Proposition 17.F.2: Suppose that the excess demand function $z(\cdot)$ is such that, for any constant returns convex technology Y, the economy formed by $z(\cdot)$ and Y has a unique (normalized) equilibrium price vector. Then $z(\cdot)$ satisfies the weak axiom. Conversely, if $z(\cdot)$ satisfies the weak axiom then, for any constant returns convex technology Y, the set of equilibrium price vectors is convex (and so, if the set of normalized price equilibria is finite, there can be at most one normalized price equilibrium).

Proof: The first part has already been shown. To verify the convexity of the set of equilibrium prices, suppose that p and p' are equilibrium price vectors for the constant returns convex technology Y; that is, $z(p) \in Y$, $z(p') \in Y$, and, for any $y \in Y$, $p \cdot y \leq 0$ and $p' \cdot y \leq 0$. Let $p'' = \alpha p + (1 - \alpha)p'$ for $\alpha \in [0, 1]$. Note, first, that $p'' \cdot y = \alpha p \cdot y + (1 - \alpha)p' \cdot y \leq 0$ for any $y \in Y$. To show that p'' is an equilibrium, we therefore need only establish that $z(p'') \in Y$. Because $0 = p'' \cdot z(p'') = \alpha p \cdot z(p'') + (1 - \alpha)p' \cdot z(p'')$, we have that either $p \cdot z(p'') \leq 0$ or $p' \cdot z(p'') \leq 0$. Suppose that the first possibility holds, so that $p \cdot z(p'') \leq 0$ [a parallel argument applies if, instead, $p' \cdot z(p'') \leq 0$]. Since $z(p) \in Y$ we have $p'' \cdot z(p) \leq 0$. But with $p'' \cdot z(p) \leq 0$ and $p \cdot z(p'') \leq 0$, a contradiction to the WA can be avoided only if $z(p'') = z(p)$. Hence $z(p'') \in Y$.[39] ■

We are therefore led to focus attention on conditions on preferences and endowments of the I consumers guaranteeing that the aggregate excess demand function $z(p)$ fulfills the WA. To begin with a relatively simple case, suppose that all the endowment vectors ω_i are proportional among themselves; that is, that $\omega_i = \alpha_i \bar{\omega}$, where $\bar{\omega}$ is the vector of total endowments and $\alpha_i \geq 0$ are shares with $\sum_i \alpha_i = 1$. In such an economy, the distribution of wealth across consumers is independent of prices. Normalizing prices to $p \cdot \bar{\omega} = 1$, the wealth of consumer 1 is α_i and $z_i(p) = x_i(p, \alpha_i) - \omega_i$. The aggregate demand behavior of a population of consumers with fixed wealth levels was studied in Section 4.C. We repeat our qualitative conclusion from there: if individual wealth levels remain fixed, the satisfaction of the WA by aggregate demand (or excess demand), although restrictive, is not implausible.[40]

A proportionality assumption on initial endowments is not very tenable in a general equilibrium context. It is important, therefore, to ask which new effects are at work (relative to those studied in Section 4.C) when the distribution of endowments does not satisfy this hypothesis. Unfortunately, it turns out that nonproportionality of endowments can reduce the likelihood of satisfaction of the weak axiom by aggregate excess demand. To see this, consider the relatively simple situation in which preferences are homothetic. Recall from Sections 4.C and 4.D that, when endowments are proportional, this case is extremely well behaved; not only is the WA satisfied, but the model even admits a representative consumer. Yet, as we proceed to discuss

39. Observe that we have established that either $z(p'') = z(p)$ or $z(p'') = z(p')$. Since this is true for any $\alpha \in [0, 1]$, and since the function $z(\cdot)$ is continuous, this implies that $z(p) = z(p')$ for any two equilibrium price vectors p and p'; that is, if the WA holds for $z(\cdot)$, then every Walrasian equilibrium for the given endowments must have the same aggregate consumption vector and, hence, the same aggregate production vector.

40. On this point, consult also the references given in Chapter 4, especially Hildenbrand (1994).

below (in small type), even with homothetic preferences, the WA can easily be violated when endowments are not proportional.[41]

In Section 2.F we offered a differential version of the WA for the case of demand functions. In a parallel fashion we can also do so for excess demand functions. It can be shown that a sufficient differential condition for the WA is

$$dp \cdot Dz(p)\, dp < 0 \text{ whenever } dp \cdot z(p) = 0 \text{ (i.e., whenever the price change is compensated) and } dp \text{ is not proportional to } p \text{ (i.e., relative prices change).} \tag{17.F.1}$$

Allowing for the first inequality to be weak, expression (17.F.1) constitutes also a necessary condition.[42]

Under the homotheticity assumption, we have

$$D_{w_i} x_i(p, p \cdot \omega_i) = \frac{1}{p \cdot \omega_i} x_i(p, p \cdot \omega_i).$$

Denoting $S_i = S_i(p, p \cdot \omega_i)$, $x_i = x_i(p, p \cdot \omega_i)$, $\bar{x} = \sum_i x_i$ and $\bar{\omega} = \sum_i \omega_i$, this implies (recall 17.E.3)

$$\begin{aligned} Dz(p) &= \sum_i S_i(p, p \cdot \omega_i) - \sum_i \frac{1}{p \cdot \omega_i} x_i(p, p \cdot \omega_i) z_i(p, p \cdot \omega_i)^{\mathrm{T}} \\ &= \sum_i S_i - \sum_i \frac{1}{p \cdot \omega_i} \left[x_i - \frac{p \cdot \omega_i}{p \cdot \bar{\omega}} \bar{x} \right] \left[x_i - \frac{p \cdot \omega_i}{p \cdot \bar{\omega}} \bar{x} \right]^{\mathrm{T}} \\ &\quad + \sum_i \frac{1}{p \cdot \omega_i} \left[x_i - \frac{p \cdot \omega_i}{p \cdot \bar{\omega}} \bar{x} \right] \left[\omega_i - \frac{p \cdot \omega_i}{p \cdot \bar{\omega}} \bar{\omega} \right]^{\mathrm{T}} - \frac{1}{p \cdot \bar{\omega}} \bar{x} z(p)^{\mathrm{T}}. \end{aligned} \tag{17.F.2}$$

For any direction of price change dp with $dp \cdot z(p) = 0$, the first two terms on the right-hand side of equation (17.F.2) generate an effect of the appropriate sign [the $L \times L$ substitution matrices $S_i(p, p \cdot \omega_i)$ and variance matrices

$$-\left[x_i - \frac{p \cdot \omega_i}{p \cdot \bar{\omega}} \bar{x} \right] \left[x_i - \frac{p \cdot \omega_i}{p \cdot \bar{\omega}} \bar{x} \right]^{\mathrm{T}}$$

are negative semidefinite], the fourth is null, *but the third is ambiguous.* It is quite possible for this covariance term to have the wrong sign (positive) and even for it to overcome the other two terms.[43] The situation to worry about is when $(1/(p \cdot \omega_i)) x_i(p, p \cdot \omega_i)$ and ω_i are positively associated within the population of consumers; that is, when the consumers who consume (per dollar) more than the average (per dollar) consumption of some commodities tend to be those that are relatively well endowed (per dollar) with those commodities. It makes sense that this case will cause difficulties: If the price of a good increases, the consumers who are (net) sellers of the good (who are likely to be those relatively well endowed with it) experience a positive wealth effect, whereas the consumers who are (net) buyers experience a negative wealth effect. Hence, an increase in the total demand for the good will ensue if (net) sellers consume relatively more of the good (per dollar) than (net) buyers.

41. To reinforce this point, it is also worth mentioning that, in fact, if we are free to choose initial endowments, then the class of homothetic preferences imposes no restrictions on aggregate demand. Indeed, as we noted in Section 17.E, the basic conclusion of Proposition 17.E.2 can still be obtained with the further restriction that preferences be homothetic. See Mantel (1976) and the survey of Shafer and Sonnenschein (1982).

42. Suppose that $\Delta p \cdot z(p) = (p' - p) \cdot z(p) = 0$. Definition 17.F.1 implies then that $\Delta p \cdot \Delta z = (p' - p) \cdot (z(p') - z(p)) \leq 0$. Going to the differential limit and using the chain rule, it follows that $dp \cdot Dz(p)\, dp \leq 0$ whenever $dp \cdot z(p) = 0$.

43. But this cannot happen if the $x_i(p, p \cdot \omega_i)$ are collinear among themselves or if the ω_i are collinear among themselves. See Exercise 17.F.1.

Example 17.F.1: This is an example of a failure of the WA compatible with homotheticity and even with the property of gross substitution, which we will discuss shortly. Consider a four-commodity economy with two consumers. Consumer 1 has preferences and endowments for only the first two goods; that is, he has an excess demand function $z_1(p) = z_1(p_1, p_2)$ that does not depend on p_3 and p_4 and, further, is such that $z_{31}(p) = z_{41}(p) = 0$ for all p. Similarly, consumer 2 has preferences and endowments for only the last two goods.[44] We claim that if there is a price vector p' at which the excess demand of the two consumers is nonzero [i.e., $z_1(p') \neq 0$ and $z_2(p') \neq 0$], then the aggregate excess demand cannot satisfy the WA. To see this, choose $(\hat{p}_1, \hat{p}_2)$ and $(\hat{p}_3, \hat{p}_4)$ arbitrarily, except that $\hat{p}_1 z_{11}(p') + \hat{p}_2 z_{21}(p') < 0$ and $\hat{p}_3 z_{32}(p') + \hat{p}_4 z_{42}(p') < 0$. For $\alpha > 0$, take $q = (p'_1, p'_2, \alpha\hat{p}_3, \alpha\hat{p}_4)$ and $q' = (\alpha\hat{p}_1, \alpha\hat{p}_2, p'_3, p'_4)$. Then if $\alpha > 0$ is sufficiently large, we have $q \cdot z(q') < 0$ and $q' \cdot z(q) < 0$ (Exercise 17.F.2). ■

See Exercise 17.F.3 for yet another example.

Gross Substitution

We now investigate the implications of a condition of a different nature from the WA. We shall see that it yields a uniqueness result for situations that are reducible to formalization as exchange economies.

To motivate the concept (and justify its name), consider the demand function of a consumer in a two-commodity situation. At given prices the demand substitution matrix has negative diagonal entries and, as a consequence, positive off-diagonal entries: if the price of one good is raised, the compensated demand for the other good increases. However, if we do not net out the wealth effects (i.e., if we look at the effect of prices on uncompensated demand), then it is possible for an increase in the price of one good to decrease the demand of the two goods: in gross terms, the two goods may be complements. We say that the two goods are *gross substitutes* if this does not happen, that is, if an increase in the price of one good decreases the (uncompensated or gross) demand for that good and increases the (uncompensated or gross) demand for the other good. By extension, the same term is used in the L-commodity case for the property that asserts that when a price of one good increases, the demand of *every* other good increases (and, therefore, the demand of that good decreases). For $L > 2$, however, this is not by any means a necessary property of even compensated demand. In fact, the gross substitute property is very restrictive. Nonetheless, it can make sense for problems with a few very aggregated commodities or for those where commodities possess special symmetries (see Exercise 17.F.4).

Definition 17.F.2: The function $z(\cdot)$ has the *gross substitute* (GS) property if whenever p' and p are such that, for some ℓ, $p'_\ell > p_\ell$ and $p'_k = p_k$ for $k \neq \ell$, we have $z_k(p') > z_k(p)$ for $k \neq \ell$.

If, as is the case here, we are dealing with the aggregate excess demand of an economy, then the fact that $z(\cdot)$ is also homogeneous of degree zero has the consequence that with gross substitution we also have $z_\ell(p') < z_\ell(p)$ whenever p' and p are related as in Definition 17.F.2. To see this, let $\bar{p} = \alpha p$, where $\alpha = p'_\ell / p_\ell$. Note that $\bar{p}_\ell = p'_\ell$ and $\bar{p}_k > p'_k$ for $k \neq \ell$. Then the homogeneity of degree zero of $z(\cdot)$

44. Thus, this example can also be seen as a case of positive association between endowments and demands.

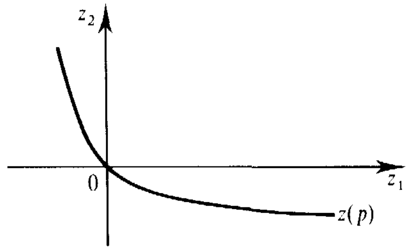


Figure 17.F.2
The offer curve of a gross substitute excess demand function.

tells us that $0 = z_\ell(\bar{p}) - z_\ell(p) = z_\ell(\bar{p}) - z_\ell(p') + z_\ell(p') - z_\ell(p)$. However, gross substitution implies that $z_\ell(\bar{p}) - z_\ell(p') > 0$ (change sequentially each price p'_k for $k \neq \ell$ to $\bar{p}_k$, applying the GS property at each step), and so $z_\ell(p') - z_\ell(p) < 0$.

The differential version of gross substitution is clear enough: At every p, it must be that $\partial z_k(p)/\partial p_\ell > 0$ for $k \neq \ell$; that is, the $L \times L$ matrix $Dz(p)$ has positive off-diagonal entries. In addition, when $z(\cdot)$ is an aggregate excess demand function, homogeneity of degree zero implies that $Dz(p)p = 0$, and so $\partial z_\ell(p)/\partial p_\ell < 0$ for all $\ell = 1, \ldots, L$: the diagonal entries of $Dz(p)$ are all negative.

If in these definitions the inequalities are weak, one speaks of *weak gross substitution*.[45]

Figure 17.F.2 represents the offer curve of a gross substitute excess demand function $L = 2$. As the relative price of good 1 increases, the excess demand for good 1 decreases and the excess demand for good 2 increases.

An important characteristic of the gross substitute property, which follows directly from its definition, is that *it is additive across excess demand functions*. In particular, if the individual excess demand functions satisfy it, then the aggregate function does also.

Example 17.F.2: Consider a utility function of the form $u_i(x_i) = \sum_\ell u_{\ell i}(x_{\ell i})$. If $-[x_{\ell i} u''_{\ell i}(x_{\ell i})/u'_{\ell i}(x_{\ell i})] < 1$ for all ℓ and $x_{\ell i}$, then the resulting excess demand function $z_i(p)$ has the gross substitute property for any initial endowments (Exercise 17.F.5). This condition is satisfied by $u_i(x_i) = (\sum_\ell \alpha_{\ell i} x_{\ell i}^\rho)^{1/\rho}$ for $0 < \rho < 1$ (Exercise 17.F.5). The limits of these preferences as $\rho \to 1$ and $\rho \to 0$ are preferences representable, respectively, by linear functions and by Cobb–Douglas utility functions (recall Exercise 3.C.6). As far as the gross substitution property is concerned, Cobb–Douglas preferences constitute a borderline case. Indeed, the excess demand function for good ℓ is then $z_{\ell i}(p) = \alpha_{\ell i}(p \cdot \omega_i)/p_\ell - \omega_{\ell i}$. If $\omega_{ki} > 0$, the excess demand for good

45. It is worth mentioning that functions satisfying the GS property arise naturally in many economic contexts. For example, if A is an $(L-1) \times (L-1)$ input–output matrix and $c \in \mathbb{R}^{L-1}_+$, then $c - (I - A)\alpha$ satisfies the (weak) GS property as a function of $\alpha \in \mathbb{R}^{L-1}_+$ (see Appendix A of Chapter 5 for the interpretation of these concepts). More generally, the equation system $g(\alpha) - \alpha$ associated with the fixed-point problem [i.e., find α such that $g(\alpha) = \alpha$] of an *increasing* function $g\colon \mathbb{R}^N_+ \to \mathbb{R}^N_+$ [i.e., $g(\alpha) \geq g(\alpha')$ whenever $\alpha \geq \alpha'$] satisfies it (perhaps, again, in its weak version). Note that in these cases there is no homogeneity of degree zero or Walras' law—conditions specific to general equilibrium applications—to complement the GS property. This is significant because these conditions add substantially to the power of the GS property. See Exercise 17.F.16 for an exploration of the implications of the GS property without homogeneity of degree zero or Walras' law.

ℓ will respond positively to an increase in p_k. But if $\omega_{ki} = 0$, there will be no response.[46] ■

In the special case of exchange economies if the gross substitute property holds for aggregate excess demand then equilibrium is unique.

Proposition 17.F.3: An aggregate excess demand function $z(\cdot)$ that satisfies the gross substitute property has at most one exchange equilibrium; that is, $z(p) = 0$ has at most one (normalized) solution.

Proof: It suffices that we show that $z(p) = z(p')$ cannot occur whenever p and p' are two price vectors that are not collinear. By homogeneity of degree zero, we can assume that $p' \geq p$ and $p_\ell = p'_\ell$ for some ℓ. Now consider altering the price vector p' to obtain the price vector p in $L - 1$ steps, lowering (or keeping unaltered) the price of every commodity $k \neq \ell$ one at a time. By gross substitution, the excess demand of good ℓ cannot decrease in any step, and, because $p \neq p'$, it will actually increase in at least one step. Hence, $z_\ell(p) > z_\ell(p')$. ■

One might hope to establish uniqueness in economies with production by applying the GS property to the production inclusive excess demand $\tilde{z}(\cdot)$. However, the direct use of the GS property in a production context is limited. Imagine, for example, a situation in which inputs and outputs are distinct goods. If the price of an input increases, the demand for every other input may decrease, not increase as the GS property would require, simply because the optimal level of output decreases. Indirectly, though, the gross substitute concept may still be quite helpful. Recall, in particular, that at the end of Section 17.B, we argued that it is always possible to reduce a production economy to an exchange economy in which, in effect, consumers exchange factor inputs and then engage in home production using a freely available constant returns technology. The aggregate excess demand in this derived exchange economy for factor inputs combines elements of both consumption and production and may well satisfy the GS property.[47]

What is the relationship between gross substitution and the weak axiom? Clearly, the WA does not imply the GS property (the latter can be violated even in quasilinear, one-consumer economies). The converse relationship is not as obvious, but it is nevertheless true that the GS property does not imply the WA. In fact, Example 17.F.1, which violated the WA, could perfectly well satisfy GS.[48] There is, however, one connection that is important. The gross substitute property implies that

$$\text{If } z(p) = 0 \text{ and } z(p') \neq 0, \text{ then } p \cdot z(p') > 0. \tag{17.F.3}$$

We shall not prove condition (17.F.3) here. For the case in which $L = 2$, you are asked for a proof in Exercise 17.F.7. To understand (17.F.3), note that if p is the price vector of an

46. See also Grandmont (1992) for an interesting result where a Cobb–Douglas positive representative consumer, and therefore GS excess demand, is derived from a requirement that at any given price, the choice behavior is widely dispersed (in a certain precise sense) across consumers. Grandmont's is an example of a model in which the individual excess demand functions may not satisfy the gross substitute property but the aggregate function does.

47. See Mas-Colell (1991) and Exercise 17.F.6 for further elaborations on this point.

48. Therefore, in view of Proposition 17.F.1, we know that in a constant returns economy the fulfillment of the GS property by the excess demand of the consumers does *not* imply the uniqueness of equilibrium.

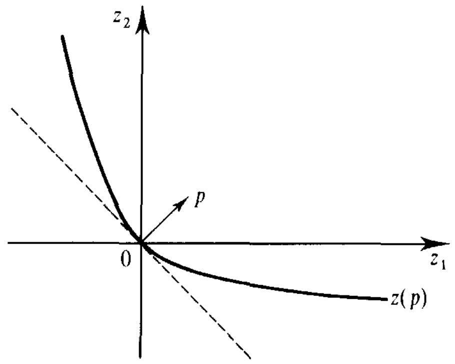


Figure 17.F.3
The revealed preference property of gross substitution.

(exchange) equilibrium and p' is not, then, since $z(p) = 0$, we have $p' \cdot z(p) = 0$, and therefore any nonequilibrium p' is revealed preferred to p. Hence, the requirement in (17.F.3) that $p \cdot z(p') > 0$ amounts to a restricted version of the WA asserting that no equilibrium price vector p can be revealed preferred to a nonequilibrium price vector p'. Geometrically, it says that the range of the excess demand function, $\{z(p'): p' \gg 0\} \subset \mathbb{R}^L$ (i.e., the offer curve), lies entirely above the hyperplane through the origin with normal vector p (see Figure 17.F.3). In parallel to Proposition 17.F.2, condition (17.F.3) implies the convexity of the equilibrium price set of the exchange economy, that is, of $\{p \in \mathbb{R}^L_{++}: z(p) = 0\} \subset \mathbb{R}^L$ (in Exercise 17.F.8, you are asked to show this). Interestingly, condition (17.F.3) is satisfied not only in the WA and the GS cases but also in the no-trade case, to be reviewed shortly.

In the differentiable case, there is a parallel way to explore the connection between the WA and gross substitution. Let $z(p) = 0$. The sufficient differential condition (17.F.1) for the WA tells us that $dp \cdot Dz(p)\, dp < 0$ for any dp not proportional to p. Suppose now that instead of the WA, we require that $Dz(p)$ has the gross substitute sign pattern. Because $z(p) = 0$, we have $p \cdot Dz(p) = 0$ and $Dz(p)p = 0$ [recall (17.E.1) and (17.E.2)]. Using these two properties it can then be shown that again we obtain $dp \cdot Dz(p)\, dp < 0$ for any dp not proportional to p (see Section M.D of the Mathematical Appendix). Hence, we can conclude that *at an exchange equilibrium price vector*, the GS property yields every local restriction implied by the WA. This is summarized in Proposition 12.F.4.

Proposition 17.F.4: If $z(\cdot)$ is an aggregate excess demand function, $z(p) = 0$, and $Dz(p)$ has the gross substitute sign pattern, then we also have $dp \cdot Dz(p)\, dp < 0$ whenever $dp \neq 0$ is not proportional to p.

Uniqueness as an Implication of Pareto Optimality

We now present a result that is not of great significance in itself but that is nonetheless interesting because it highlights a uniqueness implication of Pareto optimality. For simplicity, we restrict ourselves again to an exchange economy (see Exercise 17.F.9 for a generalization allowing for production).

Proposition 17.F.5: Suppose that the initial endowment allocation $(\omega_1, \ldots, \omega_I)$ constitutes a Walrasian equilibrium allocation for an exchange economy with strictly convex and strongly monotone consumer preferences (i.e., no-trade is an equilibrium). Then this is the unique equilibrium allocation.

Proof: Let an allocation $x = (x_1, \ldots, x_I)$ and price vector p constitute a Walrasian equilibrium when consumers' endowments are $(\omega_1, \ldots, \omega_I)$. Since ω_i is affordable at price vector p for each consumer i, we have $x_i \succsim_i \omega_i$ for all i. However, by the assumption of the proposition and the first welfare theorem, $(\omega_1, \dots, \omega_I)$ is a Pareto optimal allocation and so we must have $x_i \sim_i \omega_i$ for all i. But then we can conclude that $x_i = \omega_i$ for all i, because otherwise, by the strict convexity of preferences, the allocation $(\frac{1}{2}x_1 + \frac{1}{2}\omega_1, \dots, \frac{1}{2}x_I + \frac{1}{2}\omega_I)$ would be Pareto superior to $(\omega_1, \dots, \omega_I)$. ■

Index Analysis and Uniqueness (... and Nonuniqueness)

The index theorem (Proposition 17.D.2) provides a device to test for uniqueness in any given model. The idea is that if merely from the general maintained assumptions of the model we can attach a definite sign to the determinant of the Jacobian matrix of the equilibrium equations at any solution point, then the equilibrium must be unique. After all, the index theorem implies that sign uniformity across equilibria is impossible if there is multiplicity.

As a matter of fact, we could have proceeded by means of this index methodology for many of our previous uniqueness results. Take, for example, an exchange economy. In both the WA and the GS cases, whenever $z(p) = 0$, the matrix $Dz(p)$ is necessarily negative semidefinite [see the small-type discussion of expression (17.F.1) and Proposition 17.F.4]. Moreover, if an equilibrium is regular (i.e., if rank $Dz(p) = L - 1$), the negative semidefiniteness of $Dz(p)$ can be shown to imply that the index of the equilibrium is necessarily $+1$ (see Exercise 17.F.11). Hence, we can conclude that in both the WA and GS cases, any regular economy must have a unique (normalized) equilibrium price vector.

Although the index methodology provides a good research tool, it is often the case that, as here, uniqueness conditions lend themselves to direct proofs. It is a notable fact that some of the more subtle uses of index analysis are not to establish uniqueness but rather to establish nonuniqueness [the first usage of this type was made by Varian (1977)]. This is illustrated in Example 17.F.3.

Example 17.F.3: Suppose we have two one-consumer countries, $i = 1, 2$. Countries are symmetrically positioned relative to the home (H) and the foreign (F) good. To be specific, let each country have one unit of the home good as an endowment and none of the foreign good, and utility functions $u_i(x_{Hi}, x_{Fi}) = x_{Hi} - x_{Fi}^{\rho}$ for $-1 < \rho < 0$. Merely from symmetry considerations, it follows that there is a symmetric equilibrium $p = (1, 1)$. But we may be interested in knowing whether there are asymmetric equilibria. One way to proceed is as follows: compute the index of the symmetric equilibrium; a sufficient (but not necessary) condition for the existence of an asymmetric equilibrium is that this index be negative (i.e., -1).[49] If we carry out the computation for the present example (you are asked to do so in Exercise 17.F.13), we see that the index is negative if at prices $p = (1, 1)$ the wealth effects in each country are so biased toward the home good that an increase in the price of the good of country 1, say, actually increases the demand for this good in country 1 by more than it decreases the demand from country 2. ■

49. In this, as typically in any example, the excess demand function fails to be differentiable at prices at which demand just "hits" the boundary. Typically (we *could* say "generically"), these prices will not be equilibrium prices and the validity of the index theorem is not affected by these nondifferentiabilities.

17.G Comparative Statics Analysis

Comparative statics is the analytical methodology that concerns itself with the study of how the equilibria of a system are affected by changes (often described as "shocks") in various environmental parameters. In this section, we examine the comparative static properties of Walrasian equilibria.

To be concrete, we consider an exchange economy formalized by a system of aggregate excess demand equations for the first $L - 1$ commodities:

$$\hat{z}(p; q) = (z_1(p; q), \ldots, z_{L-1}(p; q)).$$

Here, $q \in \mathbb{R}^N$ is a vector of N parameters influencing preferences or endowments (or both). Throughout, we normalize $p_L = 1$.

Suppose the value of the parameters is given initially by the vector $\bar{q}$ and that $\bar{p}$ is an equilibrium price vector for $\bar{q}$; that is, $\hat{z}(\bar{p}; \bar{q}) = 0$. We wish to analyze the effect of a shock in the exogenous parameters q on the endogenous variable p solving the system. A first difficulty for doing so is the possibility of multiplicity of equilibrium: the system of $L - 1$ equations in $L - 1$ unknowns $\hat{z}(\cdot\,; q) = 0$ may have more than one solution for the relevant values of q, and thus we may need to decide which equilibrium to single out after a shock.

If the change in the values of the parameters from $\bar{q}$ is small, then a familiar approach to this problem is available. It consists of focusing on the *local* effects on p, that is, on the solutions that remain near $\bar{p}$. Assuming the differentiability of $\hat{z}(p; q)$, we may determine those effects by applying the implicit function theorem (see Section M.E of the Mathematical Appendix). Indeed, if the system $\hat{z}(\cdot\,; \bar{q}) = 0$ is regular at the solution $\bar{p}$, that is, if the $(L - 1) \times (L - 1)$ matrix $D_p\hat{z}(\bar{p}; \bar{q})$ has rank $L - 1$,[50] then for a neighborhood of $(\bar{p}; \bar{q})$ we can express the equilibrium price vector as a function $p(q) = (p_1(q), \ldots, p_{L-1}(q))$ whose $(L - 1) \times N$ derivative matrix at $\bar{q}$ is

$$Dp(\bar{q}) = -[D_p\hat{z}(\bar{p}; \bar{q})]^{-1} D_q\hat{z}(\bar{p}; \bar{q}). \tag{17.G.1}$$

What can we say about the first-order effects $Dp(\bar{q})$? Expression (17.G.1) and Proposition 17.E.2 [which told us that the matrix of price effects $D_p\hat{z}(\bar{p}; \bar{q})$ is unrestricted when $I \geq L$] strongly suggest that, without further assumptions, the "anything goes" principle applies to the comparative statics of equilibrium in the same manner that in Section 17.E it applied to the closely related issue of the effects of price changes on excess demand. We now elaborate on this point in the context of a specific example.

Let the list of parameters under consideration be the vector $\hat{\omega}_1 = (\omega_{11}, \ldots, \omega_{L-1,1})$ of initial endowments of the first consumer for the first $L - 1$ commodities. All of the remaining endowments are kept fixed. As before we assume that $\hat{z}(\cdot\,; \hat{\bar{\omega}}_1) = 0$ is regular at the solution $\bar{p}$. It can be shown (see Exercise 17.G.1) that if the demand function of the first consumer satisfies a strict normality condition, then rank $Dp(\hat{\bar{\omega}}_1) = L - 1$, where $p(\cdot)$ is the locally defined solution function with $p(\hat{\bar{\omega}}_1) = \bar{p}$. Proposition 17.G.1 tells us that if there are enough consumers then this is all that we can say.

50. In a slight abuse of notation, we let $D_p\hat{z}(\bar{p}; \bar{q})$ stand for the matrix obtained from $D_p z(\bar{p}; \bar{q})$ by deleting the last row and column.

Proposition 17.G.1: Given any price vector $\bar{p}$, endowments for the first consumer of the first $L-1$ commodities $\hat{\bar{\omega}}_1 = (\bar{\omega}_{11}, \ldots, \bar{\omega}_{L-1,1})$, and a $(L-1) \times (L-1)$ nonsingular matrix B, there is an exchange economy formed by $L+1$ consumers in which the first consumer has the prescribed endowments of the first $L-1$ commodities, $\hat{z}(\bar{p}; \hat{\bar{\omega}}_1) = 0$, $\hat{z}(\cdot, \hat{\bar{\omega}}_1) = 0$ is regular at $\bar{p}$ and $Dp(\hat{\bar{\omega}}_1) = B$.

Proof: Let the first consumer have endowments with the prescribed amounts of the first $L-1$ commodities, and give to this consumer arbitrary preferences, with the single restriction that $D_{\hat{\omega}_1}\hat{z}_1(\bar{p}; \hat{\bar{\omega}}_1)$ be nonsingular (it suffices for this that the demand function of consumer 1 satisfies a strict normality condition; again see Exercise 17.G.1). Since $D_{\hat{\omega}_1}\hat{z}(\bar{p}; \hat{\bar{\omega}}_1) = D_{\hat{\omega}_1}\hat{z}_1(\bar{p}; \hat{\bar{\omega}}_1)$, expression (17.G.1) tells us that we are looking for an additional collection of L consumers such that the resulting $(L+1)$-consumer economy has $\hat{z}(\bar{p}; \hat{\bar{\omega}}_1) = 0$ and

$$D_p\hat{z}(\bar{p}; \hat{\bar{\omega}}_1) = -D_{\hat{\omega}_1}\hat{z}_1(\bar{p}; \hat{\bar{\omega}}_1)B^{-1} \tag{17.G.2}$$

Note that the $(L-1) \times (L-1)$ matrix defined in (17.G.2) is nonsingular. Thus, we have reduced our problem to the following: can we find L consumers whose aggregate excess demand at p is $-\hat{z}_1(\bar{p}; \hat{\bar{\omega}}_1)$ and whose aggregate $(L-1) \times (L-1)$ matrix of price effects is $\hat{A} = -D_{\hat{\omega}_1}\hat{z}_1(\bar{p}; \hat{\bar{\omega}}_1)B^{-1} - D_p\hat{z}_1(\bar{p}; \hat{\bar{\omega}}_1)$? It follows from Proposition 17.E.2 that the answer to this question is "yes" (note that the restrictions that Proposition 17.E.2 imposes on the $L \times L$ matrix A place no restriction on the matrix obtained by deleting one row and one column of A). ■

Proposition 17.G.1 shows that any first-order effect is possible. As in Section 17.E (recall Figure 17.E.3), it is also the case here that if there are prior restrictions on initial endowments and if consumption must be nonnegative, then there are again comparative statics restrictions of a global character. [See Brown and Matzkin (1993) for a recent investigation of this point.]

There are a number of comparative static effects that, ideally, we would like to have and that seem economically intuitive: For example, that if the endowment of one good increases, then its equilibrium price decreases. Nevertheless, strong conditions are required for them to hold. By now this should not surprise us: We already know that wealth effects and/or the lack of sufficient substitutability can undermine intuitive comparative static effects. The latest instance we have seen of this occurring has been precisely Proposition 17.G.1.

The analysis of uniqueness in Section 17.E may lead us to suspect that good comparative statics effects can hold if aggregate excess demand satisfies either weak-axiom-like conditions (recall Definition 17.F.1) or gross substitution properties (see Definition 17.F.2). This is in fact so. We consider first the implications of a weak-axiom-like restriction on aggregate excess demand.

Proposition 17.G.2: Suppose that $\hat{z}(\bar{p}; \bar{q}) = 0$, where $\hat{z}(\cdot)$ is differentiable. If $D_q\hat{z}(\bar{p}; \bar{q})$ is negative definite,[51] then

$$(D_q\hat{z}(\bar{p}; \bar{q})\, dq) \cdot (Dp(\bar{q})\, dq) \geq 0 \text{ for any } dq. \tag{17.G.3}$$

51. This condition is independent of which particular commodity has been labeled as L (see Section M.D of the Mathematical Appendix).

Proof: The inverse of a negative definite matrix is negative definite. Therefore $[D_q\hat{z}(\bar{p};\bar{q})]^{-1}$ is negative definite (see Section M.D of the Mathematical Appendix). Hence, by (17.G.1) we have

$$(D_q\hat{z}(\bar{p};\bar{q})\,dq)\cdot(Dp(\bar{q})\,dq) = -D_q\hat{z}(\bar{p};\bar{q})\,dq\cdot[D_p\hat{z}(\bar{p};\bar{q})]^{-1}D_q\hat{z}(\bar{p};\bar{q})\,dq \geq 0$$

which is precisely (17.G.3). ∎

The weak axiom implies the negative semidefiniteness of $D_p\hat{z}(\bar{p};\bar{q})$ whenever $\hat{z}(\bar{p};\bar{q}) = 0$ [see expression (17.F.1) and the remark following it]. Therefore, the assumption of Proposition 17.G.2 amounts to a small strengthening of this implication. Its conclusion says that for any infinitesimal shock dq in q, the induced shock to excess demand at prices fixed at $\bar{p}$, $D_q\hat{z}(\bar{p};\bar{q})\,dq$, and the induced shock in equilibrium prices, $D_q p(\bar{q})\,dq$, move "in the same direction" (more precisely, as vectors in $\mathbb{R}^{L-1}$ they form an acute angle). For example, a shock that at fixed prices affects only the aggregate excess demand of the first good,[52] say by decreasing it, will necessarily decrease the equilibrium price of this good. Note that this does *not* say that if ω_{11} increases then the equilibrium price of good 1 decreases. Under an assumption of normal demand, this change in ω_{11} does indeed decrease the excess demand for good 1 at $\bar{p}$ but it also affects the excess demand for all other goods (see Exercise 17.G.2).

We next consider in Proposition 17.G.3 the implications of gross substitution (or, more precisely, of gross substitution holding locally at $(\bar{p};\bar{q})$).

Proposition 17.G.3: Suppose that $\hat{z}(\bar{p};\bar{q}) = 0$, where $\hat{z}(\cdot;\cdot)$ is differentiable. If the $L \times L$ matrix $D_p z(\bar{p};\bar{q})$ has negative diagonal entries and positive off-diagonal entries, then $[D_p\hat{z}(\bar{p};\bar{q})]^{-1}$ has all its entries negative.

Proof: Because of the homogeneity of degree zero of excess demand (recall Exercise 17.E.1), we have $D_p z(\bar{p};\bar{q})\bar{p} = 0$, and so $D_p\hat{z}(\bar{p};\bar{q})\hat{\bar{p}} \ll 0$, where $\hat{\bar{p}} = (\bar{p}_1,\ldots,\bar{p}_{L-1})$. Denote by I the $(L-1)\times(L-1)$ identity matrix and take an $r > 0$ large enough for the matrix $A = (1/r)D_p\hat{z}(\bar{p};\bar{q}) + I$ to have all its entries positive. Then $D_p\hat{z}(\bar{p};\bar{q}) = -r[I - A]$, and therefore $D_p\hat{z}(\bar{p};\bar{q})\hat{\bar{p}} \ll 0$ yields $(I - A)\hat{\bar{p}} \gg 0$; that is, the positive matrix A, viewed formally as an input–output matrix, is productive (see Appendix A of Chapter 5; the fact that the diagonal entries of A are not zero is inessential). Hence, as we showed in the proof of Proposition 5.AA.1, the matrix $[I - A]^{-1}$ exists and has all its entries positive. From $[D_p\hat{z}(\bar{p};\bar{q})]^{-1} = -(1/r)[I - A]^{-1}$ we have our conclusion. ∎

It follows from Proposition 17.G.3 and expression (17.G.1) that, given gross substitution, if $D_q\hat{z}(\bar{p};\bar{q})\,dq \ll 0$, that is, if the excess demand for all of the first $L-1$ goods decreases as a consequence of the shock (and therefore the excess demand for the Lth good increases), then $Dp(\bar{q})\,dq \ll 0$. That is, the equilibrium prices of the

52. What this means is that the excess demand of good 2 to $L-1$ is not changed. By Walras' law, the excess demand of good L must change.

first $L-1$ goods (relative to the price of the Lth good) decrease.[53] In particular, suppose again that consumer 1's initial endowment of some good decreases. By labelling commodities appropriately, we can let this good be commodity L. Under the assumption of normal demand for consumer 1, a decrease in ω_{L1}, at the fixed price vector $\bar{p}$, will decrease the excess demand for the first $L-1$ goods. Therefore, the prices of the first $L-1$ goods decrease and so we now reach the conclusion that we could not obtain by means of Proposition 17.G.2: if the endowments of a single good decrease then its price (relative to the price of any other good) increases. This suggests, incidentally, that the assumptions of Proposition 17.G.3 are strictly stronger than those of Proposition 17.G.2. Indeed, as we saw in Proposition 17.F.4, if $z(\bar{p}; \bar{q}) = 0$ and the $L \times L$ matrix $D_p z(\bar{p}; \bar{q})$ satisfies the gross substitute property, then $dp \cdot D_p z(\bar{p}; \bar{q})\, dp < 0$ whenever $dp \neq 0$ is not proportional to $\bar{p}$. In particular, by letting $dp_L = 0$ we have that the matrix $D_p \hat{z}(\bar{p}; \bar{q})$ is negative definite.

Expression (17.G.1) allows us to explicitly compute the effects of an infinitesimal shock. In fact, it also offers a practical computational method to estimate the local effects of small (but perhaps not infinitesimal) shocks. Suppose that the value of the vector of parameters after the shock is $\bar{\bar{q}}$ and, for $t \in [0, 1]$, consider a continuous function $\hat{z}(\cdot, t)$ that, as t ranges from $t = 0$ to $t = 1$, distorts $\hat{z}(\cdot; \bar{q})$ into $\hat{z}(\cdot; \bar{\bar{q}})$. An example of such a function, called a *homotopy*, is

$$\hat{z}(\cdot, t) = (1-t)\hat{z}(\cdot; \bar{q}) + t\hat{z}(\cdot; \bar{\bar{q}}).$$

Denote the solution set by $E = \{(t, p)\colon \hat{z}(p, t) = 0\}$. Then we may attempt to determine $p(\bar{\bar{q}})$ by following a segment in the solution set that starts at $(0, \bar{p})$.[54] If $\bar{\bar{q}}$ is close to $\bar{q}$, and the initial situation p is regular, then we are in the simple case of Figure 17.G.1(a): there is a unique segment that connects $(0, \bar{p})$ to some $(1, \bar{\bar{p}})$.[55] Naturally, we then put $p(\bar{\bar{q}}) = \bar{\bar{p}}$.

If $\bar{\bar{q}}$ is not close to $\bar{q}$ but nevertheless $\hat{z}(\cdot, t)$ is a regular excess demand function for every t [this will be the case if, for example, $z(\cdot, t)$ satisfies, for every t, any of the uniqueness conditions covered in Section 17.F], then this procedure will still succeed in going from $t = 0$ to $t = 1$ and, therefore, in determining an equilibrium for $\bar{\bar{q}}$.[56] Unfortunately, if the shock is large, we can easily find ourselves in situations such as Figures 17.G.1(b) and 17.G.1(c), where at some t' the economy $\hat{z}(\cdot, t')$ is not regular and at $(t', p_{t'})$ there is no natural

53. This conclusion holds for nonlocal shocks as well. To see this let $Dz(p; q)$ have the gross substitute sign pattern throughout its domain and suppose that $\hat{z}(p; \bar{\bar{q}}) \ll \hat{z}(p; \bar{q})$ for all p. For $t \in [0, 1]$, define $\hat{z}(p; t) = t\hat{z}(p; \bar{\bar{q}}) + (1-t)\hat{z}(p; \bar{q})$. Denote by $p(t)$ the solution to $\hat{z}(p; t) = 0$. Note that $D_t \hat{z}(p(t); t)\, dt = \hat{z}(p(t); \bar{\bar{q}}) - \hat{z}(p(t); \bar{q}) \ll 0$ for all t and therefore, by Proposition 17.G.3, $Dp(t)\, dt \ll 0$ for all t. But then, for any $\ell = 1, \ldots, L-1$, we have

$$p_\ell(\bar{\bar{q}}) - p_\ell(\bar{q}) = \int_0^1 \left[\frac{\partial p_\ell(t)}{dt}\right] dt < 0.$$

In Exercise 17.G.3 you can find a more direct approach to the global theory. See also Milgrom and Shannon (1994) for much more on the latter approach.

54. In practice, "following" a segment involves the application of appropriate numerical techniques; see Garcia-Zangwill (1981), Kehoe (1991), and references therein.

55. Moreover, if the shock is sufficiently small, the $\bar{\bar{p}}$ so obtained is independent of the particular homotopy used.

56. However, if there are multiple equilibria at $\bar{\bar{q}}$, then which equilibrium we find may now depend on the homotopy.

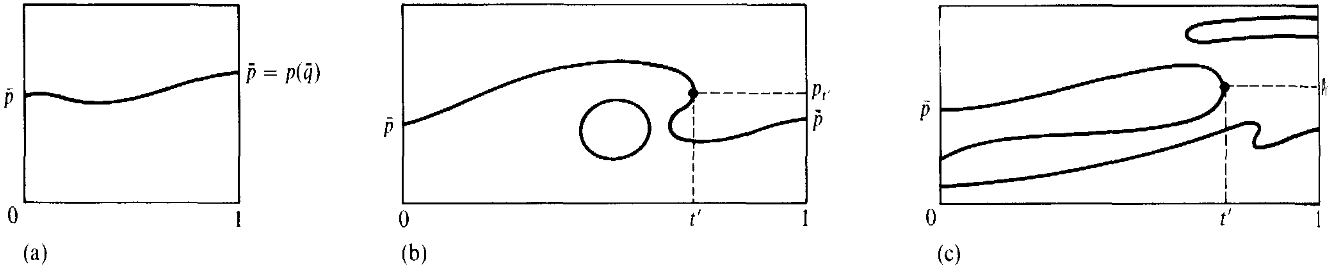


Figure 17.G.1 Comparative statics in the large: the general case.

continuation of the path as t increases.[57] To obtain an equilibrium $\bar{p}$ for $\bar{q}$ there is then no real alternative but to appeal to general algorithms for the solution of the system of equations $\hat{z}(\cdot\,;\bar{q}) = 0$. It is a sobering thought that which solution we come up with at $\bar{q}$ may be dictated more by our numerical technology than by our initial position $(\bar{p};\bar{q})$. This is most unsatisfactory, and it is a manifestation of a serious shortcoming—the lack of a theory of equilibrium selection.

17.H Tâtonnement Stability

We have, so far, carried out an extensive analysis of equilibrium equations. A characteristic feature that distinguishes economics from other scientific fields is that, for us, the equations of equilibrium constitute the center of our discipline. Other sciences, such as physics or even ecology, put comparatively more emphasis on the determination of dynamic laws of change. In contrast, up to now, we have hardly mentioned dynamics. The reason, informally speaking, is that economists are good (or so we hope) at recognizing a state of equilibrium but are poor at predicting precisely how an economy in disequilibrium will evolve. Certainly there are intuitive dynamic principles: if demand is larger than supply then the price will increase, if price is larger than marginal cost then production will expand, if industry profits are positive and there are no barriers to entry, then new firms will enter, and so on. The difficulty is in translating these informal principles into precise dynamic laws.[58]

The most famous attempt at this translation was made by Walras (1874), and the modern version of his ideas have come to be known as the theory of *tâtonnement stability*. In this section, we review two tâtonnement-style models, one of pure price adjustment and the other of pure quantity adjustment. We should emphasize,

57. Note that by reversing the direction of change of t we can continue to move along the segments in these two figures (this is actually quite a general fact). If $\bar{p}$ is the only solution at $t = 0$, as in 17.G.1(b), then the segment necessarily ends with a $(1, \bar{p})$. Thus, in some sense we have succeeded in finding an equilibrium for $\bar{q}$ that is associated with our initial $\bar{p}$. But the association is very weak: it may depend on the particular homotopy and it requires the parameter-reversal procedure. If, as in Figure 17.G.1(c), $\bar{p}$ is not the only equilibrium at $t = 0$, then the procedure may simply not work: the segment that starts at $(0, \bar{p})$ goes back to $t = 0$.

58. Refer to Hahn (1982) for a general review.

however, that those are just two examples. Indeed, one of the difficulties in this area is the plethora of plausible disequilibrium models. Although there is a single way to be in equilibrium, there are many different ways to be in disequilibrium.

Price Tâtonnement

We consider an exchange economy formalized by means of an excess demand function $z(\cdot)$. Suppose that we have an initial p that is not an equilibrium price vector, so that $z(p) \neq 0$. For example, the economy may have undergone a shock and p may be the preshock equilibrium price vector. Then the demand-and-supply principle suggests that prices will adjust upward for goods in excess demand and downward for those in excess supply. This is what was proposed by Walras; in a differential equation version put forward by Samuelson (1947), it takes the specific form

$$\frac{dp_\ell}{dt} = c_\ell z_\ell(p) \qquad \text{for every } \ell, \tag{17.H.1}$$

where dp_ℓ/dt is the rate of change of the price for the ℓth good and $c_\ell > 0$ is a constant affecting the speed of adjustment.

Simple as (17.H.1) is, its interpretation is fraught with difficulties. Which economic agent is in charge of prices? For that matter, why must the "law of one price" hold out of equilibrium (i.e., why must identical goods have identical prices out of equilibrium)? What sort of time does "t" represent? It cannot possibly be *real* time because, as the model stands, a disequilibrium p is not compatible with feasibility (i.e., not all consumption plans can be simultaneously realized).

Perhaps the most sensible answer to all these questions is that (17.H.1) is best thought of not as modeling the actual evolution of a demand-and-supply driven economy, but rather as a tentative trial-and-error process taking place in fictional time and run by an abstract market agent bent on finding the equilibrium level of prices (or, more modestly, bent on restoring equilibrium after a disturbance).[59]

The hope is that, in spite of its idealized nature, the analysis of (17.H.1) will provide further insights into the properties of equilibria. Even perhaps some help in distinguishing good from poorly behaved equilibria. The analysis is at its most suggestive in the two-commodity case. For this case, Figure 17.H.1 represents the excess demand of the first good as a function of the relative price p_1/p_2. The actual dynamic trajectory of relative prices depends both on the initial levels of absolute prices and on the differential price changes prescribed by (17.H.1).[60] But note that, whatever the initial levels of absolute prices, $p_1(t)/p_2(t)$ increases at t if and only if $z_1(p_1(t)/p_2(t), 1) > 0$. In Figure 17.H.1 we see the following two features of the adjustment equations (17.H.1).

(a) Call an equilibrium $(\bar{p}_1, \bar{p}_2)$ *locally stable* if, whenever the initial price vector is sufficiently close to it, the dynamic trajectory causes relative prices to converge to the equilibrium relative prices $\bar{p}_1/\bar{p}_2$ (the equilibrium is *locally totally unstable* if any

59. This is, in essence, the idea of Walras (tâtonnement means "groping" in French), who took inspiration from the functioning of the auctioneer-directed markets of the Paris stock exchange. The idea was made completely explicit by Barone (1908) and by Lange (1938), who went so far as to propose the tâtonnement procedure as an actual computing device for a centrally planned economy.

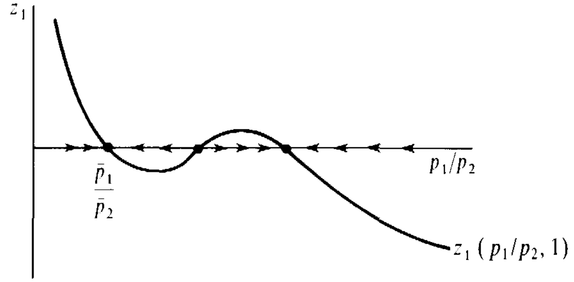


Figure 17.H.1
Tâtonnement trajectories for $L = 2$.

disturbance leads the relative prices to diverge from $\bar{p}_1/\bar{p}_2$). Then a (regular) equilibrium $\bar{p}_1/\bar{p}_2$ is *locally stable or locally totally unstable according to the sign of the slope of excess demand at the equilibrium*, that is, according to the index of the equilibrium (recall Definition 17.D.2). If excess demand slopes downward at $\bar{p}_1/\bar{p}_2$ (as in Figure 17.H.1), then a slight displacement of p_1/p_2 above $\bar{p}_1/\bar{p}_2$ will generate excess supply for good 1 (and excess demand for good 2), and therefore the relative price will move back toward the equilibrium level $\bar{p}_1/\bar{p}_2$. The effect is the reverse if excess demand slopes upward at $\bar{p}_1/\bar{p}_2$.

(b) There is *system stability*, that is, *for any initial position* $(p_1(0), p_2(0))$, *the corresponding trajectory of relative prices* $p_1(t)/p_2(t)$ *converges to some equilibrium arbitrarily closely as* $t \to \infty$.

For regular, two-commodity, economies, properties (a) and (b) give a complete picture of the dynamics. It is very satisfactory picture that accounts for the persistency of tâtonnement stability analysis: a theory yielding properties (a) and (b) must be saying something with economic content.

Unfortunately, as soon as $L > 2$ neither the local conclusions (a) nor the global conclusions (b) of the two-commodity case generalize. This should not surprise us, since the price dynamics in (17.H.1) are entirely driven by the excess demand function, and we know (Propositions 17.E.2 and 17.E.3) that the latter is not restricted in any way (beyond the boundary conditions). Consider an example for $L = 3$ and $c_1 = c_2 = c_3 = 1$. In Figure 17.H.2 we represent the normalized set of prices $S = \{p \gg 0 : (p_1)^2 + (p_2)^2 + (p_3)^2 = 1\}$. This normalization has the virtue that, for any excess demand function $z(p)$, the dynamic flow $p(t)$ generated by the differential equation $dp_\ell/dt = z_\ell(p)$, $\ell = 1, 2, 3$, remains in S [i.e., if $p(0) \in S$ then $p(t) \in S$ for all t]. This is a consequence of Walras' law:

$$\frac{d(p_1(t)^2 + p_2(t)^2 + p_3(t)^2)}{dt} = 2p_1(t)z_1(p(t)) + 2p_2(t)z_2(p(t)) + 2p_3(t)z_3(p(t)) = 0.$$

Thus, the dynamics of p can be represented by trajectories in S, the direction vector of the trajectory at any $p(t)$ being the direction of the excess demand vector $z(p(t))$. We conclude, therefore, that the only restrictions on the trajectories imposed by the general theory are those derived from the boundary behavior of excess demand. In Figure 17.H.2 we represent a possible field of trajectories. In the figure, when the

60. Note that although the change in p_ℓ at t prescribed by (17.H.1) depends only on the relative prices p_1/p_2 for $\ell = 1, 2$, the change in the price ratio p_1/p_2 at t depends both on the current price ratio and on the current absolute levels of p_1 and p_2.

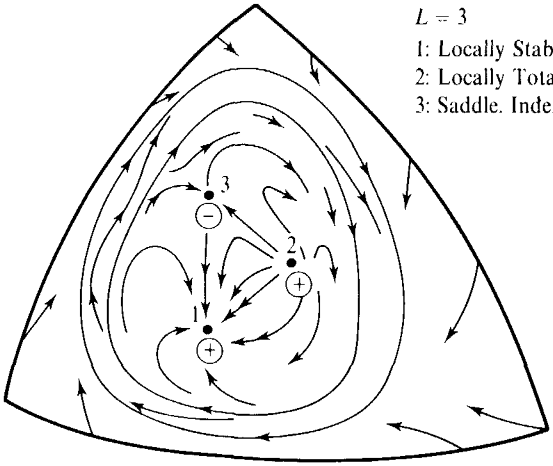


Figure 17.H.2
An example of tâtonnement trajectories for $L = 3$.

price of a good goes to zero the excess demand for the good becomes positive (thus, in particular, the trajectories point inward near the boundary). However, properties (a) and (b) are both violated: There are (regular) equilibria that are neither locally stable nor locally totally unstable (they are "saddle points," such as the equilibrium labeled 3 in the figure), and from some initial positions prices may not converge to any equilibrium.[61]

In a more positive spirit, we now argue that for the cases where we have succeeded in proving the uniqueness of Walrasian equilibrium, we are also able to establish the convergence of any price trajectory to this equilibrium (this property is called *global stability*).[62] The next proposition covers, in particular, the weak axiom, the gross substitute, and the no-trade cases studied in Section 17.F.[63] These three cases have in common that they satisfy the weak axiom when we restrict ourselves to comparisons between equilibrium and nonequilibrium prices [see the discussion of condition (17.F.3) in Section 17.F]. That is, for the unique (normalized) equilibrium price vector p^* arising in these cases we have: "If $z(p^*) = 0$ then $p^* \cdot z(p) > 0$ for any p not proportional to p^*."

Proposition 17.H.1: Suppose that $z(p^*) = 0$ and $p^* \cdot z(p) > 0$ for every p not proportional to p^*. Then the relative prices of any solution trajectory of the differential equation (17.H.1) converge to the relative prices of p^*.

Proof: Consider the (Euclidean) distance function $f(p) = \sum_\ell (1/c_\ell)(p_\ell - p_\ell^*)^2$. For any trajectory $p(t)$ let us then focus on the distance $f(p(t))$ at points t along the trajectory. We have

61. We should warn against deriving any comfort when prices converge to a limit cycle. Recall that this price tâtonnement is not happening in real time. The dynamic analysis has a hope of telling us something significant only if it converges.

62. Warning: uniqueness by itself does not imply stability—except for $L = 2$. You should try to draw a counterexample in the style of Figure 17.H.2.

63. For a proof specific to the gross substitute case, see Exercise 17.H.1.

$$\frac{df(p(t))}{dt} = 2\sum_{\ell} \frac{1}{c_\ell}(p_\ell(t) - p_\ell^*)\frac{dp_\ell(t)}{dt}$$

$$= \sum_{\ell} \frac{1}{c_\ell}(p_\ell(t) - p_\ell^*)\, c_\ell z_\ell(p(t))$$

$$= -p^* \cdot z(p(t)) \leq 0,$$

where the last inequality is strict if and only if $p(t)$ is not proportional to p^*. We conclude that the price vector $p(t)$ monotonically approaches the price vector p^* [in fact, since the same argument applies to αp^*, $p(t)$ must be monotonically approaching any αp^*]. This does not mean that $p(t)$ reaches a vicinity of p^*. Typically it will not: the rate of approach of $p(t)$ to p^* will go to zero before $p(t)$ gets near p^*. But the rate of approach can go to zero only if $p(t)$ becomes nearly proportional to p^* as $t \to \infty$, in which case the relative prices do converge.[64] ∎

We can gain further insight into the dynamics of tâtonnement by carrying out a local analysis. It will be more convenient now if we fix $p_L = 1$ and, consequently, we limit (17.H.1) to the first $L - 1$ coordinates. Accordingly we denote $\hat{z}(p) = (\hat{z}_1(p), \ldots, \hat{z}_{L-1}(p))$. Suppose that $\hat{z}(p^*) = 0$. A standard result of differential equation theory tells us that if the $(L-1) \times (L-1)$ matrix $D\hat{z}(p^*)$ is nonsingular (i.e., if the equilibrium is regular), then the behavior of the trajectories in a neighborhood of p^* is controlled by the linearization of the system at p^*, that is, by $CD\hat{z}(p^*)$, where C is the $(L-1) \times (L-1)$ diagonal matrix whose ℓth diagonal entry is the constant c_ℓ. One says that p^* is *locally stable* if there is $\varepsilon > 0$ such that $p(t) \to p^*$ whenever $\|p(0) - p^*\| < \varepsilon$ (i.e., for small perturbations the equilibrium will tend to restore itself). It then turns out that p^* is locally stable if and only if all the eigenvalues of $CD\hat{z}(p^*)$ have negative real parts. In addition, p^* is locally stable irrespective of the speeds of adjustment[65] (i.e., for all positive diagonal matrices C) if $D\hat{z}(p^*)$ is negative definite (see Section M.D of the Mathematical Appendix).[66]

One way to understand why the previous local stability result for the tâtonnement dynamics requires strong conditions on $D\hat{z}(p^*)$ is to note that we are in fact imposing the condition that the price of a commodity reacts *only* to the excess demand or supply for the same commodity. An ideal market agent may want to adjust these prices with an eye also to the effects of the adjustment on the excess demand for the *other* commodities. One concrete possibility is the following: if excess demand at time t is $\hat{z}(p(t))$, then the market agent adjusts prices by some amount $dp/dt = (dp_1/dt, \ldots, dp_{L-1}/dt)$ so as to cause a *proportional* decrease in the magnitude of all excess demands and supplies. That is, $D\hat{z}(p(t))(dp/dt) = -\lambda\hat{z}(p(t))$ for some $\lambda > 0$, or, if the relevant inverse exists,

$$\frac{dp}{dt} = -\lambda[D\hat{z}(p)]^{-1}\hat{z}(p) \tag{17.H.2}$$

This adjustment equation is known as *Newton's method* and is a standard technique of numerical analysis. If $D\hat{z}(p^*)$ is nonsingular, so that $[D\hat{z}(p^*)]^{-1}$ exists, then (17.H.2) always

64. Continuous real-valued functions that take decreasing values along any dynamic trajectory and the value zero only at stationary points are known as *Lyapunov functions*.

65. How could we pretend to know much about speeds of adjustments?

66. Note that this fits nicely with Proposition 17.H.1 because the revealed-preference-like property postulated there implies the negative (semi)definiteness of $D\hat{z}(p)$ at the equilibrium price vector p^*.

succeeds in restoring equilibrium after a small disturbance. Thus we see the contrast: for tâtonnement stability, we impose few informational restrictions on the adjustment process [to determine the change in p we only need to know $\hat{z}(p)$; in particular, no knowledge of the derivatives of $\hat{z}(\cdot)$ is required], but convergence is guaranteed only in special circumstances. For the Newton method, local convergence always obtains, but to determine the directions of price change at any p we need to know all the excess demands $\hat{z}(p)$ and all the price effects $D\hat{z}(p)$. See Smale (1976) and Saari and Simon (1978) for classic contributions to this type of Newton price dynamics.

Quantity Tâtonnement

In the analysis so far, prices could be out of equilibrium but quantities, that is to say the amounts demanded and supplied, are always at their equilibrium (i.e., utility and profit-maximizing) values. We now briefly consider a model in which quantities rather than prices may be in disequilibrium.[67] This is best done in a production context.

To be very concrete, suppose that there is a single production set Y.[68] At any moment of time, we assume that there is given a single, fixed production vector $y \in Y$. Prices, however, are always in equilibrium in the sense that the general equilibrium system of the economy, conditional on y, generates some equilibrium price system $p(y)$ (that is to say, we proceed in the short run as if the short-run production set were $\{y\} - \mathbb{R}^L_+$). This describes the short-run equilibrium of the economy.

What is an appropriate dynamics for this economy? It makes sense to think that, whatever it is, the change in production at time t, $dy(t)/dt \in \mathbb{R}^L$, moves production in a direction that increases profits *when the price vector at time t, $p(y(t))$, is taken as given*:

Definition 17.H.1: We say that the differentiable trajectory $y(t) \in Y$ is *admissible* if $p(y(t)) \cdot (dy(t)/dt) \geq 0$ for every t, with equality only if $y(t)$ is profit maximizing for $p(y(t))$ (in which case we could say that we are at a long-run equilibrium).

A difference between the price and the quantity tâtonnement approaches that adds appeal to the second is that feasibility is now insured at any t and that, as a result, we can interpret the dynamics as happening in real time.[69,70]

Will an admissible trajectory necessarily take us to long-run equilibrium? We cannot really explore this matter here in any detail. As usual, the answer is "only

67. We could also look at the general case where both could be in disequilibrium; see, for example, Mas-Colell (1986).

68. There is no difficulty in considering several. Also, Y can be interpreted as an individual or as an aggregate production set.

69. Nonetheless, it is important to realize that, even then, this is not a fully dynamic model: The optimization problems of the consumers remain static and free of expectational feedbacks and firms follow naive, short-run rules of adjustment (in a more positive spirit one might call this *adaptive*, rather than naive, behavior). For an extensive analysis of market adjustment procedures in real time, see Fisher (1983).

70. The quantity dynamics of Definition 17.H.1 are reminiscent of Marshall (1920) and are often referred to as *Marshallian dynamics*, especially in a partial equilibrium context. In contrast, the price dynamics are frequently called *Walrasian dynamics*.

under special circumstances." A limited, but important example (it covers the short-run/long-run model of Section 10.F) is described in Proposition 17.H.2.

Proposition 17.H.2: If there is a single strictly convex consumer, then any admissible trajectory converges to the (unique) equilibrium.

Proof: Consider $u(y(t) + \omega)$ where $u(\cdot)$ and ω are respectively the utility function and the endowments of the consumer. The unique equilibrium production vector is the single production vector $\bar{y}$ that maximizes $u(y + \omega)$ on Y; recall the one-consumer, one-firm example of Section 15.C.

The argument is much simpler if we assume that $u(\cdot)$ is differentiable. We claim that utility must then be increasing along any admissible trajectory. Indeed,

$$\frac{du(y(t) + \omega)}{dt} = \nabla u(y(t) + \omega) \cdot \frac{dy(t)}{dt}$$

$$= \mu(t) p(y(t)) \cdot \frac{dy(t)}{dt} > 0,$$

with equality only at equilibrium. Here we have used the fact that at a short-run (interior) equilibrium, the price vector $p(y(t))$ weighted by the marginal utility of wealth $\mu(t)$ must be equal to the vector of marginal utilities of the consumer. Now, since utility is increasing, we must necessarily reach the production vector $\bar{y}$ at which utility is maximized in the feasible production set (i.e., the equilibrium). This is illustrated in Figure 17.H.3. (We are sidestepping minor technicalities: to proceed completely rigorously, we should argue that the dynamics cannot be so sluggish that we never reach the equilibrium. To do so we would need, strictly speaking, to strengthen slightly the concept of an admissible trajectory). ■

Note that the single consumer of Proposition 17.H.2 could be a (positive) representative consumer standing for a population of consumers.

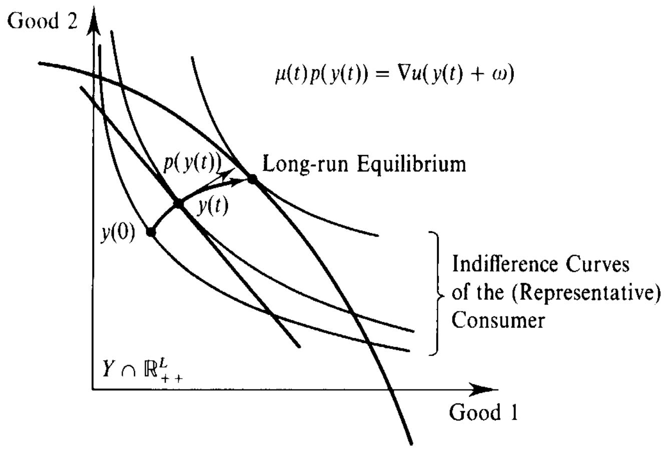


Figure 17.H.3
An example of quantity tâtonnement.

17.I Large Economies and Nonconvexities

As we have mentioned repeatedly, especially in Chapters 10 and 12, a central justification of the price-taking hypothesis is the assumption that every economic agent constitutes an insignificant part of the whole economy. Literally speaking, however, this cannot be satisfied in the model of this chapter because, formally, we allow for no more than a finite number I of consumers. (This is particularly true of our examples, where we typically have $I = 2$.) A straightforward reinterpretation is possible, however. We illustrate it for the case of a pure exchange economy.

Suppose we consider economies whose consumers have characteristics (preferences and endowments) that fall into I given types, with r consumers of each type (a generalization to unequal numbers per type is possible; see Exercise 17.I.1). That is, the set of consumers is formed by r *replicas* of a basic reference set of consumers. Furthermore, an allocation denoted by $(x_1, \dots, x_I)$ is understood now to specify that *each* consumer of type i consumes x_i (so the totality of consumers of type i consume rx_i). We observe then that the analysis and results presented up to this point are not modified by this reinterpretation; they simply do not depend in any way on the parameter r. In this way, we can conclude informally that the theory so far covers cases with an arbitrarily large number, even an infinity, of consumers; in particular, we see that any equilibrium of our earlier model is an equilibrium of the r-replica economy (for any integer $r \geq 1$).

There is, however, an important qualification. The ability to interpret the model and results in a manner that is fully independent of the number of consumers depends *crucially* on the convexity assumption on preferences. Without this assumption, it is not justified to neglect allocations that assign different consumption bundles to different consumers of the same type. Consider, for example, the Edgeworth box of Figure 17.I.1. If there is only one consumer of each type, then no equilibrium exists; but if we have two of each type, then there *is* an equilibrium. To see this, give ω_2 to the convex consumers, let one of the two nonconvex consumers receive the bundle x_1, and let the other receive the different bundle x_1'. Thus, in the nonconvex case, the

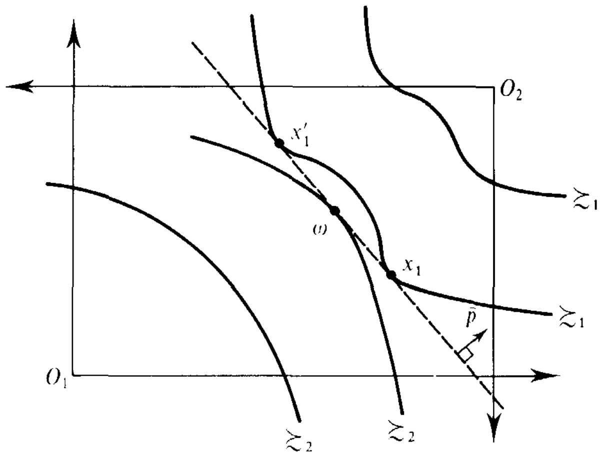


Figure 17.I.1
Equilibrium with nonconvex preferences in economies of changing size.

behavior of the economy can depend on the number of replicas: when we replicate an economy, new equilibria may emerge.[71]

The discussion of the previous example suggests an interesting observation: replication may actually *help* in the analysis of economies with nonconvexities, in the sense that an increase in the size of the economy (in terms of the number of replicas) may help insure existence of an equilibrium. Indeed, we devote the rest of this section to develop the argument that, if the economy is large enough, then the existence of an equilibrium is assured, or nearly so, even if preferences are not convex.[72]

To see this, suppose that we have an exchange economy with I types. Consider a consumer of type i. If preferences are not convex (perhaps goods are indivisible), then the excess demand of this type is a correspondence $z_i(p)$. For $p \gg 0$, $z_i(p)$ is a compact set that may not be convex (as is the case for consumer 1 at $p = \bar{p}$ in Figure 17.I.1). Measuring in average, *per-replica*, terms, the excess demand correspondence of type i when there are r replicas is

$$z_{ir}(p) = \frac{1}{r}(z_i(p) + \cdots + z_i(p)) \qquad \text{(the sum has } r \text{ terms)}$$

$$= \frac{1}{r}\{z_{i1} + \cdots + z_{ir}: z_{i1} \in z_i(p), \ldots, z_{ir} \in z_i(p)\}.$$

If we examine Figure 17.I.1 again, we see that, as $r \to \infty$, the set $z_{1r}(\bar{p}) + \{\omega_1\}$ fills the entire segment between the demand points x_1 and x_1'. In particular, for any $\alpha \in [0, 1]$ and integer r we can find an integer $a_r \in [0, r]$ such that $|a_r/r - \alpha| \leq 1/r$ (note that the r numbers $\{1/r, \ldots, r/r\}$ are evenly spaced in the interval $[0, 1]$). By putting a_r consumers at x_1 and $r - a_r$ consumers at x_1', we get a per-replica consumption of

$$\frac{a_r}{r}x_1 + \left(1 - \frac{a_r}{r}\right)x_1' \in z_{ir}(p),$$

which, by taking r large enough, comes as close to $\alpha x_1 + (1 - \alpha)x_1'$ as we wish. It turns out that this convexifying property is completely general. For any $p \gg 0$ and whatever the number of commodities, as $r \to \infty$ the per-replica excess demand $z_{ir}(p)$ of type i converges as a set of the convex hull of $z_i(p)$. In the limit, the average per-replica excess demand correspondence of type i becomes $z_{i\infty}(p) =$ convex hull $z_i(p)$. In the limit, therefore, the excess demand correspondence is convex valued and the existence of an equilibrium can be established as in Section 17.C.[73] In this sense,

71. That is, if we let $E(r)$ denote the equilibrium price set of the r replica economy, we have $E(1) \subset E(r)$, but the converse need not be true. Note, moreover, that for arbitrary $r'' > r' > 1$, there need not be any inclusion relationship between $E(r'')$ and $E(r')$ (except if $r'' = mr'$ for some integer $m > 1$, in which case $E(r') \subset E(r'')$).

72. See Starr (1969) for a classic contribution to this topic.

73. See the comment after the proof of Proposition 17.C.2 regarding demand correspondences, and also Exercise 17.C.1.

then, as r grows large, the economy must possess an allocation and price vector that constitute a "near" equilibrium.[74]

In the previous reasoning, the convexification of aggregate excess demand, with its existence implication, depends on our ability to prescribe very carefully which of several indifferent consumptions each consumer has to choose. Only in this way can we make sure that the aggregate consumption will be precisely right. Whatever we may think about the possible processes that may lead consumers to select among indifferent optimal choices in the right proportions, there can be little doubt that it would be better if we did not have to worry about this; that is, if, given any price, practically every consumer had a single optimal choice. It is therefore of interest to point out that, while not a necessity, this is a most plausible occurrence if the number of consumers is large. Indeed, *if the distribution of individual preferences is dispersed across the population* (so that, in particular, no two consumers are exactly identical[75]), *then even if the individual excess demands are true correspondences, the limit average may well be a* (continuous) *function*. This is because, at any p, only a vanishingly small proportion of consumers may display a nonconvexity at p. We commented on this point in Appendix A to Chapter 4 and we illustrate it further in Figure 17.I.2. Consider the Edgeworth box of Figure

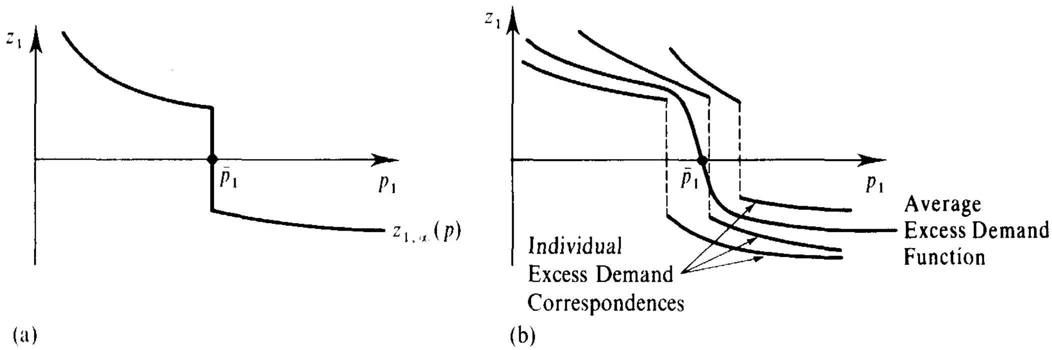


Figure 17.I.2 The aggregate demand from dispersed individual demand is a continuous function. (a) Aggregate excess demand for the Edgeworth box of Figure 17.I.1 with a continuum of consumers of each type (i.e., $r = \infty$). (b) Individual excess demands are dispersed.

17.I.1, but with a continuum of consumers of each type. Since in this economy all the consumers of type 1 are identical, all of them exhibit a "consumption switch" (a nonconvexity) at precisely the same $\bar{p}$. The excess demand correspondence for the first good in this economy is represented in Figure 17.I.2(a) as $z_{1,\infty}(\cdot)$.[76] But if tastes of type 1 consumers exhibit variation, even if slight, then we would expect that no significant group of consumers simultaneously switches at any p and therefore that, as in Figure 17.I.2(b), average demand will be well defined at any p and will change only gradually with p.

74. Roughly speaking, by a "near" equilibrium we mean an allocation and price vector that is close to satisfying the conditions of an equilibrium. A precise technical definition of this concept can be given, but we shall not do so here.

75. We note, as an incidental matter, that in the limit with an infinity of agents this requirement is incompatible with the existence of only a finite number of types. To deal with this case, the formal setting would need to be extended.

76. Precisely, $z_{1,\infty}(\cdot)$ is the correspondence whose graph is the limit graph, as r goes to ∞, of the correspondences $z_r(\cdot)$ defined by $z_r(p) = (1/r)(z(p) + \cdots + z(p))$, where the sum has r terms and $z(\cdot)$ is the excess demand correspondence of the two-consumer economy in the Edgeworth box of Figure 17.I.1.

We comment briefly on economies with production. Suppose that the consumption side of the economy is generated, as before, as the r-replica of a basic reference set of (possibly nonconvex) consumers. There are also J production sets Y_j. Each Y_j is closed, contains the origin, and satisfies free disposal (these are all standard assumptions). In addition, we assume that there is an upper bound (a capacity bound perhaps) on every Y_j; that is, there is a number s such that $y_{\ell j} \leq s$ for all ℓ and $y_j \in Y_j$. The production sets may be nonconvex.

It is then possible to argue that the economy will possess a near equilibrium if r is large relative to the bound s (i.e., if the size of the consumption side of the economy is large relative to the maximal size of a *single* firm). On the average, the production side of the economy is also being convexified, so to speak (see the small-type discussion of Section 5.E for a related point).[77] Note that the boundedness property of the production sets is important. Suppose, for example, that every firm has the technology represented in Figure 15.C.3. Then no matter how many consumers there are, the potential profits of every firm are infinite (as long as $p_2 > 0$). Thus, there is no reasonable sense in which a near equilibrium exists. For the averaging-out effect to work the nonconvexity in production has to be of bounded size (see Exercise 17.I.2).

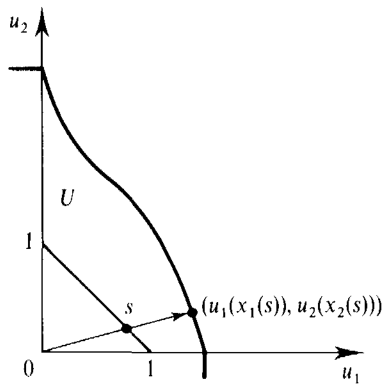
APPENDIX A: CHARACTERIZING EQUILIBRIUM THROUGH WELFARE EQUATIONS

We have seen, beginning in Section 17.B, that if our economy satisfies sufficiently nice properties (e.g., strict convexity of preferences) then we can resort, for the purposes of the analysis, to formalizing our theory by means of highly reduced systems of equilibrium equations. In the text of this chapter, we have focused on excess demand equations. But this is not the only possibility. In this appendix, we briefly illustrate a second approach that builds on the welfare properties of equilibria.

We again concentrate on a pure exchange economy in which each consumer $i = 1, \dots, I$ has the consumption set $\mathbb{R}^L_+$ and continuous, strongly monotone, and strictly convex preferences. We also assume that $\omega_i \geq 0$ for all i and $\sum_i \omega_i \gg 0$.

We know from Chapter 16 that a Walrasian equilibrium of this economy is a Pareto optimum (Proposition 16.C.1). Therefore, to identify an equilibrium, we can as well restrict ourselves to Pareto optimal allocations. To this effect, suppose we fix continuous utility functions $u_i(\cdot)$ for the I consumers with $u_i(0) = 0$. Then to every vector $s = (s_1, \dots, s_I)$ in the simplex $\Delta = \{s' \in \mathbb{R}^I_+ : \sum_i s'_i = 1\}$ we can associate a unique Pareto optimal allocation $x(s) \in \mathbb{R}^{LI}_+$ such that $(u_1(x_1(s)), \dots, u_I(x_I(s)))$ is

77. Observe that the average is with respect to r (the size of the economy in terms of the number of consumers), not with respect to J. If, as r increases, J is made to vary and is kept in some approximate fixed proportion with r, then from the qualitative point of view it does not matter how we measure size (this is a possible way to interpret, in the current context, the discussion of Section 5.E). But for the validity of the convexifying effect there is no need to vary J with r. The number J may be kept fixed and, thus, J could well be small relative to r (in which case the "averaged" economy is practically one of pure exchange) or it could be large; it could even be that $J = \infty$. The last case corresponds to a model with free entry, where the equilibrium – or the near equilibrium – determines, endogenously, the set of active firms. Typically, with free entry the set of the active firms increases as the number of consumers, measured by r, grows (this point has also been discussed in Section 10.F in a partial equilibrium context; there is not much more to add here).



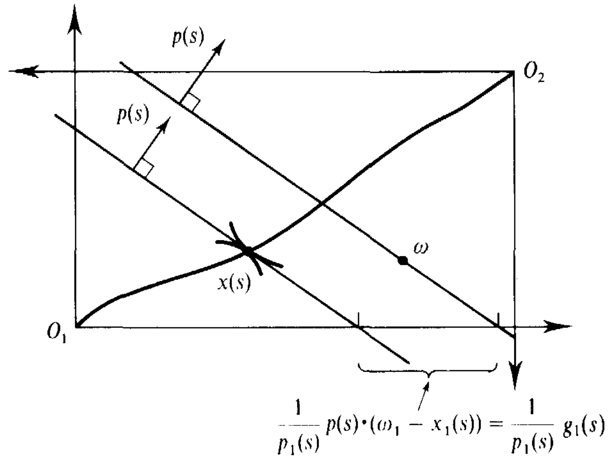


Figure 17.AA.1 (left) Construction of the welfare-theoretic equation system: first step.

Figure 17.AA.2 (right) Construction of the welfare-theoretic equation system: second step.

proportional to $s \in \Delta$ (see Exercise 17.AA.1). In words, $s \in \Delta$ stands for the values of utility distribution parameters and the determined allocation distributes "welfare" in accordance with the "shares" $s = (s_1, \ldots, s_I)$. Figure 17.AA.1 illustrates the construction.

An arbitrary $s \in \Delta$ will typically not correspond to an equilibrium. How can we recognize those $s \in \Delta$ that do? To answer this question we can resort to the second welfare theorem. From Propositions 16.D.1 (and the discussion in small type following Proposition 16.D.3), we know that, under our assumptions, associated with $x(s)$ there is a price vector $p(s) \in \mathbb{R}^L$ that supports the allocation in the sense that, for every i, $x'_i \succ_i x_i(s)$ implies $p(s)\cdot x'_i > p(s)\cdot x_i(s)$. Therefore, $(x(s^*), p(s^*))$ constitutes a Walrasian equilibrium if and only if $s^* \in \Delta$ solves the system of equations

$$g_i(s^*) = p(s^*)\cdot[\omega_i - x_i(s^*)] = 0 \qquad \text{for every } i = 1, \ldots, I. \qquad (17.AA.1)$$

The Edgeworth box example of Figure 17.AA.2 explains the point that we are currently making.

This Pareto-based equation system was first put forward by Negishi (1960), and was the approach taken by Arrow and Hahn (1971) in their proof of existence of equilibrium. It can be quite useful when the number of consumers (say, the number of countries in an international trade model) is small relative to the number of commodities. In contrast, if the number of consumers is large relative to the number of commodities, then an approach via excess demand functions will be superior. A limitation of the Negishi approach is that it is very dependent on the fact that an equilibrium must be a Pareto optimum. The excess demand approach is more easily adaptable to situations where this is not so (for example, because of tax distortions; see Exercise 17.C.3).[78]

78. The systems of equations (17.B.2) and (17.AA.1) can be formally contrasted as follows. In both of them, at any point of the domain of the equations, consumers and firms satisfy the utility maximization conditions for some prices and distribution of wealth. In (17.B.2) this distribution of wealth is always the one induced by the initial endowments, but feasibility (i.e., the equality of demand and supply) is insured only at the solution. In (17.AA.1) it is the other way around: feasibility is always satisfied, but the agreement of the wealth distribution with that induced by the initial endowments is insured only at the solution.

APPENDIX B: A GENERAL APPROACH TO THE EXISTENCE OF WALRASIAN EQUILIBRIUM

The purpose of this appendix is to offer a treatment of the existence question at the level of generality of the model of Chapter 16. The results presented correspond roughly to those of Arrow and Debreu (1954) and McKenzie (1959).

As when dealing with the second welfare theorem in Section 16.D, and for exactly the same technical reasons, it is useful to concentrate on establishing the existence of a *Walrasian quasiequilibrium*. This is a weaker notion than Walrasian equilibrium in that consumers are required to maximize preferences only relative to consumptions that cost strictly less than the available amount of wealth.

Definition 17.BB.1: An allocation $(x_1^*, \ldots, x_I^*, y_1^*, \ldots, y_J^*)$ and a price system $p \neq 0$ constitute a *Walrasian quasiequilibrium* if

(i) For every j, $p \cdot y_j \leq p \cdot y_j^*$ for all $y_j \in Y_j$.

(ii') For every i, $p \cdot x_i^* \leq p \cdot \omega_i + \sum_j \theta_{ij} p \cdot y_j^*$, and

$$\text{if } x_i \succ_i x_i^* \text{ then } p \cdot x_i \geq p \cdot \omega_i + \sum_j \theta_{ij} p \cdot y_j^*.$$

(iii) $\sum_i x_i^* = \sum_i \omega_i + \sum_j y_j^*$.

Definition 17.BB.1 is identical to Definition 17.B.1 of a Walrasian equilibrium except that the preference maximization condition (ii) of Definition 17.B.1 has been replaced by the weaker condition (ii'). With local nonsatiation, condition (ii') is equivalent to the requirement that x_i^* minimizes expenditure for the price vector p in the set $\{x_i \in X_i : x_i \succsim_i x_i^*\}$. The expenditure minimization problem has better continuity properties with respect to prices than does the preference maximization problem. Thus, in Figure 17.BB.1 we have $x_i^n = x_i^n(p^n, p^n \cdot x_i^n)$ and $p^n \cdot x_i^n = e(p^n, u_i(x_i^n))$; that is, x_i^n is preference maximizing relative to the price–wealth pair $(p^n, p^n \cdot x_i^n)$ and expenditure minimizing relative to the price–utility pair $(p^n, u_i(x_i^n))$. However, as $p^n \to p$ and $x_i^n \to \bar{x}_i$, we see that $\bar{x}_i$ fails to maximize $u_i(\cdot)$ relative to $(p, p \cdot \bar{x}_i)$, whereas $\bar{x}_i$ still minimizes expenditure relative to $(p, u_i(\bar{x}_i))$; that is, $p \cdot \bar{x}_i = e(p, u_i(\bar{x}_i))$. Because continuity is an important requirement for existence analysis, it is more convenient to prove the existence of a quasiequilibrium than that of an equilibrium. Ultimately, however, we are interested in equilibrium. As it turns out, it is relatively easy to

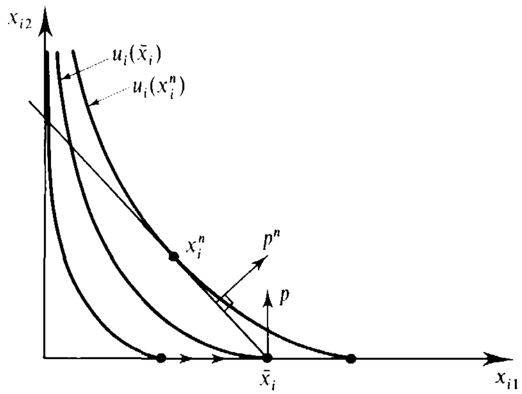


Figure 17.BB.1
Discontinuity of preference maximization.

state conditions implying that a quasiequilibrium is automatically an equilibrium. We devote the next few paragraphs to elaborating on this point.

We begin with Definition 17.BB.2

Definition 17.BB.2: The Walrasian quasiequilibrium (x^*, y^*, p) satisfies the *cheaper consumption condition for consumer i* if there is $x_i \in X_i$ such that $p \cdot x_i < p \cdot \omega_i + \sum_j \theta_{ij} p \cdot y_j^*$.

We then have Proposition 17.BB.1.

Proposition 17.BB.1: Suppose that consumption sets are convex and preferences are continuous. Then any consumer who at the Walrasian quasiequilibrium (x^*, y^*, p) satisfies the cheaper consumption condition must be preference maximizing in his budget set. Hence, if the cheaper consumption condition is satisfied for all i, (x^*, y^*, p) is also a Walrasian equilibrium.

Proof: Suppose that i satisfies the cheaper consumption condition; that is, there is $x_i \in X_i$ with $p \cdot x_i < p \cdot \omega_i + \sum_j \theta_{ij} p \cdot y_j^*$. If x_i^* fails to be preference maximizing, then there is x_i' such that $x_i' \succ_i x_i^*$ and x_i' is in the budget set of consumer i. Denote $x_i^n = (1 - (1/n))x_i' + (1/n)x_i$. Then $x_i^n \in X_i$ and $p \cdot x_i^n < p \cdot \omega_i + \sum_j \theta_{ij} p \cdot y_j^*$ for all n, and $x_i^n \to x_i'$ as $n \to \infty$. By continuity of preferences, for large enough n we will have $x_i^n \succ_i x_i^*$. But then consumer i violates condition (ii′) of the definition of quasi-equilibrium. ■

Suppose that, for every j, $0 \in Y_j$, and for every i, X_i is convex and $\omega_i \geq \hat{x}_i$ for some $\hat{x}_i \in X_i$. Suppose, in addition, that the weak condition "$\sum_i \omega_i + \sum_j \hat{y}_j \gg \sum_i \hat{x}_i$ for some $(\hat{y}_1, \ldots, \hat{y}_J) \in Y_1 \times \cdots \times Y_J$" is satisfied. Then the quasiequilibrium (x^*, y^*, p) satisfies the cheaper consumption condition for consumer i in, for example, either of the following situations (Exercise 17.BB.1):

(a) $p \geq 0$, $p \neq 0$, and $\omega_i \gg \hat{x}_i$,
(b) $p \gg 0$ and $\omega_i \neq \hat{x}_i$.

To have $p \geq 0$ at quasiequilibrium it suffices that one production set satisfies the free-disposal condition. Guaranteeing $p \gg 0$ is more difficult. It will happen if, for every i, $X_i + \mathbb{R}_+^L \subset X_i$ and preferences are continuous and strongly monotone. To see this, note that by the monotonicity of preferences and the expenditure-minimization property of quasiequilibrium we must have $p \geq 0$, $p \neq 0$, at any quasiequilibrium. It follows that $p \cdot (\sum_i \omega_i + \sum_j y_j^*) \geq p \cdot (\sum_i \omega_i + \sum_j \hat{y}_j) > p \cdot (\sum_i \hat{x}_i)$. Hence, there is at least one consumer with wealth larger than $p \cdot \hat{x}_i$. But this consumer must be maximizing preferences (by Proposition 17.BB.1), which, by the strong monotonicity property, can only occur if every price is positive (i.e., if no good is free).

Although convenient, neither condition (a) nor (b) can be regarded as extremely weak. It would be unfortunate if the validity of the theory were restricted to them. But this is not so: much weaker conditions are available. In particular, McKenzie (1959) has developed a theory of *indecomposable* economies that guarantees that at a quasiequilibrium the cheaper consumption condition is satisfied for every consumer (and therefore the quasiequilibrium is an equilibrium). The basic idea, informally described, is that an economy is indecomposable if, no matter how we partition the economy into two groups, each of the groups has something for which the other group is willing to exchange something of its own. (See Exercise 17.BB.2.)

We now turn to the existence of a Walrasian quasiequilibrium. The aim is to establish the general existence result in Proposition 17.BB.2.

Proposition 17.BB.2: Suppose that for an economy with $I > 0$ consumers and $J > 0$ firms we have

(i) For every i,
  (i.1) $X_i \subset \mathbb{R}^L$ is closed and convex;
  (i.2) $\succsim_i$ is a rational, continuous, locally nonsatiated, and convex preference relation defined on X_i;
  (i.3) $\omega_i \geq \hat{x}_i$ for some $\hat{x}_i \in X_i$.
(ii) Every $Y_j \subset \mathbb{R}^L$ is closed, convex, includes the origin, and satisfies the free-disposal property.
(iii) The set of feasible allocations

$$A = \{(x, y) \in \mathbb{R}^{LI} \times \mathbb{R}^{LJ}: x_i \in X_i \text{ for all } i,\ y_j \in Y_j \text{ for all } j, \text{ and } \sum_i x_i \leq \sum_i \omega_i + \sum_j y_j\}$$

is compact.

Then a Walrasian quasiequilibrium exists.

We comment briefly on the assumptions. As we have repeatedly illustrated (in Chapters 10, 15, and 16), the convexity assumptions on individual preferences and technologies cannot be dispensed with.[79] The Edgeworth box example of Figure 17.BB.2 shows that the local nonsatiation condition is also required.[80] In contrast, the assumption of rational preferences is entirely dispensable (see the comments at the end of this appendix). The free disposal condition $-\mathbb{R}^L_+ \subset Y_j$ is also only a matter of convenience.[81] Allowing for negative prices, we could simply drop it from our list of conditions (see Exercise 17.BB.3). The assumption (i.3) says that ω_i may not belong to the consumption set but that it is possible to reach the consumption set by just eliminating some amounts of commodities from ω_i.[82] Finally, in Appendix A to Chapter 16 we have already investigated the conditions under which the set of feasible allocations is compact.

79. Recall, however, the important qualification of Section 17.I, and see also the discussion at the end of this appendix.

80. In Figure 17.BB.2, the second consumer has conventional strongly monotone preferences; but for the first consumer both commodities are bads and, therefore, he is satiated at the origin. Also $\omega_1 \gg 0$ and $\omega_2 \gg 0$. Suppose that $x^* = (x_1^*, x_2^*)$ and price vector $p \neq 0$ constitute a Walrasian quasiequilibrium. Because the preferences of the second consumer are strongly monotone, we must have $p \gg 0$. By profit maximization (using the free-disposal technology) and the possibility of inaction, we have $p \cdot (x_1^* + x_2^* - \omega_1 - \omega_2) \geq 0$. Since $p \cdot x_2^* \leq p \cdot \omega_2$, this yields $p \cdot x_1^* \geq p \cdot \omega_1 > 0$. But then (x^*, p) cannot be a Walrasian quasiequilibrium because consuming nothing costs zero and is preferred by the first consumer to any other consumption.

81. Because Y_j is convex and closed, $-\mathbb{R}^L_+ \subset Y_j$ implies $Y_j - \mathbb{R}^L_+ \subset Y_j$ (Exercise 5.B.5).

82. A stronger condition would require that $\omega_i \gg \hat{x}_i$ for every i. With this assumption, Proposition 17.BB.2 yields the existence of a true equilibrium, not just of a quasiequilibrium. Economically, however, the latter assumption is considerably stronger: $\omega_i \geq \hat{x}_i$ can be interpreted (keeping in mind the possibility of free disposal) simply as saying that consumer i could survive without entering the markets of the economy, while $\omega_i \gg \hat{x}_i$ says that the consumer can supply to the market a strictly positive amount of *every* good.

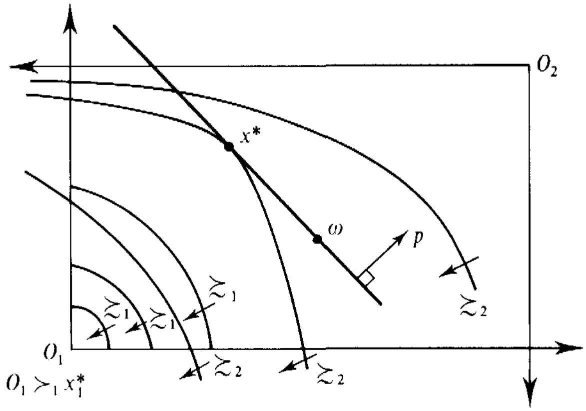


Figure 17.BB.2
Equilibrium does not exist: the preferences of the first consumer are satiated.

Proof of Proposition 17.BB.2

The approach we follow takes advantage of the fact that the reader may already have been exposed in Chapter 8 to the notion of the Nash equilibrium of a normal form game and, more particularly, to the existence results for Nash equilibrium using best-response correspondences contained in Appendix A to Chapter 8. A game-theoretic approach to the existence of Walrasian equilibrium was taken in the classic paper of Arrow and Debreu (1954). Here we follow Gale and Mas-Colell (1975).

Definition 17.BB.3: An allocation (x^*, y^*) and a price system $p \neq 0$ constitute a *free-disposal quasiequilibrium* if

(i) for every j, $p \cdot y_j \leq p \cdot y_j^*$ for all $y_j \in Y_j$.

(ii') For every i, $p \cdot x_i^* \leq p \cdot \omega_i + \sum_j \theta_{ij} p \cdot y_j^*$, and

$$\text{if } x_i \succ_i x_i^* \text{ then } p \cdot x_i \geq p \cdot \omega_i + \sum_j \theta_{ij} p \cdot y_j^*.$$

(iii') $\sum_i x_i^* \leq \sum_i \omega_i + \sum_j y_j^*$ and $p \cdot (\sum_i x_i^* - \sum_i \omega_i - \sum_j y_j^*) = 0$.

Thus, all we have done is replace in Definition 17.BB.1 of a quasiequilibrium the exact feasibility condition "$\sum_i x_i^* = \sum_i \omega_i + \sum_j y_j^*$" by (iii') above. That is, we allow the excess supply of some goods provided that they are free. In Exercise 17.BB.4 you are asked to show that if one production set, say Y_1, satisfies the free-disposal property and if $(x_1^*, \ldots, x_I^*, y_1^*, \ldots, y_J^*, p)$ is a free-disposal quasiequilibrium, then there is $y_1'^* \leq y_1^*$ such that $(x_1^*, \ldots, x_I^*, y_1'^*, y_2^*, \ldots, y_J^*, p)$ is a Walrasian quasiequilibrium. Therefore, to establish Proposition 17.BB.2, it is enough for us to show that a free-diposal quasiequilibrium exists.

We proceed to formalize the free-disposal quasiequilibrium notion as a kind of noncooperative equilibrium for a certain game among $I + J + 1$ players. The I and J players are the consumers and the firms, respectively, and their strategies are demand–supply vectors. The extra player is a fictitious *market agent* (a "grand coordinator") having as his strategy the prices of the L different goods.

Since the set A of feasible allocations is bounded, there is $r > 0$ such that whenever $(x_1, \ldots, x_I, y_1, \ldots, y_J) \in A$ we have $|x_{\ell i}| < r$ and $|y_{\ell j}| < r$ for all i, j, and ℓ. Because we need to have compactness of strategy sets to establish existence, we begin by replacing every X_i and every Y_j by a truncated version:

$$\hat{X}_i = \{x_i \in X_i \colon |x_{\ell i}| \le r \text{ for all } \ell\},$$

$$\hat{Y}_j = \{y_j \in Y_j : |y_{\ell j}| \le r \text{ for all } \ell\}.$$

Note that $A \subset \hat{X}_1 \times \cdots \times \hat{X}_I \times \hat{Y}_1 \times \cdots \times \hat{Y}_J$. Because $(\hat{x}_1, \ldots, \hat{x}_I, \ldots, 0, \ldots, 0) \in A$, it follows that $\hat{x}_i \in \hat{X}_i$ for every i, and $0 \in \hat{Y}_j$ for every j. In particular, all the strategy sets are nonempty. Lemma 17.BB.1 shows that in our search for a free-disposal quasi-equilibrium we can limit ourselves to the truncated economy.

Lemma 17.BB.1: If all X_i and Y_j are convex and (x^*, y^*, p) is a free-disposal quasi-equilibrium in the truncated economy, that is, if (x^*, y^*, p) satisfies Definition 17.BB.3 of free-disposal quasiequilibrium with the consumption and production sets replaced by their truncated versions, then (x^*, y^*, p) is also a free-disposal quasiequilibrium for the original untruncated economy.

Proof of Lemma 17.BB.1: Consider a consumer i (the reasoning is similar for a firm). Because $(x^*, y^*) \in A$, we have $|x^*_{\ell i}| < r$ for all ℓ; that is, the consumption bundle of consumer i is interior to the truncation bound. Suppose now that x_i^* fails to satisfy condition (ii′) of Definition 17.BB.3 in the nontruncated economy, that is, that there is an $x_i \in X_i$ such that $x_i \succ_i x_i^*$, and $p \cdot x_i < p \cdot \omega_i + \sum_j \theta_{ij} p \cdot y_j^*$. Denote $x_i^n = (1 - (1/n))x_i^* + (1/n)x_i$. For all n we have $p \cdot x_i^n < p \cdot \omega_i + \sum_j \theta_{ij} p \cdot y_j^*$ and, by the convexity of preferences, $x_i^n \succsim_i x_i^*$. Also, we can choose an n large enough to have $|x^n_{\ell i}| < r$ for all ℓ. By local nonsatiation there must then be an $x_i' \in \hat{X}_i$ such that $x_i' \succ_i x_i^n$ and $p \cdot x_i' < p \cdot \omega_i + \sum_j \theta_{ij} p \cdot y_j^*$. But then $x_i' \in \hat{X}_i$ and $x_i' \succ_i x_i^n \succsim_i x_i^*$, and so in the truncated economy x_i^* fails to satisfy condition (ii′) of Definition 17.BB.3. Thus, (x^*, y^*, p) must not be a free-disposal quasiequilibrium in the truncated economy. This contradiction establishes the result. ■

We are now ready to set up a simultaneous-move noncooperative game. To do so we need to specify the players' strategy sets and payoff functions. To simplify notation we assign to every consumer i, price vector p and production profile $y = (y_1, \ldots, y_J)$, a limited liability amount of wealth

$$w_i(p, y) = p \cdot \omega_i + \operatorname{Max}\left\{0, \sum_j \theta_{ij} p \cdot y_j\right\}.$$

The strategy sets are:

For consumer i: $\hat{X}_i$
For firm j: $\hat{Y}_j$
For the market agent: $\Delta = \{p \in \mathbb{R}^L \colon p_\ell \ge 0 \text{ for all } \ell \text{ and } \sum_\ell p_\ell = 1\}$.

Given a strategy profile $(x, y, p) = (x_1, \ldots, x_I, y_1, \ldots, y_J, p)$, the payoff functions and best-responses of the different agents are:

Consumer i: Chooses consumption vectors $x_i' \in \hat{X}_i$ such that
(1) $p \cdot x_i' \le w_i(p, y)$ and
(2) $x_i' \succsim_i x_i''$ for all $x_i'' \in \hat{X}_i$ satisfying $p \cdot x_i'' < w_i(p, y)$.
(Consumer i's payoff function can be thought of as giving a payoff 1 if he chooses a consumption vector satisfying this condition, and 0 otherwise.)

Denote by $\tilde{x}_i(x, y, p) \subset \hat{X}_i$ the set of consumption bundles x_i' so defined.

Firm j: Chooses productions $y_j' \in \hat{Y}_j$ that are profit maximizing for p on $\hat{Y}_j$. (Firm j's payoff function is simply its profit.)

Denote by $\tilde{y}_j(x, y, p) \subset \hat{Y}_j$ the set of production plans y_j' so defined.

Market Agent: Chooses prices $q \in \Delta$ so as to solve

$$\underset{q \in \Delta}{\text{Max}} \left(\sum_i x_i - \sum_i \omega_i - \sum_j y_j\right) \cdot q. \tag{17.BB.1}$$

Denote by $\tilde{p}(x, y, p)$ the set of price vectors q so defined.

Only the behavior of the market agent needs comment. Given the total excess demand vector, the market agent chooses prices so as to maximize the value of this vector. Hence, he puts the whole weight of prices (which, recall, have been normalized to lie in the unit simplex) into the commodities with maximal excess demand. As we have already observed when doing the same thing in the proof of Proposition 17.C.1, this is in accord with economic logic: if the objective is to eliminate the excess demand of some commodities, try raising their prices as much as possible.

Lemma 17.BB.2 says that an equilibrium of this noncooperative game yields a free-disposal quasiequilibrium for the truncated economy.

Lemma 17.BB.2: Suppose that (x^*, y^*, p) is such that $x_i^* \in \tilde{x}_i(x^*, y^*, p)$ for all i, $y_j^* \in \tilde{y}_j(x^*, y^*, p)$ for all j, and $p \in \tilde{p}(x^*, y^*, p)$. Then (x^*, y^*, p) is a free-disposal quasiequilibrium for the truncated economy.

Proof of Lemma 17.BB.2: We note first that $p \cdot y_j^* \geq 0$ for every j (because $0 \in \hat{Y}_j$). By the definition of $\tilde{x}_i(\cdot)$ and $\tilde{y}_j(\cdot)$, conditions (i) and (ii′) of Definition 17.BB.3 are then automatically satisfied. Hence, the only property that remains to be established is (iii′), that is,

$$\sum_i x_i^* - \sum_i \omega_i - \sum_j y_j^* \leq 0 \qquad \text{and} \qquad p \cdot \left(\sum_i x_i^* - \sum_i \omega_i - \sum_j y_j^*\right) = 0.$$

We have $p \cdot x_i^* \leq w_i(p, y^*) = p \cdot \omega_i + \sum_j \theta_{ij} p \cdot y_j^*$ for all i and therefore

$$p \cdot \left(\sum_i x_i^* - \sum_i \omega_i - \sum_j y_j^*\right) \leq 0.$$

This implies $\sum_i x_i^* - \sum_i \omega_i - \sum_j y_j^* \leq 0$ because otherwise the value of the solution to problem (17.BB.1) would be positive and so p (which as we have just seen has $p \cdot (\sum_i x_i^* - \sum_i \omega_i - \sum_j y_j^*) \leq 0$) could not be a maximizing solution vector, that is, a member of $\tilde{p}(x^*, y^*, p)$. It follows that $(x^*, y^*) \in A$ and so, $x_{\ell i}^* < r$ for all i and ℓ. From this we get that the budget equations are satisfied with equality (i.e., $p \cdot x_i^* = p \cdot \omega_i + \sum_j \theta_{ij} p \cdot y_j^*$ for all i) because otherwise local nonsatiation yields that for some consumer i there is a preferred consumption strictly interior to consumer i's budget set in the truncated economy, implying $x_i^* \notin \tilde{x}_i(x^*, y^*, p)$. We therefore conclude that we also have $p \cdot (\sum_i x_i^* - \sum_i \omega_i - \sum_j y_j^*) = 0$. This completes the proof. ■

Now, as we discussed in Appendix A to Chapter 8 (see the proof of Proposition 8.D.3 presented there), under appropriate conditions on the best-response correspondences, this noncooperative game has an equilibrium.

Lemma 17.BB.3: Suppose that the correspondences $\tilde{x}_i(\cdot)$, $\tilde{y}_j(\cdot)$, and $\tilde{p}(\cdot)$ are nonempty, convex valued, and upper hemicontinuous. Then there is (x^*, y^*, p) such that $x_i^* \in \tilde{x}_i(x^*, y^*, p)$ for all i, $y_j^* \in \tilde{y}_j(x^*, y^*, p)$ for all j, and $p \in \tilde{p}(x^*, y^*, p)$.

Proof of Lemma 17.BB.3: We are simply looking for a fixed point of the correspondence $\Psi(\cdot)$ from $X_1 \times \cdots \times X_I \times Y_1 \times \cdots \times Y_J \times \Delta$ to itself defined by

$$\Psi(x, y, p) = \tilde{x}_1(x, y, p) \times \cdots \times \tilde{x}_I(x, y, p) \times \tilde{y}_1(x, y, p) \times \cdots \times \tilde{y}_J(x, y, p) \times \tilde{p}(x, y, p).$$

The correspondence $\Psi(\cdot)$ is nonempty, convex valued, and upper hemicontinuous. The existence of a fixed point follows directly from Kakutani's fixed point theorem (see Section M.I of the Mathematical Appendix). ■

Lemmas 17.BB.4 to 17.BB.6 verify that the best-response correspondence of this noncooperative game is nonempty, convex valued, and upper hemicontinuous.[83]

Lemma 17.BB.4: For all strategy profiles (x, y, p), the sets $\tilde{x}_i(x, y, p)$, $\tilde{y}_j(x, y, p)$, and $\tilde{p}(x, y, p)$ are nonempty.

Proof of Lemma 17.BB.4: For $\tilde{y}_j(x, y, p)$ and $\tilde{p}(x, y, p)$ the claim is clear enough since we are maximizing a continuous (in fact, linear) function on, respectively, the nonempty, compact sets $\hat{Y}_j$ and Δ. For $\tilde{x}_i(x, y, p)$, recall that the continuity of $\succsim_i$ implies the existence of a continuous utility representation $u_i(\cdot)$ for $\succsim_i$.[84] Let x_i' be a maximizer of the continuous function $u_i(x_i)$ on the nonempty compact budget set $\{x_i \in \hat{X}_i: p \cdot x_i \leq w_i(p, y^*)\}$. Then $x_i' \in \tilde{x}_i(x, y, p)$. The budget set is nonempty because $\hat{x}_i \in \hat{X}_i$ and $\hat{x}_i \leq \omega_i$. With $p \geq 0$, this implies that $p \cdot \hat{x}_i \leq p \cdot w_i \leq w_i(p, y^*)$. ■

Lemma 17.BB.5: For all strategy profiles the sets $\tilde{x}_i(x, y, p)$, $\tilde{y}_j(x, y, p)$, and $\tilde{p}(x, y, p)$ are convex.

Proof of Lemma 17.BB.5: We establish the claim for $x_i(x, y, p)$. You are asked to complete the proof in Exercise 17.BB.6.

Suppose that $x_i, x_i' \in \tilde{x}_i(x, y, p)$ and consider $x_{i\alpha} = \alpha x_i + (1 - \alpha)x_i'$, for any $\alpha \in [0, 1]$. Note first that $p \cdot x_{i\alpha} \leq w_i(p, y)$. In addition, by the convexity of preferences we cannot have $x_i \succ_i x_{i\alpha}$ and $x_i' \succ_i x_{i\alpha}$ (Exercise 17.BB.5). So suppose that $x_{i\alpha} \succsim_i x_i$. Consider now any $x_i'' \in \hat{X}_i$ with $p \cdot x_i'' < w_i(p, y)$. Then since $x_i \in \tilde{x}_i(x, y, p)$ we have $x_i \succsim_i x_i''$, and so $x_{i\alpha} \succsim_i x_i''$. We conclude that $x_{i\alpha} \in \tilde{x}_i(x, y, p)$. A similar conclusion follows if $x_{i\alpha} \succsim_i x_i'$. Hence, $\tilde{x}_i(x, y, p)$ is a convex set. ■

Lemma 17.BB.6: The correspondences $\tilde{x}_i(\cdot)$, $\tilde{y}_j(\cdot)$, and $\tilde{p}(\cdot)$ are upper hemicontinuous.

Proof of Lemma 17.BB.6: Again, we limit ourselves to $\tilde{x}_i(\cdot)$. Exercise 17.BB.7 asks you to complete the proof for $\tilde{y}_j(\cdot)$ and $\tilde{p}(\cdot)$.

83. For the firms and the market game this result is covered by Proposition 8.D.3, but for the consumers we need a special argument (as defined, the payoff functions of the consumers are not continuous).

84. This was proved in Proposition 3.C.1 for monotone preferences on $\mathbb{R}^L_+$. As we pointed out there, however, the conclusion actually depends only on the continuity of the preference relation.

Let $p^n \to p$, $y^n \to y$, $x^n \to x$, and $x_i'^n \to x_i'$ as $n \to \infty$, and suppose that $x_i'^n \in \tilde{x}_i(x^n, y^n, p^n)$. We need to show that $x_i' \in \tilde{x}_i(p, x, y)$.

From $p^n \cdot x_i'^n \leq w_i(p^n, y^n)$ we get $p \cdot x_i' \leq w_i(p^n, y^n)$. Consider now any $x_i'' \in \hat{X}_i$ with $x_i'' \succ_i x_i'$. Then, by the continuity of preferences, $x_i'' \succ_i x_i'^n$ for n large enough. Hence, $p^n \cdot x_i'' \geq w_i(p^n, y^n)$. Going to the limit we get $p \cdot x_i'' \geq w_i(p, y)$. Thus, we conclude that, as we wanted, $x_i' \in \tilde{x}_i(x, y, p)$. It is, incidentally, because of the need to establish this closed-graph property that we have replaced preference maximization by the weaker objective of expenditure minimization in the definition of the objectives of the consumer. ∎

The combination of Lemmas 17.BB.4 to 17.BB.6 establishes that the given best-response correspondences satisfy the properties required in Lemma 17.BB.3 for the existence of a fixed-point, which completes the proof of Proposition 17.BB.2. ∎

The assumptions on preferences and technologies can be weakened in an important respect. Our existence argument requires only that the best-response correspondence $\tilde{x}_i(x, y, p)$ and $\tilde{y}_j(x, y, p)$ be convex valued and upper hemicontinuous. Beyond this, the proof imposes no restrictions whatsoever on the dependence of consumers' and firms' choices on the "state" variables (x, y, p). Thus we could allow consumers' tastes, or firms' technologies, to depend on prices (money illusion?), on the choices of other consumers or firms (a form of externalities), or even on own consumption (e.g., tastes could depend on a current reference point—a source of incompleteness or nontransitivity of preferences already illustrated in Chapter 1).[85,86] The following is an example of the sort of generality that can be accommodated: Suppose that consumer preferences are given to us by means of utility functions $u_i(\cdot\,; x, y, p)$ defined on X_i but dependent, in principle, on the state of the economy. If for every (x, y, p) the conditions of Proposition 17.BB.2 are satisfied, and the parametric dependence on (x, y, p) is continuous, then a Walrasian quasiequilibrium still exists. The proof does not need any change. We can make a similar point with respect to the possibility that firms' technologies depend on external effects, with, then, an added theoretical payoff. It allows us to see that equilibrium exists if the technology of the firm is convex: *it does not matter if the "aggregate" technology of the economy is convex.* See Exercise 17.BB.8 for more on this.

The existence proof we have given in this appendix is an example of a "large space" proof. The fixed-point argument (in our case phrased as a Nash equilibrium existence argument) has been developed in a disaggregated domain where all the equilibrating variables have been listed separately. The advantage of proceeding this way is that the argument remains very flexible and allows us to incorporate the weakest possible conditions without extra effort (as the last paragraph has illustrated). The disadvantage, of course, is that the fixed point may be

85. Suppose, for example, that the utility function of a consumer is given to us in the form $u_i(\cdot\,; x_i)$; that is, the evaluation of possible consumptions depends on the current consumption. Without loss of generality we can normalize $u_i(x_i; x_i) = 0$ for every x_i. Define the induced weak and strict preference relations $\succsim_i$ and $\succ_i$ on X_i by, respectively, "$x_i' \succsim_i x_i$ if $u_i(x_i'; x_i) \geq 0$" and "$x_i' \succ_i x_i$ if $u_i(x_i; x_i) > 0$." Then the relations $\succsim_i$ and $\succ_i$ contain all the relevant information for equilibrium analysis. Note, however, that it is perfectly possible for $\succsim_i$ not to be complete and for neither $\succsim_i$ nor $\succ_i$ to be transitive. See Shafer (1974) and Gale and Mas-Colell (1975) for more on this.

86. Another example of dependence on the overall consumption vector of the economy arises if, for example, we are considering equilibrium at a given point in time. Then current consumptions in the economy (e.g., purchases of physical or financial assets) will typically affect future prices; these, in turn, will influence current preferences via expectations.

hard to compute and cumbersome to analyze. Usually, as we have seen in Section 17.C and in Appendix A of this chapter, it is possible to work with more aggregated, reduced systems. In fact, the general point duly made, it is worthwhile to observe that this is so even under the assumptions of Proposition 17.BB.2.[87] We elaborate briefly on this.

We can prove Proposition 17.BB.2 by setting up a two-player game instead of an $I + J + 1$ one.[88] The first player is an aggregate consumer–firm that has $\sum_i \hat{X}_i - \{\sum_i \omega_i\} - \sum_j \hat{Y}_j$ as its strategy set; the second is, as before, a market agent having Δ as its strategy set. Given $p \in \Delta$, the first agent responds with the set of vectors z expressible as $z = \sum_i x_i - \sum_i \omega_i - \sum_j y_j$, where y_j is profit maximizing in $\hat{Y}_j$ for every j, and $x_i \in \hat{X}_i$ is such that (1) $p \cdot x_i \leq p \cdot \omega_i + \sum_j \theta_{ij} p \cdot y_j$ and (2) $x_i \succsim_i x_i'$ whenever $p \cdot x_i' < p \cdot \omega_i + \sum_j \theta_{ij} p \cdot y_j$. As before, the market agent responds with the set of $q \in \Delta$ that maximize $z \cdot q$ on Δ. Once this two-person game has been set up, the proof proceeds as for Proposition 17.BB.2. You should check this in Exercise 17.BB.9.

If for any $p \in \Delta$ the preference-maximizing choices of consumers, $x_i(p)$, and the profit-maximizing choices of firms, $y_j(p)$, were single valued, we could go one step further and consider a game with a single player (the market agent). Given p, we would then let the best response of the market agent be the set of price vectors $q \in \Delta$ that maximizes $[\sum_i x_i(p) - \sum_i \omega_i - \sum_j y_j(p)] \cdot q$ on Δ. In essence, this is what we did in the proof of Proposition 17.C.1.

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87. But it is not so for the generalizations described in the previous paragraph.
88. This was the approach taken in Debreu (1959).

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EXERCISES

17.B.1^A Show that for a pure exchange economy with $J = 1$ and $Y_1 = -\mathbb{R}^L_+$, "$y_1^* \leq 0$, $p \cdot y_1^* = 0$, and $p \geq 0$" if and only if "$y_1^* \in Y_1$ and $p \cdot y_1^* \geq p \cdot y_1$ for all $y_1 \in Y_1$."

17.B.2[B] Prove property (v) of Proposition 17.B.2. The proof of Proposition 17.B.2 in the text contains a hint. Recall also the following technical fact: any bounded sequence in $\mathbb{R}^L$ has a convergent subsequence.

17.B.3[B] Suppose that $z(\cdot)$ is an aggregate excess demand function satisfying conditions (i) to (v) of Proposition 17.B.2. Let $p^n \to p$ with some, but not all, of the components of p being zero.

(a) Show that as n becomes large, the maximal excess demand is always obtained for some commodity whose price goes to zero.

(b) Argue (if possible by example) that a commodity whose price goes to zero may actually remain in excess supply for all n. [*Hint*: Relative prices matter.]

17.B.4[B] Suppose that there are J firms whose production sets $Y_1, \ldots, Y_J \subset \mathbb{R}^L$ are closed, strictly convex, and bounded above. Suppose also that a strictly positive consumption bundle is producible using the initial endowments and the economy's aggregate production set $Y = \sum_j Y_j$ (i.e., there is an $\bar{x} \gg 0$ such that $\bar{x} \in \{\sum_i \omega_i\} + Y$). Show that the production inclusive aggregate excess demand function $\tilde{z}(p)$ in (17.B.3) satisfies properties (i) to (v) of Proposition 17.B.2.

17.B.5[A] Suppose that there are J firms. Each firm produces a single output under conditions of constant returns. The unit cost function of firm j is $c_j(p)$, which we assume to be differentiable. The consumption side of the economy is expressed by an aggregate excess demand function $z(p)$. Write down an equation system similar to (17.B.4)–(17.B.5) for the equilibria of this economy.

17.B.6[C] [Rader (1972)] Suppose that there is a single production set Y and that Y is a closed, convex cone satisfying free disposal. Consider the following exchange equilibrium problem. Given prices $p = (p_1, \ldots, p_L)$, every consumer i chooses a vector $v_i \in \mathbb{R}^L$ so as to maximize $\succsim_i$ on the set $\{x_i \in X_i\colon p \cdot v_i \leq p \cdot \omega_i$, and $x_i = v_i + y$ for some $y \in Y\}$. The price vector p and the choices $v^* = (v_1^*, \ldots, v_I^*)$ are in equilibrium if $\sum_i v_i^* = \sum_i \omega_i$. Show that, under the standard assumptions on preferences and consumption sets, the price vector and the individual consumptions constitute a Walrasian equilibrium for the economy with production. Interpret.

17.C.1[A] Verify that the correspondence $f(\cdot)$ introduced in the proof of Proposition 17.C.1 is convex-valued.

17.C.2[C] Show that a convex-valued correspondence $z(\cdot)$ defined on $\mathbb{R}^L_{++}$ and satisfying the conditions (i) to (v) listed below (parallel to the corresponding conditions in Proposition 17.C.1) admits a solution; that is, there is a p with $0 \in z(p)$.

(i) $z(\cdot)$ is upper-hemicontinuous.
(ii) $z(\cdot)$ is homogeneous of degree zero.
(iii) For every p and $z \in z(p)$ we have $p \cdot z = 0$ (Walras' law).
(iv) There is $s \in \mathbb{R}$ such that $z_\ell > -s$ for any $z \in z(p)$ and p.
(v) If $p^n \to p \neq 0$, $z^n \in z(p^n)$ and $p_\ell = 0$ for some ℓ, then $\text{Max}\,\{z_1^n, \ldots, z_L^n\} \to \infty$.

[*Hint*: If you try to replicate exactly the proof of Proposition 17.C.1 you will run into difficulties with the upper-hemicontinuity condition. A possible three-step approach goes as follows: (1) Show that for $\varepsilon > 0$ small enough the solutions must be contained in $\Delta_\varepsilon = \{p \in \Delta\colon p_\ell \geq \varepsilon$ for all $\ell\}$; (2) argue then that for $r > 0$ large enough, one has $z(p) \subset [-r, r]^L$ for every $p \in \Delta_\varepsilon$; finally, (3) carry out a fixed-point argument in the domain $\Delta_\varepsilon \times [-r, r]^L$. For an easier result, you could limit yourself to prove the convex-valued parallel to Proposition 17.C.2. The suggested domain for the fixed-point argument is then $\Delta \times [-r, r]^L$.

17.C.3$^{\mathrm{B}}$ Consider an exchange economy in which every consumer i has continuous, strongly monotone, strictly convex preferences, and $\omega_i \gg 0$. The peculiarity of the equilibrium problem to be considered is that the consumer will now pay a type of tax on his gross consumption; moreover, this tax can differ across commodities and consumers. We will also assume that total tax receipts are rebated equally across consumers and in a lump-sum fashion. Specifically, for every i there is a vector of given tax rates $t_i = (t_{1i}, \ldots, t_{Li}) \geq 0$ and for every price vector $p \gg 0$ the budget set of consumer i is

$$B_i(p, w_i) = \left\{x_i \in \mathbb{R}^L_+ : \sum_\ell (1 + t_{\ell i})p_\ell x_{\ell i} \leq w_i\right\}.$$

An *equilibrium with taxes* is then a price vector $p \gg 0$ and an allocation $(x_1^*, \ldots, x_I^*)$ with $\sum x_i^* = \sum_i \omega_i$ such that every i maximizes preferences in $B_i(p, p\cdot\omega_i + (1/I)(\sum_{\ell i} t_{\ell i} p_\ell x_{\ell i}))$.

(a) Illustrate the notion of an equilibrium with taxes in an Edgeworth box. Verify that an equilibrium with taxes need not be a Pareto optimum.

(b) Apply Proposition 17.C.1 to show that an equilibrium with taxes exists.

(c) As formulated here, the taxes are on gross consumptions. If they were imposed instead on net consumptions, that is, on amounts purchased or sold, then (assuming the same rate for buying or selling) the budget set would be

$$B_i(p, T_i) = \left\{x_i \in \mathbb{R}^L_+ : p\cdot(x_i - \omega_i) + \sum_\ell t_\ell |p_{\ell i}(x_{\ell i} - \omega_{\ell i})| \leq T_i\right\},$$

where the T_i are the lump-sum rebates. In what way does this budget set differ from that described previously for the case of taxes on gross consumptions? Represent graphically. Notice the kinks.

(d) Write down a budget set for the situation similar to **(c)** except that the tax rates for amounts bought or sold may be different.

(e) (More advanced) How would you approach the existence issue for the modification described in **(c)**?

17.C.4$^{\mathrm{A}}$ Consider a pure exchange economy. The only novelty is that a progressive tax system is instituted according to the following rule: individual wealth is no longer $p\cdot\omega_i$; instead, anyone with wealth above the mean of the population must contribute half of the excess over the mean into a fund, and those below the mean receive a contribution from the fund in proportion to their deficiency below the mean.

(a) For a two-consumer society with endowments $\omega_1 = (1, 2)$ and $\omega_2 = (2, 1)$, write the after-tax wealths of the two consumers as a function of prices.

(b) If the consumer preferences are continuous, strictly convex, and strongly monotone, will the excess demand functions satisfy the conditions required for existence in Proposition 17.C.1 given that wealth is being redistributed in this way?

17.C.5$^{\mathrm{B}}$ Consider a population of I consumers. Every consumer i has consumption set $\mathbb{R}^L_+$ and continuous, strictly convex preferences $\succsim_i$. Suppose, in addition, that every i has a household technology $Y_i \subset \mathbb{R}^L$ satisfying $0 \in Y_i$. We can then define the *induced preferences* $\succsim_i^*$ on $\mathbb{R}^L_+$ by $x_i \succsim_i^* x_i'$ if and only if for any $y_i' \in Y_i$ with $x_i' + y_i' \geq 0$ there is $y_i \in Y_i$ with $x_i + y_i \geq 0$ and $x_i + y_i \succsim_i x_i' + y_i'$ (i.e., whatever can be done from x_i', something at least as good can be obtained from x_i).

(a) Show that induced preferences are rational, that is, complete and transitive.

(b) Show that if Y_i is convex then induced preferences $\succsim_i^*$ are convex.

(c) Suppose that goods are of two kinds: marketed goods and nonmarketed household goods. Initial preferences $\succsim_i$ care only about household goods, and initial endowments ω_i have nonzero entries only for marketed goods. Use the concept of induced preferences to set up the equilibrium problem as one that is formally a problem of pure exchange among marketed goods. Discuss.

17.C.6[B] Let $L = 2$. Consider conditions (i), (iii), and (iv) of Proposition 17.B.2. Exhibit four examples such that in each of the examples only one condition fails and yet the system of equations $z(p) = 0$ has no solution. Why is condition (ii) not included in the list?

17.D.1[B] Consider an exchange economy with two commodities and two consumers. Both consumers have homothetic preferences of the constant elasticity variety. Moreover, the elasticity of substitution is the same for both consumers and is small (i.e., goods are close to perfect complements). Specifically,

$$u_1(x_{11}, x_{21}) = (2x_{11}^{\rho} + x_{21}^{\rho})^{1/\rho} \quad \text{and} \quad u_2(x_{12}, x_{22}) = (x_{12}^{\rho} + 2x_{22}^{\rho})^{1/\rho},$$

and $\rho = -4$. The endowments are $\omega_1 = (1, 0)$ and $\omega_2 = (0, 1)$.

Compute the excess demand function of this economy and verify that there are multiple equilibria.

17.D.2[A] Apply the implicit function theorem to show that if $f(v) = 0$ is a system of M equations in N unknowns and if at $\bar{v}$ we have $f(\bar{v}) = 0$ and rank $Df(\bar{v}) = M$, then in a neighborhood of $\bar{v}$ the solution set of $f(\cdot) = 0$ can be parameterized by means of $N - M$ parameters.

17.D.3[A] Carry out explicitly the computations for Proposition 17.D.4.

17.D.4[C] Consider a two-commodity, two-consumer exchange economy satisfying the appropriate differentiability conditions on utility and demand functions. There is a total endowment vector $\bar{\omega} \gg 0$. Show that for almost every $\omega_1 \ll \bar{\omega}$ the economy defined by the initial endowments ω_1 and $\omega_2 = \bar{\omega} - \omega_1$ has a finite number of equilibria. This differs from the situation in Proposition 17.D.2 in that total endowments are kept fixed. [*Hint*: You should use the properties of the Slutsky matrix.]

17.D.5[B] Consider a two-commodity, two-consumer exchange economy satisfying the appropriate differentiability conditions on utility and demand functions. Set the equilibrium problem as an equation system in the consumption variables $x_1 \in \mathbb{R}^2_+$ and $x_2 \in \mathbb{R}^2_+$, the price variables $p \in \mathbb{R}^2_+$, and the reciprocals of the marginal utilities of wealth $\lambda_1 \in \mathbb{R}$ and $\lambda_2 \in \mathbb{R}$ (neglect the possibility of boundary equilibria). The parameters of the system are the initial endowments $(\omega_1, \omega_2) \in \mathbb{R}^4_+$. Prove without further aggregation that (after deleting one equation and one unknown) the system satisfies the full rank condition of the transversality theorem.

17.D.6[B] The setup is identical to Exercise 17.D.5 except that an externality is allowed: The (differentiable) utility function of consumer 1 may depend on the consumption of consumer 2; that is, it has the form $u_1(x_1, x_2)$ where x_i is consumer i's consumption bundle [but we still have $u_2(x_2)$]. Equilibrium is defined as usual, with the proviso that consumer 1 takes consumer 2's consumption as given. Show that, generically on initial endowments $(\omega_1, \omega_2) \in \mathbb{R}^4_+$, the number of equilibria is finite.

17.D.7[B] Suppose the agents of an overall exchange economy are distributed across N islands with no communication among them. Each island economy has three equilibria.

(a) Argue that the number of equilibria in the overall economy is 3^N.

(b) Suppose now that the islands' economies are identical and that there is a possibility of communication across the islands: free and costless transportation of commodities. Show that then the number of equilibria is 3.

17.D.8^A Show by explicit computation that the index of the equilibrium of a one-consumer Cobb–Douglas pure exchange economy is $+1$.

17.E.1^A Derive expressions (17.E.1) and (17.E.2).

17.E.2^A Derive expression (17.E.3).

17.E.3^B Provide explicit utility functions rationalizing at a given price vector p the individual excess demands $z_i(p)$ and matrices of price effects $Dz_i(p)$ constructed in the proof of Proposition 17.E.2.

17.E.4^B Consider the two-commodity case. Give an example of a function $z(p)$ defined on $P_\varepsilon = \{(p_1, p_2) \gg 0: \varepsilon < (p_1/p_2) < (1/\varepsilon)\}$, and with values in $\mathbb{R}^2$, that is continuous, is homogeneous of degree zero, satisfies Walras' law, and cannot be generated from a rational preference relation. Represent graphically the offer curve associated with this function. Note that it goes through the initial endowment point and compare with the construction used in Figure 17.E.2.

17.E.5^A Show that the choices represented in Figure 17.E.3 cannot be generated from consumers with endowment vectors bounded above by $(1, 1)$ and nonnegative consumption.

17.E.6^A Show that the excess demand function $z_i(p) = e^i - p_i p$, defined for $\|p\| = 1$, is proportionally one-to-one in the sense used in the general proof of Proposition 17.E.3 (at the end of Section 17.E).

17.E.7^B Show directly that the excess demand function $z_i(p) = e^i - p_i p$ used in the general proof of Proposition 17.E.3 satisfies the strong axiom of revealed preference.

17.F.1^C Show that expression (17.F.2) gives rise to a negative semidefinite matrix of price effects, $Dz(p)$, if initial endowments are proportional among themselves *or* if consumptions are proportional among themselves.

17.F.2^A Complete the requested verification of Example 17.F.1.

17.F.3^B There are four goods and two consumers. The endowments of the consumers are $\omega_1 = (\omega_{11}, \omega_{21}, 0, 0)$ and $\omega_2 = (\omega_{12}, \omega_{22}, 0, 0)$. Consumer 1 spends all his wealth on good 3 while consumer 2 does the same on good 4. Specify some values of ω_1 and ω_2 for which the corresponding excess demand of this economy does not satisfy the weak axiom of revealed preference.

17.F.4^A Suppose that there are L goods but that for every consumer there is a good such that at any price the consumer spends all his wealth on that good (perhaps goods are distinguished by their location). Show that the aggregate excess demand will satisfy the (weak) gross substitute property.

17.F.5^C Complete the missing steps of Example 17.F.2.

17.F.6^C Consider a two-consumption-good, two-factor model with constant returns and no joint production. In fact, suppose that the production functions for the two consumption goods are Cobb–Douglas. Consumers have holdings of factors and have preferences only for the two consumption goods. The economy is a closed economy (at equilibrium, consumption must equal production). Suppose that the two goods are normal and gross substitutes in the *demand function* of the consumers. Define an induced exchange economy for factors of production by assuming that at any vector of factor prices the two goods are priced at average cost and the final demand for them is met. Show that the resulting aggregate excess demand for factors of production has the gross substitute property and, consequently, that there is a unique equilibrium for the overall economy.

17.F.7$^{\text{A}}$ Prove expression (17.F.3) for $L = 2$.

17.F.8$^{\text{A}}$ Show that expression (17.F.3) implies that the set of solutions to $z(p) = 0$ is convex.

17.F.9$^{\text{B}}$ Consider an economy with a single constant returns production set Y. Preferences are continuous, strictly convex, and strongly monotone. Suppose that the feasible consumptions $(x_1, \ldots, x_I)$ are associated with a Walrasian equilibrium. Assume, moreover, that no trade is required to attain these consumptions if Y is freely available to all consumers; that is $x_i - \omega_i \in Y$ for all i. Show then that those are the only possible equilibrium consumptions.

17.F.10$^{\text{A}}$ Show that expression (17.F.3) implies that $Dz(p)$ is negative semidefinite at an equilibrium p.

17.F.11$^{\text{B}}$ Show that if $z(p) = 0$, rank $Dz(p) = L - 1$, and $Dz(p)$ is negative semidefinite, then, for any ℓ, the $(L - 1) \times (L - 1)$ matrix obtained from $Dz(p)$ by deleting the ℓth row and column has a determinant of sign $(-1)^{L-1}$. [*Hint*: From Section M.D of the Mathematical Appendix you know that rank $Dz(p) = L - 1$ implies that the $(L - 1) \times (L - 1)$ matrix under study is nonsingular. Consider then $Dz(p) - \alpha I$.]

17.F.12$^{\text{B}}$ Show that if $z(p) = 0$ and $Dz(p)$ has the gross substitute sign pattern, then the $(L - 1) \times (L - 1)$ matrix obtained from $Dz(p)$ by deleting the ℓth row and column has a *negative dominant diagonal* (see Section M.D of the Mathematical Appendix for this concept) and is therefore negative definite.

17.F.13$^{\text{A}}$ Provide the missing computation for Example 17.F.3.

17.F.14$^{\text{B}}$ Consider a firm that produces good 1 out of goods $\ell = 2, \ldots, L$ by means of a production function $f(v_2, \ldots, v_L)$. Assume that $f(\cdot)$ is concave, increasing, and twice continuously differentiable. We say that ℓ and ℓ' are complements at the input combination $v = (v_2, \ldots, v_L)$ if $\partial^2 f(v)/\partial v_\ell\, \partial v_{\ell'} > 0$.

(a) Verify that for the Cobb–Douglas production function $f(v_2, \ldots, v_L) = v_2^{\alpha_2} \times \cdots \times v_L^{\alpha_L}$, $\alpha_2 + \cdots + \alpha_L \leq 1$, any two inputs are complements at any v.

(b) Suppose that $f(\cdot)$ is of the constant returns type. Show that at any v and for any ℓ there is an ℓ' that is a complement to ℓ at v.

(c) Suppose now that $f(\cdot)$ is strictly concave and that any two inputs are complements at any v. Let $v_\ell(p_1, \ldots, p_L)$ be the input demand functions. Show that, for any ℓ, $\partial v_\ell/\partial p_1 > 0$, $\partial v_\ell/\partial p_\ell < 0$, and $\partial v_\ell/\partial p_{\ell'} < 0$ for $\ell' \neq \ell$.

(d) Discuss the implications of **(a)** to **(c)** for uniqueness theorems that rely on the gross substitute property.

17.F.15$^{\text{B}}$ Consider a one-consumer economy with production and strictly convex preferences. There is a system of ad valorem taxes $t = (t_1, \ldots, t_L)$ creating a wedge between consumer and producer prices; that is, $p_\ell = (1 + t_\ell)q_\ell$ where p_ℓ and q_ℓ are, respectively, the consumer and producer price for good ℓ. Tax receipts are turned back in lump-sum fashion. Write the definition of (distorted) equilibrium. Show that the equilibrium is unique if the production sector is of the Leontief type (a single primary factor, no joint production, constant returns) and all goods are normal in consumption. Can you argue by example the nondispensability of the last normality condition? If this is simpler, you can limit your discussion to the case of two commodities (one input and one output).

17.F.16$^{\text{C}}$ Suppose that $g(p) = (g_1(p), \ldots, g_N(p))$ is defined in the domain $[0, r]^N$ and that $g(0, \ldots, 0) \gg (0, \ldots, 0)$, $g(r, \ldots, r) \ll (0, \ldots, 0)$. Note that we do not assume Walras' law, homogeneity of degree zero, or, for that matter, continuity. The function $g(\cdot)$ could, for example, be the system of excess demands corresponding to a subgroup of markets with the prices of commodities outside the group kept fixed.

(a) We say that $g(\cdot)$ satisfies the *strong gross substitute property* (SGS) if for some $\alpha > 0$ every coordinate of the function $\alpha g(p) + p$ is strictly increasing in p and $(\alpha g(p) + p) \in [0, r]^N$ for every $p \in [0, r]^N$. Show that if $g(p)$ has the SGS property then it also has the GS property.

(b) Show by example that the GS property does not imply the SGS property. Establish, however, that if $g(\cdot)$ is continuously differentiable and the GS property is satisfied then the SGS property holds.

From now on we assume that $g(\cdot)$ satisfies the SGS property.

(c) Show that there is an equilibrium, that is, a p with $g(p) = 0$. Illustrate graphically for the case $N = 1$. [*Hint*: Quote the Tarski fixed point theorem from Section M.I of the Mathematical Appendix, or, if you prefer, assume continuity and apply Brouwer's fixed point theorem.]

(d) Give an example for $N = 2$ where the equilibrium is not unique.

(e) Suppose that $g(p) = g(p') = 0$. Show that there must be an equilibrium p'' such that $p'' \geq p$ and $p'' \geq p'$. Similarly, there is an equilibrium p''' such that $p''' \leq p$ and $p''' \leq p'$. [*Hint*: Apply the argument in **(c)** to the domain $[\text{Max}\,\{p_1, p_1'\}, r] \times \cdots \times [\text{Max}\,\{p_N, p_N'\}, r]$.]

(f) Argue (you can assume continuity here) that the equilibrium set satisfies a strong and very special property, namely, that it has a maximal and a minimal equilibrium. That is, there are $p^{\max}$ and $p^{\min}$ such that $g(p^{\max}) = g(p^{\min}) = 0$ and $p^{\min} \leq p \leq p^{\max}$ whenever $g(p) = 0$.

(g) Assume now that $g(\cdot)$ is also differentiable. Suppose that we know that at equilibrium, that is, whenever $g(p) = 0$, the matrix $Dg(p)$ has a *negative dominant diagonal*; that is, $Dg(p)v \ll 0$ for a $v \gg 0$. Argue (perhaps nonrigorously) that the equilibrium must then be unique.

(h) Suppose that $g(\cdot)$ is the usual excess demand system for the first N goods of an economy with $N + 1$ goods in which the last price has been fixed to equal 1 *and* the overall $(N + 1)$-good excess demand system satisfies the gross substitute property. Apply **(g)** to show that the equilibrium is unique.

17.F.17$^{\text{A}}$ [Becker (1962), Grandmont (1992)] Suppose that $L = 2$ and you have a continuum of consumers. All consumers have the same initial endowments; they are not rational, however. Given a budget set, they choose at random from consumption bundles on the budget line using a uniform distribution among the nonnegative consumptions. Let $z(p)$ be the average excess demand (=expected value of a single consumer's choice). Show that $z(\cdot)$ can be generated from preference maximization of a Cobb–Douglas utility function (thus the economy admits a positive representative consumer in the sense of Section 4.D).

17.G.1$^{\text{B}}$ Suppose that in an exchange economy (and with the normalization $p_L = 1$) we are given equilibrium prices $p(\hat{\omega}_1)$ as a differentiable function defined as an open domain of the endowments of the first $L - 1$ goods of the first consumer, $\hat{\omega}_1 = (\omega_{11}, \ldots, \omega_{L-1,1})$. All the remaining endowments are kept fixed. Suppose that the demand function of the first consumer is strictly normal in the sense that $D_{w_1} x_1(p, w_1) \gg 0$ through the relevant domain of (p, w_1). Show then that for any $\hat{\bar{\omega}}_1$ and $\bar{p} = p(\hat{\bar{\omega}}_1)$, we have rank $D_{\hat{\omega}_1} \hat{z}_1(\bar{p}; \hat{\bar{\omega}}_1) = L - 1$ and rank $Dp(\hat{\bar{\omega}}_1) = L - 1$, where $\hat{z}_1(p; \hat{\omega}_1)$ is the excess demand function of the first consumer for the first $L - 1$ goods.

17.G.2$^{\text{B}}$ The setting is as in Exercise 17.G.1 or as in Proposition 17.G.2. Suppose that $\hat{z}(\bar{p}; \hat{\bar{\omega}}_1) = 0$. Show that there are economies with $D_p \hat{z}(\bar{p}; \hat{\bar{\omega}}_1)$ an $(L - 1) \times (L - 1)$ negative definite matrix but where $\partial p_1(\hat{\bar{\omega}}_1)/\partial \omega_{11} > 0$. [*Hint*: Use Proposition 17.G.1 and the arguments employed in its proof.]

17.G.3[C] The setting is as in Exercise 17.F.16, except that now we have two functions $g(p) \in \mathbb{R}^N$ and $\hat{g}(p) \in \mathbb{R}^N$. Each of these functions satisfies the conditions of Exercise 17.F.16 (in particular the SDS property). In addition, we assume that $\hat{g}(\cdot)$ is an upward shift of $g(\cdot)$; that is, $\hat{g}(p) \geq g(p)$ for every $p \in [0, r]^N$. Prove that if $(p^{\min}, p^{\max})$ and $(\hat{p}^{\min}, \hat{p}^{\max})$ are the minimal and maximal equilibrium price vectors (see Exercise 17.F.16) for $g(\cdot)$ and $\hat{g}(\cdot)$, respectively, then $\hat{p}^{\min} \geq p^{\min}$ and $\hat{p}^{\max} \geq p^{\max}$. [You can assume that $g(\cdot)$ and $\hat{g}(\cdot)$ are continuous; if this makes things simpler, assume also that both functions have a unique solution.] Represent graphically for the case $N = 1$.

17.H.1[C] Suppose that the system of excess demand functions $z(p)$ satisfies the gross substitute property. Consider the tâtonnement price dynamics

$$\frac{dp_\ell}{dt} = z_\ell(p) \qquad \text{for every } \ell. \tag{*}$$

For any price vector p let $\psi(p) = \text{Max}\,\{z_1(p)/p_1, \ldots, z_L(p)/p_L\}$.

(a) Argue that if $p(t)$ is a solution for the above tâtonnement dynamics (i.e., $dp_\ell(t)/dt = z_\ell(p(t))$ for every ℓ and t) and $z(p(0)) \neq 0$ then $\psi(p(t))$ should be decreasing through time. [*Hint*: If $z_\ell(p(t))/p_\ell(t) = \psi(p(t))$ then $p_\ell(t)/p_{\ell'}(t)$ cannot decrease at t for any ℓ'. Hence, $z_\ell(p(t))$ cannot increase, whereas p_ℓ surely increases.]

(b) Argue that $p(t)$ converges to an equilibrium price as $t \to \infty$. [*Hint*: Recall that for the dynamics (*) Walras' law implies that $\sum_\ell p_\ell^2(t) = \text{constant}$.]

17.H.2[B] There is an output good and a numeraire. The price of the output good is p. The data of our problem are given by two functions: The consumption side of the economy provides an excess demand function $z(p)$ for the output good, and the production side an increasing inverse output supply function $p(z)$. Both functions are differentiable. In addition, their graphs cross at $(1, 1)$, which is the equilibrium we will concentrate on in this exercise.

Given this setting we can define two one-variable dynamics:

(i) In *Walras price dynamics* we assume that at p the price increases or decreases according to the sign of the difference between excess demand and (direct) supply at p.

(ii) *In Marshall quantity dynamics* we assume that at z production increases or decreases according to the sign of the difference between the demand price (i.e., the inverse excess demand) and the supply price (i.e., $p(z)$) at z.

(a) Write the above formally and interpret economically.

(b) Suppose that the technology is nearly of the constant returns type. Show then that around the equilibrium $(1, 1)$ the system is always Walrasian stable but that Marshallian stability depends on the slope of the excess demand function (in what way?).

(c) Write general price and quantity dynamics where prices move à la Walras and quantities à la Marshall. Draw a (p, z) phase diagram and argue that in the typical case dynamic trajectories will spiral around the equilibrium.

(d) Go back to the technology specification of **(b)**. Show that the system in **(c)** is locally stable if and only if the equilibrium is Marshallian stable.

(e) Consider the simplest price and quantity dynamics in the limit case where there are constant returns and excess demand is also a constant function. Draw the phase diagram. Suppose now that the quantity dynamics is modified by making the quantity responses depend not only on price and cost but also on the "expectation of sales," that is, on the excess demand. Will this have a stabilizing or a destabilizing effect?

17.H.3[A] For $L = 3$ draw an example similar to Figure 17.H.2 but in which there is a single equilibrium that, moreover, is locally totally unstable. Could you make it a saddle?

17.I.1$^{\text{A}}$ Argue that the replica procedure described at the beginning of Section 17.I does effectively include the case where the numbers of consumers of different types are not the same (assume, for simplicity, that the proportions of the different types are rational numbers). [*Hint*: Redefine the size of the original economy.]

17.I.2$^{\text{A}}$ Consider for a one-input, one-output problem the production function $q = v^2$, where v is the amount of input. Show that the corresponding production set Y is additive but that the smallest cone containing it, Y^*, is not closed. Discuss in what sense the nonconvexity in Y is large. Argue that, whatever the number of consumers, there is no useful sense in which an equilibrium (nearly) exists.

17.I.3$^{\text{B}}$ There are three commodities: the first is a high-quality good, the second is a low-quality good, and the third is labor. The first and second goods can be produced from labor according to the production functions $f_1(v) = \text{Min}\,\{v, 1\}$ and $f_2(v) = \text{Min}\,\{v^\beta, 1\}$ for $0 < \beta < 1$. The economy has one unit of labor in the aggregate. Labor has no utility value. There are two equally sized classes of agents, with a very large number of each. "Rich" and "poor" have identical endowments, but the rich own all the shares in the firms of the economy. The rich spend all their wealth on the high-quality good; the poor must buy either one quality or the other—they cannot buy both. The utility function of the poor is $u(x_1, x_2) = x_1 + \frac{1}{2}x_2$, defined for (x_1, x_2) not both positive.

(a) Which standard hypothesis of the general model does this economy fail to satisfy?

(b) Show that there can be no equilibria other than one in which both qualities of product are produced.

(c) Show that an equilibrium exists.

17.AA.1$^{\text{A}}$ Consider an exchange economy in which the preferences of consumers are monotone, strictly convex, and represented by the utility functions $(u_1(\cdot), \dots, u_I(\cdot))$. Show that for any $(s_1, \dots, s_I) \gg 0$ there can be at most one Pareto optimal allocation $x = (x_1, \dots, x_I)$ such that $(u_1(x_i), \dots, u_I(x_I))$ is proportional to $(s_1, \dots, s_I)$.

17.AA.2$^{\text{B}}$ Consider the welfare-theoretic approach to the equilibrium equations described in Appendix A (the Negishi approach). The existence of a solution to the system of equations $g(s) = 0$ defined there follows from a fixed-point argument similar to the one carried out in Proposition 17.C.2. Assume that you are in an exchange economy with continuous, strictly convex and strongly monotone preferences, and that $\omega_i \gg 0$ for every i. Assume also that $g(s)$ turns out to be a function rather than a correspondence (a sufficient condition for this is that preferences be representable by differentiable utility functions and that at every Pareto optimal allocation at least one consumer gets a strictly positive consumption of every good).

(a) Show that $g(s)$ is continuous.

(b) Show that $g(s)$ satisfies a sort of Walras' law: "$\sum_i g_i(s) = 0$, for every s."

(c) Show that if $s_i = 0$ then $g_i(s) > 0$. [*Hint*: If $s_i = 0$ then $u_i(x_i(s)) = 0$ and so $p(s)\cdot x_i(s) = 0$.]

(d) Complete the existence proof. (Note that $g(s)$ is also defined for s with zero components. This makes matters simpler.)

17.AA.3$^{\text{B}}$ Suppose that, in an exchange economy, consumption sets are $\mathbb{R}^L_+$ and preferences are representable by concave, increasing utility functions $u_i(\cdot)$. Let $\Delta = \{\lambda \in \mathbb{R}^L_+ : \sum_i \lambda_i = 1\}$ be a simplex of utility weights. Suggest an equation system for Walrasian equilibrium that proceeds by associating with every λ a linear social welfare function.

17.BB.1$^{\text{A}}$ Give a graphical example (for $L = 2$) of a Walrasian quasiequilibrium with strictly positive prices that is not an equilibrium for an economy in which:

(i) For every j, $Y_j = -\mathbb{R}^L_+$.
(ii) For every i, X_i is nonempty, closed, convex and satisfies $X_i + \mathbb{R}^L_+ \subset X_i$.
(iii) For every i, preferences are continuous, convex, and strongly monotone.
(iv) For every i, $\omega_i \in X_i$.

Why does this example not contradict any result given in the text (see the small-type discussion after the proof of Proposition 17.BB.1)?

17.BB.2[B] Consider an economy in which every consumer desires only a subset of goods and has holdings of only some goods. For the commodities desired, however, the preferences of the consumer are strongly monotone (they are also continuous) on the corresponding nonnegative orthant. Suppose in addition that $\sum_i \omega_i \gg 0$ and that the economy satisfies the following *indecomposability* condition:

> It is not possible to divide consumers into two (nonempty) groups so that the consumers of one of the groups do not desire any of the commodities owned by the consumers of the other group.

Show then that any Walrasian quasiequilibrium is an equilibrium.

17.BB.3[C] Consider an Edgeworth box where preferences are continuous, strictly convex and locally nonsatiated (but not necessarily monotone). Suppose also that free disposal of commodities is not possible. Argue that, nonetheless, the offer curves must cross and, therefore, that an equilibrium exists. Show that at equilibrium the two prices cannot be negative. In fact, at least one price must be positive (this is harder to show).

17.BB.4[A] Prove that if (x^*, y^*, p) is a free-disposal quasiequilibrium and Y_1 satisfies free disposal, then we can get a true quasiequilibrium by changing only the production of firm 1.

17.BB.5[A] Provide the missing step in the proof of Lemma 17.BB.5 (that is, show that the convexity of preferences implies that $x_i \succ_i x_{i\alpha}$ and $x_i' \succ_i x_{i\alpha}$ cannot both occur for $x_{i\alpha} = \alpha x_i + (1-\alpha)x_i'$).

17.BB.6[A] Complete the proof of Lemma 17.BB.5 by verifying the convexity of $\tilde{y}_j(x, y, p)$ and of $\tilde{p}(x, y, p)$.

17.BB.7[A] Complete the proof of Lemma 17.BB.6 by verifying the upper hemicontinuity of the correspondences $\tilde{y}_j(\cdot)$ and $\tilde{p}(\cdot)$.

17.BB.8[B] [Existence with production externalities; see Chipman (1970) for more on this topic.] There are L goods. Good L is labor and it is the single factor of production. Consumers have consumption set $\mathbb{R}^L_+$, continuous, strongly monotone, and strictly convex preferences, and endowments only of labor. Good $\ell = 1, \ldots, L-1$ is produced in sector ℓ, which is composed of J_ℓ identical firms. The production function of a firm in sector ℓ is $f_\ell(v_\ell) = \alpha_\ell v_\ell^{\beta_\ell}$ for $0 < \beta_\ell \le 1$. The peculiarity of the model is that the productivity coefficient α_ℓ will not be a constant but will depend on the aggregate use of labor in sector ℓ. Precisely,

$$\alpha_\ell = \gamma_\ell \left(\sum_j v_{\ell j} \right)^{\rho_\ell}, \qquad \gamma_\ell > 0 \text{ and } \rho_\ell \ge 0.$$

(a) Define the notion of Walrasian equilibrium. Assume in doing so that individual firms neglect the effect on α_ℓ of their use of labor. To save on notation, suppose also that profit shares are equal across consumers.

(b) Prove the existence of a Walrasian equilibrium for the current model (make the standard additional assumptions that you find necessary). [*Hint*: The general proof of Appendix B needs very few adaptations.]

(c) Derive and represent the aggregate production set of each sector. Which conditions on the parameters β_ℓ, γ_ℓ, ρ_ℓ guarantee that the aggregate production set of sector ℓ exhibits increasing, constant, or decreasing returns to scale?

(d) Note that the existence conditions of (b) may be satisfied while the aggregate production set is not convex. What would happen if the externality of sector ℓ were internalized by putting all the firms of the sector under joint management?

(e) Suppose that $L = 2$, $\beta_\ell = 1$ and individual preferences are quasilinear in labor; that is, they admit a utility function $u_i(x_{1i}) + x_{2i}$. Discuss, both analytically and graphically, the bias of the equilibrium level of production relative to the social optimum.

17.BB.9^B Carry out the existence argument for the two-player-game approach described at the end of Appendix B.

18 Some Foundations for Competitive Equilibria

18.A Introduction

Up to this point of Part IV, the existence of markets in which prices are quoted and taken as given by economic agents has been assumed. In this chapter, we discuss four topics that, in essence, have two features in common: The first is that they all try to single out and characterize the Walrasian allocations from considerations more basic than those stated in its definition. The second is that they all emphasize the role of a large number of traders in accomplishing this task.

In Section 18.B we introduce the concept of the *core*, which can be viewed as embodying a notion of unrestricted competition. We then present the important *core equivalence theorem.*

Section 18.C examines a more restricted concept of competition: that taking place through well-specified trading mechanisms. The analysis of this section amounts to a reexamination in the general equilibrium context of the models of noncooperative competition that were presented in Section 12.F.

The motivation of the remaining two sections is more normative. In Section 18.D we show how informational limitations on the part of a policy authority (constrained to use policy tools relying on *self-selection*, or *envy freeness*) may make the Walrasian allocations the only implementable Pareto optimal allocations.

In Section 18.E the objective is to characterize the Walrasian allocations, among the Pareto optimal ones, in terms of their distributional properties. In particular, we ask to what extent it can be asserted that at the Walrasian allocation everyone gets her "marginal contribution" to the collective economic well-being of society.

A number of the ideas of this chapter (especially those related to the core, but also some in Section 18.E) have come to economics from the cooperative theory of games. This therefore seems a good place to present a brief introduction to this theory; we do it in Appendix A.

18.B Core and Equilibria

The theory to be reviewed in this section was proposed by Edgeworth (1881). His aim was to explain how the presence of many interacting competitors would lead to the emergence of a system of prices taken as given by economic agents, and consequently to a Walrasian equilibrium outcome. Edgeworth's work had no immediate impact. The modern versions of his theory follow the rediscovery of his solution concept (known now as the *core*) in the theory of cooperative games. Appendix A contains a brief introduction to the theory of cooperative games; this section, however, is self-contained. For further, and very accessible, reading on the material of this section, we refer to Hildenbrand and Kirman (1988).

The theory of the core is distinguished by its parsimony. Its conceptual apparatus does not appeal to any specific trading mechanism nor does it assume any particular institutional setup. Informally, the notion of competition that the theory explores is one in which traders are well informed of the characteristics (endowments and preferences) of other traders, and in which the members of any group of traders can bind themselves to any mutually advantageous agreement. The simplest example is a buyer and a seller exchanging a good for money, but we can also have more complex arrangements involving many individuals and goods.

Formally, we consider an economy with I consumers. Every consumer i has consumption set $\mathbb{R}^L_+$, and endowment vector $\omega_i \geq 0$, and a continuous, strictly convex, strongly monotone preference relation $\succsim_i$. There is also a publicly available constant returns convex technology $Y \subset \mathbb{R}^L$.[1] For example, we could have $Y = -\mathbb{R}^L_+$, that is, a pure exchange economy. All of these assumptions are maintained for the rest of the section.

As usual, we say that an allocation $x = (x_1, \dots, x_I) \in \mathbb{R}^{LI}_+$ is *feasible* if $\sum_i x_i = y + \sum_i \omega_i$ for some $y \in Y$.

With a slight abuse of notation, we let the symbol I stand for both the number of consumers and the *set* of consumers. Any nonempty subset of consumers $S \subset I$ is then called a *coalition*. Central to the concept of the core is the identification of circumstances under which a coalition of consumers can reach an agreement that makes every member of the coalition better off. Definition 18.B.1 provides a formal statement of these circumstances.

Definition 18.B.1: A coalition $S \subset I$ *improves upon*, or ***blocks***, the feasible allocation $x^* = (x_1^*, \dots, x_I^*) \in \mathbb{R}^{LI}_+$ if for every $i \in S$ we can find a consumption $x_i \geq 0$ with the properties:

(i) $x_i \succ_i x_i^*$ for every $i \in S$.

(ii) $\sum_{i \in S} x_i \in Y + \{\sum_{i \in S} \omega_i\}$.

Definition 18.B.1 says that a coalition S can improve upon a feasible allocation x^* if there is some way that, by using *only* their endowments $\sum_{i \in S} \omega_i$ and the publicly available technology Y, the coalition can produce an aggregate commodity bundle that can then be distributed to the members of S so as to make each of them better off.

1. The constant returns assumption is important. With general production sets the difficulty is that we cannot avoid being explicit about ownership shares. However, these have been defined to be *profit* shares, which makes our conceptual apparatus dependent on the very notion of prices whose emergence we are currently trying to explain. Thus we stick here to the case of constant returns. This is not a serious restriction: recall from Section 5.B (Proposition 5.B.2) that it is always possible to reduce general technologies to the constant returns case by reinterpreting the ownership shares as endowments of an additional "managerial" input.

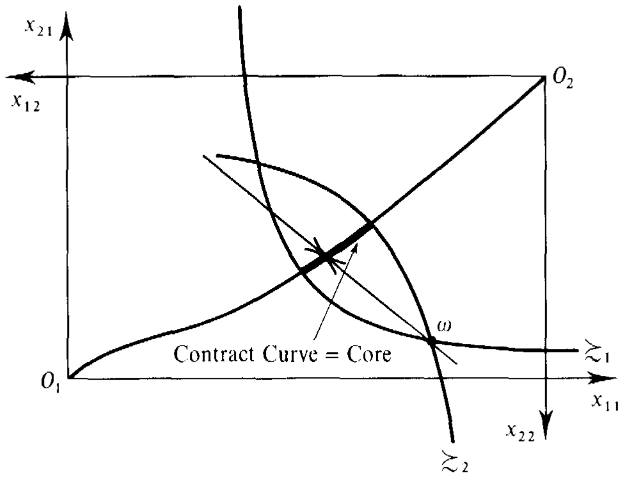

Figure 18.B.1
The core equals the contract curve in the two-consumer case.

Definition 18.B.2: We say that the feasible allocation $x^* = (x_1^*, \ldots, x_I^*) \in \mathbb{R}_+^{LI}$ has the *core property* if there is no coalition of consumers $S \subset I$ that can improve upon x^*. The *core* is the set of allocations that have the core property.

We can see in the Edgeworth box of Figure 18.B.1 that for the case of two consumers the core coincides with the *contract* curve. With two consumers there are only three possible coalitions: $\{1, 2\}$, $\{1\}$, and $\{2\}$. Any allocation that is not a Pareto optimum will be blocked by coalition $\{1, 2\}$.[2] Any allocation in the Pareto set that is not in the contract curve will be blocked by either $\{1\}$ or $\{2\}$. With more than two consumers there are other potential blocking coalitions, but the fact that the coalition of the whole is always one of them means that *all allocations in the core are Pareto optimal.*

We also observe in Figure 18.B.1 that the Walrasian equilibrium allocations, which belong to the contract curve, have the core property. Proposition 18.B.1 tells us that this is true with complete generality. The proposition amounts to an extension of the first welfare theorem. Indeed, in the current terminology, the first welfare theorem simply says that a Walrasian equilibrium cannot be blocked by the coalition of the whole.[3] The following result, Proposition 18.B.1, shows that it also cannot be blocked by any other coalition.

Proposition 18.B.1: Any Walrasian equilibrium allocation has the core property.

Proof: We simply duplicate the proof of the first welfare theorem (Proposition 16.C.1). We present it for the exchange case. See Exercise 18.B.1 for the case of a general constant returns technology.

Let $x^* = (x_1^*, \ldots, x_I^*)$ be a Walrasian allocation with corresponding equilibrium

2. With continuity and strong monotonicity of preferences, if a feasible allocation is Pareto dominated, then it is Pareto dominated by a feasible allocation that strictly improves the utility of *every* consumer. To accomplish this we simply transfer a very small amount of any good from the consumer that is made better off to every other consumer. If the amount transferred is sufficiently small then, by the continuity of preferences, the transferring consumer is still better off, while, by strong monotonicity, every other consumer is made strictly better off.

3. Keep in mind the point made in footnote 2.

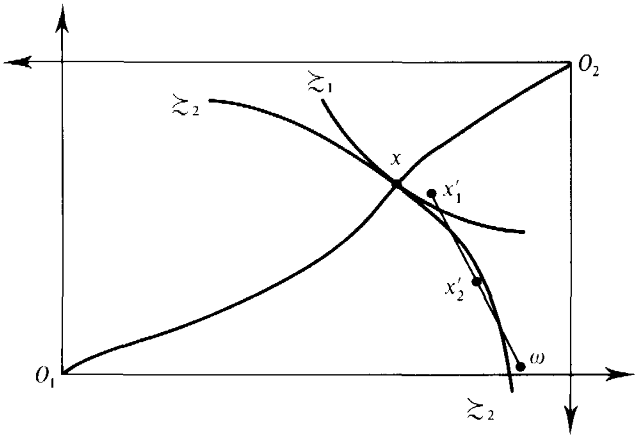

Figure 18.B.2
An allocation in the contract curve that can be blocked with two replicas.

price vector $p \geq 0$. Consider an arbitrary coalition $S \subset I$ and suppose that the consumptions $\{x_i\}_{i \in S}$ are such that $x_i \succ_i x_i^*$ for every $i \in S$. Then $p \cdot x_i > p \cdot \omega_i$ for every $i \in S$ and therefore $p \cdot (\sum_{i \in S} x_i) > p \cdot (\sum_{i \in S} \omega_i)$. But then $\sum_{i \in S} x_i \leq \sum_{i \in S} \omega_i$ cannot hold and so condition (ii) of Definition 18.B.1 is not satisfied (recall that we are in the pure exchange case). Hence coalition S cannot block the allocation x^*. ■

The converse of Proposition 18.B.1 is, of course, not true. In the two-consumer economy of Figure 18.B.1 every allocation in the contract curve is in the core, but only one is a Walrasian allocation. The core equivalence theorem, of which we will soon give a version, argues that the converse *does* hold (approximately) if consumers are numerous. Quite remarkably, it turns out that as we increase the size of the economy the non-Walrasian allocations gradually drop from the core until, in the limit, only the Walrasian allocations are left. The basic intuition for this result can perhaps be grasped by examining the Edgeworth box in Figure 18.B.2. Take an allocation such as x where consumer 1 receives a very desirable consumption within the contract curve. Consumer 2 cannot do anything about this: She could not end up better by going alone. But suppose now that the preferences and endowments in the figure represent not individual consumers but *types* of consumers and that the economy is actually composed of four consumers, two of each type. Consider again the allocation x, interpreted now as a symmetric allocation, that is, with each consumer of type 1 receiving x_1 and each consumer of type 2 receiving x_2. Then matters are quite different because a new possibility arises: The two members of type 2 can form a coalition with *one* member of type 1. In Figure 18.B.2, we see that the allocation x can indeed be blocked by giving x'_1 to the one consumer of type 1 in the coalition and x'_2 to the two consumers of type 2 [note that $-2(x'_2 - \omega_2) = (x'_1 - \omega_1)$].[4]

4. Observe that all this has the flavor of Bertrand competition, as reviewed in Section 12.C. Indeed, we can look at what happens with this three-member coalition as the following: One of the consumers of type 1 bids away the transactions of the consumers of type 2 with the other consumer of type 1. Although this is a topic we shall not get into, we remark that, in fact, there are strong parallels between Bertrand price competition and core competition. Note, in particular, that core competition is as shortsighted as Bertrand competition. By undercutting the other consumer of her type, the consumer of type 1 is only initiating a process of blocking and counterblocking (mutual underbidding in the Bertrand setting) that eventually leads to a result (perhaps the Walrasian allocation) where she will be worse off than at the initial position.

The ability to do this depends, of course, on the way we have drawn the indifference curves. Nonetheless, as we will see, we are always able to form a blocking coalition of this sort if we have sufficiently many consumers of each type.

The version of the core equivalence theorem that we will present is in essence the original of Edgeworth, as generalized by Debreu and Scarf (1963). It builds on the intuition we have just discussed.

To begin, let the set $H = \{1, \ldots, H\}$ stand for a set of *types* of consumers, with each type h having preferences $\succsim_h$ and endowments ω_h. For every integer $n > 0$, we then define the N-*replica* economy as an economy composed of N consumers of each type, for a total number of consumers $I_N = NH$.

We refer to the allocations in which consumers of the same type get the same consumption bundles as *equal-treatment allocations*. Proposition 18.B.2 shows that any allocation in the core must be an equal-treatment allocation. (We hasten to add that this is true for the current replica structure, where there are equal numbers of consumers of each type. It does not hold in general; see Exercise 18.B.2.)

Proposition 18.B.2: Denoting by hn the nth individual of type h, suppose that the allocation

$$x^* = (x_{11}^*, \ldots, x_{1n}^*, \ldots, x_{1N}^*, \ldots, x_{H1}^*, \ldots, x_{Hn}^*, \ldots, x_{HN}^*) \in \mathbb{R}_+^{LHN}$$

belongs to the core of the N-replica economy. Then x^* has the *equal-treatment property*, that is, all consumers of the same type get the same consumption bundle:

$$x_{hm}^* = x_{hn}^* \qquad \text{for all } 1 \le m,\, n \le N \text{ and } 1 \le h \le H.$$

Proof: Suppose that the feasible allocation $x = (x_{11}, \ldots, x_{HN}) \in \mathbb{R}_+^{LHN}$ does not have the equal-treatment property because, say, $x_{1m} \neq x_{1n}$ for some $m \neq n$. We show that x does not have the core property. In particular, we claim that x can be improved upon by any coalition of H members formed by choosing from every type a worst-treated individual among the consumers of that type. Suppose without loss of generality that, for every h, consumer $h1$ is one such worse-off individual, that is, $x_{hn} \succsim_h x_{h1}$ for all h and n. Define now the average consumption for each type: $\hat{x}_h = (1/N) \sum_n x_{hn}$. By the strict convexity of preferences we have (recall that consumers of type 1 are not treated identically)

$$\hat{x}_h \succsim_h x_{h1} \quad \text{for all } h \qquad \text{and} \qquad \hat{x}_1 \succ_1 x_{11}. \tag{18.B.1}$$

We claim that the coalition $S = \{11, \ldots, h1, \ldots, H1\}$, formed by H members, can attain by itself the consumptions $(\hat{x}_1, \ldots, \hat{x}_H) \in \mathbb{R}_+^{LH}$. Therefore, by (18.B.1), the original nonequal-treatment allocation can be blocked by S.[5] To check the feasibility of $(\hat{x}_1, \ldots, \hat{x}_H) \in \mathbb{R}_+^{LH}$ for S, note that, because of the feasibility of $x = (x_{11}, \ldots, x_{HN}) \in \mathbb{R}_+^{LHN}$, there is $y \in Y$ such that $\sum_h \sum_n x_{hn} = y + N(\sum_h \omega_h)$, and therefore

$$\sum_h \hat{x}_h = \frac{1}{N} \sum_h \left(\sum_n x_{hn} \right) = \frac{1}{N} y + \sum_h \omega_h.$$

5. Recall that preferences are strongly monotone and continuous, so that if S can achieve an allocation that does strictly better than x^* for some of its members, and at least as well as x^* for all of them, then it can also achieve an allocation that does strictly better for all of its members.

But by the constant returns assumption on Y, $(1/N)y \in Y$ and so we conclude that $(\hat{x}_1, \ldots, \hat{x}_H) \in \mathbb{R}^{LH}_+$ is feasible for coalition S. ■

Proposition 18.B.2 allows us to regard the core allocations as vectors of fixed size LH, irrespective of the replica that we are concerned with. As a matter of terminology, we call a vector $(x_1, \ldots, x_H) \in \mathbb{R}^{LH}_+$ a *type allocation* and, for any replica N, interpret it as the equal-treatment allocation to consumers where each consumer of type h gets x_h. A type allocation $(x_1, \ldots, x_H) \in \mathbb{R}^{LH}_+$ is feasible if $\sum_h x_h = y + \sum_h \omega_h$ for some $y \in Y$. Note that for any replica N the corresponding equal-treatment allocation is feasible because

$$\sum_h N x_h = N y + N\left(\sum_h \omega_h\right)$$

and $Ny \in Y$ by the constant returns assumption on Y.

By Proposition 18.B.2 the core allocations of a replica economy can be viewed as feasible type allocations. Define by $C_N \subset \mathbb{R}^{LH}_+$ the set of feasible type allocations for which the equal-treatment allocations induced in the N-replica have the core property. Note that C_N does depend on N. Nonetheless, we always have $C_{N+1} \subset C_N$ because a type allocation blocked in the N-replica will be blocked also in the $(N + 1)$-replica by a coalition having exactly the same composition as the one that blocked in the N-replica. Thus, as a subset of $\mathbb{R}^{LH}$ the core can only get smaller when $N \to \infty$. At the same time, we know from Proposition 18.B.1 that the core cannot vanish because the Walrasian equilibrium allocations belong to C_N for all N. More precisely, the set of Walrasian type allocations is independent of N (see Exercise 18.B.3) and contained in all C_N. The core equivalence theorem (which, in the current replica context, is the formal term for the combination of Propositions 18.B.1, 18.B.2 and the forthcoming Proposition 18.B.3) asserts that the Walrasian equilibrium allocations are the only surviving allocations in the core when $N \to \infty$.

Proposition 18.B.3: If the feasible type allocation $x^* = (x_1^*, \ldots, x_H^*) \in \mathbb{R}^{LH}_4$ has the core property for all $N = 1, 2, \ldots$, that is, $x^* \in C_N$ for all N, then x^* is a Walrasian equilibrium allocation.

Proof: To make the proof as intuitive as possible we restrict ourselves to a special case: a pure exchange economy in which, for every h, $\succsim_h$ admits a continuously differentiable utility representation $u_h(\cdot)$ [with $\nabla u_h(x_h) \gg 0$ for all x_h]. In addition, the initial endowments vector ω_h is preferred to any consumption x_h that is not strictly positive. This guarantees that any core allocation is interior. We emphasize that these simplifying assumptions are not required for the validity of the result.

Suppose that $x = (x_1, \ldots, x_H) \in \mathbb{R}^{LH}$ is a feasible type allocation that is not a Walrasian equilibrium allocation. Our aim is to show that if N is large enough then x can be blocked.

We may as well assume that x is Pareto optimal (otherwise the coalition of the whole blocks and we are done) and that $x_h \gg 0$ (otherwise a consumer of type h alone could block). Because of Pareto optimality we can apply the second welfare theorem (Proposition 16.D.1) and conclude that x is a price equilibrium with transfers with respect to some $p = (p_1, \ldots, p_L)$. If x is not Walrasian then there must be some h, say $h = 1$, with $p \cdot (x_1 - \omega_1) > 0$. Informally, type 1 receives a positive net transfer from the rest of the economy and is thus relatively favored (interpretatively, think of type 1 as the most favored). We shall show that, as long as N is large enough, it would pay for the members of all the other types in the economy to form a coalition with $N - 1$ consumers of type 1 (i.e., to throw out one consumer of type 1).

More precisely, if a member of type 1 is eliminated then to attain feasibility the rest of the economy must absorb her net trade $x_1 - \omega_1$. That, of course, presents no difficulty for the positive entries (those commodities for which the rest of the economy is a net contributor to this consumer of type 1), but it is not so simple for the negative ones (the commodities where the rest of the economy is the net beneficiary). The most straightforward methodology is to simply distribute the gains and losses equally. In summary, our coalition is formed by $(N - 1) + N(H - 1)$ members and, for every type h, every member of type h gets

$$x'_h = x_h + \frac{1}{(N-1) + N(H-1)}(x_1 - \omega_1).$$

Note that

$$\begin{aligned}(N-1)x'_1 + Nx'_2 + \cdots + Nx'_H &= (N-1)x_1 + Nx_2 + \cdots + Nx_H + (x_1 - \omega_1)\\ &= N\omega_1 + \cdots + N\omega_H - x_1 + x_1 - \omega_1\\ &= (N-1)\omega_1 + N\omega_2 + \cdots + N\omega_I.\end{aligned}$$

Hence, the proposed consumptions are feasible for the proposed coalition. Note also that the consumptions are nonnegative if N is large enough. For every h, every consumer of type h in the coalition moves from x_h to x'_h. Is this an improvement or a loss? The answer is that if N is large enough then it is an unambiguous gain. To see this, observe that $p\cdot(x_1 - \omega_1) > 0$ implies $\nabla u_h(x_h)\cdot(x_1 - \omega_1) > 0$ for every h because p and $\nabla u_h(x_h)$ are proportional. As we can then see in Figure 18.B.3 (or, analytically, from Taylor's formula; see Exercise 18.B.4) there is $\bar{\alpha} > 0$ with the property that, for every h, $u_h(x_h + \alpha(x_1 - \omega_1)) > u_h(x_h)$ whenever $0 < \alpha < \bar{\alpha}$. Hence, for any N with $(1/[(N-1) + N(H-1)]) < \bar{\alpha}$ the coalition will actually be blocking.

Intuitively, we have done the following. The coalition needs to absorb $x_1 - \omega_1$. Evaluated at the marginal shadow prices of the economy, this is a favorable "project" for the coalition since $p\cdot(x_1 - \omega_1) > 0$. If the coalition is numerous then we can make sure that every member will have to absorb only a very small piece of the project. Hence the individual portions of the project will all be "at the margin"

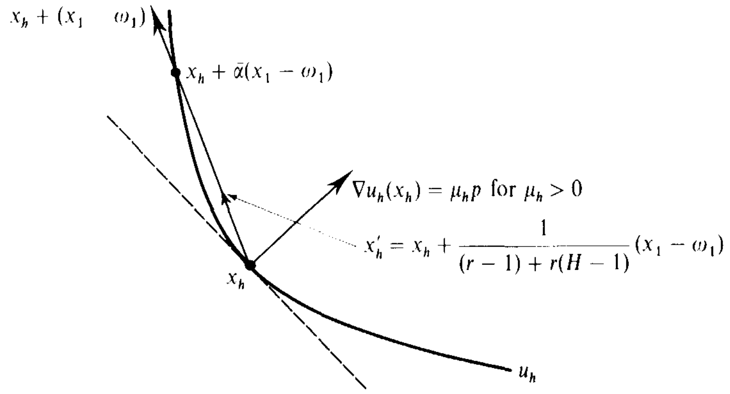

Figure 18.B.3 The consumption change of a consumer of type h in the blocking coalition.

and, therefore, will also be individually favorable (recall Section 3.I for similar arguments).[6] ■

We saw in Proposition 18.B.1 that the half of the core equivalence theorem that asserts that Walrasian allocations have the core property generalizes the first welfare theorem. In its essence, the half asserting that, provided the economy is large, core allocations are Walrasian constitutes a version of the second welfare theorem. To understand this it may be useful to go back to the general (nonreplica) setup and formulate the property of a core allocation being Walrasian in terms of the existence of a price support for a certain set. For simplicity, we restrict ourselves to the pure exchange case.

Given a core allocation $x = (x_1^*, \ldots, x_I^*) \in \mathbb{R}_+^{LI}$ then, in analogy with the construction used in the proof of the second welfare theorem (Proposition 16.D.1) we can define the setsy

$$V_i = \{x_i : x_i \succ_i x_i^*\} \cup \{\omega_i\} \subset \mathbb{R}^L$$

$$V = \sum_{i \in I} V_i \subset \mathbb{R}^L$$

We have $\sum_i \omega_i \in V$. But there is more: *the core property for x^* implies that $\sum_i \omega_i$ belongs to the boundary of V.* To see this, note that if $\sum_i \omega_i$ is in the interior of V then there is $z \in V$ such that $z \ll \sum_i \omega_i$; that is, there is $x' = (x'_1, \ldots, x'_I)$ with $x'_i \in V_i$ for every i and $\sum_i x'_i = z \ll \sum_i \omega_i$. Hence, x' is feasible, $x' \neq (\omega_1, \ldots, \omega_I)$, and, for every i, either $x'_i \succ_i x_i^*$ or $x'_i = \omega_i$. It follows that the set of consumers $S = \{i : x'_i \neq \omega_i\}$ is nonempty, that $x'_i \succ_i x_i^*$ for every $i \in S$, and that

$$\sum_{i \in S} x'_i \ll \sum_{i \in I} \omega_i - \sum_{i \notin S} x'_i = \sum_{i \in I} \omega_i - \sum_{i \notin S} \omega_i = \sum_{i \in S} \omega_i.$$

Thus S is a blocking coalition.

The next claim is that if $p = (p_1, \ldots, p_L) \neq 0$ supports V at $\sum_i \omega_i$, that is, $p \cdot z \geq p \cdot (\sum_i \omega_i)$ for all $z \in V$, then p must be a Walrasian price vector for $x^* = (x_1^*, \ldots, x_I^*)$. To verify this, note first that, for every i, we have $x'_i \succ_i x_i^*$ for some x'_i arbitrarily close to x_i^*. Therefore, $x'_i + \sum_{k \neq i} \omega_k \in V$ and so $p \cdot (x'_i + \sum_{k \neq i} \omega_k) \geq p \cdot (\omega_i + \sum_{k \neq i} \omega_k)$. Going to the limit (i.e., letting $x'_i \to x_i^*$), this yields $p \cdot x_i^* \geq p \cdot \omega_i$ for all i. Because $\sum_i x_i^* \leq \sum_i \omega_i$, we must therefore have $p \cdot x_i^* = p \cdot \omega_i$ for all i. In addition, whenever $x'_i \succ_i x_i^*$ we have $p \cdot (x'_i + \sum_{k \neq i} \omega_k) \geq p \cdot (\omega_i + \sum_{k \neq i} \omega_k)$ and so $p \cdot x'_i \geq p \cdot \omega_i$. If we exploit the continuity and strong monotonicity of preferences as we did in Section 16.D (or in Appendix B of Chapter 17), we can strengthen the last conclusion to $p \cdot x'_i > p \cdot \omega_i$.

The key difference from the case of the second welfare theorem (studied in Section 16.D) is that *$V \subset \mathbb{R}^L$ does not need to be convex* and that therefore a nonzero $p \in \mathbb{R}_+^L$ supporting V at $\sum_i \omega_i$ may not exist. The reason for the lack of convexity is that the individual sets $V_i \subset \mathbb{R}^L$ need not be convex: V_i is the union of the preferred set at x_i^*, which is convex, and the initial endowment vector ω_i, which will typically be outside this preferred set and therefore disconnected from it. However, *if the (possibly nonconvex) sets $V_i \subset \mathbb{R}^L$ being added are numerous, then the sum $\sum_i V_i \subset \mathbb{R}^L$ is "almost" convex.* Thus, the existence of (almost) supporting prices for core allocations can be seen as yet another instance of the convexifying effects of aggregation.

We end by mentioning an elegant approach to core theory pioneered by Aumann (1964) and Vind (1964). It consists of looking at a model where there is an actual continuum of consumers and where we replace all the summations by integrals. The beauty of the approach is that all the approximate results then hold exactly. The core equivalence theorem, for example,

6. See Anderson (1978) for a different line of proof that makes minimal assumptions on the economy.

takes the form: An allocation belongs to the core if and only if it is a Walrasian equilibrium allocation.

18.C Noncooperative Foundations of Walrasian Equilibria

The idea of competition that underlies the theory of the core is very unstructured; there are no trading institutions and, in principle, any conceivable profitable opportunity can be taken advantage of. It is because of this, for example, that core allocations are guaranteed to be Pareto optimal.

In many economic applications, however, the structure of competition is given. Trade takes place through some type of market mechanism making explicit use of prices. The set of instruments and the information available to competitors are then limited. Yet, we also expect that price taking will emerge if individual competitors are small relative to the size of the market. We have already investigated this topic in Section 12.F. We reexamine it here because there are a few general equilibrium qualifications worth taking into account.

There are many models of price-mediated competition arising in applications. We will describe three of them, but before doing so, we present an abstract treatment emphasizing the main issues.

Suppose there are I economic agents (abstract competitors, perhaps firms). There are also a set $P \subset \mathbb{R}^L$ of possible price vectors and a set A of "market actions." Every i has a set $A_i \subset A$ and an endowment vector $\omega_i \in \mathbb{R}^L$. For every $a_i \in A_i$ and $p \in P$, a *trading rule* defined on $A \times P$ and with values in $\mathbb{R}^L$, assigns a net trade vector $g(a_i; p)$ to agent i, satisfying $p \cdot g(a_i; p) = 0$. Given an array $a = (a_1, \ldots, a_I)$ of actions, there is then a market clearing process that generates a price vector $p(a) \in P$. We also assume that every i has a utility function $u_i(g(a_i; p) + \omega_i)$, thus indirectly defined on $A_i \times P$.

The previous setup suggests treating the problem by the methodology of noncooperative games, as presented in Chapter 8.

Definition 18.C.1: The profile of actions $a^* = (a_1^*, \ldots, a_I^*) \in A_1 \times \cdots \times A_I$ is a *trading equilibrium* if, for every i,

$$u_i(g(a_i^*; p(a^*)) + \omega_i) \geq u_i(g(a_i; p(a_i, a_{-i}^*)) + \omega_i) \quad \text{for all } a_i \in A_i.^7$$

The concept of noncooperative equilibrium incorporated in Definition 18.C.1 is the same as the one we used in Chapters 8 and 12. As there, and in contrast to the analysis of the core in Section 18.B, such equilibria need not be Pareto optimal. The question we now pose ourselves is: Under what conditions is it the case that, if individual traders are small relative to the size of the economy, the system of markets approximates a price-taking environment in which, effectively, every trader optimizes given a competitive budget set (and in which, therefore, the equilibria will be nearly Pareto optimal).

7. As it has become customary, we follow the notation $(a_i, a_{-i}^*) = (a_1^*, \ldots, a_{i-1}^*, a_i, a_{i+1}^*, \ldots, a_I^*)$.

Given $a = (a_1, \ldots, a_I) \in A_1 \times \cdots \times A_I$, define

$$B_i(a) = \{x_i \in \mathbb{R}^L_+ : x_i - \omega_i \leq g(a_i'; p(a_i', a_{-i})) \text{ for some } a_i' \in A_i\}$$

as the *effective budget set* of trader i at a. In words, $B_i(a)$ is the set of net trades that trader i can achieve through *some* choice of a_i, given that the remaining traders are choosing a_{-i}. This set of achievable net trades will be close to the Walrasian budget

$$B(p(a), p(a)\cdot\omega_i) = \{x_i \in \mathbb{R}^L_+ : p(a)\cdot x_i \leq p(a)\cdot\omega_i\}$$

if the following two types of conditions both hold:

(1) *Insensitivity of prices to own actions.* For the boundary of $B_i(a)$ to be (almost) contained in a hyperplane, we need the price-clearing function $p(a_i, a_{-i})$ to be very insensitive to a_i.[8] Often this will be guaranteed if the economy is large and, consequently, every competitor is of small relative size. Suppose, for example, that $p(a)$ has the form $p((1/r)\sum_i a_i)$ where r is a size parameter (perhaps the number of consumers). It is actually quite common that the problem be given as, or can be transformed into, one in which actions affect price only through some average. At any rate, if this is the case, then the essential fact is clear: *As long as $p(\cdot)$ depends continuously on the average action $(1/r)\sum_i a_i$ the dependence of prices on individual actions* (assume the A_i are bounded) *will become negligible as the economy becomes large*; that is, as $r \to \infty$. Continuity of $p(\cdot)$ is therefore a key property.

(2) *Individual spanning.* Even if $p(a)$ is practically independent of the actions of an individual, we could still have a failure of *individual spanning*. That is, the boundary of the set $B_i(a)$, while flat, may be "too short," as in Figure 18.C.1(a), or even lower-dimensional, as in Figure 18.C.1(b) where it reduces to the initial endowment vector (no trade at all is possible). Individual spanning will have to be checked in every case. Verifying it will typically involve showing, first, that $g(a_i, p)$ is sensitive enough to a_i and, second, that A_i is large enough.

We now briefly, and informally, discuss three examples illustrating these ideas.[9]

Example 18.C.1: *General Equilibrium, Single-Good Cournot Competition.* This is in essence the same model studied in Section 12.C, except that we now admit a completely general form of "inverse demand function," that is, of the correspondence that assigns market-clearing prices to aggregate production decisions. This is meant to reflect the possibility of wealth effects (a hallmark of the general equilibrium approach).

To be concrete, suppose that we have two goods: the first is a consumption good and the second is "money" (which is also the unit of account with price equal to 1). There are r identical consumers, each endowed with a unit of money. For a price $p \in \mathbb{R}$ of the consumption good the demand of a consumer for this good is $x(p) \in \mathbb{R}$. There are also J firms producing the consumption good out of money. Firms set quantities. Marginal cost, up to a unit of capacity, is zero. To minimize complexity,

8. Recall that $p(a_i', a_{-i})\cdot g(a_i'; p(a_i', a_{-i})) = 0$ for all a_i'. Therefore, if $p(a_i', a_{-i})$ is (almost) independent of a_i' then $B_i(a)$ is (almost) contained in the hyperplane perpendicular to $p(a)$.

9. For more on general equilibrium Cournotian models in the style of Examples 18.C.1 and 18.C.2, see Gabszewicz and Vial (1972) and Novshek and Sonnenschein (1978). For a survey of the general area, see Mas-Colell (1982).

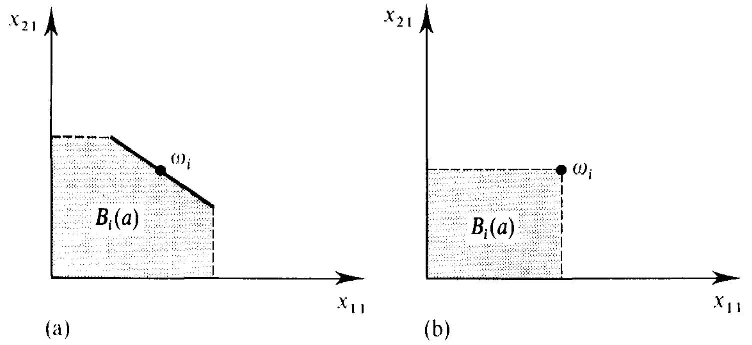

Figure 18.C.1
(a) and (b): Two nonspanning effective budget sets.

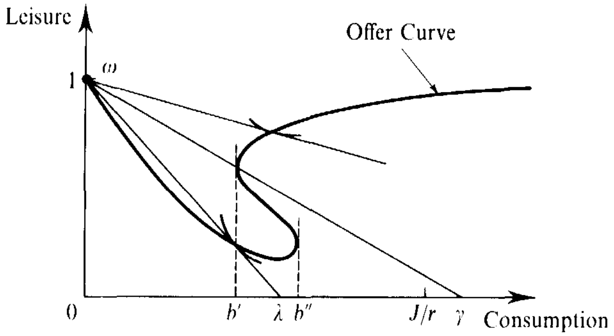

Figure 18.C.2
An economy with no continuous price selection.

assume that the owners of the firms are a separate group of agents interested only in the consumption of money. Therefore, for any total production $q = \sum_j q_j$ and size parameter r, the market price of the consumption good must solve the general equilibrium system $rx(p) = q$, or $x(p) = q/r$. We suppose that for any q/r the market selects a solution $p(q/r)$ to this equation.[10]

In the quasilinear, partial equilibrium context of Chapter 12, $x(\cdot)$ is a decreasing continuous function and therefore its inverse $p(\cdot)$ exists and *is continuous* (and decreasing). It then follows that, when r is large, $p(\sum_j q_j/r)$ is quite insensitive to the decision of any particular firm. Hence, firms are almost price takers and, as a consequence, the Cournot equilibria are almost Walrasian. Yet in the current general equilibrium context, $p(\cdot)$ may be unavoidably discontinuous. This is illustrated in Figure 18.C.2, where we represent the offer curve of a consumer. In the figure, there is no way to select money demands, and therefore prices, continuously over the offer curve as consumption per capita q/r ranges from 0 to J/r.[11] The location of potential Cournot equilibria will depend on how the market selects $p(\cdot)$ in the domain $[b', b'']$ of consumptions per capita, but the possibility of Cournot equilibria bounded away from the Walrasian equilibrium irrespective of the size of the economy is quite real. A particular price selection $p(\cdot)$ has been chosen in Figure 18.C.3. Note, first, that at the Walrasian equilibria of this model we must have every firm producing at capacity (and so the Walrasian equilibrium price is p^*). Yet, provided $r > (J\lambda/\gamma b')$,

10. To view this example as a particular case of the abstract trading model described above, you should think of the J firms as the players. Firm j is "endowed" with a unit of the first good and its strategy variable is $q_j \in [0, 1] = A_j$. Finally, the trading rule is $g(q_j; p, 1) = (-q_j, pq_j)$.

11. The example is contrived in that "money" is a Giffen good. If consumers were not identical, this feature would not be required.

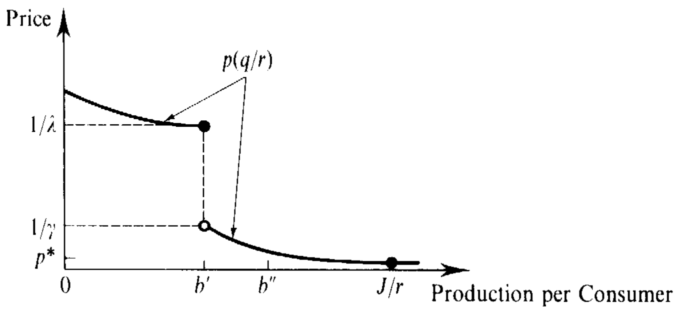

Figure 18.C.3
A price equilibrium selection.

every firm producing $rb'/J < 1$ (for a consumption per capita of b') constitutes a Cournot equilibrium: because $p(\cdot)$ is very elastic in the domain $[0, b']$ it will not pay any firm to contract production; and if any firm expands production, no matter how slightly, a precipitous and unprofitable drop in prices ensues.[12] [See Roberts (1980) for more on this point.] ■

Example 18.C.2: *Cournot Competition among Complements.* We modify the previous example in only two respects: (1) There are two consumption goods; (2) firms are producers of either the first or the second of these. The respective number of firms is J_1 and J_2. To be very simple we assume that the consumer has a quasilinear utility with money as numeraire. If the concave, strictly increasing utility function for the two consumption goods is $\psi(x_1, x_2)$ then, for any total productions (q_1, q_2), market clearing prices are

$$p(q_1, q_2) = \nabla\psi(q_1/r, q_2/r) = \left(\frac{\partial\psi(q_1/r, q_2/r)}{\partial x_1}, \frac{\partial\psi(q_1/r, q_2/r)}{\partial x_2}\right) \gg 0.$$

The Walrasian equilibrium productions are (J_1, J_2). Suppose, now, to take an extreme situation, that the two consumption goods are complements in the sense that the consumption of one is absolutely necessary for the enjoyment of the other, or $\psi(0, x_2) = \psi(x_1, 0) = 0$ for any x_1 and x_2. Then we claim that lack of activity (i.e., the null production of the two goods) is an equilibrium. The reason is clear enough: If $q_2 = 0$ then any positive supply $q_1 > 0$ of good one can only be absorbed by the market at $p_1 = \nabla_1\psi(q_1/r, 0) = 0$. Thus, no firm has an incentive to produce any amount of good 1 (and similarly for good 2). Economically, the difficulty is that the cooperation of at least two firms is needed to activate a market. Technically, we have a failure of continuity of clearing prices at (0, 0) since $p(\varepsilon, 0) = 0$ for all $\varepsilon > 0$ but the limit of $p(\varepsilon, \varepsilon)$ as ε goes to zero remains bounded away from zero.[13] ■

Example 18.C.3: *Trading Posts.* This example belongs to a family proposed by Shapley and Shubik (1977). It is not particularly realistic but it has at least three

12. When every firm produces rb'/J, the profits of one firm are $rb'/J\lambda > 1/\gamma$. But $1/\gamma$ is an upper bound for the profits of any firm that deviates from the suggested production by producing more. Hence an output level of rb'/J for every firm constitutes an equilibrium.

13. The complementarity makes it impossible for ψ to be continuously differentiable at the origin. Therefore, $p(\cdot)$ fails to be continuous. This is the crucial aspect for the example. Note that discontinuity at the origin is a natural occurrence: it will arise, for example, whenever the indifference map of $\psi(\cdot)$ is homothetic (but not linear). See Hart (1980) for more on this issue.

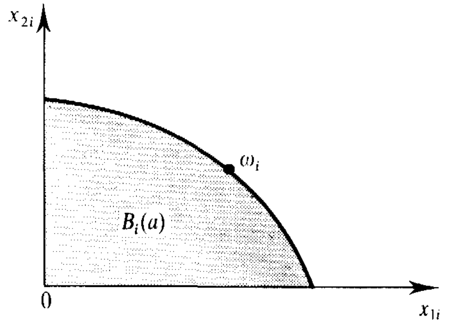

Figure 18.C.4
An effective budget set for the trading post Example 18.C.3.

18

virtues: it constitutes a complete general equilibrium model, all of the participants interact strategically (in the two previous examples, consumers adjust passively), and it is analytically simple to manipulate.

There are L goods and I consumers. Consumer i has endowment $\omega_i \in \mathbb{R}^L_+$. The Lth commodity, to be called "money," is treated asymmetrically. For each of the first $L - 1$ goods there is a *trading post* exchanging money for the good. At each trading post $\ell \leq L - 1$, each consumer i can place nonnegative bids $a_{\ell i} = (a'_{\ell i}, a''_{\ell i}) \in \mathbb{R}^2_+$. The interpretation is that an amount $a'_{\ell i}$ of good ℓ is placed at the offer side of the trading post to be exchanged for money. Similarly, an amount $a''_{\ell i}$ of money is placed in the demand side to be exchanged for good ℓ. Accordingly, the bids are also constrained by $a'_{\ell i} \leq \omega_{\ell i}$ and $\sum_{\ell \leq L-1} a''_{\ell i} \leq \omega_{Li}$.

Given the bids of consumer i in the trading posts $\ell \leq L - 1$ and prices $(p_1, \ldots, p_{L-1}, 1)$ the mechanism is completed by the trading rule:

$$g_\ell(a_{1i}, \ldots, a_{L-1,i}; p_1, \ldots, p_{L-1}, 1) = \frac{a''_{\ell i}}{p_\ell} - a'_{\ell i}$$

for all $\ell < L - 1$. The trade for the money good is derived from the budget constraint of the consumer.

Given a vector $a = (a_{11}, \ldots, a_{L-1,1}, \ldots, a_{1I}, \ldots, a_{L-1,I})$ of bids for all consumers, the clearing prices in terms of money are determined as the ratio of the amount of money offered to the amount of good offered:

$$p_\ell(a) = \frac{\sum_i a''_{\ell i}}{\sum_i a'_{\ell i}} \qquad \ell = 1, \ldots, L - 1. \tag{18.C.1}$$

Note that $p_\ell(a)$ is well defined and continuous *except* when there are no offers at the trading post ℓ [i.e., except when $a'_{\ell i} = 0$ for all i].[14]

A typical effective budget set for agent i is convex and, provided that $\sum_{k \neq i} a'_{\ell k} \neq 0$ and $\sum_{k \neq i} a''_{\ell k} \neq 0$ for all $\ell \leq L - 1$, it has an upper boundary containing no straight segments (you are asked to formally verify this in Exercise 18.C.1). This reflects the fact that as a consumer increases her bid in one side of a market the terms of trade turn against her. Figure 18.C.4 gives an illustration for the case $L = 2$.

14. For the special, but important, case in which there is a single trading post (i.e., $L = 2$), we can go a bit farther. When $\sum_i a''_{\ell i} > 0$ and $\sum_i a'_{\ell i} = 0$, the relative price of money is still well defined: it is zero. The essential difficulty in defining relative prices arises when $\sum_i a'_{\ell i} = 0$ and $\sum_i a''_{\ell i} = 0$.

It follows from expression (18.C.1) that approximate price taking will prevail in any trading post that is *thick* in the sense that the aggregate positions taken on the two sides of the market are large relative to the size of the initial endowments of any consumer. A necessary condition for thickness is that there be many consumers. But this is not sufficient: it is possible even in a large economy to have equilibrium where some market is *thin* and, as a consequence, a trading equilibrium may be far from a Walrasian equilibrium. In fact, any trading equilibrium for a model where a trading post ℓ is closed (i.e., the trading post does not exist) will remain an equilibrium if the trading post is open but stays inactive. That is, if we put $a_{\ell i} = (a'_{\ell i}, a''_{\ell i}) = 0$ for all i. Economically, this is related to Example 18.C.2: it takes at least two agents (here a buyer and a seller) to activate a market. Mathematically, the difficulty is again the impossibility of assigning prices continuously when $a_{\ell i} = 0$ for all i.

Up to now, in this and previous examples, all of the instances of trading equilibria not approaching a Walrasian outcome when individual competitors are small have been related to failures of continuity of market equilibrium prices. But the current example also lends itself to illustration of the individual spanning problem. Indeed, even if markets are thick and therefore prices, from the individual point of view, are almost fixed, it remains true that the trading post structure imposes the restriction that *goods can only be exchanged for money on hand* (in macroeconomics this restriction is called the cash-in-advance, or the Clower, constraint). Money obtained by selling goods cannot be applied to buy goods. Therefore, for a given individual the Walrasian budget set will be (almost) attainable only if the initial endowments of money are sufficient, that is, only if at the solution of the individual optimization problem the constraint $\sum_{\ell < L-1} a''_{\ell i} \leq \omega_{Li}$ is not binding. But there is no general reason why this should be so. Suppose, to take an extreme case, that $\omega_{Li} = 0$. Then consumer i simply cannot buy goods at all. ■

18.D The Limits to Redistribution

In Section 16.D we saw that, under appropriate convexity conditions and provided that wealth can be transferred in a lump-sum manner, Pareto optimal allocations can be supported by means of prices. However, as we also pointed out there, a necessary condition for lump-sum payments to be possible is the ability of the policy authority to tell who is who—that is, to be able to precisely identify the characteristics (preferences and endowments) of every consumer in the economy. In this section, we shall explore the implications of assuming that this cannot be done to any extent; that is, we shall postulate that individual characteristics are private and become public only if revealed by economic agents through their choices. We will then see that under very general conditions the second welfare theorem fails dramatically: the only Pareto optimal allocations that can be supported involve no transfers, that is, they are precisely the Walrasian allocations. Thus, if no personal information of any sort is available to the policy authority, then there may be a real conflict between equity and efficiency: if transfers have to be implemented we must give up Pareto optimality. The nature of this trade-off is further explored in Sections 22.B and 22.C.

We place ourselves in an exchange economy with I consumers. Each consumer i has the consumption set $\mathbb{R}^L_+$, the endowment vector $\omega_i \geq 0$, and the continuous, monotone, and strictly quasiconcave utility function $u_i(\cdot)$.

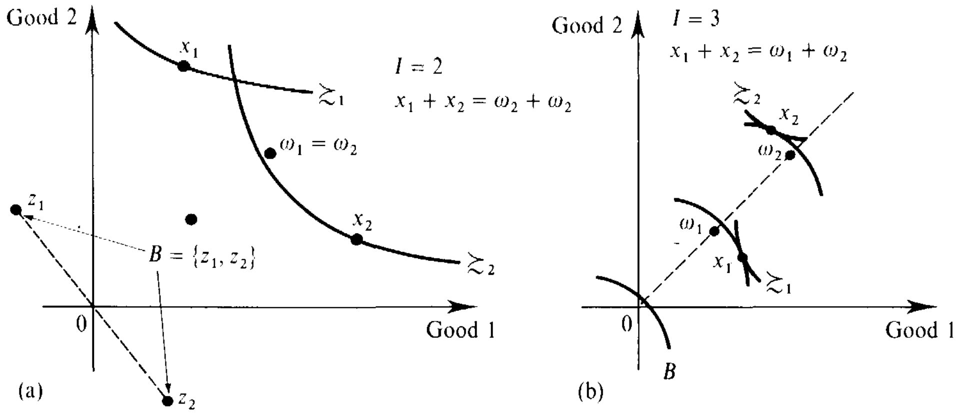

Figure 18.D.1
(a) and (b): Two self-selective allocations.

We begin by stating a restriction on feasible allocations designed to capture the possibility that the allocation is the result of a process in which every consumer maximizes utility subject to market opportunities that are identical across consumers.

Definition 18.D.1: The feasible allocation $x^* = (x_1^*, \ldots, x_I^*) \in \mathbb{R}^{LI}$ if *self-selective* (or *anonymous*, or *envy-free in net trades*) if there is a set of net trades $B \subset \mathbb{R}^L$, to be called a *generalized budget set* or a *tax system*, such that, for every i, $z_i^* = x_i^* - \omega_i$ solves the problem

$$\begin{aligned} \text{Max} \quad & u_i(z_i + \omega_i) \\ \text{s.t. } & z_i \in B, \\ & z_i + \omega_i \geq 0. \end{aligned}$$

Figures 18.D.1(a) and 18.D.1(b) present two examples of self-selective allocations.[15] In the figures the preferences and endowments of the different consumers are depicted in the same orthant.

Note that if $x^* = (x_1^*, \ldots, x_I^*)$ is self-selective then it is enough to take $B = \{x_1^*, \ldots, x_I^*\}$. Thus, we could read Definition 18.D.1 as saying that no consumer i envies the net trade of any other individual; among all the net trades present in the economy, the consumer is happy enough with the one assigned to her. Expositionally, we have preferred to keep separate the reality of B because we have in mind the limit situation in which there is, on the one hand, a multitude of consumers whose actions are individually imperceptible and, on the other, a policy authority that has perfect statistical information (i.e., knows perfectly the *distribution* of individual characteristics), but no information at all on who is who. In such a world, a viable policy instrument is to choose a set B and let each consumer select her most preferred point in it. Because this is what an income tax schedule amounts to, we also call B a tax system. We remark that, as a policy tool, the notion of a generalized budget presumes the ability to prevent individuals from choosing several times. Hence it models the income tax, adequately, but not a commodity tax.

We now pose a question of the second welfare theorem type: Which Pareto optimal allocations can be supported by means of a common budget? That is, which feasible allocations are simultaneously Pareto optimal and self-selective? This

15. The concept of a nonenvy allocation was introduced by Foley (1967) and that of a nonenvy net-trade allocation by Schmeidler and Vind (1972). See Thomson and Varian (1985) for a survey of these notions (with an emphasis on the ethical aspects).

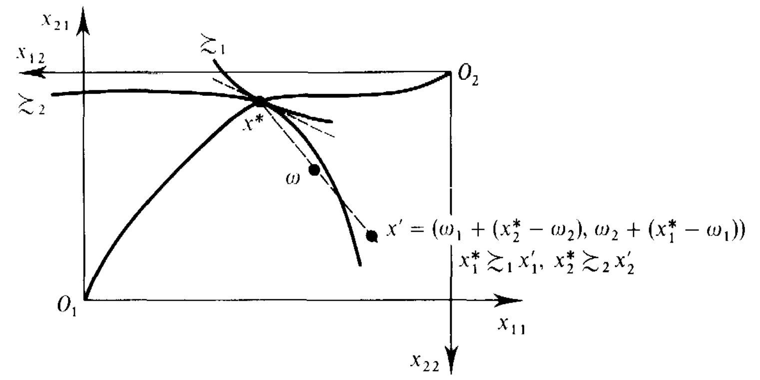

Figure 18.D.2
A Pareto optimal, self-selective allocation that is not Walrasian.

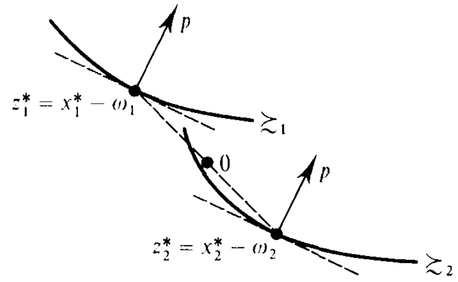

Figure 18.D.3
Another representation of the example of Figure 18.D.2.

question is both broader and more limited than the one associated with the second welfare theorem. It is broader because we allow supportability by general (nonlinear) budget sets and not only by linear hyperplanes. It is narrower because it demands that *all* consumers face the *same* budget for their net trades.

The first observation is that if $x^* = (x_1^*, \ldots, x_I^*)$ is a Walrasian allocation with equilibrium price vector $p \in \mathbb{R}^L$, then the allocation is Pareto optimal because of the first welfare theorem, and self-selective because we can take $B = \{z: p \cdot z = 0\}$.

The allocation marked x^* in the Edgeworth box of Figure 18.D.2 provides an example of a self-selective allocation in the Pareto set that is not Walrasian. Figure 18.D.3 shows transparently why x^* is self-selective. In this figure we bring the origin of the consumption sets of the two consumers to their initial endowment vector so that the preferences of the two consumers are expressed over net trades. Then with $z_i^* = x_i^* - \omega_i$, $i = 1, 2$, we see that $z_1^* \succsim_1 z_2^*$ and $z_2^* \succsim_2 z_1^*$.

The previous Edgeworth box example suggests that there may be ample room for anonymous redistribution. The example, however, is special in that there are only two consumers or, more generally, in that all the consumers fall into two preference–endowment types. We will now investigate the situation in which there is a multitude of consumers who *moreover* fall into a rich variety of types. Intuitively, this should make the compatibility of Pareto optimality and transfers more difficult because it is likely that the opportunities for envy will then be many (Pareto optimality will force the net trades to vary across consumers), and therefore the freedom in constructing the generalized budget will be limited.

In Figure 18.D.4 we represent an allocation for an economy with two commodities and a continuum of types (and therefore with a continuum of consumers).[16]

16. Approximate versions of the following results exist for economies with a finite but large number of types.

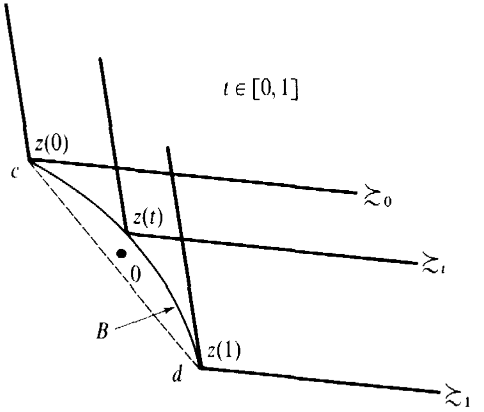

Figure 18.D.4
Pareto optimal allocation with a continuum of traders that is self-selective but not Walrasian.

Consumer types are indexed by $t \in [0, 1]$ *and their preferences* $\succsim_t$ *depend continuously on* t. For simplicity we take their endowments to be the same.[17] An implication of this continuity assumption is that the set of characteristics (preferences–endowment pairs) of the consumers present in the economy cannot be split into two disconnected classes. The consumptions of the different types are distributed along the curvilinear segment cd. This allocation has the following properties:

(i) It is not Walrasian. If it were then all the consumptions would lie in a straight line; in fact, different types end up exchanging the two goods at different ratios.

(ii) It is self-selective. We see in Figure 18.D.4 that the consumption chosen by each consumer maximizes her utility in the generalized budget set B. Note that the frontier of any admissible budget set has to include the segment cd comprised by the consumptions actually chosen by some consumer.

(iii) It is Pareto optimal. Indeed, the price vector $p = (1, 1)$ will make the allocation into a price equilibrium with transfers, hence a Pareto optimum.

Observe that fact (iii) depends crucially on the indifference curves of every consumer exhibiting a kink at the assigned bundle. If we tried to smooth these kinks out then, because cd is curved and the preferences change continuously, the result would be the existence of two consumers with different marginal rates of substitution at their chosen point, which is a violation of Pareto optimality (there would be room for profitable exchange among these two consumers). Only if B were rectilinear could we retain Pareto optimality, but then the allocation would be a Walrasian equilibrium. Thus, it appears that if the indifference curves are smooth at the consumption points then a Pareto optimal, self-selective allocation can be something other than a Walrasian equilibrium only if the characteristics of the consumers present in the economy can be split into disconnected classes. With this motivation we can state Proposition 18.D.1.[18]

17. What is important is that they change continuously with t.
18. For results of this type, see, for example, Varian (1976) or Champsaur and Laroque (1981).

Proposition 18.D.1: Suppose we have an exchange economy with a continuum of consumer types. Assume:

(i) The preferences of all consumers are representable by differentiable utility functions.

(ii) The set of characteristics of consumers present in the economy[19] cannot be split into two disconnected classes. Formally, if $(u(\cdot), \omega)$, $(u'(\cdot), \omega')$ are two preference–endowment pairs present in the economy then there is a continuous function $(u(\cdot\,; t), \omega(t))$ of $t \in [0, 1]$ such that

$$(u(\cdot\,; 0), \omega(0)) = (u(\cdot), \omega), \; (u(\cdot\,; 1), \omega(1)) = (u'(\cdot), \omega),$$

and $(u(\cdot\,; t), \omega(t))$ is present in the economy for every t.

Then any allocation $x^* = \{x_i^*\}_{i \in I}$ that is Pareto optimal, self-selective, and interior (i.e., $x_i^* \gg 0$ for all i) must be a Walrasian equilibrium allocation. Here I is an infinite set of names.

Proof: The proof is far from rigorous. It is also limited to $L = 2$.

Let $p = (p_1, p_2)$ be the price vector supporting x^* as a Pareto optimal allocation. Because of differentiability of the utility functions and interiority of the allocation, the relative prices p_1/p_2 are uniquely determined. We want to show that $p \cdot (x_i^* - \omega_i) = 0$ for all i.

The first observation is that at x^* the equal-treatment property holds: if $(u_i(\cdot), \omega_i) = (u_k(\cdot), \omega_k)$ then $x_i^* = x_k^*$. Indeed, neither i envies k nor k envies i. Hence x_i^* and x_k^* must lie in the same indifference curve of the common preference relation of i and k. By the strict convexity of preferences, the price vector p can support only one point in this indifference curve. Hence x_i^* and x_k^* must be equal.

If the set of net trades present in the economy consists of a single point, then this point has to be the vector 0 (otherwise the aggregate of the net trades could not be zero) and the result follows.

Suppose, therefore, that there are at least two different net-trade vectors present in the economy, z_0 and z_1. In Figure 18.D.5, we represent them as well as the net trades $z(t)$ of all consumers captured by the continuous parametrization given by assumption (ii), where $t = 0, 1$ correspond, respectively, to consumers underlying z_0 and z_1.

A key fact is that $z(t)$ is continuous as a function of t. This is intuitive. We have already seen that the equal-treatment property holds: Identical individuals are treated identically. Technicalities aside, the logic of the continuity of $z(t)$ is the same: If envy is to be prevented, then similar individuals must be treated similarly.

Thus as we go from $t = 0$ to $t = 1$, the net trades are moving continuously from z_0 to z_1. Hence the frontier of any generalized budget set B must actually connect z_0 to z_1. If so, then either this frontier is a straight segment with normal p between these two points [in which case $p \cdot (z_0 - z_1) = 0$], or somewhere between them there is one point ($z(t')$ in Figure 18.D.5) where the "slope" of the frontier of B, and therefore the MRS of the consumer choosing this point, is different from p_1/p_2 [in which case p would not be a supporting price vector].

We conclude that the portion M of the frontier of B containing all the net trades present in the economy is a nontrivial straight segment with normal p (hence, a convex set). Since the net trades add up to zero, we must have $0 \in M$. Thus, $p \cdot z = p \cdot (z - 0) = 0$ for every $z \in M$. In particular, $p \cdot (x_i^* - \omega_i) = 0$ for every i. See Figure 18.D.6. ■

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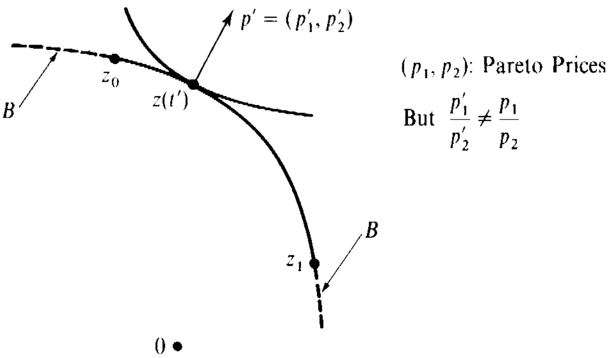
19. The expression "characteristics of consumers present in the economy" means technically "contained in the support of the distribution of characteristics induced by the population of consumers."



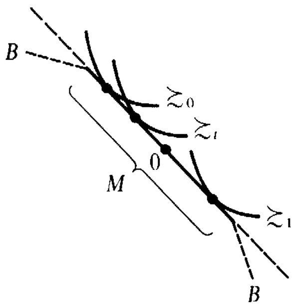


Figure 18.D.5. (left) If the net trade frontier is not flat, then the allocation is not Pareto optimal (assuming self-selectivity).

Figure 18.D.6 (right) A Pareto optimal, self-selective allocation that is Walrasian.

18.E Equilibrium and the Marginal Productivity Principle

In this section, we investigate the extent to which Walrasian equilibria can be characterized by the idea that individuals get exactly what they contribute to the economic welfare of society (at the margin). We will see that, once again, the assumption of a large number of consumers is crucial to this characterization. For an extensive analytical treatment of this topic we refer to the seminal contribution of Ostroy (1980).

To remain as simple as possible, we restrict ourselves to the case of quasilinear exchange economies. The Lth good is the numeraire.

Suppose that our economy has H types. The concave, differentiable, strictly increasing utility function of type h is

$$u_h(x_h) = \psi_h(x_{1h}, \ldots, x_{L-1,h}) + x_{Lh}.$$

It is defined on $\mathbb{R}_+^{L-1} \times \mathbb{R}$. We take ψ_h to be strictly concave. The initial endowment vector of type h is $\omega_h \geq 0$.

An economy is defined by a profile $(I_1, \ldots, I_H)$ of consumers of the different types, for a grand total of $I = \sum_h I_h$. For any economy $(I_1, \ldots, I_H)$ we define the maximal amount of "social utility" that can be generated, as in Section 10.D.[20]

$$v(I_1, \ldots, I_H) = \text{Max} \quad I_1 u_1(x_1) + \cdots + I_H u_H(x_H) \tag{18.E.1}$$
$$\text{s.t. (i) } I_1 x_1 + \cdots + I_H x_H \leq I_1 \omega_1 + \cdots + I_H \omega_H,$$
$$\text{(ii) } x_{\ell h} \geq 0 \text{ for all } \ell \leq L-1 \text{ and } h.$$

The function $v(I_1, \ldots, I_H)$ is homogeneous of degree one in its arguments: $v(rI_1, \ldots, rI_H) = rv(I_1, \ldots, I_H)$ for all r. In particular,

$$v(I_1/I, \ldots, I_H/I) = \frac{1}{I} v(I_1, \ldots, I_H).$$

That is, the per-capita social utility only depends on the type-composition and not on the size of the economy. Because of this we can extend the analysis to a situation with a continuum of consumers by defining $v(\mu_1, \ldots, \mu_H)$ for any nonnegative vector

20. Because utility functions are concave the maximum utility can be reached while treating consumers of the same type equally.

$\mu = (\mu_1, \ldots, \mu_H) \in \mathbb{R}^H_+$ of masses of the different types. Precisely,

$$v(\mu_1, \ldots, \mu_H) = \text{Max} \quad \mu_1 u_1(x_1) + \cdots + \mu_H u_H(x_H) \tag{18.E.2}$$
$$\text{s.t. (i) } \mu_1 x_1 + \cdots + \mu_H x_H \leq \mu_1 \omega_1 + \cdots + \mu_H \omega_H,$$
$$\text{(ii) } x_{\ell h} \geq 0 \text{ for all } \ell \leq L - 1 \text{ and } h.$$

If we have a sequence of finite economies $(I^n_1, \ldots, I^n_H)$ such that $I^n = \sum_h I^n_h \to \infty$ and $(1/I^n)I^n_h \to \mu_h$ for every h, then we can properly regard $(\mu_1, \ldots, \mu_H)$ as the continuum limit of the sequence of increasingly large finite economies.

n

Exercise 18.E.1: Show that the function $v(\cdot)\colon \mathbb{R}^H_+ \to \mathbb{R}$ is concave and homogeneous of degree one.

The function $v(\cdot)$ is a sort of production function whose output is social utility and whose inputs are the individual consumers themselves. Further, in the limit, every individual of type h becomes an input of infinitesimal size. For the time being, we concentrate our discussion on the continuum limit. We assume also that $v(\cdot)$ is differentiable.[21]

Definition 18.E.1: Given a continuum population $\mu = (\mu_1, \ldots, \mu_H) \in \mathbb{R}^H_+$ a feasible allocation[22] $(x^*_1, \ldots, x^*_H)$ is a *marginal product*, or *no-surplus*, *allocation* if

$$u_h(x^*_h) = \frac{\partial v(\mu)}{\partial \mu_h} \quad \text{for all } h. \tag{18.E.3}$$

In words: at a no-surplus allocation everyone is getting exactly what she contributes at the margin.

Proposition 18.E.1: For any *continuum* population $\bar{\mu} = (\bar{\mu}, \ldots, \bar{\mu}_H) \gg 0$ a feasible allocation $(x^*_1, \ldots, x^*_H) \gg 0$ is a marginal product allocation if and only if it is a Walrasian equilibrium allocation.

Proof: If $x^* = (x^*_1, \ldots, x^*_H)$ is a marginal product allocation then, using Euler's formula (see Section M.B of the Mathematical Appendix), we have

$$v(\bar{\mu}) = \sum_h \bar{\mu}_h \frac{\partial v(\bar{\mu})}{\partial \mu_h} = \sum_h \bar{\mu}_h u_h(x^*_h).$$

Hence, x^* solves problem (18.E.2) for $\mu = \bar{\mu}$.

Suppose now that $x^* = (x^*_1, \ldots, x^*_H)$ is a feasible allocation that gives rise to social utility $v(\bar{\mu})$; that is, it constitutes a solution to problem (18.E.2) for $\mu = \bar{\mu}$. Denote by p_ℓ, $\ell = 1, \ldots, L$, the values of the multipliers of the first-order conditions associated with the constraints $\sum_h \bar{\mu}_h(x_{\ell h} - \omega_{\ell h}) \leq 0, \ell = 1, \ldots, L$, in the optimization problem (18.E.2); see Section M.K of the Mathematical Appendix. By the quasilinear form of $u_h(\cdot)$ we have

$$p_L = 1 \qquad \text{and} \qquad p_\ell = \nabla_\ell \psi_h(x^*_{1h}, \ldots, x^*_{L-1,h}) \tag{18.E.4}$$

for all $\ell \leq L - 1$ and all $1 \leq h \leq H$.

21. This could be derived from more primitive assumptions.

22. We assume that consumers of the same type are treated equally. Feasibility means therefore that $\sum_h \mu_h x^*_h \leq \sum_h \mu_h \omega_h$.

It follows from (18.E.4) that the vector of multipliers $p = (p_1, \ldots, p_L)$ is the vector of Walrasian equilibrium prices of this quasilinear economy (recall the analysis of Section 10.D). In addition, by the envelope theorem (see Section M.L of the Mathematical Appendix), applied to problem (18.E.2), we have (Exercise 18.E.2):

$$\frac{\partial v(\bar{\mu})}{\partial \mu_h} = u_h(x_h^*) + p \cdot (\omega_h - x_h^*). \tag{18.E.5}$$

Therefore, we conclude that x^* is Walrasian if and only if x^* solves problem (18.E.2) for $\mu = \bar{\mu}$ and (18.E.3) is satisfied, that is, if and only if x^* is a marginal product allocation. ■

Expression (18.E.5) is intuitive. The left-hand side measures how much the maximum sum of utilities increases if we add one extra individual of type h. The right-hand side tells us that there are two effects. On the one hand, the extra consumer of type h receives from the rest of the economy the consumption bundle x_h^*, and so she directly adds her utility $u_h(x_h^*)$ to the social utility sum. On the other, while receiving x_h^*, she contributes her endowment vector ω_h. Hence the net change for the rest of the economy is $\omega_h - x_h^*$. How much is this worth to the rest of the economy? The vector of social shadow prices is precisely $p = (p_1, \ldots, p_L)$, and so the total change for the rest of the economy comes to $p \cdot (\omega_h - x_h^*)$. Note that the Walrasian allocations are thus characterized by this second effect being null: the utility of the consumer equals her entire marginal contribution to social utility.

In Exercise 18.E.4 you are asked to verify that the smoothness assumption on utility functions is essential to the validity of Proposition 18.E.1.

Let us now consider a finite economy $(I_1, \ldots, I_H) \gg 0$. We can define the marginal contribution of an individual of type h as

$$\Delta_h v(I_1, \ldots, I_H) = v(I_1, \ldots, I_h, \ldots, I_H) - v(I_1, \ldots, I_h - 1, \ldots, I_H).$$

Typically, there does not exist a feasible allocation $(x_1^*, \ldots, x_H^*)$ with $u_h(x_h^*) = \Delta_h v(I_1, \ldots, I_H)$ for all h. To see this, note that by the concavity of $v(\cdot)$ we have $\Delta_h v \geq \partial v / \partial \mu_h$ [both expressions evaluated at $(I_1, \ldots, I_H)$]. Except for degenerate cases, this inequality will be strict. Moreover, $\sum_h I_h(\partial v / \partial \mu_h) = v(I_1, \ldots, I_H)$ by Euler's formula (see Section M.B of the Mathematical Appendix), and thus we conclude that $\sum_l I_h(\Delta_h v) > v(I_1, \ldots, I_H)$; that is, it is impossible to give to each consumer the full extent of her marginal contribution while maintaining feasibility. In contrast with the continuum case, individuals are not now of negligible size: their whole contribution is not entirely at the margin. In particular, you should note that in a finite economy the Walrasian allocation is typically *not* a marginal product allocation. It follows from expression (18.E.5) that an allocation $(x_1^*, \ldots, x_H^*)$ that solves problem (18.E.2) for $(\mu_1, \ldots, \mu_H) = (I_1, \ldots, I_H)$ is a Walrasian equilibrium allocation if and only if

$$u_h(x_h^*) = \frac{\partial v}{\partial \mu_h}(I_1, \ldots, I_H).$$

But we have just argued that normally $\Delta_h v(I_1, \ldots, I_H) > \partial v(I_1, \ldots, I_H)/\partial \mu_h$. In words: At the Walrasian equilibrium consumers are compensated according to prices determined by the marginal unit of their endowments. But they lose the extra social surplus provided by the inframarginal units. This is yet another indication that the concept of Walrasian equilibrium stands on firmer ground in large economies.

We have just seen that in the context of economies with finitely many consumers it is not possible to feasibly distribute the gains of trade while adhering literally to the marginal productivity principle. The cooperative theory of games provides a possibility for a sort of reconciliation between feasibility and the marginal productivity principle. It is known as the *Shapley value*. In Appendix A, devoted to cooperative game theory, we offer a detailed presentation of this solution concept.

For an economy with profile $(I_1, \ldots, I_H)$ the Shapley value is a certain utility vector $(Sh_1, \ldots, Sh_H) \in \mathbb{R}^H$ that satisfies $\sum_h I_h Sh_h = v(I_1, \ldots, I_H)$. For every type h, the utility Sh_h can be viewed as an *average of marginal utilities* $\Delta_h v(I'_1, \ldots, I'_H)$. The average is taken over profiles $(I'_1, \ldots, I'_H) \leq (I_1, \ldots, I_H)$, where the probability weight given to $(I'_1, \ldots, I'_H)$ equals $1/I$, interpreted as the probability assigned to sample size $I'_1 + \cdots + I'_H$, times the probability of getting the profile $(I'_1, \ldots, I'_H)$ when independently sampling $I'_1 + \cdots + I'_H$ consumers out of the original population with I consumers and profile $(I_1, \ldots, I_H)$. See Appendix A for more on this formula.

An allocation that yields the Shapley value (let us call it a *Shapley allocation*) is not related in any particular way to the Walrasian equilibrium allocation (or for that matter to the core). Except by chance, they will be different allocations. Yet, remarkably, we also have a convergence of these concepts in economies with many consumers: the Walrasian and the Shapley allocations are then close to each other. This result is known as the *value equivalence theorem*. A rigorous proof of this theorem is too advanced to be given here [see Aumann (1975) and his references], but the basic intuition is relatively straightforward.

There are two key facts. First, if the entries of $(I'_1, \ldots, I'_H)$ are large, then subtracting a consumer of type h amounts to very little, and so

$$\Delta_h v(I'_1, \ldots, I'_H) \approx \partial v(I'_1, \ldots, I'_H)/\partial \mu_h.$$

Second, if the entries of $(I_1, \ldots, I_H)$ are large then, by the law of large numbers, most profiles $(I'_1, \ldots, I'_H)$ constitute a good sample of $(I_1, \ldots, I_H)$ and are therefore almost proportional to $(I_1, \ldots, I_H)$.

Using the homogeneity of degree one of $v(\cdot)$ (hence the homogeneity of degree zero of $\partial v/\partial \mu_h$), the combination of the previous two facts implies

$$\Delta_h v(I'_1, \ldots, I'_H) \approx \frac{\partial v(I'_1, \ldots, I'_H)}{\partial \mu_h} \approx \frac{\partial v(I_1, \ldots, I_H)}{\partial \mu_h}$$

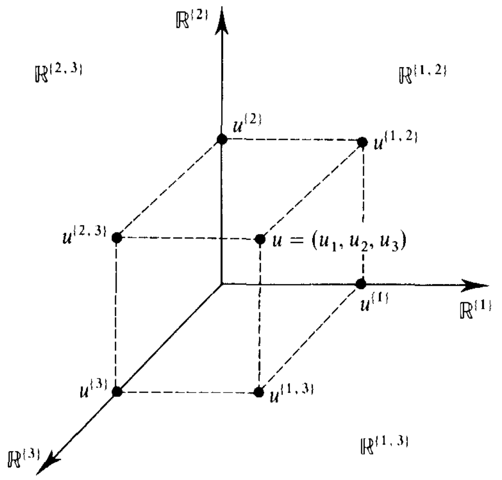
for most $(I'_1, \ldots, I'_H)$. Therefore, $Sh_h \approx \partial v(I_1, \ldots, I_H)/\partial \mu_h$, which is the utility payoff of type h at the Walrasian equilibrium allocation of the economy $(I_1, \ldots, I_H)$.

APPENDIX A: COOPERATIVE GAME THEORY

In this appendix, we offer a brief introduction to the *cooperative theory of games*. For more extensive recent accounts see Moulin (1988), Myerson (1991), or Osborne and Rubinstein (1994).[23]

In Chapter 7, we presented the normal and the extensive forms of a game. The starting point for the cooperative theory is a classical third description: that of a

23. The text by Owen (1982), although not so recent, is nevertheless strong in its coverage of cooperative theory. Another useful reference is Shubik (1984), which is encyclopedic in spirit and contains a wealth of information.



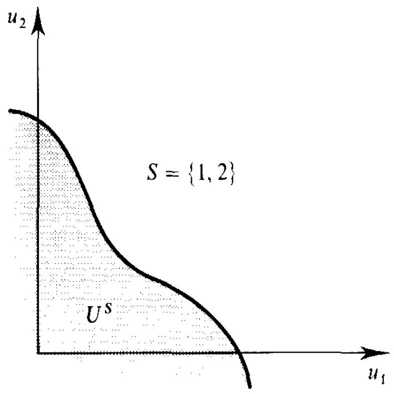


Figure 18.AA.1 (left)
Utility outcome $u \in \mathbb{R}^3$ and its projections.

Figure 18.AA.2 (right)
A utility possibility set for $S = \{1, 2\}$.

game in characteristic form. The characteristic form is meant to be a summary of the payoffs available to each group of players in a context where binding commitments among the players of the group are feasible. Although, in principle, it should be possible to derive the characteristic form from the normal or the extensive forms, the viewpoint of cooperative game theory is that it is often analytically desirable to avoid detail and to go as directly as possible to a summary description of the strategic position of the different groups of players.[24]

After defining the characteristic form, we will discuss two of the main solution concepts of cooperative game theory: the *core* and the *Shapley value.*

The set of players is denoted $I = \{1, \ldots, I\}$. We abuse notation slightly by using the same symbol to denote the set and its cardinality. Nonempty subsets S, $T \subset I$ are called *coalitions*.

An *outcome* is a list of utilities $u = (u_1, \ldots, u_I) \in \mathbb{R}^I$. Given $u = (u_1, \ldots, u_I)$, the relevant coordinates to a coalition S are $u^S = (u_i)_{i \in S}$. Mathematically, u^S is the restriction (or projection) of $u \in \mathbb{R}^I$ to the coordinates corresponding to S. We can therefore view u^S as a member of the Euclidean space $\mathbb{R}^S$ spanned by these coordinates. Figure 18.AA.1 shows how outcomes for three players are evaluated by all six proper subsets: $S = \{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$, $\{1, 3\}$, and $\{2, 3\}$.

Definition 18.AA.1: A nonempty, closed set $U^S \subset \mathbb{R}^S$ is a *utility possibility set for* $S \subset I$ if it is *comprehensive*:

$$u^S \in U^S \quad \text{and} \quad u'^S \leq u^S \text{ implies } u'^S \in U^S.$$

See Figure 18.AA.2 for an illustration.[25]

24. Nevertheless, we note that there is a school of thought within game theory of the opinion that the condensation of information that a characteristic form represents may not do justice to the strategic complexities inherent in the making of binding commitments. Despite the cogency of this position, the analytical power of games in characteristic forms for the study of normative issues in economics has been amply demonstrated. This is more than enough reason to welcome the parsimony it brings to the analysis.

25. Note that, as we did in Section 16.E, we build free disposability of utility into the definition of a utility possibility set.

Definition 18.AA.2: A *game in characteristic form* (I, V) is a set of players I and a rule $V(\cdot)$ that associates to every coalition $S \subset I$ a utility possibility set $V(S) \subset \mathbb{R}^S$.

The elements of $V(S)$ are to be interpreted as the payoffs the players in S can achieve by themselves if they jointly commit to a certain course of action. It is important to observe the expression "can achieve" is not free of subtlety. This is because the course of action undertaken by the members of $I \backslash S$ will typically affect the payoffs of the members of S. In applications, therefore, one should be explicit as to how $V(S)$ is constructed.

Example 18.AA.1: *Economies.* Consider an economy with I consumers having continuous, increasing, concave utility functions $u_i\colon \mathbb{R}^L_+ \to \mathbb{R}$ and endowments $\omega_i \geq 0$. There is also a publicly available convex, constant returns technology $Y \subset \mathbb{R}^L$. We can then define a game in characteristic form by letting

$$V(S) = \left\{(u_i(x_i))_{i \in S}\colon \sum_{i \in S} x_i = \sum_{i \in S} \omega_i + y, y \in Y\right\} - \mathbb{R}^S_+ .$$

That is, $V(S)$ is the set of payoffs that the consumers in coalition S can achieve by trading among themselves and using the technology Y. Every set $V(S)$ is convex (recall Exercise 16.E.2). Figure 18.AA.3 depicts these sets for the case $I = 3$. ■

Example 18.AA.2: *Majority Voting.* Consider a three-player situation in which any two out of the three players can form a majority and select among a set of social alternatives A. If $a \in A$ is selected, the payoffs are $u_i(a) \geq 0$, $i = 1, 2, 3$. In addition, any player i has the right to unilaterally withdraw from the group and get a payoff of zero.

Then we can define a game in characteristic form (I, V) as

$$V(I) = \{(u_1(a), u_2(a), u_3(a))\colon a \in A\} - \mathbb{R}^I_+ .$$

$$V(\{i, h\}) = \{(u_i(a), u_h(a))\colon a \in A\} - \mathbb{R}^{\{i,h\}}_+ \text{ for all distinct pairs } \{i, h\}.$$

$$V(\{i\}) = -\mathbb{R}^{\{i\}}_+ .$$

Figure 18.AA.4 shows this characteristic form for a case with three alternatives, $A = \{a_1, a_2, a_3\}$. In the figure we suppose that the three decisions yield, respectively,

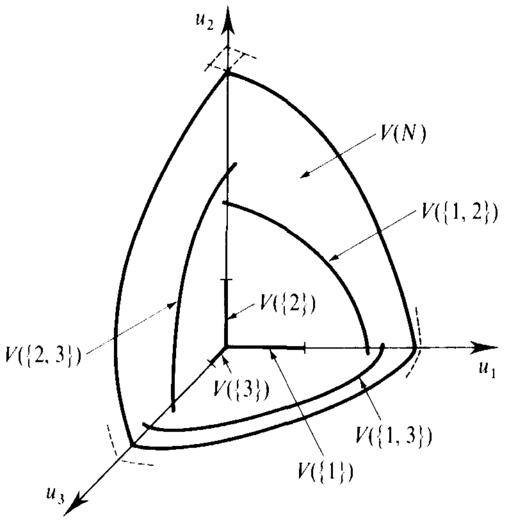


Figure 18.AA.3 (left)
A family of utility possibility sets.

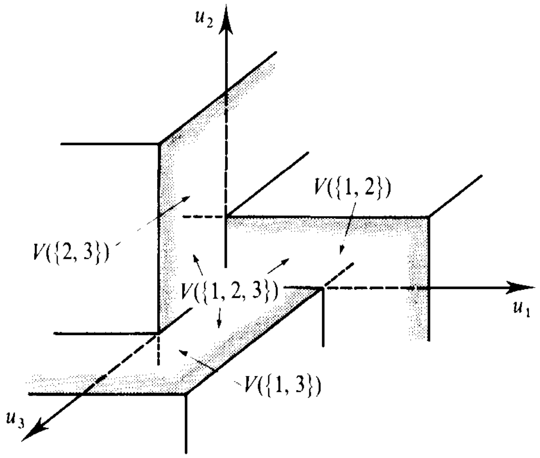


Figure 18.AA.4 (right)
The utility possibility sets for the majority voting in Example 18.AA.2.

the utility vectors (2, 1, 0), (1, 0, 2), (0, 2, 1). Note that $V(\cdot)$ need not be convex when, as here, decisions are discrete and there are no possibilities for either randomization or any form of side transfers. ■

Definition 18.AA.3: A game in characteristic form (I, V) is *superadditive* if for any coalitions S, $T \subset I$ that are disjoint (i.e., such that that $S \cap T = \varnothing$), we have

$$\text{If } u^S \in V(S) \text{ and } u^T \in V(T), \text{ then } (u^S, u^T) \in V(S \cup T).$$

Superadditivity means that coalitions S and T are able to do at least as well acting together as they could do acting separately. It is an assumption we will commonly make (it is satisfied by Examples 18.AA.1 and 18.AA.2). If one of the possibilities open to the union of two disjoint coalitions is to agree to act *as if* they were still separated coalitions, then superadditivity should hold.

It has been a constant theme of this book that often the analysis becomes much simpler when individual utility functions are quasilinear, that is, when there is a commodity ("numeraire") that can be used to effect unit-per-unit transfers of utility across economic agents. The same is true in the theory of cooperative games. Its history is, in fact, replete with instances of concepts first formulated for the transferable utility case that have later been extended to the general setting without an essential loss of intuition and analytical power.

For a situation described by a game in characteristic form, what the quasilinearity, or transferable utility, hypothesis amounts to is the assumption that the sets $V(S)$ are half-spaces (as they were, for example, in Section 10.D); that is, they are sets whose boundaries are hyperplanes in $\mathbb{R}^S$. Moreover, by choosing the units of utility, we can take the hyperplanes defining $V(S)$ to have normals $(1, \ldots, 1) \in \mathbb{R}^S$.[26] Thus, the sets $V(S)$ will now have the form

$$V(S) = \left\{ u^S \in \mathbb{R}^S \colon \sum_{i \in S} u_i^S \le v(S) \right\}$$

for some $v(S) \in \mathbb{R}$. In other words, we can view coalition S as choosing a joint action so as to maximize the total utility, denoted $v(S)$, which then can be allocated to the members of S in any desired manner through transfers of the numeraire. Figure 18.AA.5 depicts the sets $V(S)$ for the case $I = 3$.

The number $v(S)$ is called the *worth* of coalition S. Since the numbers $v(S)$, $S \subset I$, constitute a complete description of (I, V) we provide Definition 18.AA.4.

Definition 18.AA.4: A *transferable utility game in characteristic form*, (or *TU-game*), is defined by (I, v), where I is a set of players and $v(\cdot)$ is a function, called the *characteristic function*, that assigns to every nonempty coalition $S \subset I$ a number $v(S)$ called the *worth of S*.

Example 18.AA.3: *TU Majority Voting.* Suppose that in Example 18.AA.2 (with the values of Figure 18.AA.4) we add the possibility that utility be freely transferable across players (there may be a numeraire commodity with respect to which

26. This choice of units of utility is legitimate because all the solutions to be considered are invariant to normalizations of units. See Chapter 21 for more on this point.

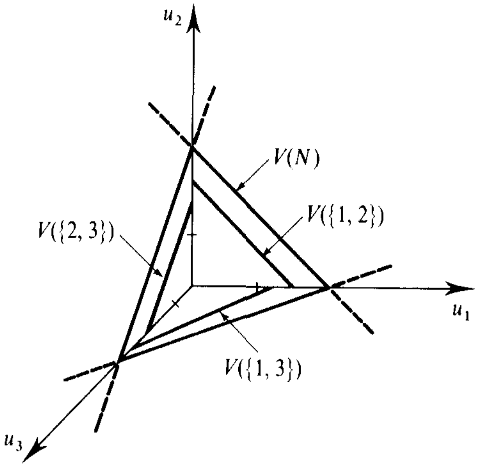


Figure 18.AA.5
Utility possibility sets for a transferable utility game.

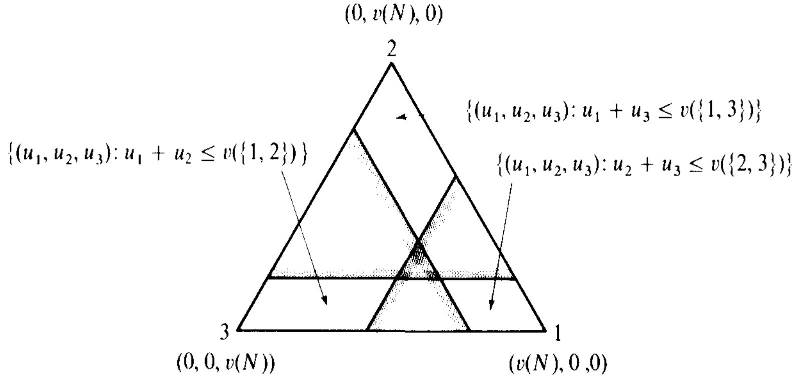


Figure 18.AA.6
Using a simplex to represent a three-player TU-game with utilities normalized so that $V(\{i\}) = 0$.

preferences are quasilinear). Then the characteristic function is

$$v(I) = 3, \qquad v(\{1, 2\}) = v(\{1, 3\}) = v(\{2, 3\}) = 3 \qquad \text{and} \qquad v(\{i\}) = 0,\ i = 1, 2, 3. \quad \blacksquare$$

All the developments in this appendix will be invariant to changes in the origins of individual utilities; thus we can fix these arbitrarily. The usual convention is to put $v(\{i\}) = 0$ for every i.

In Figure 18.AA.6 we represent a diagrammatic device for the case of three-player games that is particularly useful. Instead of working in three dimensions, we look at the two-dimensional simplex that exhibits all possible divisions of $v(I)$ subject to the condition that $u_i \geq 0$ for all i [which means, in the normalization just discussed, that $u_i \geq v(\{i\})$]. The other sets in the diagram represent, for every two-person coalition $\{i, h\}$, the utility combinations in the simplex satisfying $u_i + u_h \leq v(\{i, h\})$.

We now turn to a presentation of two well-known solution concepts for cooperative games: the *core* and the *Shapley value.*

The Core

The first solution concept we review is the *core*: the set of feasible utility outcomes with the property that no coalition could on its own improve the payoffs of all its members. An empty core is indicative of competitive instability in the situation being modeled. If the core is nonempty and small, then we could say that coalitional

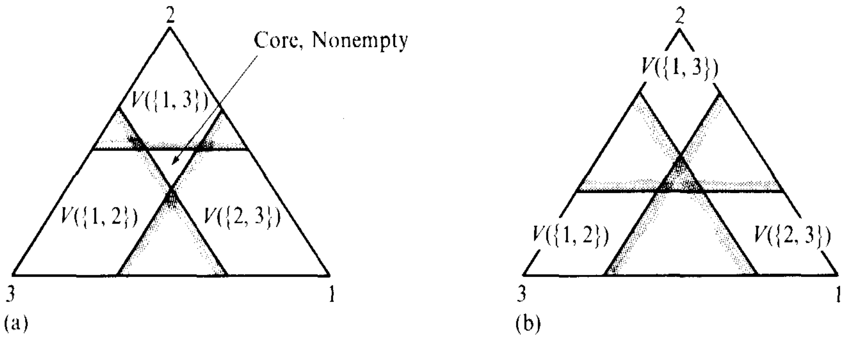


Figure 18.AA.7
(a) A TU-game with a nonempty core.
(b) A TU game with an empty core.

competition by itself brings about a sharply determined outcome. If it is nonempty and large, then coalitional competition alone does not narrow down the possible outcomes very much.

Definition 18.AA.5: Given a game in characteristic form (I, V), the utility outcome $u \in \mathbb{R}^I$ is *blocked*, or *improved upon*, by a coalition $S \subset I$ if there exists $u'^S \in V(S)$ such that $u_i^S < u_i'^S$ for all $i \in S$.

If the game is a TU game (I, v) then the outcome $u = (u_1, \ldots, u_I)$ is blocked by S if and only if $\sum_{i \in S} u_i < v(S)$.

Definition 18.AA.6: A utility outcome $u = (u_1, \ldots, u_I)$ that is feasible for the grand coalition [i.e., $u \in V(I)$] is in the *core* of the characteristic form game (I, V) if there is no coalition S that blocks u.

In TU games the core is the set of utility vectors $u = (u_1, \ldots, u_I)$ satisfying the linear inequalities

$$\sum_{i \in S} u_i \geq v(S) \text{ for all } S \subset I \qquad \text{and} \qquad \sum_{i \in I} u_i \leq v(I).$$

Figure 18.AA.7(a) depicts a three-player game with nonempty core. In contrast, in Figure 18.AA.7(b) the core is empty. See Exercise 18.AA.1 for a set of necessary and sufficient conditions in the TU case for the nonemptiness of the core.

Exercise 18.AA.2: Show that any TU game with a nonempty core must satisfy: For any two coalitions $S, T \subset I$ such that $S \cap T = \varnothing$ and $S \cup T = I$, we have $v(S) + v(T) \leq v(I)$.

Example 18.AA.4: *Majority Voting, Once Again.* For the majority voting games described in Examples 18.AA.2 and 18.AA.3, the core is empty. In the latter, which is a TU game, this is clear enough: if $u_1 + u_2 + u_3 = 3$ then $u_i + u_h < 3$ for some i, h. Hence the coalition $\{i, h\}$ will block. For the former (nontransferable utility) game, note that the outcomes $(2, 1, 0)$, $(1, 0, 2)$, $(0, 2, 1)$ are blocked, respectively, by the coalitions $\{2, 3\}$ using a_3, $\{1, 2\}$ using a_1, and $\{1, 3\}$ using a_2. These examples constitute instances of the so called *Condorcet paradox* (which we have already encountered in Section 1.B and will see again in Section 21.C). They are illustrative of an inherent instability of majority voting. ■

Example 18.AA.5: *Economies, Again.* The economic example in Example 18.AA.1 was extensively studied in Section 18.B. Note that the concept of the core discussed there is identical with the concept considered here for games in characteristic form.[27] We conclude, therefore that if a Walrasian equilibrium exists then the core is nonempty. ■

Example 18.AA.6: *Single-Input, Increasing Returns Production Function.* Consider a one-input, one-output world in which there is a publicly available technology $f(z)$ which is continuous and satisfies $f(0) = 0$. There are I players. Each player i cares only about the consumption of the output good and owns an amount ω_i of input. Assuming that utility is transferable, we can define a TU characteristic function by $v(S) = f(\sum_{i \in S} \omega_i)$. The core of this game will be nonempty whenever the technology exhibits nondecreasing returns to scale, that is, whenever average product $f(z)/z$ is *nondecreasing*. [In particular, if $f(\cdot)$ is convex, that is, if we have a nondecreasing marginal product, then $f(z)/z$ is nondecreasing.] To verify this, suppose that we distribute the product proportionally:

$$u_h = \frac{\omega_h}{\sum_{i \in I} \omega_i} f\left(\sum_{i \in I} \omega_i\right)$$

for every $h \in I$. Then, for any $S \subset I$ we have

$$\sum_{h \in S} u_h = \frac{\sum_{h \in S} \omega_h}{\sum_{i \in I} \omega_i} f\left(\sum_{i \in I} \omega_i\right) \geq f\left(\sum_{h \in S} \omega_h\right) = v(S),$$

where the inequality follows because average product is nondecreasing. We conclude that this proportional distribution of output belongs to the core. In Exercise 18.AA.3 you are asked to show that if average product is constant then the proportional allocation is the only allocation in the core. It is also intuitively clear that the more pronounced is the degree of increasing returns, the more difficult it will be for proper subgroups to do better by their own means (they will have relatively low average product) and therefore the more we could depart from the proportional allocation while remaining in the core. Hence, for this sort of one-dimensional distribution problem, the larger the degree of increasing returns, the larger the core will be.[28] ■

The Shapley Value

The core tries to capture how the possible outcomes of a game may be shaped by coalitional competitive forces. It is the simplest solution concept in what could be called the *descriptive* side of cooperative game theory. We shall now investigate a solution concept, the *value*, whose motivation is normative. It attempts to describe

27. The connection of the solution concept proposed by Edgeworth (1881) with the modern game-theoretic notion of the core was made in Shubik (1959).

28. On the other hand, if $f(\cdot)$ exhibits strictly decreasing returns then it follows from Exercise 18.AA.2 that the core is empty [indeed, $v(S) + v(T) > v(I)$ for any partition of I into two coalitions S, T].

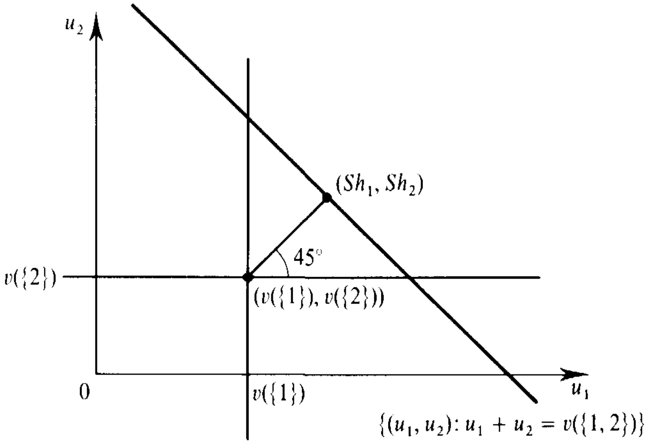


Figure 18.AA.8
Egalitarian division for two-player games.

a reasonable, or "fair," way to divide the gains from cooperation, *taking as a given* the strategic realities captured by the characteristic form.[29]

We study only the TU case, for which the theory is particularly simple and well established. The central concept is then a certain solution called the *Shapley value*.[30]

Suppose that individual utilities are measured in dollars and that, so to speak, society has decided that dollars of utility of different participants are of comparable social worth. The criterion of fairness to which value theory adheres is *egalitarianism*: the aim is to distribute the gains from trade equally.

To see what the egalitarian principle could mean in the current TU context let us begin with a two-player game $(I, v) = (\{1, 2\}, v)$. Then the gains (or losses, if super-additivity fails) from cooperation are

$$v(I) - v(\{1\}) - v(\{2\}).$$

Therefore, the obvious egalitarian solution, which we denote $(Sh_1(I, v), Sh_2(I, v))$, is (see Figure 18.AA.8)

$$Sh_i(I, v) = v(\{i\}) + \tfrac{1}{2}(v(I) - v(\{1\}) - v(\{2\})), \qquad i = 1, 2. \tag{18.AA.1}$$

How should we define the egalitarian solution $(Sh_1(I, v), \ldots, Sh_I(I, v))$ for an arbitrary TU-game (I, v)? We have already solved the problem for two-player games. It is suggestive to *rewrite* expression (18.AA.1) as

$$Sh_1(I, v) - Sh_1(\{1\}, v) = Sh_2(I, v) - Sh_2(\{2\}, v),$$
$$Sh_1(I, v) + Sh_2(I, v) = v(I),$$

where we put $Sh_i(\{i\}, v) = v(\{i\})$. In words, this says that utility differences are preserved: *What player 1 gets out of the presence of player 2 is the same as player 2 gets out of the presence of player 1.* This points to a recursive definition: Given $S \subset I$, denote by (S, v) the TU-game obtained by restricting $v(\cdot)$ to the subsets of S [this is called a *subgame* of (I, v)]. Then we could say that a family of numbers $\{Sh_i(S, v)\}_{S \subset I, i \in S}$ constitutes an *egalitarian solution* if, for every subgame (S, v) and

29. Thus, the redistributional fairness considerations that we will discuss in Sections 22.B and 22.C, based on notions of absolute justice, are alien to the value.

30. It is named after L. Shapley, who proposed it in his Ph.D. dissertation at Princeton (in 1953).

players $i, h \in S$, utility differences are preserved in a manner similar to the two-player case:

$$Sh_i(S, v) - Sh_i(S\backslash\{h\}, v) = Sh_h(S, v) - Sh_h(S\backslash\{i\}, v)$$
$$\text{for all } S \subset I,\ i, h \in S, \qquad (18.AA.2)$$
$$\sum_{i \in S} Sh_i(S, v) = v(S) \qquad \text{for all } S \subset I,$$

Expressions (18.AA.2) determine the numbers $Sh_i(S, v)$, $i \in S$, uniquely. This is clear for $Sh_i(\{i\}, v)$. From here we can then proceed inductively. Suppose that we have defined $Sh_i(S, v)$ for all $S \subset I$, $S \neq I$, $i \in S$. We show that there is one and only one way to define $Sh_i(I, v)$, $i \in I$. To this effect, note that (18.AA.2) allows us to express every $Sh_i(I, v)$ as a function of $Sh_1(I, v)$ and of already determined numbers:

$$Sh_i(I, v) = Sh_1(I, v) + Sh_i(I\backslash\{1\}, v) - Sh_1(I\backslash\{i\}, v) \qquad \text{for all } i \neq 1.$$

Then to determine $Sh_1(I, v)$ use $\sum_{i \in I} Sh_i(I, v) = v(I)$. Specifically,

$$Sh_1(I, v) = \frac{1}{I}\left[v(I) - \sum_{i \neq 1} Sh_i(I\backslash\{1\}, v) + \sum_{i \neq 1} Sh_1(I\backslash\{i\}, v)\right].$$

Definition 18.AA.7: The *Shapley value* of a game (I, v), denoted

$$Sh(I, v) = (Sh_1(I, v), \ldots, Sh_I(I, v)),$$

is the single outcome consistent with expression (18.AA.2).

We can compute $Sh_i(I, v)$ in a direct and interesting manner as follows. For any $S \subset I$ and $i \notin S$ let $m(S, i) = v(S \cup \{i\}) - v(S)$ be the *marginal contribution* of i to coalition S.[31] For any ordering π of the players in I (technically, π is a one-to-one function from I to I) denote by $S(\pi, i) \subset I$ the set of players that come before i in the ordering π [technically, $S(\pi, i) = \{h\colon \pi(h) < \pi(i)\}$]. Note that, for any given ordering π, if we consider the marginal contributions of every player i to the set of the predecessors of i in the ordering π, then the sum of these marginal contributions must exactly exhaust $v(I)$; that is, $\sum_{i \in I} m(S(\pi, i), i) = v(I)$. It then turns out that $Sh_i(I, v)$ is the *average marginal contribution* of i to the set of her predecessors, where the average is taken over all orderings (held to be equally likely). Since the total number of orderings is $I!$ this gives

$$Sh_i(I, v) = \frac{1}{I!}\sum_{\pi} m(S(\pi, i), i). \qquad (18.AA.3)$$

where the sum is taken over all orderings π of the players in I.

Example 18.AA.7: *Glove Market.* Consider the three-player game defined by

$$v(\{1, 2, 3\}) = 1,$$
$$v(\{1, 3\}) = v(\{2, 3\}) = 1, \qquad v(\{1, 2\}) = 0,$$
$$v(\{1\}) = v(\{2\}) = v(\{3\}) = 0.$$

If the utility of a matched pair of gloves is 1, while an unmatched pair is worth nothing, then this game could arise from a situation in which players 1 and 2 own

31. Whenever we compute marginal contributions we follow the convention $v(\varnothing) = 0$. Therefore, $m(S, i) = v(\{i\})$ whenever $S = \varnothing$.

one right-hand glove each, while player 3 owns a left-hand glove. Let us compute $Sh_3(I, v)$. There are six possible orderings of the players:

$$\{1, 2, 3\}, \{1, 3, 2\}, \{2, 1, 3\}, \{2, 3, 1\}, \{3, 1, 2\}, \text{ and } \{3, 2, 1\}.$$

The marginal contribution of player 3 to its predecessors in each of these orderings is, respectively: 1, 1, 1, 1, 0, and 0. The average of these numbers is $\frac{2}{3}$; hence $Sh_3(I, v) = \frac{2}{3}$. Similarly, we get $Sh_1(I, v) = Sh_2(I, v) = \frac{1}{6}$. Note that these numbers satisfy (18.AA.2). For example,

$$Sh_3(I, v) - Sh_3(I\setminus\{1\}, v) = \tfrac{2}{3} - \tfrac{1}{2} = \tfrac{1}{6} - 0 = Sh_1(I, v) - Sh_1(I\setminus\{3\}, v). \quad \blacksquare$$

We can give a more explicit formula than (18.AA.3) for $Sh_i(I, v)$. The probability that in a random ordering a given coalition $T \subset I$, $i \in T$, arises as the union of i and its predecessors equals the probability that i is in the Tth place,[32] which is simply $1/I$, multiplied by the probability that $T\setminus\{i\}$ arises when we randomly select $\# T - 1$ members from the population $I\setminus\{i\}$, which is $(I - \# T)!(\# T - 1)!/(I - 1)!$. Hence, we can rewrite (18.AA.3) as

$$Sh_i(I, v) = \sum_{T \subset I, i \in T} [(I - \# T)!(\# T - 1)!/I!](v(T) - v(T\setminus\{i\})). \tag{18.AA.4}$$

In Exercise 18.AA.4 you are asked to verify that if we were to define the Shapley value by (18.AA.4), or (18.AA.3), then equations (18.AA.2) would be satisfied; this means that, indeed, (18.AA.3) or (18.AA.4) provide correct formulas for the Shapley value.

We now put on record, rather informally, some of the basic properties of the Shapley value.

(a) *Efficiency.* $\sum_i Sh_i(I, v) = v(I)$; that is, no utility is wasted.

(b) *Symmetry.* If the games (I, v) and (I, v') are identical, except that the roles of players i and h are permuted,[33] then $Sh_i(I, v) = Sh_h(I, v')$. In words: The Shapley values do not depend on how we label players; only their position in the game, as summarized by the characteristic function, matters.

(c) *Linearity.* Note from (18.AA.3) or (18.AA.4) that the Shapley values depend linearly on the data, that is, on the coefficients $v(S)$ defining the game.

(d) *Dummy axiom.* Suppose that a player i contributes nothing to the game; that is, $v(S \cup \{i\}) - v(S) = 0$ for *all* $S \subset I$. Then $Sh_i(I, v) = 0$. This important property follows directly from (18.AA.3): The marginal contribution of player i to *any* coalition is null; hence its average is also null.

These four properties fully characterize the Shapley value. Although the proof of this fact is not difficult, we shall not give it here. See Exercises 18.AA.5 and 18.AA.6 for the discussion of some examples.

Given a game, the Shapley value assigns to it a single outcome. In contrast, the core solution assigns a set. We point out that the Shapley value need not belong to the

32. The symbol $\# T$ denotes the number of players in a coalition T.

33. Precisely, $v(S) = v'(S)$ whenever $i \in S$ and $h \in S$, $v(S) = v'(S)$ whenever $i \notin S$ and $h \notin S$, $v(S) = v'((S\setminus\{i\}) \cup \{h\})$ whenever $i \in S$ and $h \notin S$, and $v(S) = v'((S\setminus\{h\}) \cup \{i\})$ whenever $h \in S$ and $i \notin S$.

core. In a sense, we already know this because the Shapley value is defined for all games and there are games for which the core is empty. But the phenomenon can also occur if the core is nonempty. To see this, let us reexamine the glove market of Example 18.AA.7.

Example 18.AA.7 continued: In the glove market example a core utility outcome is $(0, 0, 1)$. Moreover, this is the *only* outcome in the core. Indeed, if (u_1, u_2, u_3) with $\sum_i u_i = 1$ has, say, $u_1 > 0$, then the coalition $\{2, 3\}$ can block by means of $(0, u_2 + \frac{1}{2}u_1, u_3 + \frac{1}{2}u_1)$. In effect, at the core the two owners of right-hand gloves undercut each other until they charge a price of zero. In contrast, the Shapley value, while heavily skewed towards player 3, nonetheless leaves something to the other two players ($\frac{1}{6}$ to each of them). ■

There is an important class of games for which the Shapley value belongs to the core. They are games characterized by the presence of a type of pronounced increasing returns to scale.

Definition 18.AA.8: A game (I, v) is *convex* if for every i the marginal contribution of i is larger to larger coalitions. Precisely, if $S \subset T$ and $i \in I \backslash T$, then

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T).$$

Example 18.AA.8: *Complementary Inputs.* Let $f(z_1, \ldots, z_N)$ be a production function displaying increasing marginal productivities with respect to all inputs; that is, $\partial^2 f(z)/\partial z_h \partial z_k \geq 0$ for all z and h, k. Suppose that every player $i \in I$ is endowed with a vector of inputs $\omega_i \in \mathbb{R}^n_+$. Then we can define a TU-game by $v(S) = f(\sum_{i \in S} \omega_i)$. In Exercise 18.AA.8 you are asked to show that this game is convex. A warning on terminology: if $N = 1$ the previous condition simply says that $f(\cdot)$ is convex and, thus, the convexity of $f(\cdot)$ suffices for the convexity of the game; but for $N > 1$ the condition $\partial^2 f(z)/\partial z_h \partial z_k \geq 0$ for all z and h, k, is neither necessary nor sufficient for the convexity of $f(\cdot)$. In fact, the convexity of $f(\cdot)$ is far from sufficient for the convexity of the game (see Exercise 18.AA.8). ■

We can then show the result in Proposition 18.AA.1.

Proposition 18.AA.1: If a game (I, v) is convex then its Shapley value utility outcome $Sh(I, v) = (Sh_1(I, v), \ldots, Sh_I(I, v))$ belongs to the core (in particular, the core is nonempty).

Proof: It is enough to show that if $i \in S \subset T$ then $Sh_i(S, v) \leq Sh_i(T, v)$. Indeed, for any $S \subset I$ this implies that $v(S) = \sum_{i \in S} Sh_i(S, v) \leq \sum_{i \in S} Sh_i(I, v)$ and therefore the coalition S cannot block.

To prove the claimed property it suffices to consider $i \in S$ and $T = S \cup \{h\}$. Given an ordering π of S denote by $m(\pi, i)$ the marginal contribution of i to its predecessors in S and according to the ordering π by $m'(\pi, i)$ the average marginal contribution of i to its predecessors in T when the average is taken over the $\# T$ orderings of T differing from the given ordering π of S only by the placement of h. Then

$$Sh_i(S, v) = \frac{1}{\# S!} \sum_\pi m(\pi, i) \quad \text{and} \quad Sh_i(T, v) = \frac{1}{\# S!} \sum_\pi m'(\pi, i).$$

Note that for every ordering π of S we must have $m'(\pi, i) \geq m(\pi, i)$: If we place h after i then the marginal contribution of i to its predecessors in T is still $m(\pi, i)$; if we place h before i then, by the convexity condition, this marginal contribution is at least $m(\pi, i)$. We conclude that $Sh_i(T, v) \geq Sh_i(S, v)$, as we wanted to show. ■

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EXERCISES

18.B.1^{A} Show that Walrasian allocations are in the core for the model with a constant returns technology described in Section 18.B.

18.B.2[B] Exhibit an example of a nonequal-treatment core allocation in a three-consumer exchange economy with continuous, strictly convex, strongly monotone preferences. Can an example be given with only two consumers?

18.B.3[A] Give a direct proof (i.e., not using properties of the core) that a Walrasian allocation of an economy with continuous, strictly convex preferences has the equal-treatment property.

18.B.4[A] Use Taylor's formula to complete the proof of Proposition 18.B.3.

18.B.5[B] Consider an economy composed of $2I + 1$ consumers. Of these, I each own one right shoe and $I + 1$ each own a left shoe. Shoes are indivisible. Everyone has the same utility function, which is Min $\{R, L\}$, where R and L are, respectively, the quantities of right and left shoes consumed.

(a) Show that any allocation of shoes that is matched (i.e., every individual consumes the same number of shoes of each kind) is a Pareto optimum, and conversely.

(b) Which Pareto optima are in the core of this economy? (This time, in the definition of the core allow for weak dominance in blocking.)

(c) Let p_R and p_L be the respective prices of the two kinds of shoes. Find the Walrasian equilibria of this economy.

(d) Comment on the relationship between the core and the Walrasian equilibria in this economy.

18.C.1[C] Establish the properties of effective budget sets claimed in the discussion of Example 18.C.3. You can restrict yourself to the case $L = 2$.

18.D.1[B] Consider an Edgeworth box with continuous, strictly convex and monotone preferences. Show that every feasible allocation where both consumers are at least as well off as at their initial endowments is self-selective.

18.E.1[B] In text.

18.E.2[A] Use the envelope theorem (see Section M.L of the Mathematical Appendix) to derive expression (18.E.5).

18.E.3[B] By considering an example with L-shaped preferences for two non-numeraire goods (hence, the utility function cannot be differentiable), argue that it is possible that at a Walrasian allocation with a continuum of traders every trader gets less than her marginal contribution.

18.AA.1[B] A collection of coalitions $S_1, \ldots, S_N \subset I$ is a *generalized partition* if we can assign a weight $\delta_n \in [0, 1]$ to every $1 \leq n \leq N$ such that, for every player $i \in I$, we have $\sum_{\{n:\, i \in S_n\}} \delta_n = 1$. Exhibit examples of generalized partitions, with the corresponding weights.

We say that a TU-game (I, v) is *balanced* if for every generalized partition we have $\sum_n \delta_n v(S_n) \leq v(I)$, where δ_n are the corresponding partition weights. Show that the game has a nonempty core if and only if it is balanced. [*Hint*: Appeal to the duality theorem of linear programming (see Section M.M of the Mathematical Appendix).]

18.AA.2[A] In text.

18.AA.3[A] Show that the proportional allocation of Example 18.AA.6 is the only allocation in the core if average product is constant.

18.AA.4[C] Show that if the Shapley value is defined by formula (18.AA.4)—or, equivalently, by (18.AA.3)—then the preservation of differences expression (18.AA.2) is satisfied.

18.AA.5$^{\text{B}}$ We say that a game (I, v) is a *unanimity* game if there is a nonempty $S \subset I$ such that $v(T) = v(S)$ if $S \subset T$ and $v(T) = 0$ otherwise. Show then that under the efficiency, symmetry, and dummy axioms we are led to distribute $v(S)$ equally across the members of S.

18.AA.6$^{\text{B}}$ Show that any TU-game (I, v) can be expressed as a linear combination of unanimity games. Then use the Exercise 18.AA.5 and the linearity axiom to show that there is a unique solution satisfying the efficiency, symmetry, dummy, and linearity axioms. Connect your discussion with the Shapley value.

18.AA.7$^{\text{C}}$ Show that the production game described in Example 18.AA.8 is convex.

18.AA.8$^{\text{B}}$ In the context of the production example of Example 18.AA.8, give an example of a two-input production function that is convex (as a function) but for which, nonetheless, the core is empty (thus, the induced game cannot be convex).

18.AA.9$^{\text{B}}$ Consider the game with four players defined by $v(\{i\}) = 0$, $v(\{12\}) = v(\{34\}) = 0$, $v(\{13\}) = v(\{14\}) = v(\{23\}) = v(\{24\}) = 1$, $v(\{ijk\}) = 1$ for all three-player coalitions $\{ijk\}$, and $v(\{1234\}) = 2$.

(a) Show that this is the game that you would get from the utility production technology Min $\{z_1, z_2\}$, where z_1 and z_2 are the amounts of two factors, if the factor endowments of the four consumers are $\omega_1 = \omega_2 = (1, 0)$ and $\omega_3 = \omega_4 = (0, 1)$. 1

(b) Show that the core of this game contains all points of the form $(\alpha, \alpha, 1 - \alpha, 1 - \alpha)$ for $\alpha \in [0, 1]$.

(c) Show that if $v(\{134\})$ is increased to 2, holding all other coalition values constant, there is then only one point in the core. Compare the welfare of player 1 at this point to what she would get at all the points in the core before the increase in $v(\{134\})$.

(d) Compute the Shapley value of the game [before the modification in **(c)**] without using the brute-force enumeration technique. [*Hint*: Use symmetry considerations and other axiomatically based simplifications to go part of the way to the answer.]

(e) How does the Shapley value change under the modification of part **(c)**?. Discuss the difference between the changes in the Shapley value and in the core.

18.AA.10$^{\text{B}}$ Consider a firm constituted by two divisions. The firm must provide overhead in the form of space, (x_1, x_2), to each of them. The cost of *aggregate* amounts of space is given by $C(x_1 + x_2) = (x_1 + x_2)^\gamma$, $0 < \gamma < 1$.

(a) Suppose that, whatever the usage of space (x_1, x_2), the total cost must be *exactly* allocated between the two divisions. Propose a cost allocation system based on the Shapley value to accomplish this.

(b) Compute the marginal cost imposed on each of the two divisions [according to the cost allocation system identified in **(a)**] whenever a division increases its usage of space.

(c) Suppose now that the profits accruing to the two divisions are $\alpha_1 x_1$ and $\alpha_2 x_2$, respectively (we assume that $\alpha_1 > 0$ and $\alpha_2 > 0$), and that each division uses space to the point where marginal profits equal own marginal costs [as determined in **(b)**]. Will this lead to an efficient (that is, profit-maximizing) choice of overhead?

(d) Is there any distribution rule $\psi_1(x_1, x_2)$, $\psi_2(x_1, x_2)$, with $\psi_1(x_1, x_2) + \psi_2(x_1, x_2) = C(x_1 + x_2)$ for all (x_1, x_2), that leads to efficient decentralized choice [in the sense of **(c)**] for all α_1, α_2? [*Hint*: Consider the externality imposed by each division on the other.]

19 General Equilibrium Under Uncertainty

19.A Introduction

In this chapter, we apply the general equilibrium framework developed in Chapters 15 to 18 to economic situations involving the exchange and allocation of resources under conditions of uncertainty. In a sense, this chapter offers the equilibrium counterpart of the decision theory presented in Chapter 6 (and which we recommend you review at this point).

We begin, in Section 19.B, by formalizing uncertainty by means of *states of the world* and then introducing the key idea of a *contingent commodity*: a commodity whose delivery is conditional on the realized state of the world. In Section 19.C we use these tools to define the concept of an *Arrow–Debreu equilibrium*. This is simply a Walrasian equilibrium in which contingent commodities are traded. It follows from the general theory of Chapter 16 that an Arrow–Debreu equilibrium results in a Pareto optimal allocation of risk.

In Section 19.D, we provide an important reinterpretation of the concept of Arrow–Debreu equilibrium. We show that, under the assumptions of *self-fulfilling*, or *rational, expectations*, Arrow–Debreu equilibria can be implemented by combining trade in a certain restricted set of contingent commodities with *spot trade* that occurs *after* the resolution of uncertainty. This results in a significant reduction in the number of ex ante (i.e., before uncertainty) markets that must operate.

In Section 19.E, we generalize our analysis. Instead of trading contingent commodities prior to the resolution of uncertainty, agents now trade *assets*; and instead of an Arrow–Debreu equilibrium we have the notion of a *Radner equilibrium*. We also discuss here the important notion of *arbitrage* among assets. The material of this section lies at the foundations of a very rich body of finance theory [good introductions are Duffie (1992) and Huang and Litzenberger (1988)].

In Section 19.F, we briefly illustrate some of the welfare difficulties raised by the possibility of *incomplete markets*, that is, by the possibility of there being too few asset markets to guarantee a fully Pareto optimal allocation of risk.

Section 19.G is devoted to the issue of the objectives of the firm under conditions of uncertainty. In particular, it gives sufficient conditions for shareholders to agree unanimously on the objective of *market value maximization*.

Section 19.H takes a close look at the informational requirements of the theory developed in this chapter. We see that the theory applies well to situations of *symmetric* information across consumers (reviewed in Section 19.H); but its applicability is more problematic in situations of *asymmetric* information. This provides a further argument for the techniques developed in Chapters 13 and 14 for the study of asymmetric information problems.

For additional material and references on the topic of this chapter, see the textbooks of Huang and Litzenberger (1988) and Duffie (1992) already mentioned, or, at a more advanced level, Radner (1982) and Magill and Shafer (1991).

19.B A Market Economy with Contingent Commodities: Description

As in our previous chapters, we contemplate an environment with L physical commodities, I consumers, and J firms. The new element is that technologies, endowments, and preferences are now *uncertain*.

Throughout this chapter, we represent uncertainty by assuming that technologies, endowments, and preferences depend on the *state of the world*. The concept of state of the world was already introduced in Section 6.E. A state of the world is to be understood as a complete description of a possible outcome of uncertainty, the description being sufficiently fine for any two distinct states of the world to be mutually exclusive. We assume that an exhaustive set S of states of the world is given to us. For simplicity we take S to be a finite set with (abusing notation slightly) S elements. A typical element is denoted $s = 1, \dots, S$.

We state in Definition 19.B.1 the key concepts of a *(state-)contingent commodity* and a *(state-)contingent commodity vector*. Using these concepts we shall then be able to express the dependence of technologies, endowments, and preferences on the realized states of the world.

Definition 19.B.1: For every physical commodity $\ell = 1, \dots, L$ and state $s = 1, \dots, S$, a unit of *(state-)contingent commodity* ℓs is a title to receive a unit of the physical good ℓ if and only if s occurs. Accordingly, a *(state-)contingent commodity vector* is specified by

$$x = (x_{11}, \dots, x_{L1}, \dots, x_{1S}, \dots, x_{LS}) \in \mathbb{R}^{LS},$$

and is understood as an entitlement to receive the commodity vector $(x_{1s}, \dots, x_{Ls})$ if state s occurs.[1]

We can also view a contingent commodity vector as a collection of L *random variables*, the ℓth random variable being $(x_{\ell 1}, \dots, x_{\ell S})$.

With the help of the concept of contingent commodity vectors, we can now describe how the characteristics of economic agents depend on the state of the world. To begin, we let the endowments of consumer $i = 1, \dots, I$ be a contingent commodity vector:

$$\omega_i = (\omega_{11i}, \dots, \omega_{L1i}, \dots, \omega_{1Si}, \dots, \omega_{LSi}) \in \mathbb{R}^{LS}.$$

1. As usual, a negative entry is understood as an obligation to deliver.

The meaning of this is that if state s occurs then consumer i has endowment vector $(\omega_{1si}, \ldots, \omega_{Lsi}) \in \mathbb{R}^L$.

The preferences of consumer i may also depend on the state of the world (e.g., the consumer's enjoyment of wine may well depend on the state of his health). We represent this dependence formally by defining the consumer's preferences over contingent commodity vectors. That is, we let the preferences of consumer i be specified by a rational preference relation $\succsim_i$ defined on a consumption set $X_i \subset \mathbb{R}^{LS}$.

Example 19.B.1: As in Section 6.E, the consumer evaluates contingent commodity vectors by first assigning to state s a probability π_{si} (which could have an objective or a subjective character), then evaluating the physical commodity vectors at state s according to a Bernoulli state-dependent utility function $u_{si}(x_{1si}, \ldots, x_{Lsi})$, and finally computing the expected utility.[2] That is, the preferences of consumer i over two contingent commodity vectors $x_i, x_i' \in X_i \subset \mathbb{R}^{LS}$ satisfy

$$x_i \succsim_i x_i' \quad \text{if and only if} \quad \sum_s \pi_{si} u_{si}(x_{1si}, \ldots, x_{Lsi}) \geq \sum_s \pi_{si} u_{si}(x'_{1si}, \ldots, x'_{Lsi}).$$

■

It should be emphasized that the preferences $\succsim_i$ are in the nature of ex ante preferences: the random variables describing possible consumptions are evaluated before the resolution of uncertainty.

Similarly, the technological possibilities of firm j are represented by a production set $Y_j \subset \mathbb{R}^{LS}$. The interpretation is that a *(state-)contingent production plan* $y_j \in \mathbb{R}^{LS}$ is a member of Y_j if for every s the input–output vector $(y_{1sj}, \ldots, y_{Lsj})$ of physical commodities is feasible for firm j when state s occurs.

Example 19.B.2: Suppose there are two states, s_1 and s_2, representing good and bad weather. There are two physical commodities: seeds ($\ell = 1$) and crops ($\ell = 2$). In this case, the elements of Y_j are four-dimensional vectors. Assume that seeds must be planted before the resolution of the uncertainty about the weather and that a unit of seeds produces a unit of crops if and only if the weather is good. Then

$$y_j = (y_{11j}, y_{21j}, y_{12j}, y_{22j}) = (-1, 1, -1, 0)$$

is a feasible plan. Note that since the weather is unknown when the seeds are planted, the plan $(-1, 1, 0, 0)$ is not feasible: the seeds, if planted, are planted in both states. Thus, in this manner we can imbed into the structure of Y_j constraints on production related to the timing of the resolution of uncertainty.[3] ■

To complete the description of an economy in a manner parallel to Chapters 16 and 17 it only remains to specify ownership shares for every consumer i and firm j. In principle, these shares could also be state-contingent. It will be simpler, however, to let $\theta_{ji} \geq 0$ be the share of firm j owned by consumer i whatever the state. Of course $\sum_j \theta_{ji} = 1$ for every i.

2. The discussion in Section 6.E was for $L = 1$. It extends straightforwardly to the current case of $L \geq 1$.

3. A similar point could be made on the consumption side. If, for a particular commodity ℓ, any vector $x_i \in X_i$ is such that all entries $x_{\ell si}$, $s = 1, \ldots, S$, are equal, then we can interpret this as asserting that the consumption of ℓ takes place before the resolution of uncertainty.

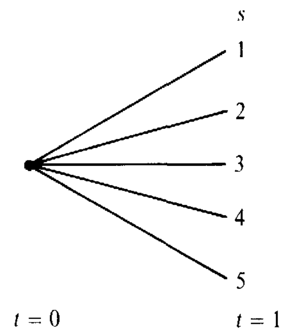

Figure 19.B.1
Two periods. Perfect information at $t = 1$.

Information and the Resolution of Uncertainty

In the setting just described, time plays no explicit formal role. In reality, however, states of the world unfold over time. Figure 19.B.1 captures the simplest example. In the figure, we have a period 0 in which there is no information whatsoever on the true state of the world and a period 1 in which this information has been completely revealed.

We have already seen (Example 19.B.2) how, by conveniently defining consumption and production sets, we can accomodate within our setup the temporal structure of Figure 19.B.1: a commodity that has as part of its physical description its availability at $t = 0$ should never appear in differing amounts across states.

The same methodology can be used to incorporate into the formalism a much more general temporal structure. Suppose we have $T + 1$ dates $t = 0, 1, \ldots, T$ and, as before, S states, but assume that the states emerge gradually through a *tree*, as in Figure 19.B.2. These trees are

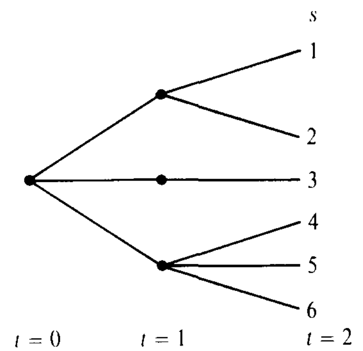

Figure 19.B.2
An information tree: gradual release of information.

similar to those described in Chapter 7. Here final nodes stand for the possible states realized by time $t = T$, that is, for complete histories of the uncertain environment. When the path through the tree coincides for two states, s and s', up to time t, this means that in all periods up to and including period t, s and s' cannot be distinguished.

Subsets of S are called *events*. A collection of events $\mathscr{S}$ is an *information structure* if it is a partition, that is, if for every state s there is $E \in \mathscr{S}$ with $s \in E$ and for any two $E, E' \in \mathscr{S}$, $E \neq E'$, we have $E \cap E' = \varnothing$. The interpretation is that if s and s' belong to the same event in $\mathscr{S}$ then s and s' cannot be distinguished in the information structure $\mathscr{S}$.

To capture formally a situation with sequential revelation of information we look at a *family* of information structures: $(\mathscr{S}_0, \ldots, \mathscr{S}_t, \ldots, \mathscr{S}_T)$. The process of information revelation makes the $\mathscr{S}_t$ increasingly fine: once one has information sufficient to distinguish between two states, the information is not forgotten.

Example 19.B.3: Consider the tree in Figure 19.B.2. We have

$$\mathcal{S}_0 = (\{1, 2, 3, 4, 5, 6\}),$$
$$\mathcal{S}_1 = (\{1, 2\}, \{3\}, \{4, 5, 6\}),$$
$$\mathcal{S}_2 = (\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}). \quad \blacksquare$$

The partitions could in principle be different across individuals. However, except in the last section of this chapter (Section 19.H), we shall assume that the information structure is the same for all consumers.

A pair (t, E) where t is a date and $E \in \mathcal{S}_t$ is called a *date-event*. Date-events are associated with the nodes of the tree. Each date-event except the first has a *unique predecessor*, and each date-event not at the end of the tree has one or more *successors*.

With this temporal modeling it is now necessary to be explicit about the time at which a physical commodity is available. Suppose there is a number H of basic physical commodities (bread, leisure, etc.). We will use the double index ht to indicate the time at which a commodity h is produced, appears as endowment, or is available for consumption. Then x_{hts} stands for an amount of the physical commodity h available at time t along the path of state s.

Fortunately, this multiperiod model can be formally reduced to the timeless structure introduced above. To see this, we define a new set of $L = H(T + 1)$ physical commodities, each of them being one of these double-indexed (i.e., ht) commodities. We then say that a vector $z \in \mathbb{R}^{LS}$ is *measurable* with respect to the family of information partitions $(\mathcal{S}_0, \ldots, \mathcal{S}_T)$ if, for every hts and hts', we have that $z_{hts} = z_{hts'}$ whenever s,s' belong to the same element of the partition $\mathcal{S}_t$. That is, whenever s and s' cannot be distinguished at time t, the amounts assigned to the two states cannot be different. Finally, we impose on endowments $\omega_i \in \mathbb{R}^{LS}$, consumption sets $X_i \subset \mathbb{R}^{LS}$ and production sets $Y_j \subset \mathbb{R}^{LS}$ the restriction that all their elements be measurable with respect to the family of information partitions. With this, we have reduced the multiperiod structure to our original formulation.

19.C Arrow–Debreu Equilibrium

We have seen in Section 19.B how an economy where uncertainty matters can be described by means of a set of states of the world S, a consumption set $X_i \subset \mathbb{R}^{LS}$, an endowment vector $\omega_i \in \mathbb{R}^{LS}$, and a preference relation $\succsim_i$ on X_i for every consumer i, together with a production set $Y_j \subset \mathbb{R}^{LS}$ and profit shares $(\theta_{j1}, \ldots, \theta_{jI})$ for every firm j.

We now go a step further and make a strong assumption. Namely, we postulate the existence of a market for every contingent commodity ℓs. These markets open *before* the resolution of uncertainty, at date 0 we could say. The price of the commodity is denoted $p_{\ell s}$. What is being purchased (or sold) in the market for the contingent commodity ℓs is commitments to receive (or to deliver) amounts of the physical good ℓ if, and when, state of the world s occurs. Observe that although deliveries are contingent, the payments are not. Notice also that for this market to be well defined it is indispensable that all economic agents be able to recognize the occurrence of s. That is, information should be *symmetric* across economic agents. This informational issue will be discussed further in Section 19.H.

Formally, the market economy just described is nothing but a particular case of the economies we have studied in previous chapters. We can, therefore, apply to our market economy the concept of Walrasian equilibrium and, with it, all the theory developed so far. When dealing with contingent commodities it is customary to call the Walrasian equilibrium an *Arrow–Debreu equilibrium.*[4]

Definition 19.C.1: An allocation

$$(x_1^*, \ldots, x_I^*, y_1^*, \ldots, y_J^*) \in X_1 \times \cdots \times X_I \times Y_1 \times \cdots \times Y_J \subset \mathbb{R}^{LS(I+J)}$$

and a system of prices for the contingent commodities $p = (p_{11}, \ldots, p_{LS}) \in \mathbb{R}^{LS}$ constitute an *Arrow–Debreu equilibrium* if:

(i) For every j, y_j^* satisfies $p \cdot y_j^* \geq p \cdot y_j$ for all $y_j \in Y_j$.

(ii) For every i, x_i^* is maximal for $\succsim_i$ in the budget set

$$\{x_i \in X_i \colon p \cdot x_i \leq p \cdot \omega_i + \textstyle\sum_j \theta_{ij} p \cdot y_j^*\}.$$

(iii) $\sum_i x_i^* = \sum_j y_j^* + \sum_i \omega_i$.

The welfare and positive theorems of Chapters 16 and 17 apply without modification to the Arrow–Debreu equilibrium. Recall from Chapter 6, especially Sections 6.C and 6.E, that, in the present context, the convexity assumption takes on an interpretation in terms of risk aversion. For example, in the expected utility setting of Example 19.B.1, the preference relation $\succsim_i$ is convex if the Bernoulli utilities $u_{si}(x_{si})$ are concave (see Exercise 19.C.1).

The Pareto optimality implication of Arrow–Debreu equilibrium says, effectively, that the possibility of trading in contingent commodities leads, at equilibrium, to an efficient allocation of risk.

It is important to realize that at any production plan the profit of a firm, $p \cdot y_j$, is a nonrandom amount of dollars. Productions and deliveries of goods do, of course, depend on the state of the world, but the firm is active in all the contingent markets and manages, so to speak, to insure completely. This has important implications for the justification of profit maximization as the objective of the firm. We will discuss this point further in Section 19.G.

Example 19.C.1: Consider an exchange economy with $I = 2$, $L = 1$, and $S = 2$. This lends itself to an Edgeworth box representation because there are precisely two contingent commodities. In Figures 19.C.1(a) and 19.C.1(b) we have $\omega_1 = (1, 0)$, $\omega_2 = (0, 1)$, and utility functions of the form $\pi_{1i} u_i(x_{1i}) + \pi_{2i} u_i(x_{2i})$, where (π_{1i}, π_{2i}) are the subjective probabilities of consumer i for the two states. Since $\omega_1 + \omega_2 = (1, 1)$ there is no aggregate uncertainty, and the state of the world determines only which consumer receives the endowment of the consumption good. Recall from Section 6.E (especially the discussion preceding Example 6.E.1) that for this model [in which the $u_i(\cdot)$ do not depend on s], the marginal rate of substitution of consumer i at any point where the consumption is the same in the two states equals the probability ratio π_{1i}/π_{2i}.

In Figure 19.C.1(a) the subjective probabilities are the same for the two consumers (i.e., $\pi_{11} = \pi_{12}$) and therefore the Pareto set coincides with the diagonal of the box (the box is a square and so the diagonal coincides with the 45-degree line, where the marginal rates of substitution for the two consumers are equal: $\pi_{11}/\pi_{21} = \pi_{12}/\pi_{22}$). Hence, at equilibrium, the two consumers insure completely; that is, consumer i's equilibrium consumption does not vary across the two states. In Figure 19.C.1(b)

4. See Chapter 7 of Debreu (1959) for a succinct development of these ideas.

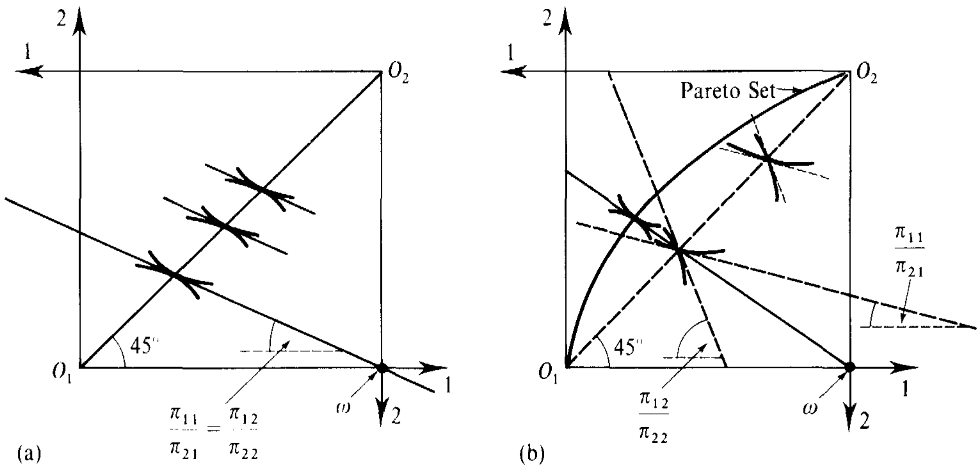

Figure 19.C.1
(a) No aggregate risk: same probability assessments.
(b) No aggregate risk: different probability assessments.

the consumer's subjective probabilities are different. In particular, $\pi_{11} < \pi_{12}$ (i.e., the second consumer gives more probability to state 1). In this case, each consumer's equilibrium consumption is higher in the state he thinks comparatively more likely (relative to the beliefs of the other consumer). ■

Example 19.C.2: The basic framework is as in Example 19.G.1. The difference is that now there is aggregate risk: $\omega_1 + \omega_2 = (2, 1)$. The utilities are state independent and the probability assessments are the same for the two traders: (π_1, π_2). The corresponding Edgeworth box is represented in Figure 19.C.2. We see that at any point of the Pareto set the common marginal rate of substitution is smaller than the ratio of probabilities (see Exercise 19.C.2). Hence at an equilibrium we must have $p_1/p_2 < \pi_1/\pi_2$, or $p_1/\pi_1 < p_2/\pi_2$. If, say, $\pi_1 = \pi_2 = \frac{1}{2}$, then $p_1 < p_2$: The price of one contingent unit of consumption is larger for the state for which the consumption good is scarcer. This constitutes the simplest version of a powerful theme of finance theory: that contingent instruments (in our case, a unit of contingent consumption) are comparatively more valuable if their returns (in our case, the amount of consumption they give in the different states) are negatively correlated with the "market return" (in our case, the random variable representing the aggregate initial endowment). ■

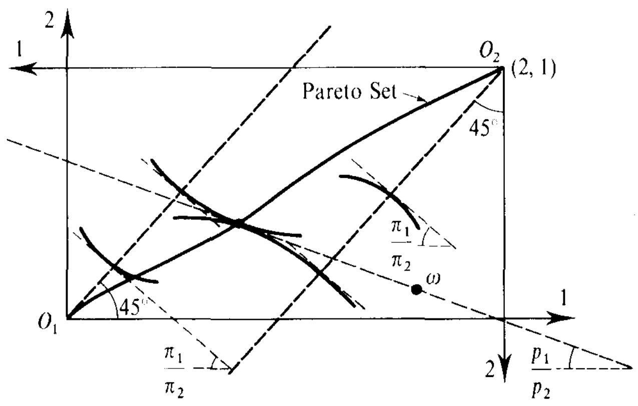

Figure 19.C.2
There is aggregate risk: p_ℓ/π_ℓ negatively correlated with total endowment of commodity ℓ.

19.D Sequential Trade

The Arrow–Debreu framework provides a remarkable illustration of the power of general equilibrium theory. Yet, it is hardly realistic. Indeed, at an Arrow– Debreu equilibrium all trade takes place simultaneously and before the uncertainty is resolved. Trade, so to speak, is a one-shot affair. In reality, however, trade takes place to a large extent sequentially over time, and frequently as a consequence of information disclosures. The aim of this section is to introduce a first model of sequential trade and show that Arrow–Debreu equilibria can be reinterpreted by means of trading processes that actually unfold through time.

To be as simple as possible we consider only exchange economies (see Section 19.G for some discussion of production). In addition, we take $X_i = \mathbb{R}^{LS}_+$ for every i. To begin with, we assume that there are two dates, $t = 0$ and $t = 1$, that there is no information whatsoever at $t = 0$, and that the uncertainty has resolved completely at $t = 1$. Thus, the date-event tree is as in Figure 19.B.1. Again for simplicity, we assume that there is no consumption at $t = 0$. (We refer to Exercise 19.D.3 for the more general situation.)

Suppose that markets for the LS possible contingent commodities are set up at $t = 0$, and that $(x_1^*, \ldots, x_I^*) \in \mathbb{R}^{LSI}$ is an Arrow–Debreu equilibrium allocation with prices $(p_{11}, \ldots, p_{LS}) \in \mathbb{R}^{LS}$. Recall that these markets are for delivery of goods at $t = 1$ (they are commonly called *forward markets*). When period $t = 1$ arrives, a state of the world s is revealed, contracts are executed, and every consumer i receives $x_{si}^* = (x_{1si}^*, \ldots, x_{Lsi}^*) \in \mathbb{R}^L$. Imagine now that, after this but before the actual consumption of x_{si}^*, markets for the L physical goods were to open at $t = 1$ (these are called *spot markets*). Would there be any incentive to trade in these markets? The answer is "no." To see why, suppose that there were potential gains from trade among the consumers. That is, that there were $x_{si} = (x_{1si}, \ldots, x_{Lsi})$ for $i = 1, \ldots, I$, such that $\sum_i x_{si} \le \sum_i \omega_{si}$ and $(x_{1i}^*, \ldots, x_{si}, \ldots, x_{Si}^*) \succsim_i (x_{1i}^*, \ldots, x_{si}^*, \ldots, x_{Si}^*)$ for all i, with at least one preference strict. It then follows from the definition of Pareto optimality that the Arrow–Debreu equilibrium allocation $(x_1^*, \ldots, x_I^*) \in \mathbb{R}^{LSI}$ is not Pareto optimal, contradicting the conclusion of the first welfare theorem.[5] In summary, at $t = 0$ the consumers can trade directly to an overall Pareto optimal allocation; hence there is no reason for further trade to take place. In other words, ex ante Pareto optimality implies ex post Pareto optimality and thus no ex post trade.

Matters are different if not all the LS contingent commodity markets are available at $t = 0$. Then the initial trade to a Pareto optimal allocation may not be feasible and it is quite possible that ex post (i.e., after the revelation of the state s) the resulting consumption allocation is not Pareto optimal. There would then be an incentive to reopen the markets and retrade.

5. Alternatively, consider the Arrow–Debreu equilibrium prices for the L contingent commodities corresponding to state s: $p_s = (p_{1s}, \ldots, p_{Ls})$. Then p_s, viewed as a system of spot prices at s, induces, for the initial endowment vector $(x_{s1}^*, \ldots, x_{sI}^*)$, a null excess demand for all traders (and therefore clears markets). Indeed, if $U_i(x_{1i}, \ldots, x_{Si})$ is a utility function for $\succsim_i$ and $(x_{1i}^*, \ldots, x_{Si}^*) \in \mathbb{R}^{LS}$ maximizes $U_i(x_{1i}, \ldots, x_{Si})$ subject to $\sum_s p_s \cdot (x_{si} - \omega_{si}) \le 0$, then, for any particular s, x_{is}^* maximizes $U_i(x_{1i}^*, \ldots, x_{si}, \ldots, x_{Si}^*)$ subject to $p_s \cdot (x_{si} - \omega_{si}) \le p_s \cdot (x_{si}^* - \omega_{si})$, that is, subject to $p_s \cdot (x_{si} - x_{si}^*) \le 0$.

A most interesting possibility, first observed by Arrow (1953), is that, even if not all the contingent commodities are available at $t = 0$, it may still be the case under some conditions that the retrading possibilities at $t = 1$ guarantee that Pareto optimality is reached, nevertheless. That is, the possibility of ex post trade can make up for an absence of some ex ante markets. In what follows, we shall verify that this is the case whenever at least one physical commodity can be traded contingently at $t = 0$ if, in addition, spot markets occur at $t = 1$ *and* the spot equilibrium prices are correctly anticipated at $t = 0$. The intuition for this result is reasonably straightforward: if spot trade can occur within each state, then the only task remaining at $t = 0$ is to transfer the consumer's overall purchasing power efficiently across states. This can be accomplished using contingent trade in a single commodity. By such a procedure we are able to reduce the number of required forward markets for LS to S.

Let us be more specific. At $t = 0$ consumers have *expectations* regarding the spot prices prevailing at $t = 1$ for each possible state $s \in S$. Denote the price vector expected to prevail in state s spot market by $p_s \in \mathbb{R}^L$, and the overall expectation vector[6] by $p = (p_1, \ldots, p_S) \in \mathbb{R}^{LS}$. Suppose that, in addition, at date $t = 0$ there is trade in the S contingent commodities denoted by 11 to $1S$; that is, there is contingent trade only in the physical good with the label 1. We denote the vector of prices for these contingent commodities traded at $t = 0$ by $q = (q_1, \ldots, q_S) \in \mathbb{R}^S$.

Faced with prices $q \in \mathbb{R}^S$ at $t = 0$ and expected spot prices $(p_1, \ldots, p_S) \in \mathbb{R}^{LS}$ at $t = 1$, every consumer i formulates a consumption, or trading, plan $(z_{1i}, \ldots, z_{Si}) \in \mathbb{R}^S$ for contingent commodities at $t = 0$, as well as a set of spot market consumption plans $(x_{1i}, \ldots, x_{Si}) \in \mathbb{R}^{LS}$ for the different states that may occur at $t = 1$. Of course, these plans must satisfy a budget constraint. Let $U_i(\cdot)$ be a utility function for $\succsim_i$. Then the problem of consumer i can be expressed formally as

$$\underset{\substack{(x_{1i}, \ldots, x_{Si}) \in \mathbb{R}^{LS}_+ \\ (z_{1i}, \ldots, z_{Si}) \in \mathbb{R}^S}}{\text{Max}} \quad U_i(x_{1i}, \ldots, x_{Si}) \tag{19.D.1}$$

$$\text{s.t.} \quad \text{(i)} \ \textstyle\sum_s q_s z_{si} \leq 0,$$

$$\text{(ii)} \ p_s \cdot x_{si} \leq p_s \cdot \omega_{si} + p_{1s} z_{si} \quad \text{for every } s.$$

Restriction (i) is the budget constraint corresponding to trade at $t = 0$. The family of restrictions (ii) are the budget constraints for the different spot markets. Note that the value of wealth at a state s is composed of two parts: the market value of the initial endowments, $p_s \cdot \omega_{si}$, and the market value of the amounts z_{si} of good 1 bought or sold forward at $t = 0$. Observe that we are not imposing any restriction on the sign or the magnitude of z_{si}. If $z_{si} < -\omega_{1si}$ then one says that at $t = 0$ consumer i is selling good 1 *short*. This is because he is selling at $t = 0$, contingent on state s occurring, more than he has at $t = 1$ if s occurs. Hence, if s occurs he will actually have to buy in the spot market the extra amount of the first good required for the fulfillment of his commitments. The possibility of selling short is, however, indirectly

6. In principle, expectations could differ across consumers, but under the assumption of correct expectations (soon to be introduced) they will not.

limited by the fact that consumption, and therefore ex post wealth, must be nonnegative for every s.[7]

To define an appropriate notion of sequential trade we shall impose a key condition: Consumers' expectations must be *self-fulfilled*, or *rational*; that is, we require that consumers' expectations of the prices that will clear the spot markets for the different states s do actually clear them once date $t = 1$ has arrived and a state s is revealed.

Definition 19.D.1: A collection formed by a price vector $q = (q_1, \ldots, q_S) \in \mathbb{R}^S$ for contingent first good commodities at $t = 0$, a spot price vector

$$p_s = (p_{1s}, \ldots, p_{Ls}) \in \mathbb{R}^L$$

for every s, and, for every consumer i, consumption plans $z_i^* = (z_{1i}^*, \ldots, z_{Si}^*) \in \mathbb{R}^S$ at $t = 0$ and $x_i^* = (x_{1i}^*, \ldots, x_{Si}^*) \in \mathbb{R}^{LS}$ at $t = 1$ constitutes a *Radner equilibrium* [see Radner (1982)] if:

(i) For every i, the consumption plans z_i^*, x_i^* solve problem (19.D.1).

(ii) $\sum_i z_{si}^* \le 0$ and $\sum_i x_{si}^* \le \sum_i \omega_{si}$ for every s.

At a Radner equilibrium, trade takes place through time and, in contrast to the Arrow–Debreu setting, economic agents face a *sequence of budget sets*, one at each date-state (more generally, at every date-event).

We can see from an examination of problem (19.D.1) that all the budget constraints are homogeneous of degree zero with respect to prices. This means that the budget sets remain unaltered if the price of one physical commodity in each date-state (that is, one price for every budget set) is arbitrarily normalized to equal 1. It is natural to choose the first commodity and to put $p_{1s} = 1$ for every s, so that a unit of the s contingent commodity then pays off 1 dollar in state s.[8] Note that this still leaves one degree of freedom, that corresponding to the forward trades at date 0 (so we could put $q_1 = 1$, or perhaps $\sum_s q_s = 1$).

In Proposition 19.D.1, which is the key result of this section, we show that for this model the set of Arrow–Debreu equilibrium allocations (induced by the arrangement of one-shot trade in LS contingent commodities) and the set of Radner equilibrium allocations (induced by contingent trade in only one commodity, sequentially followed by spot trade) are identical.

7. Observe also that we have taken the wealth at $t = 0$ to be zero (that is, there are no initial endowments of the contingent commodities). This is simply a convention. Suppose, for example, that we regard ω_{1si}, the amount of good 1 available at $t = 1$ in state s, as the amount of the s contingent commodity that i owns at $t = 0$ (to avoid double counting, the initial endowment of commodity 1 in the spot market s at $t = 1$ should simultaneously be put to zero). The budget constraints are then: (i) $\sum_s q_s(z'_{si} - \omega_{1si}) \le 0$ and (ii) $p_s \cdot x_{si} \le \sum_{\ell \ne 1} p_{\ell s}\omega_{\ell 1} + p_{1s}z'_{si}$ for every s. But letting $z'_{si} = z_{si} + \omega_{1si}$, we see that these are exactly the constraints of (19.D.1).

8. It follows from the possibility of making this normalization that, without loss of generality, we could as well suppose that our contingent commodity pays directly in dollars (see Exercise 19.D.1 for more on this).

Proposition 19.D.1: We have:

(i) If the allocation $x^* \in \mathbb{R}^{LSI}$ and the contingent commodities price vector $(p_1, \ldots, p_S) \in \mathbb{R}^{LS}_{++}$ constitute an Arrow–Debreu equilibrium, then there are prices $q \in \mathbb{R}^S_{++}$ for contingent first good commodities and consumption plans for these commodities $z^* = (z_1^*, \ldots, z_I^*) \in \mathbb{R}^{SI}$ such that the consumptions plans x^*, z^*, the prices q, and the spot prices $(p_1, \ldots, p_S)$ constitute a Radner equilibrium.

(ii) Conversely, if the consumption plans $x^* \in \mathbb{R}^{LSI}$, $z^* \in \mathbb{R}^{SI}$ and prices $q \in \mathbb{R}^S_{++}$, $(p_1, \ldots, p_S) \in \mathbb{R}^{LS}_{++}$ constitute a Radner equilibrium, then there are multipliers $(\mu_1, \ldots, \mu_S) \in \mathbb{R}^S_{++}$ such that the allocation x^* and the contingent commodities price vector $(\mu_1 p_1, \ldots, \mu_S p_S) \in \mathbb{R}^{LS}_{++}$ constitute an Arrow–Debreu equilibrium. (The multiplier μ_s is interpreted as the value, at $t = 0$, of a dollar at $t = 1$ and state s.)

Proof: (i) It is natural to let $q_s = p_{1s}$ for every s. With this we claim that, for every consumer i, the budget set of the Arrow–Debreu problem,

$$B_i^{AD} = \{(x_{1i}, \ldots, x_{Si}) \in \mathbb{R}^{LS}_+ : \textstyle\sum_s p_s \cdot (x_{si} - \omega_{si}) \leq 0\},$$

is identical to the budget set of the Radner problem,

$$B_i^R = \{(x_{1i}, \ldots, x_{Si}) \in \mathbb{R}^{LS}_+ : \text{there are } (z_{1i}, \ldots, z_{Si}) \text{ such that } \textstyle\sum_s q_s z_{si} \leq 0 \text{ and } p_s \cdot (x_{si} - \omega_{si}) \leq p_{1s} z_{si} \text{ for every } s\}.$$

To see this, suppose that $x_i = (x_{1i}, \ldots, x_{Si}) \in B_i^{AD}$. For every s, denote $z_{si} = (1/p_{1s}) p_s \cdot (x_{si} - \omega_{si})$. Then $\sum_s q_s z_{si} = \sum_s p_{1s} z_{si} = \sum_s p_s \cdot (x_{si} - \omega_{si}) \leq 0$ and $p_s \cdot (x_{si} - \omega_{si}) = p_{1s} z_{si}$ for every s. Hence, $x_1 \in B_i^R$. Conversely, suppose that $x_i = (x_{1i}, \ldots, x_{Si}) \in B_i^R$; that is, for some $(z_{1i}, \ldots, z_{Si})$ we have $\sum_s q_s z_{si} \leq 0$ and $p_s \cdot (x_{si} - \omega_{si}) \leq p_{1s} z_{si}$ for every s. Summing over s gives $\sum_s p_s \cdot (x_{si} - \omega_{si}) \leq \sum_s p_{1s} z_{si} = \sum_s q_s z_{si} \leq 0$. Hence, $x_i \in B_i^{AD}$.

We conclude that our Arrow–Debreu equilibrium allocation is also a Radner equilibrium allocation supported by $q = (p_{11}, \ldots, p_{1S}) \in \mathbb{R}^S$, the spot prices $(p_1, \ldots, p_S)$, and the contingent trades $(z_{1i}^*, \ldots, z_{Si}^*) \in \mathbb{R}^S$ defined by $z_{si}^* = (1/p_{1s}) p_s \cdot (x_{si}^* - \omega_{si})$. Note that the contingent markets clear since, for every s, $\sum_i z_{si}^* = (1/p_{1s}) p_s \cdot [\sum_i (x_{si}^* - \omega_{si})] \leq 0$.

(ii) Choose μ_s so that $\mu_s p_{1s} = q_s$. Then we can rewrite the Radner budget set of every consumer i as

$$B_i^R = \{(x_{1i}, \ldots, x_{Si}) \in \mathbb{R}^{LS} : \text{there are } (z_{1i}, \ldots, z_{Si}) \text{ such that } \textstyle\sum_s q_s z_{si} \leq 0 \text{ and } \mu_s p_s \cdot (x_{si} - \omega_{si}) \leq q_s z_{si} \text{ for every } s\}.$$

But from this we can proceed as we did in part (i) and rewrite the constraints, and therefore the budget set, in the Arrow–Debreu form:

$$B_i^R = B_i^{AD} = \{(x_{1i}, \ldots, x_{Si}) \in \mathbb{R}^{LS} : \textstyle\sum_s \mu_s p_s \cdot (x_{si} - \omega_{si}) \leq 0\}.$$

Hence, the consumption plan x_i^* is also preference maximizing in the budget set B_i^{AD}. Since this is true for every consumer i, we conclude that the price vector $(\mu_1 p_1, \ldots, \mu_S p_S) \in \mathbb{R}^{LS}$ clears the markets for the LS contingent commodities. ■

Example 19.D.1: Consider a two-good, two-state, two-consumer pure exchange economy. Suppose that the two states are equally likely and that every consumer has the same, state-independent, Bernoulli utility function $u(x_{si})$. The consumers differ only in their initial endowments. The aggregate endowment vectors in the two states

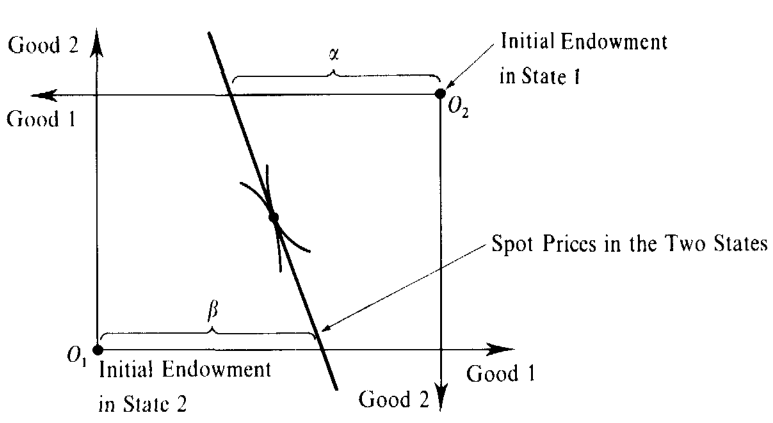

Figure 19.D.1
Reaching the Arrow–Debreu equilibrium by means of contingent trade in the first good only.

are the same; however, endowments are distributed so that consumer 1 gets everything in state 1 and consumer 2 gets everything in state 2. (See Figure 19.D.1.)

By the symmetry of the problem, at an Arrow–Debreu equilibrium each consumer gets, in each state, half of the total endowment of each good. In Figure 19.D.1, we indicate how these consumptions will be reached by means of contingent trade in the first commodity and spot markets. The spot prices will be the same in the two states. The first consumer will sell an amount α of the first good contingent on the occurrence of the first state and will in exchange buy an amount β of the same good contingent on the second state. (You are asked to provide the details in Exercise 19.D.2.) ■

It is important to emphasize that, although the concept of Radner equilibrium cuts down the number of contingent commodities required to attain optimality (from LS to S), this reduction is not obtained free of charge. With the smaller number of forward contracts, the correct anticipation of future spot prices becomes crucial.

Up to this point we have discussed the sequential implementation of an Arrow–Debreu equilibrium when there are two dates,[9] that is, for the date–event tree of Figure 19.B.1. Except for notational complications, the same ideas carry over to a tree such as that in Figure 19.B.2 where there are $T + 1$ periods and information is released gradually. (See the small-type discussion at the end of Section 19.B for basic concepts and notation.) We would then have spot markets at every admissible date–event pair tE (i.e., those tE where $E \in \mathscr{S}_t$, the information partition at t). With H the set of basic physical commodities, we denote the spot prices by $p_{tE} \in \mathbb{R}^H$. At every tE we could also have trade for the contingent delivery of physical good 1 at each of the successor date–events to tE. Denote by $q_{tE}(t+1, E')$ the price at tE of one unit of good 1 delivered at $t+1$ if event E' is revealed (of course, we require $E' \in \mathscr{S}_{t+1}$ and $E' \subset E$). The problem of the consumer consists of forming utility-maximizing plans by choosing, at every admissible tE, a vector of consumption of goods $x_{tEi} \in \mathbb{R}^H_+$ and, for every successor $(t+1, E')$, a contingent trade $z_{tEi}(t+1, E')$ of good 1 deliverable at $(t+1, E')$. Overall, the budget constraint to be satisfied at tE is

$$p_{tE} \cdot x_{tEi} + \sum_{\{E' \in \mathscr{S}_{t+1} : E' \subset E\}} q_{tE}(t+1, E') z_{tEi}(t+1, E') \leq p_{tE} \cdot \omega_{tEi} + p_{1tE} z_{t-1, E^-, i}(t, E)$$

where E^- is the event at the date $t - 1$ predecessor to event E at t.

9. To be as simple as possible, we have also assumed that there is no consumption at $t = 0$.

One can then proceed to define a corresponding concept of Radner equilibrium and to show that the Arrow–Debreu equilibrium allocations for the model with $H(T+1)S$ contingent commodity markets[10] at $t = 0$ are the same as the Radner equilibrium allocations obtained from a model with sequential trade in which, at each date-event, consumers trade only current goods and contingent claims for delivery of good 1 at successor nodes. Exercises 19.D.3 and 19.D.4 discuss this topic further.

19.E Asset Markets

The S contingent commodities studied in the previous section serve the purpose of transferring wealth across the states of the world that will be revealed in the future. They are, however, only theoretical constructs that rarely have exact counterparts in reality. Nevertheless, in reality there are *assets*, or *securities*, that to some extent perform the wealth-transferring role that we have assigned to the contingent commodities. It is therefore important to develop a theoretical structure that allows us to study the functioning of these asset markets. We accomplish the task in this section by extending the formal notion of a contingent commodity and then generalizing the theory of Radner equilibrium to the extended environment.[11]

We begin again with the simplest situation, in which we have two dates, $t = 0$ and $t = 1$, and all the information is revealed at $t = 1$. Further, for notational simplicity we assume that consumption takes place only at $t = 1$.

We view an asset, or, more precisely, a unit of an asset, as a title to receive either physical goods or dollars at $t = 1$ in amounts that may depend on which state occurs.[12] The payoffs of an asset are known as its *returns*. If the returns are in physical goods, the asset is called *real* (a durable piece of machinery or a futures contract for the delivery of copper would be examples). If they are in paper money, they are called *financial* (a government bond, for example). Mixed cases are also possible. Here we deal only with the real case and, moreover, to save on notation we assume that the returns of assets are only in amounts of physical good 1.[13] It is then convenient to normalize the spot price of that good to be 1 in every state, so that, in effect, we are using it as numeraire.

Definition 19.E.1: A unit of an *asset*, or *security*, is a title to receive an amount r_s of good 1 at date $t = 1$ if state s occurs. An asset is therefore characterized by its *return vector* $r = (r_1, \dots, r_S) \in \mathbb{R}^S$.

10. A contingent commodity is a promise to deliver a unit of physical commodity h at date t if state s occurs. Recall from Section 19.B that the consumption sets have to be defined imbedding in them the information measurability restrictions, that is, making sure that at date t no consumption is dependent on information not yet available.

11. See Radner (1982) and Kreps (1979) [complemented by Marimon (1987)] for treatments in the spirit of this section.

12. As usual "title to receive" means "duty to deliver" if the amount is negative. Although negative returns present no particular difficulty, we will avoid them.

13. This assumption also has an important simplifying feature: At any given state the returns of all assets are in units of the same physical good. Therefore, the relative spot prices of the various physical goods in any given state do not affect the relative returns of the different assets in that state.

Example 19.E.1: Examples of assets include the following:

(i) $r = (1, \dots, 1)$. This asset promises the future noncontingent delivery of one unit of good 1. Its real-world counterparts are the markets for *commodity futures*. In the special case where there is a single consumption good (i.e., $L = 1$), we call this asset the *safe* (or *riskless*) *asset*. It is important to realize that with more than one physical good a futures contract is not riskless: its return in terms of purchasing power depends on the spot prices of all the goods.[14]

(ii) $r = (0, \dots, 0, 1, 0, \dots, 0)$. This asset pays one unit of good 1 if and only if a certain state occurs. These were the assets considered in Section 19.D. In the current theoretical setting they are often called *Arrow securities*.

(iii) $r = (1, 2, 1, 2, \dots, 1, 2)$. This asset pays one unit unconditionally and, in addition, another unit in even-labeled states. ■

Example 19.E.2: *Options*. This is an example of a so-called *derivative asset*, that is, of an asset whose returns are somehow derived from the returns of another asset. Suppose there is a *primary asset* with return vector $r \in \mathbb{R}^S$. Then a (*European*) *call option* on the primary asset at the *strike price* $c \in \mathbb{R}$ is itself an asset. A unit of this asset gives the option to buy, *after the state is revealed* (but before the returns are paid), a unit of the primary asset at price c (the price c is in units of the "numeraire," that is, of good 1).

What is the return vector $r(c)$ of the option? In a given state s, the option will be exercised if and only if $r_s > c$ (we neglect the case $r_s = c$). Hence

$$r(c) = (\text{Max}\,\{0, r_1 - c\}, \dots, \text{Max}\,\{0, r_S - c\}).$$

For a primary asset with returns $r = (4, 3, 2, 1)$ specific examples are

$$\begin{aligned} r(3.5) &= (\ .5,\ \ 0\ ,\ \ 0\ ,\ \ 0), \\ r(2.5) &= (1.5,\ 0.5,\ \ 0\ ,\ \ 0), \\ r(1.5) &= (2.5,\ 1.5,\ 0.5,\ 0).\ \blacksquare \end{aligned}$$

We proceed to extend the analysis of Section 19.D by assuming that there is a given set of assets, known as an *asset structure*, and that these assets can be freely traded at date $t = 0$. We postpone to the next section a discussion of the important issue of the origin of the particular set of assets. Each asset k is characterized by a vector of returns $r_k \in \mathbb{R}^S$. The number of assets is K. As before, we assume that there are no initial endowments of assets and that short sales are possible. The price vector for the assets traded at $t = 0$ is denoted $q = (q_1, \dots, q_K)$. A vector of trades in these assets, denoted by $z = (z_1, \dots, z_K) \in \mathbb{R}^K$, is called a *portfolio*.

The next step is to generalize the definition of a Radner equilibrium to the current environment. In Definition 19.E.2, $U_i(\cdot)$ is a utility function for the preferences $\succsim_i$ of consumer i over consumption plans $(x_{1i}, \dots, x_{Si}) \in \mathbb{R}^{LS}_+$.

14. Strictly speaking, for the term "riskless" to be meaningful we need, in addition to $L = 1$, that utility functions be uniform across states.

Definition 19.E.2: A collection formed by a price vector $q = (q_1, \ldots, q_K) \in \mathbb{R}^K$ for assets traded at $t = 0$, a spot price vector $p_s = (p_{1s}, \ldots, p_{Ls}) \in \mathbb{R}^L$ for every s, and, for every consumer i, portfolio plans $z_i^* = (z_{1i}^*, \ldots, z_{Ki}^*) \in \mathbb{R}^K$ at $t = 0$ and consumption plans $x_i^* = (x_{1i}^*, \ldots, x_{Si}^*) \in \mathbb{R}^{LS}$ at $t = 1$ constitutes a *Radner equilibrium* if:

(i) For every i, the consumption plans z_i^*, x_i^* solve the problem

$$\begin{aligned} \underset{\substack{(x_{1i}, \ldots, x_{Si}) \in \mathbb{R}_+^{LS} \\ (z_{1i}, \ldots, z_{Ki}) \in \mathbb{R}^K}}{\text{Max}} \quad & U_i(x_{1i}, \ldots, x_{Si}) \\ \text{s.t.} \quad & \text{(a) } \textstyle\sum_k q_k \cdot z_{ki} \leq 0 \\ & \text{(b) } p_s \cdot x_{si} \leq p_s \cdot \omega_{si} + \textstyle\sum_k p_{1s} z_{ki} r_{sk} \quad \text{for every } s. \end{aligned}$$

(ii) $\sum_i z_{ki}^* \leq 0$ and $\sum_i x_{si}^* \leq \sum_i \omega_{si}$ for every k and s.

In the budget set of Definition 19.E.2, the wealth of consumer i at state s is the sum of the spot value of his initial endowment and the spot value of the return of his portfolio. Note that, without loss of generality, we can put $p_{1s} = 1$ for all s. From now on we will do so. It is convenient at this point to introduce the concept the *return matrix* R. This is an $S \times K$ matrix whose kth column is the return vector of the kth asset. Hence, its generic sk entry is r_{sk}, the return of asset k in state s. With this notation, the budget constraint of consumer i becomes

$$B_i(p, q, B) = \left\{ x \in \mathbb{R}_+^{LS} : \text{for some portfolio } z_i \in \mathbb{R}^K \text{ we have } q \cdot z_i \leq 0 \text{ and} \right.$$
$$\left. \begin{pmatrix} p_1 \cdot (x_{1i} - \omega_{1i}) \\ \vdots \\ p_S \cdot (x_{Si} - \omega_{Si}) \end{pmatrix} \leq \begin{bmatrix} r_{11}, \ldots, r_{1K} \\ \ddots \\ r_{S1}, \ldots, r_{SK} \end{bmatrix} z_i = Rz_i \right\}$$

We now present a very important implication, rich in ramifications, of the assumption that unlimited short sales are possible. Namely, we will establish that knowledge of the return matrix R suffices to place significant restrictions on the asset price vector $q = (q_1, \ldots, q_K)$ that could conceivably arise at equilibrium.

Proposition 19.E.1: Assume that every return vector is nonnegative and nonzero; that is, $r_k \geq 0$ and $r_k \neq 0$ for all k.[15] Then, for every (column) vector $q \in \mathbb{R}^K$ of asset prices arising in a Radner equilibrium, we can find multipliers $\mu = (\mu_1, \ldots, \mu_S) \geq 0$, such that $q_k = \sum_s \mu_s r_{sk}$ for all k (in matrix notation, $q^{\mathrm{T}} = \mu \cdot R$).

In words, Proposition 19.E.1 says that we can assign values $(\mu_1, \ldots, \mu_S)$ to units of wealth in the different states so that the price of a unit of asset k is simply equal to the sum, in value terms, of the returns across states.[16] Because this is an important

15. This assumption can be weakened substantially.

16. As we shall see shortly, the value μ_s can also be interpreted as the implicit price of the state-contingent commodity that pays one unit of good 1 if state s occurs and nothing otherwise.

result, we shall give two proofs of it. The first, which we give in small type, is based on convexity theory and uses only one implication of equilibrium: the fact that q must be *arbitrage free* (we will give a definition of this concept shortly). The second proof uses the first-order conditions of the utility maximization problem and provides further insight into the nature of the multipliers.

Proof 1 of Proposition 19.E.1: Call the system $q \in \mathbb{R}^K$ of asset prices *arbitrage free* if there is no portfolio $z = (z_1, \ldots, z_K)$ such that $q \cdot z \leq 0$, $Rz \geq 0$, and $Rz \neq 0$. In words, there is no portfolio that is budgetarily feasible and that yields a nonnegative return in *every state* and a strictly positive return in some state. Note that whether an asset price vector is arbitrage free or not depends only on the returns of the assets and not on preferences.

If, as usual, we assume that preferences are strongly monotone, then an equilibrium asset price vector $q \in \mathbb{R}^K$ must be arbitrage free: if it were not, it would be possible to increase utility merely by adding to any current portfolio a portfolio yielding an arbitrage opportunity. Because there are no restrictions on short sales this addition is always feasible.

In Lemma 19.E.1 we establish a result which in view of the observation just made is formally stronger than the statement of Proposition 19.E.1.

Lemma 19.E.1: If the asset price vector $q \in \mathbb{R}^K$ is arbitrage free, then there is a vector of multipliers $\mu = (\mu_1, \ldots, \mu_S) \geq 0$ satisfying $q^{\mathrm{T}} = \mu \cdot R$.

Proof of Lemma 19.E.1: Note to begin with that since we deal only with assets having nonnegative, nonzero returns, an arbitrage-free price vector q must have $q_k > 0$ for every k. Also, without loss of generality, we assume that no row of the return matrix R has all of its entries equal to zero.[17]

Given an arbitrage-free asset price vector $q \in \mathbb{R}^K$, consider the convex set

$$V = \{v \in \mathbb{R}^S \colon v = Rz \text{ for some } z \in \mathbb{R}^K \text{ with } q \cdot z = 0\}.$$

The arbitrage freeness of q implies that $V \cap \{\mathbb{R}^S_+ \setminus \{0\}\} = \varnothing$. Since both V and $\mathbb{R}^S_+ \setminus \{0\}$ are convex sets and the origin belongs to V, we can apply the separating hyperplane theorem (see Section M.G of the Mathematical Appendix) to obtain a nonzero vector $\mu' = (\mu'_1, \ldots, \mu'_S)$ such that $\mu' \cdot v \leq 0$ for any $v \in V$ and $\mu' \cdot v \geq 0$ for any $v \in \mathbb{R}^S_+$. Note that it must be that $\mu' \geq 0$. Moreover, because $v \in V$ implies $-v \in V$, it follows that $\mu' \cdot v = 0$ for any $v \in V$. Figure 19.E.1(a) depicts this construction for the two-state case.

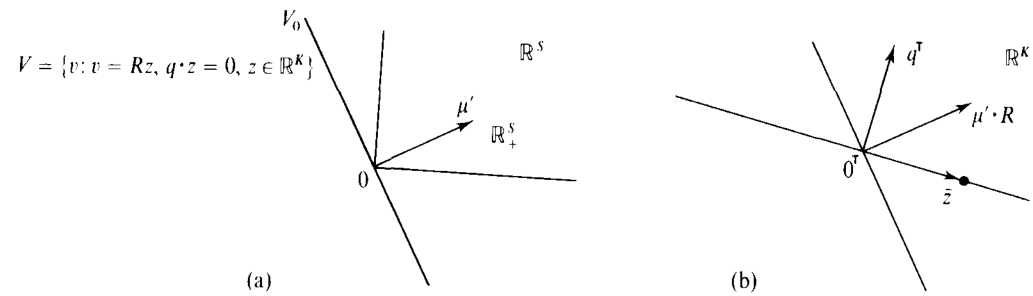

Figure 19.E.1
(a) Construction of the no-arbitrage weights.
(b) Existence of an inadmissible $\bar{z}$ if q^{T} is not proportional to $\mu' \cdot R$.

17. If there is such a row, set μ_s arbitrarily for the state s corresponding to that row, drop s from the list of states, and proceed with the remaining states.

We now argue that the row vector q^{T} must be proportional to the row vector $\mu' \cdot R \in \mathbb{R}^K$. The entries of μ' and of R are all nonnegative and no row of R is null. Therefore $\mu' \cdot R \geq 0^{\mathrm{T}}$ and $\mu' \cdot R \neq 0^{\mathrm{T}}$. If q^{T} is not proportional to $\mu' \cdot R$ then we can find $\bar{z} \in \mathbb{R}^K$ such that $q \cdot \bar{z} = 0$ and $\mu' \cdot R\bar{z} > 0$ [see Figure 19.E.1(b)]. But letting $v = R\bar{z}$, we would then have $v \in V$ and $\mu' \cdot v \neq 0$, which we have just seen cannot happen. Hence q^{T} must be proportional to $\mu' \cdot R$; that is, $q^{\mathrm{T}} = \alpha\mu' \cdot R$ for some real number $\alpha > 0$. Letting $\mu = \alpha\mu'$, we have the conclusion of the lemma. ■

As we have already argued, if short sales of assets are possible and preferences are strongly monotone (e.g., if preferences admit an expected utility representation with strictly positive subjective probabilities for the states), then equibrium asset prices must be arbitrage free and, therefore, Proposition 19.E.1 follows from Lemma 19.E.1. ■

Proof 2 of Proposition 19.E.1: For this proof we assume that preferences are represented by utility functions of the expected utility form $U_i(x_{1i}, \ldots, x_{Si}) = \sum_s \pi_{si} u_{si}(x_{si})$ and that the Bernoulli utilities $u_{si}(\cdot)$ are concave, strictly increasing, and differentiable. We denote by $v_{si}(p_s, w_{si})$ the indirect utility function [derived from $u_{si}(\cdot)$] of consumer i in state s.

Suppose that in the Radner equilibrium with asset prices $q = (q_1, \ldots, q_K)$ the equilibrium spot prices are $p = (p_1, \ldots, p_S) \in \mathbb{R}^{LS}$. Because unlimited short sales are possible, the optimal portfolio choice $z_i^* \in \mathbb{R}^K$ of any consumer i is necessarily interior and, denoting $w_{si}^* = p_s \cdot \omega_{si} + \sum_k r_{sk} z_{ki}^*$, it must satisfy the following first-order conditions for some $\alpha_i > 0$:

$$\alpha_i q_k = \sum_s \pi_{si} \frac{\partial v_{si}(p_s, w_{si}^*)}{\partial w_{si}} r_{sk} \qquad \text{for every } k = 1, \ldots, K.$$

That is, the vector of expected marginal utilities of the K assets must be proportional to the vector of asset prices.[18] With this we have attained our result, since by taking

$$\mu_{si} = \frac{\pi_{si}}{\alpha_i} \frac{\partial v_{si}(p_s, w_{si}^*)}{\partial w_{si}}$$

we have $q_j = \sum_s \mu_{si} r_{sk}$. Hence, we could determine $\mu = (\mu_1, \ldots, \mu_s)$ by choosing *any* consumer i and letting $\mu_s = \mu_{si}$, the marginal utility of wealth at state s of consumer i weighted by π_{si}/α_i. The multiplier α_i is the Lagrange multiplier of the budget constraint at $t = 0$ and can therefore be viewed as the marginal utility of wealth at $t = 0$. Hence, for any consumer i, μ_{si} equals the ratio of the (expected) utility at $t = 0$ of one extra unit of wealth at $t = 1$ and state s, and the utility of one extra unit of wealth at $t = 0$. See Exercise 19.E.1 for more on this point. Note also that different consumers may lead to different $\mu_i = (\mu_{i1}, \ldots, \mu_{is})$ and therefore to different μ's. The uniqueness of μ is assured only when rank $R = S$. ■

18. Recall that we always take $p_{1s} = 1$. Therefore, r_{sk} is the extra amount of wealth in state s derived from an extra unit of asset k. The proportionality factor α_i is the Lagrange multiplier of the problem

$$\begin{aligned} \text{Max} \quad & \sum_s \pi_{si} v_{si}(p_s, p_s \cdot \omega_{si} + \sum_k r_{sk} z_{ki}) \\ \text{s.t.} \quad & \textstyle\sum_k q_k z_{ki} \leq 0. \end{aligned}$$

Example 19.E.3: Suppose that there is available an asset with noncontingent returns; for example, $r_1 = (1, \ldots, 1)$. Normalize the price of this asset to be 1, that is, $q_1 = 1$. Then if $\mu = (\mu_1, \ldots, \mu_S)$ is the vector of multipliers given by Proposition 19.E.1, we must have $\mu \geq 0$ and $\sum_s \mu_s = \mu \cdot r_1 = q_1 = 1$. For any other asset k we then obtain the intuitive conclusion that $q_k = \sum_s \mu_s r_{sk} \geq \text{Min}_s\, r_{sk}$ and, similarly, $q_k \leq \text{Max}_s\, r_{sk}$. ■

In Section 19.D, we proved that for the set of assets consisting of the S contingent markets in a single physical commodity we have an equivalence result between Arrow Debreu and Radner equilibrium allocations (Proposition 19.D.1). We now generalize this result. In particular, we show that this equivalence holds for *any* family of S or more assets, *provided* that at least S of them have returns that are linearly independent (i.e., provided the effective number of assets is at least S). We begin with Definition 19.E.2.

Definition 19.E.3: An asset structure with an $S \times K$ return matrix R is *complete* if rank $R = S$, that is, if there is some subset of S assets with linearly independent returns.

Example 19.E.4: In the case of S contingent commodities discussed in Section 19.D, and also in Example 19.E.1(ii), the return matrix R is the $S \times S$ identity matrix. This is the canonical example of complete markets. But there are many other ways for a matrix to be nonsingular. Thus, with three states and three assets, we could have the return matrix

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix},$$

which has rank equal to 3, the number of states. ■

Example 19.E.5: *Spanning through Options.* Suppose that $S = 4$ and there is a primary asset with returns $r = (4, 3, 2, 1)$. We have seen in Example 19.E.2 that, for every strike price c, the option defined by c constitutes an asset with return vector $r(c) = (\text{Max}\,\{0, r_1 - c\}, \ldots, \text{Max}\,\{0, r_4 - c\})$. Using options we can create a complete asset structure supported entirely on the primary asset r. For example, the return vectors $r(3.5)$, $r(2.5)$, $r(1.5)$, and r and linearly independent (the matrix R has all its entries below the diagonal equal to zero). Thus, the asset structure consisting of the primary asset plus three options with strike prices 3.5, 2.5, and 1.5 is complete. More generally, whenever the primary asset is such that $r_s \neq r_{s'}$ for all $s \neq s'$, it is possible to generate a complete asset structure by means of options (see Exercise 19.E.2). If $r_s = r_{s'}$ for some distinct s and s', then we cannot do so: If the primary asset does not distinguish between two states, no derived asset can do so either. ■

To repeat, the importance of the concept of completeness derives from the fact that with it we can generalize Proposition 19.D.1. With a complete asset structure, economic agents are, in effect, unrestricted in their wealth transfers across states (except, of course, by their budget constraints). Therefore, at the equilibrium, their portfolio choices induce the same second-period consumptions as in the Arrow-Debreu equilibrium, and so full Pareto optimality is reached. This is the content of Proposition 19.E.2.

Proposition 19.E.2: Suppose that the asset structure is complete. Then:

(i) If the consumption plans $x^* = (x_1^*, \ldots, x_I^*) \in \mathbb{R}^{LSI}$ and the price vector

$$(p_1, \ldots, p_S) \in \mathbb{R}^{LS}_{++}$$

constitute an Arrow Debreu equilibrium, then there are asset prices $q \in \mathbb{R}^K_{++}$ and portfolio plans $z^* = (z_1^*, \ldots, z_I^*) \in \mathbb{R}^{KI}$ such that the consumption plans x^*, portfolio plans z^*, asset prices q, and spot prices $(p_1, \ldots, p_S)$ constitute a Radner equilibrium.

(ii) Conversely, if the consumption plans $x^* \in \mathbb{R}^{LSI}$, portfolio plans $z^* \in \mathbb{R}^{KI}$, and prices $q \in \mathbb{R}^K_{++}$, $(p_1, \ldots, p_S) \in \mathbb{R}^{LS}_{++}$ constitute a Radner equilibrium, then there are multipliers $(\mu_1, \ldots, \mu_S) \in \mathbb{R}^S_{++}$ such that the consumption plans x^* and the contingent commodities price vector $(\mu_1 p_1, \ldots, \mu_S p_S) \in \mathbb{R}^{LS}$ constitute an Arrow–Debreu equilibrium. (The multiplier μ_s is interpreted as the value, at $t = 0$, of a dollar at $t = 1$ and state s; recall that $p_{1s} = 1$.)

Proof: It is entirely similar to the proof of Proposition 19.D.1.

(i) Define $q_k = \sum_s p_{1s} r_{sk}$ for every k. Denote by Λ the $S \times S$ diagonal matrix whose s diagonal entry is p_{1s}. Then $q^{\mathrm{T}} = e \cdot \Lambda R$, where $e \in \mathbb{R}^S$ is a column vector with all its entries equal to 1. For every i the (column) vector of wealth transfers across states (at the Arrow Debreu equilibrium) is

$$m_i = (p_1 \cdot (x_{1i}^* - \omega_{1i}), \ldots, p_S \cdot (x_{Si}^* - \omega_{Si}))^{\mathrm{T}}.$$

We have $e \cdot m_i = 0$ for every i and $\sum_i m_i = 0$. By completeness, rank $\Lambda R = S$ and, therefore, we can find vectors $z_i^* \in \mathbb{R}^K$ such that $m_i = \Lambda R z_i^*$ for $i = 1, \ldots, I - 1$. Letting

$$z_I^* = -(z_1^* + \cdots + z_{I-1}^*)$$

we also have $m_I = -(m_1 + \cdots + m_{I-1}) = \Lambda R z_I^*$. Therefore, for each i, the portfolio z_i^* allows consumer i to reach the Arrow–Debreu consumptions in the different states at the spot prices $(p_1, \ldots, p_S)$. To verify budget feasibility note that $q \cdot z_i^* = e \cdot \Lambda R z_i^* = e \cdot m_i = 0$. In Exercise 19.E.3 you are asked to complete the proof by showing that the consumption and portfolio plans x_i^* and z_i^* are not just budget feasible but also utility maximizing in the budget set.

(ii) Assume, without loss of generality, that $p_{1s} = 1$ for all s. By Propositon 19.E.1 we have $q^{\mathrm{T}} = \mu \cdot R$ for some arbitrage weights $\mu = (\mu_1, \ldots, \mu_S)$. We show that x^* is an Arrow–Debreu equilibrium with respect to $(\mu_1 p_1, \ldots, \mu_S p_S)$. To this effect, suppose that $x_i \in \mathbb{R}^{LS}$ satisfies the Arrow Debreu single budget constraint, that is, $\sum_s \mu_s p_s \cdot (x_{si} - \omega_{si}) \le 0$. Then by the completeness assumption there is $z_i \in \mathbb{R}^K$ such that $(p_1 \cdot (x_{1i} - \omega_{1i}), \ldots, p_S \cdot (x_{Si} - \omega_{Si}))^{\mathrm{T}} = R z_i$ and, therefore, $q \cdot z_i = \mu \cdot R z_i \le 0$. Hence x_i also satisfies the budget constraints of the Radner equilibrium. Observe next that the Radner equilibrium consumption x_i^* is Arrow–Debreu budget feasible since $(p_1 \cdot (x_{1i}^* - \omega_{1i}), \ldots, p_S \cdot (x_{Si}^* - \omega_{Si}))^{\mathrm{T}} \le R z_i^*$ and $q^{\mathrm{T}} = \mu \cdot R$ yields

$$\sum_s \mu_s p_s \cdot (x_{si}^* - \omega_{si}) \le \mu \cdot R z_i^* = q \cdot z_i^* \le 0.$$

Therefore, x_i^* is utility maximizing in the Arrow–Debreu budget constraint. ■

It is important to realize that in discussing Radner equilibria what matters is not so much the particular asset structure but the linear space,

$$\text{Range } R = \{v \in \mathbb{R}^S \colon v = Rz \text{ for some } z \in \mathbb{R}^K\} \subset \mathbb{R}^S,$$

the set of wealth vectors that can be *spanned* by means of the existing assets. It is quite possible for two different asset structures to give rise to the same linear space. Our next result, Proposition 19.E.3, tells us that, whenever this is so, the set of Radner equilibrium allocations for the two asset structures is the same.

Proposition 19.E.3: Suppose that the asset price vector $q \in \mathbb{R}^K$, the spot prices $p = (p_1, \ldots, p_S) \in \mathbb{R}^{LS}$, the consumption plans $x^* = (x_1^*, \ldots, x_I^*) \in \mathbb{R}_+^{LSI}$, and the portfolio plans $(z_1^*, \ldots, z_I^*) \in \mathbb{R}^{KI}$ constitute a Radner equilibrium for an asset structure with $S \times K$ return matrix R. Let R' be the $S \times K'$ return matrix of a second asset structure. If Range R' = Range R, then x^* is still the consumption allocation of a Radner equilibrium in the economy with the second asset structure.

Proof: By Proposition 19.E.1, the asset prices satisfy the arbitrage condition $q^{\mathrm{T}} = \mu \cdot R$, for some $\mu \in \mathbb{R}_{++}^S$. Denote $q' = [\mu \cdot R']^{\mathrm{T}}$. We claim that if Range R = Range R' then

$$B_i(p, q', R') = B_i(p, q, R) \qquad \text{for every } i. \tag{19.E.1}$$

We show that if $x_i \in B_i(p, q, R)$ then $x_i \in B_i(p, q', R')$. To see this, let

$$(p_1 \cdot (x_{1i} - \omega_{1i}), \ldots, p_S \cdot (x_{Si} - \omega_{Si}))^{\mathrm{T}} \leq Rz_i$$

and $q \cdot z_i \leq 0$. Since Range R = Range R', we can find $z_i' \in \text{Range } R'$ such that $Rz_i = R'z_i'$. But then $q' \cdot z_i' = \mu \cdot R'z_i' = \mu \cdot Rz_i = q \cdot z_i \leq 0$, and therefore we can conclude that $x_i \in B_i(p, q', R')$. The converse statement [if $x_i \in B_i(p, q', R')$ then $x_i \in B_i(p, q, R)$] is proved in exactly the same way.

It follows from (19.E.1) that, for every consumer i, x_i^* is preference maximal in the budget set $B_i(p, q', R')$.

To argue that the asset prices q', the spot prices $p = (p_1, \ldots, p_S)$, and the consumption allocation x^* are part of a Radner equilibrium in the economy with an asset structure having return matrix R', it suffices to find portfolios $(z_1', \ldots, z_I') \in \mathbb{R}^{KI}$ such that, first, $\sum_i z_i' = 0$ and, second, for every consumer i, the vector of across-states wealth transfers

$$m_i = (p_1 \cdot (x_{1i}^* - \omega_{1i}), \ldots, p_S \cdot (x_{Si}^* - \omega_{Si}))^{\mathrm{T}}$$

satisfies $m_i = R'z_i'$. This is simple to accomplish. By strong monotonicity of preferences we have $m_i = Rz_i^*$, for every i. Hence, $m_i \in \text{Range } R$ and therefore $m_i \in \text{Range } R'$ for every i. Choose then $z_1', \ldots, z_{I-1}'$ such that $m_i = R'z_i'$ for every $i = 1, \ldots, I-1$. Finally, let $z_I' = -z_1' - \cdots - z_{I-1}'$. Then $\sum_i z_i' = 0$ and also

$$m_I = -(m_1 + \cdots + m_{I-1}) = -R'(z_1' + \cdots + z_{I-1}') = R'z_I'. \blacksquare$$

One says that an asset is *redundant* if its deletion does not affect the linear space Range R of spannable wealth transfers, that is, if its return vector is a linear combination of the return vectors of the remaining assets. It follows from Proposition 19.E.3 that the set of consumption allocations obtainable as part of a Radner equilibrium is not changed by the addition or deletion of a redundant asset. Another important fact is that a redundant asset can be priced merely by knowing the matrix of returns and the prices of the other assets.

Exercise 19.E.4: (*Pricing by Arbitrage*). Suppose that $r_3 = \alpha_1 r_1 + \alpha_2 r_2$. Show that at equilibrium we must have $q_3 = \alpha_1 q_1 + \alpha_2 q_2$. Recall that unlimited short sales are possible. (Assume also that the return vectors are nonnegative and nonzero.)

An implication of Exercise 19.E.4 is that, if the asset structure is complete, then we can deduce the prices of all assets from knowing the prices of a subset formed by S of them with linearly independent returns. A related way to see this is to note that from the prices of S assets with linearly independent returns we can uniquely deduce the state multipliers $\mu = (\mu_1, \dots, \mu_S)$ of Proposition 19.E.1; indeed, for this we just have to solve a linear system of S independent equations in S unknowns. These multipliers can be interpreted as the (arbitrage) prices of the Arrow securities [Example 19.E.1(ii)]. Once we have these multipliers, we can obtain the price of any other asset k with return vector r_k as $q_k = \sum_s \mu_s r_{sk}$.

Example 19.E.6: *Pricing an Option.* Suppose that, with $S = 2$, there is an asset with uncontingent returns, say $r_1 = (1, 1)$ and a second asset $r_2 = (3 + \alpha, 1 - \alpha)$, with $\alpha > 0$. The asset prices are $q_1 = 1$ and q_2. We now consider an option on the second asset that has strike price $c \in (1, 3)$. Then

$$r_2(c) = (3 + \alpha - c, 0) = \frac{3 + \alpha - c}{2 + 2\alpha} r_2 - \frac{(1 - \alpha)(3 + \alpha - c)}{2 + 2\alpha} r_1.$$

Therefore, the arbitrage price of the option (the only price compatible with equilibrium in the asset market) must be

$$q_2(c) = \frac{3 + \alpha - c}{2 + 2\alpha} [q_2 - (1 - \alpha)]. \tag{19.E.2}$$

An equivalent way to get the same formula is to note that, since the 2×2 return matrix R is nonsingular, the multipliers $\mu = (\mu_1, \mu_2)$ of Proposition 19.E.1 can be determined uniquely from $(1, q_2) = \mu \cdot R$. They are $\mu_1 = (q_2 - (1 - \alpha))/(2 + 2\alpha)$ and $\mu_2 = 1 - \mu_1$. But, again from Proposition 19.E.1, we have

$$q_2(c) = \mu \cdot r_2(c) = \mu_1(3 + \alpha - c),$$

which is precisely expression (19.E.2).

Note that if the prices of the two assets r_1 and r_2 are themselves arbitrage free, then we must have $3 + \alpha \geq q_2 \geq 1 - \alpha$ (recall Example 19.E.3). Therefore, we learn from formula (19.E.2) that $q_2(c)$ is nonnegative, decreasing in c, and increasing in q_2.

We can also show that if the asset price q_2 stays constant but the dispersion parameter represented by α increases, then the option becomes more valuable. Suppose, in effect, that $\alpha' > \alpha$ and $r'_2(c)$, $r_2(c)$ are the corresponding returns of the option. Then $r_3 = r'_2(c) - r_2(c)$ is itself an asset with nonnegative returns. We can also price it by arbitrage from r_1 and r_2 to get a $q_3 \geq 0$ (typically, $q_3 > 0$). But then, again by arbitrage, $q'_2(c) = q_3 + q_2(c) \geq q_2(c)$. ■

Everything generalizes to the case of $T + 1$ periods and gradual release of information. With several periods, an asset can take many forms. For example, we could have *short-term assets* available for trade in a given period and having positive returns only in the next period. Or we could have *long-term assets* available at $t = 0$, tradeable at every period, and having positive returns only at the final period $t = T$. And, of course, there could be mixed cases of assets tradeable in a subset of periods and providing returns also in a subset of periods (not necessarily the same).

Again, the equivalence result of Proposition 19.D.1 generalizes if the asset structure is *complete*. Suppose, for example, that our asset structure is composed only of collections of short-term assets available and tradeable at any admissible date-event pair tE and paying contingent amounts of physical good 1 at the immediately succeeding date-event pairs. Denote by $S(tE)$ the number of successors at tE. If the number of assets available at tE is $K(tE)$ we can view the return matrix at tE as an $S(tE) \times K(tE)$ matrix $R(tE)$. The completeness condition is then the requirement that rank $R(tE) = S(tE)$ for all admissible date–events pairs tE. In

Section 19.D, the matrices $R(tE)$ were identity matrices, and so the asset structure there was complete. But, to repeat, the results of Section 19.D generalize to the complete, nondiagonal case.

A very interesting, and new, phenomenon is that if assets are long lived, and therefore repeatedly tradeable as information is gradually disclosed, it may be possible to implement the Arrow–Debreu equilibrium with much fewer than S assets. This is illustrated in Example 19.E.7.

Example 19.E.7: Suppose that $T = 2$ and the date–event tree unfolds as in Figure 19.E.2. In

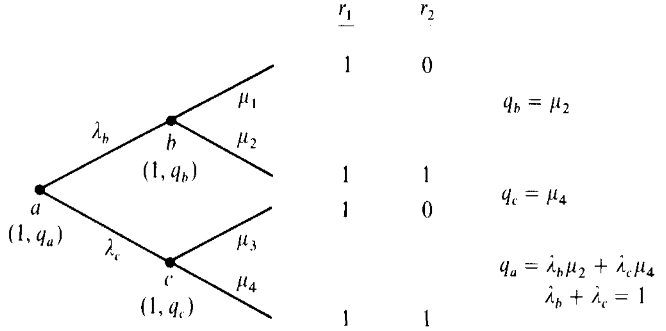

Figure 19.E.2 Construction of Arrow–Debreu prices from the equilibrium values of the asset prices of two sequentially traded assets.

the figure we have seven admissible date–events corresponding to the four terminal nodes, or states, to the initial node, denoted a, and to two intermediate nodes, denoted b and c. In particular, there are no more than two branches from any node. In this case we claim that, typically, two long-lived assets should suffice to guarantee that the Radner and the Arrow–Debreu equilibrium consumption allocations are the same.[19] Suppose, to take a simple instance, that $L = 1$ and that our two assets have return vectors $r_1 = (1, 1, 1, 1)$ and $r_2 = (0, 1, 0, 1)$, payable at the terminal nodes. Consumption takes place only at the terminal nodes, but the assets may be traded both at the node a corresponding to $t = 0$ and at the nodes b and c correponding to $t = 1$. We can normalize the price of the first asset (as well as the price of final consumption) to be 1 at every node.[20] Denote by q_a, q_b, and q_c the prices of the second asset at the respective nodes. By arbitrage (Proposition 19.E.1), applied at $t = 1$, there are $(\mu_1, \mu_2) \geq 0$ such that $\mu_1 + \mu_2 = 1$, $\mu_2 = q_b$, and $(\mu_3, \mu_4) \geq 0$ such that $\mu_3 + \mu_4 = 1$ and $\mu_4 = q_c$. Again by arbitrage (applied this time at $t = 0$), we must have $(\lambda_b, \lambda_c) \geq 0$ such that $\lambda_b + \lambda_c = 1$ and $q_a = \lambda_b q_b + \lambda_c q_c = \lambda_b \mu_2 + \lambda_c \mu_4$. This suggests considering the following Arrow–Debreu prices:

$$p = (\lambda_b\mu_1, \lambda_b\mu_2, \lambda_c\mu_3, \lambda_c\mu_4).$$

In Exercise 19.E.5 you are asked to show that, under the weak condition $q_b \neq q_c$, the set of final consumptions achievable through sequential trade with asset prices (q_a, q_b, q_c) is indeed the same as the set of final consumptions achievable with the four Arrow–Debreu contingent commodities at prices p.[21] ■

19. We say "typically" because the existence of two assets is a necessary condition for completeness at every node but, strictly speaking, not a sufficient condition.

20. Note that, as should be the case, we do not normalize more than one price per budget constraint.

21. Incidentally, the assets prices provide a specific instance of what is called the *martingale property of asset prices*: at any node the price of an asset is the conditional expectation of the final returns, where the expectation is taken with respect to some probabilities, in our case the Arrow–Debreu prices.