19.F Incomplete Markets

In this section we explore the implications of having fewer than S assets, that is, of having an asset structure that is necessarily incomplete. We pursue this point in the two-period framework of the previous sections.[22]

Markets may fail to exist for a number of reasons. One class of reasons refers to the informational asymmetries to be covered in Section 19.H: contracts for the delivery of goods can only be made contingent on states whose occurrence can be verified to the satisfaction of all contracting parties. Another class of reasons stems from transaction costs: the availability of a market is, after all, in the nature of a public good. Yet another variety of reasons comes from enforceability constraints: a promise to deliver one unit of good is worthless if delivery cannot be enforced.[23] This said, we shall not delve further into a theory of asset determination. We will rest content for the moment with taking the incomplete situation as a reasonable description of reality.

We begin by observing that when $K < S$ a Radner equilibrium need not be Pareto optimal. This is not surprising: if the possibilities of transferring wealth across states are limited, then there may well be a welfare loss due to the inability to diversify risks to the extent that would be convenient. Just consider the extreme case where there are no assets at all. Example 19.F.1 provides another interesting illustration of this type of failure.

Example 19.F.1: *Sunspots.* Suppose that preferences admit an expected utility representation and that the set of states S is such that, first, the probability estimates for the different states are the same across consumers (i.e., $\pi_{si} = \pi_{si'} = \pi_s$ for all i, i', and s) and, second, that the states do not affect the fundamentals of the economy; that is, the Bernoulli utility functions and the endowments of every consumer i are uniform across states [i.e., $u_{si}(\cdot) = u_i(\cdot)$ and $\omega_{si} = \omega_i$ for all s]. Such a set of states is called a *sunspot* set. The question we shall address [first posed by Cass and Shell (1983)] is whether in these circumstances the Radner equilibrium allocations can assign varying consumptions across states. An equilibrium where this happens is called a *sunspot equilibrium*.[24]

Under the assumption that consumers are strictly risk averse, so that the utility functions $u_i(\cdot)$ are strictly concave, *any Pareto optimal allocation* $(x_1, \ldots, x_I) \in \mathbb{R}^{LSI}$ *must be uniform across states* (*or state independent*); that is, for every i we must have

22. For a general and advanced treatment, see Magill and Shafer (1991).

23. An example of a situation where enforceability would be helped is when we are dealing with the shares of a firm (the total endowments of this asset is, therefore, positive) and no short sales are possible. Enforceability is then guaranteed because the physical shares—the ownership claims to the firm—are actually transacted at $t = 0$. This sort of consideration, which, unfortunately, clashes with the very convenient assumption that unlimited short sales are possible, could explain why some assets may exist and others not: in order for the asset to exist, it helps if the random variable has a physical reality that can be exchanged at $t = 0$.

24. The term "sunspots" is old, but the current meaning is recent. In the XIX century, research on the "sunspot problem" tried to determine the influence on the fundamentals (e.g., on agriculture) that could make an unobservable signal (sunspots) have an effect on prices. The modern problem is to determine how an observable signal with no influence on fundamentals can nonetheless have, via expectations, an effect on prices.

$x_{1i} = x_{2i} = \cdots = x_{si} = \cdots = x_{Si}$. To see this, suppose that, for every i and s, we replace the consumption bundle of consumer i in state s, $x_{si} \in \mathbb{R}^L_+$, by the expected consumption bundle of this consumer: $\bar{x}_i = \sum_s \pi_s x_{si} \in \mathbb{R}^L_+$. The new allocation is state independent, and it is also feasible because

$$\sum_i \bar{x}_i = \sum_i \sum_s \pi_s x_{si} = \sum_s \pi_s \left(\sum_i x_{si} \right) \leq \sum_s \pi_s \left(\sum_i \omega_i \right) = \sum_i \omega_i .$$

By the concavity of $u_i(\cdot)$ it follows that no consumer is worse off:

$$\sum_s \pi_s u_i(\bar{x}_i) = u_i(\bar{x}_i) = u_i\left(\sum_s \pi_s x_{si} \right) \geq \sum_s \pi_s u_i(x_{si}) \qquad \text{for every } i.$$

Because of the Pareto optimality of $(x_1, \ldots, x_I)$, the above weak inequalities must in fact be equalities; that is, $u_i(\bar{x}_i) = \sum_s \pi_s u_i(x_{si})$ for every i. But, if so, then the strict concavity of $u_i(\cdot)$ yields $x_{si} = \bar{x}_i$ for every s. In summary: the Pareto optimal allocation $(x_1, \ldots, x_I) \in \mathbb{R}^{LSI}$ is state independent.

From the state independence of Pareto optimal allocations and the first welfare theorem we reach the important conclusion that if a system of complete markets over the states S can be organized, then the equilibria are *sunspot free*, that is, consumption is uniform across states. In effect, traders wish to insure completely and have instruments to do so.

It turns out, however, that if there is not a complete set of insurance opportunities, then the above conclusion does not hold true. Sunspot-free, Pareto optimal equilibria always exist (just make the market "not pay attention" to the sunspot; see Exercise 19.F.1). But it is now possible for the consumption allocation of some Radner equilibria to depend on the state, and consequently to fail the Pareto optimality test. In such an equilibrium consumers expect different prices in different states, and their expectations end up being self-fulfilling. The simplest, and most trivial, example is when there are no assets whatsoever ($K = 0$). Then a system of spot prices $(p_1, \ldots, p_S) \in \mathbb{R}^{LS}$ is a Radner equilibrium if and only if every p_s is a Walrasian equilibrium price vector for the spot economy defined by $\{(u_i(\cdot), \omega_i)\}_{i=1}^{i=I}$. If, as is perfectly possible, this economy admits several distinct Walrasian equilibria, then by selecting different equilibrium price vectors for different states, we obtain a sunspot equilibrium, and hence a Pareto inefficient Radner equilibrium. ■

We have seen that Radner equilibrium allocations need not be Pareto optimal, and so, in principle, there may exist reallocations of consumption that make all consumers at least as well off, and at least one consumer strictly better off. It is important to recognize, however, that this *need not* imply that a welfare authority who is "as constrained in interstate transfers as the market is" can achieve a Pareto optimum. An allocation that cannot be Pareto improved by such an authority is called a *constrained Pareto optimum*. A more significant and reasonable welfare question to ask is, therefore, whether Radner equilibrium allocations are constrained Pareto optimal. We now address this matter.[25]

25. This is a typical instance of a *second-best* welfare issue. We have already encountered problems of this kind in Chapters 13 and 14, and we shall do so again in Chapter 22.

To proceed with the analysis we need a precise description of the constrained feasible set and of the corresponding notion of constrained Pareto optimality. This is most simply done in the context where there is a single commodity per state, that is, $L = 1$. The important implication of this assumption is that then the amount of consumption good that any consumer i gets in the different states is entirely determined by the portfolio z_i. Indeed, $x_{si} = \sum_k z_{ki} r_{sk} + \omega_{si}$. Hence, we can let

$$U_i^*(z_i) = U_i^*(z_{1i}, \ldots, z_{Ki}) = U_i(\sum_k z_{ki} r_{1k} + \omega_{1i}, \ldots, \sum_k z_{ki} r_{Sk} + \omega_{Si})$$

denote the utility induced by the portfolio z_i. The definition of constrained Pareto optimality is then quite natural.

Definition 19.F.1: The asset allocation $(z_1, \ldots, z_I) \in \mathbb{R}^{KI}$ is constrained Pareto optimal if it is feasible (i.e., $\sum_i z_i \leq 0$) and if there is no other feasible asset allocation $(z'_1, \ldots, z'_I) \in \mathbb{R}^{KI}$ such that

$$U_i^*(z'_1, \ldots, z'_I) \geq U_i^*(z_1, \ldots, z_I) \qquad \text{for every } i,$$

with at least one inequality strict.

In this $L = 1$ context the utility maximization problem of consumer i becomes

$$\begin{aligned} \underset{z_i \in \mathbb{R}^K}{\text{Max}} \quad & U_i^*(z_{1i}, \ldots, z_{Ki}) \\ \text{s.t. } & q \cdot z_i \leq 0. \end{aligned}$$

Suppose that $z_i^* \in \mathbb{R}^K$ for $i = 1, \ldots, I$, is a family of solutions to these individual problems, for the asset price vector $q \in \mathbb{R}^K$. Then $q \in \mathbb{R}^K$ is a Radner equilibrium price if and only if $\sum_i z_i^* \leq 0$.[26] Note that this has become now a perfectly conventional equilibrium problem with K commodities [see Exercise 19.F.2 for a discussion of the properties of $U_i^*(\cdot)$]. To it we can apply the first welfare theorem (Proposition 16.C.1) and reach the conclusion of Proposition 19.F.1.

Proposition 19.F.1: Suppose that there two periods and only one consumption good in the second period. Then any Radner equilibrium is *constrained Pareto optimal* in the sense that there is no possible redistribution of assets in the first period that leaves every consumer as well off and at least one consumer strictly better off.[27,28]

The situation considered in Proposition 19.F.1 is very particular in that once the initial asset portfolio of a consumer is determined, his overall consumption is fully determined: with only one consumption good, there are no possibilities for trade once the state occurs. In particular, second-period relative prices do not matter, simply because there are no such prices. Things change if there is more than one

26. Recall that, given z_i, the consumptions in every state are determined. Also, the price of consumption good in every state is formally fixed to be 1.

27. We reemphasize that the term *constrained* is appropriate. Wealth can be transferred across individuals and states only by trade in the given set of assets. To see how restrictive this can be, suppose that there are no assets. Then the welfare authority has no policy instrument whatsoever.

28. In our current discussion, all consumption takes place in the second period. This is a simplification that does not affect the validity of the proposition. If the welfare authority can also redistribute consumption that takes place in the first period, then the Radner equilibrium allocations will still be constrained Pareto optimal.

consumption good in the second period, or if there are more than two periods. Consider the two-period case with $L > 1$: Then we cannot summarize the individual decision problem by means of an indirect utility of the asset portfolio. The relative prices expected in the second period[29] also matter. This substantially complicates the formulation of a notion of constrained Pareto optimality. Be that as it may, there appears not to be a useful generalization of the "constrained Pareto optimal" concept in which we could assert the constrained Pareto optimality of Radner equilibrium allocations. Example 19.F.2, due to Hart (1975), makes the point. In it we have an economy with several Radner equilibria where two of them are Pareto ordered. That is, we have a Radner equilibrium that is Pareto dominated by another Radner equilibrium. To the extent that it seems natural to allow a welfare authority, at the very least, to select equilibria, it follows that the first equilibrium is not constrained Pareto optimal.[30]

Example 19.F.2: *Pareto Ordered Equilibria.* Let $I = 2$, $L = 2$, and $S = 2$. There are no assets ($K = 0$). The two consumers have, as endowments, one unit of every good in every state. The utility functions are of the form $\pi_{1i}u_i(x_{11i}, x_{21i}) + \pi_{2i}u_i(x_{12i}, x_{22i})$. Note that although the probability assessments are different for the two consumers (these probabilities will be specified in a moment), the spot economies are identical in the two states. Suppose that this spot economy has several distinct equilibria (e.g., it could be the exchange economy in Figure 15.B.9). Let $p', p'' \in \mathbb{R}^2$ be the Walrasian prices for two of these equilibria and let $v_i(p)$ be the spot market utility associated with $u_i(\cdot, \cdot)$ and the spot price vector $p \in \mathbb{R}^2$. Suppose that $v_1(p') > v_1(p'')$. By Pareto optimality in the spot market, $v_2(p') < v_2(p'')$.

We now define two Radner equilibria. The first has equilibrium prices $(p_1, p_2) = (p', p'') \in \mathbb{R}^4$ and the second has $(p_1, p_2) = (p'', p') \in \mathbb{R}^4$. Because there is no possibility of transferring wealth across states, these are indeed Radner equilibrium prices and, moreover, they are so for any probability estimates π_i. However, the expected utility of these Radner equilibria for the different consumers depends on the π_i. We can see now that if consumer 1 believes that the first state is more likely than the second, that is, he has $\pi_{11} > \frac{1}{2}$, then he will prefer the first equilibrium to the second. Indeed, $\pi_{11} > \frac{1}{2}$ and $v_1(p') > v_1(p'')$ imply $\pi_{11}v_1(p') + \pi_{21}v_1(p'') > \pi_{11}v_1(p'') + \pi_{21}v_1(p')$. Similarly, if the second consumer believes that the second state is more likely than the first, that is, he has $\pi_{22} > \frac{1}{2}$, then he will *also* prefer the first equilibrium to the second: $\pi_{22} > \frac{1}{2}$ and $v_2(p') < v_2(p'')$ imply $\pi_{12}v_2(p') + \pi_{22}v_2(p'') > \pi_{12}v_2(p'') + \pi_{22}v_2(p')$. Thus, the Radner equilibrium with prices (p', p'') Pareto dominates the one with prices (p'', p'). ■

The consensus emerging in the literature seems to be that failures of restricted Pareto optimality (for natural meanings of this concept) are not only possible but even typical [Geanakoplos and Polemarchakis (1986)]. In Exercise 19.F.3 you are asked to develop a related optimality paradox: it is possible for the set of assets to expand and for everybody to be worse off at the new equilibrium! We shall not pursue the constrained optimality analysis

29. Or the relative prices of goods between the second and third period, if we are considering more than two dates.

30. That is, the first equilibrium is not Pareto optimal relative to any set of constrained feasible allocations that includes all Radner equilibrium allocations.

in any greater depth. At some point the analysis runs into the difficulty that it is hard to proceed sensibly without tackling the difficult problem of the determination of the asset structure.

We could also analyze the positive issues studied in Chapter 17 within an incomplete market setting. For existence, there is a new set of complexities related to the fact that unbounded short sales are possible. In some contexts this may lead to existence failures (see Exercise 19.F.4).[31] New subtleties also arise for the issue of the determinacy of equilibria (i.e., the number and local uniqueness of equilibria). As we have seen in Section 17.D, with a complete asset structure we have generic finiteness. But with incomplete markets the nature of the assets (e.g., whether real or financial) matters, as may the size of S.

19.G Firm Behavior in General Equilibrium Models under Uncertainty

In the previous sections we have concentrated on the study of exchange economies. For once, this has not been just for simplicity. The consideration of production and firms is genuinely more difficult in a context of possibly incomplete markets. The reason relates to the issue of the objectives of the firm.[32]

As before, we consider a setting with two periods, $t = 0$ and $t = 1$, and S possible states at $t = 1$. There are L physical commodities traded in the spot markets of period $t = 1$ and K assets traded at $t = 0$. There is no consumption at $t = 0$. The returns of the assets are in physical amounts of the good 1 (which we call the numeraire). The $S \times K$ return matrix is denoted R.

We introduce into our model a firm that produces a random amount of numeraire at date $t = 1$ (perhaps by means of inputs used at time $t = 0$, but we do not formalize this part explicitly). We let $(a_1, \dots, a_S)$ denote the state-contingent levels of production of the firm. There are also shares $\theta_i \geq 0$, with $\sum_i \theta_i = 1$, giving the proportion of the firm that belongs to consumer i. We take, for the rest of this section (except in the small-type paragraphs at the end) the natural point of view that the firm is an asset with return vector $a = (a_1, \dots, a_S)$ whose shares are tradeable in the financial markets at $t = 0$.[33]

Suppose now that the firm can actually choose, within a range, its (random) production plan. Say, therefore, that there is a set $A \subset \mathbb{R}^S$ of possible choices of return

31. Unbounded short sales are at the origin of a discontinuity in the dependence on asset returns of the space of attainable wealth transfers across states. No matter how close asset returns (in dollar terms) may be to displaying a linear dependence, consumers can plan to attain, by using trades of very large magnitude, any wealth transfer in the subspace spanned by the asset returns. But when returns become exactly linearly dependent, this attainable subspace suddenly drops in dimension. As indicated, this can lead to an existence failure in some contexts. The model we have analyzed in this chapter is not, however, one of those. If, as here, in every state all assets have returns in a single good, which, moreover, is the same across assets, then the discontinuity does not arise.

32. The classic paper on this topic is Diamond (1967). For a more recent survey see Merton (1982).

33. A minor difference with the setting so far is that the firm does really produce the vector $(a_1, \dots, a_S)$, and therefore the total endowment of this asset is not zero. In fact, by putting $\sum_i \theta_i = 1$ we have normalized this total endowment to be 1.

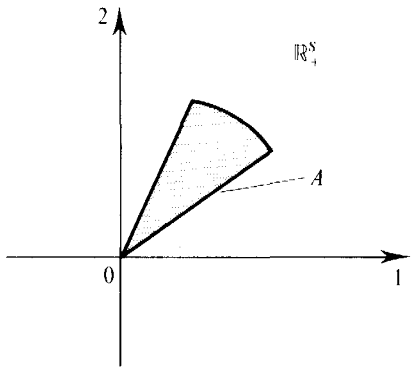

Figure 19.G.1
An example of possible production choices of the firm.

vectors $(a_1, \ldots, a_S) \in A$ of the firm. See Figure 19.G.1 for the case where $S = 2$. We assume that the return vector $a \in A$ is chosen before the financial markets of period $t = 0$ open. Thus, the decision is made by the *initial* shareholders (since shares may be sold in period $t = 0$, the shareholders at the end of period $t = 0$ may be a different set). Which production plan should these initial owners choose? It turns out that the answer is very simple if A can be spanned by the existing assets and is very difficult if it cannot.

Definition 19.G.1: A set $A \subset \mathbb{R}^S$ of random variables is *spanned* by a given asset structure if every $a \in A$ is in the range of the return matrix R of the asset structure, that is, if every $a \in A$ can be expressed as a linear combination of the available asset returns.

If we assume, first, that A is spanned by R and, second, that we are dealing with a small project (i.e., all the possible productions $a \in A$ are small relative to the size of the economy; e.g., $a_s/\|\sum_i \omega_{si}\|$ is small for all s), then we are (almost) justified in taking the equilibrium spot prices $p = (p_1, \ldots, p_S) \in \mathbb{R}^{LS}$ and asset prices $q = (q_1, \ldots, q_K) \in \mathbb{R}^K$ as constants independent of the particular production plan chosen by the firm.[34] For the asset prices $q \in \mathbb{R}^K$ the *market value* $v(a, q)$ of any production plan $a \in A$ can be computed by arbitrage: if $a = \sum_k \alpha_k r_k$ then $v(a, q) = \sum_k \alpha_k q_k$. In Exercise 19.G.1 you are asked to show that if the firm is added as a new asset to the given list of assets, and each production plan $a \in A$ is priced at its arbitrage value $v(a)$, then any budget-feasible consumption plan of any consumer can actually be reached without purchasing any shares of the firm (the fact can be deduced from Proposition 19.E.3). Thus, for fixed asset prices $q \in \mathbb{R}^K$ and spot prices $p = (p_1, \ldots, p_S) \in \mathbb{R}^{LS}$, the budget constraint of consumer i is[35]

34. Both assumptions are important for this conclusion. Suppose for a moment that there are zero total endowments of the asset. Then, since the asset is redundant, Proposition 19.E.3 (see also Exercise 19.E.4) implies that at the Radner equilibrium the new asset is absorbed without any change in prices. What we are now assuming is that this remains approximately true if the total endowment of the asset is small (i.e., if the project is small).

35. Note that the value of the initial endowments at $t = 0$ is the value of the shares of the firm $\theta_i v(a, q)$.

$$B_{ai} = \{(x_{1i}, \ldots, x_{Si}) \in \mathbb{R}^{LS}_+ : \text{there is a portfolio } z_i \in \mathbb{R}^K \text{ such that}$$
$$p_s \cdot (x_{is} - \omega_{is}) \leq \sum_k p_{1s} r_{sk} z_{ki} \text{ for every } s, \text{ and } q \cdot z_i \leq \theta_i v(a, q)\}. \quad (19.G.1)$$

It follows from the form of this budget constraint that at constant prices every consumer–owner (i.e., any i with $\theta_i > 0$) faced with the choice between two production plans $a, a' \in A$, will prefer the one with higher market value. Indeed, if $v(a, q) \geq v(a', q)$ then $B_{a'i} \subset B_{ai}$. Thus, the objective of market value maximization will be the *unanimous* desire of the firm's initial owners.[36]

If A is not spannable by the given asset structure we run into at least two serious difficulties. The first difficulty has to do with price *quoting* and is common to any commodity innovation problem. Without spanning, the value of a production plan $a \in A$ cannot be computed from current asset prices simply by arbitrage. The value is not, so to speak, implicitly quoted in the economy. Therefore, it would need to be anticipated by the agents of the economy from their understanding of the workings of the overall economy—no mean task.

The second difficulty, more specific to the financial context, has to do with price *taking*. Due to the possibility of unlimited short sales there is a discontinuity in the plausibility of the price-taking assumption. With spanning we can argue, as we did, that if the project is small then the effect of production decisions on asset prices, or on spot prices at $t = 1$, is also small. But if a new asset $a \in A$, no matter how small, is not generated by the current asset structure, then its availability increases the span of available wealth transfers by one whole extra dimension. The impact is therefore substantial, and may well have a dramatic effect on prices.[37] There is then no reason for owners' preferences over different production plans to be dictated merely by the increase in wealth at the prices prior to the introduction of the firm (see Exercise 19.G.2). These two difficulties, to repeat, are serious. There is no easy way out.

A variation of the above model entirely eliminates the asset role of the firm at $t = 0$. Let us assume that the firm's shares cannot be traded at $t = 0$.[38] If owners at $t = 0$ choose $a \in A$, this simply means that their endowments at $t = 1$ are modified by the random variable $\theta_i a$ that, recall, pays in good 1 (i.e., the new endowment of consumer i becomes $(\omega_{si} + (\theta_i a_s, 0, \ldots, 0)) \in \mathbb{R}^L$ for every state s).

If $a \in A$ can be spanned, then we are as in the previous model. It does not matter whether shares of the firm can be sold or not at $t = 0$. In either case consumers can take positions in the asset markets that will guarantee that the resulting final consumptions at $t = 1$ are the same (Exercise 19.G.3).

If $a \in A$ cannot be spanned, matters are different. The good news is that, because no new tradeable asset is created at $t = 0$, the price-taking discontinuity problem disappears. The bad news is that there is now another difficulty: Because there is no market for the shares at $t = 0$, the value of the asset cannot be computed as a deterministic amount at $t = 0$. It is rather a

36. As long as the project is small and, consequently, the prices are almost constant.

37. Recall that short sales are possible. One way out of the dilemma is to put a bound on short sales of roughly the size of the possible production vectors. The cost of this assumption is that we have then to give up the theory of arbitrage pricing.

38. Or perhaps we are now at the end of period $t = 0$ and the financial markets have already closed.

random variable at $t = 1$ and therefore the risk attitudes of the consumer–owners will be essential to the determination of the preferred $a \in A$. In particular, unanimity of the consumer–owners should not be expected (see Exercise 19.G.4).

19.H Imperfect Information

Up to now we have concentrated the analysis on a model where spot trading for goods occurs under conditions of perfect information about the state of the world. In this section, we relax this feature by considering the possibility that this information is not perfect. In doing so we shall see that there is a key difference between the case of *symmetric* information (all traders have the same information), which is largely reducible to the previous theory, and the case of *asymmetric* information, where a host of new and difficult conceptual problems arise.

To focus on essentials, we deal with trade in a single period. You can think of it as the period $t = 1$ of the previous treatments. In this period, one of several states $s = 1, \dots, S$ can arise. Once a state occurs, we consider the simplest case in which there is a single spot market. In this market a first commodity (good, service, ...) is traded against a second good, to be thought of as money (thus, $L = 2$). The price of the second good is normalized to 1. We reserve the symbol $p \in \mathbb{R}$ for the price of the nonmonetary commodity.

There are I consumers. Given probabilities $\pi = (\pi_{1i}, \dots, \pi_{Si})$ over the states, a random consumption vector $x_i = (x_{1i}, \dots, x_{Si}) \in \mathbb{R}^{2S}$ is evaluated by consumer i according to an extended von Neumann–Morgenstern utility function:

$$U_i(x_i) = \sum_s \pi_{si} u_{si}(x_{si}),$$

where $u_{si}(\cdot)$ is consumer i's Bernoulli utility function in state s. Consumer i also has an initial, state-dependent, endowment vector $\omega_i = (\omega_{1i}, \dots, \omega_{Si}) \in \mathbb{R}^{2S}$, and a *signal function* $\sigma_i(\cdot)$ assigning a real number $\sigma_i(s) \in \mathbb{R}$ to every state $s \in S$.

The state s occurs at the beginning of the period. We assume that, once this has happened, consumer i receives the initial endowment ω_{si} and the signal $\sigma_i(s) \in \mathbb{R}$. The interpretation is that consumer i can distinguish two states $s, s' \in S$ if and only if $\sigma_i(s) \neq \sigma_i(s')$.[39] Consistent with this interpretation, we require that the endowments be *measurable* with respect to the signal function, that is, $\omega_{si} = \omega_{s'i}$ whenever $\sigma_i(s) = \sigma_i(s')$ (thus, we can write ω_{si} as $\omega_{\sigma_i(s)i}$). In this manner the endowments of goods of consumer i do not reveal to him information about the state of the world that is not already revealed by the signal. After every consumer gets his signal, the spot market operates. Finally, at the end of the period, the state is revealed and consumption takes place.[40]

39. Equivalently, as we did (in small type) in Section 19.B, we could use *information partitions* instead of explicit signal functions. The information partition $\mathscr{S}_i$ associated with the signal $\sigma_i(\cdot)$ is composed of the events $\{s \in S: \sigma_i(s) = c\}$ obtained by letting $c \in \mathbb{R}$ run over all possible values.

40. In a more general, multistage, situation, some information is revealed at the end of the period and the economy moves on to the next period.

Symmetric Information

We say that information is *symmetric* if any two states, $s, s' \in S$, are distinguishable by one consumer i if and only if they are distinguishable by every other consumer k; that is, $\sigma_i(s) \neq \sigma_i(s')$ if and only if $\sigma_k(s) \neq \sigma_k(s')$. Thus, with symmetric information we can as well assume that all consumers share the same signal function. We therefore write $\sigma_i(\cdot) = \sigma(\cdot)$ for all i. We can think of $\sigma(\cdot)$ as a public signal.

With symmetric information the determination of the spot prices proceeds in a manner entirely parallel to what we have seen so far. Suppose that state s occurs. Then every consumer i receives the signal $\sigma(s)$ and initial endowments $\omega_{\sigma(s)i}$.[41] From the signal and the prior probabilities $\pi_i = (\pi_{1i}, \ldots, \pi_{Si})$, which we take to be strictly positive, he computes his posterior probabilities over the different states s' as

$$\pi_{s'i} \mid \sigma(s) = \frac{\pi_{s'i}}{\sum_{\{s'':\, \sigma(s'') = \sigma(s)\}} \pi_{s''i}}$$

for any s' with $\sigma(s') = \sigma(s)$, and $\pi_{s'i} \mid \sigma(s) = 0$ otherwise. The utility of a consumption bundle $x_i \in \mathbb{R}^2$ conditional on the signal $\sigma(s)$ is then

$$u_i(x_i \mid \sigma(s)) = \sum_{s'} (\pi_{s'i} \mid \sigma(s)) u_{s'i}(x_i).$$

Therefore we have, conditional on s, a perfectly well-specified spot economy. Under the usual assumption of price-taking behavior, an equilibrium price will be generated. We write this price as $p(\sigma(s)) \in \mathbb{R}$.[42]

The concept of an information signal function lends itself to interesting comparative statics exercises.

Definition 19.H.1: The signal function $\sigma': S \to \mathbb{R}$ is *at least as informative* as $\sigma: S \to \mathbb{R}$ if $\sigma(s) \neq \sigma(s')$ implies $\sigma'(s) \neq \sigma'(s')$ for any pair s, s'. It is *more informative* if, in addition, $\sigma'(s) \neq \sigma'(s')$ for some pair s, s' with $\sigma(s) = \sigma(s')$.[43]

Two arbitrary signal functions $\sigma(\cdot)$, $\sigma'(\cdot)$ may well not be comparable by the "at least as informative" concept. If they are, it is natural to ask if the more informative signal leads to a welfare improvement. We pose the "improvement" question in an ex ante sense (see Exercise 19.H.1 for an interim and an ex post sense); that is, we want to compare the expected utility of the different consumers under $\sigma(\cdot)$ and under $\sigma'(\cdot)$ when the expectation is evaluated *before* s occurs.

41. The endowments could, of course, be the result of the execution of forward trades entered upon in the past. The measurability requirement, namely, the restriction that endowments depend on the state only through the signal, then captures the restriction that a forward contract can only be made contingent on information available at the time of the execution of the contract (strictly speaking, it should be conditional on information available at that time to the contract enforcing authority).

42. Because updated probabilities and utility functions depend only on the values of the signal, we have imposed the natural requirement that the clearing price depends also only on the signal; that is, we write $p(\sigma(s))$ rather than $p(s)$. Indeed, how could the (unmodeled) market mechanism manage to distinguish states that no consumer can distinguish?

43. In terms of the corresponding information partitions, the signal $\sigma'(\cdot)$ is more informative than $\sigma(\cdot)$ if the information partition of $\sigma'(\cdot)$ refines the information partition of $\sigma(\cdot)$.

Consider first the decision problem of a single consumer i. Suppose, for simplicity, that the spot price $p \in \mathbb{R}$ and the consumer wealth $w_i \in \mathbb{R}$ are given and are independent of s (a more general case is presented in Exercise 19.H.2). For any signal function $\sigma(\cdot)$ the consumer forms a consumption plan $x_i^{\sigma(\cdot)} \in \mathbb{R}^{2S}$ as follows: Subject to the restriction that $x_{si}^{\sigma(\cdot)} = x_{s'i}^{\sigma(\cdot)}$ whenever $\sigma(s') = \sigma(s)$, the consumer chooses, for every possible state s, a consumption $x_{si}^{\sigma(\cdot)}$ in his budget set that maximizes expected utility conditional on the signal $\sigma(s)$. The ex ante utility of the information signal function $\sigma(\cdot)$ is therefore $\sum_s \pi_{si} u_{si}(x_{si}^{\sigma(\cdot)})$.

Proposition 19.H.1: In the single-consumer problem, if the signal function $\sigma'(\cdot)$ is at least as informative as the signal function $\sigma(\cdot)$, then the ex ante utility derived from $\sigma'(\cdot)$, $\sum_s \pi_{si} u_{si}(x_{si}^{\sigma'(\cdot)})$, is at least as large as the ex ante utility derived from $\sigma(\cdot)$, $\sum_s \pi_{si} u_{si}(x_{si}^{\sigma(\cdot)})$.

Proof: The first observation is that for any $\sigma(\cdot)$, $x_i^{\sigma(\cdot)}$ solves the problem

$$\text{Max} \quad \sum_s \pi_{si} u_{si}(x_{si})$$

$$\text{s.t. } x_i \in B_i^{\sigma(\cdot)} = \{x_i \in \mathbb{R}^{2S} : p x_{1si} + x_{2si} \le w_i \text{ for every } s, \text{ and } x_{si} = x_{s'i} \text{ whenever } \sigma(s) = \sigma(s')\}.$$

You are asked to verify this formally in Exercise 19.H.3.

The second observation is that if $\sigma'(\cdot)$ is at least as informative as $\sigma(\cdot)$, then $B_i^{\sigma(\cdot)} \subset B_i^{\sigma'(\cdot)}$. Again you should verify this in Exercise 19.H.3.

It follows that moving from $\sigma(\cdot)$ to $\sigma'(\cdot)$ we only expand the constraint set of the consumer's problem. In doing so, the maximum value cannot decrease. ■

The claim of Proposition 19.H.1 is intuitive: in an isolated single-person decision problem, a more informative signal does not upset the feasibility of any decision plan (hence, it brings about an expansion of the feasible set) because the decision maker always has the option not to act on the extra information provided by $\sigma'(\cdot)$ over $\sigma(\cdot)$. Unfortunately, this line of argument does *not* apply to a system with interacting decision makers. At the equilibrium the budget set of a single consumer may well be affected by the signal, even if the consumer does not use it. It suffices that the other consumers use it and that, as a result, the new information finds its way into the spot prices. Example 19.H.1 shows that, because of this, it is even possible for an increase of information to make *everybody* (ex ante) worse off.

Example 19.H.1: Suppose there are two consumers, two commodities, and two equally probable states $s = 1, 2$. In both states the two consumers' endowments of the two physical goods are $\omega_1 = (1, 0)$ for consumer 1 and $\omega_2 = (0, 1)$ for consumer 2. In total, therefore, there is one unit of every commodity in every state. The two consumers have the same von Neumann–Morgenstern expected utility function. Their state dependent Bernoulli utility function is

$$u_{si}(x_{1si}, x_{2si}) = \beta_s \sqrt{x_{1si}} + (1 - \beta_s)\sqrt{x_{2si}}$$

where $\beta_1 = 1$ and $\beta_2 = 0$. Thus, in state 1 the second good is worthless, while in state 2 the first good is worthless.

Suppose first that there is no information (i.e., there is no signal function distinguishing the two states). Then there is a single spot market where every consumer chooses amounts (x_{1i}, x_{2i}) of the two commodities so as to maximize the expected utility function

$$\tfrac{1}{2}\sqrt{x_{1i}} + \tfrac{1}{2}\sqrt{x_{2i}}.$$

By symmetry (but nonetheless compute the first order conditions) we see that at equilibrium every consumer will get half of each commodity and have an expected utility of $1/\sqrt{2}$. Hence, in this no-information equilibrium every consumer has managed to insure against the possibility that the good he owns turns out to be worthless.

Suppose that instead we have a perfectly informative signal function revealing the state prior to the opening of the spot market. Then the spot market equilibrium will be different under the two states. What happens now is that in each state one good is known to be worthless and, therefore, there is no possibility of trade in the spot market: Each consumer consumes his endowment, receiving a utility of 1 in one state and of 0 in the other. Ex ante this means that under perfect information every consumer has an expected utility of $\frac{1}{2} < 1/\sqrt{2}$. Thus, we see that the availability of a more informative signal function makes everybody worse off. The reason is that the availability of information destroys insurance opportunities [a possibility first pointed out by Hirshleifer (1973)].[44] ■

Asymmetric Information

Suppose now that information is not symmetric; that is, the signal functions $\sigma_i(\cdot)$ are private and not necessarily the same across consumers. How to proceed then? A first thought is to proceed exactly as before. When s occurs every consumer observes $\sigma_i(s)$ and uses his signal function $\sigma_i(\cdot)$ to update probabilities and utility functions. This defines a spot economy to which we can associate in the usual way a spot market clearing price written as $p(\sigma_1(s), \ldots, \sigma_I(s))$. Note that the price $p(\sigma_1, \ldots, \sigma_S)$ depends on *all* the individual signals: One says that the price *aggregates* the information of the market participants. In particular, the *price function* $p(s) = p(\sigma_1(s), \ldots, \sigma_I(s))$ need not be measurable with respect to the individual signal functions $\sigma_i(\cdot)$; that is, it may be that two states $s, s' \in S$ are *not* distinguishable by consumer i (i.e., $\sigma_i(s) = \sigma_I(s')$), but *are* distinguished by the market [i.e., $p(\sigma_1(s), \ldots, \sigma_I(s)) \neq p(\sigma_1(s'), \ldots, \sigma_I(s'))$]. This raises an important difficulty that we discuss by means of Example 19.H.2.

44. Suppose that our period is period 1 and that previous to it there is a period 0 in which forward trade could conceivably take place. Under the no-information scenario there can be no contingent trade at $t = 0$ since the two states cannot be distinguished at $t = 1$. The model considered, therefore, is as complete as it can be (hence, the equilibrium is Pareto optimal relative to the no-information structure). This is not so for the model with perfect information. There is then no informational impediment to the creation of a complete set of contingent markets at $t = 0$. With them the possibility of insurance would be restored. In general, if markets are complete [relative to the information signal function $\sigma(\cdot)$] then the equilibrium is a Pareto optimum [relative to $\sigma(\cdot)$] and, therefore, if information improves (*and* the corresponding additional markets are created) some traders may gain and some may lose (i.e., there may be distribution effects) but overall the new vector of ex ante expected utilities is at the frontier of an expanded utility possibility set. We can conclude, therefore, that if markets are always complete relative to the information signal (i.e., a forward market contingent on every signal takes place at $t = 0$), then not everyone can end up worse off if the information signal function improves.

Example 19.H.2: The economy has two goods and two consumers. The two consumers have identical utility functions $u_i(x_{1i}, x_{2i}) = \beta \ln x_{1i} + x_{2i}$. The parameter β is the same for the two consumers and it is uncertain. It can take the values $\beta = 1$ and $\beta = 2$ with equal probability. (Hence, we can think that there are two equally probable states: One yields $\beta = 1$ and the other $\beta = 2$.) The two consumers have deterministic endowments of one unit of the first good (because of quasilinearity with respect to the second good we do not need to specify endowments of the latter).

The first consumer has an informative signal $\sigma_1(\beta) = \beta$ that allows him to distinguish the two possible values of β. The second consumer is not informed: his signal function has $\sigma_2(\beta) = k$ for some constant k.

After nature has determined the value of β and the information $\sigma_1(\beta)$, $\sigma_2(\beta)$ has been transmitted to the two consumers, a spot market takes place (as usual, the price of the numeraire commodity is fixed to be 1). Since the first consumer knows β, his demand, given the price $p \in \mathbb{R}$ of the first good, is $x_{11}(p; \beta) = \beta/p$. The second, uninformed, consumer will equalize his expected marginal utility to the price. Hence, his demand function, which does not depend on β, is

$$x_{12}(p) = \frac{1}{p}\left[\frac{1}{2}1 + \frac{1}{2}2\right] = \frac{3}{2p}. \tag{19.H.1}$$

Solving the market equilibrium equation $x_{11}(p; \beta) + x_{12}(p) = 2$ we get the equilibrium price function

$$p(\beta) = \tfrac{1}{4}(3 + 2\beta).$$

Note now that $p(1) \neq p(2)$. This means *that the price reveals the informed consumer's information.* If so, then it is logical to suppose that the uninformed consumer will try to use the observed market price to infer the unobserved value of β. There is really no good reason to prevent him from doing so. But, once he does, his demand will no longer be given by (19.H.1), and the price function $p(\beta)$ specified above will no longer clear markets for every possible value of β. This is the difficulty we wanted to illustrate. It suggests that what is needed is an equilibrium notion embodying a consistency requirement between the information revealed by prices and the information used by consumers. ■

We have argued in Example 19.H.2 that it makes sense to require that the information revealed by prices be taken into account by the consumers in making their consumption plans in the different spot markets. Suppose, therefore, that $p(s) = p(\sigma_1(s), \ldots, \sigma_I(s))$ is an arbitrary price function. Interpret it as a specification of the prices expected to hold by the consumers at the different states. We could now view this price function as a public signal function and let any consumer use it in combination with his private signal. That is, when state s occurs, consumer i knows that the event $E_{p(s),\sigma_i(s)} = \{s' : p(s') = p(s) \text{ and } \sigma_i(s') = \sigma_i(s)\}$ has occurred and updates his probability estimates of any state $s' \in E_{p(s),\sigma_i(s)}$ to

$$\pi_{s'i} \mid p(s), \sigma_i(s) = \frac{\pi_{s'i}}{\sum_{\{s'' : s'' \in E_{p(s),\sigma_i(s)}\}} \pi_{s''i}}.$$

If, for the updated utility functions, the price $p(s)$ clears the spot market for every s, then we say that the price function $p(\cdot)$ is a *rational expectations equilibrium price function.*[45,46] This is expressed formally in Definition 19.H.2.

Definition 19.H.2: The price function $p(\cdot)$ is a *rational expectations equilibrium price function* if, for every s, $p(s)$ clears the spot market when every consumer i knows that $s \in E_{p(s),\sigma_i(s)}$ and, therefore, evaluates commodity bundles $x_i \in \mathbb{R}^2$ according to the updated utility function

$$\sum_{s'} (\pi_{s'i} \mid p(s), \sigma_i(s)) u_{s'i}(x_i).$$

We saw in Example 19.H.2 a situation in which all privately observed information is revealed by the spot market price. This suggests the following approach to the determination of a rational expectations equilibrium price function. Imagine (this is merely a hypothetical experiment) that all the individual signal functions are known to all consumers and that for every state the vector of signal values $(\sigma_1(s), \dots, \sigma_I(s))$ is made public and is, therefore, usable by all consumers to update probabilities and utilities. The market-clearing price function $\hat{p}(s) = \hat{p}(\sigma_1(s), \dots, \sigma_I(s))$ thus generated is called the *pooled information equilibrium price function.* If the values of $\hat{p}(\cdot)$ distinguish all possible values of $(\sigma_1, \dots, \sigma_I)$, that is, if $\hat{p}(s) \neq \hat{p}(s')$ whenever $\sigma_i(s) \neq \sigma_i(s')$ for some s, s', and i, then we say the price function $\hat{p}(\cdot)$ is *fully revealing.* In other words, the price function is fully revealing if it distinguishes the occurrence of any two states that can be distinguished by some consumer.

We argue now that if the pooled information equilibrium price function $\hat{p}(\cdot)$ is fully revealing, then it must be a rational expectations equilibrium price function. For any s, $\hat{p}(s)$ is determined under the assumption that every i knows that $s \in \{s' \colon \sigma_k(s') = \sigma_k(s) \text{ for all } k\}$. Because the pooled information equilibrium price function $\hat{p}(\cdot)$ is fully revealing, it follows that $\{s' \colon \sigma_k(s') = \sigma_k(s) \text{ for all } k\} = \{s' \colon \hat{p}(s') = \hat{p}(s)\}$. Hence, for any s, $\hat{p}(s)$ is a market-clearing price when every i knows that $s \in E_{\hat{p}(s),\sigma_i(s)}$. We conclude that $\hat{p}(\cdot)$ is a rational expectations equilibrium price function. In other words: If a pooled information equilibrium price function is fully revealing, then the pooled information used by consumers need not be obtained by violating any privacy constraint but can simply be derived from the public price signals.

Example 19.H.2 continued: If both consumers are fully informed, we have the demand functions $x_{11}(p) = x_{12}(p) = \beta/p$. Thus, in this case, the pooled information equilibrium price function is $\hat{p}(\beta) = \beta$. This pooled information equilibrium price function is fully revealing and therefore a rational expectations equilibrium. ■

45. For the concept of rational expectations equilibrium (including additional references), see Green (1973), Grossman (1977) and (1981), Lucas (1972), and Allen (1986).

46. In this section we concentrate on issues relating to information transmission more than on matters concerning spanning or completeness. But note that, as we pointed out in the case of symmetric information (in footnote 44), there is no conceptual difficulty in imagining that previous to the period $t = 1$ under consideration there has been contingent trade for the delivery at $t = 1$ of amounts of physical good conditional on the values of the public signals at $t = 1$ (we call the overall situation complete if such contingent markets exist for every possible value of the public signals). Observe that because the spot price constitutes a public signal, a possible instrument of contingent trade is an asset with returns conditional on the realized value of the spot market price at $t = 1$; options are instances of such assets.

In Example 19.H.3 the pooled information equilibrium price function is not fully revealing and fails to be a rational expectations equilibrium. In fact, in the example no rational expectations equilibrium price function exists.

Example 19.H.3: [Kreps (1977)] There are two commodities and two consumers with utility functions $u_1(x_{11}, x_{21}) = \beta \ln x_{11} + x_{21}$ and $u_2(x_{12}, x_{22}) = (3 - \beta) \ln x_{12} + x_{22}$. As in Example 19.H.2, there are two states yielding values $\beta = 1,2$ with equal probability. Consumer 1 is completely informed [i.e., $\sigma_1(1) \neq \sigma_1(2)$] while consumer 2 is uninformed [i.e., $\sigma_2(\beta) =$ constant]. In the two states there is a total endowment of the first good of 3 units.

Given a rational expectations equilibrium price function, which we can write as $p(\beta)$, we have two possibilities. Either $p(1) \neq p(2)$, so that the information is revealed and as a consequence $p(\beta)$ coincides with the pooled information equilibrium price function $\hat{p}(\beta)$, or $p(1) = p(2)$ so that the information is not revealed.

The first possibility cannot arise because, if the information is pooled, then for the values $\beta = 1$ and $\beta = 2$ the utility functions are the same in the two states (except that the utility functions of consumers 1 and 2 are switched) and so the spot equilibrium price is also the same in the two states. In fact, $p = 1$ is the price that clears the market in the two states.

But the second possibility cannot arise either. With a constant, nonrevealing, price function, the uninformed consumer has a demand that is independent of the state, whereas the demand of the first (informed) consumer is state dependent. Hence, the same price cannot clear the market in both states.

In summary: If we assume that information is transmitted at equilibrium, then in fact it is not. And if we assume that it is not transmitted, then it is. As a result, we can only conclude that no rational expectation equilibrium price function exists. ■

As we have seen, the concept of a fully revealing equilibrium provides a useful tool for the study of markets with asymmetric information. In applications it is more common that we encounter a slightly weaker and more natural version of the full revelation idea. In effect, in order for the pooled information equilibrium price function $\hat{p}(\cdot)$ to be a rational expectations equilibrium price function, we do not need that, for every s, $\hat{p}(\cdot)$ reveals precisely the vector of signals $(\sigma_1(s), \dots, \sigma_I(s))$; it is enough that it reveals a *sufficient statistic* for this vector [or a statistic that is sufficient for every consumer i in conjunction with the private signal $\sigma_i(\cdot)$]. More generally, all we need is that for every possible state s the expressed demand of every consumer i at the price $\hat{p}(s)$ is the same whether the consumer knows the pooled information signed functions $(\sigma_1(\cdot), \dots, \sigma_I(\cdot))$, and receives the signal vector $(\sigma_1(s), \dots, \sigma_I(s))$, or instead knows only the signal function $\hat{p}(\cdot)$ [or only $\hat{p}(\cdot)$ and $\sigma_i(\cdot)$].

Example 19.H.4: The basic economy is as in Example 19.H.2. Now, however, the signal of each consumer $i = 1, 2$ is $\sigma_i = \beta^2 + \varepsilon_i$. The ε_i, for $i = 1$ and 2, are noise variables independently distributed and taking the values $\varepsilon_i = -2, -1, 0, 1, 2$ with equal probability.[47]

47. All of this could be expressed in terms of underlying states $s = (\beta, \varepsilon_1, \varepsilon_2)$. We would need $2 \times 5 \times 5 = 50$ of them.

Suppose that information is pooled. Then:

(i) If $\sigma_i = 4$, 5, or 6 for either $i = 1$ or $i = 2$, we know that $\beta = 2$ with probability 1 and therefore $\hat{p}(\sigma_1, \sigma_2) = \beta/2 = 1$.
(ii) If $\sigma_i = -1$, 0, or 1 for either $i = 1$ or $i = 2$, we know that $\beta = 1$ with probability 1 and therefore $\hat{p}(\sigma_1, \sigma_2) = \beta/2 = \frac{1}{2}$.
(iii) In the remaining cases, $\sigma_i = 2$ or 3 for both $i = 1$ and $i = 2$, the updated probabilities on the two values of β remain $\frac{1}{2}$: No useful information is transmitted. Hence the clearing price is $\hat{p}(\sigma_1, \sigma_2) = \frac{3}{4}$ (Exercise 19.H.4).

The price function $\hat{p}(\cdot)$ defined by (i) to (iii) is not fully revealing: Given the value of $\hat{p}(\cdot)$ we cannot deduce from it the specific values of σ_1 and σ_2.[48] Yet, the price function $\hat{p}(\cdot)$ is sufficient to distinguish among cases (i) to (iii), and therefore the knowledge of the single function $\hat{p}(\cdot)$ can replace for every consumer the knowledge of the vector of functions $(\sigma_1(\cdot), \sigma_2(\cdot))$. We can say, therefore, that $\hat{p}(\cdot)$ is a sufficient statistic for the signals, and conclude that $\hat{p}(\cdot)$ is a rational expectations price function. ∎

In Example 19.H.5 price function is not a sufficient statistic but it becomes so when combined with the private signal of any consumer.

Example 19.H.5: The basic economy and the signals are as in Example 19.H.4, but with three differences. First, there are I consumers. Second, the noise terms ε_i are now payoff relevant: in particular, $u_i(x_i) = (\beta + \varepsilon_i)\ln x_{1i} + x_{2i}$. Third, half the consumers have their noise uniformly distributed in the interval $[-\frac{2}{3}, \frac{2}{3}]$, whereas the other half is perfectly informed about β, that is, $\varepsilon_i = 0$.

The pooled information equilibrium price function is $\hat{p}(\beta, \varepsilon_1, \ldots, \varepsilon_I) = \beta + (1/I)(\sum_i \varepsilon_i)$. Note that this price function reveals β [if $\beta = 1$ then $\hat{p}(\cdot) < 1.5$ with probability 1; if $\beta = 2$ then $\hat{p}(\cdot) > 1.5$ with probability 1] but not the individual values of ε_i. However, a consumer i that knows β *and* $\sigma_i = \beta^2 + \varepsilon_i$ also knows ε_i, and therefore, at any given price, expresses a demand that coincides with the pooled information demand. We conclude that the pooled information equilibrium is a rational expectation equilibrium. It is important to observe that, in contrast with Example 19.H.4, the equilibrium price function alone does not now provide a sufficient statistic. At the rational expectations equilibrium the individual utility maximization problems of half the consumers make essential use of the private signal functions. ∎

Example 19.H.5 allows us to address another issue. Suppose that to our general model we add the feature that acquiring the information signal function $\sigma_i(\cdot)$ costs some small amount of money $\delta > 0$. Suppose also that I is large, so that, plausibly, the pooled information price function $\hat{p}(\cdot)$ is not very sensitive to any single consumer i failing to acquire his signal function $\sigma_i(\cdot)$. Then we have the following paradox [see Grossman and Stiglitz (1976)]: If the price function $\hat{p}(\cdot)$ is fully revealing (or is a sufficient statistic by itself), why will any consumer i pay δ for the signal function $\sigma_i(\cdot)$? Any one consumer would rather not do so and attempt to free ride on the information transmitted by the price system. But if everybody proceeds in this manner, then the price function cannot be fully revealing (there is nothing to reveal)! Example 19.H.5 suggests one way out of this paradox: it can be verified that in the example there is a sufficiently small $\delta > 0$, not dependent on the number I, such that at any fixed price p, a consumer i with nontrivial ε_i, even if he already knows β, has an incentive to pay δ for the improvement of information provided by his private signal (Exercise 19.H.5). ∎

48. Recall that "fully revealing" does not mean that we get to know the value of β, only that we get to know the value of the signals.

Up to now, prices may have conveyed information about an exogenously ocurring state. But in a world of asymmetric information, prices could also convey information on the consumers' endogenously chosen actions, and those actions could matter for individual utilities. For example, the final utility of a consumer may depend not only of the number of units consumed and on exogenous states but also on some other statistic depending on other consumers' actions. If we regard this statistic as a "state" then it is as if states were determined endogenously. To illustrate this point we consider another example—the market for used cars (also referred to as the "lemons market"). With this, we connect with the analysis of adverse selection in Section 13.B and bring this section to a close.

Example 19.H.6: *The Lemons Market.* Suppose that consumers fall into two types: potential buyers and potential sellers (of, say, used cars). There are many consumers and twice as many potential sellers as potential buyers. Potential sellers have one unit of the good and potential buyers buy either one unit or none. The peculiarity of this market is that commodities are of two kinds: good and bad. Half the potential sellers have a good product and half have a bad one. The quality, known to the potential sellers, is unrecognizable to the buyers at the moment of trade. A good commodity is worth 1 to the potential seller and 4 to the buyer. A bad commodity is worth nothing to every consumer.

We could call the *state of the market* the fraction $\alpha \in [0, 1]$ of the commodities supplied that are of good quality. If the state of the market is α, then a buyer paying p gets expected utility $4\alpha - p$. The problem is that the state of the market depends on the price (thus, as before, the price provides information about the utility derived from consuming one unit of the good). Indeed, for any $p > 0$, every unit of bad commodity will be supplied to the market. But for $p < 1$ no unit of the good commodity will be supplied, whereas for $p > 1$ every unit of the good commodity will be supplied. Therefore, we must have $\alpha = 0$ for $p < 1$, $\alpha \in [0, \frac{1}{2}]$ for $p = 1$, and $\alpha = \frac{1}{2}$ for $p > 1$. If a pair (α, p) satisfies these inequalities then we say that the pair (α, p) is admissible. Note that the inequalities can be equivalently expressed as $p \leq 1$ for $\alpha = 0$, $p = 1$ for $\alpha \in (0, \frac{1}{2})$ and $p \geq 1$ for $\alpha = \frac{1}{2}$.

A potential buyer will deduce $\alpha = 0$ if he observes $p < 1$, and $\alpha = \frac{1}{2}$ if $p > 1$. Potential buyers may or may not express a demand depending on this inference. It is natural to say that if at the admissible pair (α, p) the total demand is not larger than the total supply then we are at a *rational expectations* equilibrium. In fact, in our case any admissible pair (α, p) turns out to be a rational expectations equilibrium. Note, however, that for some (α, p) the supply is larger than the demand (e.g. at $\alpha = \frac{1}{2}$, $p = 3$ no demand is expressed).[49] ■

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49. For simplicity we have chosen an example where it is always the case that the forthcoming demand is not larger than the forthcoming supply. It is because of this that all admissible pairs (α, p) may appear as equilibria. More generally, some of these pairs may be eliminated because forthcoming demand is larger than supply. It is also worth observing that it would not be legitimate to impose as an equilibrium condition that demand be larger than or equal to supply. Suppose, for example, that $\alpha = \frac{1}{2}$ and $p = 1.5$. Then total supply is 2 while total demand is 1. The usual argument (underlying the tâtonnement dynamics) for downward pressure on demand is that some frustrated seller would attempt to sell to some buyer at a price $1.5 - \varepsilon$. But, in the current context, how is the buyer to know that the commodity being offered is not of bad quality? Note that things would look different if it were the buyer who approached a random seller (this is the reason why the requirement that the demand be no larger than the supply is a natural equilibrium condition.) In contrast, with symmetric information it is of no consequence who approaches whom when a buyer and a seller meet. The lesson to be learned from this discussion is that with asymmetric information the particular disequilibrium story matters a lot. To push the analysis forward it is therefore appropriate to refer back to Chapter 13 where, in a more limited, partial equilibrium setting, we have already studied asymmetric information problems with the help of a methodology well suited to the consideration of this type of microstructure.

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EXERCISES

19.C.1[A] There are S states. A consumer has, in every state s, a Bernoulli utility function $u_s(x_s)$, where $x_s \in \mathbb{R}^L_+$. Suppose that, for every s, $u_s(\cdot)$ is concave. Show that the expected utility function $U(x_1, \ldots, x_S) = \sum_s \pi_s u_s(x_s)$ defined on $\mathbb{R}^{LS}_+$ is concave.

19.C.2[A] For the model described in Example 19.C.2, show that the marginal rates of substitution along the Pareto set are as drawn in Figure 19.C.2; that is, at any point of the Pareto set the marginal rate of substitution is smaller than the ratio of probabilities.

19.C.3[A] Consider an Arrow–Debreu equilibrium of the economy described in Section 19.C. Suppose that $L = 1$ and that preferences of every consumer i admit an expected utility representation with continuous, strictly concave, and strictly increasing Bernoulli utility functions (identical across states). For every state s denote by p_s, π_{si}, and x_{si} the equilibrium price for the s-contingent commodity, the subjective probability of consumer i for state s, and the consumption of consumer i in state s, respectively.

Denote by $\bar{p} = \sum_s p_s$ the price of uncontingent delivery of one unit of consumption.

Show that $\sum_s (\pi_{si}\bar{p} - p_s)x_{si} \geq 0$ for every i. [*Hint*: Use a revealed preference argument.] Interpret.

19.C.4[B] There are a single consumption good, two states, and two consumers. Note that this allows the use of Edgeworth boxes.

Utility functions are of the expected utility form. Bernoulli utility functions are identical across states. That is,

$$U_1(x_{11}, x_{21}) = \pi_{11}u_1(x_{11}) + \pi_{21}u_1(x_{21})$$

and

$$U_2(x_{12}, x_{22}) = \pi_{12}u_2(x_{12}) + \pi_{22}u_2(x_{22}),$$

where x_{si} is the amount of s-contingent good consumed by consumer i and π_{si} is the subjective probability of consumer i for state s. We assume that every $u_i(\cdot)$ is strictly concave and differentiable.

The total initial endowments of the two contingent commodities are $\bar{\omega} = (\bar{\omega}_1, \bar{\omega}_2) \gg 0$. We assume that every consumer gets half of the random variable $\bar{\omega}$, that is, $(\omega_{11}, \omega_{21}) = \frac{1}{2}\bar{\omega}$ and $(\omega_{12}, \omega_{22}) = \frac{1}{2}\bar{\omega}$.

(a) Suppose that consumer 1 is risk neutral, consumer 2 is not, and both consumers have the same subjective probabilities. Show that at an interior Arrow–Debreu equilibrium consumer 2 insures completely.

(b) Suppose now that consumer 1 is risk neutral, consumer 2 is not, and the subjective probabilities of the two consumers are not the same. Show then that at an interior (Arrow–Debreu) equilibrium consumer 2 will not insure completely. Which is the direction of the bias in terms of the differences in subjective probabilities? Argue also that consumer 1 (the risk-neutral agent) will not gain from trade.

19.C.5[A] Consider an economy such as the one introduced in Section 19.C but with only one commodity in each state. There is a number I of risk-averse consumers. Preferences admit an expected utility representation. Suppose that the Bernoulli utility functions of a consumer for the good are identical across states and that subjective probabilities are the same across individuals. Individual endowments vary from state to state. However, we assume that the total endowment is nonstochastic, that is, uniform across states (if, say, I is large and the realizations of individual endowments are identically and independently distributed, then the total endowment per capita is almost nonstochastic).

Set up the Arrow–Debreu trading problem. Show that the allocation in which every individual's consumption in every state is the average across states of his endowments is an equilibrium allocation.

19.D.1[A] Consider the model of sequential trade of Section 19.D. The only difference is that we now assume that, for every s, the s-contingent commodity pays 1 dollar (rather than one unit of physical good 1) if state s occurs (and nothing otherwise). Write down the budget constraints corresponding to this model and discuss which price normalizations are possible.

19.D.2[A] Show that in Example 19.D.1 the contingent trades of the two consumers are as claimed in the discussion of the example.

19.D.3[A] Formulate a model similar to the two-period model of Section 19.D with the difference that consumption also takes place at period $t = 0$. Show that the result of Proposition 19.D.1 remains valid.

19.D.4[B] Consider a three-period economy, $t = 0, 1, 2$, in which at $t = 0$ the economy splits into two branches and at $t = 1$ every branch splits again into two. There are H physical commodities and consumption can take place at the three dates.

(a) Describe the Arrow–Debreu equilibrium problem for this economy.

(b) Describe the Radner equilibrium problem. Suppose that at $t = 0$ and $t = 1$ there are contingent markets for the delivery of one unit of the first physical good at the following date.

(c) Argue that the conclusion of Proposition 19.D.1 remains valid.

19.E.1[B] Consider an asset trading model such as that considered in Section 19.E. The only difference is that consumption is also possible at date $t = 0$. Assume for simplicity that the Bernoulli utility functions on consumption are state independent and additively separable across time; that is, $u_i(x_{0i}, x_{1i}) = u_{0i}(x_{0i}) + u_{1i}(x_{1i})$, where $x_{0i}, x_{1i} \in \mathbb{R}^L$.

(a) Argue along the lines of the second proof of Proposition 19.E.1 that the conclusion of Proposition 19.E.1 remains valid.

(b) Suppose now that there is a single physical good in every period. Express the multipliers μ_s in terms of marginal utilities of consumption.

19.E.2[A] Show that if a primary asset with return vector $r \in \mathbb{R}^S$ *separates* states, that is, if $r_s \neq r_{s'}$ whenever $s \neq s'$, then it is possible to create a complete asset structure by using only options on this primary asset. You can assume that $r_s > 0$ for every s.

19.E.3[A] Complete part (i) of the proof of Proposition 19.E.2 in the manner requested in the text.

19.E.4[B] In text.

19.E.5[B] Complete the verification that at the prices specified in Example 19.E.7 the set of consumptions achievable through sequential trade is the same as the set of consumptions achievable through ex ante trade of the four Arrow–Debreu commodities.

19.E.6[A] There are two dates: At date 1 there are three states; at date 0 there is trade in assets. There are two basic assets whose return vectors in current dollars are

$$r_1 = (64, 16, 4) \qquad \text{and} \qquad r_2 = (0, 0, 1).$$

The market prices of these assets are $q_1 = 32$ and $q_2 = 1$, respectively. In the following you are asked to price by arbitrage a variety of derived assets.

(a) Suppose that one unit of a derived asset is described as "One unit of this asset confers the right to buy one unit of asset 1 at 75% of its spot value in period 1 (after the state of the world occurs)." Write the return vector of this asset and price it.

(b) The situation is the same as in (a) except that the asset is modified to read "One unit of this asset confers the right to buy one unit of asset 1 at 75% of its spot value in period 1 (after the state of the world occurs) provided the spot value is at least 10."

(c) Suppose that the asset is as in (b) except that "at least 10" is replaced by "at least 19." Write down the return vector and argue that this asset cannot be priced by arbitrage with the available primary assets.

(d) How would the analysis in (c) differ if we had in addition a riskless asset with a price equal to 1? (You do not need to compute the price explicitly.)

(e) Suppose that now the asset is further complicated to read "One unit of this asset confers, at the choosing of the holder, either 1 dollar in period 1 or the right to buy one unit of asset

1 at 75% of its spot value in period 1 (after the state of the world occurs)." Write the return vector of this asset and price it.

(f) The situation is the same as in **(e)** except that the asset is modified to read "One unit of this asset confers, at the choosing of the holder, either 1 dollar in period 1 or the right to buy one unit of asset 1 at 75% of its spot value in period 1 (after the state of the world occurs) provided this value is at least 10."

19.F.1[B] Consider the sunspot model of Example 19.F.1. Argue that, under the standard conditions on preferences, there is a sunspot-free equilibrium, whatever the asset structure.

19.F.2[A] Consider (for the case $L = 1$) the utility function $U_i^*(\cdot)$ on asset portfolios defined in Section 19.F. Give sufficient conditions for $U_i^*(\cdot)$ to be continuous and concave. Show also that if returns are strictly positive then $U_i^*(\cdot)$ is strictly increasing.

19.F.3[C] The aim of this exercise is to show that it is possible in an incomplete market situation for the number of assets to increase while at the same time *everybody* becomes worse off at the new Radner equilibrium. We do this in steps.

(a) Construct a two-consumer economy with two equally likely states and in which the distributional effects of trade are so biased that the sum of the utilities at the equilibrium with complete markets is smaller than the sum of the utilities at the equilibrium with incomplete markets.

(b) Now construct an economy that has four equally likely states and in which in the first two states the economy is as in **(a)**, while in the other two states it is also as in **(a)** *except* that the roles of the two consumers are reversed.

(c) Show then that the asset structure in which there is a single asset allowing a transfer of wealth from the first two states to the other two yields an equilibrium that is better for every consumer than the equilibrium obtained if we add two new assets, one allowing a transfer of wealth between states 1 and 2 and the other doing the same between states 3 and 4.

19.F.4[C] Exhibit an example in which with unlimited short sales a Radner equilibrium may not exist. This example requires returns denominated in more than one commodity. [*Hint*: Recall the no-arbitrage necessary condition.]

19.F.5[B] Consider an economy with a single period, a single consumption good, and a single input (labor). All workers are identical ex ante. Each of them has a probability $\frac{1}{2}$ of being able to work (in which case the production is k units of output and work causes no disutility). With probability $\frac{1}{2}$, the worker is unable to work (is disabled). The utility of an amount c of consumption if able or disabled is $U_a(c)$, $U_d(c)$, respectively. Assume that the probability of disability is independent across workers and that there are sufficient workers that society operates on the expected value production possibility set.

(a) If there is a full set of Arrow–Debreu markets that are open before the disability is known, what is the equilibrium allocation of resources? (You may need to allow for the possibility of infinitely many states.)

(b) Assume now that it is impossible for others to observe whether an individual really is disabled or just claims to be so and stops working. Assume that insurance markets continue to exist as a competitive industry. Assume also that the condition "$U_a'(c_a) = U_d'(c_d)$ implies $c_a > c_d$" is satisfied and that each individual purchases insurance only from a single company. Show that the competitive equilibrium (you should also define what this means) is the same as the one derived in **(a)**.

(c) Continue to assume that it is impossible for others to observe whether a claimed disability is real. But assume now that $U_a'(c_a) = U_d'(c_d)$ implies $c_a < c_d$. Show then that the equilibrium described in the answer to **(b)** is not reachable. Continuing to assume that each worker can purchase insurance from no more than one insurance company, determine the competitive equilibrium. Is it optimal relative to the allocations the government can achieve, supposing that the government also cannot observe disability?

19.G.1^A Justify expression (19.G.1). That is, suppose that every possible production plan of the firm can be spanned and that prices (in the asset and spot markets) are given. Now introduce the firm as a new asset. Show that any consumption plan of any consumer can be reached without purchasing any shares of the firm.

19.G.2^A Suppose that, in a two-consumer economy with $L = 1$, initially there is a single asset (which, therefore, goes untraded; recall that we do not allow consumption at $t = 0$). Now a firm is introduced that can produce the return vector $\varepsilon a \in \mathbb{R}^S_+$. The firm is owned, with equal shares, by the two consumers. Give an example in which, no matter how small ε may be (and letting the vector $a \in \mathbb{R}^S_+$ remain fixed), the introduction of the firm as an asset tradeable at $t = 0$ has the property that at the new equilibrium one consumer is significantly better off and the other is significantly worse off.

19.G.3^A Suppose that, in an economy with $L = 1$, a firm is introduced that can produce the single return vector $a \in \mathbb{R}^S$. The ownership share of consumer i is θ_i. Suppose also that the returns a can be spanned by existing assets. Show that the consumption bundles reachable at an equilibrium are the same under the following two scenarios. In the first the shares of the firm are traded at $t = 0$. In the second the asset is not traded and for every i the vector $\theta_i a$ is added to the initial endowments of consumer i.

19.G.4^A Suppose there are two consumers and $L = 1$. Initially there are no assets. We now introduce a firm with two possible return vectors $A = \{a^1, a^2\} \subset \mathbb{R}^S_+$. Ownership shares are the same for the two consumers. The firm's shares are not traded at $t = 0$. The production choice of the firm, a^1 or a^2, is added to the endowments at $t = 1$ (half to each consumer). Show that it is possible for the two consumers not to be unanimous in their preference for a^1 or a^2.

19.H.1^A We place ourselves in the framework of the single-consumer decision problem considered in Proposition 19.H.1.

(a) Show that if $\sigma(\cdot)$ is completely revealing [i.e., $\sigma(s) \neq \sigma(s')$ whenever $s \neq s'$] then $x_i^{\sigma(\cdot)}$ is ex post optimal in the sense that for every $s \in S$ the consumption allocation in state s is optimal for the spot economy that obtains at state s. Show also that this need not be true if $\sigma(\cdot)$ is not completely revealing.

(b) Show that $x_i^{\sigma(\cdot)}$ is interim optimal in the following sense: there is no allocation x_i measurable with respect to $\sigma(\cdot)$ and such that for some possible signal $\sigma(s)$ the expected utility of x_i [conditional on $\sigma(s)$] is larger than that corresponding to $x_i^{\sigma(\cdot)}$.

(c) Show that if $\sigma'(\cdot)$ is at least as informative as $\sigma(\cdot)$ then for every s the expected utility generated by $x_i^{\sigma'(\cdot)}$, conditional on $\sigma'(s)$, cannot be inferior to the expected utility generated by $x^{\sigma(\cdot)}$, conditional on $\sigma'(s)$.

(d) Show, similarly, that if $\sigma'(\cdot)$ is at least as informative as $\sigma(\cdot)$ then for every s the expected utility generated by $x_i^{\sigma'(\cdot)}$, conditional on $\sigma(s)$, cannot be inferior to the expected utility generated by $x^{\sigma(\cdot)}$, conditional on $\sigma(\cdot)$.

19.H.2^A Argue that for the validity of Proposition 19.H.1 we may allow p and w_i to depend on the state. What is required is that, as functions, they be measurable with respect to the original $\sigma(\cdot)$ and that they remain unaltered as $\sigma(\cdot)$ is replaced by $\sigma'(\cdot)$.

19.H.3^A Complete the requested steps of the proof of Proposition 19.H.1.

19.H.4[A] Complete the missing step in Example 19.H.4.

19.H.5[B] Carry out the requested verification in Example 19.H.5.

19.H.6[C] Consider the following two-consumer, two-commodity general equilibrium model. The (Bernoulli) utility functions of the two consumers are

$$u_1(x_{11}, x_{21}) = x_{11} + x_{21},$$

$$u_2(x_{12}, x_{22}) = (x_{12})^{1/2} + x_{22}.$$

Consumer 1's endowment of the second good is ω_{21}. He has no endowment of the first good. Consumer 2 has no endowments of the second good and his endowments of the first good depend on which of three equally likely states occurs. The respective levels in the three states are $\omega_{112}, \omega_{122}, \omega_{132}$.

(a) Determine the Arrow Debreu equilibrium of this economy. You can assume that the parameters are such that the equilibrium is interior.

(b) Suppose that the only possible markets are for the noncontingent delivery of the two goods. Set up the equilibrium problem. Is the equilibrium allocation a Pareto optimum?

(c) Suppose now that before any trade takes place, and before the endowments are revealed, the two consumers are told whether or not state 1 has occurred. After the revelation of this information (and before the values of the endowments are disclosed) non-contingent trade can take place. Set up the equilibrium problem as it depends on the information available.

(d) The setting is as in **(c)**; the only difference is that contingent trade (after the revelation of the information on state 1) is now permitted.

(e) Compare ex ante (i.e., before any announcement is known) the expected utilities attained by the two consumers in the equilibria of **(a)**, **(b)**, **(c)**, and **(d)**; assume that all of these equilibria are interior. When can you assert that the information available in parts **(c)** and **(d)** is socially valuable?

19.H.7[A] Suppose that there are two equally likely states. In every state there is a spot market where a consumption good (good 1) is exchanged against the numeraire, which we denote as good 2. There are two consumers. Their utilities are

	State 1	*State 2*
Consumer 1	$2 \ln x_{11} + x_{21}$	$4 \ln x_{11} + x_{21}$
Consumer 2	$4 \ln x_{12} + x_{22}$	$2 \ln x_{12} \times x_{22}$

The total endowment of the first good equals 6 in the first state and $6 + \varepsilon$ in the second state. All the endowments of this good are received by the second consumer. Assume also that the endowments of numeraire for the two consumers are sufficient for us to neglect the possibility of boundary equilibria. The price of the numeraire is fixed to be one in the two states. The prices of the non-numeraire good in the two states are denoted (p_1, p_2).

(a) Suppose that when uncertainty is resolved, both consumers know which state of the world has occurred. Determine the spot equilibrium prices $(\hat{p}_1(\varepsilon), \hat{p}_2(\varepsilon))$ in the two states (as a function of the parameter ε).

(b) Assume now that when a state occurs, consumer 2 knows the state but consumer 1 remains uninformed (i.e., he must keep thinking of the two states as equally likely). Determine, under this information setup (and assuming that prices cannot be used as signals), the spot equilibrium prices $(\bar{p}_1(\varepsilon), \bar{p}_2(\varepsilon))$ in the two states.

(c) The situation is as in **(b)**, except that now we allow consumer 1 to deduce the state of the world from prices. That is, if $p_1 \neq p_2$ then consumer 1 is actually informed, but if $p_1 = p_2$ he is not informed. A pair of spot prices $(p_1^*(\varepsilon), p_2^*(\varepsilon))$ constitute rational expectations equilibrium prices if they clear the two spot markets when consumer 1 derives information from $((p_1^*(\varepsilon), p_2^*(\varepsilon))$ in the manner just described. Suppose that $\varepsilon \neq 0$ and derive a rational expectations equilibrium pair of prices.

(d) Show that if $\varepsilon = 0$ then there is no rational expectations equilibrium pair of prices. Compare with Example 19.H.3.

CHAPTER

20

Equilibrium and Time

20.A Introduction

In this chapter, we present the basic elements of the extension of competitive equilibrium theory to an intertemporal setting. In the presentation, we try to maintain a balance between two possible approaches to the theory varying by the degree of emphasis on time.

A first approach contemplates equilibrium in time merely as the particular case of the general theory developed in the previous chapters in which commodities are indexed by time as one of the many characteristics defining them. This is a useful point of view (the display of the underlying unity of seemingly disparate phenomena is one of the prime roles of theory), and to a point we build on it. However, exclusive reliance on this approach would, in the limit, be self-defeating. It would reduce this chapter to a footnote to the preceding ones.

A second approach proceeds by stressing, rather than deemphasizing, the special structure of time. Again, we follow this line to some degree. Thus, every model discussed in this chapter accepts the open-ended infinity of time, or the fact that production takes time. Also, at the cost of some generality, we pursue our treatment under assumptions of stationarity and time separability that allow for a sharp presentation of the dynamic aspects of the theory.

Sections 20.B and 20.C are concerned with the description of, respectively, the consumption and the production sides of the economy.

Section 20.D is the heart of this chapter. It deals with the basic properties of equilibria (including definitions, existence, optimality, and computability) in the context of a single-consumer economy.

Section 20.E (which concentrates on steady states) and Section 20.F (which is general) study the dynamics of the single-consumer model.

Section 20.G considers economies with several consumers. The message of this section is that, as long as the Pareto optimality of equilibrium is guaranteed, the qualitative aspects of the positive theory of Chapter 17 extend to the more general situation and, moreover, that the properties of individual equilibria identified by the single-consumer methodology remain valid in the broader context.

Section 20.H gives an extremely succinct account of overlapping-generations economics, a model of central importance in modern macroeconomic theory. Our interest in it is twofold: on the one hand, we want to display it as yet another instance of a useful equilibrium model; on the other hand, we want to point out that it is an example that, because of the infinity of generations, does not fit the general model of Section 20.G, and one that gives rise to some new and interesting issues having to do with the optimality and the multiplicity of equilibria.

Section 20.I gathers some remarks on nonequilibrium considerations (short-run equilibrium and tâtonnement stability, learning, and so on).

For pedagogical purposes, the entire chapter deals only with the deterministic version of the theory. The unfolding of time is a line, not a tree. A full synthesis of the approaches of Chapter 19 (on uncertainty) and the current one (on time) is possible. However, we view its presentation as advanced material beyond the scope of this textbook. The account of Stokey and Lucas with Prescott (1989) constitutes an excellent introduction to the general theory.

A point of notation: in this chapter $\sum_t$ always means $\sum_{t=0}^{t=\infty}$, that is, $\lim_{T\to\infty}\sum_{t=0}^{t=T}$. When the sum does not run from $t = 0$ to $t = \infty$ the two end-points of the sum are explicitly indicated.

20.B Intertemporal Utility

In this chapter, we assume that there are infinitely many dates $t = 0, 1, \ldots,$ and that the objects of choice for consumers are *consumption streams* $c = (c_0, \ldots, c_t, \ldots)$ where $c_t \in \mathbb{R}^L_+$, $c_t \geq 0$.[1] To keep things simple, we will consider only consumption streams that are *bounded*, that is, that have $\sup_t \|c_t\| < \infty$.

Rather than proceed from the most general form of preferences over consumption streams to the more specific, we instead introduce first the very special form that we assume throughout this chapter (except for Sections 20.H and 20.I); we subsequently discuss its special properties from a general point of view.

It is customary in intertemporal economies to assume that preferences over consumption streams $c = (c_0, \ldots, c_t, \ldots)$ can be represented by a utility function $V(c)$ having the special form

$$V(c) = \sum_{t=0}^{\infty} \delta^t u(c_t) \tag{20.B.1}$$

where $\delta < 1$ is a *discount* factor and $u(\cdot)$, which is defined on $\mathbb{R}^L_+$, is strictly increasing and concave. This chapter will be no exception to this rule: Throughout it we assume that preferences over consumption streams take this form. However, we comment here, at some length, on six aspects of this utility function. As a matter of notation, given a consumption stream $c = (c_0, \ldots, c_t, \ldots)$, we let $c^T = (c_0^T, c_1^T, \ldots)$ denote the T-period "backward shift" consumption stream, that is, the stream $(c_0^T, c_1^T, \ldots)$ with $c_t^T = c_{t+T}$ for all $t \geq 0$.

(1) *Time impatience*. The requirement that future utility is discounted (i.e., that $\delta < 1$), implies *time impatience*. That is, if $c = (c_0, c_1, \ldots, c_t, \ldots)$ is a nonzero consumption stream, then the (forward-) shifted consumption stream $c' = (0, c_0, c_1, \ldots, c_{t-1}, \ldots)$ is strictly worse than c (see Exercise 20.B.1). It is an

1. We use the terms "stream," "trajectory," "program," and "path" synonymously.

assumption that is very helpful in guaranteeing that a bounded consumption stream has a finite utility value [i.e., guarantees that the sum in (20.B.1) converges], thus allowing us to compare any two such consumption streams[2] and making possible the application of the machinery of the calculus. There is a strand of opinion that views this technical convenience as the real reason for the fundamental role that the assumption of time discounting plays in economics. This skeptical view on the existence of substantive reasons[3] is excessive. An implication of time discounting is that the distant future does not matter much for current decisions, and this feature seems more realistic than its opposite.

A possible interpretation, and defense, of the discount factor δ views it as a probability of survival to the next period. Then $V(c)$ is the expected value of lifetime utility. For another interpretation, see (6) below.

(2) *Stationarity.* A more general form of the utility function would be

$$V(c) = \sum_{t=0}^{\infty} u_t(c_t). \tag{20.B.2}$$

The form (20.B.1) corresponds to the special case of (20.B.2) in which $u_t(c_t) = \delta^t u(c_t)$. This special form can be characterized in terms of *stationarity*. Consider two consumption streams $c \neq c'$ such that $c_t = c'_t$ for $t \leq T - 1$: that is, the two streams c and c' are one and the same up to period $T - 1$ and differ only after $T - 1$. Observe that the problem of choosing at $t = T$ between the current and future consumptions in c and c' is the same problem that a consumer would face at $t = 0$ in choosing between the consumption streams c^T and c'^T, the T backward shifts of c and c', respectively. Then *stationarity* requires that

$$V(c) \geq V(c') \quad \text{if and only if} \quad V(c^T) \geq V(c'^T).$$

It is a good exercise to verify that (20.B.1) satisfies the stationarity property and that the property can be violated by utility functions of the form $V(c) = \sum_t \delta_t^t u(c_t)$, that is, with a time-dependent discount factor (Exercise 20.B.2).

The property of stationarity should *not* be confused with the statement asserting that if the consumption streams c and c' coincide in the first $T - 1$ periods and a consumer chooses one of these streams at $t = 0$, then she will not change her mind at T. This "property" is tautologically true: at both dates we are comparing $V(c)$ and $V(c')$.[4] The stationarity experiment compares $V(c)$ and $V(c')$ at $t = 0$, but at period T it compares the utility values of the future streams shifted to $t = 0$, that is, $V(c^T)$ and $V(c'^T)$. Thus, stationarity says that in the context of the form (20.B.2), the preferences over the future are independent of the *age* of the decision maker.

Time stationarity is not essential to the analysis of this chapter (except for Sections 20.E and 20.F on dynamics), but it saves substantially on the use of subindices.

2. Hence, the completeness of the preference relation on consumption streams is guaranteed.

3. Ramsey (1928) called the assumption a "weakness of the imagination."

4. This property is often called *time consistency*. Time inconsistency is possible if tastes change through time (recall the example of Ulysses and the Sirens in Section 1.B!), but, as we have just argued, it must necessarily hold if the preference ordering over consumption streams $(c_0, \ldots, c_t, \ldots)$ does not change as time passes. In line with the entire treatment of Part IV, we maintain the assumption of unchanging tastes throughout the chapter.

(3) *Additive separability.* Two implications of the additive form of the utility function are that at any date T we have, first, that the induced ordering on consumption streams that begin at $T + 1$ is independent of the consumption stream followed from 0 to T, and, second, that the ordering on consumption streams from 0 to T is independent of whatever (fixed) consumption expectation we may have from $T + 1$ onward (see Exercise 20.B.3). In turn, these two separability properties imply additivity; that is, if the preference ordering over consumption streams satisfies these separability properties, then it can be represented by a utility function of the form $V(c) = \sum_t u_t(c_t)$ [this is not easy to prove, see Blackorby, Primont and Russell (1978)].

How restrictive is the assumption of additive separability? We can make two arguments in its favor: the first is technical convenience; the second is a vague sense that what happens far in the future or in the past should be irrelevant to the relative welfare appreciation of current consumption alternatives. Against it we have obvious counter-examples: Past consumption creates habits and addictions, the appreciation of a particularly wonderful dish may depend on how many times it has been consumed in the last week, and so on. There is, however, a very natural way to accommodate these phenomena within an additively separable framework. We could, for example, allow for the form $V(c) = \sum_t u_t(c_{t-1}, c_t)$. Here the utility at period t depends not only on consumption at date t but also on consumption at date $t - 1$ (or, more generally, on consumption at several past dates). We can formulate this in a slightly different way. Define a vector z_t of "habit" variables and a *household production technology* that uses an input vector c_{t-1} at $t - 1$ to jointly produce an output vector c_{t-1} of consumption goods at $t - 1$ and a vector $z_t = c_{t-1}$ of "habit" variables at t. Then, formally, u_t depends only on time t variables and total utility is $\sum_t u_t(z_t, c_t)$. In summary: additive separability is less restrictive than it appears if we allow for household production and a suitable number (typically larger than 1) of current variables.

(4) *Length of period.* The plausibility of the separability assumption, which makes the enjoyment of current consumption independent of the consumption in other periods, depends on the length of the period. Because even the most perishable consumption goods have elements of durability in them (in the form, for example, of a flow of "services" after the act of consumption), the assumption is quite strained if the length of the elementary period is very short. What determines the length of the period? To the extent that our model is geared to competitive theory, this period is institutionally determined: it should be an interval of time for which prices can be taken as constant. On a related point, note that the value of δ also depends, implicitly, on the length of the period. The shorter the period, the closer δ should be to 1.

(5) *Recursive utility.* With the form (20.B.1) for the utility function, we have $V(c) = u(c_0) + \delta V(c^1)$ for any consumption stream $c = (c_0, c_1, \ldots, c_t, \ldots)$. If we think of $u = u(c_0)$ as current utility and of $V = V(c^1)$ as future utility, we see that the marginal rate of substitution of current for future utility equals δ and is therefore independent of the levels of current and future utility. The *recursive utility model* [due to Koopmans (1960)] is a useful generalization of (20.B.1) that combines two features: it allows this rate to be variable but, as in the additively separable case, it has the property that the ordering of future consumption streams is independent of the consumption stream followed in the past.

The recursive model goes as follows. Denote current utility by $u \geq 0$ and future utility by $V \geq 0$. Then we are given a current utility function $u(c_t)$ and an *aggregator* function $G(u, V)$ that combines current and future utility into overall utility. For example, in the separable additive case we have $G(u, V) = u + \delta V$. More generally we could also have, for example, $G(u, V) = u^\alpha + \delta V^\alpha$, $0 < \alpha \leq 1$. In this case, the indifference curves in the (u, V) plane are not straight lines. The utility of a consumption stream $c = (c_0, \ldots, c_t, \ldots)$ could then be computed recursively from

$$V(c) = G(u(c_0), V(c^1)) = G(u(c_0), G(u(c_1), V(c^2))) = \cdots. \tag{20.B.3}$$

For (20.B.3) to make sense we must be able to argue that the influence of $V(c^T)$ on $V(c)$ will become negligible as $T \to \infty$ [so that $V(c)$ can be approximately determined by taking a large T and letting $V(c^T)$ have an arbitrary value]. This amounts to an assumption of time impatience. In applications, it will typically not be necessary to compute $V(c)$ explicitly. See Exercise 20.B.4 for more on recursive utility.

(6) *Altruism*. The expression $V(c) = u(c_0) + \delta V(c^1)$ suggests a multigeneration interpretation of the single-consumer problem (20.B.1). Indeed, if generations live a single period and we think of generation 0 as enjoying her consumption according to $u(c_0)$, but caring also about the *utility* $V(c^1)$ of the next generation according to $\delta V(c^1)$, then $V(c) = u(c_0) + \delta V(c^1)$ is her overall utility. If every generation is similarly altruistic, then we conclude, by recursive substitution, that the objective function of generation 0 is precisely (20.B.1). The entire "dynasty" behaves as a single individual. With this we also have another justification for $\delta < 1$. The inequality means then that the members of the current generation care for their children, but not quite as much as for themselves. See Barro (1989) for more on these points.

20.C Intertemporal Production and Efficiency

Assume that there is an infinite sequence of dates $t = 0, 1, \ldots$. In each period t, there are L commodities. If it facilitates reading, you can take $L = 2$ and interpret the commodities as labor services and a generalized consumption–investment good (see Example 20.C.1). One of the great advantages of vector notation, however, is that in some cases—and this is one—there is no novelty involved in the general case. Thus, while you think you are understanding the simple problem, you are at the same time understanding the most general one.

We shall adopt the convention that goods are *nondurables*. This is a convention because, in order to make a good durable, it suffices to specify a storage technology whose role is, so to speak, to transport the commodity through time.

If we were exogenously endowed with some amount of resources (e.g., some initial capital and some amount of labor every period), we would ask what we could do with them. To give an answer, we need to specify the *production technology*. We already know from Chapter 5 how to do this formally by means of the concept of a *production set* (or a production transformation function, or a production function). With minimal loss of generality, we will restrict our technologies to be of the following form: the production possibilities at time t are entirely determined by the production decisions at the most recent past, that is, at time $t - 1$. If we keep in mind that we can always define new intermediate goods (such as different vintages of a machine), and also that we can always define periods to be very long, we see that the restriction is minor.

Thus, the technological possibilities at t will be formally specified by a production set $Y \subset \mathbb{R}^{2L}$ whose generic entries, or *production plans*, are written $y = (y_b, y_a)$. The indices b and a are mnemonic for "before" and "after." The interpretation is that the production plans in Y cover two periods (the "initial" and the "last" period) with $y_b \in \mathbb{R}^L$ and $y_a \in \mathbb{R}^L$ being, respectively, the production plans for the initial and the last periods. As usual, negative entries represent inputs and positive entries represent outputs.

We impose some assumptions on Y that are familiar from Section 5.B:

(i) Y is *closed and convex.*
(ii) $Y \cap \mathbb{R}^{2L}_+ = \{0\}$ *(no free lunch).*
(iii) $Y - \mathbb{R}^{2L}_+ \subset Y$ *(free disposal).*

An assumption specific to the temporal setting is the requirement that inputs not be used *later* than outputs are produced (i.e., production takes time). This is captured by

(iv) If $y = (y_b, y_a) \in Y$ then $(y_b, 0) \in Y$ *(possibility of truncation).*

In words, (iv) says that, whatever the production plans for the initial period, not producing in the last period is a possibility. A simple case is when $y_{at} \geq 0$ for every $y \in Y$, that is, when all inputs are used in the initial period. Then (iv) is implied by the free-disposal property (iii).

Example 20.C.1: *Ramsey–Solow Model.*[5] Assume that there are only two commodities: A consumption–investment good and labor. It will be convenient to describe the technology by a production function $F(k, l)$. To any amounts of capital investment $k \geq 0$ and of labor input $l \geq 0$, applied in the initial period, the production function assigns the *total* amount $F(k, l)$ of consumption–investment good available at the last period. Then

$$Y = \{(-k, -l, x, 0): k \geq 0, l \geq 0, x \leq F(k, l)\} - \mathbb{R}^4_+.$$

Note that labor is a primary factor; that is, it cannot be produced. ∎

Example 20.C.2: *Cost-of-Adjustment Model.* Suppose that there are three goods: capacity, a consumption good, and labor. With the amounts k and l of invested capacity and labor at the initial period, one gets $F(k, l)$ units of consumption good output at the last period. This output can be transformed into invested capacity at the last period at a cost of $k' + \gamma(k' - k)$ units of consumption good for k' units of capacity, where $\gamma(\cdot)$ is a convex function satisfying $\gamma(k' - k) = 0$ for $k' < k$ and $\gamma(k' - k) > 0$ for $k' > k$. The term $\gamma(k' - k)$ represents the cost of adjusting capacity upward in a given period relative to the previous period. (Note the marginal cost of doing so increases with invested capacity of the period.) Formally, the production set Y is

$$Y = \{(-k, 0, -l, k', x, 0): k \geq 0, l \geq 0, k' \geq 0, x \leq F(k, l) - k' - \gamma(k' - k)\} - \mathbb{R}^6_+. \ ∎$$

5. See Ramsey (1928) and Solow (1956). The same model was also introduced in Swan (1956).

Example 20.C.3: *Two-Sector Model.* We could make a more general distinction between an investment and a consumption good than the one embodied in Examples 20.C.1 and 20.C.2. Indeed, we could let the production set be

$$Y = \{(-k, 0, -l, k', x, 0): k \geq 0, l \geq 0, k' \geq 0, x \leq G(k, l, k')\} - \mathbb{R}^6_+,$$

where k, k' are, respectively, the investments in the initial and the last periods. Note that the investment and the consumption good need not be perfectly substitutable [they are produced in two separate sectors, so to speak; see Uzawa (1964)]. If they are [i.e., if the transformation function $G(k, l, k')$ has the form $F(k, l) - k'$] then this example is equivalent to the Ramsey–Solow model of Example 20.C.1. If it has the form $G(k, l, k') = F(k, l) - k' - \gamma(k' - k)$ then we have the cost-of-adjustment model of Example 20.C.2. ■

Example 20.C.4: $(N + 1)$*-Sector Model.* As in Example 20.C.3, we have a consumption good and labor, but we now interpret k and k' as N-dimensional vectors. For simplicity of exposition, in Example 20.C.3 we have taken $G(k, l, k')$ to be defined for any $k \geq 0$, $k' \geq 0$. In general, however, this could lead to the production of negative amounts of consumption good. To avoid this it is convenient to complete the specification by means of an admissible domain A of (k, l, k') combinations. Then

$$Y = \{(-k, 0, -l, k', x, 0): (k, l, k') \in A \text{ and } x \leq G(k, l, k')\} - \mathbb{R}^{2(N+2)}_+. \qquad \blacksquare$$

Once we have specified our technology, we can define what constitutes a path of production plans.

Definition 20.C.1: The list $(y_0, y_1, \ldots, y_t, \ldots)$ is a *production path*, or *trajectory*, or *program*, if $y_t \in Y \subset \mathbb{R}^{2L}$ for every t.

Note that along a production path $(y_0, \ldots, y_t, \ldots)$ there is overlap in the time indices over which the production plans y_{t-1} and y_t are defined. Indeed, both $y_{a,t-1} \in \mathbb{R}^L$ and $y_{bt} \in \mathbb{R}^L$ represent plans, made respectively at dates $t - 1$ and t, for input use or output production at date t. Thus, we have, at every t, a net input–output vector equal to $y_{a,t-1} + y_{bt} \in \mathbb{R}^L$ (at $t = 0$, we put $y_{a,-1} = 0$; this convention is kept throughout the chapter).[6] The negative entries of this vector stand for amounts of inputs that have to be injected from the outside at period t if the path is to be realized, that is, amounts of input required at period t for the operation of y_{t-1} and y_t in excess of the amounts provided as outputs by the operation of y_{t-1} and y_t. Similarly, the positive entries represent the amounts of goods left over after input use and thus available for final consumption at time t.

The situation is entirely analogous to the description of the production side of an economy in Chapter 5. If we think of the technology at every t as being run by a distinct firm (or as an aggregate of distinct firms) and of $\hat{y}_t$ as an infinite sequence with nonzero entries (equal to y_t) only in the t and $t + 1$ places, then $\sum_t \hat{y}_t$ is the aggregate production path; and it is also precisely the sequence that assigns the net input–output vector $y_{a,t-1} + y_{bt} \in \mathbb{R}^L$ to period t. If we had a finite horizon, the current setting would thus be a particular case of the description of production in

6. A minor point of notation: when there is any possibility of confusion or ambiguity in the reading of indices, we insert commas; for example, we write $y_{a,t-1}$ instead of $y_{a\,t-1}$.

Chapter 5. With an infinite horizon there is a difference: we now have a countable infinity of commodities and of firms instead of only a finite number. As we shall see, this is not a minor difference. It will, however, be most helpful to arrange our discussion around the exploration of the analogy with the finite horizon case by asking the same questions we posed in Section 5.F regarding the relationship between efficient production plans and price equilibria.

Definition 20.C.2: The production path $(y_0, \ldots, y_t, \ldots)$ is *efficient* if there is no other production path $(y'_0, \ldots, y'_t, \ldots)$ such that

$$y_{a,t-1} + y_{bt} \leq y'_{a,t-1} + y'_{bt} \quad \text{for all } t,$$

and equality does not hold for at least one t.

In words: the path $(y_0, \ldots, y_t, \ldots)$ is efficient if there is no way that we can produce at least as much final consumption in every period using at most the same amount of inputs in every period (with at least one inequality strict). The definition is exactly parallel to Definition 5.F.1.

What constitutes a *price vector* in the current intertemporal context? It is natural to define it as a sequence $(p_0, p_1, \ldots, p_t, \ldots)$, where $p_t \in \mathbb{R}^L$. For the moment we shall not ask where this sequence comes from. We assume that it is somehow given and that it is available to any possible production unit. The prices should be thought of as present-value prices. We shall discuss further the nature of these prices in the next section.

Given a path $(y_0, \ldots, y_t, \ldots)$ and a price sequence $(p_0, \ldots, p_t, \ldots)$, the profit level associated with the production plan at t is

$$p_t \cdot y_{bt} + p_{t+1} \cdot y_{at}.$$

We now pursue the implications of profit maximization on the production plans made period by period.

Definition 20.C.3: The production path $(y_0, \ldots, y_t, \ldots)$ is *myopically*, or *short-run, profit maximizing for the price sequence* $(p_0, \ldots, p_t, \ldots)$ if for every t we have

$$p_t \cdot y_{bt} + p_{t+1} \cdot y_{at} \geq p_t \cdot y'_{bt} + p_{t+1} \cdot y'_{at} \quad \text{for all } y'_t \in Y.$$

Prices $(p_0, \ldots, p_t, \ldots)$ capable of sustaining a path $(y_1, \ldots, y_t, \ldots)$ as myopically profit-maximizing are often called *Malinvaud prices* for the path [because of Malinvaud (1953)].[7]

Does the first welfare theorem hold for myopic profit maximization? That is, if $(y_0, \ldots, y_t, \ldots)$ is myopically profit maximizing with respect to strictly positive prices, does it follow that $(y_0, \ldots, y_t, \ldots)$ is efficient? In a finite-horizon economy this conclusion holds true because of Proposition 5.F.1, but a little thought reveals that in the infinite-horizon context it need not. The intuition for a negative answer rests on the phenomenon of *capital overaccumulation*. Suppose that prices increase through

7. Observe that we do not require that $\sum_t p_t \cdot (y_{a,t-1} + y_{bt}) < \infty$. In principle, a production path may have an infinite present value. We saw in Sections 5.E and 5.F, where we had a finite number of commodities and firms that individual, decentralized profit maximization and overall profit maximization amounted to the same thing. Because of the possibility of an infinite present value, the existence of a countable number of commodities *and* production sets makes this a more delicate matter in the current context. See Exercises 20.C.2 to 20.C.5 for a discussion.

time fast enough. Then it may very well happen that at every single period it always pays to invest everything at hand. Along such a path, consumption never takes place hardly an efficient outcome.

Example 20.C.5: With $L = 1$, let $Y = \{(-k, k'): k \geq 0, k' \leq k\} \subset \mathbb{R}^2$. This is just a trivial storage technology. Consider the path where $y_t = (-1, 1)$ for all t; that is, we always carry forward one unit of good. Then $y_{a,-1} + y_{b0} = -1$ and $y_{a,t-1} + y_{bt} = 0$ for all $t > 0$. This is not efficient; just consider the path $y'_t = (0, 0)$ for all t, which has $y'_{a,t-1} + y'_{bt} = 0$ for all $t \geq 0$. But for the stationary price sequence where $p_t = 1$ for all t, $(y_0, \ldots, y_t, \ldots)$ is myopically profit maximizing. ■

Efficiency will obtain if, in addition to myopic profit maximization, the (present) value of the production path becomes insignificant as $t \to \infty$. Precisely, efficiency obtains if the (present) value of the period t production plan for period $t + 1$ goes to zero, that is, if $p_{t+1} \cdot y_{at} \to 0$ as $t \to \infty$. This is the so-called *transversality condition*. Note that the condition is violated in the storage illustration of Example 20.C.5.

Proposition 20.C.1: Suppose that the production path $(y_0, \ldots, y_t, \ldots)$ is myopically profit maximizing with respect to the price sequence $(p_0, \ldots, p_t, \ldots) \gg 0$. Suppose also that the production path and the price sequence satisfy the *transversality condition* $p_{t+1} \cdot y_{at} \to 0$. Then the path $(y_0, \ldots, y_t, \ldots)$ is efficient.

Proof: Suppose that the path $(y'_0, \ldots, y'_t, \ldots)$ is such that $y_{a,t-1} + y_{bt} \leq y'_{a,t-1} + y'_{bt}$ for all t, with equality not holding for at least one t. Then there is $\varepsilon > 0$ such that if we take a T sufficiently large for some strict inequality to correspond to a date previous to T, we must have

$$\sum_{t=0}^{T} p_t \cdot (y'_{a,t-1} + y'_{bt}) > \sum_{t=0}^{T} p_t \cdot (y_{a,t-1} + y_{bt}) + \varepsilon.$$

In fact, if T is very large then $p_{T+1} \cdot y_{aT}$ is very small (because of the transversality condition) and therefore

$$\sum_{t=0}^{T} p_t \cdot (y'_{a,t-1} + y'_{bt}) > p_{T+1} \cdot y_{aT} + \sum_{t=0}^{T} p_t \cdot (y_{a,t-1} + y_{bt}).$$

By rearranging terms—a standard trick in dynamic economics—this can be rewritten as (recall the convention $y_{a,-1} = y'_{a,-1} = 0$)

$$p_T \cdot y'_{bT} + \sum_{t=0}^{T-1} (p_{t+1} \cdot y'_{at} + p_t \cdot y'_{bt}) > \sum_{t=0}^{T} (p_{t+1} \cdot y_{at} + p_t \cdot y_{bt}).$$

We must thus have either $p_{t+1} \cdot y'_{at} + p_t \cdot y'_{bt} > p_{t+1} \cdot y_{at} + p_t \cdot y_{bt}$ for some $t \leq T - 1$ or $p_T \cdot y'_{bT} > p_{T+1} \cdot y_{aT} + p_T \cdot y_{bT}$. In either case we obtain a violation of the myopic profit-maximization assumption [recall that by the possibility of truncation we have $(y'_{bT}, 0) \in Y$]. Therefore, no such path $(y'_0, \ldots, y'_t, \ldots)$ can exist.

Note that the essence of the argument is very simple. The key fact is that if the transversality condition holds, then for T large enough we can approximate the overall profits of the truncated path $(y_0, \ldots, y_T)$ by the sum of the net values of period-by-period input output realizations (up to period T). It does not matter whether we match the inputs and the outputs per period or per firm (that is, "per production plan"). If the horizon is far enough away, either method will come down to Profits = Total Revenue − Total Cost. ■

Proposition 20.C.1 tells us that a modified version of the first welfare theorem holds in the dynamic production setting. Let us now ask about the second welfare theorem: *Given an efficient path* $(y_0, \ldots, y_t, \ldots)$, *can it be price supported*? In Proposition 5.F.2 we gave a positive answer to this question which applies to the finite-horizon case. In the current infinite-horizon situation we could decompose the question into two parts:

(i) *Is there a system of Malinvaud prices* $(p_0, \ldots, p_t, \ldots)$ *for* $(y_0, \ldots, y_t, \ldots)$, that is, a sequence $(p_0, \ldots, p_t, \ldots)$ with respect to which $(y_0, \ldots, y_t, \ldots)$ is myopically profit maximizing?

(ii) *If the answer to (i) is yes, can we conclude that the pair* $(y_0, \ldots, y_t, \ldots)$, $(p_0, \ldots, p_t, \ldots)$ *satisfies the transversality condition*?

The answer to (ii) *is "not necessarily."* In Section 20.E we will see, by means of an example, that the transversality condition is definitely not a necessary property of Malinvaud prices.

The answer to (i) *is "Essentially yes."* We illustrate the matter by means of two examples and then conclude this section by a small-type discussion of the general situation.

Example 20.C.6: *Ramsey–Solow Model Continued.* In this model, we can summarize a path by the sequence (k_t, l_t, c_t) of total capital usage, labor usage, and amount available for consumption. From now on we assume that $k_{t+1} + c_{t+1} = F(k_t, l_t)$ and that the sequence l_t of labor inputs is exogenously given. Then it is enough to specify the capital path $(k_0, \ldots, k_t, \ldots)$. Denoting by (q_t, w_t) the prices of the two commodities at t, we have that profits at t are $q_{t+1}F(k_t, l_t) - q_t k_t - w_t l_t$ and, therefore, the necessary and sufficient conditions for short-run profit maximization at t are

$$\frac{q_t}{q_{t+1}} = \nabla_1 F(k_t, l_t) \qquad \text{and} \qquad \frac{w_t}{q_{t+1}} = \nabla_2 F(k_t, l_t).$$

Note that, up to a normalization (we could put $q_0 = 1$), these first-order conditions determine supporting prices for *any* feasible capital path (see Exercise 20.C.6).

The transversality condition says that $q_{t+1}F(k_t, l_t) \to 0$. If the sequence of productions $F(k_t, l_t)$ is bounded, then it suffices that $q_t \to 0$. In view of Proposition 20.C.1, we can conclude that a set of sufficient conditions for efficiency of a feasible and bounded capital path $(k_0, \ldots, k_t, \ldots)$ is that there exist a sequence of output prices $(q_0, \ldots, q_t, \ldots)$ such that

$$\frac{q_t}{q_{t+1}} = \nabla_1 F(k_t, l_t) \qquad \text{for all } t \tag{20.C.1}$$

and

$$q_t \to 0 \qquad (\text{equivalently, } 1/q_t \to \infty). \tag{20.C.2}$$

Because of the possibility of capital overaccumulation, (20.C.1), which is necessary, is not alone sufficient for efficiency. On the other hand, (20.C.2) is not necessary (see Section 20.E). Cass (1972) obtained a weakened version of (20.C.2) that, with (20.C.1), is both necessary and sufficient.[8] The condition is

$$\sum_{t=0}^{\infty} \frac{1}{q_t} = \infty. \qquad (20.C.2')$$

■

Example 20.C.7: *Cost of Adjustment Model continued.* In the cost of adjustment model, a production plan at time $t-1$ involves the variables k_{t-1}, l_{t-1}, k_t, c_t. We associate with these variables the prices q_{t-1}, w_{t-1}, q_t, s_t. Profits are then

$$s_t(F(k_{t-1}, l_{t-1}) - k_t - \gamma(k_t - k_{t-1})) + q_t k_t - q_{t-1}k_{t-1} - w_{t-1}l_{t-1}.$$

Using the first-order profit-maximization conditions with respect to k_t and k_{t-1} we get the following two conditions:

(i) $q_t = s_t(1 + \gamma'(k_t - k_{t-1}))$; that is, the price of capacity at t equals the investment cost in extra capacity at t.

(ii) $q_{t-1} = s_t(\nabla_1 F(k_{t-1}, l_{t-1}) + \gamma'(k_t - k_{t-1}))$; that is, the price of capacity at $t-1$ equals the return at t of one unit of extra capacity at $t-1$ (the return has two parts: the increased production at t and the saving in the cost of capacity adjustment at t).

Combining (i) and (ii),

$$\frac{q_{t-1}}{q_t} = \frac{\nabla_1 F(k_{t-1}, l_{t-1}) + \gamma'(k_t - k_{t-1})}{1 + \gamma'(k_t - k_{t-1})}. \qquad (20.C.3)$$

Note that if there are no adjustment costs [i.e., if $\gamma(\cdot)$ is identically equal to zero], then (20.C.3) is precisely (20.C.1). Observe also that, in parallel to Example 20.C.6, condition (20.C.3) determines short-run supporting prices for any feasible capital path. ■

In a general smooth model it is not difficult to explain how the supporting prices $(p_0, \ldots, p_t, \ldots)$ for an efficient path $(y_0, \ldots, y_t, \ldots)$ can be constructed. Note that, because of efficiency, every y_t belongs to the boundary of Y. The smoothness property that we require is that, for every t, the production set Y has a single (normalized) outward normal $q_t = (q_{bt}, q_{at})$ at y_t (we could, for example, normalize q_t to have unit length); see Figure 20.C.1. Less geometrically, smoothness means that at $y_t \in Y$ all the marginal rates of transformation (MRT) of inputs for inputs, inputs for outputs, and outputs for outputs are uniquely defined.

We claim that the efficiency property implies that for every t we have that $q_{a,t-1} = \beta q_{bt}$ for some $\beta > 0$. Heuristically: for any two commodities their MRT as outputs at t for the production decision taken at time $t-1$ must be the same as their MRT as inputs at t for the production decision taken at time t. If this were not so, it would be possible to generate a surplus of goods. The argument is standard (recall the analysis of Section 16.F). Consider, for example, Figure 20.C.2, where in panel (a) we have drawn the output transformation frontier through $y_{a,t-1}$ (i.e., keeping $y_{b,t-1}$ fixed) and in panel (b) the input isoquant through y_{bt} (i.e., keeping y_{at} fixed; recall the sign conventions for inputs). We see that if the slopes at these points are not the same, then it is possible to move from $y_{a,t-1}$ to $y'_{a,t-1}$ and from y_{bt} to y'_{bt} in such a way that $y'_{a,t-1} + y'_{bt} > y_{a,t-1} + y_{bt}$, thus contradicting efficiency.

8. Some additional, very minor, regularity conditions on the production function $F(\cdot)$ are required for the validity of this equivalence.

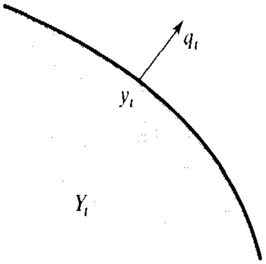


Figure 20.C.1 (left) Smooth production set.

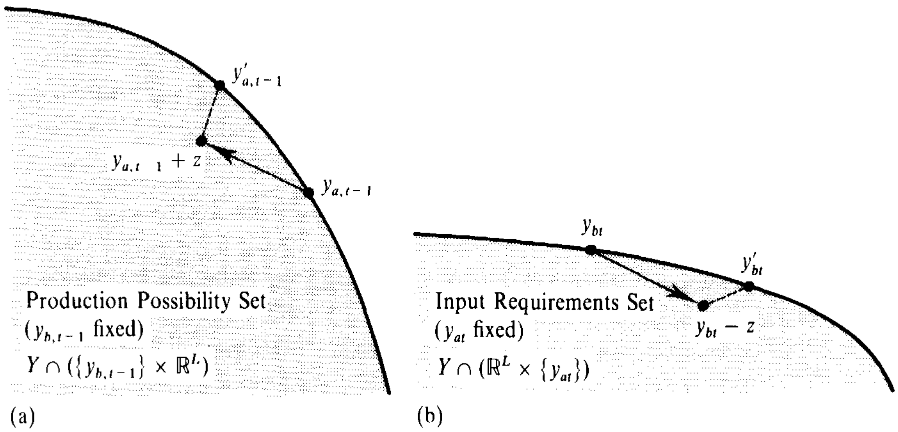


Figure 20.C.2 (right) A production path that is inefficient at T.

We construct the desired price sequence $(p_0, \ldots, p_t, \ldots)$ by induction. Put $p_0 = q_{b0}$ (i.e., the relative prices at $t = 0$ are the MRTs between goods at the initial part of the production plan $y_0 \in Y \subset \mathbb{R}^{2L}$). Suppose now that the prices $(p_0, \ldots, p_T)$ have already been determined, and that every y_t up to $t = T - 1$ is myopically profit maximizing for these prices. Because of the first-order conditions for profit maximization at $T - 1$, we have that $p_T = \alpha q_{a,T-1}$ for some $\alpha > 0$. We know that $q_{a,T-1} = \beta q_{bT}$ for some $\beta > 0$. Then $p_T = \alpha\beta q_{bT}$. Therefore, if we put $p_{T+1} = \alpha\beta q_{aT}$, we have that $(p_T, p_{T+1}) = (\alpha\beta q_{bT}, \alpha\beta q_{aT})$ is proportional to $q_T = (q_{bT}, q_{aT})$, which means that y_T is profit maximizing for (p_T, p_{T+1}). Hence we have extended our sequence to $(p_0, \ldots, p_{T+1})$ and we can keep going.

Note that, as in Examples 20.C.6 and 20.C.7, the construction of the supporting short-run prices does not make full use of the efficiency. What is used is that the production path is "short-run efficient" (that is, the production path cannot be shown inefficient by changes in the production plans at a finite number of dates).

The above observations can be made into a perfectly rigorous argument for the existence of Malinvaud prices in the smooth case. The proof for the nonsmooth case is more complex. It must combine an appeal to the separating hyperplane theorem (to get prices for truncated horizons) with a limit operation as the horizon goes to infinity. With a minor technical condition (call *nontightness* in the literature), this limit operation can be carried out.

20.D Equilibrium: The One-Consumer Case

In this section, we bring the consumption and the production sides together and begin the study of equilibrium in the intertemporal setting. We shall start with the one-consumer case. As we will see in Section 20.G, the relevance of this case goes beyond the domain of applicability of the representative consumer theory of Chapter 4.

An economy is specified by a *short-term production technology* $Y \subset \mathbb{R}^{2L}$, *a utility function* $u(\cdot)$ defined on $\mathbb{R}^L_+$, a *discount factor* $\delta < 1$, and, finally, a (bounded) sequence of *initial endowments* $(\omega_0, \ldots, \omega_t, \ldots)$, $\omega_t \in \mathbb{R}^L_+$.

We assume that Y *satisfies hypotheses* (i) to (iv) of Section 20.C and that $u(\cdot)$ *is strictly concave, differentiable, and has strictly positive marginal utilities throughout its domain.*

Prices are given to us as sequences $(p_0, \ldots, p_t, \ldots)$ with $p_t \in \mathbb{R}^L_+$. As in Chapter 19 we can interpret these prices either as the prices of a complete system of forward markets occurring simultaneously at $t = 0$ or as the correctly anticipated (present value) prices of a sequence of spot markets. We will consider only bounded price sequences. In fact, most of the time we will have $\|p_t\| \to 0$.[9]

Given a production path $(y_0, \ldots, y_t, \ldots)$, $y_t \in Y$, the induced stream of consumptions $(c_0, \ldots, c_t, \ldots)$ is given by

$$c_t = y_{a,t-1} + y_{bt} + \omega_t.$$

If $c_t \geq 0$ for every t, then we say that the production path $(y_0, \ldots, y_t, \ldots)$ is *feasible*: Given the initial endowment stream the production path is capable of sustaining nonnegative consumptions at every period.

To keep the exposition manageable *from now on we restrict all our production paths and consumption streams to be bounded.* Delicate points come up in the general case, which are better avoided in a first approach. Alternatively, we could simply assume that our technology is such that any feasible production path is bounded.

Given a production path $(y_0, \ldots, y_t, \ldots)$ and a price sequence $(p_0, \ldots, p_t, \ldots)$, the induced stream of profits $(\pi_0, \ldots, \pi_t, \ldots)$ is given by

$$\pi_t = p_t \cdot y_{bt} + p_{t+1} \cdot y_{at} \qquad \text{for every } t.$$

Fixing T and rearranging the terms of $\sum_{t \leq T} p_t \cdot c_t = \sum_{t \leq T} p_t \cdot (y_{a,t-1} + y_{bt} + \omega_t)$ we get

$$\sum_{t \leq T} (\pi_t + p_t \cdot \omega_t) - \sum_{t \leq T} p_t \cdot c_t = p_{T+1} \cdot y_{aT} \tag{20.D.1}$$

Expression (20.D.1) is an important identity. It tells us that *the transversality condition is equivalent to the overall value of consumption not being strictly inferior to wealth* (i.e., there is no escape of purchasing power at infinity).

The definition of a Walrasian equilibrium is now as in the previous chapters. One only has to make sure that a few infinite sums make sense.

Definition 20.D.1: The (bounded) production path $(y_0^*, \ldots, y_t^*, \ldots)$, $y_t^* \in Y$, and the (bounded) price sequence $p = (p_0, \ldots, p_t, \ldots)$ constitute a *Walrasian* (or *competitive*) equilibrium if:

(i) $c_t^* = y_{a,t-1}^* + y_{bt}^* + \omega_t \geq 0$ for all t. (20.D.2)

(ii) For every t,

$$\pi_t = p_t \cdot y_{bt}^* + p_{t+1} \cdot y_{at}^* \geq p_t \cdot y_b + p_{t+1} \cdot y_a \tag{20.D.3}$$

for all $y = (y_b, y_a) \in Y$.

(iii) The consumption sequence $(c_0^*, \ldots, c_t^*, \ldots) \geq 0$ solves the problem

$$\begin{aligned} \text{Max} \quad & \sum_t \delta^t u(c_t) \\ \text{s.t.} \quad & \textstyle\sum_t p_t \cdot c_t \leq \sum_t \pi_t + \sum_t p_t \cdot \omega_t. \end{aligned} \tag{20.D.4}$$

Condition (i) is the *feasibility* requirement. Condition (ii) is the short-run, or myopic, profit-maximization condition already considered in Section 20.C (Definition 20.C.3). The form of the budget constraint in part (iii) deserves comment. Note first that there is a single budget constraint. As in Chapter 19, this amounts to an assumption of *completeness*, which means, in one interpretation, that at time $t = 0$

9. Keep in mind that prices are to be thought of as measured in current-value terms.

there is a forward market for every commodity at every date, or, in another, that assets (e.g., money) are available that are capable of transferring purchasing power through time (see Exercise 20.D.1 for more on this). Secondly, observe that the strict monotonicity of $u(\cdot)$ implies that if we have reached utility maximization then, a fortiori, total wealth (denoted w) must be finite; that is,

$$w = \sum_t \pi_t + \sum_t p_t \cdot \omega_t < \infty.$$

Moreover, at the equilibrium consumptions the budget constraint of (20.D.4) must hold with equality.

An important consequence of the last observation is that at equilibrium the transversality condition is satisfied. Formally, we have Proposition 20.D.1.

Proposition 20.D.1: Suppose that the (bounded) production path $(y_0^*, \ldots, y_t^*, \ldots)$ and the (bounded) price sequence $(p_0, \ldots, p_t, \ldots)$ constitute a Walrasian equilibrium. Then the transversality condition $p_{t+1} \cdot y_{at}^* \to 0$ holds.

Proof: Denote $c_t^* = y_{a,t-1}^* + y_{bt}^* + \omega_t$. By expression (20.D.1) we have

$$\sum_{t \leq T} (\pi_t + p_t \cdot \omega_t) - \sum_{t \leq T} p_t \cdot c_t = p_{T+1} \cdot y_{aT}.$$

Since each of the sums in the left-hand side converges to $w < \infty$ as $T \to \infty$, we conclude that $p_{T+1} \cdot y_{aT}^* \to 0$. ■

Another implication of $w < \infty$ is the possibility of replacing condition (ii) of Definition 20.D.1 by

(ii′) The production path $(y_0^*, \ldots, y_t^*, \ldots)$ maximizes total profits, in the sense that for any other path $(y_0, \ldots, y_t, \ldots)$ and any T we have

$$\sum_{t=0}^{t=T} (p_t \cdot y_{bt} + p_{t+1} \cdot y_{at}) \leq \sum_t (p_t \cdot y_{bt}^* + p_{t+1} \cdot y_{at}^*) < \infty.$$

Clearly, (ii′) implies (ii), and (ii) with $w < \infty$ implies (ii′) (see Exercise 20.D.2). Thus, at equilibrium prices, the test of myopic and of overall profit maximization coincide. Could a similar statement be made for an appropriate concept of myopic utility maximization? We now investigate this question.

Definition 20.D.2: We say that the consumption stream $(c_0, \ldots, c_t, \ldots)$ is *myopically, or short-run, utility maximizing* in the budget set determined by $(p_0, \ldots, p_t, \ldots)$ and $w < \infty$ if utility cannot be increased by a new consumption stream that merely transfers purchasing power between some two consecutive periods.

The key fact is presented in Exercise 20.D.3.

Exercise 20.D.3: Show that a consumption stream $(c_0, \ldots, c_t, \ldots) \gg 0$ is short-run utility maximizing for $p = (p_0, \ldots, p_t, \ldots)$ and $w < \infty$ if and only if it satisfies $\sum_t p_t \cdot c_t = w$ and the collection of first-order conditions:

For every t there is $\lambda_t > 0$ such that

$$\lambda_t p_t = \nabla u(c_t) \qquad \text{and} \qquad \lambda_t p_{t+1} = \delta \nabla u(c_{t+1}). \tag{20.D.5}$$

It follows from (20.D.5) that $\lambda_t p_t = \nabla u(c_t)$ and $\lambda_{t-1} p_t = \delta \nabla u(c_t)$. Therefore, $\lambda_{t-1} = \delta \lambda_t$ and so $\lambda_0 = \delta^t \lambda_t$. Hence letting $\lambda = \lambda_0$, we see that (20.D.5) is actually equivalent to

$$\text{For some } \lambda, \qquad \lambda p_t = \delta^t \nabla u(c_t) \quad \text{for all } t. \tag{20.D.6}$$

Once we realize that myopic utility maximization in a budget set amounts to (20.D.6), we can verify that overall utility maximization follows. This is done in Proposition 20.D.2.

Proposition 20.D.2: If the consumption stream $(c_0, \ldots, c_t, \ldots)$ satisfies $\sum_t p_t \cdot c_t = w < \infty$ and condition (20.D.6), then it is utility maximizing in the budget set determined by $(p_0, \ldots, p_t, \ldots)$ and w.

Proof: We first note that we cannot improve upon $(c_0, \ldots, c_t, \ldots)$ by transferring purchasing power only through a finite number of dates. Indeed, (20.D.6) implies that the first-order sufficient conditions for any such constrained utility maximization problem are satisfied.

Suppose now that $(c'_0, \ldots, c'_t, \ldots)$ is a consumption stream satisfying the budget constraint and yielding higher total utility. Then for a sufficiently large T, consider the stream $(c''_0, \ldots, c''_t, \ldots)$ with $c''_t = c'_t$ for $t \le T$ and $c''_t = c_t$ for $t > T$. Because $\delta < 1$, there is $\varepsilon > 0$ such that if T is large enough then there is an improvement of utility of more than 2ε in going from $(c_0, \ldots, c_t, \ldots)$ to $(c''_0, \ldots, c''_t, \ldots)$. Since $w < \infty$, the amount $\sum_{t>T} |p_t \cdot (c_t - c'_t)|$ can be made arbitrarily small. Hence, for large T the stream $(c''_0, \ldots, c''_t, \ldots)$ is almost budget feasible. It follows that it can be made budget feasible by a small sacrifice of consumption in the first period resulting in a utility loss not larger than ε. Overall, it still results in an improvement. But this yields a contradiction because only the consumption in a finite number of periods has been altered in the process. ■

Example 20.D.1: In this example we illustrate the use of conditions (20.D.6) for the computation of equilibrium prices. Suppose that we are in a one-commodity world with utility function $\sum_t \delta^t \ln c_t$. Given a price sequence $(p_0, \ldots, p_t, \ldots)$ and wealth w, the first-order conditions for utility maximization (20.D.6) are

$$\lambda p_t = \frac{\delta^t}{c_t} \quad \text{for all } t, \quad \text{and} \quad \sum_t p_t c_t = w.$$

Hence, $w = \sum_t p_t c_t = (1/\lambda) \sum_t \delta^t = (1/\lambda)[1/(1-\delta)]$ and so $p_t c_t = \delta^t/\lambda = \delta^t(1-\delta)w$ for all t. Note that this implies a *constant rate of savings* because $p_T c_T / (\sum_{t \ge T} p_t c_t) = 1 - \delta$, for all T (Exercise 20.D.4).[10]

We now discuss three possible production scenarios.

(i) The economy is of the exchange type; that is, there is no possibility of production and we are given an initial endowment sequence $(\omega_0, \ldots, \omega_t, \ldots) \gg 0$. Then the equilibrium must involve $c_t^* = \omega_t$ for every t, and therefore, normalizing to $\sum_t p_t \omega_t = 1$, the equilibrium prices should be

$$p_t = \frac{\delta^t(1-\delta)}{\omega_t} \qquad \text{for every } t.$$

10. Logarithmic utility functions facilitate computation and are very important in applications. However, they are not continuous at the boundary ($\ln c_t \to -\infty$ as $c_t \to 0$) and therefore violate one of our maintained assumptions. This does not affect the current analysis but should be kept in mind.

(ii) Suppose instead that $\omega_0 = 1$ and $\omega_t = 0$ for $t > 0$. There is, however, a linear production technology transforming every unit of input at t into $\alpha > 0$ units of output at $t + 1$. Because of the boundary behavior of the utility function, consumption will be positive in every period, and therefore the technology will be in operation at every period. The linearity of the technologies then has the important implication that the equilibrium price sequence is completely determined by the technology. Putting $p_0 = 1$, we must have $p_t = 1/\alpha^t$. Wealth is $w = p_0\omega_0 = 1$, and therefore the equilibrium consumptions must be $c_t^* = [\delta^t(1 - \delta)]/p_t = (\alpha\delta)^t(1 - \delta)$. Note that, as long as $1 \leq \alpha < 1/\delta$, both the price and the consumption sequences are bounded. Observe also the interesting fact that for this example we have been able to compute the equilibrium without explicitly solving for the sequence of capital investments.

(iii) We are as in (ii) except that we now have a general technology $F(k)$ transforming every unit k_t, of investment at t into $F(k_t)$ units of output at $t + 1$. This output can then be used indistinctly for consumption or investment purposes at $t + 1$. That is, $c_{t+1} = F(k_t) - k_{t+1}$. The logarithmic form of the utility function allows for a shortcut to the computation of equilibrium prices. Indeed, say that $(p_0, \ldots, p_t, \ldots)$ are equilibrium prices and $(c_0^*, \ldots, c_t^*, \ldots)$, $(k_0^*, \ldots, k_t^*, \ldots)$ equilibrium paths of consumption and capital investment. Then we know that at any T a constant fraction δ of remaining wealth is invested. That is,

$$p_{T+1}k_{T+1}^* = \delta\left(\sum_{t \geq T+1} p_t c_t^*\right) = \delta p_{T+1}F(k_T^*).$$

Therefore, we must have $k_{t+1}^* = \delta F(k_t^*)$ for every t. With $k_0 = \omega_0 = 1$ given, this allows us to iteratively compute the sequence of equilibrium capital investments. The sequence of prices is then obtained from the profit-maximization conditions $p_{t+1}F'(k_t^*) - p_t = 0$. ■

Since a Walrasian equilibrium is myopically profit maximizing and satisfies the transversality condition (Proposition 20.D.1), we know from Proposition 20.C.1 that it is production efficient (assuming $p_t \gg 0$ for all t). Can we strengthen this to the claim that the full first welfare theorem holds? We will now verify that we can. In the current one-consumer problem, Pareto optimality simply means that the equilibrium solves the utility-maximization problem under the technological and endowment constraints:

$$\text{Max} \sum_t \delta^t u(c_t), \tag{20.D.7}$$

$$\text{s.t. } c_t = y_{a,t-1} + y_{bt} + \omega_t \geq 0 \quad \text{and} \quad y_t \in Y \text{ for all } t.$$

Proposition 20.D.3: Any Walrasian equilibrium path $(y_0^*, \ldots, y_t^*, \ldots)$ solves the planning problem (20.D.7).

Proof: Denote by B the budget set determined by the Walrasian equilibrium price sequence $(p_0, \ldots, p_t, \ldots)$ and wealth $w = \sum_t \pi_t + \sum_t p_t \cdot \omega_t$, where

$$\pi_t = p_t \cdot y_{bt}^* + p_{t+1} \cdot y_{a,t+1}^*$$

for all t. That is,

$$B = \{(c'_0, \dots, c'_t, \dots): c'_t \geq 0 \text{ for all } t \text{ and } \textstyle\sum_t p_t \cdot c'_t \leq w\}.$$

By the definition of Walrasian equilibrium, the utility of the stream $(c^*_0, \dots, c^*_t, \dots)$ defined by $c^*_t = y^*_{a,t-1} + y^*_{bt} + \omega_t$ is maximal in this budget set. It suffices, therefore, to show that any feasible path $(y''_0, \dots, y''_t, \dots)$, that is, any path for which $y''_t \in Y$ and $c''_t = y''_{a,t-1} + y''_{bt} + \omega_t \geq 0$ for all t, must yield a consumption stream in B. To see this note that, for any T,

$$\sum_{t \leq T} p_t \cdot c''_t = \sum_{t \leq T-1} (p_t \cdot y''_{bt} + p_{t+1} \cdot y''_{at}) + p_T \cdot y''_{bT} + \sum_{t \leq T} p_t \cdot \omega_t.$$

By the possibility of truncation of production plans, we have $(y''_{bT}, 0) \in Y$. Therefore, by short-run profit maximization, $p_t \cdot y''_{bT} \leq \pi_T$ and $p_t \cdot y''_{bt} + p_{t+1} \cdot y''_{at} \leq \pi_t$ for all $t \leq T - 1$. Hence,

$$\sum_{t \leq T} p_t \cdot c''_t \leq \sum_{t \leq T} \pi_t + \sum_{t \leq T} p_t \cdot \omega_t \leq w \quad \text{for all } T,$$

which implies $\sum_t p_t \cdot c''_t \leq w$. ■

Let us now ask for the converse of Proposition 20.D.3 (i.e., for the second welfare theorem question; see chapter 16): Is any solution $(y_0, \dots, y_t, \dots)$ to the planning problem (20.D.7) a Walrasian equilibrium? In essence, the answer is "yes," but the precise theorems are somewhat technical because, to obtain a well-behaved price system (i.e., a price system as we understand it: a sequence of nonzero prices), one needs some regularity condition on the path. We give an example of one such result.[11]

Proposition 20.D.4: Suppose that the (bounded) path $(y^*_0, \dots, y^*_t, \dots)$ solves the planning problem (20.D.7) and that it yields strictly positive consumption (in the sense that, for some $\varepsilon > 0$, $c_{\ell t} = y^*_{\ell a,t-1} + y^*_{\ell bt} + \omega_{\ell t} > \varepsilon$ for all ℓ and t). Then the path is a Walrasian equilibrium with respect to some price sequence $(p_0, \dots, p_t, \dots)$.

Proof: We provide only a sketch of the proof. A possible candidate for an equilibrium price system is suggested by expression (20.D.6):

$$p_t = \delta^t \nabla u(c^*_t) \qquad \text{for all } t,$$

where $c^*_t = y^*_{a,t-1} + y^*_{bt} + \omega_t$. Because $(c^*_0, \dots, c^*_t, \dots)$ is bounded above and bounded away from the boundary (uniformly in t) we have $\sum_t \|p_t\| < \infty$, which implies the transversality condition. In turn, by expression (20.D.1) this yields $\sum_t p_t \cdot c^*_t = \sum_t (\pi_t + p_t \cdot \omega_t) = w < \infty$. Therefore, by Proposition 20.D.2, the utility-maximization condition holds.

It remains to establish that short-run profit maximization also holds. To that effect suppose that this is not so, that is, that for some T there is $y' \in Y$ with

$$p_T \cdot y'_b + p_{T+1} \cdot y'_a > p_T \cdot y^*_{bT} + p_{T+1} \cdot y^*_{aT} = \pi_T.$$

Let $(y'_1, \dots, y'_t, \dots)$ be the path with $y'_T = y'$ and $y'_t = y^*_t$ for any $t \neq T$. Let $(c'_0, \dots, c'_t, \dots)$ be the associated consumption stream. Because of the convexity of Y and the strict positivity property of $(c^*_0, \dots, c^*_t, \dots)$ we can assume that $y'_T = y'$ is sufficiently close to y^*_T for us to

11. A general treatment would involve, as in Sections 15.C or 16.D, the application of a suitable version (here infinite-dimensional) of the separating hyperplane theorem. The next result gets around this by exploiting the differentiability of $u(\cdot)$. It is thus parallel to the discussion in Section 16.F.

have $c'_t \gg 0$ for all t and, moreover, for it to be legitimate to determine the sign of

$$\sum_t \delta^t(u(c'_t) - u(c_t^*)) = \delta^T(u(c'_T) - u(c_T^*)) + \delta^{T+1}(u(c'_{T+1}) - u(c_{T+1}^*))$$

by signing the first-order term

$$\begin{aligned}&\delta^T\, \nabla u(c_T^*)\cdot(c'_T - c_T^*) + \delta^{T+1}\, \nabla u(c_{T+1}^*)\cdot(c'_{T+1} - c_{T+1}^*)\\ &= p_T\cdot(y'_{bT} - y_{bT}^*) + p_{T+1}\cdot(y'_{aT} - y_{aT}^*)\\ &= p_T\cdot y'_{bT} + p_{T+1}\cdot y'_{aT} - p_T\cdot y_{bT}^* - p_{T+1}\cdot y_{aT}^* > 0.\end{aligned}$$

But this conclusion contradicts the assumption that $(y_0^*, \ldots, y_t^*, \ldots)$ solves (20.D.7). ∎

The close connection between the solutions of the equilibrium and the planning problem (20.D.7) has three important implications for, respectively, the existence, uniqueness, and computation of equilibria.

The first implication is that it reduces the question of the *existence* of an equilibrium to the possibility of solving a single optimization problem, albeit an infinite-dimensional one.

Proposition 20.D.5: Suppose that there is a uniform bound on the consumption streams generated by all the feasible paths. Then the planning problem (20.D.7) attains a maximum; that is, there is a feasible path that yields utility at least as large as the utility corresponding to any other feasible paths.

The proof, which is purely technical and which we skip, involves simply establishing that, in a suitable infinite-dimensional sense, the objective function of problem (20.D.7) is continuous and the constraint set is compact.

The second implication is that it allows us to assert the *uniqueness* of equilibrium.

Proposition 20.D.6: The planning problem (20.D.7) has at most one consumption stream solution.

Proof: The proof consists of the usual argument showing that the maximum of a strictly concave function in a convex set is unique. Suppose that $(y_0, \ldots, y_t, \ldots)$ and $(y'_0, \ldots, y'_t, \ldots)$ are feasible paths with $\sum_t \delta^t u(c_t) = \sum_t \delta^t u(c'_t) = \gamma$, where $(c_0, \ldots, c_t, \ldots)$ and $(c'_0, \ldots, c'_t, \ldots)$ are the consumption streams associated with the two production paths. Consider $y''_t = \frac{1}{2}y_t + \frac{1}{2}y'_t$. Then the path $(y''_0, \ldots, y''_t, \ldots)$ is feasible and at every t the consumption level is $c''_t = \frac{1}{2}c_t + \frac{1}{2}c'_t$. Hence, $\sum_t \delta^t u(c''_t) \geq \gamma$, with the inequality strict if $c_t \neq c'_t$ for some t. Thus, if $c_t \neq c'_t$ for some t, the paths $(y_0, \ldots, y_t, \ldots)$, $(y'_0, \ldots, y'_t, \ldots)$ could not both solve (20.D.7). ∎

The third implication is that Proposition 20.D.3 provides a workable approach to the *computation* of the equilibrium. We devote the rest of this section to elaborating on this point.

The Computation of Equilibrium and Euler Equations

It will be convenient to pursue the discussion of computational issues in the slightly restricted setting of Example 20.C.4, the $(N + 1)$-sector model. To recall, we have N capital goods, labor, and a consumption good. We fix the endowments of labor to a constant level through time. A function $G(k, k')$ gives the total amount of consumption good obtainable at any t if the investment in capital goods at $t - 1$ is given by the vector $k \in \mathbb{R}^N$, the investment at t is required to be $k' \in \mathbb{R}^N_+$, and the labor usage at $t-1$ and t is fixed at the level exogenously given by the initial endowments. We denote by $A \subset \mathbb{R}^N \times \mathbb{R}^N$ the region of pairs $(k, k') \in \mathbb{R}^{2N}$ compatible with nonnegative consumption [i.e., $A = \{(k, k') \in \mathbb{R}^{2N}: G(k, k') \geq 0\}$]. For notational convenience, we write $u(G(k, k'))$ as $u(k, k')$. We assume that A is convex and that $u(\cdot, \cdot)$ is strictly concave. Also, at $t = 0$ there is some already installed capital investment $\bar{k}_0$ and this is the only initial endowment of capital in the economy.

In this economy the planning problem (20.D.7) becomes[12]

$$\text{Max} \quad \sum_t \delta^t u(k_{t-1}, k_t) \qquad (20.D.8)$$
$$\text{s.t. } (k_{t-1}, k_t) \in A \text{ for every } t, \text{ and } k_0 = \bar{k}_0.$$

From now on we assume that (20.D.8) has a (bounded) solution. Because of the strict concavity of $u(\cdot, \cdot)$ this solution is unique.

For every $t \geq 1$ the vector of variables $k_t \in \mathbb{R}^N$ enters the objective function of (20.D.8) only through the two-term sum $\delta^t u(k_{t-1}, k_t) + \delta^{t+1} u(k_t, k_{t+1})$. Therefore, differentiating with respect to these N variables, we obtain the following necessary conditions for an interior path $(k_0, \ldots, k_t, \ldots)$ to be a solution of the problem (20.D.8):[13]

$$\frac{\partial u(k_{t-1}, k_t)}{\partial k'_n} + \delta \frac{\partial u(k_t, k_{t+1})}{\partial k_n} = 0 \qquad \text{for every } n \leq N \text{ and } t \geq 1.$$

In vector notation,

$$\nabla_2 u(k_{t-1}, k_t) + \delta\, \nabla_1 u(k_t, k_{t+1}) = 0 \qquad \text{for every } t \geq 1. \qquad (20.D.9)$$

Conditions (20.D.9) are known as the *Euler equations* of the problem (20.D.8).

Example 20.D.2: Consider the Ramsey–Solow technology of Example 20.C.1 (with $l_t = 1$ for all t). Then, $u(k, k') = u(F(k) - k')$ and $A = \{(k, k'): k' \leq F(k)\}$. Therefore, the Euler equations take the form

$$-u'(F(k_{t-1}) - k_t) + \delta u'(F(k_t) - k_{t+1})F'(k_t) = 0, \quad \text{for all } t \geq 1$$

or

$$\frac{u'(c_t)}{\delta u'(c_{t+1})} = F'(k_t) \qquad \text{for all } t \geq 1.$$

In words: the marginal utilities of consuming at t or of investing and postponing consumption one period are the same. ■

Example 20.D.3: Consider the cost-of-adjustment technology of Example 20.C.2 (except that as in Example 20.D.2 we fix $l_t = 1$ for all t and drop labor as an explicitly considered commodity) and suppose we have an overall firm that tries to maximize the infinite discounted sum of profits by means of a suitable investment policy in capacity. Output can be sold at a constant unitary price that, with a constant rate

12. By convention we put $u(k_{-1}, k_0) = 0$.

13. The expression "interior path" means that (k_t, k_{t+1}) is in the interior of A for all t. For the interpretation of the expression to come, recall also that k_n and k'_n stand, respectively, for the nth and the $(N + n)$th argument of $u(k, k')$.

of interest, gives a present value price of δ^t. Thus the problem becomes that of maximizing $\sum_t \delta^t[F(k_{t-1}) - k_t - \gamma(k_t - k_{t-1})]$. The Euler equations are then

$$-1 - \gamma'(k_t - k_{t-1}) + \delta[F'(k_t) + \gamma'(k_{t+1} - k_t)] = 0 \qquad \text{for all } t \geq 1.$$

In words: the marginal cost of a unit of investment in capacity at t equals the discounted value of the marginal product of capacity at t *plus* the marginal saving in the cost of capacity expansion at $t + 1$. Note that, iterating from $t = 1$, we get

$$1 + \gamma'(k_1 - k_0) = \sum_{t \geq 1} \delta^t(F'(k_t) - 1).$$

In words: At the optimum, the cost of investing in an extra unit of capacity at $t = 1$ equals the discounted sum of the marginal products of a *maintained* increase of a unit of capacity.[14] See Exercise 20.D.5 for more detail.[15] ■

Suppose that a path $(k_0, \ldots, k_t, \ldots)$ satisfies the Euler necessary equations (20.D.9). From their own definition, and the concavity of $u(\cdot, \cdot)$, it follows that the Euler equations are also sufficient to guarantee that the trajectory cannot be improved upon by a trajectory involving changes in a single k_t. In fact, the same is true if the changes are limited to any finite number of periods (see Exercise 20.D.6). Thus, we can say that the Euler equations are necessary and sufficient for short-run optimization. The question is then: Do the Euler equations (or, equivalently, short-run optimization) imply long-run optimization? We shall see that, under a regularity property on the path (related, in a manner we shall not make explicit, to the transversality condition[16]), they do.

We say that the path $(k_0, \ldots, k_t, \ldots)$ is *strictly interior* if it stays strictly away from the boundary of the admissible region A. [More precisely, the path is strictly interior if there is $\varepsilon > 0$ such that for every t there is an ε neighborhood of (k_t, k_{t+1}) entirely contained in A.]

Proposition 20.D.7: Suppose that the path $(\bar{k}_0, \ldots, k_t, \ldots)$ is bounded, is strictly interior, and satisfies the Euler equations (20.D.9). Then it solves the optimization problem (20.D.8).

Proof: The basic argument is familiar. If $(\bar{k}_0, \ldots, k_t, \ldots)$ does not solve (20.D.8), then there is a feasible trajectory $(\bar{k}_0, \ldots, k'_t, \ldots)$ that gives a higher utility. To simplify the reasoning suppose that this trajectory is bounded. Then, by the concavity of the objective function, the boundedness of $(\bar{k}_0, \ldots, k_t, \ldots)$ and its strict interiority, we can assume that, for every t, k'_t is so close to k_t that $(k'_t, k_{t+1}) \in A$. We can now take T large enough for $\sum_{t<T} \delta^t u(k'_{t-1}, k'_t) > \sum_{t<T} \delta^t u(k_{t-1}, k_t)$ and define then a new trajectory $(\bar{k}_0, \ldots, k''_t, \ldots)$ by

14. That is to say, the extra unit of capacity available at $t = 1$ produces $F'(k_1)$ at $t = 2$. Of this amount, one unit is devoted to additional investment at $t = 2$. With this, at $t = 2$ the net addition of capacity has not changed (the initial and final capacities at $t = 2$ expand by one unit) and therefore there is no change in the adjustment cost paid. Consequently, the net gain at $t = 2$ in terms of commodity is $F'(k_1) - 1$. But this is not all the gain because the extra unit of capacity available at $t = 2$ produces $F'(k_2)$ at $t = 3$, and so on.

15. The ideas of this example are related to what is known in macroeconomic theory as the q-theory of investment. See, for example, Chapter 2 of Blanchard and Fischer (1989).

16. We refer to the storage illustration of Example 20.C.5 for the need to appeal to a regularity property.

$k_t'' = k_t'$ for $t \leq T$ and $k_t'' = k_t$ for $t > T$. The new trajectory is admissible [note that $(k_T', k_{T+1}) \in A$]; it coincides with $(\bar{k}_0, \ldots, k_t', \ldots)$ up to T and with $(\bar{k}_0, \ldots, k_t, \ldots)$ after T. Moreover, if T is large enough, it still gives higher utility than $(\bar{k}_0, \ldots, k_t, \ldots)$. But this is impossible because, as we have already indicated, the Euler equations imply short-run optimization, that is, they are the first-order conditions for the optimization problem where we are constrained to adjust only the variables corresponding to a finite number of periods (see Exercise 20.D.6). ■

It may be helpful at this stage to introduce the concept of the *value function* $V(k)$ and the *policy function* $\psi(k)$. Given an initial condition $k_0 = k$, the maximum value attained by (20.D.8) is denoted $V(k)$, and if $(k_0, k_1, \ldots, k_t, \ldots)$ is the (unique) trajectory solving (20.D.8) with $k_0 = k$, then we put $\psi(k) = k_1$. That is, $\psi(k) \in \mathbb{R}^N$ is the vector of optimal levels of investment, hence of capital, at $t = 1$ when the levels of capital at $t = 0$ are given by k.

What accounts for the importance of the policy function is the observation that if the path $(\hat{k}_0, \ldots, \hat{k}_t, \ldots)$ solves (20.D.8) for $k_0 = \hat{k}_0$ then, for any T, the path $(\hat{k}_T, \ldots, \hat{k}_{T+t}, \ldots)$ solves (20.D.8) for $k_0 = \hat{k}_T$. Thus, if $(k_0, \ldots, k_t, \ldots)$ solves (20.D.8) we must have

$$k_{t+1} = \psi(k_t) \text{ for every } t, \tag{20.D.10}$$

and we see that the optimal path can be computed from knowledge of k_0 and the policy function $\psi(\cdot)$. But how do we determine $\psi(\cdot)$? We now describe two approaches to the computation of $\psi(\cdot)$. The first exploits the Euler equations; the second rests on the method of *dynamic programming*.

The Euler equations (20.D.9) suggest an iterative procedure for the computation of $\psi(k)$. Fix $k_0 = k$ and consider the equations corresponding to k_1. With k_0 given, we have N equations in the $2N$ unknowns $k_1 \in \mathbb{R}^N$ and $k_2 \in \mathbb{R}^N$. There are therefore N degrees of freedom. Suppose that we try to fix k_1 arbitrarily [equivalently, we try to fix $-\nabla_2 u(k_0, k_1)$, the marginal costs of investment at $t = 1$] and then use the N Euler equations at $t = 1$ to solve for the remaining k_2 unknowns [equivalently, we adjust the commitments for investment at $t = 2$ so that the discounted marginal payoff of investment at $t = 1$, $\delta\nabla_1 u(k_1, k_2)$, equals the preestablished marginal cost of investment at $t = 1$, i.e. $-\nabla_2 u(k_0, k_1)$]. Suppose that such a solution k_2 is found [by the strict concavity of $u(\cdot)$, if there is one solution then it has to be unique]. We can then repeat the process. The N Euler equations for period 2 are now exactly determined: Both k_1 and k_2 are given, but we still have the N variables k_3 corresponding to $t = 3$ with which we can try to satisfy the N equations of period 2. Suppose that we reiterate in this fashion. There are three possibilities. The first is that the process breaks down somewhere, that is, that given k_{t-1} and k_t there is no solution k_{t+1} [or, more precisely, no solution with $(k_t, k_{t+1}) \in A$]; the second is that we generate a sequence that is unbounded (or nonstrictly interior); the third is that we generate a bounded (and strictly interior) sequence $(k_0, k_1, \ldots, k_t, \ldots)$. In the third case, by Proposition 20.D.7 we have obtained an optimum, and since by Proposition 20.D.6 the optimum is unique, we can conclude that *given k_0, the third possibility (the trajectory starting at k_0 and k_1 is strictly interior and bounded) can occur for at most one value of k_1. If it occurs, this value of k_1 is precisely $\psi(k_0)$*. Thus, the computational method is: Solve the difference equation induced by the Euler equations with initial condition (k_0, k_1) and then for fixed k_0 search for an initial condition k_1 generating a bounded infinite path.

Example 20.D.4: Consider a Ramsey–Solow model with linear technology $F(k) = 2k$ and utility function $\sum_t (1/2)^t \ln c_t$. Then $u(k_{t-1}, k_t) = \ln(2k_{t-1} - k_t)$ and the period-t Euler equation is (see Exercise 20.D.7)

$$k_{t+1} = 3k_t - 2k_{t-1}.$$

This difference equation has the solution $k_t = k_0 + (k_1 - k_0)(2^t - 1)$. If $k_1 < k_0$, then k_t eventually becomes negative. If $k_1 > k_0$, then k_t is unbounded. The only value of k_1 generating a bounded k_t is $k_1 = k_0$. Therefore, $\psi(k_0) = k_0$ for any k_0. It is instructive to see what happens if we try $k_1 \geq k_0$. Then, the path induced by the difference equation is feasible and, in fact, we have a constant level of consumption $c_t = 2k_{t-1} - k_t = 2k_0 - k_1$. Thus, for $k_1 > k_0$, we have here an example of a path that is compatible with the Euler equations but that is not optimal, because at $k_1 = k_0$ we get a higher level of constant consumption.[17] ■

The dynamic programming approach exploits the recursivity of the optimum problem (20.D.8), namely, the fact that

$$V(k) = \underset{k' \text{ with } (k,k') \in A}{\text{Max}} \quad u(k, k') + \delta V(k'), \tag{20.D.11}$$

and obtains $\psi(k)$ as the vector k' that solves (20.D.11). This, of course, only transforms the problem into one of computing the value function $V(\cdot)$. However, it turns out that, first, under some general conditions [e.g., if $V(\cdot)$ is bounded] the value function is the *only* function that solves (20.D.11) when viewed as a functional equation, that is, $V(\cdot)$ is the only function for which (20.D.11) is true for every k, and, second, that there are some well-known and quite effective algorithms for solving equations such as (20.D.11) for the unknown function $V(\cdot)$. (See Section M.M. of the Mathematical Appendix.)

We end this section by pointing out two implications of the definition of the value function (see Exercise 20.D.8):

(i) *The value function $V(k)$ is concave.*

(ii) *For every perturbation parameter $z \in \mathbb{R}^N$ with $(k + z, \psi(k)) \in A$ we have*

$$V(k + z) \geq u(k + z, \psi(k)) + \delta V(\psi(k)). \tag{20.D.12}$$

Suppose that $N = 1$ and $(k, \psi(k))$ is interior to A. For later reference we point out that from (i), (ii), and $V(k) = u(k, \psi(k)) + \delta V(\psi(k))$ we obtain

$$V'(k) = \nabla_1 u(k, \psi(k))$$

and, if $V(\cdot)$ is twice-differentiable,

$$V''(k) \geq \nabla^2_{11} u(k, \psi(k)).$$

(See Figure 20.D.1 and Exercise 20.D.9.[18])

17. Hence, when $k_1 > k_0$, the Euler equations lead to capital overaccumulation. We note, without further elaboration, that given a path satisfying the Euler equations we could use the equations themselves to determine a myopically supporting price sequence. However, if $k_1 > k_0$ this sequence will violate the transversality condition.

18. The expression $\nabla^2_{ij} f(\cdot)$ denotes the ij second partial derivative of the real-value function $f(\cdot)$.

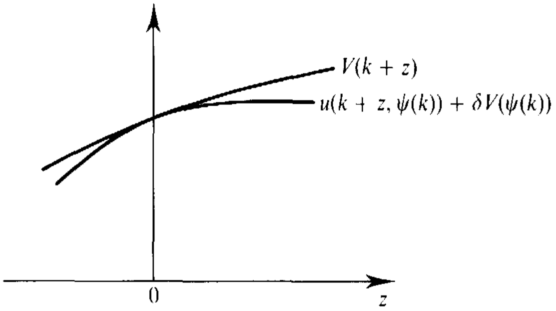


Figure 20.D.1
Along an optimal path the value function is majorized by the utilities of single-period adjustments.

20.E Stationary Paths, Interest Rates, and Golden Rules

In this section, we concentrate on the study of steady states. This study constitutes a first step towards the analysis of the dynamics of equilibrium paths. We refer to Bliss (1975), Gale (1973), or Weizsäcker (1971) for further analysis of steady-state theory.

We begin with a production set $Y \subset \mathbb{R}^{2L}$ satisfying the properties considered in Section 20.C. Recall that a production path is a sequence $(y_0, \ldots, y_t, \ldots)$ with $y_t \in Y$ for every t.

Definition 20.E.1: A production path $(y_0, \ldots, y_t, \ldots)$ is *stationary*, or a *steady state*, if there is a production plan $\bar{y} = (\bar{y}_b, \bar{y}_a) \in Y$ such that $y_t = \bar{y}$ for all $t > 0$.

Abusing terminology slightly, we refer to the "stationary path $(\bar{y}, \ldots, \bar{y}, \ldots)$" as simply the "stationary path $\bar{y}$."

The first important observation is that stationary paths *that are also efficient* are supportable by proportional prices.[19] This is shown in Proposition 20.E.1.

Proposition 20.E.1: Suppose that $\bar{y} \in Y$ defines a stationary and efficient path. Then, there is a price vector $p_0 \in \mathbb{R}^L$ and an $\alpha > 0$ such that the path is myopically profit maximizing for the price sequence $(p_0, \alpha p_0, \ldots, \alpha^t p_0, \ldots)$.

Proof: A complete proof is too delicate an affair, but the basic intuition may be grasped from the case in which production sets have smooth boundaries. For this case we can, in fact, show that *every* (myopically) supporting price sequence must be proportional.

By the efficiency of the path $(\bar{y}, \ldots, \bar{y}, \ldots)$, the vector $\bar{y}$ must lie at the boundary of Y. Let $\bar{q} = (\bar{q}_0, \bar{q}_1)$ be the unique (up to normalization) vector perpendicular to Y at $\bar{y}$. Also, by the small type discussion at the end of Section 20.C, there exists a price sequence $(p_0, \ldots, p_t, \ldots)$ that myopically supports this efficient path. Because $\bar{y} \in Y$ is short-run profit maximizing at every t we must have $(p_t, p_{t+1}) = \lambda_t(\bar{q}_0, \bar{q}_1)$ for some $\lambda_t > 0$. Therefore, $p_t = \lambda_t \bar{q}_0$ and $p_{t+1} = \lambda_t \bar{q}_1$ for all t. In particular, $p_t = \lambda_{t-1}\bar{q}_1$ and $p_{t+1} = \lambda_{t+1}\bar{q}_0$. Combining, we obtain $p_{t+1} = (\lambda_t/\lambda_{t-1})p_t$ and

19. To prevent possible misunderstanding, we warn that establishing the inefficiency of a given stationary path will typically require the consideration of nonstationary paths.

$p_{t+1} = (\lambda_{t+1}/\lambda_t)p_t$. From this we get $\lambda_t/\lambda_{t-1} = \lambda_{t+1}/\lambda_t$ for all $t \geq 1$. Hence, denoting this quotient by α, we have $p_{t+1} = \alpha p_t = \alpha^2 p_{t-1} = \cdots = \alpha^{t+1} p_0$.

The factor α has a simple interpretation. Indeed, $r = (1-\alpha)/\alpha$ [so that $p_t = (1+r)p_{t+1}$] can be viewed as a *rate of interest* implicit in the price sequence (see Exercise 20.E.1).

Proposition 20.E.1 is a sort of second welfare theorem result for stationary paths. We could also pose the parallel first welfare theorem question. Namely, suppose that $(\bar{y}, \ldots, \bar{y}, \ldots)$ is a stationary path myopically supported by a proportional price sequence with rate of interest r. If $r > 0$, then $p_t = (1/(1+r))^t p_0 \to 0$ and therefore the transversality condition $p_t \cdot \bar{y}_a \to 0$ is satisfied. We conclude from Proposition 20.C.1 that the path is efficient. If $r \leq 0$, the transversality condition is not satisfied (p_t does not go to zero), but this does not automatically imply inefficiency because the transversality condition is sufficient but not necessary for efficiency. Suppose that $r < 0$ and, to make things simple, let us be in the smooth case again. Consider the stationary candidate paths defined by the constant production plan $\bar{y}_\varepsilon = (\bar{y}_b + \varepsilon e, \bar{y}_a - \varepsilon e)$, where $e = (1, \ldots, 1) \in \mathbb{R}^L$. This candidate path uses fewer inputs (or produces more outputs) at $t = 0$ and generates exactly the same net input output vector at every other t. Therefore, if for some $\varepsilon > 0$, the candidate path is in fact a feasible path; that is, if $\bar{y}_\varepsilon \in Y$, then the stationary path $\bar{y}$ is not efficient (it overaccumulates). But if Y has a smooth boundary at $\bar{y}$, the feasibility of $\bar{y}_\varepsilon$ for some $\varepsilon > 0$ can be tested by checking whether $\bar{y}_\varepsilon - \bar{y} = \varepsilon(e, -e)$ lies below the hyperplane determined by the supporting prices $(p_0, [1/(1+r)]p_0)$. Evaluating, we have $\varepsilon(1 - 1/(1+r))p_0 \cdot e < 0$, because $r < 0$. Conclusion: For ε small enough, the stationary path $\bar{y}$ is dominated by the stationary path $\bar{y}_\varepsilon$. We record these facts for later reference in Proposition 20.E.2.

Proposition 20.E.2: Suppose that the stationary path $(\bar{y}, \ldots, \bar{y} \ldots)$, $\bar{y} \in Y$, is myopically supported by proportional prices with rate of interest r, then the path is efficient if $r > 0$ and inefficient if $r < 0$.

We have not yet dealt with the case $r = 0$, which as we shall see, is very important.[20] We will later verify in a more specific setup that efficiency obtains in this case.

Let us now bring in the consumption side of the economy and consider *stationary equilibrium paths*. Assuming differentiability and interiority, a stationary path $(\bar{y}, \ldots, \bar{y}, \ldots)$ that is also an equilibrium can be supported only (up to a normalization) by the price sequence $p_t = \delta^t \nabla u(\bar{c})$, where $\bar{c} = \bar{y}_b + \bar{y}_a$; recall Proposition 20.D.4 and expression (20.D.6). That is, *a stationary equilibrium is supported by a price sequence embodying a proportionality factor equal to the discount factor* δ, or, equivalently, with rate of interest $r = (1-\delta)/\delta$.

Definition 20.E.2: A stationary production path that is myopically supported by proportional prices $p_t = \alpha^t p_0$ with $\alpha = \delta$ is called a *modified golden rule path*. A stationary production path myopically supported by constant prices $p_t = p_0$ is called a *golden rule path*.

20. Note that 0 is the rate of growth implicit in the path $(\bar{y}, \ldots, \bar{y}, \ldots)$. In a more general treatment we could allow for a constant returns technology and for the production path to be proportional (but not necessarily stationary). Then Proposition 20.E.2 remains valid with 0 replaced by the corresponding rate of growth.

Depending on the technology and on the discount factor δ, there may be a single or there may be several modified golden rule paths (see the small-type discussion at the end of this section). But in any case we have just seen that a *stationary equilibrium path is necessarily a modified golden rule path.* Thus, we have the important implication that the *candidates for stationary equilibrium paths* $(\bar{y}, \dots, \bar{y}, \dots)$ *are completely determined by the technology and the discount factor and are independent of the utility function* $u(\cdot)$.

To pursue the analysis it will be much more convenient to reduce the level of abstraction. Consider an extremely simple case, the Ramsey–Solow model technology of Example 20.C.1. We study trajectories with $l_t = 1$ for all t (imagine that there is available one unit of labor at every point in time). We can then identify a production path with the sequence of capital investments $(k_0, \dots, k_t, \dots)$.

Given $(k_0, \dots, k_t, \dots)$, denote $r_t = \nabla_1 F(k_t, 1) - 1$. Thus, r_t is the *net* (i.e., after replacing capital) *marginal productivity of capital.* Suppose that $k_t > 0$ and that the sequence of output prices $(q_0, \dots, q_t, \dots)$ and wages $(w_0, \dots, w_t, \dots)$ myopically price supports the given path. Then, by the first-order condition for profit maximization, we have $q_{t+1}(1 + r_t) - q_t = 0$. Hence r_t is the output rate of interest at time t implicit in the output price sequence $(q_0, \dots, q_t, \dots)$.

Let us now focus on the stationary paths of this example. Any $k \geq 0$ fixed through time constitutes a *steady state.* With any such steady state we can associate a constant surplus level $c(k) = F(k, 1) - k$ and a rate of interest $r(k) = \nabla_1 F(k, 1) - 1$, also constant through time.[21] Therefore, the supporting price–wage sequence is

$$(q_t, w_t) = \left(\frac{1}{1 + r(k)}\right)^t (q_0, w_0), \qquad \text{with } \frac{w_0}{q_0} = \frac{\nabla_2 F(k, 1)}{\nabla_1 F(k, 1)}.$$

Denote by $w(k)$ the real wage w_0/q_0 so determined. It is instructive to analyze how the steady-state levels of consumption $c(k)$, the rate of interest $r(k)$, and the real wage $w(k)$ depend on k.

Let $\bar{k}$ be the level of capital at which the steady-state consumption level is maximized [i.e., $\bar{k}$ solves $\text{Max}\, F(k, 1) - k$]. Note that $\bar{k}$ is characterized by $r(\bar{k}) = \nabla_1 F(\bar{k}, 1) - 1 = 0$. Thus $\bar{k}$ is precisely the *golden rule* steady state. The construction is illustrated in Figure 20.E.1, where we also represent the modified golden rule k_δ [characterized by $r(k_\delta) = \nabla_1 F(k_\delta, 1) - 1 = (1 - \delta)/\delta$]. Observe that if $k < \bar{k}$ then $r(k) > 0$. As we saw in Proposition 20.E.2, $r(k) > 0$ implies that the steady state k is efficient (thus, in particular, the modified golden rule is efficient: it gives less consumption than the golden rule but it also uses less capital). Similarly, if $k > \bar{k}$ then $r(k) < 0$ and we have inefficiency of the steady state k. What about $\bar{k}$?[22] We now argue that *the golden rule steady state $\bar{k}$ is efficient.* A graphic proof will be quickest. Suppose we try to dominate the constant path $\bar{k}$ by starting with $k_0 < \bar{k}$, so that consumption at $t = 0$ is raised. Since the surplus at $t = 1$ must be at least

21. Thus, $c(k)$ is the amount of good constantly available through time and usable as a flow for consumption purposes.

22. Recall that the associated price sequence is constant and that the transversality condition is therefore violated.

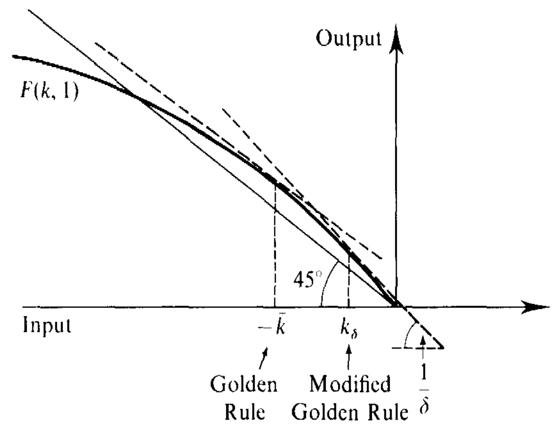


Figure 20.E.1 The production technology of the Ramsey–Solow model and the golden rule.

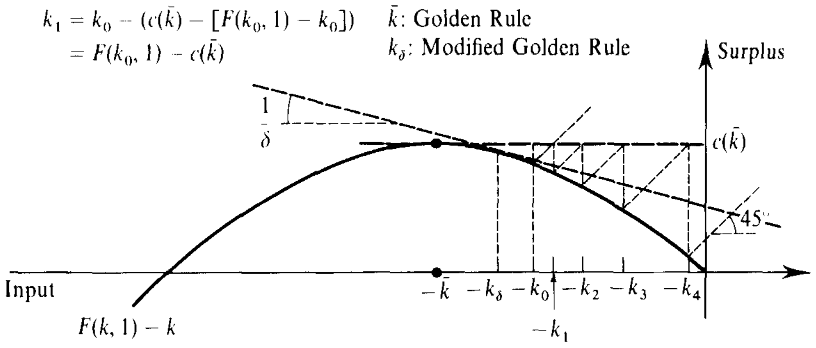


Figure 20.E.2 Ramsey–Solow model: the golden rule is efficient.

$c(\bar{k})$, the best we can do for k_1 is

$$k_1 = F(k_0, 1) - c(\bar{k}) = F(k_0, 1) - k_0 + k_0 - c(\bar{k}) < k_0,$$

because $F(k_0, 1) - k_0 < c(\bar{k})$. This new best possible value of k_1 is represented in Figure 20.E.2. In the figure we also see that as the process of determination of k_1 is iterated to obtain k_2, k_3 and so on we will, at some point get a $k_t < 0$. Hence, the path is not feasible, and we conclude that a constant $\bar{k}$ cannot be dominated from the point of view of efficiency: the attempt to use less capital at some stage will inexorably lead to capital depletion in finite time.

From the form of the production function, three "neoclassical" properties follow immediately (you are asked to prove them in Exercise 20.E.4):

(i) As k increases, the level $c(k)$ increases monotonically up to the golden rule level and then decreases monotonically.
(ii) The rate of interest $r(k)$ decreases monotonically with the level of capital k.
(iii) The real wage $w(k)$ increases monotonically with the level of capital. (For the validity of this property you should also assume that production function $F(k, l)$ is homogeneous of degree one.)

From the study of the steady states of the Ramsey–Solow model we have learnt at least six new things: First, the rate of interest is equal to the net marginal productivity of capital; second, the golden rule (i.e., zero rate of interest) path is characterized by a surplus-maximizing property among steady states; third, the golden rule is efficient; fourth, fifth, and sixth, we have the three neoclassical properties.

How general is all of this? That is, can we make similar claims for the general model with any number of goods? The answer, in short, is that the three neoclassical properties may or may not hold in a world with several capital goods, but the other three, duly interpreted, remain valid with great generality. Attempting to give proofs of all this would take us into too advanced material [see Bliss (1975) or Brock and Burmeister (1976)], but perhaps we can provide some intuition.

Suppose we consider the general $(N + 1)$-sector technology of Example 20.C.4. That is, $G(k, k')$ is the amount of consumption good available at any period if $k \in \mathbb{R}^N$ is the vector of levels of capital used in the previous period and the investment required in the period is $k' \in \mathbb{R}^N$ (we also let $l_t = 1$ for all t). At a steady-state path we have $k' = k$. Denote by $\hat{G}(k) = G(k, k)$ the level of consumption associated with the steady state k. If $G(\cdot, \cdot)$ is a concave function then so is $\hat{G}(\cdot)$. In particular, $\nabla \hat{G}(k) = 0$ characterizes the steady state with maximal level of consumption.

Consider a steady steady k. By Proposition 20.E.1, this steady state can be myopically supported by a proportional price sequence $s_t \in \mathbb{R}$, $q_t \in \mathbb{R}^N$. Here s_t is the price of the consumption good in period t, and q_t is the vector of prices of investment in period t. Because of proportionality there is an $r(k)$ such that $s_t = (1 + r(k))s_{t+1}$, $q_t = (1 + r(k))q_{t+1}$ for all t. Because of profit maximization,

$$\nabla_1 G(k, k) = \frac{1}{s_t} q_{t-1} \qquad \text{and} \qquad \nabla_2 G(k, k) = -\frac{1}{s_t} q_t \qquad \text{for all } t \tag{20.E.1}$$

(you are asked to verify this in Exercise 20.E.5). Therefore,

$$\nabla \hat{G}(k) = \nabla_1 G(k, k) + \nabla_2 G(k, k) = \frac{1}{s_t}(q_{t-1} - q_t) = \frac{r(k)}{s_t} q_t,$$

that is, at any time *an extra dollar invested in a permanent increase of any capital good yields $r(k)$ dollars in extra value of (permanent) consumption.* In this precise sense the rate of interest measures the marginal productivity of capital. We see again that $\nabla \hat{G}(k) = 0$ (the necessary and sufficient condition for maximum steady-state consumption) is equivalent to $r(k) = 0$. Hence, the golden rule property holds: a steady-state level k yields maximal consumption across steady states if and only if it has associated with it a zero rate of interest. We add that we could also prove that the golden rule path is efficient.

As we have already indicated, the neoclassical properties do not carry over to the general setting. A taste of the possible difficulties can be given even if $N = 1$, that is, for the two-sector model of Example 20.C.3. In Figure 20.E.3 we represent the level curves of $G(k, k')$. The steady states correspond to the diagonal, where $k = k'$. Every steady state k can be myopically

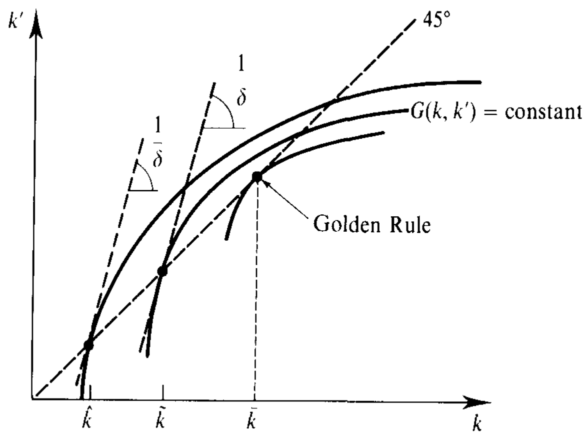


Figure 20.E.3
An example with several modified golden rules.

supported by proportional prices $q_t = (1 + r(k))q_{t+1}$ where, to insure profit maximization, q_t/q_{t+1} must be equal to the slope of the level curve through (k, k) (you should verify this in Exercise 20.E.6). Therefore, the efficient steady states, those with $r(k) \geq 0$, correspond to the subset of the diagonal that goes from the origin to the golden rule, where $r(\bar{k}) = 0$. In the special case of the Ramsey–Solow model we have $G(k, k') = F(k, 1) - k'$ and therefore the level curves of $G(k, k')$ admit a quasilinear representation with respect to k' (i.e., they can be generated from each other by parallel displacement along the k' axis). In Exercise 20.E.7 you are asked to show that this guarantees the satisfaction of the neoclassical properties. In general, however, it is clear from Figure 20.E.3 that we may, for example, have two different $\hat{k}, \tilde{k} < \bar{k}$ such that, at the diagonal, the corresponding level curves have the same slope and therefore $r(\hat{k}) = r(\tilde{k})$ (contradicting the second neoclassical property). In particular, while the golden rule is unique [if the function $G(k, k')$ is strictly concave], there may be several modified golden rules [this is the case if, say, the discount factor δ is equal to the interest rate $r(\hat{k})$].

20.F Dynamics

In this section, we offer a few observations on the vast topic of the dynamic properties of equilibria. The basic framework is as in the previous section: a one-consumer economy with stationary technology and utility.

The arbitrarily given initial conditions[23] will typically not be compatible with a stationary equilibrium situation (e.g., the steady-state level of capital may be higher than the initial availability of capital). Therefore, the typical equilibrium path will be nonstationary. How complicated can the equilibrium dynamics be? Can we, for example, expect convergence to a modified golden rule? This would be nice, as it would tell us that our models carry definite long-run predictions.

We can gain much insight into these matters by considering a variation of the two-sector model of Example 20.C.3. We assume that the technology produces consumption goods (possibly of more than one kind) out of labor and a capital good. There is, as initial endowment, one unit of labor in each period, and we let $u(k, k')$ stand for the maximum utility that can be attained in any given period if in the previous period $k \in \mathbb{R}$ units of capital were installed and the current investment is required to be k' (and, in both periods, a unit of labor is used). There is a positive initial endowment of capital only at $t = 0$. Also, we take $u(\cdot, \cdot)$ to be strictly concave and differentiable.

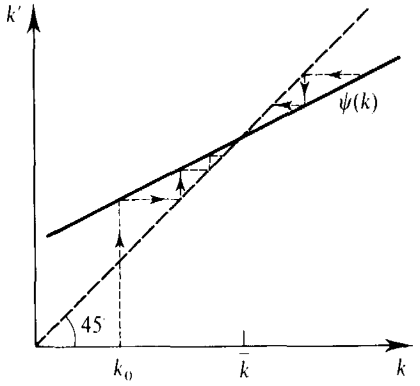
We know from Proposition 20.D.3 and 20.D.4 that the equilibrium paths can be determined by means of the following planning problem:

$$\text{Max} \sum_t \delta^t u(k_{t-1}, k_t) \tag{20.F.1}$$
$$\text{s.t. } k_t \geq 0 \text{ and } k_0 = k \text{ is given.}$$

Suppose that $V(k)$ and $\psi(k)$ are value and policy functions, respectively, for the problem (20.F.1). These concepts were introduced in Section 20.D. As we explained there, the equilibrium dynamics are entirely determined by iterating the policy function [see expression (20.D.10)]. That is, given k_0, the equilibrium trajectory is

$$(k_0, k_1, k_2, \ldots) = (k_0, \psi(k_0), \psi(\psi(k_0)), \ldots).$$

23. That is, the initial endowment sequence $(\omega_0, \ldots, \omega_t, \ldots)$.



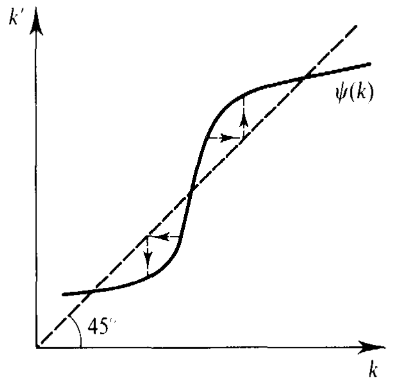
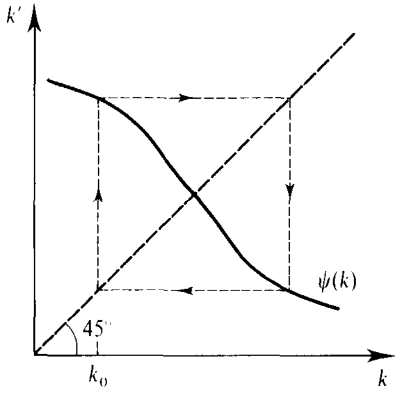


Figure 20.F.1 (left)
A single, stable steady state.

Figure 20.F.2 (right)
Several steady states, no cycles.



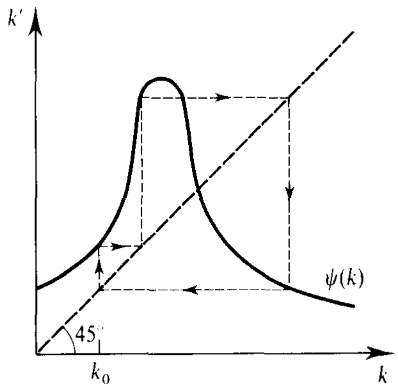


Figure 20.F.3 (left)
A single steady state and a cycle of period 2.

Figure 20.F.4 (right)
A cycle of period 3: chaos.

Note that a steady-state path $(\bar{k}, \ldots, \bar{k}, \ldots)$ is an equilibrium path (for $k_0 = \bar{k}$), and therefore a modified golden rule steady state path for discount factor δ (see Definition 20.E.2 and the discussion surrounding it), if and only if $\bar{k} = \psi(\bar{k})$.

Figures 20.F.1 through 20.F.4 represent four mathematical possibilities for this equilibrium dynamics. In Figure 20.F.1, we have the simplest possible situation: a monotonically increasing policy function with a single steady state $\bar{k}$. The steady state is then necessarily globally stable; that is, $k_t \to \bar{k}$ for any k_0. In Figure 20.F.2, the policy function is again monotonically increasing, but now there are several steady states. They have different stability properties, but it is still true that from any initial point we converge to some steady state. In Figure 20.F.3, the steady state is unique, but now the policy function is not increasing and cycles are possible. Finally, in Figure 20.F.4 we have a policy function that generates a cycle of period 3. It is known that a one-dimensional dynamical system exhibiting a nontrivial cycle of period 3 is necessarily *chaotic* [see Grandmont (1986) for an exposition of the mathematical theory]. We cannot go here into an explanation of the term "chaotic" in this context. It suffices to say that the equilibrium trajectory may wander in a complicated way and that its location in the distant future is very sensitive to initial conditions. The theoretical possibility of chaotic equilibrium trajectories is troublesome from the economic point of view. How is it to be expected that an auctioneer will succeed in computing them; or even worse, how would a consumer exactly anticipate such a sequence?

Unfortunately, the "anything goes" principle that haunted us in Chapter 17 in the form of the Sonnenschein–Mantel–Debreu theorem (Section 17.E) reemerges here in the guise of the Boldrin–Montruccio theorem [see Boldrin and Montruccio (1986)]: *Any candidate policy function $\psi(k)$ can be generated from some concave $u(k, k')$ and $\delta > 0$.* We will not state or demonstrate this theorem precisely, but the main idea of its proof is quite accessible. We devote the next few paragraphs to it.

Suppose for a moment that for a given $u(\cdot, \cdot)$ our candidate $\psi(\cdot)$ is such that $\psi(k)$ solves, for every k, the following "complete impatience" problem:

$$\underset{k' \geq 0}{\text{Max}} \quad u(k, k'). \tag{20.F.2}$$

This would be the problem of a decision maker who did not care about the future. While this is not quite the problem that we want to solve, it approximates it if we take $\delta > 0$ to be very low. Then the decision maker cares very little about the future and therefore its optimal action k' will, by continuity, be very close to $\psi(k)$. Hence, in an approximate sense, we are done if we can find a $u(\cdot, \cdot)$ such that $\psi(k)$ solves (20.F.2) for every k.

In order for a $\psi(k) > 0$ to solve (20.F.2), $u(k, \cdot)$ cannot be everywhere decreasing in its second argument (the optimal decision would then be $k' = 0$). In the simplest version of the Ramsey–Solow model (Example 20.C.1), the returns of k', the investment in the current period, accrue only in the next period, and therefore the utility function $u(k, k')$ is decreasing in k'. But in the current, more general, two-sector model there is no reason that forces this conclusion. Suppose, for example, that there are two consumption goods. The first is the usual consumption–investment good, while the second is a pure consumption good not perfectly substitutable with the first. Say that with an amount k of investment at time $t - 1$ one gets, jointly, k units of the consumption–investment good at time t and k units of the second consumption good at time $t - 1$. Accordingly, with k' units of the consumption–investment good invested at t one gets, jointly, k' units of the consumption–investment good at $t + 1$ and k' units of the second consumption good at t. Thus, if k and k' are the amounts of investment at $t - 1$ and t, respectively, then the bundle of consumption goods available at t is $(k - k', k')$. Hence, the utility function $u(\cdot, \cdot)$ has the form $u(k, k') = \hat{u}(k - k', k')$, where $\hat{u}(\cdot, \cdot)$ is a utility function for bundles of the two consumption goods.

Therefore, our problem is reduced to the following: Given $\psi(k)$ can we find $\hat{u}(\cdot, \cdot)$ such that $\psi(k)$ solves $\text{Max}_{k'}\, \hat{u}(k - k', k')$ for all k in some range? The problem is represented in Figure 20.F.5.[24] We see from the figure that the problem has formally become one of finding a concave utility function with a prespecified Engel curve at some given prices (in our case, the two prices are equal). Such a utility function can always be obtained. It is a well-known, and most plausible fact that the concavity of $\hat{u}(\cdot)$ imposes no restrictions on the shape that a single Engel curve may exhibit (see Exercise 20.F.1).

The news is not uniformly bad, however. In principle, as we have seen, everything may be possible; yet there are interesting and useful sufficient conditions implying a

24. We also assume that $\psi(k) < k$ for all k.

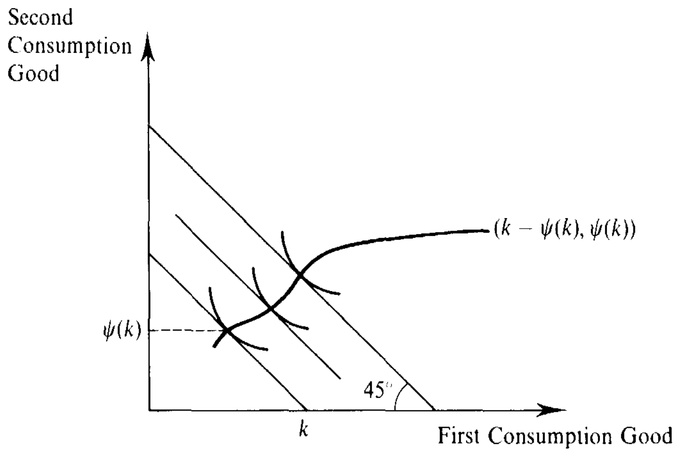


Figure 20.F.5 Construction of an arbitrary policy function in the completely impatient case.

well-behaved dynamic behavior. We discuss two types of conditions: a *low discount of time* and *cross derivatives of uniform positive sign.*

Low Discount of Time

One of the most general results of dynamic economics is the *turnpike theorem*, which, informally, asserts that *if the one-period utility function is strictly concave and the decision maker is very patient, then there is a single modified golden rule steady state that, moreover, attracts the optimal trajectories from any initial position.*

In the context of the two-sector model studied in this section, we can give some intuition for the turnpike theorem. Suppose that the value function $V(k)$, which is concave, is twice-differentiable.[25] At the end of Section 20.D, we saw that since by definition,

$$V(k + z) \geq u(k + z, \psi(k)) + \delta V(\psi(k))$$

for all z and k (with equality for $z = 0$), we must have

$$V'(k) = \nabla_1 u(k, \psi(k)) \qquad \text{and} \qquad V''(k) \geq \nabla^2_{11} u(k, \psi(k)) \qquad \text{for all } k.$$

Also for all k, $\psi(k)$ solves the first-order condition

$$\nabla_2 u(k, \psi(k)) + \delta V'(\psi(k)) = 0. \tag{20.F.3}$$

Differentiating this first-order condition, we have (all the derivatives are evaluated at k, $\psi(k)$ and assumed to be nonzero)

$$\psi'(\cdot) = -\frac{\nabla^2_{21} u(\cdot)}{\nabla^2_{22} u(\cdot) + \delta V''(\cdot)}.$$

Because $\nabla^2_{22} u(\cdot) \leq 0$ and $\delta \nabla^2_{11} u(\cdot) \leq \delta V''(\cdot) \leq 0$, it follows that

$$|\psi'(\cdot)| \leq \left| \frac{\nabla^2_{21} u(\cdot)}{\nabla^2_{22} u(\cdot) + \delta \nabla^2_{11} u(\cdot)} \right|.$$

By the concavity of $u(\cdot)$ we have (see Sections M.C and M.D of the Mathematical Appendix)

$$(\nabla^2_{21} u(\cdot))^2 \leq \nabla^2_{11} u(\cdot)\, \nabla^2_{22} u(\cdot) < (\nabla^2_{11} u(\cdot) + \nabla^2_{22} u(\cdot))^2.$$

25. For a (very advanced) discussion of this assumption, see Santos (1991).

Hence, if the discount factor δ is close to 1, it is a plausible conclusion that $|\psi'(k)| < 1$ for all k. In technical language: $\psi(\cdot)$ is a *contraction*, and this implies global convergence to a unique steady state.[26] In Exercise 20.F.2 you are invited to draw the policy functions and the arrow diagrams for this case. A particular instance of a contraction is exhibited in Figure 20.F.1.

The turnpike theorem is valid for any number of goods. The precise statement and the proof of the theorem are subtle and technical [see McKenzie (1987) for a brief survey], but the main logic is simply conveyed. Consider the extreme case where there is complete patience, that is, "only the long-run matters." A difficulty is that it is not clear what this means for arbitrary paths; but at least for paths that are not too "wild," say for those that from some time become cyclical, it is natural to assume that it means that the paths are evaluated by taking the average utility over the cycle. Observe now that *for any cyclical nonconstant path, the strict concavity of the utility function implies that the constant path equal to the mean level of capital over the cycle yields a higher utility.* It may take some time to carry out a transition from the cycle to the constant path (e.g., it may be necessary to build up capital) but, as long as this can be done in a finite number of periods, the cost of the transition will not show up in the long run. Hence the cyclical nonconstant path cannot be optimal for a completely patient optimizer. By continuity, all this remains valid if δ is very close to 1. We can conclude, therefore, that if a path tends to a nonconstant cycle then we can always implement a finite transition to a suitable constant "long-run average," for a relatively large long-run gain of utility and a relatively low short-run cost. In fact, this conclusion remains valid whenever a path does not stabilize in the long-run. It follows that the optimal path must be asymptotically almost constant, which can only be the case if the path reaches and remains in a neighborhood of a modified golden rule steady state (recall from Section 20.E that those are the only constant paths that can be equilibria, and therefore optimal).[27]

Cross Derivative of Uniform Positive Sign

We shall concern ourselves here with the particular case of the two-sector model studied so far where $\nabla_1 u(k, k') > 0$ and $\nabla_2 u(k, k') < 0$ for all (k, k'). By a *cross derivative of uniform positive sign we mean that* $\nabla_{12} u(k, k') > 0$, again at *all points of the domain*. In words: An increase in investment requirements at one date leads to a situation of increased productivity (in terms of current utility) of the capital installed the previous date. Examples are the classical Ramsey–Solow model $u(F(k) - k')$ and the cost-of-adjustment model $u(F(k) - k' - \gamma(k' - k))$ (see Exercise 20.F.3). We shall argue that *under this cross derivative condition the policy function is increasing* (as in Figures 20.F.1 or 20.F.2), and therefore the optimal path converges to a stationary path.

To prove the claim, it is useful to express $\psi(k)$ as the k' solution to

$$\begin{aligned} \operatorname*{Max}_{(k',V)} \quad & u(k, k') + \delta V \\ \text{s.t. } & V \leq V(k'), \end{aligned} \tag{20.F.4}$$

26. We note that $\psi(\cdot)$ need not be monotone and the convergence may be cyclical, although the cycles will dampen through time.

27. Also, with δ close to 1, the modified golden rule will typically retain the uniqueness property of the golden rule.

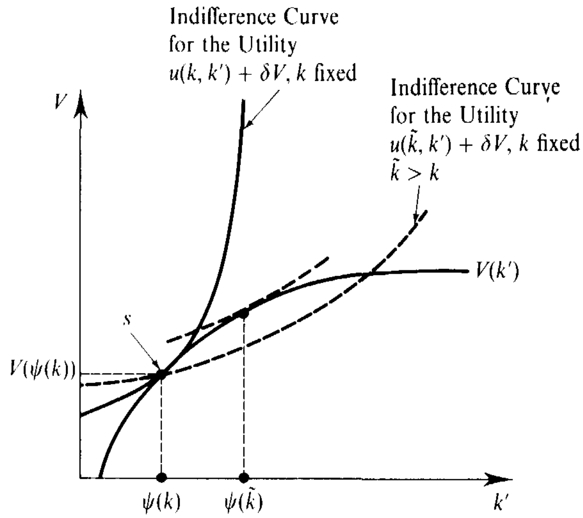


Figure 20.F.6 With the uniform positive sign cross derivative condition, the policy function is increasing.

where $V(\cdot)$ is the value function. For fixed k, problem (20.F.4) is represented in Figure 20.F.6. The marginal rate of substitution (MRS) between current investment k' and future utility V at $s = (\psi(k), V(\psi(k)))$ is $(1/\delta)\nabla_2 u(k, \psi(k)) < 0$. Suppose now that we take $\tilde{k} > k$. Then the indifference map in Figure 20.F.6 changes. Because $\nabla_{12} u(k, \psi(k)) > 0$, the MRS at s is altered in the manner displayed in the figure, that is, the indifference curve becomes flatter. But we can see then that necessarily $\psi(\tilde{k}) > \psi(k)$, as we wanted to show.

The cross derivative condition does not, by itself, imply the existence of a single modified golden rule. Thus, we could be in Figure 20.F.2 rather than in Figure 20.F.1. Note, however, that in many cases of interest it may be possible to show directly that the modified golden rule is unique. Thus, in both the classical Ramsey–Solow model of Example 20.C.1 and in the cost-of-adjustment model [with $\gamma'(0) = 0$] of Example 20.C.2, the modified golden rule is characterized by $F'(k) = 1/\delta$. Hence it is unique and, because the policy function is increasing, we conclude that every optimal path converges to it.

We also point out that if the cross derivative is of uniform *negative* sign, then, by the same arguments, $\psi(\cdot)$ is *decreasing*. While this allows for cycles, the dynamics are still relatively simple. In particular, the nonmonotonic shape associated with the possibility of chaotic paths (Figure 20.F.4) cannot rise. See Deneckere and Pelikan (1986) for more on these points.

Figure 20.F.6 is also helpful in illuminating the distinction between *transitory* and *permanent* shocks. One of the important uses of dynamic analysis in general, and of global convergence turnpike results in particular, is in the examination of how an economy at long-run rest reacts to a perturbation of the data at time $t = 1$. In an extremely crude classification, these perturbations can be of two types:

(i) *Transitory* shocks affect the environment of the economy only at $t = 1$; that is, they alter k_0 or, more generally, $u(k_0, \cdot)$, the utility function at $t = 1$. Then Figure 20.F.6 allows us to see how the equilibrium path will be displaced. The (k', V) indifference curve of $u(k_0, k') + \delta V$ changes, but the constraint function $V(k')$ remains unaltered. Hence, after the (transitory) shock, the new k_1'' corresponds to the solution of the optimum problem depicted in Figure

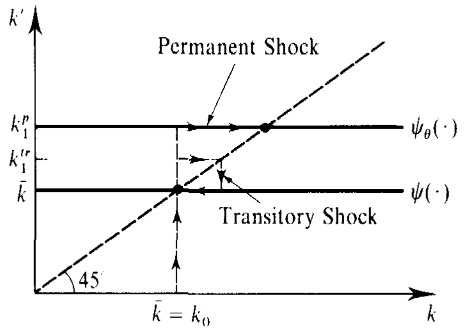


Figure 20.F.7
An example of dynamic adjustment under transitional and permanent shocks.

20.F.6 but with the new indifference map. From $t = 2$ on we simply follow the old policy function.

(ii) *Permanent* shocks move the economy to a new utility function $\hat{u}(k, k')$ constant over time. Then the entire policy function changes to a new $\hat{\psi}(\cdot)$. In terms of Figure 20.F.6 there would be a change in both the indifference curves *and* the constraint. The new k_1^p is now harder to determine and to compare with the preshock k_1 or, for the same shock at period 1, with k_1^{tr}; but it can often be done. We pursue the matter through Example 20.F.1.

Example 20.F.1: Consider the separable utility $u(k, k') = g(k) + h(k')$. This could be the investment problem of a firm: $g(k)$ is the maximal revenue obtainable with k, and $-h(k')$ is the cost of investment. Then $\nabla_{12}^2 u(k, k') = 0$ at all (k, k'). Our previous analysis of Figure 20.F.6, tells us that in this case $\psi(\cdot)$ is constant; that is, from any k_0 the economy goes in one step to its steady-state value $\bar{k}$. This is illustrated in Figure 20.F.7.

Suppose now there is a shock variable θ such that $u(k, k', \theta) = g(k, \theta) + h(k', \theta)$, with the preshock value being $\theta = 0$. The economy is initially at its steady state $\bar{k}$.

If there is a transitory shock to a small $\theta > 0$, then from the analysis of Figure 20.F.6 we can see that $k_1^{tr} \gtrless \bar{k}$ according to $\partial^2 h(\bar{k}, 0)/\partial k' \, \partial\theta \gtrless 0$. (Exercise 20.F.4 asks you to verify this.)

To evaluate the effects of a permanent shock to a small $\theta > 0$ (and therefore to a new $\psi_\theta(\cdot)$) the term

$$\partial^2 V(\bar{k}, 0)/\partial k \, \partial\theta = \partial^2 g(\bar{k}, 0)/\partial k \, \partial\theta$$

also matters [the previous equality follows from expression (20.F.3)]. Suppose, for example, that the shock is unambiguously favourable, in the sense that $\partial^2 g(\bar{k}, 0)/\partial k \, \partial\theta > 0$ and $\partial^2 h(\bar{k}, 0)/\partial k' \, \partial\theta > 0$. Then a careful analysis of Figure 20.F.6, would allow us to conclude that $k_1^p > k_1^{tr} > \bar{k}$. (Exercise 20.F.5 asks you to verify this. Note that the indifference map of Figure 20.F.6 is quasilinear with respect to V.) Figure 20.F.7 illustrates this case further. ■

20.G Equilibrium: Several Consumers

Up to now we have had a single consumer, or, to be more precise, a single type of consumer. The extension of the definition of equilibrium to economies with several consumers, say I, presents no particular difficulty. We simply have to rewrite Definition 20.D.1 as in Definition 20.G.1.

Definition 20.G.1: The (bounded) production path $(y_0^*, \ldots, y_t^*, \ldots)$, $y_t^* \in Y$, the (bounded) price sequence $(p_0, \ldots, p_t, \ldots) \geq 0$, and the consumption streams

$(c^*_{0i}, \ldots, c^*_{ti}, \ldots) \geq 0$, $i = 1, \ldots, I$, constitute a *Walrasian* (or *competitive*) equilibrium if:

(i)
$$\sum_i c^*_{ti} = y^*_{a,t-1} + y^*_{bt} + \sum_i \omega_{ti}, \text{ for all } t. \tag{20.G.1}$$

(ii) For every t,
$$\pi_t = p_t \cdot y^*_{bt} + p_{t+1} \cdot y^*_{at} \geq p_t \cdot y_{bt} + p_{t+1} \cdot y_{at} \tag{20.G.2}$$
for all $y = (y_{bt}, y_{at}) \in Y$.

(iii) For every i, the consumption stream $(c^*_{0i}, \ldots, c^*_{ti}, \ldots) \geq 0$ solves the problem
$$\text{Max } \sum_t \delta_i^t u_i(c_i) \tag{20.G.3}$$
$$\text{s.t. } \textstyle\sum_t p_t \cdot c_{ti} \leq \sum_t \theta_{ti} \pi_t + \sum_t p_t \cdot \omega_{ti} = w_i.$$
where θ_{ti} is consumer i's given share of period t profits.

The first, and very important, observation to make is that, in complete analogy with the finite-horizon case (see Section 16.C), the first welfare theorem holds.[28]

Proposition 20.G.1: A Walrasian equilibrium allocation is Pareto optimal.

Proof: The proof is as in Proposition 16.C.1. Let the Walrasian equilibrium path under consideration be given by the production path $(y^*_0, \ldots, y^*_t, \ldots)$, the consumption streams $(c^*_{0i}, \ldots, c^*_{ti}, \ldots)$, $i = 1, \ldots, I$, and the price sequence $(p_0, \ldots, p_t, \ldots)$. Suppose now that the paths $(y_0, \ldots, y_t, \ldots)$, and $(c_{0i}, \ldots, c_{ti}, \ldots) \geq 0$, $i = 1, \ldots, I$, are feasible [i.e., they satisfy condition (i) of Definition 20.G.1] and are Pareto preferred to the Walrasian equilibrium.

By the utility-maximization condition we have $\sum_t p_t \cdot c_{ti} \geq w_i$ for all i, with at least one inequality strict. Hence,
$$\sum_t p_t \cdot \left(\sum_i c_{ti}\right) = \sum_i \left(\sum_t p_t \cdot c_{ti}\right) > \sum_i w_i. \tag{20.G.4}$$

Because of the profit maximization condition we get[29]
$$\begin{aligned}
\sum_t p_t \cdot \left(\sum_i c_{ti}\right) &= \sum_t p_t \cdot \left(y_{a,t-1} + y_{bt} + \sum_i \omega_{ti}\right) \\
&= \sum_t p_t \cdot y_{a,t-1} + \sum_t p_t \cdot y_{bt} + \sum_t \sum_i p_t \cdot \omega_{ti} \\
&= \sum_{t \geq 1} (p_{t-1} \cdot y_{b,t-1} + p_t \cdot y_{a,t-1}) + \sum_i \sum_t p_t \cdot \omega_{ti} \\
&\leq \sum_t \pi_t + \sum_i \sum_t p_t \cdot \omega_{ti} = \sum_i w_i.
\end{aligned}$$

But this conclusion contradicts (20.G.4). ∎

28. Note also that, in the terminology of Chapter 19, the market structure is complete: Every consumer has a single budget constraint and, therefore, only prices limit the possibilities of transferring wealth across periods.

29. Recall that, by convention, $y_{a,-1} = 0$.

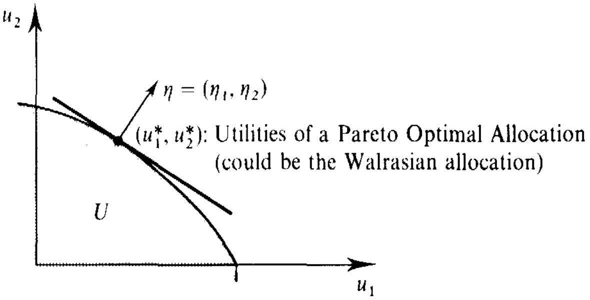


Figure 20.G.1
The Walrasian equilibrium as a solution of a planning problem.

We saw in Sections 16.E and 16.F that, under the assumption of concave utility functions, a Pareto optimal allocation of an economy with a finite number of commodities can be viewed as the solution of a planning problem. As described in Figure 20.G.1, the objective function of the planner is a weighted sum of the utilities of the different consumers (the weights being the reciprocal of the marginal utilities of wealth at the equilibrium with transfers associated with the particular Pareto optimum). The arguments of Section 16.E (in particular, Proposition 16.E.2) apply as well to the current infinite-horizon case. Therefore, Proposition 20.G.1 has, besides its substantive interest, a significant methodological implication. It tells us that the prices, productions, and aggregate consumptions of a given Walrasian equilibrium correspond exactly to those of a certain single-consumer economy. We give a more precise statement in Proposition 20.G.2. In it we restrict ourselves to the case of a common discount factor, namely, $\delta_i = \delta$ for all i.

Proposition 20.G.2: Suppose that $(y_0^*, \ldots, y_t^*, \ldots)$ is the production path and $(p_0, \ldots, p_t, \ldots)$ is the price sequence of a Walrasian equilibrium of an economy with I consumers. Then there are weights $(\eta_1, \ldots, \eta_I) \gg 0$ such that $(y_0^*, \ldots, y_t^*, \ldots)$ and $(p_0, \ldots, p_t, \ldots)$ constitute a Walrasian equilibrium for the one-consumer economy defined by the utility $\sum_t \delta^t u(c_t)$, where $u(c_t)$ is the solution to Max $\sum_i \eta_i u_i(c_{ti})$ s.t. $\sum_i c_{ti} \le c_t$.

Proof: We will not give a rigorous proof, but the result is intuitive from Figure 20.G.1. From there we see (technically this involves, as in Proposition 16.E.2, an application of the separating hyperplane theorem) that there are weights $(\eta_1, \ldots, \eta_I) \gg 0$ such that the equilibrium consumption streams maximize $\sum_i \eta_i (\sum_t \delta^t u_i(c_{ti}))$ over all feasible consumption streams, or, equivalently (it is here that the assumption of a common discount factor matters), the aggregate equilibrium consumption stream, solves the two-step planning problem specified by the definition of $u(c_t)$ and the maximization of $\sum_t \delta^t u(c_t)$. Because we already know (Proposition 20.D.4) that this is tantamount to the one-consumer equilibrium problem, we are done. ■

Proposition 20.G.2 allows us to conclude that the one-consumer theory developed in the last three sections remains highly relevant to the several-consumer case.[30] Somewhat informally, we can distinguish two types of properties of an equilibrium.

30. More generally, it remains highly relevant to any equilibrium model that guarantees the Pareto optimality of equilibria.

The *internal* properties are those that refer only to the structure of an equilibrium viewed solely in reference to itself (e.g., convergence to a steady state); the *external* properties refer to how the equilibrium relates to other possible equilibrium trajectories of the economy (e.g., uniqueness or local uniqueness). The message of Proposition 20.G.2 is that, because of Pareto optimality, the internal properties of an equilibrium of an economy with several consumers are those of its associated one-consumer economy. The implications of the one-consumer theory should not, however, be pushed beyond the internal properties. The reason is that *the weights defining the planning problem depend on the particular equilibrium considered.* For example, it is perfectly possible for there to be more than one equilibrium, each a Pareto optimum but supported by different weights.

What can be said about the determinacy properties of equilibrium; for example, about the finiteness of the number of equilibria? We will not be able to give a precise treatment of this matter, in part because it is very technical and in part because it is still an active area of research where the ultimate results may not yet be at hand. The basic intuition, however, can be transmitted. We begin by pointing out another implication of Proposition 20.G.1. Formally, our infinite-horizon model involves infinitely many variables (prices, say) and infinitely many equations (Euler equations, say). This is most unpleasant, as the mathematical theory described in Section 17.D applies only (and for good reasons, as we shall see in Section 20.H) to systems with a *finite* number of equations and unknowns. However, Proposition 20.G.1 allows us to view the equilibrium problem as one of finding not equilibrium prices but *equilibrium weights* η. If we do this then the equilibrium equations in our system are $I - 1$ in number, the same as the number of unknowns. More precisely, the ith equation would associate with the vector of weights $\eta = (\eta_1, \dots, \eta_I)$, $\sum_i \eta_i = 1$, the wealth "gap" of consumer i:

$$\sum_t p_t(\eta)\cdot c_{ti}(\eta) - \sum_t (\theta_{ti}\pi_t(\eta) + p_t(\eta)\cdot\omega_{ti}) = 0,$$

where $p_t(\eta)$, $c_{ti}(\eta)$, and $\pi_t(\eta)$ correspond to the Pareto optimum indexed by η. See Appendix A of Chapter 17 for a construction similar to this. At any rate, once looked at as a wealth-equilibrating problem across a finite number of consumers, the central conjecture should be that, as in Chapter 17, the equilibrium set is nonempty and generically finite. That is, equilibrium exists and, except for pathological cases, there are only a finite number of weights solving the equilibrium equations (we could similarly go on to formulate an index theorem). Technical difficulties[31] aside, this central conjecture can be established in a wide variety of cases [see Exercise 20.G.3 and Kehoe and Levine (1985)].

We end this section with two remarks. The first derives from the question: Is there a relationship, a "correspondence," between internal and external properties? At least in a first approximation the answer is "no." We have seen that in a one-consumer economy the equilibrium is unique, but the equilibrium path may be complicated. Similarly, in a several-consumer economy there may be several equilibria, or even a continuum, each of them nicely converging to a steady state.[32]

31. These have to do with guaranteeing the differentiability of the relevant functions.

32. The simplest, trivial, example is the following. Suppose that $L = 2$, $I = 2$ and that there is no possibility of intertemporal production. Individual endowments are constant through time and the utility functions are concave. Then the intertemporal Walrasian equilibria correspond exactly to the infinite, constant repetitions of the one-period Walrasian equilibria (you are asked to prove this in Exercise 20.G.4). Because there are may be several of those, we obtain our conclusion.

The second remark brings home the point that Pareto optimality is key to an expectation of generic determinacy. Consider, as an example, a model of identical consumers but with an externality. The utility function, $u(k, k', e)$, now has three arguments: k and k' are the capital investments in the previous and the current periods, respectively, and e is the level of currently felt externality. Given, for the moment, an exogenously fixed externality path $(e_0, \ldots, e_t, \ldots)$, the (bounded, strictly interior) capital trajectory k_t is an equilibrium if it solves the planning problem for the utility functions $u(\cdot, \cdot, e_t)$, that is, if it satisfies the Euler equations:

$$\nabla_2 u(k_{t-1}, k_t, e_t) + \delta \nabla_1 u(k_t, k_{t+1}, e_{t+1}) = 0 \qquad \text{for all } t.$$

An overall equilibrium must take into account the technology determining the externality. Say that this is $e_t = k_t$; that is, the externality is a side product of current investment. Hence, the equilibrium conditions are

$$\nabla_2 u(k_{t-1}, k_t, k_t) + \delta \nabla_1 u(k_t, k_{t+1}, k_{t+1}) = 0 \qquad \text{for all } t. \tag{20.G.5}$$

Suppose that starting from an equilibrium steady state ($k_t = \bar{k}$ for all t), we try, as we did in Section 20.D, to generate a different equilibrium by fixing $k_0 = \bar{k}$, taking k_1 to be slightly different from $\bar{k}$, and then iteratively solving (20.G.5) for k_{t+1}. A sufficient (but not necessary) condition for this method to succeed is that $|dk_{t+1}/dk_t| < \frac{1}{2}$ and $|dk_{t+1}/dk_{t-1}| < \frac{1}{2}$, where the values dk_{t+1}/dk_t and dk_{t+1}/dk_{t-1} are obtained by applying the implicit function theorem to (20.G.5) and evaluating at the steady state. Indeed, if this condition holds, then the initial perturbation of k_1 induces a sequence of adjustments that dampen over time and that will, therefore, never become unfeasible (and, in fact, will remain bounded and strictly interior). Explicitly:

$$\frac{dk_{t+1}}{dk_t} = -\frac{\nabla_{22}^2 u(\cdot) + \nabla_{23}^2 u(\cdot) + \delta \nabla_{11}^2 u(\cdot)}{\delta(\nabla_{12}^2 u(\cdot) + \nabla_{13}^2 u(\cdot))}. \tag{20.G.6}$$

If there are no externalities [i.e., if $\nabla_{23}^2 u(\cdot) = \nabla_{13}^2 u(\cdot) = 0$] then the concavity of $u(\cdot, \cdot)$ implies that expression (20.G.6) is larger than 1 in absolute value (you should verify this in Exercise 20.G.5). Thus, in agreement with the discussion of Section 20.D, we are not then able to find a non-steady-state solution of the Euler equations. But if the externality effects are significant enough, inspection of expression (20.G.6) tells us immediately that dk_{t+1}/dk_t can perfectly well be less than $\frac{1}{2}$ in absolute value. The same is true for $dk_{t+1}/\delta k_{t-1}$, and therefore we can conclude that robust examples with a continuum of equilibria are possible.

20.H Overlapping Generations

In the previous sections we have studied economies that, formally, have an overlapping structure of firms but only one (or, in Section 20.G, several), infinitely long-lived, consumer. We pointed out in Section 20.B that in the presence of suitable forms of altruism it may be possible to interpret an infinitely long-lived agent as a dynasty. We will now describe a model where this cannot be done, and where, as a consequence, the consumption side of the economy consists of an infinite succession of consumers in an essential manner. To make things interesting, these consumers, to be called *generations*, will overlap, so that intergenerational trade is possible. The model originates in Allais (1947) and Samuelson (1958) and has become a workhorse of macroeconomics, monetary theory, and public finance. The literature on it is very extensive; see Geanakoplos (1987) or Woodford (1984) for an overview. Here we will limit ourselves to discussing a simple case with the purpose of highlighting, first, the extent to which the model can be analyzed with the Walrasian equilibrium methodology and, second, the departures from the broad lessons of the previous sections. We shall classify these departures into two categories: issues relating to optimality and issues relating to the multiplicity of equilibria.

We begin by describing an economy that, except for the infinity of generations, is as simple as possible. We have an infinite succession of dates $t = 0, 1, \ldots$ and in every period a single consumption good. For every t there is a generation born at time t, living for two periods, and having utility function $u(c_{bt}, c_{at})$ where c_{bt} and c_{at} are, respectively, the consumption of the tth generation when young (i.e., in period t), and its consumption when old (i.e., in period $t + 1$); the indices b and a are mnemonic symbols for "before" and "after." Note that the utility functions of the different generations over consumption in their lifespan are identical. We assume that $u(\cdot, \cdot)$ is quasiconcave, differentiable and strictly increasing.

Every generation t is endowed when young with a unit of a primary factor (e.g., labor). This primary factor does not enter the utility function and can be used to produce consumption goods contemporaneously by means of some production function $f(z)$.[33] Say that $f(1) = 1$. Under the competitive price-taking assumption, total profits at t, in terms of period-t good, will be $\varepsilon = 1 - f'(1)$ and, correspondingly, labor payments will be $1 - \varepsilon$. Thus, we may as well directly assume that the initial endowments of generation $t \geq 0$ are specified to us as a vector of consumption goods $(1 - \varepsilon, 0)$. In addition, we assign the infinite stream of profits to generation 0. That is, the technology $f(\cdot)$ is an infinitely long-lived asset owned at $t = 0$ by the only generation alive in that period and yielding a permanent profit stream of $\varepsilon > 0$ units of consumption good.

Now let $(p_0, \ldots, p_t, \ldots)$ be an infinite sequence of (anticipated) prices. We do not require that it be bounded. For the budget constraint of the different generations we take

$$p_t c_{bt} + p_{t+1} c_{at} \leq (1 - \varepsilon) p_t \qquad \text{for } t > 0 \tag{20.H.1}$$

and

$$p_0 c_{b0} + p_1 c_{a0} \leq (1 - \varepsilon) p_0 + \varepsilon \left(\sum_t p_t \right) + M. \tag{20.H.2}$$

These budget constraints deserve comment. For $t > 0$, (20.H.1) is easy to interpret. The value of the initial endowments, available at t, is $(1 - \varepsilon)p_t$. Part of this amount is spent at time t and the rest, $(1 - \varepsilon)p_t - p_t c_{bt}$, is saved for consumption at $t + 1$. The saving instrument could be the title to the technology, which would thus be bought from the old by the young at t and then sold at $t + 1$ to the new young (after collecting the period $t + 1$ return). The price paid for the asset is the amount saved, that is, $(1 - \varepsilon)p_t - p_t c_{bt}$. The direct return at $t + 1$ is εp_{t+1} and so, if the asset market is to be in equilibrium, the selling price at $t + 1$ should be $(1 - \varepsilon)p_t - p_t c_{bt} - \varepsilon p_{t+1}$. In summary, in agreement with the budget constraint (20.H.1) this leaves $(1 - \varepsilon)p_t - p_t c_{bt}$ to be spent at $t + 1$.

The constraint (20.H.2) for $t = 0$ is more interesting. Its right-hand side is the value of the asset to generation 0. Note that asset market equilibrium requires that

33. The assumption that production is contemporary with input usage fits well with the length of the period being long.

this value should be at least the *fundamental* value, that is, $\varepsilon(\sum_t p_t)$.[34] Indeed, the value of the asset at $t = 0$ equals the profit return εp_0 plus the price paid by the young of generation 1. At any T, the price paid by the young of generation T should not be inferior to the direct return εp_{T+1}. In turn, at $T - 1$ it should not be inferior to the direct return plus the value at T; that is, it should be at least $\varepsilon(p_T + p_{T+1})$. Iterating, we get the lower bound $\varepsilon(p_1 + \cdots + p_{T+1})$ for the price paid by generation 1, which, going to the limit and adding εp_0, gives $\varepsilon(\sum_t p_t)$ as a lower bound for the value to generation 0. Thus, in terms of expression (20.H.2) a necessary condition for equilibrium is $M \geq 0$. In principle, however, we should allow for the possibility of a *bubble* in the value of the asset (i.e., of $M > 0$). We did not do so in Sections 20.D or 20.G because with a *finite* number of consumers, bubbles are impossible at equilibrium. The equality of demand and supply implies that the (finite) value of total endowments plus total profits equals the value of total consumption, and therefore at equilibrium no individual value of consumption can be larger than the corresponding individual value of endowments and profit wealth (you should verify this in Exercise 20.H.1). We will see shortly that under some circumstances bubbles can occur at equilibrium with infinitely many consumers. It would therefore not be legitimate to eliminate them by definition.

The definition of a *Walrasian equilibrium* is now the natural one presented in Definition 20.H.1.

Definition 20.H.1: A sequence of prices $(p_0, \ldots, p_t, \ldots)$, an $M \geq 0$, and a family of consumptions $\{(c^*_{bt}, c^*_{at})\}_{t=0}^{\infty}$ constitute a *Walrasian* (or *competitive*) *equilibrium* if:

(i) Every (c^*_{bt}, c^*_{at}) solves the individual utility maximization problem subject to the budget constraints (20.H.1) and (20.H.2).

(ii) The feasibility requirement $(c^*_{a,t-1} + c^*_{bt} = 1)$ is satisfied for all $t \geq 0$ (we put $c^*_{a,-1} = 0$).

In a process reminiscent of the iterative procedure (presented in Section 20.D) for the determination of the policy function from the Euler equations, Figures 20.H.1 and 20.H.2 describe how we could attempt to construct an equilibrium. Normalize to $p_0 = 1$. Suppose that we now try to arbitrarily fix c_{a0}. At equilibrium, $c_{b0} = 1$, and thus p_1 is determined by the fact that p_1/p_0 must equal the marginal rate of substitution of $u(\cdot, \cdot)$ at $(1, c_{a0})$. Also, $c_{b1} = 1 - c_{a0}$. This now determines p_2. Indeed, p_2, the price at period 2, should be fixed at a value that induces a level of demand by generation 1 in period 1 of precisely c_{b1} [under the budget set given by p_1, p_2 and wealth $(1 - \varepsilon)p_1$]. With this, the demand of generation 1 in period 2, and therefore the residual amount c_{b2} left in that period for generation 2, has also been determined. But then we may be able to fix p_3 at a value that precisely induces the right amount of demand by generation 2 in period 2, that is, c_{b2}. If we can pursue this construction indefinitely so as to generate an infinite sequence $(p_1, \ldots, p_t, \ldots)$, then we have found an equilibrium. In Figure 20.H.1, where $\varepsilon > 0$, there is a single price sequence (with $p_0 = 1$) that can be continued indefinitely, and therefore a single equilibrium path.

34. Strictly speaking, we are saying that if the consumption good prices are given by $(p_0, \ldots, p_t, \ldots)$ *and* the asset prices present no arbitrage opportunity, then the price of the asset should be at least as large as its fundamental value.

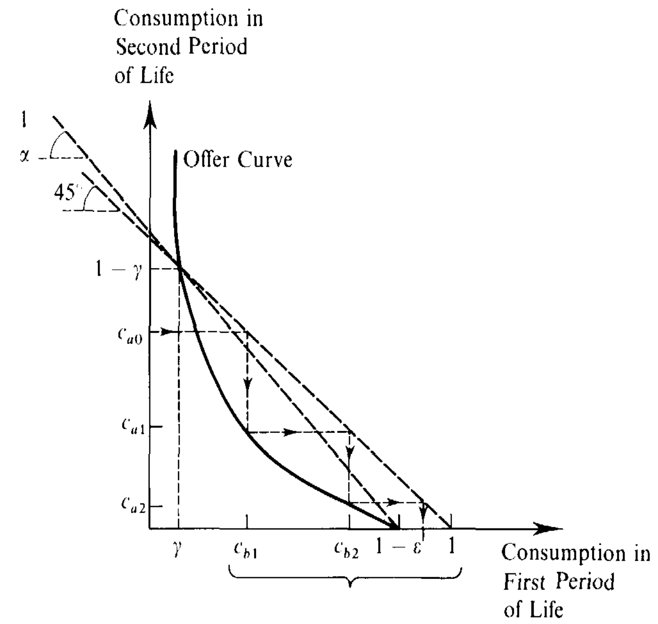


Figure 20.H.1 Overlapping generations: construction of the equilibrium (case $\varepsilon > 0$).

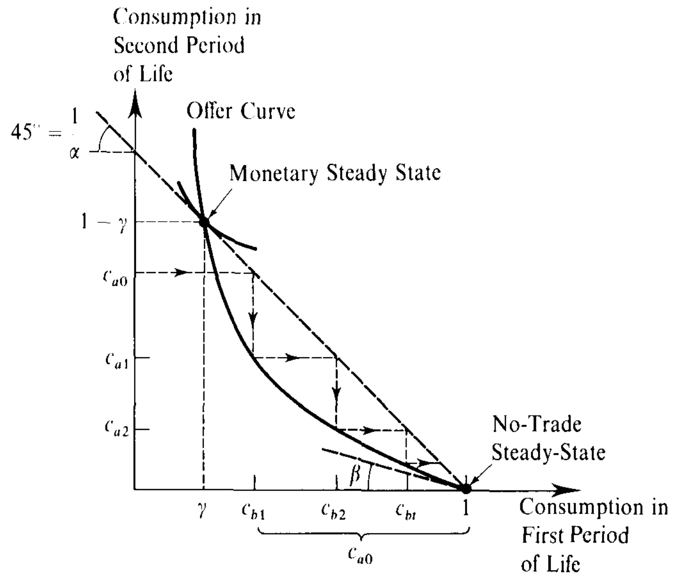


Figure 20.H.2 Overlapping generations: construction of equilibria (case $\varepsilon = 0$).

It corresponds to the stationary consumptions $(\gamma, 1-\gamma)$ and the price sequence $p_t = \alpha^t$, where $\alpha = (1-\varepsilon-\gamma)/(1-\gamma) < 1$. Note that the iterates that begin at a value $c_{a0} \neq 1-\gamma$ unavoidably "leave the picture," that is, become unfeasible. In Figure 20.H.2, where $\varepsilon = 0$, there is a continuum of equilibria: any initial condition $c_{a0} \leq 1-\gamma$ can be continued indefinitely.

It is plausible from Figures 20.H.1 and 20.H.2 that the existence of an equilibrium can be guaranteed under general conditions. This is indeed the case [see Wilson (1981)].

Pareto Optimality

Suppose first that $\varepsilon > 0$. We say then that the asset is *real* (it has "real" returns). At an equilibrium the wealth of generation 0, $(1-\varepsilon)p_0 + \varepsilon(\sum_t p_t) + M$, must be finite (how could this generation be in equilibrium otherwise?). Therefore, if $\varepsilon > 0$, it follows that $\sum_t p_t < \infty$.[35] An important implication of this is that the *aggregate* (i.e., added over all generations) *wealth of society*, which is precisely $\sum_t p_t$, *is finite*. In Proposition 20.H.1 we now show that, as a consequence, the first welfare theorem applies for the model with $\varepsilon > 0$.

Proposition 20.H.1: Any Walrasian equilibrium $(p_0, \ldots, p_t, \ldots)$, $\{(c^*_{bt}, c^*_{at})\}_{t=0}^{\infty}$, with $\sum_t p_t < \infty$ is a Pareto optimum; that is, there are no other feasible consumptions $\{(c_{bt}, c_{at})\}_{t=0}^{\infty}$ such that $u(c_{bt}, c_{at}) \geq u(c^*_{bt}, c^*_{at})$ for all $t \geq 0$, with strict inequality for some t.

Proof: We repeat the standard argument. Suppose that $\{(c_{bt}, c_{at})\}_{t=0}^{\infty}$ Pareto dominates $\{(c^*_{bt}, c^*_{at})\}_{t=0}^{\infty}$. From feasibility, we have $c^*_{bt} + c^*_{a,t-1} = 1$ and $c_{bt} + c_{a,t-1} \leq 1$ for every t. Therefore, $\sum_t p_t(c^*_{bt} + c^*_{a,t-1}) = \sum_t p_t$ and $\sum_t p_t(c_{bt} + c_{a,t-1}) \leq \sum_t p_t$. Because $\sum_t p_t < \infty$, we can rearrange terms and get

$$\sum_t (p_t c_{bt} + p_{t+1} c_{at}) \leq \sum_t (p_t c^*_{bt} + p_{t+1} c^*_{at}) = \sum_t p_t < \infty.$$

Because the utility function is increasing and (c^*_{bt}, c^*_{at}) maximizes utility in the budget set we conclude that $p_t c_{bt} + p_{t+1} c_{at} \geq p_t c^*_{bt} + p_{t+1} c^*_{at}$ for every t, with at least one strict inequality. Therefore, $\sum_t (p_t, c_{bt} + p_{t+1} c_{at}) > \sum_t (p_t c^*_{bt} + p_{t+1} c^*_{at})$. Contradiction. ■

Proposition 20.H.1 is important but it is not the end of the story. Suppose now that the asset is purely *nominal* (i.e., $\varepsilon = 0$; for example, the asset could be fiat money, or ownership of a constant returns technology). Then *it is possible to have equilibria that are not optimal*. In fact, it is easy to see that we can sustain autarchy (i.e., no trade) as an equilibrium. Just put $M = 0$ (no bubble, worthless fiat money) and choose $(p_0, \ldots, p_t, \ldots)$ so that, for every t, the relative prices p_t/p_{t+1} equal the marginal rate of substitution of $u(\cdot, \cdot)$ at $(1, 0)$, denoted by β. This no-trade stationary state (also called *the nonmonetary steady state*) where every generation consumes $(1, 0)$ is represented in Figure 20.H.2. As it is drawn (with $\beta < 1$), we can also see that the no-trade outcome is strictly Pareto dominated by the steady state $(\gamma, 1-\gamma)$ [or, more precisely, by the consumption path in which generation 0 consumes $(1, 1-\gamma)$ and every other generation consumes $(\gamma, 1-\gamma)$]. What is going on is simple: in this example the open-endedness of the horizon makes it possible for the members of every generation t to pass an extra amount of good to the older generation at t and, at the same time, be more then compensated by the amount passed to them at $t+1$ by the next generation. Note that, in agreement with Proposition 20.H.1, the lack of optimality of this no-trade equilibrium entails $p_t/p_{t+1} = \beta < 1$ for all t; that is, prices increase through time.

It is also possible in the purely nominal case for an equilibrium with $M > 0$ not to be Pareto optimal. Note first if $\{(c^*_{bt}, c^*_{at})\}_{t=0}^{\infty}$, $(p_0, \ldots, p_t, \ldots)$ and M constitute an

35. You can also verify this graphically by examining Figure 20.H.1.

equilibrium, then we have (recall that $c_{b0}^* = 1$)

$$p_{t+1}c_{at}^* = p_t(1 - c_{bt}^*) = p_t c_{a,t-1}^* = \cdots = p_1 c_{a0}^* = M \qquad \text{for every } t.$$

Thus, $M = 0$ can occur only at a no-trade equilibrium. In Figure 20.H.2, there is a continuum of equilibria indexed by c_{b1} for $\gamma \leq c_{b1} \leq 1$. The no-trade equilibrium corresponds to $c_{b1} = 1$. But for every $c_{b1} < 1$ with $c_{b1} > \gamma$ we have a nonstationary equilibrium trajectory with trade (hence $M > 0$) which is also strictly Pareto dominated by the steady state $(\gamma, 1 - \gamma)$. Nonetheless, it is still true that for any equilibrium with $c_{b1} > \gamma$ we have $M/p_t \to 0$; that is, in real terms the value of the asset becomes vanishingly small with time. For $c_{b1} = \gamma$, matters are quite different. We have a steady-state equilibrium (called *the monetary steady state*) in which the price sequence p_t is constant and therefore the real value of money remains constant and positive. This monetary steady state is the analog of the *golden rule* of Section 20.E and, as was the case there, we have that, in spite of $\sum_t p_t < \infty$ being violated, the *monetary steady state is Pareto optimal.* We will not give a rigorous proof of this. The basic argument is contained in Figure 20.H.3. There we represent the indifference curve through $(\gamma, 1 - \gamma)$ and check that any attempt at increasing the utility of generation 0 by putting $c_{b1} < \gamma$ leads to an unfeasible chain of compensations; that is, it cannot be done.

The discussion just carried out of the examples in Figures 20.H.2 and 20.H.3 suggests and confirms the following claim, which we leave without proof: *In the purely nominal case, of all equilibrium paths the Pareto optimal ones are those, and only those, that exhibit a bubble whose real value is bounded away from zero throughout time.*

It is certainly interesting that a bubble can serve the function of guaranteeing the optimality of the equilibria of an economy, but one should keep in mind that this happens only because an asset is needed to transfer wealth through time. If a real asset exists then this asset can do the job. If one does not exist then the economy, so to speak, needs to invent an asset. To close the circle, we point out that if there is a real asset then not only is a bubble not needed but, in fact, it cannot occur.

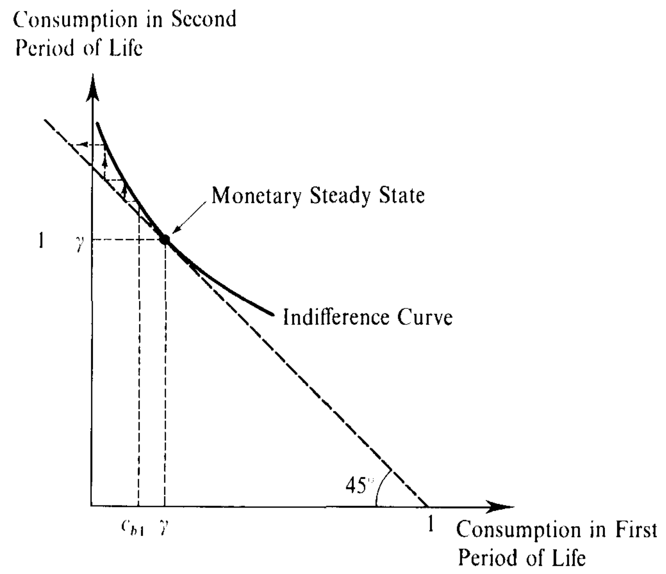


Figure 20.H.3
The monetary steady state is Pareto optimal.

Proposition 20.H.2: Suppose that at an equilibrium we have $\sum_t p_t < \infty$. Then $M = 0$.

Proof: The sum of wealths over generations is $\sum_t p_t + M < \infty$. The value of total consumption is $\sum_t p_t < \infty$. The second amount cannot be smaller than the first (otherwise some generation is not spending its entire wealth). Therefore $M = 0$. ■

Multiplicity of Equilibria

We have already seen, in Figure 20.H.2, a model with a purely nominal asset (i.e., $\varepsilon = 0$) and very nicely shaped preferences (the offer curve is of the gross substitute type) for which there is a continuum of equilibria. Of those, one is the Pareto optimal monetary steady state and the rest are nonoptimal equilibria where the real value of money goes to zero asymptotically. The existence of this sort of indeterminacy is clearly related to the ability to fix with some arbitrariness the real value of money (the "bubble") at $t = 0$, that is M/p_0. It cannot occur if bubbles are impossible, as, for example, in the model with a real asset (i.e., $\varepsilon > 0$) where, in addition, we know that the equilibrium is Pareto optimal.

One may be led by the above observation to suspect that the failure of Pareto optimality is a precondition for the presence of a robust indeterminacy (i.e., of a continuum of equilibria not associated with any obvious coincidence in the basic data of the economy). This suspicion may be reinforced by the discussion of Section 20.G, where we saw that the Pareto optimality of equilibria was key to our ability to claim the generic determinacy of equilibria in models with a finite number of consumers. Unfortunately, with overlapping generations the number of consumers is infinite in a fundamental way,[36] and this complicates matters. Whereas with a real asset the Pareto optimality of equilibria is guaranteed and the type of indeterminacy of Figure 20.H.2 disappears, it is nevertheless possible to construct nonpathological examples with a continuum of equilibria.

The simplest example is illustrated in Figure 20.H.4. The figure describes a real-asset model with the steady state $(\gamma, 1 - \gamma)$. Suppose that, in a procedure we have resorted to repeatedly, we tried to construct an equilibrium with c_{a0} slightly different from $1 - \gamma$. Then, normalizing to $p_0 = 1$, we would need to use p_1 to clear the market *of period* 0, p_2 to do the same for period 1, and so on. In the leading case of Figure 20.H.1, we have seen that this eventually becomes unfeasible. A change in p_t that takes care of a disequilibrium at $t - 1$ creates an even larger disequilibrium at t, which then has to be compensated by a change of a larger magnitude in p_{t+1} in an explosive process that finally becomes impossible. But in Figure 20.H.4, the utility function is such that, at the relative prices of the steady state, a change in the price of the second-period good has a larger impact on the demand for the first-period good than on the demand for the second-period good. Hence, the successive adjustments necessitated by an initial disturbance from $c_{a0} = 1 - \gamma$ dampen with each iteration and can be pursued indefinitely. We conclude that an equilibrium exists with the new initial condition. As a matter of terminology, the

36. By this vague statement we mean that there is no way we could assert that the infinitely many consumers are sufficiently similar for them to be "approximated" by a finite number of representatives.

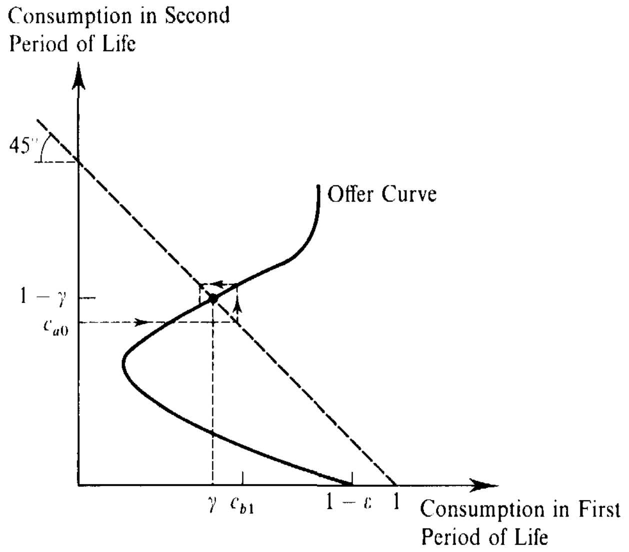


Figure 20.H.4
An example of a continuum of (Pareto optimal) equilibria in the real asset case.

locally isolated steady state equilibrium of Figure 20.H.1 is called *determinate*, and the one of Figure 20.H.4 is called *indeterminate*.[37]

It is interesting to point out that the leading case of unique equilibrium (Figure 20.H.1) in a real-asset model corresponds to a gross substitute excess demand function, while Figure 20.H.4 represents the sort of pronounced complementaries that were sources of nonuniqueness in the examples of Sections 15.B (recall also the discussion of uniqueness in Section 17.F). The connection between nonuniqueness and indeterminateness is actually quite close, and you are asked to explore it in Exercise 20.H.2. Here we simply mention that gross substitution is not a necessary condition for uniqueness. It can be checked, for example, that in the real asset model the steady state remains the only equilibrium if consumption in both periods is normal in the demand function of $u(\cdot, \cdot)$ and if the corresponding excess demand $(z_b(p_b, p_a), z_a(p_b, p_a))$ satisfies

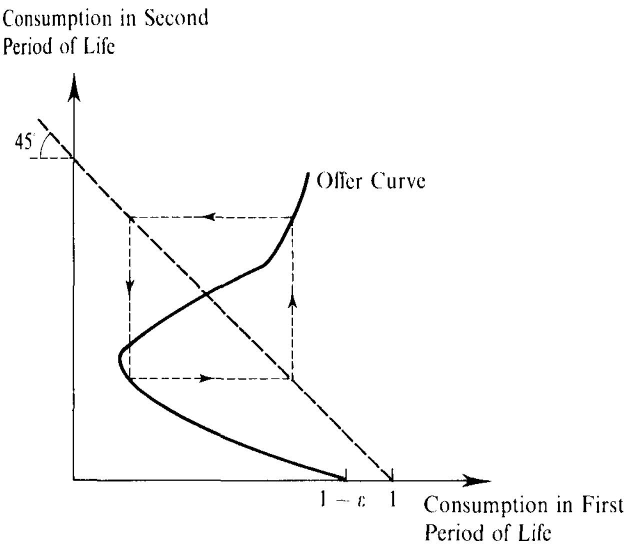
$$\nabla_1 z_b(p_b, p_a) < \nabla_1 z_a(p_b, p_a) \qquad \text{for all } p_b, p_a. \tag{20.H.3}$$

Expression (20.H.3) permits a price increase in the first period of life to lead to an increase in demand in this period (a possibility ruled out by gross substitution); but, if so, it requires the increase of demand in the second period of life to be larger. Geometrically speaking, the condition is that the slope of the offer curve in the (c_b, c_a) plane should never be positive and less than 1. Note that in Figure 20.H.4 this is violated at the steady state. Condition (20.H.3) is known as the *determinacy condition.* If the reverse inequality holds at the steady state, then, as in Figure 20.H.4, there is a continuum of equilibria all converging to the steady state (the steady state is therefore *indeterminate*).

37. Observe that, at least in the context of the relatively simple model we are now discussing, there is little room for cases intermediate between uniqueness or the existence of a continuum of equilibria.

In Chapter 17 (see Section 17.D and Appendix A of Chapter 17) we argued that, with Pareto optimality, an equilibrium problem with a finite number of consumers could be represented by means of a finite number of equations with the same number of unknowns. From this we claimed that generic determinacy was the logical conjecture to make for this case. In Section 20.G we extended this argument to the model with a finite number of infinitely long-lived consumers. However, the current overlapping generations problem has a basic difference in formal structure: there is no natural trick allowing us to see the equilibrium as anything but the zeros of an infinite system of equations (of the excess demand type, say). Mathematically, this is significant. To give an example, intimately related to the issues we are discussing, suppose that $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear map that is onto (i.e., $f(x) = Ax$, where A is a nonsingular matrix). Then 0 is the unique solution to $f(x) = 0$. But suppose now that $f(\cdot)$ maps bounded sequences into bounded sequences and that it is linear and onto. Then $f(x) = 0$, or, equivalently, $f_t(x_1, \dots, x_t, \dots) = 0$ for all t, need not have a unique solution. A simple example is the backward shift, that is, $f_t(x_1, \dots, x_t, \dots) = x_{t+1}$, where any $(\alpha, 0, \dots, 0, \dots)$ is sent to zero.

What can we say about the dynamics of an equilibrium? We saw that the "anything goes" principle applied to the one-consumer model. It would be surprising if it did not apply here; indeed, in Figures 20.H.5 and 20.H.6 we provide nonpathological examples with cycles.[38] Note



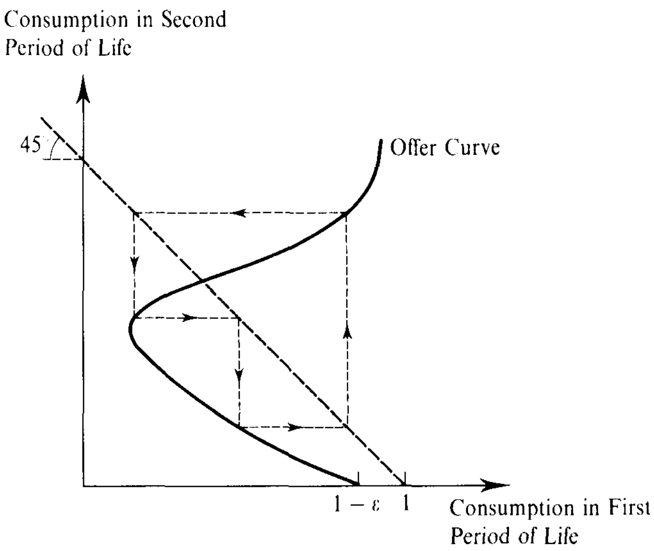


Figure 20.H.5 (left)
Complementary consumptions: example of a period-2 equilibrium path.

Figure 20.H.6 (right)
Complementary consumptions: example of a period-3 equilibrium path.

that in Figure 20.H.6 we have a three-period cycle: chaos rears its head. In the gross substitute example of Figure 20.H.1 we have monotone convergence to the steady-state. In a sense, the gross substitute case is the analog of the approach based on the sign of the second derivatives described in Section 20.F. Note that in the overlapping generations situation the factor of discount is not a meaningful concept and, therefore, there is no analog of a dynamic theory based on patience. In Section 20.G we also mentioned, quite loosely, that there did not seem to be, for the case of a finite number of agents with Pareto optimality, a close relation between the determinacy and the dynamic properties of equilibrium. In the current setting the connection is closer, at least in the following sense: If equilibrium trajectories with cycles can occur, then there are infinitely many equilibria.

38. In particular, no inferior goods are required for these examples.

20.I Remarks on Nonequilibrium Dynamics: Tâtonnement and Learning

The dynamic analysis that has concerned us so far in this chapter is of a different nature from, and should not be confused with, the dynamics studied in Section 17.H. The dynamics here display the temporal unfolding of an equilibrium (an internal property of the equilibrium, in the terminology of Section 20.G), whereas in Section 17.H we were trying to assess the dynamic forces that, in real or in fictional time, would buffet an economy disturbed from its equilibrium (hence, we were looking at an external property). As we saw, nonequilibrium dynamic analysis raises a host of conceptual problems, yet it may offer useful insight into the plausibility of the occurrence of particular equilibria. This remains valid in the setting of intertemporal equilibrium.

Abstracting from technical complexities, the analysis and the results of Section 17.H can be adapted and hold true for the infinite-horizon, finite number of consumers model of Section 20.G. On the other hand, as we have seen, the temporal framework has its own special theory, which could conceivably be illuminated by specific nonequilibrium considerations. We make three remarks in this direction.

Short-Run Equilibrium and Permanent Income[39]

Suppose that $(p_0, \ldots, p_t, \ldots)$ is the equilibrium price sequence of an economy with L goods and I consumers. Consumers are as in Section 20.D. Then at the equilibrium consumptions we have (assuming interiority)

$$\delta^t \nabla u_i(c_{ti}) = \lambda_i p_t \qquad \text{for all } t \text{ and every } i. \tag{20.I.1}$$

This is just (20.D.6). The variable λ_i is the marginal utility of income, or wealth, and the vector of reciprocals $(\eta_1, \ldots, \eta_I) = (1/\lambda_1, \ldots, 1/\lambda_I)$ can serve as the weights for which the given equilibrium maximizes the weighted sum of utilities (see Section 20.G).

It follows from (20.I.1) that the short-run demands (i.e., the demands at $t = 0$) are entirely determined by p_0 and the marginal utilities of wealth λ_i. Denote this demand by $c_{0i}(p, \lambda_i)$. In the spirit of tâtonnement dynamics, suppose that p_0 is perturbed to some p_0'. What will happen to demand at $t = 0$? If the λ_i remain fixed, then (20.I.1) implies that short-run demand behaves as the demand for non-numeraire goods in a quasilinear utility model with concave utility functions. In particular, differentiating (20.I.1) we see that the $L \times L$ matrix of short-run price effects

$$D_{p_0} c_{0i}(p_0, \lambda_i) = \lambda_i [D^2 u_i(c_{0i})]^{-1}$$

is negative definite (by the concavity of $u_i(\cdot)$) and, therefore, so is the aggregate $\sum_i D_{p_0} c_{0i}(p_0, \lambda_i)$. In more economic terms, as long as the λ_i stay fixed there are no wealth effects present in the short-run demands. Substitution prevails and, consequently, the short-run equilibrium is unique and globally tâtonnement stable.

In reality, however, after a change in p_0 we should expect that λ_i will have changed at the new consumer optimum. But if the rate of discount is close to 1 (i.e., if agents

39. See Bewley (1977) for more on this topic. The term "permanent income" is standard and so we use it rather than "permanent wealth."

are patient) then the change in λ_i should be small: The current period is not significantly more important than any other period and, therefore, it will account for only a small fraction of total utility and expenditure. Hence, we could say that partial equilibrium analysis is justified in the short run (recall the discussion of partial equilibrium analysis in Section 10.G). In summary: *If consumers are sufficiently patient, then the short-run equilibrium is unique and globally stable (for the tâtonnement dynamics).*

The (Short-Run) Law of Demand in Overlapping Generations Models

We now look at the short-run equilibrium of the overlapping generations model of Section 20.H. This is an example of a model where wealth effects matter in the short run and, therefore, the permanent income approach does not apply. We consider the version of the model with a real asset and normal goods and ask whether the stability of the fictional-time tâtonnement dynamics at a given date t helps us to distinguish among types of equilibria. Because there is a single good per period, the stability criterion for a single period is simple enough—it amounts to the law of demand at time t. That is, we say that an equilibrium $(p_0, \ldots, p_t, \ldots)$ is tâtonnement stable at time t if an (anticipated) increase in p_t, all other prices remaining fixed, results in excess supply in that period (note that only generations $t-1$ and t will alter their consumption plans).

We know that if the excess demand function of the generations is of the gross substitute type, then there is a unique equilibrium (which is in steady state). (See Figure 20.H.1.) Moreover, the definition of gross substitution tells us that the law of demand is satisfied at any t. This gives us a first link between the notions of determinate equilibrium and tâtonnement stability. This link can be pushed beyond the gross substitute case. Take a steady-state equilibrium price sequence $(1, \rho, \ldots, \rho^t, \ldots)$. By the homogeneity of degree zero of excess demand functions $(z_a(\cdot,\cdot), z_b(\cdot,\cdot))$, which implies the homogeneity of degree -1 of $\nabla z_a(\cdot,\cdot)$ and $\nabla z_b(\cdot,\cdot)$, we have (you should verify this in Exercise 20.I.1)

$$\nabla_2 z_a(1/\rho, 1) + \nabla_1 z_b(1, \rho) = \rho \nabla_2 z_a(1, \rho) + \nabla_1 z_b(1, \rho) = -\nabla_1 z_a(1, \rho) + \nabla_1 z_b(1, \rho).$$

The negativity of the left-hand side is the tâtonnement stability criterion, that is, the law of demand at a single market,[40] while the negativity of the right-hand side (i.e., the requirement that wealth effects are not so askew that a decrease in the price in one period increases the demand of the young in that period by less than it increases the demand of these same young for their consumption in the next period) is the criterion for the determinacy of the steady state [see expression (20.H.3)]. Recall that determinate means that there is no other equilibrium trajectory that remains in an arbitrarily small neighborhood of the steady state. We conclude that there is an exact correspondence: *a steady-state equilibrium is (short-run, locally) tâtonnement stable at any t if and only if it is determinate.*[41]

40. If p_t is changed infinitesimally then the demand of the old changes by $\nabla_2 z_a(\rho^{t-1}, \rho^t)$ while the demand of the young changes by $\nabla_1 z_b(\rho^t, \rho^{t+1})$. Because $\nabla_2 z(\cdot,\cdot)$ and $\nabla_1 z(\cdot,\cdot)$ are homogeneous of degree -1, the total change equals $(1/\rho^t)\,\nabla_2 z_a(1/\rho, 1) + (1/\rho^t)\,\nabla_1 z_b(1, \rho)$

41. In this "if and only if" statement we neglect borderline cases.

We have confined ourselves to the real asset case to avoid a complication. With a purely nominal asset the previous concept of tâtonnement stability loses the power to discriminate between determinate and indeterminate steady-state equilibria, unless we restrict ourselves a priori to monetary steady states (to see this, consider the simplest gross substitute case). The learning concept to be presented in the remainder of this section does not suffer from this limitation.

Learning

We now briefly discuss a nonequilibrium dynamics that takes place in real time and that can be interpreted in terms of learning. The framework is that of the overlapping generations of Section 20.H and, to be as simple as possible, we focus on the purely nominal asset case.

We describe first how the short-run equilibrium (i.e., the equilibrium at a given period t) is determined. We suppose that there is a certain fixed amount of fiat money M (denominated, say, in dollars). The excess demand of the older generation at date $t \geq 1$ is then M/p_t. The excess demand of the younger generation at the same date depends on p_t but also on the *expectation* p_{t+1}^e of the price at $t+1$. Given p_{t+1}^e, the price p_t is a *temporary equilibrium at time* $t \geq 1$ if $z_b(p_t, p_{t+1}^e) + (M/p_t) = 0$. Thus, given a sequence of price expectations $(p_1^e, \ldots, p_t^e, \ldots)$, we generate a sequence of temporary equilibrium prices $(p_1, \ldots, p_t, \ldots)$.

But, how are expected prices determined? To take them as given does not make much sense. The sequence of realizations should feed back into the sequence of expectations. The self-fulfilled, or rational, expectations approach (which we have implicitly adhered to in this chapter) imposes a correct expectations condition: $p_{t+1}^e = p_{t+1}$ for every t.[42] An alternative is to require that p_{t+1}^e (the price expected at t to prevail at $t+1$) be an extrapolation of the past (and current) realizations $p_0, \ldots, p_t$. In this approach we think of consumers as engaged in some sort of learning and of expectations responding *adaptively* to experienced outcomes.[43]

To be specific, let us take a not very realistic, but very simple, extrapolation rule: $p_{t+1}^e = p_{t-1}$ (i.e., the price at $t+1$ expected by the young at $t \geq 1$ is the price that ruled in the most recent past). Equivalently (given the fixed amount of fiat money M), the young at t expect to consume at $t+1$, when old, the same amount consumed by the old at $t-1$. The equation for the determination of p_t is then

$$z_b(p_t, p_{t-1}) = -\frac{M}{p_t}. \qquad (20.I.2)$$

By Walras' law, (20.I.2) can equivalently be written as

$$z_a(p_t, p_{t-1}) = \frac{M}{p_{t-1}}. \qquad (20.I.3)$$

42. The term "self-fulfilled" is justified because the sequence of expectations $(p_1^e, \ldots, p_t^e, \ldots)$ induces a sequence of realizations identical to itself. The term "rational" is justified by the fact that, given $(p_1^e, \ldots, p_t^e, \ldots)$, a member of generation t should, in principle, be able to compute the price realization p_{t+1} and therefore verify the correctness of p_{t+1}^e.

43. We should emphasize, first, that all this is a nonequilibrium story and, second, that we cannot rigorously discuss learning without explicitly introducing an uncertain environment.

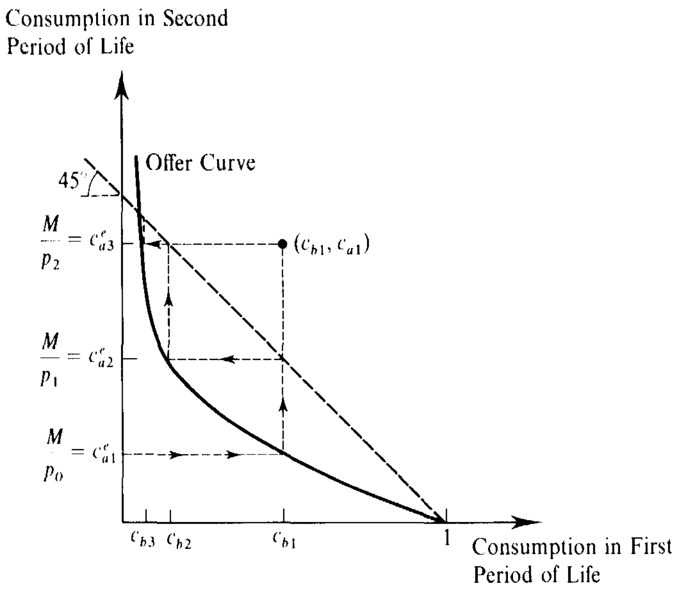


Figure 20.I.1
Learning dynamics.

Given an arbitrary initial condition p_0, we can then compute the sequence of temporary equilibrium realizations $(p_1, \ldots, p_t, \ldots)$ by iteratively using (20.I.2) or (20.I.3). Note that in doing so, the planned excess demands in (20.I.2) will be realized but those in (20.I.3) may not (because p_{t+1} may not be equal to p_{t-1}). We represent the dynamic process in Figure 20.I.1. In the figure, c_{bt} and c^e_{at} stand for the planned consumptions of generation t at times t and $t+1$, respectively. Given M/p_{t-1} we get c^e_{at} from (20.I.3), and c_{bt} from the fact that planned consumptions are in the offer curve. Finally (20.I.2) moves us to the next value M/p_t. For generation 1 we also show the actual consumption vector (c_{b1}, c_{a1}).

From Figure 20.I.1 we can see an interesting fact: The learning dynamics exactly reverses the equilibrium dynamics (compare with Figure 20.H.2).[44] For the gross substitute case shown in the figure, this means that all the trajectories tend to the monetary steady state. Hence, in the limit we have a true self-fulfilled expectations equilibrium. Consumers have learned their way into equilibrium, so to speak. For the crude learning dynamics we are considering, this need not be so for the case of a general offer curve (an infinite sequence with systematic prediction error is quite possible), but the property of exact reversal of equilibrium dynamics suffices to provide, once again, a test for the well-behavedness of steady states that reinforces the intuitions developed earlier: *A steady state is (locally) stable for the learning dynamics if and only if it is determinate (i.e., "locally isolated").*

44. More precisely, if $(p_1, \ldots, p_t, \ldots)$ is the sequence of realizations of the adaptive expectations dynamics, then for any T there is an equilibrium sequence $(p'_0, \ldots, p'_t, \ldots)$ such that $p'_t = p_{T-t}$ for every $t < T$.

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EXERCISES

20.B.1[B] Adopting the definition of *time impatience* given in comment (1) of Section 20.B, show that a utility function of the form (20.B.1) exhibits time impatience.

20.B.2[B] Verify that a utility function of the form (20.B.1) is stationary according to the definition given in comment (2) of Section 20.B. Also, exhibit a violation of stationarity with a utility function of the form $V(c) = \sum_{t=0}^{\infty} \delta_t^t u(c_t)$.

20.B.3[B] With reference to comment (3) of Section 20.B, write $c = (c', c'')$ where $c' = (c_0, \ldots, c_t)$, $c'' = (c_{t+1}, \ldots)$. Suppose that the utility function $V(\cdot)$ is additively separable. Show that if $V(\bar{c}', c'') \geq V(\bar{c}', \hat{c}'')$ for some $\bar{c}'$, then $V(c', c'') \geq V(c', \hat{c}'')$ for all c'. Show that if $V(c', \bar{c}'') \geq V(\hat{c}', \bar{c}'')$ for some c'', then $V(c', c'') \geq V(\hat{c}', c'')$ for all c''. Interpret.

20.B.4[C] Show that in a recursive utility model with aggregator function $G(u, V) = u^\alpha + \delta V^\alpha$, $0 < \alpha < 1$, $\delta < 1$, and increasing, continuous one-period utility $u(c_t)$, the utility $V(c)$ of a bounded consumption stream is well defined. [*Hint*: Use (20.B.3) to compute the utility for consumption streams truncated at a finite horizon. Then show that a limit exists as $T \to \infty$. Finally, argue that the limit satisfies the aggregator equation.]

20.B.5[A] Show that the utility function $V(c)$ on consumption streams given by (20.B.1) is concave. Show also that the additively separable form of $V(\cdot)$ is a cardinal property.

20.C.1[A] Given the price sequence $(p_0, p_1, \ldots, p_t, \ldots)$, $p_t \in \mathbb{R}^L$, define for every t and every commodity ℓ the rate of interest from t to $t + 1$ in terms of commodity ℓ (this is known as the *own rate of interest* of commodity ℓ at t).

20.C.2[A] Show that if the path $(y_0, \ldots, y_t, \ldots)$, is myopically profit maximizing for $(p_0, p_1, \ldots, p_t, \ldots) \geq 0$, then $(y_0, \ldots, y_t, \ldots)$ is also profit maximizing for $(p_0, p_1, \ldots, p_t, \ldots)$ over any finite horizon, in the sense that, for any T, the total profits over the first T periods cannot be increased by any coordinated move involving only these periods.

20.C.3[A] Define an appropriate concept of weak efficiency and reprove Proposition 20.C.1, requiring only that $(p_0, \ldots, p_t, \ldots)$ is a nonnegative sequence with some nonzero entry.

20.C.4[B] Suppose that the production path $(y_0, \ldots, y_t, \ldots)$ is bounded (i.e., there is a fixed α such that $\|y_t\| \leq \alpha$ for all t), that $(p_0, \ldots, p_t, \ldots) \gg 0$, and that $\sum_{t=0}^{\infty} p_t < \infty$. We say that the path $(y_0, \ldots, y_t, \ldots)$ is overall profit maximizing with respect to $(p_0, \ldots, p_t, \ldots)$ if

$$\sum_{t=0}^{\infty} (p_t \cdot y_{bt} + p_{t+1} \cdot y_{a,t}) \geq \sum_{t=0}^{\infty} (p_t \cdot y'_{bt} + p_{t+1} \cdot y'_{a,t})$$

for any other production path $(y'_0, \ldots, y'_t, \ldots)$.

(a) Show that if $(y_0, \ldots, y_t, \ldots)$ is overall profit maximizing with respect to $(p_0, \ldots, p_t, \ldots) \gg 0$, then it is efficient.

(b) Show that if $(y_0, \ldots, y_t, \ldots)$ is myopically profit maximizing with respect to $(p_0, \ldots, p_t, \ldots) \gg 0$, then it is also overall profit maximizing.

20.C.5[C] Say that a production path $(y_0, \ldots, y_t, \ldots)$, is T-efficient, for $T < \infty$, if there is no other production path $(y'_0, \ldots, y'_t, \ldots)$ that, first, dominates $(y_0, \ldots, y_t, \ldots)$ in the sense of efficiency and, second, is such that the cardinality of $\{t\colon y_t \neq y'_t\}$ is at most T.

(a) Show that if $(y_0, \ldots, y_t, \ldots)$ is myopically profit maximizing with respect to $(p_0, \ldots, p_t, \ldots) \gg 0$, then $(y_0, \ldots, y_t, \ldots)$ is T-efficient for all $T < \infty$.

(b) Show that if the technology is smooth (in the sense used in the small-type discussion at the end of Section 20.C; assume also that the outward unit normals to the production frontiers are strictly positive), then 2-efficiency implies T-efficiency for all $T < \infty$.

(c) (Harder) Show that the conclusion of **(b)** fails for general linear activity technologies. Exhibit an example. [*Hint*: Rely on chains of intermediate goods.]

20.C.6[A] Consider the Ramsey–Solow technology of Example 20.C.1, as continued in Example 20.C.6. The exogenous path of labor endowments is $(l_0, \ldots, l_t, \ldots)$. Given a production path $(k_0, \ldots, k_t, \ldots)$, we determine a sequence of consumption good prices $(q_0, \ldots, q_t, \ldots)$ by the requirement that $(q_t/q_{t+1}) = \nabla_1 F(k_t, l_t)$ for all t. Show then that a sequence of wages w_t can be found so that the path determined by $(k_0, \ldots, k_t, \ldots)$ is myopically profit maximizing for the price sequence determined by $((q_0, w_0), \ldots, (q_t, w_t), \ldots)$.

20.D.1^{A} Consider the budget constraint of problem (20.D.3). To simplify, suppose that we are in a pure exchange situation. Write the budget constraint as a sequence of budget constraints, one for each date. To this effect, assume that money can be borrowed and lent at a zero nominal rate of interest.

20.D.2^{A} Show that condition (ii′) in Section 20.D (it is stated just before Definition 20.D.2) implies condition (ii) of Definition 20.D.1. Show that, conversely, condition (ii), together with $w = \sum_t p_t \cdot \omega_t + \sum_t \pi_t < \infty$, implies condition (ii′).

20.D.3^{A} in text.

20.D.4^{A} Complete the computations requested in Example 20.D.1.

20.D.5^{C} In the context of Example 20.D.3, compute the Euler equations for the optimal investment policy when the production function has the form $F(k) = k^{\alpha}$, $0 < \alpha < 1$, and the adjustment cost function is given by $g(k' - k) = (k' - k)^{\beta}$, with $\beta > 1$, for $k' > k$, and by $g(k' - k) = 0$ for $k' \le k$. Say as much as you can about the policy. In particular, determine the steady-state trajectory of investment.

20.D.6^{B} Verify the claim made in the proof of Proposition 20.D.7 that the Euler equations (20.D.9) are the first-order necessary and sufficient conditions for short-run optimization. In other words: they are necessary and sufficient for the nonexistence of an improving trajectory differing from the given one at only a finite number of dates.

20.D.7^{A} With reference to Example 20.D.4, show that, for the functional forms given, the Euler equations are as indicated in the example: $k_{t+1} = 3k_t - 2k_{t-1}$ for every t. Also verify that the solution to this difference equation given in the text is indeed a solution, that is, that it satisfies the equation.

20.D.8^{A} Verify that the value function $V(k)$ does satisfy the properties (i) and (ii) claimed for it at the end of Section 20.D.

20.D.9^{A} Argue that the properties (i) and (ii) of the value function referred to in Exercise 20.D.8 yield the two consequences, concerning $V'(k)$ and $V''(k)$, claimed at the end of Section 20.D.

20.E.1^{A} Discuss in what sense the term r defined after the proof of Proposition 20.E.1 can be interpreted as the rate of interest implicit in the proportional price sequence.

20.E.2^{B} Suppose that the production set $Y \subset \mathbb{R}^L$ is of the constant return type and consider production paths that are *proportional* (but not necessarily stationary), that is, paths $(y_0, \ldots, y_t, \ldots)$ that satisfy $y_t = (1 + n) y_{t-1}$ for all t and some n.

(a) Argue that the conclusion of Proposition 20.E.1 remains valid for proportional paths.

(b) State and prove the result parallel to Proposition 20.E.2 for proportional paths.

20.E.3^{B} Suppose that in the Ramsey–Solow model $\bar{k}$ solves Max $(F(k, 1) - k)$ (see Figure 20.E.2). Show that if $k_t \le \bar{k} - \varepsilon$ for all t, then the path determined by $(k_0, \ldots, k_t, \ldots)$ is efficient. [*Hint*: Compute prices and verify the transversality condition.]

20.E.4^{A} Prove the three neoclassical properties stated at the end of the regular type part of Section 20.E.

20.E.5^{A} Carry out the requested verification of expression (20.E.1).

20.E.6^A Carry out the verification requested in the discussion of Figure 20.E.3.

20.E.7^A In the Ramsey Solow model, two different steady states are associated with different rates of interest. This is not so in the example illustrated in Figure 20.E.3, at first sight very similar. The key difference is that in the Ramsey–Solow model the consumption and investment goods are perfect substitutes in production. Clarify this by proving, in the context of the example underlying Figure 20.E.3, that if the two goods are perfect substitutes then $r(\bar{k}) \neq r(\bar{\bar{k}})$ whenever $\bar{k} \neq \bar{\bar{k}}$. [*Hint*: Their being perfect substitutes means that $G(k, k' + \alpha) = G(k, k') - \alpha$ for any $\alpha < F(k, k')$.]

20.E.8^A Consider the proportional production paths with rate of growth equal to $n > 0$ (recall Exercise 20.E.2) in the context of a Ramsey–Solow technology of constant returns. Show that among these paths the one that maximizes surplus (at $t = 1$, or, equivalently, normalized surplus or surplus "per capita") is characterized by having the rate of interest equal to n. This path is also called the *golden rule steady state path*.

20.E.9^A Argue that, for the one-consumer model of Section 20.D, the golden rule path cannot arise as part of a competitive equilibrium. [*Hint*: The key fact is that $\delta < 1$.]

20.F.1^C Consider two arbitrary functions $\gamma_1(w)$ and $\gamma_2(w)$ that are defined for $w > 0$, take nonnegative values, and satisfy $\gamma_1(w) + \gamma_2(w) = w$ for all w. Suppose also that they are twice continuously differentiable.

Show that for any $\alpha > 0$ there is a utility function for two commodities, $u(x_1, x_2)$, that is increasing and concave on the domain $\{(x_1, x_2): x_1 + x_2 \leq \alpha\}$ and is such that $(\gamma_1(w), \gamma_2(w))$ coincides with the Engel curve functions for prices $p_1 = 1$, $p_2 = 1$ and wealth $w < \alpha$. [*Hint*: Let $u(x_1, x_2) = (x_1 + x_2)^{1/2} - \varepsilon[(x_1 - \gamma_1(x_1 + x_2))^2 + (x_2 - \gamma_2(x_1 + x_2))^2]$ and take ε to be small enough. Verify then that $\nabla u(x_1, x_2)$ is strictly positive and $D^2u(x_1, x_2)$ is negative definite for any (x_1, x_2) such that $0 < x_1 + x_2 \leq \alpha$, and that the Engel curve is as required.]

20.F.2^A Suppose that, for $k \in \mathbb{R}_+$, the policy function $\psi(k)$ is a contraction (see the definition in the part of Section 20.F headed by "Low discount of time"). Draw several possible graphs for such a policy function and argue that there is always a unique steady state. Also, carry out the graphical dynamic analysis for your graphs and establish in this way that the steady states are always globally stable.

20.F.3^A Verify that for the classical Ramsey–Solow technology and for the cost-of-adjustment technology the cross derivative of uniform positive sign condition is satisfied.

20.F.4^A Carry out the verification concerning transitory shocks requested in Example 20.F.1.

20.F.5^A Carry out the verification concerning permanent shocks requested in Example 20.F.1.

20.G.1^B Analyze the equilibrium problem for the exchange case with two consumers (i.e., $I = 2$), and a single physical commodity (i.e., $L = 1$). Both consumers have the same discount factor [utility functions are of the form (20.B.1)]. In addition, assume that $\omega_{t1} + \omega_{t2} = 1$ for all t. Show in particular that the equilibrium consumption streams must be stationary, that the sequence of equilibrium prices is proportional (what is the rate of interest?), and that therefore there is only one stream of equilibrium consumptions.

20.G.2^A Consider an exchange model with two consumers. Utility functions are of the form (20.B.1) and both consumers have the same discount factor. There are no restrictions on the number of commodities L or on the total endowments at any t. Show that at a Pareto optimal allocation the following holds: for every consumer, the in-period marginal utilities of wealth of the consumer is the same across periods (and equal to the overall marginal utility of wealth of the consumer). Interpret and discuss what this means in terms of intertemporal and interindividual transfers of wealth.

20.G.3[B] The situation is the same as that of Exercise 20.G.2.

(a) Parametrize the Pareto frontier of the utility possibility set by the ratio of marginal utilities of wealth of the two consumers.

(b) Then express the equations of equilibrium à la Negishi (see Appendix A of Chapter 17). That is, write down one equation in one unknown (the ratio of marginal utilities of wealth) whose zeros are precisely the equilibria of the model.

(c) Argue in terms of the methodology discussed in Section 17.D that generically there are only a finite number of equilibria. Be as precise as you can.

20.G.4[A] Prove the claim made in footnote 32. Be explicit about the form of the equilibrium price sequences.

20.G.5[B] Verify that the concavity of the utility function implies that the expression (20.G.6) is larger than one in absolute value if there is no externality (i.e., if $\nabla^2_{23}u(\cdot) = \nabla^2_{13}u(\cdot) = 0$).

20.H.1[B] Show that in the context of Sections 20.D or 20.G (a finite number of consumers) no bubbles can arise at equilibrium.

20.H.2[B] In the framework of Section 20.H do the following (diagrammatic proofs are permissible).

(a) Show that if condition (20.H.3) is satisfied then, in the real asset case, the steady state is the only equilibrium.

(b) Show that if condition (20.H.3) is satisfied then, in the purely nominal asset case, the monetary steady state is the only equilibrium that is a Pareto optimum.

(c) Conversely, suppose that condition (20.H.3) is violated with strict inequality at $p_b = p_a$. Show then that, for the purely nominal asset case, there is more than one Pareto optimal equilibrium.

(d) (Harder) Suppose that the utility function is of the form $v(c_b) + \delta v(c_a)$. Investigate which conditions on $v(\cdot)$ and δ imply that the excess demand function satisfies condition (20.H.3). [*Hint*: Recall Example 17.F.2 for a special case.]

20.I.1[A] Verify the computation requested in the part of Section 20.I headed "The (short-run) law of demand in overlapping generations models."

PART FIVE

Welfare Economics and Incentives

Part V is devoted to a systematic presentation of a number of issues related to the foundations of welfare economics, a topic that we have encountered repeatedly throughout the text. The point of view is that of a policy maker engaged in the design and implementation of collective decisions.

In Chapter 21, we review classical social choice theory. The central question of this theory concerns the possibility of deriving the objectives of the policy maker as an aggregation of the preferences of the agents in the economy, and of doing so in a manner that could be deemed as satisfactory according to a number of desiderata. The difficulties of accomplishing this task are dramatically illustrated in *Arrow's impossibility theorem*, which we present and discuss in detail. On a more positive note, we also discuss the assumption of single-peaked preferences and analyze the performance of majority voting under it.

In Chapter 22, we admit, to a variety of extents, the possibility of explicit value judgments as to the comparability of individuals' utility levels. Most of the chapter is devoted to a presentation of welfare economics in the Bergson–Samuelson tradition. Towards this end, we develop the apparatus of utility possibility sets and social welfare functions and emphasize the distinction between first-best and second-best problems. We also offer an account of axiomatic bargaining theory, an approach that emphasizes the compromise, rather than the constrained optimality, nature of social decisions.

In Chapter 23, we recognize that, in actuality, a policy maker rarely knows individuals' preferences with certainty; rather, this information is typically only *privately* known by the individuals themselves. The presence of information that is observed privately by rational, self-interested actors generates severe incentive-compatibility, second-best constraints. We offer a detailed analysis of what can and cannot be implemented collectively when these incentive constraints are taken into consideration. The content of Chapter 23 links with the game theory covered in Part II and revisits a number of themes first broached in Part III.

CHAPTER

21

Social Choice Theory

21.A Introduction

In this chapter, we analyze the extent to which individual preferences can be aggregated into social preferences, or more directly into social decisions, in a "satisfactory" manner that is, in a manner compatible with the fulfillment of a variety of desirable conditions.

Throughout the chapter, we contemplate a set of possible social alternatives and a population of individuals with well-defined preferences over these alternatives.

In Section 21.B, we start with the simplest case: that in which the set of alternatives has only two elements. There are then many satisfactory solutions to the aggregation problem. In our presentation, we focus on a detailed analysis of the properties of aggregation by means of majority voting.

In Section 21.C, we move to the case of many alternatives and the discussion takes a decidedly negative turn. We state and prove the celebrated *Arrow's impossibility theorem*. In essence, this theorem tells us that we cannot have everything: If we want our aggregation rule (which we call a *social welfare functional*) to be defined for any possible constellation of individual preferences, to always yield Pareto optimal decisions, and to satisfy the convenient, and key, property that social preferences over any two alternatives depend only on individual preferences over these alternatives (the *pairwise independence condition*), then we have a dilemma. Either we must give up the hope that social preferences could be rational in the sense introduced in Chapter 1 (i.e., that society behaves as an individual would) or we must accept dictatorship.

Section 21.D describes two ways out of the conclusion of the impossibility theorem. In one we allow for partial relaxations of the degree of rationality demanded of social preferences. In the other, we settle for aggregation rules that perform satisfactorily on restricted domains of individual preferences. In particular, we introduce the important notion of *single-peaked* preferences and, for populations with preferences in this class, we analyze the role of a *median voter* in the workings of pairwise majority voting as an aggregation method.

Section 21.E sets the aggregation problem more directly as one of aggregating individual preferences into social decisions. It introduces the concept of a *social choice* *function*, and proceeds to give a version of the impossibility result for the latter. Essentially, this result is obtained by replacing the pairwise independence condition (which is meaningless in the context of this section) by a *monotonicity* condition on the social choice function. This condition provides an important link to the incentive-based theory of Chapter 23.

General references and surveys for the topics of this chapter are Arrow (1963), Moulin (1988), and Sen (1970) and (1986).

21.B A Special Case: Social Preferences over Two Alternatives

We begin our analysis of social choice by considering the simplest possible case: that in which there are only two alternatives over which to decide. We call these alternative x and alternative y. Alternative x, for example, could be the "status-quo," and alternative y might be a particular public project whose implementation is being contemplated.

The data for our problem are the individual preferences of the members of society over the two alternatives. We assume that there is a number $I < \infty$ of individuals, or *agents*. The family of individual preferences between the two alternatives can be described by a profile

$$(\alpha_1, \ldots, \alpha_I) \in \mathbb{R}^I,$$

where α_i takes the value 1, 0, or -1 according to whether agent i prefers alternative x to alternative y, is indifferent between them, or prefers alternative y to alternative x, respectively.[1]

Definition 21.B.1: A *social welfare functional* (or *social welfare aggregator*) is a rule $F(\alpha_1, \ldots, \alpha_I)$ that assigns a social preference, that is, $F(\alpha_1, \ldots, \alpha_I) \in \{-1, 0, 1\}$, to every possible profile of individual preferences $(\alpha_1, \ldots, \alpha_I) \in \{-1, 0, 1\}^I$.

All the social welfare functionals to be considered respect individual preferences in the weak sense of Definition 21.B.2.

Definition 21.B.2: The social welfare functional $F(\alpha_1, \ldots, \alpha_I)$ is *Paretian*, or has the *Pareto property*, if it respects unanimity of strict preference on the part of the agents, that is, if $F(1, \ldots, 1) = 1$ and $F(-1, \ldots, -1) = -1$.

Example 21.B.1: Paretian social welfare functionals between two alternatives abound. Let $(\beta_1, \ldots, \beta_I) \in \mathbb{R}^I_+$ be a vector of nonnegative numbers, not all zero. Then we

1. In the whole of this chapter we make the restriction that only the agents' rankings between the two alternatives matter for the social decision between them. In Section 21.C we will state formally the principle involved. Note, in particular, that this specification precludes the use of any "cardinal" or "intensity" information between the two alternatives because this intensity can only be calibrated (perhaps using lotteries) by appealing to some third alternative. A fortiori, the specification also precludes the comparison of feelings of pleasure or pain across individuals. In Chapter 22, we discuss in some detail matters pertaining to the issue of interpersonal comparability of utilities.

could define

$$F(\alpha_1, \dots, \alpha_I) = \text{sign} \sum_i \beta_i \alpha_i,$$

where, recall, for any $a \in \mathbb{R}$, sign a equals 1, 0, or -1 according to whether $a > 0$, $a = 0$, or $a < 0$, respectively.

An important particular case is *majority voting*, where we take $\beta_i = 1$ for every i. Then $F(\alpha_1, \dots, \alpha_I) = 1$ if and only if the number of agents that prefer alternative x to alternative y is larger than the number of agents that prefer y to x. Similarly, $F(\alpha_1, \dots, \alpha_I) = -1$ if and only if those that prefer y to x are more numerous than those that prefer x to y. Finally, in case of equality of these two numbers, we have $F(\alpha_1, \dots, \alpha_I) = 0$, that is, social indifference. ■

Example 21.B.2: *Dictatorship.* We say that a social welfare functional is *dictatorial* if there is an agent h, called a *dictator*, such that, for any profile $(\alpha_1, \dots, \alpha_I)$, $\alpha_h = 1$ implies $F(\alpha_1, \dots, \alpha_I) = 1$ and, similarly, $\alpha_h = -1$ implies $F(\alpha_1, \dots, \alpha_I) = -1$. That is, the strict preference of the dictator prevails as the social preference. A dictatorial social welfare functional is Paretian in the sense of Definition 21.B.2. For the social welfare functionals of Example 21.B.1, we have dictatorship whenever $\alpha_h > 0$ for some agent h and $\alpha_i = 0$ for $i \neq h$, since then $F(\alpha_1, \dots, \alpha_I) = \alpha_h$. ■

The majority voting social welfare functional plays a leading benchmark role in social choice theory. In addition to being Paretian it has three important properties, which we proceed to state formally. The first (symmetry among agents) says that the social welfare functional treats all agents on the same footing. The second (neutrality between alternatives) says that, similarly, the social welfare functional does not a priori distinguish either of the two alternatives. The third (positive responsiveness) says, more strongly than the Paretian property of Definition 21.B.2, that the social welfare functional is sensitive to individual preferences.

Definition 21.B.3: The social welfare functional $F(\alpha_1, \dots, \alpha_I)$ is *symmetric among agents* (or *anonymous*) if the names of the agents do not matter, that is, if a permutation of preferences across agents does not alter the social preference. Precisely, let $\pi: \{1, \dots, I\} \to \{1, \dots, I\}$ be an onto function (i.e., a function with the property that for any i there is h such that $\pi(h) = i$). Then for any profile $(\alpha_1, \dots, \alpha_I)$ we have $F(\alpha_1, \dots, \alpha_I) = F(\alpha_{\pi(1)}, \dots, \alpha_{\pi(I)})$.

Definition 21.B.4: The social welfare functional $F(\alpha_1, \dots, \alpha_I)$ is *neutral between alternatives* if $F(\alpha_1, \dots, \alpha_I) = -F(-\alpha_1, \dots, -\alpha_I)$ for every profile $(\alpha_1, \dots, \alpha_I)$, that is, if the social preference is reversed when we reverse the preferences of all agents.

Definition 21.B.5: The social welfare functional $F(\alpha_1, \dots, \alpha_I)$ is *positively responsive* if, whenever $(\alpha_1, \dots, \alpha_I) \geq (\alpha'_1, \dots, \alpha'_I)$, $(\alpha_1, \dots, \alpha_I) \neq (\alpha'_1, \dots, \alpha'_I)$, and $F(\alpha'_1, \dots, \alpha'_I) \geq 0$, we have $F(\alpha_1, \dots, \alpha_I) = +1$. That is, if x is socially preferred or indifferent to y and some agents raise their consideration of x, then x becomes socially preferred.

It is simple to verify that majority voting satisfies the three properties of symmetry among agents, neutrality between alternatives, and positive responsiveness (see Exercise 21.B.1). As it turns out, these properties entirely characterize majority voting. The result given in Proposition 21.B.1 is due to May (1952).

Proposition 21.B.1: (*May's Theorem*) A social welfare functional $F(\alpha_1, \dots, \alpha_I)$ is a majority voting social welfare functional if and only if it is symmetric among agents, neutral between alternatives, and positive responsive.

Proof: We have already argued that majority voting satisfies the three properties. To establish sufficiency note first that the symmetry property among agents means that the social preference depends only on the total number of agents that prefer alternative x to y, the total number that are indifferent, and the total number that prefer y to x. Given $(\alpha_1, \dots, \alpha_I)$, denote

$$n^+(\alpha_1, \dots, \alpha_I) = \#\{i: \alpha_i = 1\}, \text{and } n^-(\alpha_1, \dots, \alpha_I) = \#\{i: \alpha_i = -1\}.^2$$

Then symmetry among agents allows us to express $F(\alpha_1, \dots, \alpha_I)$ in the form

$$F(\alpha_1, \dots, \alpha_I) = G(n^+(\alpha_1, \dots, \alpha_I), n^-(\alpha_1, \dots, \alpha_I)).$$

Now suppose that $(\alpha_1, \dots, \alpha_I)$ is such that $n^+(\alpha_1, \dots, \alpha_I) = n^-(\alpha_1, \dots, \alpha_I)$. Then $n^+(-\alpha_1, \dots, -\alpha_I) = n^-(\alpha_1, \dots, \alpha_I) = n^+(\alpha_1, \dots, \alpha_I) = n^-(-\alpha_1, \dots, -\alpha_I)$, and so

$$\begin{aligned} F(\alpha_1, \dots, \alpha_I) &= G(n^+(\alpha_1, \dots, \alpha_I), n^-(\alpha_1, \dots, \alpha_I)) \\ &= G(n^+(-\alpha_1, \dots, -\alpha_I), n^-(-\alpha_1, \dots, -\alpha_I)) \\ &= F(-\alpha_1, \dots, -\alpha_I) \\ &= -F(\alpha_1, \dots, \alpha_I). \end{aligned}$$

The last equality follows from the neutrality between alternatives. Since the only number that equals its negative is zero, we conclude that if $n^+(\alpha_1, \dots, \alpha_I) = n^-(\alpha_1, \dots, \alpha_I)$ then $F(\alpha_1, \dots, \alpha_I) = 0$.

Suppose next that $n^+(\alpha_1, \dots, \alpha_I) > n^-(\alpha_1, \dots, \alpha_I)$. Denote $H = n^+(\alpha_1, \dots, \alpha_I)$, $J = n^-(\alpha_1, \dots, \alpha_I)$; then $J < H$. Say, without loss of generality, that $\alpha_i = 1$ for $i \le H$ and $\alpha_i \le 0$ for $i > H$. Consider a new profile $(\alpha'_1, \dots, \alpha'_I)$ defined by $\alpha'_i = \alpha_i = 1$ for $i \le J < H$, $\alpha'_i = 0$ for $J < i \le H$, and $\alpha'_i = \alpha_i \le 0$ for $i > H$. Then $n^+(\alpha'_1, \dots, \alpha'_I) = J$ and $n^-(\alpha'_1, \dots, \alpha'_I) = n^-(\alpha_1, \dots, \alpha_I) = J$. Hence $F(\alpha'_1, \dots, \alpha'_I) = 0$. But by construction, the alternative x has lost strength in the new individual preference. Indeed, $(\alpha_1, \dots, \alpha_I) \ge (\alpha'_1, \dots, \alpha'_I)$ and $\alpha_{J+1} = 1 > 0 = \alpha'_{J+1}$. Therefore, by the positive responsiveness property, we must have $F(\alpha_1, \dots, \alpha_I) = 1$.

In turn, if $n^-(\alpha_1, \dots, \alpha_I) > n^+(\alpha_1, \dots, \alpha_I)$ then $n^+(-\alpha_1, \dots, -\alpha_I) > n^-(-\alpha_1, \dots, -\alpha_I)$ and so $F(-\alpha_1, \dots, -\alpha_I) = 1$. Therefore, by neutrality among alternatives:

$$F(\alpha_1, \dots, \alpha_I) = -F(-\alpha_1, \dots, -\alpha_I) = -1.$$

We conclude that $F(\alpha_1, \dots, \alpha_I)$ is indeed a majority voting social welfare functional. ■

In Exercise 21.B.2, you are asked to find examples different from majority voting that satisfy any two of the three properties of Proposition 21.B.1.

21.C The General Case: Arrow's Impossibility Theorem

We now proceed to study the problem of aggregating individual preferences over any number of alternatives. We denote the set of alternatives by X, and assume that

2. Recall the notation $\# A$ = cardinality of the set A = number of elements in the set A.

there are I agents, indexed by $i = 1, \ldots, I$. Every agent i has a rational preference relation $\succsim_i$ defined on X. The strict preference and the indifference relation derived from $\succsim_i$ are denoted by $\succ_i$ and $\sim_i$, respectively.[3] In addition, it will often be convenient to assume that no two distinct alternatives are indifferent in an individual preference relation $\succsim_i$. It is therefore important, for clarity of exposition, to have a symbol for the set of all possible rational preference relations on X and for the set of all possible preference relations on X having the property that no two distinct alternatives are indifferent. We denote these sets, respectively, by $\mathscr{R}$ and $\mathscr{P}$. Observe that $\mathscr{P} \subset \mathscr{R}$.[4]

In parallel to Section 21.B, we can define a social welfare functional as a rule that assigns social preferences to profiles of individual preferences $(\succsim_i, \ldots, \succsim_I) \in \mathscr{R}^I$. Definition 21.C.1 below generalizes Definition 21.B.1 in two respects: it allows for any number of alternatives and it permits the aggregation problem to be limited to some given domain $\mathscr{A} \subset \mathscr{R}^I$ of individual profiles. In this section, however, we focus on the largest domains, that is, $\mathscr{A} = \mathscr{R}^I$ and $\mathscr{A} = \mathscr{P}^I$.

Definition 21.C.1: A *social welfare functional* (or *social welfare aggregator*) defined on a given subset $\mathscr{A} \subset \mathscr{R}^I$ is a rule $F: \mathscr{A} \to \mathscr{R}$ that assigns a rational preference relation $F(\succsim_1, \ldots, \succsim_I) \in \mathscr{R}$, interpreted as the social preference relation, to any profile of individual rational preference relations $(\succsim_1, \ldots, \succsim_I)$ in the admissible domain $\mathscr{A} \subset \mathscr{R}^I$.

Note that, as we did in Section, 21.B the problem of social aggregation is being posed as one in which individuals are described exclusively by their preference relations over alternatives.[5]

For any profile $(\succsim_1, \ldots, \succsim_I)$, we denote by $F_p(\succsim_1, \ldots, \succsim_I)$ the strict preference relation derived from $F(\succsim_1, \ldots, \succsim_I)$. That is, we let $x\, F_p(\succsim_1, \ldots, \succsim_I)\, y$ if $x\, F(\succsim_1, \ldots, \succsim_I)\, y$ holds but $y\, F(\succsim_1, \ldots, \succsim_I)\, x$ does not. We say then that "x is socially preferred to y." We read $x\, F(\succsim_1, \ldots, \succsim_I)\, y$ as "x is socially at least as good as y."

Definition 21.C.2. (which generalizes Definition 21.B.2) isolates the social welfare functionals that satisfy a minimal condition of respect for individual preferences.

3. Recall from Section 1.B that $\succ_i$ is formally defined by letting $x \succ_i y$ if $x \succsim_i y$ holds but $y \succsim_i x$ does not. That is, x is preferred to y if x is at least as good as y but y is not as good as x. Also, the indifference relation $\sim_i$ is defined by letting $x \sim_i y$ if $x \succsim_i y$ and $y \succsim_i x$. From Proposition 1.B.1 we know that if $\succsim_i$ is rational, that is, complete and transitive, then $\succ_i$ is irreflexive ($x \succ_i x$ cannot occur) and transitive ($x \succ_i y$ and $y \succ_i z$ implies $x \succ_i z$). Similarly, $\sim_i$ is reflexive ($x \sim_i x$ for all $x \in X$), transitive ($x \sim_i y$ and $y \sim_i z$ implies $x \sim_i z$) and symmetric ($x \sim_i y$ implies $y \sim_i x$).

4. Formally, the preference relation $\succsim_i$ belongs to $\mathscr{P}$ if it is reflexive ($x \succsim_i x$ for every $x \in X$), transitive ($x \succsim_i y$ and $y \succsim_i z$ implies $x \succsim_i z$) and *total* (if $x \neq y$ then either $x \succsim_i y$ or $y \succsim_i x$, but not both). Such preference relations are often referred to as *strict preferences* (although *strict-total preferences* would be less ambiguous) or even as *linear orders*, because these are the properties of the usual "larger than or equal to" order in the real line.

5. In particular, there are no individual utility levels and, therefore, there is no meaningful sense in which any conceivable information on individual utility levels could be compared and matched up. We refer again to Chapter 22 (especially Section 22.D) for an analysis of the problem that focuses on the information used in the aggregation process.

Definition 21.C.2: The social welfare functional $F: \mathscr{A} \to \mathscr{R}$ is *Paretian* if, for any pair of alternatives $\{x, y\} \subset X$ and any preference profile $(\succsim_1, \ldots, \succsim_I) \in \mathscr{A}$, we have that x is socially preferred to y, that is, $x \, F_p(\succsim_1, \ldots, \succsim_I) \, y$, whenever $x \succ_i y$ for every i.

In Example 21.C.1 we describe an interesting class of Paretian social welfare functionals.

Example 21.C.1: *The Borda Count.* Suppose that the number of alternatives is finite. Given a preference relation $\succsim_i \in \mathscr{R}$ we assign a number of points $c_i(x)$ to every alternative $x \in X$ as follows. Suppose for a moment that in the preference relation $\succsim_i$ no two alternatives are indifferent. Then we put $c_i(x) = n$ if x is the nth ranked alternative in the ordering of $\succsim_i$. If indifference is possible in $\succsim_i$ then $c_i(x)$ is the average rank of the alternatives indifferent to x.[6] Finally, for any profile $(\succsim_1, \ldots, \succsim_I) \in \mathscr{R}^I$ we determine a social ordering by adding up points. That is, we let $F(\succsim_1, \ldots, \succsim_I) \in \mathscr{R}$ be the preference relation defined by $x \, F(\succsim_1, \ldots, \succsim_I) \, y$ if $\sum_i c_i(x) \leq \sum_i c_i(y)$. This preference relation is complete and transitive [it is represented by the utility function $-c(x) = -\sum_i c_i(x)$]. Moreover, it is Paretian since if $x \succ_i y$ for every i then $c_i(x) < c_i(y)$ for every i, and so $\sum_i c_i(x) < \sum_i c_i(y)$. ■

We next state an important restriction on social welfare functionals first suggested by Arrow (1963). The restriction says that the social preferences between any two alternatives depend only on the individual preferences between the same two alternatives. There are three possible lines of justification for this assumption. The first is strictly normative and has considerable appeal: it argues that in settling on a social ranking between x and y, the presence or absence of alternatives other than x and y should not matter. They are irrelevant to the issue at hand. The second is one of practicality. The assumption enormously facilitates the task of making social decisions because it helps to separate problems. The determination of the social ranking on a subset of alternatives does not need any information on individual preferences over alternatives outside this subset. The third relates to incentives and belongs to the subject matter of Chapter 23 (see also Proposition 21.E.2). Pairwise independence is intimately connected with the issue of providing the right inducements for the truthful revelation of individual preferences.

Definition 21.C.3: The social welfare functional $F: \mathscr{A} \to \mathscr{R}$ defined on the domain $\mathscr{A}$ satisfies the *pairwise independence condition* (or the *independence of irrelevant alternatives condition*) if the social preference between any two alternatives $\{x, y\} \subset X$ depends only on the profile of individual preferences over the same alternatives. Formally[7], for any pair of alternatives $\{x, y\} \subset X$, and for any pair of preference profiles $(\succsim_1, \ldots, \succsim_I) \in \mathscr{A}$ and $(\succsim'_1, \ldots, \succsim'_I) \in \mathscr{A}$ with the property that, for every i,

$$x \succsim_i y \iff x \succsim'_i y \quad \text{and} \quad y \succsim_i x \iff y \succsim'_i x,$$

6. Thus if $X = \{x, y, z\}$ and $x \succsim_i y \sim_i z$ then $c_i(x) = 1$, and $c_i(y) = c_i(z) = 2.5$.

7. The expressions that follow are a bit cumbersome, We emphasize therefore that they do nothing more than to capture formally the statement just made. An equivalent formulation would be: for any $\{x, y\} \subset X$, if $\succsim_i |\{x, y\} = \succsim'_i |\{x, y\}$ for all i, then $F(\succsim_1, \ldots, \succsim_I)|\{x, y\} = F(\succsim'_1, \ldots, \succsim'_I)|\{x, y\}$. Here $\succsim |\{x, y\}$ stands for the restriction of the preference ordering $\succsim$ to the set $\{x, y\}$.

we have that

$$x\,F(\succsim_1,\ldots,\succsim_I)\,y \quad\Leftrightarrow\quad x\,F(\succsim'_1,\ldots,\succsim'_I)\,y$$

and

$$y\,F(\succsim_1,\ldots,\succsim_I)\,x \quad\Leftrightarrow\quad y\,F(\succsim'_1,\ldots,\succsim'_I)\,x.$$

Example 21.C.1: *continued.* Alas, the Borda count does not satisfy the pairwise independence condition. The reason is simple: the rank of an alternative depends on the placement of *every* other alternative. Suppose, for example, that there are two agents and three alternatives $\{x, y, z\}$. For the preferences

$$x \succ_1 z \succ_1 y,$$
$$y \succ_2 x \succ_2 z$$

we have that x is socially preferred to y [indeed, $c(x) = 3$ and $c(y) = 4$]. But for the preferences

$$x \succ'_1 y \succ'_1 z,$$
$$y \succ'_2 z \succ'_2 x$$

we have that y is socially preferred to x [indeed, now $c(x) = 4$ and $c(y) = 3$]. Yet the relative ordering of x and y has not changed for either of the two agents.

For another illustration, this time with three agents and four alternatives $\{x, y, z, w\}$, consider

$$z \succ_1 x \succ_1 y \succ_1 w,$$
$$z \succ_2 x \succ_2 y \succ_2 w,$$
$$y \succ_3 z \succ_3 w \succ_3 x.$$

Here, y is socially preferred to x [$c(x) = 8$ and $c(y) = 7$]. But suppose now that alternatives z and w move to the bottom for all agents (which because of the Pareto property is a way of saying that the two alternatives are eliminated from the alternative set):

$$\begin{aligned} &x \succ'_1 y \succ'_1 z \succ'_1 w,\\ &x \succ'_2 y \succ'_2 z \succ'_2 w,\\ &y \succ'_3 x \succ'_3 z \succ'_3 w. \end{aligned} \tag{21.C.1}$$

Then x is socially preferred to y [$c(x) = 4$, $c(y) = 5$]. Thus the presence or absence of alternatives z and w matters to the social preference between x and y. Another modification would take alternative x to the bottom for agent 3:

$$x \succ''_1 y \succ''_1 z \succ''_1 w,$$
$$x \succ''_2 y \succ''_2 z \succ''_2 w,$$
$$y \succ''_3 z \succ''_3 w \succ''_3 x.$$

Now y is socially preferred to x [which, relative to the outcome with (21.C.1), is a nice result from the point of view of agent 3]. ■

The previous discussion of Example 21.C.1 teaches us that the pairwise independence condition is a substantial restriction. However, there is a way to proceed that will automatically guarantee that it is satisfied. It consists of determining the social preference between any given two alternatives by applying an aggregation rule that uses only the information about the ordering of *these two alternatives* in individual preferences. We saw in Section 21.B that, for any pair of alternatives, there are many such rules. Can we proceed in this pairwise fashion and still end up with social preferences that are rational, that is, complete and transitive? Example 21.C.2 shows that this turns out to be a real difficulty.

Example 21.C.2: *The Condorcet Paradox.*[8] Suppose that we were to try majority voting among any two alternatives (see Section 21.B for an analysis of majority voting). Does this determine a social welfare functional? We shall see in the next section that the answer is positive in some restricted domains $\mathscr{A} \subset \mathscr{R}^I$. But in general we run into the following problem, known as the Condorcet paradox. Let us have three alternatives $\{x, y, z\}$ and three agents. The preferences of the three agents are

$$x \succ_1 y \succ_1 z,$$

$$z \succ_2 x \succ_2 y,$$

$$y \succ_3 z \succ_3 x.$$

Then pairwise majority voting tells us that x must be socially preferred to y (since x has a majority against y and, a fortiori, y does not have a majority against x). Similarly, y must be socially preferred to z (two voters prefer y to z) and z must be socially preferred to x (two voters prefer z to x). But this cyclic pattern violates the transitivity requirement on social preferences. ■

The next proposition is *Arrow's impossibility theorem*, the central result of this chapter. It essentially tells us that the Condorcet paradox is not due to any of the strong properties of majority voting (which, we may recall from Proposition 21.B.1, are symmetry among agents, neutrality between alternatives, and positive responsiveness). The paradox goes to the heart of the matter: with pairwise independence there is no social welfare functonal defined on $\mathscr{R}^I$ that satisfies a minimal form of symmetry among agents (no dictatorship) and a minimal form of positive responsiveness (the Pareto property).

Proposition 21.C.1: (*Arrow's Impossibility Theorem*) Suppose that the number of alternatives is at least three and that the domain of admissible individual profiles, denoted $\mathscr{A}$, is either $\mathscr{A} = \mathscr{R}^I$ or $\mathscr{A} = \mathscr{P}^I$. Then every social welfare functional $F: \mathscr{A} \to \mathscr{R}$ that is Paretian and satisfies the pairwise independence condition is *dictatorial* in the following sense: There is an agent h such that, for any $\{x, y\} \subset X$ and any profile $(\succsim_1, \ldots, \succsim_I) \in \mathscr{A}$, we have that x is socially preferred to y, that is, $x\, F_p(\succsim_1, \ldots, \succsim_I)\, y$, whenever $x \succ_h y$.

Proof: We present here the classical proof of this result. For another approach to the demonstration we refer to Section 22.D.

It is convenient from now on to view I not only as the number but also as the *set* of agents. For the entire proof we refer to a fixed social welfare functional $F: \mathscr{A} \to \mathscr{R}$ satisfying the Pareto and the pairwise independence conditions. We begin with some definitions. In what follows, when we refer to pairs of alternatives we always mean distinct alternatives.

8. This example was already discussed in Section 1.B.

Definition 21.C.4: Given $F(\cdot)$, we say that a subset of agents $S \subset I$ is:

(i) *Decisive for x over y* if whenever every agent in S prefers x to y *and* every agent not in S prefers y to x, x is socially preferred to y.
(ii) *Decisive* if, for any pair $\{x, y\} \subset X$, S is decisive for x over y.
(iii) *Completely decisive for x over y* if whenever every agent in S prefers x to y, x is socially preferred to y.

The proof will proceed by a detailed investigation of the structure of the family of decisive sets. We do this in a number of small steps. Steps 1 to 3 show that if a subset of agents is decisive for some pair of alternatives then it is decisive for all pairs. Steps 4 to 6 establish some algebraic properties of the family of decisive sets. Steps 7 and 8 use these to show that there is a smallest decisive set formed by a single agent. Steps 9 and 10 prove that this agent is a dictator.

Step 1: If for some $\{x, y\} \subset X$, $S \subset I$ is decisive for x over y, then, for any alternative $z \neq x$, S is decisive for x over z. Similarly, for any $z \neq y$, S is decisive for z over y.

We show that if S is decisive for x over y then it is decisive for x over any $z \neq x$. The reasoning for z over y is identical (you are asked to carry it out in Exercise 21.C.1).

If $z = y$ there is nothing to prove. So we assume that $z \neq y$. Consider a profile of preferences $(\succsim_1, \ldots, \succsim_I) \in \mathcal{A}$ where

$$x \succ_i y \succ_i z \qquad \text{for every } i \in S$$

and

$$y \succ_i z \succ_i x \qquad \text{for every } i \in I \backslash S.$$

Then, because S is decisive for x over y, we have that x is socially preferred to y, that is, $x\, F_p(\succsim_1, \ldots, \succsim_I)\, y$. In addition, since $y \succsim_i z$ for every $i \in I$, and $F(\cdot)$ satisfies the Pareto property it follows that $y\, F_p(\succsim_1, \ldots, \succsim_I)\, z$. Therefore, by the transitivity of the social preference relation, we conclude that $x\, F_p(\succsim_1, \ldots, \succsim_I)\, z$. By the pairwise independence condition, it follows that x is socially preferred to z whenever every agent in S prefers x to z and every agent not in S prefers z to x. That is, S is decisive for x over z.

Step 2: If for some $\{x, y\} \subset X$, $S \subset I$ is decisive for x over y and z is a third alternative, then S is decisive for z over w and for w over z, where $w \in X$ is any alternative distinct from z.

By step 1, S is decisive for x over z and for z over y. But then, applying step 1 again, this time to the pair $\{x, z\}$ and the alternative w, we conclude that S is decisive for w over z. Similarly, applying step 1 to $\{z, y\}$ and w, we conclude that S is decisive for z over w.

Step 3: If for some $\{x, y\} \subset X$, $S \subset I$ is decisive for x over y, then S is decisive.

This is an immediate consequence of step 2 and the fact that there is some alternative $z \in X$ distinct from x or y. Indeed, take any pair $\{v, w\}$. If $v = z$ or $w = z$, then step 2 implies the result directly. If $v \neq z$ and $w \neq z$, we apply step 2 to conclude that S is decisive for z over w, and then step 1 [applied to the pair $\{z, w\}$] to conclude that S is decisive for v over w.

Step 4: If $S \subset I$ and $T \subset I$ are decisive, then $S \cap T$ is decisive.

Take any triple of distinct alternatives $\{x, y, z\} \subset X$ and consider a profile of preferences $(\succsim_1, \ldots, \succsim_I) \in \mathscr{A}$ where

$$\begin{aligned} z \succ_i y \succ_i x &\quad \text{for every } i \in S \backslash (S \cap T), \\ x \succ_i z \succ_i y &\quad \text{for every } i \in S \cap T, \\ y \succ_i x \succ_i z &\quad \text{for every } i \in T \backslash (S \cap T), \\ y \succ_i z \succ_i x &\quad \text{for every } i \in I \backslash (S \cup T). \end{aligned}$$

Then $z\, F_p(\succsim_1, \ldots, \succsim_I)\, y$ because $S\,(= [S \backslash (S \cap T)] \cup (S \cap T))$ is a decisive set. Similarly, $x\, F_p(\succsim_1, \ldots, \succsim_I)\, z$ because T is a decisive set. Therefore, by the transitivity of the social preference, we have that $x\, F_p(\succsim_1, \ldots, \succsim_I)\, y$. It follows by the pairwise independence condition that $S \cap T$ is decisive for x over y, and so, by step 3, that $S \cap T$ is a decisive set.

Step 5: For any $S \subset I$, we have that either S or its complement, $I \backslash S \subset I$, is decisive.

Take any triple of distinct alternatives $\{x, y, z\} \subset X$ and consider a profile of preferences $(\succsim_1, \ldots, \succsim_I) \in \mathscr{A}$ where

$$\begin{aligned} x \succ_i z \succ_i y &\quad \text{for every } i \in S \\ y \succ_i x \succ_i z &\quad \text{for every } i \in I \backslash S. \end{aligned}$$

Then there are two possibilities: either $x\, F_p(\succsim_1, \ldots, \succsim_I)\, y$, in which case, by the pairwise independence condition, S is decisive for x over y (hence, by step 3, decisive), or $y\, F(\succsim_1, \ldots, \succsim_I)\, x$. Because, by the Paretian condition, we have $x\, F_p(\succsim_1, \ldots, \succsim_I)\, z$, the transitivity of the social preference relation yields that $y\, F_p(\succsim_1, \ldots, \succsim_I)\, z$ in this case. But then, using the pairwise independence condition again, we conclude that $I \backslash S$ is decisive for y over z (hence, by step 3, decisive).

Step 6: If $S \subset I$ is decisive and $S \subset T$, then T is also decisive.

Because of the Pareto property the empty set of agents cannot be decisive (indeed, if no agent prefers x over y and every agent prefers y over x, then x is not socially preferred to y). Therefore $I \backslash T$ cannot be decisive because otherwise, by step 4, $S \cap (I \backslash T) = \varnothing$ would be decisive. Hence, by step 5, T is decisive.

Step 7: If $S \subset I$ is decisive and it includes more than one agent, then there is a strict subset $S' \subset S$, $S' \neq S$, such that S' is decisive.

Take any $h \in S$. If $S \backslash \{h\}$ is decisive, then we are done. Suppose, therefore, that $S \backslash \{h\}$ is not decisive. Then, by step 5, $I \backslash (S \backslash \{h\}) = (I \backslash S) \cup \{h\}$ is decisive. It follows, by step 4, that $\{h\} = S \cap [(I \backslash S) \cup \{h\}]$ is also decisive. Thus, we are again done since, by assumption, $\{h\}$ is a strict subset of S.

Step 8: There is an $h \in I$ such that $S = \{h\}$ is decisive.

This follows by iterating step 7 and taking into account, first, that the set of agents I is finite and, second, that, by the Pareto property, the set I of all agents is decisive.

Step 9: If $S \subset I$ is decisive then, for any $\{x, y\} \subset X$, S is completely decisive for x over y.

We want to prove that, for any $T \subset I \backslash S$, x is socially preferred to y whenever every agent in S prefers x to y, every agent in T regards x to be at least as good as

y, and every other agent prefers y to x. To verify this property, take a third alternative $z \in X$, distinct from x and y. By the pairwise independence condition it suffices to consider a profile of preferences $(\succsim_1, \ldots, \succsim_I) \in \mathscr{A}$ where

$$\begin{aligned} x \succ_i z \succ_i y \quad & \text{for every } i \in S, \\ x \succ_i y \succ_i z \quad & \text{for every } i \in T, \\ y \succ_i z \succ_i x \quad & \text{for every } i \in I \backslash (S \cup T). \end{aligned}$$

Then $x\ F_p(\succsim_1, \ldots, \succsim_I)\ z$ because, by step 6, $S \cup T$ is decisive, and $z\ F_p(\succsim_1, \ldots, \succsim_I)\ y$ because S is decisive. Therefore, by the transitivity of social preference, we have that $x\ F_p(\succsim_1, \ldots, \succsim_I)\ y$, as we wanted to show.

Step 10: If, for some $h \in I$, $S = \{h\}$ is decisive, then h is a dictator.

If $\{h\}$ is decisive then, by step 9, $\{h\}$ is completely decisive for any x over any y. That is, if the profile $(\succsim_1, \ldots, \succsim_I)$ is such that $x \succ_h y$, then $x\ F_p(\succsim_1, \ldots, \succsim_I)\ y$. But this is precisely what is meant by $h \in I$ being a dictator.

The combination of steps 8 and 10 completes the proof of Proposition 21.C.1. ■

21.D Some Possibility Results: Restricted Domains

The result of Arrow's impossibility theorem is somewhat disturbing, but it would be too facile to conclude from it that "democracy is impossible." What it shows is something else that we should not expect a collectivity of individuals to behave with the kind of coherence that we may hope from an individual.

It is important to observe, however, that in practice collective judgments are made and decisions are taken. What Arrow's theorem does tell us, in essence, is that the institutional detail and procedures of the political process cannot be neglected. Suppose, for example, that the decision among three alternatives $\{x, y, z\}$ is made by first choosing between x and y by majority voting, and then voting again to choose between the winner and the third alternative z. This will produce an outcome, but the outcome may depend on how the agenda is set—that is, on which alternative is taken up first and which is left for the last. [Thus, if preferences are as in the Condorcet paradox (Example 21.C.2) then the last alternative, whichever it is, will always be the survivor.] This relevance of procedures and rules to social aggregation has far-reaching implications. They have been taken up and much emphasized in modern political science; see, for example, Austen-Smith and Banks (1996) or Shepsle and Boncheck (1995).

In this section, we remain modest and retain the basic framework. We explore to what extent we can escape the dictatorship conclusion if we relax some of the demands imposed by Arrow's theorem. We will investigate two weakenings. In the first, we relax the rationality requirements made on aggregate preferences. In the second, we pose the aggregation question in a restricted domain. In particular, we will consider a restriction *single-peakedness*—that has been found to be significant and useful in applications.

Less Than Full Social Rationality

Suppose that we keep the Paretian and pairwise independence conditions but permit the social preferences to be less than fully rational. Two weakenings of the rationality of preferences are captured in Definition 21.D.1.

Definition 21.D.1: Suppose that the preference relation $\succsim$ on X is reflexive and complete. We say then that:

(i) $\succsim$ is *quasitransitive* if the strict preference $\succ$ induced by $\succsim$ (i.e. $x \succ y \Leftrightarrow x \succsim y$ but not $y \succsim x$) is transitive.

(ii) $\succsim$ is *acyclic* if $\succsim$ has a maximal element in every finite subset $X' \subset X$, that is, $\{x \in X' : x \succsim y \text{ for all } y \in X'\} \neq \varnothing$.

A quasitransitive preference relation is acyclic, but the converse may not hold. Also, a rational preference relation is quasitransitive, but, again, the converse may not hold.[9] Thus the weaker condition is acyclicity. Yet acyclicity is not a drastic weakening of rationality: Note, for example, that the social orderings of the Condorcet paradox (Example 21.C.2) also violate acyclicity. (For more on acylicity see Exercise 21.D.1.)

We will not discuss in detail the possibilities opened to us by these weakenings of social rationality. There are some but they are not very substantial. We refer to Sen (1970) for a detailed exposition. The next two examples are illustrative.

Example 21.D.1: *Oligarchy.* Let I be the set of agents, and let $S \subset I$ be a given subset of agents to be called an *oligarchy* (the possibilities $S = \{h\}$ or $S = I$ are permitted). Given any profile $(\succsim_1, \ldots, \succsim_I) \in \mathscr{R}^I$, the social preferences are formed as follows: For any $x, y \in X$, we say that *x is socially at least as good as y* [written $x\, F(\succsim_1, \ldots, \succsim_I)\, y$] *if there is at least one $h \in S$ that has $x \succsim_h y$*. Hence, x is socially *preferred* to y if and only if *every* member of the oligarchy prefers x to y. In Exercise 21.D.2 you should verify that this social preference relation is quasitransitive but not transitive (because social indifference fails to be transitive). This is the only condition of Arrow's impossibility theorem that is violated (the Paretian condition and pairwise independence conditions are clearly satisfied). Nonetheless, this is scarcely a satisfactory solution to the social aggregation problem, as the aggregator has become very sluggish. At one extreme, if the oligarchy is a single agent then we have a dictatorship. At the other, if the oligarchy is the entire population then society is able to express strict preference only if there is complete unanimity among its members. ■

Example 21.D.2: *Vetoers.* Suppose there are two agents and three alternatives $\{x, y, z\}$. Then given any profile of preferences $(\succsim_1, \succsim_2)$, we let the social preferences coincide with the preferences of agent 1 with one qualification: agent 2 can veto the possibility that alternative x be socially preferred to y. Specifically, if $y \succ_2 x$ then y is socially at least as good as x. Summarizing, for any two alternatives

9. Suppose that $\succsim$ is quasitransitive. Assume for a moment that it is not acyclic. Then there is some finite set $X' \subset X$ without a maximal element for $\succsim$. That is, for every $x \in X'$ there is some $y \in X'$ such that $y \succ x$ (i.e., such that $y \succsim x$ but not $x \succsim y$). Thus, for any integer M we can find a chain $x^1 \succ x^2 \succ \cdots \succ x^M$, where $x^m \in X'$ for every $m = 1, \ldots, M$. If M is larger than the number of alternatives in X', then there must be some repetition in this chain. Say that $x^{m'} = x^m$ for $m > m'$. By quasitransitivity, $x^{m'} \succ x^m = x^{m'}$, which is impossible because $\succ$ is irreflexive by definition. Hence, $\succsim$ must be acyclic. An example of an acyclic but not quasitransitive relation will be given in Example 21.D.2. The relation $\succ$ derived from a rational preference relation $\succsim$ is transitive (Proposition 1.B.1). An example of a quasitransitive, but not rational, preference relation is given in Example 21.D.1.

$\{v, w\} \subset \{x, y, z\}$ we have that v is socially at least as good as w if either $v \succsim_1 w$, or $v = y$, $w = x$ and $v \succ_2 w$. In Exercise 21.D.3 you should verify that the social preferences so defined are acyclic but not necessarily quasitransitive. ■

Single-Peaked Preferences

We proceed now to present the most important class of restricted domain conditions: single-peakedness. We will then see that, in this restricted domain, nondictatorial aggregation is possible. In fact, with a small qualification, we will see that on this domain pairwise majority voting gives rise on this domain to a social welfare functional.

Definition 21.D.2: A binary relation $\geq$ on the set of alternatives X is a *linear order* on X if it is *reflexive* (i.e., $x \geq x$ for every $x \in X$), transitive (i.e., $x \geq y$ and $y \geq z$ implies $x \geq z$) and *total* (i.e., for any distinct $x, y \in X$, we have that either $x \geq y$ or $y \geq x$, but not both).

Example 21.D.3: The simplest example of a linear order occurs when X is a subset of the real line, $X \subset \mathbb{R}$, and $\geq$ is the natural "greater than or equal to" order of the real numbers. ■

Definition 21.D.3: The rational preference relation $\succsim$ is *single peaked* with respect to the linear order $\geq$ on X if there is an alternative $x \in X$ with the property that $\succsim$ is increasing with respect to $\geq$ on $\{y \in X: x \geq y\}$ and decreasing with respect to $\geq$ on $\{y \in X: y \geq x\}$. That is,

$$\text{If } x \geq z > y \quad \text{then} \quad z \succ y$$

and

$$\text{If } y > z \geq x \quad \text{then} \quad z \succ y.$$

In words: There is an alternative x that represents a peak of satisfaction and, moreover, satisfaction increases as we approach this peak (so that, in particular, there cannot be any other peak of satisfaction).

Example 21.D.4: Suppose that $X = [a, b] \subset \mathbb{R}$ and $\geq$ is the "greater than or equal to" ordering of the real numbers. Then a continuous preference relation $\succsim$ on X is single peaked with respect to $\geq$ if and only if it is *strictly convex*, that is, if and only if, for every $w \in X$, we have $\alpha y + (1 - \alpha)z \succ w$ whenever $y \succsim w$, $z \succsim w$, $y \neq z$, and $\alpha \in (0,1)$. (Recall Definition 3.B.5 and also that, as a matter of definition, preference relations generated from strictly quasiconcave utility functions are strictly convex.) This fact accounts to a large extent for the importance of single-peakedness in economic applications. The sufficiency of strict convexity is actually quite simple to verify. (You are asked to prove necessity in Exercise 21.D.4.) Indeed, suppose that x is a maximal element for $\succsim$, and that, say, $x > z > y$. Then $x \succsim y$, $y \succsim y$, $x \neq y$, and $z = \alpha x + (1 - \alpha)y$ for some $\alpha \in (0, 1)$. Thus, $z \succ y$ by strict convexity. In Figures 21.D.1 and 21.D.2, we depict utility functions for two preference relations on $X = [0, 1]$. The preference relation in Figure 21.D.1 is single peaked with respect to $\geq$, but that in Figure 21.D.2 is not. ■

Definition 21.D.4: Given a linear order $\geq$ on X, we denote by $\mathscr{R}_{\geq} \subset \mathscr{R}$ the collection of all rational preference relations that are single peaked with respect to $\geq$.

Given a linear order $\geq$ and a set of agents I, from now on we consider the

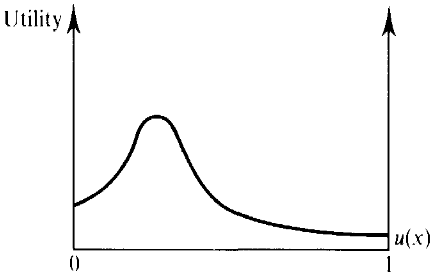


Figure 21.D.1 (left)
Preferences are single peaked with respect to $\geq$.

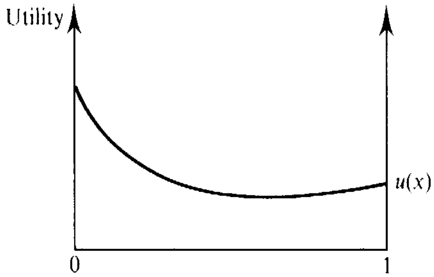


Figure 21.D.2 (right)
Preferences are not single peaked with respect to $\geq$.

restricted domain of preferences $\mathscr{R}^I_{\geq}$. This amounts to the requirement that all individuals have single-peaked preferences with respect to the *same* linear order $\geq$.

Suppose that on the domain $\mathscr{R}^I_{\geq}$ we define social preferences by means of pairwise majority voting (as introduced in Example 21.B.1). That is, given a profile $(\succsim_1, \ldots, \succsim_I) \in \mathscr{R}^I_{\geq}$ and any pair $\{x, y\} \subset X$, we put $x \hat{F}(\succsim_1, \ldots, \succsim_I) y$, to be read as "$x$ is socially at least as good as y", if the number of agents that strictly prefer x to y is larger or equal to the number of agents that strictly prefer y to x, that is, if $\#\{i \in I: x \succ_i y\} \geq \#\{i \in I: y \succ_i x\}$.

Note that, from the definition, it follows that for any pair $\{x, y\}$ we must have either $x \hat{F}(\succsim_1, \ldots, \succsim_I)\, y$ or $y \hat{F}(\succsim_1, \ldots, \succsim_I)\, x$. Thus, pairwise majority voting induces a complete social preference relation (this holds on any possible domain of preferences).

In Exercise 21.D.5 you are asked to show in a direct manner that the preferences of the Condorcet paradox (Example 21.C.2) are not single peaked with respect to any possible linear order on the alternatives. In fact, they cannot be because, as we now show, with single-peaked preferences we are always assured that the social preferences induced by pairwise majority voting have maximal elements, that is, that there are alternatives that cannot be defeated by any other alternatives under majority voting.

Let $(\succsim_1, \ldots, \succsim_I) \in \mathscr{R}^I_{\geq}$ be a fixed profile of preferences. For every $i \in I$ we denote by $x_i \in X$ the maximal alternative for $\succsim_i$ (we will say that x_i is "i's peak").

Definition 21.D.5: Agent $h \in I$ is a *median agent for the profile* $(\succsim_1, \ldots, \succsim_I) \in \mathscr{R}^I_{\geq}$ if

$$\#\{i \in I: x_i \geq x_h\} \geq \frac{I}{2} \quad \text{and} \quad \#\{i \in I: x_h \geq x_i\} \geq \frac{I}{2}.$$

A median agent always exists. The determination of a median agent is illustrated in Figure 21.D.3.

If there are no ties in peaks and $\# I$ is odd, then Definition 21.D.5 simply says that a number $(I - 1)/2$ of the agents have peaks strictly smaller than x_h and another number $(I - 1)/2$ strictly larger. In this case the median agent is unique.

Proposition 21.D.1: Suppose that $\geq$ is a linear order on X and consider a profile of preferences $(\succsim_1, \ldots, \succsim_I)$ where, for every i, $\succsim_i$ is single peaked with respect to $\geq$. Let $h \in I$ be a median agent. Then $x_h \hat{F}(\succsim_1, \ldots, \succsim_I)\, y$ for every $y \in X$. That is, the peak x_h of the median agent cannot be defeated by majority voting by any other alternative. Any alternative having this property is called a *Condorcet winner*. Therefore, a Condorcet winner exists whenever the preferences of all agents are singlepeaked with respect to the same linear order.

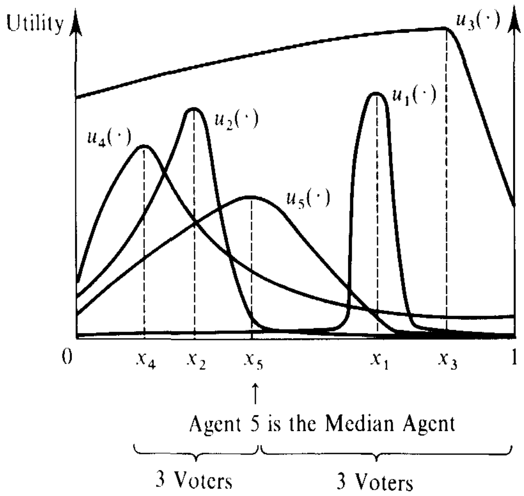


Figure 21.D.3
Determination of a median for a single-peaked family.

Proof: Take any $y \in X$ and suppose that $x_h > y$ (the argument is the same for $y > x_h$). We need to show that y does not defeat x, that is, that

$$\#\{i \in I: x_h \succ_i y\} \geq \#\{i \in I: y \succ_i x_h\}.$$

Consider the set of agents $S \subset I$ that have peaks *larger than or equal* to x_h, that is, $S = \{i \in I: x_i \geq x_h\}$. Then $x_i \geq x_h > y$ for every $i \in S$. Hence, by single-peakedness of $\succsim_i$ with respect to $\geq$, we get $x_h \succ_i y$ for every $i \in S$. On the other hand, because agent h is a median agent we have that $\#S \geq I/2$ and so $\#\{i \in I: y \succ_i x_h\} \leq \#(I \backslash S) \leq I/2 \leq \#S \leq \#\{i \in I: x_h \succ_i y\}$. ■

Proposition 21.D.1 guarantees that the preference relation $\hat{F}(\succsim_1, \ldots, \succsim_I)$ is acyclic. It may, however, not be transitive. In Exercise 21.D.6 you are asked to find an example of nontransitivity. Transitivity obtains in the special case where I is odd and, for every i, the preference relation $\succsim_i$ belongs to the class $\mathscr{P}^I_{\geq} \subset \mathscr{R}^I_{\geq}$ formed by the rational preference relations $\succsim$ that are single peaked with respect to $\geq$ and have the property that no two distinct alternatives are indifferent for $\succsim$. Note that, if I is odd and preferences are in this class, then, for any pair of alternatives, there is always a strict majority for one of them against the other. Hence, in this case, a Condorcet winner necessarily defeats any other alternative.

Proposition 21.D.2: Suppose that I is odd and that $\geq$ is a linear order on X. Then pairwise majority voting generates a well-defined social welfare functional $F: \mathscr{P}^I_{\geq} \to \mathscr{R}$. That is, on the domain of preferences that are single-peaked with respect to $\geq$ and, moreover, have the property that no two distinct alternatives are indifferent, we can conclude that the social relation $\hat{F}(\succsim_1, \ldots, \succsim_I)$ generated by pairwise majority voting is complete and transitive.

Proof: We already know that $\hat{F}(\succsim_1, \ldots, \succsim_I)$ is complete. It remains to show that it is transitive. For this purpose, suppose that $x\, \hat{F}(\succsim_1, \ldots, \succsim_I) y$ and $y\, \hat{F}(\succsim_1, \ldots, \succsim_I)\, z$. Under our assumptions (recall that I is odd and that no individual indifference is allowed) this means that x defeats y and y defeats z. Consider the set $X' = \{x, y, z\}$. If preferences are restricted to this set then, relative to X', preferences still belong to the class $\mathscr{P}^I_{\geq}$, and therefore there is an alternative in X' that is not defeated by any

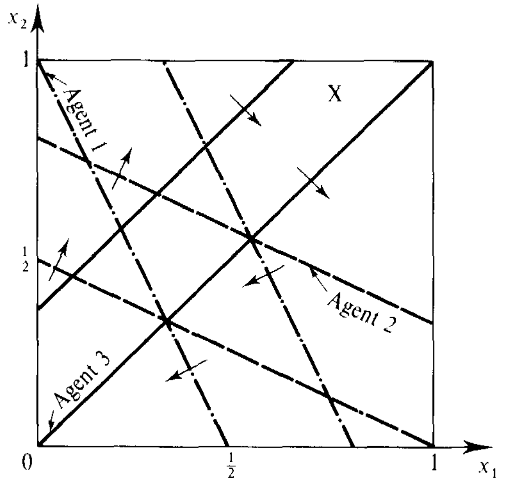


Figure 21.D.4 Indifference curves for the preferences of Example 21.D.5.

other alternative in X'. This alternative can be neither y (defeated by x) nor z (defeated by y). Hence, it has to be x and we conclude that $x\ \hat{F}(\succsim_1, \ldots, \succsim_I)\ z$, as required by transitivity. ■

In applications, the linear order on alternatives arises typically as the natural order, as real numbers, of the values of a one-dimensional parameter. Then, as we have seen, single-peakedness follows from the strict quasiconcavity of utility functions, a restriction quite often satisfied in economics. It is an unfortunate fact that the power of quasiconcavity is confined to one-dimensional problems. We illustrate the issues involved in more general cases by discussing two examples.

Example 21.D.5: Suppose that the space of alternatives is the unit square, that is, $X = [0, 1]^2$. The generic entries of X are denoted $x = (x_1, x_2)$. There are three agents $I = \{1, 2, 3\}$. The preferences of the agents are expressed by the utility functions on X:

$$u_1(x_1, x_2) = -2x_1 - x_2,$$

$$u_2(x_1, x_2) = x_1 + 2x_2,$$

$$u_3(x_1, x_2) = x_1 - x_2.$$

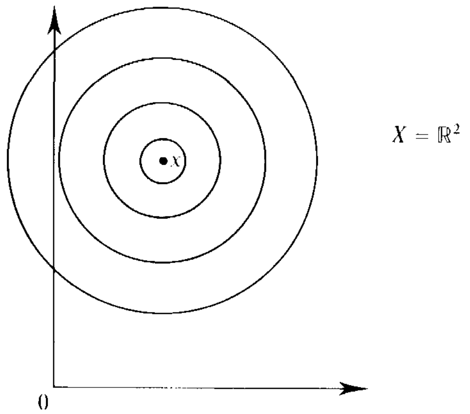
These preferences are represented in Figure 21.D.4. Every utility function is linear and therefore preferences are convex (also, they have a single maximal element on X).[10] But, we will now argue that for every $x \in X$ there is a $y \in X$ preferred by two of the agents to x. To see this we take an arbitrary $x = (x_1, x_2) \in [0, 1]^2$ and distinguish three cases:

(i) If $x_1 = 0$, then $y = (\frac{1}{2}, x_2)$ is preferred by agents 2 and 3 to x.
(ii) If $x_2 = 1$, then $y = (x_1, \frac{1}{2})$ is preferred by agents 1 and 3 to x.
(iii) If $x_1 > 0$ and $x_2 < 1$, then $y = (x_1 - \varepsilon, x_2 + \varepsilon) \in [0, 1]^2$ with $\varepsilon > 0$, is preferred by agents 1 and 2 to x.

You should verify the claims made in (i), (ii), and (iii). ■

The situation illustrated in Example 21.D.5 is not a peculiarity. The key property of the

10. The preferences of this example are not strictly convex. This is immaterial. Without changing the nature of the example we could modify them slightly so as to make the indifference curve map strictly convex.



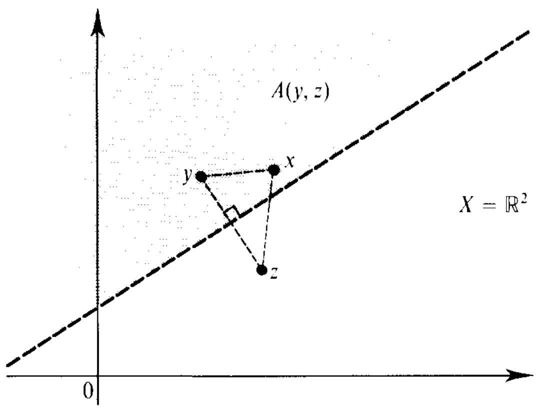


Figure 21.D.5 (left) Euclidean preferences in $\mathbb{R}^2$.

Figure 21.D.6 (right) The region of Euclidean preferences that prefer y to z.

example is that the cone spanned by the nonnegative combinations of the gradient vectors of the three utility functions equals the entire $\mathbb{R}^2$ (see Figure 21.D.4). Exercises 21.D.7 and 21.D.8 provide further elaboration on this issue.

The reason why in two (or more) dimensions, quasiconcavity does not particularly help is that, in contrast with the one-dimensional case, there is no sensible way to assign a "median" to a set of points in the plane. This will become clear in the next, classical, Example 21.D.6 which we now describe.

Example 21.D.6: *Euclidean Preferences.* Suppose that the set of alternatives is $\mathbb{R}^n$. Agents have preferences $\succsim$ represented by utility functions of the form $u(y) = -\|y - x\|$, where x is a fixed alternative in $\mathbb{R}^n$. In words: x is the most preferred alternative for $\succsim$ and other alternatives are evaluated by how close they are to x in the Euclidean distance. The indifference curves of a typical consumer in $\mathbb{R}^2$ are pictured in Figure 21.D.5.

In the current example, the set $\mathbb{R}^n$ does double duty. On the one hand, it represents the set of alternatives. On the other, it also stands for the set of all possible preferences because every $x \in \mathbb{R}^n$ uniquely identifies the preferences that have x as a peak.[11]

Given two distinct alternatives $y, z \in \mathbb{R}^n$, an agent will prefer y to z if and only if his peak is closer to y than to z. Thus, the region of peaks associated with preferences that prefer y to z is

$$A(y, z) = \{x \in \mathbb{R}^n \colon \|x - y\| < \|x - z\|\}.$$

See Figure 21.D.6 for a representation. Geometrically, the boundary of $A(y, z)$ is the hyperplane perpendicular to the segment connecting y and z and passing through its midpoint.

We will consider the idealized limit situation where there is a continuum of agents with Euclidean preferences and the population is described by a density function $g(x)$ defined on $\mathbb{R}^n$, the set of possible peaks. Then given two distinct alternatives $y, z \in \mathbb{R}^n$, the fraction of the total population that prefers y to z, denoted $m_g(y, z)$, is simply the integral of $g(\cdot)$ over the region $A(y, z) \subset \mathbb{R}^n$.

When will there exist a Condorcet winner? Suppose there is an $x^* \in \mathbb{R}^n$ with the property that *any* hyperplane through x^* divides $\mathbb{R}^n$ into two half-spaces each having a total mass of $\frac{1}{2}$ according to the density $g(\cdot)$. This point could be called a *median* for the density $g(\cdot)$; it coincides with the usual concept of a median in the case $n = 1$. A median in this sense is a Condorcet winner. It cannot be defeated by any other alternative because if $y \neq x^*$ then $A(x^*, y)$ is larger than a half-space through x^* and, therefore, $m_g(x^*, y) \geq \frac{1}{2}$. Conversely, if x^*

11. For an example in the same spirit where the two roles are kept separate, see Grandmont (1978) and Exercise 21.D.9.

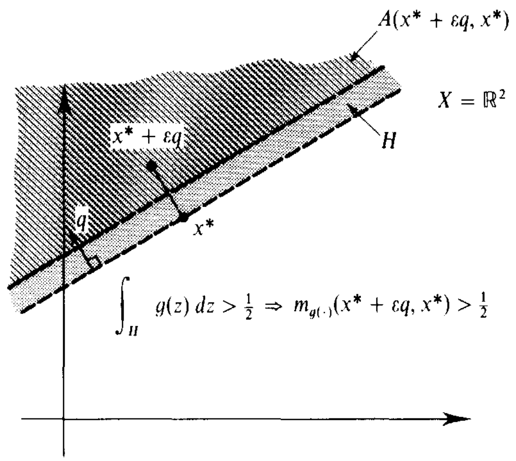


Figure 21.D.7
If x^* is not a median then it is not a Condorcet winner.

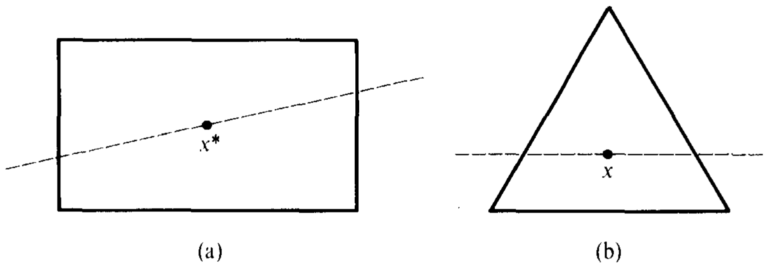


Figure 21.D.8
(a) Uniform distribution over a rectangle: The center point x^* is a median since every plane through x^* divides the rectangle into two figures of equal area. (b) Uniform distribution over a triangle: There is no median.

is not a median then there is a direction $q \in \mathbb{R}^n$ such that the mass of the half-space $\{z \in \mathbb{R}^n: q \cdot z > q \cdot x^*\}$ is larger than $\frac{1}{2}$. Thus, by continuity, if $\varepsilon > 0$ is small then the mass of the translated half-space $A(x^* + \varepsilon q, x^*)$ is larger than $\frac{1}{2}$. Hence $x^* + \varepsilon q$ defeats x^*, and x^* cannot be a Condorcet winner. (See Figure 21.D.7.)

We have seen that a Condorcet winner exists if and only if there is a median for the density $g(\cdot)$. But for $n > 1$ the existence of a median imposes so many conditions (there are many half-spaces) that it becomes a knife-edge property. Figure 21.D.8 provides examples. In Figure 21.D.8(a), the density $g(\cdot)$ is the uniform density over a rectangle [a case first studied by Tullock (1967)]. The center of the rectangle is then, indeed, a median. But the rectangle is very special. The typical case is one of nonexistence. In Figure 21.D.8(b), the density $g(\cdot)$ is the uniform density over a triangle. Then no median exists: Through any point of a triangle we can draw a line that divides it into two regions of unequal area.[12] ■

12. See Caplin and Nalebuff (1988) for further analysis. They show that under a restriction on the density function (called "logarithmic concavity" and satisfied, in particular, for uniform densities over convex sets), there are always points ("generalized medians") in $\mathbb{R}^n$ with the property that any hyperplane through the point divides $\mathbb{R}^n$ into two regions, each of which has mass larger than $1/e$. This means that these points cannot be defeated by any other alternative *if* the majority required is not $\frac{1}{2}$ but any number larger than $1 - (1/e) > \frac{1}{2}$, 64% say. Of course, a 64% rule becomes less decisive than a 50% rule: There will now be many pairs of alternatives with the property that no member of the pair defeats the other.

21.E Social Choice Functions

The task we set ourselves to accomplish in the previous sections was how to aggregate profiles of individual preference relations into a coherent (i.e. rational) social preference order. Presumably, this social preference order is then used to make decisions. In this section we focus directly on social decisions and pose the aggregation question as one of analyzing how profiles of individual preferences turn into social decisions.

The main result we obtain again yields a dictatorship conclusion. The result amounts, in a sense, to a translation of the Arrow's impossibility theorem into the language of choice functions. It also offers a reinterpretation of the condition of pairwise independence, and provides a link towards the incentive-based analysis of Chapter 23.

As before, we have a set of alternatives X and a finite set of agents I. The set of preference relations $\succsim$ on X is denoted $\mathscr{R}$. We also designate by $\mathscr{P}$ the subset of $\mathscr{R}$ consisting of the preference relations $\succsim \in \mathscr{R}$ with the property that no two distinct alternatives are indifferent for $\succsim$.

Definition 21.E.1: Given any subset $\mathscr{A} \subset \mathscr{R}^I$, a *social choice function* $f: \mathscr{A} \rightarrow X$ defined on $\mathscr{A}$ assigns a chosen element $f(\succsim_1, \ldots, \succsim_I) \in X$ to every profile of individual preferences in $\mathscr{A}$.

The notion of social choice function embodies the requirement that the chosen set be single valued. We could argue that this is, after all, in the nature of what a choice is.[13] More restrictive is the fact that we do not allow for random choice.[14]

If X is finite, every social welfare functional $F(\cdot)$ on a domain $\mathscr{A}$ induces a natural social choice function by associating with each $(\succsim_1, \ldots, \succsim_I) \in \mathscr{A}$ a most preferred element in X for the social preference relation $F(\succsim_1, \ldots, \succsim_I)$. For example, if, as in Proposition 21.D.2, $\mathscr{A} \subset \mathscr{P}^I_{\geq}$ is a domain of single-peaked preferences, I is odd, and $F(\cdot)$ is the pairwise majority voting social welfare functional defined on $\mathscr{A}$, then for every $(\succsim_1, \ldots, \succsim_I)$ the choice $f(\succsim_1, \ldots, \succsim_I)$ is the Condorcet winner in X.

We now state and prove a result parallel to Arrow's impossibility theorem. Recall that for Arrow's theorem we had two conditions: the social welfare functional had to be Paretian and had to be pairwise independent. Here we require again two conditions: first, the social choice function must be, again, (*weakly*) *Paretian*; and, second, it should be *monotonic*. We define these concepts in Definitions 21.E.2 and 21.E.4, respectively.

13. Nevertheless, allowing for multivalued choice sets (that is, allowing there to be more than one acceptable social choice) is natural in some contexts, and certain assumptions on social choice may be more plausible in the multivalued case.

14. Note also the contrast between the definition of choice function here and the similar concept of choice rule in Section 1.C. There we contemplated the possibility of there being several budgets and of the choice depending on the budget at hand. Here the budget is fixed (it is always X) but the choice may depend on the profile of underlying individual preferences. Clearly we could, but will not, consider situations that encompass both cases. Another contrast with Section 1.C is that here we limit ourselves to single-valued choice.

Definition 21.E.2: The social choice function $f: \mathscr{A} \to X$ defined on $\mathscr{A} \subset \mathscr{R}^I$ is *weakly Paretian* if for any profile $(\succsim_1, \ldots, \succsim_I) \in \mathscr{A}$ the choice $f(\succsim_1, \ldots, \succsim_I) \in X$ is a weak Pareto optimum. That is, if for some pair $\{x, y\} \subset X$ we have that $x \succ_i y$ for every i, then $y \neq f(\succsim_1, \ldots, \succsim_I)$.

In order to define monotonicity we need a preliminary concept.

Definition 21.E.3: The alternative $x \in X$ *maintains its position from* the profile $(\succsim_1, \ldots, \succsim_I) \in \mathscr{R}^I$ to the profile $(\succsim'_1, \ldots, \succsim'_I) \in \mathscr{R}^I$ if

$$x \succsim_i y \quad \text{implies} \quad x \succsim'_i y$$

for every i and every $y \in X$.

In other words, x maintains its position from $(\succsim_1, \ldots, \succsim_I)$ to $(\succsim'_1, \ldots, \succsim'_I)$ if for every i the set of alternatives inferior (or indifferent) to x expands (or remains the same) in moving from $\succsim_i$ to $\succsim'_i$. That is,

$$L(x, \succsim_i) = \{y \in X : x \succsim_i y\} \subset L(x, \succsim'_i) = \{y \in X : x \succsim'_i y\}.$$

Note that the condition stated in Definition 21.E.3 imposes no restriction on how other alternatives different from x may change their mutual order in going from $\succsim_i$ to $\succsim'_i$.[15]

Definition 21.E.4: The social choice function $f: \mathscr{A} \to X$ defined on $\mathscr{A} \subset \mathscr{R}^I$ is *monotonic* if for any two profiles $(\succsim_1, \ldots, \succsim_I) \in \mathscr{A}$, $(\succsim'_1, \ldots, \succsim'_I) \in \mathscr{A}$ with the property that the chosen alternative $x = f(\succsim_1, \ldots, \succsim_I)$ maintains its position from $(\succsim_1, \ldots, \succsim_I)$ to $(\succsim'_1, \ldots, \succsim'_I)$, we have that $f(\succsim'_1, \ldots, \succsim'_I) = x$.

In words: The social choice function is monotonic if no alternative can be dropped from being chosen unless for some agent its disirability deteriorates.

Are there social choice functions that are weakly Paretian and monotonic? The answer is "yes." For example, in Exercise 21.E.1 you are asked to verify that the pairwise majority voting social decision function defined on a domain of single-peaked preferences is weakly Paretian and monotonic. But what if we have a universal domain (i.e., $\mathscr{A} = \mathscr{R}^I$ or $\mathscr{A} = \mathscr{P}^I$)? A not very attractive class of social choice functions having the two properties in this domain are the *dictatorial* social choice functions.

Definition 21.E.5: An agent $h \in I$ is a *dictator* for the social choice function $f: \mathscr{A} \to X$ if, for every profile $(\succsim_1, \ldots, \succsim_I) \in \mathscr{A}$, $f(\succsim_1, \ldots, \succsim_I)$ is a most preferred alternative for $\succsim_h$ in X; that is,

$$f(\succsim_1, \ldots, \succsim_I) \in \{x \in X : x \succsim_h y \text{ for every } y \in X\}.$$

A social choice function that admits a dictator is called *dictatorial*.

In the domain $\mathscr{P}^I$, a dictatorial social choice function is weakly Paretian and monotonic. (This is clear enough, but at any rate you should verify it in Exercise 21.E.2, where you are also asked to discuss the case $\mathscr{A} = \mathscr{R}^I$.) Unfortunately, in the universal domain we cannot get anything better than the dictatorial social choice functions. The impossibility result of Proposition 21.E.1 establishes this.

15. As in Section 3.B, the sets $L(x, \succsim_i)$ are also referred to as *lower contour sets*.

Proposition 21.E.1: Suppose that the number of alternatives is at least three and that the domain of admissible preference profiles is either $\mathscr{A} = \mathscr{R}^I$ or $\mathscr{A} = \mathscr{P}^I$. Then every weakly Paretian and monotonic social choice function $f: \mathscr{A} \to X$ is dictatorial.

Proof: The proof of the result will be obtained as a corollary of Arrow's impossibility theorem (Proposition 21.C.1). To this effect, we proceed to derive a social welfare functional $F(\cdot)$ that rationalizes $f(\succsim_1, \ldots, \succsim_I)$ for every profile $(\succsim_1, \ldots, \succsim_I) \in \mathscr{A}$. We will then show that $F(\cdot)$ satisfies the assumptions of Arrow's theorem, hence yielding the dictatorship conclusion.

We begin with a useful definition.

Definition 21.E.6: Given a finite subset $X' \subset X$ and a profile $(\succsim_1, \ldots, \succsim_I) \in \mathscr{R}^I$, we say that the profile $(\succsim'_1, \ldots, \succsim'_I)$ *takes X' to the top from* $(\succsim_1, \ldots, \succsim_I)$ if, for every i,

$$x \succ'_i y \qquad \text{for } x \in X' \text{ and } y \notin X',$$

$$x \succsim_i y \Leftrightarrow x \succsim'_i y \qquad \text{for all } x, y \in X'.$$

In words: The preference relation $\succsim'_i$ is obtained from $\succsim_i$ by simply taking every alternative in X' to the top, while preserving the internal (weak or strict) ordering among these alternatives. The ordering among alternatives not in X' is arbitrary. For example, if $x \succ_i y \succ_i z \succ_i w$, then the preference relation $\succ'_i$ defined by $y \succ'_i w \succ'_i z \succ'_i x$ takes $\{y, w\}$ to the top from $\succsim_i$. Note also that if $(\succsim'_1, \ldots, \succsim'_I)$ takes X' to the top from $(\succsim_1, \ldots, \succsim_I)$, then every $x \in X'$ maintains its position in going from $(\succsim_1, \ldots, \succsim_I)$ to $(\succsim'_1, \ldots, \succsim'_I)$.

For the rest of the proof we proceed in steps:

Step 1: If both the profiles $(\succsim'_1, \ldots, \succsim'_I) \in \mathscr{A}$ and $(\succsim''_1, \ldots, \succsim''_I) \in \mathscr{A}$ take $X' \subset X$ to the top from $(\succsim_1, \ldots, \succsim_I)$, then $f(\succsim'_1, \ldots, \succsim'_I) = f(\succsim''_1, \ldots, \succsim''_I)$.

For every i and $x \in X'$ we have

$$\{y \in X : x \succsim'_i y\} = \{y \in X : x \succsim''_i y\} = \{y \in X : x \succsim_i y\} \cup X \backslash X'.$$

By the weak Pareto property, $f(\succsim'_1, \ldots, \succsim'_I) \in X'$. Thus, $f(\succsim'_1, \ldots, \succsim'_I) \in X'$ maintains its position in going from $(\succsim'_1, \ldots, \succsim'_I)$ to $(\succsim''_1, \ldots, \succsim''_I)$. Therefore, by the monotonicity of $f(\cdot)$, we conclude that $f(\succsim'_1, \ldots, \succsim'_I) = f(\succsim''_1, \ldots, \succsim''_I)$.

Step 2: Definition of $F(\succsim_1, \ldots, \succsim_I)$.

For every profile $(\succsim_1, \ldots, \succsim_I) \in \mathscr{A}$ we define a certain binary relation $F(\succsim_1, \ldots, \succsim_I)$ on X. Specifically, we let $x\ F(\succsim_1, \ldots, \succsim_I)\ y$, (read as "$x$ is socially at least as good as y") if $x = y$ or if $x = f(\succsim'_1, \ldots, \succsim'_I)$ when $(\succsim'_1, \ldots, \succsim'_I) \in \mathscr{A}$ is any profile that takes $\{x, y\} \subset X$ to the top from the profile $(\succsim_1, \ldots, \succsim_I)$. By step 1 this is well defined, that is, independent of the particular profile $(\succsim'_1, \ldots, \succsim'_I)$ chosen.

Step 3: For every profile $(\succsim_1, \ldots, \succsim_I) \in \mathscr{A}$, $F(\succsim_1, \ldots, \succsim_I)$ is a rational preference relation. Moreover, $F(\succsim_1, \ldots, \succsim_I) \in \mathscr{P}$; that is, no two distinct alternatives are socially indifferent.

Because $f(\cdot)$ is weakly Paretian, it follows that when $(\succsim'_1, \ldots, \succsim'_I)$ takes $\{x, y\}$ to the top from $(\succsim_1, \ldots, \succsim_I)$ we must have $f(\succsim'_1, \ldots, \succsim'_I) \in \{x, y\}$. Therefore, we conclude that either $x\ F(\succsim_1, \ldots, \succsim_I)\ y$ or $y\ F(\succsim_1, \ldots, \succsim_I)\ x$, *but*, because of step 1, *not both* (unless $x = y$). In particular, $F(\succsim_1, \ldots, \succsim_I)$ is complete.