Tirole [1991a, p. 82]) have defined games of *almost perfect* information. They use this term to refer to repeated simultaneous-move games (of the kind studied here in Chapter 5) in which at each repetition all players know the results of all the moves, including those of Nature, in previous repetitions. It is a pity they use such a general-sounding term to describe so narrow a class of games; it could be usefully extended to cover all games which have perfect information except for simultaneous moves.
- **Poker Classifications.** (1) Perfect, certain. (2) Incomplete, symmetric, certain. (3) Incomplete, asymmetric, certain. (4) Complete, asymmetric, certain. (5) Perfect, uncertain. (6) Incomplete, asymmetric, certain.
- For an explanation of von Neumann-Morgenstern utility, see Varian (1992, chapter 11) or Kreps (1990a, Chapter 3). For other approaches to utility, see Starmer (2000). Expected utility and Bayesian updating are the two foundations of standard game theory, partly because they seem realistic and partly because they are so simple to use. Sometimes they do not explain people's behavior well, however, and there exist extensive literatures (a) pointing out anomalies, and (b) suggesting alternatives. So far no alternatives have proven to be big enough improvements to justify replacing the standard techniques, given the tradeoff between descriptive realism and added complexity in modelling. The standard response is to admit and ignore the anomalies in theoretical work, and to not press any

theoretical models too hard in situations where the anomalies are likely to make a significant difference. On anomalies, see Kahneman, Slovic & Tversky (1982) (an edited collection); Thaler (1992) (essays from his *Journal of Economic Perspectives* feature); and Dawes (1988) (a good mix of psychology and business).

- Mixed strategies (to be described in Section 3.1) are allowed in a game of perfect information because they are an aspect of the game's equilibrium, not of its exogenous structure.

- Although the word "perfect," appears in both "perfect information" (Section 2.3) and "perfect equilibrium" (Section 4.1), the concepts are unrelated.

- An unobserved move by Nature in a game of symmetric information can be represented in any of three ways: (1) as the last move in the game; (2) as the first move in the game; or (3) by replacing the payoffs with the expected payoffs and not using any explicit moves by Nature.

N2.4 The Harsanyi Transformation and Bayesian Games

- Mertens & Zamir (1985) probes the mathematical foundations of the Harsanyi transformation. The transformation requires the extensive form to be common knowledge, which raises subtle questions of recursion.

- A player always has some idea of what the payoffs are, so we can always assign him a subjective probability for each possible payoff. What would happen if he had no idea? Such a question is meaningless, because people always have some notion, and when they say they do not, they generally mean that their prior probabilities are low but positive for a great many possibilities. You, for instance, probably have as little idea as I do of how many cups of coffee I have consumed in my lifetime, but you would admit it to be a nonnegative number less than 3,000,000, and you could make a much more precise guess than that. On the topic of subjective probability, the classic reference is Savage (1954).

- The term "marginal likelihood" is confusing for economists, since it refers to an unconditional likelihood. Statisticians came up with it because they start with $Prob(a, b)$ and then move to $Prob(a)$. That is like going to the margin of a graph— the a-axis– and asking how probable each value of a is.

- If two players have common priors and their information partitions are finite, but they each have private information, iterated communication between them will lead to the adoption of a common posterior. This posterior is not always the posterior they would reach if they directly pooled their information, but it is almost always that posterior (Geanakoplos & Polemarchakis [1982]).

- For formal analysis of regression to the mean and business conservatism, see Rasmusen (1992b). This can also explain why even after discounting revenues further in the future, businesses favor projects that offer quicker returns, if more distant revenue forecasts are less accurate.

Problems

2.1. The Monty Hall Problem

You are a contestant on the TV show, "Let's Make a Deal." You face three curtains, labelled A, B and C. Behind two of them are toasters, and behind the third is a Mazda Miata car. You choose A, and the TV showmaster says, pulling curtain B aside to reveal a toaster, "You're lucky you didn't choose B, but before I show you what is behind the other two curtains, would you like to change from curtain A to curtain C?" Should you switch? What is the exact probability that curtain C hides the Miata?

2.2. Elmer's Apple Pie

Mrs Jones has made an apple pie for her son, Elmer, and she is trying to figure out whether the pie tasted divine, or merely good. Her pies turn out divinely a third of the time. Elmer might be ravenous, or merely hungry, and he will eat either 2, 3, or 4 pieces of pie. Mrs Jones knows he is ravenous half the time (but not which half). If the pie is divine, then, if Elmer is hungry, the probabilities of the three consumptions are (0, 0.6, 0.4), but if he is ravenous the probabilities are (0, 0, 1). If the pie is just good, then the probabilities are (0.2, 0.4, 0.4) if he is hungry and (0.1, 0.3, 0.6) if he is ravenous.

Elmer is a sensitive, but useless, boy. He will always say that the pie is divine and his appetite weak, regardless of his true inner feelings.

a What is the probability that he will eat four pieces of pie?

b If Mrs Jones sees Elmer eat four pieces of pie, what is the probability that he is ravenous and the pie is merely good?

c If Mrs Jones sees Elmer eat four pieces of pie, what is the probability that the pie is divine?

2.3. Cancer Tests (adapted from McMillan [1992:211])

Imagine that you are being tested for cancer, using a test that is 98 percent accurate. If you indeed have cancer, the test shows positive (indicating cancer) 98 percent of the time. If you do not have cancer, it shows negative 98 percent of the time. You have heard that 1 in 20 people in the population actually have cancer. Now your doctor tells you that you tested positive, but you shouldn't worry because his last 19 patients all died. How worried should you be? What is the probability you have cancer?

2.4. The Battleship Problem (adapted from Barry Nalebuff, "Puzzles," *Journal of Economic Perspectives*,2:181-82 [Fall 1988])

The Pentagon has the choice of building one battleship or two cruisers. One battleship costs the same as two cruisers, but a cruiser is sufficient to carry out the navy's mission — if the cruiser survives to get close enough to the target. The battleship has a probability of p of carrying out its mission, whereas a cruiser only has probability $p/2$. Whatever the outcome, the war ends and any surviving ships are scrapped. Which option is superior?

2.5. Joint Ventures

Software Inc. and Hardware Inc. have formed a joint venture. Each can exert either high or low effort, which is equivalent to costs of 20 and 0. Hardware moves first, but Software cannot observe his effort. Revenues are split equally at the end, and the two firms are risk neutral. If both firms exert low effort, total revenues are 100. If the parts are defective, the total revenue is 100; otherwise, if both exert high effort, revenue is 200, but if only one player does, revenue is 100 with probability 0.9 and 200 with probability 0.1. Before they start, both players believe that the probability of defective parts is 0.7. Hardware discovers the truth about the parts by observation before he chooses effort, but Software does not.

a Draw the extensive form and put dotted lines around the information sets of Software at any nodes at which he moves.

b What is the Nash equilibrium?

c What is Software's belief, in equilibrium, as to the probability that Hardware chooses low effort?

d If Software sees that revenue is 100, what probability does he assign to defective parts if he himself exerted high effort and he believes that Hardware chose low effort?

2.6. California Drought (Adapted from McMillan [1992], page xxx)
California is in a drought and the reservoirs are running low. The probability of rainfall in 1991 is 1/2, but with probability 1 there will be heavy rainfall in 1992 and any saved water will be useless. The state uses rationing rather than the price system, and it must decide how much water to consume in 1990 and how much to save till 1991. Each Californian has a utility function of $U = log(w_{90}) + log(w_{91})$. Show that if the discount rate is zero the state should allocate twice as much water to 1990 as to 1991.

2.7. Smith's Energy Level
The boss is trying to decide whether Smith's energy level is high or low. He can only look in on Smith once during the day. He knows if Smith's energy is low, he will be yawning with a 50 percent probability, but if it is high, he will be yawning with a 10 percent probability. Before he looks in on him, the boss thinks that there is an 80 percent probability that Smith's energy is high, but then he sees him yawning. What probability of high energy should the boss now assess?

2.8. Two Games
Suppose that Column gets to choose which of the following two payoff structures applies to the simultaneous-move game he plays with Row. Row does not know which of these Column has chosen.

Table 5: Payoffs A, The Prisoner's Dilemma

		Column	
		Deny	*Confess*
	Deny	-1,-1	-10, 0
Row:			
	Confess	0,-10	-8,-8

Payoffs to: (Row,Column).

Table 6: Payoffs B, A Confession Game

		Column	
		Deny	*Confess*
Row:	*Deny*	-4,-4	-12, -200
	Confess	-200,-12	-10,-10

Payoffs to: (Row,Column).

a What is one example of a strategy for each player?

b Find a Nash equilibrium. Is it unique? Explain your reasoning.

c Is there a dominant strategy for Column? Explain why or why not.

d Is there a dominant strategy for Row?Explain why or why not.

e Does Row's choice of strategy depend on whether Column is rational or not? Explain why or why not.

2.9. Bank Runs, Coordination, and Asymmetric Information

A recent article has suggested that during the Chicago bank run of 1932, only banks that actually had negative net worth failed, even though depositors tried to take their money out of all the banks in town. (Charles Calomiris and Joseph Mason (1998), "Contagion and Bank Failures During the Great Depression: THe June 1932 Chicago Banking Panic", *American Economic Review*, December 1997, 87: 863-883.) A bank run occurs when many depositors all try to withdraw their deposits simultaneously, which creates a cash crunch for the bank since banks ordinarily do not keep much cash on hand, and have lent out most of it in business and home loans.

(a) Explain why some people might model bank runs as coordination games.

(b) Why would the prisoner's dilemma be an inappropriate model for bank runs?

(c) Suppose that some banks are owned by embezzlers who each year secretly steal some of the funds deposited in the banks, and that these thefts will all be discovered in 1940. The present year is 1931. Some depositors learn in 1932 which banks are owned by embezzlers and which are not, and the other depositors know who these depositors are. Construct a game to capture this situation and predict what would happen.

(d) How would your model change if the government introduced deposit insurance in 1931, which would pay back all depositors if the bank were unable to do so?

Bayes Rule at the Bar: A Classroom Game for Chapter 2

Your instructor has wandered into a dangerous bar in Jersey City. There are six people in there. Based on past experience, he estimates that three are cold-blooded killer and three are cowardly bullies. He also knows that 2/3 of killers are aggressive and 1/3 reasonable; but 1/3 of cowards are aggressive and 2/3 are reasonable. Unfortuntely, your instructor then spills his drink on a mean- looking rascal who responds with an aggressive remark.

In crafting his response in the two seconds he has to think, your instructor would like to know the probability he has offended a killer. Give him your estimate.

After writing the estimates and discussion, the story continues. A friend of the wet rascal comes in the door and discovers what has happened. He, too, turns aggressive. We know that the friend is just like the first rascal– a killer if the first one was a killer, a coward otherwise. Does this extra trouble change your estimate that the two of them are killers?

This game is a descendant of the game in Charles Holt & Lisa R. Anderson. "Classroom Games: Understanding Bayes Rule," *Journal of Economic Perspectives*, 10: 179-187 (Spring 1996), but I use a different heuristic for the rule, and a barroom story instead of urns. Psychologists have found that people can solve logical puzzles better if the puzzles are associated with a story involving social interactions; see Chapter 7 of Robin Dunbar's *The Trouble with Science*, which explains experiments and ideas from Cosmides & Toobey (1993). For instructors' notes, go to http://www.rasmusen.org/GI/probs/2bayesgame.pdf.

September 6, 1999. December 29, 2003. November 4, 2004. March 4, 2005. 25 March 2005 Eric Rasmusen, Erasmuse@indiana.edu. Http://www.rasmusen.org Footnotes starting with xxx are the author's notes to himself. Comments welcomed.

3 Mixed and Continuous Strategies

3.1 Mixed Strategies: The Welfare Game

The games we have looked at have so far been simple in at least one respect: the number of moves in the action set has been finite. In this chapter we allow a continuum of moves, such as when a player chooses a price between 10 and 20 or a purchase probability between 0 and 1. Chapter 3 begins by showing how to find mixed- strategy equilibria for a game with no pure-strategy equilibrium. In Section 3.2 the mixed-strategy equilibria are found by the payoff-equating method, and mixed strategies are applied to a dynamic game, the War of Attrition. Section 3.3 takes a more general look at mixed strategy equilibria and distinguishes between mixed strategies and random actions in auditing games. Section 3.4 begins the analysis of continuous action spaces, and this is continued in Section 3.5 in the Cournot duopoly model, where the discussion focusses on the example of two firms choosing output from the continuum between zero and infinity. These sections introduce other ideas that will be built upon in later chapters— dynamic games in Chapter 4, auditing and agency in Chapters 7 and 8, and Cournot oligopoly in Chapter 14. Section 3.6 looks at the Bertrand model and strategic substitutes. Section 3.7 switches gears a bit and talks about four reasons why a Nash equilibrium might not exist.

We invoked the concept of Nash equilibrium to provide predictions of outcomes without dominant strategies, but some games lack even a Nash equilibrium. It is often useful and realistic to expand the strategy space to include random strategies, in which case a Nash equilibrium almost always exists. These random strategies are called "mixed strategies."

A **pure strategy** *maps each of a player's possible information sets to one action.* $s_i : \omega_i \rightarrow a_i$.

A **mixed strategy** *maps each of a player's possible information sets to a probability distribution over actions.*

$$s_i : \omega_i \rightarrow m(a_i), \;\; \textit{where} \;\; m \geq 0 \;\; \text{and} \;\; \int_{A_i} m(a_i) da_i = 1.$$

A **completely mixed** *strategy puts positive probability on every action, so* $m > 0$.

The version of a game expanded to allow mixed strategies is called the **mixed extension** *of the game.*

A pure strategy constitutes a rule that tells the player what action to choose, while a mixed strategy constitutes a rule that tells him what dice to throw in order to choose an action. If a player pursues a mixed strategy, he might choose any of several different actions in a given situation, an unpredictability which can be helpful to him. Mixed strategies occur frequently in the real world. In American football games, for example, the offensive team has to decide whether to pass or to run. Passing generally gains more yards, but what is most important is to choose an action not expected by the other team. Teams decide to run part of the time and pass part of the time in a way that seems random to observers but rational to game theorists.

The Welfare Game

The Welfare Game models a government that wishes to aid a pauper if he searches for work but not otherwise, and a pauper who searches for work only if he cannot depend on government aid.

Table 1 shows payoffs which represent the situation. "Work" represents trying to find work, and "Loaf" represents not trying. The government wishes to help a pauper who is trying to find work, but not one who does not try. Neither player has a dominant strategy, and with a little thought we can see that no Nash equilibrium exists in pure strategies either.

Table 1: The Welfare Game

		Pauper		
		Work (γ_w)		*Loaf* ($1-\gamma_w$)
	Aid (θ_a)	3,2	$\rightarrow$	$-1,3$
Government		$\uparrow$		$\downarrow$
	No Aid ($1-\theta_a$)	$-1,1$	$\leftarrow$	0,0

Payoffs to: (Government, Pauper)

Each strategy profile must be examined in turn to check for Nash equilibria.

1 The strategy profile (*Aid, Work*) is not a Nash equilibrium, because the Pauper would respond with *Loaf* if the Government picked *Aid*.

2 (*Aid, Loaf*) is not Nash, because the Government would switch to *No Aid*.

3 (*No Aid, Loaf*) is not Nash, because the Pauper would switch to *Work*.

4 (*No Aid, Work*) is not Nash, because the Government would switch to *Aid*, which brings us back to (1).

The Welfare Game does have a mixed-strategy Nash equilibrium, which we can calculate. The players' payoffs are the expected values of the payments from Table 1. If the Government plays *Aid* with probability θ_a and the Pauper plays *Work* with probability γ_w, the Government's expected payoff is

$$\begin{aligned}
\pi_{Government} &= \theta_a[3\gamma_w + (-1)(1-\gamma_w)] + [1-\theta_a][-1\gamma_w + 0(1-\gamma_w)] \\
&= \theta_a[3\gamma_w - 1 + \gamma_w] - \gamma_w + \theta_a\gamma_w \qquad (1) \\
&= \theta_a[5\gamma_w - 1] - \gamma_w.
\end{aligned}$$

If only pure strategies are allowed, θ_a equals zero or one, but in the mixed extension of the game, the Government's action of θ_a lies on the continuum from zero to one, the pure strategies being the extreme values. If we followed the usual procedure for solving a maximization problem, we would differentiate the payoff function with respect to the choice variable to obtain the first-order condition. That procedure is actually not the best way to find mixed- strategy equilibria, which is the "payoff-equating method" I will describe in the next section. Let us use the maximization approach here, though, because it is good for helping you understand how mixed strategies work. The first order condition for the Government would be

$$\begin{aligned} 0 = \tfrac{d\pi_{Government}}{d\theta_a} &= 5\gamma_w - 1 \\ \Rightarrow \quad \gamma_w &= 0.2. \end{aligned} \tag{2}$$

In the mixed-strategy equilibrium, the Pauper selects *Work* 20 percent of the time. This is a bit strange, though: we obtained the Pauper's strategy by differentiating the Government's payoff! That is because we have not used maximization in the standard way. The problem has a corner solution, because depending on the Pauper's strategy, one of three strategies maximizes the Government's payoff: (i) Do not aid ($\theta_a = 0$) if the Pauper is unlikely enough to try to work; (ii) Definitely aid ($\theta_a = 1$) if the Pauper is likely enough to try to work;(iii) any probability of aid, if the Government is indifferent because the Pauper's probability of work is right on the border line of $\gamma_w = 0.2$.

It is possibility (iii) which allows a mixed strategy equilibrium to exist. To see this, go through the following 4 steps:

1 I assert that an optimal mixed strategy exists for the government.

2 If the Pauper selects *Work* more than 20 percent of the time, the Government always selects *Aid*. If the Pauper selects *Work* less than 20 percent of the time, the Government never selects *Aid*.

3 If a mixed strategy is to be optimal for the Government, the pauper must therefore select *Work* with probability exactly 20 percent.

To obtain the probability of the Government choosing *Aid*, we must turn to the Pauper's payoff function, which is

$$\begin{aligned} \pi_{Pauper} &= \theta_a(2\gamma_w + 3[1-\gamma_w]) + (1-\theta_a)(1\gamma_w + 0[1-\gamma_w]), \\ &= 2\theta_a\gamma_w + 3\theta_a - 3\theta_a\gamma_w + \gamma_w - \theta_a\gamma_w, \\ &= -\gamma_w(2\theta_a - 1) + 3\theta_a. \end{aligned} \tag{3}$$

The first order condition is

$$\begin{aligned} &\tfrac{d\pi_{Pauper}}{d\gamma_w} = -(2\theta_a - 1) = 0, \\ &\Rightarrow \theta_a = 1/2. \end{aligned} \tag{4}$$

If the Pauper selects *Work* with probability 0.2, the Government is indifferent among selecting *Aid* with probability 100 percent, 0 percent, or anything in between. If the strategies are to form a Nash equilibrium, however, the Government must choose $\theta_a = 0.5$. In the mixed-strategy Nash equilibrium, the Government selects *Aid* with probability 0.5 and the Pauper selects *Work* with probability 0.2. The equilibrium outcome could be any of the four entries in the outcome matrix. The entries having the highest probability of occurence are *(No Aid, Loaf)* and *(Aid, Loaf)*, each with probability 0.4 ($= 0.5[1 - 0.2]$).

Interpreting Mixed Strategies

Mixed strategies are not as intuitive as pure strategies, and many modellers prefer to restrict themselves to pure-strategy equilibria in games which have them. One objection to mixed strategies is that people in the real world do not take random actions. That is not a compelling objection, because all that a model with mixed strategies requires to be a good description of the world is that the actions appear random to observers, even if the player himself has always been sure what action he would take. Even explicitly random actions are not uncommon, however— the Internal Revenue Service randomly selects which tax returns to audit, and telephone companies randomly monitor their operators' conversations to discover whether they are being polite.

A more troubling objection is that a player who selects a mixed strategy is always indifferent between two pure strategies. In the Welfare Game, the Pauper is indifferent between his two pure strategies and a whole continuum of mixed strategies, given the Government's mixed strategy. If the Pauper were to decide not to follow the particular mixed strategy $\gamma_w = 0.2$, the equilibrium would collapse because the Government would change its strategy in response. Even a small deviation in the probability selected by the Pauper, a deviation that does not change his payoff if the Government does not respond, destroys the equilibrium completely because the Government does respond. A mixed-strategy Nash equilibrium is weak in the same sense as the (*North*, *North*) equilibrium in the Battle of the Bismarck Sea: to maintain the equilibrium, a player who is indifferent between strategies must pick a particular strategy from out of the set of strategies.

One way to reinterpret The Welfare Game is to imagine that instead of a single pauper there are many, with identical tastes and payoff functions, all of whom must be treated alike by the Government. In the mixed-strategy equilibrium, each of the paupers chooses *Work* with probability 0.2, just as in the one-pauper game. But the many-pauper game has a pure-strategy equilibrium: 20 percent of the paupers choose the pure strategy *Work* and 80 percent choose the pure strategy *Loaf*. The problem persists of how an individual pauper, indifferent between the pure strategies, chooses one or the other, but it is easy to imagine that individual characteristics outside the model could determine which actions are chosen by which paupers.

The number of players needed so that mixed strategies can be interpreted as pure strategies in this way depends on the equilibrium probability γ_w, since we cannot speak of a fraction of a player. The number of paupers must be a multiple of five in The Welfare Game in order to use this interpretation, since the equilibrium mixing probability is a multiple of 1/5. For the interpretation to apply no matter how we vary the parameters of a model we would need a *continuum* of players.

Another interpretation of mixed strategies, which works even in the single-pauper game, assumes that the pauper is drawn from a population of paupers, and the Government does not know his characteristics. The Government only knows that there are two types of paupers, in the proportions (0.2, 0.8): those who pick *Work* if the Government picks $\theta_a = 0.5$, and those who pick *Loaf*. A pauper drawn randomly from the population might be of either type. Harsanyi (1973) gives a careful interpretation of this situation.

3.2 Chicken, The War of Attrition, and Correlated Strategies

Chicken and the Payoff-Equating Method

The next game illustrates why we might decide that a mixed-strategy equilibrium is best even if pure-strategy equilibria also exist. In the game of Chicken, the players are two Malibu teenagers, Smith and Jones. Smith drives a hot rod south down the middle of Route 1, and Jones drives north. As collision threatens, each decides whether to *Continue* in the middle or *Swerve* to the side. If a player is the only one to *Swerve*, he loses face, but if neither player picks *Swerve* they are both killed, which has an even lower payoff. If a player is the only one to *Continue*, he is covered with glory, and if both *Swerve* they are both embarassed. (We will assume that to *Swerve* means by convention to *Swerve* right; if one swerved to the left and the other to the right, the result would be both death and humiliation.) Table 2 assigns numbers to these four outcomes.

Table 2: Chicken

		Jones		
		Continue (θ)		*Swerve* ($1-\theta$)
	Continue (θ)	$-3, -3$	$\rightarrow$	**2, 0**
Smith:		$\downarrow$		$\uparrow$
	Swerve ($1-\theta$)	**0, 2**	$\leftarrow$	1, 1

Payoffs to: (Smith, Jones)

Chicken has two pure-strategy Nash equilibria, (*Swerve*, *Continue*) and (*Continue*, *Swerve*), but they have the defect of asymmetry. How do the players know which equilibrium is the one that will be played out? Even if they talk before the game started, it is not clear how they could arrive at an asymmetric result. We encountered the same dilemma in choosing an equilibrium for Battle of the Sexes . As in that game, the best prediction in Chicken is perhaps the mixed-strategy equilibrium, because its symmetry makes it a focal point of sorts, and does not require any differences between the players.

The **payoff-equating** method used here to calculate the mixing probabilities for Chicken will be based on the logic followed in Section 3.1, but it does not use the calculus of maximization. In the mixed strategy equilibrium, Smith must be indifferent between *Swerve* and *Continue*. Moreover, Chicken, unlike the Welfare Game, is a symmetric game, so we can guess that in equilibrium each player will choose the same mixing probability. If that is the case, then, since the payoffs from each of Jones' pure strategies must be equal in a mixed-strategy equilibrium, it is true that

$$\begin{aligned}\pi_{Jones}(Swerve) &= (\theta_{Smith}) \cdot (0) + (1-\theta_{Smith}) \cdot (1) \\ &= (\theta_{Smith}) \cdot (-3) + (1-\theta_{Smith}) \cdot (2) = \pi_{Jones}(Continue).\end{aligned} \tag{5}$$

From equation (5) we can conclude that $1-\theta_{Smith} = 2-5\theta_{Smith}$, so $\theta_{Smith} = 0.25$. In the symmetric equilibrium, both players choose the same probability, so we can replace θ_{Smith} with simply θ. As for the question which represents the greatest interest to their mothers, the two teenagers will survive with probability $1-(\theta \cdot \theta) = 0.9375$.

The payoff-equating method is easier to use than the calculus method if the modeller is sure which strategies will be mixed, and it can also be used in asymmetric games. In the Welfare Game, it would start with $V_g(Aid) = V_g(No\ Aid)$ and $V_p(Loaf) = V_p(Work)$, yielding two equations for the two unknowns, θ_a and γ_w, which when solved give the same mixing probabilities as were found earlier for that game. The reason why the payoff-equating and calculus maximization methods reach the same result is that the expected payoff is linear in the possible payoffs, so differentiating the expected payoff equalizes the possible payoffs. The only difference from the symmetric-game case is that two equations are solved for two different mixing probabilities instead of a single equation for the one mixing probability that both players use.

It is interesting to see what happens if the payoff of -3 in the northwest corner of Table 2 is generalized to x. Solving the analog of equation (5) then yields

$$\theta = \frac{1}{1-x}. \tag{6}$$

If $x = -3$, this yields $\theta = 0.25$, as was just calculated, and if $x = -9$, it yields $\theta = 0.10$. This makes sense; increasing the loss from crashes reduces the equilibrium probability of continuing down the middle of the road. But what if $x = 0.5$? Then the equilibrium probability of continuing appears to be $\theta = 2$, which is impossible; probabilities are bounded by zero and one.

When a mixing probability is calculated to be greater than one or less than zero, the implication is either that the modeller has made an arithmetic mistake or, as in this case, that he is wrong in thinking that the game has a mixed-strategy equilibrium. If $x = 0.5$, one can still try to solve for the mixing probabilities, but, in fact, the only equilibrium is in pure strategies— *(Continue, Continue)* (the game has become a Prisoner's Dilemma). The absurdity of probabilities greater than one or less than zero is a valuable aid to the fallible modeller because such results show that he is wrong about the qualitative nature of the equilibrium— it is pure, not mixed. Or, if the modeller is not sure whether the equilibrium is mixed or not, he can use this approach to prove that the equilibrium is not in mixed strategies.

The War of Attrition

The War of Attrition is a game something like Chicken stretched out over time, where both players start with *Continue*, and the game ends when the first one picks *Swerve*. Until the game ends, both earn a negative amount per period, and when one exits, he earns zero and the other player earns a reward for outlasting him.

We will look at a war of attrition in discrete time. We will continue with Smith and Jones, who have both survived to maturity and now play games with more expensive toys: they control two firms in an industry which is a natural monopoly, with demand strong enough for one firm to operate profitably, but not two. The possible actions are to *Exit* or to *Continue*. In each period that both *Continue*, each earns -1. If a firm exits, its losses cease and the remaining firm obtains the value of the market's monopoly profit, which we set equal to 3. We will set the discount rate equal to $r > 0$, although that is inessential to the model, even if the possible length of the game is infinite (discount rates will be discussed in detail in Section 4.3).

The War of Attrition has a continuum of Nash equilibria. One simple equilibrium is for Smith to choose (*Continue* regardless of what Jones does) and for Jones to choose (*Exit* immediately), which are best responses to each other. But we will solve for a symmetric equilibrium in which each player chooses the same mixed strategy: a constant probability θ that the player picks *Exit* given that the other player has not yet exited.

We can calculate θ as follows, adopting the perspective of Smith. Denote the expected discounted value of Smith's payoffs by V_{stay} if he stays and V_{exit} if he exits immediately. These two pure strategy payoffs must be equal in a mixed strategy equilibrium (which was the basis for the payoff-equating method). If Smith exits, he obtains $V_{exit} = 0$. If Smith stays in, his payoff depends on what Jones does. If Jones stays in too, which has probability $(1 - \theta)$, Smith gets -1 currently and his expected value for the following period, which is discounted using r, is unchanged. If Jones exits immediately, which has probability θ, then Smith receives a payment of 3. In symbols,

$$V_{stay} = \theta \cdot (3) + (1 - \theta) \left(-1 + \left[\frac{V_{stay}}{1 + r} \right] \right), \tag{7}$$

which, after a little manipulation, becomes

$$V_{stay} = \left(\frac{1 + r}{r + \theta} \right) (4\theta - 1) \, . \tag{8}$$

Once we equate V_{stay} to V_{exit}, which equals zero, equation (8) tells us that $\theta = 0.25$ in equilibrium, and that this is independent of the discount rate r.

Returning from arithmetic to ideas, why does Smith *Exit* immediately with positive probability, given that Jones will exit first if Smith waits long enough? The reason is that Jones might choose to continue for a long time and both players would earn -1 each period until Jones exited. The equilibrium mixing probability is calculated so that both of them are likely to stay in long enough so that their losses soak up the gain from being the survivor. Papers on the War of Attrition include Fudenberg & Tirole (1986b), Ghemawat & Nalebuff (1985), Maynard Smith (1974), Nalebuff & Riley (1985), and Riley (1980). All are examples of "rent-seeking" welfare losses. As Posner (1975) and Tullock (1967) have pointed out, the real costs of acquiring rents can be much bigger than the second-order triangle losses from allocative distortions, and the war of attrition shows that the big loss from a natural monopoly might be not the reduced trade that results from higher prices, but the cost of the battle to gain the monopoly.

In The War of Attrition, the reward goes to the player who does not choose the move which ends the game, and a cost is paid each period that both players refuse to end it. Various other **timing games** also exist. The opposite of a war of attrition is a **preemption game**, in which the reward goes to the player who chooses the move which ends the game, and a cost is paid if both players choose that move, but no cost is incurred in a period when neither player chooses it. The game of **Grab the Dollar** is an example. A dollar is placed on the table between Smith and Jones, who each must decide whether to grab for it or not. If both grab, both are fined one dollar. This could be set up as a one-period game, a T period game, or an infinite- period game, but the game definitely ends when someone grabs the dollar. Table 3 shows the payoffs.

Table 3: Grab the Dollar

		Jones		
		Grab		*Don't Grab*
	Grab	$-1,-1$	$\rightarrow$	**1,0**
Smith:		$\downarrow$		$\uparrow$
	Don't Grab	**0,1**	$\leftarrow$	$0,0$

Payoffs to: (Smith, Jones)

Like The War of Attrition, Grab the Dollar has asymmetric equilibria in pure strategies, and a symmetric equilibrium in mixed strategies. In the infinite-period version, the equilibrium probability of grabbing is 0.5 per period in the symmetric equilibrium.

Still another class of timing games are duels, in which the actions are discrete occurrences which the players locate at particular points in continuous time. Two players with guns approach each other and must decide when to shoot. In a **noisy duel**, if a player shoots and misses, the other player observes the miss and can kill the first player at his leisure. An equilibrium exists in pure strategies for the noisy duel. In a **silent duel**, a player does not know when the other player has fired, and the equilibrium is in mixed strategies. Karlin (1959) has details on duelling games, and Chapter 4 of Fudenberg & Tirole (1991a) has an excellent discussion of games of timing in general. See also Shubik (1954) on the rather different problem of who to shoot first in a battle with three or more sides.

We will go through one more game of timing to see how to derive a continuous mixed strategies probability distribution, instead of just the single number derived earlier. In presenting this game, a new presentation scheme will be useful. If a game has a continuous strategy set, it is harder or impossible to depict the payoffs using tables or the extensive form using a tree. Tables of the sort we have been using so far would require a continuum of rows and columns, and trees a continuum of branches. A new format for game descriptions of the players, actions, and payoffs will be used for the rest of the book. The new format will be similar to the way the rules of the Dry Cleaners Game were presented in Section 1.1.

Patent Race for a New Market

Players
Three identical firms, Apex, Brydox, and Central.

The Order of Play
Each firm simultaneously chooses research spending $x_i \geq 0$, $(i = a, b, c)$.

Payoffs
Firms are risk neutral and the discount rate is zero. Innovation occurs at time $T(x_i)$ where $T' < 0$. The value of the patent is V, and if several players innovate simultaneously they share its value.

$$\pi_i = \begin{cases} V - x_i & \text{if } T(x_i) < T(x_j), \;\; (\forall j \neq i) \quad \text{(Firm } i \text{ gets the patent)} \\ \\ \frac{V}{1+m} - x_i & \text{if } T(x_i) = T(x_k), \quad \begin{array}{l}\text{(Firm } i \text{ shares the patent with} \\ m = 1 \text{ or } 2 \text{ other firms)}\end{array} \\ \\ -x_i & \text{if } T(x_i) > T(x_j) \text{ for some } j \quad \text{(Firm } i \text{ does not get the patent)} \end{cases}$$

The format first assigns the game a title, after which it lists the players, the order of play (together with who observes what), and the payoff functions. Listing the players is redundant, strictly speaking, since they can be deduced from the order of play, but it is useful for letting the reader know what kind of model to expect. The format includes very little explanation; that is postponed, lest it obscure the description. This exact format is not standard in the literature, but every good article begins its technical section by specifying the same information, if in a less structured way, and the novice is strongly advised to use all the structure he can.

The game Patent Race for a New Market does not have any pure strategy Nash equilibria, because the payoff functions are discontinuous. A slight difference in research by one player can make a big difference in the payoffs, as shown in Figure 1 for fixed values of x_b and x_c. The research levels shown in Figure 1 are not equilibrium values. If Apex chose any research level x_a less than V, Brydox would respond with $x_a + \varepsilon$ and win the patent. If Apex chose $x_a = V$, then Brydox and Central would respond with $x_b = 0$ and $x_c = 0$, which would make Apex want to switch to $x_a = \varepsilon$.

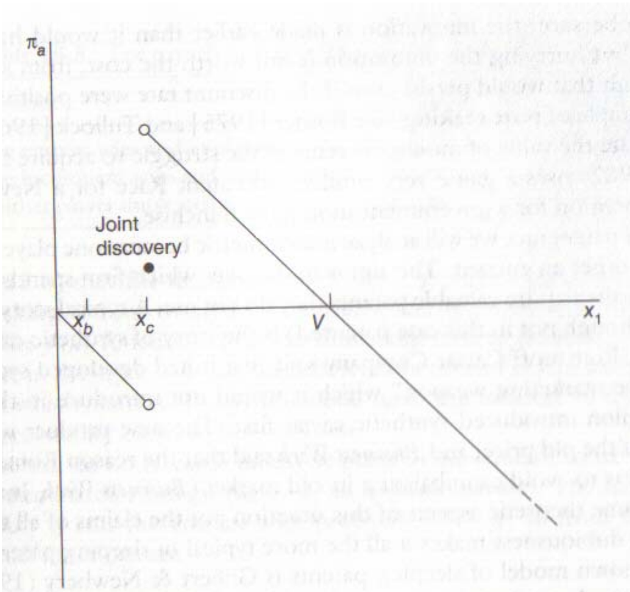


Figure 1: The Payoffs in Patent Race for a New Market

There does exist a symmetric mixed strategy equilibrium. Denote the probability that firm i chooses a research level less than or equal to x as $M_i(x)$. This function describes the firm's mixed strategy. In a mixed-strategy equilibrium a player is indifferent between any of the pure strategies among which he is mixing. Since we know that the pure strategies $x_a = 0$ and $x_a = V$ yield zero payoffs, if Apex mixes over the support $[0, V]$ then the expected payoff for every strategy mixed between must also equal zero. The expected payoff from the pure strategy x_a is the expected value of winning minus the cost of research. Letting x stand for nonrandom and X for random variables, this is

$$\pi_a(x_a) = V \cdot Pr(x_a \geq X_b, x_a \geq X_c) - x_a = 0 = \pi_a(x_a = 0), \tag{9}$$

which can be rewritten as

$$V \cdot Pr(X_b \leq x_a) Pr(X_c \leq x_a) - x_a = 0, \tag{10}$$

or

$$V \cdot M_b(x_a) M_c(x_a) - x_a = 0. \tag{11}$$

We can rearrange equation (11) to obtain

$$M_b(x_a) M_c(x_a) = \frac{x_a}{V}. \tag{12}$$

If all three firms choose the same mixing distribution M, then

$$M(x) = \left(\frac{x}{V}\right)^{1/2} \text{ for } 0 \leq x \leq V. \tag{13}$$

What is noteworthy about a patent race is not the nonexistence of a pure strategy equilibrium but the overexpenditure on research. All three players have expected payoffs of zero, because the patent value V is completely dissipated in the race. As in Brecht's *Threepenny Opera*, "When all race after happiness/Happiness comes in last."[1] To be sure, the innovation is made earlier than it would have been by a monopolist, but hurrying the innovation is not worth the cost, from society's point of view, a result that would persist even if the discount rate were positive. Rogerson (1982) uses a game very similar to Patent Race for a New Market to analyze competition for a government monopoly franchise.

Correlated Strategies

One example of a war of attrition is setting up a market for a new security, which may be a natural monopoly for reasons to be explained in Section 8.5. Certain stock exchanges have avoided the destructive symmetric equilibrium by using lotteries to determine which of them would trade newly listed stock options under a system similar to the football draft.[2] Rather than waste resources fighting, these exchanges use the lottery as a coordinating device, even though it might not be a binding agreement.

Aumann (1974) has pointed out that it is often important whether players can use the same randomizing device for their mixed strategies. If they can, we refer to the resulting strategies as **correlated strategies**. Consider the game of Chicken. The only mixed-strategy equilibrium is the symmetric one in which each player chooses *Continue* with probability 0.25 and the expected payoff is 0.75. A correlated equilibrium would be for the two players to flip a coin and for Smith to choose *Continue* if it comes up heads and for Jones to choose *Continue* otherwise. Each player's strategy is a best response to the other's, the probability of each choosing *Continue* is 0.5, and the expected payoff for each is 1.0, which is better than the 0.75 achieved without correlated strategies.

Usually the randomizing device is not modelled explicitly when a model refers to correlated equilibrium. If it is, uncertainty over variables that do not affect preferences, endowments, or production is called **extrinsic uncertainty.** Extrinsic uncertainty is the driving force behind **sunspot models**, so called because the random appearance of sunspots might cause macroeconomic changes via correlated equilibria (Maskin & Tirole [1987]) or bets made between players (Cass & Shell [1983]).

One way to model correlated strategies is to specify a move in which Nature gives each player the ability to commit first to an action such as *Continue* with equal probability. This is often realistic because it amounts to a zero probability of both players entering the industry at exactly the same time without anyone knowing in advance who will be the lucky starter. Neither firm has an a priori advantage, but the outcome is efficient.

The population interpretation of mixed strategies cannot be used for correlated strategies. In ordinary mixed strategies, the mixing probabilities are statistically independent, whereas in correlated strategies they are not. In Chicken, the usual mixed strategy can be interpreted as populations of Smiths and Joneses, each population consisting of a certain proportion of pure swervers and pure stayers. The correlated equilibrium has no such

[1] Act III, scene 7 of the *Threepenny Opera*, translated by John Willett (Berthold Brecht, *Collected Works*, London: Eyre Methuen (1987).

[2] "Big Board will Begin Trading of Options on 4 Stocks it Lists," *Wall Street Journal*,p. 15 (4 October 1985).

interpretation.

Another coordinating device, useful in games that, like Battle of the Sexes, have a coordination problem, is **cheap talk** (Crawford & Sobel [1982], Farrell [1987]). Cheap talk refers to costless communication before the game proper begins. In Ranked Coordination, cheap talk instantly allows the players to make the desirable outcome a focal point. In Chicken, cheap talk is useless, because it is dominant for each player to announce that he will choose *Continue*. But in The Battle of the Sexes, coordination and conflict are combined. Without communication, the only symmetric equilibrium is in mixed strategies. If both players know that making inconsistent announcements will lead to the wasteful mixed-strategy outcome, then they are willing to mix announcing whether they will go to the ballet or the prize fight. With many periods of announcements before the final decision, their chances of coming to an agreement are high. Thus communication can help reduce inefficiency even if the two players are in conflict.

3.3 Mixed Strategies with General Parameters and N Players: The Civic Duty Game

Having looked at a number of specific games with mixed-strategy equilibria, let us now apply the method to the general game of Table 4.

Table 4: The General 2-by-2 Game

		Column	
		Left (θ)	*Right* ($1-\theta$)
	Up (γ)	a, w	b, x
Row:			
	Down ($1-\gamma$)	c, y	d, z

Payoffs to: (Row, Column)

To find the game's equilibrium, equate the payoffs from the pure strategies. For Row, this yields

$$\pi_{Row}(Up) = \theta a + (1-\theta)b \tag{14}$$

and

$$\pi_{Row}(Down) = \theta c + (1-\theta)d. \tag{15}$$

Equating (14) and (15) gives

$$\theta(a+d-b-c) + b - d = 0, \tag{16}$$

which yields

$$\theta^* = \frac{d-b}{(d-b)+(a-c)}. \tag{17}$$

Similarly, equating the payoffs for Column gives

$$\pi_{Column}(Left) = \gamma w + (1-\gamma)y = \pi_{Column}(Right) = \gamma x + (1-\gamma)z, \tag{18}$$

which yields

$$\gamma^* = \frac{z-y}{(z-y)+(w-x)}. \tag{19}$$

The equilibrium represented by (17) and (19) illustrates a number of features of mixed strategies.

First, it is possible, but wrong, to follow the payoff-equating method for finding a mixed strategy even if no mixed strategy equilibrium actually exists. Suppose, for example, that *Down* is a strongly dominant strategy for Row, so $c > a$ and $d > b$. Row is unwilling to mix, so the equilibrium is not in mixed strategies. Equation (17) would be misleading, though some idiocy would be required to stay misled for very long since the equation implies that $\theta^* > 1, or theta^* <= 0$ in this case.

Second, the exact features of the equilibrium in mixed strategies depend heavily on the cardinal values of the payoffs, not just on their ordinal values like the pure strategy equilibria in other 2-by-2 games. Ordinal rankings are all that is needed to know that an equilibrium exists in mixed strategies, but cardinal values are needed to know the exact mixing probabilities. If the payoff to Column from *(Confess, Confess*) is changed slightly in the Prisoner's Dilemma it makes no difference at all to the equilibrium. If the payoff of z to Column from *(Down, Right)* is increased slightly in the General 2-by-2 Game, equation (19) says that the mixing probability γ^* will change also.

Third, the payoffs can be changed by affine transformations without changing the game substantively, even though cardinal payoffs do matter (which is to say that monotonic but non-affine transformations do make a difference). Let each payoff π in Table 4 become $\alpha + \beta\pi$. Equation (19) then becomes

$$\begin{aligned} \gamma^* &= \tfrac{\alpha+\beta z-\alpha-\beta y}{(\alpha+\beta z-\alpha-\beta y)+(\alpha+\beta w-\alpha-\beta x)} \\ &= \tfrac{z-y}{(z-y)+(w-x)}. \end{aligned} \tag{20}$$

The affine transformation has left the equilibrium strategy unchanged.

Fourth, as was mentioned earlier in connection with the Welfare Game, each player's mixing probability depends only on the payoff parameters of the other player. Row's strategy γ^* in equation (19) depends on the parameters w, x, y and z, which are the payoff parameters for Column, and have no direct relevance for Row.

Categories of Games with Mixed Strategies

Table 5 uses the players and actions of Table 4 to depict three major categories of 2-by-2 games in which mixed-strategy equilibria are important. Some games fall in none of these categories— those with tied payoffs, such as the Swiss Cheese Game in which all eight payoffs equal zero— but the three games in Table 5 encompass a wide variety of economic phenomena.

Table 5: 2-by-2 Games with Mixed Strategy Equilibria [3]

[3] xxx Put equilibria in bold fonts in Coord and Cont games

$$\begin{array}{|ccc|}\hline a,w & \rightarrow & b,x \\ \uparrow & & \downarrow \\ c,y & \leftarrow & d,z \\ \hline\end{array} \quad \begin{array}{|ccc|}\hline a,w & \leftarrow & b,x \\ \downarrow & & \uparrow \\ c,y & \rightarrow & d,z \\ \hline\end{array} \quad \begin{array}{|ccc|}\hline a,w & \leftarrow & b,x \\ \uparrow & & \downarrow \\ c,y & \rightarrow & d,z \\ \hline\end{array} \quad \begin{array}{|ccc|}\hline a,w & \rightarrow & b,x \\ \downarrow & & \uparrow \\ c,y & \leftarrow & d,z \\ \hline\end{array}$$

Discoordination Games — Coordination Games — Contribution Games

Discoordination games have a single equilibrium, in mixed strategies. The payoffs are such that either (a) $a > c$, $d > b$, $x > w$, and $y > z$, or (b) $c > a$, $b > d$, $w > x$, and $z > y$. The Welfare Game is a discoordination game, as is Auditing Game I in the next section and Matching Pennies in problem 3.3.

Coordination games have three equilibria: two symmetric equilibria in pure strategies and one symmetric equilibrium in mixed strategies. The payoffs are such that $a > c$, $d > b$, $w > x$, and $z > y$. Ranked Coordination and the Battle of the Sexes are two varieties of coordination games in which the players have the same and opposite rankings of the pure-strategy equilibria.

Contribution games have three equilibria: two asymmetric equilibria in pure strategies and one symmetric equilibrium in mixed strategies. The payoffs are such that $c > a$, $b > d$, $x > w$, and $y > z$. Also, it must be true that $c < b$ and $y > x$.

I have invented the name "contribution game" for the occasion, since the type of game described by this term is often used to model a situation in which two players each have a choice of taking some action that contributes to the public good, but would each like the other player to bear the cost. The difference from the Prisoner's Dilemma is that each player in a contribution game is willing to bear the cost alone if necessary.

Contribution games appear to be quite different from the Battle of the Sexes, but they are essentially the same. Both of them have two pure-strategy equilibria, ranked oppositely by the two players. In mathematical terms, the fact that contribution games have the equilibria in the southwest and northeast corners of the outcome matrix whereas coordination games have them in the northwest and southeast, is unimportant; the location of the equilibria could be changed by just switching the order of Row's strategies. We do view real situations differently, however, depending on whether players choose the same actions or different actions in equilibrium.

Let us take a look at a particular contribution game to show how to extend two-player games to games with several players. A notorious example in social psychology is the murder of Kitty Genovese, who was killed in New York City in 1964 despite the presence of numerous neighbors. "For more than half an hour 38 respectable, law-abiding citizens in Queens watched a killer stalk and stab a woman in three separate attacks in Kew Gardens.... Twice the sound of their voices and the sudden glow of their bedroom lights interrupted him and frightened him off. Each time he returned, sought her out, and stabbed her again. Not one person telephoned the police during the assault; one witness called after the woman was dead." (Martin Gansberg, "38 Who Saw Murder Didn't Call Police," *The New York Times*, March 27, 1964, p. 1.) Even as hardened an economist as myself finds it somewhat distasteful to call this a "game," but game theory does explain what happened.

I will use a less appalling story for our model. In the Civic Duty Game of Table 6, Smith and Jones observe a burglary taking place. Each would like someone to call the police and stop the burglary because having it stopped adds 10 to his payoff, but neither wishes to make the call himself because the effort subtracts 3. If Smith can be assured that Jones will call, Smith himself will ignore the burglary. Table 6 shows the payoffs.

Table 6: The Civic Duty Game

		Jones		
		Ignore (γ)		*Telephone* ($1-\gamma$)
	Ignore (γ)	$0,0$	$\rightarrow$	**10,7**
Smith:		$\downarrow$		$\uparrow$
	Telephone ($1-\gamma$)	**7,10**	$\leftarrow$	$7,7$

Payoffs to: (Row, Column)

The Civic Duty Game has two asymmetric pure-strategy equilibria and a symmetric mixed-strategy equilibrium. In solving for the mixed- strategy equilibrium, let us move from two players to N players. In the N-player version of the game, the payoff to Smith is 0 if nobody calls, 7 if he himself calls, and 10 if one or more of the other $N-1$ players calls. This game also has asymmetric pure-strategy and a symmetric mixed-strategy equilibrium. If all players use the same probability γ of $Ignore$, the probability that the other $N-1$ players besides Smith all choose $Ignore$ is γ^{N-1}, so the probability that one or more of them chooses $Telephone$ is $1-\gamma^{N-1}$. Thus, equating Smith's pure-strategy payoffs using the payoff-equating method of equilibrium calculation yields

$$\pi_{Smith}(Telephone) = 7 = \pi_{Smith}(Ignore) = \gamma^{N-1}(0) + (1-\gamma^{N-1})(10). \tag{21}$$

Equation (21) tells us that

$$\gamma^{N-1} = 0.3 \tag{22}$$

and

$$\gamma^* = 0.3^{\frac{1}{N-1}}. \tag{23}$$

If $N=2$, Smith chooses $Ignore$ with a probability of 0.30. As N increases, Smith's expected payoff remains equal to 7 whether $N=2$ or $N=38$, since his expected payoff equals his payoff from the pure strategy of $Telephone$. The probability of $Ignore$, γ^*, however, increases with N. If $N=38$, the value of γ^* is about 0.97. When there are more players, each player relies more on somebody else calling.

The probability that nobody calls is γ^{*N}. Equation (22) shows that $\gamma^{*N-1} = 0.3$, so $\gamma^{*N} = 0.3\gamma^*$, which is increasing in N because γ^* is increasing in N. If $N=2$, the probability that neither player phones the police is $\gamma^{*2} = 0.09$. When there are 38 players, the probability rises to γ^{*38}, about 0.29. The more people that watch a crime, the less likely it is to be reported.

As in the Prisoner's Dilemma, the disappointing result in the Civic Duty Game suggests a role for active policy. The mixed- strategy outcome is clearly bad. The expected payoff per player remains equal to 7 whether there is 1 player or 38, whereas if the equilibrium played out was the equilibrium in which one and only one player called the police, the average payoff would rise from 7 with 1 player to about 9.9 with 38 (=[1(7) + 37(10)]/38). A situation like this requires something to make one of the pure-strategy equilibria a focal point. The problem is divided responsibility. One person must be made responsible for calling the police, whether by tradition (e.g., the oldest person on the block always calls the police) or direction (e.g., Smith shouts to Jones: "Call the police!").

3.4 Different Uses of Mixing and Randomizing: Minimax and the Auditing Game

A pure strategy may be strictly dominated by a mixed strategy, even if it is not dominated by any of the other pure strategies in the game.

Table 7: Pure Strategies Dominated by a Mixed Strategy

		Column	
		North	*South*
	North	0,0	4,-4
Row	*South*	4,-4	0,0
	Defense	1,-1	1,-1

Payoffs to: (Row, Column)

In the zero-sum game of Table 7, Row's army can attack in the North, attack in the South, or remain on the defense. An unexpected attack gives Row a payoff of 4, an expected attack a payoff of 0, and defense a payoff of 1. Column can respond by preparing to defend in the North or in the South.

Row could guarantee himself a payoff of 1 if he chose Defense. But suppose he plays North with probability .5 and South with probability .5. His expected payoff from this mixed strategy if Column plays North with probability N is $.5(N)(0) + .5(1-N)(4) + .5(N)(4) + .5(1-N)(0) = 2$, so whatever response Column picks, Row's expected payoff is higher than his payoff of 1 from Defense. Defense is thus strictly dominated for Row by (.5 North, .5 South).[4] [5]

The next three games will illustrate the difference between mixed strategies and random actions, a subtle but important distinction. In all three games, the Internal Revenue Service must decide whether to audit a certain class of suspect tax returns to discover whether they are accurate or not. The goal of the IRS is to either prevent or catch cheating at minimum cost. The suspects want to cheat only if they will not be caught. Let us

[4]xxx In the unique Nash equilibrium, Column would choose North with probability N=.5 and south with probability .5; this is his unique equilibrium action because any other choice of N would cause Row to deviate to whatever direction Column is not guarding as heavily.

[5]xxx Use this as an example of the minimax theorem at work. It really is good for 2-person zero sum games. Fundamental insight of the Minimax theorem: With mixing, you can protect yourself even if you aren't smart. Mention this.

assume that the benefit of preventing or catching cheating is 4, the cost of auditing is C, where $C < 4$, the cost to the suspects of obeying the law is 1, and the cost of being caught is the fine $F > 1$.

Even with all of this information, there are several ways to model the situation. Table 8 shows one way: a 2-by-2 simultaneous-move game.

Table 8: Auditing Game I

		Suspects		
		Cheat (θ)		*Obey* ($1-\theta$)
	Audit (γ)	$4-C, -F$	$\rightarrow$	$4-C, -1$
IRS:		$\uparrow$		$\downarrow$
	Trust ($1-\gamma$)	0,0	$\leftarrow$	$4, -1$

Payoffs to: (IRS, Suspects)

Auditing Game I is a discoordination game, with only a mixed strategy equilibrium. Equations (17) and (19) or the payoff-equating method tell us that

$$\begin{aligned} Probability(Cheat) = \theta^* &= \tfrac{4-(4-C)}{(4-(4-C))+((4-C)-0)} \\ &= \tfrac{C}{4} \end{aligned} \tag{24}$$

and

$$\begin{aligned} Probability(Audit) = \gamma^* &= \tfrac{-1-0}{(-1-0)+(-F--1)} \\ &= \tfrac{1}{F}. \end{aligned} \tag{25}$$

Using (24) and (25), the payoffs are

$$\begin{aligned} \pi_{IRS}(Audit) = \pi_{IRS}(Trust) &= \theta^*(0) + (1-\theta^*)(4) \\ &= 4 - C. \end{aligned} \tag{26}$$

and

$$\begin{aligned} \pi_{Suspect}(Obey) = \pi_{Suspect}(Cheat) &= \gamma^*(-F) + (1-\gamma^*)(0) \\ &= -1. \end{aligned} \tag{27}$$

A second way to model the situation is as a sequential game. Let us call this Auditing Game II. The simultaneous game implicitly assumes that both players choose their actions without knowing what the other player has decided. In the sequential game, the IRS chooses government policy first, and the suspects react to it. The equilibrium in Auditing Game II is in pure strategies, a general feature of sequential games of perfect information. In equilibrium, the IRS chooses *Audit*, anticipating that the suspect will then choose *Obey*. The payoffs are $4-C$ for the IRS and -1 for the suspects, the same for both players as in Auditing Game I, although now there is more auditing and less cheating and fine-paying.

We can go a step further. Suppose the IRS does not have to adopt a policy of auditing or trusting every suspect, but instead can audit a random sample. This is not necessarily a mixed strategy. In Auditing Game I, the equilibrium strategy was to audit all suspects with probability $1/F$ and none of them otherwise. That is different from announcing in advance that the IRS will audit a random sample of $1/F$ of the suspects. For Auditing Game III, suppose the IRS move first, but let its move consist of the choice of the proportion α of tax returns to be audited.

We know that the IRS is willing to deter the suspects from cheating, since it would be willing to choose $\alpha = 1$ and replicate the result in Auditing Game II if it had to. It chooses α so that

$$\pi_{suspect}(Obey) \geq \pi_{suspect}(Cheat), \tag{28}$$

i.e.,

$$-1 \geq \alpha(-F) + (1-\alpha)(0). \tag{29}$$

In equilibrium, therefore, the IRS chooses $\alpha = 1/F$ and the suspects respond with *Obey*. The IRS payoff is $4 - \alpha C$, which is better than the $4 - C$ in the other two games, and the suspect's payoff is -1, exactly the same as before.

The equilibrium of Auditing Game III is in pure strategies, even though the IRS's action is random. It is different from Auditing Game I because the IRS must go ahead with the costly audit even if the suspect chooses *Obey*. Auditing Game III is different in another way also: its action set is continuous. In Auditing Games I and Auditing Game II the action set is $\{$*Audit, Trust*$\}$, although the strategy set becomes $\gamma \in [0,1]$ once mixed strategies are allowed. In Auditing Game III, the action set is $\alpha \in [0,1]$, and the strategy set would allow mixing of any of the elements in the action set, although mixed strategies are pointless for the IRS because the game is sequential.

Games with mixed strategies are like games with continuous strategies since a probability is drawn from the continuum between zero and one. Auditing Game III also has a strategy drawn from the interval between zero and one, but it is not a mixed strategy to pick an audit probability of, say, 70 percent. A mixed strategy in that game would be something like a choice of a probability 0.5 of an audit probability of 60 percent and 0.5 of 80 percent. The big difference between the pure strategy choice of an audit probability of .70 and the mixed strategy of (.7-audit, .3-don't audit) is that the pure strategy is an irreversible choice that might be used even when the player is not indifferent between pure strategies, but the mixed strategy is the result of a player who in equilibrium is indifferent as to what he does. The next section will show another difference between mixed strategies and continuous strategies: the payoffs are linear in the mixed-strategy probability, as is evident from payoff equations (14) and (15), but they can be nonlinear in continuous strategies generally.

I have used auditing here mainly to illustrate what mixed strategies are and are not, but auditing is interesting in itself and optimal auditing schemes have many twists to them. An example is the idea of **cross-checking**. Suppose an auditor is supposed to check the value of some variable $x \in [0,1]$, but his employer is worried that he will not report the true value. This might be because the auditor will be lazy and guess rather than go to the effort of finding x, or because some third party will bribe him, or that certain values of

x will trigger punishments or policies the auditor dislikes (this model applies even if x is the auditor's own performance on some other task). The idea of cross-checking is to hire a second auditor and ask him to simultaneously report x. Then, if both auditors report the same x, they are both rewarded, but if they report different values they are both punished. There will still be multiple equilibria, because anything in which they report the same value is an equilibrium. But at least truthful reporting becomes a possible equilibrium. See Kandori & Matsushima (1998) for details.

3.5 Continuous Strategies: The Cournot Game

Most of the games so far in the book have had discrete strategy spaces: *Aid* or *No Aid*, *Confess* or *Deny*. Quite often when strategies are discrete and moves are simultaneous, no pure- strategy equilibrium exists. The only sort of compromise possible in the Welfare Game, for instance, is to choose *Aid* sometimes and *No Aid* sometimes, a mixed strategy. If "*A Little Aid*" were a possible action, maybe there would be a pure-strategy equilibrium. The simultaneous-move game we discuss next, the Cournot Game, has a continuous strategy space even without mixing. It models a duopoly in which two firms choose output levels in competition with each other.

The Cournot Game

Players
Firms Apex and Brydox

The Order of Play
Apex and Brydox simultaneously choose quantities q_a and q_b from the set $[0, \infty)$.

Payoffs
Marginal cost is constant at $c = 12$. Demand is a function of the total quantity sold, $Q = q_a + q_b$, and we will assume it to be linear (for generalization see Chapter 14), and, in fact, will use the following specific function:

$$p(Q) = 120 - q_a - q_b. \tag{30}$$

Payoffs are profits, which are given by a firm's price times its quantity minus its costs, i.e.,

$$\begin{array}{l} \pi_{Apex} = (120 - q_a - q_b)q_a - cq_a = (120 - c)q_a - q_a^2 - q_a q_b; \\ \pi_{Brydox} = (120 - q_a - q_b)q_b = (120 - c)q_b - q_a q_b - q_b^2. \end{array} \tag{31}$$

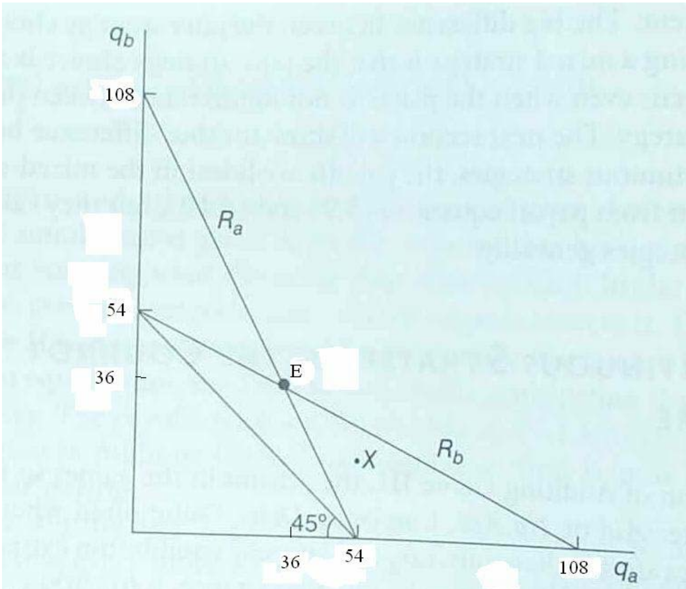


Figure 2: Reaction Curves in The Cournot Game

If this game were cooperative (see Section 1.2), firms would end up producing somewhere on the 45° line in Figure 2, where total output is the monopoly output and maximizes the sum of the payoffs. The monopoly output maximizes $pQ - cQ = (120 - Q - c)Q$ with respect to the total output of Q, resulting in the first-order condition

$$120 - c - 2Q = 0, \tag{32}$$

which implies a total output of $Q = 54$ and a price of 66. Deciding how much of that output of 54 should be produced by each firm—where the firm's output should be located on the 45 ° line—would be a zero-sum cooperative game, an example of bargaining. But since the Cournot Game is noncooperative, the strategy profiles such that $q_a + q_b = 54$ are not necessarily equilibria despite their Pareto optimality (where Pareto optimality is defined from the point of view of the two players, not of consumers, and under the implicit assumption that price discrimination cannot be used).

Cournot noted in Chapter 7 of his 1838 book that this game has a unique equilibrium when demand curves are linear. To find that "Cournot-Nash" equilibrium, we need to refer to the **best response functions** for the two players. If Brydox produced 0, Apex would produce the monopoly output of 54. If Brydox produced $q_b = 108$ or greater, the market price would fall to 12 and Apex would choose to produce zero. The best response function is found by maximizing Apex's payoff, given in equation (31), with respect to his strategy, q_a. This generates the first order condition $120 - c - 2q_a - q_b = 0$, or

$$q_a = 60 - \frac{q_b + c}{2} = 54 - \frac{q_b}{2}. \tag{33}$$

Another name for the best response function, the name usually used in the context of the Cournot Game, is the **reaction function**. Both names are somewhat misleading since the players move simultaneously with no chance to reply or react, but they are useful in imagining what a player would do if the rules of the game did allow him to move second. The reaction functions of the two firms are labelled R_a and R_b in Figure 2. Where they cross, point E, is the **Cournot-Nash equilibrium**, which is simply the Nash equilibrium when the strategies consist of quantities. Algebraically, it is found by solving the two reaction functions for q_a and q_b, which generates the unique equilibrium, $q_a = q_b = 40 - c/3 = 36$. The equilibrium price is then 48 (= 120-36-36).

In the Cournot Game, the Nash equilibrium has the particularly nice property of **stability**: we can imagine how starting from some other strategy profile the players might reach the equilibrium. If the initial strategy profile is point X in Figure 2, for example, Apex's best response is to decrease q_a and Brydox's is to increase q_b, which moves the profile closer to the equilibrium. But this is special to The Cournot Game, and Nash equilibria are not always stable in this way.

Stackelberg Equilibrium

There are many ways to model duopoly. The three most prominent are Cournot, Stackelberg, and Bertrand. Stackelberg equilibrium differs from Cournot in that one firm gets to choose its quantity first. If Apex moved first, what output would it choose? Apex knows how Brydox will react to its choice, so it picks the point on Brydox's reaction curve that maximizes Apex's profit (see Figure 3).

The Stackelberg Game

Players
Firms Apex and Brydox

The Order of Play
1. Apex chooses quantity q_a from the set $[0, \infty)$.
2. Brydox chooses quantity q_b from the set $[0, \infty)$.

Payoffs
Marginal cost is constant at $c = 12$. Demand is a function of the total quantity sold, $Q = q_a + q_b$:

$$p(Q) = 120 - q_a - q_b. \tag{34}$$

Payoffs are profits, which are given by a firm's price times its quantity minus its costs, i.e.,

$$\begin{array}{l} \pi_{Apex} = (120 - q_a - q_b)q_a - cq_a = (120 - c)q_a - q_a^2 - q_a q_b; \\ \pi_{Brydox} = (120 - q_a - q_b)q_b = (120 - c)q_b - q_a q_b - q_b^2. \end{array} \tag{35}$$

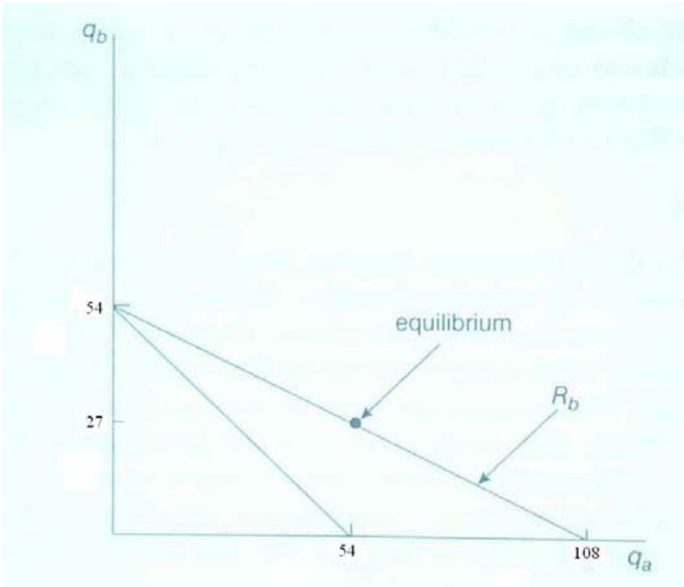


Figure 3: Stackelberg Equilibrium

Apex, moving first, is called the **Stackelberg leader** and Brydox is the **Stackelberg follower.** The distinguishing characteristic of a Stackelberg equilibrium is that one player gets to commit himself first. In Figure 3, Apex moves first intertemporally. If moves were simultaneous but Apex could commit himself to a certain strategy, the same equilibrium would be reached as long as Brydox was not able to commit himself. Algebraically, since Apex forecasts Brydox's output to be $q_b = 60 - \frac{q_a+c}{2}$ from the analog of equation (33), Apex can substitute this into his payoff function in (31), obtaining

$$\pi_a = (120 - c)q_a - q_a^2 - q_a(60 - \frac{q_a + c}{2}). \tag{36}$$

Maximizing with respect to q_a yields the first order condition

$$(120 - c) - 2q_a - 60 + q_a + \frac{c}{2} = 0, \tag{37}$$

which generates $q_a = 60 - \frac{c}{2} = 54$ (which only equals the monopoly output by coincidence, due to the particular numbers in this example). Once Apex chooses this output, Brydox chooses his output to be $q_b = 27$. (That Brydox chooses exactly half the monopoly output is also accidental.) The market price is 120-54-27= 39 for both firms, so Apex has benefited from his status as Stackelberg leader, but industry profits have fallen compared to the Cournot equilibrium.

3.6 Continuous Strategies: The Bertrand Game, Strategic Complements, and Strategic Substitutes (formerly 14.2, new)

A natural alternative to a duopoly model in which the two firms pick outputs simultaneously is a model in which they pick prices simultaneously. This is known as **Bertrand equilibrium**, because the difficulty of choosing between the two models was stressed in

Bertrand (1883), a review discussion of Cournot's book. We will use the same two-player linear-demand world as before, but now the strategy spaces will be the prices, not the quantities. We will also use the same demand function, equation (30), which implies that if p is the lowest price, $q = 120 - p$. In the Cournot model, firms chose quantities but allowed the market price to vary freely. In the Bertrand model, they choose prices and sell as much as they can.

The Bertrand Game

Players
Firms Apex and Brydox

The Order of Play
Apex and Brydox simultaneously choose prices p_a and p_b from the set $[0, \infty)$.

Payoffs
Marginal cost is constant at $c = 12$. Demand is a function of the total quantity sold, $Q(p) = 120 - p$. The payoff function for Apex (Brydox's would be analogous) is

$$\pi_a = \begin{cases} (120 - p_a)(p_a - c) & \text{if } p_a \leq p_b \\ \frac{(120-p_a)(p_a-c)}{2} & \text{if } p_a = p_b \\ 0 & \text{if } p_a > p_b \end{cases}$$

The Bertrand Game has a unique Nash equilibrium: $p_a = p_b = c = 12$. That this is a weak Nash equilibrium is clear: if either firm deviates to a higher price, it loses all its customers and so fails to increase its profits to above zero. (In fact, this is an example of a Nash equilibrium in weakly dominated strategies.) That it is unique is less clear. To see why, divide the strategy profiles into four groups:

> <u>$p_a < c$ or $p_b < c$.</u> In either of these cases, the firm with the lowest price will earn negative profits, and could profitably deviate to a price high enough to reduce its demand to zero.
>
> <u>$p_a > p_b > c$ or $p_b > p_a > c$.</u> In either of these cases the firm with the higher price could deviate to a price below its rival and increase its profits from zero to some positive value.
>
> <u>$p_a = p_b > c$.</u> In this case, Apex could deviate to a price ϵ less than Brydox and its profit would rise, because it would go from selling half the market quantity to selling all of it with an infinitesimal decline in profit per unit sale.
>
> <u>$p_a > p_b = c$ or $p_b > p_a = c$.</u> In this case, the firm with the price of c could move from zero profits to positive profits by increasing its price slightly while keeping it below the other firm's price.

This proof is a good example of one common method of proving uniqueness of equilibrium in game theory: partition the strategy profile space and show area by area that deviations would occur. It is such a good example that I recommend it to anyone teaching from this book as a good test question.[6]

Like the surprising outcome of Prisoner's Dilemma, the Bertrand equilibrium is less surprising once one thinks about the model's limitations. What it shows is that duopoly profits do not arise just because there are two firms. Profits arise from something else, such as multiple periods, incomplete information, or differentiated products.

Both the Bertrand and Cournot models are in common use. The Bertrand model can be awkward mathematically because of the discontinuous jump from a market share of 0 to 100 percent after a slight price cut. The Cournot model is useful as a simple model that avoids this problem and which predicts that the price will fall gradually as more firms enter the market. There are also ways to modify the Bertrand model to obtain intermediate prices and gradual effects of entry. Let us proceed to look at one such modification.

The Differentiated Bertrand Game

The Bertrand model generates zero profits because only slight price discounts are needed to bid away customers. The assumption behind this is that the two firms sell identical goods, so if Apex's price is slightly higher than Brydox's all the customers go to Brydox. If customers have brand loyalty or poor price information, the equilibrium is different. Let us now move to a different duopoly market, where the demand curves facing Apex and Brydox are

$$q_a = 24 - 2p_a + p_b \tag{38}$$

and

$$q_b = 24 - 2p_b + p_a, \tag{39}$$

and they have constant marginal costs of $c = 3$.

The greater the difference in the coefficients on prices in demand curves like these, the less substitutable are the products. As with standard demand curves like (30), we have made implicit assumptions about the extreme points of (38) and (39). These equations only apply if the quantities demanded turn out to be nonnegative, and we might also want to restrict them to prices below some ceiling, since otherwise the demand facing one firm becomes infinite as the other's price rises to infinity. A sensible ceiling here is 12, since if $p_a > 12$ and $p_b = 0$, equation (38) would yield a negative quantity demanded for Apex. Keeping in mind these limitations, the payoffs are

$$\pi_a = (24 - 2p_a + p_b)(p_a - c) \tag{40}$$

and

$$\pi_b = (24 - 2p_b + p_a)(p_b - c). \tag{41}$$

[6] Is it still a good question given that I have just provided a warning to the students? Yes. First, it will prove a filter for discovering which students have even skimmed the assigned reading. Second, questions like this are not always easy even if one knows they are on the test. Third, and most important, even if in equilibrium every student answers the question correctly, that very fact shows that the incentive to learn this particular item has worked – and that is our main goal, is it not?

The order of play is the same as in The Bertrand Game (or Undifferentiated Bertrand Game, as we will call it when that is necessary to avoid confusion): Apex and Brydox simultaneously choose prices p_a and p_b from the set $[0, \infty)$.

Maximizing Apex's payoff by choice of p_a, we obtain the first- order condition,

$$\frac{d\pi_a}{dp_a} = 24 - 4p_a + p_b + 2c = 0, \tag{42}$$

and the reaction function,

$$p_a = 6 + \frac{c}{2} + \frac{1}{4}p_b = 7.5 + \frac{1}{4}p_b. \tag{43}$$

Since Brydox has a parallel first-order condition, the equilibrium occurs where $p_a = p_b = 10$. The quantity each firm produces is 14, which is below the 21 each would produce at prices of $p_a = p_b = c = 3$. Figure 4 shows that the reaction functions intersect. Apex's demand curve has the elasticity

$$\left(\frac{\partial q_a}{\partial p_a}\right) \cdot \left(\frac{p_a}{q_a}\right) = -2\left(\frac{p_a}{q_a}\right), \tag{44}$$

which is finite even when $p_a = p_b$, unlike in the undifferentiated-good Bertrand model.

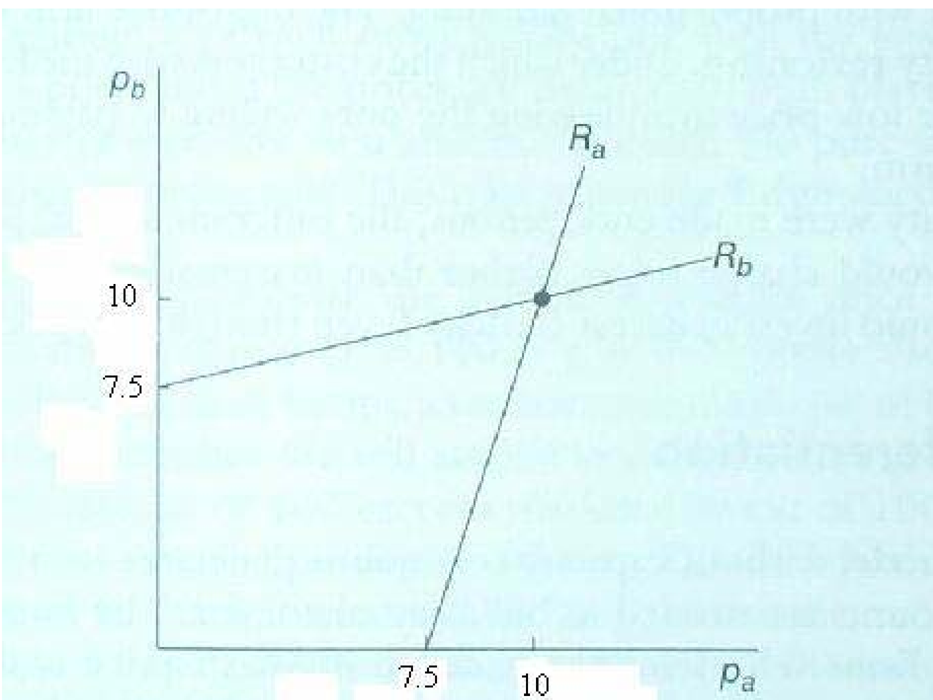


Figure 4: Bertrand Reaction Functions with Differentiated Products

The differentiated-good Bertrand model is important because it is often the most descriptively realistic model of a market.[7] A basic idea in marketing is that selling depends

[7]xxx Maybe I should do Dixit-Stiglitz in Chapter 14, taking off from this.

on "The Four P's": Product, Place, Promotion, and Price. Economists have concentrated heavily on price differences between products, but we realize that differences in product quality and characteristics, where something is sold, and how the sellers get information about it to the buyers also matter. Sellers use their prices as control variables more often than their quantities, but the seller with the lowest price does not get all the customers.

Why, then, did I bother to even describe the Cournot and Undifferentiated Bertrand models? Aren't they obsolete? No, because descriptive realism is not the *summum bonum* of modelling. Simplicity matters a lot too. The Cournot and Undifferentiated Bertrand models are simpler, especially when we go to three or more firms, so they are better models in many applications.

Strategic Substitutes and Strategic Complements

You may have noticed an interesting difference between the Cournot and Differentiated Bertrand reaction curves in Figures 2 and 4: the reaction curves have opposite slopes. Figure 5 puts the two together for easier comparison.

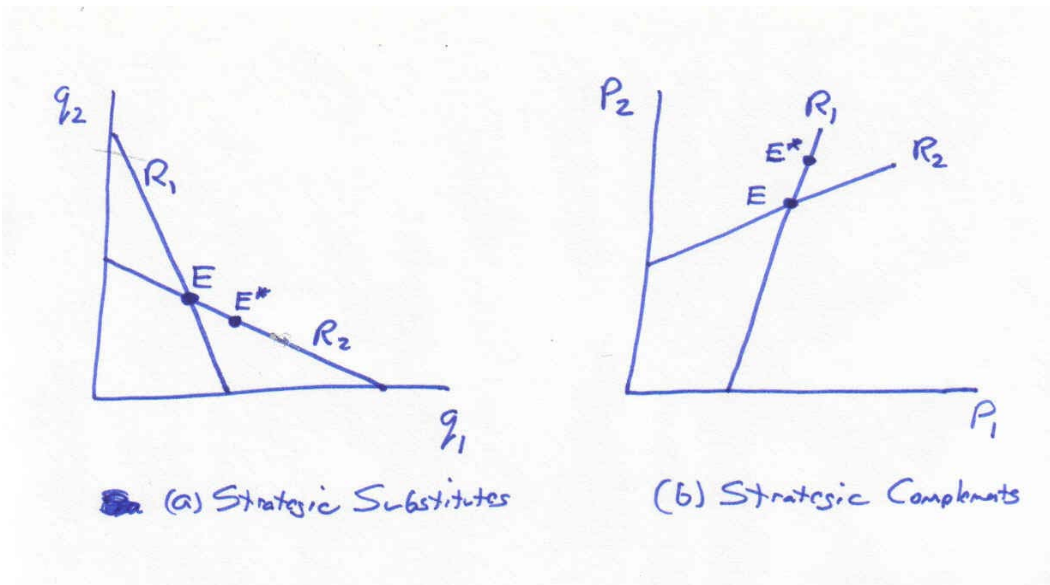


Figure 5: Cournot vs. Differentiated Bertrand Reaction Functions (Strategic Substitutes vs. Strategic Complements)

In both models, the reaction curves cross once, so there is a unique Nash equilibrium. Off the equilibrium path, though, there is an interesting difference. If a Cournot firm increases its output, its rival will do the opposite and reduce its output. If a Bertrand firm increases its price, its rival will do the same thing, and increase its price too.

We can ask of any game: "If the other players do more of their strategy, will I do more of my own strategy, or less?" In some games, the answer is "do more" and in others it is "do less". Jeremy Bulow, John Geanakoplos & Paul Klemperer (1985) call the strategies in the "do more" kind of game,"strategic complements," because when Player 1 does more of his strategy that increases Player 2's marginal payoff from 2's strategy, just as when I buy more bread it increases my marginal utility from buying more butter. If strategies are strategic complements, then their reaction curves are upward sloping, as in the Differentiated Bertrand Game.

On the other hand, in the "do less" kind of game, when Player 1 does more of his strategy that *reduces* Player 2's marginal payoff from 2's strategy, just as my buying potato chips reduces my marginal utility from buying more corn chips. The strategies are therefore "strategic substitutes" and their reaction curves are downward sloping, as in the Cournot Game.

Which way the reaction curves slope also affects whether a player wants to move first or second. Esther Gal-Or (1985) notes that if reaction curves slope down (as with strategic substitutes and Cournot) there is a first-mover advantage, whereas if they slope upwards (as with strategic complements and Differentiated Bertrand) there is a second-mover advantage.

We can see that in Figure 5. The Cournot Game in which Player 1 moves first is simply the Stackelberg Game, which we have already analyzed using Figure 3. The equilibrium moves from E to E^* in Figure 5a, Player 1's payoff increases, and Player 2's payoff falls. Note, too, that the total industry payoff is lower in Stackelberg than in Cournot– not only does one player lose, but he loses more than the other player gains.

We have not analyzed the Differentiated Bertrand Game when Player 1 moves first, but since price is a strategic complement, the effect of sequentiality is very different from in the Cournot Game (and, actually, from the sequential undifferentiated Bertrand Game– see the end-of-chapter notes). We cannot tell what Player 1's optimal strategy is from the diagram alone, but Figure 5 illustrates one possibility. Player 1 chooses a price p^* higher than he would in the simultaneous-move game, predicting that Player 2's response will be a price somewhat lower than p^*, but still greater than the simultaneous Bertrand price at E. The result is that Player 2's payoff is higher than Player 1's—a second-mover advantage. Note, however, that both players are better off at E^* than at E, so both players would favor converting the game to be sequential.

Both sequential games could be elaborated further by adding moves beforehand which would determine which player would choose his price or quantity first, but I will leave that to you. The important point for now is that whether a game has strategic complements or strategic substitutes is hugely important to the incentives of the players.

The point is simple enough and important enough that I devote an entire session of my MBA game theory course to strategic complements and strategic substitutes. In the practical game theory that someone with a Master of Business Administration degree ought to know, the most important thing is to learn how to describe a situation in terms of players, actions, information, and payoffs. Often there is not enough data to use a specific functional form, but it is possible to figure out with a mixture of qualitative and quantitative information whether the relation between actions and payoffs is one of strategic substitutes or strategic complements. The businessman then knows whether, for example, he should try to be a first mover or a second mover, and whether he should keep his action secret or proclaim his action to the entire world.

To understand the usefulness of the idea of strategic complements and substitutes, think about how you would model situations like the following (note that there is no universally right answer for any of them):

1. Two firms are choosing their research and development budgets. Are the budgets strategic complements or strategic substitutes?
2. Smith and Jones are both trying to be elected President of the United States. Each must decide how much he will spend on advertising in California. Are the advertising budgets strategic complements or strategic substitutes?
3. Seven firms are each deciding whether to make their products more special, or more suited to the average consumer. Is the degree of specialness a strategic complement or a strategic substitute?
4. Iran and Iraq are each deciding whether to make their armies larger or smaller. Is army size a strategic complement or a strategic substitute?

3.7 Existence of Equilibrium

One of the strong points of Nash equilibria is that they exist in practically every game one is likely to encounter. There are four common reasons why an equilibrium might not exist or might only exist in mixed strategies.

(1) An unbounded strategy space

Suppose in a stock market game that Smith can borrow money and buy as many shares x of stock as he likes, so his strategy set, the amount of stock he can buy, is $[0, \infty)$, a set which is unbounded above. (Note, by the way, that we thus assume that he can buy fractional shares, e.g. $x = 13.4$, but cannot sell short, e.g. $x = -100$.)

If Smith knows that the price is lower today than it will be tomorrow, his payoff function will be $\pi(x) = x$ and he will want to buy an infinite number of shares, which is not an equilibrium purchase. If the amount he buys is restricted to be less than or equal to 1,000, however, then the strategy set is bounded (by 1,000), and an equilibrium exists– $x =, 000$.

Sometimes, as in the Cournot Game discussed earlier in this chapter, the unboundedness of the strategy sets does not matter because the optimum is an interior solution. In other games, though, it is important, not just to get a determinate solution but because the real world is a rather bounded place. The solar system is finite in size, as is the amount of human time past and future.

(2) An open strategy space. Again consider Smith. Let his strategy be $x \in [0, 1, 000)$, which is the same as saying that $0 \leq x < 1, 000$, and his payoff function be

$\pi(x) = x$. Smith's strategy set is bounded (by 0 and 1,000), but it is open rather than closed, because he can choose any number less than 1,000, but not 1, 000 itself. This means no equilibrium will exist, because he wants to buy 999.999... shares. This is just a technical problem; we ought to have specified Smith strategy space to be $[0, 1, 000]$, and then an equilibrium would exist, at $x = 1, 000$.

(3) A discrete strategy space (or, more generally, a nonconvex strategy space). Suppose we start with an arbitrary pair of strategies s_1 and s_2 for two players. If the players' strategies are strategic complements, then if player 1 increases his strategy in response to s_2, then player 2 will increase his strategy in response to that. An equilibrium will occur where the players run into diminishing returns or increasing costs, or where they hit the upper bounds of their strategy sets. If, on the other hand, the strategies are strategic substitutes, then if player 1 increases his strategy in response to s_2, player 2 will in turn want to reduce his strategy. If the strategy spaces are continuous, this can lead to an equilibrium, but if they are discrete, player 2 cannot reduce his strategy just a little bit– he has to jump down a discrete level. That could then induce Player 1 to increase his strategy by a discrete amount. This jumping of responses can be never- ending–there is no equilibrium.

That is what is happening in The Welfare Game of Table 1 in this chapter. No compromise is possible between a little aid and no aid, or between working and not working– until we introduce mixed strategies. That allows for each player to choose a continuous amount of his strategy.

This problem is not limited to games such as 2-by-2 games that have discrete strategy spaces. Rather, it is a problem of "gaps" in the strategy space. Suppose we had a game in which the Government was not limited to amount 0 or 100 of aid, but could choose any amount in the space $\{[0, 10], [90, 100]\}$. That is a continuous, closed, and bounded strategy space, but it is non-convex– there is gap in it. (For a space $\{x\}$ to be convex, it must be true that if x_1 and x_2 are in the space, so is $\theta x_1 + (1 - \theta)x_2$ for any $\theta \in [0, 1]$.) Without mixed strategies, an equilibrium to the game might well not exist.

(4) A discontinuous reaction function arising from nonconcave or discontinuous payoff functions.

Even if the strategy spaces are closed, bounded, and convex, a problem remains. For a Nash equilibrium to exist, we need for the reaction functions of the players to intersect. If the reaction functions are discontinuous, they might not intersect.

Figure 6 shows this for a two-player game in which each player chooses a strategy from the interval between 0 and 1. Player 1's reaction function, $s_1(s_2)$, must pick one or more value of s_1 for each possible value of s_2, so it must cross from the bottom to the top of the diagram. Player 2's reaction function, $s_2(s_1)$, must pick one or more value of s_2 for each possible value of s_1, so it must cross from the left to the right of the diagram. If the strategy sets were unbounded or open, the reaction functions might not exist, but that is not a problem here: they do exist. And in Panel (a) a Nash equilibrium exists, at the point, E, where the two reaction functions intersect.

In Panel (b), however, no Nash equilibrium exists. The problem is that Firm 2's reaction function $s_2(s_1)$ is discontinuous at the point $s_1 = 0.5$. It jumps down from $s_2(0.5) = 0.6$ to $s_2(0.50001) = 0.4$. As a result, the reaction curves never intersect, and no equilibrium exists.

If the two players can use mixed strategies, then an equilibrium will exist even for the game in Panel (b), though I will not prove that here. I would, however, like to say why it is that the reaction function might be discontinuous. A player's reaction functions, remember, is derived by maximizing his payoff as a function of his own strategy given the strategies of the other players.

Thus, a first reason why Player 1's reaction function might be discontinuous in the other players' strategies is that his payoff function is discontinuous in either his own or the other players' strategies. This is what happens in the Hotelling Pricing Game, where if Player 1's price drops enough (or Player 2's price rises high enough), all of Player 2's customers suddenly rush to Player 1.

A second reason why Player 1's reaction function might be discontinuous in the other players' strategies is that his payoff function is not concave. The intuition is that if an objective function is not concave, then there might be a number of maxima that are local but not global, and as the parameters change, which maximum is the global one can suddenly change. This means that the reaction function will suddenly jump from one maximizing choice to another one that is far-distant, rather than smoothly changing as it would in a more nicely behaved problem.

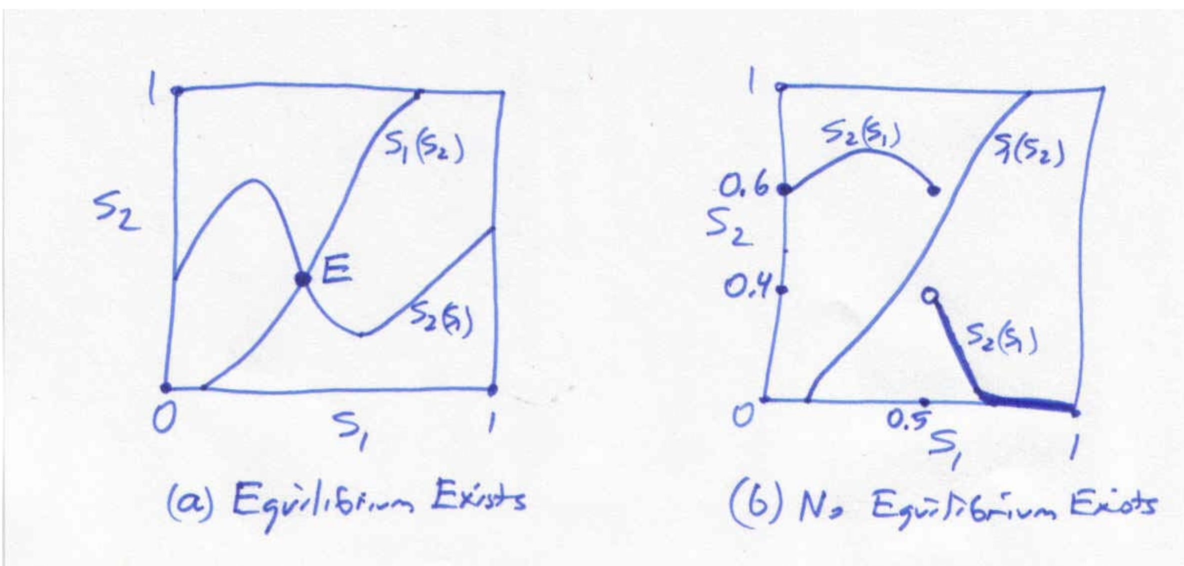


Figure 6: Continuous and Discontinuous Reaction Functions

Problems (1) and (2) are really problems in decision theory, not game theory, because unboundedness and openness lead to nonexistence of the solution to even a one-player maximization problem. Problems (3) and (4) are special to game theory. They arise because although each player has a best response to the other players, no profile of best choices is such that everybody has chosen his best response to everybody else. They are similar to the decision theory problem of nonexistence of an interior solution, but if only one player were involved, we would at least have a corner solution.

In this chapter, I have introduced a number of seemingly disparate ideas– mixed strategies, auditing, continuous strategy spaces, reaction curves, complentary substitutes and complements, existence of equilibrium... What ties them together? The unifying theme is the possibility of reaching equilibrium by small changes in behavior, whether that be by changing the probability in a mixed strategy or an auditing game or by changing the level of a continuous price or quantity. Continuous strategies free us from the need to use n-by-n tables to predict behavior in games, and with a few technical assumptions they guarantee we will find equilibria.

NOTES

N3.1 Mixed Strategies: The Welfare Game

- Waldegrave (1713) is a very early reference to mixed strategies.

- The January 1992 issue of *Rationality and Society* is devoted to attacks on and defenses of the use of game theory in the social sciences, with considerable discussion of mixed strategies and multiple equilibria. Contributors include Harsanyi, Myerson, Rapaport, Tullock, and Wildavsky. The Spring 1989 issue of the *RAND Journal of Economics* also has an exchange on the use of game theory, between Franklin Fisher and Carl Shapiro. I also recommend the Peltzman (1991) attack on the game theory approach to industrial organization in the spirit of the "Chicago School".

- In this book it will always be assumed that players remember their previous moves. Without this assumption of **perfect recall**, the definition in the text is not that for a mixed strategy, but for a **behavior strategy**. As historically defined, a player pursues a mixed strategy when he randomly chooses between pure strategies at the starting node, but he plays a pure strategy thereafter. Under that definition, the modeller cannot talk of random choices at any but the starting node. Kuhn (1953) showed that the definition of mixed strategy given in the text is equivalent to the original definition if the game has perfect recall. Since all important games have perfect recall and the new definition of mixed strategy is better in keeping with the modern spirit of sequential rationality, I have abandoned the old definition.

  The classic example of a game without perfect recall is **bridge**, where the four players of the actual game can be cutely modelled as two players who forget what half their cards look like at any one time in the bidding. A more useful example is a game that has been simplified by restricting players to Markov strategies (see Section 5.4), but usually the modeller sets up such a game with perfect recall and then rules out non-Markov equilibria after showing that the Markov strategies form an equilibrium for the general game.

- For more examples of calculating mixed strategy equilibria, see Sections 5.6, 11.6, 12.5, and 12.6.

- It is *not* true that when two pure-strategy equilibria exist a player would be just as willing to use a strategy mixing the two even when the other player is using a pure strategy. In Battle of the Sexes, for instance, if the man knows the woman is going to the ballet he is not indifferent between the ballet and the prize fight.

- A continuum of players is useful not only because the modeller need not worry about fractions of players, but because he can use more modelling tools from calculus— taking the integral of the quantities demanded by different consumers, for example, rather than the sum. But using a continuum is also mathematically more difficult: see Aumann (1964a, 1964b).

- There is an entire literature on the econometrics of estimating game theory models. Suppose we would like to estimate the payoff numbers in a 2-by-2 game, where we observe the actions taken by each of the two players and various background variables. The two actions might be, for example, to enter or not enter, and the background variables might be such things as the size of the market or the cost conditions facing one of the players. We will of course need multiple repetitions of the situation to generate enough data to use econometrics. There is an identification problem, because there are eight payoffs in a 2-by-2 payoff matrix, but only four possible action profiles– and if mixed strategies are being used, the four mixing probabilities have to add up to one, so there are really only three independent observed outcomes. How can we estimate 8 parameters with only 3 possible outcomes? For identification, it must be that some environmental variables affect only one of the players, as Bajari, Hong & Ryan (2004) note. In addition, there is the problem that there may be multiple equilibria being played out, so that additional identifying assumptions are needed to help us know which equilibria are being played out in which observations. The foundational articles in this literature are Bresnahan & Reiss (1990, 1991a), and it is an active area of research.

N3.2 Chicken, The War of Attrition, and Correlated Strategies

- The game of Chicken discussed in the text is simpler than the game acted out in the movie *Rebel Without a Cause,* in which the players race towards a cliff and the winner is the player who jumps out of his car last. The pure-strategy space in the movie game is continuous and the payoffs are discontinuous at the cliff's edge, which makes the game more difficult to analyze technically. (Recall, too, the importance in the movie of a disastrous mistake– the kind of "tremble" that Section 4.1 will discuss.)

- Technical difficulties arise in some models with a continuum of actions and mixed strategies. In the Welfare Game, the Government chose a single number, a probability, on the continuum from zero to one. If we allowed the Government to mix over a continuum of aid levels, it would choose a function, a probability density, over the continuum. The original game has a finite number of elements in its strategy set, so its mixed extension still has a strategy space in $\mathbf{R}^n$. But with a continuous strategy set extended by a continuum of mixed strategies for each pure strategy, the mathematics become difficult. A finite number of mixed strategies can be allowed without much problem, but usually that is not satisfactory.

  Games in continuous time frequently run into this problem. Sometimes it can be avoided by clever modelling, as in Fudenberg & Tirole's (1986b) continuous-time war of attrition with asymmetric information. They specify as strategies the length of time firms would proceed to *Continue* given their beliefs about the type of the other player, in which case there is a pure strategy equilibrium.

- **Differential games** are played in continuous time. The action is a function describing the value of a state variable at each instant, so the strategy maps the game's past history to such a function. Differential games are solved using dynamic optimization. A book-length treatment is Bagchi (1984).

- Fudenberg & Levine (1986) show circumstances under which the equilibria of games with infinite strategy spaces can be found as the limits of equilibria of games with finite strategy spaces.

N3.4 Randomizing Versus Mixing: The Auditing Game

- Auditing Game I is similar to a game called the Police Game. Care must be taken in such games that one does not use a simultaneous- move game when a sequential game is appropriate. Also, discrete strategy spaces can be misleading. In general, economic analysis assumes that costs rise convexly in the amount of an activity and benefits rise concavely.

Modelling a situation with a 2-by-2 game uses just two discrete levels of the activity, so the concavity or convexity is lost in the simplification. If the true functions are linear, as in auditing costs which rise linearly with the probability of auditing, this is no great loss. If the true costs rise convexly, as in the case where the hours a policeman must stay on the street each day are increased, then a 2-by-2 model can be misleading. Be especially careful not to press the idea of a mixed-strategy equilibrium too hard if a pure-strategy equilibrium would exist when intermediate strategies are allowed. See Tsebelis (1989) and the criticism of it in J. Hirshleifer & Rasmusen (1992).

- Douglas Diamond (1984) shows the implications of monitoring costs for the structure of financial markets. A fixed cost to monitoring investments motivates the creation of a financial intermediary to avoid repetititive monitoring by many investors.

- Baron & Besanko (1984) study auditing in the context of a government agency which can at some cost collect information on the true production costs of a regulated firm.

- Mookherjee & Png (1989) and Border & Sobel (1987) have examined random auditing in the context of taxation. They find that if a taxpayer is audited he ought to be more than compensated for his trouble if it turns out he was telling the truth. Under the optimal contract, the truth-telling taxpayer should be delighted to hear that he is being audited. The reason is that a reward for truthfulness widens the differential between the agent's payoff when he tells the truth and when he lies.

  Why is such a scheme not used? It is certainly practical, and one would think it would be popular with the voters. One reason might be the possibility of corruption; if being audited leads to a lucrative reward, the government might purposely choose to audit its friends. The current danger seems even worse, though, since the government can audit its enemies and burden them with the trouble of an audit even if they have paid their taxes properly.

- Government action strongly affects what information is available as well as what is contractible. In 1988, for example, the United States passed a law sharply restricting the use of lie detectors for testing or monitoring. Previous to the restriction, about two million workers had been tested each year. ("Law Limiting Use of Lie Detectors is Seen Having Widespread Effect" *Wall Street Journal*, p. 13, 1 July 1988), "American Polygraph Association," http: //www.polygraph.org/betasite/menu8.html (Viewed August 31,2003), Eric Rasmusen, "Bans on Lie Detector Tests," http://mypage.iu.edu/∼erasmuse/archives1.htm#august10a (Viewed August 31, 2003).)

- Section 3.4 shows how random actions come up in auditing and in mixed strategies. Another use for randomness is to reduce transactions costs. In 1983, for example, Chrysler was bargaining over how much to pay Volkswagen for a Detroit factory. The two negotiators locked themselves into a hotel room and agreed not to leave till they had an agreement. When they narrowed the price gap from $100 million to $5 million, they agreed to flip a coin. (Chrysler won.) How would you model that? "Chrysler Hits Brakes, Starts Saving Money After Shopping Spree, " *Wall Street Journal*, p. 1, 12 January 1988. See also David Friedman's ingenious idea in Chapter 15 of *Law's Order* of using a 10% probability of death to replace a 6-year prison term (http: //www.daviddfriedman.com/Academic/Course_Pages/L_and_E_LS_9 8 /Why_Is _aw/ Why_Is_Law_Chapter_15/Why_Is_Law_Chapter_15.ht ml [viewed August 31, 2003])

N3.5 Continuous Strategies: The Cournot Game

- An interesting class of simple continuous payoff games are the **Colonel Blotto games** (Tukey [1949], McDonald & Tukey [1949]). In these games, two military commanders allocate their forces to m different battlefields, and a battlefield contributes more to the payoff of the commander with the greater forces there. A distinguishing characteristic is that player i's payoff increases with the value of player i's particular action relative to player j's, and i's actions are subject to a budget constraint. Except for the budget constraint, this is similar to the tournaments of Section 8.2.

- Considerable work has been done characterizing the Cournot model. A representative article is Gaudet & Salant (1991) on conditions which ensure a unique equilibrium.

- "Stability" is a word used in many different ways in game theory and economics. The natural meaning of a stable equilibrium is that it has dynamics which cause the system to return to that point after being perturbed slightly, and the discussion of the stability of Cournot equilibrium was in that spirit. The uses of the term by von Neumann & Morgenstern (1944) and Kohlberg & Mertens (1986) are entirely different.

- The term "Stackelberg equilibrium" is not clearly defined in the literature. It is sometimes used to denote equilibria in which players take actions in a given order, but since that is just the perfect equilibrium (see Section 4.1) of a well-specified extensive form, I prefer to reserve the term for the Nash equilibrium of the duopoly quantity game in which one player moves first, which is the context of Chapter 3 of Stackelberg (1934).

  An alternative definition is that a Stackelberg equilibrium is a strategy profile in which players select strategies in a given order and in which each player's strategy is a best response to the fixed strategies of the players preceding him and the yet-to-be-chosen strategies of players succeeding him, i.e., a situation in which players precommit to strategies in a given order. Such an equilibrium would not generally be either Nash or perfect.

- Stackelberg (1934) suggested that sometimes the players are confused about which of them is the leader and which the follower, resulting in the disequilibrium outcome called **Stackelberg warfare**.

- With linear costs and demand, total output is greater in Stackelberg equilibrium than in Cournot. The slope of the reaction curve is less than one, so Apex's output expands more than Brydox's contracts. Total output being greater, the price is less than in the Cournot equilibrium.

- A useful application of Stackelberg equilibrium is to an industry with a dominant firm and a **competitive fringe** of smaller firms that sell at capacity if the price exceeds their marginal cost. These smaller firms act as Stackelberg leaders (not followers), since each is small enough to ignore its effect on the behavior of the dominant firm. The oil market could be modelled this way with OPEC as the dominant firm and producers such as Britain on the fringe.

N3.6 Continuous Strategies: The Bertrand Game, Strategic Complements, and Strategic Subsitutes (formerly Section 14.2)

- The text analyzed the simultaneous undifferentiated Bertrand game but not the sequential one. $p_a = p_c = c$ remains an equilibrium outcome, but it is no longer unique. Suppose Apex moves first, then Brydox, and suppose, for a technical reason to be apparent shortly, that if $p_a = p_b$ Brydox captures the entire market. Apex cannot achieve more than a payoff of zero, because either $p_a = c$ or Brydox will choose $p_b = p_a$ and capture the entire market. Thus, Apex is indifferent between any $p_a \geq c$.

The game needs to be set up with this tiebreaking rule because if split the market between Apex and Brydox when $p_a = p_b$, Brydox's best response to $p_a > c$ would be to choose p_b to be the biggest number less than p_a– but with a continuous space, no such number exists, so Brydox's best response is ill-defined. Giving all the demand to Brydox in case of price ties gets around this problem.

- We can also work out the Cournot equilibrium for demand functions (38) and (39), but product differentiation does not affect it much. Start by expressing the price in the demand curve in terms of quantities alone, obtaining

$$p_a = 12 - \frac{1}{2}q_a + \frac{1}{2}p_b \tag{45}$$

and

$$p_b = 12 - \frac{1}{2}q_b + \frac{1}{2}p_a. \tag{46}$$

After substituting from (46) into (45) and solving for p_a, we obtain

$$p_a = 24 - \frac{2}{3}q_a - \frac{1}{3}q_b. \tag{47}$$

The first-order condition for Apex's maximization problem is

$$\frac{d\pi_a}{dq_a} = 24 - 3 - \frac{4}{3}q_a - \frac{1}{3}q_b = 0, \tag{48}$$

which gives rise to the reaction function

$$q_a = 15.75 - \frac{1}{4}q_b. \tag{49}$$

We can guess that $q_a = q_b$. It follows from (49) that $q_a = 12.6$ and the market price is 11.4. On checking, you would find this to indeed be a Nash equilibrium. But reaction function (49) has much the same shape as if there were no product differentiation, unlike when we moved from undifferentiated Bertrand to differentiated Bertrand competition.

- For more on the technicalities of strategic complements and strategic substitutes, see Bulow, Geanakoplos & Klemperer (1985) and Milgrom & Roberts (1990). If the strategies are strategic complements, Milgrom & Roberts (1990) and Vives (1990) show that pure-strategy equilibria exist. These models often explain peculiar economic phenomenon nicely, as in Peter Diamond (1982) on search and business cycles and Douglas Diamond and P. Dybvig (1983) on bank runs. If the strategies are strategic substitutes, existence of pure-strategy equilibria is more troublesome; see Pradeep Dubey, Ori Haimanko & Andriy Zapechelnyuk (2002).

Problems

3.1. Presidential Primaries

Smith and Jones are fighting it out for the Democratic nomination for President of the United States. The more months they keep fighting, the more money they spend, because a candidate must spend one million dollars a month in order to stay in the race. If one of them drops out, the other one wins the nomination, which is worth 11 million dollars. The discount rate is r per month. To simplify the problem, you may assume that this battle could go on forever if neither of them drops out. Let θ denote the probability that an individual player will drop out each month in the mixed-strategy equilibrium.

(a) In the mixed-strategy equilibrium, what is the probability θ each month that Smith will drop out? What happens if r changes from 0.1 to 0.15?

(b) What are the two pure-strategy equilibria?

(c) If the game only lasts one period, and the Republican wins the general election (for Democrat payoffs of zero) if both Democrats refuse to exit, what is the probability γ with which each candidate exits in a symmetric equilibrium?

3.2. Running from the Police

Two risk-neutral men, Schmidt and Braun, are walking south along a street in Nazi Germany when they see a single policeman coming to check their papers. Only Braun has his papers (unknown to the policeman, of course). The policeman will catch both men if both or neither of them run north, but if just one runs, he must choose which one to stop— the walker or the runner. The penalty for being without papers is 24 months in prison. The penalty for running away from a policeman is 24 months in prison, on top of the sentences for any other charges, but the conviction rate for this offense is only 25 percent. The two friends want to maximize their joint welfare, which the policeman wants to minimize. Braun moves first, then Schmidt, then the policeman.

(a) What is the outcome matrix for outcomes that might be observed in equilibrium? (Use θ for the probability that the policeman chases the runner and γ for the probability that Braun runs.)

(b) What is the probability that the policeman chases the runner, (call it θ^*)?

(c) What is the probability that Braun runs, (call it γ^*)?

(d) Since Schmidt and Braun share the same objectives, is this a cooperative game?

(a) Draw the outcome matrix for Matching Pennies.

(b) Show that there is no Nash equilibrium in pure strategies.

(c) Find the mixed-strategy equilibrium, denoting Smith's probability of *Heads* by γ and Jones's by θ.

(d) Prove that there is only one mixed-strategy equilibrium.

3.4. Mixed Strategies in The Battle of the Sexes
Refer back to The Battle of the Sexes and Ranked Coordination in Section 1.4. Denote the probabilities that the man and woman pick *Prize Fight* by γ and θ.

(a) Find an expression for the man's expected payoff.

(b) What are the equilibrium values of γ and θ, and the expected payoffs?

(c) Find the most likely outcome and its probability.

(d) What is the equilibrium payoff in the mixed-strategy equilibrium for Ranked Coordination ?

(e) Why is the mixed-strategy equilibrium a better focal point in the Battle of the Sexes than in Ranked Coordination?

3.5. A Voting Paradox
Adam, Karl, and Vladimir are the only three voters in Podunk. Only Adam owns property. There is a proposition on the ballot to tax property-holders 120 dollars and distribute the proceeds equally among all citizens who do not own property. Each citizen dislikes having to go to the polling place and vote (despite the short lines), and would pay 20 dollars to avoid voting. They all must decide whether to vote before going to work. The proposition fails if the vote is tied. Assume that in equilibrium Adam votes with probability θ and Karl and Vladimir each vote with the same probability γ, but they decide to vote independently of each other.

(a) What is the probability that the proposition will pass, as a function of θ and γ?

(b) What are the two possible equilibrium probabilities γ_1 and γ_2 with which Karl might vote? Why, intuitively, are there two symmetric equilibria?

(c) What is the probability θ that Adam will vote in each of the two symmetric equilibria?

(d) What is the probability that the proposition will pass?

3.6. Industry Output
Industry output is
(a) lowest with monopoly, highest with a Cournot equilibrium
(b) lowest with monopoly, highest with a Stackelberg equilibrium.
(c) lowest with a Cournot, highest with a Stackelberg equilibrium.
(d) lowest with a Stackelberg, highest with a Cournot equilibrium.

3.7. Duopoly Output
Three firms producing an identical product face the demand curve $P = 240 - \alpha Q$, and produce at marginal cost β. Each firm picks its quantity simultaneously. If $\alpha = 1$ and $\beta = 40$, the equilibrium output of the industry is in the interval
(a) $[0, 20]$
(b) $[20, 100]$
(c) $[100, 130]$

(d) $[130, 200]$
(e) $[200, \infty]$

3.8. Mixed Strategies
If a player uses mixed strategies in equilibrium,
(a) All players are indifferent among all their strategies.
(b) That player is indifferent among all his strategies.
(c) That player is indifferent among the strategies he has a positive probability of choosing in equilibrium.
(d) That player is indifferent among all his strategies except the ones that are weakly dominated.
(e) None of the above.

3.9. Nash Equilibrium
Find the unique Nash equilibrium of the game in Table 9.

Table 9: A Game for the 1996 Midterm

		Column		
		Left	*Middle*	*Right*
	Up	1,0	$10, -1$	$0, 1$
Row:	*Sideways*	$-1, 0$	-2,-2	$-12, 4$
	Down	$0, 2$	823,-1	$2, 0$

Payoffs to: (Row, Column).

3.10. Triopoly
Three companies provide tires to the Australian market. The total cost curve for a firm making Q tires is $TC = 5 + 20Q$, and the demand equation is $P = 100 - N$, where N is the total number of tires on the market.

According to the Cournot model, in which the firms's simultaneously choose quantities, what will the total industry output be?

3.11 (hard). Cournot with Heterogeneous Costs
On his job visit, Professor Schaffer of Michigan told me that in a Cournot model with a linear demand curve $P = \alpha - \beta Q$ and constant marginal cost C_i for firm i, the equilibrium industry output Q depends on $\Sigma_i C_i$, but not on the individual levels of C_i. I may have misremembered. Prove or disprove this assertion. Would your conclusion be altered if we made some other assumption on demand? Discuss.

(a) With what probability θ would the Curiatii give chase if Horatius were to run?

(b) With what probability γ does Horatius run?

(c) How would θ and γ be affected if the Curiatii falsely believed that the probability of Horatius being panic-stricken was 1? What if they believed it was 0.9?

3.13. Finding Nash Equilibria
Find all of the Nash equilibria for the game of Table 10.

Table 10: A Takeover Game

		Target		
		Hard	*Medium*	*Soft*
	Hard	-3, -3	$-1, 0$	$4, 0$
Raider:	*Medium*	$0, 0$	2,2	$3, 1$
	Soft	0,0	$2, 4$	$3, 3$

Payoffs to: (Raider, Target).

3.14. Risky Skating
Elena and Mary are the two leading figure skaters in the world. Each must choose during her training what her routine is going to look like. They cannot change their minds later and try to alter any details of their routines. Elena goes first in the Olympics, and Mary goes next. Each has five minutes for her performance. The judges will rate the routines on three dimensions, beauty, how high they jump, and whether they stumble after they jump. A skater who stumbles is sure to lose, and if both Elena and Mary stumble, one of the ten lesser skaters will win, though those ten skaters have no chance otherwise.

Elena and Mary are exactly equal in the beauty of their routines, and both of them know this, but they are not equal in their jumping ability. Whoever jumps higher without stumbling will definitely win. Elena's probability of stumbling is $P(h)$, where h is the height of the jump, and P is increasing smoothly and continuously in h. (In calculus terms, let P' and P'' both exist, and P' be positive) Mary's probability is $P(h) - .1$— that is, it is 10 percent less for equal heights.

Let us define as h=0 the maximum height that the lesser skaters can achieve, and assume that P(0) = 0.

(a) Show that it cannot be an equilibrium for both Mary and Elena to choose the same value for h (Call them M and E).

(b) Show for any pair of values (M, E) that it cannot be an equilibrium for Mary and Elena to choose those values.

(c) Describe the optimal strategies to the best of your ability. (Do not get hung up on trying to answer this question; my expectations are not high here.)

(d) What is a business analogy? Find some situation in business or economics that could use this same model.

3.15. The Kosovo War
Senator Robert Smith of New Hampshire said of the US policy in Serbia of bombing but promising not to use ground forces, "It's like saying we'll pass on you but we won't run the football." (*Human Events*, p. 1, April 16, 1999.) Explain what he meant, and why this is a strong criticism of U.S. policy, using the concept of a mixed strategy equilibrium. (Foreign students: in American football, a team can choose to throw the football (to pass it) or to hold it and run with it to move towards the goal.) Construct a numerical example to compare the U.S. expected payoff in (a) a mixed strategy equilibrium in which it ends up not using ground forces, and (b) a pure strategy equilibrium in which the U.S. has committed not to use ground forces.

3.16. IMF Aid
Consider the game of Table 11.

Table 11: IMF Aid.

		Debtor	
		Reform	Waste
IMF	Aid	3,2	-1,3
	No Aid	-1,1	0,0

Payoffs to: (IMF, Debtor).

(a) What is the exact form of every Nash equilibrium?

(b) For what story would this matrix be a good model?

3.17. Coupon Competition
Two marketing executives are arguing. Smith says that reducing our use of coupons will make us a less aggressive competitor, and that will hurt our sales. Jones says that reducing our use of coupons will make us a less aggressive competitor, but that will end up helping our sales.

Discuss, using the effect of reduced coupon use on your firm's reaction curve, under what circumstance each executive could be correct.

3.18. Rent Seeking
I mentioned that Rogerson (1982) uses a game very similar to "Patent Race for a New Market" on p. 374 to analyze competition for a government monopoly franchise.[8] See if you can do this too. What can you predict about the welfare results of such competition?

3.19. Does not exist. xxx

3.20. Entry for Buyout
Find the equilibrium in "Entry for Buyout" if all the parameters of the numerical example on page 388[9] are the same except that the marginal cost of output is $c = 20$ instead of $c = 10$.

[8]xxx Fix page reference.
[9]xxx Fix this.

The War of Attrition: A Classroom Game for Chapter 3

Each firm consists of 3 students. Each year a firm must decide whether to stay in the industry or to exit. If it stays in, it incurs a fixed cost of 300 and a marginal cost of 2, and it chooses an integer price at which to sell. The firms can lose unlimited amounts of money; they are backed by large corporations who will keep supplying them with capital indefinitely.

Demand is inelastic at 60 up to a threshold price of $10/unit, above which the quantity demanded falls to zero.

Each firm writes down its price (or the word "EXIT") on a piece of paper and gives it to the instructor. The instructor then writes the strategies of each firm on the blackboard (EXIT or price), and the firms charging the lowest price split the 60 consumers evenly.

The game then starts with a new year, but any firm that has exited is out permanently and cannot re-enter. The game continues until only one firm is active, in which case it is awarded a prize of $2,000, the capitalized value of being a monopolist. This means the game can continue forever, in theory. The instructor may wish to cut it off at some point, however.

The game can then be restarted and continued for as long as class time permits.

For instructors' notes, go to http://www.rasmusen.org/GI/probs/03_attritiongame.pdf.

September 22, 1999. December 7, 2003. November 11, 2004. 25 March 2005. Eric Rasmusen, Erasmuse@indiana.edu. Http://www.rasmusen.org. Footnotes starting with xxx are the author's notes to himself. Comments welcomed.

4 Dynamic Games with Symmetric Information

4.1 Subgame Perfectness

In this chapter we will make heavy use of the extensive form to study games with moves that occur in sequence. We start in Section 4.1 with a refinement of the Nash equilibrium concept called perfectness that incorporates sensible implications of the order of moves. Perfectness is illustrated in Section 4.2 with a game of entry deterrence. Section 4.3 expands on the idea of perfectness using the example of nuisance suits, meritless lawsuits brought in the hopes of obtaining a settlement out of court. Nuisance suits show the importance of a threat being made credible and how sinking costs early or having certain nonmonetary payoffs can benefit a player. This example will also be used to discuss the open-set problem of weak equilibria in games with continuous strategy spaces, in which a player offering a contract chooses its terms to make the other player indifferent about accepting or rejecting. The last perfectness topic will be renegotiation: the idea that when there are multiple perfect equilibria, the players will coordinate on equilibria that are Pareto optimal in subgames but not in the game as a whole.

The Perfect Equilibrium of Follow the Leader I

Subgame perfectness is an equilibrium concept based on the ordering of moves and the distinction between an equilibrium path and an equilibrium. The **equilibrium path** is the path through the game tree that is followed in equilibrium, but the equilibrium itself is a strategy combination, which includes the players' responses to other players' deviations from the equilibrium path. These off-equilibrium responses are crucial to decisions on the equilibrium path. A threat, for example, is a promise to carry out a certain action if another player deviates from his equilibrium actions, and it has an influence even if it is never used.

Perfectness is best introduced with an example. In Section 2.1, a flaw of Nash equilibrium was revealed in the game Follow the Leader I, which has three pure strategy Nash equilibria of which only one is reasonable. The players are Smith and Jones, who choose disk sizes. Both their payoffs are greater if they choose the same size and greatest if they coordinate on *Large*. Smith moves first, so his strategy set is {*Small, Large*}. Jones' strategy is more complicated, because it must specify an action for each information set, and Jones's information set depends on what Smith chose. A typical element of Jones's strategy set is (*Large*, *Small*), which specifies that he chooses *Large* if Smith chose *Large*, and *Small* if Smith chose *Small*. From the strategic form we found the following three Nash equilibria.

Equilibrium	Strategies	Outcome
E_1	{*Large*, (*Large*, *Large*)}	Both pick *Large*.
E_2	{*Large*, (*Large*, *Small*)}	Both pick *Large*.
E_3	{*Small*, (*Small*, *Small*)}	Both pick *Small*.

Only Equilibrium E_2 is reasonable, because the order of the moves should matter to the decisions players make. The problem with the strategic form, and thus with simple Nash equilibrium, is that it ignores who moves first. Smith moves first, and it seems reasonable that Jones should be allowed— in fact should be required— to rethink his strategy after Smith moves.

Figure 1: *Follow the Leader I*

Consider Jones's strategy of $(Small, Small)$ in equilibrium E_3. If Smith deviated from equilibrium by choosing *Large*, it would be unreasonable for Jones to stick to the response *Small.* Instead, he should also choose *Large.* But if Smith expected a response of $Large$, he would have chosen *Large* in the first place, and E_3 would not be an equilibrium. A similar argument shows that it would be irrational for Jones to choose (*Large, Large*), and we are left with E_2 as the unique equilibrium.

We say that equilibria E_1 and E_3 are Nash equilibria but not "perfect" Nash equilibria. A strategy combination is a perfect equilibrium if it remains an equilibrium on all possible paths, including not only the equilibrium path but all the other paths, which branch off into different "subgames."

A **subgame** *is a game consisting of a node which is a singleton in every player's information partition, that node's successors, and the payoffs at the associated end nodes.*[1]

[1] Technically, this is a *proper* subgame because of the information qualifier, but no economist is so ill-bred as to use any other kind of subgame.

A strategy combination is a **subgame perfect Nash equilibrium** *if (a) it is a Nash equilibrium for the entire game; and (b) its relevant action rules are a Nash equilibrium for every subgame.*

The extensive form of Follow the Leader I in Figure 1 (a reprise of Figure 1 from Chapter 2) has three subgames: (1) the entire game, (2) the subgame starting at node J_1, and (3) the subgame starting at node J_2. Strategy combination E_1 is not a subgame perfect equilibrium because it is only Nash in subgames (1) and (3), not in subgame (2). Strategy combination E_3 is not a subgame perfect equilibrium because it is only Nash in subgames (1) and (2), not in subgame (3). Strategy combination E_2 is perfect because it is Nash in all three subgames.

The term **sequential rationality** is often used to denote the idea that a player should maximize his payoffs at each point in the game, re- optimizing his decisions at each point and taking into account the fact that he will re-optimize in the future. This is a blend of the economic ideas of ignoring sunk costs and rational expectations. Sequential rationality is so standard a criterion for equilibrium now that often I will speak of "equilibrium" without the qualifier when I wish to refer to an equilibrium that satisfies sequential rationality in the sense of being a "subgame perfect equilibrium" or, in a game of asymmetric information, a "perfect Bayesian equilibrium."

One reason why perfectness (the word "subgame" is usually left off) is a good equilibrium concept is because it represents the idea of sequential rationality. A second reason is that a weak Nash equilibrium is not robust to small changes in the game. So long as he is certain that Smith will not choose *Large*, Jones is indifferent between the never-to-be-used responses (*Small* if *Large*) and (*Large* if *Large*). Equilibria E_1, E_2, and E_3 are all weak Nash equilibria because of this. But if there is even a small probability that Smith will choose *Large*— perhaps by mistake— then Jones would prefer the response (*Large* if *Large*), and equilibria E_1 and E_3 are no longer valid. Perfectness is a way to eliminate some of these less robust weak equilibria. The small probability of a mistake is called a **tremble**, and Section 6.1 returns to this **trembling hand** approach as one way to extend the notion of perfectness to games of asymmetric information.

For the moment, however, the reader should note that the tremble approach is distinct from sequential rationality. Consider the Tremble Game in Figure 2. This game has three Nash equilibria, all weak: *(Exit, Down)*, *(Exit, Up)*, and *(Remain, Up)*. Only *(Exit, Up)* and *(Remain, Up)* are subgame perfect, because although *Down* is weakly Jones's best response to Smith's *Exit*, it is inferior if Smith chooses *Remain*. In the subgame starting with Jones's move, the only subgame perfect equilibrium is for Jones to choose *Up*. The possibility of trembles, however, rules out *(Remain, Up)* as an equilibrium. If Jones has even an infinitesimal chance of trembling and choosing *Down*, Smith will choose *Exit* instead of *Remain*. Also, Jones will choose *Up*, not *Down*, because if Smith trembles and chooses *Remain*, Jones prefers *Up* to *Down*. This leaves only *(Exit, Up)* as an equilibrium, despite the fact that it is weakly Pareto dominated by *(Remain, Up)*.

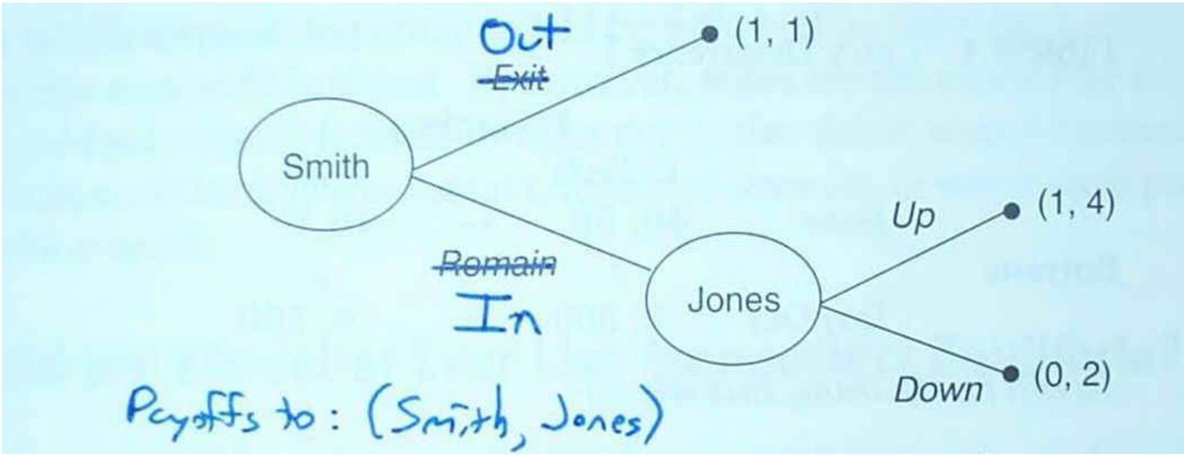


Figure 2: The *Tremble Game*: **Trembling Hand Versus Subgame Perfectness**

4.2 An Example of Perfectness: Entry Deterrence I

We turn now to a game in which perfectness plays a role just as important as in Follow the Leader I but in which the players are in conflict. An old question in industrial organization is whether an incumbent monopolist can maintain his position by threatening to wage a price war against any new firm that enters the market. This idea was heavily attacked by Chicago School economists such as McGee (1958) on the grounds that a price war would hurt the incumbent more than collusion with the entrant. Game theory can present this reasoning very cleanly. Let us consider a single episode of possible entry and price warfare, which nobody expects to be repeated. We will assume that even if the incumbent chooses to collude with the entrant, maintaining a duopoly is difficult enough that market revenue drops considerably from the monopoly level.

Entry Deterrence I

Players
Two firms, the entrant and the incumbent.

The Order of Play

1 The entrant decides whether to *Enter* or *Stay Out.*

2 If the entrant enters, the incumbent can *Collude* with him, or *Fight* by cutting the price drastically.

Payoffs
Market profits are 300 at the monopoly price and 0 at the fighting price. Entry costs are 10. Duopoly competition reduces market revenue to 100, which is split evenly.

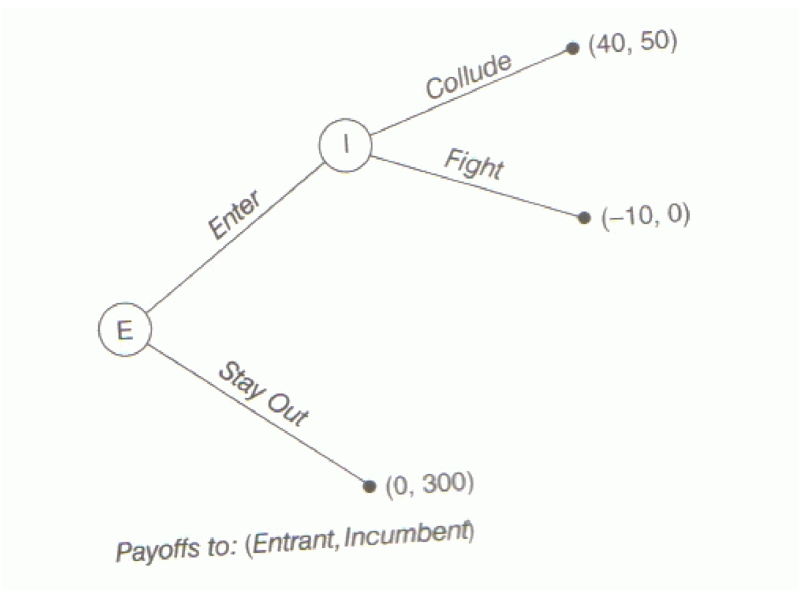
Table 1: Entry Deterrence I

		Incumbent		
		Collude		*Fight*
	Enter	**40,50**	←	−10, 0
Entrant:		↑		↓
	Stay Out	0, 300	↔	**0,300**

Payoffs to: (Entrant, Incumbent).

The strategy sets can be discovered from the order of play. They are {*Enter*, *Stay Out*} for the entrant, and {*Collude* if entry occurs, *Fight* if entry occurs} for the incumbent. The game has the two Nash equilibria indicated in boldface in Table 1, (*Enter*, *Collude*) and (*Stay Out*, *Fight*). The equilibrium (*Stay Out*, *Fight*) is weak, because the incumbent would just as soon *Collude* given that the entrant is staying out.

Figure 3: Entry Deterrence I



A piece of information has been lost by condensing from the extensive form, Figure 3, to the strategic form, Table 1: the fact that the entrant gets to move first. Once he has chosen *Enter*, the incumbent's best response is *Collude*. The threat to fight is not credible and would be employed only if the incumbent could bind himself to fight, in which case he never does fight, because the entrant chooses to stay out. The equilibrium (*Stay Out*, *Fight*) is Nash but not subgame perfect, because if the game is started after the entrant has already entered, the incumbent's best response is *Collude*. This does not prove that collusion is inevitable in duopoly, but it is the equilibrium for Entry Deterrence I.

The trembling hand interpretation of perfect equilibrium can be used here. So long as it is certain that the entrant will not enter, the incumbent is indifferent between *Fight* and *Collude*, but if there were even a small probability of entry— perhaps because of a lapse of good judgement by the entrant— the incumbent would prefer *Collude* and the Nash equilibrium would be broken.

Perfectness rules out threats that are not credible. Entry Deterrence I is a good example because if a communication move were added to the game tree, the incumbent might tell the entrant that entry would be followed by fighting, but the entrant would ignore this noncredible threat. If, however, some means existed by which the incumbent could precommit himself to fight entry, the threat would become credible. The next section will look at one context, nuisance lawsuits, in which such precommitment might be possible

Should the Modeller Ever Use Nonperfect Equilibria?

A game in which a player can commit himself to a strategy can be modelled in two ways:

1 As a game in which nonperfect equilibria are acceptable, or
2 By changing the game to replace the action *Do X* with *Commit to Do X* at an earlier node.

An example of (2) in Entry Deterrence I is to reformulate the game so the incumbent moves first, deciding in advance whether or not to choose *Fight* before the entrant moves. Approach (2) is better than (1) because if the modeller wants to let players commit to some actions and not to others, he can do this by carefully specifying the order of play. Allowing equilibria to be nonperfect forbids such discrimination and multiplies the number of equilibria. Indeed, the problem with subgame perfectness is not that it is too restrictive but that it still allows too many strategy combinations to be equilibria in games of asymmetric information. A subgame must start at a single node and not cut across any player's information set, so often the only subgame will be the whole game and subgame perfectness does not restrict equilibrium at all. Section 6.1 discusses perfect Bayesian equilibrium and other ways to extend the perfectness concept to games of asymmetric information.

4.3 Credible Threats, Sunks Costs, and the Open-set Problem in the Game of Nuisance Suits

Like the related concepts of sunks costs and rational expectations, sequential rationality is a simple idea with tremendous power. This section will show that power in another simple game, one which models nuisance suits. We have already come across one application of game theory to law, in the Png (1983) model of Section 2.5. In some ways, law is particularly well suited to analysis by game theory because the legal process is so concerned with conflict and the provision of definite rules to regulate that conflict. In what other field could an article be titled: "An Economic Analysis of Rule 68," as Miller (1986) does in his discussion of the federal rule of procedure that penalizes a losing litigant who had refused to accept a settlement offer. The growth in the area can be seen by comparing the overview in the Ayres's (1990) review of the first edition of the present book with the entire book by Baird, Gertner & Picker (1994). In law, even more clearly than in business, a major objective is to avoid inefficient outcomes by restructuring the rules, and nuisance suits are one of the inefficiencies that a good policy maker hopes to eliminate.

Nuisance suits are lawsuits with little chance of success, whose only possible purpose seems to be the hope of a settlement out of court. In the context of entry deterrence people commonly think large size is an advantage and a large incumbent will threaten a small entrant, but in the context of nuisance suits people commonly think large size is a disadvantage and a wealthy corporation is vulnerable to extortionary litigation. Nuisance Suits I models the essentials of the situation: bringing suit is costly and has little chance of success, but because defending the suit is also costly the defendant might pay a generous amount to settle it out of court. The model is similar to the Png Settlement Game of Chapter 2 in many respects, but here the model will be one of symmetric information and we will make explicit the sequential rationality requirement that was implicit the discussion in Chapter 2.

Nuisance Suits I: Simple Extortion

Players
A plaintiff and a defendant.

The Order of Play

1 The plaintiff decides whether to bring suit against the defendant at cost c.

2 The plaintiff makes a take-it-or-leave-it settlement offer of $s > 0$.

3 The defendant accepts or rejects the settlement offer.

4 If the defendant rejects the offer, the plaintiff decides whether to give up or go to trial at a cost p to himself and d to the defendant.

5 If the case goes to trial, the plaintiff wins amount x with probability γ and otherwise wins nothing.

Payoffs
Figure 4 shows the payoffs. Let $\gamma x < p$, so the plaintiff's expected winnings are less than his marginal cost of going to trial.

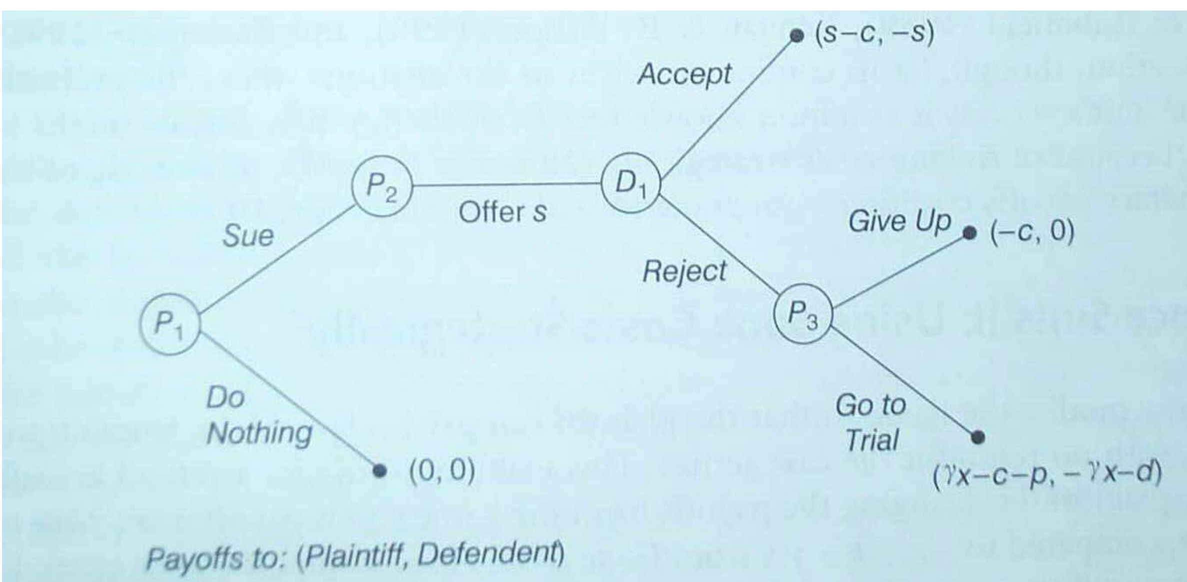


Figure 4: The Extensive Form for *Nuisance Suits*

The perfect equilibrium is

Plaintiff: *Do nothing*, *Offer s*, *Give up*
Defendant: *Reject*
Outcome: The plaintiff does not bring a suit.

The equilibrium settlement offer s can be any positive amount. Note that the equilibrium specifies actions at all four nodes of the game, even though only the first is reached in equilibrium.

To find a perfect equilibrium the modeller starts at the end of the game tree, following the advice of Dixit & Nalebuff (1991, p. 34) to "Look ahead and reason back." At node P_3, the plaintiff will choose *Give up*, since by assumption $\gamma x - c - p < -c$. This is because the suit is brought only in the hope of settlement, not in the hope of winning at trial. At node D_1, the defendant, foreseeing that the plaintiff will give up, rejects any positive settlement offer. This makes the plaintiff's offer at P_2 irrelevant, and, looking ahead to a payoff of $-c$ from choosing *Sue* at P_1, the plaintiff chooses *Do nothing.*

Thus, if nuisance suits are brought, it must be for some reason other than the obvious one, the plaintiff's hope of extracting a settlement offer from a defendant who wants to avoid trial costs. This is fallacious because the plaintiff himself bears trial costs and hence cannot credibly make the threat. It is fallacious even if the defendant's legal costs would be much higher than the plaintiff's (d much bigger than p), because the relative size of the costs does not enter into the argument.

One might wonder how risk aversion affects this conclusion. Might not the defendant settle because he is more risk averse than the plaintiff? That is a good question, but Nuisance Suits I can be adapted to risk-averse players with very little change. Risk would enter at the trial stage, as a final move by Nature to decide who wins. In Nuisance Suits I, γx represented the expected value of the award. If both the defendant and the plaintiff are equally risk averse, γx can still represent the expected payoff from the award— one simply interprets x and 0 as the utility of the cash award and the utility of an award of 0, rather than as the actual cash amounts. If the players have different degrees of risk aversion, the expected loss to the defendant is not the same as the expected gain to the plaintiff, and the payoffs must be adjusted. If the defendant is more risk averse, the payoffs from *Go to trial* would change to $(-c - p + \gamma x, -\gamma x - y - d)$, where y represents the extra disutility of risk to the defendant. This, however, makes no difference to the equilibrium. The crux of the game is that the plaintiff is unwilling to go to trial because of the cost to himself, and the cost to the defendant, including the cost of bearing risk, is irrelevant.

If nuisance suits are brought, it must therefore be for some more complicated reason. Already, in chapter 2, we looked at one reason for litigation to reach trial in the *Png Settlement Game:* incomplete information. That is probably the most important explanation and it has been much studied, as can be seen from the surveys by Cooter & Rubinfeld (1989) and Kennan and R. Wilson (1993). In this section, though, let us confine ourselves to explanations where the probability of the suit's success is common knowledge. Even then, costly threats might be credible because of sinking costs strategically (*Nuisance Suits II*), or because of the nonmonetary payoffs resulting from going to trial (*Nuisance Suits*

III) .

Nuisance Suits II : Using Sunk Costs Strategically [2]

Let us now modify the game so that the plaintiff can pay his lawyer the amount p in advance, with no refund if the case settles. This inability to obtain a refund actually helps the plaintiff, by changing the payoffs from the game so his payoff from *Give up* is $-c-p$, compared to $-c-p+\gamma x$ from *Go to trial.* Having sunk the legal costs, he will go to trial if $\gamma x > 0$— that is, if he has any chance of success at all.[3]

This, in turn, means that the plaintiff would only prefer settlement to trial if $s > \gamma x$. The defendant would prefer settlement to trial if $s < \gamma x + d$, so there is a positive **settlement range** of $[\gamma x, \gamma x + d]$ within which both players are willing to settle. The exact amount of the settlement depends on the bargaining power of the parties, something to be examined in chapter 11. Here, allowing the plaintiff to make a take-it-or-leave-it offer means that $s = \gamma x + d$ in equilibrium, and if $\gamma x + d > p + c$, the nuisance suit will be brought even though $\gamma x < p + c$. Thus, the plaintiff is bringing the suit only because he can extort d, the amount of the defendant's legal costs.

Even though the plaintiff can now extort a settlement, he does it at some cost to himself, so an equilibrium with nuisance suits will require that

$$-c - p + \gamma x + d \geq 0 \tag{1}$$

If inequality (1) is false, then, even if the plaintiff could extract the maximum possible settlement of $s = \gamma x + d$, he would not do so, because he would then have to pay $c + p$ before reaching the settlement stage. This implies that a totally meritless suit (with $\gamma = 0$), would not be brought unless the defendant had higher legal costs than the plaintiff ($d > p$). If inequality (1) is satisfied, however, the following strategy combination is a perfect equilibrium:

Plaintiff: *Sue*, *Offer* $s = \gamma x + d$, *Go to trial*
Defendant: *Accept* $s \leq \gamma x + d$
Outcome: Plaintiff sues and offers to settle, to which the defendant agrees.

An obvious counter to the plaintiff's ploy would be for the defendant to also sink his costs, by paying d before the settlement negotiations, or even before the plaintiff decides to file suit. Perhaps this is one reason why large corporations use in house counsel, who are paid a salary regardless of how many hours they work, as well as outside counsel, hired by the hour. If so, nuisance suits cause a social loss—the wasted time of the lawyers, d—even if nuisance suits are never brought, just as aggressor nations cause social loss in the form of world military expenditure even if they never start a war.[4]

[2] The inspiration for this model is Rosenberg & Shavell (1985).

[3] Have a figure with the previous NS I Extensive Form and the new one side by side, with the differences highlighted.

[4] Nonrefundable lawyer's fees, paid in advance, have traditionally been acceptable, but a New York court recently ruled they were unethical. The court thought that such fees unfairly restricted the client's ability to fire his lawyer, an example of how ignorance of game theory can lead to confused rule-making.

Two problems, however, face the defendant who tries to sink the cost d. First, although it saves him γx if it deters the plaintiff from filing suit, it also means the defendant must pay the full amount d. This is worthwhile if the plaintiff has all the bargaining power, as in Nuisance Suits II, but it might not be if s lay in the middle of the settlement range because the plaintiff was not able to make a take-it-or-leave-it offer. If settlement negotiations resulted in s lying exactly in the middle of the settlement range, so $s = \gamma x + \frac{d}{2}$, then it might not be worthwhile for the defendant to sink d to deter nuisance suits that would settle for $\gamma x + \frac{d}{2}$.

Second, there is an asymmetry in litigation: the plaintiff has the choice of whether to bring suit or not. Since it is the plaintiff who has the initiative, he can sink p and make the settlement offer before the defendant has the chance to sink d. The only way for the defendant to avoid this is to pay d well in advance, in which case the expenditure is wasted if no possible suits arise. What the defendant would like best would be to buy legal insurance which, for a small premium, would pay all defense costs in future suits that might occur. As we will see in Chapters 7 and 9, however, insurance of any kind faces problems arising from asymmetric information. In this context, there is the "moral hazard" problem, in that once the defendant is insured he has less incentive to avoid causing harm to the plaintiff and provoking a lawsuit.

The Open-set Problem in Nuisance Suits II

Nuisance Suits II illustrates a technical point that arises in a great many games with continuous strategy spaces and causes great distress to novices in game theory. The equilibrium in Nuisance Suits II is only a weak Nash equilibrium. The plaintiff proposes $s = \gamma x + d$, and the defendant has the same payoff from accepting or rejecting, but in equilibrium the defendant accepts the offer with probability one, despite his indifference. This seems arbitrary, or even silly. Should not the plaintiff propose a slightly lower settlement to give the defendant a strong incentive to accept it and avoid the risk of having to go to trial? If the parameters are such that $s = \gamma x + d = 60$, for example, why does the plaintiff risk holding out for 60 when he might be rejected and most likely receive 0 at trial, when he could offer 59 and give the defendant a strong incentive to accept?

One answer is that no other equilibrium exists besides $s = 60$. Offering 59 cannot be part of an equilibrium because it is dominated by offering 59.9; offering 59.9 is dominated by offering 59.99, and so forth. This is known as the **open-set problem**, because the set of offers that the defendant strongly wishes to accept is open and has no maximum— it is bounded at 60, but a set must be bounded *and closed* to guarantee that a maximum exists.

A second answer is that under the assumptions of rationality and Nash equilibrium the objection's premise is false because the plaintiff bears no risk whatsoever in offering $s = 60$. It is fundamental to Nash equilibrium that each player believe that the others will follow equilibrium behavior. Thus, if the equilibrium strategy combination says that the defendant will accept $s \leq 60$, the plaintiff can offer 60 and believe it will be accepted.

See "Nonrefundable Lawyers' Fees, Paid in Advance, are Unethical, Court Rules," *Wall Street Journal*, January 29, 1993, p. B3, citing *In the matter of Edward M. Cooperman, Appellate Division of the Supreme Court, Second Judicial Department, Brooklyn, 90-00429.*

This is really just to say that a weak Nash equilibrium is still a Nash equilibrium, a point emphasized in chapter 3 in connection with mixed strategies.

A third answer is that the problem is an artifact of using a model with a continuous strategy space, and it disappears if the strategy space is made discrete. Assume that s can only take values in multiples of 0.01, so it could be 59.0, 59.01, 59.02, and so forth, but not 59.001 or 59.002. The settlement part of the game will now have two perfect equilibria. In the strong equilibrium E1, $s = 59.99$ and the defendant accepts any offer $s < 60$. In the weak equilibrium E2, $s = 60$ and the defendant accepts any offer $s \leq 60$. The difference is trivial, so the discrete strategy space has made the model more complicated without any extra insight.[5]

One can also specify a more complicated bargaining game to avoid the issue of how exactly the settlement is determined. Here one could say that the settlement is not proposed by the plaintiff, but simply emerges with a value halfway through the settlement range, so $s = \gamma x + \frac{d}{2}$. This seems reasonable enough, and it adds a little extra realism to the model at the cost of a little extra complexity. It avoids the open-set problem, but only by avoiding being clear about how s is determined. I call this kind of modelling **blackboxing**, because it is as if at some point in the game, variables with certain values go into a black box and come out the other side with values determined by an exogenous process. Blackboxing is perfectly acceptable as long as it neither drives nor obscures the point the model is making. Nuisance Suits III will illustrate this method.

Fundamentally, however, the point to keep in mind is that games are models, not reality. They are meant to clear away the unimportant details of a real situation and simplify it down to the essentials. Since a model is trying to answer a question, it should focus on what answers that question. Here, the question is why nuisance suits might be brought, so it is proper to exclude details of the bargaining if they are irrelevant to the answer. Whether a plaintiff offers 59.99 or 60, and whether a rational person accepts an offer with probability 0.99 or 1.00, is part of the unimportant detail, and whatever approach is simplest should be used. If the modeller really thinks that these are important matters, they can indeed be modelled, but they are not important in this context.

One source of concern over the open-set problem, I think, is that perhaps that the payoffs are not quite realistic, because the players should derive utility from hurting "unfair" players. If the plaintiff makes a settlement offer of 60, keeping the entire savings from avoiding the trial for himself, everyday experience tells us that the defendant will indignantly refuse the offer. Guth *et al.* (1982) have found in experiments that people turn down bargaining offers they perceive as unfair, as one might expect. If indignation is truly important, it can be explicitly incorporated into the payoffs, and if that is done, the open-set problem returns. Indignation is not boundless, whatever people may say. Suppose that accepting a settlement offer that benefits the plaintiff more than the defendant gives a disutility of x to the defendant because of his indignation at his unjust treatment. The plaintiff will then offer to settle for exactly $60 - x$, so the equilibrium is still weak and

[5] A good example of the ideas of discrete money values and sequential rationality is in Robert Louis Stevenson's story, "The Bottle Imp" (Stevenson [1987]). The imp grants the wishes of the bottle's owner but will seize his soul if he dies in possession of it. Although the bottle cannot be given away, it can be sold, but only at a price less than that for which it was purchased.

the defendant is still indifferent between accepting and rejecting the offer. The open-set problem persists, even after realistic emotions are added to the model.

I have spent so much time on the open-set problem not because it is important but because it arises so often and is a sticking point for people unfamiliar with modelling. It is not a problem that disturbs experienced modellers, unlike other basic issues we have already encountered—for example, the issue of how a Nash equilibrium comes to be common knowledge among the players— but it is important to understand why it is not important.

Nuisance Suits III: Malice

One of the most common misconceptions about game theory, as about economics in general, is that it ignores non-rational and non-monetary motivations. Game theory does take the basic motivations of the players to be exogenous to the model, but those motivations are crucial to the outcome and they often are not monetary, although payoffs are always given numerical values. Game theory does not call somebody irrational who prefers leisure to money or who is motivated by the desire to be world dictator. It does require the players' emotions to be carefully gauged to determine exactly how the actions and outcomes affect the players' utility.

Emotions are often important to lawsuits, and law professors tell their students that when the cases they study seem to involve disputes too trivial to be worth taking to court, they can guess that the real motivations are emotional. Emotions could enter in a variety of distinct ways. The plaintiff might simply like going to trial, which can be expressed as a value of $p < 0$. This would be true of many criminal cases, because prosecutors like news coverage and want credit with the public for prosecuting certain kinds of crime. The Rodney King trials of 1992 and 1993 were of this variety; regardless of the merits of the cases against the policemen who beat Rodney King, the prosecutors wanted to go to trial to satisfy the public outrage, and when the state prosecutors failed in the first trial, the federal government was happy to accept the cost of bringing suit in the second trial. A different motivation is that the plaintiff might derive utility from the fact of winning the case quite separately from the monetary award, because he wants a public statement that he is in the right. This is a motivation in bringing libel suits, or for a criminal defendant who wants to clear his good name.

A different emotional motivation for going to trial is the desire to inflict losses on the defendant, a motivation we will call "malice," although it might as inaccurately be called "righteous anger." In this case, d enters as a positive argument in the plaintiff's utility function. We will construct a model of this kind, called Nuisance Suits III, and assume that $\gamma = 0.1$, $c = 3$, $p = 14$, $d = 50$, and $x = 100$, and that the plaintiff receives additional utility of 0.1 times the defendant's disutility. Let us also adopt the blackboxing technique discussed earlier and assume that the settlement s is in the middle of the settlement range. The payoffs conditional on suit being brought are

$$\pi_{plaintiff}(Defendant\ accepts) = s - c + 0.1s = 1.1s - 3 \qquad (2)$$

and

$$\pi_{plaintiff}(Go\ to\ trial) = \gamma x - c - p + 0.1(d + \gamma x) \quad (3)$$

7

$$= 10 - 3 - 14 + 6 = -1.$$

Now, working back from the end in accordance with sequential rationality, note that since the plaintiff's payoff from *Give Up* is -3, he will go to trial if the defendant rejects the settlement offer. The overall payoff from bringing a suit that eventually goes to trial is still -1, which is worse than the payoff of 0 from not bringing suit in the first place, but if s is high enough, the payoff from bringing suit and settling is higher still. If s is greater than 1.82 ($= \frac{-1+3}{1.1}$, rounded), the plaintiff prefers settlement to trial, and if s is greater than about 2.73 ($= \frac{0+3}{1.1}$, rounded), he prefers settlement to not bringing the suit at all.

In determining the settlement range, the relevant payoff is the expected incremental payoff since the suit was brought. The plaintiff will settle for any $s \geq 1.82$, and the defendant will settle for any $s \leq \gamma x + d = 60$, as before. The settlement range is $[1.82, 60]$, and $s = 30.91$. The settlement offer is no longer the maximizing choice of a player, and hence is moved to the outcome in the equilibrium description below.

Plaintiff: *Sue*, *Go to Trial*
Defendant: *Accept any* $s \leq 60$
Outcome: The plaintiff sues and offers $s = 30.91$, and the defendant accepts the settlement.

Perfectness is important here because the defendant would like to threaten never to settle and be believed. The plaintiff would not bring suit given his expected payoff of -1 from bringing a suit that goes to trial, so a believable threat would be effective. But such a threat is not believable. Once the plaintiff does bring suit, the only Nash equilibrium in the remaining subgame is for the defendant to accept his settlement offer. This is interesting because the plaintiff, despite his willingness to go to trial, ends up settling out of court. When information is symmetric, as it is here, there is a tendency for equilibria to be efficient. Although the plaintiff wants to hurt the defendant, he also wants to keep his expenses low. Thus, he is willing to hurt the defendant less if it enables him to save on his own legal costs.

One final point before leaving these models is that much of the value of modelling comes simply from setting up the rules of the game, which helps to show what is important in a situation. One problem that arises in setting up a model of nuisance suits is deciding what a "nuisance suit" really is. In the game of Nuisance Suits, it has been defined as a suit whose expected damages do not repay the plaintiff's costs of going to trial. But having to formulate a definition brings to mind another problem that might be called the problem of nuisance suits: that the plaintiff brings suits he knows will not win unless the court makes a mistake. Since the court might make a mistake with very high probability, the games above would not be appropriate models— γ would be high, and the problem is not that the plaintiff's expected gain from trial is low, but that it is high. This, too, is an important problem, but having to construct a model shows that it is different.

4.4 Recoordination to Pareto-dominant Equilibria in Subgames: Pareto Perfection

One simple refinement of equilibrium that was mentioned in chapter 1 is to rule out any strategy combinations that are Pareto dominated by Nash equilibria. Thus, in the game of Ranked Coordination, the inferior Nash equilibrium would be ruled out as an acceptable equilibrium. The idea behind this is that in some unmodelled way the players discuss their situation and coordinate to avoid the bad equilibria. Since only Nash equilibria are discussed, the players' agreements are self-enforcing and this is a more limited suggestion than the approach in cooperative game theory according to which the players make binding agreements.

The coordination idea can be taken further in various ways. One is to think about coalitions of players coordinating on favorable equilibria, so that two players might coordinate on an equilibrium even if a third player dislikes it. Bernheim, Peleg, & Whinston (1987) and Bernheim & Whinston (1987) define a Nash strategy combination as a **coalition-proof Nash equilibrium** if no coalition of players could form a self-enforcing agreement to deviate from it. They take the idea further by subordinating it to the idea of sequential rationality. The natural way to do this is to require that no coalition would deviate in future subgames, a notion called by various names, including **renegotiation proofness**, **recoordination** (e.g., Laffont & Tirole [1993], p. 460), and **Pareto perfection** (e.g., Fudenberg & Tirole (1991a), p. 175). The idea has been used extensively in the analysis of infinitely repeated games, which are particularly subject to the problem of multiple equilibria; Abreu, Pearce & Stachetti (1986) is an example of this literature. Whichever name is used, the idea is distinct from the renegotiation problem in the principal-agent models to be studied in Chapter 8, which involves the rewriting of earlier binding contracts to make new binding contracts.

The best way to demonstrate the idea of Pareto perfection is by an illustration, the Pareto Perfection Puzzle , whose extensive form is shown in Figure 5. In this game Smith chooses *In* or *Outside Option 1*, which yields payoffs of 10 to each player. Jones then chooses *Outside Option 2*, which yields 20 to each player, or initiates either a coordination game or a prisoner's dilemma. Rather than draw the full subgames in extensive form, Figure 5 inserts the payoff matrix for the subgames.

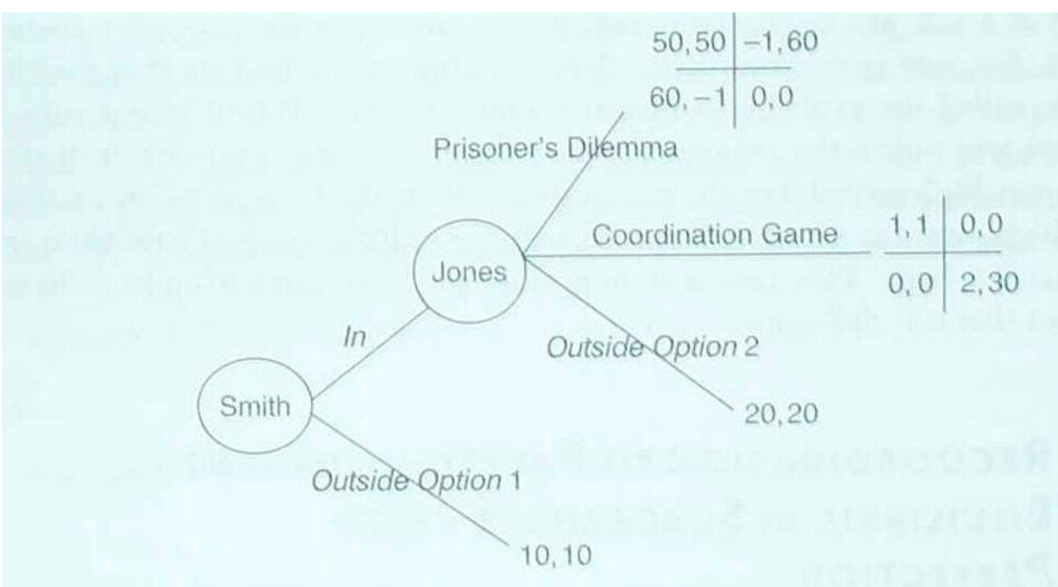


Figure 5: The Pareto Perfection Puzzle

The Pareto Perfection Puzzle illustrates the complicated interplay between perfectness and Pareto dominance. The pareto-dominant strategy combination is *(In, Prisoner's Dilemma|In, any actions in the coordination subgame, the actions yielding (50,50) in the Prisoner's Dilemma subgame)*. Nobody expects this strategy combination to be an equilibrium, since it is neither perfect nor Nash. Perfectness tells us that if the *Prisoner's Dilemma* subgame is reached, the payoffs will be (0,0), and if the coordination subgame is reached they will be either (1,1) or (2,30). In light of this, the perfect equilibria of the Pareto Perfection Puzzle are:

E1: (In, outside option 2|In, the actions yielding (1,1) in the coordination subgame, the actions yielding (0,0) in the Prisoner's Dilemma subgame). The payoffs are (20,20).

E2: (outside option 1, coordination game|In, the actions yielding (2,30) in the coordination subgame, the actions yielding (0,0) in the Prisoner's Dilemma subgame). The payoffs are (10,10).

If one applies Pareto dominance without perfection, E1 will be the equilibrium, since both players prefer it. If the players can recoordinate at any point and change their expectations, however, then if play of the game reaches the coordination subgame, the players will recoordinate on the actions yielding (2,30). Pareto perfection thus knocks out E1 as an equilibrium. Not only does it rule out the Pareto-dominant strategy combination that yields (50,50) as an equilibrium, it also rules out the Pareto-dominant perfect strategy combination that yields (20,20) as an equilibrium. Rather, the payoff is (10,10). Thus, Pareto perfection is not the same thing as simply picking the Pareto-dominant perfect strategy combination.

It is difficult to say which equilibrium is best here, since this is an abstract game and we cannot call upon details from the real world to refine the model. The approach of applying an equilibrium refinement is not as likely to yield results as using the intuition behind the refinement. The intuition here is that the players will somehow coordinate on Pareto-dominant equilibria, perhaps finding open discussion helpful. If we ran an experiment on student players using the Pareto Perfection Puzzle , I would expect to reach different equilibria depending on what communication is allowed. If the players are allowed to talk only before the game starts, it seems more likely that E1 would be the equilibrium, since players could agree to play it and would have no chance to explicitly recoordinate later. If the players could talk at any time as the game proceeded, E2 becomes more plausible. Real-world situations arise with many different communications technologies, so there is no one right answer.

Notes

N4.1 Subgame perfectness

- The terms "perfectness" and "perfection" are used synonymously. Selten (1965) proposed the equilibrium concept in an article written in German. "Perfectness" is used in Selten (1975) and conveys an impression of completeness more appropriate to the concept than the goodness implied by "perfection." "Perfection," however, is more common.

- It is debatable whether the definition of subgame ought to include the original game. Gibbon (1992, p. 122) does not, for example, and modellers usually do not in their conversation.

- Perfectness is not the only way to eliminate weak Nash equilibria like *(Stay Out, Collude)*. In Entry Deterrence I, (*Enter*, *Collude*) is the only iterated dominance equilibrium, because *Fight* is weakly dominated for the incumbent.

- The distinction between perfect and non-perfect Nash equilibria is like the distinction between **closed loop** and **open loop** trajectories in dynamic programming. Closed loop (or **feedback**) trajectories can be revised after they start, like perfect equilibrium strategies, while open loop trajectories are completely prespecified (though they may depend on state variables). In dynamic programming the distinction is not so important, because prespecified strategies do not change the behavior of other players. No threat, for example, is going to alter the pull of the moon's gravity on a rocket.

- A subgame can be infinite in length, and infinite games can have non-perfect equilibria. The infinitely repeated *Prisoner's Dilemma* is an example; here every subgame looks exactly like the original game, but begins at a different point in time.

- **Sequential rationality in macroeconomics.** In macroeconomics the requirement of **dynamic consistency** or **time consistency** is similar to perfectness. These terms are less precisely defined than perfectness, but they usually require that strategies need only be best responses in subgames starting from nodes on the equilibrium path, instead of all subgames. Under this interpretation, time consistency is a less stringent condition than perfectness.

  The Federal Reserve, for example, might like to induce inflation to stimulate the economy, but the economy is stimulated only if the inflation is unexpected. If the inflation is expected, its effects are purely bad. Since members of the public know that the Fed would like to fool them, they disbelieve its claims that it will not generate inflation (see Kydland & Prescott [1977]). Likewise, the government would like to issue nominal debt, and promises lenders that it will keep inflation low, but once the debt is issued, the government has an incentive to inflate its real value to zero. One reason the US Federal Reserve Board was established to be independent of Congress in the United States was to diminish this problem.

  The amount of game theory used in macroeconomics has been increasing at a fast rate. For references see Canzoneri & Henderson's 1991 book, which focusses on international coordination and pays particular attention to trigger strategies.

- Often, irrationality— behavior that is automatic rather than strategic— is an advantage. The Doomsday Machine in the movie *Dr Strangelove* is one example. The Soviet Union decides that it cannot win a rational arms race against the richer United States, so it creates a bomb which automatically blows up the entire world if anyone explodes a nuclear bomb. The movie also illustrates a crucial detail without which such irrationality is worse than useless: you have to tell the other side that you have the Doomsday Machine.

President Nixon reportedly told his aide H.R. Haldeman that he followed a more complicated version of this strategy: "I call it the Madman Theory, Bob. I want the North Vietnamese to believe that I've reached the point where I might do *anything* to stop the war. We'll just slip the word to them that 'for God's sake, you know Nixon is obsessed about Communism. We can't restrain him when he's angry— and he has his hand on the nuclear button'— and Ho Chi Minh himself will be in Paris in two days begging for peace"(Haldeman & DiMona [1978] p. 83). The Gang of Four model in section 6.4 tries to model a situation like this.

- The "lock-up agreement" is an example of a credible threat: in a takeover defense, the threat to destroy the firm is made legally binding. See Macey & McChesney (1985) p. 33.

N4.3 An example of perfectness: Entry Deterrence I

- The Stackelberg equilibrium of a duopoly game (section 3.4) can be viewed as the perfect equilibrium of a Cournot game modified so that one player moves first, a game similar to Entry Deterrence I. The player moving first is the Stackelberg leader and the player moving second is the Stackelberg follower. The follower could threaten to produce a high output, but he will not carry out his threat if the leader produces a high output first.

- Perfectness is not so desirable a property of equilibrium in biological games. The reason the order of moves matters is because the rational best reply depends on the node at which the game has arrived. In many biological games the players act by instinct and unthinking behavior is not unrealistic.

- Reinganum & Stokey (1985) is a clear presentation of the implications of perfectness and commitment illustrated with the example of natural resource extraction.

Problems

4.1. Repeated Entry Deterrence
Consider two repetitions without discounting of the game Entry Deterrence I from Section 4.2. Assume that there is one entrant, who sequentially decides whether to enter two markets that have the same incumbent.

(a) Draw the extensive form of this game.

(b) What are the 16 elements of the strategy sets of the entrant?

(c) What is the subgame perfect equilibrium?

(d) What is one of the nonperfect Nash equilibria?

4.2. The Three-Way Duel (after Shubik (1954)
Three gangsters armed with pistols, Al, Bob, and Curly, are in a room with a suitcase containing 120 thousand dollars. Al is the least accurate, with a 20 percent chance of killing his target. Bob has a 40 percent probability. Curly is slow but sure; he kills his target with 70 percent probability. For each, the value of his own life outweighs the value of any amount of money. Survivors split the money.

(a) Suppose each gangster has one bullet and the order of shooting is first Al, then Bob, then Curly. Assume also that each gangster must try to kill another gangster when his turn comes. What is an equilibrium strategy combination and what is the probability that each of them dies in that equilibrium? Hint: Do not try to draw a game tree.

(b) Suppose now that each gangster has the additional option of shooting his gun at the ceiling, which may kill somebody upstairs but has no direct effect on his payoff. Does the strategy combination that you found was an equilibrium in part (a) remain an equilibrium?

(c) Replace the three gangsters with three companies, Apex, Brydox, and Costco, which are competing with slightly different products. What story can you tell about their advertising strategies?

(d) In the United States, before the general election a candidate must win the nomination of his party. It is often noted that candidates are reluctant to be seen as the frontrunner in the race for the nomination of their party, Democrat or Republican. In the general election, however, no candidate ever minds being seen to be ahead of his rival from the other party. Why?

(e) In the 1920's, several men vied for power in the Soviet Union after Lenin died. First Stalin and Zinoviev combined against Trotsky. Then Stalin and Bukharin combined against Zinoviev. Then Stalin turned on Bukharin. Relate this to Curly, Bob, and Al.

4.3. Heresthetics in Pliny and the freedmens' trial (Pliny, 1963, pp. 221-4, Riker, 1986, pp. 78-88)
Afranius Dexter died mysteriously, perhaps dead by his own hand, perhaps killed by his freedmen (servants a step above slaves), or perhaps killed by his freedmen by his own orders. The freedmen

went on trial before the Roman Senate. Assume that 45 percent of the senators favor acquittal, 35 percent favor banishment, and 20 percent favor execution, and that the preference rankings in the three groups are $A \succ B \succ E$, $B \succ A \succ E$, and $E \succ B \succ A$. Also assume that each group has a leader and votes as a bloc.

(a) Modern legal procedure requires the court to decide guilt first and then assign a penalty if the accused is found guilty. Draw a tree to represent the sequence of events (this will not be a game tree, since it will represent the actions of groups of players, not of individuals). What is the outcome in a perfect equilibrium?

(b) Suppose that the acquittal bloc can pre-commit to how they will vote in the second round if guilt wins in the first round. What will they do, and what will happen? What would the execution bloc do if they could control the second-period vote of the acquittal bloc?

(c) The normal Roman procedure began with a vote on execution versus no execution, and then voted on the alternatives in a second round if execution failed to gain a majority. Draw a tree to represent this. What would happen in this case?

(d) Pliny proposed that the Senators divide into three groups, depending on whether they supported acquittal, banishment, or execution, and that the outcome with the most votes should win. This proposal caused a roar of protest. Why did he propose it?

(e) Pliny did not get the result he wanted with his voting procedure. Why not?

(f) Suppose that personal considerations made it most important to a senator that he show his stand by his vote, even if he had to sacrifice his preference for a particular outcome. If there were a vote over whether to use the traditional Roman procedure or Pliny's procedure, who would vote with Pliny, and what would happen to the freedmen?

4.4. DROPPED

4.5. Garbage Entry

Mr. Turner is thinking of entering the garbage collection business in a certain large city. Currently, Cutright Enterprises has a monopoly, earning 40 million dollars from the 40 routes the city offers up for bids. Turner thinks he can take away as many routes as he wants from Cutright, at a profit of 1.5 million per route for him. He is worried, however, that Cutright might resort to assassination, killing him to regain their lost routes. He would be willing to be assassinated for profit of 80 million dollars, and assassination would cost Cutright 6 million dollars in expected legal costs and possible prison sentences.

How many routes should Turner try to take away from Cutright?

4.6. [No problem– a placeholder]

4.7. Voting Cycles

Uno, Duo, and Tres are three people voting on whether the budget devoted to a project should be Increased, kept the Same, or Reduced. Their payoffs from the different outcomes, given in Table 3, are not monotonic in budget size. Uno thinks the project could be very profitable if its budget were increased, but will fail otherwise. Duo mildly wants a smaller budget. Tres likes the budget as it is now.

	Uno	Duo	Tres
Increase	100	2	4
Same	3	6	9
Reduce	9	8	1

Table 3: Payoffs from Different Policies

Each of the three voters writes down his first choice. If a policy gets a majority of the votes, it wins. Otherwise, *Same* is the chosen policy.

(a) Show that (*Same*, *Same*, *Same*) is a Nash equilibrium. Why does this equilibrium seem unreasonable to us?

(b) Show that (*Increase*, *Same*, *Same*) is a Nash equilibrium.

(c) Show that if each player has an independent small probability ϵ of "trembling" and choosing each possible wrong action by mistake, (*Same*, *Same*, *Same*) and (*Increase*, *Same*, *Same*) are no longer equilibria.

(d) Show that (*Reduce*, *Reduce*, *Same*) is a Nash equilibrium that survives each player has an independent small probability ϵ of "trembling" and choosing each possible wrong action by mistake.

(e) Part (d) showed that if Uno and Duo are expected to choose *Reduce*, then Tres would choose *Same* if he could hope they might tremble– not *Increase*. Suppose, instead, that Tres votes first, and publicly. Construct a subgame perfect equilibrium in which Tres chooses *Increase*. You need not worry about trembles now.

(f) Consider the following voting procedure. First, the three voters vote between *Increase* and *Same*. In the second round, they vote between the winning policy and *Reduce*. If, at that point, *Increase* is not the winning policy, the third vote is between *Increase* and whatever policy won in the second round.

What will happen? (watch out for the trick in this question!)

(g) Speculate about what would happen if the payoffs are in terms of dollar willingness to pay by each player and the players could make binding agreements to buy and sell votes. What, if anything, can you say about which policy would win, and what votes would be bought at what price?

5 Reputation and Repeated Games with Symmetric Information

September 11, 1999. November 29, 2003. December 13, 2004. 25 March 2005. Eric Rasmusen, Erasmuse@indiana.edu. Http://www.rasmusen.org. Footnotes starting with xxx are the author's notes to himself. Comments welcomed.

5.1 Finitely Repeated Games and the Chainstore Paradox

Chapter 4 showed how to refine the concept of Nash equilibrium to find sensible equilibria in games with moves in sequence over time, so- called dynamic games. An important class of dynamic games is repeated games, in which players repeatedly make the same decision in the same environment. Chapter 5 will look at such games, in which the rules of the game remain unchanged with each repetition and all that changes is the "history" which grows as time passes, and, if the number of repetitions is finite, the approach of the end of the game. It is also possible for asymmetry of information to change over time in a repeated game since players' moves may convey their private information, but Chapter 5 will confine itself to games of symmetric information.

Section 5.1 will show the perverse unimportance of repetition for the games of Entry Deterrence and the Prisoner's Dilemma, a phenomenon known as the Chainstore Paradox. Neither discounting, probabilistic end dates, infinite repetitions, nor precommitment are satisfactory escapes from the Chainstore Paradox. This is summarized in the Folk Theorem of Section 5.2. Section 5.2 will also discuss strategies which punish players who fail to cooperate in a repeated game— strategies such as the Grim Strategy, Tit-for-Tat, and Minimax. Section 5.3 builds a framework for reputation models based on the Prisoner's Dilemma, and Section 5.4 presents one particular reputation model, the Klein-Leffler model of product quality. Section 5.5 concludes the chapter with an overlapping generations model of consumer switching costs which uses the idea of Markov strategies to narrow down the number of equilibria.

The Chainstore Paradox

Suppose that we repeat Entry Deterrence I 20 times in the context of a chainstore that is trying to deter entry into 20 markets where it has outlets. We have seen that entry into just one market would not be deterred, but perhaps with 20 markets the outcome is different because the chainstore would fight the first entrant to deter the next 19.

The repeated game is much more complicated than the **one-shot game**, as the unrepeated version is called. A player's action is still to *Enter* or *Stay Out*, to *Fight* or *Collude*, but his strategy is a potentially very complicated rule telling him what action to choose depending on what actions both players took in each of the previous periods. Even the five-round repeated Prisoner's Dilemma has a strategy set for each player with over two billion strategies, and the number of strategy profiles is even greater (Sugden [1986], p. 108).

The obvious way to solve the game is from the beginning, where there is the least past history on which to condition a strategy, but that is not the easy way. We have to follow Kierkegaard, who said, "Life can only be understood backwards, but it must be lived forwards" (Kierkegaard 1938, p. 465). In picking his first action, a player looks ahead to its implications for all the future periods, so it is easiest to start by understanding the end of a multi-period game, where the future is shortest.

Consider the situation in which 19 markets have already been invaded (and maybe the chainstore fought, or maybe not). In the last market, the subgame in which the two players find themselves is identical to the one-shot Entry Deterrence I, so the entrant will *Enter* and the chainstore will *Collude*, regardless of the past history of the game. Next, consider the next-to-last market. The chainstore can gain nothing from building a reputation for ferocity, because it is common knowledge that he will *Collude* with the last entrant anyway. So he might as well *Collude* in the 19th market. But we can say the same of the 18th market and— by continuing backward induction— of every market, including the first. This result is called the **Chainstore Paradox** after Selten (1978) .

Backward induction ensures that the strategy profile is a subgame perfect equilibrium. There are other Nash equilibria— (*Always Fight, Never Enter*), for example— but because of the Chainstore Paradox they are not perfect.

The Repeated Prisoner's Dilemma

The Prisoner's Dilemma is similar to Entry Deterrence I. Here the prisoners would like to commit themselves to *Deny*, but, in the absence of commitment, they *Confess*. The Chainstore Paradox can be applied to show that repetition does not induce cooperative behavior. Both prisoners know that in the last repetition, both will *Confess*. After 18 repetitions, they know that no matter what happens in the 19th, both will *Confess* in the 20th, so they might as well *Confess* in the 19th too. Building a reputation is pointless, because in the 20th period it is not going to matter. Proceeding inductively, both players *Confess* in every period, the unique perfect equilibrium outcome.

In fact, as a consequence of the fact that the one-shot Prisoner's Dilemma has a dominant strategy equilibrium, confessing is the only Nash outcome for the repeated Prisoner's Dilemma, not just the only perfect outcome. The argument of the previous paragraph did not show that confessing was the unique Nash outcome. To show subgame perfectness, we worked back from the end using longer and longer subgames. To show that confessing is the only Nash outcome, we do not look at subgames, but instead rule out successive classes of strategies from being Nash. Consider the portions of the strategy which apply to the equilibrium path (that is, the portions directly relevant to the payoffs). No strategy in the class that calls for *Deny* in the last period can be a Nash strategy, because the same strategy with $Confess$ replacing *Deny* would dominate it. But if both players have strategies calling for confessing in the last period, then no strategy that does not call for confessing in the next-to-last period is Nash, because a player should deviate by replacing *Deny* with $Confess$ in the next-to-last period. The argument can be carried back to the first period, ruling out any class of strategies that does not call for confessing everywhere along the equilibrium path.

The strategy of always confessing is not a dominant strategy, as it is in the one-shot game, because it is not the best response to various suboptimal strategies such as (*Deny until the other player Confesses, then Deny for the rest of the game*). Moreover, the uniqueness is only on the equilibrium path. Nonperfect Nash strategies could call for cooperation at nodes far away from the equilibrium path, since that action would never have to be taken. If Row has chosen (*Always Confess*), one of Column's best responses is (*Always Confess unless Row has chosen Deny ten times; then always Deny*).

5.2 Infinitely Repeated Games, Minimax Punishments, and the Folk Theorem

The contradiction between the Chainstore Paradox and what many people think of as real world behavior has been most successfully resolved by adding incomplete information to the model, as will be seen in Section 6.4. Before we turn to incomplete information, however, we will explore certain other modifications. One idea is to repeat the Prisoner's Dilemma an infinite number of times instead of a finite number (after all, few economies have a known end date). Without a last period, the inductive argument in the Chainstore Paradox fails.

In fact, we can find a simple perfect equilibrium for the infinitely repeated Prisoner's Dilemma in which both players cooperate-a game in which both players adopt the Grim Strategy.

Grim Strategy
1 Start by choosing Deny.
2 Continue to choose Deny unless some player has chosen $Confess$, in which case choose $Confess$ forever.

Notice that the Grim Strategy says that even if a player is the first to deviate and choose *Confess*, he continues to choose *Confess* thereafter.

If Column uses the Grim Strategy, the Grim Strategy is weakly Row's best response. If Row cooperates, he will continue to receive the high (*Deny*, *Deny*) payoff forever. If he confesses, he will receive the higher ($Confess$, *Deny*) payoff once, but the best he can hope for thereafter is the ($Confess$, $Confess$) payoff.

Even in the infinitely repeated game, cooperation is not immediate, and not every strategy that punishes confessing is perfect. A notable example is the strategy of Tit-for-Tat.

Tit-for-Tat
1 Start by choosing Deny.
2 Thereafter, in period n choose the action that the other player chose in period $(n-1)$.

If Column uses Tit-for-Tat, Row does not have an incentive to $Confess$ first, because if Row cooperates he will continue to receive the high (*Deny*, *Deny*) payoff, but if he confesses and then returns to Tit-for-Tat, the players alternate $(Confess, Deny)$ with $(Deny, Confess)$ forever. Row's average payoff from this alternation would be lower than if he had stuck to $(Deny, Deny)$, and would swamp the one-time gain. But Tit-for-Tat is almost never perfect in the infinitely repeated Prisoner's Dilemma without discounting, because it is not rational for Column to punish Row's initial $Confess$. Adhering to Tit-for-Tat's punishments results in a miserable alternation of $Confess$ and $Deny$, so Column would rather ignore Row's first $Confess$. The deviation is not from the equilibrium path action of *Deny*, but from the off-equilibrium action rule of *Confess in response to a Confess*. Tit-for-Tat, unlike the Grim Strategy, cannot enforce cooperation.[1]

Unfortunately, although eternal cooperation is a perfect equilibrium outcome in the infinite game under at least one strategy, so is practically anything else, including eternal confessing. The multiplicity of equilibria is summarized by the Folk Theorem, so called because its origins are hazy.

Theorem 1 (the Folk Theorem)
In an infinitely repeated n-person game with finite action sets at each repetition, any profile of actions observed in any finite number of repetitions is the unique outcome of some subgame perfect equilibrium given
Condition 1: *The rate of time preference is zero, or positive and sufficiently small;*
Condition 2: *The probability that the game ends at any repetition is zero, or positive and sufficiently small; and*
Condition 3: *The set of payoff profiles that strictly Pareto dominate the minimax payoff profiles in the mixed extension of the one-shot game is n-dimensional.*

What the Folk Theorem tells us is that claiming that particular behavior arises in a perfect equilibrium is meaningless in an infinitely repeated game. This applies to any game that meets conditions 1 to 3, not just to the Prisoner's Dilemma. If an infinite amount of time always remains in the game, a way can always be found to make one player willing to punish some other player for the sake of a better future, even if the punishment currently hurts the punisher as well as the punished. Any finite interval of time is insignificant compared to eternity, so the threat of future reprisal makes the players willing to carry out the punishments needed.

We will next discuss conditions 1 to 3.

Condition 1: Discounting

The Folk Theorem helps answer the question of whether discounting future payments lessens the influence of the troublesome Last Period. Quite to the contrary, with discounting, the present gain from confessing is weighted more heavily and future gains from cooperation more lightly. If the discount rate is very high the game almost returns to being one-shot. When the real interest rate is 1,000 percent, a payment next year is little better than a payment a hundred years hence, so next year is practically irrelevant. Any model that relies on a large number of repetitions also assumes that the discount rate is not too high.

Allowing a little discounting is none the less important to show there is no discontinuity

[1] See Kalai, Samet & Stanford (1988) and Problem 5.5 for elaboration of this point.

at the discount rate of zero. If we come across an undiscounted, infinitely repeated game with many equilibria, the Folk Theorem tells us that adding a low discount rate will not reduce the number of equilibria. This contrasts with the effect of changing the model by having a large but finite number of repetitions, a change which often eliminates all but one outcome by inducing the Chainstore Paradox.

A discount rate of zero supports many perfect equilibria, but if the rate is high enough, the only equilibrium outcome is eternal confessing. We can calculate the critical value for given parameters. The Grim Strategy imposes the heaviest possible punishment for deviant behavior. Using the payoffs for the Prisoner's Dilemma from Table 2a in the next section, the equilibrium payoff from the Grim Strategy is the current payoff of 5 plus the value of the rest of the game, which from Table 2 of Chapter 4 is $\frac{5}{r}$. If Row deviated by confessing, he would receive a current payoff of 10, but the value of the rest of the game would fall to 0. The critical value of the discount rate is found by solving the equation $5 + \frac{5}{r} = 10 + 0$, which yields $r = 1$, a discount rate of 100 percent or a discount factor of $\delta = 0.5$. Unless the players are extremely impatient, confessing is not much of a temptation.

Condition 2: A probability of the game ending

Time preference is fairly straightforward, but what is surprising is that assuming that the game ends in each period with probability θ does not make a drastic difference. In fact, we could even allow θ to vary over time, so long as it never became too large. If $\theta > 0$, the game ends in finite time with probability one; or, put less dramatically, the expected number of repetitions is finite, but it still behaves like a discounted infinite game, because the expected number of future repetitions is always large, no matter how many have already occurred. The game still has no Last Period, and it is still true that imposing one, no matter how far beyond the expected number of repetitions, would radically change the results.

The following two situations are different from each other.
"1 The game will end at some uncertain date before T."
"2 There is a constant probability of the game ending."

In situation (1), the game is like a finite game, because, as time passes, the maximum length of time still to run shrinks to zero. In situation (2), even if the game will end by T with high probability, if it actually lasts until T the game looks exactly the same as at time zero. The fourth verse from the hymn "Amazing grace" puts this stationarity very nicely (though I expect it is supposed to apply to a game with $\theta = 0$).

When we've been there ten thousand years,
Bright shining as the sun,
We've no less days to sing God's praise
Than when we'd first begun.

Condition 3: Dimensionality

The "minimax payoff" mentioned in theorem 5.1 is the payoff that results if all the other players pick strategies solely to punish player i, and he protects himself as best he can.

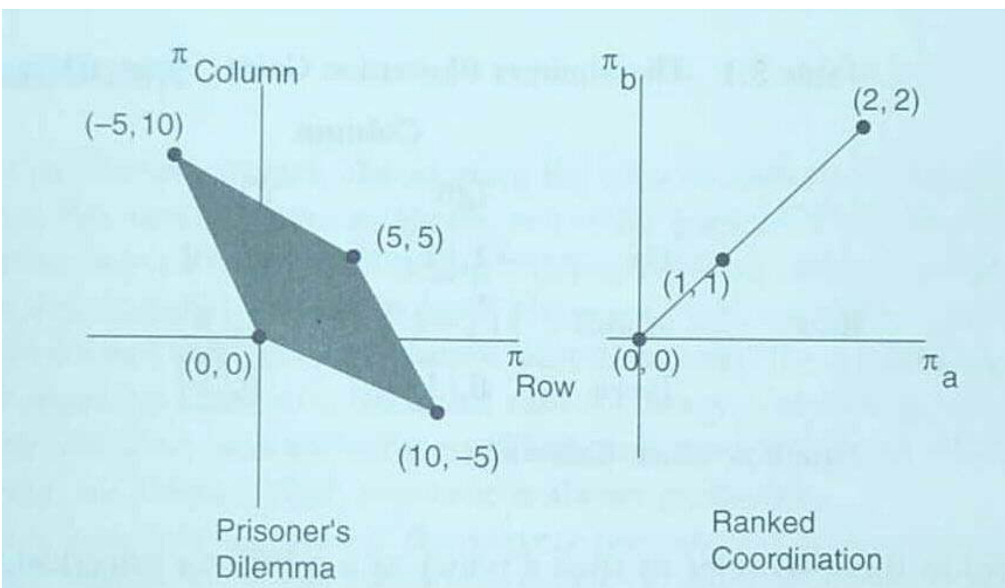
The set of strategies s_{-i}^{i*} is a set of $(n-1)$ **minimax strategies** *chosen by all the players except i to keep i's payoff as low as possible, no matter how he responds. s_{-i}^{i*} solves*

$$\underset{s_{-i}}{Minimize} \ \underset{s_i}{Maximum} \ \pi_i(s_i, s_{-i}). \tag{1}$$

Player i's **minimax payoff**, **minimax value**, *or* **security value** *is his payoff from the solution of (1).*

The dimensionality condition is needed only for games with three or more players. It is satisfied if there is some payoff profile for each player in which his payoff is greater than his minimax payoff but still different from the payoff of every other player. Figure 1 shows how this condition is satisfied for the two-person Prisoner's Dilemma of Table 2a a few pages beyond this paragraph, but not for the two-person Ranked Coordination game. It is also satisfied by the n-person Prisoner's Dilemma in which a solitary confesser gets a higher payoff than his cooperating fellow-prisoners, but not by the n-person Ranked Coordination game, in which all the players have the same payoff. The condition is necessary because establishing the desired behavior requires some way for the other players to punish a deviator without punishing themselves.

Figure 1: The Dimensionality Condition



An alternative to the dimensionality condition in the Folk Theorem is

Condition 3′: *The repeated game has a "desirable" subgame-perfect equilibrium in which the strategy profile $\overline{s}$ played each period gives player i a payoff that exceeds his payoff from some other "punishment" subgame-perfect equilibrium in which the strategy profile $\underline{s}^i$ is played each period:*

$$\exists \overline{s} : \forall i, \ \exists \underline{s}^i : \pi_i(\underline{s}^i) < \pi_i(\overline{s}).$$

Condition 3′ is useful because sometimes it is easy to find a few perfect equilibria. To enforce the desired pattern of behavior, use the "desirable" equilibrium as a carrot and the "punishment" equilibrium as a self-enforcing stick (see Rasmusen [1992a]).

Minimax and Maximin

In discussions of strategies which enforce cooperation, the question of deciding on the maximum severity of punishment strategies frequently arises. The idea of the minimax strategy is useful for this in that the minimax strategy is defined as the most severe sanction possible if the offender does not cooperate in his own punishment. The corresponding strategy for the offender, trying to protect himself from punishment, is the maximin strategy:

The strategy s_i^* is a **maximin strategy** *for player i if, given that the other players pick strategies to make i's payoff as low as possible, s_i^* gives i the highest possible payoff. In our notation, s_i^* solves*

$$\underset{s_i}{Maximize} \ \underset{s_{-i}}{Minimum} \ \pi_i(s_i, s_{-i}). \tag{2}$$

The following formulae show how to calculate the minimax and maximin strategies for a two-player game with Player 1 as i.

$$\text{Maximin:} \quad \underset{s_1}{Maximum} \ \underset{s_2}{Minimum} \ \pi_1.$$

$$\text{Minimax:} \quad \underset{s_2}{Minimum} \ \underset{s_1}{Maximum} \ \pi_1.$$

In the Prisoner's Dilemma, the minimax and maximin strategies are both *Confess.* Although the Welfare Game (table 3.1) has only a mixed strategy Nash equilibrium, if we restrict ourselves to the pure strategies[2] the Pauper's maximin strategy is *Try to Work*, which guarantees him at least 1, and his strategy for minimaxing the Government is *Be Idle*, which prevents the Government from getting more than zero.

Under minimax, Player 2 is purely malicious but must move first (at least in choosing a mixing probability) in his attempt to cause player 1 the maximum pain. Under maximin, Player 1 moves first, in the belief that Player 2 is out to get him. In variable-sum games, minimax is for sadists and maximin for paranoids. In zero-sum games, the players are merely neurotic: minimax is for optimists and maximin for pessimists.

The maximin strategy need not be unique, and it can be in mixed strategies. Since maximin behavior can also be viewed as minimizing the maximum loss that might be suffered, decision theorists refer to such a policy as a **minimax criterion,** a catchier phrase (Luce & Raiffa [1957], p. 279).

It is tempting to use maximin strategies as the basis of an equilibrium concept. A **maximin equilibrium** is made up of a maximin strategy for each player. Such a strategy might seem reasonable because each player then has protected himself from the worst harm possible. Maximin strategies have very little justification, however, for a rational player. They are not simply the optimal strategies for risk-averse players, because risk aversion is accounted for in the utility payoffs. The players' implicit beliefs can be inconsistent in a

[2]xxx what if we don't?

maximin equilibrium, and a player must believe that his opponent would choose the most harmful strategy out of spite rather than self-interest if maximin behavior is to be rational.

The usefulness of minimax and maximin strategies is not in directly predicting the best strategies of the players, but in setting the bounds of how their strategies affect their payoffs, as in condition 3 of Theorem 1.

It is important to remember that minimax and maximin strategies are not always pure strategies. In the Minimax Illustration Game of Table 1, which I take from page 150 of Row can guarantee himself a payoff of 0 by choosing *Down*, so that is his maximin strategy. Column cannot hold Row's payoff down to 0, however, by using a pure minimax strategy. If Column chooses *Left*, Row can choose *Middle* and get a payoff of 1; if Column chooses *Right*, Row can choose *Up* and get a payoff of 1. Column can, however, hold Row's payoff down to 0 by choosing a mixed minimax strategy of *(Probability 0.5 of Left, Probability 0.5 of Right)*. Row would then respond with *Down*, for a minimax payoff of 0, since either *Up*, *Middle*, or a mixture of the two would give him a payoff of -0.5 $(= 0.5(-2) + 0.5(1))$.[3]

Table 1 The Minimax Illustration Game

		Column	
		Left	*Right*
	Up	$-2, \boxed{2}$	$\boxed{1}, -2$
Row:	*Middle*	$\boxed{1}, -2$	$-2, \boxed{2}$
	Down	$0, \boxed{1}$	$0, \boxed{1}$

Payoffs to: (Row, Column).

In two-person zero-sum games, minimax and maximin strategies are more directly useful, because when player 1 reduces player 2's payoff, he increases his own payoff. Punishing the other player is equivalent to rewarding yourself. This is the origin of the celebrated **Minimax Theorem** (von Neumann [1928]), which says that a minimax equilibrium exists in pure or mixed strategies for every two-person zero-sum game and is identical to the maximin equilibrium. Unfortunately, the games that come up in applications are almost never zero-sum games, so the Minimax Theorem is of limited applicability.

Precommitment

What if we use metastrategies, abandoning the idea of perfectness by allowing players to commit at the start to a strategy for the rest of the game? We would still want to keep the game noncooperative by disallowing binding promises, but we could model it as a game with simultaneous choices by both players, or with one move each in sequence.

If precommitted strategies are chosen simultaneously, the equilibrium outcome of the finitely repeated Prisoner's Dilemma calls for always confessing, because allowing commitment is the same as allowing equilibria to be nonperfect, in which case, as was shown earlier, the unique Nash outcome is always confessing.

[3] Column's maximin and minimax strategies can also be computed. The strategy for minimaxing Column is *(Probability 0.5 of Up, Probability 0.5 of Middle)*, his maximin strategy is *(Probability 0.5 of Left, Probability 0.5 of Right)*, and his minimax payoff is 0.

A different result is achieved if the players precommit to strategies in sequence. The outcome depends on the particular values of the parameters, but one possible equilibrium is the following: Row moves first and chooses the strategy (*Deny* until Column $Confess$s; thereafter always $Confess$), and Column chooses (*Deny* until the last period; then $Confess$). The observed outcome would be for both players to deny until the last period, and then for Row to again deny, but for Column to confess. Row would submit to this because if he chose a strategy that initiated confessing earlier, Column would choose a strategy of starting to confess earlier too. The game has a second-mover advantage.

5.3 Reputation: The One-sided Prisoner's Dilemma

Part II of this book will analyze moral hazard and adverse selection. Under moral hazard, a player wants to commit to high effort, but he cannot credibly do so. Under adverse selection, a player wants to prove he is high ability, but he cannot. In both, the problem is that the penalties for lying are insufficient. Reputation seems to offer a way out of the problem. If the relationship is repeated, perhaps a player is willing to be honest in early periods in order to establish a reputation for honesty which will be valuable to himself later.

Reputation seems to play a similar role in making threats to punish credible. Usually punishment is costly to the punisher as well as the punished, and it is not clear why the punisher should not let bygones be bygones. Yet in 1988 the Soviet Union paid off 70-year-old debt to dissuade the Swiss authorities from blocking a mutually beneficial new bond issue ("Soviets Agree to Pay Off Czarist Debt to Switzerland," *Wall Street Journal*, January 19, 1988, p. 60). Why were the Swiss so vindictive towards Lenin?

The questions of why players do punish and do not cheat are really the same questions that arise in the repeated Prisoner's Dilemma, where the fact of an infinite number of repetitions allows cooperation. That is the great problem of reputation. Since everyone knows that a player will *Confess*, choose low effort, or default on debt in the last period, why do they suppose he will bother to build up a reputation in the present? Why should past behavior be any guide to future behavior?

Not all reputation problems are quite the same as the Prisoner's Dilemma, but they have much the same flavor. Some games, like duopoly or the original Prisoner's Dilemma, are **two-sided** in the sense that each player has the same strategy set and the payoffs are symmetric. Others, such as the game of Product Quality (see below), are what we might call **one-sided Prisoner's Dilemmas**, which have properties similar to the Prisoner's Dilemma, but do not fit the usual definition because they are asymmetric. Table 2 shows the normal forms for both the original Prisoner's Dilemma and the one-sided version.[4] The important difference is that in the one-sided Prisoner's Dilemma at least one player really does prefer the outcome equivalent to ($Deny, Deny$), which is (*High Quality, Buy)* in Table 2b, to anything else. He confesses defensively, rather than both offensively and defensively. The payoff (0,0) can often be interpreted as the refusal of one player to interact

[4]The exact numbers are different from the Prisoner's Dilemma in Table 1 in Chapter 1, but the ordinal rankings are the same. Numbers such as those in Table 2 of the present chapter are more commonly used, because it is convenient to normalize the (*Confess, Confess*) payoffs to (0,0) and to make most of the numbers positive rather than negative.

with the other, for example, the motorist who refuses to buy cars from Chrysler because he knows they once falsified odometers. Table 3 lists examples of both one-sided and two-sided games. Versions of the Prisoner's Dilemma with three or more players can also be classified as one-sided or two-sided, depending on whether or not all players find *Confess* a dominant strategy.

Table 2 Prisoner's Dilemmas

(a) Two-Sided (conventional)

		Column		
		Deny		*Confess*
	Deny	5,5	→	-5,10
Row:		↓		↓
	Confess	10,-5	→	**0,0**

Payoffs to: (Row,Column)

(b) One-Sided

		Consumer (Column)		
		Buy		*Boycott*
	High Quality	5,5	←	0,0
Seller (Row):		↓		↕
	Low Quality	10, -5	→	**0,0**

Payoffs to: (Seller, Consumer)

Table 3 Some repeated games in which reputation is important

Application	Sidedness	Players	Actions
Prisoner's Dilemma	two-sided	Row	*Deny/Confess*
		Column	*Deny/Confess*
Duopoly	two-sided	Firm	*High price/Low price*
		Firm	*High price/Low price*
Employment	two-sided	Employer	*Bonus/No bonus*
		Employee	*Work/Shirk*
Product Quality	one-sided	Consumer	*Buy/Boycott*
		Seller	*High quality/low quality*
Entry Deterrence	one-sided	Incumbent	*Low price/High price*
		Entrant	*Enter/Stay out*
Financial Disclosure	one-sided	Corporation	*Truth/Lies*
		Investor	*Invest/Refrain*
Borrowing	one-sided	Lender	*Lend/Refuse*
		Borrower	*Repay/Default*

The Nash and iterated dominance equilibria in the one-sided Prisoner's Dilemma are still (*Confess, Confess*), but it is not a dominant-strategy equilibrium. Column does not have a dominant strategy, because if Row were to choose *Deny*, Column would also choose *Deny*, to obtain the payoff of 5; but if Row chooses $Confess$, Column would choose $Confess$, for a payoff of zero. $Confess$ is however, weakly dominant for Row, which makes ($Confess$,$Confess$) the iterated dominant strategy equilibrium. In both games, the players would like to persuade each other that they will cooperate, and devices that induce cooperation in the one-sided game will usually obtain the same result in the two-sided game.

5.4 Product Quality in an Infinitely Repeated Game

The Folk Theorem tells us that some perfect equilibrium of an infinitely repeated game— sometimes called an **infinite horizon model**— can generate any pattern of behavior observed over a finite number of periods. But since the Folk Theorem is no more than a mathematical result, the strategies that generate particular patterns of behavior may be unreasonable. The theorem's value is in provoking close scrutiny of infinite horizon models so that the modeller must show why his equilibrium is better than a host of others. He must go beyond satisfaction of the technical criterion of perfectness and justify the strategies on other grounds.

In the simplest model of product quality, a seller can choose between producing costly high quality or costless low quality, and the buyer cannot determine quality before he purchases. If the seller would produce high quality under symmetric information, we have a one-sided Prisoner's Dilemma, as in Table 2b. Both players are better off when the seller produces high quality and the buyer purchases the product, but the seller's weakly dominant strategy is to produce low quality, so the buyer will not purchase. This is also an example of moral hazard, the topic of chapter 7.

A potential solution is to repeat the game, allowing the firm to choose quality at each repetition. If the number of repetitions is finite, however, the outcome stays the same because of the Chainstore Paradox. In the last repetition, the subgame is identical to the one- shot game, so the firm chooses low quality. In the next-to-last repetition, it is foreseen that the last period's outcome is independent of current actions, so the firm also chooses low quality, an argument that can be carried back to the first repetition.

If the game is repeated an infinite number of times, the Chainstore Paradox is inapplicable and the Folk Theorem says that a wide range of outcomes can be observed in equilibrium. Klein & Leffler (1981) construct a plausible equilibrium for an infinite period model. Their original article, in the traditional verbal style of UCLA, does not phrase the result in terms of game theory, but we will recast it here, as I did in Rasmusen (1989b). In equilibrium, the firm is willing to produce a high quality product because it can sell at a high price for many periods, but consumers refuse to ever buy again from a firm that has once produced low quality. The equilibrium price is high enough that the firm is unwilling to sacrifice its future profits for a one- time windfall from deceitfully producing low quality and selling it at a high price. Although this is only one of a large number of subgame perfect equilibria, the consumers' behavior is simple and rational: no consumer can benefit by deviating from the equilibrium.

Product Quality

Players
An infinite number of potential firms and a continuum of consumers.

The Order of Play
1 An endogenous number n of firms decide to enter the market at cost F.
2 A firm that has entered chooses its quality to be *High* or *Low*, incurring the constant marginal cost c if it picks *High* and zero if it picks *Low*. The choice is unobserved by consumers. The firm also picks a price p.
3 Consumers decide which firms (if any) to buy from, choosing firms randomly if they are indifferent. The amount bought from firm i is denoted q_i.
4 All consumers observe the quality of all goods purchased in that period.
5 The game returns to (2) and repeats.

Payoffs
The consumer benefit from a product of low quality is zero, but consumers are willing to buy quantity $q(p) = \sum_{i=1}^{n} q_i$ for a product believed to be high quality, where $\frac{dq}{dp} < 0$.
If a firm stays out of the market, its payoff is zero.
If firm i enters, it receives $-F$ immediately. Its current end-of- period payoff is $q_i p$ if it produces *Low* quality and $q_i(p-c)$ if it produces *High* quality. The discount rate is $r \geq 0$.

That the firm can produce low quality items at zero marginal cost is unrealistic, but it is only a simplifying assumption. By normalizing the cost of producing low quality to zero, we avoid having to carry an extra variable through the analysis without affecting the result.

The Folk Theorem tells us that this game has a wide range of perfect outcomes, including a large number with erratic quality patterns like (*High, High, Low, High, Low, Low...*). If we confine ourselves to pure-strategy equilibria with the stationary outcome of constant quality and identical behavior by all firms in the market, then the two outcomes are low quality and high quality. Low quality is always an equilibrium outcome, since it is an equilibrium of the one-shot game. If the discount rate is low enough, high quality is also an equilibrium outcome, and this will be the focus of our attention. Consider the following strategy profile:

Firms. $\tilde{n}$ firms enter. Each produces high quality and sells at price $\tilde{p}$. If a firm ever deviates from this, it thereafter produces low quality (and sells at the same price $\tilde{p}$). The values of $\tilde{p}$ and $\tilde{n}$ are given by equations (4) and (8) below.

Buyers. Buyers start by choosing randomly among the firms charging $\tilde{p}$. Thereafter, they remain with their initial firm unless it changes its price or quality, in which case they switch randomly to a firm that has not changed its price or quality.

This strategy profile is a perfect equilibrium. Each firm is willing to produce high quality and refrain from price-cutting because otherwise it would lose all its customers. If it has deviated, it is willing to produce low quality because the quality is unimportant, given the absence of customers. Buyers stay away from a firm that has produced low quality because they know it will continue to do so, and they stay away from a firm that has cut the price because they know it will produce low quality. For this story to work, however, the equilibrium must satisfy three constraints that will be explained in more depth in Section 7.3: incentive compatibility, competition, and market clearing.

The **incentive compatibility** constraint says that the individual firm must be willing to produce high quality. Given the buyers' strategy, if the firm ever produces low quality it receives a one-time windfall profit, but loses its future profits. The tradeoff is represented by constraint (3), which is satisfied if the discount rate is low enough.

$$\frac{q_i p}{1+r} \leq \frac{q_i(p-c)}{r} \qquad (\textit{incentive compatibility}). \tag{3}$$

Inequality (3) determines a lower bound for the price, which must satisfy

$$\tilde{p} \geq (1+r)c. \tag{4}$$

Condition (4) will be satisfied as an equality, because any firm trying to charge a price higher than the quality-guaranteeing $\tilde{p}$ would lose all its customers.

The second constraint is that competition drives profits to zero, so firms are indifferent between entering and staying out of the market.

$$\frac{q_i(p-c)}{r} = F. \qquad (\textit{competition}) \tag{5}$$

Treating (3) as an equation and using it to replace p in equation (5) gives

$$q_i = \frac{F}{c}. \tag{6}$$

We have now determined p and q_i, and only n remains, which is determined by the equality of supply and demand. The market does not always clear in models of asymmetric information (see Stiglitz [1987]), and in this model each firm would like to sell more than its equilibrium output at the equilibrium price, but the market output must equal the quantity demanded by the market.

$$nq_i = q(p). \qquad (market\ \ clearing) \tag{7}$$

Combining equations (3), (6), and (7) yields

$$\tilde{n} = \frac{cq([1+r]c)}{F}. \tag{8}$$

We have now determined the equilibrium values, the only difficulty being the standard existence problem caused by the requirement that the number of firms be an integer (see note N5.4).

The equilibrium price is fixed because F is exogenous and demand is not perfectly inelastic, which pins down the size of firms. If there were no entry cost, but demand were still elastic, then the equilibrium price would still be the unique p that satisfied constraint (3), and the market quantity would be determined by $q(p)$, but F and q_i would be undetermined. If consumers believed that any firm which might possibly produce high quality paid an exogenous dissipation cost F, the result would be a continuum of equilibria. The firms' best response would be for $\tilde{n}$ of them to pay F and produce high quality at price $\tilde{p}$, where $\tilde{n}$ is determined by the zero profit condition as a function of F. Klein & Leffler note this indeterminacy and suggest that the profits might be dissipated by some sort of brand-specific capital. This is especially plausible when there is asymmetric information, so firms might wish to use capital spending to signal that they intend to be in the business for a long time; Rasmusen & Perri (2001) shows a way to model this. Another good explanation for which firms enjoy the high profits of good reputation is simply the history of the industry. Schmalensee (1982) shows how a pioneering brand can retain a large market share because consumers are unwilling to investigate the quality of new brands.

The repeated-game model of reputation for product quality can be used to model many other kinds of reputation too. Even before Klein & Leffler, Telser titled his 1980 article "A Theory of Self- Enforcing Agreements," looked at a number of situations in which repeated play could balance the short-run gain from cheating again the long-run gain from cooperation. We will see the idea later in this book in in Section 8.1 as part of the idea of the "efficiency wage".

*5.5 Markov Equilibria and Overlapping Generations in the Game of Customer Switching Costs

The next model demonstrates a general modelling technique, the **overlapping generations model**, in which different cohorts of otherwise identical players enter and leave the game with overlapping "lifetimes," and a new equilibrium concept, "Markov equilibrium." The best-known example of an overlapping-generations model is the original consumption-loans model of Samuelson (1958). The models are most often used in macroeconomics, but they can also be useful in microeconomics. Klemperer (1987) has stimulated considerable interest in customers who incur costs in moving from one seller to another. The model used here will be that of Farrell & C. Shapiro (1988).

Customer Switching Costs

Players
Firms Apex and Brydox, and a series of customers, each of whom is first called a youngster and then an oldster.

The Order of Play
1a Brydox, the initial incumbent, picks the incumbent price p_1^i.
1b Apex, the initial entrant, picks the entrant price p_1^e.
1c The oldster picks a firm.
1d The youngster picks a firm.
1e Whichever firm attracted the youngster becomes the incumbent.
1f The oldster dies and the youngster becomes an oldster.
2a Return to (1a), possibly with new identities for entrant and incumbent.

Payoffs
The discount factor is δ. The customer reservation price is R and the switching cost is c. The per period payoffs in period t are, for $j = (i, e)$,

$$\pi_{firm\ j} = \begin{cases} 0 & \text{if no customers are attracted.} \\ p_t^j & \text{if just oldsters or just youngsters are attracted.} \\ 2p_t^j & \text{if both oldsters and youngsters are attracted.} \end{cases}$$

$$\pi_{oldster} = \begin{cases} R - p_t^i & \text{if he buys from the incumbent.} \\ R - p_t^e - c & \text{if he switches to the entrant.} \end{cases}$$

$$\pi_{youngster} = \begin{cases} R - p_t^i & \text{if he buys from the incumbent.} \\ R - p_t^e & \text{if he buys from the entrant.} \end{cases}$$

Finding all the perfect equilibria of an infinite game like this one is difficult, so we will follow Farrell and Shapiro in limiting ourselves to the much easier task of finding the perfect Markov equilibrium, which is unique.

A **Markov strategy** *is a strategy that, at each node, chooses the action independently of the history of the game except for the immediately preceding action (or actions, if they were simultaneous).*

Here, a firm's Markov strategy is its price as a function of whether the particular is the incumbent or the entrant, and not a function of the entire past history of the game.

There are two ways to use Markov strategies: (1) just look for equilibria that use Markov strategies, and (2) disallow nonMarkov strategies and then look for equilibria.

Because the first way does not disallow non-Markov strategies, the equilibrium must be such that no player wants to deviate by using any other strategy, whether Markov or not. This is just a way of eliminating possible multiple equilibria by discarding ones that use non-Markov strategies. The second way is much more dubious, because it requires the players not to use non-Markov strategies, even if they are best responses. A **perfect Markov equilibrium** uses the first approach: it is a perfect equilibrium that happens to use only Markov strategies.

Brydox, the initial incumbent, moves first and chooses p^i low enough that Apex is not tempted to choose $p^e < p^i - c$ and steal away the oldsters. Apex's profit is p^i if it chooses $p^e = p^i$ and serves just youngsters, and $2(p^i - c)$ if it chooses $p^e = p^i - c$ and serves both oldsters and youngsters. Brydox chooses p^i to make Apex indifferent between these alternatives, so

$$p^i = 2(p^i - c), \tag{9}$$

and

$$p^i = p^e = 2c. \tag{10}$$

In equilibrium, Apex and Brydox take turns being the incumbent and charge the same price.

Because the game lasts forever and the equilibrium strategies are Markov, we can use a trick from dynamic programming to calculate the payoffs from being the entrant versus being the incumbent. The equilibrium payoff of the current entrant is the immediate payment of p^e plus the discounted value of being the incumbent in the next period:

$$\pi_e^* = p^e + \delta\pi_i^*. \tag{11}$$

The incumbent's payoff can be similarly stated as the immediate payment of p^i plus the discounted value of being the entrant next period:

$$\pi_i^* = p^i + \delta\pi_e^*. \tag{12}$$

We could use equation (10) to substitute for p^e and p^i, which would leave us with the two equations (11) and (12) for the two unknowns π_i^* and π_e^*, but an easier way to compute the payoff is to realize that in equilibrium the incumbent and the entrant sell the same amount at the same price, so $\pi_i^* = \pi_e^*$ and equation (12) becomes

$$\pi_i^* = 2c + \delta\pi_i^*. \tag{13}$$

It follows that

$$\pi_i^* = \pi_e^* = \frac{2c}{1-\delta}. \tag{14}$$

Prices and total payoffs are increasing in the switching cost c, because that is what gives the incumbent market power and prevents ordinary competition of the ordinary Bertrand kind to be analyzed in section 13.2. The total payoffs are increasing in δ for the usual reason that future payments increase in value as δ approaches one.

*5.6 Evolutionary Equilibrium: Hawk-Dove

For most of this book we have been using the Nash equilibrium concept or refinements of it based on information and sequentiality, but in biology such concepts are often inappropriate. The lower animals are less likely than humans to think about the strategies of their opponents at each stage of a game. Their strategies are more likely to be preprogrammed and their strategy sets more restricted than the businessman's, if perhaps not more so than his customer's. In addition, behavior evolves, and any equilibrium must take account of the possibility of odd behavior caused by the occasional mutation. That the equilibrium is common knowledge, or that players cannot precommit to strategies, are not compelling assumptions. Thus, the ideas of Nash equilibrium and sequential rationality are much less useful than when game theory is modelling rational players.

Game theory has grown to some importance in biology, but the style is different than in economics. The goal is not to explain how players would rationally pick actions in a given situation, but to explain how behavior evolves or persists over time under exogenous shocks. Both approaches end up defining equilibria to be strategy profiles that are best responses in some sense, but biologists care much more about the stability of the equilibrium and how strategies interact over time. In section 3.5, we touched briefly on the stability of the Cournot equilibrium, but economists view stability as a pleasing by-product of the equilibrium rather than its justification. For biologists, stability is the point of the analysis.

Consider a game with identical players who engage in pairwise contests. In this special context, it is useful to think of an equilibrium as a strategy profile such that no player with a new strategy can enter the environment (**invade**) and receive a higher expected payoff than the old players. Moreover, the invading strategy should continue to do well even if it plays itself with finite probability, or its invasion could never grow to significance. In the commonest model in biology, all the players adopt the same strategy in equilibrium, called an evolutionarily stable strategy. John Maynard Smith originated this idea, which is somewhat confusing because it really aims at an equilibrium concept, which involves a strategy profile, not just one player's strategy. For games with pairwise interactions and identical players, however, the evolutionarily stable strategy can be used to define an equilibrium concept.

A strategy s^* is an **evolutionarily stable strategy***, or* **ESS***, if, using the notation $\pi(s_i, s_{-i})$ for player i's payoff when his opponent uses strategy s_{-i}, for every other strategy s' either*

$$\pi(s^*, s^*) > \pi(s', s^*) \tag{15}$$

or

$$\begin{array}{l} (a)\;\; \pi(s^*, s^*) = \pi(s', s^*) \\ \text{and} \\ (b)\;\; \pi(s^*, s') > \pi(s', s'). \end{array} \tag{16}$$

If condition (15) holds, then a population of players using s^* cannot be invaded by a deviant using s'. If condition (16) holds, then s' does well against s^*, but badly against itself, so that if more than one player tried to use s' to invade a population using s^*, the invaders would fail.

We can interpret ESS in terms of Nash equilibrium. Condition (15) says that s^* is a strong Nash equilibrium (although not every strong Nash strategy is an ESS). Condition

(16) says that if s^* is only a weak Nash strategy, the weak alternative s' is not a best response to itself. ESS is a refinement of Nash, narrowed by the requirement that ESS not only be a best response, but that (a) it have the highest payoff of any strategy used in equilibrium (which rules out equilibria with asymmetric payoffs), and (b) it be a strictly best response to itself.

The motivations behind the two equilibrium concepts are quite different, but the similarities are useful because even if the modeller prefers ESS to Nash, he can start with the Nash strategies in his efforts to find an ESS.

As an example of (a), consider the Battle of the Sexes. In it, the mixed strategy equilibrium is an ESS, because a player using it has as high a payoff as any other player. The two pure strategy equilibria are not made up of ESS's, though, because in each of them one player's payoff is higher than the other's. Compare with Ranked Coordination, in which the two pure strategy equilibria and the mixed strategy equilibrium are all made up of ESS's. (The dominated equilibrium strategy is nonetheless an ESS, because given that the other players are using it, no player could do as well by deviating.)

As an example of (b), consider the Utopian Exchange Economy game in Table 4, adapted from problem 7.5 of Gintis (forthcoming). In Utopia, each citizen can produce either one or two units of individualized output. He will then go into the marketplace and meet another citizen. If either of them produced only one unit, trade cannot increase their payoffs. If both of them produced two, however, they can trade one unit for one unit, and both end up happier with their increased variety of consumption.

Table 4 The Utopian Exchange Economy Game

			Jones	
		Low Output		*HighOutput*
	LowOutput	**1, 1**	$\leftrightarrow$	1, 1
Smith:		$\updownarrow$		$\downarrow$
	High Output	1,1	$\rightarrow$	**2,2**

Payoffs to: (Smith,Jones).

This game has three Nash equilibria, one of which is in mixed strategies. Since all strategies but *High Output* are weakly dominated, that alone is an ESS. *Low Output* fails to meet condition (16b), because it is not the strictly best response to itself. If the economy began with all citizens choosing *Low Output*, then if Smith deviated to *High Output* he would not do any better, but if *two* people deviated to *High Output*, they would do better in expectation because they might meet each other and receive the payoff of (2,2).

An Example of ESS: Hawk-Dove

The best-known illustration of the ESS is the game of Hawk-Dove . Imagine that we have a population of birds, each of whom can behave as an aggressive Hawk or a pacific Dove. We will focus on two randomly chosen birds, Bird One and Bird Two. Each bird has a choice of what behavior to choose on meeting another bird. A resource worth $V = 2$ "fitness units" is at stake when the two birds meet. If they both fight, the loser incurs a cost of $C = 4$, which means that the expected payoff when two Hawks meet is -1 $(= 0.5[2] + 0.5[-4])$ for each of them. When two Doves meet, they split the resource, for a payoff of 1 apiece. When a Hawk meets a Dove, the Dove flees for a payoff of 0, leaving the Hawk with a payoff of 2. Table 5 summarizes this.

Table 5 Hawk-Dove: Economics Notation

		Bird Two		
		Hawk		*Dove*
	Hawk	-1,-1	$\rightarrow$	**2,0**
Bird One:		$\downarrow$		$\uparrow$
	Dove	**0, 2**	$\leftarrow$	1,1

Payoffs to: (Bird One, Bird Two)

These payoffs are often depicted differently in biology games. Since the two players are identical, one can depict the payoffs by using a table showing the payoffs only of the row player. Applying this to Hawk-Dove generates Table 6.

Table 6 Hawk-Dove: Biology Notation

		Bird Two	
		Hawk	*Dove*
	Hawk	-1	2
Bird One:			
	Dove	0	1

Payoffs to: (Bird One)

Hawk-Dove is Chicken with new feathers. The two games have the same ordinal ranking of payoffs, as can be seen by comparing Table 5 with Chapter 3's Table 2, and their equilibria are the same except for the mixing parameters. Hawk-Dove has no symmetric pure-strategy Nash equilibrium, and hence no pure-strategy ESS, since in the two asymmetric Nash equilibria, *Hawk* gives a bigger payoff than *Dove*, and the doves would disappear from the population. In the ESS for this game, neither hawks nor doves completely take over the environment. If the population consisted entirely of hawks, a dove could invade and obtain a one-round payoff of 0 against a hawk, compared to the -1 that a hawk obtains against itself. If the population consisted entirely of doves, a hawk could invade and obtain a one-round payoff of 2 against a dove, compared to the 1 that a dove obtains against a dove.

In the mixed-strategy ESS, the equilibrium strategy is to be a hawk with probability 0.5 and a dove with probability 0.5, which can be interpreted as a population 50 percent hawks and 50 percent doves. As in the mixed-strategy equilibria in chapter 3, the players are indifferent as to their strategies. The expected payoff from being a hawk is the 0.5(2) from meeting a dove plus the $0.5(-1)$ from meeting another hawk, a sum of 0.5. The expected payoff from being a dove is the 0.5(1) from meeting another dove plus the 0.5(0) from meeting a hawk, also a sum of 0.5. Moreover, the equilibrium is stable in a sense similar to the Cournot equilibrium. If 60 percent of the population were hawks, a bird would have a higher fitness level as a dove. If "higher fitness" means being able to reproduce faster, the number of doves increases and the proportion returns to 50 percent over time.

The ESS depends on the strategy sets allowed the players. If two birds can base their behavior on commonly observed random events such as which bird arrives at the resource first, and $V < C$ (as specified above), then a strategy called the **bourgeois strategy** is an ESS. Under this strategy, the bird respects property rights like a good bourgeois; it behaves as a hawk if it arrives first, and a dove if it arrives second, where we assume the order of arrival is random. The bourgeois strategy has an expected payoff of 1 from meeting itself, and behaves exactly like a 50:50 randomizer when it meets a strategy that ignores the order of arrival, so it can successfully invade a population of 50:50 randomizers. But the bourgeois strategy is a correlated strategy (see section 3.3), and requires something like the order of arrival to decide which of two identical players will play *Hawk*.

The ESS is suited to games in which all the players are identical and interacting in pairs. It does not apply to games with non-identical players— wolves who can be wily or big and deer who can be fast or strong— although other equilibrium concepts of the same flavor can be constructed. The approach follows three steps, specifying (1) the initial population proportions and the probabilities of interactions, (2) the pairwise interactions, and (3) the dynamics by which players with higher payoffs increase in number in the population. Economics games generally use only the second step, which describes the strategies and payoffs from a single interaction.

The third step, the evolutionary dynamics, is especially foreign to economics. In specifying dynamics, the modeller must specify a difference equation (for discrete time) or differential equation (for continuous time) that describes how the strategies employed change over iterations, whether because players differ in the number of their descendants or because they learn to change their strategies over time. In economics games, the adjustment process is usually degenerate: the players jump instantly to the equilibrium. In biology games, the adjustment process is slower and cannot be derived from theory. How quickly the population of hawks increases to relative to doves depends on the metabolism of the bird and the length of a generation.

Slow dynamics also makes the starting point of the game important, unlike the case when adjustment is instantaneous. Figure 2, taken from D. Friedman (1991), shows a way to graphically depict evolution in a game in which all three strategies of *Hawk*, *Dove*, and *Bourgeois* are used. A point in the triangle represents a proportion of the three strategies in the population. At point E_3, for example, half the birds play *Hawk*, half play *Dove*, and none play *Bourgeois*, while at E_4 all the birds play *Bourgeois*.

Figure 2: Evolutionary Dynamics in the Hawk-Dove- Bourgeois Game

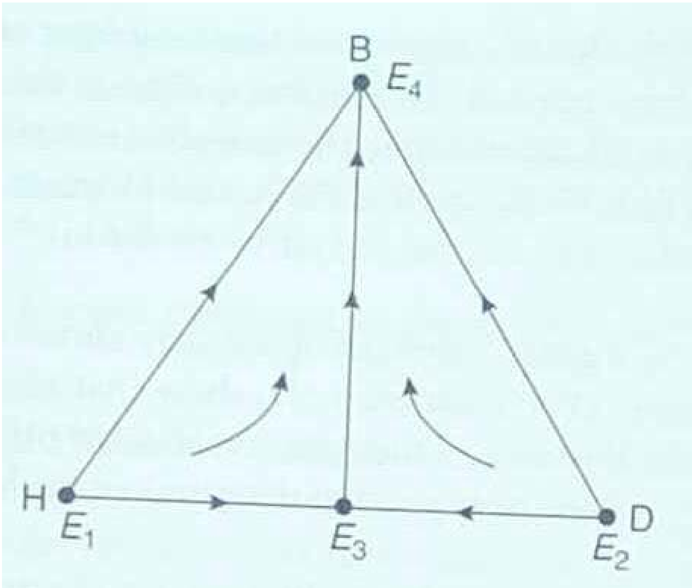


Figure 2 shows the result of dynamics based on a function specified by Friedman that gives the rate of change of a strategy's proportion based on its payoff relative to the other two strategies. Points E_1, E_2, E_3, and E_4 are all fixed points in the sense that the proportions do not change no matter which of these points the game starts from. Only point E_4 represents an evolutionarily stable equilibrium, however, and if the game starts with any positive proportion of birds playing *Bourgeois*, the proportions tend towards E_4. The original Hawk-Dove which excluded the bourgeois strategy can be viewed as the HD line at the bottom of the triangle, and E_3 is evolutionarily stable in that restricted game.

Figure 2 also shows the importance of mutation in biological games. If the population of birds is 100 percent dove, as at E_2, it stays that way in the absence of mutation, since if there are no hawks to begin with, the fact that they would reproduce at a faster rate than doves becomes irrelevant. If, however, a bird could mutate to play *Hawk* and then pass this behavior on to his offspring, then eventually some bird would do so and the mutant strategy would be successful. The technology of mutations can be important to the ultimate equilibrium. In more complicated games than Hawk-Dove, it can matter whether mutations happen to be small, accidental shifts to strategies similar to those that are currently being played, or can be of arbitrary size, so that a superior strategy quite different from the existing strategies might be reached.

The idea of mutation is distinct from the idea of evolutionary dynamics, and it is possible to use one without the other. In economics models, a mutation would correspond to the appearance of a new action in the action set of one of the players in a game. This is one way to model innovation: not as research followed by stochastic discoveries, but as accidental learning. The modeller might specify that the discovered action becomes available to players slowly through evolutionary dynamics, or instantly, in the usual style of economics. This style of research has promise for economics, but since the technologies of dynamics and mutation are important there is a danger of simply multiplying models without reliable results unless the modeller limits himself to a narrow context and bases his technology on empirical measurements.

Notes

N5.1 Finitely repeated games and the Chainstore Paradox

- The Chainstore Paradox does not apply to all games as neatly as to Entry Deterrence and the Prisoner's Dilemma. If the one-shot game has only one Nash equilibrium, the perfect equilibrium of the finitely repeated game is unique and has that same outcome. But if the one-shot game has multiple Nash equilibria, the perfect equilibrium of the finitely repeated game can have not only the one-shot outcomes, but others besides. See Benoit & Krishna (1985), Harrington (1987), and Moreaux (1985).

- John Heywood is Bartlett's source for the term "tit-for-tat," from the French "tant pour tant."

- A realistic expansion of a game's strategy space may eliminate the Chainstore Paradox. D. Hirshleifer & Rasmusen (1989), for example, show that allowing the players in a multi-person finitely repeated Prisoner's Dilemma to ostracize offenders can enforce cooperation even if there are economies of scale in the number of players who cooperate and are not ostracized.

- The peculiarity of the unique Nash equilibrium for the repeated Prisoner's Dilemma was noticed long before Selten (1978) (see Luce & Raiffa [1957] p. 99), but the term Chainstore Paradox is now generally used for all unravelling games of this kind.

- *An* **epsilon-equilibrium** *is a strategy profile s^* such that no player has more than an ϵ incentive to deviate from his strategy given that the other players do not deviate. Formally,*

$$\forall i, \;\; \pi_i(s_i^*, s_{-i}^*) \geq \pi_i(s_i', s_{-i}^*) - \epsilon, \;\; \forall s_i' \in S_i. \tag{17}$$

  Radner (1980) has shown that cooperation can arise as an ϵ-equilibrium of the finitely repeated Prisoner's Dilemma. Fudenberg & Levine (1986) compare the ϵ-equilibria of finite games with the Nash equilibria of infinite games. Other concepts besides Nash can also use the ϵ-equilibrium idea.

- A general way to decide whether a mathematical result is a trick of infinity is to see if the same result is obtained as the limit of results for longer and longer finite models. Applied to games, a good criterion for picking among equilibria of an infinite game is to select one which is the limit of the equilibria for finite games as the number of periods gets longer. Fudenberg & Levine (1986) show under what conditions one can find the equilibria of infinite-horizon games by this process. For the Prisoner's Dilemma, (*Always Confess*) is the only equilibrium in all finite games, so it uniquely satisfies the criterion.

- Defining payoffs in games that last an infinite number of periods presents the problem that the total payoff is infinite for any positive payment per period. Ways to distinguish one infinite amount from another include the following.

  1 Use an **overtaking criterion**. Payoff stream π is preferred to $\tilde{\pi}$ if there is some time T^* such that for every $T \geq T^*$,

$$\sum_{t=1}^{T} \delta^t \pi_t > \sum_{t=1}^{T} \delta^t \tilde{\pi}_t.$$

  2 Specify that the discount rate is strictly positive, and use the present value. Since payments in distant periods count for less, the discounted value is finite unless the payments are growing faster than the discount rate.

3 Use the average payment per period, a tricky method since some sort of limit needs to be taken as the number of periods averaged goes to infinity.

Whatever the approach, game theorists assume that the payoff function is **additively separable** over time, which means that the total payoff is based on the sum or average, possibly discounted, of the one-shot payoffs. Macroeconomists worry about this assumption, which rules out, for example, a player whose payoff is very low if any of his one-shot payoffs dips below a certain subsistence level. The issue of separability will arise again in section 13.5 when we discuss durable monopoly.

- Ending in finite time with probability one means that the limit of the probability the game has ended by date t approaches one as t tends to infinity; the probability that the game lasts till infinity is zero. Equivalently, the expectation of the end date is finite, which it could not be were there a positive probability of an infinite length.

N5.2 Infinitely Repeated Games, Minimax Punishments, and the Folk Theorem

- References on the Folk Theorem include Aumann (1981), Fudenberg & Maskin (1986), Fudenberg & Tirole (1991a, pp. 152-62), and Rasmusen (1992a). The most commonly cited version of the Folk Theorem says that if conditions 1 to 3 are satisfied, then:

  Any payoff profile that strictly Pareto dominates the minimax payoff profiles in the mixed extension of an n-person one-shot game with finite action sets is the average payoff in some perfect equilibrium of the infinitely repeated game.

- The evolutionary approach can also be applied to the repeated Prisoner's Dilemma. Boyd & Lorberbaum (1987) show that no pure strategy, including Tit-for-Tat, is evolutionarily stable in a population-interaction version of the Prisoner's Dilemma. J. Hirshleifer & Martinez-Coll (1988) have found that Tit-for-Tat is no longer part of an ESS in an evolutionary Prisoner's Dilemma if (1) more complicated strategies have higher computation costs; or (2) sometimes a *Deny* is observed to be a *Confess* by the other player.

- **Trigger strategies** of *trigger-price strategies* are an important kind of strategies for repeated games. Consider the oligopolist facing uncertain demand (as in Stigler [1964]). He cannot tell whether the low demand he observes facing him is due to Nature or to price cutting by his fellow oligopolists. Two things that could trigger him to cut his own price in retaliation are a series of periods with low demand or one period of especially low demand. Finding an optimal trigger strategy is a difficult problem (see Porter [1983a]). Trigger strategies are usually not subgame perfect unless the game is infinitely repeated, in which case they are a subset of the equilibrium strategies. Recent work has looked carefully at what trigger strategies are possible and optimal for players in infinitely repeated games; see Abreu, Pearce & Staccheti (1990).

  Empirical work on trigger strategies includes Porter (1983b), who examines price wars between railroads in the 19th century, and Slade (1987), who concluded that price wars among gas stations in Vancouver used small punishments for small deviations rather than big punishments for big deviations.

- A macroeconomist's technical note related to the similarity of infinite games and games with a constant probability of ending is Blanchard (1979), which discusses speculative bubbles.

- In the repeated Prisoner's Dilemma, if the end date is infinite with positive probability and only one player knows it, cooperation is possible by reasoning similar to that of the Gang of Four theorem in Section 6.4.

- Any Nash equilibrium of the one-shot game is also a perfect equilibrium of the finitely or infinitely repeated game.

N5.3 Reputation: The One-sided Prisoner's Dilemma

- *A game that is repeated an infinite number of times without discounting is called a* **supergame**.

  There is no connection between the terms "supergame" and "subgame."

- The terms, "one-sided" and "two-sided" Prisoner's Dilemma, are my inventions. Only the two-sided version is a true Prisoner's Dilemma according to the definition of note N1.2.

- Empirical work on reputation is scarce. One worthwhile effort is Jarrell & Peltzman (1985), which finds that product recalls inflict costs greatly in excess of the measurable direct costs of the operations. The investigations into actual business practice of Macaulay (1963) is much cited and little imitated. He notes that reputation seems to be more important than the written details of business contracts.

- **Vengeance and Gratitude.** Most models have excluded these feelings (although see J. Hirshleifer [1987]), which can be modelled in two ways.

  1 A player's current utility from $Confess$ or *Deny* depends on what the other player has played in the past; or
  2 A player's current utility depends on current actions and the other players' current utility in a way that changes with past actions of the other player.

  The two approaches are subtly different in interpretation. In (1), the joy of revenge is in the action of confessing. In (2), the joy of revenge is in the discomfiture of the other player. Especially if the players have different payoff functions, these two approaches can lead to different results.

N5.4 Product Quality in an infinitely repeated game

- The game of Product Quality game may also be viewed as a principal agent model of moral hazard (see chapter 7). The seller (an agent), takes the action of choosing quality that is unobserved by the buyer (the principal), but which affects the principal's payoff, an interpretation used in much of the Stiglitz (1987) survey of the links between quality and price.

  The intuition behind the Klein & Leffler model is similar to the explanation for high wages in the Shapiro & Stiglitz (1984) model of involuntary unemployment (section 8.1). Consumers, seeing a low price, realize that with a price that low the firm cannot resist lowering quality to make short-term profits. A large margin of profit is needed for the firm to decide on continuing to produce high quality.

- A paper related to Klein & Leffler (1981) is Shapiro (1983), which reconciles a high price with free entry by requiring that firms price under cost during the early periods to build up a reputation. If consumers believe, for example, that any firm charging a high price for any of the first five periods has produced a low quality product, but any firm charging a high price thereafter has produced high quality, then firms behave accordingly and the beliefs are confirmed. That the beliefs are self-confirming does not make them irrational; it only means that many different beliefs are rational in the many different equilibria.

- An equilibrium exists in the Product Quality model only if the entry cost F is just the right size to make n an integer in equation (8). Any of the usual assumptions to get around the integer problem could be used: allowing potential sellers to randomize between entering and staying out; assuming that for historical reasons, n firms have already entered; or assuming that firms lie on a continuum and the fixed cost is a uniform density across firms that have entered.

N5.5 Markov equilibria and overlapping generations in the game of Customer Switching Costs

- We assumed that the incumbent chooses its price first, but the alternation of incumbency remains even if we make the opposite assumption. The natural assumption is that prices are chosen simultaneously, but because of the discontinuity in the payoff function, that subgame has no equilibrium in pure strategies.

N5.6 Evolutionary equilibrium: the Hawk-Dove Game

- Dugatkin & Reeve (1998) is an edited volume of survey articles on different applications of game theory to biology. Dawkins (1989) is a good verbal introduction to evolutionary conflict. See also Axelrod & Hamilton (1981) for a short article on biological applications of the Prisoner's Dilemma, Hines (1987) for a survey, and Maynard Smith (1982) for a book. Weibull (1995) is a more recent treeatment. J. Hirshleifer (1982) compares the approaches of economists and biologists. Boyd & Richerson (1985) uses evolutionary game theory to examine cultural transmission, which has important differences from purely genetic transmission.

Problems

5.1:.Overlapping Generations (see Samuelson [1958])
There is a long sequence of players. One player is born in each period t, and he lives for periods t and $t+1$. Thus, two players are alive in any one period, a youngster and an oldster. Each player is born with one unit of chocolate, which cannot be stored. Utility is increasing in chocolate consumption, and a player is very unhappy if he consumes less than 0.3 units of chocolate in a period: the per-period utility functions are $U(C) = -1$ for $C < 0.3$ and $U(C) = C$ for $C \geq 0.3$, where C is consumption. Players can give away their chocolate, but, since chocolate is the only good, they cannot sell it. A player's action is to consume X units of chocolate as a youngster and give away $1 - X$ to some oldster. Every person's actions in the previous period are common knowledge, and so can be used to condition strategies upon.

5.1a If there is finite number of generations, what is the unique Nash equilibrium?

5.1b If there are an infinite number of generations, what are two Pareto-ranked perfect equilibria?

5.1c If there is a probability θ at the end of each period (after consumption takes place) that barbarians will invade and steal all the chocolate (leaving the civilized people with payoffs of -1 for any X), what is the highest value of θ that still allows for an equilibrium with $X = 0.5$?

5.2. Product Quality with Lawsuits
Modify the Product Quality game of section 5.4 by assuming that if the seller misrepresents his quality he must, as a result of a class-action suit, pay damages of x per unit sold, where $x \in (0, c]$ and the seller becomes liable for x at the time of sale.

5.2a What is $\tilde{p}$ as a function of x, F, c, and r? Is $\tilde{p}$ greater than when $x = 0$?

5.2b What is the equilibrium output per firm? Is it greater than when $x = 0$?

5.2c What is the equilibrium number of firms? Is it greater than when $x = 0$?

5.2d If, instead of x per unit, the seller pays X to a law firm to successfully defend him, what is the incentive compatibility constraint?

5.3. Repeated Games (see Benoit & Krishna [1985])
Players Benoit and Krishna repeat the game in Table 7 three times, with discounting:

Table 7: A Benoit-Krishna Game

		Krishna		
		Deny	*Waffle*	*Confess*
	Deny	10,10	$-1, -12$	$-1, 15$
Benoit:	*Waffle*	$-12, -1$	8,8	$-1, -1$
	Confess	$15, -1$	8,-1	$0, 0$

Payoffs to: (Benoit, Krishna).

(a) Why is there no equilibrium in which the players play *Deny* in all three periods?

5.3b Describe a perfect equilibrium in which both players pick *Deny* in the first two periods.

5.3c Adapt your equilibrium to the twice-repeated game.

5.3d Adapt your equilibrium to the T-repeated game.

5.3e What is the greatest discount rate for which your equilibrium still works in the three-period game?

5.4. Repeated Entry Deterrence
Assume that Entry Deterrence I is repeated an infinite number of times, with a tiny discount rate and with payoffs received at the start of each period. In each period, the entrant chooses *Enter* or *Stay out*, even if he entered previously.

5.4a What is a perfect equilibrium in which the entrant enters each period?

5.4b Why is (*Stay out, Fight*) not a perfect equilibrium?

5.4c What is a perfect equilibrium in which the entrant never enters?

5.4d What is the maximum discount rate for which your strategy profile in part (c) is still an equilibrium?

5.5. The Repeated Prisoner's Dilemma
Set $P = 0$ in the general Prisoner's Dilemma in Table 1.9, and assume that $2R > S + T$.

5.5a Show that the Grim Strategy, when played by both players, is a perfect equilibrium for the infinitely repeated game. What is the maximum discount rate for which the Grim Strategy remains an equilibrium?

5.5b Show that Tit-for-Tat is not a perfect equilibrium in the infinitely repeated Prisoner's Dilemma with no discounting.

Table 8 Evolutionarily stable strategies

		Scholar 2	
		Football (θ)	*Economics* ($1-\theta$)
	Football (θ)	1,1	0, 0
Scholar 1			
	Economics ($1-\theta$)	0, 0	5,5

Payoffs to: (Scholar 1, Scholar 2)

5.6a) There are three Nash equilibria: *(Football, Football)*, *(Economics, Economics)*, and a mixed-strategy equilibrium. What are the evolutionarily stable strategies?

5.6b Let $N_t(s)$ be the number of scholars playing a particular strategy in period t and let $\pi_t(s)$ be the payoff. Devise a Markov difference equation to express the population dynamics from period to period: $N_{t+1}(s) = f(N_t(s), \pi_t(s))$. Start the system with a population of 100,000, half the scholars talking football and half talking economics. Use your dynamics to finish Table 9.

Table 9: Conversation dynamics

t	$N_t(F)$	$N_t(E)$	θ	$\pi_t(F)$	$\pi_t(E)$
-1	50,000	50,000	0.5	0.5	2.5
0					
1					
2				.	

5.6c Repeat part (b), but specifying non-Markov dynamics, in which $N_{t+1}(s) = f(N_t(s), \pi_t(s), \pi_{t-1}(s))$.

5.7. Grab the Dollar

Table 10 shows the payoffs for the simultaneous-move game of Grab the Dollar. A silver dollar is put on the table between Smith and Jones. If one grabs it, he keeps the dollar, for a payoff of 4 utils. If both grab, then neither gets the dollar, and both feel bitter. If neither grabs, each gets to keep something.

Table 10: Grab the Dollar

		Jones	
		Grab (θ)	*Wait* ($1-\theta$)
Smith:	*Grab* (θ)	$-1, -1$	$4, 0$
	Wait ($1-\theta$)	$0, 4$	1,1

Payoffs to: (Smith, Jones)

(5.7a What are the evolutionarily stable strategies?

5.7b Suppose each player in the population is a point on a continuum, and that the initial amount of players is 1, evenly divided between *Grab* and *Wait*. Let $N_t(s)$ be the amount of players playing a particular strategy in period t and let $\pi_t(s)$ be the payoff. Let the population dynamics be $N_{t+1}(i) = (2N_t(i)) \left(\frac{\pi_t(i)}{\sum_j \pi_t(j)} \right)$. Find the missing entries in Table 11.

Table 11: Grab the Dollar: dynamics

t	$N_t(G)$	$N_t(W)$	$N_t(total)$	θ	$\pi_t(G)$	$\pi_t(w)$
0	0.5	0.5	1	0.5	1.5	0.5
1						
2						

5.7c Repeat part (b), but with the dynamics $N_{t+t}(s) = [1 + \frac{\pi_t(s)}{\sum_j \pi_t(j)}][2N_t(s)]$.

5.7d Which three games that have appeared so far in the book resemble Grab the Dollar?

September 8, 1999. January 3, 2000. November 29, 2003. December 21, 2004. 25 March 2005. Eric Rasmusen, Erasmuse@indiana.edu. Http://www.rasmusen.org. Footnotes starting with xxx are the author's notes to himself. Comments welcomed.

6 Dynamic Games with Incomplete Information

6.1 Perfect Bayesian Equilibrium: Entry Deterrence II and III

Asymmetric information, and, in particular, incomplete information, is enormously important in game theory. This is particularly true for dynamic games, since when the players have several moves in sequence, their earlier moves may convey private information that is relevant to the decisions of players moving later on. Revealing and concealing information are the basis of much of strategic behavior and are especially useful as ways of explaining actions that would be irrational in a nonstrategic world.

Chapter 4 showed that even if there is symmetric information in a dynamic game, Nash equilibrium may need to be refined using subgame perfectness if the modeller is to make sensible predictions. Asymmetric information requires a somewhat different refinement to capture the idea of sunk costs and credible threats, and Section 6.1 sets out the standard refinement of perfect Bayesian equilibrium. Section 6.2 shows that even this may not be enough refinement to guarantee uniqueness and discusses further refinements based on out-of-equilibrium beliefs. Section 6.3 uses the idea to show that a player's ignorance may work to his advantage, and to explain how even when all players know something, lack of common knowledge still affects the game. Section 6.4 introduces incomplete information into the repeated Prisoner's Dilemma and shows the Gang of Four solution to the Chainstore Paradox of Chapter 5. Section 6.5 describes the celebrated Axelrod tournament, an experimental approach to the same paradox.[1]

Subgame Perfectness is Not Enough

[1] xxx Somewhere put in the table for the Prisoner dilemma payoffs.

Table 2: The Prisoner's Dilemma

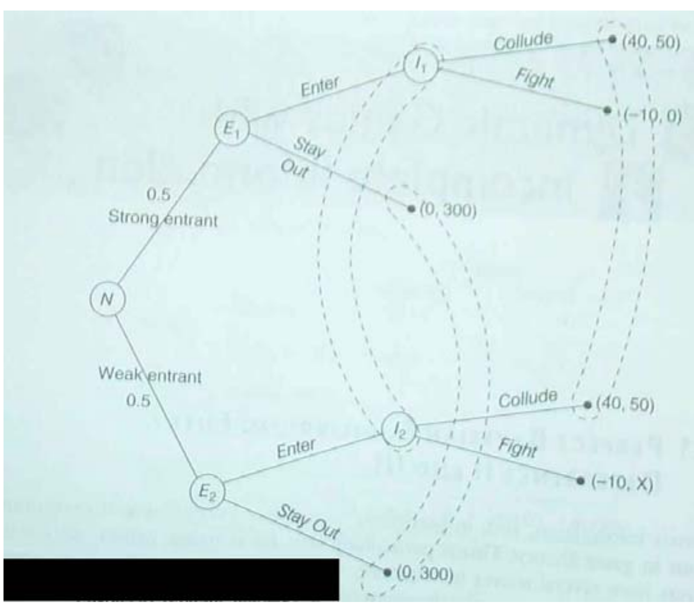
		Column		
		Deny		*Confess*
	Deny	-1,-1	→	-10, 0
Row		↓		↓
	Confess	0,-10	→	**- 8,-8**

Payoffs to: (Row,Column)

In games of asymmetric information, we will still require that an equilibrium be subgame perfect, but the mere forking of the game tree might not be relevant to a player's decision, because with asymmetric information he does not know which fork the game has taken. Smith might know he is at one of two different nodes depending on whether Jones has high or low production costs, but if he does not know the exact node, the "subgames" starting at each node are irrelevant to his decisions. In fact, they are not even subgames as we have defined them, because they cut across Smith's information sets. This can be seen in an asymmetric information version of Entry Deterrence I (Section 4.2). In Entry Deterrence I, the incumbent colluded with the entrant because fighting him was more costly than colluding once the entrant had entered. Now, let us set up the game to allow some entrants to be *Strong* and some $Weak$ in the sense that it is more costly for the incumbent to choose $Fight$ against a *Strong* entrant than a $Weak$ one. The incumbent's payoff from $Fight|Strong$ will be 0, as before, but his payoff from $Fight|Weak$ will be X, where X will take values ranging from 0 (Entry Deterrence I) to 300 (Entry Deterrence IV and V) in different versions of the game.

Entry Deterrence II, III, and IV will all have the extensive form shown in Figure 1. With 50 percent probability, the incumbent's payoff from *Fight* is X rather than the 0 in Entry Deterrence I, but the incumbent does not know which payoff is the correct one in the particular realization of the game. This is modelled as an initial move by Nature, who chooses between the entrant being $Weak$ or *Strong*, unobserved by the incumbent.

Figure 1 Entry Deterrence II, III, and IV



Entry Deterrence II: Fighting is Never Profitable

In Entry Deterrence II, $X = 1$, so information is not very asymmetric. It is common knowledge that the incumbent never benefits from *Fight*, even though his exact payoff might be zero or might be one. Unlike in the case of Entry Deterrence I, however, subgame perfectness does not rule out any Nash equilibria, because the only subgame is the subgame starting at node N, which is the entire game. A subgame cannot start at nodes E_1 or E_2, because neither of those nodes are singletons in the information partitions. Thus, the implausible Nash equilibrium, (*Stay Out*, *Fight*), escapes elimination by a technicality.

The equilibrium concept needs to be refined in order to eliminate the implausible equilibrium. Two general approaches can be taken: either introduce small "trembles" into the game, or require that strategies be best responses given rational beliefs. The first approach takes us to the "trembling hand-perfect" equilibrium, while the second takes us to the "perfect Bayesian" and "sequential" equilibrium. The results are similar whichever approach is taken.

Trembling-Hand Perfectness

Trembling-hand perfectness is an equilibrium concept introduced by Selten (1975) according to which a strategy that is to be part of an equilibrium must continue to be optimal for the player even if there is a small chance that the other player will pick an out-of-equilibrium action (i.e., that the other player's hand will "tremble").

Trembling-hand perfectness is defined for games with finite action sets as follows.

The strategy profile s^* is a **trembling-hand perfect** *equilibrium if for any ϵ there is a vector of positive numbers $\delta_1, \ldots, \delta_n \in [0,1]$ and a vector of completely mixed strategies $\sigma_1, \ldots \sigma_n$ such that the perturbed game where every strategy is replaced by $(1-\delta_i)s_i + \delta_i \sigma_i$ has a Nash equilibrium in which every strategy is within distance ϵ of s^*.*

Every trembling-hand perfect equilibrium is subgame perfect; indeed, Section 4.1 justified subgame perfectness using a tremble argument. Unfortunately, it is often hard to tell whether a strategy profile is trembling-hand perfect, and the concept is undefined for games with continuous strategy spaces because it is hard to work with mixtures of a continuum (see note N3.1). Moreover, the equilibrium depends on which trembles are chosen, and deciding why one tremble should be more common than another may be difficult.

Perfect Bayesian Equilibrium and Sequential Equilibrium

The second approach to asymmetric information, introduced by Kreps & Wilson (1982b) in the spirit of Harsanyi (1967), is to start with prior beliefs, common to all players, that specify the probabilities with which Nature chooses the types of the players at the beginning of the game. Some of the players observe Nature's move and update their beliefs, while other players can update their beliefs only by deductions they make from observing the actions of the informed players.

The deductions used to update beliefs are based on the actions specified by the equilibrium. When players update their beliefs, they assume that the other players are following the equilibrium strategies, but since the strategies themselves depend on the beliefs, an equilibrium can no longer be defined based on strategies alone. Under asymmetric information, an equilibrium is a strategy profile and a set of beliefs such that the strategies are best responses. The profile of beliefs and strategies is called an **assessment** by Kreps and Wilson.

On the equilibrium path, all that the players need to update their beliefs are their priors and Bayes' s Rule, but off the equilibrium path this is not enough. Suppose that in equilibrium, the entrant always enters. If for whatever reason the impossible happens and the entrant stays out, what is the incumbent to think about the probability that the entrant is weak? Bayes' s Rule does not help, because when $Prob(data) = 0$, which is the case for data such as *Stay Out* which is never observed in equilibrium, the posterior belief cannot be calculated using Bayes' s Rule. From section 2.4,

$$Prob(Weak|Stay\ Out) = \frac{Prob(Stay\ Out|Weak)Prob(Weak)}{Prob(Stay\ Out)}. \tag{1}$$

The posterior $Prob(Weak|Stay\ Out)$ is undefined, because (6.1) requires dividing by zero.

A natural way to define equilibrium is as a strategy profile consisting of best responses given that equilibrium beliefs follow Bayes' s Rule and out-of-equilibrium beliefs follow a specified pattern that does not contradict Bayes' s Rule.

A **perfect Bayesian equilibrium** *is a strategy profile s and a set of beliefs μ such that at each node of the game:*

(1) The strategies for the remainder of the game are Nash given the beliefs and strategies of the other players.
(2) The beliefs at each information set are rational given the evidence appearing thus far in the game (meaning that they are based, if possible, on priors updated by Bayes' s Rule, given the observed actions of the other players under the hypothesis that they are in equilibrium).

Kreps & Wilson (1982b) use this idea to form their equilibrium concept of sequential equilibrium, but they impose a third condition, defined only for games with discrete strategies, to restrict beliefs a little further:

(3) The beliefs are the limit of a sequence of rational beliefs, i.e., if (μ^*, s^*) is the equilibrium assessment, then some sequence of rational beliefs and completely mixed strategies converges to it:

$$(\mu^*, s^*) = Lim_{n\to\infty}(\mu^n, s^n)\ for\ some\ sequence\ (\mu^n, s^n)\ in\ \{\mu, s\}.$$

Condition (3) is quite reasonable and makes sequential equilibrium close to trembling-hand perfect equilibrium, but it adds more to the concept's difficulty than to its usefulness. If players are using the sequence of completely mixed strategies s^n, then every action is taken with some positive probability, so Bayes'Rule can be applied to form the beliefs μ^n after any action is observed. Condition (3) says that the equilibrium assessment has to be the limit of some such sequence (though not of every such sequence). For the rest of the book we will use perfect Bayesian equilibrium and dispense with condition (3), although it usually can be satisfied.

Sequential equilibria are always subgame perfect (condition (1) takes care of that). Every trembling-hand perfect equilibrium is a sequential equilibrium, and "almost every" sequential equilibrium is trembling hand perfect. Every sequential equilibrium is perfect Bayesian, but not every perfect Bayesian equilibrium is sequential.

Back to Entry Deterrence II

Armed with the concept of the perfect Bayesian equilibrium, we can find a sensible equilibrium for Entry Deterrence II .

> Entrant: $Enter|Weak$, $Enter|Strong$
> Incumbent: *Collude*
> Beliefs: *Prob*(*Strong*| *Stay Out*) = 0.4

In this equilibrium the entrant enters whether he is $Weak$ or *Strong*. The incumbent's strategy is $Collude$, which is not conditioned on Nature's move, since he does not observe it. Because the entrant enters regardless of Nature's move, an out-of-equilibrium belief for the incumbent if he should observe *Stay Out* must be specified, and this belief is arbitrarily chosen to be that the incumbent's subjective probability that the entrant is *Strong* is 0.4 given his observation that the entrant deviated by choosing *Stay Out*. Given this strategy profile and out-of-equilibrium belief, neither player has incentive to change his strategy.

There is no perfect Bayesian equilibrium in which the entrant chooses *Stay Out*. $Fight$ is a bad response even under the most optimistic possible belief, that the entrant is $Weak$ with probability 1. Notice that perfect Bayesian equilibrium is not defined structurally, like subgame perfectness, but rather in terms of optimal responses. This enables it to come closer to the economic intuition which we wish to capture by an equilibrium refinement.

Finding the perfect Bayesian equilibrium of a game, like finding the Nash equilibrium, requires intelligence. Algorithms are not useful. To find a Nash equilibrium, the modeller thinks about his game, picks a plausible strategy profile, and tests whether the strategies are best responses to each other. To make it a perfect Bayesian equilibrium, he notes which actions are never taken in equilibrium and specifies the beliefs that players use to interpret those actions. He then tests whether each player's strategies are best responses given his beliefs at each node, checking in particular whether any player would like to take an out-of-equilibrium action in order to set in motion the other players' out-of-equilibrium beliefs and strategies. This process does not involve testing whether a player's beliefs are beneficial to the player, because players do not choose their own beliefs; the priors and out-of-equilibrium beliefs are exogenously specified by the modeller.

One might wonder why the beliefs have to be specified in Entry Deterrence II. Does not the game tree specify the probability that the entrant is $Weak$? What difference does it make if the entrant stays out? Admittedly, Nature does choose each type with probability 0.5, so if the incumbent had no other information than this prior, that would be his belief. But the entrant's action might convey additional information. The concept of perfect Bayesian equilibrium leaves the modeller free to specify how the players form beliefs from that additional information, so long as the beliefs do not violate Bayes' Rule. (A technically valid choice of beliefs by the modeller might still be met with scorn, though, as with any silly assumption.) Here, the equilibrium says that if the entrant stays out, the incumbent believes he is *Strong* with probability 0.4 and $Weak$ with probability 0.6, beliefs that are arbitrary but do not contradict Bayes' s Rule.

In Entry Deterrence II the out-of-equilibrium beliefs do not and should not matter.

If the entrant chooses *Stay Out*, the game ends, so the incumbent's beliefs are irrelevant. Perfect Bayesian equilibrium was only introduced as a way out of a technical problem. In the next section, however, the precise out-of- equilibrium beliefs will be crucial to which strategy profiles are equilibria.

6.2 Refining Perfect Bayesian Equilibrium: the PhD Admissions Game

Entry Deterrence III: Fighting is Sometimes Profitable

In Entry Deterrence III, assume that $X = 60$, not $X = 1$. This means that fighting is more profitable for the incumbent than collusion if the entrant is $Weak$. As before, the entrant knows if he is $Weak$, but the incumbent does not. Retaining the prior after observing out- of-equilibrium actions, which in this game is $Prob(Strong) = 0.5$, is a convenient way to form beliefs that is called **passive conjectures.** The following is a perfect Bayesian equilibrium which uses passive conjectures.

> **A plausible pooling equilibrium for Entry Deterrence III**
> Entrant: $Enter|Weak$, $Enter|Strong$
> Incumbent:] *Collude*
> Beliefs: $Prob(Strong|\ Stay\ Out) = 0.5$

In choosing whether to enter, the entrant must predict the incumbent's behavior. If the probability that the entrant is $Weak$ is 0.5, the expected payoff to the incumbent from choosing *Fight* is 30 ($= 0.5[0] + 0.5[60]$), which is less than the payoff of 50 from $Collude$. The incumbent will collude, so the entrant enters. The entrant may know that the incumbent's payoff is actually 60, but that is irrelevant to the incumbent's behavior.

The out-of-equilibrium belief does not matter to this first equilibrium, although it will in other equilibria of the same game. Although beliefs in a perfect Bayesian equilibrium must follow Bayes' s Rule, that puts very little restriction on how players interpret out-of-equilibrium behavior. Out-of-equilibrium behavior is "impossible," so when it does occur there is no obvious way the player should react. Some beliefs may seem more reasonable than others, however, and Entry Deterrence III has another equilibrium that requires less plausible beliefs off the equilibrium path.

> **An implausible equilibrium for Entry Deterrence III**
> Entrant: $Stay\ Out|Weak$, $Stay\ Out|Strong$
> Incumbent: *Fight*
> Beliefs: $Prob(Strong|Enter) = 0.1$

This is an equilibrium because if the entrant were to deviate and enter, the incumbent would calculate his payoff from fighting to be 54 ($= 0.1[0] + 0.9[60]$), which is greater than the *Collude* payoff of 50. The entrant would therefore stay out.

The beliefs in the implausible equilibrium are different and less reasonable than in the plausible equilibrium. Why should the incumbent believe that weak entrants would enter mistakenly nine times as often as strong entrants? The beliefs do not violate Bayes' s Rule, but they have no justification.

The reasonableness of the beliefs is important because if the incumbent uses passive conjectures, the implausible equilibrium breaks down. With passive conjectures, the incumbent would want to change his strategy to *Collude*, because the expected payoff from *Fight* would be less than 50. The implausible equilibrium is less robust with respect to beliefs than the plausible equilibrium, and it requires beliefs that are harder to justify.

Even though dubious outcomes may be perfect Bayesian equilibria, the concept does have some bite, ruling out other dubious outcomes. There does not, for example, exist an equilibrium in which the entrant enters only if he is *Strong* and stays out if he is *Weak* (called a "separating equilibrium" because it separates out different types of players). Such an equilibrium would have to look like this:

A conjectured separating equilibrium for Entry Deterrence III
Entrant: *Stay Out*$|Weak$, *Enter*$|Strong$
Incumbent: *Collude*

No out-of-equilibrium beliefs are specified for the conjectures in the separating equilibrium because there is no out-of-equilibrium behavior about which to specify them. Since the incumbent might observe either *Stay out* or *Enter* in equilibrium, the incumbent will always use Bayes' s Rule to form his beliefs. He will believe that an entrant who stays out must be weak and an entrant who enters must be strong. This conforms to the idea behind Nash equilibrium that each player assumes that the other follows the equilibrium strategy, and then decides how to reply. Here, the incumbent's best response, given his beliefs, is *Collude*$|$*Enter*, so that is the second part of the proposed equilibrium. But this cannot be an equilibrium, because the entrant would want to deviate. Knowing that entry would be followed by collusion, even the weak entrant would enter. So there cannot be an equilibrium in which the entrant enters only when strong.

The PhD Admissions Game

Passive conjectures may not always be the most satisfactory belief, as the next example shows. Suppose that a university knows that 90 percent of the population hate economics and would be unhappy in its PhD program, and 10 percent love economics and would do well. In addition, it cannot observe the applicant's type. If the university rejects an application, its payoff is 0 and the applicant's is -1 because of the trouble needed to apply. If the university accepts the application of someone who hates economics, the payoffs of both university and student are -10, but if the applicant loves economics, the payoffs are $+20$ for each player. Figure 2 shows this game in extensive form. The population proportions are represented by a node at which Nature chooses the student to be a *Lover* or *Hater* of economics.

Figure 2 PhD Admissions

The PhD Admissions Game is a signalling game of the kind we will look at in Chapter 10. It has various perfect Bayesian equilibria that differ in their out-of-equilibrium beliefs, but the equilibria can be divided into two distinct categories, depending on the outcome: the **separating equilibrium**, in which the lovers of economics apply and the haters do not, and the **pooling equilibrium**, in which neither type of student applies.

A separating equilibrium for PhD Admissions
Student: *Apply* |*Lover*, *Do Not Apply* | *Hater*
University: *Admit*

The separating equilibrium does not need to specify out-of-equilibrium beliefs, because Bayes' s Rule can always be applied whenever both of the two possible actions *Apply* and *Do Not Apply* can occur in equilibrium.

A pooling equilibrium for PhD Admissions
Student: *Do Not Apply* |*Lover*, *Do Not Apply* |*Hater*
University: *Reject*
Beliefs: $Prob(Hater|Apply) = 0.9$ (passive conjectures)

The pooling equilibrium is supported by passive conjectures. Both types of students refrain from applying because they believe correctly that they would be rejected and receive a payoff of -1; and the university is willing to reject any student who foolishly applied, believing that he is a *Hater* with 90 percent probability.

Because the perfect Bayesian equilibrium concept imposes no restrictions on out-of-equilibrium beliefs, researchers starting with McLennan (1985) have come up with a variety of exotic refinements of the equilibrium concept. Let us consider whether various alternatives to passive conjectures would support the pooling equilibrium in PhD Admissions.

Passive Conjectures. $Prob(Hater|Apply) = 0.9$

This is the belief specified above, under which out-of-equilibrium behavior leaves beliefs unchanged from the prior. The argument for passive conjectures is that the student's application is a mistake, and that both types are equally likely to make mistakes, although *Haters* are more common in the population. This supports the pooling equilibrium.

The Intuitive Criterion. $Prob(Hater|Apply) = 0$

Under the Intuitive Criterion of Cho & Kreps (1987), if there is a type of informed player who could not benefit from the out-of-equilibrium action no matter what beliefs were held by the uninformed player, the uninformed player's belief must put zero probability on that type. Here, the *Hater* could not benefit from applying under any possible beliefs of the university, so the university puts zero probability on an applicant being a *Hater*. This argument will not support the pooling equilibrium, because if the university holds this belief, it will want to admit anyone who applies.

Complete Robustness. $Prob(Hater|Apply) = m$, $0 \leq m \leq 1$

Under this approach, the equilibrium strategy profile must consist of responses that are best, given any and all out-of-equilibrium beliefs. Our equilibrium for Entry Deterrence II satisfied this requirement. Complete robustness rules out a pooling equilibrium in PhD Admissions, because a belief like $m = 0$ makes accepting applicants a best response, in which case only the *Lover* will apply. A useful first step in analyzing conjectured pooling equilibria is to test whether they can be supported by extreme beliefs such as $m = 0$ and $m = 1$.

An ad hoc specification. $Prob(Hater|Apply) = 1$

Sometimes the modeller can justify beliefs by the circumstances of the particular game. Here, one could argue that anyone so foolish as to apply knowing that the university would reject them could not possibly have the good taste to love economics. This supports the pooling equilibrium also.

An alternative approach to the problem of out-of-equilibrium beliefs is to remove its origin by building a model in which every outcome is possible in equilibrium because different types of players take different equilibrium actions. In PhD Admissions, we could assume that there are a few students who both love economics and actually enjoy writing applications. Those students would always apply in equilibrium, so there would never be a pure pooling equilibrium in which nobody applied, and Bayes' s Rule could always be used. In equilibrium, the university would always accept someone who applied, because applying is never out-of-equilibrium behavior and it always indicates that the applicant is a *Lover*. This approach is especially attractive if the modeller takes the possibility of trembles literally, instead of just using it as a technical tool.

The arguments for different kinds of beliefs can also be applied to Entry Deterrence III, which had two different pooling equilibria and no separating equilibrium. We used passive conjectures in the "plausible" equilibrium. The intuitive criterion would not restrict beliefs at all, because both types would enter if the incumbent's beliefs were such as to make him collude, and both would stay out if they made him fight. Complete robustness would rule out as an equilibrium the strategy profile in which the entrant stays out regardless of type, because the optimality of staying out depends on the beliefs. It would support the strategy profile in which the entrant enters and out-of-equilibrium beliefs do not matter.

The Importance of Common Knowledge: Entry Deterrence IV and V

To demonstrate the importance of common knowledge, let us consider two more versions of Entry Deterrence. We will use passive conjectures in both. In Entry Deterrence III, the incumbent was hurt by his ignorance. Entry Deterrence IV will show how he can benefit from it, and Entry Deterrence V will show what can happen when the incumbent has the same information as the entrant but the information is not common knowledge.

Entry Deterrence IV: the incumbent benefits from ignorance

To construct Entry Deterrence IV, let $X = 300$ in Figure 1, so fighting is even more profitable than in Entry Deterrence III but the game is otherwise the same: the entrant knows his type, but the incumbent does not. The following is the unique perfect Bayesian equilibrium in pure strategies.[2]

Equilibrium for Entry Deterrence IV
Entrant: *Stay Out* $|Weak$, *Stay Out* $|Strong$
Incumbent: *Fight*
Beliefs: $Prob(Strong|Enter) = 0.5$ (passive conjectures)

This equilibrium can be supported by other out-of-equilibrium beliefs, but no equilibrium is possible in which the entrant enters. There is no pooling equilibrium in which both types of entrant enter, because then the incumbent's expected payoff from *Fight* would be 150 ($= 0.5[0] + 0.5[300]$), which is greater than the *Collude* payoff of 50. There is no separating equilibrium, because if only the strong entrant entered and the incumbent always colluded, the weak entrant would be tempted to imitate him and enter as well.

In Entry Deterrence IV, unlike Entry Deterrence III, the incumbent benefits from his own ignorance, because he would always fight entry, even if the payoff were (unknown to himself) just zero. The entrant would very much like to communicate the costliness of fighting, but the incumbent would not believe him, so entry never occurs.

Entry Deterrence V: Lack of Common Knowledge of Ignorance

In Entry Deterrence V, it may happen that both the entrant and the incumbent know the payoff from (*Enter, Fight*), but the entrant does not know whether the incumbent knows. The information is known to both players, but is not common knowledge.

[2] There exists a plausible mixed-strategy equilibrium too: *Entrant: Enter if Strong, Enter with probability* $m = .2$ *if Weak; Incumbent: Collude with probability* $n = .2$. The payoff from this is only 150, so if the equilibrium were one in mixed strategies, ignorance would *not* help.

Figure 3 depicts this somewhat complicated situation. The game begins with Nature assigning the entrant a type, *Strong* or *Weak* as before. This is observed by the entrant but not by the incumbent. Next, Nature moves again and either tells the incumbent the entrant's type or remains silent. This is observed by the incumbent, but not by the entrant. The four games starting at nodes G_1 to G_4 represent different profiles of payoffs from (*Enter, Fight*) and knowledge of the incumbent. The entrant does not know how well informed the incumbent is, so the entrant's information partition is $(\{G_1, G_2\}, \{G_3, G_4\})$.

Figure 3 Entry Deterrence V

Equilibrium for Entry Deterrence V
Entrant: *Stay Out|Weak*, *Stay Out|Strong*
Incumbent: *Fight|Nature said "Weak"*, *Collude |Nature said "Strong"*, *Fight |Nature said nothing*
Beliefs: *Prob(Strong|Enter, Nature said nothing)* $= 0.5$ (passive conjectures)

Since the entrant puts a high probability on the incumbent not knowing, the entrant should stay out, because the incumbent will fight for either of two reasons. With probability 0.9, Nature has said nothing and the incumbent calculates his expected payoff from *Fight* to be 150, and with probability 0.05 ($= 0.1[0.5]$) Nature has told the incumbent that the entrant is weak and the payoff from *Fight* is 300. Even if the entrant is strong and Nature tells this to the incumbent, the entrant would choose *Stay Out*, because he does not know that the incumbent knows, and his expected payoff from *Enter* would be -5 ($= [0.9][-10] + 0.1[40]$).

If it were common knowledge that the entrant was strong, the entrant would enter and the incumbent would collude. If it is known by both players, but not common knowledge, the entrant stays out, even though the incumbent would collude if he entered. Such is the importance of common knowledge.

6.4 Incomplete Information in the Repeated Prisoner's Dilemma: The Gang of Four Model

Chapter 5 explored various ways to steer between the Scylla of the Chainstore Paradox and the Charybdis of the Folk Theorem to find a resolution to the problem of repeated games. In the end, uncertainty turned out to make little difference to the problem, but incomplete information was left unexamined in chapter 5. One might imagine that if the players did not know each others' types, the resulting confusion might allow cooperation. Let us investigate this by adding incomplete information to the finitely repeated Prisoner's Dilemma and finding the perfect Bayesian equilibria.

One way to incorporate incomplete information would be to assume that a large number of players are irrational, but that a given player does not know whether any other player is of the irrational type or not. In this vein, one might assume that with high probability Row is a player who blindly follows the strategy of Tit-for-Tat. If Column thinks he is playing against a Tit-for-Tat player, his optimal strategy is to *Deny* until near the last period (how near depending on the parameters), and then *Confess*. If he were not certain of this, but the probability were high that he faced a Tit-for-Tat player, Row would choose that same strategy. Such a model begs the question, because it is not the incompleteness of the information that drives the model, but the high probability that one player blindly uses Tit-for- Tat. Tit-for-Tat is not a rational strategy, and to assume that many players use it is to assume away the problem. A more surprising result is that a small amount of incomplete information can make a big difference to the outcome.[3]

The Gang of Four Model

One of the most important explanations of reputation is that of Kreps, Milgrom, Roberts & Wilson (1982), hereafter referred to as the Gang of Four. In their model, a few players are genuinely unable to play any strategy but Tit-for-Tat, and many players pretend to be of that type. The beauty of the model is that it requires only a small amount of incomplete information, and a low probability γ that player Row is a Tit-for-Tat player. It is not unreasonable to suppose that a world which contains Neo-Ricardians and McGovernites contains a few mildly irrational tit-for-tat players, and such behavior is especially plausible among consumers, who are subject to less evolutionary pressure than firms.

It may even be misleading to call the Tit-for-Tat "irrational", because they may just have unusual payoffs, particularly since we will assume that they are rare. The unusual players have a small direct influence, but they matter because other players imitate them. Even if Column knows that with high probability Row is just pretending to be Tit-for-Tat, Column does not care what the truth is so long as Row keeps on pretending. Hypocrisy is not only the tribute vice pays to virtue; it can be just as good for deterring misbehavior.

Theorem 6.1: The Gang of Four theorem
Consider a T-stage, repeated prisoner's dilemma, without discounting but with a probability

[3] Begging the question is not as illegitimate in modelling as in rhetoric, however, because it may indicate that the question is a vacuous one in the first place. If the payoffs of the Prisoner's Dilemma are not those of most of the people one is trying to model, the Chainstore Paradox becomes irrelevant.

γ of a Tit-for-Tat player. In any perfect Bayesian equilibrium, the number of stages in which either player chooses Confess *is less than some number M that depends on γ but not on T.*

The significance of the Gang of Four theorem is that while the players do resort to $Confess$ as the last period approaches, the number of periods during which they $Confess$ is independent of the total number of periods. Suppose $M = 2,500$. If $T = 2,500$, there might be a $Confess$ every period. But if $T = 10,000$, there are 7,500 periods without a $Confess$. For reasonable probabilities of the unusual type, the number of periods of cooperation can be much larger. Wilson (unpublished) has set up an entry deterrence model in which the incumbent fights entry (the equivalent of *Deny* above) up to seven periods from the end, although the probability the entrant is of the unusual type is only 0.008.

The Gang of Four Theorem characterizes the equilibrium outcome rather than the equilibrium. Finding perfect Bayesian equilibria is difficult and tedious, since the modeller must check all the out-of-equilibrium subgames, as well as the equilibrium path. Modellers usually content themselves with describing important characteristics of the equilibrium strategies and payoffs. Section 14.3 contains a somewhat more detailed description of what happens in a model of repeated entry deterrence with incomplete information.

To get a feeling for why Theorem 6.1 is correct, consider what would happen in a 10,001 period game with a probability of 0.01 that Row is playing the Grim Strategy of *Deny* until the first $Confess$, and $Confess$ every period thereafter. If the payoffs are as in table 5.2a, a best response for Column to a known grim player is ($Confess$ only in the last period, unless Row chooses $Confess$ first, in which case respond with $Confess$). Both players will choose *Deny* until the last period, and Column's payoff will be 50,010 (= (10,000)(5) + 10) . Suppose for the moment that if Row is not grim, he is highly aggressive, and will choose $Confess$ every period. If Column follows the strategy just described, the outcome will be ($Confess$, *Deny*) in the first period and *(Confess, Confess*) thereafter, for a payoff to Column of $-5(= -5 + (10,000)(0))$. If the probabilities of the two outcomes are 0.01 and 0.99, Column's expected payoff from the strategy described is 495.15. If instead he follows a strategy of ($Confess$ every period), his expected payoff is just 0.1 $(= 0.01(10) + 0.99(0))$. It is clearly in Column's advantage to take a chance by cooperating with Row, even if Row has a 0.99 probability of following a very aggressive strategy.

The aggressive strategy, however, is not Row's best response to Column's strategy. A better response is for Row to choose *Deny* until the second-to-last period, and then to choose $Confess$. Given that Column is cooperating in the early periods, Row will cooperate also. This argument has not described what the Nash equilibrium actually is, since the iteration back and forth between Row and Column can be continued, but it does show why Column chooses *Deny* in the first period, which is the leverage the argument needs: the payoff is so great if Row is actually the grim player that it is worthwhile for Column to risk a low payoff for one period.

The Gang of Four Theorem provides a way out of the Chainstore Paradox, but it creates a problem of multiple equilibria in much the same way as the infinitely repeated game. For one thing, if the asymmetry is two-sided, so both players might be unusual types, it becomes much less clear what happens in threat games such as Entry Deterrence. Also, what happens depends on which unusual behaviors have positive, if small, probability. Theorem 6.2 says that the modeller can make the average payoffs take any particular values by making the game last long enough and choosing the form of the irrationality carefully.

Theorem 6.2: The Incomplete Information Folk Theorem(Fudenberg & Maskin [1986] p. 547)
For any two-person repeated game without discounting, the modeller can choose a form of irrationality so that for any probability $\epsilon > 0$ there is some finite number of repetitions such that with probability $(1 - \epsilon)$ a player is rational and the average payoffs in some sequential equilibrium are closer than ϵ to any desired payoffs greater than the minimax payoffs.

6.5 The Axelrod Tournament

Another way to approach the repeated Prisoner's Dilemma is through experiments, such as the round robin tournament described by political scientist Robert Axelrod in his 1984 book. Contestants submitted strategies for a 200-repetition Prisoner's Dilemma . Since the strategies could not be updated during play, players could precommit, but the strategies could be as complicated as they wished. If a player wanted to specify a strategy which simulated subgame perfectness by adapting to past history just as a noncommitted player would, he was free to do so, but he could also submit a non-perfect strategy such as Tit-for-Tat or the slightly more forgiving Tit-for-Two-Tats. Strategies were submitted in the form of computer programs that were matched with each other and played automatically. In Axelrod's first tournament, 14 programs were submitted as entries. Every program played every other program, and the winner was the one with the greatest sum of payoffs over all the plays. The winner was Anatol Rapoport, whose strategy was Tit-for-Tat.

The tournament helps to show which strategies are robust against a variety of other strategies in a game with given parameters. It is quite different from trying to find a Nash equilibrium, because it is not common knowledge what the equilibrium is in such a tournament. The situation could be viewed as a game of incomplete information in which Nature chooses the number and cognitive abilities of the players and their priors regarding each other.

After the results of the first tournament were announced, Axelrod ran a second tournament, adding a probability $\theta = 0.00346$ that the game would end each round so as to avoid the Chainstore Paradox. The winner among the 62 entrants was again Anatol Rapoport, and again he used Tit-for-Tat.

Before choosing his tournament strategy, Rapoport had written an entire book on the Prisoner's Dilemma in analysis, experiment, and simulation (Rapoport & Chammah [1965]). Why did he choose such a simple strategy as Tit-for-Tat? Axelrod points out that Tit-for-Tat has three strong points.

1. It never initiates confessing (**niceness**);
2. It retaliates instantly against confessing (**provokability**);
3. It forgives a confesser who goes back to cooperating (it is **forgiving**).

Despite these advantages, care must be taken in interpreting the results of the tournament. It does not follow that Tit-for-Tat is the best strategy, or that cooperative behavior should always be expected in repeated games.

First, Tit-for-Tat never beats any other strategy in a one-on-one contest. It won the tournament by piling up points through cooperation, having lots of high score plays and very few low score plays. In an elimination tournament, Tit-for-Tat would be eliminated very early, because it scores *high* payoffs but never the *highest* payoff.

Second, the other players' strategies matter to the success of Tit-for-Tat. In neither tournament were the strategies submitted a Nash equilibrium. If a player knew what strategies he was facing, he would want to revise his own. Some of the strategies submitted in the second tournament would have won the first, but they did poorly because the environment had changed. Other programs, designed to try to probe the strategies of their opposition, wasted too many $(Confess, Confess)$ episodes on the learning process, but if the games had lasted a thousand repetitions they would have done better.

Third, in a game in which players occasionally confessed because of trembles, two Tit-for-Tat players facing each other would do very badly. The strategy instantly punishes a confessing player, and it has no provision for ending the punishment phase.

Optimality depends on the environment. When information is complete and the payoffs are all common knowledge, confessing is the only equilibrium outcome, but in practically any imaginable situation, information is slightly incomplete, so cooperation becomes more plausible. Tit-for-Tat is suboptimal for any given environment, but it is robust across environments, and that is its advantage.

6.6 Credit and the Age of the Firm: the Diamond Model

An example of another way to look at reputation is Diamond's model of credit terms, which seeks to explain why older firms get cheaper credit using a game similar to the Gang of Four model. Telser (1966) suggested that predatory pricing would be a credible threat if the incumbent had access to cheaper credit than the entrant, and so could hold out for more periods of losses before going bankrupt. While one might wonder whether this is effective protection against entry— what if the entrant is a large old firm from another industry?— we shall focus on how better-established firms might get cheaper credit.

D. Diamond (1989) aims to explain why old firms are less likely than young firms to default on debt. His model has both adverse selection, because firms differ in type, and moral hazard, because they take hidden actions. The three types of firms, R, S, and RS, are "born" at time zero and borrow to finance projects at the start of each of T periods. We must imagine that there are overlapping generations of firms, so that at any point in time a variety of ages are coexisting, but the model looks at the lifecycle of only one generation. All the players are risk neutral. Type RS firms can choose independently risky projects with negative expected values or safe projects with low but positive expected values. Although the risky projects are worse in expectation, if they are successful the return is much higher than from safe projects. Type R firms can only choose risky projects, and type S firms only safe projects. At the end of each period the projects bring in their profits and loans are repaid, after which new loans and projects are chosen for the next period. Lenders cannot tell which project is chosen or what a firm's current profits are, but they can seize the firm's assets if a loan is not repaid, which always happens if the risky project was chosen and turned out unsuccessfully.

This game foreshadows two other models of credit that will be described in this book, the Repossession Game of section 8.4 and the Stiglitz-Weiss model of section 9.6. Both will be one-shot games in which the bank worried about not being repaid; in the Repossession Game because the borrower did not exert enough effort, and in the Stiglitz-Weiss model because he was of an undesirable type that could not repay. The Diamond model is a mixture of adverse selection and moral hazard: the borrowers differ in type, but some borrowers have a choice of action.

The equilibrium path has three parts. The RS firms start by choosing risky projects. Their downside risk is limited by bankruptcy, but if the project is successful the firm keeps large residual profits after repaying the loan. Over time, the number of firms with access to the risky project (the RS's and R's) diminishes through bankruptcy, while the number of S's remains unchanged. Lenders can therefore maintain zero profits while lowering their interest rates. When the interest rate falls, the value of a stream of safe investment profits minus interest payments rises relative to the expected value of the few periods of risky returns minus interest payments before bankruptcy. After the interest rate has fallen enough, the second phase of the game begins when the RS firms switch to safe projects at a period we will call t_1. Only the tiny and diminishing group of type R firms continue to choose risky projects. Since the lenders know that the RS firms switch, the interest rate can fall sharply at t_1. A firm that is older is less likely to be a type R, so it is charged a lower interest rate. Figure 4 shows the path of the interest rate over time.

Figure 4 The interest rate over time

Towards period T, the value of future profits from safe projects declines and even with a low interest rate the RS's are again tempted to choose risky projects. They do not all switch at once, however, unlike in period t_1. In period t_1, if a few RS's had decided to switch to safe projects, the lenders would have been willing to lower the interest rate, which would have made switching even more attractive. If a few firms switch to risky projects at some time t_2, on the other hand, the interest rate rises and switching to risky projects becomes more attractive— a result that will also be seen in the Lemons model in Chapter 9. Between t_2 and t_3, the RS's follow a mixed strategy, an increasing number of them choosing risky projects as time passes. The increasing proportion of risky projects causes the interest rate to rise. At t_3, the interest rate is high enough and the end of the game is close enough that the RS's revert to the pure strategy of choosing risky projects. The interest rate declines during this last phase as the number of RS's diminishes because of failed risky projects.

One might ask, in the spirit of modelling by example, why the model contains three types of firms rather than two. Types S and RS are clearly needed, but why type R? The little extra detail in the game description allows simplification of the equilibrium, because with three types bankruptcy is never out-of-equilibrium behaviour, since the failing firm might be a type R. Bayes's Rule can therefore always be applied, elminating the problem of ruling out peculiar beliefs and absurd perfect bayesian equilibria.

This is a Gang of Four model but differs from previous examples in an important respect: the Diamond model is not stationary, and as time progresses, some firms of types R and RS go bankrupt, which changes the lenders' payoff functions. Thus, it is not, strictly speaking, a repeated game.

Notes

N6.1 Perfect Bayesian equilibrium: Entry Deterrence I and II

- Section 4.1 showed that even in games of perfect information, not every subgame perfect equilibrium is trembling-hand perfect. In games of perfect information, however, every subgame perfect equilibrium is a perfect Bayesian equilibrium, since no out-of-equilibrium beliefs need to be specified.

N6.2 Refining Perfect Bayesian equilibrium: the PhD Admissions Game

- Fudenberg & Tirole (1991b) is a careful analysis of the issues involved in defining perfect Bayesian equilibrium.

- Section 6.2 is about debatable ways of restricting beliefs such as passive conjectures or equilibrium dominance, but less controversial restrictions are sometimes useful. In a three-player game, consider what happens when Smith and Jones have incomplete information about Brown, and then Jones deviates. If it was Brown himself who had deviated, one might think that the other players might deduce something about Brown's type. But should they update their priors on Brown because Jones has deviated? Especially, should Jones updated his beliefs, just because he himself deviated? Passive conjectures seems much more reasonable.

  If, to take a second possibility, Brown himself does deviate, is it reasonable for the out-of-equilibrium beliefs to specify that Smith and Jones update their beliefs about Brown in different ways? This seems dubious in light of the Harsanyi doctrine that everyone begins with the same priors.

  On the other hand, consider a tremble interpretation of out-of- equilibrium moves. Maybe if Jones trembles and picks the wrong strategy, that really does say something about Brown's type. Jones might tremble more often, for example, if Brown's type is strong than if it is weak. Jones himself might learn from his own trembles. Once we are in the realm of non-Bayesian beliefs, it is hard to know what to do without a real-world context.

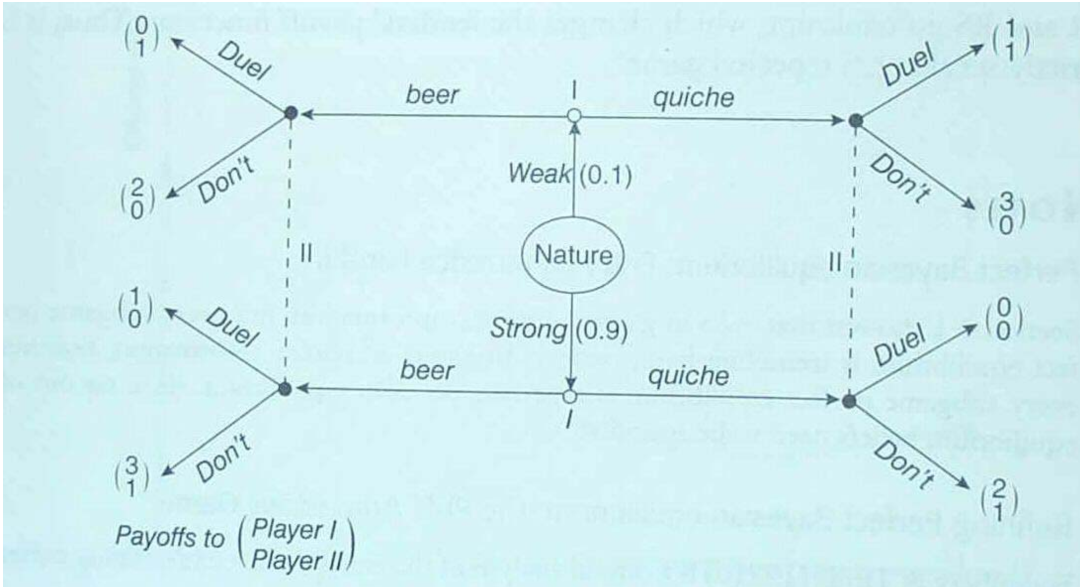
- For discussions of the appropriateness of different equilibrium concepts in actual economic models see Rubinstein (1985b) on bargaining, Shleifer & Vishny (1986) on greenmail and D. Hirshleifer & Titman (1990) on tender offers.

- **Exotic refinements.** Binmore (1990) and Kreps (1990b) are booklength treatments of rationality and equilibrium concepts.

- **The Beer-Quiche Game** of Cho & Kreps (1987). To illustrate their "intuitive criterion", Cho and Kreps use the Beer-Quiche Game. In this game, Player I might be either weak or strong in his duelling ability, but he wishes to avoid a duel even if he thinks he can win. Player II wishes to fight a duel only if player I is weak, which has a probability of 0.1. Player II does not know player I's type, but he observes what player I has for breakfast. He knows that weak players prefer quiche for breakast, while strong players prefer beer. The payoffs are shown in Figure 5.

  Figure 5 illustrates a few twists on how to draw an extensive form. It begins with Nature's choice of $Strong$ or $Weak$ in the middle of the diagram. Player I then chooses whether to breakfast on *beer* or *quiche*. Player II's nodes are connected by a dotted line if they are in the same information set. Player II chooses $Duel$ or $Don't$, and payoffs are then received.

Figure 5 The Beer-Quiche Game



This game has two perfect Bayesian equilibrium outcomes, both of which are pooling. In E_1, player I has beer for breakfast regardless of type, and Player II chooses not to duel. This is supported by the out-of-equilibrium belief that a quiche-eating player I is weak with probability over 0.5, in which case player II would choose to duel on observing quiche. In E_2, player I has quiche for breakfast regardless of type, and player II chooses not to duel. This is supported by the out-of-equilibrium belief that a beer-drinking player I is weak with probability greater than 0.5, in which case player II would choose to duel on observing beer.

Passive conjectures and the intuitive criterion both rule out equilibrium E_2. According to the reasoning of the intuitive criterion, player I could deviate without fear of a duel by giving the following convincing speech,

> I am having beer for breakfast, which ought to convince you I am strong. The only conceivable benefit to me of breakfasting on beer comes if I am strong. I would never wish to have beer for breakfast if I were weak, but if I am strong and this message is convincing, then I benefit from having beer for breakfast.

N6.5 The Axelrod tournament

- Hofstadter (1983) is a nice discussion of the Prisoner's Dilemma and the Axelrod tournament by an intelligent computer scientist who came to the subject untouched by the preconceptions or training of economics. It is useful for elementary economics classes. Axelrod's 1984 book provides a fuller treatment.

Problems

6.1. Cournot Duopoly under Incomplete Information about Costs
This problem introduces incomplete information into the Cournot model of Chapter 3 and allows for a continuum of player types.

6.1a Modify the Cournot Game of Chapter 3 by specifying that Apex's average cost of production be c per unit, while Brydox's remains zero. What are the outputs of each firm if the costs are common knowledge? What are the numerical values if $c = 10$?

6.1b Let Apex' cost c be c_{max} with probability θ and 0 with probability $1 - \theta$, so Apex is one of two types. Brydox does not know Apex's type. What are the outputs of each firm?

6.1c Let Apex's cost c be drawn from the interval $[0, c_{max}]$ using the uniform distribution, so there is a continuum of types. Brydox does not know Apex's type. What are the outputs of each firm?

6.1d Outputs were 40 for each firm in the zero-cost game in chapter 3. Check your answers in parts (b) and (c) by seeing what happens if $c_{max} = 0$.

6.1e Let $c_{max} = 20$ and $\theta = 0.5$, so the expectation of Apex's average cost is 10 in parts (a), (b), and (c). What are the average outputs for Apex in each case?

6.1f Modify the model of part (b) so that $c_{max} = 20$ and $\theta = 0.5$, but somehow $c = 30$. What outputs do your formulas from part (b) generate? Is there anything this could sensibly model?

Problem 6.2. Limit Pricing (see Milgrom and Roberts [1982a])
An incumbent firm operates in the local computer market, which is a natural monopoly in which only one firm can survive. The incumbent knows his own operating cost c, which is 20 with probability 0.2 and 30 with probability 0.8.

In the first period, the incumbent can price *Low*, losing 40 in profits, or *High*, losing nothing if his cost is $c = 20$. If his cost is $c = 30$, however, then pricing *Low* he loses 180 in profits. (You might imagine that all consumers have a reservation price that is *High*, so a static monopolist would choose that price whether marginal cost was 20 or 30.)

A potential entrant knows those probabilities, but not the incumbent's exact cost. In the second period, the entrant can enter at a cost of 70, and his operating cost of 25 is common knowledge. If there are two firms in the market, each incurs an immediate loss of 50, but one then drops out and the survivor earns the monopoly revenue of 200 and pays his operating cost. There is no discounting: $r = 0$.

6.2a In a perfect bayesian equilibrium in which the incumbent prices *High* regardless of its costs (a pooling equilibrium), about what do out-of-equilibrium beliefs have to be specified?

6.2b Find a pooling perfect bayesian equilibrium, in which the incumbent always chooses the same price no matter what his costs may be.

6.2c What is a set of out-of-equilibrium beliefs that do not support a pooling equilibrium at a *High* price?

6.2d What is a separating equilibrium for this game?

6.3. Symmetric Information and Prior Beliefs

In the Expensive-Talk Game of Table 1, the Battle of the Sexes is preceded by by a communication move in which the man chooses *Silence* or *Talk*. *Talk* costs 1 payoff unit, and consists of a declaration by the man that he is going to the prize fight. This declaration is just talk; it is not binding on him.

Table 1: Subgame payoffs in the Expensive-Talk Game

		Woman	
		Fight	*Ballet*
Man:	*Fight*	3,1	0, 0
	Ballet	0, 0	1,3

Payoffs to: (Man, Woman)

6.3a Draw the extensive form for this game, putting the man's move first in the simultaneous-move subgame.

6.3b What are the strategy sets for the game? (Start with the woman's.)

6.3c What are the three perfect pure-strategy equilibrium outcomes in terms of observed actions? (Remember: strategies are not the same thing as outcomes.)

6.3d Describe the equilibrium strategies for a perfect equilibrium in which the man chooses to talk.

6.3e The idea of "forward induction" says that an equilibrium should remain an equilibrium even if strategies dominated in that equilibrium are removed from the game and the procedure is iterated. Show that this procedure rules out SBB as an equilibrium outcome.(See Van Damme [1989]. In fact, this procedure rules out TFF *(Talk, Fight, Fight)* also.)

6.4. Lack of common knowledge

This problem looks at what happens if the parameter values in Entry Deterrence V are changed.

6.4a If the value for the belief, $Pr(Strong|Enter, Nature\ said\ nothing)$, were .05 or .95, would such beliefs support the equilibrium in section 6.3?

6.4b Why is the equilibrium in section 6.3 not an equilibrium if 0.7 is the probability that Nature tells the incumbent?

6.4c Describe the equilibrium if 0.7 is the probability that Nature tells the incumbent. For what out-of-equilibrium beliefs does this remain the equilibrium?

September 4, 1999. February 3, 2000. February 6, 2000. November 29, 2003. 25 March 2005. Eric Rasmusen, Erasmuse@indiana.edu. Http://www.rasmusen.org. Footnotes starting with xxx are the author's notes to himself. Comments welcomed.

Part II Asymmetric Information

7 Moral Hazard: Hidden Actions

7.1 Categories of Asymmetric Information Models

It used to be that the economist's generic answer to someone who brought up peculiar behavior which seemed to contradict basic theory was "It must be some kind of price discrimination." Today, we have a new answer: "It must be some kind of asymmetric information." In a game of asymmetric information, player Smith knows something that player Jones does not. This covers a broad range of models (including price discrimination nowadays), so perhaps it is not surprising that so many situations come under its rubric. We will divide games of asymmetric information into five categories, to be studied in four chapters.

1 **Moral hazard with hidden actions** (Chapters 7 and 8).
Smith and Jones begin with symmetric information and agree to a contract, but then Smith takes an action unobserved by Jones. Information is complete.

2 **Adverse selection** (Chapter 9).
Nature begins the game by choosing Smith's type (his payoff and strategies), unobserved by Jones. Smith and Jones then agree to a contract. Information is incomplete.

3 **Mechanism Design in Adverse Selection and in Moral Hazard with Hidden Information**) (Chapter 10).
Jones is designing a contract for Smith designed to elicit Smith's private information. This may happen under adverse selection– in which case Smith knows the information prior to contracting– or moral hazard with hidden information–in which case Smith will learn it after contracting.

4,5 **Signalling** and **Screening** (Chapter 11).
Nature begins the game by choosing Smith's type, unobserved by Jones. To demonstrate his type, Smith takes actions that Jones can observe. If Smith takes the action before they agree to a contract, he is signalling; if he takes it afterwards, he is being screened. Information is incomplete.

Signalling and screening are special cases of adverse selection, which is itself a situation of hidden knowledge. Information is complete in either kind of moral hazard, and incomplete in adverse selection, signalling, and screening.

Note that some people may say that information *becomes* incomplete in a model of moral hazard with hidden knowledge, even though it is complete at the start of the game. That statement runs contrary to the definition of complete information in Chapter 2, however. The most important distinctions to keep in mind are whether or not the players agree to a contract before or after information becomes asymmetric and whether their own actions are common knowledge.

We will make heavy use of the principal-agent model to analyze asymmetric information. Usually this term is applied to moral hazard models, since the problems studied in the law of agency usually involve an employee who disobeys orders by choosing the wrong actions, but the paradigm will be useful in all of these contexts. The two players are the principal and the agent, who are usually representative individuals. The principal hires an agent to perform a task, and the agent acquires an informational advantage about his type, his actions, or the outside world at some point in the game. It is usually assumed that the players can make a binding **contract** at some point in the game, which is to say that the principal can commit to paying the agent an agreed sum if he observes a certain outcome. In the implicit background of such models are courts which will punish any player who breaks a contract in a way that can be proven with public information.

The **principal** *(or* **uninformed player***) is the player who has the coarser information partition.*

The **agent** *(or* **informed player***) is the player who has the finer information partition.*

Figure 1: Categories of Asymmetric Information Models

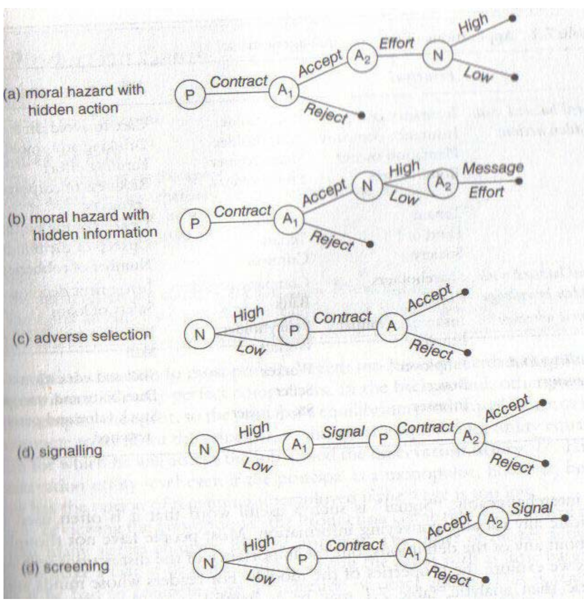


Figure 1 shows the game trees for five principal-agent models. In each model, the principal (P) offers the agent (A) a contract, which he accepts or rejects. In some, Nature (N) makes a move or the agent chooses an effort level, message, or signal. The moral hazard models are games of complete information with uncertainty. The principal offers a contract, and after the agent accepts, Nature adds noise to the task being performed. In moral hazard with hidden actions, (a) in Figure 1, the agent moves before Nature and in moral hazard with hidden knowledge, (b) in Figure 1, the agent moves after Nature and conveys a "message" to the principal about Nature's move.

Adverse selection models have incomplete information, so Nature moves first and picks the type of the agent, generally on the basis of his ability to perform the task. In the simplest model, Figure 1(c), the agent simply accepts or rejects the contract. If the agent can send a "signal" to the principal, as in Figures 1(d) and 1(e), the model is signalling if he sends the signal before the principal offers a contract, and is screening otherwise. A "signal" is different from a "message" because it is not a costless statement, but a costly action. Some adverse selection models contain uncertainty and some do not.

A problem we will consider in detail is that of an employer (the principal) hiring a worker (the agent). If the employer knows the worker's ability but not his effort level, the problem is moral hazard with hidden actions. If neither player knows the worker's ability at first, but the worker discovers it once he starts working, the problem is moral hazard with hidden knowledge. If the worker knows his ability from the start, but the employer does not, the problem is adverse selection. If, in addition to the worker knowing his ability from the start, he can acquire credentials before he makes a contract with the employer, the problem is signalling. If the worker acquires his credentials in response to a wage offer made by the employer, the problem is screening.

The five categories are only gradually rising from the swirl of the literature on agency models, and the definitions are not well established. In particular, some would argue that what I have called moral hazard with hidden knowledge and screening are essentially the same as adverse selection. Myerson (1991, p. 263), for example, suggests calling the problem of players taking the wrong action "moral hazard" and the problem of misreporting information "adverse selection." Many economists do not realize that screening and signalling are different and use the terms interchangeably. "Signal" is such a useful word that it is often used simply to indicate any variable conveying information. Most people have not thought very hard about any of the definitions, but the importance of the distinctions will become clear as we explore the properties of the models. For readers whose minds are more synthetic than analytic, Table 1 may be as helpful as anything in clarifying the categories.

Table 1: Applications of the Principal-Agent Model

	Principal	**Agent**	**Effort or type and signal**
Moral hazard with hidden actions	Insurance company	Policyholder	Care to avoid theft
	Insurance company	Policyholder	Drinking and smoking
	Plantation owner	Sharecropper	Farming effort
	Bondholders	Stockholders	Riskiness of corporate projects
	Tenant	Landlord	Upkeep of the building
	Landlord	Tenant	Upkeep of the building
	Society	Criminal	Number of robberies
Moral hazard with hidden knowledge	Shareholders	Company president	Investment decision
	FDIC	Bank	Safety of loans
Adverse selection	Insurance company	Policyholder	Infection with HIV virus
	Employer	Worker	Skill
Signalling and screening	Employer	Worker	Skill and education
	Buyer	Seller	Durability and warranty
	Investor	Stock issuer	Stock value and percentage retained

Section 7.2 discusses the roles of uncertainty and asymmetric information in a principal-

agent model of moral hazard with hidden actions, called the Production Game, and Section 7.3 shows how various constraints are satisfied in equilibrium. Section 7.4 collects several unusual contracts produced under moral hazard and discusses the properties of optimal contracts using the example of the Broadway Game.

7.2 A Principal-Agent Model: The Production Game

In the archetypal principal-agent model, the principal is a manager and the agent a worker. In this section we will devise a series of these games, the last of which will be the standard principal-agent model.

Denote the monetary value of output by $q(e)$, which is increasing in effort, e. The agent's utility function $U(e, w)$ is decreasing in effort and increasing in the wage, w, while the principal's utility $V(q - w)$ is increasing in the difference between output and the wage.

The Production Game

Players
The principal and the agent.

The order of play
1 The principal offers the agent a wage w.
2 The agent decides whether to accept or reject the contract.
3 If the agent accepts, he exerts effort e.
4 Output equals $q(e)$, where $q' > 0$.

Payoffs
If the agent rejects the contract, then $\pi_{agent} = \bar{U}$ and $\pi_{principal} = 0$.
If the agent accepts the contract, then $\pi_{agent} = U(e, w)$ and $\pi_{principal} = V(q - w)$.

An assumption common to most principal-agent models is that either the principal or the agent is one of many perfect competitors. In the background, other principals compete to employ the agent, so the principal's equilibrium profit equals zero; or many agents compete to work for the principal, so the agent's equilibrium utility equals the minimum for which he will accept the job, called the **reservation utility**, $\bar{U}$. There is some reservation utility level even if the principal is a monopolist, however, because the agent has the option of remaining unemployed if the wage is too low.

One way of viewing the assumption in the Production Game that the principal moves first is that many agents compete for one principal. The order of moves allows the principal to make a take-it-or-leave-it offer, leaving the agent with as little bargaining room as if he had to compete with a multitude of other agents. This is really just a modelling convenience, however, since the agent's reservation utility, $\bar{U}$, can be set at the level a principal would have to pay the agent in competition with other principals. This level of $\bar{U}$ can even be calculated, since it is the level at which the principal's payoff from profit maximization using the optimal contract is driven down to the principal's reservation utility by competition with other principals. Here the principal's reservation utility is zero, but that too can be chosen to fit the situation being modelled. As in the game of Nuisance Suits in Section 4.3, the main concern in choosing who makes the offer is to avoid getting caught up in a bargaining subgame.

Refinements of the equilibrium concept will not be important in this chapter. Nash equilibrium will be sufficient, because information is complete and the concerns of perfect Bayesian equilibrium will not arise. Subgame perfectness will be required, since otherwise the agent might commit to reject any contract that does not give him all of the gains from trade, but it will not drive the important results.

We will go through a series of five versions of the Production Game in this chapter.

Production Game I: Full Information. In the first version of the game, every move is common knowledge and the contract is a function $w(e)$.

Finding the equilibrium involves finding the best possible contract from the point of view of the principal, given that he must make the contract acceptable to the agent and that he foresees how the agent will react to the contract's incentives. The principal must decide what he wants the agent to do and what incentive to give him to do it.

The agent must be paid some amount $\tilde{w}(e)$ to exert effort e, where $\tilde{w}(e)$ is defined to be the w that solves the participation constraint

$$U(e, w(e)) = \overline{U}. \tag{1}$$

Thus, the principal's problem is

$$\underset{e}{Maximize} \quad V(q(e) - \tilde{w}(e)) \tag{2}$$

The first-order condition for this problem is

$$V'(q(e) - \tilde{w}(e)) \left(\frac{\partial q}{\partial e} - \frac{\partial \tilde{w}}{\partial e} \right) = 0, \tag{3}$$

which implies that

$$\frac{\partial q}{\partial e} = \frac{\partial \tilde{w}}{\partial e}. \tag{4}$$

From the implicit function theorem (see section 13.4) and the participation constraint,

$$\frac{\partial \tilde{w}}{\partial e} = -\left(\frac{\frac{\partial U}{\partial e}}{\frac{\partial U}{\partial \tilde{w}}} \right). \tag{5}$$

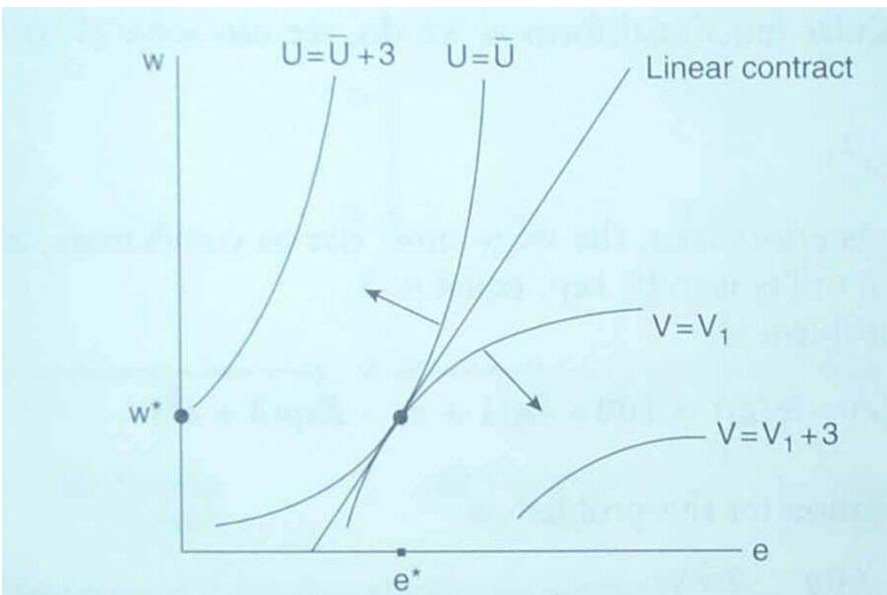
Combining equations (4) and (5) yields

$$\frac{\partial U}{\partial \tilde{w}} \frac{\partial q}{\partial e} = -\frac{\partial U}{\partial e}. \tag{6}$$

Equation (6) says that at the optimal effort level, e^*, the marginal utility to the agent which would result if he kept all the marginal output from extra effort equals the marginal disutility to him of that effort.

Figure 2 shows this graphically. The agent has indifference curves in effort-wage space that slope upwards, since if his effort rises his wage must increase also to keep his utility the same. The principal's indifference curves also slope upwards, because although he does not care about effort directly, he does care about output, which rises with effort. The principal might be either risk averse or risk neutral; his indifference curve is concave rather than linear in either case because Figure 2 shows a technology with diminishing returns to effort. If effort starts out being higher, extra effort yields less additional output so the wage cannot rise as much without reducing profits.

Figure 2: The Efficient Effort Level in Production Game I



Under perfect competition among the principals the profits are zero, so the reservation utility, $\overline{U}$, will be at the level such that at the profit-maximizing effort e^*, $\tilde{w}(e^*) = q(e^*)$, or

$$U(e^*, q(e^*)) = \overline{U}. \tag{7}$$

The principal selects the point on the $U = \overline{U}$ indifference curve that maximizes his profits, at effort e^* and wage w^*. He must then design a contract that will induce the agent to choose this effort level. The following three contracts are equally effective under full information.

1 The **forcing contract** sets $w(e^*) = w^*$ and $w(e \neq e^*) = 0$. This is certainly a strong incentive for the agent to choose exactly $e = e^*$.

2 The **threshold contract** sets $w(e \geq e^*) = w^*$ and $w(e < e^*) = 0$. This can be viewed as a flat wage for low effort levels, equal to 0 in this contract, plus a bonus if effort reaches e^*. Since the agent dislikes effort, the agent will choose exactly $e = e^*$.

3 The **linear contract** shown in Figure 2 sets $w(e) = \alpha + \beta e$, where α and β are chosen so that $w^* = \alpha + \beta e^*$ and the contract line is tangent to the indifference curve $U = \bar{U}$ at e^*. The most northwesterly of the agent's indifference curves that touch this contract line touches it at e^*.

Let's now fit out Production Game I with specific functional forms. Suppose the agent exerts effort $e \in [0, \infty]$, and output equals $q(e) = 100 * log(1 + e)$. If the agent rejects the contract, let $\pi_{agent} = \bar{U} = 3$ and $\pi_{principal} = 0$, whereas if the agent accepts the contract, let $\pi_{agent} = U(e, w) = log(w) - e^2$ and $\pi_{principal} = q(e) - w(e)$.

The agent must be paid some amount $\tilde{w}(e)$ to exert effort e, where $\tilde{w}(e)$ is defined to solve the participation constraint,

$$U(e, w(e)) = \overline{U}, \quad so \;\; log(\tilde{w}(e)) - e^2 = 3. \tag{8}$$

Knowing the particular functional form as we do, we can solve (8) for the wage function:

$$\tilde{w}(e) = Exp(3 + e^2). \tag{9}$$

This makes sense. As effort rises, the wage must rise to compensate, and rise more than exponentially if utility is to be kept equal to 3.

The principal's problem is

$$\underset{e}{Maximize} \quad V(q(e) - \tilde{w}(e)) = 100 * log(1 + e) - Exp(3 + e^2) \tag{10}$$

The first order condition for this problem is

$$V'(q(e) - \tilde{w}(e)) \left(\frac{\partial q}{\partial e} - \frac{\partial \tilde{w}}{\partial e} \right) = 0, \tag{11}$$

or, for our problem, since the firm is risk-neutral and $V' = 1$,

$$\frac{100}{1 + e} - 2e(Exp(3 + e^2)) = 0, \tag{12}$$

We can simplify the first order condition a little to get

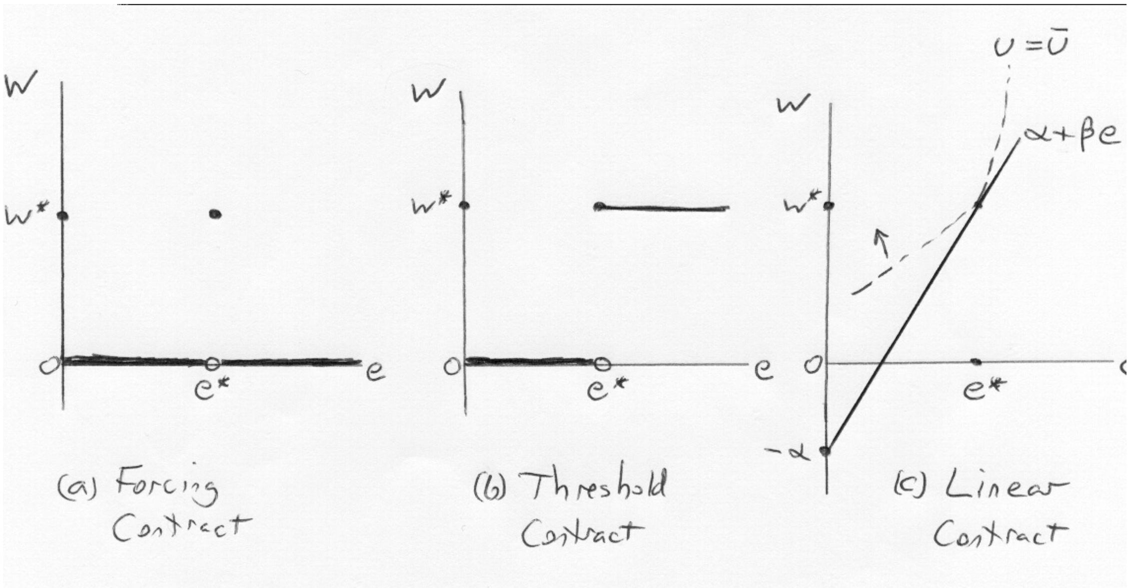
$$(2e + 2e^2)Exp(3 + e^2) = 100, \tag{13}$$

but this cannot be solved analytically. Using the computer program Mathematica, I found that $e^* \approx .77$, from which, using the formulas above, we get $q^* \approx 100 * log(1 + .77) \approx 57.26$ and $w^* \approx 36.50$.

Here, the implicit function theorem was not needed, because specifying the functional forms allowed us to find the solution using algebra instead.

Note that if $\overline{U}$ were high enough, the principal's payoff would be zero. If the market for agents were competitive, this is what would happen, since the agent's reservation payoff would be from working for another principal.

Figure 3: Three contracts that induce effort e^* for wage w^*



To implement the contract, a number of types of contracts could be used, as shown in Figure 3.

1 The **forcing contract** sets $w(e^*) = w^*$ and $w(e \neq .77) = 0$. Here, $w(.77) = 37$ (rounding up) and $w(e \neq e^*) = 0$.

2 The **threshold contract** sets $w(e \geq e^*) = w^*$ and $w(e < e^*) = 0$. Here, $w(e \geq .77) = 37$ and $w(e < .77) = 0$.

3 The **linear contract** sets $w(e) = \alpha + \beta e$, where α and β are chosen so that $w^* = \alpha + \beta e^*$ and the contract line is tangent to the indifference curve $U = \bar{U}$ at e^*. The slope of that indifference curve is the derivative of the $\tilde{w}(e)$ function, which is

$$\frac{\partial \tilde{w}(e)}{\partial e} = 2e * Exp(3 + e^2). \tag{14}$$

At $e^* = .77$, this takes the value 56. That is the β for the linear contract. The α must solve $w(e^*) = 37 = \alpha + 56(.77)$, so $\alpha = -7$.

You ought to be a little concerned as to whether the linear contract satisfies the incentive compatibility constraint. We constructed it so that it satisfied the participation constraint, because if the agent chooses $e = 0.77$, his utility will be 3. But might he prefer to choose some larger or smaller e and get even more utility? No, because his utility is concave. That makes the indifference curve convex, so its slope is always increasing and no preferable indifference curve touches the equilibrium contract line.

Before going on to versions of the game with asymmetric information, it will be useful to look at one other version of the game with full information, in which the agent, not the principal, proposes the contract. This will be called Production Game II.

Production Game II: Full Information. Agent Moves First.

In this version, every move is common knowledge and the contract is a function $w(e)$. The order of play, however, is now as follows

The order of play
1 The agent offers the principal a contract $w(e)$.
2 The principal decides whether to accept or reject the contract.
3 If the principal accepts, the agent exerts effort e.
4 Output equals $q(e)$, where $q' > 0$.

In this game, the agent has all the bargaining power, not the principal. The participation constraint is now that the principal must earn zero profits, so $q(e) - w(e) \geq 0$. The agent will maximize his own payoff by driving the principal to exactly zero profits, so $w(e) = q(e)$. Substituting $q(e)$ for $w(e)$ to account for the participation constraint, the maximization problem for the agent in proposing an effort level e at a wage $w(e)$ can therefore be written as

$$\underset{e}{Maximize} \quad U(e, q(e)) \tag{15}$$

The first-order condition is

$$\frac{\partial U}{\partial e} + \left(\frac{\partial U}{\partial q}\right)\left(\frac{\partial q}{\partial e}\right) = 0. \tag{16}$$

Since $\frac{\partial U}{\partial q} = \frac{\partial U}{\partial w}$ when the wages equals output, equation (16) implies that

$$\frac{\partial U}{\partial w}\frac{\partial q}{\partial e} = -\frac{\partial U}{\partial e}. \tag{17}$$

Comparing this with equation (6), the equation when the principal had the bargaining power, it is clear that e^* is identical in Production Games I and II. It does not matter who has the bargaining power; the efficient effort level stays the same.

Figure 2 (a few pages back) can be used to illustrate this game. Suppose that $V_1 = 0$. The agent must choose a point on the $V = 0$ indifference curve that maximizes his own utility, and then provide himself with contract incentives to choose that point. The agent's payoff is highest at effort e^* given that he must make $V = 0$, and all three contracts described in Production Game I provide him with the correct incentives.

The efficient-effort level is independent of which side has the bargaining power because the gains from efficient production are independent of how those gains are distributed so long as each party has no incentive to abandon the relationship. This as the same lesson as that of the Coase theorem, which says that under general conditions the activities undertaken will be efficient and independent of the distribution of property rights (Coase [1960]). This property of the efficient-effort level means that the modeller is free to make the assumptions on bargaining power that help to focus attention on the information problems he is studying.

Production Game III: A Flat Wage Under Certainty

In this version of the game, the principal can condition the wage neither on effort nor on output. This is modelled as a principal who observes neither effort nor output, so information is asymmetric.

It is easy to imagine a principal who cannot observe effort, but it seems very strange that he cannot observe output, especially since he can deduce the output from the value of his payoff. It is not ridiculous that he cannot base wages on output, however, because a contract must be enforceable by some third party such as a court. Law professors complain about economists who speak of "unenforceable contracts." In law school, a contract is defined as an enforceable agreement, and most of a contracts class is devoted to discovering which agreements are contracts. A simple promise to give someone money without any obligation on his part, for example, is not something a court will enforce, nor is it possible in practice for a court to enforce a contract in which someone agrees to pay a barber \$50 "if the haircut is especially good" but \$10 otherwise. A court can only enforce contingencies it can observe. In the extreme, Production Game III is appropriate. Output is not **contractible** (the court will not enforce a contract) or **verifiable** (the court cannot observe output), which usually leads to the same outcome as when output is unobservable to the two parties to the agreement.

The outcome of Production Game III is simple and inefficient. If the wage is non-negative, the agent accepts the job and exerts zero effort, so the principal offers a wage of zero.

If there is nothing on which to condition the wage, the agency problem cannot be solved by designing the contract carefully. If it is to be solved at all, it will be by some other means such as reputation or repetition of the game, the solutions of Chapter 5. Typically, however, there is some contractible variable such as output upon which the principal can condition the wage. Such is the case in Production Game IV.

Production Game IV: an output-based wage under certainty

In this version, the principal cannot observe effort but can observe output and specify the contract to be $w(q)$.

Now the principal picks not a number w but a function $w(q)$. His problem is not quite so straightforward as in Production Game I, where he picked the function $w(e)$, but here, too, it is possible to achieve the efficient effort level e^* despite the unobservability of effort. The principal starts by finding the optimal effort level e^*, as in Production Game I. That effort yields the efficient output level $q^* = q(e^*)$. To give the agent the proper incentives, the contract must reward him when output is q^*. Again, a variety of contracts could be used. The forcing contract, for example, would be any wage function such that $U(e^*, w(q^*)) = \bar{U}$ and $U(e, w(q)) < \bar{U}$ for $e \neq e^*$.

Production Game IV shows that the unobservability of effort is not a problem in itself, if the contract can be conditioned on something which is observable and perfectly correlated with effort. The true agency problem occurs when that perfect correlation breaks down, as in Production Game V.

Production Game V: Output-Based Wage under Uncertainty.

In this version, the principal cannot observe effort but can observe output and specify the contract to be $w(q)$. Output, however, is a function $q(e, \theta)$ both of effort and the state of the world $\theta \in \mathbf{R}$, which is chosen by Nature according to the probability density $f(\theta)$ as the new move (5) of the game. Move (5) comes just after the agent chooses effort, so the agent cannot choose a low effort knowing that Nature will take up the slack. (If the agent can observe Nature's move before his own, the game becomes moral hazard with hidden knowledge and hidden actions).

Because of the uncertainty about the state of the world, effort does not map cleanly onto the observed output in Production Game V. A given output might have been produced by any of several different effort levels, so a forcing contract will not necessarily achieve the desired effort. Unlike the case in Production Game IV, here the principal cannot deduce that $e = e^*$ from the fact that $q = q^*$. Moreover, even if the contract does induce the agent to choose e^*, if it does so by penalizing him heavily when $q \neq q^*$ it will be expensive for the principal. The agent's expected utility must be kept equal to $\bar{U}$ because of the participation constraint, and if the agent is sometimes paid a low wage because output happens not to equal q^*, he must be paid more when output does equal q^* to make up for it. If the agent is risk averse, this variability in his wage requires that his expected wage be higher than the w^* found earlier, because he must be compensated for the extra risk. There is a tradeoff between incentives and insurance against risk.

Moral hazard becomes a problem when $q(e)$ is not a one-to-one function because a single value of e might result in any of a number of values of q, depending on the value of θ. In this case the output function is not invertible; knowing q, the principal cannot deduce the value of e perfectly without assuming equilibrium behavior on the part of the agent.

The combination of unobservable effort and lack of invertibility in Production Game V means that no contract can induce the agent to put forth the efficient effort level without incurring extra costs, which usually take the form of an extra risk imposed on the agent. We will still try to find a contract that is efficient in the sense of maximizing welfare given the informational constraints. The terms "first-best" and "second-best" are used to distinguish these two kinds of optimality.

A **first-best contract** *achieves the same allocation as the contract that is optimal when the principal and the agent have the same information set and all variables are contractible.*

A **second-best contract** *is Pareto optimal given information asymmetry and constraints on writing contracts.*

The difference in welfare between the first-best world and the second-best world is the cost of the agency problem.

The first four production games were easier because the principal could find a first-best contract without searching very far. But even defining the strategy space in a game like Production Game V is tricky, because the principal may wish to choose a very complicated function $w(q)$. Finding the optimal contract when a forcing contract cannot be used becomes a difficult problem without general answers, because of the tremendous variety of possible contracts. The rest of the chapter will show how the problem may at least be approached, if not actually solved.

7.3 The Incentive Compatibility, Participation, and Competition Constraints

The principal's objective in Production Game V is to maximize his utility knowing that the agent is free to reject the contract entirely and that the contract must give the agent an incentive to choose the desired effort. These two constraints arise in every moral hazard problem, and they are named the **participation constraint** and the **incentive compatibility contraint**. Mathematically, the principal's problem is

$$\underset{w(\cdot)}{Maximize} \quad EV(q(\tilde{e},\theta) - w(q(\tilde{e},\theta))) \tag{18}$$

subject to

$$\tilde{e} = \overset{argmax}{e} \quad EU(e, w(q(e,\theta))) \quad \text{(incentive compatibility constraint) (18a)}$$

$$EU(\tilde{e}, w(q(\tilde{e},\theta))) \geq \bar{U} \quad \text{(participation constraint) (18b)}$$

The incentive-compatibility constraint takes account of the fact that the agent moves second, so the contract must induce him to voluntarily pick the desired effort. The participation constraint, also called the **reservation utility** or **individual rationality** constraint, requires that the worker prefer the contract to leisure, home production, or alternative jobs.

Expression (18) is the way an economist instinctively sets up the problem, but setting it up is often as far as he can get with the **first-order condition approach**. The difficulty is not just that the maximizer is choosing a wage function instead of a number, because control theory or the calculus of variations can solve such problems. Rather, it is that the constraints are nonconvex— they do not rule out a nice convex set of points in the space of wage functions such as the constraint "$w \geq 4$" would, but rather rule out a very complicated set of possible wage functions.

A different approach, developed by Grossman & Hart (1983) and called the **three-step procedure** by Fudenberg & Tirole (1991a), is to focus on contracts that induce the agent to pick a particular action rather than to directly attack the problem of maximizing profits. The first step is to find for each possible effort level the set of wage contracts that induce the agent to choose that effort level. The second step is to find the contract which supports that effort level at the lowest cost to the principal. The third step is to choose the effort level that maximizes profits, given the necessity to support that effort with the costly wage contract from the second step.

To support the effort level e, the wage contract $w(\cdot)$ must satisfy the incentive compatibility and participation constraints. Mathematically, the problem of finding the least cost $C(\tilde{e})$ of supporting the effort level $\tilde{e}$ combines steps one and two.

$$C(\tilde{e}) = \underset{w(\cdot)}{Minimum} \quad Ew(q(\tilde{e},\theta)) \tag{19}$$

subject to constraints (18a) and (18b).

Step three takes the principal's problem of maximizing his payoff, expression (18), and restates it as

$$\underset{\tilde{e}}{Maximize} \quad EV(q(\tilde{e},\theta) - C(\tilde{e})). \tag{20}$$

After finding which contract most cheaply induces each effort, the principal discovers the optimal effort by solving problem (20).

Breaking the problem into parts makes it easier to solve. Perhaps the most important lesson of the three-step procedure, however, is to reinforce the points that the goal of the contract is to induce the agent to choose a particular effort level and that asymmetric information increases the cost of the inducements.

7.4 Optimal Contracts: The Broadway Game

The next game, inspired by Mel Brooks's offbeat film *The Producers*, illustrates a peculiarity of optimal contracts: sometimes the agent's reward should not increase with his output. Investors advance funds to the producer of a Broadway show that might succeed or might fail. The producer has the choice of embezzling or not embezzling the funds advanced to him, with a direct gain to himself of 50 if he embezzles. If the show is a success, the revenue is 500 if he did not embezzle and 100 if he did. If the show is a failure, revenue is -100 in either case, because extra expenditure on a fundamentally flawed show is useless.

Broadway Game I

Players
Producer and investors.

The order of play
1 The investors offer a wage contract $w(q)$ as a function of revenue q.
2 The producer accepts or rejects the contract.
3 The producer chooses to *Embezzle* or *Do not embezzle*.
4 Nature picks the state of the world to be *Success* or *Failure* with equal probability. Table 2 shows the resulting revenue q.

Payoffs
The producer is risk averse and the investors are risk neutral. The producer's payoff is $U(100)$ if he rejects the contract, where $U' > 0$ and $U'' < 0$, and the investors' payoff is 0. Otherwise,

$$\pi_{producer} = \begin{cases} U(w(q)+50) & if\ he\ embezzles \\ U(w(q)) & if\ he\ is\ honest \end{cases}$$

$$\pi_{investors} = q - w(q)$$

Table 2: Profits in Broadway Game I

		State of the World	
		Failure (0.5)	*Success* (0.5)
Effort	*Embezzle*	-100	$+100$
	Do not embezzle	-100	$+500$

Another way to tabulate outputs, shown in Table 3, is to put the probabilities of outcomes in the boxes, with effort in the rows and output in the columns.

Table 3: Probabilities of Profits in Broadway Game I

		Profit			
		-100	**+100**	**+500**	**Total**
Effort	*Embezzle*	0.5	0.5	0	1
	Do not embezzle	0.5	0	0.5	1

The investors will observe q to equal either $-100, +100$, or $+500$, so the producer's contract will specify at most three different wages: $w(-100), w(+100)$, and $w(+500)$. The producer's expected payoffs from his two possible actions are

$$\pi(\textit{Do not embezzle}) = 0.5U(w(-100)) + 0.5U(w(+500)) \tag{21}$$

and

$$\pi(\textit{Embezzle}) = 0.5U(w(-100) + 50) + 0.5U(w(+100) + 50). \tag{22}$$

The incentive compatibility constraint is $\pi(\textit{Do not embezzle}) \geq \pi(\textit{Embezzle})$, so

$$0.5U(w(-100)) + 0.5U(w(+500)) \geq 0.5U(w(-100) + 50) + 0.5U(w(+100) + 50), \tag{23}$$

and the participation constraint is

$$\pi(\textit{Do not embezzle}) = 0.5U(w(-100)) + 0.5U(w(+500)) \geq U(100). \tag{24}$$

The investors want the participation constraint (24) to be satisfied at as low a dollar cost as possible. This means they want to impose as little risk on the producer as possible, since he requires a higher expected value for his wage if the risk is higher. Ideally, $w(-100) = w(+500)$, which provides full insurance. The usual agency tradeoff is between smoothing out the agent's wage and providing him with incentives. Here, no tradeoff is required, because of a special feature of the problem: there exists an outcome that could not occur unless the producer chooses the undesirable action. That outcome is $q = +100$, and it means that the following **boiling-in-oil contract** provides both riskless wages and effective incentives.

$w(+500) = 100$
$w(-100) = 100$
$w(+100) = -\infty$

Under this contract, the producer's wage is a flat 100 when he does not embezzle, so the participation constraint is satisfied. It is also binding, because it is satisfied as an equality, and the investors would have a higher payoff if the constraint were relaxed. If the producer does embezzle, he faces the payoff of $-\infty$ with probability 0.5, so the incentive compatibility constraint is satisfied. It is nonbinding, because it is satisfied as a strong inequality and the investors' equilibrium payoff does not fall if the constraint is tightened a little by making the producer's earnings from embezzlement slightly higher. Note that the cost of the contract to the investors is 100 in equilibrium, so that their overall expected payoff is $0.5(-100) + 0.5(+500) - 100 = 100$, which is greater than zero and thus gives the investors enough return to be willing to back the show.

The boiling-in-oil contract is an application of the **sufficient statistic condition**, which states that for incentive purposes, if the agent's utility function is separable in effort and money, wages should be based on whatever evidence best indicates effort, and only incidentally on output (see Holmstrom [1979] and note N7.2). In the spirit of the three-step procedure, what the principal wants is to induce the agent to choose the appropriate effort, *Do not embezzle*, and his data on what the agent chose is the output. In equilibrium (though not out of it), the datum $q = +500$ contains exactly the same information as the datum $q = -100$. Both lead to the same posterior probability that the agent chose *Do not embezzle*, so the wages conditioned on each datum should be the same. We need to insert the qualifier "in equilibrium," because to form the posterior probabilities the principal needs to have some beliefs as to the agent's behavior. Otherwise, the principal could not interpret $q = -100$ at all.

Milder contracts than this would also be effective. Two wages will be used in equilibrium, a low wage $\underline{w}$ for an output of $q = 100$ and a high wage $\overline{w}$ for any other output. The participation and incentive compatibility constraints provide two equations to solve for these two unknowns. To find the mildest possible contract, the modeller must also specify a function for $U(w)$ which, interestingly enough, was unnecessary for finding the first boiling-in- oil contract. Let us specify that

$$U(w) = 100w - 0.1w^2. \tag{25}$$

A quadratic utility function like this is only increasing if its argument is not too large, but since the wage will not exceed $w = 1000$, it is a reasonable utility function for this model. Substituting (25) into the participation constraint (24) and solving for the full-insurance high wage $\overline{w} = w(-100) = w(+500)$ yields $\overline{w} = 100$ and a reservation utility of 9000. Substituting into the incentive compatibility constraint, (23), yields

$$9000 \geq 0.5U(100 + 50) + 0.5U(\underline{w} + 50). \tag{26}$$

When (26) is solved using the quadratic equation, it yields (with rounding error), $\underline{w} \leq 5.6$. A low wage of $-\infty$ is far more severe than what is needed.

If both the producer and the investors were risk averse, risk sharing would change the part of the contract that applied in equilibrium. The optimal contract would then provide for $w(-100) < w(+500)$ to share the risk. The principal would have a lower marginal utility of wealth when output was +500, so he would be better able to pay an extra dollar of wages in that state than when output was -100.

One of the oddities of Broadway Game I is that the wage is higher for an output of −100 than for an output of +100. This illustrates the idea that the principal's aim is to reward not output, but input. If the principal pays more simply because output is higher, he is rewarding Nature, not the agent. People usually believe that higher pay for higher output is "fair," but Broadway Game I shows that this ethical view is too simple. Higher effort usually leads to higher output, so higher pay is usually a good incentive, but this is not invariably true.

The decoupling of reward and result has broad applications. Becker (1968) in criminal law and Polinsky & Che (1991) in tort law note that if society's objective is to keep the amount of enforcement costs and harmful behavior low, the penalty applied should not simply be matched to the harm. Very high penalties that are seldom inflicted will provide the proper incentives and keep enforcement costs low, even though a few unlucky offenders will receive penalties all out of proportion to the harm they have caused.

A less gaudy name for a boiling-in-oil contract is the alliterative "**shifting support scheme**," so named because the contract depends on the support of the output distribution being different when effort is optimal than when effort is other than optimal. Put more simply, the set of possible outcomes under optimal effort must be different from the set of possible outcomes under any other effort level. As a result, certain outputs show without doubt that the producer embezzled. Very heavy punishments inflicted only for those outputs achieve the first-best because a non-embezzling producer has nothing to fear.

Figure 4: Shifting Supports in an Agency Model

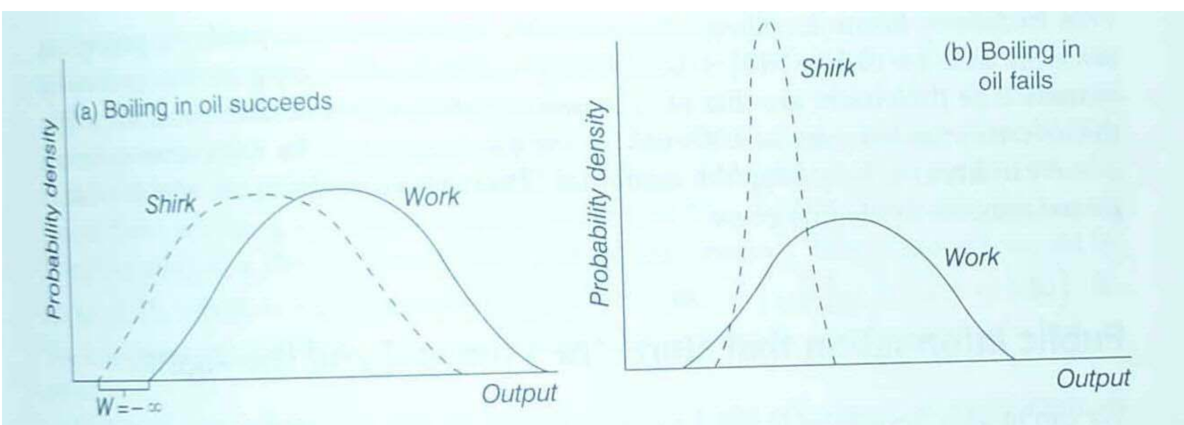


Figure 4 shows shifting supports in a model where output can take not three, but a continuum of values. If the agent shirks instead of working, certain low outputs become possible and certain high outputs become impossible. In a case like this, where the support of the output shifts when behavior changes, boiling-in-oil contracts are useful: the wage is $-\infty$ for the low outputs possible only under shirking. In Figure 4b, on the other hand, the support just shrinks under shirking, so boiling in oil is appropriate. When there is a limit to the amount the agent can be punished, or the support is the same under all actions, the threat of boiling-in-oil might not achieve the first-best contract, but similar contracts can still be used. The conditions favoring boiling-in-oil contracts are

1 The agent is not very risk averse.

2 There are outcomes with high probability under shirking that have low probability under optimal effort.

3 The agent can be severely punished.

4 It is credible that the principal will carry out the severe punishment.

Selling the Store

Another first-best contract that can sometimes be used is **selling the store.** Under this arrangement, the agent buys the entire output for a flat fee paid to the principal, becoming the **residual claimant**, since he keeps every additional dollar of output that his extra effort produces. This is equivalent to fully insuring the principal, since his payoff becomes independent of the moves of the agent and of Nature.

In Broadway Game I, selling the store takes the form of the producer paying the investors 100 ($= 0.5[-100] + 0.5[+500] - 100$) and keeping all the profits for himself. The drawbacks are that (1) the producer might not be able to afford to pay the investors the flat price of 100; and (2) the producer might be risk averse and incur a heavy utility cost in bearing the entire risk. These two drawbacks are why producers go to investors in the first place.

Public Information that Hurts the Principal and the Agent

We can modify Broadway Game I to show how having more public information available can hurt both players. This will also provide a little practice in using information sets. Let us split *Success* into two states of the world, *Minor Success* and *Major Success*, which have probabilities 0.3 and 0.2 as shown in Table 4.

Table 4: Profits in Broadway Game II

		State of the World		
		Failure (0.5)	*Minor Success* (0.3)	*Major Success* (0.2)
Effort	*Embezzle*	−100	−100	+400
	Do not embezzle	−100	+450	+575

Under the optimal contract,

$$w(-100) = w(+450) = w(+575) > w(+400) + 50. \quad (27)$$

This is so because the producer is risk averse and only the datum $q = +400$ is proof that the producer embezzled. The optimal contract must do two things: deter embezzlement and pay the producer as predictable a wage as possible. For predictability, the wage is made constant unless $q = +400$. To deter embezzlement, the producer must be punished if $q = +400$. As in Broadway Game I, the punishment would not have to be infinitely severe, and the minimum effective punishment could be calculated in the same way as in that game. The investors would pay the producer a wage of 100 in equilibrium and their expected payoff would be 100 $(= 0.5(-100) + 0.3(450) + 0.2(575) - 100)$. Thus, a contract can be found for Broadway Game II such that the agent would not embezzle.

But consider what happens when the information set is refined so that before the agent takes his action both he and the principal can tell whether the show will be a major success or not. Let us call this Broadway Game III. Under the refinement, each player's initial information partition is

$$(\{Failure, \textit{Minor Success}\}, \{\textit{Major Success}\}),$$

instead of the original coarse partition

$$(\{Failure, \textit{Minor Success}, \textit{Major Success}\}).$$

If the information sets were refined all the way to singletons, this would be very useful to the investors because they could abstain from investing in a failure and they could easily determine whether the producer embezzled or not. As it is, however, the refinement does not help the investors decide when to finance the show. If they could still hire the producer and prevent him from embezzling at a cost of 100, the payoff from investing in a major success would be 475 $(= 575 - 100)$. But the payoff from investing in a show given the information set $\{Failure, \textit{Minor Success}\}$ would be about 6.25, which is still positive $\left(\left(\frac{0.5}{0.5+0.3}\right)(-100) + \left(\frac{0.3}{0.5+0.3}\right)(450) - 100\right)$. So the improvement in information is no help with respect to the decision of when to invest.

Although the refinement has no direct effect on the efficiency of investment, it ruins the producer's incentives. If he observes $\{Failure, \textit{Minor Success}\}$, he is free to embezzle without fear of the oil-boiling output of +400. He would still refrain from embezzling if he observed $\{\textit{Major Success}\}$, but no contract that does not impose risk on a nonembezzling producer can stop him from embezzling if he observes $\{Failure, \textit{Minor Success}\}$. Whether a risky contract can be found that would prevent the producer from embezzling at a cost of less than 6.25 to the investors depends on the producer's risk aversion. If he is very risk averse, the cost of the incentive is more than 6.25, and the investors will give up investing in shows that might be minor successes. Better information reduces welfare, because it increases the producer's temptation to misbehave.

Notes

N7.1 Categories of asymmetric information models

- The separation of asymmetric information into hidden actions and hidden knowledge is suggested in Arrow (1985) and commented upon in Hart & Holmstrom (1987). The term "hidden knowledge" seems to have become more popular than "hidden information," which I used in the first edition.

- Empirical work on agency problems includes Joskow (1985, 1987) on coal mining, Masten & Crocker (1985) on natural gas contracts, Monteverde & Teece (1982) on auto components, Murphy (1986) on executive compensation, Rasmusen (1988b) on the mutual organization in banking, Staten & Umbeck (1982) on air traffic controllers and disability payments, and Wolfson (1985) on the reputation of partners in oil-drilling.

- A large literature of nonmathematical theoretical papers looks at organizational structure in the light of the agency problem. See Alchian & Demsetz (1972), Fama (1980), and Klein, Crawford, & Alchian (1978). Milgrom & Roberts (1992) have written a book on organization theory that describes what has been learned about the principal-agent problem at a technical level that MBA students can understand. There may be much to be learned from intelligent economists of the past also; note that part III, chapter 8, section 12 of Pigou's *Economics of Welfare* (1932/1920) has an interesting discussion of the advantage of piece-rate work, which can more easily induce each worker to choose the correct effort when abilities differ as well as efforts.

- For examples of agency problems, see "Many Companies Now Base Workers' Raises on Their Productivity," *Wall Street Journal*, November 15,1985, pp. 1, 15; "Big Executive Bonuses Now Come with a Catch: Lots of Criticism," *Wall Street Journal*, May 15, 1985, p. 33; "Bribery of Retail Buyers is Called Pervasive," *Wall Street Journal*, April 1, 1985, p. 6; "Some Employers Get Tough on Use of Air-Travel Prizes," *Wall Street Journal*, March 22,1985, p. 27.

- We have lots of "prinsipuls" in economics. I find this paradigm helpful for remembering spelling: "The principal's principal principle was to preserve his principal."

- "Principal" and "Agent" are legal terms, and agency is an important area of the law. Economists have focussed on quite different questions than lawyers. Economists focus on effort: how the principal induces the agent to do things. Lawyers focus on malfeasance and third parties: how the principal stops the agent from doing the wrong things and who bears the burden if he fails. If, for example, the manager of a tavern enters into a supply contract against the express command of the owner, who must be disappointed— the owner or the third party supplier?

- *Double-sided moral hazard*. The text described one-sided moral hazard. Moral hazard can also be double-sided, as when each player takes actions unobservable by the other that affect the payoffs of both of them. An example is tort negligence by both plaintiff and defendant: if a careless auto driver hits a careless pedestrian, and they go to law, the court must try to allocate blame, and the legislature must try to set up laws to induce the proper amount of care. Landlords and tenants also face double moral hazard, as implied in Table 1.

- A common convention in principal-agent models is to make one player male and the other female, so that "his" and "her" can be used to distinguish between them. I find this distracting, since gender is irrelevant to most models and adds one more detail for the reader to keep track of. If readers naturally thought "male" when they saw "principal," this would not be a problem— but they do not.

N7.2 A principal-agent model: the Production Game

- In Production Game III, we could make the agent's utility depend on the state of the world as well as on effort and wages. Little would change from the simpler model.

- The model in the text uses "effort" as the action taken by the agent, but effort is used to represent a variety of real-world actions. The cost of pilferage by employees is an estimated \$8 billion a year in the USA. Employers have offered rewards for detection, one even offering the option of a year of twice-weekly lottery tickets instead of a lump sum. The Chicago department store Marshall Field's, with 14,000 workers, in one year gave out 170 rewards of \$500 each, catching almost 500 dishonest employees. ("Hotlines and Hefty Rewards: Retailers Step Up Efforts to Curb Employee Theft," *Wall Street Journal*, September 17,1987, p. 37.)

  For an illustration of the variety of kinds of "low effort," see "Hermann Hospital Estate, Founded for the Poor, has Benefited the Wealthy, Investigators Allege," *Wall Street Journal*, March 13, 1985, p. 4, which describes such forms of misbehavior as pleasure trips on company funds, high salaries, contracts for redecorating awarded to girlfriends, phony checks, kicking back real estate commissions, and investing in friendly companies. Nonprofit enterprises, often lacking both principles and principals, are especially vulnerable, as are governments, for the same reason.

- The Production Game assumes that the agent dislikes effort. Is this realistic? People differ. My father tells of his experience in the navy when the sailors were kept busy by being ordered to scrape loose paint. My father found it a way to pass the time but says that other sailors would stop chipping when they were not watched, preferring to stare into space. *De gustibus non est disputandum* ("About tastes there can be no arguing"). But even if effort has positive marginal utility at low levels, it has negative marginal utility at high enough levels— including, perhaps, at the efficient level. This is as true for professors as for sailors.

- Suppose that the principal does not observe the variable θ (which might be effort), but he does observe t and x (which might be output and profits). From Holmstrom (1979) and Shavell (1979) we have, restated in my words,

  The Sufficient Statistic Condition. *If t is a sufficient statistic for θ relative to x, then the optimal contract needs to be based only on t if both principal and agent have separable utility functions.*

  The variable t *is a* **sufficient statistic** *for θ relative to x if, for all t and x,*

  $$Prob(\theta|t,x) = Prob(\theta|t). \tag{28}$$

  This implies, from Bayes' Rule, that $Prob(t,x|\theta) = Prob(x|t)Prob(t|\theta)$; that is, x depends on θ only because x depends on t and t depends on θ.

The sufficient statistic condition is closely related to the Rao-Blackwell Theorem (see Cox & Hinkley [1974] p. 258), which says that the decision rule for nonstrategic decisions ought not to be random.

Gjesdal (1982) notes that if the utility functions are not separable, the theorem does not apply and randomized contracts may be optimal. Suppose there are two actions the agent might take. The principal prefers action X, which reduces the agent's risk aversion, to action Y, which increases it. The principal could offer a randomized wage contract, so the agent would choose action X and make himself less risk averse. This randomization is not a mixed strategy. The principal is not indifferent to high and low wages; he prefers to pay a low wage, but we allow him to commit to a random wage earlier in the game.

N7.3 The Incentive Compatibility, Participation, and Competition Constraints

- Discussions of the first-order condition approach can be found in Grossman & Hart (1983) and Hart & Holmstrom (1987).
- The term, "individual rationality constraint," is perhaps more common, but "participation constraint" is more sensible. Since in modern modelling every constraint requires individuals to be rational, the name is ill-chosen.
- **Paying the agent more than his reservation wage.** If agents compete to work for principals, the participation constraint is binding whenever there are only two possible outcomes or whenever the agent's utility function is separable in effort and wages. Otherwise, it might happen that the principal picks a contract giving the agent more expected utility than is necessary to keep him from quitting. The reason is that the principal not only wants to keep the agent working, but to choose a high effort.
- If the distribution of output satisfies the **monotone likelihood ratio property** (MLRP), the optimal contract specifies higher pay for higher output. Let $f(q|e)$ be the probability density of output. The MLRP is satisfied if

$$\forall e' > e, \text{ and } q' > q, \qquad f(q'|e')f(q|e) - f(q'|e)f(q|e') > 0, \tag{29}$$

  or, in other words, if when $e' > e$, the ratio $f(q|e')/f(q|e)$ is increasing in q. Alternatively, f satisfies the MLRP if $q' > q$ implies that q' is a more favorable message than q in the sense of Milgrom (1981b). Less formally, the MLRP is satisfied if the ratio of the likelihood of a high effort to a low effort rises with observed output. The distributions in the Broadway Game of Section 7.4 violate the MLRP, but the normal, exponential, Poisson, uniform, and chi-square distributions all satisfy it. Stochastic dominance does not imply the MLRP. If effort of 0 produces outputs of 10 or 12 with equal probability, and effort of 1 produces outputs of 11 or 13 also with equal probability, the second distribution is stochastically dominant, but the MLRP is not satisfied.
- Finding general conditions that allow the modeller to characterize optimal contracts is difficult. Much of Grossman & Hart (1983) is devoted to the rather obscure Spanning Condition or Linear Distribution Function Condition (LDFC), under which the first order condition approach is valid. The survey by Hart & Holmstrom (1987) makes a valiant attempt at explaining the LDFC.

N7.4 Optimal Contracts: The Broadway Game

- Daniel Asquith suggested the idea behind Broadway Game II.

- Franchising is one compromise between selling the store and paying a flat wage. See Mathewson & Winter (1985), Rubin (1978), and Klein & Saft (1985).

- Mirrlees (1974) is an early reference on the idea of the boiling-in-oil contract.

- Broadway Game II shows that improved information could reduce welfare by increasing a player's incentive to misbehave. This is distinct from the nonstrategic insurance reason why improved information can be harmful. Suppose that Smith is insuring Jones against hail ruining Jones' wheat crop during the next year, increasing Jones' expected utility and giving a profit to Smith. If someone comes up with a way to forecast the weather before the insurance contract is agreed upon, both players will be hurt. Insurance will break down, because if it is known that hail will ruin the crop, Smith will not agree to share the loss, and if it is known there will be no hail, Jones will not pay a premium for insurance. Both players prefer not knowing the outcome in advance.

Problems

7.1: First-Best Solutions in a Principal-Agent Model

Suppose an agent has the utility function of $U = \sqrt{w} - e$, where e can assume the levels 0 or 1. Let the reservation utility level be $\overline{U} = 3$. The principal is risk neutral. Denote the agent's wage, conditioned on output, as $\underline{w}$ if output is 0 and $\overline{w}$ if output is 100. Table 5 shows the outputs.

Table 5: A Moral Hazard Game

Effort	**Probability of Output of** 0	100	Total
Low ($e = 0$)	0.3	0.7	1
High ($e = 1$)	0.1	0.9	1

a What would the agent's effort choice and utility be if he owned the firm?

b If agents are scarce and principals compete for them, what will the agent's contract be under full information? His utility?

c If principals are scarce and agents compete to work for them, what would the contract be under full information? What will the agent's utility and the principal's profit be in this situation?

d Suppose that $U = w - e$. If principals are the scarce factor and agents compete to work for principals, what would the contract be when the principal cannot observe effort? (Negative wages are allowed.) What will be the agent's utility and the principal's profit be in this situation?

7.2: The Principal-Agent Problem

Suppose the agent has a utility function of $U = \sqrt{w} - e$, where e can assume the levels 0 or 7, and a reservation utility of $\overline{U} = 4$. The principal is risk neutral. Denote the agent's wage, conditioned on output, as $\underline{w}$ if output is 0 and $\overline{w}$ if output is 1,000. Only the agent observes his effort. Principals compete for agents. Table 6 shows the output.

Table 6: Output from Low and High Effort

Effort	**Probability of output of** 0	1,000	Total
Low($e = 0$)	0.9	0.1	1
High ($e = 7$)	0.2	0.8	1

a What are the incentive compatibility, participation, and zero-profit constraints for obtaining high effort?

b What would utility be if the wage were fixed and could not depend on output or effort?

c What is the optimal contract? What is the agent's utility?

d What would the agent's utility be under full information? Under asymmetric information, what is the agency cost (the lost utility) as a percentage of the utility the agent receives?

Table 7: Entrepreneurs Selling Out

	Probability of output of				
Method	0	49	100	225	Total
Safe ($e = 0$)	0.1	0.1	0.8	0	1
Risky ($e = 2.4$)	0	0.5	0	0.5	1

a What would the agent's effort choice and utility be if he owned the firm?

b If agents are scarce and principals compete for them, what will the agent's contract be under full information? His utility?

c If principals are scarce and agents compete to work for principals, what will the contract be under full information? What will the agent's utility and the principal's profit be in this situation?

d If agents are the scarce factor, and principals compete for them, what will the contract be when the principal cannot observe effort? What will the agent's utility and the principal's profit be in this situation?

a What is the first-best level of effort, X_a?

b If the boss has the authority to block the salesman from selling to this customer, but cannot force him to sell, what value will X take?

c If the salesman has the authority over the decision on whether to sell to this customer, and can bargain for higher pay, what will his effort be?

d Rank the effort levels X_a, X_b, and X_c in the previous three sections.

7.5. Worker Effort

A worker can be *Careful* or *Careless*, efforts which generate mistakes with probabilities 0.25 and 0.75. His utility function is $U = 100 - 10/w - x$, where w is his wage and x takes the value 2 if he is careful, and 0 otherwise. Whether a mistake is made is contractible, but effort is not. Risk-neutral employers compete for the worker, and his output is worth 0 if a mistake is made and 20 otherwise. No computation is needed for any part of this problem.

a Will the worker be paid anything if he makes a mistake?

b Will the worker be paid more if he does not make a mistake?

c How would the contract be affected if employers were also risk averse?

d What would the contract look like if a third category, "slight mistake," with an output of 19, occurs with probability 0.1 after *Careless* effort and with probability zero after *Careful* effort?

7.6. The Source of Inefficiency
In the hidden actions problem facing an employer, inefficiency arises because
(a) The worker is risk averse.
(b) The worker is risk neutral.
(c) No contract can induce high effort.
(d) The type of the worker is unknown.
(e) The level of risk aversion of the worker is unknown.

7.7. Optimal Compensation
An agent's utility function is U =(log(wage) - effort). What should his compensation scheme be if different (output,effort) pairs have the probabilities in Table 8?
(a) The agent should be paid exactly his output.
(b) The same wage should be paid for outputs of 1 and 100.
(c) The agent should receive more for an output of 100 than of 1, but should receive still lower pay if output is 2.
(d) None of the above.

Table 8: Output Probabilities

		Output		
		1	2	100
Effort	*High*	0.5	0	0.5
	Low	0.1	0.8	0.1

7.8. Effort and Output, Multiple Choices
The utility function of the agents whose situation is depicted in Table 9 is $U = w + \sqrt{w} - \alpha e$, and his reservation utility is 0. Principals compete for agents, and have reservation profits of zero. Principals are risk neutral.

Table 9: Output Probabilities

		Effort	
		Low $(e = 0)$	High $(e = 5)$
Output	$y = 0$	0.9	0.5
	$y = 100$	0.1	0.5

a If $\alpha = 2$, then if the agent's action can be observed by the principal, his equilibrium utility is in the interval

(a) $[-\infty, 0.5]$
(b) $[0.5, 5]$
(c) $[5, 10]$
(d) $[10, 40]$
(e) $[40, \infty]$

b If $\alpha = 10$, then if the agent's action can be observed by the principal, his equilibrium utility is in the interval
(a) $[-\infty, 0.5]$
(b) $[0.5, 5]$
(c) $[5, 10]$
(d) $[10, 40]$
(e) $[40, \infty]$

c If $\alpha = 5$, then if the agent's action can be observed by the principal, his equilibrium effort level is
(a) Low
(b) High
(c) A mixed strategy effort, sometimes low and sometimes high

d If $\alpha = 2$, then if the agent's action cannot be observed by the principal, and he must be paid a flat wage, his wage will be in the interval
(a) $[-\infty, 2]$
(b) $[2, 5]$
(c) $[5, 8]$
(d) $[8, 12]$
(e) $[12, \infty]$

e If the agent owns the firm, and $\alpha = 2$, will his utility be higher or lower than in the case where he works for the principal and his action can be observed?
(a) Higher
(b) Lower
(c) Exactly the same.

f If the agent owns the firm, and $\alpha = 2$, his equilibrium utility is in the interval
(a) $[-\infty, 0.5]$
(b) $[0.5, 5]$
(c) $[5, 10]$
(d) $[10, 40]$
(e) $[40, \infty]$

g If the agent owns the firm, and $\alpha = 8$, his equilibrium utility is in the interval
(a) $[-\infty, 0.5]$
(b) $[0.5, 5]$
(c) $[5, 10]$
(d) $[10, 40]$
(e) $[40, \infty]$

7.9. Hiring a Lawyer

A one-man firm with concave utility function $U(X)$ hires a lawyer to sue a customer for breach of contract. The lawyer is risk-neutral and effort averse, with a convex disutility of effort. What can you say about the optimal contract? What would be the practical problem with such a contract, if it were legal?

7.10. Constraints

An agent has the utility function $U = log(w) - e$, where e can take the levels 0 and 4, and his reservation utility is $\overline{U} = 4$. His principal is risk-neutral. Denote the agent's wage conditioned on output as $\underline{w}$ if output is 0 and $\overline{w}$ if output is 10. Only the agent observes his effort. Principals compete for agents. Output is as shown in Table 10.

Table 10: Effort and Outputs

	Probability of Outputs		
Effort	0	10	Total
Low$(e = 0)$	0.9	0.1	1
High $(e = 4)$	0.2	0.8	1

What are the incentive compatibility and participation constraints for obtaining high effort?

7.11. Constraints Again

Suppose an agent has the utility function $U = log(w) - e$, where e can take the levels 1 or 3, and a reservation utility of $\overline{U}$. The principal is risk-neutral. Denote the agent's wage conditioned on output as $\underline{w}$ if output is 0 and $\overline{w}$ if output is 100. Only the agent observes his effort. Principals compete for agents, and outputs occur according to Table 11.

Table 11: Efforts and Outputs

	Probability of Outputs	
Effort	0	100
Low$(e = 1)$	0.9	0.1
High $(e = 3)$	0.5	0.5

What conditions must the optimal contract satisfy, given that the principal can only observe output, not effort? You do not need to solve out for the optimal contract– just provide the equations which would have to be true. Do not just provide inequalities– if the condition is a binding constraint, state it as an equation.

7.12. Bankruptcy Constraints

A risk-neutral principal hires an agent with utility function $U = w - e$ and reservation utility $\overline{U} = 5$. Effort is either 0 or 10. There is a bankruptcy constraint: $w \geq 0$. Output is given by Table 12.

Table 12: Bankruptcy

	Probability of Outputs		
Effort	0	400	Total
Low $(e = 0)$	0.5	0.5	1
High $(e = 10)$	0.1	0.9	1

(a) What would be the agent's effort choice and utility if he owned the firm?

(b) If agents are scarce and principals compete for them what will be the agent's contract under full information? His utility?

(c) If principals are scarce and agents compete to work for them, what will the contract be under full information? What will the agent's utility be?

(d) If principals are scarce and agents compete to work for them, what will the contract be when the principal cannot observe effort? What will the payoffs be for each player?

(e) Suppose there is no bankruptcy constraint. If principals are the scarce factor and agents compete to work for them, what will the contract be when the principal cannot observe effort? What will the payoffs be for principal and agent?

7.13. The Game Wizard

A high-tech firm is trying to develop the game Wizard 1.0. It will have revenues of $200,000 if its succeeds, and 0$ if it fails. Success depends on the programmer. If he exerts high effort, the probability of success is .8. If he exerts low effort, it is .6. The programmer requires wages of at least $50,000 if he can exert low effort, but 70$,000 if he must exert high effort. (Let's just use payoffs in thousands of dollars, so 70,000 dollars will be written as 70.)

(a) Prove that high effort is first-best efficient.

(b) Explain why high effort would be inefficient if the probability of success when effort is low were .75.

(c) Let the probability of success with low effort go back to .6 for the remainder of the problem. If you cannot monitor the programmer and cannot pay him a wage contingent on success, what should you do?

(d) Now suppose you can make the wage contingent on success. Let the wage be S if Wizard is successful, and F if it fails. S and F will have to satisfy two conditions: a participation constraint and an incentive compatibility constraint. What are they?

(e) What is a contract that will achieve the first best?

(f) What is the optimal contract if you cannot pay a programmer a negative wage?

7.14. The Supercomputer Salesman

If a salesman exerts high effort, he will sell a supercomputer this year with probability .9. If he exerts low effort, he will succeed with probability .5. The company will make a profit of 2 million dollars if the sale is made. The salesman would require a wage of \$50,000 if he had to exert low effort, but \$70,000 if he had to exert high effort, he is risk neutral, and his utility is separable in effort and money. (Let's just use payoffs in thousands of dollars, so 70,000 dollars will be written as 70, and 2 million dollars will be 2000)

(a) Prove that high effort is first-best efficient.

(b) How high would the probability of success with low effort have to be for high effort to be inefficient?

(c) If you cannot monitor the programmer and cannot pay him a wage contingent on success, what should you do?

(d) Now suppose you can make the wage contingent on success. Let the wage be S if he makes a sale and F if he does not. S and F will have to satisfy two conditions: a participation constraint and an incentive compatibility constraint. What are they?

(e) What is a contract that will achieve the first best?

(f) Now suppose the salesman is risk averse, and his utility from money is $log(w)$. Set up the participation and incentive compatibility constraints again.

(g) You do not need to solve for the optimal contract. Using the $log(w)$ utility function assumption, however, will the expected payment by the firm in the optimal contract rise, fall, or stay the same, compared with what it was in part (e) for the risk neutral salesman?

(h) You do not need to solve for the optimal contract. Using the $log(w)$ utility function assumption, however, will the gap between S and F in the optimal contract rise, fall, or stay the same, compared with what it was in part (e) for the risk neutral salesman?

September 19, 1999. February 6, 2000. November 30, 2003. 25 March 2005. Eric Rasmusen, Erasmuse@indiana.edu. Http://wwww.rasmusen.org. Footnotes starting with xxx are the author's notes to himself. Comments welcomed.

8 Further Topics in Moral Hazard

Moral hazard deserves two chapters of this book. As we will see, adverse selection will sneak in with two also, since signalling is really just an elaboration of the adverse selection model, but moral hazard is perhaps even more important. It is really just the study of incentives, one of the central concepts of economics. And so in the chapter we will be going through a hodge-podge of special situations of moral hazard in which chapter 7's paradigm of providing the right incentives for effort by satisfying a participation constraint and an incentive compatibility constraint do not apply so straightforwardly.

The chapter begins with efficiency wages– high wages provided in situations where the it is so important to provide incentive compatibility that the principal is willing to abandon a tight participation constraint. Section 8.2 will be about tournaments– situations where competition between two agents can be used to simplify the optimal contract. After an excursion into various institutions, we will go to a big problem for incentive contracts: how does the principal restrain himself from being too merciful to a wayward agent, when mercy is not only kind but profitable? Section 8.5 then abandons the algebraic paradigm altogether to pursue a diagrammatic approach to the classic problem of moral hazard in insurance, and section 8.6 concludes with another special case: the teams problem, in which the unobservable efforts of many agents produce one observable output.

8.1 Efficiency Wages

One's first thought is that the basic idea of an incentive contract is to punish the agent if he chooses the wrong action. That is not quite right. Rather, the basic idea of an incentive contract is to create a difference between the agent's expected payoff from right and wrong actions. That can be done either with the stick of punishment or the carrot of reward.

It is important to keep this in mind, because sometimes punishments are simply not available. Consider the following game.

The Lucky Executive Game

Players
A corporation and an executive.

The Order of play
1 The corporation offers the executive a contract which pays $w(q) \geq 0$ depending on profit, q.
2 The executive accepts the contract, or rejects it and receives his reservation utility of $\overline{U} = 5$
3 The executive exerts effort e of either 0 or 10.
4 Nature chooses profit according to Table 1.

Payoffs
Both players are risk neutral. The corporation's payoff is $q - w$. The executive's payoff is $w - e$ if he accepts the contract.

Table 1: Output in the Lucky Executive Game

	Probability of Outputs		
Effort	0	400	Total
Low ($e = 0$)	0.5	0.5	1
High ($e = 10$)	0.1	0.9	1

Since both players are risk neutral, you might think that the first-best can be achieved by selling the store, putting the entire risk on the agent. The participation constraint if the executive exerts high effort is

$$0.1[w(0) - 10] + 0.9[w(400) - 10] \geq 5, \tag{1}$$

so his expected wage must equal 15. The incentive compatility constraint is

$$0.5w(0) + 0.5w(400) \leq 0.1w(0) + 0.9w(400) - 10, \tag{2}$$

which can be rewritten as $w(400) - w(0) \geq 25$, so the gap between the executive's wage for high output and low output must equal at least 25.

A contract that satisfies both constraints is $\{w(0) = -345, w(400) = 55\}$. But this contract is not feasible, because the game requires $w(q) \geq 0$. This is an example of the common and realistic **bankruptcy constraint**; the principal cannot punish the agent by taking away more than the agent owns in the first place. The worst the boss can do is fire the worker. (In fact, the same problem would arise in a slavery regime that allowed the owner to kill his slaves– there, the worst the boss can do is kill the worker.) So what can be done?

What can be done is to use the carrot instead of the stick and abandon satisfying the participation constraint as an equality. All that is needed from constraint (2) is a gap between the high wage and the low wage of 25. Setting the low wage as low as is feasible, the corporation can use the contract $\{w(0) = 0, w(400) = 25\}$, and this will induce high effort. Notice, however, that the executive's expected utility will be $.1(0) + .9(25) - 10 = 12.5$, more than double his reservation utility of 5. He is very happy in this equilibrium– but the corporation is reasonably happy, too. The corporation's payoff is $337.5(= 0.1(0 - 0) +$

$0.9(400 - 25)$, compared with the $195 (= 0.5(0 - 5) + 0.5(400 - 5))$ it would get if it paid a lower expected wage. Since high enough punishments are infeasible, the corporation has to use higher rewards.

Executives, of course, will now be lining up to work for this corporation, since they can get an expected utility of 12.5 there and only 5 elsewhere. If, in fact, there was some chance of the current executive dying and his job opening up, potential successors would be willing to pass up alternative jobs in order to be in position to get this unusually attractive job. Thus, the model generates unemployment. These are the two parts of the idea of the **efficiency wage**: the employer pays a wage higher than that needed to attract workers, and workers are willing to be unemployed in order to get a chance at the efficiency-wage job.

Shapiro & Stiglitz (1984) showed in more detail how involuntary unemployment can be explained by a principal-agent model. When all workers are employed at the market wage, a worker who is caught shirking and fired can immediately find another job just as good. Firing is ineffective and effective penalties like boiling-in-oil are excluded from the strategy spaces of legal businesses. Becker & Stigler (1974) suggested that workers post performance bonds, but if workers are poor this is impractical. Without bonds or boiling-in-oil, the worker chooses low effort and receives a low wage.

To induce a worker not to shirk, the firm can offer to pay him a premium over the market-clearing wage, which he loses if he is caught shirking and fired. If one firm finds it profitable to raise the wage, however, so do all firms. One might think that after the wages equalized, the incentive not to shirk would disappear. But when a firm raises its wage, its demand for labor falls, and when all firms raise their wages, the market demand for labor falls, creating unemployment. Even if all firms pay the same wage, a worker has an incentive not to shirk, because if he were fired he would stay unemployed, and even if there is a random chance of leaving the unemployment pool, the unemployment rate rises sufficiently high that workers choose not to risk being caught shirking. The equilibrium is not first-best efficient, because even though the marginal revenue of labor equals the wage, it exceeds the marginal disutility of effort, but it is efficient in a second-best sense. By deterring shirking, the hungry workers hanging around the factory gates are performing a socially valuable function (but they mustn't be paid for it!).

The idea of paying high wages to increase the threat of dismissal is old, and can even be found in *The Wealth of Nations* (Smith [1776] p. 207). What is new in Shapiro & Stiglitz (1984) is the observation that unemployment is generated by these "efficiency wages." These firms behave paradoxically. They pay workers more than necessary to attract them, and outsiders who offer to work for less are turned away. Can this explain why "overqualified" jobseekers are unsuccessful and mediocre managers are retained? Employers are unwilling to hire someone talented, because he could find another job after being fired for shirking, and trustworthiness matters more than talent in some jobs.

This discussion should remind you of the game of Product Quality of section 5.4. There too, purchasers paid more than the reservation price in order to give the seller an incentive to behave properly, because a seller who misbehaved could be punished by termination of the relationship. The key characteristics of such models are a constraint on the amount of contractual punishment for misbehavior and a participation constraint that is not binding in equilibrium. In addition, although the Lucky Executive Game works even with just one period, many versions, including the Product Quality Game, rely on there being a repeated game (infinitely repeated, or otherwise avoiding the Chainstore Paradox). Repetition allows for a situation in which the agent could considerably increase his payoff in one period by misbehavior such as stealing or low quality, but refrains because he would lose his position and lose all the future efficiency wage payments.

8.2 Tournaments

Games in which relative performance is important are called **tournaments**. Tournaments are similar to auctions, the difference being that the actions of the losers matter directly, unlike in auctions. Like auctions, they are especially useful when the principal wants to elicit information from the agents. A principal-designed tournament is sometimes called a **yardstick competition** because the agents provide the measure for their wages.

Farrell (2001) uses a tournament to explain how "slack" might be the major source of welfare loss from monopoly, an old idea usually prompted by faulty reasoning. The usual claim is that monopolists are inefficient because , unlike competitive firms, they do not have to maximize profits to survive. This relies on the dubious assumption that firms care about survival, not profits. Farrell makes a subtler point: although the shareholders of a monopoly maximize profit, the managers maximize their own utility, and moral hazard is severe without the benchmark of other firms' performances.

Let firm Apex have two possible production techniques, $Fast$ and $Careful$. Independently for each technique, Nature chooses production cost $c = 1$ with probability θ and $c = 2$ with probability $1 - \theta$. The manager can either choose a technique at random or investigate the costs of both techniques at a utility cost to himself of α. The shareholders can observe the resulting production cost, but not whether the manager investigates. If they see the manager pick $Fast$ and a cost of $c = 2$, they do not know whether he chose it without investigating, or investigated both techniques and found they were both costly. The wage contract is based on what the shareholders can observe, so it takes the form (w_1, w_2), where w_1 is the wage if $c = 1$ and w_2 if $c = 2$. The manager's utility is log w if he does not investigate, log $w - \alpha$ if he does, and the reservation utility of log $\bar{w}$ if he quits.

If the shareholders want the manager to investigate, the contract must satisfy the self-selection constraint

$$U(\text{not investigate}) \leq U(\text{investigate}). \tag{3}$$

If the manager investigates, he still fails to find a low-cost technique with probability $(1-\theta)^2$, so (3) is equivalent to

$$\theta \log w_1 + (1-\theta)\log w_2 \leq [1-(1-\theta)^2]\log w_1 + (1-\theta)^2 \log w_2 - \alpha. \tag{4}$$

The self-selection constraint is binding, since the shareholders want to keep the manager's compensation to a minimum. Turning inequality (4) into an equality and simplifying yields

$$\theta(1-\theta)\log\frac{w_1}{w_2}=\alpha. \tag{5}$$

The participation constraint, which is also binding, is $U(\bar{w}) = U(\text{investigate})$, or

$$\log\bar{w}=[1-(1-\theta)^2]\log w_1+(1-\theta)^2\log w_2-\alpha. \tag{6}$$

Solving equations (5) and (6) together for w_1 and w_2 yields

$$\begin{aligned} w_1 &= \bar{w}e^{\alpha/\theta}. \\ w_2 &= \bar{w}e^{-\alpha/(1-\theta)}. \end{aligned} \tag{7}$$

The expected cost to the firm is

$$[1-(1-\theta)^2]\bar{w}e^{\alpha/\theta}+(1-\theta)^2\bar{w}e^{-\alpha/(1-\theta)}. \tag{8}$$

If the parameters are $\theta = 0.1$, $\alpha = 1$, and $\bar{w} = 1$, the rounded values are $w_1 = 22{,}026$ and $w_2 = 0.33$, and the expected cost is $4{,}185$. Quite possibly, the shareholders decide it is not worth making the manager investigate.

But suppose that Apex has a competitor, Brydox, in the same situation. The shareholders of Apex can threaten to boil their manager in oil if Brydox adopts a low-cost technology and Apex does not. If Brydox does the same, the two managers are in a prisoner's dilemma, both wishing not to investigate, but each investigating from fear of the other. The forcing contract for Apex specifies $w_1 = w_2$ to fully insure the manager, and boiling-in-oil if Brydox has lower costs than Apex. The contract need satisfy only the participation constraint that $\log w - \alpha = \log\bar{w}$, so $w = 2.72$ and the cost of learning to Apex is only 2.72, not $4{,}185$. Competition raises efficiency, not through the threat of firms going bankrupt but through the threat of managers being fired.

8.3 Institutions and Agency Problems (formerly section 8.6)

Ways to Alleviate Agency Problems

Usually when agents are risk averse, the first-best cannot be achieved, because some tradeoff must be made between providing the agent with incentives and keeping his compensation from varying too much between states of the world, or because it is not possible to punish him sufficiently. We have looked at a number of different ways to solve the problem, and at this point a listing might be useful. Each method is illustrated by application to the particular problem of executive compensation, which is empirically important, and interesting both because explicit incentive contracts are used and because they are not used more often (see Baker, Jensen & Murphy [1988]).

1 **Reputation** (sections 5.3, 5.4, 6.4, 6.6).
Managers are promoted on the basis of past effort or truthfulness.

2 **Risk-sharing contracts** (sections 7.2, 7.3, 7.4).
The executive receives not only a salary, but call options on the firm's stock. If he reduces the stock value, his options fall in value.

3 **Boiling in oil** (section 7.4).
If the firm would only become unable to pay dividends if the executive shirked and was unlucky, the threat of firing him when the firm skips a dividend will keep him working hard.

4 **Selling the store** (section 7.4).
The managers buy the firm in a leveraged buyout.

5 **Efficiency wages** (section 8.1).
To make him fear losing his job, the executive is paid a higher salary than his ability warrants (cf. Rasmusen [1988b] on mutual banks).

6 **Tournaments** (section 8.2).
Several vice presidents compete and the winner succeeds the president.

7 **Monitoring** (section 3.4).
The directors hire a consultant to evaluate the executive's performance.

8 **Repetition**.
Managers are paid less than their marginal products for most of their career, but are rewarded later with higher salaries or generous pensions if their career record has been good.

9 **Changing the type of the agent**
Older executives encourage the younger by praising ambition and hard work.

We have talked about all but the last two solutions. Repetition enables the contract to come closer to the first-best if the discount rate is low (Radner [1985]). Production Game V failed to attain the first-best in section 7.2 because output depended on both the agent's effort and random noise. If the game were repeated 50 times with independent drawings of the noise, the randomness would average out and the principal could form an accurate estimate of the agent's effort. This is, in a sense, begging the question, by saying that in the long run effort can be deduced after all.

Changing the agent's type by increasing the direct utility from desirable or decreasing that from undesirable behavior is a solution that has received little attention from economists, who have focussed on changing the utility by changing monetary rewards. Akerlof (1983), one of the few papers on the subject of changing type, points out that the moral education of children, not just their intellectual education, affects their productivity and success. The attitude of economics, however, has been that while virtuous agents exist, the rules of an organization need to be designed with the unvirtuous agents in mind. As the Chinese thinker Han Fei Tzu said some two thousand years ago,

> Hardly ten men of true integrity and good faith can be found today, and yet the offices of the state number in the hundreds. If they must be filled by men of integrity and good faith, then there will never be enough men to go around; and if the offices are left unfilled, then those whose business it is to govern will dwindle in numbers while disorderly men increase. Therefore the way of the enlighted ruler is to unify the laws instead of seeking for wise men, to lay down firm policies instead of longing for men of good faith. (Han Fei Tzu [1964], p. 109 from his chapter, "The Five Vermin")

The number of men of true integrity has probably not increased as fast as the size of government, so Han Fei Tzu's observation remains valid, but it should be kept in mind that honest men do exist and honesty can enter into rational models. There are tradeoffs between spending to foster honesty and spending for other purposes, and there may be tradeoffs between using the second-best contracts designed for agents indifferent about the truth and using the simpler contracts appropriate for honest agents.

Government Institutions and Agency Problems

The field of law is well suited to analysis by principal-agent models. Even in the nineteenth century, Holmes (1881, p. 31) conjectured in *The Common Law* that the reason why sailors at one time received no wages if their ship was wrecked was to discourage them from taking to the lifeboats too early instead of trying to save it. The reason why such a legal rule may have been suboptimal is not that it was unfair— presumably sailors knew the risk before they set out— but because incentive compatibility and insurance work in opposite directions. If sailors are more risk averse than ship owners, and pecuniary advantage would not add much to their effort during storms, then the owner ought to provide insurance to the sailors by guaranteeing them wages whether the voyage succeeds or not.

Another legal question is who should bear the cost of an accident: the victim (for example, a pedestrian hit by a car) or the person who caused it (the driver). The economist's answer is that it depends on who has the most severe moral hazard. If the pedestrian could have prevented the accident at the lowest cost, he should pay; otherwise, the driver. This idea of the **least-cost avoider** is extremely useful in the economic analysis of law, and is a major theme of Posner's classic treatise on law and economics (Posner [1992]). Insurance or wealth transfer may also enter as considerations. If pedestrians are more risk averse, drivers should bear the cost, and, according to some political views, if pedestrians are poorer, drivers should bear the cost. Note that this last consideration— wealth transfer— is not relevant to private contracts. If a principal earning zero profits is required to bear the cost of work accidents, for example, the agent's wage will be lower than if he bore them instead.

Criminal law is also concerned with tradeoffs between incentives and insurance. Holmes (1881, p. 40) also notes, approvingly, that Macaulay's draft of the Indian Penal Code made breach of contract for the carriage of passengers a criminal offense. The reason is that the palanquin-bearers were too poor to pay damages for abandoning their passengers in desolate regions, so the power of the State was needed to provide for heavier punishments than bankruptcy. In general, however, the legal rules actually used seem to diverge more from optimality in criminal law than civil law. If, for example, there is no chance that an innocent man can be convicted of embezzlement, boiling embezzlers in oil might be good policy, but most countries would not allow this. Taking the example a step further, if the evidence for murder is usually less convincing than for embezzling, our analysis could easily indicate that the penalty for murder should be less, but such reasoning offends the common notion that the severity of punishment should be matched with harm from the crime.

Private Institutions and Agency Problems

While agency theory can be used to explain and perhaps improve government policy, it also helps explain the development of many curious private institutions. Agency problems are an important hindrance to economic development, and may explain a number of apparently irrational practices. Popkin (1979, pp. 66, 73, 157) notes a variety of these. In Vietnam, for example, absentee landlords were more lenient than local landlords, but improved the land less, as one would expect of principals who suffer from informational disadvantages *vis-à-vis* their agents. Along the pathways in the fields, farmers would plant early-harvesting rice that the farmer's family could harvest by itself in advance of the regular crop, so that hired labor could not grab handfuls as they travelled. In thirteenth century England, beans were seldom grown, despite their nutritional advantages, because they were too easy to steal. Some villages tried to solve the problem by prohibiting anyone from entering the beanfields except during certain hours marked by the priest's ringing the church bell, so everyone could tend and watch their beans at the same official time.

In less exotic settings, moral hazard provides another reason besides tax benefits why employees take some of their wages in fringe benefits. Professors are granted some of their wages in university computer time because this induces them to do more research. Having a zero marginal cost of computer time is a way around the moral hazard of slacking on research, despite being a source of moral hazard in wasting computer time. A less typical but more imaginative example is that of the bank in Minnesota which, concerned about its image, gave each employee $100 in credit at certain clothing stores to upgrade their style of dress. By compromising between paying cash and issuing uniforms the bank could hope to raise both its profits and the utility of its employees. ("The $100 Sounds Good, but What do They Wear on the Second Day?" *Wall Street Journal*, October 16, 1987, p. 17.)

Longterm contracts are an important occasion for moral hazard, since so many variables are unforeseen, and hence noncontractible. The term **opportunism** has been used to describe the behavior of agents who take advantage of noncontractibility to increase their payoff at the expense of the principal (see Williamson [1975] and Tirole [1986]). Smith may be able to extract a greater payment from Jones than was agreed upon in their contract, because when a contract is incomplete, Smith can threaten to harm Jones in some way. This is called **hold-up potential** (Klein, Crawford, & Alchian [1978]). Hold-up potential can even make an agent introduce competing agents into the game, if competition is not so extreme as to drive rents to zero. Michael Granfield tells me that Fairchild once developed a new patent on a component of electronic fuel injection systems that it sought to sell to another firm, TRW. TRW offered a much higher price if Fairchild would license its patent to other producers, fearing the hold-up potential of buying from just one supplier. TRW could have tried writing a contract to prevent hold-up, but knew that it would be difficult to prespecify all the ways that Fairchild could cause harm, including not only slow delivery, poor service, and low quality, but also sins of omission like failing to sufficiently guard the plant from shutdown due to accidents and strikes.

It should be clear from the variety of these examples that moral hazard is a common problem. Now that the first flurry of research on the principal-agent problem has finished, researchers are beginning to use the new theory to study institutions that were formerly relegated to descriptive "soft" scholarly work.

*8.4 Renegotiation: the Repossession Game

Renegotiation comes up in two very different contexts in game theory. Chapter 4 looked at the situation where players can coordinate on Pareto-superior subgame equilibria that might be Pareto inferior for the entire game, an idea linked to the problem of selecting among multiple equilibria. This section looks at a completely different context, one in which the players have signed a binding contract, but in a subsequent subgame, both players might agree to scrap the old contract and write a new one using the old contract as a starting point in their negotiations. Here, the questions are not about equilibrium selection, but instead concern which strategies should be allowed in the game. This is an issue that frequently arises in principal-agent models, and it was first pointed out in the context of hidden knowledge by Dewatripont (1989). Here we will use a model of hidden actions to illustrate renegotiation, a model in which a bank wants to lend money to a consumer so that he can buy a car, and must worry whether the consumer will work hard enough to repay the loan.

The Repossession Game

Players
A bank and a consumer.

The Order of Play
1 The bank can do nothing or it can offer the consumer an auto loan which allows him to buy a car that costs 11, but requires him to pay back L or lose possession of the car to the bank.
2 The consumer accepts or rejects the loan.
3 The consumer chooses to $Work$, for an income of 15, or $Play$, for an income of 8. The disutility of work is 5.
4 The consumer repays the loan or defaults.
4a In one version of the game, the bank offers to settle for an amount S and leave possession of the car to the consumer.
4b The consumer accepts or rejects the settlement S.
5 If the bank has not been paid L or S, it repossesses the car.

Payoffs
If the bank does not make any loan or the consumer rejects it, both players' payoffs are zero. The value of the car is 12 to the consumer and 7 to the bank, so the bank's payoff if a loan is made is

$$\pi_{bank} = \begin{cases} L - 11 & \text{if the original loan is repaid} \\ S - 11 & \text{if a settlement is made} \\ 7 - 11 & \text{if the car is repossessed.} \end{cases}$$

If the consumer chooses $Work$ his income W is 15 and his disutility of effort D is -5. If he chooses $Play$, then $W = 8$ and $D = 0$. His payoff is

$$\pi_{consumer} = \begin{cases} W + 12 - L - D & \text{if the original loan is repaid} \\ W + 12 - S - D & \text{if a settlement is made} \\ W - D & \text{if the car is repossessed.} \end{cases}$$

We will consider two versions of the game, both of which allow commitment in the sense of legally binding agreements over transfers of money and wealth but do not allow the consumer to commit directly to $Work$. If the consumer does not repay the loan, the bank has the legal right to repossess the car, but the bank cannot have the consumer thrown into prison for breaking a promise to choose $Work$. Where the two versions of the game will differ is in whether they allow the renegotiation moves (4a) and (4b).

Repossession Game I

The first version of the game does not allow renegotiation, so moves (4a) and (4b) are dropped from the game. In equilibrium, the bank will make the loan at a rate of $L = 12$, and the consumer will choose $Work$ and repay the loan. Working back from the end of the game in accordance with sequential rationality, the consumer is willing to repay because by repaying 12 he receives a car worth 12.[1] He will choose $Work$ because he can then repay the loan and his payoff will be 10 $(= 15 + 12 - 12 - 5)$, but if he chooses $Play$ he will not be able to repay the loan and the bank will repossess the car, reducing his payoff to 8 $(= 8 - 0)$. The bank will offer a loan at $L = 12$ because the consumer will repay it and that is the maximum repayment to which the consumer will agree. The bank's equilibrium payoff is 1 $(= 12 - 11)$. This is an efficient outcome because the consumer does buy the car, which he values at more than its cost to the car dealer, although it is the bank rather than the consumer that gains the surplus, because of the bank's bargaining power over the terms of the loan.

Repossession Game II

The second version of the game does allow renegotiation, so moves (4a) and (4b) are added back into the game. Renegotiation turns out to be harmful, because it results in an equilibrium in which the bank refuses to make a loan, reducing the payoffs of bank and consumer to (0,10) instead of (1,10); the gains from trade are lost.

[1] As usual, we could change the model slightly to make the consumer strongly desire to repay the loan, by substituting a bargaining subgame that splits the gains from trade between bank and consumer rather than specifying that the bank make a take-it-or-leave- it offer. See section 4.3.

The equilibrium in Repossession Game I breaks down because the consumer would deviate by choosing *Play*. In Repossession Game I, this would result in the bank repossessing the car, and in Repossession Game II, the bank still has the right to do this, for a payoff of -4 $(= 7 - 11)$. If the bank chooses to renegotiate and offer $S = 8$, however, this settlement will be accepted by the consumer, since in exchange he gets to keep a car worth 12, and the payoffs of bank and consumer are -3 $(= 8 - 11)$ and 12 $(= 8 + 12 - 8)$. Thus, the bank will renegotiate, and the consumer will have increased his payoff from 10 to 12 by choosing *Play*. Looking ahead to this from move (1), however, the bank will see that it can do better by refusing to make the loan, resulting in the payoffs (0,10). The bank cannot even break even by raising the loan rate L. If $L = 30$, for instance, the consumer will still happily accept, knowing that when he chooses *Play* and defaults the ultimate amount he will pay will be just $S = 8$.

Renegotiation has a paradoxical effect. In the subgame starting with consumer default it increases efficiency, by allowing the players to make a Pareto improvement over an inefficient punishment. In the game as a whole, however, it reduces efficiency by preventing players from using punishments to deter inefficient actions. This is true of any situation in which punishment imposes a deadweight loss instead of being simply a transfer from the punished to the punisher. This may be why American judges are less willing than the general public to impose punishments on criminals. By the time a criminal reaches the courtroom, extra years in jail have no beneficial effect (incapacitation aside) and impose real costs on both criminal and society, and judges are unwilling to impose sentences which in each particular case are inefficient.

The renegotiation problem also comes up in principal-agent models because of risk bearing by a risk-averse agent when the principal is risk neutral. Optimal contracts impose risk on risk-averse agents to provide incentives for high effort or self selection. If at some point in the game it is common knowledge that the agent has chosen his action or report, but Nature has not yet moved, the agent bears needless risk. The principal knows the agent has already moved, so the two of them are willing to recontract to put the risk from Nature's move back on the principal. But the expected future recontracting makes a joke of the original contract and reduces the agent's incentives for effort or truthfulness.

The Repossession Game illustrates other ideas besides renegotiation. Note that it is a game of perfect information but has the feel of a game of moral hazard with hidden actions. This is because the game has an implicit bankruptcy constraint, so that the contract cannot sufficiently punish the consumer for an inefficient choice of effort. Restricting the strategy space has the same effect as restricting the information available to a player. It is another example of the distinction between observability and contractibility— the consumer's effort is observable, but it is not really contractible, because the bankruptcy constraint prevents him from being punished for his low effort.

This game also illustrates the difficulty of deciding what "bargaining power" means. This is a term that is very important to how many people think about law and public policy but which they define hazily. Chapter 11 will analyze bargaining in great detail, using the paradigm of splitting a pie. The natural way to think of bargaining power is to treat it as the ability to get a bigger share of the pie. Here, the pie to be split is the surplus of 1 from the consumer's purchase of a car at cost 11 which will yield him 12 in utility. Both versions of the Repossession Game give all the bargaining power to the bank in the sense that where there is a surplus to be split, the bank gets 100 percent of it. But this does not help the bank in Repossession Game II, because the consumer can put himself in a position where the bank ends up a loser from the transaction despite its bargaining power.

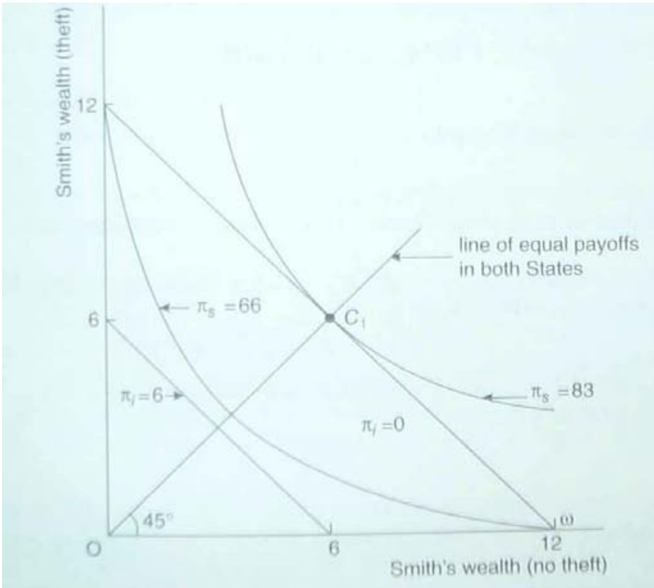
*8.5 State-space Diagrams: Insurance Games I and II (formerly section 7.5)

The principal-agent models so far in the chapter have been presented in terms of algebraic equations or outcome matrices. Another approach, especially useful when the strategy space is continuous, is to use diagrams. The term "moral hazard" comes from the insurance industry. Suppose Mr Smith (the agent) is considering buying theft insurance for a car with a value of 12. Figure 1, which illustrates his situation, is an example of a **State-space diagram**, a diagram whose axes measure the values of one variable in two different states of the world. Before Smith buys insurance, his dollar wealth is 0 if there is a theft and 12 otherwise, depicted as his endowment, $\omega = (12, 0)$. The point (12,0) indicates a wealth of 12 in one state and 0 in the other, while the point (6,6) indicates a wealth of 6 in each state.

One cannot tell the probabilities of each state just by looking at the state-space diagram. Let us specify that if Smith is careful where he parks, the state *Theft* occurs with probability 0.5, but if he is careless the probability rises to 0.75. He is risk averse, and, other things equal, he has a mildly preference to be careless, a preference worth only ϵ to him. Other things are not equal, however, and he would choose to be careful were he uninsured because of the high correlation of carlessness with carelessness.

The insurance company (the principal) is risk neutral, perhaps because it is owned by diversified shareholders. We assume that no transaction costs are incurred in providing insurance and that the market is competitive, a switch from Production Game V, where the principal collected all the gains from trade. If the insurance company can require Smith to park carefully, it offers him insurance at a premium of 6, with a payout of 12 if theft occurs, leaving him with an allocation of $C_1 = (6, 6)$. This satisfies the competition constraint because it is the most attractive contract any company can offer without making losses. Smith, whose allocation is 6 no matter what happens, is **fully insured**. In state-space diagrams, allocations like C_1 which fully insure one player are on the 45° line through the origin, the line along which his allocations in the two states are equal.

Figure 1 Insurance Game I



The game is described below in a specification that includes two insurance companies to simulate a competitive market. For Smith, who is risk averse, we must distinguish between dollar *allocations* such as (12,0) and utility *payoffs* such as $0.5U(12) + 0.5U(0)$. The curves in Figure 1 are labelled in units of utility for Smith and dollars for the insurance company.

Insurance Game I: observable care

Players
Smith and two insurance companies.

The Order of Play
1 Smith chooses to be either *Careful* or *Careless*, observed by the insurance company.
2 Insurance company 1 offers a contract (x, y), in which Smith pays premium x and receives compensation y if there is a theft.
3 Insurance company 2 also offers a contract of the form (x, y).
4 Smith picks a contract.
5 Nature chooses whether there is a theft, with probability 0.5 if Smith is $Careful$ or 0.75 if Smith is $Careless$.

Payoffs
Smith is risk averse and the insurance companies are risk neutral. The insurance company not picked by Smith has a payoff of zero.
Smith's utility function U is such that $U' > 0$ and $U'' < 0$. If Smith picks contract (x, y), the payoffs are:

If Smith chooses *Careful*,

$$\pi_{Smith} = 0.5U(12-x) + 0.5U(0+y-x)$$
$$\pi_{company} = 0.5x + 0.5(x-y), \text{ for his insurer.}$$

If Smith chooses *Careless*,

$$\pi_{Smith} = 0.25U(12-x) + 0.75U(0+y-x) + \epsilon$$
$$\pi_{company} = 0.25x + 0.75(x-y), \text{ for his insurer.}$$

In the equilibrium of Insurance Game I Smith chooses to be $Careful$ because he foresees that otherwise his insurance will be more expensive. Figure 1 is the corner of an Edgeworth box which shows the indifference curves of Smith and his insurance company given that Smith's care keeps the probability of a theft down to 0.5. The company is risk neutral, so its indifference curve, $\pi_i = 0$, is a straight line with slope $-1/1$. Its payoffs are higher on indifference curves such as $\pi_i = 6$ that are closer to the origin and thus have smaller expected payouts to Smith. The insurance company is indifferent between ω and C_1, at both of which its profits are zero. Smith is risk averse, so if he is $Careful$ his indifference curves are closest to the origin on the 45 degree line, where his wealth in the two states is equal. Picking the numbers 66 and 83 for concreteness, I have labelled his original indifference curve $\pi_s = 66$ and drawn the preferred indifference curve $\pi_s = 83$ through the equilibrium contract C_1. The equilibrium contract is C_1, which satisfies the competition constraint by generating the highest expected utility for Smith that allows nonnegative profits to the company.

Insurance Game I is a game of symmetric information. Insurance Game II changes that. Suppose that

1. The company cannot observe Smith's action; or
2. The state insurance commission does not allow contracts to require Smith to be careful; or
3. A contract requiring Smith to be careful is impossible to enforce because of the cost of proving carelessness.

In each case Smith's action is a noncontractible variable, so we model all three the same way by putting Smith's move second. The new game is like Production Game V, with uncertainty, unobservability, and two levels of output, *Theft* and *No Theft*. The insurance company may not be able to directly observe Smith's action, but his dominant strategy is to be $Careless$, so the company knows the probability of a theft is 0.75. Insurance Game II is the same as Insurance Game I except for the following.

Insurance Game II: unobservable care

The Order of Play

1. Insurance company 1 offers a contract of form (x, y), under which Smith pays premium x and receives compensation y if there is a theft.

2. Insurance company 2 offers a contract of form (x, y)

3. Smith picks a contract.

4. Smith chooses either *Careful* or *Careless*.

5. Nature chooses whether there is a theft, with probability 0.5 if Smith is $Careful$ or 0.75 if Smith is $Careless$.

Smith's dominant strategy is $Careless$, so in contrast to Insurance Game I, the insurance company must offer a contract with a premium of 9 and a payout of 12 to prevent losses, which leaves Smith with an allocation $C_2 = (3, 3)$. Making thefts more probable reduces the slopes of both players' indifference curves, because it decreases the utility of points to the southeast of the 45 degree line and increases utility to the northwest. In Figure 2, the insurance company's isoprofit curve swivels from the solid line $\pi_i = 0$ to the dotted line $\tilde{\pi}_i = 0$. It swivels around ω because that is the point at which the company's profit is independent of how probable it is that Smith's car will be stolen, since the company is not insuring him at point ω. Smith's indifference curve also swivels, from the solid curve $\pi_s = 66$ to the dotted curve $\tilde{\pi}_s = 66 + \epsilon$. It swivels around the intersection of the $\pi_s = 66$ curve with the 45 degree line, because on that line the probability of theft does not affect Smith's payoff. The ϵ difference appears because Smith gets to choose the action $Careless$, which he slightly prefers.

Figure 2: Insurance Game II with full and partial insurance

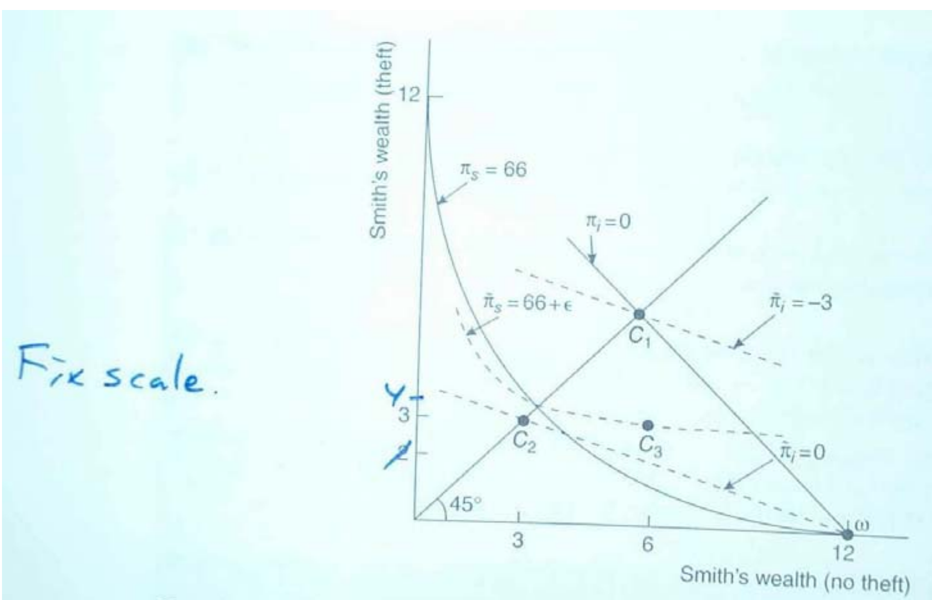


Figure 2 shows that no full insurance contract will be offered. The contract C_1 is acceptable to Smith, but not to the insurance company, because it earns negative profits, and the contract C_2 is acceptable to the insurance company, but not to Smith, who prefers ω. Smith would like to commit himself to being careful, but he cannot make his commitment credible. If the means existed to prove his honesty, he would use them even if they were costly. He might, for example, agree to buy off-street parking even though locking his car would be cheaper, if verifiable.

Although no full insurance contract such as C_1 or C_2 is mutually agreeable, other contracts can be used. Consider the partial insurance contract C_3 in Figure 2, which has a premium of 6 and a payout of 8. Smith would prefer C_3 to his endowment of $\omega = (12, 0)$ whether he chooses $Careless$ or $Careful$. We can think of C_3 in two ways:

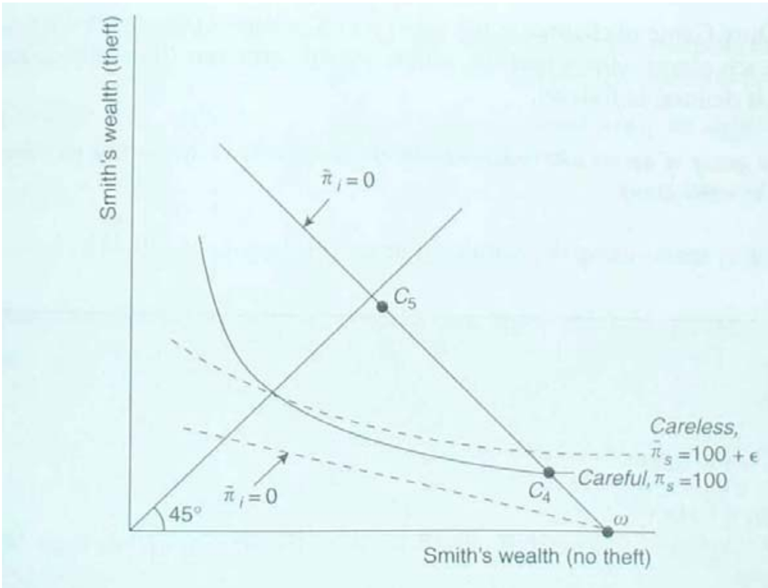
1. Full insurance except for a **deductible** of four. The insurance company pays for all losses in excess of four.

2. Insurance with a **coinsurance** rate of one-third. The insurance company pays two-thirds of all losses.

The outlook is bright, because Smith chooses *Careful* under a partial insurance contract like C_3. The moral hazard is "small" in the sense that Smith barely prefers $Careless$. With even a small deductible, Smith would choose $Careful$ and the probability of theft would fall to 0.5, allowing the company to provide much more generous insurance. The solution of full insurance is "almost" reached. In reality, we rarely observe truly full insurance, because insurance contracts repay only the price of the car and not the bother of replacing it, which is great enough to deter owners from leaving their cars unlocked.

Figure 3 illustrates effort choice under partial insurance. Smith has a choice between dashed indifference curves ($Careless$) and solid ones ($Careful$). To the southeast of the 45 degree line, the dashed indifference curve for a particular utility level is always above that utility's solid indifference curve. Offered contract C_4, Smith chooses $Careful$, remaining on the solid indifference curve, so C_4 yields zero profit to the insurance company. In fact, the competing insurance companies will offer contract C_5 in equilibrium, which is almost full insurance, but just almost, so that Smith will choose $Careful$ to avoid the small amount of risk he still bears.

Figure 3: More on Partial Insurance in Insurance Game II



Thus, as in the principal-agent model there is a tradeoff between efficient effort and efficient risk allocation. Even when the ideal of full insurance and efficient effort cannot be reached, there exists some best choice like C_5 in the set of feasible contracts, a second-best insurance contract that recognizes the constraints of informational asymmetry.

*8.6 Joint Production by Many Agents: The Holmstrom Teams Model

To conclude this chapter, let us switch our focus from the individual agent to a group of agents. We have already looked at tournaments, which involve more than one agent, but a tournament still takes place in a situation where each agent's output is distinct. The tournament is a solution to the standard problem, and the principal could always fall back on other solutions such as individual risk-sharing contracts. In this section, the existence of a group of agents results in destroying the effectiveness of the individual risk-sharing contracts, because observed output is a joint function of the unobserved effort of many agents. Even though there is a group, a tournament is impossible, because only one output is observed. The situation has much of the flavor of the Civic Duty Game of chapter 3: the actions of a group of players produce a joint output, and each player wishes that the others would carry out the costly actions. A teams model is defined as follows.

A **team** *is a group of agents who independently choose effort levels that result in a single output for the entire group.*

We will look at teams using the following game.

Teams

(Holmstrom [1982])

Players
A principal and n agents.

The order of play
1 The principal offers a contract to each agent i of the form $w_i(q)$, where q is total output.
2 The agents decide whether or not to accept the contract.
3 The agents simultaneously pick effort levels e_i, $(i = 1, \ldots, n)$.
4 Output is $q(e_1, \ldots e_n)$.

Payoffs
If any agent rejects the contract, all payoffs equal zero. Otherwise,

$$\begin{array}{ll} \pi_{principal} & = q - \sum_{i=1}^n w_i; \\ \pi_i & = w_i - v_i(e_i), \text{ where } v_i' > 0 \text{ and } v_i'' > 0. \end{array}$$

Despite the risk neutrality of the agents, "selling the store" fails to work here, because the team of agents still has the same problem as the employer had. The team's problem is cooperation between agents, and the principal is peripheral.

Denote the efficient vector of actions by e^*. An efficient contract is

$$w_i(q) = \begin{cases} b_i & \text{if } q \geq q(e^*) \\ 0 & \text{if } q < q(e^*) \end{cases} \tag{9}$$

where $\sum_{i=1}^n b_i = q(e^*)$ and $b_i > v_i(e_i^*)$.

Contract (9) gives agent i the wage b_i if all agents pick the efficient effort, and nothing if any of them shirks, in which case the principal keeps the output. The teams model gives one reason to have a principal: he is the residual claimant who keeps the forfeited output. Without him, it is questionable whether the agents would carry out the threat to discard the output if, say, output were 99 instead of the efficient 100. There is a problem of dynamic consistency. The agents would like to commit in advance to throw away output, but only because they never have to do so in equilibrium. If the modeller wishes to disallow discarding output, he imposes the **budget-balancing constraint** that the sum of the wages exactly equal the output, no more and no less. But budget balancing creates a problem for the team that is summarized in Proposition 1.

Proposition 1. *If there is a budget-balancing constraint, no differentiable wage contract $w_i(q)$ generates an efficient Nash equilibrium.*

Agent i's problem is

$$\underset{e_i}{Maximize} \quad w_i(q(e)) - v_i(e_i). \tag{10}$$

His first-order condition is

$$\left(\frac{dw_i}{dq}\right)\left(\frac{dq}{de_i}\right) - \frac{dv_i}{de_i} = 0. \tag{11}$$

With budget balancing and a linear utility function, the Pareto optimum maximizes the sum of utilities (something not generally true), so the optimum solves

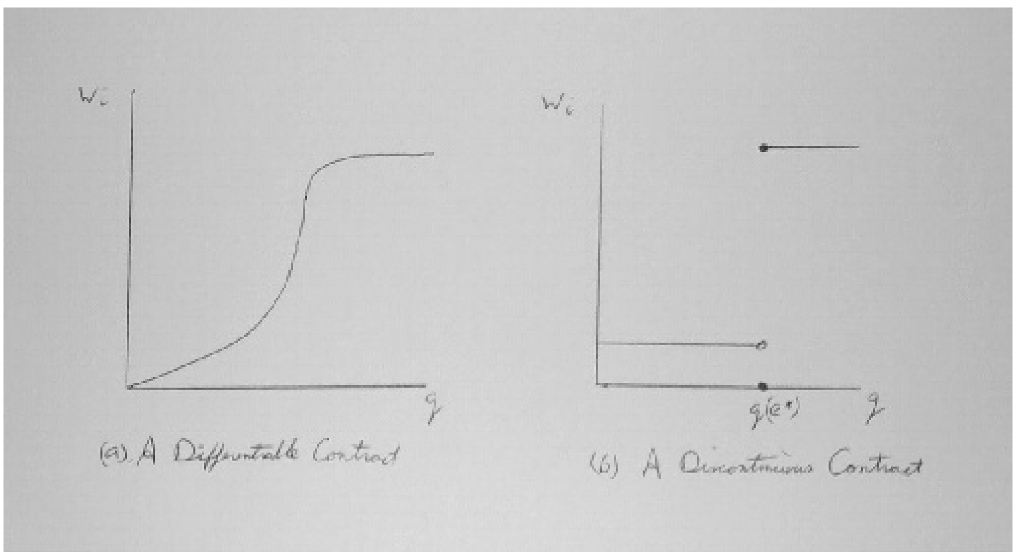
$$\underset{e_1, \ldots, e_n}{Maximize} \quad q(e) - \sum_{i=1}^n v_i(e_i) \tag{12}$$

The first-order condition is that the marginal dollar contribution to output equal the marginal disutility of effort:

$$\frac{dq}{de_i} - \frac{dv_i}{de_i} = 0. \tag{13}$$

Equation (13) contradicts (11), the agent's first- order condition, because $\frac{dw_i}{dq}$ is not equal to one. If it were, agent i would be the residual claimant and receive the entire marginal increase in output— but under budget balancing, not every agent can do that. Because each agent bears the entire burden of his marginal effort and only part of the benefit, the contract does not achieve the first-best. Without budget balancing, on the other hand, if the agent shirked a little he would gain the entire leisure benefit from shirking, but he would lose his entire wage under the optimal contract.

Figure 4 (new): xxx Contracts in the Holmstrom Teams Model



Discontinuities in Public Good Payoffs

Ordinarily, there is a free rider problem if several players each pick a level of effort which increases the level of some public good whose benefits they share. Noncooperatively, they choose effort levels lower than if they could make binding promises. Mathematically, let identical risk-neutral players indexed by i choose effort levels e_i to produce amount $q(e_1, \ldots, e_n)$ of the public good, where q is a continuous function. Player i's problem is

$$\underset{e_i}{Maximize} \quad q(e_1, \ldots, e_n) - e_i, \tag{14}$$

which has first order condition

$$\frac{\partial q}{\partial e_i} - 1 = 0, \tag{15}$$

whereas the greater, first-best effort n-vector e^* is characterized by

$$\sum_{i=1}^{n} \frac{\partial q}{\partial e_i} - 1 = 0. \tag{16}$$

If the function q were discontinuous at e^* (for example, if $q = 0$ if $e_i < e_i^*$ for any i), the strategy profile e^* could be a Nash equilibrium. In the game of Teams, the same effect is at work. Although output is not discontinuous, contract (9) is constructed as if it were (as if $q = 0$ if $e_i \neq e_i{}^*$ for any i), in order to obtain the same incentives.

The first-best can be achieved because the discontinuity at e^* makes every player the marginal, decisive player. If he shirks a little, output falls drastically and with certainty. Either of the following two modifications restores the free rider problem and induces shirking:

1 Let q be a function not only of effort but of random noise— Nature moves after the players. Uncertainty makes the *expected* output a continuous function of effort.
2 Let players have incomplete information about the critical value—Nature moves before the players and chooses e^*. Incomplete information makes the estimated output a continuous function of effort.

The discontinuity phenomenon is common. Examples, not all of which note the problem, include:

1 Effort in teams (Holmstrom [1982], Rasmusen [1987]).
2 Entry deterrence by an oligopoly (Bernheim [1984b], Waldman [1987]).
3 Output in oligopolies with trigger strategies (Porter [1983a]).
4 Patent races (Section 14.1).
5 Tendering shares in a takeover (Grossman & Hart [1980], Section 14.2).
6 Preferences for levels of a public good.

Notes

N8.1 Efficiency wages

- Which is the better, carrot or stick, is an interesting question. Two misconceptions that might lead one to think sticks are more powerful should be cleared up. First, if the agent is risk averse, equal dollar punishments and rewards lead to the punishment disutility being greater than the reward utility. Second, regression to the mean can easily lead a principal to think sticks work better than carrots in practice. Suppose a teacher assigns equal utility rewards and punishments to a student depending on his performance on tests, and that the student's effort is, in fact, constant. If the student is lucky on a test, he will do well and be rewarded, but will probably do worse on the next test. If the student is unlucky, he will be punished, and will do better on the next test. The naive teacher will think that rewards hurt performance and punishments help it. See Robyn Dawes's 1988 book, *Rational Choice in an Uncertain World* for a good exposition of this and other pitfalls of reasoning (especially pages 84-87). Kahneman, Slovic & Tversky (1982) covers similar material.

- For surveys of the efficiency wage literature, see the article by L. Katz (1986), the book of articles edited by Akerlof & Yellen (1986), and the book-length survey by Weiss (1990).

- While the efficiency wage model does explain involuntary unemployment, it does not explain cyclical changes in unemployment. There is no reason for the unemployment needed to control moral hazard to fluctuate widely and create a business cycle.

- The efficiency wage idea is essentially the same idea as in the Klein & Leffler (1981) model of product quality formalized in section 5.3. If no punishment is available for player who is tempted to misbehave, a punishment can be created by giving him something to take away. This something can be a high-paying job or a loyal customer. It is also similar to the idea of **co-opting** opponents familiar in politics and university administration. To tame the radical student association, give them an office of their own which can be taken away if they seize the dean's office. Rasmusen (1988b) shows yet another context: when depositors do not know which investments are risky and which are safe, mutual bank managers can be highly paid to deter them from making risky investments that might cost them their jobs.

- Adverse selection can also drive an efficiency wage model. We will see in Chapter 9 that a customer might be willing to pay a high price to attract sellers of high-quality cars when he cannot detect quality directly.

N8.2 Tournaments

- An article which stimulated much interest in tournaments is Lazear & Rosen (1981), which discusses in detail the importance of risk aversion and adverse selection.

- One example of a tournament is the two-year, three-man contest for the new chairman of Citicorp. The company named three candidates as vice-chairmen: the head of consumer banking, the head of corporate banking, and the legal counsel. Earnings reports were even split into three components, two of which were the corporate and consumer banking (the third was the "investment" bank, irrelevant to the tournament). See "What Made Reed Wriston's Choice at Citicorp," *Business Week*, July 2, 1984, p. 25.

- General Motors has tried a tournament among its production workers. During a depressed year, management credibly threatened to close down the auto plant with the lowest productivity. Reportedly, this did raise productivity. Such a tournament is interesting because it helps explain why a firm's supply curve could be upward sloping even if all its plants are identical, and why it might hold excess capacity. Should information on a plant's current performance have been released to other plants? See "Unions Say Auto Firms Use Interplant Rivalry to Raise Work Quotas," *Wall Street Journal*, November 8, 1983, p. 1.

- Under adverse selection, tournaments must be used differently than under moral hazard because agents cannot control their effort. Instead, tournaments are used to deter agents from accepting contracts in which they must compete for a prize with other agents of higher ability.

- Interfirm management tournaments run into difficulties when shareholders want managers to cooperate in some arenas. If managers collude in setting prices, for example, they can also collude to make life easier for each other.

- Antle & Smith (1986) is an empirical study of tournaments in managers' compensation. Rosen (1986) is a theoretical model of a labor tournament in which the prize is promotion.

- Suppose a firm conducts a tournament in which the best-performing of its vice-presidents becomes the next president. Should the firm fire the most talented vice-president before it starts the tournament? The answer is not obvious. Maybe in the tournament's equilibrium, Mr Talent works less hard because of his initial advantage, so that all of the vice-presidents retain the incentive to work hard.

- A tournament can reward the winner, or shoot the loser. Which is better? Nalebuff & Stiglitz (1983) say to shoot the loser, and Rasmusen (1987) finds a similar result for teams, but for a different reason. Nalebuff & Stiglitz's result depends on uncertainty and a large number of agents in the tournament, while Rasmusen's depends on risk aversion. If a utility function is concave because the agent is risk averse, the agent is hurt more by losing a given sum than he would benefit by gaining it. Hence, for incentive purposes the carrot is inferior to the stick, a result unfortunate for efficiency since penalties are often bounded by bankruptcy or legal constraints.

- Using a tournament, the equilibrium effort might be greater in a second-best contract than in the first-best, even though the second-best is contrived to get around the problem of inducing sufficient effort. Also, a pure tournament, in which the prizes are distributed solely according to the ordinal ranking of output by the agents, is often inferior to a tournament in which an agent must achieve a significant margin of superiority over his fellows in order to win (Nalebuff & Stiglitz [1983]). Companies using sales tournaments sometimes have prizes for record yearly sales besides ordinary prizes, and some long distance athletic races have nonordinal prizes to avoid dull events in which the best racers run "tactical races."

- Organizational slack of the kind described in the Farrell model has important practical implications. In dealing with bureaucrats, one must keep in mind that they are usually less concerned with the organization's prosperity than with their own. In complaining about bureaucratic ineptitude, it may be much more useful to name particular bureaucrats and send them copies of the complaint than to stick to the abstract issues at hand. Private firms, at least, are well aware that customers help monitor agents.

N8.3 Institutions and agency problems

- Gaver & Zimmerman (1977) describes how a performance bond of 100 percent was required for contractors building the BART subway system in San Francisco. "Surety companies" generally bond a contractor for five to 20 times his net worth, at a charge of 0.6 percent of the bond per year, and absorption of their bonding capacity is a serious concern for contractors in accepting jobs.

- Even if a product's quality need not meet government standards, the seller may wish to bind himself to them voluntarily. Stroh's *Erlanger* beer proudly announces on every bottle that although it is American, "Erlanger is a special beer brewed to meet the stringent requirements of Reinheitsgebot, a German brewing purity law established in 1516." Inspection of household electrical appliances by an independent lab to get the "U_L" listing is a similarly voluntary adherence to standards.

- The stock price is a way of using outside analysts to monitor an executive's performance. When General Motors bought EDS, they created a special class of stock, GM-E, which varied with EDS performance and could be used to monitor it.

*N8.6 Joint production by many agents: the Holmstrom Teams Model

- **Team theory**, as developed by Marschak & Radner (1972) is an older mathematical approach to organization. In the old usage of "team" (different from the current, Holmstrom [1982] usage), several agents who have different information but cannot communicate it must pick decision rules. The payoff is the same for each agent, and their problem is coordination, not motivation.

- The efficient contract (9) supports the efficient Nash equilibrium, but it also supports a continuum of inefficient Nash equilibria. Suppose that in the efficient equilibrium all workers work equally hard. Another Nash equilibrium is for one worker to do no work and the others to work inefficiently hard to make up for him.

- **A teams contract with hidden knowledge.** In the 1920s, National City Co. assigned 20 percent of profits to compensate management as a group. A management committee decided how to share it, after each officer submitted an unsigned ballot suggesting the share of the fund that Chairman Mitchell should have, and a signed ballot giving his estimate of the worth of each of the other eligible officers, himself excluded. (Galbraith [1954] p. 157)

- **A First-best, budget-balancing contract when agents are risk averse**. Proposition 8.1 can be shown to hold for any contract, not just for differentiable sharing rules, but it does depend on risk neutrality and separability of the utility function.Consider the following contract from Rasmusen (1987):

$$w_i = \begin{cases} b_i & \text{if } q \geq q(e^*). \\ \left.\begin{array}{ll} 0 & \text{with probability } (n-1)/n \\ q & \text{with probability } 1/n \end{array}\right\} & \text{if } q < q(e^*) \end{cases}$$

  If the worker shirks, he enters a lottery. If his risk aversion is strong enough, he prefers the riskless return b_i, so he does not shirk. If agents' wealth is unlimited, then for any positive risk aversion we could construct such a contract, by making the losers in the lottery accept negative pay.

- A teams contract like (9) is not a tournament. Only absolute performance matters, even though the level of absolute performance depends on what all the players do.

- **The budget-balancing constraint.** The legal doctrine of "consideration" makes it difficult to make binding, Pareto-suboptimal promises. An agreement is not a legal contract unless it is more than a promise: both parties have to receive something valuable for the courts to enforce the agreement.

- Adverse selection can be incorporated into a teams model. A team of workers who may differ in ability produce a joint output, and the principal tries to ensure that only high-ability workers join the team. (See Rasmusen & Zenger [1990]).

Problems

8.1. Monitoring with error

An agent has a utility function $U = \sqrt{(w)} - \alpha e$, where $\alpha = 1$ and e is either 0 or 5. His reservation utility level is $\overline{U} = 9$, and his output is 100 with low effort and 250 with high effort. Principals are risk neutral and scarce, and agents compete to work for them. The principal cannot condition the wage on effort or output, but he can, if he wishes, spend five minutes of his time, worth 10 dollars, to drop in and watch the agent. If he does that, he observes the agent *Daydreaming* or *Working*, with probabilities that differ depending on the agent's effort. He can condition the wage on those two things, so the contract will be $\{\underline{w}, \overline{w}\}$. The probabilities are given by Table 1.

Table 1: Monitoring with Error

	Probability of	
Effort	*Daydreaming*	*Working*
$Low(e = 0)$	0.6	0.4
$High(e = 5)$	0.1	0.9

a What are profits in the absence of monitoring, if the agent is paid enough to make him willing to work for the principal?

b Show that high effort is efficient under full information.

c If $\alpha = 1.2$, is high effort still efficient under full information?

d Under asymmetric information, with $\alpha = 1$, what are the participation and incentive compatibility constraints?

e Under asymmetric information, with $\alpha = 1$, what is the optimal contract?

8.2. Monitoring with Error: Second Offenses (see Rubinstein [1979]).

Individuals who are risk-neutral must decide whether to commit zero, one, or two robberies. The cost to society of robbery is 10, and the benefit to the robber is 5. No robber is ever convicted and jailed, but the police beat up any suspected robber they find. They beat up innocent people mistakenly sometimes, as shown by Table 2, which shows the probabilities of zero or more beatings for someone who commits zero, one, or two robberies.

Table 2: Crime

	Beatings		
Robberies	0	1	2
0	0.81	0.18	0.01
1	0.60	0.34	0.06
2	0.49	0.42	0.09

a How big should p^*, the disutility of a beating, be made to deter crime completely while inflicting a minimum of punishment on the innocent?

b In equilibrium, what percentage of beatings are of innocent people? What is the payoff of an innocent man?

c Now consider a more flexible policy, which inflicts heavier beatings on repeat offenders. If such flexibility is possible, what are the optimal severities for first- and second-time offenders? (call these p_1 and p_2). What is the expected utility of an innocent person under this policy?

d Suppose that the probabilities are as given in Table 3. What is an optimal policy for first and second offenders?

Table 3: More Crime

	Beatings		
Robberies	0	1	2
0	0.9	0.1	0
1	0.6	0.3	0.1
2	0.5	0.3	0.2

8.3: Bankruptcy Constraints. A risk-neutral principal hires an agent with utility function $U = w - e$ and reservation utility $\overline{U} = 7$. Effort is either 0 or 20. There is a bankruptcy constraint: $w \geq 0$. Output is given by Table 4.

Table 4: Bankruptcy

	Probability of output of		
Effort	0	400	Total
Low ($e = 0$)	0.5	0.5	1
High ($e = 10$)	0.2	0.8	1

a What would the agent's effort choice and utility be if he owned the firm?

b If agents are scarce and principals compete for them, what will the agent's contract be under full information? His utility?

c If principals are scarce and agents compete to work for them, what will the contract be under full information? What will the agent's utility be?

d If principals are scarce and agents compete to work for them, what will the contract be when the principal cannot observe effort? What will the payoffs be for each player?

e Suppose there is no bankruptcy constraint. If principals are the scarce factor and agents compete to work for them, what will the contract be when the principal cannot observe effort? What will the payoffs be for principal and agent?

8.4: Teams. A team of two workers produces and sells widgets for the principal. Each worker chooses high or low effort. An agent's utility is $U = w - 20$ if his effort is high, and $U = w$ if it is low, with a reservation utility of $\overline{U} = 0$. Nature chooses business conditions to be excellent, good, or bad, with probabilities θ_1, θ_2, and θ_3. The principal observes output but not business conditions, as shown in Table 5.