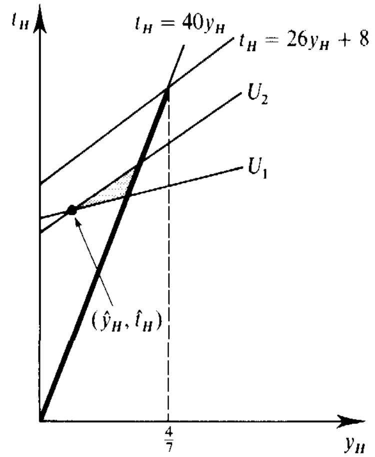

Figure 23.F.3
The set of ex ante incentive efficient social choice functions corresponds to those *interim incentive* efficient social choice functions with (y_H, t_H) lying in the heavily traced line segment.

is therefore precisely the heavily traced line segment in Figure 23.F.3.[54] Notice that in every such social choice function the interim individual rationality constraint for a high-quality seller binds: the high-quality seller receives no gains from trade. ■

APPENDIX A: IMPLEMENTATION AND MULTIPLE EQUILIBRIA

The notion of implementation that we have employed throughout the chapter (e.g., in Definition 23.B.4) is weaker in one potentially important respect than what we might want: Although a mechanism Γ may implement the social choice function $f(\cdot)$ in the sense of having *an* equilibrium whose outcomes coincide with $f(\cdot)$ for all $\theta \in \Theta$, there may be *other* equilibria of Γ whose outcomes do *not* coincide with $f(\cdot)$. In essence, we have implicitly assumed that the agents will play the equilibrium that the mechanism designer wants if there is more than one.[55]

This suggests that if a mechanism designer wishes to be fully confident that the mechanism Γ does indeed yield the outcomes associated with $f(\cdot)$, he might instead want to insist upon the stronger notion of implementation given in Definition 23.AA.1 (as in Definition 23.B.4, we are deliberately vague here about the equilibrium concept to be employed).

Definition 23.AA.1: The mechanism $\Gamma = (S_1, \ldots, S_I, g(\cdot))$ *strongly implements* social choice function $f\colon \Theta_1 \times \cdots \times \Theta_I \to X$ if every equilibrium strategy

54. Note that $y_H \leq 4/7 < 1$ in any ex ante incentive efficient social choice function. This may seem at odds with our conclusion in Section 13.B that the ex ante efficient outcome that gives firms zero expected profits has all workers accepting employment in a firm (the structure of the model in Section 13.B parallels that here). The difference is that in Section 13.B we did not impose any interim individual rationality constraints on the workers, effectively supposing that the government could compel workers to participate (pay any taxes, etc.). See Exercise 23.F.10.

55. One possible argument for this assumption is that in a direct revelation mechanism that truthfully implements social choice function $f(\cdot)$, the truth-telling equilibrium may be focal (in the sense discussed in Section 8.D).

profile $(s_1^*(\cdot), \ldots, s_I^*(\cdot))$ of the game induced by Γ has the property that $g(s_1^*(\theta_1), \ldots, s_I^*(\theta_I)) = f(\theta_1, \ldots, \theta_I)$ for all $(\theta_1, \ldots, \theta_I)$.[56]

Let us consider first the implications of this stronger concept for implementation in dominant strategy equilibria. In Exercise 23.C.8 we have already seen a case where, in the direct revelation mechanism that truthfully implements a social choice function $f(\cdot)$ in dominant strategies, some player has more than one dominant strategy, and when he plays one of these dominant strategies the outcome in $f(\cdot)$ does not result (see also Exercises 23.AA.1 and 23.AA.2). Thus, with dominant strategy implementation we may have mechanisms that implement a social choice function $f(\cdot)$ but that do not strongly implement it.

Nevertheless, there are at least two reasons why the multiple equilibrium problem may not be too severe with dominant strategy implementation. First, whenever each agent's dominant strategy in a mechanism Γ that implements $f(\cdot)$ is in fact a *strictly* dominant strategy (rather than just a weakly dominant one), mechanism Γ also strongly implements $f(\cdot)$. This is always the case, for example, in any environment in which agents' preferences never involve indifference between any two elements of X. Second, when a player has two weakly dominant strategies, he is of necessity indifferent between them for any strategies than the other agents choose. Thus, to play the "right" equilibrium in this case, it is only necessary that each agent be willing to resolve his indifference in the way we desire.

In contrast, with Nash-based equilibrium concepts such as Bayesian Nash equilibrium, if mechanism Γ has two equilibria, then in each equilibrium each player may have a strict preference for his equilibrium strategy given that the other agents are playing their respective equilibrium strategies. Having agents play the "right" equilibrium is then not just a matter of resolving indifference but rather of generating expectations that the desired equilibrium is the one that will occur. Example 23.AA.1 illustrates the problem.

Example 23.AA.1: *Multiple Equilibria in the Expected Externality Mechanism.* Consider again the expected externality mechanism of Section 23.D. Suppose that we are in a setting with two agents ($I = 2$) in which a decision must be made regarding a public project (see Example 23.B.3). The project may be either done ($k = 1$) or not done ($k = 0$). Each agent's valuation (net of funding the project) is either θ_L or θ_H (so $\Theta_i = \{\theta_L, \theta_H\}$ for $i = 1, 2$), where $\theta_H > 0 > \theta_L$ and $\theta_L + \theta_H > 0$. The agents' valuations are statistically independent with Prob $(\theta_i = \theta_L) = \lambda \in (0, 1)$ for $i = 1, 2$.

In the expected externality mechanism, each agent i announces his valuation and agent i's transfer when the announced types are (θ_1, θ_2) has the form $t_i(\theta_i, \theta_{-i}) = E_{\tilde{\theta}_{-i}}[\tilde{\theta}_{-i} k^*(\theta_i, \tilde{\theta}_{-i})] + h_i(\theta_{-i})$, where $k^*(\theta_1, \theta_2) = 0$ if $\theta_1 = \theta_2 = \theta_L$, and $k^*(\theta_1, \theta_2) = 1$ otherwise.

As we saw in Section 23.D, in one Bayesian Nash equilibrium of this mechanism, truth telling is each agent's equilibrium strategy. But this truth-telling equilibrium is *not* the only Bayesian Nash equilibrium. In particular, there is an equilibrium in which both agents always claim that θ_H is their type. To see this, consider agent i's optimal

56. The "strong" terminology is not standard; in the literature it is not uncommon, for example, to see the strong implementation concept simply referred to as "implementation."

strategy if agent $-i$ will always announce θ_H. Whichever announcement agent i makes, the project is done. Thus, regardless of his actual type, agent i's direct benefit (i.e., $\theta_i k^*(\theta_1, \theta_2)$) is not affected by his announcement (it is θ_L if he is of type θ_L, and θ_H if he is of type θ_H). It follows that agent i's optimal strategy is to make an announcement that maximizes his expected transfer. Now, agent i's expected transfer if he announces θ_H is $(\lambda\theta_L + (1-\lambda)\theta_H) + h_i(\theta_H)$, whereas if he announces θ_L his expected transfer is $(1-\lambda)\theta_H + h_i(\theta_H)$. Hence, agent i will prefer to announce θ_H regardless of his type if agent $-i$ is doing the same. It follows that both agents always announcing θ_H, and the project consequently always being done, constitutes a second Bayesian Nash equilibrium of this mechanism. ■

We shall not pursue here the characterization of social choice functions that can be strongly implemented in Bayesian Nash equilibria. A good source of further reading on this subject is Palfrey (1992). We also refer to Appendix B, where we discuss many of these issues for the special context of complete information environments.

There are, however, two important points about strong implementation that we wish to stress here. First, when trying to strongly implement a social choice function $f(\cdot)$, we *cannot* generally restrict attention to direct revelation mechanisms. The reason is that when we replace a mechanism $\Gamma = (S_1, \ldots, S_I, g(\cdot))$ with a direct revelation mechanism, as envisioned by the revelation principle, we may introduce new, undesirable equilibria. (See Exercises 23.C.8 and 23.AA.1 for an illustration.) Second, because a social choice function $f(\cdot)$ can be strongly implemented only if it can be implemented in the weaker sense studied in the text of the chapter, all of the necessary conditions for implementation that we have derived are *still* necessary for $f(\cdot)$ to be strongly implemented. Thus, for example, the conclusions of the Gibbard–Satterthwaite theorem (Proposition 23.C.3), the revenue equivalence theorem (Proposition 23.D.3), and the Myerson–Satterthwaite theorem (Proposition 23.E.1) all continue to be valid when we seek strong implementation.

Throughout the chapter we have restricted attention to single-valued social choice functions. It is sometimes natural, however, to consider social choice *correspondences* that can specify more than one acceptable alternative for a given profile of agent types. In this case, we would say that mechanism $\Gamma = (S_1, \ldots, S_I, g(\cdot))$ strongly implements the social choice correspondence $f(\cdot)$ if every equilibrium $s^*(\cdot)$ of the game induced by Γ has the property that $g(s^*(\theta)) \in f(\theta)$, that is, if, for every θ, all possible equilibrium outcomes are acceptable alternatives according to $f(\cdot)$.

APPENDIX B: IMPLEMENTATION IN ENVIRONMENTS WITH COMPLETE INFORMATION

In this appendix we provide a brief discussion of implementation in complete information environments. An excellent source for further reading on this subject is the survey by Moore (1992) [see also Maskin (1985)].

In the complete information case we suppose that each agent will observe not only his own preference parameter θ_i, but also the preference parameters θ_{-i} of all other agents. However, while the agents will observe each others' preference parameters, we suppose that no outsider does. Thus, despite the agents' abilities to observe θ, there is still an implementation problem: Because no outsider (such as a court) will observe θ, the agents cannot write an enforceable ex ante agreement saying that they will choose outcome $f(\theta)$ when agents' preferences are θ. Rather, they can only agree to participate in some mechanism in which equilibrium play yields $f(\theta)$ if θ is realized.[57]

Note that a complete information setting can be viewed as a special case of the general environment considered throughout this chapter, in which the probability density $\phi(\cdot)$ on Θ has the (degenerate) property that each agent's observation of his "type" is completely informative about the types of the other agents.[58]

To begin, observe that the set of social choice functions that are dominant strategy implementable is unaffected by the presence of complete information because an agent's belief about the types of other agents does not affect his set of dominant strategies.[59] [Indeed, recall our comment in Section 23.C that if mechanism Γ implements social choice function $f(\cdot)$ in dominant strategies, then it does so for *any* $\phi(\cdot)$.]

This is *not* the case, however, for implementation in Nash-based equilibrium concepts, such as Bayesian Nash equilibrium. Recall that in complete information environments, the Bayesian Nash equilibrium concept reduces to our standard notion of Nash equilibrium. This motivates Definition 23.BB.1.

Definition 23.BB.1: The mechanism $\Gamma = (S_1, \ldots, S_I, g(\cdot))$ *implements the social choice function $f(\cdot)$ in Nash equilibrium* if, for each profile of the agents' preference parameters $\theta = (\theta_1, \ldots, \theta_I) \in \Theta$, there is a Nash equilibrium of the game induced by Γ, $s^*(\theta) = (s_1^*(\theta), \ldots, s_I^*(\theta))$, such that $g(s^*(\theta)) = f(\theta)$. The mechanism $\Gamma = (S_1, \ldots, S_I, g(\cdot))$ *strongly implements the social choice function $f(\cdot)$ in Nash equilibrium* if, for each profile of the agents' preference parameters $\theta = (\theta_1, \ldots, \theta_I) \in \Theta$, every Nash equilibrium of the game induced by Γ results in outcome $f(\theta)$.

The first point to note about implementation in Nash equilibria is that if we are satisfied with the weaker notion of implementation that we have employed throughout the text of the chapter (as opposed to the strong implementation concept discussed in Appendix A), then *any* social choice function can be implemented in Nash equilibrium as long as $I \geq 3$. To see this, consider the following mechanism: each agent i simultaneously announces a profile of types for each of the I agents. If at least $I - 1$ agents announce the same profile, say $\hat{\theta}$, then we select outcome $f(\hat{\theta})$.

57. This type of setting is often natural in contracting problems, where it is frequently reasonable to suppose that the parties will come to know a lot about each other that is not verifiable by any outside enforcer of their contract.

58. Thus, we can think of the complete information environment as a case in which agents receive signals that are perfectly correlated. There are several ways to formalize this. Perhaps the simplest is to suppose that each agent i's preference parameter is drawn from some set Θ_i. An agent's *signal* (or *type*), which is now represented by $\bar{\theta}_i = (\theta_{i1}, \ldots, \theta_{iI}) \in \Theta$, is a vector giving agent i's observation of his and every other agent's preference parameters. Thus, the set of possible "types" for agent i in the sense in which we have used this term throughout the chapter, is now $\bar{\Theta}_i = \Theta$ for each $i = 1, \ldots, I$. The probability density $\phi(\cdot)$ on the set of possible types $\bar{\Theta}_1 \times \cdots \times \bar{\Theta}_I$ then satisfies the property that $\phi(\bar{\theta}_1, \ldots, \bar{\theta}_I) > 0$ if and only if $\bar{\theta}_1 = \cdots = \bar{\theta}_I$, and agent i's Bernoulli utility function has the form $u_i(x, \bar{\theta}_i) = \tilde{u}_i(x, \theta_{ii})$.

59. The same is true for strong implementation in dominant strategies.

Otherwise we select outcome $x_0 \in X$ (x_0 is arbitary). With this mechanism, for every profile θ, there is a Nash equilibrium in which every agent announces θ, and the resulting outcome is $f(\theta)$, because no agent can affect the outcome by unilaterally deviating.

Although this mechanism implements $f(\cdot)$ in Nash equilibrium, it is obviously not a very attractive mechanism [i.e., its implementation of $f(\cdot)$ is not very convincing] because there are so many other Nash equilibria that do not result in outcome $f(\theta)$ when the preference profile is θ. Indeed, with this mechanism, given the profile of preference parameters θ, there is a Nash equilibrium resulting in x for every $x \in f(\Theta) = \{x \in X\colon \text{there is a } \theta \in \Theta \text{ such that } f(\theta) = x\}$. We see then that for Nash implementation with $I \geq 3$, the *entire* problem of satisfactorily implementing a given social choice function revolves around the issue of successfully dealing with the multiple equilibrium problem discussed in Appendix A.

Given this observation, what social choice functions can we strongly implement in Nash equilibrium? The simple but powerful result in Proposition 23.BB.1 comes from Maskin's (1977) path-breaking paper on Nash implementation.

Proposition 23.BB.1: If the social choice function $f(\cdot)$ can be strongly implemented in Nash equilibrium, then $f(\cdot)$ is monotonic.[60]

Proof: Suppose that $\Gamma = (S_1, \ldots, S_I, g(\cdot))$ strongly implements $f(\cdot)$. Then when the preference parameter profile is θ, there is a Nash equilibrium resulting in outcome $f(\theta)$; that is, there is a strategy profile $s^* = (s_1^*, \ldots, s_I^*)$ having the properties that $g(s^*) = f(\theta)$ and $g(\hat{s}_i, s_{-i}^*) \in L_i(f(\theta), \theta_i)$ for all $\hat{s}_i \in S_i$ and all i.[61] Now suppose that $f(\cdot)$ is not monotonic. Then there exists another profile of preference parameters $\theta' \in \Theta$ such that $L_i(f(\theta), \theta_i) \subset L_i(f(\theta), \theta_i')$ for all i, but $f(\theta') \neq f(\theta)$. But s^* is also a Nash equilibrium under preference parameter profile θ', because $g(\hat{s}_i, s_{-i}^*) \in L_i(f(\theta), \theta_i')$ for all $\hat{s}_i \in S_i$ and all i. Hence, Γ does not strongly implement $f(\cdot)$—a contradiction. ■

As an illustration of the restriction imposed by monotonicity, Proposition 23.BB.2 records one implication of this result.

Proposition 23.BB.2: Suppose that X is finite and contains at least three elements, that $\mathscr{R}_i = \mathscr{P}$ for all i, and that $f(\Theta) = X$. Then the social choice function $f(\cdot)$ is strongly implementable in Nash equilibrium if and only if it is dictatorial.

Proof: To strongly implement a dictatorial social choice function in Nash equilibrium, we need only let the dictator choose an alternative from X. In the other direction, the result follows from steps 2 and 3 of the proof of the Gibbard–Satterthwaite theorem (Proposition 23.C.3). ■

One lesson to be learned from Propositions 23.BB.1 and 23.BB.2 is that dealing with the multiple equilibrium problem can potentially impose very significant restrictions on the set of implementable social choice functions.

60. See Definition 23.C.5.

61. Recall that $L_i(x, \theta_i) \subset X$ is agent i's lower contour set for outcome $x \in X$ when agent i's preference parameter is θ_i.

Maskin (1977) also showed that, when $I > 2$, monotonicity is *almost*, but not quite, a sufficient condition for strong implementation [we omit the proof; see Moore (1992) for a discussion of this and more recent results, including consideration of the case $I = 2$]. Maskin's added condition, known as *no veto power*, requires that if $I - 1$ agents all rank some alternative x as their best alternative then $x = f(\theta)$.

Proposition 23.BB.3: If $I \geq 3$, $f(\cdot)$ is monotonic, and $f(\cdot)$ satisfies no veto power, then $f(\cdot)$ is strongly implementable in Nash equilibrium.

No veto power should be thought of as a very weak additional requirement; indeed, in any setting in which there is a desirable and transferable private good, it is trivially satisfied: no two agents then ever have the same top-ranked alternative (each wants to get all of the available transferable private good). Thus, in these commonly studied environments, monotonicity of $f(\cdot)$ is a necessary and sufficient condition for $f(\cdot)$ to be strongly implementable in Nash equilibrium.

A multivalued social choice correspondence $f(\cdot)$ is said to be monotonic if whenever $x \in f(\theta)$ and $L_i(x, \theta_i) \subset L_i(x, \theta_i')$ for all i then $x \in f(\theta')$. The necessary and sufficient conditions for Nash implementation in Propositions 23.BB.1 and 23.BB.3 carry over to the multivalued case. [In fact, for Proposition 23.BB.3, Maskin's result actually establishes that there is a mechanism that has, for each preference profile θ, a set of Nash equilibrium outcomes exactly equal to the set $f(\theta)$; this type of implementation of a social choice correspondence is commonly called *full implementation.*] It may be verified, for example, that if $f(\theta)$ is equal to the Pareto set for all θ (the set of all ex post efficient outcomes in X given preference profile θ), then $f(\cdot)$ is monotonic. Thus, in any setting with a transferable good, the Pareto set social choice correspondence is strongly implementable in Nash equilibrium.

Implementation using Extensive Form Games: Subgame Perfect Implementation

So far we have seen that the need to "knock out" undesirable equilibria (formalized through the notion of strong implementation) can significantly restrict the set of implementable social choice functions. This suggests the possibility that we may be more successful if we use instead a refinement of the Nash equilibrium concept. Indeed, recent work has shown that such refinements can be very powerful. Here we briefly illustrate how, by considering *dynamic* mechanisms and employing the equilibrium concepts for dynamic games discussed in Section 9.B, we can expand the set of strongly implementable social choice functions.

Example 23.BB.1: [Adapted from Moore and Repullo (1988)]. Consider a pure exchange economy (see Example 23.B.2) with two consumers in which each consumer has two possible individualistic preference relations: if $\theta_i = \theta_i^C$, then consumer i has Cobb-Douglas preferences; if $\theta_i = \theta_i^L$, then consumer i has Leontief preferences. These two possible preference relations for consumers 1 and 2 are depicted in Figure 23.BB.1.

Suppose that we wish to strongly implement the social choice function

$$f(\theta) = \begin{cases} x^C & \text{if } \theta_1 = \theta_1^C, \\ x^L & \text{if } \theta_1 = \theta_1^L, \end{cases}$$

where x^C and x^L are the allocations depicted in Figure 23.BB.1. Note that consumer 1 always prefers x^C to x^L, and the reverse is true for consumer 2. By inspection of

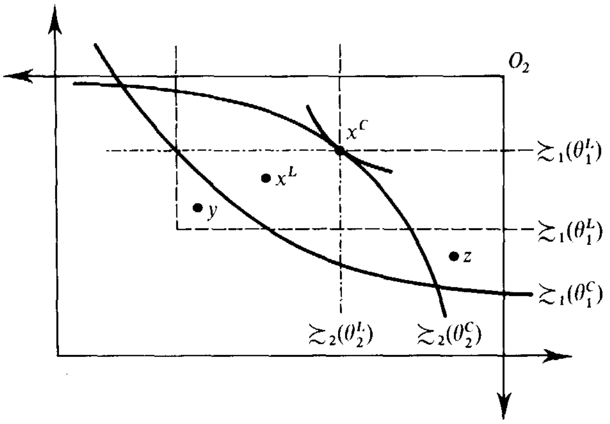

Figure 23.BB.1 Preferences and outcomes in Example 23.BB.1.

Figure 23.BB.1, we see that $f(\cdot)$ is not monotonic, because $L_i(x^C, \theta_i^C) \subset L_i(x^C, \theta_i^L)$ for $i = 1, 2$ but $f(\theta_1^L, \theta_2^L) \neq f(\theta_1^C, \theta_2^C)$. Hence, by Proposition 23.BB.1, $f(\cdot)$ cannot be strongly implemented in Nash equilibrium.

Suppose, instead, that we construct the following three-stage dynamic mechanism:

Stage 1: Agent 1 announces either "L" or "C." If he announces "L," x^L is immediately chosen. If he announces "C," we go to stage 2.

Stage 2: Agent 2 says either "agree" or "challenge." If he says "agree," then x^C is immediately chosen. If he says "challenge," then we go to stage 3.

Stage 3: Agent 1 chooses between the allocations y and z depicted in Figure 23.BB.1.

It is straightforward to verify that, for each possible profile of preferences $\theta = (\theta_1, \theta_2)$, the unique subgame perfect Nash equilibrium of this dynamic game of perfect information results in outcome $f(\theta)$ (see Exercise 23.BB.1). Thus, $f(\cdot)$ *can* be strongly implemented if we consider dynamic mechanisms and take subgame perfect Nash equilibrium as the appropriate solution concept for the games induced by these mechanisms. ∎

In fact, Moore and Repullo (1988) [see also Moore (1990)] show that the use of dynamic mechanisms and subgame perfection expands the set of strongly implementable social choice functions dramatically compared to the use of the Nash equilibrium concept. Even stronger results are possible with other refinements; see, for example, Palfrey and Srivastava (1991) for a study of strong implementation in undominated Nash equilibrium (i.e., Nash equilibria in which no agent is playing a weakly dominated strategy).[62]

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EXERCISES

23.B.1[A] Consider the setting explored in Example 23.B.1, where $\mathscr{R}_1 = \{\succsim_1(\bar{\theta}_1)\}$ and $\mathscr{R}_2 = \{\succsim_2(\theta_2'), \succsim_2(\theta_2'')\}$. For each of the following social choice functions $f(\cdot)$, will agent 2 be willing to truthfully reveal his preferences?

(a) $f(\theta_1, \theta_2') = y$, $f(\bar{\theta}_1, \theta_2'') = y$.

(b) $f(\theta_1, \theta_2') = z$, $f(\bar{\theta}_1, \theta_2'') = x$.

(c) $f(\bar{\theta}_1, \theta_2') = z$, $f(\theta_1, \theta_2'') = y$.

(d) $f(\theta_1, \theta_2') = z$, $f(\bar{\theta}_1, \theta_2'') = z$.

(e) $f(\theta_1, \theta_2') = y$, $f(\bar{\theta}_1, \theta_2'') = z$.

23.B.2[A] Consider a bilateral trade setting (see Example 23.B.4) in which both the seller's (agent 1) and the buyer's (agent 2) types are drawn independently from the uniform distribution on $[0, 1]$. Suppose that we try to implement the social choice function $f(\cdot) = (y_1(\cdot), y_2(\cdot), t_1(\cdot), t_2(\cdot))$ such that

$$y_1(\theta_1, \theta_2) = 1 \text{ if } \theta_1 \geq \theta_2; \ = 0 \text{ if } \theta_1 < \theta_2.$$

$$y_2(\theta_1, \theta_2) = 1 \text{ if } \theta_2 > \theta_1; \ = 0 \text{ if } \theta_2 \leq \theta_1.$$

$$t_1(\theta_1, \theta_2) = \tfrac{1}{2}(\theta_1 + \theta_2) y_2(\theta_1, \theta_2).$$

$$t_2(\theta_1, \theta_2) = -\tfrac{1}{2}(\theta_1 + \theta_2) y_2(\theta_1, \theta_2).$$

Suppose that the seller truthfully reveals his type for all $\theta_1 \in [0, 1]$. Will the buyer find it worthwhile to reveal his type? Interpret.

23.B.3[B] Show that $b_i(\theta_i) = \theta_i$ for all $\theta_i \in [0, 1]$ is a weakly dominant strategy for each agent i in the second-price sealed-bid auction.

23.B.4[C] Consider a bilateral trade setting (see Example 23.B.4) in which both the seller's and the buyer's types are drawn independently from the uniform distribution on $[0, 1]$.

(a) Consider the *double auction* mechanism in which the seller (agent 1) and buyer (agent 2) each submit a sealed bid, $b_i \geq 0$. If $b_1 \geq b_2$, the seller keeps the good and no monetary transfer is made; while if $b_2 > b_1$, the buyer gets the good and pays the seller the amount $\frac{1}{2}(b_1 + b_2)$. (The interpretation is that the seller's bid is his minimum acceptable price, while the buyer's is his maximum acceptable price; if trade occurs, the price splits the difference between these amounts.) Solve for a Bayesian Nash equilibrium of this game in which each agent i's strategy takes the form $b_i(\theta_i) = \alpha_i + \beta_i \theta_i$. What social choice function does this equilibrium of this mechanism implement? Is it ex post efficient?

(b) Show that the social choice function derived in **(a)** is incentive compatible; that is, that it can be truthfully implemented in Bayesian Nash equilibrium.

23.C.1[A] Verify that if the preference reversal property [condition (23.C.6)] is satisfied for all i and all θ_i', θ_i'', and θ_{-i}, then $f(\cdot)$ is truthfully implementable in dominant strategies.

23.C.2[B] Show that, for any I, when X contains two elements (say, $X = \{x_1, x_2\}$), then any majority voting social choice function [i.e., a social choice function that always chooses alternative x_i if more agents prefer x_i over x_j than prefer x_j over x_i (it may select either x_1 or x_2 if the number of agents preferring x_1 over x_2 equals the number preferring x_2 over x_1)] is truthfully implementable in dominant strategies.

23.C.3$^{\text{A}}$ Show that when $\mathscr{R}_i = \mathscr{P}$ for all i, any ex post efficient social choice function $f(\cdot)$ has $f(\Theta) = X$.

23.C.4$^{\text{A}}$ Show that if $f\colon \Theta \to X$ is truthfully implementable in dominant strategies when the set of possible types is Θ_i for $i = 1, \ldots, I$, then when each agent i's set of possible types is $\hat{\Theta}_i \subset \Theta_i$ (for $i = 1, \ldots, I$) the social choice function $\hat{f}\colon \hat{\Theta} \to X$ satisfying $\hat{f}(\theta) = f(\theta)$ for all $\theta \in \hat{\Theta}$ is truthfully implementable in dominant strategies.

23.C.5$^{\text{C}}$ Show that in an environment with single-peaked preferences having the property that no two alternatives are indifferent (see Section 21.D) and an odd number of agents, the (unique) social choice function that always selects a Condorcet winner (see Section 21.D) is truthfully implementable in dominant strategies.

23.C.6$^{\text{C}}$ The property of a social choice function identified in Proposition 23.C.2 is called *independent person-by-person monotonicity* (IPM) by Dasgupta, Hammond, and Maskin (1979). In this exercise, we investigate its relationship with the *monotonicity* property defined in Definition 23.C.5.

(a) Show by means of an example that if $f(\cdot)$ satisfies IPM, it need not be monotonic (this can be done with a very simple example).

(b) Show by means of an example that if $f(\cdot)$ is monotonic, it need not satisfy IPM.

(c) Prove that if $f(\cdot)$ satisfies IPM, and if $\mathscr{R}_i \subset \mathscr{P}$ for all i, then $f(\cdot)$ is monotonic.

(d) Prove that if $f(\cdot)$ is monotonic, and $\mathscr{R}_i = \mathscr{P}$ for all i, then $f(\cdot)$ satisfies IPM.

23.C.7$^{\text{C}}$ A social welfare functional $F(\cdot)$ (see Section 21.C) satisfies the property of *nonnegative responsiveness* if for all $x, y \in X$, and for any two pairs of profiles of preferences for the I agents $(\succsim_1, \ldots, \succsim_I)$ and $(\succsim'_1, \ldots, \succsim'_I)$ such that $x \succsim_i y \Rightarrow x \succsim'_i y$ and $x \succ_i y \Rightarrow x \succ'_i y$ for all i, we have

$$x\, F(\succsim_1, \ldots, \succsim_I)\, y \Rightarrow x\, F(\succsim'_1, \ldots, \succsim'_I)\, y$$

and

$$x\, F_p(\succsim_1, \ldots, \succsim_I)\, y \Rightarrow x\, F_p(\succsim'_1, \ldots, \succsim'_I)\, y,$$

where $x\, F_p(\cdot)\, y$ means "$x\, F(\cdot)\, y$ and not $y\, F(\cdot)\, x$." Show that if the social choice function $f(\cdot)$ maximizes a social welfare functional $F(\cdot)$ satisfying nonnegative responsiveness [in the sense that for all $(\theta_1, \ldots, \theta_I)$ we have $f(\theta_1, \ldots, \theta_I) = \{x \in X\colon x\, F(\succsim_1(\theta_1), \ldots, \succsim_I(\theta_I))\, y$ for all $y \in X\}$], then $f(\cdot)$ is truthfully implementable in dominant strategies.

23.C.8$^{\text{A}}$ Suppose that $I = 2$, $X = \{a, b, c, d, e\}$, $\Theta_1 = \{\theta'_1, \theta''_1\}$, and $\Theta_2 = \{\theta'_2, \theta''_2\}$, and that the agents' possible preferences are (a–b means that alternatives a and b are indifferent):

$\succsim_1(\theta'_1)$	$\succsim_1(\theta''_1)$	$\succsim_2(\theta'_2)$	$\succsim_2(\theta''_2)$
a–b	a	a–b	a
c	b	c	b
d	d	d	d
e	c	e	c
	e		e

Consider the social choice function

$$f(\theta) = \begin{cases} b & \text{if } \theta = (\theta_1', \theta_2'), \\ a & \text{otherwise.} \end{cases}$$

(a) Is $f(\cdot)$ ex post efficient?

(b) Does it satisfy the property identified in Proposition 23.C.2?

(c) Examine the direct revelation mechanism that truthfully implements $f(\cdot)$. Is truth telling each agent's *unique* (weakly) dominant strategy? Show that if an agent chooses his untruthful (weakly) dominant strategy, then $f(\cdot)$ is not implemented.

23.C.9[C] Suppose that $K = \mathbb{R}$, the $v_i(\cdot, \theta_i)$ functions are assumed to be twice continuously differentiable, θ_i is drawn from an interval $[\underline{\theta}_i, \bar{\theta}_i]$, $\partial^2 v_i(k, \theta_i)/\partial k^2 < 0$, and $\partial^2 v_i(k, \theta_i)/\partial k\, \partial\theta_i > 0$. Show that the continuously differentiable social choice function $f(\cdot) = (k(\cdot), t_1(\cdot), \ldots, t_I(\cdot))$ is truthfully implementable in dominant strategies if and only if, for all $i = 1, \ldots, I$,

$$k(\theta) \text{ is nondecreasing in } \theta_i$$

and

$$t_i(\theta_i, \theta_{-i}) = t_i(\underline{\theta}_i, \theta_{-i}) - \int_{\underline{\theta}_i}^{\theta_i} \frac{\partial v_i(k(s, \theta_{-i}), s)}{\partial k} \frac{\partial k(s, \theta_{-i})}{\partial s}\, ds.$$

23.C.10[B] (B. Holmstrom) Consider the quasilinear environment studied in Section 23.C. Let $k^*(\cdot)$ denote any project decision rule that satisfies (23.C.7). Also define the function $V^*(\theta) = \sum_i v_i(k^*(\theta), \theta_i)$.

(a) Prove that there exists an ex post efficient social choice function [i.e., one that satisfies condition (23.C.7) *and* the budget balance condition (23.C.12)] that is truthfully implementable in dominant strategies if and only if the function $V^*(\cdot)$ can be written as $V^*(\theta) = \sum_i V_i(\theta_{-i})$ for some functions $V_1(\cdot), \ldots, V_I(\cdot)$ having the property that $V_i(\cdot)$ depends only on θ_{-i} for all i.

(b) Use the result in part **(a)** to show that when $I = 3$, $K = \mathbb{R}$, $\Theta_i = \mathbb{R}_+$ for all i, and $v_i(k, \theta_i) = \theta_i k - (\frac{1}{2})k^2$ for all i an ex post efficient social choice function exists that is truthfully implementable in dominant strategies. (This result extends to any $I > 2$.)

(c) Now suppose that the $v_i(k, \theta_i)$ functions are such that $V^*(\cdot)$ is an I-times continuously differentiable function. Argue that a necessary condition for an ex post efficient social choice function to exist is that, at all θ,

$$\frac{\partial^I V^*(\theta)}{\partial\theta_1 \ldots \partial\theta_I} = 0.$$

(In fact, this is a sufficient condition as well.)

(d) Use the result in **(c)** to verify that, under the assumptions made in the small type discussion at the end of Section 23.C, when $I = 2$ no ex post efficient social choice function is truthfully implementable in dominant strategies.

23.C.11[A] Consider a quasilinear environment, but now suppose that each agent i has a Bernoulli utility function of the form $u_i(v_i(k, \theta_i) + \bar{m}_i + t_i)$ with $u_i'(\cdot) > 0$. That is, preferences over certain outcomes take a quasilinear form, but risk preferences are unrestricted. Verify that Proposition 23.C.4 is unaffected by this change.

23.D.1[B] [Based on an example in Myerson (1991)] A buyer and a seller are bargaining over the sale of an indivisible good. The buyer's valuation is $\theta_b = 10$. The seller's valuation takes one of two values: $\theta_s \in \{0, 9\}$. Let t be the period in which trade occurs ($t = 1, 2, \ldots$) and let p be the price agreed. Both the buyer and the seller have discount factor $\delta < 1$.

(a) What is the set X of alternatives in this setting?

(b) Suppose that in a Bayesian Nash equilibrium of this bargaining process, trade occurs immediately when the seller's valuation is 0 and the price agreed to when the seller has valuation θ_s is $(10 + \theta_s)/2$. What is the earliest possible time at which trade can occur when the seller's valuation is 9?

23.D.2[B] Consider a bilateral trade setting in which each θ_i ($i = 1, 2$) is independently drawn from a uniform distribution on $[0, 1]$.

(a) Compute the transfer functions in the expected externality mechanism.

(b) Verify that truth telling is a Bayesian Nash equilibrium.

23.D.3[A] Reconsider the first-price and second-price sealed-bid auctions studied in Examples 23.B.5 and 23.B.6. Verify that the revenue equivalence theorem holds for the equilibria identified there.

23.D.4[C] Consider a first-price sealed-bid auction with I symmetric buyers. Each buyer's valuation is independently drawn from the interval $[\underline{\theta}, \bar{\theta}]$ according to the strictly positive density $\phi(\cdot)$.

(a) Show that the buyer's equilibrium bid function is nondecreasing in his type.

(b) Argue that in any symmetric equilibrium $(b^*(\cdot), \ldots, b^*(\cdot))$ there can be no interval of types (θ', θ''), $\theta' \neq \theta''$, such that $b^*(\theta)$ is the same for all $\theta \in (\theta', \theta'')$. Conclude that $b^*(\cdot)$ must therefore be strictly increasing.

(c) Argue, using the revenue equivalence theorem, that any symmetric equilibrium of such an auction must yield the seller the same expected revenue as in the (dominant strategy) equilibrium of the second-price sealed-bid auction.

23.D.5[C] For the same assumptions as in Exercise 23.D.4, consider a sealed-bid *all-pay* auction in which every buyer submits a bid, the highest bidder receives the good, and *every* buyer pays the seller the amount of his bid *regardless of whether he wins*. Argue that any symmetric equilibrium of this auction also yields the seller the same expected revenue as the sealed-bid second-price auction. [*Hint*: Follow similar steps as in Exercise 23.D.4.]

23.D.6[C] Suppose that I symmetric individuals wish to acquire the single remaining ticket to a concert. The ticket office opens at 9 a.m. on Monday. Each individual must decide what time to go to get on line: the first individual to get on line will get the ticket. An individual who waits t hours incurs a (monetary equivalent) disutility of βt. Suppose also that an individual showing up after the first one can go home immediately and so incurs no waiting cost. Individual i's value of receiving the ticket is θ_i, and each individual's θ_i is independently drawn from a uniform distribution on $[0, 1]$. What is the expected value of the number of hours that the first individual in line will wait? [*Hint*: Note the analogy to a first-price sealed-bid auction and use the revenue equivalence theorem.] How does this vary when β doubles? When I doubles?

23.E.1[B] Consider again a bilateral trade setting in which each θ_i ($i = 1, 2$) is independently drawn from a uniform distribution on $[0, 1]$. Suppose now that by refusing to participate in the mechanism a seller with valuation θ_1 receives expected utility θ_1 (he simply consumes the good), whereas a buyer with valuation θ_2 receives expected utility 0 (he simply consumes his endowment of the numeraire, which we have normalized to equal 0). Show that in the expected externality mechanism there is a type of buyer or seller who will strictly prefer not to participate.

23.E.2[A] Argue that when the assumptions of Proposition 23.E.1 hold in the bilateral trade setting:

(a) There is no social choice function $f(\cdot)$ that is dominant strategy incentive compatible and interim individually rational (i.e., that gives each agent i nonnegative gains from participation conditional on his type θ_i, for all θ_i).

(b) There is no social choice function $f(\cdot)$ that is Bayesian incentive compatible and ex post individually rational [i.e., that gives each agent nonnegative gains from participation for every pair of types (θ_1, θ_2)].

23.E.3$^{\text{B}}$ Show by means of an example that when the buyer and seller in a bilateral trade setting both have a discrete set of possible valuations, social choice functions may exist that are Bayesian incentive compatible, ex post efficient, and individually rational. [*Hint*: It suffices to let each have two possible types.] Conclude that the assumption of a strictly positive density is required for the Myerson–Satterthwaite theorem.

23.E.4$^{\text{B}}$ A seller ($i = 1$) and a buyer ($i = 2$) are bargaining over the sale of an indivisible good. Trade can occur at discrete periods $t = 1, 2, \ldots$. Both the buyer and the seller have discount factor $\delta < 1$. The buyer's and seller's valuations are drawn independently with positive densities from $[\underline{\theta}_2, \bar{\theta}_2]$ and $[\underline{\theta}_1, \bar{\theta}_1]$, respectively. Assume that $(\underline{\theta}_2, \bar{\theta}_2) \cap (\underline{\theta}_1, \bar{\theta}_1) \neq \varnothing$. Note that in this setting ex post efficiency requires that trade occur in period 1 whenever $\theta_2 > \theta_1$, and that trade not occur whenever $\theta_1 > \theta_2$. Use the Myerson–Satterthwaite theorem to show that, in this setting with discounting, no voluntary trading process can achieve ex post efficiency.

23.E.5$^{\text{B}}$ Suppose there is a *continuum* of buyers and sellers (with quasilinear preferences). Each seller initially has one unit of an indivisible good and each buyer initially has none. A seller's valuation for consumption of the good is $\theta_1 \in [\underline{\theta}_1, \bar{\theta}_1]$, which is independently and identically drawn from distribution $\Phi_1(\cdot)$ with associated strictly positive density $\phi_1(\cdot)$. A buyer's valuation from consumption of the good is $\theta_2 \in [\underline{\theta}_2, \bar{\theta}_2]$, which is independently and identically drawn from distribution $\Phi_2(\cdot)$ with associated strictly positive density $\phi_2(\cdot)$.

(a) Characterize the trading rule in an ex post efficient social choice function. Which buyers and sellers end up with a unit of the good?

(b) Exhibit a social choice function that has the trading rule you identified in **(a)**, is Bayesian incentive compatible, and is individually rational. [*Hint*: Think of a "competitive" mechanism.] Conclude that the inefficiency identified in the Myerson–Satterthwaite theorem goes away as the number of buyers and sellers grows large. [For a formal examination showing that, with a finite number of traders, the efficiency loss goes to zero as the number of traders grows large, see Gresik and Satterthwaite (1989).]

23.E.6$^{\text{B}}$ Consider a bilateral trading setting in which *both* agents initially own one unit of a good. Each agent i's ($i = 1, 2$) valuation per unit consumed of the good is θ_i. Assume that θ_i is independently drawn from a uniform distribution on $[0, 1]$.

(a) Characterize the trading rule in an ex post efficient social choice function.

(b) Consider the following mechanism: Each agent submits a bid; the highest bidder buys the other agent's unit of the good and pays him the amount of his bid. Derive a symmetric Bayesian Nash equilibrium of this mechanism. [*Hint*: Look for one in which an agent's bid is a linear function of his type.]

(c) What is the social choice function that is implemented by this mechanism? Verify that it is Bayesian incentive compatible. Is it ex post efficient? Is it individually rational [which here requires that $U_i(\theta_i) \geq \theta_i$ for all θ_i and $i = 1, 2$]? Intuitively, why is there a difference from the conclusion of the Myerson–Satterthwaite theorem? [See Cramton, Gibbons, and Klemperer (1987) for a formal analysis of these "partnership division" problems.]

23.E.7$^{\text{B}}$ Consider a bilateral trade setting in which the buyer's and seller's valuations are drawn independently from the uniform distribution on $[0, 1]$.

(a) Show that if $f(\cdot)$ is a Bayesian incentive compatible and interim individually rational social choice function that is ex post efficient, the sum of the buyer's and seller's expected utilities under $f(\cdot)$ cannot be less than 5/6.

(b) Show that, in fact, there is no social choice function (whether Bayesian incentive compatible and interim individually rational or not) in which the sum of the buyer's and seller's expected utilities exceeds 2/3.

23.F.1^{C} Consider the quasilinear setting studied in Sections 23.C and 23.D. Show that if the social choice function $f(\cdot) \in F^*$ is ex post classically efficient in F_{IR} then it is both ex ante and interim incentive efficient in F^*. [From this fact, we see that if an ex post classically efficient social choice function *can* be implemented in a setting with privately observed types (i.e., if it is incentive feasible), then no other incentive feasible social choice function can welfare dominate it. Note, however, that there may be *other* ex ante or interim incentive efficient social choice functions that are not ex post efficient; for example, you can verify that in Example 23.F.1 there is an ex post classically efficient social choice function that is incentive feasible, but the particular interim incentive efficient social choice function derived in the example is not ex post efficient.]

23.F.2^{B} [Based on Maskin and Riley (1984)] A monopolist seller produces a good with constant returns to scale at a cost of $c > 0$ per unit. The monopolist sells to a consumer whose preference for the product the monopolist cannot observe. A consumer of type $\theta > 0$ derives a utility of $\theta v(x) - t$ when he consumes x units of the monopolist's product and pays the monopolist a total of t dollars for these units. Assume that $v'(\cdot) > 0$ and $v''(\cdot) < 0$. The set of possible consumer types is $[\underline{\theta}, \bar{\theta}]$ with $\bar{\theta} > \underline{\theta} > 0$, and the distribution of types is $\Phi(\cdot)$, with an associated strictly positive density function $\phi(\cdot) > 0$. Assume that $[\theta - ((1 - \Phi(\theta))/\phi(\theta))]$ is nondecreasing in θ.

Characterize the monopolist's optimal selling mechanism to this consumer, assuming that a consumer of type θ can always choose not to buy at all, thereby deriving a utility of 0.

23.F.3^{C} An auction with a *reserve price* is an auction in which there is a minimum allowable bid. Suppose that in the auction setting of Example 23.F.2 the I buyers are symmetric and that $\underline{\theta} = 0$. Argue that a second-price sealed-bid auction with a reserve price is an optimal auction in this case. What is the optimal reserve price? Can you think of a modified second-price sealed-bid auction that is optimal in the general (nonsymmetric) case?

23.F.4^{B} Derive the optimal $y_i(\cdot)$ functions in the auction setting of Example 23.F.2 when the seller's valuation for the object is $\theta_0 > 0$.

23.F.5^{B} Suppose that a monopolist seller who has two potential buyers has a total of one divisible unit to sell; that is, production costs are zero up to one unit, and infinite beyond that. The demand function of buyer i is the decreasing function $x_i(p)$ for $i = 1, 2$. The monopolist can name distinct prices for the two buyers.

(a) Characterize the monopolist's optimal prices.

(b) Relate your answer in **(a)** to the optimal auction derived in Example 23.F.2. [For more on this, see Bulow and Roberts (1989).]

23.F.6^{C} [Based on Baron and Myerson (1982)]. Consider the optimal regulatory scheme for a regulator of a monopolist who has known demand function $x(p)$, with $x'(p) < 0$, and a privately observed constant marginal cost of production θ. The regulator can set the monopolist's price and can make a transfer from or to the monopolist, so the set of outcomes is $X = \{(p, t): p > 0 \text{ and } t \in \mathbb{R}\}$. The regulator must guarantee the monopolist a nonnegative profit regardless of his production costs to prevent the monopolist from shutting down. The monopolist's marginal cost θ is drawn from $[\underline{\theta}, \bar{\theta}]$ with $\bar{\theta} > \underline{\theta} > 0$ according to the distribution function $\Phi(\cdot)$, which has an associated strictly positive density function $\phi(\cdot) > 0$. Assume that

$\Phi(\theta)/\phi(\theta)$ is nondecreasing in θ. Denote a type-θ monopolist's profit from outcome (p, t) by $\pi(p, t, \theta) = (p - \theta)x(p) + t$.

(a) Adapt the characterization in Proposition 23.D.2 to this application.

(b) Suppose that the regulator wants to design a direct revelation regulatory scheme $(p(\cdot), t(\cdot))$ that maximizes the expected value of a weighted sum of consumer and producer surplus,

$$\int_{p(\theta)}^{\infty} x(s)\, ds + \alpha\pi(p(\theta), t(\theta), \theta),$$

where $\alpha < 1$. Characterize the regulator's optimal regulatory scheme. What if $\alpha \geq 1$?

23.F.7^{C} [Based on Dana and Spier (1994)] Two firms, $j = 1, 2$, compete for the right to produce in a given market. A social planner designs an optimal auction of production rights to maximize the expected value of social welfare as measured by

$$W = \sum_j \pi_j + S + (\lambda - 1) \sum_j t_j,$$

where t_j denotes the transfer from firm j to the planner, S is consumer surplus, π_j is the gross (pretransfer) profit of firm j, and $\lambda > 1$ is the shadow cost of public funds. The auction specifies transfers for each of the firms and a market structure; that is, it either awards neither firm production rights, awards only one firm production rights (thereby making that firm an unregulated monopolist), or gives production rights to both firms (thereby making them compete as unregulated duopolists).

Each firm j privately observes its fixed cost of production θ_j. The fixed cost levels θ_1 and θ_2 are independently distributed on $[\underline{\theta}, \bar{\theta}]$ with continuously differentiable density function $\phi(\cdot)$ and distribution function $\Phi(\cdot)$. Assume that $\Phi(\cdot)/\phi(\cdot)$ is increasing in θ. The firms have common marginal cost $c < 1$ and produce a homogeneous product for which the market inverse demand function is $p(x) = 1 - x$ (this is publicly known). If both firms are awarded production rights, they interact as Cournot competitors (see Section 12.C).

Characterize the planner's optimal auction of production rights.

23.F.8^{A} Show that any ex post classically efficient social choice function in Example 23.F.3 has $y_L = y_H = 1$.

23.F.9^{B} Show that in the model of Example 23.F.3:

(a) No feasible social choice function is ex post efficient.

(b) In any feasible social choice function, $y_H \leq y_L$ and $t_H \leq t_L$.

(c) In any feasible social choice function, the expected gains from trade of a low-quality seller are at least as large as the expected gains from trade of a high-quality seller; that is, $t_L - 20y_L \geq t_H - 40y_H$.

23.F.10^{B} Characterize the sets of interim and ex ante incentive efficient social choice functions in the model of Example 23.F.3 when trade is not voluntary for the seller (but it is voluntary for the buyer).

23.AA.1^{B} Reconsider Exercise 23.C.8. Exhibit a mechanism $\Gamma = (S_1, \ldots, S_I, g(\cdot))$ that is not a direct revelation mechanism that truthfully implements $f(\cdot)$ in dominant strategies and for which each agent has a *unique* (weakly) dominant strategy.

23.AA.2^{B} Let $K = \{k_0, k_1, \ldots, k_N\}$ be the set of possible projects and suppose that, for each agent i, $\{v_i(\cdot, \theta_i): \theta_i \in \Theta_i\} = \mathscr{V}$, that is, that every possible valuation function from K to $\mathbb{R}$ arises for some $\theta_i \in \Theta_i$. Do players in a Groves mechanism have a unique (weakly) dominant strategy? Consider instead a mechanism in which each agent i is allowed to announce a *normalized* valuation function, that is, a function such that $v_i(k_0) = 0$. Suppose that $k^*(\cdot)$ and the Groves transfers are calculated using these announcements. Does each agent have a unique (weakly) dominant strategy in this normalized Groves mechanism?

23.BB.1[A] Consider the dynamic mechanism in Example 23.BB.1.

(a) For each possible preference profile, write down its normal form and identify its Nash equilibria.

(b) For each possible preference profile, identify this mechanism's subgame perfect Nash equilibria.

23.BB.2[B] Is a social choice function that is implementable in dominant strategies necessarily implementable in Nash equilibrium? What if we are interested in strong implementation instead?

23.BB.3[C] Consider a setting of public project choice (see Example 23.B.3) in which $K = \{0, 1\}$. Let θ_i denote agent i's benefit if the project is done (i.e., if $k = 1$); normalize the value from $k = 0$ to equal zero. Assume that $\Theta_i = \mathbb{R}$. In this setting, the only mechanisms that involve an ex post efficient project choice are Groves mechanisms. Let $k^*(\cdot)$ denote the project choice rule in such a mechanism. Also, suppose that $I \geq 3$. The transfers in a Groves mechanism are characterized by two properties:

(i) if $k^*(\theta_i, \theta_{-i}) = k^*(\theta'_i, \theta_{-i})$, then $t_i(\theta_i, \theta_{-i}) = t_i(\theta'_i, \theta_{-i})$;
(ii) if $k^*(\theta_i, \theta_{-i}) = 1$ and $k^*(\theta'_i, \theta_{-i}) = 0$, then $t_i(\theta_i, \theta_{-i}) - t_i(\theta'_i, \theta_{-i}) = \sum_{j \neq i} \theta_j$.

Which, if any, of these two properties must be satisfied by any Nash implementable social choice function that involves an ex post efficient project choice?

Mathematical Appendix

This appendix contains a quick and unsystematic review of some of the mathematical concepts and techniques used in the text.

The formal results are quoted as "Theorems" and they are fairly rigorously stated. It seems useful in a technical appendix such as this to provide motivational remarks, examples, and general ideas for some proofs. This we often do under the label of the "Proof" of the mathematical theorem under discussion. Nonetheless, no rigor of any sort is intended here. Perhaps the heading "Discussion of Theorem" would be more accurate.

It goes without saying that this appendix is no substitute for a more extensive and systematic, book-length, treatment. Good references for some or most of the material covered in this appendix, as well as for further background reading, are Simon and Blume (1993), Sydsaeter and Hammond (1994), Novshek (1993), Dixit (1990), Chang (1984), and Intriligator (1971).

M.A Matrix Notation for Derivatives

We begin by reviewing some matters of notation. The first and most important is that formally and mathematically a "vector" in $\mathbb{R}^N$ is a *column*. This applies to any vector; it does not matter, for example, if the vector represents quantities or prices. It applies also to the *gradient* vector $\nabla f(\bar{x}) \in \mathbb{R}^N$ of a function at a point $\bar{x}$; this is the vector whose nth entry is the partial derivative with respect to the nth variable of the real-valued function $f\colon \mathbb{R}^N \to \mathbb{R}$, evaluated at the point $\bar{x} \in \mathbb{R}^N$. Expositionally, however, because rows take less space to display, we typically describe vectors horizontally in the text, as in $x = (x_1, \ldots, x_N)$. But the rule has no exception: all vectors are mathematically columns.

The inner product of two N vectors $x \in \mathbb{R}^N$ and $y \in \mathbb{R}^N$ is written as $x \cdot y = \sum_n x_n y_n$. If we view these vectors as $N \times 1$ matrices, we see that $x \cdot y = x^{\mathrm{T}} y$, where $^{\mathrm{T}}$ is the matrix transposition operator. An expression such as "$x\cdot$" can always be read as "x^{T}"; for example, the expression $x \cdot A$, where A is an $N \times M$ matrix, is the same as $x^{\mathrm{T}} A$.

If $f\colon \mathbb{R}^N \to \mathbb{R}^M$ is a vector-valued differentiable function, then at any $x \in \mathbb{R}^N$ we denote by $Df(x)$ the $M \times N$ matrix whose mnth entry is $\partial f_m(x)/\partial x_n$. Note, in particular, that if $M = 1$ (so that $f(x) \in \mathbb{R}$) then $Df(x)$ is a $1 \times N$ matrix; in fact $\nabla f(x) = [Df(x)]^{\mathrm{T}}$. To avoid ambiguity, in some cases we write $D_x f(x)$ to indicate explicitly the variables with respect to which the function $f(\cdot)$ is being differentiated. For example, with this notation, if $f\colon \mathbb{R}^{N+K} \to \mathbb{R}^M$ is a function whose arguments are the vectors $x \in \mathbb{R}^N$ and $y \in \mathbb{R}^K$, the matrix $D_x f(x, y)$ is the $M \times N$ matrix whose mnth entry is $\partial f_m(x, y)/\partial x_n$. Finally, for a real-valued differentiable function $f\colon \mathbb{R}^N \to \mathbb{R}$, the *Hessian matrix* $D^2 f(x)$ is the derivative matrix of the vector-valued gradient function $\nabla f(x)$; i.e., $D^2 f(x) = D[\nabla f(x)]$.

In the remainder of this section, we consider differentiable functions and examine how two well-known rules of calculus—the chain rule and the product rule—come out in matrix notation.

The Chain Rule

Suppose that $g\colon \mathbb{R}^S \to \mathbb{R}^N$ and $f\colon \mathbb{R}^N \to \mathbb{R}^M$ are differentiable functions. The *composite* function $f(g(\cdot))$ is also differentiable. Consider any point $x \in \mathbb{R}^S$. The chain rule allows us to evaluate the $M \times S$ derivative matrix of the composite function with respect to x, $D_x f(g(x))$ by matrix multiplication of the $N \times S$ derivative matrix of $g(\cdot)$, $Dg(x)$, and the $M \times N$ derivative matrix of $f(\cdot)$ evaluated at $g(x)$, that is, $Df(y)$, where $y = g(x)$. Specifically,

$$D_x f(g(x)) = Df(g(x))\, Dg(x). \tag{M.A.1}$$

The Product Rule

Here we simply provide a few illustrations.

(i) Suppose that $f\colon \mathbb{R}^N \to \mathbb{R}$ has the form $f(x) = g(x)h(x)$, where both $g(\cdot)$ and $h(\cdot)$ are real-valued functions of the N variables $x = (x_1, \dots, x_N)$ (so that $g\colon \mathbb{R}^N \to \mathbb{R}$ and $h\colon \mathbb{R}^N \to \mathbb{R}$). Then the product rule of calculus tells us that

$$Df(x) = g(x)\, Dh(x) + h(x)\, Dg(x). \tag{M.A.2}$$

which, transposing, can also be written as

$$\nabla f(x) = g(x)\, \nabla h(x) + h(x)\, \nabla g(x).$$

(ii) Suppose that $f\colon \mathbb{R}^N \to \mathbb{R}$ has the form $f(x) = g(x) \cdot h(x)$ where both $g(\cdot)$ and $h(\cdot)$ are vector-valued functions which map the N variables $x = (x_1, \dots, x_N)$ into $\mathbb{R}^M$. Then

$$Df(x) = g(x) \cdot Dh(x) + h(x) \cdot Dg(x). \tag{M.A.3}$$

Note that $h(x) \cdot Dg(x) = [h(x)]^{\mathrm{T}}\, Dg(x)$ is a $1 \times N$ matrix, as is the other term in the right-hand side. Thus, the vector-valued case (M.A.3) implies the scalar-valued formula (M.A.2).

(iii) Suppose that $f\colon \mathbb{R} \to \mathbb{R}^M$ has the form $f(x) = \alpha(x)g(x)$, where $\alpha(\cdot)$ is a real-valued function of one variable (i.e., $\alpha\colon \mathbb{R} \to \mathbb{R}$) and $g\colon \mathbb{R} \to \mathbb{R}^M$. Then

$$Df(x) = \alpha(x)\, Dg(x) + \alpha'(x)g(x). \tag{M.A.4}$$

(iv) Suppose that $f\colon \mathbb{R}^N \to \mathbb{R}^M$ has the form $f(x) = h(x)g(x)$ where $h\colon \mathbb{R}^N \to \mathbb{R}$ and $g\colon \mathbb{R}^N \to \mathbb{R}^M$. Then

$$Df(x) = h(x)\, Dg(x) + g(x)\, Dh(x). \tag{M.A.5}$$

Note that $g(x)$ is an M-element vector (i.e., an $M \times 1$ matrix) and $Dh(x)$ is a $1 \times N$ matrix. Hence, $g(x)\, Dh(x)$ is an $M \times N$ matrix (of rank 1). Observe also that (M.A.4) follows as a special case of (M.A.5).

M.B Homogeneous Functions and Euler's Formula

In this section, we consider functions of N variables, $f(x_1, \ldots, x_N)$, defined for all nonnegative values $(x_1, \ldots, x_N) \geq 0$.

Definition M.B.1: A function $f(x_1, \ldots, x_N)$ is *homogeneous of degree r* (for $r = \ldots, -1, 0, 1, \ldots$) if for every $t > 0$ we have

$$f(tx_1, \ldots, tx_N) = t^r f(x_1, \ldots, x_N).$$

As an example, $f(x_1, x_2) = x_1/x_2$ is homogeneous of degree zero and $f(x_1, x_2) = (x_1 x_2)^{1/2}$ is homogeneous of degree one.

Note that if $f(x_1, \ldots, x_N)$ is homogeneous of degree zero and we restrict the domain to have $x_1 > 0$ then, by taking $t = 1/x_1$, we can write the function $f(\cdot)$ as

$$f(1, x_2/x_1, \ldots, x_N/x_1) = f(x_1, \ldots, x_N).$$

Similarly, if the function is homogeneous of degree one then

$$f(1, x_2/x_1, \ldots, x_N/x_1) = (1/x_1) f(x_1, \ldots, x_N).$$

Theorem M.B.1: If $f(x_1, \ldots, x_N)$ is homogeneous of degree r (for $r = \ldots, -1, 0, 1, \ldots$), then for any $n = 1, \ldots, N$ the partial derivative function $\partial f(x_1, \ldots, x_N)/\partial x_n$ is homogeneous of degree $r - 1$.

Proof: Fix a $t > 0$. By the definition of homogeneity (Definition M.B.1) we have

$$f(tx_1, \ldots, tx_N) - t^r f(x_1, \ldots, x_N) = 0.$$

Differentiating this expression with respect to x_n gives

$$t \frac{\partial f(tx_1, \ldots, tx_N)}{\partial x_n} - t^r \frac{\partial f(x_1, \ldots, x_N)}{\partial x_n} = 0,$$

so that

$$\frac{\partial f(tx_1, \ldots, tx_N)}{\partial x_n} = t^{r-1} \frac{\partial f(x_1, \ldots, x_N)}{\partial x_n}.$$

By Definition M.B.1, we conclude that $\partial f(x_1, \ldots, x_N)/\partial x_n$ is homogeneous of degree $r - 1$. ■

For example, for the homogeneous of degree one function $f(x_1, x_2) = (x_1 x_2)^{1/2}$, we have $\partial f(x_1, x_2)/\partial x_1 = \frac{1}{2}(x_2/x_1)^{1/2}$, which is indeed homogeneous of degree zero in accordance with Theorem M.B.1.

Note that if $f(\cdot)$ is a homogeneous function of any degree then $f(x_1, \ldots, x_N) = f(x'_1, \ldots, x'_N)$ implies $f(tx_1, \ldots, tx_N) = f(tx'_1, \ldots, tx'_N)$ for any $t > 0$; that is, a radial expansion of a level set of $f(\cdot)$ gives a new level set of $f(\cdot)$.[1] This has an interesting implication: the slopes of the level sets of $f(\cdot)$ are unchanged along any ray through the origin. For example, suppose that $N = 2$. Then, assuming that $\partial f(\bar{x})/\partial x_2 \neq 0$, the slope of the level set containing point $\bar{x} = (\bar{x}_1, \bar{x}_2)$ at $\bar{x}$ is $-(\partial f(\bar{x})/\partial x_1)/(\partial f(\bar{x})/\partial x_2)$,

1. A *level set* of function $f(\cdot)$ is a set of the form $\{x \in \mathbb{R}^N_+ : f(x) = k\}$ for some k. A radial expansion of this set is the set of points obtained by multiplying each vector x in this level set by some positive scalar $t > 0$.

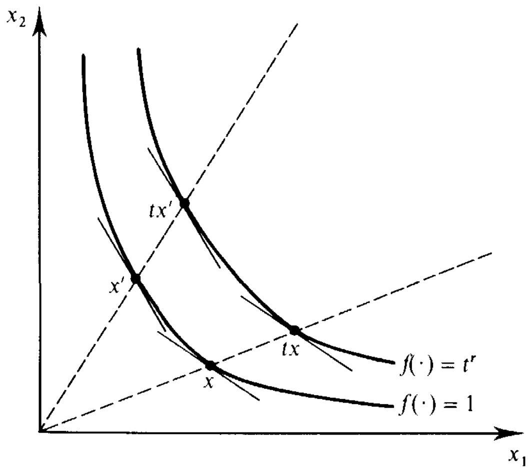


Figure M.B.1
The level sets of a homogeneous function.

and the slope of the level set containing point $t\bar{x}$ for $t > 0$ at $t\bar{x}$ is

$$-\frac{\partial f(t\bar{x})/\partial x_1}{\partial f(t\bar{x})/\partial x_2} = -\frac{t^{r-1}\,\partial f(\bar{x})/\partial x_1}{t^{r-1}\,\partial f(\bar{x})/\partial x_2} = -\frac{\partial f(\bar{x})/\partial x_1}{\partial f(\bar{x})/\partial x_2}.$$

An illustration of this fact is provided in Figure M.B.1.

Suppose that $f(\cdot)$ is homogeneous of some degree r and that $h(\cdot)$ is an increasing function of one variable. Then the function $h(f(x_1, \ldots, x_N))$ is called *homothetic*. Note that the family of level sets of $h(f(\cdot))$ coincides with the family of level sets of $f(\cdot)$. Therefore, for any homothetic function it is also true that the slopes of the level sets are unchanged along rays through the origin.

A key property of homogeneous functions is given in Theorem M.B.2.

Theorem M.B.2: (*Euler's Formula*) Suppose that $f(x_1, \ldots, x_N)$ is homogeneous of degree r (for some $r = \ldots, -1, 0, 1, \ldots$) and differentiable. Then at any $(\bar{x}_1, \ldots, \bar{x}_N)$ we have

$$\sum_{n=1}^{N} \frac{\partial f(\bar{x}_1, \ldots, \bar{x}_N)}{\partial x_n}\bar{x}_n = rf(\bar{x}_1, \ldots, \bar{x}_N),$$

or, in matrix notation, $\nabla f(\bar{x})\cdot\bar{x} = rf(\bar{x})$.

Proof: By definition we have

$$f(t\bar{x}_1, \ldots, t\bar{x}_N) - t^r f(\bar{x}_1, \ldots, \bar{x}_N) = 0.$$

Differentiating this expression with respect to t gives

$$\sum_{n=1}^{N} \frac{\partial f(t\bar{x}_1, \ldots, t\bar{x}_N)}{\partial x_n}\bar{x}_n - rt^{r-1}f(\bar{x}_1, \ldots, \bar{x}_N) = 0.$$

Evaluating at $t = 1$, we obtain Euler's formula. ■

For a function that is homogeneous of degree zero, Euler's formula says that

$$\sum_{n=1}^{N} \frac{\partial f(x_1, \ldots, x_N)}{\partial x_n}\bar{x}_n = 0.$$

As an example, note that for the function $f(x_1, x_2) = x_1/x_2$, we have $\partial f(\bar{x}_1, \bar{x}_2)/\partial x_1 = 1/\bar{x}_2$ and $\partial f(\bar{x}_1, \bar{x}_2)/\partial x_2 = -(\bar{x}_1/(\bar{x}_2)^2)$, and so

$$\sum_{n=1}^{N} \frac{\partial f(x_1, \ldots, x_N)}{\partial x_n} \bar{x}_n = \frac{1}{\bar{x}_2} \bar{x}_1 - \frac{\bar{x}_1}{(\bar{x}_2)^2} \bar{x}_2 = 0,$$

in accordance with Euler's formula.

For a function that is homogeneous of degree one, Euler's formula says that

$$\sum_{n=1}^{N} \frac{\partial f(x_1, \ldots, x_N)}{\partial x_n} \bar{x}_n = f(\bar{x}_1, \ldots, \bar{x}_N).$$

For example, when $f(x_1, x_2) = (x_1 x_2)^{1/2}$, we have $\partial f(\bar{x}_1, \bar{x}_2)/\partial x_1 = \frac{1}{2}(\bar{x}_2/\bar{x}_1)^{1/2}$ and $\partial f(\bar{x}_1, \bar{x}_2)/\partial x_2 = \frac{1}{2}(\bar{x}_1/\bar{x}_2)^{1/2}$, and so

$$\begin{aligned}\sum_{n=1}^{N} \frac{\partial f(x_1, \ldots, x_N)}{\partial x_n} \bar{x}_n &= \frac{1}{2}\left(\frac{\bar{x}_2}{\bar{x}_1}\right)^{1/2} \bar{x}_1 + \frac{1}{2}\left(\frac{\bar{x}_1}{\bar{x}_2}\right)^{1/2} \bar{x}_2 \\ &= (\bar{x}_1 \bar{x}_2)^{1/2} \\ &= f(\bar{x}_1, \bar{x}_2).\end{aligned}$$

M.C Concave and Quasiconcave Functions

In this section, we consider functions of N variables $f(x_1, \ldots, x_N)$ defined on a domain A that is a convex subset of $\mathbb{R}^N$ (such as $A = \mathbb{R}^N$ or $A = \mathbb{R}^N_+ = \{x \in \mathbb{R}^N : x \geq 0\}$).[2] We denote $x = (x_1, \ldots, x_N)$.

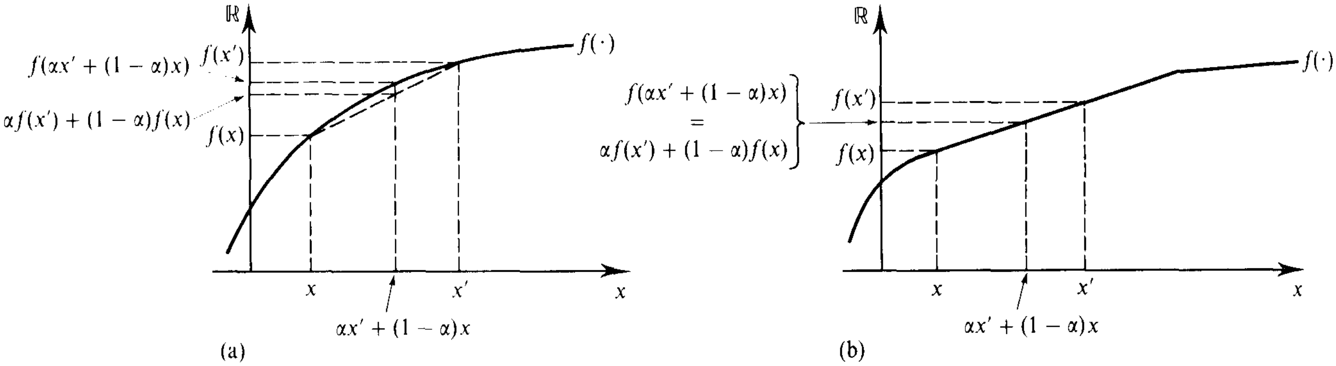
Definition M.C.1: The function $f: A \to \mathbb{R}$, defined on the convex set $A \subset \mathbb{R}^N$, is *concave* if

$$f(\alpha x' + (1 - \alpha)x) \geq \alpha f(x') + (1 - \alpha) f(x) \qquad \text{(M.C.1)}$$

for all x and $x' \in A$ and all $\alpha \in [0, 1]$. If the inequality is strict for all $x' \neq x$ and all $\alpha \in (0, 1)$, then we say that the function is *strictly concave*.

Figure M.C.1(a) illustrates a strictly concave function of one variable. For this case, condition (M.C.1) says that the straight line connecting any two points in the graph of $f(\cdot)$ lies entirely below this graph.[3] In Figure M.C.1(b), we show a function

Figure M.C.1 (a) A strictly concave function. (b) A concave but not strictly concave function.



2. For basic facts about convex sets, see Section M.G.
3. The *graph* of the function $f: A \to \mathbb{R}$ is the set $\{(x, y) \in A \times \mathbb{R} : y = f(x)\}$.

that is concave but not strictly concave; note that in this case the straight line connecting points x and x' lies on the graph of the function, so that condition (M.C.1) holds with equality.

We note that condition (M.C.1) is equivalent to the seemingly stronger property that

$$f(\alpha_1 x^1 + \cdots + \alpha_K x^K) \geq \alpha_1 f(x^1) + \cdots + \alpha_K f(x^K) \qquad \text{(M.C.2)}$$

for any collection of vectors $x^1 \in A, \ldots, x^K \in A$ and numbers $\alpha_1 \geq 0, \ldots, \alpha_K \geq 0$ such that $\alpha_1 + \cdots + \alpha_K = 1$.

Let us consider again the one-variable case. We could view each number α_k in condition (M.C.2) as the "probability" that x^k occurs. Then condition (M.C.2) says that the value of the expectation is not smaller than the expected value. Indeed, a concave function $f\colon \mathbb{R} \to \mathbb{R}$ is characterized by the condition that

$$f\left(\int x \, dF\right) \geq \int f(x) \, dF \qquad \text{(M.C.3)}$$

for any distribution function $F\colon \mathbb{R} \to [0, 1]$. Condition (M.C.3) is known as *Jensen's inequality*.

The properties of *convexity* and *strict convexity* for a function $f(\cdot)$ are defined analogously but with the inequality in (M.C.1) reversed. In particular, for a strictly convex function $f(\cdot)$, a straight line connecting any two points in its graph should lie entirely *above* its graph, as shown in Figure M.C.2. Note also that $f(\cdot)$ is concave if and only if $-f(\cdot)$ is convex.

Theorem M.C.1 provides a useful alternative characterization of concavity and strict concavity.

Theorem M.C.1: The (continuously differentiable) function $f\colon A \to \mathbb{R}$ is concave if and only if

$$f(x + z) \leq f(x) + \nabla f(x) \cdot z \qquad \text{(M.C.4)}$$

for all $x \in A$ and $z \in \mathbb{R}^N$ (with $x + z \in A$). The function $f(\cdot)$ is strictly concave if inequality (M.C.4) holds strictly for all $x \in A$ and all $z \neq 0$.

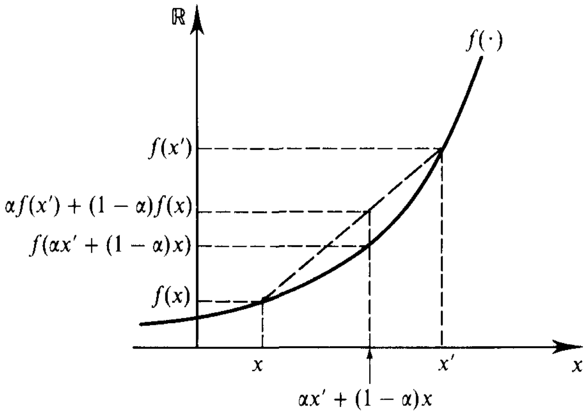


Figure M.C.2
A strictly convex function.

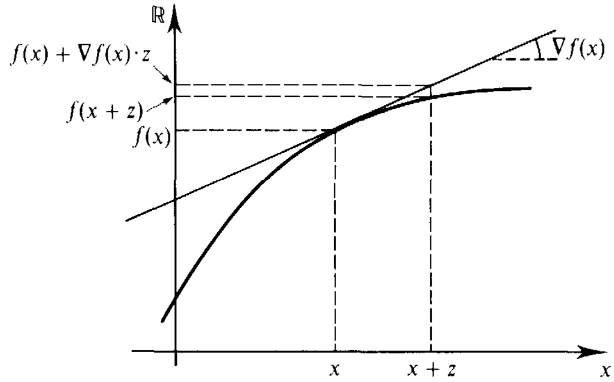


Figure M.C.3
Any tangent to the graph of a concave function lies above the graph of the function.

Proof: We argue only the necessity of condition (M.C.4) for concave functions. For all $\alpha \in (0, 1]$, the condition $f(\alpha x' + (1-\alpha)x) \geq \alpha f(x') + (1-\alpha)f(x)$ for all $x, x' \in A$, can be rewritten (think of $z = x' - x$) as

$$f(x+z) \leq f(x) + \frac{f(x+\alpha z) - f(x)}{\alpha}$$

for all $x \in A$, $z \in \mathbb{R}^N$ (with $x + z \in A$), and $\alpha \in (0, 1]$. Taking the limit as $\alpha \to 0$, we conclude that condition (M.C.4) must hold for a (continuously differentiable) concave function $f(\cdot)$. ■

Condition (M.C.4) is shown graphically in Figure M.C.3. It says that any tangent to the graph of a concave function $f(\cdot)$ must lie (weakly) above the graph of $f(\cdot)$.

The corresponding characterization of convex and strictly convex functions simply entails reversing the direction of the inequality in condition (M.C.4); that is, a convex function is characterized by the condition that $f(x+z) \geq f(x) + \nabla f(x)\cdot z$ for all $x \in A$ and $z \in \mathbb{R}^N$ (with $x + z \in A$).

We next develop a third characterization of concave and strictly concave functions.

Definition M.C.2: The $N \times N$ matrix M is *negative semidefinite* if

$$z\cdot Mz \leq 0 \tag{M.C.5}$$

for all $z \in \mathbb{R}^N$. If the inequality is strict for all $z \neq 0$, then the matrix M is *negative definite*. Reversing the inequalities in condition (M.C.5), we get the concepts of *positive semidefinite* and *positive definite* matrices.

We refer to Section M.E for further details on these properties of matrices. Here we put on record their intimate connection with the properties of the Hessian matrices $D^2 f(\cdot)$ of concave functions.[4]

4. For theorems M.C.2, M.C.3, and M.C.4, the set A is assumed to be open (see Section M.F) so as to avoid boundary problems.

Theorem M.C.2: The (twice continuously differentiable) function $f: A \to \mathbb{R}$ is concave if and only if $D^2f(x)$ is negative semidefinite for every $x \in A$. If $D^2f(x)$ is negative definite for every $x \in A$, then the function is strictly concave.

Proof: We argue only necessity. Suppose that $f(\cdot)$ is concave. Consider a fixed $x \in A$ and a direction of displacement from x, $z \in \mathbb{R}^N$ with $z \neq 0$. Taking a Taylor expansion of the function $\phi(\alpha) = f(x + \alpha z)$, where $\alpha \in \mathbb{R}$, around the point $\alpha = 0$ gives

$$f(x + \alpha z) - f(x) - \nabla f(x) \cdot (\alpha z) = \frac{\alpha^2}{2} z \cdot D^2 f(x + \beta z) z$$

for some $\beta \in [0, \alpha]$. By Theorem M.C.1, the left-hand side of the above expression is nonpositive. Therefore, $z \cdot D^2 f(x + \beta z) z \leq 0$. Since α, hence β, can be taken to be arbitrarily small, this gives the conclusion $z \cdot D^2 f(x) z \leq 0$. ■

In the special case in which $N = 1$ [so $f(\cdot)$ is a function of a single variable], negative semidefiniteness of $D^2 f(x)$ amounts to the condition that $d^2 f(x)/dx^2 \leq 0$, whereas with negative definiteness we have $d^2 f(x)/dx^2 < 0$ [to see this note that then $z \cdot D^2 f(x) z = z^2(d^2 f(x)/dx^2)$]. Theorem M.C.2 tells us that in this case $f(\cdot)$ is concave if and only if $d^2 f(x)/dx^2 \leq 0$ for all x, and that if $d^2 f(x)/dx^2 < 0$ for all x, then $f(\cdot)$ is strictly concave. Note that Theorem M.C.2 does *not* assert that negative definiteness of $D^2 f(x)$ must hold whenever $f(\cdot)$ is strictly concave. Indeed, this is not true: For example, when $N = 1$ the function $f(x) = -x^4$ is strictly concave, but $d^2 f(0)/dx^2 = 0$.

For convex and strictly convex functions the analogous result to Theorem M.C.2 holds by merely replacing the word "negative" with "positive."

The remainder of this section is devoted to discussion of *quasiconcave* and *strictly quasiconcave* functions.

Definition M.C.3: The function $f: A \to \mathbb{R}$, defined on the convex set $A \subset \mathbb{R}^N$, is *quasiconcave* if its upper contour sets $\{x \in A: f(x) \geq t\}$ are convex sets; that is, if

$$f(x) \geq t \text{ and } f(x') \geq t \quad \text{implies that} \quad f(\alpha x + (1 - \alpha)x') \geq t \qquad \text{(M.C.6)}$$

for any $t \in \mathbb{R}$, x, $x' \in A$, and $\alpha \in [0, 1]$.[5] If the concluding inequality in (M.C.6) is strict whenever $x \neq x'$ and $\alpha \in (0, 1)$, then we say that $f(\cdot)$ is *strictly quasiconcave*.

Analogously, we say that the function $f(\cdot)$ is *quasiconvex* if its *lower* contour sets are *convex*; that is, if $f(x) \leq t$ and $f(x') \leq t$ implies that $f(\alpha x + (1 - \alpha)x') \leq t$ for any $t \in \mathbb{R}$, x, $x' \in A$, and $\alpha \in [0, 1]$. For strict quasiconvexity, the final inequality must hold strictly whenever $x \neq x'$ and $\alpha \in (0, 1)$. Note that $f(\cdot)$ is quasiconcave if and only if the function $-f(\cdot)$ is quasiconvex.

The level sets of a strictly quasiconcave function are illustrated in Figure M.C.4(a);

5. For more on convex sets, see Section M.G.

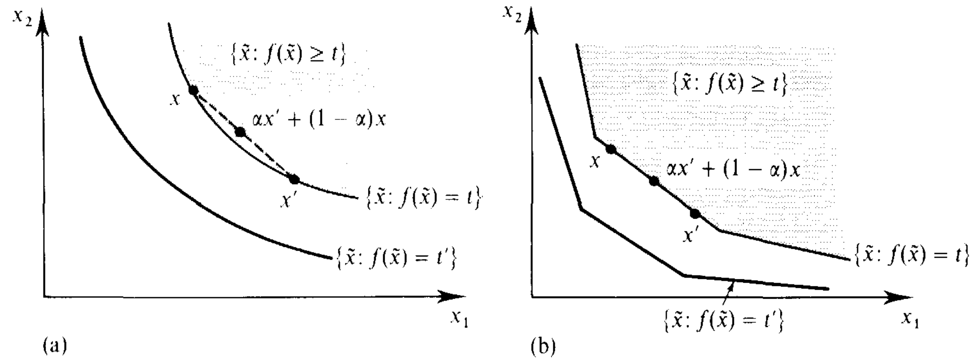


Figure M.C.4
(a) The level sets of a strictly quasiconcave function.
(b) The level sets of a quasiconcave function that is not strictly quasiconcave.

in Figure M.C.4(b) we show a function that is quasiconcave, but not strictly quasiconcave.

It follows from Definition M.C.3 that $f(\cdot)$ is quasiconcave if and only if

$$f(\alpha x + (1 - \alpha)x') \geq \text{Min}\,\{f(x), f(x')\} \qquad \text{(M.C.7)}$$

for all $x, x' \in A$ and $\alpha \in [0, 1]$. From this, or directly from (M.C.6), we see that a concave function is automatically quasiconcave. The converse is not true: For example, *any* increasing function of one variable is quasiconcave. Thus, concavity is a stronger property that quasiconcavity. It is also stronger in a different sense: concavity is a "cardinal" property in that it will *not* generally be preserved under an increasing transformation of $f(\cdot)$. Quasiconcavity, in contrast, will be preserved.

Theorems M.C.3 and M.C.4 are the quasiconcave counterparts of Theorems M.C.1 and M.C.2, respectively.

Theorem M.C.3: The (continuously differentiable) function $f: A \to \mathbb{R}$ is quasiconcave if and only if

$$\nabla f(x) \cdot (x' - x) \geq 0 \quad \text{whenever} \quad f(x') \geq f(x) \qquad \text{(M.C.8)}$$

for *all* $x, x' \in A$. If $\nabla f(x) \cdot (x' - x) > 0$ whenever $f(x') \geq f(x)$ and $x' \neq x$, then $f(\cdot)$ is strictly quasiconcave. In the other direction, if $f(\cdot)$ is strictly quasiconcave and if $\nabla f(x) \neq 0$ for all $x \in A$, then $\nabla f(x) \cdot (x' - x) > 0$ whenever $f(x') \geq f(x)$ and $x' \neq x$.

Proof: Again, we argue only the necessity of (M.C.8) for quasiconcave functions. If $f(x') \geq f(x)$ and $\alpha \in (0, 1]$ then, using condition (M.C.7), we have that

$$\frac{f(\alpha(x' - x) + x) - f(x)}{\alpha} \geq 0.$$

Taking the limit as $\alpha \to 0$, we get $\nabla f(x) \cdot (x' - x) \geq 0$.

The need for the condition "$\nabla f(x) \neq 0$ for all $x \in A$" in the last part of the theorem is illustrated by the function $f(x) = x^3$ for $x \in \mathbb{R}$. This function is strictly quasiconcave (check this using the criterion in Definition M.C.3), but because $\nabla f(0) = 0$ we have $\nabla f(x) \cdot (x' - x) = 0$ whenever $x = 0$. ■

Theorem M.C.3's characterization of quasiconcave functions is illustrated in Figure M.C.5. The content of the theorem's condition (M.C.8) is that for any quasiconcave function $f(\cdot)$ and any pair of points x and x' with $f(x') \geq f(x)$, the gradient vector $\nabla f(x)$ and the vector $(x' - x)$ must form an acute angle.

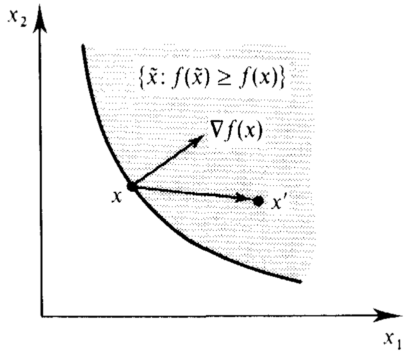


Figure M.C.5
Condition (M.C.8).

For a quasiconvex function, we reverse the direction of both inequalities in (M.C.8).

Theorem M.C.4: The (twice continuously differentiable) function $f: A \to \mathbb{R}$ is quasiconcave if and only if for every $x \in A$, the Hessian matrix $D^2f(x)$ is negative semidefinite in the subspace $\{z \in \mathbb{R}^N: \nabla f(x) \cdot z = 0\}$, that is, if and only if

$$z \cdot D^2 f(x) z \leq 0 \quad \text{whenever} \quad \nabla f(x) \cdot z = 0 \tag{M.C.9}$$

for *every* $x \in A$.[6] If the Hessian matrix $D^2f(x)$ is negative definite in the subspace $\{z \in \mathbb{R}^N: \nabla f(x) \cdot z = 0\}$ for every $x \in A$, then $f(\cdot)$ is strictly quasiconcave.

Proof: Necessity (again, we limit ourselves to this) can be argued exactly as for Theorem M.C.2. The only adjustment is that we restrict z to be such that $\nabla f(x) \cdot z = 0$ and we resort to Theorem M.C.3 instead of Theorem M.C.1. ∎

For a quasiconvex function, we replace the word "negative" with "positive" everywhere in the statement of Theorem M.C.4.

M.D Matrices: Negative (Semi)Definiteness and Other Properties

In this section, we gather various useful facts about matrices.

Definition M.D.1: The $N \times N$ matrix M is *negative semidefinite* if

$$z \cdot Mz \leq 0 \tag{M.D.1}$$

for all $z \in \mathbb{R}^N$. If the inequality is strict for all $z \neq 0$, then the matrix M is *negative definite*. Reversing the inequalities in condition (M.D.1), we get the concepts of *positive semidefinite* and *positive definite* matrices.

Note that a matrix M is positive semidefinite (respectively, positive definite) if and only if the matrix $-M$ is negative semidefinite (respectively, negative definite).

Recall that for an $N \times N$ matrix M the complex number λ is a *characteristic value* (or *eigenvalue* or *root*) if it solves the equation $|M - \lambda I| = 0$. The characteristic values of symmetric matrices are always real.

6. See Section M.E for a discussion of the properties of such matrices.

Theorem M.D.1: Suppose that M is an $N \times N$ matrix.

(i) The matrix M is negative definite if and only if the symmetric matrix $M + M^{\mathrm{T}}$ is negative definite.

(ii) If M is symmetric, then M is negative definite if and only if all of the characteristic values of M are negative.

(iii) The matrix M is negative definite if and only if M^{-1} is negative definite.

(iv) If the matrix M is negative definite, then for all diagonal $N \times N$ matrices K with positive diagonal entries the matrix KM is *stable*.[7]

Proof: Part (i) simply follows from the observation that $z \cdot (M + M^{\mathrm{T}})z = 2z \cdot Mz$ for every $z \in \mathbb{R}^N$.

The logic of part (ii) is the following. Any symmetric matrix M can be diagonalized in a simple manner: There is an $N \times N$ matrix of full rank C having $C^{\mathrm{T}} = C^{-1}$ and such that CMC^{T} is a diagonal matrix with the diagonal entries equal to the characteristic values of M. But then $z \cdot Mz = (Cz) \cdot CMC^{\mathrm{T}}(Cz)$, and for every $\hat{z} \in \mathbb{R}^N$ there is a z such that $\hat{z} = Cz$. Thus, the matrix M is negative definite if and only if the diagonal matrix CMC^{T} is. But it is straightforward to verify that a diagonal matrix is negative definite if and only if every one of its diagonal entries is negative.

Part (iii): Suppose that M^{-1} is negative definite and let $z \neq 0$. Then $z \cdot Mz = (z \cdot Mz)^{\mathrm{T}} = z \cdot M^{\mathrm{T}}z = (M^{\mathrm{T}}z) \cdot M^{-1}(M^{\mathrm{T}}z) < 0$.

Part (iv): It is known that a matrix A is stable if and only if there is a symmetric positive definite matrix E such that EA is negative definite. Thus, in our case, we can take $A = KM$ and $E = K^{-1}$. ■

For positive definite matrices, we can simply reverse the words "positive" and "negative" wherever they appear in Theorem M.D.1.

Our next result (Theorem M.D.2) provides a determinantal test for negative definiteness or negative semidefiniteness of a matrix M. Given any $T \times S$ matrix M, we denote by ${}_tM$ the $t \times S$ submatrix of M where only the first $t \le T$ rows are retained. Analogously, we let M_s be the $T \times s$ submatrix of M where the first $s \le S$ columns are retained, and we let ${}_tM_s$ be the $t \times s$ submatrix of M where only the first $t \le T$ rows and $s \le S$ columns are retained. Also, if M is an $N \times N$ matrix, then for any permutation π of the indices $\{1, \ldots, \mathrm{N}\}$ we denote by M^{π} the matrix in which rows and columns are correspondingly permuted.

Theorem M.D.2: Let M be an $N \times N$ matrix.

(i) Suppose that M is symmetric. Then M is negative definite if and only if $(-1)^r|{}_rM_r| > 0$ for every $r = 1, \ldots, N$.

(ii) Suppose that M is symmetric. Then M is negative semidefinite if and only if $(-1)^r|{}_rM_r^{\pi}| \ge 0$ for every $r = 1, \ldots, N$ *and* for every permutation π of the indices $\{1, \ldots, N\}$.

(iii) Suppose that M is negative definite (not necessarily symmetric). Then $(-1)^r|{}_rM_r^{\pi}| > 0$ for every $r = 1, \ldots, N$ and for every permutation π of the indices $\{1, \ldots, N\}$.[8]

7. A matrix M is stable if all of its characteristic values have negative real parts. This terminology is motivated by the fact that in this case the solution of the system of differential equations $dx(t)/dt = Mx(t)$ will converge to zero as $t \to \infty$ for *any* initial position $x(0)$.

8. A matrix M such that $-M$ satisfies the condition in (iii) is called a P matrix. The reason is that the determinant of any submatrix obtained by deleting some rows (and corresponding columns) is positive.

Proof: (i) The necessity part is simple. Note that by the definition of negative definiteness we have that every ${}_rM_r$ is negative definite. Thus, by Theorem M.D.1, the characteristic values of ${}_rM_r$ are negative. The determinant of a square matrix is equal to the product of its characteristic values. Hence, $|{}_rM_r|$ has the sign of $(-1)^r$. The sufficiency part requires some computation, which we shall not carry out. It is very easy to verify for the case $N = 2$ [if the conclusion of (i) holds for a 2×2 symmetric matrix, then the determinant is positive and *both* diagonal entries are negative; the combination of these two facts is well known to imply the negativity of the two characteristic values].

For (ii), we simply note the requirement to consider all permutations. For example, if M is a matrix with all its entries equal to zero except the NN entry, which is positive, then M satisfies the nonnegative version of (i) but it is not negative semidefinite according to Definition M.D.1.

Notice that in part (iii) we only claim necessity of the determinantal condition. In fact, for nonsymmetric matrices the condition is not sufficient. ■

Example M.D.1: Consider a real-valued function of two variables, $f(x_1, x_2)$. In what follows, we let subscripts denote partial derivatives; for example, $f_{12}(x_1, x_2) = \partial^2 f(x_1, x_2)/\partial x_1\, \partial x_2$. Theorem M.C.2 tells us that $f(\cdot)$ is strictly concave if

$$D^2 f(x_1, x_2) = \begin{bmatrix} f_{11}(x_1, x_2) & f_{12}(x_1, x_2) \\ f_{21}(x_1, x_2) & f_{22}(x_1, x_2) \end{bmatrix}$$

is negative definite for all (x_1, x_2). According to Theorem M.D.2, this is true if and only if

$$|f_{11}(x_1, x_2)| < 0 \qquad \text{and} \qquad \begin{vmatrix} f_{11}(x_1, x_2) & f_{12}(x_1, x_2) \\ f_{21}(x_1, x_2) & f_{22}(x_1, x_2) \end{vmatrix} > 0,$$

or equivalently, if and only if

$$f_{11}(x_1, x_2) < 0$$

and

$$f_{11}(x_1, x_2) f_{22}(x_1, x_2) - [f_{12}(x_1, x_2)]^2 > 0.$$

Theorem M.C.2 also tells us that $f(\cdot)$ is concave if and only if $D^2 f(x_1, x_2)$ is negative semidefinite for all (x_1, x_2). Theorem M.D.2 tells us that this is the case if and only if

$$|f_{11}(x_1, x_2)| \leq 0 \qquad \text{and} \qquad \begin{vmatrix} f_{11}(x_1, x_2) & f_{12}(x_1, x_2) \\ f_{21}(x_1, x_2) & f_{22}(x_1, x_2) \end{vmatrix} \geq 0,$$

and, permuting the rows and columns of $D^2 f(x_1, x_2)$,

$$|f_{22}(x_1, x_2)| \leq 0 \qquad \text{and} \qquad \begin{vmatrix} f_{22}(x_1, x_2) & f_{21}(x_1, x_2) \\ f_{12}(x_1, x_2) & f_{11}(x_1, x_2) \end{vmatrix} \geq 0.$$

Thus, $f(\cdot)$ is concave if and only if

$$f_{11}(x_1, x_2) \leq 0,$$

$$f_{22}(x_1, x_2) \leq 0,$$

and

$$f_{11}(x_1, x_2) f_{22}(x_1, x_2) - [f_{12}(x_1, x_2)]^2 \geq 0. \qquad ■$$

A similar test is available for positive definite and semidefinite matrices: The results for these matrices parallel conditions (i) to (iii) of Theorem M.D.2, but omit the factor $(-1)^r$.[9]

Theorem M.D.3: Let M be an $N \times N$ symmetric matrix and let B be an $N \times S$ matrix with $S \leq N$ and rank equal to S.

(i) M is negative definite on $\{z \in \mathbb{R}^N: Bz = 0\}$ (i.e., $z \cdot Mz < 0$ for any $z \in \mathbb{R}^N$ with $Bz = 0$ and $z \neq 0$) if and only if

$$(-1)^r \begin{vmatrix} {}_rM_r & {}_rB \\ ({}_rB)^{\mathrm{T}} & 0 \end{vmatrix} > 0$$

for $r = S + 1, \ldots, N$.

(ii) M is negative semidefinite on $\{z \in \mathbb{R}^N: Bz = 0\}$ (i.e., $z \cdot Mz \leq 0$ for any $z \in \mathbb{R}^N$ with $Bz = 0$ and $z \neq 0$) if and only if

$$(-1)^r \begin{vmatrix} {}_rM_r^\pi & {}_rB^\pi \\ ({}_rB^\pi)^{\mathrm{T}} & 0 \end{vmatrix} \geq 0$$

for $r = S + 1, \ldots, N$ *and* and every permutation π, where ${}_rB^\pi$ is the matrix formed by permuting only the *rows* of the matrix ${}_rB$ according to the permutation π (${}_rM_r^\pi$ is, as before, a matrix formed by permuting *both* the rows *and* columns of ${}_rM_r$).

Proof: We will not prove this result. Note that it is parallel to parts (i) and (ii) of Theorem M.D.2 with the bordered matrix here playing a role similar to the matrix there. ■

Example M.D.2: Suppose we have a function of two variables, $f(x_1, x_2)$. We assume that $\nabla f(x) \neq 0$ for every x. Theorem M.C.4 tells us that $f(\cdot)$ is strictly quasiconcave if the Hessian matrix $D^2 f(x_1, x_2)$ is negative definite in the subspace $\{z \in \mathbb{R}^2: \nabla f(x) \cdot z = 0\}$ for every $x = (x_1, x_2)$. By Theorem M.D.3 the latter is true if and only if

$$\begin{vmatrix} f_{11}(x_1, x_2) & f_{12}(x_1, x_2) & f_1(x_1, x_2) \\ f_{21}(x_1, x_2) & f_{22}(x_1, x_2) & f_2(x_1, x_2) \\ f_1(x_1, x_2) & f_2(x_1, x_2) & 0 \end{vmatrix} > 0,$$

or equivalently, if and only if

$$2f_1(x_1, x_2)f_2(x_1, x_2)f_{12}(x_1, x_2) - [f_1(x_1, x_2)]^2 f_{22}(x_1, x_2) - [f_2(x_1, x_2)]^2 f_{11}(x_1, x_2) > 0.$$

If we apply this test to $f(x_1, x_2) = x_1 x_2$ we get $2x_1x_2 > 0$ confirming that the function is strictly quasiconcave.

By Theorem M.C.4, $f(\cdot)$ is quasiconcave if and only if the Hessian matrix $D^2 f(x_1, x_2)$ is negative semidefinite in the subspace $\{z \in \mathbb{R}^2: \nabla f(x) \cdot z = 0\}$ for every $x = (x_1, x_2)$. By Theorem M.D.3 this is true if and only if

$$\begin{vmatrix} f_{11}(x_1, x_2) & f_{12}(x_1, x_2) & f_1(x_1, x_2) \\ f_{21}(x_1, x_2) & f_{22}(x_1, x_2) & f_2(x_1, x_2) \\ f_1(x_1, x_2) & f_2(x_1, x_2) & 0 \end{vmatrix} \geq 0,$$

9. Recall that M is positive (semi)definite if and only if $-M$ is negative (semi)definite. Moreover, $|-{}_rM_r| = (-1)^r|{}_rM_r|$.

and (performing the appropriate permutations)

$$\begin{vmatrix} f_{22}(x_1,x_2) & f_{21}(x_1,x_2) & f_2(x_1,x_2) \\ f_{12}(x_1,x_2) & f_{11}(x_1,x_2) & f_1(x_1,x_2) \\ f_2(x_1,x_2) & f_1(x_1,x_2) & 0 \end{vmatrix} \geq 0.$$

Computing these two determinants gives us the necessary and sufficient condition

$$2f_1(x_1,x_2)f_2(x_1,x_2)f_{12}(x_1,x_2) - [f_1(x_1,x_2)]^2 f_{22}(x_1,x_2) - [f_2(x_1,x_2)]^2 f_{11}(x_1,x_2) \geq 0.$$

■

To characterize matrices that are positive definite or positive semidefinite on the subspace $\{z \in \mathbb{R}^N\colon Bz = 0\}$, we need only alter Theorem M.D.3 by replacing the term $(-1)^r$ with $(-1)^S$.

Theorem M.D.4: Suppose that M is an $N \times N$ matrix and that for some $p \gg 0$ we have $Mp = 0$ and $M^{\mathrm{T}}p = 0$. Denote $T_p = \{z \in \mathbb{R}^N\colon p \cdot z = 0\}$ and let $\hat{M}$ be the $(N-1) \times (N-1)$ matrix obtained from M by deleting one row and the corresponding column.

(i) If rank $M = N - 1$, then rank $\hat{M} = N - 1$.

(ii) If $z \cdot Mz < 0$ for all $z \in T_p$ with $z \neq 0$ (i.e., if M is negative definite on T_p), then $z \cdot Mz < 0$ for any $z \in \mathbb{R}^N$ not proportional to p.

(iii) The matrix M is negative definite on T_p if and only if $\hat{M}$ is negative definite.

Proof: (i) Suppose that rank $\hat{M} < N - 1$, that is, $\hat{M}\hat{z} = 0$ for some $\hat{z} \in \mathbb{R}^{N-1}$ with $\hat{z} \neq 0$. Complete $\hat{z}$ to a vector $z \in \mathbb{R}^N$ by letting the value of the missing coordinate be zero. Then we have that, first, z is linearly independent of p (recall that $p \gg 0$) and, second, $Mz = 0$ and $Mp = 0$. Thus, rank $M < N - 1$, which contradicts the hypothesis.

(ii) Take a $z \in \mathbb{R}^N$ not proportional to p. For $\alpha_z = (p \cdot z)/(p \cdot p)$ and $z^* = z - \alpha_z p$, we have $z^* \in T_p$ and $z^* \neq 0$. Because $M^{\mathrm{T}}p = Mp = 0$, we have then

$$z \cdot Mz = (z^* + \alpha_z p) \cdot M(z^* + \alpha_z p) = z^* \cdot Mz^* < 0.$$

(iii) This is similar to part (ii). In fact, part (ii) directly implies that $\hat{M}$ is negative definite if M is negative definite on T_p (because for any $\hat{z} \in \mathbb{R}^{N-1}$, $\hat{z} \cdot \hat{M}\hat{z} = z \cdot Mz$, where z has been completed from $\hat{z}$ by placing a zero in the missing coordinate, and if $\hat{z} \neq 0$ this z is by construction not proportional to p). For the converse, let n denote the row and column dropped from M to obtain $\hat{M}$. If for every $z' \in T_p$ with $z' \neq 0$ we let $z = z' - (z'_n/p_n)p$, then $z_n = 0$ and $z \neq 0$ [if z were equal to zero, then we would have $z' = (z'_n/p_n)p$ in contradiction to $z' \cdot p = 0$]. Moreover, $z' \cdot Mz' = z \cdot Mz = \hat{z} \cdot \hat{M}\hat{z} < 0$. ■

Definition M.D.2: The $N \times N$ matrix M with generic entry a_{ij} has a *dominant diagonal* if there is $(p_1, \ldots, p_N) \gg 0$ such that, for every $i = 1, \ldots, N$, $|p_i a_{ii}| > \sum_{j \neq i} |p_j a_{ij}|$.

Definition M.D.3: The $N \times N$ matrix M has the *gross substitute sign pattern* if every nondiagonal entry is positive.

Theorem M.D.5: Suppose that M is an $N \times N$ matrix.

(i) If M has a dominant diagonal, then it is nonsingular.

(ii) Suppose that M is symmetric. If M has a negative and dominant diagonal then it is negative definite.

(iii) If M has the gross substitute sign pattern and if for some $p \gg 0$ we have $Mp \ll 0$ and $M^T p \ll 0$, then M is negative definite.

(iv) If M has the gross substitute sign pattern and we have $Mp = M^T p = 0$ for some $p \gg 0$, then $\hat{M}$ is negative definite, where $\hat{M}$ is any $(N-1) \times (N-1)$ matrix obtained from M by deleting a row and the corresponding column.

(v) Suppose that all the entries of M are nonnegative and that $Mz \ll z$ for some $z \gg 0$ (i.e., M is a *productive input–output* matrix). Then the matrix $(I - M)^{-1}$ exists. In fact, $(I - M)^{-1} = \sum_{k=0}^{k=\infty} M^k$.

Proof: (i) Assume, for simplicity, that $p = (1, \ldots, 1)$. Suppose, by way of contradiction, that $Mz = 0$ for $z \neq 0$. Choose a coordinate n such that $|z_n| \geq |z_{n'}|$ for every other coordinate n'. Then $|a_{nn} z_n| > \sum_{j \neq n} |a_{nj} z_n| \geq \sum_{j \neq n} |a_{nj} z_j|$, where a_{ij} is the generic entry of M. Hence, we cannot have $\sum_j a_{nj} z_j = 0$, and so $Mz \neq 0$. Contradiction.

(ii) If M has a negative dominant diagonal then so does the matrix $M - \alpha I$, for any value $\alpha \geq 0$. Hence, by (i) we have $(-1)^N |M - \alpha I| \neq 0$. Now if α is very large it is clear that $(-1)^N |M - \alpha I| > 0$ (since $(-1)^N |M - \alpha I| = (-1)^N \alpha^N |(M/\alpha) - I|$ and $|-I| = (-1)^N$). Moreover, since $(-1)^N |M - \alpha I|$ is continuous in α and $(-1)^N |M - \alpha I| \neq 0$ for all $\alpha \geq 0$, this tells us that $(-1)^N |M - \alpha I| > 0$ for all $\alpha \geq 0$. Hence, $(-1)^N |M| > 0$. By the same argument, $(-1)^r |{}_r M_r| > 0$ for all r. So, if M is also symmetric then by part (i) of Theorem M.D.2 it is negative definite.

(iii) The stated conditions imply that $M + M^T$ has a negative and dominant diagonal [in particular, note that $Mp \ll 0$ and $M^T p \ll 0$ implies that $p_n(2a_{nn}) < -\sum_{j \neq n} p_j(a_{jn} + a_{nj})$ for all n, where a_{ij} is the generic entry of M]. Because, by the gross substitute property, $a_{ij} > 0$ for $i \neq j$, this gives us $|p_n(2a_{nn})| > |\sum_{j \neq n} p_j(a_{jn} + a_{nj})|$ for all n. Hence, the conclusion follows from part (ii) of this theorem and part (i) of Theorem M.D.1.

(iv) If M satisfies the condition of (iv), then the fact that M has the gross substitute sign pattern implies that $\hat{M}$ does as well and that $\hat{M}p \ll 0$ and $\hat{M}^T p \ll 0$. Hence, $\hat{M}$ satisfies the conditions of (iii) and is therefore negative definite.

(v) This result was already proved in the Appendix to Chapter 5 (see the proof of Proposition 5.AA.1).

M.E The Implicit Function Theorem

The setting for the *implicit function theorem* (IFT) is as follows. We have a system of N equations depending on N endogenous variables $x = (x_1, \ldots, x_N)$ and M parameters $q = (q_1, \ldots, q_M)$:

$$
\begin{aligned}
f_1(x_1, \ldots, x_N; q_1, \ldots, q_M) &= 0 \\
&\vdots \\
f_N(x_1, \ldots, x_N; q_1, \ldots, q_M) &= 0
\end{aligned}
\tag{M.E.1}
$$

The domain of the endogenous variables is $A \subset \mathbb{R}^N$ and the domain of the parameters is $B \subset \mathbb{R}^M$.[10]

10. In what follows, we take A and B to be open sets (see Section M.F) so as to avoid boundary problems.

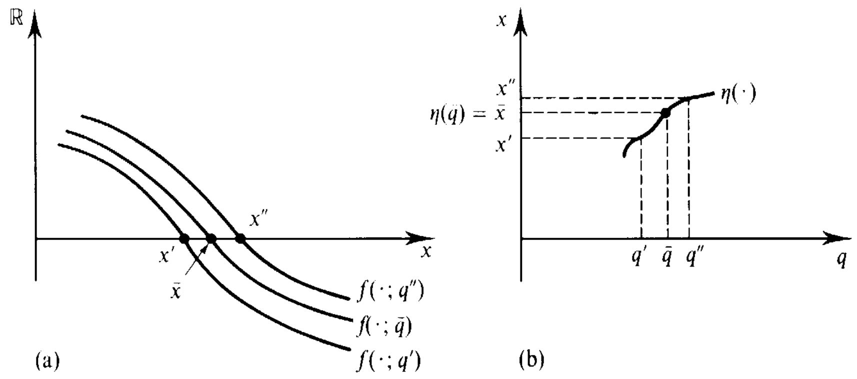


Figure M.E.1
A locally solvable equation.
(a) Solutions of $f(x; q) = 0$ near $(\bar{x}, \bar{q})$.
(b) The graph of $\eta(\cdot)$.

Suppose that $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_N) \in A$ and $\bar{q} = (\bar{q}_1, \ldots, \bar{q}_M) \in B$ satisfy equations (M.E.1). That is, $f_n(\bar{x}, \bar{q}) = 0$ for every n. We are then interested in the possibility of solving for $x = (x_1, \ldots, x_N)$ as a function of $q = (q_1, \ldots, q_M)$ *locally* around $\bar{q}$ and $\bar{x}$. Formally, we say that a set A' is an *open neighborhood* of a point $x \in \mathbb{R}^N$ if $A' = \{x' \in \mathbb{R}^N: \|x' - x\| < \varepsilon\}$ for some scalar $\varepsilon > 0$. An open neighborhood B' of a point $q \in \mathbb{R}^M$ is defined in the same way.

Definition M.E.1: Suppose that $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_N) \in A$ and $\bar{q} = (\bar{q}_1, \ldots, \bar{q}_M) \in B$ satisfy the equations (M.E.1). We say that we can *locally solve* equations (M.E.1) at $(\bar{x}, \bar{q})$ for $x = (x_1, \ldots, x_N)$ as a function of $q = (q_1, \ldots, q_M)$ if there are open neighborhoods $A' \subset A$ and $B' \subset B$, of $\bar{x}$ and $\bar{q}$, respectively, and N uniquely determined "implicit" functions $\eta_1(\cdot), \ldots, \eta_N(\cdot)$ from B' to A' such that

$$f_n(\eta_1(q), \ldots, \eta_N(q); q) = 0 \qquad \text{for every } q \in B' \text{ and every } n,$$

and

$$\eta_n(\bar{q}) = \bar{x}_n \qquad \text{for every } n.$$

In Figure M.E.1 we represent, for the case where $N = M = 1$, a situation in which the system of equations can be locally solved around a given solution.

The implicit function theorem gives a sufficient condition for the existence of such implicit functions and tells us the first-order comparative statics effects of q on x at a solution.

Theorem M.E.1: (*Implicit Function Theorem*) Suppose that every equation $f_n(\cdot)$ is continuously differentiable with respect to its $N + M$ variables and that we consider a solution $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_N)$ at parameter values $\bar{q} = (\bar{q}_1, \ldots, \bar{q}_M)$, that is, satisfying $f_n(\bar{x}; \bar{q}) = 0$ for every n. If the *Jacobian* matrix of the system (M.E.1) with respect to the endogenous variables, evaluated at $(\bar{x}, \bar{q})$, is nonsingular, that is, if

$$\begin{vmatrix} \dfrac{\partial f_1(\bar{x}, \bar{q})}{\partial x_1} & \cdots & \dfrac{\partial f_1(\bar{x}, \bar{q})}{\partial x_N} \\ & \ddots & \\ \dfrac{\partial f_N(\bar{x}, \bar{q})}{\partial x_1} & \cdots & \dfrac{\partial f_N(\bar{x}, \bar{q})}{\partial x_N} \end{vmatrix} \neq 0, \qquad \text{(M.E.2)}$$

then the system can be locally solved at $(\bar{x}, \bar{q})$ by implicitly defined functions $\eta_n: B' \to A'$ that are continuously differentiable. Moreover, the first-order effects of q on x at $(\bar{x}, \bar{q})$ are given by

$$D_q\eta(\bar{q}) = -[D_x f(\bar{x}; \bar{q})]^{-1} D_q f(\bar{x}; \bar{q}). \tag{M.E.3}$$

Proof: A proof of the existence of the implicit functions η_n: $B' \to A'$ is too technical for this appendix, but its common-sense logic is easy to grasp. Expression (M.E.2), a full rank condition, tells us that we can move the values of the system of equations in any direction by appropriate changes of the endogenous variables. Therefore, if there is a shock to the parameters and the values of the equation system are pushed away from zero, then we can adjust the endogenous variables so as to restore the "equilibrium."

Now, given a system of implicit functions $\eta(q) = (\eta_1(q), \ldots, \eta_N(q))$ defined on some neighborhood of $(\bar{x}, \bar{q})$, the first-order comparative static effects $\partial\eta_n(\bar{q})/\partial q_m$ are readily determined. Let $f(x; q) = (f_1(x; q), \ldots, f_N(x; q))$. Since we have

$$f(\eta(q); q) = 0 \qquad \text{for all } q \in B',$$

we can apply the chain rule of calculus to obtain

$$D_x f(\bar{x}; \bar{q}) D_q \eta(\bar{q}) + D_q f(\bar{x}; \bar{q}) = 0.$$

Because of (M.E.2), the $N \times N$ matrix $D_x f(\bar{x}; \bar{q})$ is invertible, and so we can conclude that

$$D_q\eta(\bar{q}) = -[D_x f(\bar{x}; \bar{q})]^{-1} D_q f(\bar{x}; \bar{q}). \qquad \blacksquare$$

Note that when $N = M = 1$ (the case of one endogenous variable and one parameter), (M.E.3) reduces to the simple expression

$$\frac{d\eta(\bar{q})}{dq} = -\frac{\partial f(\bar{x}; \bar{q})/\partial q}{\partial f(\bar{x}; \bar{q})/\partial x}.$$

The special case of the implicit function theorem where $M = N$ and every equation has the form $f_n(x, q) = g_n(x) - q_n = 0$ is known as the *inverse function theorem.*

How restrictive is condition (M.E.2)? Not very. In Figure M.E.2 we depict a situation where it fails to hold. [By contrast, in Figure M.E.1 condition (M.E.2) is satisfied.] However, the tangency displayed in Figure M.E.2 appears pathological: it would be removed by any small perturbation of the function $f(\cdot\,; \cdot)$.

An important result, the *transversality theorem*, makes this idea precise by asserting that, under a weak condition [enough first-order variability of $f(\cdot\,; \cdot)$ with respect to x *and* q], (M.E.2) holds *generically* on the parameters. We present a preliminary concept in Definition M.E.2.

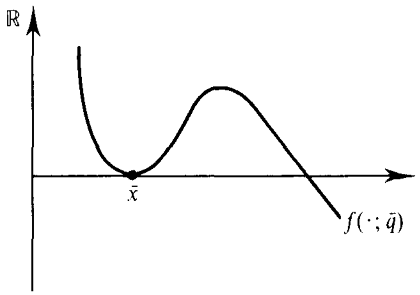


Figure M.E.2
Condition (M.E.2) is violated at the solution $(\bar{x}, \bar{q})$.

Definition M.E.2: Given open sets $A \subset \mathbb{R}^N$ and $B \subset \mathbb{R}^M$, the (continuously differentiable) system of equations $f(\cdot\,; \hat{q}) = 0$ defined on A is *regular at* $\hat{q} \in B$ if (M.E.2) holds at any solution x; that is, if $f(x; \hat{q}) = 0$ implies that $|D_x f(x; \hat{q})| \neq 0$.

With this definition we then have Theorem M.E.2.

Theorem M.E.2: (*Transversality Theorem*) Suppose that we are given open sets $A \subset \mathbb{R}^N$ and $B \subset \mathbb{R}^M$ and a (continuously differentiable) function $f: A \times B \to \mathbb{R}^N$. If $f(\cdot\,; \cdot)$ satisfies the condition

The $N \times (N + M)$ matrix $Df(x; q)$ has rank N whenever $f(x; q) = 0$,

then the system of N equations in N unknowns $f(\cdot\,; \hat{q}) = 0$ is regular for *almost every* $\hat{q} \in B$.[11]

M.F Continuous Functions and Compact Sets

In this section, we begin by formally defining the concept of a continuous function. We then develop the notion of a compact set (and, along the way, the notions of open and closed sets). Finally, we discuss some properties of continuous functions that relate to compact sets.

A *sequence* in $\mathbb{R}^N$ assigns to every positive integer $m = 1, 2, \ldots$ a vector $x^m \in \mathbb{R}^N$. We denote the sequence by $\{x^m\}_{m=1}^{m=\infty}$ or, simply, by $\{x^m\}$ or even x^m.

Definition M.F.1: The sequence $\{x^m\}$ *converges* to $x \in \mathbb{R}^N$, written as $\lim_{m \to \infty} x^m = x$, or $x^m \to x$, if for every $\varepsilon > 0$ there is an integer M_ε such that $\|x^m - x\| < \varepsilon$ whenever $m > M_\varepsilon$. The point x is then said to be the *limit point* (or simply the *limit*) of sequence $\{x^m\}$.

In words: The sequence $\{x^m\}$ converges to x if x^m approaches x arbitrarily closely as m increases.

Definition M.F.2: Consider a domain $X \subset \mathbb{R}^N$. A function $f: X \to \mathbb{R}$ is *continuous* if for all $x \in X$ and every sequence $x^m \to x$ (having $x^m \in X$ for all m), we have $f(x^m) \to f(x)$. A function $f: X \to \mathbb{R}^K$ is continuous if every coordinate function $f_k(\cdot)$ is continuous.

In words: a function is continuous if, when we take a sequence of points $x^1, x^2, \ldots$ converging to x, the corresponding sequence of function values $f(x^1), f(x^2), \ldots$ converges to $f(x)$. Intuitively, a function fails to be continuous if it displays a "jump" in its value at some point x. Examples of continuous and discontinuous functions defined on $[0, 1]$ are illustrated in Figure M.F.1.

We next develop the notions of open, closed, and compact sets.

Definition M.F.3: Fix a set $X \subset \mathbb{R}^N$. We say that a set $A \subset X$ is *open* (relative to X) if for every $x \in A$ there is an $\varepsilon > 0$ such that $\|x' - x\| < \varepsilon$ and $x' \in X$ implies $x' \in A$.

11. "Almost every" means that if, for example, we choose $\hat{q}$ according to some nondegenerate multinomial normal distribution in $\mathbb{R}^M$, then with probability 1 the equation system $f(\cdot\,; \hat{q}) = 0$ is regular. This is the concept of "genericity" in this context.

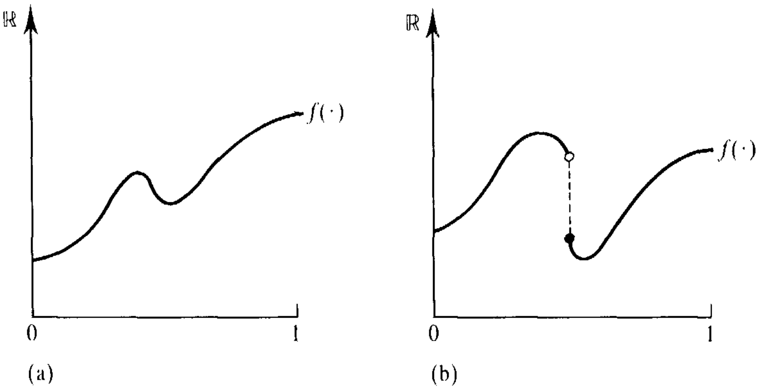


Figure M.F.1
Continuous and discontinuous functions.
(a) A continuous function.
(b) A discontinuous function.

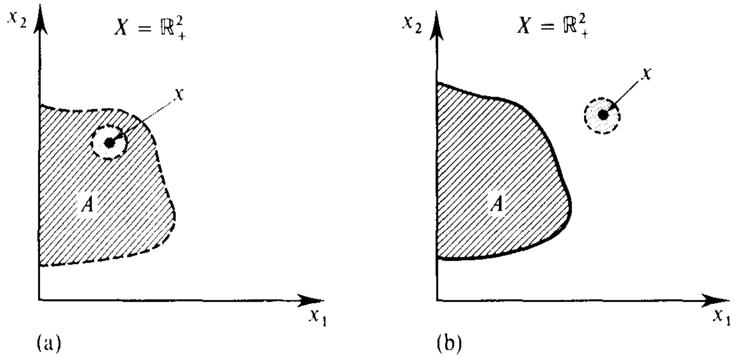


Figure M.F.2
Open and closed sets.
(a) An open set (relative to X).
(b) A closed set (relative to X).

A set $A \subset X$ is *closed* (relative to X) if its complement $X \backslash A$ is open (relative to X).[12] If $X = \mathbb{R}^N$ we simply refer to "open" and "closed" sets.

Given a point $x \in \mathbb{R}^N$, a set $B = \{x' \in \mathbb{R}^N: \|x' - x\| < \varepsilon\}$ for some scalar $\varepsilon > 0$ is called an *open ball around* x. With this notion, the idea of an open set can be put as follows: Suppose that the universe of possible vectors in $\mathbb{R}^N$ is X. A set $A \subset X$ is open (relative to X) if, for every point x in A, there is an open ball around x all of whose elements (in X) are elements of A. In Figure M.F.2(a) the hatched set A is open (relative to X). In the figure, we depict a typical point $x \in A$ and a shaded open ball around x that lies within A; points on the dashed curve do not belong to A. In contrast, the hatched set A in Figure M.F.2(b) is closed because the set $X \backslash A$ is open; note how there is an open ball around the point $x \in X \backslash A$ that lies entirely within $X \backslash A$ [in the figure, the points on the inner solid curve belong to A].

Theorem M.F.1 gathers some basic facts about open and closed sets.

Theorem M.F.1: Fix a set $X \subset \mathbb{R}^N$. In what follows, all the open and closed sets are relative to X.

(i) The union of any number, finite or infinite, of open sets is open. The intersection of a finite number of open sets is open.

(ii) The intersection of any number, finite or infinite, of closed sets is closed. The union of a finite number of closed sets is closed.

(iii) A set $A \subset X$ is closed if and only if for every sequence $x^m \to x \in X$, with $x^m \in A$ for all m, we have $x \in A$.

12. Given two sets A and B, the set $A \backslash B$ is the set containing all the elements of A that are not elements of B.

Property (iii) of Theorem M.F.1 is noteworthy because it gives us a direct way to characterize a closed set: a set A is closed if and only if the limit point of any sequence whose members are all elements of A is itself an element of A. Points (in X) that are the limits of sequences whose members are all elements of the set A are known as the *limit points* of A. Thus, property (iii) says that a set A is closed if and only if it contains all of its limit points.

Given $A \subset X$, the *interior* of A (relative to X) is the open set[13]

$$\text{Int}_X\, A = \{x \in A\colon \text{there is } \varepsilon > 0 \text{ such that } \|x' - x\| < \varepsilon \text{ and } x' \in X \text{ implies } x' \in A\}.$$

The *closure of* A (relative to X) is the closed set $\text{Cl}_X\, A = X \setminus \text{Int}_X\,(X \setminus A)$. Equivalently, $\text{Cl}_X\, A$ is the union of the set A and its limit points; it is the smallest closed set containing A. The *boundary* of A (relative to X) is the closed set $\text{Bdry}_X\, A = \text{Cl}_X\, A \setminus \text{Int}_X\, A$. The set A is closed if and only if $\text{Bdry}_X\, A \subset A$.

Definition M.F.4: A set $A \subset \mathbb{R}^N$ is *bounded* if there is $r \in \mathbb{R}$ such that $\|x\| < r$ for every $x \in A$. The set $A \subset \mathbb{R}^N$ is *compact* if it is bounded and closed relative to $\mathbb{R}^N$.

We conclude by noting two properties of continuous functions relating to compact sets. Formally, given a function $f\colon X \to \mathbb{R}^K$, the *image* of a set $A \subset X$ under $f(\cdot)$ is the set $f(A) = \{y \in \mathbb{R}^K\colon y = f(x) \text{ for some } x \in A\}$.

Theorem M.F.2: Suppose that $f\colon X \to \mathbb{R}^K$ is a continuous function defined on a nonempty set $X \subset \mathbb{R}^N$.

(i) The image of a compact set under $f(\cdot)$ is compact: That is, if $A \subset X$ is compact, then $f(A) = \{y \in \mathbb{R}^K\colon y = f(x) \text{ for some } x \in A\}$ is a compact subset of $\mathbb{R}^K$.

(ii) Suppose that $K = 1$ and X is compact. Then $f(\cdot)$ has a maximizer: That is, there is $x \in X$ such that $f(x) \geq f(x')$ for every $x' \in X$.

Part (ii) of Theorem M.F.2 asserts that any continuous function $f\colon X \to \mathbb{R}$ defined on a compact set X attains a maximum. We illustrate this result in Figure M.F.3. A maximum is not attained either in Figure M.F.3(a) or in Figure M.F.3(b). In Figure M.F.3(a), the function is continuous, but the domain is not compact. In Figure M.F.3(b), the domain is compact, but the function is not continuous.

Given a sequence $\{x^m\}$, suppose that we have a strictly increasing function $m(k)$ that assigns to each positive integer k a positive integer $m(k)$. Then the sequence $x^{m(1)}, x^{m(2)}, \ldots$ (written $\{x^{m(k)}\}$) is called a *subsequence* of $\{x^m\}$. That is, $\{x^{m(k)}\}$ is composed of an (order-preserving) subset of the sequence $\{x^m\}$. For example, if the sequence $\{x^m\}$ is $1, 2, 4, 16, 25, 36, \ldots$, then one subsequence of $\{x^m\}$ is $1, 4, 16, 36, \ldots$; another is $2, 4, 16, 25, 36, \ldots$.

Theorem M.F.3: Suppose that the set $A \subset \mathbb{R}^N$ is compact.

(i) Every sequence $\{x^m\}$ with $x^m \in A$ for all m has a convergent subsequence. Specifically, there is a subsequence $\{x^{m(k)}\}$ of the sequence $\{x^m\}$ that has a limit in A, that is, a point $x \in A$ such that $x^{m(k)} \to x$.

(ii) If, in addition to being compact, A is also *discrete*, that is, if all its points are isolated [formally, for every $x \in A$ there is $\varepsilon > 0$ such that $x' = x$ whenever $x' \in A$ and $\|x' - x\| < \varepsilon$], then A is finite.

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13. In what follows in this paragraph, all of the open and closed sets are once again relative to X.

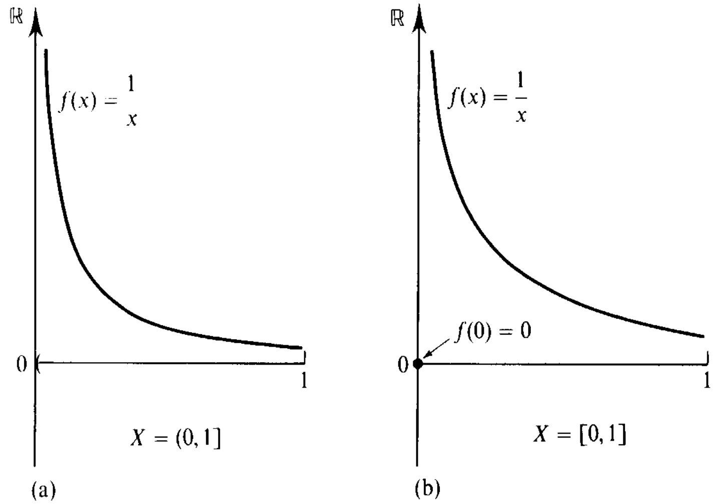


Figure M.F.3
Indispensability of the continuity and compactness assumptions for the existence of a maximizer.
(a) A continuous function with no maximizer on a noncompact domain.
(b) A discontinuous function with no maximizer on a compact domain.

M.G Convex Sets and Separating Hyperplanes

In this section, we review some basic properties of convex sets, including the important separating hyperplane theorems.

Definition M.G.1: The set $A \subset \mathbb{R}^N$ is *convex* if $\alpha x + (1 - \alpha)x' \in A$ whenever $x, x' \in A$ and $\alpha \in [0, 1]$.[14]

In words: A set in $\mathbb{R}^N$ is convex if whenever it contains two vectors x and x', it also contains the entire segment connecting them. In Figure M.G.1(a), we depict a convex set. The set in Figure M.G.1(b) is not convex.

Note that for a concave function $f: A \to \mathbb{R}$ the set $\{(z, v) \in \mathbb{R}^{N+1}: v \le f(z), z \in A\}$ is convex. Note also that the intersection of any number of convex sets is convex, but the union of convex sets need not be convex.

Definition M.G.2: Given a set $B \subset \mathbb{R}^N$, the *convex hull* of B, denoted $\text{Co}\, B$, is the smallest convex set containing B, that is, the intersection of all convex sets that contain B.

Figure M.G.2 represents a set and its convex hull. It is not difficult to verify that the convex hull can also be described as the set of all possible convex combinations of elements of B, that is,

$$\text{Co}\, B = \left\{ \sum_{j=1}^{J} \alpha_j x_j : \text{for some } x_1, \ldots, x_J \text{ with } x_j \in B \text{ for all } j, \right.$$
$$\left. \text{and some } (\alpha_1, \ldots, \alpha_J) \ge 0 \text{ with } \sum_{j=1}^{J} \alpha_j = 1 \right\}.$$

Definition M.G.3: The vector $x \in B$ is an *extreme point* of the convex set $B \subset \mathbb{R}^N$ if it cannot be expressed as $x = \alpha y + (1 - \alpha)z$ for any $y, z \in B$ and $\alpha \in (0, 1)$.

14. The set A is *strictly convex* if $\alpha x + (1 - \alpha)x'$ is an element of the *interior* of A whenever $x, x' \in A$ and $\alpha \in (0, 1)$ (see Section M.F for a definition of the interior of a set).

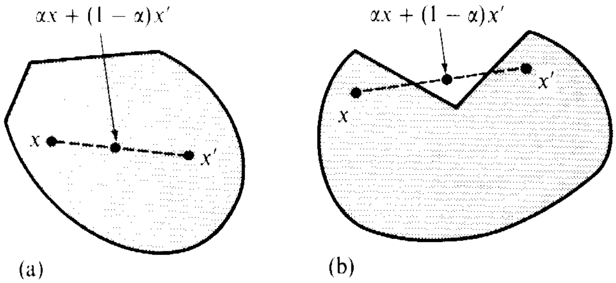


Figure M.G.1
Convex and nonconvex sets.
(a) A convex set.
(b) A nonconvex set.

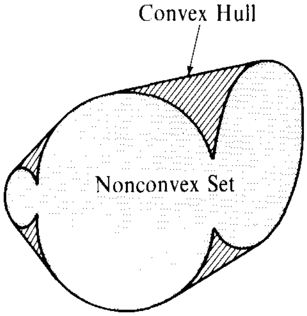


Figure M.G.2
A nonconvex set and its convex hull.

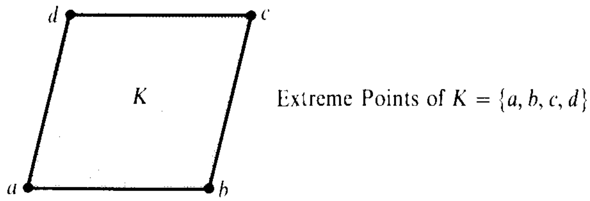


Figure M.G.3
Extreme points of a convex set.

The extreme points of the convex set represented in Figure M.G.3 are the four corners.

A very important result of convexity theory is contained in Theorem M.G.1.

Theorem M.G.1: Suppose that $B \subset \mathbb{R}^N$ is a convex set that is also compact (that is, closed and bounded; see Section M.F). Then every $x \in B$ can be expressed as a convex combination of at most $N + 1$ extreme points of B.

Proof: The proof is too technical to be given here. Note simply that the result is correct for the convex set in Figure M.G.3: Any point belongs to the triangle spanned by *some* collection of three corners. ■

We now turn to the development of the separating hyperplane theorems.

Definition M.G.4: Given $p \in \mathbb{R}^N$ with $p \neq 0$, and $c \in \mathbb{R}$, the *hyperplane generated by p and c* is the set $H_{p,c} = \{z \in \mathbb{R}^N\colon p \cdot z = c\}$. The sets $\{z \in \mathbb{R}^N\colon p \cdot z \geq c\}$ and $\{z \in \mathbb{R}^N\colon p \cdot z \leq c\}$ are called, respectively, the *half-space above* and the *half-space below* the hyperplane $H_{p,c}$.

Hyperplanes and half-spaces are convex sets. Figure M.G.4 provides illustrations.

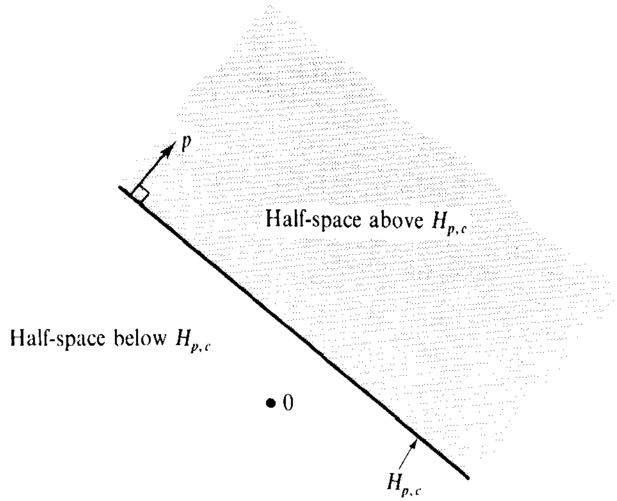


Figure M.G.4
Hyperplanes and half-spaces.

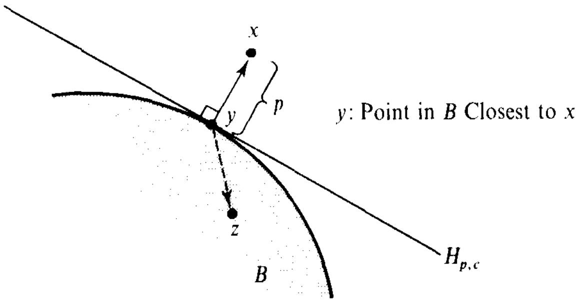


Figure M.G.5
The separating hyperplane theorem.

Theorem M.G.2: (*Separating Hyperplane Theorem*) Suppose that $B \subset \mathbb{R}^N$ is convex and closed (see Section M.F for a discussion of closed sets), and that $x \notin B$. Then there is $p \in \mathbb{R}^N$ with $p \neq 0$, and a value $c \in \mathbb{R}$ such that $p \cdot x > c$ and $p \cdot y < c$ for every $y \in B$.

More generally, suppose that the convex sets $A, B \subset \mathbb{R}^N$ are disjoint (i.e., $A \cap B = \varnothing$). Then there is $p \in \mathbb{R}^N$ with $p \neq 0$, and a value $c \in \mathbb{R}$, such that $p \cdot x \geq c$ for every $x \in A$ and $p \cdot y \leq c$ for every $y \in B$. That is, there is a hyperplane that separates A and B, leaving A and B on different sides of it.

Proof: We discuss only the first part (i.e., the separation of a point and a closed, convex set). In Figure M.G.5 we represent a closed, convex set B and a point $x \notin B$. We also indicate by $y \in B$ the point in set B closest to x.[15] If we let $p = x - y$ and $c' = p \cdot y$, we can then see, first, that $p \cdot x > c'$ [since $p \cdot x - c' = p \cdot x - p \cdot y = (x - y) \cdot (x - y) = \|x - y\|^2 > 0$] and, second, that for any $z \in B$ the vectors p and $z - y$ cannot make an acute angle, that is, $p \cdot (z - y) = p \cdot z - c' \leq 0$. Finally, let $c = c' + \varepsilon$ where $\varepsilon > 0$ is small enough for $p \cdot x > c' + \varepsilon = c$ to hold. ■

15. We use the familiar Euclidean distance measure. It is to guarantee the existence of a closest point in B that we require B to be closed.

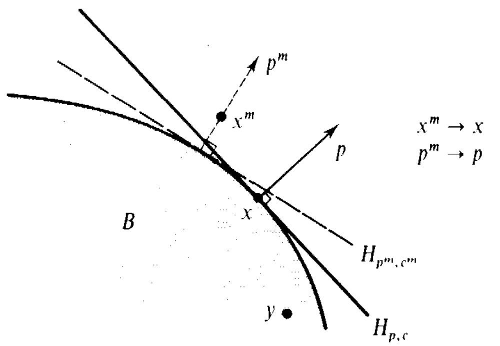


Figure M.G.6
The supporting hyperplane theorem

Theorem M.G.3: (*Supporting Hyperplane Theorem*) Suppose that $B \subset \mathbb{R}^N$ is convex and that x is not an element of the interior of set B (i.e., $x \notin \text{Int}\, B$; see Section M.F for the concept of the interior of a set). Then there is $p \in \mathbb{R}^N$ with $p \neq 0$ such that $p \cdot x \geq p \cdot y$ for every $y \in B$.

Proof: Suppose that $x \notin \text{Int}\, B$. The following argument can be followed in Figure M.G.6. It is intuitive that we can find a sequence $x^m \to x$ such that, for all m, x^m is not an element of the closure of set B (i.e., $x^m \notin \text{Cl}\, B$; see Section M.F for a discussion of sequences and the closure of a set). By the separating hyperplane theorem (Theorem M.G.2), for each m there is a $p^m \neq 0$ and a $c^m \in \mathbb{R}$ such that

$$p^m \cdot x^m > c^m \geq p^m \cdot y \tag{M.G.1}$$

for every $y \in B$. Without loss of generality we can suppose that $\|p^m\| = 1$ for every m. Thus, extracting a subsequence if necessary (see the small type discussion at the end of Section M.F), we can assume that there is $p \neq 0$ and $c \in \mathbb{R}$ such that $p^m \to p$ and $c^m \to c$. Hence, taking limits in (M.G.1), we have

$$p \cdot x \geq c \geq p \cdot y$$

for every $y \in B$. ■

Finally, for the important concept of the *support function* of a set and its properties we refer to Section 3.F of the text.

M.H Correspondences

It is common in economics to resort to a generalized concept of a function called a *correspondence*.

Definition M.H.1: Given a set $A \subset \mathbb{R}^N$, a *correspondence* $f\colon A \to \mathbb{R}^K$ is a rule that assigns a set $f(x) \subset \mathbb{R}^K$ to every $x \in A$.

Note that when, for every $x \in A$, $f(x)$ is composed of precisely one element, then $f(\cdot)$ can be viewed as a function in the usual sense. Note also that the definition allows for $f(x) = \varnothing$, but typically we consider only correspondences with $f(x) \neq \varnothing$ for every $x \in A$. Finally, if for some set $Y \subset \mathbb{R}^K$ we have $f(x) \subset Y$ for every $x \in A$, we indicate this by $f\colon A \to Y$.

We now proceed to discuss continuity notions for correspondences. Given $A \subset \mathbb{R}^N$ and $Y \subset \mathbb{R}^K$, the *graph* of the correspondence $f\colon A \to Y$ is the set $\{(x, y) \in A \times Y\colon y \in f(x)\}$.

Definition M.H.2: Given $A \subset \mathbb{R}^N$ and the closed set $Y \subset \mathbb{R}^K$, the correspondence $f\colon A \to Y$ has a *closed graph* if for any two sequences $x^m \to x \in A$ and $y^m \to y$, with $x^m \in A$ and $y^m \in f(x^m)$ for every m, we have $y \in f(x)$.

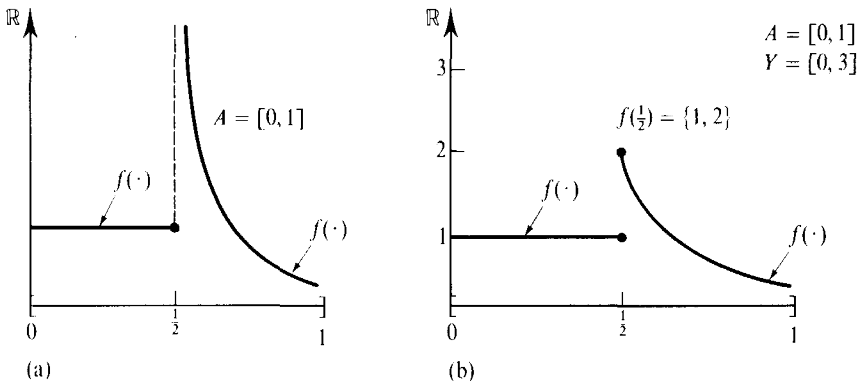


Figure M.H.1
Closed graphs and upper hemicontinuous correspondences.
(a) A closed graph correspondence that is not upper hemicontinuous.
(b) An upper hemicontinuous correspondence.

Note that the concept of a closed graph is simply our usual notion of closedness (relative to $A \times Y$) applied to the set $\{(x, y) \in A \times Y: y \in f(x)\}$ (see Section M.F).

Definition M.H.3: Given $A \subset \mathbb{R}^N$ and the closed set $Y \subset \mathbb{R}^K$, the correspondence $f: A \to Y$ is *upper hemicontinuous* (uhc) if it has a closed graph and the images of compact sets are bounded, that is, for every compact set $B \subset A$ the set $f(B) = \{y \in Y: y \in f(x) \text{ for some } x \in B\}$ is bounded.[16,17]

In many applications, the range space Y of $f(\cdot)$ is itself compact. In that case, the upper hemicontinuity property reduces to the closed graph condition. In Figure M.H.1(a), we represent a correspondence (in fact, a function) having a closed graph that is not upper hemicontinuous. In contrast, the correspondence represented in Figure M.H.1(b) is upper hemicontinuous.

The upper hemicontinuity property for correspondences can be thought of as a natural generalization of the notion of continuity for functions. Indeed, we have the result of Theorem M.H.1.

Theorem M.H.1: Given $A \subset \mathbb{R}^N$ and the closed set $Y \subset \mathbb{R}^K$, suppose that $f: A \to Y$ is a single-valued correspondence (so that it is, in fact, a function). Then $f(\cdot)$ is an upper hemicontinuous correspondence if and only if it is continuous as a function.

Proof: If $f(\cdot)$ is continuous as a function, then Definition M.F.2 implies that $f(\cdot)$ has a closed graph (relative to $A \times Y$). In addition, Theorem M.F.2 tells us that images of compact sets under $f(\cdot)$ are compact, hence bounded. Thus, $f(\cdot)$ is upper hemicontinuous as a correspondence.

Suppose now that $f(\cdot)$ is upper hemicontinuous as a correspondence and consider any sequence $x^m \to x \in A$ with $x^m \in A$ for all m. Let $S = \{x^m: m = 1, 2, \ldots\} \cup \{x\}$. Then there exists an $r > 0$ such that $\|x'\| < r$ if $x' \in S$.[18] Because S is also closed, it follows that S is compact.

16. See Section M.F for a discussion of bounded and compact sets.

17. It can be verified that Definition M.H.3 implies that the image of a compact set under an upper hemicontinuous correspondence is in fact compact (i.e., closed and bounded), a property also shared by continuous functions (see Theorem M.F.2).

18. To see this, recall that if $x^m \to x$, then for any $\varepsilon > 0$ there is a positive integer M_ε such that $\|x^m - x\| < \varepsilon$ for all $m > M_\varepsilon$. Hence, for any $r > \text{Max}\,\{\|x^1\|, \ldots, \|x^{M_\varepsilon}\|, \|x\| + \varepsilon\}$, we have $\|x'\| < r$ if $x' \in S$.

By Definition M.H.3, $f(S)$ is bounded, and so Cl $f(S)$ (relative to $\mathbb{R}^K$) is a compact set. If, contradicting continuity of the function $f(\cdot)$, the sequence $\{f(x^m)\}$ [which lies in the compact set Cl $f(S)$] did not converge to $f(x)$, then by Theorem M.F.3 we could extract a subsequence $x^{m(k)} \to x$ such that $f(x^{m(k)}) \to y$ for some $y \in \text{Cl } f(S)$ having $y \neq f(x)$. But then the graph of $f(\cdot)$ could not be closed, in contradiction to the upper hemicontinuity of $f(\cdot)$ as a correspondence. ■

Upper hemicontinuity is only one of two possible generalizations of the continuity notion to correspondences. We now state the second (for the case where the range space Y is compact).

Definition M.H.4: Given $A \subset \mathbb{R}^N$ and a compact set $Y \subset \mathbb{R}^K$, the correspondence $f: A \to Y$ is *lower hemicontinuous* (lhc) if for every sequence $x^m \to x \in A$ with $x^m \in A$ for all m, and every $y \in f(x)$, we can find a sequence $y^m \to y$ and an integer M such that $y^m \in f(x^m)$ for $m > M$.

Figure M.H.2(a) represents a lower hemicontinuous correspondence.[19] Observe that the correspondence is not upper hemicontinuous—it does not have a closed graph. Similarly, the correspondence represented in Figure M.H.2(b) is upper hemicontinuous but it fails to be lower hemicontinuous (consider the illustrated sequence $x^m \to x$ that approaches x from below and the point $y \in f(x)$). Roughly speaking, upper hemicontinuity is compatible only with "discontinuities" that appear as "explosions" of sets [as at $x = \frac{1}{2}$ in Figure M.H.2(b)], while lower hemicontinuity is compatible only with "implosions" of sets [as at $x = \frac{1}{2}$ in Figure M.H.2(a)].

As with upper hemicontinuous correspondences, if $f(\cdot)$ is a function then the concepts of lower hemicontinuity as a correspondence and of continuity as a function coincide.

Finally, when a correspondence is both upper and lower hemicontinuous, we say that it is *continuous*. An example is illustrated in Figure M.H.3.

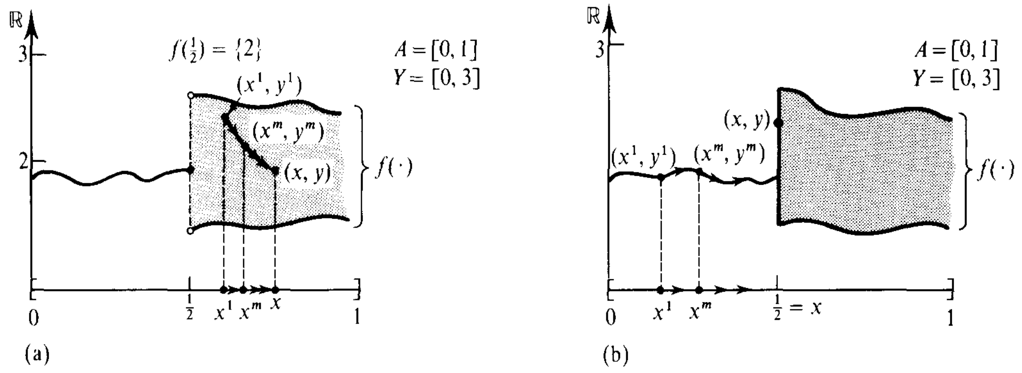


Figure M.H.2
Upper and lower hemicontinuous correspondences.
(a) A lower hemicontinuous correspondence that is not upper hemicontinuous.
(b) An upper hemicontinuous correspondence that is not lower hemicontinuous.

19. For another source of examples, note that any correspondence $f: A \to Y$ with an open graph (relative to $A \times Y$) is lower hemicontinuous.

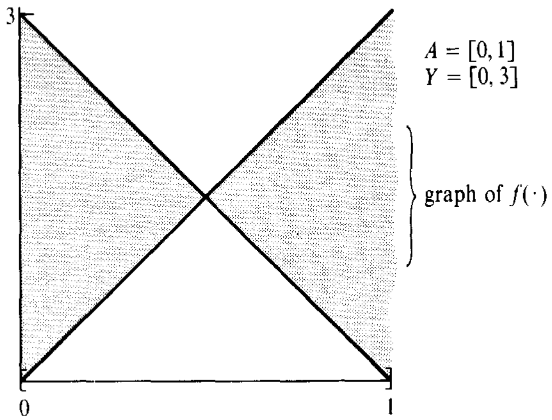


Figure M.H.3
A continuous correspondence.

M.I Fixed Point Theorems

In economics the most frequent technique for establishing the existence of solutions to an equilibrium system of equations consists of setting up the problem as the search for a *fixed point* of a suitably constructed function or correspondence $f\colon A \to A$ from some set $A \subset \mathbb{R}^N$ into itself. A vector $x \in A$ is a fixed point of $f(\cdot)$ if $x = f(x)$ [or, in the correspondence case, if $x \in f(x)$]. That is, the vector is mapped into itself and so it remains "fixed." The reason for proceeding in this, often roundabout, way is that important mathematical theorems for proving the existence of fixed points are readily available.

The most basic and well-known result is stated in Theorem M.I.1.

Theorem M.I.1: (*Brouwer's Fixed Point Theorem*) Suppose that $A \subset \mathbb{R}^N$ is a nonempty, compact, convex set, and that $f\colon A \to A$ is a continuous function from A into itself. Then $f(\cdot)$ has a fixed point; that is, there is an $x \in A$ such that $x = f(x)$.

The logic of Brouwer's fixed point theorem is illustrated in Figure M.I.1(a) for the easy case where $N = 1$ and $A = [0, 1]$. In this case, the theorem says that the graph of any continuous function from the interval $[0, 1]$ into itself must cross the diagonal, and it is then a simple consequence of the *intermediate value theorem*. In

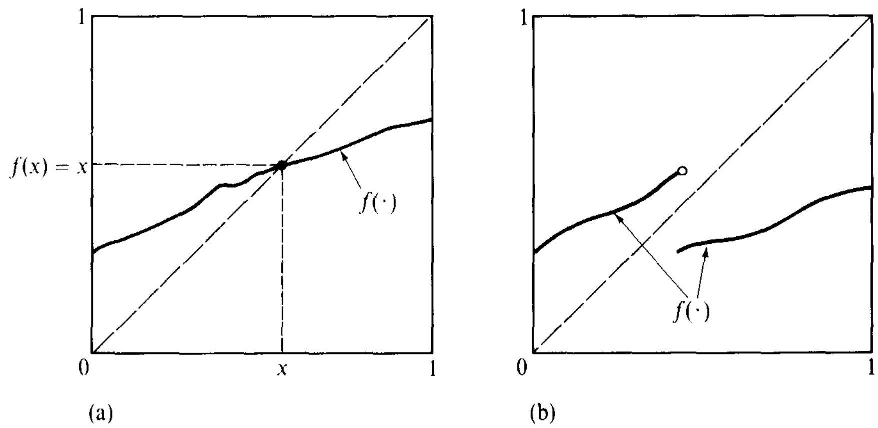


Figure M.I.1
Brouwer's fixed point theorem.
(a) A continuous function from [0, 1] to [0, 1] has a fixed point.
(b) The continuity assumption is indispensable.

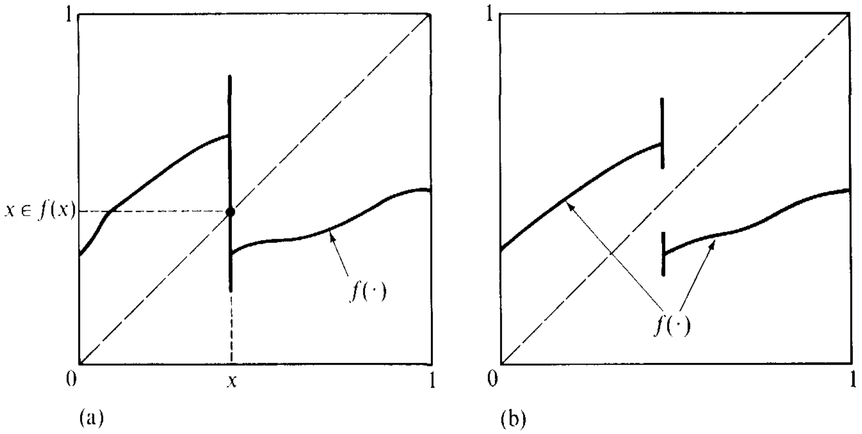


Figure M.I.2
Kakutani's fixed point theorem.
(a) A fixed point exists.
(b) The convex-valuedness assumption is indispensable.

particular, if we define the continuous function $\phi(x) = f(x) - x$, then $\phi(0) \geq 0$ and $\phi(1) \leq 0$, and so $\phi(x) = 0$ for some $x \in [0, 1]$; hence, $f(x) = x$ for some $x \in [0, 1]$. In Figure M.I.1(b) we can see that, indeed, the continuity of $f(\cdot)$ is required. As for the convexity of the domain, consider the function defined by a 90-degree clockwise rotation on the circle $S = \{x \in \mathbb{R}^2\colon \|x\| = 1\}$: It is a continuous function with no fixed point. The set S, however, is not convex.

In applications, it is often the case that the following extension of Brouwer's fixed point theorem to correspondence is most useful.

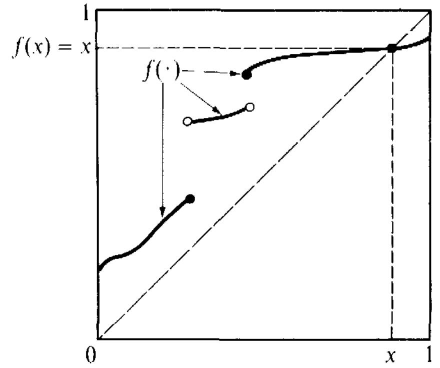
Theorem M.I.2: (*Kakutani's Fixed Point Theorem*) Suppose that $A \subset \mathbb{R}^N$ is a nonempty, compact, convex set, and that $f\colon A \to A$ is an upper hemicontinuous correspondence from A into itself with the property that the set $f(x) \subset A$ is nonempty and convex for every $x \in A$. Then $f(\cdot)$ has a fixed point; that is, there is an $x \in A$ such that $x \in f(x)$.

The logic of Kakutani's fixed point theorem is illustrated in Figure M.I.2(a) for $N = 1$. Note that the convexity of the set $f(x)$ for all x is indispensable. Without this condition we could have cases such as that in Figure M.I.2(b) where no fixed point exists.

Finally, we mention a fixed point theorem that is of a different style but that is being found of increasing relevance to economic applications.

Theorem M.I.3: (*Tarsky's Fixed Point Theorem*) Suppose that $f\colon [0, 1]^N \to [0, 1]^N$ is a nondecreasing function, that is, $f(x') \geq f(x)$ whenever $x' \geq x$. Then $f(\cdot)$ has a fixed point; that is, there is an $x \in A$ such that $x = f(x)$.

Tarsky's theorem differs from Brouwer's in three respects. First, the base set is not any compact, convex set, but rather a special one—an N-product of intervals. Second, the function is required to be nondecreasing. Third, the function is not required to be continuous. The logic of Tarsky's fixed point theorem is illustrated in Figure M.I.3 for the case $N = 1$. In the figure, the function $f(\cdot)$ is not continuous. Yet, the fact that it is nondecreasing forces its graph to intersect the diagonal.



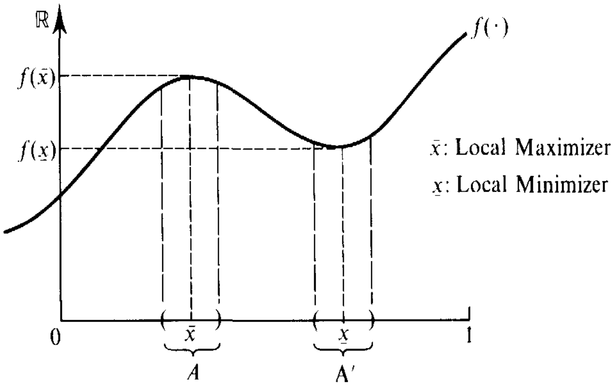


Figure M.I.3 (left) Tarski's fixed point theorem.

Figure M.J.1 (right) Local maximizers and minimizers.

M.J Unconstrained Maximization

In this section we consider a function $f: \mathbb{R}^N \to \mathbb{R}$.

Definition M.J.1: The vector $\bar{x} \in \mathbb{R}^N$ is a *local maximizer* of $f(\cdot)$ if there is an open neighborhood of $\bar{x}$, $A \subset \mathbb{R}^N$, such that $f(\bar{x}) \geq f(x)$ for every $x \in A$.[20] If $f(\bar{x}) \geq f(x)$ for every $x \in \mathbb{R}^N$ (i.e., if we can take $A = \mathbb{R}^N$), then we say that $\bar{x}$ is a *global maximizer* of $f(\cdot)$ (or simply a *maximizer*). The concepts of *local* and *global minimizers* are defined analogously.

In Figure M.J.1, we illustrate a local maximizer $\bar{x}$ and a local minimizer $\underline{x}$ (with open neighborhoods A and A', respectively) of a function for the case in which $N = 1$.

Theorem M.J.1: Suppose that $f(\cdot)$ is differentiable and that $\bar{x} \in \mathbb{R}^N$ is a local maximizer or local minimizer of $f(\cdot)$. Then

$$\frac{\partial f(\bar{x})}{\partial x_n} = 0 \quad \text{for every } n, \tag{M.J.1}$$

or, in more concise notation,

$$\nabla f(\bar{x}) = 0. \tag{M.J.2}$$

Proof: Suppose that $\bar{x}$ is a local maximizer or local minimizer of $f(\cdot)$ but that $\partial f(\bar{x})/\partial x_n = a > 0$ (the argument is analogous if $a < 0$). Denote by $e^n \in \mathbb{R}^N$ the vector having its nth entry equal to 1 and all other entries equal to 0 (i.e., having $e^n_n = 1$ and $e^n_h = 0$ for $h \neq n$). By the definition of a (partial) derivative, this means that there is an $\varepsilon > 0$ arbitrarily small such that $[f(\bar{x} + \varepsilon e^n) - f(\bar{x})]/\varepsilon > a/2 > 0$ and $[f(\bar{x} - \varepsilon e^n) - f(\bar{x})]/\varepsilon < -a/2$. Thus, $f(\bar{x} - \varepsilon e^n) < f(\bar{x}) < f(\bar{x} + \varepsilon e^n)$. In words: The function $f(\cdot)$ is locally increasing around $\bar{x}$ in the direction of the nth coordinate axis. But then $\bar{x}$ can be neither a local maximizer nor a local minimizer of $f(\cdot)$. Contradiction. ∎

The conclusion of Theorem M.J.1 can be seen in Figure M.J.1: In the figure, we have $\partial f(\bar{x})/\partial x = 0$ and $\partial f(\underline{x})/\partial x = 0$.

A vector $\bar{x} \in \mathbb{R}^N$ such that $\nabla f(\bar{x}) = 0$ is called a *critical point*. By Theorem M.J.1, every local maximizer or local minimizer is a critical point. The converse, however,

20. An open neighborhood of $\bar{x}$ is an open set that includes $\bar{x}$.

does not hold. Consider for example, the function $f(x_1, x_2) = (x_1)^2 - (x_2)^2$ defined on $\mathbb{R}^2$. At the origin we have $\nabla f(0,0) = (0,0)$. Thus, the origin is a critical point, but it is neither a local maximizer nor a local minimizer of this function. To characterize local maximizers and local minimizers of $f(\cdot)$ more completely, we must look at second-order conditions.

Theorem M.J.2: Suppose that the function $f\colon \mathbb{R}^N \to \mathbb{R}$ is twice continuously differentiable and that $f(\bar{x}) = 0$.

(i) If $\bar{x} \in \mathbb{R}^N$ is a local maximizer, then the (symmetric) $N \times N$ matrix $D^2 f(\bar{x})$ is negative semidefinite.

(ii) If $D^2 f(\bar{x})$ is negative definite, then $\bar{x}$ is a local maximizer.

Replacing "negative" by "positive," the same is true for local minimizers.

Proof: The idea is as follows. For an arbitrary direction of displacement $z \in \mathbb{R}^N$ and scalar ε, a Taylor's expansion of the function $\phi(\varepsilon) = f(\bar{x} + \varepsilon z)$ around $\varepsilon = 0$ gives

$$\begin{aligned} f(\bar{x} + \varepsilon z) - f(\bar{x}) &= \varepsilon \nabla f(\bar{x}) \cdot z + \tfrac{1}{2}\varepsilon^2 z \cdot D^2 f(\bar{x}) z + \text{Remainder} \\ &= \tfrac{1}{2}\varepsilon^2 z \cdot D^2 f(\bar{x}) z + \text{Remainder}, \end{aligned}$$

where $\varepsilon \in \mathbb{R}_+$ and $(1/\varepsilon^2)$ Remainder is small if ε is small. If $\bar{x}$ is a local maximizer, then for ε small we must have $(1/\varepsilon^2)[f(\bar{x} + \varepsilon z) - f(\bar{x})] \le 0$, and so taking limits we get

$$z \cdot D^2 f(\bar{x}) z \le 0.$$

Similarly, if $z \cdot D^2 f(\bar{x}) z < 0$ for any $z \neq 0$, then $f(\bar{x} + \varepsilon z) < f(\bar{x})$ for $\varepsilon > 0$ small, and so $\bar{x}$ is a local maximizer. ■

In the borderline case in which $D^2 f(\bar{x})$ is negative semidefinite but not negative definite, we cannot assert that $\bar{x}$ is a local maximizer. Consider, for example, the function $f(x) = x^3$ whose domain is $\mathbb{R}$. Then $D^2 f(0)$ is negative semidefinite because $d^2 f(0)/dx = 0$, but $\bar{x} = 0$ is neither a local maximizer nor a local minimizer of this function.

Finally, when is a local maximizer $\bar{x}$ of $f(\cdot)$ (or, more generally, a critical point) automatically a global maximizer? Theorem M.J.3 tells us that a sufficient condition is the concavity of the objective function $f(\cdot)$.

Theorem M.J.3: Any critical point $\bar{x}$ of a concave function $f(\cdot)$ [i.e., any $\bar{x}$ satisfying $\nabla f(\bar{x}) = 0$] is a global maximizer of $f(\cdot)$.

Proof: Recall from Theorem M.C.1 that for a concave function we have $f(x) \le f(\bar{x}) + \nabla f(\bar{x}) \cdot (x - \bar{x})$ for every x in the domain of the function. Since $\nabla f(\bar{x}) = 0$, this tells us that $\bar{x}$ is a global maximizer. ■

By analogous reasoning, any critical point of a *convex* function $f(\cdot)$ is a global *minimizer* of $f(\cdot)$.[21]

21. In fact, this follows directly from Theorem M.J.3 because $\bar{x}$ is a global minimizer of $f(\cdot)$ if and only if it is a global maximizer of $-f(\cdot)$, and $-f(\cdot)$ is concave if and only if $f(\cdot)$ is convex.

M.K Constrained Maximization

We start by considering the problem of maximizing a function $f(\cdot)$ under M equality constraints. Namely, we study the problem

$$\begin{aligned} \underset{x \in \mathbb{R}^N}{\text{Max}} \quad & f(x) \\ \text{s.t. } & g_1(x) = \bar{b}_1 \\ & \vdots \\ & g_M(x) = \bar{b}_M, \end{aligned} \tag{M.K.1}$$

where the functions $f(\cdot), g_1(\cdot), \ldots, g_M(\cdot)$ are defined on $\mathbb{R}^N$ (or, more generally, on an open set $A \subset \mathbb{R}^N$). We assume that $N \geq M$; if $M \geq N$ there will generally be no points satisfying all of the constraints.

The set of all $x \in \mathbb{R}^N$ satisfying the constraints of problem (M.K.1) is denoted

$$C = \{x \in \mathbb{R}^N : g_m(x) = \bar{b}_m \quad \text{for} \quad m = 1, \ldots, M\}$$

and is called the *constraint set*. The definitions of a *local constrained* or a *global constrained maximizer* are parallel to those given in Definition M.J.1, except that we now consider only points x that belong to the constraint set C. The feasible point $\bar{x} \in C$ is a *local constrained maximizer* in problem (M.K.1) if there exists an open neighborhood of $\bar{x}$, say $A \subset \mathbb{R}^N$, such that $f(\bar{x}) \geq f(x)$ for all $x \in A \cap C$, that is, if $\bar{x}$ solves problem (M.K.1) when we replace the condition $x \in \mathbb{R}^N$ by $x \in A$. The point $\bar{x}$ is a *global constrained maximizer* if it solves problem (M.K.1), that is, if $f(\bar{x}) \geq f(x)$ for all $x \in C$.

Our first result (Theorem M.K.1) states the first-order conditions for this constrained maximization problem.

Theorem M.K.1: Suppose that the objective and constraint functions of problem (M.K.1) are differentiable and that $\bar{x} \in C$ is a local constrained maximizer. Assume also that the $M \times N$ matrix

$$\begin{bmatrix} \dfrac{\partial g_1(\bar{x})}{\partial x_1} & \cdots & \dfrac{\partial g_1(\bar{x})}{\partial x_N} \\ & \ddots & \\ \dfrac{\partial g_M(\bar{x})}{\partial x_1} & \cdots & \dfrac{\partial g_M(\bar{x})}{\partial x_N} \end{bmatrix}$$

has rank M. (This is called the *constraint qualification*: It says that the constraints are independent at $\bar{x}$.) Then there are numbers $\lambda_m \in \mathbb{R}$, one for each constraint, such that

$$\frac{\partial f(\bar{x})}{\partial x_n} = \sum_{m=1}^{M} \lambda_m \frac{\partial g_m(\bar{x})}{\partial x_n} \quad \text{for every } n = 1, \ldots, N, \tag{M.K.2}$$

or, in more concise notation [letting $\lambda = (\lambda_1, \ldots, \lambda_M)$],

$$\nabla f(\bar{x}) = \sum_{m=1}^{M} \lambda_m \nabla g_m(\bar{x}). \tag{M.K.3}$$

The numbers λ_m are referred to as *Lagrange multipliers*.

Proof: The role of the constraint qualification is to insure that $\bar{x}$ is also a local maximizer in the linearized problem

$$\begin{aligned}\underset{x \in \mathbb{R}^N}{\text{Max}} \quad & f(\bar{x}) + \nabla f(\bar{x}) \cdot (x - \bar{x}) \\ \text{s.t. } & \nabla g_1(\bar{x}) \cdot (x - \bar{x}) = 0 \\ & \vdots \\ & \nabla g_M(\bar{x}) \cdot (x - \bar{x}) = 0,\end{aligned}$$

in which the objective function and constraints have been linearized around the point $\bar{x}$. Thus, the constraint qualification guarantees the correctness of the following intuitively sensible statement: If $\bar{x}$ is a local constrained maximizer, then every direction of displacement $z \in \mathbb{R}^N$ having no first-order effect on the constraints, that is, satisfying $\nabla g_m(\bar{x}) \cdot z = 0$ for every m, must also have no first-order effect on the objective function, that is, must have $\nabla f(\bar{x}) \cdot z = 0$ (see also the discussion after the proof and Figure M.K.1). From now on we assume that this is true.

The rest is just a bit of linear algebra. Let E be the $(M + 1) \times N$ matrix whose first row is $\nabla f(\bar{x})^{\mathrm{T}}$ and whose last M rows are the vectors $\nabla g_1(\bar{x})^{\mathrm{T}}, \ldots, \nabla g_M(\bar{x})^{\mathrm{T}}$. By the implication of the constraint qualification cited above, we have $\{z \in \mathbb{R}^N: \nabla g_m(\bar{x}) \cdot z = 0$ for all $m\} = \{z \in \mathbb{R}^N: Ez = 0\}$. Hence, these two linear spaces have the same dimension M. Therefore, by a basic result of linear algebra, rank $E = M$. Hence, $\nabla f(\bar{x})$ must be a linear combination of the linearly independent set of gradients $\nabla g_1(\bar{x}), \ldots, \nabla g_M(\bar{x})$. This is exactly what (M.K.3) says. ■

In words, Theorem M.K.1 asserts that, at a local constrained maximizer $\bar{x}$, the gradient of the objective function is a linear combination of the gradients of the constraint functions. The indispensability of the constraint qualification is illustrated in Figure M.K.1. In the figure, we wish to maximize a linear function $f(x_1, x_2)$ in the constraint set $C = \{(x_1, x_2) \in \mathbb{R}^2: g_m(x_1, x_2) = \bar{b}_m$ for $m = 1, 2\}$ [the figure shows the loci of points satisfying $g_1(x_1, x_2) = \bar{b}_1$ and $g_2(x_1, x_2) = \bar{b}_2$, as well as the level sets of the function $f(\cdot)$]. While the point $\bar{x}$ is a global constrained maximum (it is the only vector in the constraint set!), we see that $\nabla f(\bar{x})$ is *not* spanned by the vectors $\nabla g_1(\bar{x})$ and $\nabla g_2(\bar{x})$ [i.e., it cannot be expressed as a linear combination of $\nabla g_1(\bar{x})$ and $\nabla g_2(\bar{x})$]. Note, however, that $\nabla g_1(\bar{x}) = -\nabla g_2(\bar{x})$, and so the constraint qualification

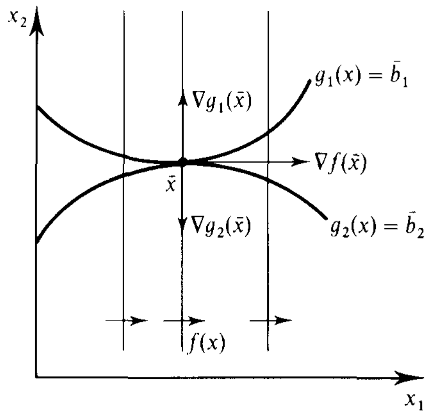


Figure M.K.1
Indispensability of the constraint qualification.

is violated. Observe that, indeed, $\bar{x}$ is *not* a local maximizer of $f(\bar{x}) + \nabla f(\bar{x})\cdot(x - \bar{x})$ [$= f(x)$ since $f(\cdot)$ is linear] in the linearized constraint set

$$C' = \{(x_1, x_2) \in \mathbb{R}^2 \colon \nabla g_m(\bar{x})\cdot(x - \bar{x}) = 0 \quad \text{for} \quad m = 1, 2\}.$$

Often the first-order conditions (M.K.2) or (M.K.3) are presented in a slightly different way. Given variables $x = (x_1, \ldots, x_N)$ and $\lambda = (\lambda_1, \ldots, \lambda_N)$, we can define the *Lagrangian function*

$$L(x, \lambda) = f(x) - \sum_m \lambda_m g_m(x).$$

Note that conditions (M.K.2) [or (M.K.3)] are the (unconstrained) first-order conditions of this function with respect to the variables $x = (x_1, \ldots, x_N)$. Similarly, the constraints $g(x) = 0$ are the first-order conditions of $L(\cdot, \cdot)$ with respect to the variables $\lambda = (\lambda_1, \ldots, \lambda_M)$. In summary, Theorem M.K.1 says that if $\bar{x}$ is a local constrained maximizer (and if the constraint qualification is satisfied), then for some values $\lambda_1, \ldots, \lambda_M$ all of the partial derivatives of the Lagrangian function are null; that is, $\partial L(\bar{x}, \lambda)/\partial x_n = 0$ for $n = 1, \ldots, N$ and $\partial L(\bar{x}, \lambda)/\partial \lambda_m = 0$ for $m = 1, \ldots, M$.

Theorem M.K.1 implies that if $\bar{x}$ is a local maximizer in problem (M.K.1), then the $N + M$ variables $(\bar{x}_1, \ldots, \bar{x}_N, \lambda_1, \ldots, \lambda_M)$ are a solution to the $N + M$ equations formed by (M.K.2) and $g_m(\bar{x}) = \bar{b}_m$ for $m = 1, \ldots, M$.

There is also a second-order theory associated with problem (M.K.1). Suppose that at $\bar{x}$ the constraint qualification is satisfied and that there are Lagrange multipliers satisfying (M.K.3). If $\bar{x}$ is a local maximizer, then

$$D_x^2 L(\bar{x}, \lambda) = D^2 f(\bar{x}) - \sum_m \lambda_m \nabla g_m(\bar{x})$$

is negative semidefinite on the subspace $\{z \in \mathbb{R}^N \colon \nabla g_m(\bar{x})\cdot z = 0 \text{ for all } m\}$. In the other direction, if the vector $\bar{x}$ is feasible (i.e., $\bar{x} \in C$) and satisfies the first-order conditions (M.K.2), and if $D_x^2 L(\bar{x}, \lambda)$ is negative definite on the subspace $\{z \in \mathbb{R}^N \colon \nabla g_m(\bar{x})\cdot z = 0$ for all $m\}$, then $\bar{x}$ is a local maximizer. These conditions can be verified using the determinantal tests provided in Theorem M.D.3.

Finally, note that a local constrained minimizer of $f(\cdot)$ is a local constrained maximizer of $-f(\cdot)$, and therefore Theorem M.K.1 and our discussion of second-order conditions above is also applicable to the characterization of local constrained minimizers.

Inequality Constraints

We now generalize our analysis to problems that may have inequality constraints. The basic problem is therefore now

$$\begin{aligned} \operatorname*{Max}_{x \in \mathbb{R}^N} \quad & f(x) \\ \text{s.t. } & g_1(x) = \bar{b}_1 \\ & \vdots \\ & g_M(x) = \bar{b}_M \\ & h_1(x) \le \bar{c}_1 \\ & \vdots \\ & h_K(x) \le \bar{c}_K, \end{aligned} \tag{M.K.4}$$

where every function is defined on $\mathbb{R}^N$ (or an open set $A \subset \mathbb{R}^N$). We assume that $N \geq M + K$. It is of course possible to have $M = 0$ (no equality constraints) or $K = 0$ (no inequality constraints).

We again denote the constraint set by $C \subset \mathbb{R}^N$, and the meaning of a local constrained maximizer or a global constrained maximizer is unaltered from above.

We now say that the constraint qualification is satisfied at $\bar{x} \in C$ if the constraints *that hold at* $\bar{x}$ *with equality* are independent; that is, if the vectors in $\{\nabla g_m(\bar{x}): m = 1, \ldots, M\} \cup \{\nabla h_k(\bar{x}): h_k(\bar{x}) = \bar{c}_k\}$ are linearly independent.

Theorem M.K.2 presents the first-order conditions for this problem. All of the functions involved are assumed to be differentiable.

Theorem M.K.2: (*Kuhn Tucker Conditions*) Suppose that $\bar{x} \in C$ is a local maximizer of problem (M.K.4). Assume also that the constraint qualification is satisfied. Then there are multipliers $\lambda_m \in \mathbb{R}$, one for each equality constraint, and $\lambda_k \in \mathbb{R}_+$, one for each inequality constraint, such that:[22]

(i) For every $n = 1, \ldots, N$,

$$\frac{\partial f(\bar{x})}{\partial x_n} = \sum_{m=1}^{M} \lambda_m \frac{\partial g_m(\bar{x})}{\partial x_n} + \sum_{k=1}^{K} \lambda_k \frac{\partial h_k(\bar{x})}{\partial x_n}, \tag{M.K.5}$$

or, in more concise notation,

$$\nabla f(\bar{x}) = \sum_{m=1}^{M} \lambda_m \nabla g_m(\bar{x}) + \sum_{k=1}^{K} \lambda_k \nabla h_k(\bar{x}). \tag{M.K.6}$$

(ii) For every $k = 1, \ldots, K$,

$$\lambda_k (h_k(\bar{x}) - \bar{c}_k) = 0, \tag{M.K.7}$$

i.e., $\lambda_k = 0$ for any constraint k that does not hold with equality.

Proof: We illustrate the proof of the result for the case in which there are only inequality constraints (i.e., $M = 0$).

As with the case of equality constraints, the role of the constraint qualification is to insure that $\bar{x}$ remains a local maximizer in the problem linearized around $\bar{x}$. More specifically, we assume from now on that the following is true: Any direction of displacement $z \in \mathbb{R}^N$ that satisfies the constraints to first order [i.e., such that $\nabla h_k(\bar{x}) \cdot z \leq 0$ for every k with $h_k(\bar{x}) = \bar{c}_k$] must not create a first-order increase in the objective function, that is, must have $\nabla f(\bar{x}) \cdot z \leq 0$.

In Figure M.K.2 we represent a problem with two variables and two constraints for which the logic of the result is illustrated and made plausible. The Kuhn–Tucker theorem says that if $\bar{x}$ is a local maximizer then $\nabla f(\bar{x})$ must be in the cone

$$\Gamma = \{y \in \mathbb{R}^2: y = \lambda_1 \nabla h_1(\bar{x}) + \lambda_2 \nabla h_2(\bar{x}) \quad \text{for some} \quad (\lambda_1, \lambda_2) \geq 0\}$$

depicted in the figure; that is, $\nabla f(\bar{x})$ must be a nonnegative linear combination of $\nabla h_1(\bar{x})$ and $\nabla h_2(\bar{x})$. Suppose now that $\bar{x}$ is a local maximizer. If starting from $\bar{x}$ we move along the boundary of the constraint set to any point $(\bar{x}_1 + z_1, \bar{x}_2 + z_2)$ with $z_1 < 0$ and $z_2 > 0$, then in the situation depicted we would have

$$h_1(\bar{x}_1 + z_1, \bar{x}_2 + z_2) = \bar{c}_1, \qquad h_2(\bar{x}_1 + z_1, \bar{x}_2 + z_2) < \bar{c}_2,$$

22. By convention, if there are no constraints (i.e., if $M = K = 0$), then the right-hand side of (M.K.5) is zero.

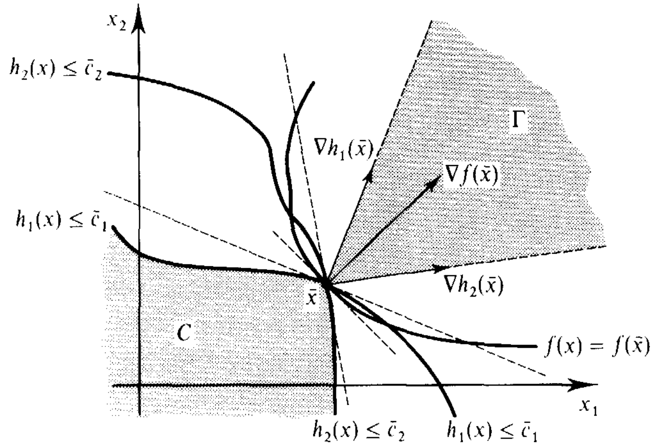


Figure M.K.2
Necessity of the Kuhn Tucker conditions.

and $f(\bar{x}_1 + z_1, \bar{x}_2 + z_2) \le f(\bar{x})$. Taking limits we conclude that in the direction z such that $\nabla h_1(\bar{x})\cdot z = 0$ and $\nabla h_2(\bar{x})\cdot z < 0$, we have $\nabla f(\bar{x})\cdot z \le 0$. Geometrically this means that the vector $\nabla f(\bar{x})$ must lie below the vector $\nabla h_1(\bar{x})$, as is shown in the figure. By similar reasoning (moving along the boundary of the constraint set C in the opposite direction), if $\bar{x}$ is a local constrained maximizer the vector $\nabla f(\bar{x})$ must lie above the vector $\nabla h_2(\bar{x})$. Hence, $\nabla f(\bar{x})$ must lie in the cone Γ. This is precisely what the Kuhn Tucker conditions require in this case.

The above intuition can be extended to the general case. Suppose that all the constraints are binding at $\bar{x}$ (if a constraint k is not binding, put $\lambda_k = 0$ and drop it from the list). We must show that $\nabla f(\bar{x})$ belongs to the convex cone

$$\Gamma = \left\{ y \in \mathbb{R}^N \colon y = \sum_k \lambda_k \nabla h_k(\bar{x}) \quad \text{for some} \quad (\lambda_1, \ldots, \lambda_K) \ge 0 \right\} \subset \mathbb{R}^N.$$

Assume for a moment that this is not so, that is, that $\nabla f(\bar{x}) \notin \Gamma$. Then, by the separating hyperplane theorem (Theorem M.G.2), there exists a nonzero vector $z \in \mathbb{R}^N$ and a number $\beta \in \mathbb{R}$ such that $\nabla f(\bar{x})\cdot z > \beta$ and $y\cdot z < \beta$ for every $y \in \Gamma$. Since $0 \in \Gamma$ we must have $\beta > 0$. Hence, $\nabla f(\bar{x})\cdot z > 0$. Also, for any $y \in \Gamma$ we have $\theta y \in \Gamma$ for all $\theta \ge 0$. But then $\theta y\cdot z < \beta$ can hold for all θ (arbitrarily large) only if $y\cdot z \le 0$. We conclude therefore that $\nabla f(\bar{x})\cdot z > 0$ and $\nabla h_k(\bar{x})\cdot z \le 0$ for all k, which contradicts the linearization implication of the constraint qualification. ∎

It is common in applications that a constraint takes the form of a nonnegativity requirement on some variable x_n; that is, $x_n \ge 0$. In this case, the appropriate first-order conditions require only a small modification of those above. In particular, we need only change the first-order condition for x_n to

$$\frac{\partial f(\bar{x})}{\partial x_n} \le \sum_{m=1}^{M} \lambda_m \frac{\partial g_m(\bar{x})}{\partial x_n} + \sum_{k=1}^{K} \lambda_k \frac{\partial h_m(\bar{x})}{\partial x_n}, \qquad \text{with equality if } \bar{x}_n > 0. \qquad \text{(M.K.8)}$$

To see why this is so, suppose that we explicitly introduced this nonnegativity requirement as our $(K + 1)$th inequality constraint [i.e., $h_{K+1}(x) = -x_n \le 0$] and let $\lambda_{K+1} \ge 0$ be the corresponding multiplier. Note that $\lambda_{K+1}(\partial h_{K+1}(\bar{x})/\partial x_n) = -\lambda_{K+1}$ and $\partial h_{K+1}(\bar{x})/\partial x_{n'} = 0$ for $n' \ne n$. Thus, if we apply condition (M.K.5) of Theorem M.K.2, the only modification to the first-order conditions is that the first-order condition for x_n is now

$$\frac{\partial f(\bar{x})}{\partial x_n} = \sum_{m=1}^{M} \lambda_m \frac{\partial g_m(\bar{x})}{\partial x_n} + \sum_{k=1}^{K} \lambda_k \frac{\partial h_m(\bar{x})}{\partial x_n} - \lambda_{K+1},$$

and we have the added condition that

$$-\lambda_{K+1} x_n = 0.$$

But these two conditions are exactly equivalent to condition (M.K.8). Given the simplicity of the adjustment required to take account of nonnegativity constraints, it is customary in applications not to explicitly introduce the nonnegativity constraint and its associated multiplier, but rather simply to modify the usual first-order conditions as in (M.K.8).

Note also that any constraint of the form $h_k(x) \geq \bar{c}_k$ can be written as $-h_k(x) \leq -\bar{c}_k$. Using this fact, we see that Theorem M.K.2 extends to constraints of the form $h_k(x) \geq \bar{c}_k$. The only modification is that the sign restriction on the multiplier of constraint k is now $\lambda_k \leq 0$. Similarly, because minimizing the function $f(\cdot)$ is equivalent to maximizing the function $-f(\cdot)$, Theorem M.K.2 applies also to local minimizers, with the only change being that the sign restriction on all of the multipliers is now $(\lambda_1, \ldots, \lambda_M) \leq 0$ [assuming that the constraints are all still written as in problem (M.K.4)].

The second-order conditions for the inequality problem (M.K.4) are exactly the same as those already mentioned for the equality problem (M.K.1). The only adjustment is that the constraints that count are those that bind, that is, those that hold with equality at the point $\bar{x}$ under consideration.

Suppose that a vector $x \in C$ satisfies the Kuhn–Tucker conditions, that is, conditions (i) and (ii) in Theorem M.K.2. When can we say that x is a global maximizer? Theorem M.K.3 gives a useful set of conditions.

Theorem M.K.3: Suppose that there are no equality constraints (i.e., $M = 0$) and that every inequality constraint k is given by a quasiconvex function $h_k(\cdot)$.[23] Suppose also that the objective function satisfies

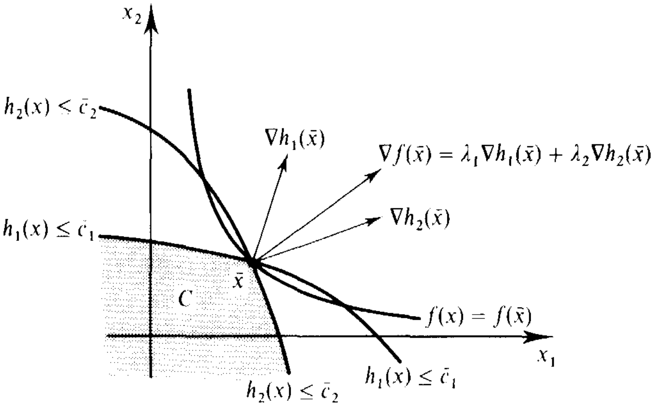
$$\nabla f(x) \cdot (x' - x) > 0 \qquad \text{for any } x \text{ and } x' \text{ with } f(x') > f(x). \qquad \text{(M.K.9)}$$

Then if $\bar{x} \in C$ satisfies the Kuhn–Tucker conditions [conditions (i) and (ii) of Theorem M.K.2], and if the constraint qualification holds at $\bar{x}$, it follows that $\bar{x}$ is a global maximizer.[24]

Proof: Suppose that this is not so, that is, that $f(x) > f(\bar{x})$ for some $x \in \mathbb{R}^N$ satisfying $h_k(x) \leq \bar{c}_k$ for every k. Denote $z = x - \bar{x}$. Then, by (M.K.9) we have $\nabla f(\bar{x}) \cdot z > 0$. If $\lambda_k > 0$, then the Kuhn–Tucker conditions imply that $h_k(\bar{x}) = \bar{c}_k$. Moreover, since $h_k(\cdot)$ is quasiconvex and $h_k(x) \leq \bar{c}_k = h_k(\bar{x})$, it follows that $\nabla h_k(\bar{x}) \cdot z \leq 0$. Hence, we have both $\nabla f(\bar{x}) \cdot z > 0$ and $\sum_k \lambda_k \nabla h_k(\bar{x}) \cdot z \leq 0$, which contradicts the Kuhn–Tucker conditions because these require that $\nabla f(\bar{x}) = \sum_k \lambda_k \nabla h_k(\bar{x})$. ■

23. More generally, equality constraints are permissible if they are linear.

24. If instead we have $\nabla f(x) \cdot (x' - x) < 0$ whenever $f(x') < f(x)$ and the multipliers have the nonpositive sign that corresponds to a minimization problem, then $\bar{x}$ is a global minimizer.



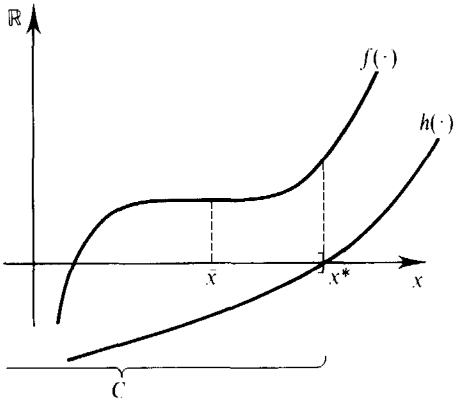


Figure M.K.3 (left) With quasiconvex constraint functions and a quasiconcave objective function satisfying $\nabla f(x) \neq 0$ for all x, satisfaction of the Kuhn–Tucker conditions at $\bar{x}$ implies that $\bar{x}$ is a global constrained maximizer.

Figure M.K.4 (right) The necessity of condition (M.K.10) for Theorem M.K.3.

Note that condition (M.K.9) of Theorem M.K.3 is satisfied if $f(\cdot)$ is concave or if $f(\cdot)$ is quasiconcave and $\nabla f(x) \neq 0$ for all $x \in \mathbb{R}^N$. The condition that the constraint functions $h_1(\cdot), \ldots, h_K(\cdot)$ are quasiconvex implies that the constraint set C is convex (check this).[25] In Figure M.K.3, we illustrate the theorem for a case in which $N = K = 2$, $M = 0$, and $f(\cdot)$ is a quasiconcave function with $\nabla f(x) \neq 0$ for all x.

The indispensability of condition (M.K.9) in Theorem M.K.3 is shown in Figure M.K.4. There we have $N = M = 1$, and the quasiconcave function $f(\cdot)$ is being maximized on the constraint set $C = \{x \in \mathbb{R}: h(x) \le 0\}$. In the figure, the point $\bar{x}$ satisfies the Kuhn–Tucker conditions for a multiplier value of $\lambda = 0$, but $\bar{x}$ is not a global maximizer of $f(\cdot)$ on C (the point x^* is the global constrained maximizer). Note, however, that condition (M.K.10) is violated when $x = \bar{x}$ and $x' = x^*$.

We observe finally in Theorem M.K.4 an important implication of the constraint set C being convex and the objective function $f(\cdot)$ being strictly quasiconcave.

Theorem M.K.4: Suppose that in problem (M.K.4) the constraint set C is convex and the objective function $f(\cdot)$ is strictly quasiconcave. Then there is a unique global constrained maximizer.[26]

Proof: If x and $x' \neq x$ were both global constrained maximizers, then the point $x'' = \alpha x + (1 - \alpha)x'$ for $\alpha \in (0, 1)$ would be feasible (i.e., $x'' \in C$) and by the strict quasiconcavity of $f(\cdot)$, would yield a higher value of $f(\cdot)$ [i.e., $f(x'') > f(x) = f(x')$]. ∎

Suppose that in the case in which only inequality constraints are present we denote by C_{-k} the relaxed constraint set arising when the kth inequality constraint is dropped. The following two facts are often useful in applications.

(i) If $f(\bar{x}) \ge f(x)$ for all $x \in C_{-k}$ and if $h_k(\bar{x}) \le \bar{c}_k$, then $\bar{x}$ is a global constrained maximizer in problem (M.K.4). That is, if we solve a constrained optimization problem ignoring a constraint, and the solution we obtain satisfies this omitted constraint as well, then it must be a solution to the fully constrained problem. This follows simply from the fact that $C \subset C_{-k}$, and so optimizing $f(\cdot)$ on C can at best yield the same value of $f(\cdot)$ as optimizing it on C_{-k}.

25. Also, under the conditions of the theorem, a sufficient constraint qualification is that the constraint set C should have a nonempty interior.

26. When $f(\cdot)$ is quasiconcave, but not strictly so, a similar argument allows us to conclude that the set of maximizers is convex.

(ii) Suppose that all of the constraint functions $h_1(\cdot), \ldots, h_K(\cdot)$ are continuous and quasiconvex and that condition (M.K.9) holds. Then if $\bar{x}$ is a solution to problem (M.K.4) in which the kth constraint is not binding [i.e., if $h_k(\bar{x}) < \bar{c}_k$], we have $f(\bar{x}) \geq f(x)$ for all $x \in C_{-k}$. That is, under the stated assumptions, if a constraint is not binding at a solution to problem (M.K.4), then ignoring it altogether should have no effect on the solution. To see this, suppose otherwise; i.e., that there is an $x' \in C_{-k}$ such that $f(x') > f(\bar{x})$. Then because the constraint functions $h_1(\cdot), \ldots, h_K(\cdot)$ are quasiconvex, we know that the point $x(\alpha) = \alpha x' + (1 - \alpha)\bar{x}$ is an element of C_{-k} for all $\alpha \in [0, 1]$. Moreover, since the kth constraint is not binding at $\bar{x}$, there is an $\bar{\alpha} > 0$ such that $h_k(x(\alpha)) < \bar{c}_k$ for all $\alpha < \bar{\alpha}$. Hence, $x(\alpha) \in C$ for all $\alpha < \bar{\alpha}$. But the derivative of $f(x(\alpha))$ at $\alpha = 0$ is $\nabla f(x) \cdot (x' - \bar{x}) > 0$ [recall that condition (M.K.9) holds and that, by assumption $f(x') > f(\bar{x})$]. Therefore, there must be a point $x(\alpha) \in C$ such that $f(x(\alpha)) > f(\bar{x})$ a contradiction to $\bar{x}$ being a global constrained maximizer in problem (M.K.4).

Comparative Statics

In our previous discussion we have treated the parameters $\bar{b} = (\bar{b}_1, \ldots, \bar{b}_M)$ and $\bar{c} = (\bar{c}_1, \ldots, \bar{c}_K)$ of problem M.K.4 as given. We will now let them vary.

Suppose that $(b, c) \in \mathbb{R}^{M+K}$ are parameters for which problem (M.K.4) has some solution $\bar{x}(b, c)$ and denote the value of this solution by $v(b, c) = f(\bar{x}(b, c))$. Under fairly general conditions (see the small type at the end of this section), the value $v(b, c)$ depends continuously on the parameters (b, c).

Theorem M.K.5 provides an interpretation for the Lagrange multipliers as the "shadow prices" of the constraints.

Theorem M.K.5: Suppose that in an open neighborhood of $(\bar{b}, \bar{c})$ the set of binding constraints remains unaltered and that $v(b, c)$ is differentiable at $(\bar{b}, \bar{c})$.[27] Then for every $m = 1, \ldots, M$ and $k = 1, \ldots, K$ we have

$$\frac{\partial v(\bar{b}, \bar{c})}{\partial b_m} = \lambda_m \quad \text{and} \quad \frac{\partial v(\bar{b}, \bar{c})}{\partial c_k} = \lambda_k.$$

Proof: This is a particular case of the envelope theorem (Theorem M.L.1) to be presented in the next section. ■

Consider a more general optimization problem. We maximize a function $f: \mathbb{R}^N \to \mathbb{R}$ subject to $x \in C(q)$ where $C(q)$ is a nonempty constraint set and $q = (q, \ldots, q_S)$ belongs to an admissible set of parameters $Q \subset \mathbb{R}^S$. Suppose that $f(\cdot)$ is continuous and that $C(q)$ is compact for every $q \in Q$. Then we know [from Theorem M.F.2, part (ii)] that the maximum problem has at least one solution. Denote by $x(q) \subset C(q)$ the *set* of solutions corresponding to q and by $v(q)$ [$= f(x)$ for any $x \in x(q)$] the associated maximum value. Theorem M.K.6 concerns the continuity of $x(\cdot)$ and $v(\cdot)$.

Theorem M.K.6: (*Theorem of the Maximum*) Suppose that the constraint correspondence $C: Q \to \mathbb{R}^N$ is continuous (see Section M.H) and that $f(\cdot)$ is a continuous function. Then the maximizer correspondence $x: Q \to \mathbb{R}^N$ is upper hemicontinuous and the value function $v: Q \to \mathbb{R}$ is continuous.

27. These are simplifying assumptions; a similar result holds more generally but requires the use of directional derivatives at points of nondifferentiability of the function $v(\cdot, \cdot)$.

The result cannot be improved upon. Suppose that we maximize $x_1 + x_2$ subject to $x_1 \in [0,1]$, $x_2 \in [0,1]$, and $q_1 x_1 + q_2 x_2 \leq q_1 q_2$ for $q = (q_1, q_2) \in Q = (0,1)^2$. Then the maximizer correspondence is given by

$$x(q) = \{(q_2, 0)\} \qquad \text{if } q_1 < q_2,$$

$$x(q) = \{(0, q_1)\} \qquad \text{if } q_2 < q_1,$$

and

$$x(q) = \{(x_1, x_2) \in [0,1]^2 \colon x_1 + x_2 = q_1\} \qquad \text{if } q_1 = q_2.$$

Both the objective function and the constraint correspondence are continuous (you should check the latter). In accordance with Theorem M.K.6, $x(\cdot)$ is upper hemicontinuous. But it is not continuous (there is an explosion along the line $q_1 = q_2$). On the other hand, suppose that we take $Q = [0,1]^2$. Then the conclusion of the theorem fails: the maximizer correspondence is not upper hemicontinuous [we have $x(2\varepsilon, \varepsilon) = \{(0, 2\varepsilon)\}$, but $x(0,0) = \{(1,1)\}$]. However, the assumptions also fail: at $q = (0,0)$ the vector $(1,1)$ belongs to the constraint set, but at $q = (\varepsilon, \varepsilon)$ no vector x with $x_1 + x_2 > \varepsilon$ belongs to the constraint set. Hence the constraint correspondence is not continuous once extended to $Q = [0,1]^2$.

M.L The Envelope Theorem

In this section, we return to the problem of maximizing a function $f(\cdot)$ under constraints, but we suppose that we want to keep track of some parameters $q = (q_1, \ldots, q_S) \in \mathbb{R}^S$ entering the objective function or the constraints of the problem. In particular, we now write the maximization problem as

$$\begin{aligned} \operatorname*{Max}_{x \in \mathbb{R}^N} \quad & f(x; q) \\ \text{s.t. } & g_1(x; q) = \bar{b}_1 \\ & \vdots \\ & g_M(x; q) = \bar{b}_M. \end{aligned} \tag{M.L.1}$$

We denote by $v(\cdot)$ the *value function* of problem (M.L.1); that is, $v(q)$ is the value attained by $f(\cdot)$ at a solution to problem (M.L.1) when the parameter vector is q. To be specific, we suppose that $v(q)$ is well-defined in the neighborhood of some reference parameter vector $\bar{q} \in \mathbb{R}^S$. It is then natural to investigate the marginal effects of changes in q on the value $v(q)$. The *envelope theorem* addresses this matter.[28]

It will be convenient from now on to assume that, at least locally (i.e., for values of q close to $\bar{q}$), the solution to problem (M.L.1) is a (differentiable) function $x(q)$. We can then write $v(q) = f(x(q); q)$.

To start with the simplest case, suppose that there is a single variable and a single parameter (i.e., $N = K = 1$) and that there are no constraints (i.e., $M = 0$). Then, by the chain rule,

$$\frac{dv(\bar{q})}{dq} = \frac{\partial f(x(\bar{q}); \bar{q})}{\partial q} + \frac{\partial f(x(\bar{q}); \bar{q})}{\partial x} \frac{dx(\bar{q})}{dq}. \tag{M.L.2}$$

28. Formally, we are presenting a case with equality constraints. But note that as long as in a neighborhood of the parameter vector under consideration the set of binding constraints does not change, the discussion applies automatically to the case with inequality constraints.

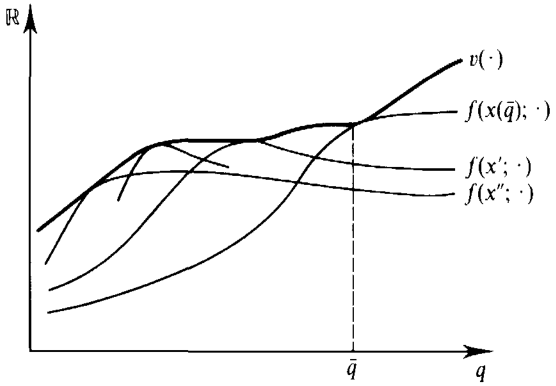


Figure M.L.1
The envelope theorem.

But note, and this is a key observation, that by the first-order conditions for unconstrained maximization (see Section M.J), we must have $\partial f(x(\bar{q}); \bar{q})/\partial x = 0$. Therefore, (M.L.2) simplifies to

$$\frac{dv(\bar{q})}{dq} = \frac{\partial f(x(\bar{q}), \bar{q})}{\partial q}. \tag{M.L.3}$$

That is, the fact that $x(q)$ is determined by maximizing the function $f(\cdot; q)$ has the implication that in computing the first-order effects of changes in q on the maximum value, we can equally well assume that the maximizer will not adjust: The only effect of any consequence is the direct effect.

This result is illustrated in Figure M.L.1, which also motivates the use of the term "envelope." In the figure we represent the function $f(x; \cdot)$ for different values of x. Because at every q we have $v(q) = \text{Max}_x f(x; q)$, the value function $v(\cdot)$ is given by the upper envelope of these functions. Suppose now that we consider a fixed $\bar{q}$. Then, denoting $\bar{x} = x(\bar{q})$, we have $f(\bar{x}; q) \leq v(q)$ for all q, and $f(\bar{x}; \bar{q}) = v(\bar{q})$. Hence, the graph of $f(\bar{x}; \cdot)$ lies weakly below the graph of $v(\cdot)$ and touches it when $q = \bar{q}$. So the two graphs have the same slope at that point. This is precisely what condition (M.L.3) says.

We now state the general envelope theorem for a problem with any number of variables, parameters, and constraints. As we will see, its conclusion is similar to (M.L.3), except that Lagrange multipliers play an important role.

Theorem M.L.1: (*Envelope Theorem*) Consider the value function $v(q)$ for the problem (M.L.1). Assume that it is differentiable at $\bar{q} \in \mathbb{R}^S$ and that $(\lambda_1, \ldots, \lambda_M)$ are values of the Lagrange multipliers associated with the maximizer solution $x(\bar{q})$ at $\bar{q}$. Then[29]

$$\frac{\partial v(\bar{q})}{\partial q_s} = \frac{\partial f(x(\bar{q}); \bar{q})}{\partial q_s} - \sum_{m=1}^{M} \lambda_m \frac{\partial g_m(x(\bar{q}); \bar{q})}{\partial q_s} \quad \text{for } s = 1, \ldots, S, \tag{M.L.4}$$

29. If we have a case with inequality constraints in which the set of binding constraints remains unaltered in a neighborhood of $\bar{q}$, then expressions (M.L.4) and (M.L.5) are still valid: Accounting for the nonbinding constraints will have no effect either on the left-hand side or on the right-hand side (because its associated multipliers are zero).

or, in matrix notation,

$$\nabla v(\bar{q}) = \nabla_q f(x(\bar{q}); \bar{q}) - \sum_{m=1}^{M} \lambda_m \nabla_q g_m(x(\bar{q}); \bar{q}). \qquad \text{(M.L.5)}$$

Proof: We proceed as in the case of a single variable and no constraints. Let $x(\cdot)$ be the maximizer function. Then $v(q) = f(x(q); q)$ for all q, and therefore, using the chain rule, we have

$$\frac{\partial v(\bar{q})}{\partial q_s} = \frac{\partial f(x(\bar{q}); \bar{q})}{\partial q_s} + \sum_{n=1}^{N} \left(\frac{\partial f(x(\bar{q}); \bar{q})}{\partial x_n} \frac{\partial x_n(\bar{q})}{\partial q_s} \right).$$

The first-order conditions (M.K.2) tell us that

$$\frac{\partial f(x(\bar{q}); \bar{q})}{\partial x_n} + \sum_{m=1}^{M} \lambda_m \frac{\partial g_m(x(\bar{q}); \bar{q})}{\partial x_n}.$$

Hence (switching the order of summation as we go),

$$\frac{\partial v(\bar{q})}{\partial q_s} = \frac{\partial f(x(\bar{q}); \bar{q})}{\partial q_s} + \sum_{m=1}^{M} \lambda_m \sum_{n=1}^{N} \left(\frac{\partial g_m(x(\bar{q}); \bar{q})}{\partial x_n} \frac{\partial x_n(\bar{q})}{\partial q_s} \right).$$

Moreover, since $g_m(x(q); q) = \bar{b}_m$ for all q, we have

$$\sum_{n=1}^{N} \left(\frac{\partial g_m(x(\bar{q}); \bar{q})}{\partial x_n} \frac{\partial x_n(\bar{q})}{\partial q_s} \right) = -\frac{\partial g_m(\bar{x}; \bar{q})}{\partial q_s} \qquad \text{for all } m = 1, \dots, M.$$

Combining, we get (M.L.4). ■

M.M Linear Programming

Linear programming problems constitute the special cases of constrained maximization problems for which both the constraints and the objective function are linear in the variables $(x_1, \dots, x_N)$.

A general linear programming problem is typically written in the form

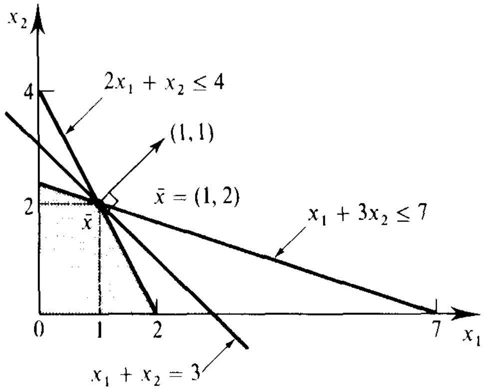
$$\begin{aligned} \underset{(x_1,\dots,x_N)\geq 0}{\text{Max}} \quad & f_1 x_1 + \cdots + f_N x_N \\ \text{s.t. } & a_{11}x_1 + \cdots + a_{1N}x_N \leq c_1 \\ & \vdots \\ & a_{K1}x_1 + \cdots + a_{KN}x_N \leq c_K, \end{aligned} \qquad \text{(M.M.1)}$$

or, in matrix notation,

$$\begin{aligned} \underset{x \in \mathbb{R}^N_+}{\text{Max}} \quad & f \cdot x \\ \text{s.t. } & Ax \leq c, \end{aligned}$$

where A is the $K \times N$ matrix with generic entry a_{kn}, and $f \in \mathbb{R}^N$, $x \in \mathbb{R}^N$, and $c \in \mathbb{R}^K$ are (column) vectors.[30]

30. We say that this is the general form of the linear programming problem because, first, an equality constraint $a \cdot x = b$ can always be expressed as two inequality constraints ($a \cdot x \leq b$ and $-a \cdot x \geq b$) and, second, a variable x_n that is unrestricted in sign can always be replaced by the difference of two variables $(x_{n+} - x_{n-})$, each restricted to be nonnegative.



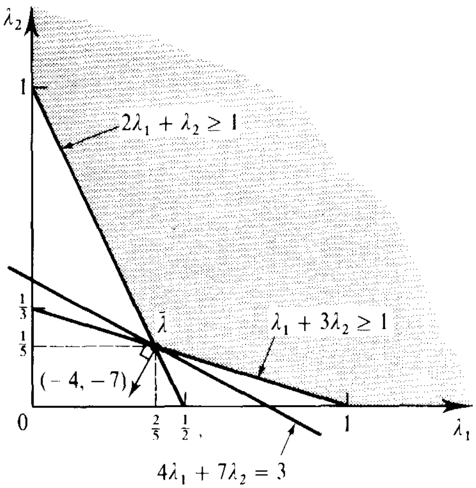


Figure M.M.1 (left)
A linear programming problem (the primal).

Figure M.M.2 (right)
A linear programming problem (the dual).

Figure M.M.1 represents a linear programming problem with $N = 2$, the two constraints $2x_1 + x_2 \le 4$ and $x_1 + 3x_2 \le 7$, and the objective function $x_1 + x_2$.

A most interesting fact about the linear programming problem (M.M.1) is that with it we can associate another linear programming problem, called the *dual* problem, that has the form of a minimization problem with K variables (one for each constraint of the original, or *primal*, problem) and N constraints (one for each variable of the primal problem):

$$\begin{aligned} \min_{(\lambda_1,\ldots,\lambda_K)\ge 0} \quad & c_1\lambda_1 + \cdots + c_K\lambda_K \\ \text{s.t. } & a_{11}\lambda_1 + \cdots + a_{K1}\lambda_K \ge f_1 \\ & \vdots \\ & a_{1N}\lambda_1 + \cdots + a_{KN}\lambda_K \ge f_N, \end{aligned} \qquad \text{(M.M.2)}$$

or, in matrix notation,

$$\begin{aligned} \max_{\lambda \in \mathbb{R}^K_+} \quad & c\cdot\lambda \\ \text{s.t. } & A^{\mathrm{T}}\lambda \ge f, \end{aligned}$$

where $\lambda \in \mathbb{R}^K$ is a column vector.

Figure M.M.2 represents the dual problem associated with Figure M.M.1. The constraints are now $2\lambda_1 + \lambda_2 \ge 1$ and $\lambda_1 + 3\lambda_2 \ge 1$, and the objective function is now $4\lambda_1 + 7\lambda_2$.

Suppose that $x \in \mathbb{R}^N_+$ and $\lambda \in \mathbb{R}^K_+$ satisfy, respectively, the constraints of the primal and the dual problems. Then

$$f\cdot x \le (A^{\mathrm{T}}\lambda)\cdot x = \lambda\cdot(Ax) \le \lambda\cdot c = c\cdot\lambda. \qquad \text{(M.M.3)}$$

Thus, the solution value to the primal problem can be no larger than the solution value to the dual problem. The *duality theorem of linear programming*, now to be stated, says that these values are actually equal. The key for an understanding of this fact is that, as the notation suggests, the dual variables $(\lambda_1, \ldots, \lambda_K)$ have the interpretation of Lagrange multipliers.

Theorem M.M.1: (*Duality Theorem of Linear Programming*) Suppose that the primal problem (M.M.1) attains a maximum value $v \in \mathbb{R}$. Then v is also the minimum value attained by the dual problem (M.M.2).

Proof: Let $\bar{x} \in \mathbb{R}^N$ be a maximizer vector for problem (M.M.1). Denote by $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_K) \geq 0$ the Lagrange multipliers associated with this problem (see Theorem M.K.2).[31] Formally, we regard $\bar{\lambda}$ as a column vector. Then, applying Theorem M.K.2, we have

$$A^{\mathrm{T}}\bar{\lambda} = f \qquad \text{and} \qquad \bar{\lambda}\cdot(c - A\bar{x}) = 0.$$

Hence, $\bar{\lambda}$ satisfies the constraints of the dual problem (since $A^{\mathrm{T}}\bar{\lambda} \geq f$) and

$$c\cdot\bar{\lambda} = \bar{\lambda}\cdot c = \bar{\lambda}\cdot A\bar{x} = (A^{\mathrm{T}}\bar{\lambda})\cdot\bar{x} = f\cdot\bar{x}. \tag{M.M.4}$$

Now, by (M.M.3), we know that $c\cdot\lambda \geq f\cdot\bar{x}$ for all $\lambda \in \mathbb{R}^K_+$ such that $A^{\mathrm{T}}\lambda \geq f$. Therefore $c\cdot\bar{\lambda} \leq c\cdot\lambda$ if $A^{\mathrm{T}}\lambda \geq f$. So (M.M.4) tell us that, in fact, $\bar{\lambda}$ solves the dual problem (M.M.2) and therefore the value of the dual problem, $c\cdot\bar{\lambda}$, equals $f\cdot\bar{x}$, the value of the primal problem. ■

We can verify the duality theorem for the primal and dual problems of Figures M.M.1 and M.M.2. The maximizer vector for the primal problem is $\bar{x} = (1, 2)$, yielding a value of $1 + 2 = 3$. The minimizer vector for the dual problem is $\bar{\lambda} = (\frac{2}{5}, \frac{1}{5})$, yielding a value of $4(\frac{2}{5}) + 7(\frac{1}{5}) = \frac{15}{5} = 3$.

M.N Dynamic Programming

Dynamic programming is a technique for the study of maximization problems defined over sequences that extend to an infinite horizon. We consider here only a very particular and simple case of what is a very general theory [an extensive review is contained in Stokey and Lucas with Prescott (1989)].

Suppose that $A \subset \mathbb{R}^N$ is a nonempty, compact, set.[32] Let $u\colon A \times A \to \mathbb{R}$ be a continuous function and let $\delta \in (0, 1)$. Given a vector $z \in A$ (interpreted as the initial condition of the variables $\{x_t\}_{t=0}^{\infty}$), the maximization problem we are now interested in is

$$\begin{aligned} \underset{\{x_t\}_{t=0}^{\infty}}{\text{Max}} \quad & \sum_{t=0}^{\infty} \delta^t u(x_t, x_{t+1}) \\ \text{s.t. } & x_t \in A \text{ for every } t, \\ & x_0 = z. \end{aligned} \tag{M.N.1}$$

It is not mathematically difficult to verify that a maximizer sequence exists for problem (M.N.1) and that, therefore, there is a maximum value $v(z)$. The function $v\colon A \to \mathbb{R}$ is called the *value function* of problem (M.N.1). As with $u(\cdot, \cdot)$ itself, the value function is continuous. If, in addition, A is convex and $u(\cdot, \cdot)$ is concave, then $v(\cdot)$ is also concave.

31. For linear programming problems the constraint qualification is not required. Put another way, the linearity of the constraints is a sufficient form of constraint qualification.

32. The compactness assumption cannot be dispensed with entirely, but it can be much weakened.

It is fairly clear that for every $z \in A$ the value function satisfies the so-called *Bellman equation* (or the *Bellman optimality principle*):

$$v(z) = \underset{z' \in A}{\text{Max}} \quad u(z, z') + \delta v(z').$$

It is perhaps more surprising that, as shown in Theorem M.N.1, the value function is the *only* function that satisfies this equation.

Theorem M.N.1: Suppose that $f: A \to \mathbb{R}$ is a continuous function such that for every $z \in A$ the Bellman equation is satisfied; that is,

$$f(z) = \underset{z' \in A}{\text{Max}} \quad u(z, z') + \delta f(z') \tag{M.N.2}$$

for all $z \in A$. Then the function $f(\cdot)$ coincides with $v(\cdot)$; that is, $f(z) = v(z)$ for every $z \in A$.

Proof: Successively applying (M.N.2) we have that, for every T,

$$\begin{aligned} f(z) = \underset{\{x_t\}_{t=0}^{T}}{\text{Max}} \quad & \sum_{t=0}^{T-1} \delta^t u(x_t, x_{t+1}) + \delta^T f(x_T) \\ \text{s.t. } & x_t \in A \text{ for all } t \leq T, \\ & x_0 = z. \end{aligned}$$

But as $T \to \infty$, the term $\delta^T f(\cdot)$ makes an increasingly negligible contribution to the sum. We conclude therefore that $f(z) = v(z)$. ∎

Theorem M.N.1 suggests a procedure for the computation of the value function. Suppose that for $r = 0$ we start with an arbitrary continuous function $f_0: A \to \mathbb{R}$. Think of $f_0(z')$ as a trial "evaluation" function giving a tentative evaluation of the value of starting with $z' \in A$. Then we can generate a new tentative evaluation function $f_1(\cdot)$ by letting, for every $z \in A$,

$$f_1(z) = \underset{z' \in A}{\text{Max}} \quad u(z, z') + \delta f_0(z').$$

If $f_1(\cdot) = f_0(\cdot)$, then $f_0(\cdot)$ satisfies the Bellman equation and Theorem M.N.1 tells us that, in fact, $f_0(\cdot) = v(\cdot)$. If $f_1(\cdot) \neq f_0(\cdot)$, then $f_0(\cdot)$ was not correct. We could then try again, starting with the new tentative $f_1(\cdot)$. This will give us a function $f_2(\cdot)$, and so on for an entire sequence of functions $\{f_r(\cdot)\}_{r=0}^{\infty}$. Does this take us anywhere? The answer is that it does: For every $z \in A$, we have $f_r(z) \to v(z)$ as $r \to \infty$. That is, as r increases, we approach the correct evaluation of z.

Suppose that the sequence $\{\bar{x}_t\}_{t=0}^{\infty}$ is a sequence (or a *trajectory*) that solves the maximization problem (M.N.1). A fortiori, for every $t \geq 1$, the decisions taken at t must be optimal. Examining the sum in (M.N.1), we see that $\bar{x}_t$ must solve

$$\underset{x_t \in A}{\text{Max}} \quad u(\bar{x}_{t-1}, x_t) + \delta u(x_t, \bar{x}_{t+1}). \tag{M.N.3}$$

Assuming that $\bar{x}_t$ is in the interior of A, (M.N.3) implies that

$$\frac{\partial u(\bar{x}_{t-1}, \bar{x}_t)}{\partial x_{N+n}} + \delta \frac{\partial u(\bar{x}_t, \bar{x}_{t+1})}{\partial x_n} = 0 \tag{M.N.4}$$

for every $n = 1, \ldots, N$.[33] The necessary conditions captured by (M.N.4) are called the *Euler equations* of problem (M.N.1).

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33. Note that the function $u(\cdot, \cdot)$ has $2N$ arguments, the N variables of the initial period and the N variables of the subsequent period. In condition (M.N.4), the variable x_n is the nth component of the initial period, and the variable x_{N+n} is the nth component of the subsequent period.

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