Proposition 6.F.1 gives us a utility representation $\sum_s u_s(x_s)$ for the preferences on state-by-state sure outcomes $(x_1, \dots, x_S) \in \mathbb{R}_+^S$ that has two properties. First, it is additively separable across states. Second, every $u_s(\cdot)$ is a Bernoulli utility function that can be used to evaluate lotteries over money payoffs in state s by means of expected utility. It is because of the second property that risk aversion (defined in exactly the same manner as in Section 6.C) is equivalent to the concavity of each $u_s(\cdot)$.

There is another approach to the extended expected utility representation that rests with the preferences $\succsim$ defined on $\mathbb{R}_+^S$ and does not appeal to preferences defined on a larger space. It is based on the so-called *sure-thing axiom.*

Definition 6.E.4: The preference relation $\succsim$ satisfies the *sure-thing axiom* if, for any subset of states $E \subset S$ (E is called an *event*), whenever $(x_1, \dots, x_S)$ and $(x'_1, \dots, x'_S)$ differ only in the entries corresponding to E (so that $x'_s = x_s$ for $s \notin E$), the preference ordering between $(x_1, \dots, x_S)$ and $(x'_1, \dots, x'_S)$ is independent of the particular (common) payoffs for states not in E. Formally, suppose that $(x_1, \dots, x_S)$, $(x'_1, \dots, x'_S)$, $(\bar{x}_1, \dots, \bar{x}_S)$, and $(\bar{x}'_1, \dots, \bar{x}'_S)$ are such that

$$\text{For all } s \notin E\text{:} \quad x_s = x'_s \quad \text{and} \quad \bar{x}_s = \bar{x}'_s.$$
$$\text{For all } s \in E\text{:} \quad x_s = \bar{x}_s \quad \text{and} \quad x'_s = \bar{x}'_s.$$

Then $(\bar{x}_1, \dots, \bar{x}_S) \succsim (\bar{x}'_1, \dots, \bar{x}'_S)$ if and only if $(x_1, \dots, x_S) \succsim (x'_1, \dots, x'_S)$.

The intuitive content of this axiom is similar to that of the independence axiom. It simply says that if two random variables cannot be distinguished in the complement of E, then the ordering among them can depend only on the values they take on E. In other words, tastes conditional on an event should not depend on what the payoffs would have been in states that have not occurred.

If $\succsim$ admits an extended expected utility representation, the sure-thing axiom holds because then $(x_1, \dots, x_S) \succsim (x'_1, \dots, x'_S)$ if and only if $\sum_s (u_s(x_s) - u_s(x'_s)) \geq 0$, and any term of the sum with $x_s = x'_s$ will cancel. In the other direction we have Proposition 6.E.2.

Proposition 6.E.2: *Suppose that there are at least three states and that the preferences $\succsim$ on $\mathbb{R}_+^S$ are continuous and satisfy the sure-thing axiom. Then $\succsim$ admits an extended expected utility representation.*

Idea of Proof: A complete proof is too advanced to be given in any detail. One wants to show that under the assumptions, preferences admit an additively separable utility representation $\sum_s u_s(x_s)$. This is not easy to show, and it is not a result particularly related to uncertainty. The conditions for the existence of an additively separable utility function for continuous preferences on the positive orthant of a Euclidean space (i.e., the context of Chapter 3) are well understood; as it turns out, they are *formally identical* to the sure-thing axiom (see Exercise 3.G.4). ■

Although the sure-thing axiom yields an extended expected utility representation $\sum_s \pi_s u_s(x_s)$, we would emphasize that randomizations over monetary payoffs in a state s have not been considered in this approach, and therefore we cannot bring the idea of risk aversion to bear on the determination of the properties of $u_s(\cdot)$. Thus, the approach via the extended independence axiom assumes a stronger basic framework (preferences are defined on the set $\mathscr{L}$ rather than on the smaller $\mathbb{R}_+^S$), but it also yields stronger conclusions.

6.F Subjective Probability Theory

Up to this point in the development of the theory, we have been assuming that risk, summarized by means of numerical probabilities, is regarded as an objective fact by the decision maker. But this is rarely true in reality. Individuals make judgments about the chances of uncertain events that are not necessarily expressible in quantitative form. Even when probabilities are mentioned, as sometimes happens when a doctor discusses the likelihood of various outcomes of medical treatment, they are often acknowledged as imprecise *subjective* estimates.

It would be very helpful, both theoretically and practically, if we could assert that choices are made *as if* individuals held probabilistic beliefs. Even better, we would like that well-defined probabilistic beliefs be revealed by choice behavior. This is the intent of *subjective probability theory.* The theory argues that even if states of the world are not associated with recognizable, objective probabilities, consistency-like restrictions on preferences among gambles still imply that decision makers behave *as if* utilities were assigned to outcomes, probabilities were attached to states of nature, and decisions were made by taking expected utilities. Moreover, this rationalization of the decision maker's behavior with an expected utility function can be done uniquely (up to a positive linear transformation for the utility functions). The theory is therefore a far-reaching generalization of expected utility theory. The classical reference for subjective probability theory is Savage (1954), which is very readable but also advanced. It is, however, possible to gain considerable insight into the theory if one is willing to let the analysis be aided by the use of lotteries with objective random outcomes. This is the approach suggested by Anscombe and Aumann (1963), and we will follow it here.

We begin, as in Section 6.E, with a set of states $\{1, \dots, S\}$. The probabilities on $\{1, \dots, S\}$ are not given. In effect, we aim to *deduce* them. As before, a random variable with monetary payoffs is a vector $x = (x_1, \dots, x_S) \in \mathbb{R}^S_+$.[26] We also want to allow for the possibility that the monetary payoffs in a state are not certain but are themselves money lotteries with objective distributions F_s. Thus, our set of risky alternatives, denoted $\mathscr{L}$, is the set of all S-tuples $(F_1, \dots, F_S)$ of distribution functions.

Suppose now that we are given a rational preference relation $\succsim$ on $\mathscr{L}$. We assume that $\succsim$ satisfies the continuity and the extended independence axioms introduced in Section 6.E. Then, by Proposition 6.E.1, we conclude that there are $u_s(\cdot)$ such that any $(x_1, \dots, x_S) \in \mathbb{R}^S_+$ can be evaluated by $\sum_s u_s(x_s)$. In addition, $u_s(\cdot)$ is a Bernoulli utility function for money lotteries in state s.

The existence of the $u_s(\cdot)$ functions does not yet allow us to identify subjective probabilities. Indeed, for *any* $(\pi_1, \dots, \pi_S) \gg 0$, we could define $\tilde{u}_s(\cdot) = (1/\pi_s)u_s(\cdot)$, and we could then evaluate $(x_1, \dots, x_S)$ by $\sum_s \pi_s \tilde{u}_s(x_s)$. What is needed is some way to disentangle utilities from probabilities.

Consider an example. Suppose that a gamble that gives one dollar in state 1 and none in state 2 is preferred to a gamble that gives one dollar in state 2 and none in state 1. Provided *there is no reason to think that the labels of the states have any*

26. To be specific, we consider monetary payoffs here. All the subsequent arguments, however, work with arbitrary sets of outcomes.

particular influence on the value of money, it is then natural to conclude that the decision maker regards state 2 as less likely than state 1.

This example suggests an additional postulate. Preferences over money lotteries within state s should be the same as those within any other state s'; that is, risk attitudes towards money gambles should be the same across states. To formulate such a property, we define the state s preferences $\succsim_s$ on state s lotteries by

$$F_s \succsim_s F'_s \quad \text{if} \quad \int u_s(x_s)\, dF_s(x_s) \geq \int u_s(x_s)\, dF'_s(x_s).$$

Definition 6.F.1: The state preferences $(\succsim_1, \ldots, \succsim_S)$ on state lotteries are *state uniform* if $\succsim_s = \succsim_{s'}$ for any s and s'.

With state uniformity, $u_s(\cdot)$ and $u_{s'}(\cdot)$ can differ only by an increasing linear transformation. Therefore, there is $u(\cdot)$ such that, for all $s = 1, \ldots, S$,

$$u_s(\cdot) = \pi_s u(\cdot) + \beta_s$$

for some $\pi_s > 0$ and β_s. Moreover, because we still represent the same preferences if we divide all π_s and β_s by a common constant, we can normalize the π_s so that $\sum_s \pi_s = 1$. These π_s are going to be our subjective probabilities.

Proposition 6.F.1: (*Subjective Expected Utility Theorem*) Suppose that the preference relation $\succsim$ on $\mathscr{L}$ satisfies the continuity and extended independence axioms. Suppose, in addition, that the derived state preferences are state uniform. Then there are probabilities $(\pi_1, \ldots, \pi_S) \gg 0$ and a utility function $u(\cdot)$ on amounts of money such that for any $(x_1, \ldots, x_S)$ and $(x'_1, \ldots, x'_S)$ we have

$$(x_1, \ldots, x_S) \succsim (x'_1, \ldots, x'_S) \quad \text{if and only if} \quad \sum_s \pi_s u(x_s) \geq \sum_s \pi_s u(x'_s).$$

Moreover, the probabilities are uniquely determined, and the utility function is unique up to origin and scale.

Proof: Existence has already been proven. You are asked to establish uniqueness in Exercise 6.F.1. ■

The practical advantages of the subjective expected utility representation are similar to those of the objective version, which we discussed in Section 6.B, and we will not repeat them here. A major virtue of the theory is that it gives a precise, quantifiable, and operational meaning to uncertainty. It is, indeed, most pleasant to be able to remain in the familiar realm of the probability calculus.

But there are also problems. The plausibility of the axioms cannot be completely dissociated from the complexity of the choice situations. The more complex these become, the more strained even seemingly innocent axioms are, For example, is the completeness axiom reasonable for preferences defined on huge sets of random variables? Or consider the implicit axiom (often those are the most treacherous) that the situation can actually be formalized as indicated by the model. This posits the ability to list all conceivable states of the world (or, at least, a sufficiently disaggregated version of this list). In summary, every difficulty so far raised against our model of the rational consumer (i.e., to transitivity, to completeness, to independence) will apply with increased force to the current model.

There are also difficulties specific to the nonobjective nature of probabilities. We devote Example 6.F.1 to this point.

Example 6.F.1: This example is a variation of the *Ellsberg paradox*.[27] There are two urns, denoted R and H. Each urn contains 100 balls. The balls are either white or black. Urn R contains 49 white balls and 51 black balls. Urn H contains an unspecified assortment of balls. A ball has been randomly picked from each urn. Call them the *R-ball* and the *H-ball*, respectively. The color of these balls has not been disclosed. Now we consider two choice situations. In both experiments, the decision maker must choose either the R-ball or the H-ball. After the choices have been made, the color will be disclosed. In the first choice situation, a prize of 1000 dollars is won if the chosen ball is black. In the second choice situation, the same prize is won if the ball is white. With the information given, most people will choose the R-ball in the first experiment. If the decision is made using subjective probabilities, this should mean that the subjective probability that the H-ball is white is larger than .49. Hence, most people should choose the H-ball in the second experiment. However, it turns out that this does not happen overwhelmingly in actual experiments. The decision maker understands that by choosing the R-ball, he has only a 49% chance of winning. However, this chance is "safe" and well understood. The uncertainties incurred are much less clear if he chooses the H-ball. ■

Knight (1921) proposed distinguishing between *risk* and *uncertainty* according to whether the probabilities are given to us objectively or not. In a sense, the theory of subjective probability nullifies this distinction by reducing all uncertainty to risk through the use of beliefs expressible as probabilities. The Example 6.F.1 suggests that there may be something to the distinction. This is an active area of research [e.g., Bewley (1986) and Gilboa and Schmeidler (1989)].

27. From Ellsberg (1961).

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EXERCISES

6.B.1[A] In text.

6.B.2[A] In text.

6.B.3[B] Show that if the set of outcomes C is finite and the rational preference relation $\succsim$ on the set of lotteries $\mathscr{L}$ satisfies the independence axiom, then there are best and worst lotteries in $\mathscr{L}$. That is, we can find lotteries $\bar{L}$ and $\underline{L}$ such that $\bar{L} \succsim L \succsim \underline{L}$ for all $L \in \mathscr{L}$.

6.B.4[B] The purpose of this exercise is to illustrate how expected utility theory allows us to make consistent decisions when dealing with extremely small probabilities by considering relatively large ones. Suppose that a safety agency is thinking of establishing a criterion under which an area prone to flooding should be evacuated. The probability of flooding is 1%. There are four possible outcomes:

(A) No evacuation is necessary, and none is performed.
(B) An evacuation is performed that is unnecessary.
(C) An evacuation is performed that is necessary.
(D) No evacuation is performed, and a flood causes a disaster.

Suppose that the agency is indifferent between the sure outcome B and the lottery of A with probability p and D with probability $1 - p$, and between the sure outcome C and the lottery of B with probability q and D with probability $1 - q$. Suppose also that it prefers A to D and that $p \in (0, 1)$ and $q \in (0, 1)$. Assume that the conditions of the expected utility theorem are satisfied.

(a) Construct a utility function of the expected utility form for the agency.

(b) Consider two different policy criteria:

Criterion 1: This criterion will result in an evacuation in 90% of the cases in which flooding will occur and an unnecessary evacuation in 10% of the cases in which no flooding occurs.

Criterion 2: This criterion is more conservative. It results in an evacuation in 95% of the cases in which flooding will occur and an unnecessary evacuation in 5% of the cases in which no flooding occurs.

First, derive the probability distributions over the four outcomes under these two criteria. Then, by using the utility function in **(a)**, decide which criterion the agency would prefer.

6.B.5[B] The purpose of this exercise is to show that the Allais paradox is compatible with a weaker version of the independence axiom. We consider the following axiom, known as the *betweenness axiom* [see Dekel (1986)]:

For all L, L' and $\lambda \in (0,1)$, if $L \sim L'$, then $\lambda L + (1-\lambda)L' \sim L$.

Suppose that there are three possible outcomes.

(a) Show that a preference relation on lotteries satisfying the independence axiom also satisfies the betweenness axiom.

(b) Using a simplex representation for lotteries similar to the one in Figure 6.B.1(b), show that if the continuity and betweenness axioms are satisfied, then the indifference curves of a preference relation on lotteries are straight lines. Conversely, show that if the indifference curves are straight lines, then the betweenness axiom is satisfied. Do these straight lines need to be parallel?

(c) Using **(b)**, show that the betweenness axiom is weaker (less restrictive) than the independence axiom.

(d) Using Figure 6.B.7, show that the choices of the Allais paradox are compatible with the betweeness axiom by exhibiting an indifference map satisfying the betweenness axiom that yields the choices of the Allais paradox.

6.B.6$^{\mathrm{B}}$ Prove that the induced utility function $U(\cdot)$ defined in the last paragraph of Section 6.B is convex. Give an example of a set of outcomes and a Bernoulli utility function for which the induced utility function is not linear.

6.B.7$^{\mathrm{A}}$ Consider the following two lotteries:

$$\text{L:} \begin{cases} 200 \text{ dollars with probability } .7. \\ \ \ 0 \text{ dollars with probability } .3. \end{cases}$$

$$\text{L}': \begin{cases} 1200 \text{ dollars with probability } .1. \\ \ \ 0 \text{ dollars with probability } .9. \end{cases}$$

Let x_L and $x_{L'}$ be the sure amounts of money that an individual finds indifferent to L and L'. Show that if his preferences are transitive and monotone, the individual must prefer L to L' if and only if $x_L > x_{L'}$. [*Note*: In actual experiments, however, a preference reversal is often observed in which L is preferred to L' but $x_L < x_{L'}$. See Grether and Plott (1979) for details.]

6.C.1$^{\mathrm{B}}$ Consider the insurance problem studied in Example 6.C.1. Show that if insurance is not actuarially fair (so that $q > \pi$), then the individual will not insure completely.

6.C.2$^{\mathrm{B}}$

(a) Show that if an individual has a Bernoulli utility function $u(\cdot)$ with the quadratic form

$$u(x) = \beta x^2 + \gamma x,$$

then his utility from a distribution is determined by the mean and variance of the distribution and, in fact, by these moments alone. [*Note*: The number β should be taken to be negative in order to get the concavity of $u(\cdot)$. Since $u(\cdot)$ is then decreasing at $x > -\gamma/2\beta$, $u(\cdot)$ is useful only when the distribution cannot take values larger than $-\gamma/2\beta$.]

(b) Suppose that a utility function $U(\cdot)$ over distributions is given by

$$U(F) = (\text{mean of } F) - r(\text{variance of } F),$$

where $r > 0$. Argue that unless the set of possible distributions is further restricted (see, e.g., Exercise 6.C.19), $U(\cdot)$ cannot be compatible with any Bernoulli utility function. Give an example of two lotteries L and L' over the same two amounts of money, say x' and $x'' > x'$, such that L gives a higher probability to x'' than does L' and yet according to $U(\cdot)$, L' is preferred to L.

6.C.3$^{\mathrm{B}}$ Prove that the four conditions of Proposition 6.C.1 are equivalent. [*Hint*: The equivalence of (i), (ii), and (iii) has already been shown. As for (iv), prove that (i) implies (iv) and that (iv) implies $u(\frac{1}{2}x + \frac{1}{2}y) \geq \frac{1}{2}u(x) + \frac{1}{2}u(y)$ for any x and y, which is, in fact, sufficient for (ii).]

6.C.4$^{\mathrm{B}}$ Suppose that there are N risky assets whose returns z_n $(n = 1, \ldots, N)$ per dollar invested are jointly distributed according to the distribution function $F(z_1, \ldots, z_N)$. Assume also that all the returns are nonnegative with probability one. Consider an individual who has a continuous, increasing, and concave Bernoulli utility function $u(\cdot)$ over $\mathbb{R}_+$. Define the utility function $U(\cdot)$ of this investor over $\mathbb{R}^N_+$, the set of all nonnegative portfolios, by

$$U(\alpha_1, \ldots, \alpha_N) = \int u(\alpha_1 z_1 + \cdots + \alpha_N z_N)\, dF(z_1, \ldots, z_N).$$

Prove that $U(\cdot)$ is (a) increasing, (b) concave, and (c) continuous (this is harder).

6.C.5$^{\mathrm{A}}$ Consider a decision maker with utility function $u(\cdot)$ defined over $\mathbb{R}^L_+$, just as in Chapter 3.

(a) Argue that concavity of $u(\cdot)$ can be interpreted as the decision maker exhibiting risk aversion with respect to lotteries whose outcomes are bundles of the L commodities.

(b) Suppose now that a Bernoulli utility function $u(\cdot)$ for wealth is derived from the maximization of a utility function defined over bundles of commodities for each given wealth level w, while prices for those commodities are fixed. Show that, if the utility function for the commodities exhibits risk aversion, then so does the derived Bernoulli utility function for wealth. Interpret.

(c) Argue that the converse of part **(b)** does not need to hold: There are nonconcave functions $u\colon \mathbb{R}^L_+ \to \mathbb{R}$ such that for any price vector the derived Bernoulli utility function on wealth exhibits risk aversion.

6.C.6$^{\mathrm{B}}$ For Proposition 6.C.2:

(a) Prove the equivalence of conditions (ii) and (iii).

(b) Prove the equivalence of conditions (iii) and (v).

6.C.7$^{\mathrm{A}}$ Prove that, in Proposition 6.C.2, condition (iii) implies condition (iv), and (iv) implies (i).

6.C.8$^{\mathrm{A}}$ In text.

6.C.9$^{\mathrm{B}}$ (M. Kimball) The purpose of this problem is to examine the implications of uncertainty and precaution in a simple consumption–savings decision problem.

In a two-period economy, a consumer has first-period initial wealth w. The consumer's utility level is given by

$$u(c_1, c_2) = u(c_1) + v(c_2),$$

where $u(\cdot)$ and $v(\cdot)$ are concave functions and c_1 and c_2 denote consumption levels in the first and the second period, respectively. Denote by x the amount saved by the consumer in the first period (so that $c_1 = w - x$ and $c_2 = x$), and let x_0 be the optimal value of x in this problem.

We now introduce uncertainty in this economy. If the consumer saves an amount x in the first period, his wealth in the second period is given by $x + y$, where y is distributed according to $F(\cdot)$. In what follows, $E[\cdot]$ always denotes the expectation with respect to $F(\cdot)$. Assume that the Bernoulli utility function over realized wealth levels in the two periods (w_1, w_2) is $u(w_1) + v(w_2)$. Hence, the consumer now solves

$$\operatorname*{Max}_x\ u(w - x) + E[v(x + y)].$$

Denote the solution to this problem by x^*.

(a) Show that if $E[v'(x_0 + y)] > v'(x_0)$, then $x^* > x_0$.

(b) Define the *coefficient of absolute prudence* of a utility function $v(\cdot)$ at wealth level x to be $-v'''(x)/v''(x)$. Show that if the coefficient of absolute prudence of a utility function $v_1(\cdot)$ is not larger than the coefficient of absolute prudence of utility function $v_2(\cdot)$ for all levels of wealth, then $E[v_1'(x_0 + y)] > v_1'(x_0)$ implies $E[v_2'(x_0 + y)] > v_2'(x_0)$. What are the implications of this fact in the context of part **(a)**?

(c) Show that if $v'''(\cdot) > 0$, and $E[y] = 0$, then $E[v'(x + y)] > v'(x)$ for all values of x.

(d) Show that if the coefficient of absolute risk aversion of $v(\cdot)$ is decreasing with wealth, then $-v'''(x)/v''(x) > -v''(x)/v'(x)$ for all x, and hence $v'''(\cdot) > 0$.

6.C.10$^{\mathbf{A}}$ Prove the equivalence of conditions (i) through (v) in Proposition 6.C.3. [*Hint*: By letting $u_1(z) = u(w_1 + z)$ and $u_2(z) = u(w_2 + z)$, show that each of the five conditions in Proposition 6.C.3 is equivalent to the counterpart in Proposition 6.C.2.]

6.C.11$^{\mathbf{B}}$ For the model in Example 6.C.2, show that if $r_R(x, u)$ is increasing in x then the proportion of wealth invested in the risky asset $\gamma = \alpha/x$ is decreasing with x. Similarly, if $r_R(x, u)$ is decreasing in x, then $\gamma = \alpha/x$ is increasing in x. [*Hint*: Let $u_1(t) = u(tw_1)$ and $u_2(t) = u(tw_2)$, and use the fact, stated in the analysis of Example 6.C.2, that if one Bernoulli utility function is more risk averse than another, then the optimal level of investment in the risky asset for the first function is smaller than that for the second function. You could also attempt a direct proof using first-order conditions.]

6.C.12$^{\mathbf{B}}$ Let $u\colon \mathbb{R}_+ \to \mathbb{R}$ be a strictly increasing Bernoulli utility function. Show that

(a) $u(\cdot)$ exhibits constant relative risk aversion equal to $\rho \neq 1$ if and only if $u(x) = \beta x^{1-\rho} + \gamma$, where $\beta > 0$ and $\gamma \in \mathbb{R}$.

(b) $u(\cdot)$ exhibits constant relative risk aversion equal to 1 if and only if $u(x) = \beta \ln x + \gamma$, where $\beta > 0$ and $\gamma \in \mathbb{R}$.

(c) $\lim_{\rho \to 1} (x^{1-\rho}/(1 - \rho)) = \ln x$ for all $x > 0$.

6.C.13$^{\mathbf{B}}$ Assume that a firm is risk neutral with respect to profits and that if there is any uncertainty in prices, production decisions are made after the resolution of such uncertainty. Suppose that the firm faces a choice between two alternatives. In the first, prices are uncertain. In the second, prices are nonrandom and equal to the expected price vector in the first alternative. Show that a firm that maximizes expected profits will prefer the first alternative over the second.

6.C.14$^{\mathbf{B}}$ Consider two risk-averse decision makers (i.e., two decision makers with concave Bernoulli utility functions) choosing among monetary lotteries. Define the utility function $u^*(\cdot)$ to be strongly more risk averse than $u(\cdot)$ if and only if there is a positive constant k and a nonincreasing and concave function $v(\cdot)$ such that $u^*(x) = ku(x) + v(x)$ for all x. The monetary amounts are restricted to lie in the interval $[0, r]$.

(a) Show that if $u^*(\cdot)$ is strongly more risk averse than $u(\cdot)$, then $u^*(\cdot)$ is more risk averse than $u(\cdot)$ in the usual Arrow–Pratt sense.

(b) Show that if $u(\cdot)$ is bounded, then there is no $u^*(\cdot)$ other than $u^*(\cdot) = ku(\cdot) + c$, where c is a constant, that is strongly more risk averse than $u(\cdot)$ on the entire interval $[0, +\infty]$. [*Hint*: in this part, disregard the assumption that the monetary amounts are restricted to lie in the interval $[0, r]$.]

(c) Using **(b)**, argue that the concept of a strongly more risk-averse utility function is stronger (i.e., more restrictive) than the Arrow–Pratt concept of a more risk-averse utility function.

6.C.15^{A} Assume that, in a world with uncertainty, there are two assets. The first is a riskless asset that pays 1 dollar. The second pays amounts a and b with probabilities of π and $1 - \pi$, respectively. Denote the demand for the two assets by (x_1, x_2).

Suppose that a decision maker's preferences satisfy the axioms of expected utility theory and that he is a risk averter. The decision maker's wealth is 1, and so are the prices of the assets. Therefore, the decision maker's budget constraint is given by

$$x_1 + x_2 = 1, \quad x_1, x_2 \in [0, 1].$$

(a) Give a simple *necessary* condition (involving a and b only) for the demand for the riskless asset to be strictly positive.

(b) Give a simple *necessary* condition (involving a, b, and π only) for the demand for the risky asset to be strictly positive.

In the next three parts, assume that the conditions obtained in **(a)** and **(b)** are satisfied.

(c) Write down the first-order conditions for utility maximization in this asset demand problem.

(d) Assume that $a < 1$. Show by analyzing the first-order conditions that $dx_1/da \leq 0$.

(e) Which sign do you conjecture for $dx_1/d\pi$? Give an economic interpretation.

(f) Can you prove your conjecture in **(e)** by analyzing the first-order conditions?

6.C.16^{A} An individual has Bernoulli utility function $u(\cdot)$ and initial wealth w. Let lottery L offer a payoff of G with probability p and a payoff of B with probability $1 - p$.

(a) If the individual owns the lottery, what is the minimum price he would sell it for?

(b) If he does not own it, what is the maximum price he would be willing to pay for it?

(c) Are buying and selling prices equal? Give an economic interpretation for your answer. Find conditions on the parameters of the problem under which buying and selling prices are equal.

(d) Let $G = 10$, $B = 5$, $w = 10$, and $u(x) = \sqrt{x}$. Compute the buying and selling prices for this lottery and this utility function.

6.C.17^{B} Assume that an individual faces a two-period portfolio allocation problem. In period $t = 0, 1$, his wealth w_t is to be divided between a safe asset with return R and a risky asset with return x. The initial wealth at period 0 is w_0. Wealth at period $t = 1, 2$ depends on the portfolio α_{t-1} chosen at period $t - 1$ and on the return x_t realized at period t, according to

$$w_t = ((1 - \alpha_{t-1})R + \alpha_{t-1}x_t)w_{t-1}.$$

The objective of this individual is to maximize the expected utility of terminal wealth w_2. Assume that x_1 and x_2 are independently and identically distributed. Prove that the individual optimally sets $\alpha_0 = \alpha_1$ if his utility function exhibits constant relative risk aversion. Show also that this fails to hold if his utility function exhibits constant absolute risk aversion.

6.C.18^{B} Suppose that an individual has a Bernoulli utility function $u(x) = \sqrt{x}$.

(a) Calculate the Arrow–Pratt coefficients of absolute and relative risk aversion at the level of wealth $w = 5$.

(b) Calculate the certainty equivalent and the probability premium for a gamble $(16, 4; \frac{1}{2}, \frac{1}{2})$.

(c) Calculate the certainty equivalent and the probability premium for a gamble $(36, 16; \frac{1}{2}, \frac{1}{2})$. Compare this result with the one in **(b)** and interpret.

6.C.19^{C} Suppose that an individual has a Bernoulli utility function $u(x) = -e^{-\alpha x}$ where $\alpha > 0$. His (nonstochastic) initial wealth is given by w. There is one riskless asset and there are N

risky assets. The return per unit invested on the riskless asset is r. The returns of the risky assets are jointly normally distributed random variables with means $\mu = (\mu_1, \ldots, \mu_N)$ and variance covariance matrix V. Assume that there is no redundancy in the risky assets, so that V is of full rank. Derive the demand function for these $N + 1$ assets.

6.C.20^A Consider a lottery over monetary outcomes that pays $x + \varepsilon$ with probability $\frac{1}{2}$ and $x - \varepsilon$ with probability $\frac{1}{2}$. Compute the second derivative of this lottery's certainty equivalent with respect to ε. Show that the limit of this derivative as $\varepsilon \to 0$ is exactly $-r_A(x)$.

6.D.1^A The purpose of this exercise is to prove Proposition 6.D.1 in a two-dimensional probability simplex. Suppose that there are three monetary outcomes: 1 dollar, 2 dollars, and 3 dollars. Consider the probability simplex of Figure 6.B.1(b).

(a) For a given lottery L over these outcomes, determine the region of the probability simplex in which lie the lotteries whose distributions first-order stochastically dominate the distribution of L.

(b) Given a lottery L, determine the region of the probability simplex in which lie the lotteries L' such that $F(x) \leq G(x)$ for every x, where $F(\cdot)$ is the distribution of L' and $G(\cdot)$ is the distribution of L. [Notice that we get the same region as in **(a)**.]

6.D.2^A Prove that if $F(\cdot)$ first-order stochastically dominates $G(\cdot)$, then the mean of x under $F(\cdot)$, $\int x\, dF(x)$, exceeds that under $G(\cdot)$, $\int x\, dG(x)$. Also provide an example where $\int x\, dF(x) > \int x\, dG(x)$ but $F(\cdot)$ does not first-order stochastically dominate $G(\cdot)$.

6.D.3^A Verify that if a distribution $G(\cdot)$ is an elementary increase in risk from a distribution $F(\cdot)$, then $F(\cdot)$ second-order stochastically dominates $G(\cdot)$.

6.D.4^A The purpose of this exercise is to verify the equivalence of the three statements of Proposition 6.D.2 in a two-dimensional probability simplex. Suppose that there are three monetary outcomes: 1, 2, and 3 dollars. Consider the probability simplex in Figure 6.B.1(b).

(a) If two lotteries have the same mean, what are their positions relative to each other in the probability simplex.

(b) Given a lottery L, determine the region of the simplex in which lie the lotteries L' whose distributions are second-order stochastically dominated by the distribution of L.

(c) Given a lottery L, determine the region of the simplex in which lie the lotteries L' whose distributions are mean preserving spreads of L.

(d) Given a lottery L, determine the region of the simplex in which lie the lotteries L' for which condition (6.D.2) holds, where $F(\cdot)$ and $G(\cdot)$ are, respectively, the distributions of L and L'.

Notice that in **(b)**, **(c)**, and **(d)**, you always have the same region.

6.E.1^B The purpose of this exercise is to show that preferences may not be transitive in the presence of regret. Let there be S states of the world, indexed by $s = 1, \ldots, S$. Assume that state s occurs with probability π_s. Define the expected regret associated with lottery $x = (x_1, \ldots, x_S)$ relative to lottery $x' = (x'_1, \ldots, x'_S)$ by

$$\sum_{s=1}^{S} \pi_s h(\text{Max}\,\{0, x'_s - x_s\}),$$

where $h(\cdot)$ is a given increasing function. [We call $h(\cdot)$ the *regret valuation function*; it measures the regret the individual has after the state of nature is known.] We define x to be at least as good as x' in the presence of regret if and only if the expected regret associated with x relative to x' is not greater than the expected regret associated with x' relative to x.

Suppose that $S = 3$, $\pi_1 = \pi_2 = \pi_3 = \frac{1}{3}$, and $h(x) = \sqrt{x}$. Consider the following three lotteries:

$$x = (0, -2, \ \ 1),$$
$$x' = (0, \ \ 2, -2),$$
$$x'' = (2, -3, -1).$$

Show that the preference ordering over these three lotteries is not transitive.

6.E.2[A] Assume that in a world with uncertainty there are two possible states of nature ($s = 1, 2$) and a single consumption good. There is a single decision maker whose preferences over lotteries satisfy the axioms of expected utility theory and who is a risk averter. For simplicity, we assume that utility is state-independent.

Two contingent commodities are available to the decision maker. The first (respectively, the second) pays one unit of the consumption good in state $s = 1$ (respectively $s = 2$) and zero otherwise. Denote the vector quantities of the two contingent commodities by (x_1, x_2).

(a) Show that the preference relation of the decision maker on (x_1, x_2) is convex.

(b) Argue that the decision maker is also a risk averter when choosing between lotteries whose outcomes are vectors (x_1, x_2).

(c) Show that the Walrasian demand functions for x_1 and x_2 are normal.

6.E.3[B] Let $g: S \to \mathbb{R}_+$ be a random variable with mean $E(g) = 1$. For $\alpha \in (0, 1)$, define a new random variable $g^*: S \to \mathbb{R}_+$ by $g^*(s) = \alpha g(s) + (1 - \alpha)$. Note that $E(g^*) = 1$. Denote by $G(\cdot)$ and $G^*(\cdot)$ the distribution functions of $g(\cdot)$ and $g^*(\cdot)$, respectively. Show that $G^*(\cdot)$ second-order stochastically dominates $G(\cdot)$. Interpret.

6.F.1[B] Prove that in the subjective expected utility theorem (Proposition 6.F.2), the obtained utility function $u(\cdot)$ on money is uniquely determined up to origin and scale. That is, if both $u(\cdot)$ and $\hat{u}(\cdot)$ satisfy the condition of the theorem, then there exist $\beta > 0$ and $\gamma \in \mathbb{R}$ such that $\hat{u}(x) = \beta u(x) + \gamma$ for all x. Prove also that the subjective probabilities are uniquely determined.

6.F.2[A] The purpose of this exercise is to explain the outcomes of the experiments described in Example 6.F.1 by means of the theory of *nonunique prior beliefs* of Gilboa and Schmeidler (1989).

We consider a decision maker with a Bernoulli utility function $u(\cdot)$ defined on $\{0, 1000\}$. We normalize $u(\cdot)$ so that $u(0) = 0$ and $u(1000) = 1$.

The probabilistic belief that the decision maker might have on the color of the H-ball being white is a number $\pi \in [0, 1]$. We assume that the decision maker has, not a single belief but a *set* of beliefs given by a subset P of $[0, 1]$. The actions that he may take are denoted R or H with R meaning that he chooses the R-ball and H meaning that he chooses the H-ball.

As in Example 6.F.1, the decision maker is faced with two different choice situations. In choice situation W, he receives 1000 dollars if the ball chosen is white and 0 dollars otherwise. In choice situation B, he receives 1000 dollars if the ball chosen is black and 0 dollars otherwise.

For each of the two choice situations, define his utility function over the actions R and H in the following way:

For situation W, $U_W: \{R, H\} \to \mathbb{R}$ is defined by

$$U_W(R) = .49 \quad \text{and} \quad U_W(H) = \text{Min}\,\{\pi: \pi \in P\}.$$

For situation B, $U_B: \{R, H\} \to \mathbb{R}$ is defined by

$$U_B(R) = .51 \quad \text{and} \quad U_B(H) = \text{Min}\,\{(1 - \pi): \pi \in P\}.$$

Namely, his utility from choice R is the expected utility of 1000 dollars with the (objective) probability calculated from the number of white and black balls in urn R. However, his utility from choice H is the expected utility of 1000 dollars with the probability associated with the most pessimistic belief in P.

(a) Prove that if P consists of only one belief, then U_W and U_B are derived from a von Neumann Morgenstern utility function and that $U_W(\mathrm{R}) > U_W(\mathrm{H})$ if and only if $U_B(\mathrm{R}) < U_B(\mathrm{H})$.

(b) Find a set P for which $U_W(\mathrm{R}) > U_W(\mathrm{H})$ and $U_B(\mathrm{R}) > U_B(\mathrm{H})$.

PART TWO

Game Theory

In Part I, we analyzed individual decision making, both in abstract decision problems and in more specific economic settings. Our primary aim was to lay the groundwork for the study of how the simultaneous behavior of many self-interested individuals (including firms) generates economic outcomes in market economies. Most of the remainder of the book is devoted to this task. In Part II, however, we study in a more general way how multiperson interactions can be modeled.

A central feature of multiperson interaction is the potential for the presence of *strategic interdependence*. In our study of individual decision making in Part I, the decision maker faced situations in which her well-being depended only on the choices she made (possibly with some randomness). In contrast, in multiperson situations with strategic interdependence, each agent recognizes that the payoff she receives (in utility or profits) depends not only on her own actions but also on the actions of *other* individuals. The actions that are best for her to take may depend on actions these other individuals have already taken, on those she expects them to be taking at the same time, and even on future actions that they may take, or decide not to take, as a result of her current actions.

The tool that we use for analyzing settings with strategic interdependence is *noncooperative game theory*. Although the term "game" may seem to undersell the theory's importance, it correctly highlights the theory's central feature: The agents under study are concerned with strategy and winning (in the general sense of utility or profit maximization) in much the same way that players of most parlor games are.

Multiperson economic situations vary greatly in the degree to which strategic interaction is present. In settings of monopoly (where a good is sold by only a single firm; see Section 12.B) or of perfect competition (where all agents act as price takers; see Chapter 10 and Part IV), the nature of strategic interaction is minimal enough that our analysis need not make any formal use of game theory.[1] In other settings, however, such as the analysis of oligopolistic markets (where there is more than one

1. However, we could well do so in both cases; see, for example, the proof of existence of competitive equilibrium in Chapter 17, Appendix B. Moreover, we shall stress how perfect competition can be viewed usefully as a limiting case of oligopolistic strategic interaction; see, for example, Section 12.F.

but still not many sellers of a good; see Sections 12.C to 12.G), the central role of strategic interaction makes game theory indispensable for our analysis.

Part II is divided into three chapters. Chapter 7 provides a short introduction to the basic elements of noncooperative game theory, including a discussion of exactly what a game is, some ways of representing games, and an introduction to a central concept of the theory, a player's *strategy*. Chapter 8 addresses how we can predict outcomes in the special class of games in which all the players move simultaneously, known as *simultaneous-move games*. This restricted focus helps us isolate some central issues while deferring a number of more difficult ones. Chapter 9 studies *dynamic games* in which players' moves may precede one another, and in which some of these more difficult (but also interesting) issues arise.

Note that we have used the modifier *noncooperative* to describe the type of game theory we discuss in Part II. There is another branch of game theory, known as *cooperative game theory*, that we do not discuss here. In contrast with noncooperative game theory, the fundamental units of analysis in cooperative theory are groups and subgroups of individuals that are assumed, as a primitive of the theory, to be able to attain particular outcomes for themselves through binding cooperative agreements. Cooperative game theory has played an important role in general equilibrium theory, and we provide a brief introduction to it in Appendix A of Chapter 18. We should emphasize that the term *noncooperative game theory* does *not* mean that noncooperative theory is incapable of explaining cooperation within groups of individuals. Rather, it focuses on how cooperation may emerge as rational behavior in the absence of an ability to make binding agreements (e.g., see the discussion of repeated interaction among oligopolists in Chapter 12).

Some excellent recent references for further study of noncooperative game theory are Fudenberg and Tirole (1991), Myerson (1992), and Osborne and Rubinstein (1994), and at a more introductory level Gibbons (1992) and Binmore (1992). Kreps (1990) provides a very interesting discussion of some of the strengths and weaknesses of the theory. Von Neumann and Morgenstern (1944), Luce and Raiffa (1957), and Schelling (1960) remain classic references.

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7 Basic Elements of Noncooperative Games

7.A Introduction

In this chapter, we begin our study of noncooperative game theory by introducing some of its basic building blocks. This material serves as a prelude to our analysis of games in Chapters 8 and 9.

Section 7.B begins with an informal introduction to the concept of a *game*. It describes the four basic elements of any setting of strategic interaction that we must know to specify a game.

In Section 7.C, we show how a game can be described by means of what is called its *extensive form representation*. The extensive form representation provides a very rich description of a game, capturing who moves when, what they can do, what they know when it is their turn to move, and the outcomes associated with any collection of actions taken by the individuals playing the game.

In Section 7.D, we introduce a central concept of game theory, a player's *strategy*. A player's strategy is a complete contingent plan describing the actions she will take in each conceivable evolution of the game. We then show how the notion of a strategy can be used to derive a much more compact representation of a game, known as its *normal* (or *strategic*) *form representation*.

In Section 7.E, we consider the possibility that a player might randomize her choices. This gives rise to the notion of a *mixed strategy*.

7.B What Is a Game?

A *game* is a formal representation of a situation in which a number of individuals interact in a setting of *strategic interdependence*. By that, we mean that each individual's welfare depends not only on her own actions but also on the actions of the other individuals. Moreover, the actions that are best for her to take may depend on what she expects the other players to do.

To describe a situation of strategic interaction, we need to know four things:

(i) *The players:* Who is involved?

(ii) *The rules:* Who moves when? What do they know when they move? What can they do?

(iii) *The outcomes:* For each possible set of actions by the players, what is the outcome of the game?

(iv) *The payoffs:* What are the players' preferences (i.e., utility functions) over the possible outcomes?

We begin by considering items (i) to (iii). A simple example is provided by the school-yard game of *Matching Pennies.*

Example 7.B.1: *Matching Pennies.* Items (i) to (iii) are as follows:

Players: There are two players, denoted 1 and 2.

Rules: Each player simultaneously puts a penny down, either heads up or tails up.

Outcomes: If the two pennies match (either both heads up or both tails up), player 1 pays 1 dollar to player 2; otherwise, player 2 pays 1 dollar to player 1. ■

Consider another example, the game of *Tick-Tack-Toe.*

Example 7.B.2: *Tick-Tack-Toe.* Items (i) to (iii) are as follows:

Players: There are two players, X and O.

Rules: The players are faced with a board that consists of nine squares arrayed with three rows of three squares each stacked on one another (see Figure 7.B.1). The players take turns putting their marks (an X or an O) into an as-yet-unmarked square. Player X moves first. Both players observe all choices previously made.

Outcomes: The first player to have three of her marks in a row (horizontally, vertically, or diagonally) wins and receives 1 dollar from the other player. If no one succeeds in doing so after all nine boxes are marked, the game is a tie and no payments are made or received by either player. ■

To complete our description of these two games, we need to say what the players' preferences are over the possible outcomes [item (iv) in our list]. As a general matter, we describe a player's preferences by a utility function that assigns a utility level for each possible outcome. It is common to refer to the player's utility function as her *payoff function* and the utility level as her *payoff.* Throughout, we assume that these utility functions take an expected utility form (see Chapter 6) so that when we consider situations in which outcomes are random, we can evaluate the random prospect by means of the player's expected utility.

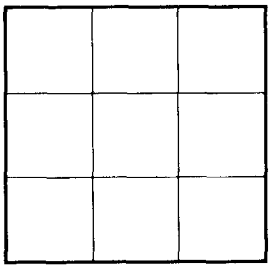

Figure 7.B.1
A Tick-Tack-Toe board.

In later references to Matching Pennies and Tick-Tack-Toe, we assume that each player's payoff is simply equal to the amount of money she gains or loses. Note that in both examples, the actions that maximize a player's payoff depend on what she expects her opponent to do.

Examples 7.B.1 and 7.B.2 involve situations of pure conflict: What one player wins, the other player loses. Such games are called *zero-sum games*. But strategic interaction and game theory are not limited to situations of pure or even partial conflict. Consider the situation in Example 7.B.3.

Example 7.B.3: *Meeting in New York.* Items (i) to (iv) are as follows:

Players: Two players, Mr. Thomas and Mr. Schelling.

Rules: The two players are separated and cannot communicate. They are supposed to meet in New York City at noon for lunch but have forgotten to specify where. Each must decide where to go (each can make only one choice).

Outcomes: If they meet each other, they get to enjoy each other's company at lunch. Otherwise, they must eat alone.

Payoffs: They each attach a monetary value of 100 dollars to the other's company (their payoffs are each 100 dollars if they meet, 0 dollars if they do not).

In this example, the two players' interests are completely aligned. Their problem is simply one of coordination. Nevertheless, each player's payoff depends on what the other player does; and more importantly, *each player's optimal action depends on what he thinks the other will do.* Thus, even the task of coordination can have a strategic nature. ■

Although the information given in items (i) to (iv) fully describe a game, it is useful for purposes of analysis to represent this information in particular ways. We examine one of these ways in Section 7.C.

7.C The Extensive Form Representation of a Game

If we know the items (i) to (iv) described in Section 7.B (the players, the rules, the outcomes, and the payoffs), then we can formally represent the game in what is called its *extensive form*. The extensive form captures who moves when, what actions each player can take, what players know when they move, what the outcome is as a function of the actions taken by the players, and the players' payoffs from each possible outcome.

We begin by informally introducing the elements of the extensive form representation through a series of examples. After doing so, we then provide a formal specification of the extensive form (some readers may want to begin with this and then return to the examples).

The extensive form relies on the conceptual apparatus known as a *game tree*. As our starting point, it is useful to begin with a very simple variation of Matching Pennies, which we call *Matching Pennies Version B*.

Example 7.C.1: *Matching Pennies Version B and Its Extensive Form.* Matching Pennies Version B is identical to Matching Pennies (see Example 7.B.1) except

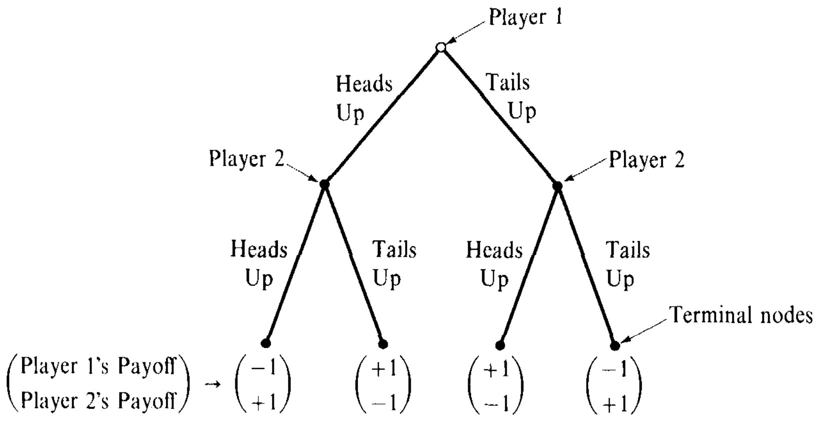

Figure 7.C.1 Extensive form for Matching Pennies Version B.

that the two players move sequentially, rather than simultaneously. In particular, player 1 puts her penny down (heads up or tails up) first. Then, after seeing player 1's choice, player 2 puts her penny down. (This is a very nice game for player 2!)

The extensive form representation of this game is depicted in Figure 7.C.1. The game starts at an *initial decision node* (represented by an open circle), where player 1 makes her move, deciding whether to place her penny heads up or tails up. Each of the two possible choices for player 1 is represented by a *branch* from this initial decision node. At the end of each branch is another decision node (represented by a solid dot), at which player 2 can choose between two actions, heads up or tails up, after seeing player 1's choice. The initial decision node is referred to as *player 1's decision node*; the latter two as *player 2's decision nodes.* After player 2's move, we reach the end of the game, represented by *terminal nodes.* At each terminal node, we list the players' payoffs arising from the sequence of moves leading to that terminal node.

Note the treelike structure of Figure 7.C.1: Like an actual tree, it has a unique connected path of branches from the initial node (sometimes also called the *root*) to each point in the tree. This type of figure is known as a *game tree.* ■

Example 7.C.2: *The Extensive Form of Tick-Tack-Toe.* The more elaborate game tree shown in Figure 7.C.2 depicts the extensive form for Tick-Tack-Toe (to conserve space, many parts are omitted). Note that every path through the tree represents a unique sequence of moves by the players. In particular, when a given board position (such as the two left corners filled by X and the two right corners filled by O) can be reached through several different sequences of moves, each of these sequences is depicted separately in the game tree. Nodes represent not only the current position but also *how it was reached.* ■

In both Matching Pennies Version B and Tick-Tack-Toe, when it is a player's turn to move, she is able to observe all her rival's previous moves. They are games of *perfect information* (we give a precise definition of this term in Definition 7.C.1). The concept of an *information set* allows us to accommodate the possibility that this is not so. Formally, the elements of an information set are a subset of a particular player's decision nodes. The interpretation is that when play has reached one of the decision nodes in the information set and it is that player's turn to move, she does

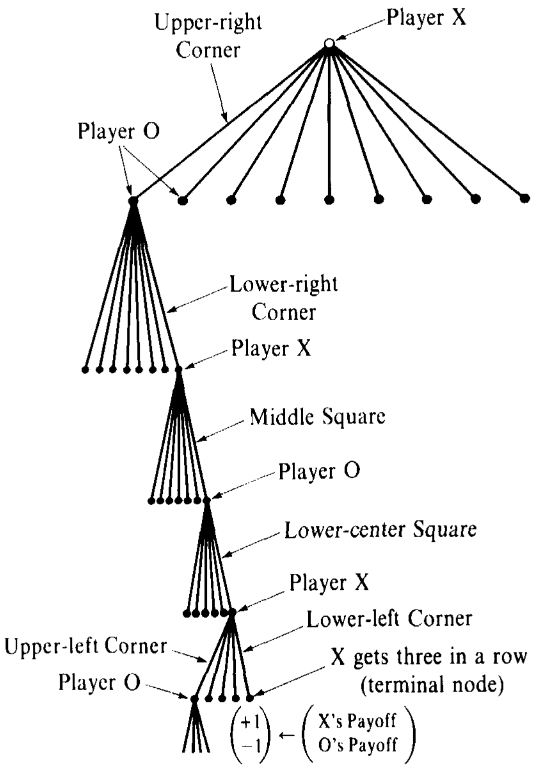

Figure 7.C.2
Part of the extensive form for Tick-Tack-Toe.

not know which of these nodes she is actually at. The reason for this ignorance is that the player does not observe something about what has previously transpired in the game. A further variation of Matching Pennies, which we call *Matching Pennies Version C*, helps make this concept clearer.

Example 7.C.3: *Matching Pennies Version C and Its Extensive Form.* This version of Matching Pennies is just like Matching Pennies Version B (in Example 7.C.1) except that when player 1 puts her penny down, she keeps it covered with her hand. Hence, player 2 cannot see player 1's choice until after player 2 has moved.

The extensive form for this game is represented in Figure 7.C.3. It is identical to Figure 7.C.1 except that we have drawn a circle around player 2's two decision nodes to indicate that these two nodes are in a single information set. The meaning of this information set is that when it is player 2's turn to move, she cannot tell which of these two nodes she is at because she has not observed player 1's previous move. Note that player 2 has the same two possible actions at each of the two nodes in her information set. This must be the case if player 2 is unable to distinguish the two nodes; otherwise, she could figure out which move player 1 had taken simply by what her own possible actions are.

In principle, we could also associate player 1's decision node with an information set. Because player 1 knows that nothing has happened before it is her turn to move, this information set has only one member (player 1 knows exactly which node she is at when she moves). To be fully rigorous, we should therefore also draw an information set circle around player 1's decision node in Figure 7.C.3. It is common, however, to

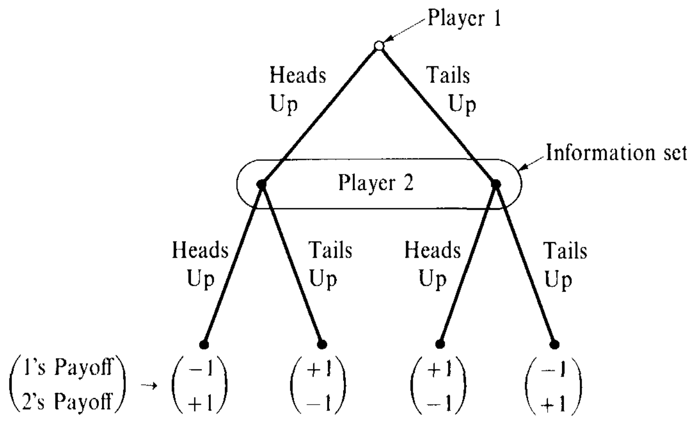

Figure 7.C.3 Extensive form for Matching Pennies Version C.

simplify the diagrammatic depiction of a game in extensive form by not drawing the information sets that contain a single node. Thus, any uncircled decision nodes are understood to be elements of *singleton* information sets. In Figures 7.C.1 and 7.C.2, for example, every decision node belongs to a singleton information set. ■

A listing of all of a player's information sets gives a listing, from the player's perspective, of all of the possible distinguishable "events" or "circumstances" in which she might be called upon to move. For example, in Example 7.C.1, from player 2's perspective there are two distinguishable events that might arise in which she would be called upon to move, each one corresponding to play having reached one of her two (singleton) information sets. By way of contrast, player 2 foresees only one possible circumstance in which she would need to move in Example 7.C.3 (this circumstance is, however, certain to arise).

In Example 7.C.3, we noted a natural restriction on information sets: At every node within a given information set, a player must have the same set of possible actions. Another restriction we impose is that players possess what is known as *perfect recall.* Loosely speaking, perfect recall means that a player does not forget what she once knew, including her own actions. Figure 7.C.4 depicts two games in which this condition is not met. In Figure 7.C.4(a), as the game progresses, player 2 forgets a move by player 1 that she once knew (namely, whether player 1 chose ℓ or r). In Figure 7.C.4(b), player 1 forgets her own previous move.[1] All the games we consider in this book satisfy the property of perfect recall.

The use of information sets also allows us to capture play that is simultaneous rather than sequential. This is illustrated in Example 7.C.4 for the game of (standard) Matching Pennies introduced in Example 7.B.1.

1. In terms of the formal specification of the extensive form given later in this section, if we denote the information set containing decision node x by $H(x)$, a game is formally characterized as one of perfect recall if the following two conditions hold: (i) If $H(x) = H(x')$, x is neither a predecessor nor a successor of x'; and (ii) if x and x' are two decision nodes for player i with $H(x) = H(x')$, and if x'' is a predecessor of x (not necessarily an immediate one) that is also in one of player i's information sets, with a'' being the action at $H(x'')$ on the path to x, then there must be a predecessor node to x' that is an element of $H(x'')$ and the action at this predecessor node that is on the path to x' must also be a''.

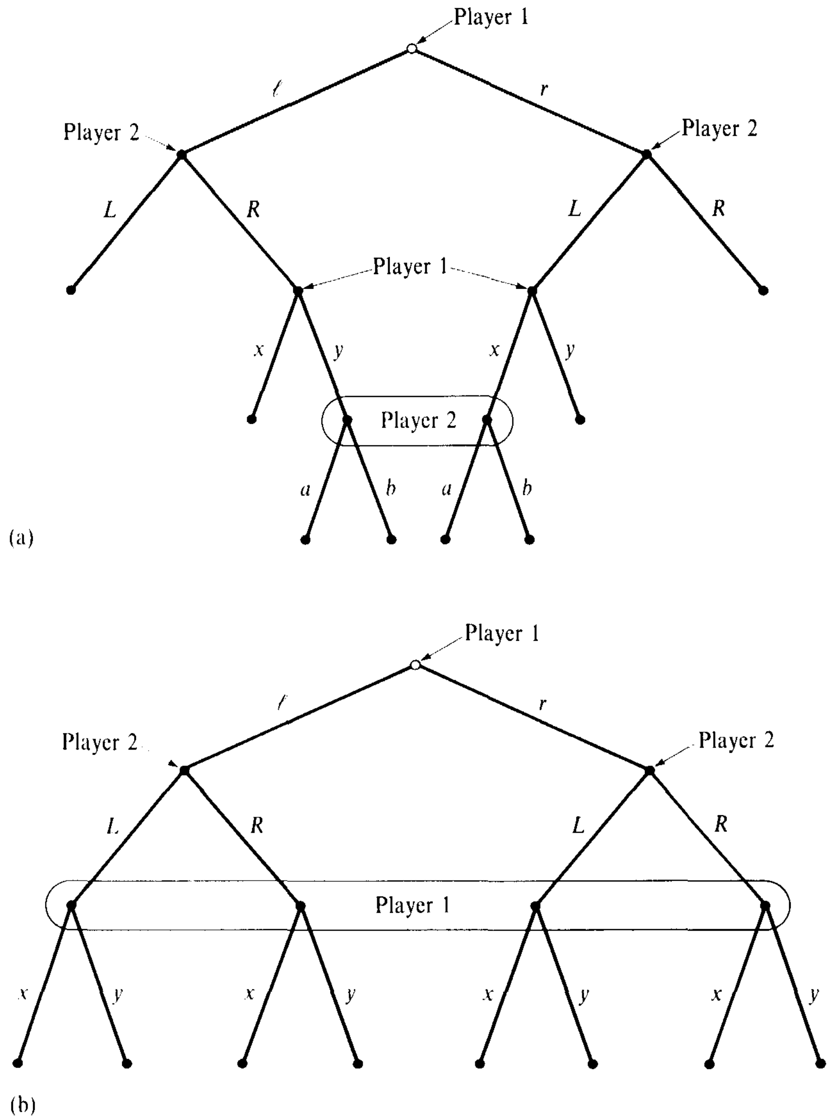

Figure 7.C.4
Two games not satisfying perfect recall.

Example 7.C.4: *The Extensive Form for Matching Pennies.* Suppose now that the players put their pennies down simultaneously. For each player, this game is strategically equivalent to the Version C game. In Version C, player 1 was unable to observe player 2's choice because player 1 moved first, and player 2 was unable to observe player 1's choice because player 1 kept it covered; here each player is unable to observe the other's choice because they move simultaneously. As long as they cannot observe each other's choices, the timing of moves is irrelevant. Thus, we can use the game tree in Figure 7.C.3 to describe the game of (standard) Matching Pennies. Note that by this logic we can also describe this game with a game tree that reverses the decision nodes of players 1 and 2 in Figure 7.C.3. ■

We can now return to the notion of a game of perfect information and offer a formal definition.

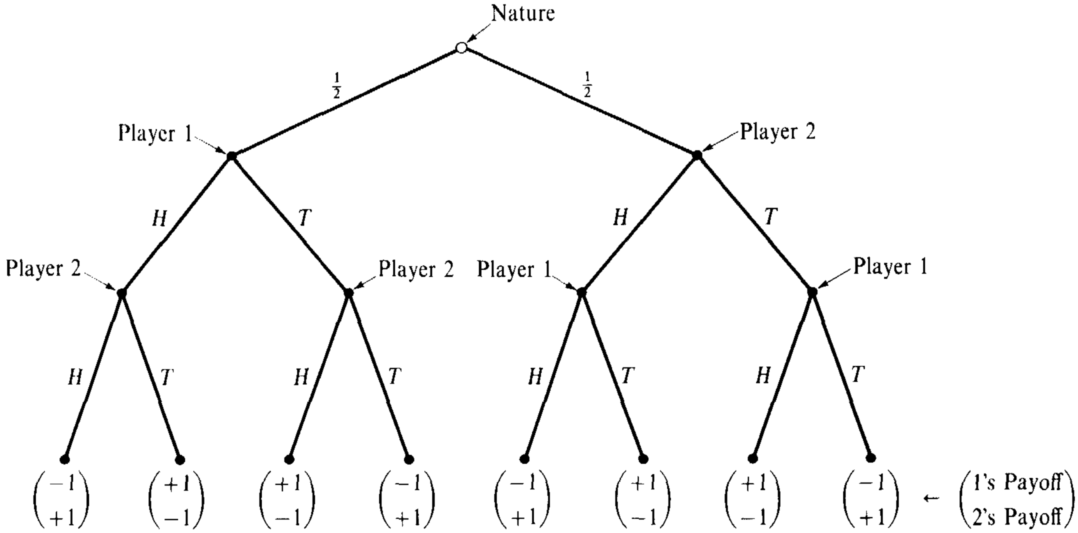

Figure 7.C.5 Extensive form for Matching Pennies Version D.

Definition 7.C.1: A game is one of *perfect information* if each information set contains a single decision node. Otherwise, it is a game of *imperfect information*.

Up to this point, the outcome of a game has been a deterministic function of the players' choices. In many games, however, there is an element of chance. This, too, can be captured in the extensive form representation by including *random moves of nature*. We illustrate this point with still another variation, *Matching Pennies Version D*.

Example 7.C.5: *Matching Pennies Version D and Its Extensive Form.* Suppose that prior to playing Matching Pennies Version B, the two players flip a coin to see who will move first. Thus, with equal probability either player 1 will put her penny down first, or player 2 will. In Figure 7.C.5, this game is depicted as beginning with a *move of nature* at the initial node that has two branches, each with probability $\frac{1}{2}$. Note that this is drawn as if nature were an additional player who must play its two actions with fixed probabilities. (In the figure, H stands for "heads up" and T stands for "tails up".) ■

It is a basic postulate of game theory that all players know the structure of the game, know that their rivals know it, know that their rivals know that they know it, and so on. In theoretical parlance, we say that the structure of the game is *common knowledge* [see Aumann (1976) and Milgrom (1981) for discussions of this concept].

In addition to being depicted graphically, the extensive form can be described mathematically. The basic components are fairly easily explained and can help you keep in mind the fundamental building blocks of a game. Formally, a game represented in extensive form consists of the following items:[2]

2. To be a bit more precise about terminology: A collection of items (i) to (vi) is formally known as an extensive *game form*; adding item (vii), the players' preferences over the outcomes, leads to a *game* represented in extensive form. We will not make anything of this distinction here. See Kuhn (1953) or Section 2 of Kreps and Wilson (1982) for additional discussion of this and other points regarding the extensive form.

(i) A finite set of nodes $\mathscr{X}$, a finite set of possible actions $\mathscr{A}$, and a finite set of players $\{1, \ldots, I\}$.

(ii) A function $p: \mathscr{X} \to \{\mathscr{X} \cup \varnothing\}$ specifying a single immediate predecessor of each node x; $p(x)$ is nonempty for all $x \in \mathscr{X}$ but one, designated as the *initial node* x_0. The immediate successor nodes of x are then $s(x) = p^{-1}(x)$, and the set of *all* predecessors and *all* successors of node x can be found by iterating $p(x)$ and $s(x)$. To have a tree structure, we require that these sets be disjoint (a predecessor of node x cannot also be a successor to it). The set of *terminal nodes* is $T = \{x \in \mathscr{X}: s(x) = \varnothing\}$. All other nodes $\mathscr{X} \backslash T$ are known as *decision nodes*.

(iii) A function $\alpha: \mathscr{X} \backslash \{x_0\} \to \mathscr{A}$ giving the action that leads to any noninitial node x from its immediate predecessor $p(x)$ and satisfying the property that if $x', x'' \in s(x)$ and $x' \neq x''$, then $\alpha(x') \neq \alpha(x'')$. The set of choices available at decision node x is $c(x) = \{a \in \mathscr{A}: a = \alpha(x') \text{ for some } x' \in s(x)\}$.

(iv) A collection of information sets $\mathscr{H}$, and a function $H: \mathscr{X} \to \mathscr{H}$ assigning each decision node x to an information set $H(x) \in \mathscr{H}$. Thus, the information sets in $\mathscr{H}$ form a partition of $\mathscr{X}$. We require that all decision nodes assigned to a single information set have the same choices available; formally, $c(x) = c(x')$ if $H(x) = H(x')$. We can therefore write the choices available at information set H as $C(H) = \{a \in \mathscr{A}: a \in c(x) \text{ for } x \in H\}$.

(v) A function $\iota: \mathscr{H} \to \{0, 1, \ldots, I\}$ assigning each information set in $\mathscr{H}$ to the player (or to nature: formally, player 0) who moves at the decision nodes in that set. We can denote the collection of player i's information sets by $\mathscr{H}_i = \{H \in \mathscr{H}: i = \iota(H)\}$.

(vi) A function $\rho: \mathscr{H}_0 \times \mathscr{A} \to [0, 1]$ assigning probabilities to actions at information sets where nature moves and satisfying $\rho(H, a) = 0$ if $a \notin C(H)$ and $\sum_{a \in C(H)} \rho(H, a) = 1$ for all $H \in \mathscr{H}_0$.

(vii) A collection of payoff functions $u = \{u_1(\cdot), \ldots, u_I(\cdot)\}$ assigning utilities to the players for each terminal node that can be reached, $u_i: T \to \mathbb{R}$. As we noted in Section 7.B, because we want to allow for a random realization of outcomes we take each $u_i(\cdot)$ to be a Bernoulli utility function.

Thus, formally, a game in extensive form is specified by the collection $\Gamma_E = \{\mathscr{X}, \mathscr{A}, I, p(\cdot), \alpha(\cdot), \mathscr{H}, H(\cdot), \iota(\cdot), \rho(\cdot), u\}$.

We should note that there are three implicit types of finiteness hidden in the formulation just presented. Because we will often encounter games not sharing these features in the economic applications discussed in later chapters, we briefly identify them here, although without any formal treatment. The formal definition of an extensive form representation of a game can be extended to these infinite cases without much difficulty, although there can be important differences in the predicted outcomes of finite and infinite economic models, as we shall see later (e.g., in Chapters 12 and 20).

First, we have assumed that players have a finite number of actions available at each decision node. This would rule out a game in which, say, a player can choose any number from some interval $[a, b] \subset \mathbb{R}$. In fact, allowing for an infinite set of actions requires that we allow for an infinite set of nodes as well. But with this change, items (i) to (vii) remain the basic elements of an extensive form representation (e.g., decision nodes and terminal nodes are still associated with a unique path through the tree).

Second, we have described the extensive form of a game that must end after a finite number of moves (because the set of decision nodes is finite). Indeed, all the examples we have considered so far fall into this category. There are, however, other types of games. For example, suppose that two players with infinite life spans (perhaps two firms) play Matching Pennies repeatedly every January 1. The players discount the money gained or lost at future dates with interest rate r and seek to maximize their discounted net gains. In this game, there are no terminal nodes. Even so, we can still associate discounted payoffs for the two players with every (infinite) sequence of moves the players make. Of course, actually drawing a complete game tree would be impossible, but the basic elements of the extensive form can nonetheless be captured as before (with payoffs being associated with paths through the tree rather than with terminal nodes).

Third, we may at times also imagine that there are an infinite number of players who take actions in a game. For example, models involving overlapping generations of players (as in various macroeconomic models) have this feature, as do models of entry in which we want to allow for an infinite number of potential firms. In the games of this type that we consider, this issue can be handled in a simple and natural manner.

Note that all three of these extensions require that we relax the assumption that there is a finite set of nodes. Games with a finite number of nodes, such as those we have been considering, are known as *finite games*.

For pedagogical purposes, we restrict our attention in Part II to finite games except where specifically indicated otherwise. The extension of the formal concepts we discuss here to the economic games studied later in the book that do not share these finiteness properties is straightforward.

7.D Strategies and the Normal Form Representation of a Game

A central concept of game theory is the notion of a player's *strategy*. A strategy is a *complete contingent plan*, or *decision rule*, that specifies how the player will act in *every possible distinguishable circumstance* in which she might be called upon to move. Recall that, from a player's perspective, the set of such circumstances is represented by her collection of information sets, with each information set representing a different distinguishable circumstance in which she may need to move (see Section 7.C). Thus, a player's strategy amounts to a specification of how she plans to move at each one of her information sets, should it be reached during play of the game. This is stated formally in Definition 7.D.1.

Definition 7.D.1: Let $\mathscr{H}_i$ denote the collection of player i's information sets, $\mathscr{A}$ the set of possible actions in the game, and $C(H) \subset \mathscr{A}$ the set of actions possible at information set H. A *strategy* for player i is a function $s_i \colon \mathscr{H}_i \to \mathscr{A}$ such that $s_i(H) \in C(H)$ for all $H \in \mathscr{H}_i$.

The fact that a strategy is a complete contingent plan cannot be overemphasized, and it is often a source of confusion to those new to game theory. When a player specifies her strategy, it is as if she had to write down an instruction book prior to play so that a representative could act on her behalf merely by consulting that book.

As a complete contingent plan, a strategy often specifies actions for a player at information sets that may not be reached during the actual play of the game.

For example, in Tick-Tack-Toe, player O's strategy describes what she will do on her first move if player X starts the game by marking the center square. But in the actual play of the game, player X might not begin in the center; she may instead mark the lower-right corner first, making this part of player O's plan no longer relevant.

In fact, there is an even subtler point: A player's strategy may include plans for actions that her *own* strategy makes irrelevant. For example, a complete contingent plan for player X in Tick-Tack-Toe includes a description of what she will do after she plays "center" and player O then plays "lower-right corner," even though her own strategy may call for her first move to be "upper-left corner." This probably seems strange; its importance will become apparent only when we talk about dynamic games in Chapter 9. Nevertheless, remember: *A strategy is a complete contingent plan that says what a player will do at each of her information sets if she is called on to play there.*

It is worthwhile to consider what the players' possible strategies are for some of the simple Matching Pennies games.

Example 7.D.1: *Strategies in Matching Pennies Version B.* In Matching Pennies Version B, a strategy for player 1 simply specifies her move at the game's initial node. She has two possible strategies: She can play heads (H) or tails (T). A strategy for player 2, on the other hand, specifies how she will play (H or T) at each of her two information sets, that is, how she will play if player 1 picks H *and* how she will play if player 1 picks T. Thus, player 2 has *four* possible strategies.

Strategy 1 (s_1): Play H if player 1 plays H; play H if player 1 plays T.
Strategy 2 (s_2): Play H if player 1 plays H; play T if player 1 plays T.
Strategy 3 (s_3): Play T if player 1 plays H; play H if player 1 plays T.
Strategy 4 (s_4): Play T if player 1 plays H; play T if player 1 plays T. ■

Example 7.D.2: *Strategies in Matching Pennies Version C.* In Matching Pennies Version C, player 1's strategies are exactly the same as in Version B; but player 2 now only has two possible strategies, "play H" and "play T", because she now has only one information set. She can no longer condition her action on player 1's previous action. ■

We will often find it convenient to represent a profile of players' strategy choices in an I-player game by a vector $s = (s_1, \dots, s_I)$, where s_i is the strategy chosen by player i. We will also sometimes write the strategy profile s as (s_i, s_{-i}), where s_{-i} is the $(I - 1)$ vector of strategies for players other than i.

The Normal Form Representation of a Game

Every profile of strategies for the players $s = (s_1, \dots, s_I)$ induces an outcome of the game: a sequence of moves actually taken and a probability distribution over the terminal nodes of the game. Thus, for any profile of strategies $(s_1, \dots, s_I)$, we can deduce the payoffs received by each player. We might think, therefore, of specifying the game directly in terms of strategies and their associated payoffs. This second way to represent a game is known as the *normal* (or *strategic*) *form*. It is, in essence, a condensed version of the extensive form.

		Player 2 s_1	s_2	s_3	s_4
Player 1	H	$-1, +1$	$-1, +1$	$+1, -1$	$+1, -1$
	T	$+1, -1$	$-1, +1$	$+1, -1$	$-1, +1$

Figure 7.D.1 The normal form of Matching Pennies Version B.

Definition 7.D.2: For a game with I players, the *normal form representation* Γ_N specifies for each player i a set of strategies S_i (with $s_i \in S_i$) and a payoff function $u_i(s_1, \ldots, s_I)$ giving the von Neumann–Morgenstern utility levels associated with the (possibly random) outcome arising from strategies $(s_1, \ldots, s_I)$. Formally, we write $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$.

In fact, when describing a game in its normal form, there is no need to keep track of the specific moves associated with each strategy. Instead, we can simply number the various possible strategies of a player, writing player i's strategy set as $S_i = \{s_{1i}, s_{2i}, \ldots\}$ and then referring to each strategy by its number.

A concrete example of a game in normal form is presented in Example 7.D.3 for Matching Pennies Version B.

Example 7.D.3: *The Normal Form of Matching Pennies Version B.* We have already described the strategy sets of the two players in Example 7.D.1. The payoff functions are

$$u_1(s_1, s_2) = \begin{cases} +1 \text{ if } (s_1, s_2) = (\text{H, strategies 3 or 4}) \text{ or } (\text{T, strategies 1 or 3}), \\ -1 \text{ if } (s_1, s_2) = (\text{H, strategies 1 or 2}) \text{ or } (\text{T, strategies 2 or 4}), \end{cases}$$

and $u_2(s_1, s_2) = -u_1(s_1, s_2)$. A convenient way to summarize this information is in the "game box" depicted in Figure 7.D.1. The different rows correspond to the strategies of player 1, and the columns to those of player 2. Within each cell, the payoffs of the two players are depicted as $(u_1(s_1, s_2), u_2(s_1, s_2))$. ■

Exercise 7.D.2: Depict the normal forms for Matching Pennies Version C and the standard version of Matching Pennies.

The idea behind using the normal form representation to study behavior in a game is that a player's decision problem can be thought of as one of choosing her strategy (her contingent plan of action) given the strategies that she thinks her rivals will be adopting. Because each player is faced with this problem, we can think of the players as simultaneously choosing their strategies from the sets $\{S_i\}$. It is as if the players each simultaneously write down their strategies on slips of paper and hand them to a referee, who then computes the outcome of the game from the players' submitted strategies.

From the previous discussion, it is clear that for any extensive form representation of a game, there is a unique normal form representation (more precisely, it is unique up to any renaming or renumbering of the strategies). The converse is not true, however. Many different extensive forms may be represented by the same normal form. For example, the normal form shown in Figure 7.D.1 represents not only the extensive form in Figure 7.C.1 but also the

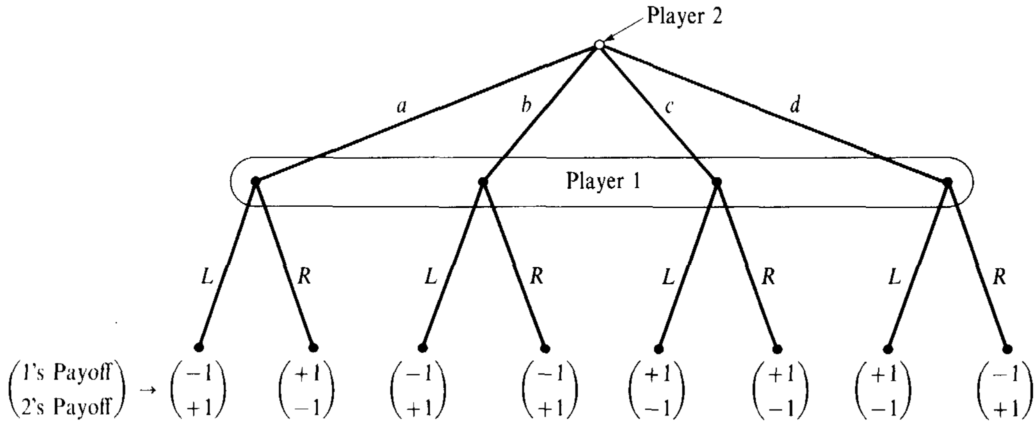

Figure 7.D.2 An extensive form whose normal form is that depicted in Figure 7.D.1.

extensive form in Figure 7.D.2. In the latter game, players move simultaneously, player 1 choosing between two strategies, L and R, and player 2 choosing among four strategies: a, b, c, and d. In terms of their representations in a game box, the only difference between the normal forms for these games lies in the "labels" given to the rows and columns.

Because the condensed representation of the game in the normal form generally omits some of the details present in the extensive form, we may wonder whether this omission is important or whether the normal form summarizes all of the strategically relevant information (as the last paragraph in regular type seems to suggest). The question can be put a little differently: Is the scenario in which players simultaneously write down their strategies and submit them to a referee really equivalent to their playing the game over time as described in the extensive form? This question is currently a subject of some controversy among game theorists. The debate centers on issues arising in dynamic games such as those studied in Chapter 9.

For the simultaneous-move games that we study in Chapter 8, in which all players choose their actions at the same time, the normal form captures *all* the strategically relevant information. In simultaneous-move games, a player's strategy is a simple noncontingent choice of an action. In this case, players' simultaneous choice of strategies in the normal form is clearly equivalent to their simultaneous choice of actions in the extensive form (captured there by having players not observing each other's choices).

7.E Randomized Choices

Up to this point, we have assumed that players make their choices with certainty. However, there is no a priori reason to exclude the possibility that a player could randomize when faced with a choice. Indeed, we will see in Chapters 8 and 9 that in certain circumstances the possibility of randomization can play an important role in the analysis of games.

As stated in Definition 7.D.1, a deterministic strategy for player i, which we now call a *pure strategy*, specifies a deterministic choice $s_i(H)$ at each of her information sets $H \in \mathscr{H}_i$. Suppose that player i's (finite) set of pure strategies is S_i. One way for the player to randomize is to choose randomly one element of this set. This kind of randomization gives rise to what is called a *mixed strategy.*

Definition 7.E.1: Given player i's (finite) pure strategy set S_i, a *mixed strategy* for player i, $\sigma_i: S_i \to [0, 1]$, assigns to each pure strategy $s_i \in S_i$ a probability $\sigma_i(s_i) \geq 0$ that it will be played, where $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$.

Suppose that player i has M pure strategies in set $S_i = \{s_{1i}, \ldots, s_{Mi}\}$. Player i's set of possible mixed strategies can therefore be associated with the points of the following simplex (recall our use of a simplex to represent lotteries in Chapter 6):

$$\Delta(S_i) = \{(\sigma_{1i}, \ldots, \sigma_{Mi}) \in \mathbb{R}^M: \sigma_{mi} \geq 0 \text{ for all } m = 1, \ldots, M \text{ and } \sum_{m=1}^{M} \sigma_{mi} = 1\}.$$

This simplex is called the *mixed extension* of S_i. Note that a pure strategy can be viewed as a special case of a mixed strategy in which the probability distribution over the elements of S_i is degenerate.

When players randomize over their pure strategies, the induced outcome is itself random, leading to a probability distribution over the terminal nodes of the game. Since each player i's normal form payoff function $u_i(s)$ is of the von Neumann–Morgenstern type, player i's payoff given a profile of mixed strategies $\sigma = (\sigma_1, \ldots, \sigma_I)$ for the I players is her expected utility $E_\sigma[u_i(s)]$, the expectation being taken with respect to the probabilities induced by σ on pure strategy profiles $s = (s_1, \ldots, s_I)$. That is, letting $S = S_1 \times \cdots \times S_I$, player i's von Neumann–Morgenstern utility from mixed strategy profile σ is

$$\sum_{s \in S} [\sigma_1(s_1)\, \sigma_2(s_2) \cdots \sigma_I(s_I)] u_i(s),$$

which, with a slight abuse of notation, we denote by $u_i(\sigma)$. Note that because we assume that each player randomizes on her own, we take the realizations of players' randomizations to be independent of one another.[3]

The basic definition of the normal form representation need not be changed to accommodate the possibility that players might choose to play mixed strategies. We can simply consider the normal form game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ in which players' strategy sets are extended to include both pure and mixed strategies.

Note that we can equivalently think of a player forming her mixed strategy as follows: Player i has access to a private signal θ_i that is uniformly distributed on the interval [0, 1] and is independent of other players' signals, and she forms her mixed strategy by making her plan of action contingent on the realization of the signal. That is, she specifies a pure strategy $s_i(\theta_i) \in S_i$ for each realization of θ_i. We shall return to this alternative interpretation of mixed strategies in Chapter 8.

If we use the extensive form description of a game, there is another way that player i could randomize. Rather than randomizing over the potentially very

3. In Chapter 8, however, we discuss the possibility that players' randomizations could be correlated.

large set of pure strategies in S_i, she could randomize separately over the possible actions at each of her information sets $H \in \mathscr{H}_i$. This way of randomizing is called a *behavior strategy*.

Definition 7.E.2: Given an extensive form game Γ_E, a *behavior strategy* for player i specifies, for every information set $H \in \mathscr{H}_i$ and action $a \in C(H)$, a probability $\lambda_i(a, H) \geq 0$, with $\sum_{a \in C(H)} \lambda_i(a, H) = 1$ for all $H \in \mathscr{H}_i$.

As might seem intuitive, for games of perfect recall (and we deal only with these), the two types of randomization are equivalent. For any behavior strategy of player i, there is a mixed strategy for that player that yields exactly the same distribution over outcomes for any strategies, mixed or behavior, that might be played by i's rivals, and vice versa [this result is due to Kuhn (1953); see Exercise 7.E.1]. Which form of randomized strategy we consider is therefore a matter of analytical convenience; we typically use behavior strategies when analyzing the extensive form representation of a game and mixed strategies when analyzing the normal form.

Because the way we introduce randomization is solely a matter of analytical convenience, we shall be a bit loose in our terminology and refer to all randomized strategies as *mixed strategies*.

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EXERCISES

7.C.1[A] Suppose that in the Meeting in New York game (Example 7.B.3), there are two possible places where the two players can meet: Grand Central Station and the Empire State Building. Draw an extensive form representation (game tree) for this game.

7.D.1[B] In a game where player i has N information sets indexed $n = 1, \ldots, N$ and M_n possible actions at information set n, how many strategies does player i have?

7.D.2[A] In text.

7.E.1[B] Consider the two-player game whose extensive form representation (excluding payoffs) is depicted in Figure 7.Ex.1.

(a) What are player 1's possible strategies? Player 2's?

(b) Show that for any behavior strategy that player 1 might play, there is a realization equivalent mixed strategy; that is, a mixed strategy that generates the same probability distribution over the terminal nodes for *any* mixed strategy choice by player 2.

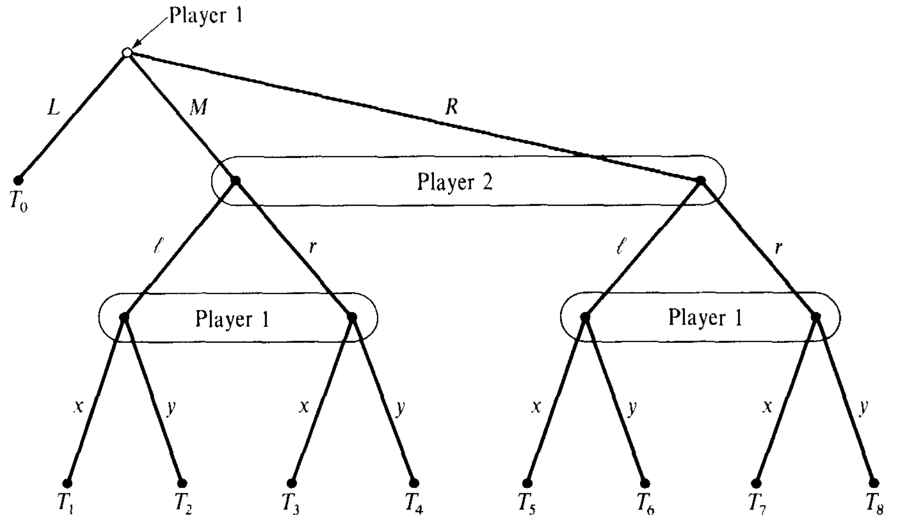


Figure 7.Ex.1

(c) Show that the converse is also true: For any mixed strategy that player 1 might play, there is a realization equivalent behavior strategy.

(d) Suppose that we change the game by merging the information sets at player 1's second round of moves (so that all four nodes are now in a single information set). Argue that the game is no longer one of perfect recall. Which of the two results in **(b)** and **(c)** still holds?

C H A P T E R

8

Simultaneous-Move Games

8.A Introduction

We now turn to the central question of game theory: What should we expect to observe in a game played by rational players who are fully knowledgeable about the structure of the game and each others' rationality? In this chapter, we study *simultaneous-move* games, in which all players move only once and at the same time. Our motivation for beginning with these games is primarily pedagogic; they allow us to concentrate on the study of strategic interaction in the simplest possible setting and to defer until Chapter 9 some difficult issues that arise in more general, dynamic games.

In Section 8.B, we introduce the concepts of *dominant* and *dominated* strategies. These notions and their extension in the concept of *iterated dominance* provide a first and compelling restriction on the strategies rational players should choose to play.

In Section 8.C, we extend these ideas by defining the notion of a *rationalizable strategy*. We argue that the implication of players' common knowledge of each others' rationality and of the structure of the game is precisely that they will play rationalizable strategies.

Unfortunately, in many games, the set of rationalizable strategies does not yield a very precise prediction of the play that will occur. In the remaining sections of the chapter, we therefore study solution concepts that yield more precise predictions by adding "equilibrium" requirements regarding players' behavior.

Section 8.D begins our study of equilibrium-based solution concepts by introducing the important and widely applied concept of *Nash equilibrium*. This concept adds to the assumption of common knowledge of players' rationality a requirement of *mutally correct expectations*. By doing so, it often greatly narrows the set of predicted outcomes of a game. We discuss in some detail the reasonableness of this requirement, as well as the conditions under which we can be assured that a Nash equilibrium exists.

In Sections 8.E and 8.F, we examine two extensions of the Nash equilibrium concept. In Section 8.E, we broaden the notion of a Nash equilibrium to cover situations with *incomplete information*, where each player's payoffs may, to some extent, be known only by the player. This yields the concept of *Bayesian Nash* *equilibrium.* In Section 8.F, we explore the implications of players entertaining the possibility that, with some small but positive probability, their opponents might make a mistake in choosing their strategies. We define the notion of a *(normal form) trembling-hand perfect Nash equilibrium,* an extension of the Nash equilibrium concept that requires that equilibria be robust to the possibility of small mistakes.

Throughout the chapter, we study simultaneous-move games using their normal form representations (see Section 7.D). Thus, we use $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ when we consider only pure (nonrandom) strategy choices and $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ when we allow for the possibility of randomized choices by the players (see Section 7.E for a discussion of randomized choices). We often denote a profile of pure strategies for player i's opponents by $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_I)$, with a similar meaning applying to the profile of mixed strategies σ_{-i}. We then write $s = (s_i, s_{-i})$ and $\sigma = (\sigma_i, \sigma_{-i})$. We also let $S = S_1 \times \cdots \times S_I$ and $S_{-i} = S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_I$.

8.B Dominant and Dominated Strategies

We begin our study of simultaneous-move games by considering the predictions that can be made based on a relatively simple means of comparing a player's possible strategies: that of *dominance.*

To keep matters as simple as possible, we initially ignore the possibility that players might randomize in their strategy choices. Hence, our focus is on games $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ whose strategy sets allow for only pure strategies.

Consider the game depicted in Figure 8.B.1, the famous *Prisoner's Dilemma.* The story behind this game is as follows: Two individuals are arrested for allegedly engaging in a serious crime and are held in separate cells. The district attorney (the DA) tries to extract a confession from each prisoner. Each is privately told that if he is the only one to confess, then he will be rewarded with a light sentence of 1 year while the recalcitrant prisoner will go to jail for 10 years. However, if he is the only one not to confess, then it is he who will serve the 10-year sentence. If both confess, they will both be shown some mercy: they will each get 5 years. Finally, if neither confesses, it will still be possible to convict both of a lesser crime that carries a sentence of 2 years. Each player wishes to minimize the time he spends in jail (or maximize the negative of this, the payoffs that are depicted in Figure 8.B.1).

What will the outcome of this game be? There is only one plausible answer: (confess, confess). To see why, note that playing "confess" is each player's best strategy *regardless of what the other player does.* This type of strategy is known as a *strictly dominant strategy.*

		Prisoner 2: Don't Confess	Prisoner 2: Confess
Prisoner 1	Don't Confess	$-2, -2$	$-10, -1$
	Confess	$-1, -10$	$-5, -5$

Figure 8.B.1
The Prisoner's Dilemma.

Definition 8.B.1: A strategy $s_i \in S_i$ is a *strictly dominant strategy* for player i in game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ if for all $s'_i \neq s_i$, we have

$$u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$$

for all $s_{-i} \in S_{-i}$.

In words, a strategy s_i is a strictly dominant strategy for player i if it maximizes uniquely player i's payoff for any strategy that player i's rivals might play. (The reason for the modifier *strictly* in Definition 8.B.1 will be made clear in Definition 8.B.3.) If a player has a strictly dominant strategy, as in the Prisoner's Dilemma, we should expect him to play it.

The striking aspect of the (confess, confess) outcome in the Prisoner's Dilemma is that although it is the one we expect to arise, it is not the best outcome for the players *jointly*; both players would prefer that neither of them confess. For this reason, the Prisoner's Dilemma is the paradigmatic example of self-interested, rational behavior *not* leading to a socially optimal result.

One way of viewing the outcome of the Prisoner's Dilemma is that, in seeking to maximize his own payoff, each prisoner has a negative effect on his partner; by moving away from the (don't confess, don't confess) outcome, a player reduces his jail time by 1 year but increases that of his partner by 8 (in Chapter 11, we shall see this as an example of an *externality*).

Dominated Strategies

Although it is compelling that players should play strictly dominant strategies if they have them, it is rare for such strategies to exist. Often, one strategy of player i's may be best when his rivals play s_{-i} and another when they play some other strategies s'_{-i} (think of the standard Matching Pennies game in Chapter 7). Even so, we might still be able to use the idea of dominance to eliminate some strategies as possible choices. In particular, we should expect that player i will not play *dominated* strategies, those for which there is some alternative strategy that yields him a greater payoff regardless of what the other players do.

Definition 8.B.2: A strategy $s_i \in S_i$ is *strictly dominated* for player i in game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ if there exists another strategy $s'_i \in S_i$ such that for all $s_{-i} \in S_{-i}$,

$$u_i(s'_i, s_{-i}) > u_i(s_i, s_{-i}).$$

In this case, we say that strategy s'_i *strictly dominates* strategy s_i.

With this definition, we can restate our definition of a strictly dominant strategy (Definition 8.B.1) as follows: Strategy s_i is a strictly dominant strategy for player i in game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ if it strictly dominates every other strategy in S_i.

Example 8.B.1: Consider the game shown in Figure 8.B.2. There is no strictly dominant strategy, but strategy D for player 1 is strictly dominated by strategy M (and also by strategy U). ■

Definition 8.D.3 presents a related, weaker notion of a dominated strategy that is of some importance.

Player 1 \ Player 2	L	R
U	1, −1	−1, 1
M	−1, 1	1, −1
D	−2, 5	−3, 2

Player 1 \ Player 2	L	R
U	5, 1	4, 0
M	6, 0	3, 1
D	6, 4	4, 4

Figure 8.B.2 (left) Strategy D is strictly dominated.

Figure 8.B.3 (right) Strategies U and M are weakly dominated.

Definition 8.B.3: A strategy $s_i \in S_i$ is *weakly dominated* in game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ if there exists another strategy $s_i' \in S_i$ such that for all $s_{-i} \in S_{-i}$,

$$u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i}),$$

with strict inequality for *some* s_{-i}. In this case, we say that strategy s_i' *weakly dominates* strategy s_i. A strategy is a *weakly dominant strategy* for player i in game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ if it weakly dominates every other strategy in S_i.

Thus, a strategy is weakly dominated if another strategy does at least as well for all s_{-i} and strictly better for some s_{-i}.

Example 8.B.2: Figure 8.B.3 depicts a game in which player 1 has two weakly dominated strategies, U and M. Both are weakly dominated by strategy D. ∎

Unlike a strictly dominated strategy, a strategy that is only weakly dominated cannot be ruled out based solely on principles of rationality. For any alternative strategy that player i might pick, there is at least one profile of strategies for his rivals for which the weakly dominated strategy does as well. In Figure 8.B.3, for example, player 1 could rationally pick M if he was *absolutely sure* that player 2 would play L. Yet, if the probability of player 2 choosing strategy R was perceived by player 1 as positive (no matter how small), then M would not be a rational choice for player 1. *Caution* might therefore rule out M. More generally, weakly dominated strategies could be dismissed if players always believed that there was at least some positive probability that any strategies of their rivals could be chosen. We do not pursue this idea here, although we return to it in Section 8.F. For now, we continue to allow a player to entertain any conjecture about what an opponent might play, even a perfectly certain one.

Iterated Deletion of Strictly Dominated Strategies

As we have noted, it is unusual for elimination of strictly dominated strategies to lead to a unique prediction for a game (e.g., recall the game in Figure 8.B.2). However, the logic of eliminating strictly dominated strategies can be pushed further, as demonstrated in Example 8.B.3.

Example 8.B.3: In Figure 8.B.4, we depict a modification of the Prisoner's Dilemma, which we call the *DA's Brother*.

The story (a somewhat far-fetched one!) is now as follows: One of the prisoners, prisoner 1, is the DA's brother. The DA has some discretion in the fervor with which

		Prisoner 2: Don't Confess	Prisoner 2: Confess
Prisoner 1	Don't Confess	$0, -2$	$-10, -1$
	Confess	$-1, -10$	$-5, -5$

Figure 8.B.4
The DA's Brother.

he prosecutes and, in particular, can allow prisoner 1 to go free if neither of the prisoners confesses. With this change, if prisoner 2 confesses, then prisoner 1 should also confess; but "don't confess" has become prisoner 1's best strategy if prisoner 2 plays "don't confess." Thus, we are unable to rule out either of prisoner 1's strategies as being dominated, and so elimination of strictly dominated (or, for that matter, weakly dominated) strategies does not lead to a unique prediction.

However, we can still derive a unique prediction in this game if we push the logic of eliminating strictly dominated strategies further. Note that "don't confess" is still strictly dominated for prisoner 2. Furthermore, once prisoner 1 eliminates "don't confess" as a possible action by prisoner 2, "confess" is prisoner 1's unambiguously optimal action; that is, it is his strictly dominant strategy once the strictly dominated strategy of prisoner 2 has been deleted. Thus, the unique predicted outcome in the DA's Brother game should still be (confess, confess). ■

Note the way players' common knowledge of each other's payoffs and rationality is used to solve the game in Example 8.B.3. Elimination of strictly dominated strategies requires only that each player be rational. What we have just done, however, requires not only that prisoner 2 be rational but also that prisoner 1 *know* that prisoner 2 is rational. Put somewhat differently, a player need not know anything about his opponents' payoffs or be sure of their rationality to eliminate a strictly dominated strategy from consideration as his own strategy choice; but for the player to eliminate one of his strategies from consideration because it is dominated if his opponents never play *their* dominated strategies *does* require this knowledge.

As a general matter, if we are willing to assume that all players are rational *and* that this fact and the players' payoffs are common knowledge (so everybody knows that everybody knows that . . . everybody is rational), then we do not need to stop after only two iterations. We can eliminate not only strictly dominated strategies and strategies that are strictly dominated after the first deletion of strategies but also strategies that are strictly dominated after this *next* deletion of strategies, and so on. Note that with each elimination of strategies, it becomes possible for additional strategies to become dominated because the fewer strategies that a player's opponents might play, the more likely that a particular strategy of his is dominated. However, each additional iteration requires that players' knowledge of each others' rationality be one level deeper. A player must now know not only that his rivals are rational but also that they know that he is, and so on.

One feature of the process of iteratively eliminating strictly dominated strategies is that the order of deletion does not affect the set of strategies that remain in the end (see Exercise 8.B.4), That is, if at any given point several strategies (of one or several players) are strictly dominated, then we can eliminate them all at once or in any sequence without changing the set of strategies that we ultimately end up with. This is fortunate, since we would worry if our prediction depended on the arbitrarily chosen order of deletion.

Exercise 8.B.5 presents an interesting example of a game for which the iterated removal of strictly dominated strategies yields a unique prediction: the *Cournot duopoly game* (which we will discuss in detail in Chapter 12).

The iterated deletion of *weakly* dominated strategies is harder to justify. As we have already indicated, the argument for deletion of a weakly dominated strategy for player i is that he contemplates the possibility that every strategy combination of his rivals occurs with positive probability. However, this hypothesis clashes with the logic of iterated deletion, which assumes, precisely, that eliminated strategies are not expected to occur. This inconsistency leads the iterative elimination of weakly dominated strategies to have the undesirable feature that it *can* depend on the order of deletion. The game in Figure 8.B.3 provides an example. If we first eliminate strategy U, we next eliminate strategy L, and we can then eliminate strategy M; (D, R) is therefore our prediction. If, instead, we eliminate strategy M first, we next eliminate strategy R, and we can then eliminate strategy U; now (D, L) is our prediction.

Allowing for Mixed Strategies

When we recognize that players may randomize over their pure strategies, the basic definitions of strictly dominated and dominant strategies can be generalized in a straightforward way.

Definition 8.B.4: A strategy $\sigma_i \in \Delta(S_i)$ is *strictly dominated* for player i in game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ if there exists another strategy $\sigma'_i \in \Delta(S_i)$ such that for all $\sigma_{-i} \in \prod_{j \neq i} \Delta(S_j)$,

$$u_i(\sigma'_i, \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i}).$$

In this case, we say that strategy σ'_i *strictly dominates* strategy σ_i. A strategy σ_i is a *strictly dominant strategy* for player i in game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ if it strictly dominates every other strategy in $\Delta(S_i)$.

Using this definition and the structure of mixed strategies, we can say a bit more about the set of strictly dominated strategies in game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$.

Note first that when we test whether a strategy σ_i is strictly dominated by strategy σ'_i for player i, we need only consider these two strategies' payoffs against the *pure* strategies of i's opponents. That is,

$$u_i(\sigma'_i, \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i}) \quad \text{for all } \sigma_{-i}$$

if and only if

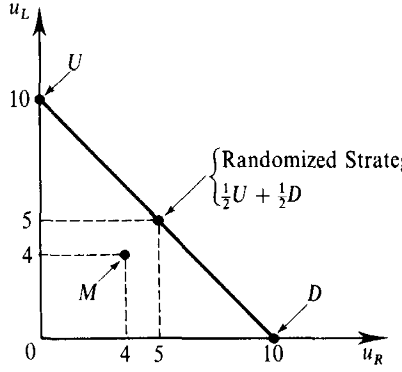
$$u_i(\sigma'_i, s_{-i}) > u_i(\sigma_i, s_{-i}) \quad \text{for all } s_{-i}.$$

This follows because we can write

$$u_i(\sigma'_i, \sigma_{-i}) - u_i(\sigma_i, \sigma_{-i}) = \sum_{s_{-i} \in S_{-i}} \left[\prod_{k \neq i} \sigma_k(s_k)\right] [u_i(\sigma'_i, s_{-i}) - u_i(\sigma_i, s_{-i})].$$

		Player 2	
		L	R
Player 1	U	10, 1	0, 4
	M	4, 2	4, 3
	D	0, 5	10, 2

(a)



(b)

Figure 8.B.5
Domination of a pure strategy by a randomized strategy.

This expression is positive for all σ_{-i} if and only if $[u_i(\sigma_i', s_{-i}) - u_i(\sigma_i, s_{-i})]$ is positive for all s_{-i}. One implication of this point is presented in Proposition 8.B.1.

Proposition 8.B.1: Player i's pure strategy $s_i \in S_i$ is strictly dominated in game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ if and only if there exists another strategy $\sigma_i' \in \Delta(S_i)$ such that

$$u_i(\sigma_i', s_{-i}) > u_i(s_i, s_{-i})$$

for all $s_{-i} \in S_{-i}$.

Proposition 8.B.1 tells us that to test whether a pure strategy s_i is dominated when randomized play is possible, the test given in Definition 8.B.2 need only be augmented by checking whether any of player i's mixed strategies does better than s_i against every possible profile of pure strategies by i's rivals.

In fact, this extra requirement can eliminate additional pure strategies because a pure strategy s_i may be dominated only by a randomized combination of other pure strategies; that is, to dominate a strategy, even a pure one, it may be necessary to consider alternative strategies that involve randomization. To see this, consider the two-player game depicted in Figure 8.B.5(a). Player 1 has three strategies: U, M, and D. We can see that U is an excellent strategy when player 2 plays L but a poor one against R and that D is excellent against R and poor against L. Strategy M, on the other hand, is a good but not great strategy against both L and R. None of these three pure strategies is strictly dominated by any of the others. But if we allow player 1 to randomize, then playing U and D each with probability $\frac{1}{2}$ yields player 1 an expected payoff of 5 regardless of player 2's strategy, strictly dominating M (remember, payoffs are levels of von Neumann–Morgenstern utilities). This is shown in Figure 8.B.5(b), where player 1's expected payoffs from playing U, D, M, and the randomized strategy $\frac{1}{2}U + \frac{1}{2}D$ are plotted as points in $\mathbb{R}^2$ (the two dimensions correspond to a strategy's expected payoff for player 1 when player 2 plays R, denoted by u_R, and L, denoted by u_L). In the figure, the payoff vectors achievable by randomizing over U and D, and that from the randomized strategy $\frac{1}{2}U + \frac{1}{2}D$ in particular, lie on the line connecting points (0, 10) and (10, 0). As can be seen, the payoffs from $\frac{1}{2}U + \frac{1}{2}D$ strictly dominate those from strategy M.

Once we have determined the set of undominated pure strategies for player i, we need to consider which mixed strategies are undominated. We can immediately eliminate any mixed strategy that uses a dominated pure strategy; if pure strategy s_i is strictly dominated for player i, then so is every mixed strategy that assigns a positive probability to this strategy.

Exercise 8.B.6: Prove that if pure strategy s_i is a strictly dominated strategy in game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$, then so is any strategy that plays s_i with positive probability.

But these are not the only mixed strategies that may be dominated. A mixed strategy that randomizes over undominated pure strategies may itself be dominated. For example, if strategy M in Figure 8.B.5(a) instead gave player 1 a payoff of 6 for either strategy chosen by player 2, then although neither strategy U nor strategy D would be strictly dominated, the randomized strategy $\frac{1}{2}U + \frac{1}{2}D$ would be strictly dominated by strategy M [look where the point (6, 6) would lie in Figure 8.B.5(b)].

In summary, to find the set of strictly dominated strategies for player i in $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$, we can first eliminate those pure strategies that are strictly dominated by applying the test in Proposition 8.B.1. Call player i's set of undominated pure strategies $S_i^u \subset S_i$. Next, eliminate any mixed strategies in set $\Delta(S_i^u)$ that are dominated. Player i's set of undominated strategies (pure and mixed) is exactly the remaining strategies in set $\Delta(S_i^u)$.

As when we considered only pure strategies, we can push the logic of removal of strictly dominated strategies in game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ further through iterative elimination. The preceding discussion implies that this iterative procedure can be accomplished with the following two-stage procedure: First iteratively eliminate dominated pure strategies using the test in Proposition 8.B.1, applied at each stage using the remaining set of pure strategies. Call the remaining sets of pure strategies $\{\bar{S}_1^u, \dots, \bar{S}_I^u\}$. Then, eliminate any mixed strategies in sets $\{\Delta(\bar{S}_1^u), \dots, \Delta(\bar{S}_I^u)\}$ that are dominated.

8.C Rationalizable Strategies

In Section 8.B, we eliminated strictly dominated strategies based on the argument that a rational player would never choose such a strategy regardless of the strategies that he anticipates his rivals will play. We then used players' common knowledge of each others' rationality and the structure of the game to justify iterative removal of strictly dominated strategies.

In general, however, players' common knowledge of each others' rationality and the game's structure allows us to eliminate more than just those strategies that are iteratively strictly dominated. Here, we develop this point, leading to the concept of a *rationalizable strategy*. The set of rationalizable strategies consists precisely of those strategies that may be played in a game where the structure of the game and the players' rationality are common knowledge among the players. Throughout this section, we focus on games of the form $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ (mixed strategies are permitted).

We begin with Definition 8.C.1.

Definition 8.C.1: In game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$, strategy σ_i is a *best response* for player i to his rivals' strategies σ_{-i} if

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$$

for all $\sigma_i' \in \Delta(S_i)$. Strategy σ_i is *never a best response* if there is no σ_{-i} for which σ_i is a best response.

Strategy σ_i is a best response to σ_{-i} if it is an optimal choice when player i conjectures that his opponents will play σ_{-i}. Player i's strategy σ_i is never a best response if there is no belief that player i may hold about his opponents' strategy choices σ_{-i} that justifies choosing strategy σ_i.[1] Clearly, a player should not play a strategy that is never a best response.

Note that a strategy that is strictly dominated is never a best response. However, as a general matter, a strategy might never be a best response even though it is not strictly dominated (we say more about this relation at the end of this section in small type). Thus, eliminating strategies that are never a best response must eliminate at least as many strategies as eliminating just strictly dominated strategies and may eliminate more.

Moreover, as in the case of strictly dominated strategies, common knowledge of rationality and the game's structure implies that we can iterate the deletion of strategies that are never a best response. In particular, a rational player should not play a strategy that is never a best response once he eliminates the possibility that any of his rivals might play a strategy that is never a best response for them, and so on.

Equally important, the strategies that remain after this iterative deletion are the strategies that a rational player can *justify*, or *rationalize*, affirmatively with some reasonable conjecture about the choices of his rivals; that is, with a conjecture that does not assume that any player will play a strategy that is never a best response or one that is only a best response to a conjecture that someone else will play such a strategy, and so on. (Example 8.C.1 provides an illustration of this point.) As a result, the set of strategies surviving this iterative deletion process can be said to be precisely the set of strategies that can be played by rational players in a game in which the players' rationality and the structure of the game are common knowledge. They are known as *rationalizable strategies* [a concept developed independently by Bernheim (1984) and Pearce (1984)].

Definition 8.C.2: In game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$, the strategies in $\Delta(S_i)$ that survive the iterated removal of strategies that are never a best response are known as player i's *rationalizable strategies*.

Note that the set of rationalizable strategies can be no larger than the set of strategies surviving iterative removal of strictly dominated strategies because, at each stage of the iterative process in Definition 8.C.2, all strategies that are strictly dominated at that stage are eliminated. As in the case of iterated deletion of strictly dominated strategies, the order of removal of strategies that are never a best response can be shown not to affect the set of strategies that remain in the end (see Exercise 8.C.2).

1. We speak here as if a player's conjecture is necessarily deterministic in the sense that the player believes it is certain that his rivals will play a particular profile of mixed strategies σ_{-i}. One might wonder about conjectures that are probabilistic, that is, that take the form of a nondegenerate probability distribution over possible profiles of mixed strategy choices by his rivals. In fact, a strategy σ_i is an optimal choice for player i given some probabilistic conjecture (that treats his opponents' choices as independent random variables) only if it is an optimal choice given some deterministic conjecture. The reason is that if σ_i is an optimal choice given some probabilistic conjecture, then it must be a best response to the profile of mixed strategies σ_{-i} that plays each possible pure strategy profile $s_{-i} \in S_{-i}$ with exactly the compound probability implied by the probabilistic conjecture.

Player 2

		b_1	b_2	b_3	b_4
Player 1	a_1	0, 7	2, 5	7, 0	0, 1
	a_2	5, 2	3, 3	5, 2	0, 1
	a_3	7, 0	2, 5	0, 7	0, 1
	a_4	0, 0	0, −2	0, 0	10, −1

Figure 8.C.1 $\{a_1, a_2, a_3\}$ are rationalizable strategies for player 1; $\{b_1, b_2, b_3\}$ are rationalizable strategies for player 2.

Example 8.C.1: Consider the game depicted in Figure 8.C.1, which is taken from Bernheim (1984). What is the set of rationalizable pure strategies for the two players? In the first round of deletion, we can eliminate strategy b_4, which is never a best response because it is strictly dominated by a strategy that plays strategies b_1 and b_3 each with probability $\frac{1}{2}$. Once strategy b_4 is eliminated, strategy a_4 can be eliminated because it is strictly dominated by a_2 once b_4 is deleted. At this point, no further strategies can be ruled out: a_1 is a best response to b_3, a_2 is a best response to b_2, and a_3 is a best response to b_1. Similarly, you can check that b_1, b_2, and b_3 are each best responses to one of a_1, a_2, and a_3. Thus, the set of rationalizable pure strategies for player 1 is $\{a_1, a_2, a_3\}$, and the set $\{b_1, b_2, b_3\}$ is rationalizable for player 2.

Note that for each of these rationalizable strategies, a player can construct a *chain of justification* for his choice that never relies on any player believing that another player will play a strategy that is never a best response.[2] For example, in the game in Figure 8.C.1, player 1 can justify choosing a_2 by the belief that player 2 will play b_2, which player 1 can justify to himself by believing that player 2 will think that he is going to play a_2, which is reasonable if player 1 believes that player 2 is thinking that he, player 1, thinks player 2 will play b_2, and so on. Thus, player 1 can construct an (infinite) chain of justification for playing strategy a_2, $(a_2, b_2, a_2, b_2, \dots)$, where each element is justified using the next element in the sequence.

Similarly, player 1 can rationalize playing strategy a_1 with the chain of justification $(a_1, b_3, a_3, b_1, a_1, b_3, a_3, b_1, a_1, \dots)$. Here player 1 justifies playing a_1 by believing that player 2 will play b_3. He justifies the belief that player 2 will play b_3 by thinking that player 2 believes that he, player 1, will play a_3. He justifies this belief by thinking that player 2 thinks that he, player 1, believes that player 2 will play b_1. And so on.

Suppose, however, that player 1 tried to justify a_4. He could do so only by a belief that player 2 would play b_4, but there is *no* belief that player 2 could have that would justify b_4. Hence, player 1 cannot justify playing the nonrationalizable strategy a_4. ■

2. In fact, this chain-of-justification approach to the set of rationalizable strategies is used in the original definition of the concept [for a formal treatment, consult Bernheim (1984) and Pearce (1984)].

It can be shown that under fairly weak conditions a player always has at least one rationalizable strategy.[3] Unfortunately, players may have many rationalizable strategies, as in Example 8.C.1. If we want to narrow our predictions further, we need to make additional assumptions beyond common knowledge of rationality. The solution concepts studied in the remainder of this chapter do so by imposing "equilibrium" requirements on players' strategy choices.

We have said that the set of rationalizable strategies is no larger than the set remaining after iterative deletion of strictly dominated strategies. It turns out, however, that for the case of two-player games ($I = 2$), these two sets are identical because in two-player games a (mixed) strategy σ_i is a best response to some strategy choice of a player's rival whenever σ_i is not strictly dominated.

To see that this is plausible, reconsider the game in Figure 8.B.5 (Exercise 8.C.3 asks you for a general proof). Suppose that the payoffs from strategy M are altered so that M is not strictly dominated. Then, as depicted in Figure 8.C.2, the payoffs from M lie somewhere above

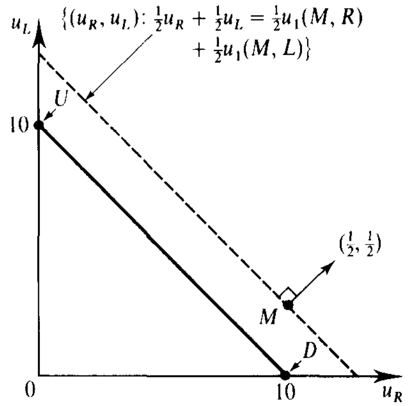


Figure 8.C.2
In a two-player game, a strategy is a best response if it is not strictly dominated.

the line connecting the points for strategies U and D. Is M a best response here? Yes. To see this, note that if player 2 plays strategy R with probability $\sigma_2(R)$, then player 1's expected payoff from choosing a strategy with payoffs $(u_R,\ u_L)$ is $\sigma_2(R)u_R + (1 - \sigma_2(R))u_L$. Points yielding the same expected payoff as strategy M therefore lie on a hyperplane with normal vector $(1 - \sigma_2(R), \sigma_2(R))$. As can be seen, strategy M is a best response to $\sigma_2(R) = \frac{1}{2}$; it yields an expected payoff strictly larger than any expected payoff achievable by playing strategies U and/or D.

With more than two players, however, there can be strategies that are never a best response and yet are not strictly dominated. The reason can be traced to the fact that players' randomizations are independent. If the randomizations by i's rivals can be correlated (we discuss how this might happen at the end of Sections 8.D and 8.E), the equivalence reemerges. Exercise 8.C.4 illustrates these points.

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3. This will be true, for example, whenever a Nash equilibrium (introduced in Section 8.D) exists.

8.D Nash Equilibrium

In this section, we present and discuss the most widely used solution concept in applications of game theory to economics, that of *Nash equilibrium* [due to Nash (1951)]. Throughout the rest of the book, we rely on it extensively.

For ease of exposition, we initially ignore the possibility that players might randomize over their pure strategies, restricting our attention to game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$. Mixed strategies are introduced later in the section.

We begin with Definition 8.D.1.

Definition 8.D.1: A strategy profile $s = (s_1, \ldots, s_I)$ constitutes a *Nash equilibrium* of game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ if for every $i = 1, \ldots, I$,

$$u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$$

for all $s'_i \in S_i$.

In a Nash equilibrium, each player's strategy choice is a best response (see Definition 8.C.1) to the strategies *actually played* by his rivals. The italicized words distinguish the concept of Nash equilibrium from the concept of rationalizability studied in Section 8.C. Rationalizability, which captures the implications of the players' common knowledge of each others' rationality and the structure of the game, requires only that a player's strategy be a best response to some reasonable conjecture about what his rivals will be playing, where *reasonable* means that the conjectured play of his rivals can also be so justified. Nash equilibrium adds to this the requirement that players be *correct* in their conjectures.

Examples 8.D.1 and 8.D.2 illustrate the use of the concept.

Example 8.D.1: Consider the two-player simultaneous-move game shown in Figure 8.D.1. We can see that (M, m) is a Nash equilibrium. If player 1 chooses M, then the best response of player 2 is to choose m; the reverse is true for player 2. Moreover, (M, m) is the only combination of (pure) strategies that is a Nash equilibrium. For example, strategy profile (U, r) cannot be a Nash equilibrium because player 1 would prefer to deviate to strategy D given that player 2 is playing r. (Check the other possibilities for yourself.) ■

Example 8.D.2: *Nash Equilibrium in the Game of Figure 8.C.1.* In this game, the unique Nash equilibrium profile of (pure) strategies is (a_2, b_2). Player 1's best response to b_2 is a_2, and player 2's best response to a_2 is b_2, so (a_2, b_2) is a Nash equilibrium.

		Player 2		
		ℓ	m	r
	U	5, 3	0, 4	3, 5
Player 1	M	4, 0	5, 5	4, 0
	D	3, 5	0, 4	5, 3

Figure 8.D.1
A Nash equilibrium.

		Mr. Schelling	
		Empire State	Grand Central
Mr. Thomas	Empire State	100, 100	0, 0
	Grand Central	0, 0	100, 100

Figure 8.D.2
Nash equilibria in the Meeting in New York game.

At any other strategy profile, one of the players has an incentive to deviate. [In fact, (a_2, b_2) is the unique Nash equilibrium even when randomization is permitted; see Exercise 8.D.1.]

This example illustrates a general relationship between the concept of Nash equilibrium and that of rationalizable strategies: *Every strategy that is part of a Nash equilibrium profile is rationalizable* because each player's strategy in a Nash equilibrium can be justified by the Nash equilibrium strategies of the other players. Thus, as a general matter, the Nash equilibrium concept offers at least as sharp a prediction as does the rationalizability concept. In fact, it often offers a *much* sharper prediction. In the game of Figure 8.C.1, for example, the rationalizable strategies a_1, a_3, b_1, and b_3 are eliminated as predictions because they cannot be sustained when players' beliefs about each other's play are required to be correct. ■

In the previous two examples, the Nash equilibrium concept yields a unique prediction. However, this is not always the case. Consider the Meeting in New York game.

Example 8.D.3: *Nash Equilibria in the Meeting in New York Game.* Figure 8.D.2 depicts a simple version of the Meeting in New York game. Mr. Thomas and Mr. Schelling each have two choices: They can meet either at noon at the top of the Empire State Building or at noon at the clock in Grand Central Station. There are two Nash equilibria (ignoring the possibility of randomization): (Empire State, Empire State) and (Grand Central, Grand Central). ■

Example 8.D.3 emphasizes how strongly the Nash equilibrium concept uses the assumption of mutually correct expectations. The theory of Nash equilibrium is silent on *which* equilibrium we should expect to see when there are many. Yet, the players are assumed to correctly forecast which one it will be.

A compact restatement of the definition of a Nash equilibrium can be obtained through the introduction of the concept of a player's *best-response correspondence*. Formally, we say that player i's best-response correspondence $b_i\colon S_{-i} \to S_i$ in the game $\Gamma_N = [I, \{S_i\} \{u_i(\cdot)\}]$, is the correspondence that assigns to each $s_{-i} \in S_{-i}$ the set

$$b_i(s_{-i}) = \{s_i \in S_i\colon u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \text{ for all } s_i' \in S_i\}.$$

With this notion, we can restate the definition of a Nash equilibrium as follows: The strategy profile $(s_1, \ldots, s_I)$ is a Nash equilibrium of game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ if and only if $s_i \in b_i(s_{-i})$ for $i = 1, \ldots, I$.

Discussion of the Concept of Nash Equilibrium

Why might it be reasonable to expect players' conjectures about each other's play to be correct? Or, in sharper terms, why should we concern ourselves with the concept of Nash equilibrium?

A number of arguments have been put forward for the Nash equilibrium concept and you will undoubtedly react to them with varying degrees of satisfaction. Moreover, one argument might seem compelling in one application but not at all convincing in another. Until very recently, all these arguments have been informal, as will be our discussion. The issue is one of the more important open questions in game theory, particularly given the Nash equilibrium concept's widespread use in applied problems, and it is currently getting some formal attention.

(i) *Nash equilibrium as a consequence of rational inference.* It is sometimes argued that because each player can think through the strategic considerations faced by his opponents, rationality alone implies that players must be able to correctly forecast what their rivals will play. Although this argument may seem appealing, it is faulty. As we saw in Section 8.C, the implication of common knowledge of the players' rationality (and of the game's structure) is precisely that each player must play a rationalizable strategy. Rationality need not lead players' forecasts to be correct.

(ii) *Nash equilibrium as a necessary condition if there is a unique predicted outcome to a game.* A more satisfying version of the previous idea argues that if there is a unique predicted outcome for a game, then rational players will understand this. Therefore, for no player to wish to deviate, this predicted outcome must be a Nash equilibrium. Put somewhat differently [as in Kreps (1990)], if players think and share the belief that there is an *obvious* (in particular, a unique) way to play a game, then it must be a Nash equilibrium.

Of course, this argument is only relevant if there is a unique prediction for how players will play a game. The discussion of rationalizability in Section 8.C, however, shows that common knowledge of rationality alone does not imply this. Therefore, this argument is really useful only in conjunction with some reason why a particular profile of strategies might be the obvious way to play a particular game. The other arguments for Nash equilibrium that we discuss can be viewed as combining this argument with a reason why there might be an "obvious" way to play a game.

(iii) *Focal points.* It sometimes happens that certain outcomes are what Schelling (1960) calls *focal.* For example, take the Meeting in New York game depicted in Figure 8.D.2, and suppose that restaurants in the Grand Central area are so much better then those around the Empire State Building that the payoffs to meeting at Grand Central are (1000, 1000) rather than (100, 100). Suddenly, going to Grand Central seems like the obvious thing to do. Focal outcomes could also be culturally determined. As Schelling pointed out in his original discussion, two people who do not live in New York will tend to find meeting at the top of the Empire State building (a famous tourist site) to be focal, whereas two native New Yorkers will find Grand

Central Station (the central railroad station) a more compelling choice. In both examples, one of the outcomes has a natural appeal. The implication of argument (ii) is that this kind of appeal can lead an outcome to be the clear prediction in a game only if the outcome is a Nash equilibrium.

(iv) *Nash equilibrium as a self-enforcing agreement.* Another argument for Nash equilibrium comes from imagining that the players can engage in nonbinding communication prior to playing the game. If players agree to an outcome to be played, this naturally becomes the obvious candidate for play. However, because players cannot bind themselves to their agreed-upon strategies, any agreement that the players reach must be self-enforcing if it is to be meaningful. Hence, any meaningful agreement must involve the play of a Nash equilibrium strategy profile. Of course, even though players have reached an agreement to play a Nash equilibrium, they could still deviate from it if they expect others to do so. In essence, this justification assumes that once the players have agreed to a choice of strategies, this agreement becomes focal.

(v) *Nash equilibrium as a stable social convention.* A particular way to play a game might arise over time if the game is played repeatedly and some stable social convention emerges. If it does, it may be "obvious" to all players that the convention will be maintained. The convention, so to speak, becomes focal.

A good example is the game played by New Yorkers every day: Walking in Downtown Manhattan. Every day, people who walk to work need to decide which side of the sidewalk they will walk on. Over time, the stable social convention is that everyone walks on the right side, a convention that is enforced by the fact that any individual who unilaterally deviates from it is sure to be severely trampled. Of course, on any given day, it is *possible* that an individual might decide to walk on the left by conjecturing that everyone else suddenly expects the convention to change. Nevertheless, the prediction that we will remain at the Nash equilibrium "everyone walks on the right" seems reasonable in this case. Note that if an outcome is to become a stable social convention, it must be a Nash equilibrium. If it were not, then individuals would deviate from it as soon as it began to emerge.

The notion of an equilibrium as a rest point for some dynamic adjustment process underlies the use and the traditional appeal of equilibrium notions in economics. In this sense, the stable social convention justification of Nash equilibrium is closest to the tradition of economic theory.

To formally model the emergence of stable social conventions is not easy. One difficulty is that the repeated one-day game may itself be viewed as a larger dynamic game. Thus, when we consider rational players choosing their strategies in this overall game, we are merely led back to our original conundrum: Why should we expect a Nash equilibrium in this larger game? One response to this difficulty currently getting some formal attention imagines that players follow simple rules of thumb concerning their opponents' likely play in situations where play is repeated (note that this implies a certain withdrawal from the assumption of complete rationality). For example, a player could conjecture that whatever his opponents did yesterday will be repeated today. If so, then each day players will play a best response to yesterday's play. If a combination of strategies arises that is a stationary point of this process (i.e., the

		Player 2	
		Heads	Tails
Player 1	Heads	−1, +1	+1, −1
	Tails	+1, −1	−1, +1

Figure 8.D.3 Matching Pennies.

play today is the same as it was yesterday), it must be a Nash equilibrium. However, it is less clear that from any initial position, the process will converge to a stationary outcome; convergence turns out to depend on the game.[4]

Mixed Strategy Nash Equilibria

It is straightforward to extend the definition of Nash equilibrium to games in which we allow the players to randomize over their pure strategies.

Definition 8.D.2: A mixed strategy profile $\sigma = (\sigma_1, \ldots, \sigma_I)$ constitutes a *Nash equilibrium* of game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ if for every $i = 1, \ldots, I$,

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$$

for all $\sigma_i' \in \Delta(S_i)$.

Example 8.D.4: As a very simple example, consider the standard version of Matching Pennies depicted in Figure 8.D.3. This is a game with no pure strategy equilibrium. On the other hand, it is fairly intuitive that there is a mixed strategy equilibrium in which each player chooses H or T with equal probability. When a player randomizes in this way, it makes his rival indifferent between playing heads or tails, and so his rival is also willing to randomize between heads and tails with equal probability. ■

It is not an accident that a player who is randomizing in a Nash equilibrium of Matching Pennies is indifferent between playing heads and tails. As Proposition 8.D.1 confirms, this indifference among strategies played with positive probability is a general feature of mixed strategy equilibria.

Proposition 8.D.1: Let $S_i^+ \subset S_i$ denote the set of pure strategies that player i plays with positive probability in mixed strategy profile $\sigma = (\sigma_1, \ldots, \sigma_I)$. Strategy profile σ is a Nash equilibrium in game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ if and only if for all $i = 1, \ldots, I$,

(i) $u_i(s_i, \sigma_{-i}) = u_i(s_i', \sigma_{-i})$ for all $s_i, s_i' \in S_i^+$;
(ii) $u_i(s_i, \sigma_{-i}) \geq u_i(s_i', \sigma_{-i})$ for all $s_i \in S_i^+$ and all $s_i' \notin S_i^+$.

Proof: For necessity, note that if either of conditions (i) or (ii) does not hold for some player i, then there are strategies $s_i \in S_i^+$ and $s_i' \in S_i$ such that $u_i(s_i', \sigma_{-i}) > u_i(s_i, \sigma_{-i})$. If so, player i can strictly increase his payoff by playing strategy s_i' whenever he would have played strategy s_i.

4. This approach actually dates to Cournot's (1838) myopic adjustment procedure. A recent example can be found in Milgrom and Roberts (1990). Interestingly, this work explains the "ultrarational" Nash outcome by *relaxing* the assumption of rationality. It also can be used to try to identify the likelihood of various Nash equilibria arising when multiple Nash equilibria exist.

For sufficiency, suppose that conditions (i) and (ii) hold but that σ is not a Nash equilibrium. Then there is some player i who has a strategy σ_i' with $u_i(\sigma_i', \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i})$. But if so, then there must be some pure strategy s_i' that is played with positive probability under σ_i' for which $u_i(s_i', \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i})$. Since $u_i(\sigma_i, \sigma_{-i}) = u_i(s_i, \sigma_{-i})$ for all $s_i \in S_i^+$, this contradicts conditions (i) and (ii) being satisfied. ■

Hence, a necessary and sufficient condition for mixed strategy profile σ to be a Nash equilibrium of game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ is that each player, given the distribution of strategies played by his opponents, is indifferent among all the pure strategies that he plays with positive probability and that these pure strategies are at least as good as any pure strategy he plays with zero probability.

An implication of Proposition 8.D.1 is that to test whether a strategy profile σ is a Nash equilibrium it suffices to consider only pure strategy deviations (i.e., changes in a player's strategy σ_i to some pure strategy s_i'). As long as no player can improve his payoff by switching to any pure strategy, σ is a Nash equilibrium. We therefore get the comforting result given in Corollary 8.D.1.

Corollary 8.D.1: Pure strategy profile $s = (s_1, \ldots, s_I)$ is a Nash equilibrium of game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ if and only if it is a (degenerate) mixed strategy Nash equilibrium of game $\Gamma_N' = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$.

Corollary 8.D.1 tells us that to identify the pure strategy equilibria of game $\Gamma_N' = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$, it suffices to restrict attention to the game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ in which randomization is not permitted.

Proposition 8.D.1 can also be of great help in the computation of mixed strategy equilibria as Example 8.D.5 illustrates.

Example 8.D.5: *Mixed Strategy Equilibria in the Meeting in New York Game.* Let us try to find a mixed strategy equilibrium in the variation of the Meeting in New York game where the payoffs of meeting at Grand Central are (1000, 1000). By Proposition 8.D.1, if Mr. Thomas is going to randomize between Empire State and Grand Central, he must be indifferent between them. Suppose that Mr. Schelling plays Grand Central with probability σ_s. Then Mr. Thomas' expected payoff from playing Grand Central is $1000\sigma_s + 0(1 - \sigma_s)$, and his expected payoff from playing Empire State is $100(1 - \sigma_s) + 0\sigma_s$. These two expected payoffs are equal only when $\sigma_s = 1/11$. Now, for Mr. Schelling to set $\sigma_s = 1/11$, he must also be indifferent between his two pure strategies. By a similar argument, we find that Mr. Thomas' probability of playing Grand Central must also be 1/11. We conclude that each player going to Grand Central with a probability of 1/11 is a Nash equilibrium. ■

Note that in accordance with Proposition 8.D.1, the players in Example 8.D.5 have no real preference over the probabilities that they assign to the pure strategies they play with positive probability. What determines the probabilities that each player uses is an equilibrium consideration: the need to make the *other* player indifferent over *his* strategies.

This fact has led some economists and game theorists to question the usefulness of mixed strategy Nash equilibria as predictions of play. They raise two concerns: First, if players always have a pure strategy that gives them the same expected payoff as their equilibrium mixed strategy, it is not clear why they will bother to randomize.

One answer to this objection is that players may not actually randomize. Rather, they may make definite choices that are affected by seemingly inconsequential variables ("signals") that only they observe. For example, consider how a pitcher for a major league baseball team "mixes his pitches" to keep batters guessing. He may have a completely deterministic plan for what he will do, but it may depend on which side of the bed he woke up on that day or on the number of red traffic lights he came to on his drive to the stadium. As a result, batters view the behavior of the pitcher as random even though it is not. We touched briefly on this interpretation of mixed strategies as behavior contingent on realizations of a signal in Section 7.E, and we will examine it in more detail in Section 8.E.

The second concern is that the stability of mixed strategy equilibria seems tenuous. Players must randomize with exactly the correct probabilities, but they have no positive incentive to do so. One's reaction to this problem may depend on why one expects a Nash equilibrium to arise in the first place. For example, the use of the correct probabilities may be unlikely to arise as a stable social convention, but may seem more plausible when the equilibrium arises as a self-enforcing agreement.

Up to this point, we have assumed that players' randomizations are independent. In the Meeting in New York game in Example 8.D.5, for instance, we could describe a mixed strategy equilibrium as follows: Nature provides *private and independently distributed* signals $(\theta_1, \theta_2) \in [0, 1] \times [0, 1]$ to the two players, and each player i assigns decisions to the various possible realizations of his signal θ_i.

However, suppose that there are also *public* signals available that both players observe. Let $\theta \in [0, 1]$ be such a signal. Then many new possibilities arise. For example, the two players could both decide to go to Grand Central if $\theta < \frac{1}{2}$ and to Empire State if $\theta \geq \frac{1}{2}$. Each player's strategy choice is still random, but the coordination of their actions is now perfect and they always meet. More importantly, the decisions have an equilibrium character. If one player decides to follow this decision rule, then it is also optimal for the other player to do so. This is an example of a *correlated equilibrium* [due to Aumann (1974)]. More generally, we could allow for correlated equilibria in which nature's signals are partly private and partly public.

Allowing for such correlation may be important because economic agents observe many public signals. Formally, a correlated equilibrium is a special case of a Bayesian Nash equilibrium, a concept that we introduce in Section 8.E; hence, we defer further discussion to the end of that section.

Existence of Nash Equilibria

Does a Nash equilibrium necessarily exist in a game? Fortunately, the answer turns out to be "yes" under fairly broad circumstances. Here we describe two of the more important existence results; their proofs, based on mathematical fixed point theorems, are given in Appendix A of this chapter. (Proposition 9.B.1 of Section 9.B provides another existence result.)

Proposition 8.D.2: Every game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ in which the sets $S_1, \ldots, S_I$ have a finite number of elements has a mixed strategy Nash equilibrium.

Thus, for the class of games we have been considering, a Nash equilibrium always exists as long as we are willing to accept equilibria in which players randomize. (If you want to be convinced without going through the proof, try Exercise 8.D.6.) Allowing for randomization is essential for this result. We have already seen in (standard) Matching Pennies, for example, that a pure strategy equilibrium may not exist in a game with a finite number of pure strategies.

Up to this point, we have focused on games with finite strategy sets. However, in economic applications, we frequently encounter games in which players have strategies naturally modeled as continuous variables. This can be helpful for the existence of a pure strategy equilibrium. In particular, we have the result given in Proposition 8.D.3.

Proposition 8.D.3: A Nash equilibrium exists in game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ if for all $i = 1, \ldots, I$,

(i) S_i is a nonempty, convex, and compact subset of some Euclidean space $\mathbb{R}^M$.

(ii) $u_i(s_1, \ldots, s_I)$ is continuous in $(s_1, \ldots, s_I)$ and quasiconcave in s_i.

Proposition 8.D.3 provides a significant result whose requirements are satisfied in a wide range of economic applications. The convexity of strategy sets and the nature of the payoff functions help to smooth out the structure of the model, allowing us to achieve a pure strategy equilibrium.[5]

Further existence results can also be established. In situations where quasiconcavity of the payoff functions $u_i(\cdot)$ fails but they are still continuous, existence of a mixed strategy equilibrium can still be demonstrated. In fact, even if continuity of the payoff functions fails to hold, a mixed strategy equilibrium can be shown to exist in a variety of cases [see Dasgupta and Maskin (1986)].

Of course, these results do not mean that we *cannot* have an equilibrium if the conditions of these existence results do not hold. Rather, we just cannot be *assured* that there is one.

8.E Games of Incomplete Information: Bayesian Nash Equilibrium

Up to this point, we have assumed that players know all relevant information about each other, including the payoffs that each receives from the various outcomes of the game. Such games are known as games of *complete information*. A moment of thought, however, should convince you that this is a very strong assumption. Do two firms in an industry necessarily know each other's costs? Does a firm bargaining with a union necessarily know the disutility that union members will feel if they go out on strike for a month? Clearly, the answer is "no." Rather, in many circumstances, players have what is known as *incomplete information*.

The presence of incomplete information raises the possibility that we may need to consider a player's beliefs about other players' preferences, his beliefs about their beliefs about his preferences, and so on, much in the spirit of rationalizability.[6]

5. Note that a finite strategy set S_i cannot be convex. In fact, the use of mixed strategies in Proposition 8.D.2 helps us to obtain existence of equilibrium in much the same way that Proposition 8.D.3's assumptions assure existence of a pure strategy Nash equilibrium: It convexifies players' strategy sets and yields well-behaved payoff functions. (See Appendix A for details.)

6. For more on this problem, see Mertens and Zamir (1985).

Fortunately, there is a widely used approach to this problem, originated by Harsanyi (1967–68), that makes this unnecessary. In this approach, one imagines that each player's preferences are determined by the realization of a random variable. Although the random variable's actual realization is observed only by the player, its ex ante probability distribution is assumed to be common knowledge among all the players. Through this formulation, the situation of incomplete information is reinterpreted as a game of imperfect information: Nature makes the first move, choosing realizations of the random variables that determine each player's preference *type*, and each player observes the realization of only his own random variable. A game of this sort is known as a *Bayesian game*.

Example 8.E.1: Consider a modification of the DA's Brother game discussed in Example 8.B.3. With probability μ, prisoner 2 has the preferences in Figure 8.B.4 (we call these *type I preferences*), while with probability $(1 - \mu)$, prisoner 2 hates to rat on his accomplice (this is *type II*). In this case, he pays a psychic penalty equal to 6 years in prison for confessing. Prisoner 1, on the other hand, always has the preferences depicted in Figure 8.B.4. The extensive form of this Bayesian game is represented in Figure 8.E.1 (in the figure, "C" and "DC" stand for "confess" and "don't confess" respectively).

In this game, a pure strategy (a complete contingent plan) for player 2 can be viewed as a function that for each possible realization of his preference type

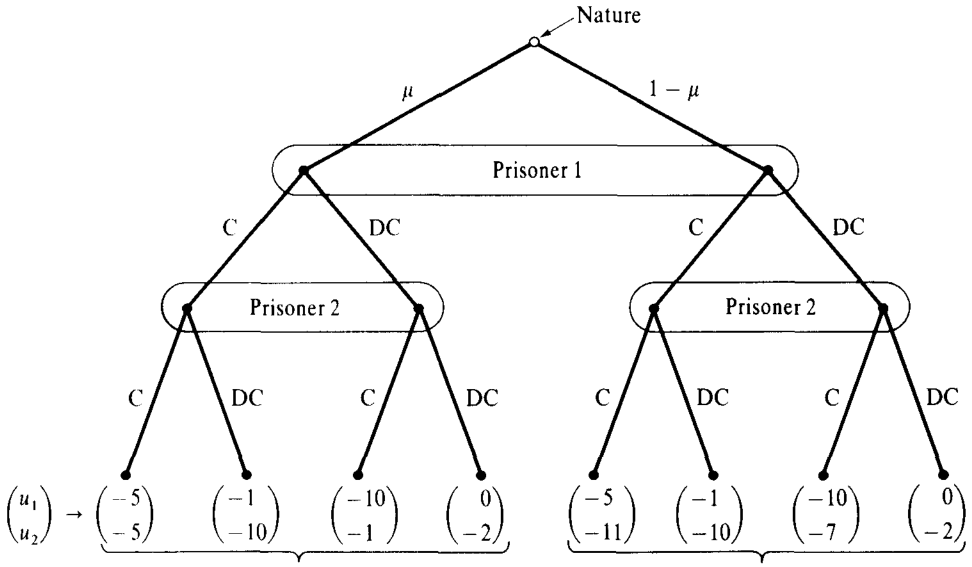

Figure 8.E.1 The DA's Brother game with incomplete information.

Simultaneous-Move Game:

Prisoner 2

Prisoner 1	DC	C
DC	0, −2	−10, −1
C	−1, −10	−5, −5

Simultaneous-Move Game:

Prisoner 2

Prisoner 1	DC	C
DC	0, −2	−10, −7
C	−1, −10	−5, −11

indicates what action he will take. Hence, prisoner 2 now has four possible pure strategies:

(confess if type I, confess if type II);
(confess if type I, don't confess if type II);
(don't confess if type I, confess if type II);
(don't confess if type I, don't confess if type II).

Notice, however, that player 1 does not observe player 2's type, and so a pure strategy for player 1 in this game is simply a (noncontingent) choice of either "confess" or "don't confess." ■

Formally, in a Bayesian game, each player i has a payoff function $u_i(s_i, s_{-i}, \theta_i)$, where $\theta_i \in \Theta_i$ is a random variable chosen by nature that is observed only by player i. The joint probability distribution of the θ_i's is given by $F(\theta_1, \ldots, \theta_I)$, which is assumed to be common knowledge among the players. Letting $\Theta = \Theta_1 \times \cdots \times \Theta_I$, a Bayesian game is summarized by the data $[I, \{S_i\}, \{u_i(\cdot)\}, \Theta, F(\cdot)]$.

A pure strategy for player i in a Bayesian game is a function $s_i(\theta_i)$, or *decision rule*, that gives the player's strategy choice for each realization of his type θ_i. Player i's pure strategy set $\mathscr{S}_i$ is therefore the set of all such functions. Player i's expected payoff given a profile of pure strategies for the I players $(s_1(\cdot), \ldots, s_I(\cdot))$ is then given by

$$\tilde{u}_i(s_1(\cdot), \ldots, s_I(\cdot)) = E_\theta[u_i(s_1(\theta_1), \ldots, s_I(\theta_I), \theta_i)]. \qquad (8.E.1)$$

We can now look for an ordinary (pure strategy) Nash equilibrium of this game of imperfect information, which is known in this context as a *Bayesian Nash equilibrium*.[7]

Definition 8.E.1: A (pure strategy) *Bayesian Nash equilibrium* for the Bayesian game $[I, \{S_i\}, \{u_i(\cdot)\}, \Theta, F(\cdot)]$ is a profile of decision rules $(s_1(\cdot), \ldots, s_I(\cdot))$ that constitutes a Nash equilibrium of game $\Gamma_N = [I, \{\mathscr{S}_i\}, \{\tilde{u}_i(\cdot)\}]$. That is, for every $i = 1, \ldots, I$,

$$\tilde{u}_i(s_i(\cdot), s_{-i}(\cdot)) \geq \tilde{u}_i(s_i'(\cdot), s_{-i}(\cdot))$$

for all $s_i'(\cdot) \in \mathscr{S}_i$, where $\tilde{u}_i(s_i(\cdot), s_{-i}(\cdot))$ is defined as in (8.E.1).

A very useful point to note is that in a (pure strategy) Bayesian Nash equilibrium each player must be playing a best response to the conditional distribution of his opponents' strategies *for each type that he might end up having.* Proposition 8.E.1 provides a more formal statement of this point.

Proposition 8.E.1: A profile of decision rules $(s_1(\cdot), \ldots, s_I(\cdot))$ is a Bayesian Nash equilibrium in Bayesian game $[I, \{S_i\}, \{u_i(\cdot)\}, \Theta, F(\cdot)]$ if and only if, for all i and

7. We shall restrict our attention to pure strategies here; mixed strategies involve randomization over the strategies in $\mathscr{S}_i$. Note also that we have not been very explicit about whether the Θ_i's are finite sets. If they are, then the strategy sets $\mathscr{S}_i$ are finite; if they are not, then the sets $\mathscr{S}_i$ include an infinite number of possible functions $s_i(\cdot)$. Either way, however, the basic definition of a Bayesian Nash equilibrium is the same.

all $\bar{\theta}_i \in \Theta_i$ occurring with positive probability[8]

$$E_{\theta_{-i}}[u_i(s_i(\bar{\theta}_i), s_{-i}(\theta_{-i}), \bar{\theta}_i) \mid \bar{\theta}_i] \geq E_{\theta_{-i}}[u_i(s'_i, s_{-i}(\theta_{-i}), \bar{\theta}_i) \mid \bar{\theta}_i] \tag{8.E.2}$$

for all $s'_i \in S_i$, where the expectation is taken over realizations of the other players' random variables conditional on player i's realization of his signal $\bar{\theta}_i$.

Proof: For necessity, note that if (8.E.2) did not hold for some player i for some $\bar{\theta}_i \in \Theta_i$ that occurs with positive probability, then player i could do better by changing his strategy choice in the event he gets realization $\bar{\theta}_i$, contradicting $(s_1(\cdot), \ldots, s_I(\cdot))$ being a Bayesian Nash equilibrium. In the other direction, if condition (8.E.2) holds for all $\bar{\theta}_i \in \Theta_i$ occurring with positive probability, then player i cannot improve on the payoff he receives by playing strategy $s_i(\cdot)$. ■

Proposition 8.E.1 tells us that, in essence, we can think of each type of player i as being a separate player who maximizes his payoff given his conditional probability distribution over the strategy choices of his rivals.

Example 8.E.1 Continued: To solve for the (pure strategy) Bayesian Nash equilibrium of this game, note first that type I of prisoner 2 must play "confess" with probability 1 because this is that type's dominant strategy. Likewise, type II of prisoner 2 also has a dominant strategy: "don't confess." Given this behavior by prisoner 2, prisoner 1's best response is to play "don't confess" if $[-10\mu + 0(1-\mu)] > [-5\mu - 1(1-\mu)]$, or equivalently, if $\mu < \frac{1}{6}$, and is to play "confess" if $\mu > \frac{1}{6}$. (He is indifferent if $\mu = \frac{1}{6}$.) ■

Example 8.E.2: The Alphabeta research and development consortium has two (noncompeting) members, firms 1 and 2. The rules of the consortium are that any independent invention by one of the firms is shared fully with the other. Suppose that there is a new invention, the "Zigger," that either of the two firms could potentially develop. To develop this new product costs a firm $c \in (0, 1)$. The benefit of the Zigger to each firm i is known only by that firm. Formally, each firm i has a type θ_i that is independently drawn from a uniform distribution on $[0, 1]$, and its benefit from the Zigger when its type is θ_i is $(\theta_i)^2$. The timing is as follows: The two firms each privately observe their own type. Then they each simultaneously choose either to develop the Zigger or not.

Let us now solve for the Bayesian Nash equilibrium of this game. We shall write $s_i(\theta_i) = 1$ if type θ_i of firm i develops the Zigger and $s_i(\theta_i) = 0$ if it does not. If firm i develops the Zigger when its type is θ_i, its payoff is $(\theta_i)^2 - c$ regardless of whether firm j does so. If firm i decides not to develop the Zigger when its type is θ_i, it will have an expected payoff equal to $(\theta_i)^2 \text{ Prob } (s_j(\theta_j) = 1)$. Hence, firm i's best response is to develop the Zigger if and only if its type θ_i is such that (we assume firm i develops the Zigger if it is indifferent):

$$\theta_i \geq \left[\frac{c}{1 - \text{Prob } (s_j(\theta_j) = 1)}\right]^{1/2}. \tag{8.E.3}$$

8. The formulation given here (and the proof) is for the case in which the sets Θ_i are finite. When a player i has an infinite number of possible types, condition (8.E.2) must hold on a subset of Θ_i that is of full measure (i.e., that occurs with probability equal to one). It is then said that (8.E.2) holds for *almost every* $\bar{\theta}_i \in \Theta_i$.

Note that for any given strategy of firm j, firm i's best response takes the form of a *cutoff rule*: It optimally develops the Zigger for all θ_i above the value on the right-hand side of (8.E.3) and does not for all θ_i below it. [Note that if firm i existed in isolation, it would be indifferent about developing the Zigger when $\theta_i = \sqrt{c}$. But (8.E.3) tells us that when firm i is part of the consortium, its cutoff is always (weakly) above this. This is true because each firm hopes to *free-ride* on the other firm's development effort; see Chapter 11 for more on this.]

Suppose then that $\hat{\theta}_1, \hat{\theta}_2 \in (0, 1)$ are the cutoff values for firms 1 and 2 respectively in a Bayesian Nash equilibrium (it can be shown that $0 < \hat{\theta}_i < 1$ for $i = 1, 2$ in any Bayesian Nash equilibrium of this game). If so, then using the fact that $\text{Prob}(s_j(\theta_j) = 1) = 1 - \hat{\theta}_j$, condition (8.E.3) applied first for $i = 1$ and then for $i = 2$ tells us that we must have

$$(\hat{\theta}_1)^2 \hat{\theta}_2 = c$$

and

$$(\hat{\theta}_2)^2 \hat{\theta}_1 = c.$$

Because $(\hat{\theta}_1)^2 \hat{\theta}_2 = (\hat{\theta}_2)^2 \hat{\theta}_1$ implies that $\hat{\theta}_1 = \hat{\theta}_2$, we see that any Bayesian Nash equilibrium of this game involves an identical cutoff value for the two firms, $\theta^* = (c)^{1/3}$. In this equilibrium, the probability that neither firm develops the Zigger is $(\theta^*)^2$, the probability that exactly one firm develops it is $2\theta^*(1 - \theta^*)$, and the probability that both do is $(1 - \theta^*)^2$. ■

The exercises at the end of this chapter consider several other examples of Bayesian Nash equilibria. Another important application arises in the theory of implementation with incomplete information, studied in Chapter 23.

In Section 8.D, we argued that mixed strategies could be interpreted as situations where players play deterministic strategies conditional on seemingly irrelevant signals (recall the baseball pitcher). We can now say a bit more about this. Suppose we start with a game of complete information that has a unique mixed strategy equilibrium in which players actually randomize. Now consider changing the game by introducing many different types (formally, a continuum) of each player, with the realizations of the various players' types being statistically independent of one another. Suppose, in addition, that all types of a player have *identical* preferences. A (pure strategy) Bayesian Nash equilibrium of this Bayesian game is then precisely equivalent to a mixed strategy Nash equilibrium of the original complete information game. Moreover, in many circumstances, one can show that there are also "nearby" Bayesian games in which preferences of the different types of a player differ only slightly from one another, the Bayesian Nash equilibria are close to the mixed strategy distribution, and each type has a strict preference for his strategy choice. Such results are known as *purification theorems* [see Harsanyi (1973)].

We can also return to the issue of *correlated equilibria* raised in Section 8.D. In particular, if we allow the realizations of the various players' types in the previous paragraph to be statistically correlated, then a (pure strategy) Bayesian Nash equilibrium of this Bayesian game is a correlated equilibrium of the original complete information game. The set of all correlated equilibria in game $[I, \{S_i\}, \{u_i(\cdot)\}]$ is identified by considering all possible Bayesian games of this sort (i.e., we allow for all possible signals that the players might observe).

8.F The Possibility of Mistakes: Trembling-Hand Perfection

In Section 8.B, we noted that although rationality per se does not rule out the choice of a weakly dominated strategy, such strategies are unappealing because they are dominated unless a player is absolutely sure of what his rivals will play. In fact, as the game depicted in Figure 8.F.1 illustrates, the Nash equilibrium concept also does not preclude the use of such strategies. In this game, (D, R) is a Nash equilibrium in which both players play a weakly dominated strategy with certainty.

Here, we elaborate on the idea, raised in Section 8.B, that *caution* might preclude the use of such strategies. The discussion leads us to define a refinement of the concept of Nash equilibrium, known as a *(normal form) trembling-hand perfect Nash equilibrium*, which identifies Nash equilibria that are robust to the possibility that, with some very small probability, players make mistakes.

Following Selten (1975), for any normal form game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$, we can define a *perturbed* game $\Gamma_\varepsilon = [I, \{\Delta_\varepsilon(S_i)\}, \{u_i(\cdot)\}]$ by choosing for each player i and strategy $s_i \in S_i$ a number $\varepsilon_i(s_i) \in (0, 1)$, with $\sum_{s_i \in S_i} \varepsilon_i(s_i) < 1$, and then defining player i's perturbed strategy set to be

$$\Delta_\varepsilon(S_i) = \{\sigma_i\colon \sigma_i(s_i) \geq \varepsilon_i(s_i) \text{ for all } s_i \in S_i \text{ and } \sum_{s_i \in S_i} \sigma_i(s_i) = 1\}.$$

That is, perturbed game Γ_ε is derived from the original game Γ_N by requiring that each player i play every one of his strategies, say s_i, with at least some minimal positive probability $\varepsilon_i(s_i)$; $\varepsilon_i(s_i)$ is interpreted as the unavoidable probability that strategy s_i gets played by mistake.

Having defined this perturbed game, we want to consider as predictions in game Γ_N only those Nash equilibria σ that are robust to the possibility that players make mistakes. The robustness test we employ can be stated roughly as: To consider σ as a robust equilibrium, we want there to be at least some slight perturbations of Γ_N whose equilibria are close to σ. The formal definition of a *(normal form) trembling-hand perfect Nash equilibrium* (the name comes from the idea of players making mistakes because of their trembling hands) is presented in Definition 8.F.1.

Definition 8.F.1: A Nash equilibrium σ of game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ is *(normal form) trembling-hand perfect* if there is *some* sequence of perturbed games $\{\Gamma_{\varepsilon^k}\}_{k=1}^{\infty}$ that converges to Γ_N [in the sense that $\lim_{k\to\infty} \varepsilon_i^k(s_i) = 0$ for all i and $s_i \in S_i$], for which there is *some* associated sequence of Nash equilibria $\{\sigma^k\}_{k=1}^{\infty}$ that converges to σ (i.e., such that $\lim_{k\to\infty} \sigma^k = \sigma$).

We use the modifier *normal form* because Selten (1975) also proposes a slightly different form of trembling-hand perfection for dynamic games; we discuss this version of the concept in Chapter 9.[9]

Note that the concept of a (normal form) trembling-hand perfect Nash equilibrium provides a relatively mild test of robustness: We require only that *some* perturbed games exist that have equilibria arbitrarily close to σ. A stronger test would

9. In fact, Selten (1975) is primarily concerned with the problem of identifying desirable equilibria in dynamic games. See Chapter 9, Appendix B for more on this.

	L	R
U	1, 1	0, −3
D	−3, 0	0, 0

Figure 8.F.1
(D, R) is a Nash equilibrium involving play of weakly dominated strategies.

require that the equilibrium σ be robust to *all* perturbations close to the original game.

In general, the criterion in Definition 8.F.1 can be difficult to work with because it requires that we compute the equilibria of many possible perturbed games. The result presented in Proposition 8.F.1 provides a formulation that makes checking whether a Nash equilibrium is trembling-hand perfect much easier (in its statement, a *totally mixed* strategy is a mixed strategy in which every pure strategy receives positive probability).

Proposition 8.F.1: A Nash equilibrium σ of game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ is (normal form) trembling-hand perfect if and only if there is some sequence of totally mixed strategies $\{\sigma^k\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} \sigma^k = \sigma$ and σ_i is a best response to every element of sequence $\{\sigma^k_{-i}\}_{k=1}^{\infty}$ for all $i = 1, \ldots, I$.

You are asked to prove this result in Exercise 8.F.1 [or consult Selten (1975)]. The result presented in Proposition 8.F.2 is an immediate consequence of Definition 8.F.1 and Proposition 8.F.1.

Proposition 8.F.2: If $\sigma = (\sigma_1, \ldots, \sigma_I)$ is a (normal form) trembling-hand perfect Nash equilibrium, then σ_i is not a weakly dominated strategy for any $i = 1, \ldots, I$. Hence, in any (normal form) trembling-hand perfect Nash equilibrium, no weakly dominated pure strategy can be played with positive probability.

The converse, that any Nash equilibrium not involving play of a weakly dominated strategy is necessarily trembling-hand perfect, turns out to be true for two-player games but not for games with more than two players. Thus, trembling-hand perfection can rule out more than just Nash equilibria involving weakly dominated strategies. The reason is tied to the fact that when a player's rivals make mistakes with small probability, this can give rise to only a limited set of probability distributions over their nonequilibrium strategies. For example, if a player's two rivals each have a small probability of making a mistake, there is a much greater probability that one will make a mistake than that both will. If the player's equilibrium strategy is a unique best response only when both of his rivals make a mistake, his strategy may not be a best response to any local perturbation of his rivals' strategies even though his strategy is not weakly dominated. (Exercise 8.F.2 provides an example.) However, if players' trembles are allowed to be correlated (e.g., as in the correlated equilibrium concept), then the converse of Proposition 8.F.2 would hold regardless of the number of players.

Selten (1975) also proves an existence result that parallels Proposition 8.D.2: Every game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ with finite strategy sets $S_1, \ldots, S_I$ has a trembling-hand perfect Nash equilibrium. An implication of this result is that every such game has at least one Nash equilibrium in which no player plays any weakly dominated strategy with positive probability. Hence, if we decide to accept only Nash equilibria that do not involve the play of weakly dominated strategies, with great generality there is at least one such equilibrium.[10]

Myerson (1978) proposes a refinement of Selten's idea in which players are less likely to make more costly mistakes (the idea is that they will try harder to avoid these mistakes). He establishes that the resulting solution concept, called a *proper Nash equilibrium*, exists under the conditions described in the previous paragraph for trembling-hand perfect Nash equilibria. van Damme (1983) presents a good discussion of this and other refinements of trembling-hand perfection.

APPENDIX A: EXISTENCE OF NASH EQUILIBRIUM

In this appendix, we prove Propositions 8.D.2 and 8.D.3. We begin with Lemma 8.AA.1, which provides a key technical result.

Lemma 8.AA.1: If the sets $S_1, \ldots, S_I$ are nonempty, S_i is compact and convex, and $u_i(\cdot)$ is continuous in $(s_1, \ldots, s_I)$ and quasiconcave in s_i, then player i's best-response correspondence $b_i(\cdot)$ is nonempty, convex-valued, and upper hemicontinuous.[11]

Proof: Note first that $b_i(s_{-i})$ is the set of maximizers of the continuous function $u_i(\cdot, s_{-i})$ on the compact set S_i. Hence, it is nonempty (see Theorem M.F.2 of the Mathematical Appendix). The convexity of $b_i(s_{-i})$ follows because the set of maximizers of a quasiconcave function [here, the function $u_i(\cdot, s_{-i})$] on a convex set (here, S_i) is convex. Finally, for upper hemicontinuity, we need to show that for any sequence $(s_i^n, s_{-i}^n) \to (s_i, s_{-i})$ such that $s_i^n \in b_i(s_{-i}^n)$ for all n, we have $s_i \in b_i(s_{-i})$. To see this, note that for all n, $u_i(s_i^n, s_{-i}^n) \geq u_i(s_i', s_{-i}^n)$ for all $s_i' \in S_i$. Therefore, by the continuity of $u_i(\cdot)$, we have $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$. ■

It is convenient to prove Proposition 8.D.3 first.

Proposition 8.D.3: A Nash equilibrium exists in game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ if for all $i = 1, \ldots, I$,

(i) S_i is a nonempty, convex, and compact subset of some Euclidean space $\mathbb{R}^M$.

(ii) $u_i(s_1, \ldots, s_I)$ *is continuous in* $(s_1, \ldots, s_I)$ and quasiconcave in s_i.

10. The Bertrand duopoly game discussed in Chapter 12 provides one example of a game in which this is not the case; its unique Nash equilibrium involves the play of weakly dominated strategies. The problem arises because the strategies in that game are continuous variables (and so the sets S_i are not finite). Fortunately, this equilibrium can be viewed as the limit of undominated equilibria in "nearby" discrete versions of the game. (See Exercise 12.C.3 for more on this point.)

11. See Section M.H of the Mathematical Appendix for a discussion of upper hemicontinuous correspondences.

Proof: Define the correspondence $b: S \to S$ by

$$b(s_1, \ldots, s_I) = b_1(s_{-1}) \times \cdots \times b_I(s_{-I}).$$

Note that $b(\cdot)$ is a correspondence from the nonempty, convex, and compact set $S = S_1 \times \cdots \times S_I$ to itself. In addition, by Lemma 8.AA.1, $b(\cdot)$ is a nonempty, convex-valued, and upper hemicontinuous correspondence. Thus, all the conditions of the Kakutani fixed point theorem are satisfied (see Section M.I of the Mathematical Appendix). Hence, there exists a fixed point for this correspondence, a strategy profile $s \in S$ such that $s \in b(s)$. The strategies at this fixed point constitute a Nash equilibrium because by construction $s_i \in b_i(s_{-i})$ for all $i = 1, \ldots, I$. ■

Now we move to the proof of Proposition 8.D.2.

Proposition 8.D.2: Every game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ in which the sets $S_1, \ldots, S_I$ have a finite number of elements has a mixed strategy Nash equilibrium.

Proof: The game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$, viewed as a game with strategy sets $\{\Delta(S_i)\}$ and payoff functions $u_i(\sigma_1, \ldots, \sigma_I) = \sum_{s \in S}[\prod_{k=1}^{I} \sigma_k(s_k)]u_i(s)$ for all $i = 1, \ldots, I$, satisfies all the assumptions of Proposition 8.D.3. Hence, Proposition 8.D.2 is a direct corollary of that result. ■

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EXERCISES

8.B.1^{A} There are I firms in an industry. Each can try to convince Congress to give the industry a subsidy. Let h_i denote the number of hours of effort put in by firm i, and let $c_i(h_i) = w_i(h_i)^2$, where w_i is a positive constant, be the cost of this effort to firm i. When the effort levels of the firms are $(h_1, \ldots, h_I)$, the value of the subsidy that gets approved is $\alpha \sum_i h_i + \beta(\prod_i h_i)$, where α and β are constants.

Consider a game in which the firms decide simultaneously and independently how many hours they will each devote to this effort. Show that each firm has a strictly dominant strategy if and only if $\beta = 0$. What is firm i's strictly dominant strategy when this is so?

8.B.2^{B} **(a)** Argue that if a player has two weakly dominant strategies, then for every strategy choice by his opponents, the two strategies yield him equal payoffs.

(b) Provide an example of a two-player game in which a player has two weakly dominant pure strategies but his opponent prefers that he play one of them rather than the other.

8.B.3^{B} Consider the following auction (known as a *second-price*, or *Vickrey*, auction). An object is auctioned off to I bidders. Bidder i's valuation of the object (in monetary terms) is v_i. The auction rules are that each player submit a bid (a nonnegative number) in a sealed envelope. The envelopes are then opened, and the bidder who has submitted the highest bid gets the object but pays the auctioneer the amount of the *second-highest* bid. If more than one bidder submits the highest bid, each gets the object with equal probability. Show that submitting a bid of v_i with certainty is a weakly dominant strategy for bidder i. Also argue that this is bidder i's unique weakly dominant strategy.

8.B.4^{C} Show that the order of deletion does not matter for the set of strategies surviving a process of iterated deletion of strictly dominated strategies.

8.B.5^{C} Consider the Cournot duopoly model (discussed extensively in Chapter 12) in which two firms, 1 and 2, simultaneously choose the quantities they will sell on the market, q_1 and q_2. The price each receives for each unit given these quantities is $P(q_1, q_2) = a - b(q_1 + q_2)$. Their costs are c per unit sold.

(a) Argue that successive elimination of strictly dominated strategies yields a unique prediction in this game.

(b) Would this be true if there were three firms instead of two?

8.B.6^{B} In text.

8.B.7^{B} Show that any strictly dominant strategy in game $[I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ must be a pure strategy.

8.C.1^{A} Argue that if elimination of strictly dominated strategies yields a unique prediction in a game, this prediction also results from eliminating strategies that are never a best response.

8.C.2^{C} Prove that the order of removal does not matter for the set of strategies that survives a process of iterated deletion of strategies that are never a best response.

8.C.3^{C} Prove that in a two-player game (with finite strategy sets), if a pure strategy s_i for player i is never a best response for any mixed strategy by i's opponent, then s_i is strictly dominated by some mixed strategy $\sigma_i \in \Delta(S_i)$. [*Hint:* Try using the supporting hyperplane theorem presented in Section M.G of the Mathematical Appendix.]

have to walk a full mile). If more than one vendor is at the same location, they split the business evenly.

(a) Consider a game in which two ice-cream vendors pick their locations simultaneously. Show that there exists a unique pure strategy Nash equilibrium and that it involves both vendors locating at the midpoint of the boardwalk.

(b) Show that with three vendors, no pure strategy Nash equilibrium exists.

8.D.6^{B} Consider any two-player game of the following form (where letters indicate arbitrary payoffs):

		Player 2: b_1	b_2
Player 1	a_1	u, v	ℓ, m
	a_2	w, x	y, z

Show that a mixed strategy Nash equilibrium always exists in this game. [*Hint:* Define player 1's strategy to be his probability of choosing action a_1 and player 2's to be his probability of choosing b_1; then examine the best-response correspondences of the two players.]

8.D.7^{C} (*The Minimax Theorem*) A two-player game with finite strategy sets $\Gamma_N = [I, \{S_1, S_2\}, \{u_1(\cdot), u_2(\cdot)\}]$ is a *zero-sum* game if $u_2(s_1, s_2) = -u_1(s_1, s_2)$ for all $(s_1, s_2) \in S_1 \times S_2$.

Define i's *maximin* expected utility level $\underline{w}_i$ to be the level he can guarantee himself in game $[I, \{\Delta(S_1), \Delta(S_2)\}, \{u_1(\cdot), u_2(\cdot)\}]$:

$$\underline{w}_i = \underset{\sigma_i}{\text{Max}} \left[\underset{\sigma_{-i}}{\text{Min}}\ u_i(\sigma_i, \sigma_{-i}) \right].$$

Define player i's *minimax* utility level $\underline{v}_i$ to be the worst expected utility level he can be forced to receive if he gets to respond to his rival's actions:

$$\underline{v}_i = \underset{\sigma_{-i}}{\text{Min}} \left[\underset{\sigma_i}{\text{Max}}\ u_i(\sigma_i, \sigma_{-i}) \right].$$

(a) Show that $\underline{v}_i \geq \underline{w}_i$ in any game.

(b) Prove that in any mixed strategy Nash equilibrium of the zero-sum game $\Gamma_N = [I, \{\Delta(S_1), \Delta(S_2)\}, \{u_1(\cdot), u_2(\cdot)\}]$, player i's expected utility u_i° satisfies $u_i^{\circ} = \underline{v}_i = \underline{w}_i$. [*Hint:* Such an equilibrium must exist by Proposition 8.D.2.]

(c) Show that if (σ_1', σ_2') and (σ_1'', σ_2'') are both Nash equilibria of the zero-sum game $\Gamma_N = [I, \{\Delta(S_1), \Delta(S_2)\}, \{u_1(\cdot), u_2(\cdot)\}]$, then so are (σ_1', σ_2'') and (σ_1'', σ_2').

8.D.8^{C} Consider a simultaneous-move game with normal form $[I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$. Suppose that, for all i, S_i is a convex set and $u_i(\cdot)$ is strictly quasiconvex. Argue that any mixed strategy Nash equilibrium of this game must be degenerate, with each player playing a single pure strategy with probability 1.

8.D.9^{B} Consider the following game [based on an example from Kreps (1990)]:

		Player 2: LL	L	M	R
Player 1	U	100, 2	−100, 1	0, 0	−100, −100
	D	−100, −100	100, −49	1, 0	100, 2

(a) If you were player 2 in this game and you were playing it once without the ability to engage in preplay communication with player 1, what strategy would you choose?

(b) What are all the Nash equilibria (pure and mixed) of this game?

(c) Is your strategy choice in **(a)** a component of any Nash equilibrium strategy profile? Is it a rationalizable strategy?

(d) Suppose now that preplay communication were possible. Would you expect to play something different from your choice in **(a)**?

8.E.1[B] Consider the following strategic situation. Two opposed armies are poised to seize an island. Each army's general can choose either "attack" or "not attack." In addition, each army is either "strong" or "weak" with equal probability (the draws for each army are independent), and an army's type is known only to its general. Payoffs are as follows: The island is worth M if captured. An army can capture the island either by attacking when its opponent does not or by attacking when its rival does if it is strong and its rival is weak. If two armies of equal strength both attack, neither captures the island. An army also has a "cost" of fighting, which is s if it is strong and w if it is weak, where $s < w$. There is no cost of attacking if its rival does not.

Identify all pure strategy Bayesian Nash equilibria of this game.

8.E.2[C] Consider the first-price sealed-bid auction of Exercise 8.D.3, but now suppose that each bidder i observes only his own valuation v_i. This valuation is distributed uniformly and independently on $[0, \bar{v}]$ for each bidder.

(a) Derive a symmetric (pure strategy) Bayesian Nash equilibrium of this auction. (You should now suppose that bids can be any real number.) [*Hint:* Look for an equilibrium in which bidder i's bid is a linear function of his valuation.]

(b) What if there are I bidders? What happens to each bidder's equilibrium bid function $s(v_i)$ as I increases?

8.E.3[B] Consider the linear Cournot model described in Exercise 8.B.5. Now, however, suppose that each firm has probability μ of having unit costs of c_L and $(1 - \mu)$ of having unit costs of c_H, where $c_H > c_L$. Solve for the Bayesian Nash equilibrium.

8.F.1[C] Prove Proposition 8.F.1.

8.F.2[B] Consider the following three-player game [taken from van Damme (1983)], in which player 1 chooses rows ($S_1 = \{U, D\}$), player 2 chooses columns ($S_2 = \{L, R\}$), and player 3 chooses boxes ($S_3 = \{B_1, B_2\}$):

B_1

	L	R
U	(1, 1, 1)	(1, 0, 1)
D	(1, 1, 1)	(0, 0, 1)

B_2

	L	R
U	(1, 1, 0)	(0, 0, 0)
D	(0, 1, 0)	(1, 0, 0)

Each cell describes the payoffs to the three players (u_1, u_2, u_3) from that strategy combination. Both (D, L, B_1) and (U, L, B_1) are pure strategy Nash equilibria. Show that (D, L, B_1) is not (normal form) trembling-hand perfect even though none of these three strategies is weakly dominated.

8.F.3^C Prove that every game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ in which the S_i are finite sets has a (normal form) trembling-hand perfect Nash equilibrium. [*Hint:* Show that every perturbed game has an equilibrium and that for any sequence of perturbed games converging to the original game Γ_N and corresponding sequence of equilibria, there is a subsequence that converges to an equilibrium of Γ_N.]

CHAPTER

9 Dynamic Games

9.A Introduction

In Chapter 8, we studied simultaneous-move games. Most economic situations, however, involve players choosing actions over time.[1] For example, a labor union and a firm might make repeated offers and counteroffers to each other in the course of negotiations over a new contract. Likewise, firms in a market may invest today in anticipation of the effects of these investments on their competitive interactions in the future. In this chapter, we therefore shift our focus to the study of *dynamic games*.

One way to approach the problem of prediction in dynamic games is to simply derive their normal form representations and then apply the solution concepts studied in Chapter 8. However, an important new issue arises in dynamic games: the *credibility* of a player's strategy. This issue is the central concern of this chapter.

Consider a vivid (although far-fetched) example: You walk into class tomorrow and your instructor, a sane but very enthusiastic game theorist, announces, "This is an important course, and I want exclusive dedication. Anyone who does not drop every other course will be barred from the final exam and will therefore flunk." After a moment of bewilderment and some mental computation, your first thought is, "Given that I indeed prefer this course to all others, I had better follow her instructions" (after all, you have studied Chapter 8 carefully and know what a best response is). But after some further reflection, you ask yourself, "Will she really bar me from the final exam if I do not obey? This is a serious institution, and she will surely lose her job if she carries out the threat." You conclude that the answer is "no" and refuse to drop the other courses, and indeed, she ultimately does not bar you from the exam. In this example, we would say that your instructor's announced strategy, "I will bar you from the exam if you do not drop every other course," is not credible. Such empty threats are what we want to rule out as equilibrium strategies in dynamic games.

In Section 9.B, we demonstrate that the Nash equilibrium concept studied in Chapter 8 does not suffice to rule out noncredible strategies. We then introduce a stronger solution concept, known as *subgame perfect Nash equilibrium*, that helps

1. As do most parlor games.

to do so. The central idea underlying this concept is the *principle of sequential rationality*: equilibrium strategies should specify optimal behavior from any point in the game onward, a principle that is intimately related to the procedure of *backward induction*.

In Section 9.C, we show that the concept of subgame perfection is not strong enough to fully capture the idea of sequential rationality in games of imperfect information. We then introduce the notion of a *weak perfect Bayesian equilibrium* (also known as a *weak sequential equilibrium*) to push the analysis further. The central feature of a weak perfect Bayesian equilibrium is its explicit introduction of a player's *beliefs* about what may have transpired prior to her move as a means of testing the sequential rationality of the player's strategy. The modifier *weak* refers to the fact that the weak perfect Bayesian equilibrium concept imposes a *minimal* set of consistency restrictions on players' beliefs. Because the weak perfect Bayesian equilibrium concept can be too weak, we also examine some related equilibrium notions that impose stronger consistency restrictions on beliefs, discussing briefly stronger notions of *perfect Bayesian equilibrium* and, in somewhat greater detail, the concept of *sequential equilibrium*.

In Section 9.D, we go yet further by asking whether certain beliefs can be regarded as "unreasonable" in some situations, thereby allowing us to further refine our predictions. This leads us to consider the notion of *forward induction*.

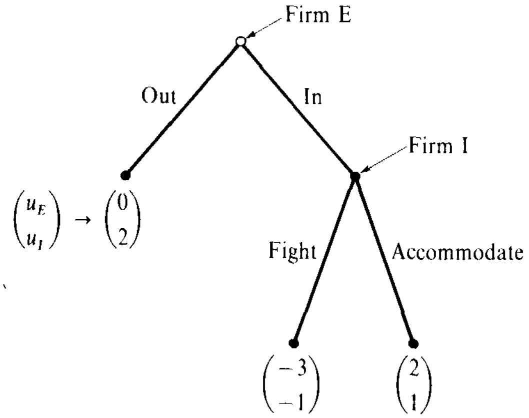
Appendix A studies finite and infinite horizon models of bilateral bargaining as an illustration of the use of subgame perfect Nash equilibrium in an important economic application. Appendix B extends the discussion in Section 9.C by examining the notion of an *extensive form trembling-hand perfect Nash equilibrium*.

We should note that—following most of the literature on this subject—all the analysis in this chapter consists of attempts to "refine" the concept of Nash equilibrium; that is, we take the position that we want our prediction to be a Nash equilibrium, and we then propose additional conditions for such an equilibrium to be a "satisfactory" prediction. However, the issues that we discuss here are not confined to this approach. We might, for example, be concerned about noncredible strategies even if we were unwilling to impose the mutually correct expectations condition of Nash equilibrium and wanted to focus instead only on rationalizable outcomes. See Bernheim (1984) and, especially, Pearce (1984) for a discussion of nonequilibrium approaches to these issues.

9.B Sequential Rationality, Backward Induction, and Subgame Perfection

We begin with an example to illustrate that in dynamic games the Nash equilibrium concept may not give sensible predictions. This observation leads us to develop a strengthening of the Nash equilibrium concept known as *subgame perfect Nash equilibrium*.

Example 9.B.1: Consider the following *predation* game. Firm E (for entrant) is considering entering a market that currently has a single incumbent (firm I). If it does so (playing "in"), the incumbent can respond in one of two ways: It can either accommodate the entrant, giving up some of its sales but causing no change in



		Firm I	
		Fight if Firm E Plays "In"	Accommodate if Firm E Plays "In"
Firm E	Out	0, 2	0, 2
	In	−3, −1	2, 1

Figure 9.B.1 Extensive and normal forms for Example 9.B.1. The Nash equilibrium (σ_E, σ_I) = (out, fight if firm E plays "in") involves a noncredible threat.

the market price, or it can fight the entrant, engaging in a costly war of predation that dramatically lowers the market price. The extensive and normal form representations of this game are depicted in Figure 9.B.1.

Examining the normal form, we see that this game has two pure strategy Nash equilibria: (σ_E, σ_I) = (out, fight if firm E plays "in") and (σ_E, σ_I) = (in, accommodate if firm E plays "in"). Consider the first of these strategy profiles. Firm E prefers to stay out of the market if firm I will fight after it enters. On the other hand, "fight if firm E plays 'in'" is an optimal choice for the incumbent if firm E is playing "out." Similar arguments show that the second pair of strategies is also a Nash equilibrium.

Yet, we claim that (out, fight if firm E plays "in") is not a sensible prediction for this game. As in the example of your instructor that we posed in Section 9.A, firm E can foresee that if it does enter, the incumbent will, in fact, find it optimal to accommodate (by doing so, firm I earns 1 rather than -1). Hence, the incumbent's strategy "fight if firm E plays 'in'" is not credible. ■

Example 9.B.1 illustrates a problem with the Nash equilibrium concept in dynamic games. In this example, the concept is, in effect, permitting the incumbent to make an empty threat that the entrant nevertheless takes seriously when choosing its strategy. The problem with the Nash equilibrium concept here arises from the fact that when the entrant plays "out," actions at decision nodes that are unreached by play of the equilibrium strategies (here, firm I's action at the decision node following firm E's unchosen move "in") do not affect firm I's payoff. As a result, firm I can plan to do *absolutely anything* at this decision node: Given firm E's strategy of choosing "out," firm I's payoff is still maximized. *But*—and here is the crux of the matter—what firm I's strategy says it will do at the unreached node can actually *insure* that firm E, taking firm I's strategy as given, wants to play "out."

To rule out predictions such as (out, fight if firm E plays "in"), we want to insist that players' equilibrium strategies satisfy what might be called the *principle of sequential rationality*: A player's strategy should specify optimal actions *at every point in the game tree*. That is, given that a player finds herself at some point in the tree, her strategy should prescribe play that is optimal from that point on given her opponents' strategies. Clearly, firm I's strategy "fight if firm E plays 'in'" does not: after entry, the only optimal strategy for firm I is "accommodate."

In Example 9.B.1, there is a simple procedure that can be used to identify the

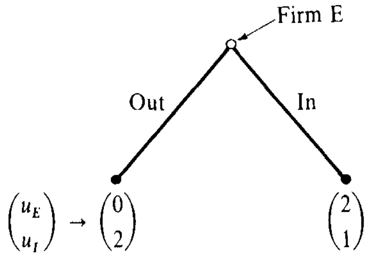


Figure 9.B.2
Reduced game after solving for post-entry behavior in Example 9.B.1.

desirable (i.e., sequentially rational) Nash equilibrium (σ_E, σ_I) = (in, accommodate if firm E plays "in"). We first determine optimal behavior for firm I in the post-entry stage of the game; this is "accommodate." Once we have done this, we then determine firm E's optimal behavior earlier in the game given the anticipation of what will happen after entry. Note that this second step can be accomplished by considering a *reduced* extensive form game in which firm I's post-entry decision is replaced by the payoffs that will result from firm I's optimal post-entry behavior. See Figure 9.B.2. This reduced game is a very simple single-player decision problem in which firm E's optimal decision is to play "in." In this manner, we identify the sequentially rational Nash equilibrium strategy profile (σ_E, σ_I) = (in, accommodate if firm E plays "in").

This type of procedure, which involves solving first for optimal behavior at the "end" of the game (here, at the post-entry decision node) and then determining what optimal behavior is earlier in the game given the anticipation of this later behavior, is known as *backward induction* (or *backward programming*). It is a procedure that is intimately linked to the idea of sequential rationality because it insures that players' strategies specify optimal behavior at every decision node of the game.

The game in Example 9.B.1 is a member of a general class of games in which the backward induction procedure can be applied to capture the idea of sequential rationality with great generality and power: *finite games of perfect information.* These are games in which every information set contains a single decision node and there is a finite number of such nodes (see Chapter 7).[2] Before introducing a formal equilibrium concept, we first discuss the general application of the backward induction procedure to this class of games.

Backward Induction in Finite Games of Perfect Information

To apply the idea of backward induction in finite games of perfect information, we start by determining the optimal actions for moves at the final decision nodes in the tree (those for which the only successor nodes are terminal nodes). Just as in firm I's post-entry decision in Example 9.B.1, play at these nodes involves no further strategic interactions among the players, and so the determination of optimal behavior at these decision nodes involves a simple single-person decision problem. Then, given that these will be the actions taken at the final decision nodes, we can proceed to the next-to-last decision nodes and determine the optimal actions to be

2. The assumption of finiteness is important for some aspects of this analysis. We discuss this point further toward the end of the section.

taken there by players who correctly anticipate the actions that will follow at the final decision nodes, and so on backward through the game tree.

This procedure is readily implemented using reduced games. At each stage, after solving for the optimal actions at the current final decision nodes, we can derive a new reduced game by deleting the part of the game following these nodes and assigning to these nodes the payoffs that result from the already determined continuation play.

Example 9.B.2: Consider the three-player finite game of perfect information depicted in Figure 9.B.3(a). The arrows in Figure 9.B.3(a) indicate the optimal play at the final decision nodes of the game. Figure 9.B.3(b) is the reduced game formed by replacing these final decision nodes by the payoffs that result from optimal play once these nodes have been reached. Figure 9.B.3(c) represents the reduced game derived

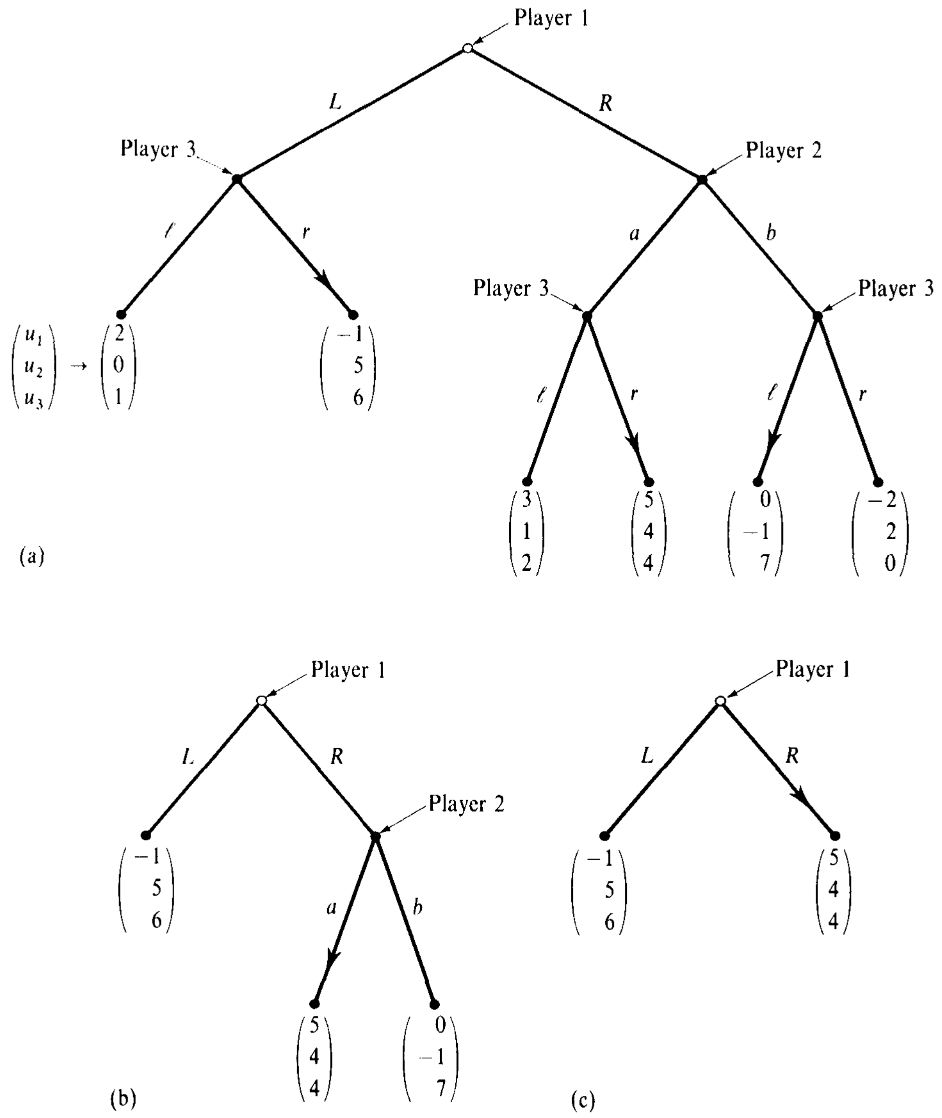


Figure 9.B.3 Reduced games in a backward induction procedure for a finite game of perfect information. (a) Original game. (b) First reduced game. (c) Second reduced game.

in the next stage of the backward induction procedure, when the final decision nodes of the reduced game in Figure 9.B.3(b) are replaced by the payoffs arising from optimal play at these nodes (again indicated by arrows). The backward induction procedure therefore identifies the strategy profile $(\sigma_1, \sigma_2, \sigma_3)$ in which $\sigma_1 = R$, $\sigma_2 =$ "a if player 1 plays R," and

$$\sigma_3 = \begin{cases} r \text{ if player 1 plays } L \\ r \text{ if player 1 plays } R \text{ and player 2 plays } a \\ \ell \text{ if player 1 plays } R \text{ and player 2 plays } b. \end{cases}$$

Note that this strategy profile is a Nash equilibrium of this three-player game but that the game also has other pure strategy Nash equilibria. (Exercise 9.B.3 asks you to verify these two points and to argue that these other Nash equilibria do not satisfy the principle of sequential rationality.) ■

In fact, for finite games of perfect information, we have the general result presented in Proposition 9.B.1.

Proposition 9.B.1: (*Zermelo's Theorem*) Every finite game of perfect information Γ_E has a pure strategy Nash equilibrium that can be derived through backward induction. Moreover, if no player has the same payoffs at any two terminal nodes, then there is a unique Nash equilibrium that can be derived in this manner.

Proof: First, note that in finite games of perfect information, the backward induction procedure is well defined: The player who moves at each decision node has a finite number of possible choices, so optimal actions necessarily exist at each stage of the procedure (if a player is indifferent, we can choose any of her optimal actions). Moreover, the procedure fully specifies all of the players' strategies after a finite number of stages. Second, note that if no player has the same payoffs at any two terminal nodes, then the optimal actions must be *unique* at every stage of the procedure, and so in this case the backward induction procedure identifies a unique strategy profile for the game.

What remains is to show that a strategy profile identified in this way, say $\sigma = (\sigma_1, \ldots, \sigma_I)$, is necessarily a Nash equilibrium of Γ_E. Suppose that it is not. Then there is some player i who has a deviation, say to strategy $\hat{\sigma}_i$, that strictly increases her payoff given that the other players continue to play strategies σ_{-i}. That is, letting $u_i(\sigma_i, \sigma_{-i})$ be player i's payoff function,[3]

$$u_i(\hat{\sigma}_i, \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i}). \tag{9.B.1}$$

We argue that this cannot be. The proof is inductive. We shall say that decision node x has *distance* n if, among the various paths that continue from it to the terminal nodes, the maximal number of decision nodes lying between it and a terminal node is n. We let N denote the maximum distance of any decision node in the game; since Γ_E is a finite game, N is a finite number. Define $\hat{\sigma}_i(n)$ to be the strategy that plays in accordance with strategy σ_i at all nodes with distances $0, \ldots, n$, and plays in accordance with strategy $\hat{\sigma}_i$ at all nodes with distances greater than n.

By the construction of σ through the backward induction procedure, $u_i(\hat{\sigma}_i(0), \sigma_{-i}) \geq u_i(\hat{\sigma}_i, \sigma_{-i})$. That is, player i can do at least as well as she does with strategy $\hat{\sigma}_i$ by instead playing the moves specified in strategy σ_i at all nodes with distance 0 (i.e., at the final decision nodes in the game) and following strategy $\hat{\sigma}_i$ elsewhere.

3. To be precise, $u_i(\cdot)$ is player i's payoff function in the normal form derived from extensive form game Γ_E.

We now argue that if $u_i(\hat{\sigma}_i(n-1), \sigma_{-i}) \geq u_i(\hat{\sigma}_i, \sigma_{-i})$, then $u_i(\hat{\sigma}_i(n), \sigma_{-i}) \geq u_i(\hat{\sigma}_i, \sigma_{-i})$. This is straightforward. The only difference between strategy $\hat{\sigma}_i(n)$ and strategy $\hat{\sigma}_i(n-1)$ is in player i's moves at nodes with distance n. In both strategies, player i plays according to σ_i at all decision nodes that follow the distance-n nodes and in accordance with strategy $\hat{\sigma}_i$ before them. But given that all players are playing in accordance with strategy profile σ after the distance-n nodes, the moves derived for the distance-n decision nodes through backward induction, namely those in σ_i, must be optimal choices for player i at these nodes. Hence, $u_i(\hat{\sigma}_i(n), \sigma_{-i}) \geq u_i(\hat{\sigma}_i(n-1), \sigma_{-i})$.

Applying induction, we therefore have $u_i(\hat{\sigma}_i(N), \sigma_{-i}) \geq u_i(\hat{\sigma}_i, \sigma_{-i})$. But $\hat{\sigma}_i(N) = \sigma_i$, and so we have a contradiction to (9.B.1). Strategy profile σ must therefore constitute a Nash equilibrium of Γ_E. ■

Note, incidentally, that Proposition 9.B.1 establishes the existence of a pure strategy Nash equilibrium in all finite games of perfect information.

Subgame Perfect Nash Equilibria

It is clear enough how to apply the principle of sequential rationality in Example 9.B.1 and, more generally, in finite games of perfect information. Before distilling a general solution concept, however, it is useful to discuss another example. This example suggests how we might identify Nash equilibria that satisfy the principle of sequential rationality in more general games involving imperfect information.

Example 9.B.3: We consider the same situation as in Example 9.B.1 except that firms I and E now play a simultaneous-move game after entry, each choosing either "fight" or "accommodate." The extensive and normal form representations are depicted in Figure 9.B.4.

Examining the normal form, we see that in this game there are three pure strategy Nash equilibria (σ_E, σ_I):[4]

((out, accommodate if in), (fight if firm E plays "in")),

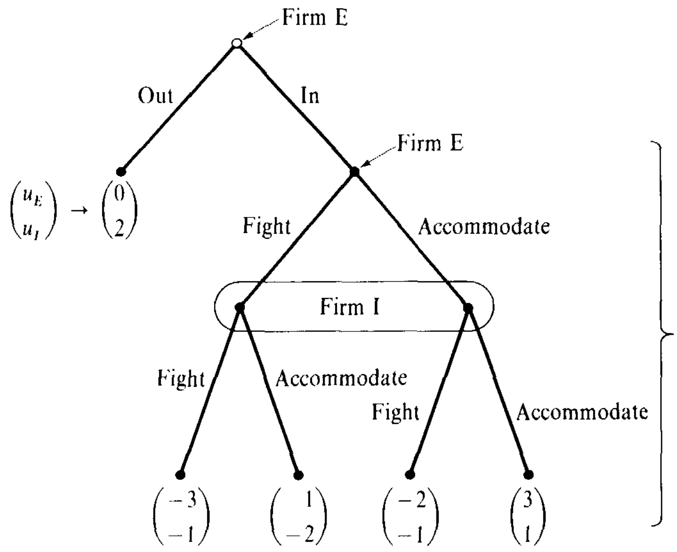
((out, fight if in), (fight if firm E plays "in")),

((in, accommodate if in), (accommodate if firm E plays "in")).

Notice, however, that (accommodate, accommodate) is the sole Nash equilibrium in the simultaneous-move game that follows entry. Thus, the firms should expect that they will both play "accommodate" following firm E's entry.[5] But if this is so, firm E

4. The entrant's strategy in the first two equilibria may appear odd. Firm E is planning to take an action conditional on entering while at the same time planning not to enter. Recall from Section 7.D, however, that a strategy is a *complete contingent plan*. Indeed, the reason we have insisted on this requirement is precisely the need to test the sequential rationality of a player's strategy.

5. Recall that throughout this chapter we maintain the assumption that rational players always play some Nash equilibrium in any strategic situation in which they find themselves (i.e., we assume that players will have mutually correct expectations). Two points about this assumption are worth noting. First, some justifications for a Nash equilibrium may be less compelling in the context of dynamic games. For example, if players never reach certain parts of a game, the stable social convention argument given in Section 8.D may no longer provide a good reason for believing that a Nash equilibrium would be played *if* that part of the game tree were reached. Second, the idea of sequential rationality can still have force even if we do not make this assumption. For example, here we would reach the same conclusion even if we assumed only that neither player would play an iteratively strictly dominated strategy in the post-entry simultaneous-move game.



A Simultaneous-Move Game:

Firm E \ Firm I	Accommodate	Fight
Accommodate	3, 1	−2, −1
Fight	1, −2	−3, −1

Firm E \ Firm I	Accommodate if E Plays "In"	Fight if E Plays "In"
Out, Accommodate if In	0, 2	0, 2
Out, Fight if In	0, 2	0, 2
In, Accommodate if In	3, 1	−2, −1
In, Fight if In	1, −2	−3, −1

Figure 9.B.4 Extensive and normal forms for Example 9.B.3. A sequentially rational Nash equilibrium must have both firms play "accommodate" after entry.

should enter. The logic of sequential rationality therefore suggests that only the last of the three equilibria is a reasonable prediction in this game. ■

The requirement of sequential rationality illustrated in this and the preceding examples is captured by the notion of a *subgame perfect Nash equilibrium* [introduced by Selten (1965)]. Before formally defining this concept, however, we need to specify what a *subgame* is.

Definition 9.B.1: A *subgame* of an extensive form game Γ_E is a subset of the game having the following properties:

(i) It begins with an information set containing a single decision node, contains all the decision nodes that are successors (both immediate and later) of this node, and contains *only* these nodes.

(ii) If decision node x is in the subgame, then every $x' \in H(x)$ is also, where $H(x)$ is the information set that contains decision node x. (That is, there are no "broken" information sets.)

Note that according to Definition 9.B.1, the game as a whole is a subgame, as may be some strict subsets of the game.[6] For example, in Figure 9.B.1, there are two subgames: the game as a whole and the part of the game tree that begins with and follows firm I's decision node. The game in Figure 9.B.4 also has two subgames: the game as a whole and the part of the game beginning with firm E's post-entry decision node. In Figure 9.B.5, the dotted lines indicate three parts of the game of Figure 9.B.4 that are *not* subgames.

Finally, note that in a finite game of perfect information, every decision node initiates a subgame. (Exercise 9.B.1 asks you to verify this fact for the game of Example 9.B.2.)

The key feature of a subgame is that, contemplated in isolation, it is a game in its own right. We can therefore apply to it the idea of Nash equilibrium predictions. In the discussion that follows, we say that a strategy profile σ in extensive form game Γ_E *induces* a Nash equilibrium in a particular subgame of Γ_E if the moves specified in σ for information sets within the subgame constitute a Nash equilibrium when this subgame is considered in isolation.

Definition 9.B.2: A profile of strategies $\sigma = (\sigma_1, \ldots, \sigma_I)$ in an I-player extensive form game Γ_E is a *subgame perfect Nash equilibrium* (SPNE) if it induces a Nash equilibrium in every subgame of Γ_E.

Note that any SPNE is a Nash equilibrium (since the game as a whole is a subgame) but that not every Nash equilibrium is subgame perfect.

Exercise 9.B.2: Consider a game Γ_E in extensive form. Argue that:

(a) If the only subgame is the game as a whole, then every Nash equilibrium is subgame perfect.
(b) A subgame perfect Nash equilibrium induces a subgame perfect Nash equilibrium in every subgame of Γ_E.

The SPNE concept isolates the reasonable Nash equilibria in Examples 9.B.1 and 9.B.3. In Example 9.B.1, any subgame perfect Nash equilibrium must have firm I playing "accommodate if firm E plays 'in'" because this is firm I's strictly dominant strategy in the subgame following entry. Likewise, in Example 9.B.3, any SPNE must have the firms both playing "accommodate" after entry because this is the unique Nash equilibrium in this subgame.

Note also that in finite games of perfect information, such as the games of Examples 9.B.1 and 9.B.2, the set of SPNEs coincides with the set of Nash equilibria that can be derived through the backward induction procedure. Recall, in particular, that in finite games of perfect information every decision node initiates a subgame. Thus, in any SPNE, the strategies must specify actions at each of the final decision nodes of the game that are optimal in the single-player subgame that begins there. Given that this must be the play at the final decision nodes in any SPNE, consider play in the subgames starting at the next-to-last decision nodes. Nash equilibrium play in these subgames, which is required in any SPNE, must have the players who

6. In the literature, the term *proper subgame* is sometimes used with the same meaning we assign to *subgame*. We choose to use the unqualified term *subgame* here to make clear that the game itself qualifies.

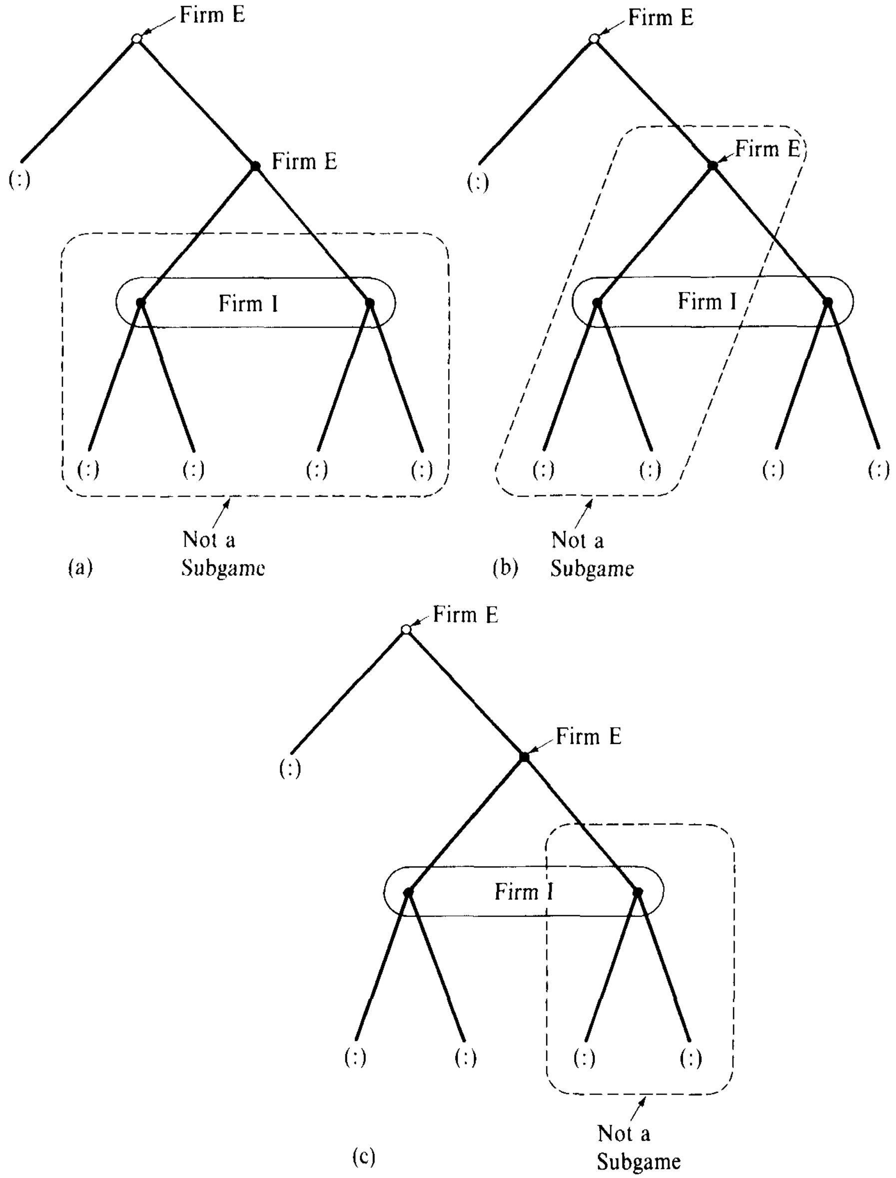


Figure 9.B.5
Three parts of the game in Figure 9.B.4 that are not subgames.

move at these next-to-last nodes choosing optimal strategies given the play that will occur at the last nodes. And so on. An implication of this fact and Proposition 9.B.1 is therefore the result stated in Proposition 9.B.2.

Proposition 9.B.2: Every finite game of perfect information Γ_E has a pure strategy subgame perfect Nash equilibrium. Moreover, if no player has the same payoffs at any two terminal nodes, then there is a unique subgame perfect Nash equilibrium.[7]

7. The result can also be seen directly from Proposition 9.B.1. Just as the strategy profile derived using the backward induction procedure constitutes a Nash equilibrium in the game as a whole, it is also a Nash equilibrium in every subgame.

In fact, to identify the set of subgame perfect Nash equilibria in a general (finite) dynamic game Γ_E, we can use a generalization of the backward induction procedure. This *generalized backward induction procedure* works as follows:

1. Start at the end of the game tree, and identify the Nash equilibria for each of the *final* subgames (i.e., those that have no other subgames nested within them).
2. Select one Nash equilibrium in each of these final subgames, and derive the reduced extensive form game in which these final subgames are replaced by the payoffs that result in these subgames when players use these equilibrium strategies.
3. Repeat steps 1 and 2 for the reduced game. Continue the procedure until every move in Γ_E is determined. This collection of moves at the various information sets of Γ_E constitutes a profile of SPNE strategies.
4. If multiple equilibria are never encountered in any step of this process, this profile of strategies is the unique SPNE. If multiple equilibria are encountered, the full set of SPNEs is identified by repeating the procedure for each possible equilibrium that could occur for the subgames in question.

The formal justification for using this generalized backward induction procedure to identify the set of SPNEs comes from the result shown in Proposition 9.B.3.

Proposition 9.B.3: Consider an extensive form game Γ_E and some subgame S of Γ_E. Suppose that strategy profile σ^S is an SPNE in subgame S, and let $\hat{\Gamma}_E$ be the reduced game formed by replacing subgame S by a terminal node with payoffs equal to those arising from play of σ^S. Then:

(i) In any SPNE σ of Γ_E in which σ^S is the play in subgame S, players' moves at information sets outside subgame S must constitute an SPNE of reduced game $\hat{\Gamma}_E$.

(ii) If $\hat{\sigma}$ is an SPNE of $\hat{\Gamma}_E$, then the strategy profile σ that specifies the moves in σ^S at information sets in subgame S and that specifies the moves in $\hat{\sigma}$ at information sets not in S is an SPNE of Γ_E.

Proof: (i) Suppose that strategy profile σ specifies play at information sets outside subgame S that does not constitute an SPNE of reduced game $\hat{\Gamma}_E$. Then there exists a subgame of $\hat{\Gamma}_E$ in which σ does not induce a Nash equilibrium. In this subgame of $\hat{\Gamma}_E$, some player has a deviation that improves her payoff, taking as given the strategies of her opponents. But then it must be that this player also has a profitable deviation in the corresponding subgame of game Γ_E. She makes the same alterations in her moves at information sets not in S and leaves her moves at information sets in S unchanged. Hence, σ could not be an SPNE of the overall game Γ_E.

(ii) Suppose that $\hat{\sigma}$ is an SPNE of reduced game $\hat{\Gamma}_E$, and let σ be the strategy in the overall game Γ_E formed by specifying the moves in σ^S at information sets in subgame S and the moves in $\hat{\sigma}$ at information sets not in S. We argue that σ induces a Nash equilibrium in every subgame of Γ_E. This follows immediately from the construction of σ for subgames of Γ_E that either lie entirely in subgame S or never intersect with subgame S (i.e., that do not have subgame S nested within them). So consider any subgame that has subgame S nested within it. If some player i has a profitable deviation in this subgame given her opponent's strategies, then she must also have a profitable deviation that leaves her moves within subgame S unchanged because, by hypothesis, a player does best within subgame S by playing the moves specified in strategy profile σ^S given that her opponents do so. But if she has such a profitable deviation, then she must have a profitable deviation in the corresponding subgame of reduced game $\hat{\Gamma}_E$, in contradiction to $\hat{\sigma}$ being an SPNE of $\hat{\Gamma}_E$. ■

Note that for the final subgames of Γ_E, the set of Nash equilibria and SPNEs coincide, because these subgames contain no nested subgames. Identifying Nash equilibria in these final subgames therefore allows us to begin the inductive application of Proposition 9.B.3.

This generalized backward induction procedure reduces to our previous backward induction procedure in the case of games of perfect information. But it also applies to games of imperfect information. Example 9.B.3 provides a simple illustration. There we can identify the unique SPNE by first identifying the unique Nash equilibrium in the post-entry subgame: (accommodate, accommodate). Having done this, we can replace this subgame with the payoffs that result from equilibrium play in it. The reduced game that results is then much the same as that shown in Figure 9.B.2, the only difference being that firm E's payoff from playing "in" is now 3 instead of 2. Hence, in this manner, we can derive the unique SPNE of Example 9.B.3: $(\sigma_E, \sigma_I) =$ ((in, accommodate if in), (accommodate if firm E plays "in")).

The game in Example 9.B.3 is simple to solve in two respects. First, there is a unique equilibrium in the post-entry subgame. If this were not so, behavior earlier in the game could depend on *which* equilibrium resulted after entry. Example 9.B.4 illustrates this point:[8]

Example 9.B.4: *The Niche Choice Game.* Consider a modification of Example 9.B.3 in which instead of having the two firms choose whether to fight or accommodate each other, we suppose that there are actually two niches in the market, one large and one small. After entry, the two firms decide simultaneously which niche they will be in. For example, the niches might correspond to two types of customers, and the firms may be deciding to which type they are targeting their product design. Both firms lose money if they choose the same niche, with more lost if it is the small niche. If they choose different niches, the firm that targets the large niche earns a profit, and the firm with the small niche incurs a loss, but a smaller loss than if the two firms targeted the same niche. The extensive form of this game is depicted in Figure 9.B.6.

To determine the SPNE of this game, consider the post-entry subgame first. There are two pure strategy Nash equilibria of this simultaneous-move game: (large niche, small niche) and (small niche, large niche).[9] In any pure strategy SPNE, the firms' strategies must induce one of these two Nash equilibria in the post-entry subgame. Suppose, first, that the firms will play (large niche, small niche). In this case, the payoffs from reaching the post-entry subgame are $(u_E, u_I) = (1, -1)$, and the reduced game is as depicted in Figure 9.B.7(a). The entrant optimally chooses to enter in this

8. Similar issues can arise in games of perfect information when a player is indifferent between two actions. However, the presence of multiple equilibria in subgames involving simultaneous play is, in a sense, a more robust phenomenon. Multiple equilibria are generally robust to small changes in players' payoffs, but ties in games of perfect information are not.

9. We restrict attention here to pure strategy SPNEs. There is also a mixed strategy Nash equilibrium in the post-entry subgame. Exercise 9.B.6 asks you to investigate the implications of this mixed strategy play being the post-entry equilibrium behavior.

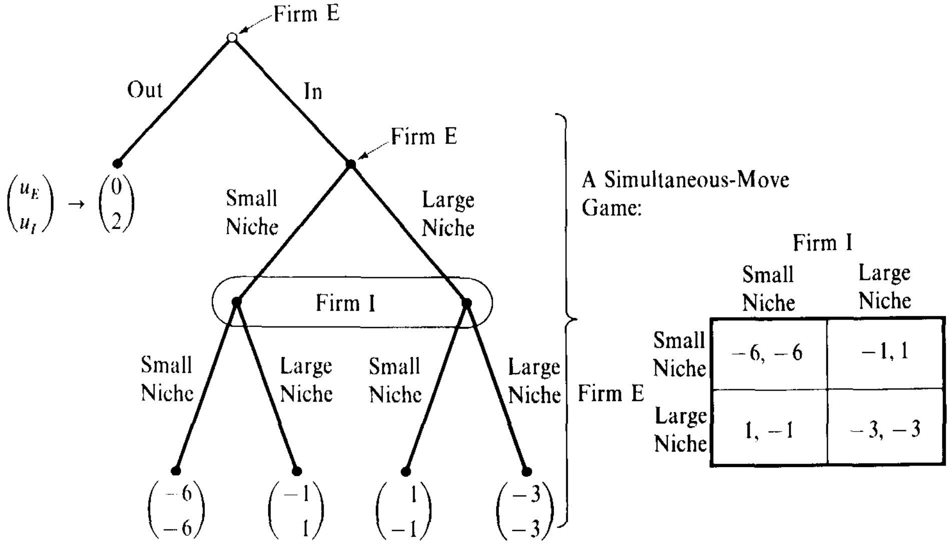


Figure 9.B.6 Extensive form for the Niche Choice game. The post-entry subgame has multiple Nash equilibria.

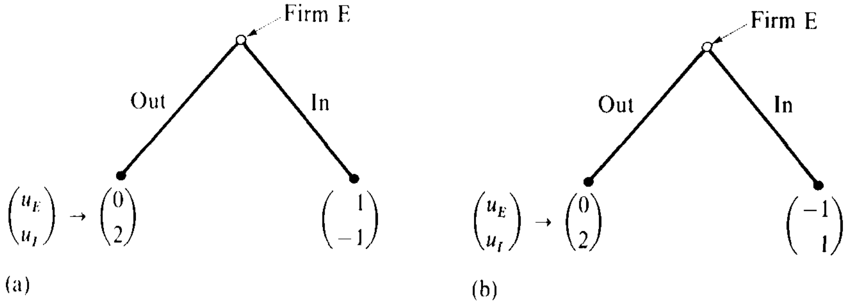


Figure 9.B.7 Reduced games after identifying (pure strategy) Nash equilibria in the post-entry subgame of the Niche Choice game. (a) Reduced game if (large niche, small niche) is post-entry equilibrium. (b) Reduced game if (small niche, large niche) is post-entry equilibrium.

case. Hence, one SPNE is $(\sigma_E, \sigma_I) =$ ((in, large niche if in), (small niche if firm E plays "in")).

Now suppose that the post-entry play is (small niche, large niche). Then the payoffs from reaching the post-entry game are $(u_E, u_I) = (-1, 1)$, and the reduced game is that depicted in Figure 9.B.7(b). The entrant optimally chooses not to enter in this case. Hence, there is a second pure strategy SPNE: $(\sigma_E, \sigma_I) =$ ((out, small niche if in), (large niche if firm E plays "in")). ■

A second sense in which the game in Example 9.B.3 is simple to solve is that it involves only one subgame other than the game as a whole. Like games of perfect information, a game with imperfect information may in general have *many* subgames, with one subgame nested within another, and that larger subgame nested within a still larger one, and so on.

One interesting class of imperfect information games in which the generalized backward induction procedure gives a very clean conclusion is described in Proposition 9.B.4.

Proposition 9.B.4: Consider an I-player extensive form game Γ_E involving successive play of T simultaneous-move games, $\Gamma_N^t = [I, \{\Delta(S_i^t)\}, \{u_i^t(\cdot)\}]$ for $t = 1, \ldots, T$, with the players observing the pure strategies played in each game immediately after its play is concluded. Assume that each player's payoff is equal to the sum of her payoffs in the plays of the T games. If there is a unique Nash equilibrium in each game Γ_N^t, say $\sigma^t = (\sigma_1^t, \ldots, \sigma_I^t)$, then there is a unique SPNE of Γ_E and it consists of each player i playing strategy σ_i^t in each game Γ_N^t regardless of what has happened previously.

Proof: The proof is by induction. The result is clearly true for $T = 1$. Now suppose it is true for all $T \leq n - 1$. We will show that it is true for $T = n$.

We know by hypothesis that in any SPNE of the overall game, after play of game Γ_N^1 the play in the remaining $n - 1$ simultaneous-move games must simply involve play of the Nash equilibrium of each game (since any SPNE of the overall game induces an SPNE in each of its subgames). Let player i earn G_i from this equilibrium play in these $n - 1$ games. Then in the reduced game that replaces all the subgames that follow Γ_N^1 with their equilibrium payoffs, player i earns an overall payoff of $u_i(s_1^1, \ldots, s_I^1) + G_i$ if $(s_1^1, \ldots, s_I^1)$ is the profile of pure strategies played in game Γ_N^1. The unique Nash equilibrium of this reduced game is clearly σ^1. Hence, the result is also true for $T = n$. ■

The basic idea behind Proposition 9.B.4 is an application of backward induction logic: Play in the last game must result in the unique Nash equilibrium of that game being played because at that point players essentially face just that game. But if play in the last game is predetermined, then when players play the next-to-last game, it is again as if they were playing just *that* game in isolation (think of the case where $T = 2$). And so on.

An interesting aspect of Proposition 9.B.4 is the way the SPNE concept rules out history dependence of strategies in the class of games considered there. In general, a player's strategy could potentially promise later rewards or punishments to other players if they take particular actions early in the game. But as long as each of the component games has a unique Nash equilibrium, SPNE strategies cannot be history dependent.[10]

Exercises 9.B.9 to 9.B.11 provide some additional examples of the use of the subgame perfect Nash equilibrium concept. In Appendix A we also study an important economic application of subgame perfection to a finite game of perfect information (albeit one with an infinite number of possible moves at some decision nodes): a finite horizon model of bilateral bargaining.

Up to this point, our analysis has assumed that the game being studied is finite. This has been important because it has allowed us to identify subgame perfect Nash equilibria by starting at the end of the game and working backward. As a general matter, in games in which there can be an infinite sequence of moves (so that some paths through the tree never reach a terminal node), the definition of a subgame perfect Nash equilibrium remains that given in Definition 9.B.2: Strategies must induce a Nash equilibrium in every subgame. Nevertheless, the lack of a definite finite point of termination of the game can reduce the power of the SPNE concept because we can no longer use the end of the game to pin down behavior. In games in which there is always a future, a wide range of behaviors can sometimes be justified as sequentially rational (i.e., as part of an SPNE). A striking example of this sort arises in

10. This lack of history dependence depends importantly on the uniqueness assumption of Proposition 9.B.4. With multiple Nash equilibria in the component games, we can get outcomes that are not merely the repeated play of the static Nash equilibria. (See Exercise 9.B.9 for an example.)

Chapter 12 and its Appendix A when we consider *infinitely repeated games* in the context of studying oligopolistic pricing.

Nevertheless, it is not always the case that an infinite horizon weakens the power of the subgame perfection criterion. In Appendix A of this chapter, we study an infinite horizon model of bilateral bargaining in which the SPNE concept predicts a unique outcome, and this outcome coincides with the limiting outcome of the corresponding finite horizon bargaining model as the horizon grows long.

The methods used to identify subgame perfect Nash equilibria in infinite horizon games are varied. Sometimes, the method involves showing that the game can effectively be truncated because after a certain point it is obvious what equilibrium play must be (see Exercise 9.B.11). In other situations, the game possesses a stationarity property that can be exploited; the analysis of the infinite horizon bilateral bargaining model in Appendix A is one example of this kind.

After the preceding analysis, the logic of sequential rationality may seem unassailable. But things are not quite so clear. For example, nowhere could the principle of sequential rationality seem on more secure footing than in finite games of perfect information. But chess is a game of this type (the game ends if 50 moves occur without a piece being taken or a pawn being moved), and so its "solution" should be simple to predict. Of course, it is exactly players' *inability* to do so that makes it an exciting game to play. The same could be said even of the much simpler game of Chinese checkers. It is clear that in practice, players may be only boundedly rational. As a result, we might feel more comfortable with our rationality hypotheses in games that are relatively simple, in games where repetition helps players learn to think through the game, or in games where large stakes encourage players to do so as much as possible. Of course, the possibility of bounded rationality is not a concern limited to dynamic games and subgame perfect Nash equilibria; it is also relevant for simultaneous-move games containing many possible strategies.

There is, however, an interesting tension present in the SPNE concept that is related to this bounded rationality issue and that does not arise in the context of simultaneous-move games. In particular, the SPNE concept insists that players should play an SPNE wherever they find themselves in the game tree, even after a sequence of events that is contrary to the predictions of the theory. To see this point starkly, consider the following example due to Rosenthal (1981), known as the *Centipede game*.

Example 9.B.5: *The Centipede Game.* In this finite game of perfect information, there are two players, 1 and 2. The players each start with 1 dollar in front of them. They alternate saying "stop" or "continue," starting with player 1. When a player says "continue," 1 dollar is taken by a referee from her pile and 2 dollars are put in her opponent's pile. As soon as either player says "stop," play is terminated, and each player receives the money currently in her pile. Alternatively, play stops if both players' piles reach 100 dollars. The extensive form for this game is depicted in Figure 9.B.8.

The unique SPNE in this game has both players saying "stop" whenever it is their turn, and the players each receive 1 dollar in this equilibrium. To see this, consider player 2's move at the final decision node of the game (after the players have said "continue" a total of 197 times). Her optimal move if play reaches this point is to say "stop"; by doing so, she receives 101 dollars compared with a payoff of 100 dollars if she says "continue." Now consider what happens if play reaches the next-to-last decision node. Player 1, anticipating player 2's move at the final decision node, also says "stop"; doing so, she earns 99 dollars, compared with 98 dollars if she says "continue." Continuing backward through the tree in this fashion, we identify saying "stop" as the optimal move at every decision node.

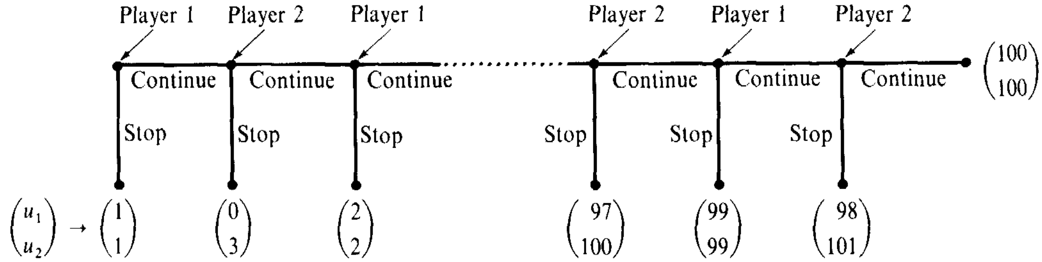


Figure 9.B.8 The Centipede game.

A striking aspect of the SPNE in the Centipede game is how bad it is for the players. They each get 1 dollar, whereas they might get 100 dollars by repeatedly saying "continue."

Is this (unique) SPNE in the Centipede game a reasonable prediction? Consider player 1's initial decision to say "stop." For this to be rational, player 1 must be pretty sure that if instead she says "continue," player 2 will say "stop" at her first turn. Indeed, "continue" would be better for player 1 as long as she could be sure that player 2 would say "continue" at her next move. Why might player 2 respond to player 1 saying "continue" by also saying "continue"? First, as we have pointed out, player 2 might not be fully rational, and so she might not have done the backward induction computation assumed in the SPNE concept. More interestingly, however, once she sees that player 1 has chosen "continue"—an event that should never happen according to the SPNE prediction—she might entertain the possibility that player 1 is not rational in the sense demanded by the SPNE concept. If, as a result, she thinks that player 1 would say "continue" at her next move if given the chance, then player 2 would want to say "continue" herself. The SPNE concept denies this possibility, instead assuming that at any point in the game, players will assume that the remaining play of the game will be an SPNE even if play up to that point has contradicted the theory. One way of resolving this tension is to view the SPNE theory as treating any deviation from prescribed play as the result of an extremely unlikely "mistake" that is unlikely to occur again. In Appendix B, we discuss one concept that makes this idea explicit. ■

9.C Beliefs and Sequential Rationality

Although subgame perfection is often very useful in capturing the principle of sequential rationality, sometimes it is not enough. Consider Example 9.C.1's adaptation of the entry game studied in Example 9.B.1.

Example 9.C.1: We now suppose that there are two strategies firm E can use to enter, "in_1" and "in_2," and that the incumbent is unable to tell which strategy firm E has used if entry occurs. Figure 9.C.1 depicts this game and its payoffs.

As in the original entry game in Example 9.B.1, there are two pure strategy Nash equilibria here: (out, fight if entry occurs) and (in_1, accommodate if entry occurs). Once again, however, the first of these does not seem very reasonable; regardless of what entry strategy firm E has used, the incumbent prefers to accommodate once entry has occurred. *But the criterion of subgame perfection is of absolutely no use here*: Because the only subgame is the game as a whole, both pure strategy Nash equilibria are subgame perfect. ■

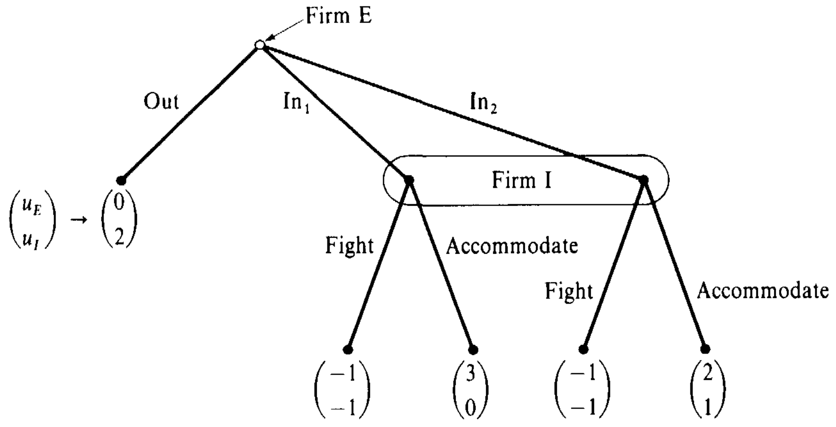


Figure 9.C.1
Extensive form for Example 9.C.1. The SPNE concept may fail to insure sequential rationality.

How can we eliminate the unreasonable equilibrium here? One possibility, which is in the spirit of the principle of sequential rationality, might be to insist that the incumbent's action after entry be optimal for *some belief* that she might have about which entry strategy was used by firm E. Indeed, in Example 9.C.1, "fight if entry occurs" is not an optimal choice for *any* belief that firm I might have. This suggests that we may be able to make some progress by formally considering players' beliefs and using them to test the sequential rationality of players' strategies.

We now introduce a solution concept, which we call a *weak perfect Bayesian equilibrium* [Myerson (1991) refers to this same concept as a *weak sequential equilibrium*], that extends the principle of sequential rationality by formally introducing the notion of beliefs.[11] It requires, roughly, that at any point in the game, a player's strategy prescribe optimal actions from that point on given her opponents' strategies and her beliefs about what has happened so far in the game and that her beliefs be consistent with the strategies being played.

To express this notion formally, we must first formally define the two concepts that are its critical components: the notions of a *system of beliefs* and the *sequential rationality of strategies*. Beliefs are simple.

Definition 9.C.1: A *system of beliefs* μ in extensive form game Γ_E is a specification of a probability $\mu(x) \in [0, 1]$ for each decision node x in Γ_E such that

$$\textstyle\sum_{x \in H} \mu(x) = 1$$

for all information sets H.

A system of beliefs can be thought of as specifying, for each information set, a probabilistic assessment by the player who moves at that set of the relative likelihoods of being at each of the information set's various decision nodes, conditional upon play having reached that information set.

11. The concept of a *perfect Bayesian equilibrium* was first developed to capture the requirements of sequential rationality in dynamic games with incomplete information, that is (using the terminology introduced in Section 8.E), in dynamic Bayesian games. The *weak perfect Bayesian equilibrium* concept is a variant that is introduced here primarily for pedagogic purposes (the reason for the modifier *weak* will be made clear later in this section). Myerson (1991) refers to this same concept as a *weak sequential equilibrium* because it may also be considered a weak variant of the *sequential equilibrium* concept introduced in Definition 9.C.4.

To define sequential rationality, it is useful to let $E[u_i \mid H, \mu, \sigma_i, \sigma_{-i}]$ denote player i's expected utility starting at her information set H if her beliefs regarding the conditional probabilities of being at the various nodes in H are given by μ, if she follows strategy σ_i, and if her rivals use strategies σ_{-i}. [We will not write out the formula for this expression explicitly, although it is conceptually straightforward: Pretend that the probability distribution $\mu(x)$ over nodes $x \in H$ is generated by nature; then player i's expected payoff is determined by the probability distribution that is induced on the terminal nodes by the combination of this initial distribution plus the players' strategies from this point on.]

Definition 9.C.2: A strategy profile $\sigma = (\sigma_1, \ldots, \sigma_I)$ in extensive form game Γ_E is *sequentially rational at information set H given a system of beliefs* μ if, denoting by $\iota(H)$ the player who moves at information set H, we have

$$E[u_{\iota(H)} \mid H, \mu, \sigma_{\iota(H)}, \sigma_{-\iota(H)}] \geq E[u_{\iota(H)} \mid H, \mu, \tilde{\sigma}_{\iota(H)}, \sigma_{-\iota(H)}]$$

for all $\tilde{\sigma}_{\iota(H)} \in \Delta(S_{\iota(H)})$. If strategy profile σ satisfies this condition for *all* information sets H, then we say that σ is *sequentially rational given belief system* μ.

In words, a strategy profile $\sigma = (\sigma_1, \ldots, \sigma_I)$ is sequentially rational if no player finds it worthwhile, once one of her information sets has been reached, to revise her strategy given her beliefs about what has already occurred (as embodied in μ) and her rivals' strategies.

With these two notions, we can now define a weak perfect Bayesian equilibrium. The definition involves two conditions: First, strategies must be sequentially rational given beliefs. Second, whenever possible, beliefs must be consistent with the strategies. The idea behind the consistency condition on beliefs is much the same as the idea underlying the concept of Nash equilibrium (see Section 8.D): In an equilibrium, players should have correct beliefs about their opponents' strategy choices.

To motivate the specific consistency requirement on beliefs to be made in the definition of a weak perfect Bayesian equilibrium, consider how we might define the notion of consistent beliefs in the special case in which each player's equilibrium strategy assigns a strictly positive probability to each possible action at every one of her information sets (known as a *completely mixed strategy*).[12] In this case, every information set in the game is reached with positive probability. The natural notion of beliefs being consistent with the play of the equilibrium strategy profile σ is in this case straightforward: For each node x in a given player's information set H, the player should compute the probability of reaching that node given play of strategies σ, Prob $(x \mid \sigma)$, and she should then assign conditional probabilities of being at each of these nodes given that play has reached this information set using *Bayes' rule*:[13]

$$\text{Prob}\,(x \mid H, \sigma) = \frac{\text{Prob}\,(x \mid \sigma)}{\sum_{x' \in H} \text{Prob}\,(x' \mid \sigma)}.$$

12. Equivalently, a completely mixed strategy can be thought of as a strategy that assigns a strictly positive probability to each of the player's pure strategies in the normal form derived from extensive form game Γ_E.

13. Bayes' rule is a basic principle of statistical inference. See, for example, DeGroot (1970), where it is referred to as *Bayes' theorem*.

As a concrete example, suppose that in the game in Example 9.C.1, firm E is using the completely mixed strategy that assigns a probability of $\frac{1}{4}$ to "out," $\frac{1}{2}$ to "in$_1$," and $\frac{1}{4}$ to "in$_2$." Then the probability of reaching firm I's information set given this strategy is $\frac{3}{4}$. Using Bayes' rule, the probability of being at the left node of firm I's information set conditional on this information set having been reached is $\frac{2}{3}$, and the conditional probability of being at the right node in the set is $\frac{1}{3}$. For firm I's beliefs following entry to be consistent with firm E's strategy, firm I's beliefs should assign exactly these probabilities.

The more difficult issue arises when players are not using completely mixed strategies. In this case, some information sets may no longer be reached with positive probability, and so we cannot use Bayes' rule to compute conditional probabilities for the nodes in these information sets. At an intuitive level, this problem corresponds to the idea that even if players were to play the game repeatedly, the equilibrium play would generate no experience on which they could base their beliefs at these information sets. The weak perfect Bayesian equilibrium concept takes an agnostic view toward what players should believe if play were to reach these information sets unexpectedly. In particular, it allows us to assign *any* beliefs at these information sets. It is in this sense that the modifier *weak* is appropriately attached to this concept.

We can now give a formal definition.

Definition 9.C.3: A profile of strategies and system of beliefs (σ, μ) is a *weak perfect Bayesian equilibrium* (weak PBE) in extensive form game Γ_E if it has the following properties:

(i) The strategy profile σ is sequentially rational given belief system μ.

(ii) The system of beliefs μ is derived from strategy profile σ through Bayes' rule whenever possible. That is, for any information set H such that Prob $(H \mid \sigma) > 0$ (read as "the probability of reaching information set H is positive under strategies σ"), we must have

$$\mu(x) = \frac{\text{Prob}\,(x \mid \sigma)}{\text{Prob}\,(H \mid \sigma)} \quad \text{for all } x \in H.$$

It should be noted that the definition formally incorporates beliefs as part of an equilibrium by identifying a *strategy–beliefs pair* (σ, μ) as a weak perfect Bayesian equilibrium. In the literature, however, it is not uncommon to see this treated a bit loosely: a set of strategies σ will be referred to as an equilibrium with the meaning that there is at least one associated set of beliefs μ such that (σ, μ) satisfies Definition 9.C.3. At times, however, it can be very useful to be more explicit about what these beliefs are, such as when testing them against some of the "reasonableness" criteria that we discuss in Section 9.D.

A useful way to understand the relationship between the weak PBE concept and that of Nash equilibrium comes in the characterization of Nash equilibrium given in Proposition 9.C.1.

Proposition 9.C.1: A strategy profile σ is a Nash equilibrium of extensive form game Γ_E if and only if there exists a system of beliefs μ such that

(i) The strategy profile σ is sequentially rational given belief system μ *at all information sets H such that Prob* $(H \mid \sigma) > 0$.

(ii) The system of beliefs μ is derived from strategy profile σ through Bayes' rule whenever possible.

Exercise 9.C.1 asks you to prove this result. The italicized portion of condition (i) is the only change from Definition 9.C.3: For a Nash equilibrium, we require sequential rationality only on the equilibrium path. Hence, a weak perfect Bayesian equilibrium of game Γ_E is a Nash equilibrium, but not every Nash equilibrium is a weak PBE.

We now illustrate the application of the weak PBE concept in several examples. We first consider how the concept performs in Example 9.C.1.

Example 9.C.1 Continued: Clearly, firm I must play "accommodate if entry occurs" in any weak perfect Bayesian equilibrium because that is firm I's optimal action starting at its information set for *any* system of beliefs. Thus, the Nash equilibrium strategies (out, fight if entry occurs) cannot be part of any weak PBE.

What about the other pure strategy Nash equilibrium, (in_1, accommodate if entry occurs)? To show that this strategy profile *is* part of a weak PBE, we need to supplement these strategies with a system of beliefs that satisfy criterion (ii) of Definition 9.C.3 and that lead these strategies to be sequentially rational. Note first that to satisfy criterion (ii), the incumbent's beliefs must assign probability 1 to being at the left node in her information set because this information set is reached with positive probability given the strategies (in_1, accommodate if entry occurs) [a specification of beliefs at this information set fully describes a system of beliefs in this game because the only other information set is a singleton]. Moreover, these strategies are, indeed, sequentially rational given this system of beliefs. In fact, this strategy–beliefs pair is the unique weak PBE in this game (pure or mixed). ■

Examples 9.C.2 and 9.C.3 provide further illustrations of the application of the weak PBE concept.

Example 9.C.2: Consider the following "joint venture" entry game: Now there is a second potential entrant E2. The story is as follows: Firm E1 has the essential capability to enter the market but lacks some important capability that firm E2 has. As a result, E1 is considering proposing a joint venture with E2 in which E2 shares its capability with E1 and the two firms split the profits from entry. Firm E1 has three initial choices: enter directly on its own, propose a joint venture with E2, or stay out of the market. If it proposes a joint venture, firm E2 can either accept or decline. If E2 accepts, then E1 enters with E2's assistance. If not, then E1 must decide whether to enter on its own. The incumbent can observe whether E1 has entered, but not whether it is with E2's assistance. Fighting is the best response for the incumbent if E1 is unassisted (E1 can then be wiped out quickly) but is not optimal for the incumbent if E1 is assisted (E1 is then a tougher competitor). Finally, if E1 is unassisted, it wants to enter only if the incumbent accommodates; but if E1 is assisted by E2, then because it will be such a strong competitor, its entry is profitable regardless of whether the incumbent fights. The extensive form of this game is depicted in Figure 9.C.2.

To identify the weak PBE of this game note first that, in any weak PBE, firm E2 must accept the joint venture if firm E1 proposes it because E2 is thereby assured of a positive payoff regardless of firm I's strategy. But if so, then in any weak PBE

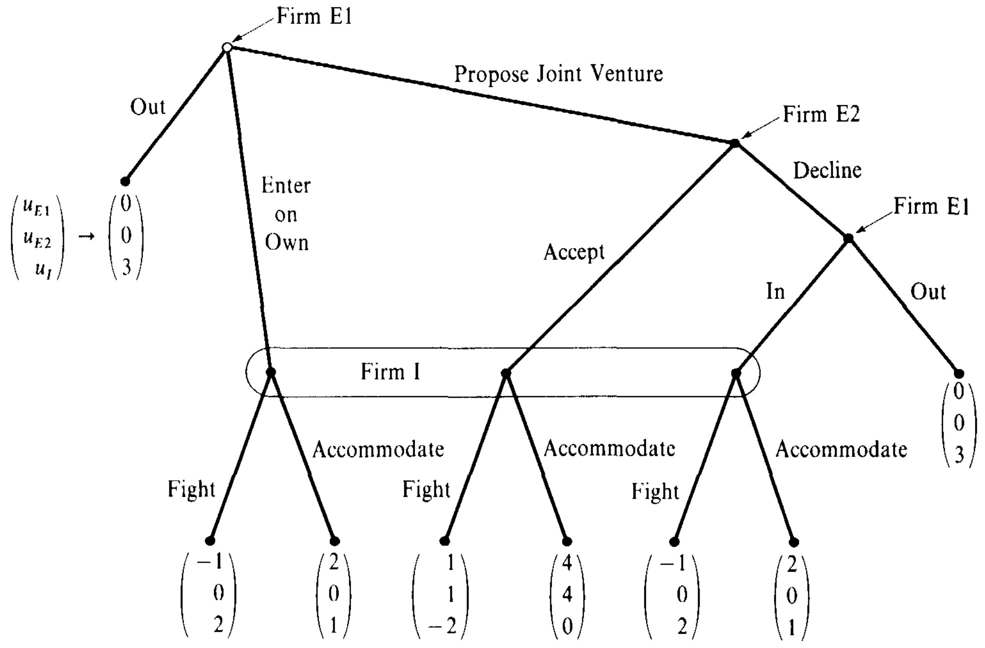


Figure 9.C.2
Extensive form for Example 9.C.2.

firm E1 must propose the joint venture since if firm E2 will accept its proposal, then firm E1 does better proposing the joint venture than it does by either staying out or entering on its own, regardless of firm I's post-entry strategy. Next, these two conclusions imply that firm I's information set is reached with positive probability (in fact, with certainty) in any weak PBE. Applying Bayesian updating at this information set, we conclude that the beliefs at this information set must assign a probability of 1 to being at the middle node. Given this, in any weak PBE firm I's strategy must be "accommodate if entry occurs." Finally, if firm I is playing "accommodate if entry occurs," then firm E1 must enter if it proposes a joint venture that firm E2 then rejects.

We conclude that the unique weak PBE in this game is a strategy–beliefs pair with strategies of $(\sigma_{E1}, \sigma_{E2}, \sigma_I)$ = ((propose joint venture, in if E2 declines), (accept), (accommodate if entry occurs)) and a belief system of μ (middle node of incumbent's information set) $= 1$. Note that this is not the only Nash equilibrium or, for that matter, the only SPNE. For example, $(\sigma_{E1}, \sigma_{E2}, \sigma_I)$ = ((out, out if E2 declines), (decline), (fight if entry occurs)) is an SPNE in this game. ■

Example 9.C.3: In the games of Examples 9.C.1 and 9.C.2 the trick to identifying the weak PBEs consisted of seeing that some player had an optimal strategy that was independent of her beliefs and/or the future play of her opponents. In the game depicted in Figure 9.C.3, however, this is not so for either player. Firm I is now willing to fight if she thinks that firm E has played "in$_1$," and the optimal strategy for firm E depends on firm I's behavior (note that $\gamma > -1$).

To solve this game, we look for a *fixed point* at which the behavior generated by beliefs is consistent with these beliefs. We restrict attention to the case where $\gamma > 0$. [Exercise 9.C.2 asks you to determine the set of weak PBEs when $\gamma \in (-1, 0)$.] Let σ_F be the probability that firm I fights after entry, let μ_1 be firm I's belief that

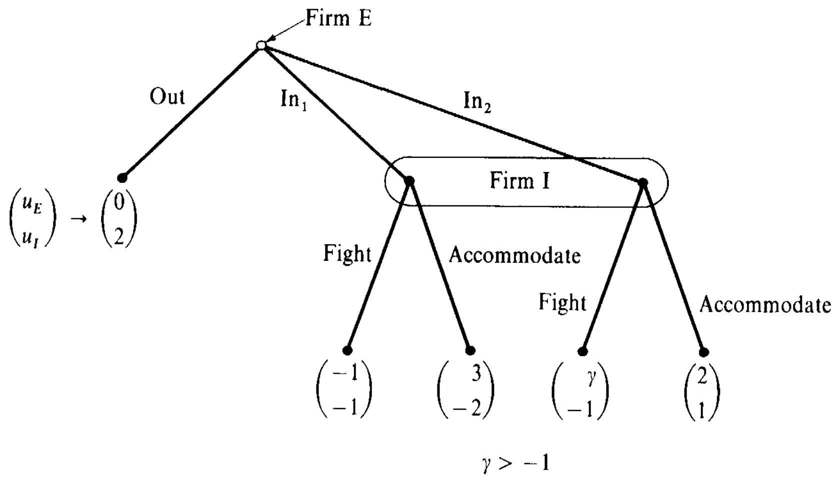


Figure 9.C.3
Extensive form for Example 9.C.3.

"in_1" was E's entry strategy if entry has occurred, and let $\sigma_0, \sigma_1, \sigma_2$ denote the probabilities with which firm E actually chooses "out," "in_1," and "in_2," respectively.

Note, first, that firm I is willing to play "fight" with positive probability if and only if $-1 \geq -2\mu_1 + 1(1-\mu_1)$, or $\mu_1 \geq \frac{2}{3}$.

Suppose, first, that $\mu_1 > \frac{2}{3}$ in a weak PBE. Then firm I must be playing "fight" with probability 1. But then firm E must be playing "in_2" with probability 1 (since $\gamma > 0$), and the weak PBE concept would then require that $\mu_1 = 0$, which is a contradiction.

Suppose, instead, that $\mu_1 < \frac{2}{3}$ in a weak PBE. Then firm I must be playing "accommodate" with probability 1. But, if so, then firm E must be playing "in_1" with probability 1, and the weak PBE concept then requires that $\mu_1 = 1$, another contradiction.

Hence, in any weak PBE of this game, we must have $\mu_1 = \frac{2}{3}$. If so, then firm E must be randomizing in the equilibrium with positive probabilities attached to both "in_1" and "in_2" and with "in_1" twice as likely as "in_2." This means that firm I's probability of playing "fight" must make firm E indifferent between "in_1" and "in_2." Hence, we must have $-1\sigma_F + 3(1-\sigma_F) = \gamma\sigma_F + 2(1-\sigma_F)$, or $\sigma_F = 1/(\gamma+2)$. Firm E's payoff from playing "in_1" or "in_2" is then $(3\gamma+2)/(\gamma+2) > 0$, and so firm E must play "out" with zero probability. Therefore, the unique weak PBE in this game when $\gamma > 0$ has $(\sigma_0, \sigma_1, \sigma_2) = (0, \frac{2}{3}, \frac{1}{3})$, $\sigma_F = 1/(\gamma+2)$, and $\mu_1 = \frac{2}{3}$. ■

Strengthenings of the Weak Perfect Bayesian Equilibrium Concept

We have referred to the concept defined in Definition 9.C.3 as a *weak* perfect Bayesian equilibrium because the consistency requirements that it puts on beliefs are very minimal: The *only* requirement for beliefs, other than that they specify nonnegative probabilities which add to 1 within each information set, is that they are consistent with the equilibrium strategies on the equilibrium path, in the sense of being derived from them through Bayes' rule. *No restrictions at all are placed on beliefs off the equilibrium path* (i.e., at information sets not reached with positive probability with play of the equilibrium strategies). In the literature, a number of strengthenings of this concept that put additional consistency restrictions on off-the-equilibrium-path

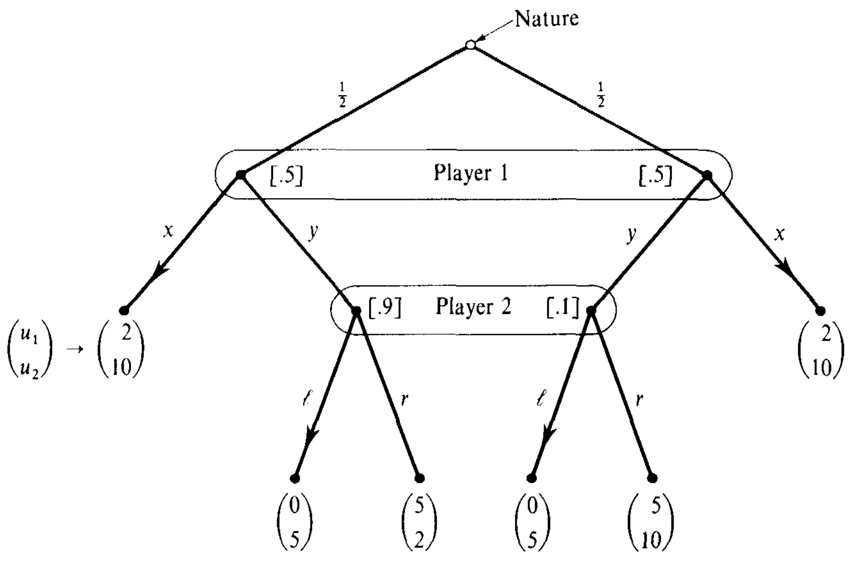


Figure 9.C.4
Extensive form for Example 9.C.4. Beliefs in a weak PBE may not be structurally consistent.

beliefs are used. Examples 9.C.4 and 9.C.5 illustrate why a strengthening of the weak PBE concept is often needed.

Example 9.C.4: Consider the game shown in Figure 9.C.4. The pure strategies and beliefs depicted in the figure constitute a weak PBE (the strategies are indicated by arrows on the chosen branches at each information set, and beliefs are indicated by numbers in brackets at the nodes in the information sets). The beliefs satisfy criterion (ii) of Definition 9.C.3; only player 1's information set is reached with positive probability, and player 1's beliefs there do reflect the probabilities assigned by nature. But the beliefs specified for player 2 in this equilibrium are not very sensible; player 2's information set can be reached only if player 1 deviates by instead choosing action y with positive probability, a deviation that must be independent of nature's actual move, since player 1 is ignorant of it. Hence, player 2 could reasonably have only beliefs that assign an equal probability to the two nodes in her information set. Here we see that it is desirable to require that beliefs at least be "structurally consistent" off the equilibrium path in the sense that there is *some* subjective probability distribution over strategy profiles that could generate probabilities consistent with the beliefs. ■

Example 9.C.5: A second and more significant problem is that a weak perfect Bayesian equilibrium need not be subgame perfect. To see this, consider again the entry game in Example 9.B.3. One weak PBE of this game involves strategies of $(\sigma_E, \sigma_I) =$ ((out, accommodate if in), (fight if firm E plays "in")) combined with beliefs for firm I that assign probability 1 to firm E having played "fight." This weak PBE is shown in Figure 9.C.5. But note that these strategies are not subgame perfect; they do not specify a Nash equilibrium in the post-entry subgame.

The problem is that firm I's post-entry belief about firm E's post-entry play is unrestricted by the weak PBE concept because firm I's information set is off the equilibrium path. ■

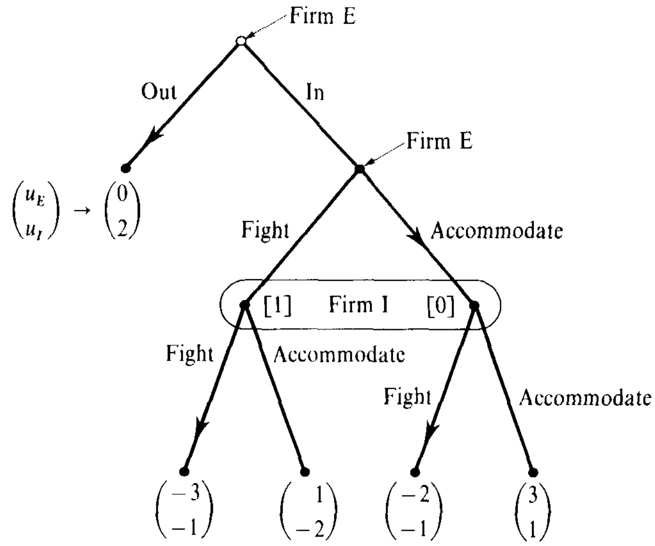


Figure 9.C.5
Extensive form for Example 9.C.5. A weak PBE may not be subgame perfect.

These two examples indicate that the weak PBE concept can be too weak. Thus, in applications in the literature, extra consistency restrictions on beliefs are often added to the weak PBE concept to avoid these problems, with the resulting solution concept referred to as a *perfect Bayesian equilibrium.* (As a simple example, restricting attention to equilibria that induce a weak PBE in every subgame insures subgame perfection.) We shall also do this when necessary later in the book; see, in particular, the discussion of signaling in Section 13.C. For formal definitions and discussion of some notions of perfect Bayesian equilibrium, see Fudenberg and Tirole (1991a) and (1991b).

An important closely related equilibrium notion that also strengthens the weak PBE concept by embodying additional consistency restrictions on beliefs is the *sequential equilibrium* concept developed by Kreps and Wilson (1982). In contrast to notions of perfect Bayesian equilibrium (such as the one we develop in Section 13.C), the sequential equilibrium concept introduces these consistency restrictions indirectly through the formalism of a limiting sequence of strategies. Definition 9.C.4 describes its requirements.

Definition 9.C.4: A strategy profile and system of beliefs (σ, μ) is a *sequential equilibrium* of extensive form game Γ_E if it has the following properties:

(i) Strategy profile σ is sequentially rational given belief system μ.

(ii) There exists a sequence of completely mixed strategies $\{\sigma^k\}_{k=1}^{\infty}$, with $\lim_{k\to\infty} \sigma^k = \sigma$, such that $\mu = \lim_{k\to\infty} \mu^k$, where μ^k denotes the beliefs derived from strategy profile σ^k using Bayes' rule.

In essence, the sequential equilibrium notion requires that beliefs be justifiable as coming from some set of totally mixed strategies that are "close to" the equilibrium strategies σ (i.e., a small perturbation of the equilibrium strategies). This can be viewed as requiring that players can (approximately) justify their beliefs by some story in which, with some small probability, players make mistakes in choosing their strategies. Note that every sequential equilibrium is a weak perfect Bayesian equilibrium because the limiting beliefs in Definition 9.C.4 exactly coincide with the beliefs derived from the equilibrium strategies σ via Bayes' rule on the outcome path of strategy profile σ. But, in general, the reverse is not true.

As we now show, the sequential equilibrium concept strengthens the weak perfect Bayesian equilibrium concept in a manner that avoids the problems identified in Examples 9.C.4 and 9.C.5.

Example 9.C.4 Continued: Consider again the game in Figure 9.C.4. In this game, all beliefs that can be derived from any sequence of totally mixed strategies assign equal probability to the two nodes in player 2's information set. Given this fact, in any sequential equilibrium player 2 must play r and player 1 must therefore play y. In fact, strategies (y, r) and beliefs giving equal probability to the two nodes in both players' information sets constitute the unique sequential equilibrium of this game. ■

Example 9.C.5 Continued: The unique sequential equilibrium strategies in the game in Example 9.C.5 (see Figure 9.C.5) are those of the unique SPNE: ((in, accommodate if in), (accommodate if firm E plays "in")). To verify this point, consider any totally mixed strategy $\bar{\sigma}$ and any node x in firm I's information set, which we denote by H_I. Letting z denote firm E's decision node following entry (the initial node of the subgame following entry), the beliefs $\mu_{\bar{\sigma}}$ associated with $\bar{\sigma}$ at information set H_I are equal to

$$\mu_\sigma(x) = \frac{\text{Prob}\,(x \mid \bar{\sigma})}{\text{Prob}\,(H_I \mid \bar{\sigma})} = \frac{\text{Prob}\,(x \mid z, \bar{\sigma})\;\text{Prob}\,(z \mid \bar{\sigma})}{\text{Prob}\,(H_I \mid z, \bar{\sigma})\;\text{Prob}\,(z \mid \bar{\sigma})},$$

where $\text{Prob}\,(x \mid z, \bar{\sigma})$ is the probability of reaching node x under strategies $\bar{\sigma}$ conditional on having reached node z. Canceling terms and noting that $\text{Prob}\,(H_I \mid z, \bar{\sigma}) = 1$, we then have $\mu_\sigma(x) = \text{Prob}\,(x \mid z, \bar{\sigma})$. But this is exactly the probability that firm E plays the action that leads to node x in strategy $\bar{\sigma}$. Thus, any sequence of totally mixed strategies $\{\bar{\sigma}^k\}_{k=1}^{\infty}$ that converge to σ must generate limiting beliefs for firm I that coincide with the play at node z specified in firm E's actual strategy σ_E. It is then immediate that the strategies in any sequential equilibrium must specify Nash equilibrium behavior in this post-entry subgame and thus must constitute a subgame perfect Nash equilbrium. ■

Proposition 9.C.2 gives a general result on the relation between sequential equilibria and subgame perfect Nash equilibria.

Proposition 9.C.2: In every sequential equilibrium (σ, μ) of an extensive form game Γ_E, the equilibrium strategy profile σ constitutes a subgame perfect Nash equilibrium of Γ_E.

Thus, the sequential equilibrium concept strengthens both the SPNE and the weak PBE concepts; every sequential equilibrium is both a weak PBE and an SPNE.

Although the concept of sequential equilibrium restricts beliefs that are off the equilibrium path enough to take care of the problems with the weak PBE concept illustrated in Examples 9.C.4 and 9.C.5, there are some ways in which the requirements on off-equilibrium-path beliefs embodied in the notion of sequential equilibrium may be too strong. For example, they imply that any two players with the same information must have exactly the same beliefs regarding the deviations by other players that have caused play to reach a given part of the game tree.

In Appendix B, we briefly describe another related (and still stronger) solution concept, an *extensive form trembling-hand perfect Nash equilibrium*, first proposed by Selten (1975).[14]

9.D Reasonable Beliefs and Forward Induction

In Section 9.C, we saw the importance of beliefs at unreached information sets for testing the sequential rationality of a strategy. Although the weak perfect Bayesian equilibrium concept and the related stronger concepts discussed in Section 9.C can help rule out noncredible threats, in many games we can nonetheless justify a large range of off-equilibrium-path behavior by picking off-equilibrium-path beliefs appropriately (we shall see some examples shortly). This has led to a considerable amount of recent research aimed at specifying additional restrictions that "reasonable" beliefs should satisfy. In this section, we provide a brief introduction to these ideas. (We shall encounter them again when we study signaling models in Chapter 13, particularly in Appendix A of that chapter.)

To start, consider the two games depicted in Figure 9.D.1. The first is a variant of the entry game of Figure 9.C.1 in which firm I would now find it worthwhile to fight if it knew that the entrant chose strategy "in_1"; the second is a variant of the Niche Choice game of Example 9.B.4, in which firm E now targets a niche at the time of its entry. Also shown in each diagram is a weak perfect Bayesian equilibrium (arrows denote pure strategy choices, and the numbers in brackets in firm I's information set denote beliefs).

One can argue that in neither game is the equilibrium depicted very sensible.[15] Consider the game in Figure 9.D.1(a). In the weak PBE depicted, if entry occurs, firm I plays "fight" because it believes that firm E has chosen "in_1." But "in_1" is strictly dominated for firm E by "in_2." Hence, it seems reasonable to think that if firm E decided to enter, it must have used strategy "in_2." Indeed, as is commonly done in this literature, one can imagine firm E making the following speech upon entering: "I have entered, but notice that I would never have used 'in_1' to do so because 'in_2' is always a better entry strategy for me. Think about this carefully before you choose your strategy."

A similar argument holds for the weak PBE depicted in Figure 9.D.1(b). Here "small niche" is strictly dominated for firm E, not by "large niche", but by "out." Once again, firm I could not reasonably hold the beliefs that are depicted. In this case, firm I should recognize that if firm E entered rather than playing "out," it must have chosen the large niche. Now you can imagine firm E saying: "Notice that the only way I could ever do better by entering than by choosing 'out' is by targeting the large niche."

14. Selten actually gave it the name *trembling-hand perfect Nash equilibrium*; we add the modifier *extensive form* to help distinguish it from the normal form concept introduced in Section 8.F.

15. For simplicity, we focus on weak perfect Bayesian equilibria here. The points to be made apply as well to the stronger related notions discussed in Section 9.C. In fact, all the weak perfect Bayesian equilibria discussed here are also sequential equilibria; indeed, they are even extensive form trembling-hand perfect.

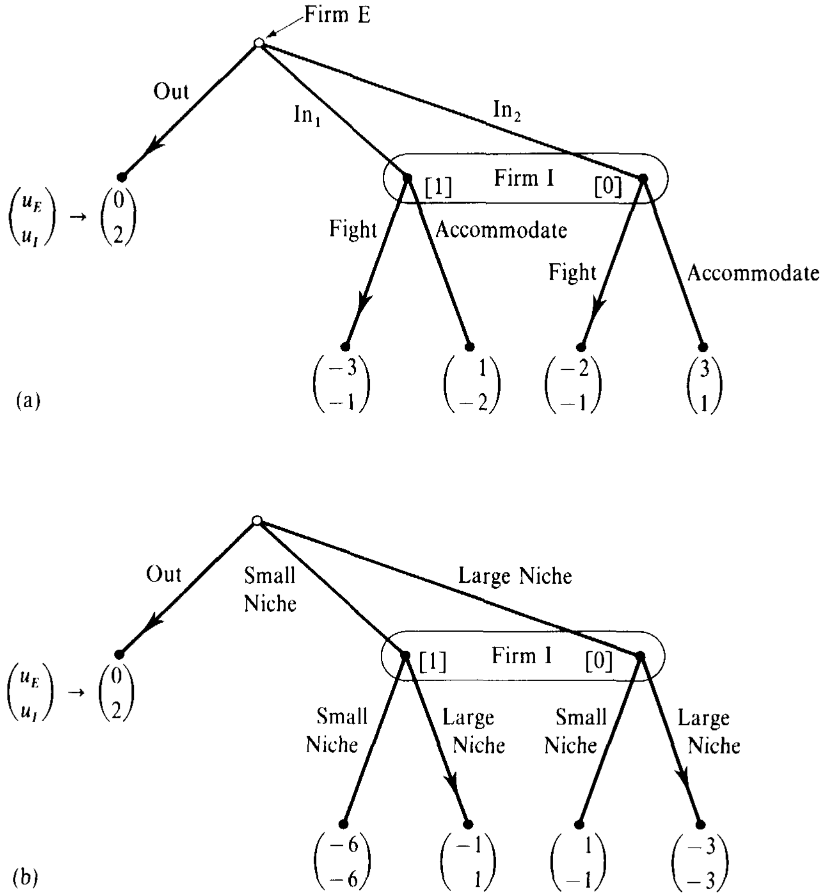


Figure 9.D.1
Two weak PBEs with unreasonable beliefs.

These arguments make use of what is known as *forward induction* reasoning [see Kohlberg (1989) and Kohlberg and Mertens (1986)]. In using backward induction, a player decides what is an optimal action for her at some point in the game tree based on her calculations of the actions that her opponents will rationally play at *later* points of the game. In contrast, in using forward induction, a player reasons about what could have rationally happened *previously*. For example, here firm I decides on its optimal post-entry action by assuming that firm E must have behaved rationally in its entry decision.

This type of idea is sometimes extended to include arguments based on *equilibrium domination*. For example, suppose that we augment the game in Figure 9.D.1(b) by also giving firm I a move after firm E plays "out," as depicted in Figure 9.D.2 (perhaps "out" really involves entry into some alternative market of firm I's in which firm E has only one potential entry strategy).

The figure depicts a weak PBE of this game in which firm E plays "out" and firm I believes that firm E has chosen "small niche" whenever its post-entry information set is reached. In this game, "small niche" is no longer strictly dominated for firm E by "out," so our previous argument does not apply. Nevertheless, if firm E deviates from this equilibrium by entering, we can imagine firm I thinking that since firm E could have received a payoff of 0 by following its equilibrium strategy, it must be hoping to do better than that by entering, and so it must

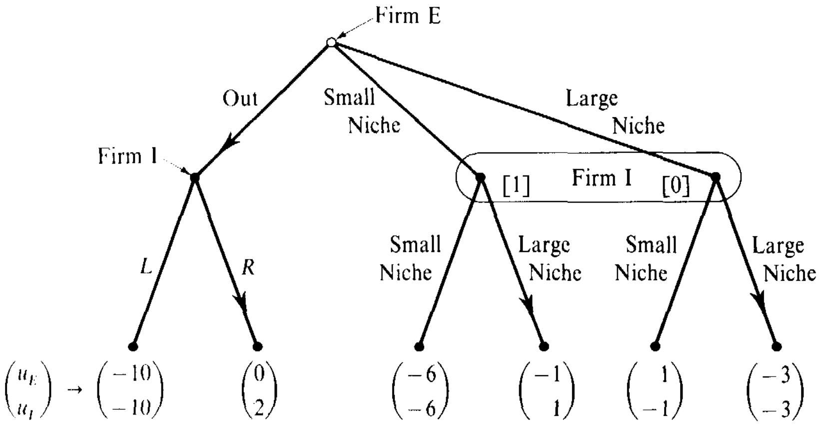


Figure 9.D.2
Strategy "small niche" is equilibrium dominated for firm E.

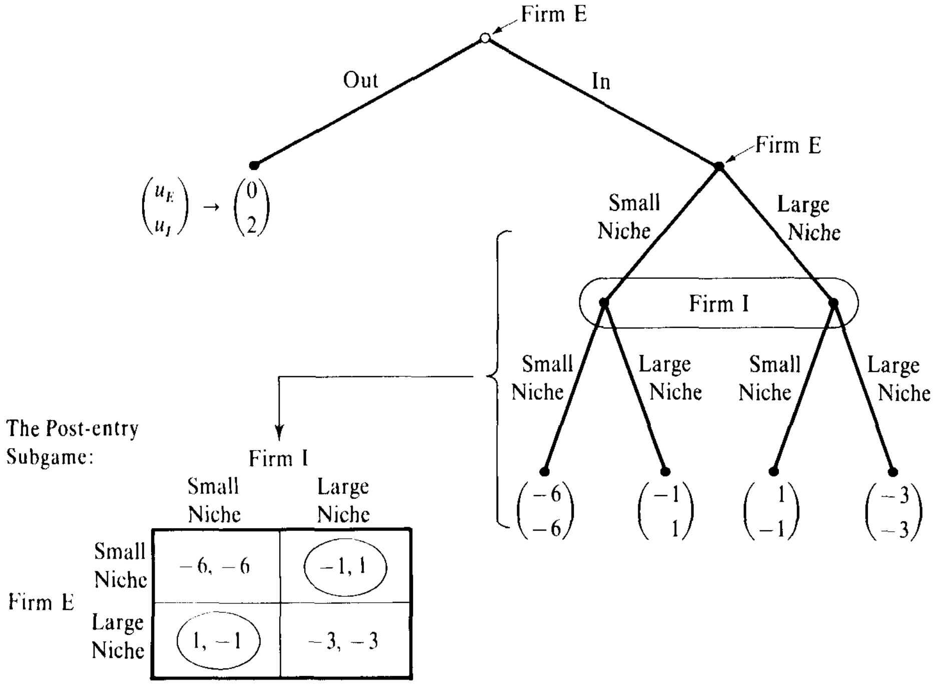


Figure 9.D.3
Forward induction selects equilibrium (large niche, small niche) in the post-entry subgame.

have chosen to target the large niche. In this case, we say that "small niche" is *equilibrium dominated* for firm E; that is, it is dominated if firm E treats its equilibrium payoff as something that it can achieve with certainty by following its equilibrium strategy. (This type of argument is embodied in the *intuitive criterion* refinement that we discuss in Section 13.C and Appendix A of Chapter 13 in the context of signaling models.)

Forward induction can be quite powerful. For example, reconsider the original Niche Choice game depicted in Figure 9.D.3. Recall that there are two (pure strategy) Nash equilibria in the post-entry subgame: (large niche, small niche) and (small niche, large niche). However, the force of the forward induction argument for the game in Figure 9.D.1(b) seems to apply equally well here: Strategy (in, small niche if in) is strictly dominated for firm E by playing "out." As a result, the incumbent should reason that if firm E has played "in," it intends to target the large niche in the post-entry game. If so, firm I is better off targeting the small niche. Thus, forward induction rules out one of the two Nash equilibria in the post-entry subgame.

Although these arguments may seem very appealing, there are also some potential problems. For example, suppose that we are in a world where players make mistakes with some small probability. In such a world, are the forward induction arguments just given convincing? Perhaps not. To see why, suppose that firm E enters in the game shown in Figure 9.D.1(a) when it was supposed to play "out." Now firm I can explain the deviation to itself as being the result of a mistake on firm E's part, a mistake that might equally well have led firm E to pick "in_1" as "in_2." And firm E's speech may not fall on very sympathetic ears: "Of course, firm E is telling me this," reasons the incumbent, "it has made a mistake and now is trying to make the best of it by convincing me to accommodate."

To see this in an even more striking manner, consider the game in Figure 9.D.3. Now, after firm E has entered and the two firms are about to play the simultaneous-move post-entry game, firm E makes its speech. But the incumbent retorts: "Forget it! I think you just made a mistake and even if you did not, I'm going to target the large niche!"

Clearly, the issues here, although interesting and important, are also tricky.

A noticeable feature of these forward induction arguments is how they use the normal form notion of dominance to restrict predicted play in dynamic games. This stands in sharp contrast with our discussion earlier in this chapter, which relied exclusively on the extensive form to determine how players should play in dynamic games. This raises a natural question: Can we somehow use the normal form representation to predict play in dynamic games?

There are at least two reasons why we might think we can. First, as we discussed in Chapter 7, it seems appealing as a matter of logic to think that players simultaneously choosing their strategies in the normal form (e.g., submitting contingent plans to a referee) is equivalent to their actually playing out the game dynamically as represented in the extensive form. Second, in many circumstances, it seems that the notion of weak dominance can get at the idea of sequential rationality. For example, for finite games of perfect information in which no player has equal payoffs at any two terminal nodes, any strategy profile surviving a process of iterated deletion of weakly dominated strategies leads to the same predicted outcome as the SPNE concept (take a look at Example 9.B.1, and see Exercise 9.D.1).

The argument for using the normal form is also bolstered by the fact that extensive form concepts such as weak PBE can be sensitive to what may seem like irrelevant changes in the extensive form. For example, by breaking up firm E's decision in the game in Figure 9.D.1(a) into an "out" or "in" decision followed by an "in_1" or "in_2" decision [just as we did in Figure 9.D.3 for the game in Figure 9.D.1(b)], the unique SPNE (and, hence, the unique sequential equilibrium) becomes firm E entering and playing "in_2" and firm I accommodating. However, the reduced normal form associated with these two games (i.e., the normal form where we eliminate all but one of a player's strategies that have identical payoffs) is invariant to this change in the extensive form; therefore, any solution based on the (reduced) normal form would be unaffected by this change.

These points have led to a renewed interest in the use of the normal form as a device for predicting play in dynamic games [see, in particular, Kohlberg and Mertens (1986)]. At the same time, this issue remains controversial. Many game theorists believe that there is a loss of some information of strategic importance in going from the extensive form to the more condensed normal form. For example, are the games in Figures 9.D.3 and 9.D.1(b) really the same? If you were firm I, would you be as likely to rely on the forward induction argument in the game in Figure 9.D.3 as in that in Figure 9.D.1(b)? Does it matter for your answer whether in the game in Figure 9.D.3 a minute or a month passes between firm E's two decisions? These issues remain to be sorted out.

APPENDIX A: FINITE AND INFINITE HORIZON BILATERAL BARGAINING

In this appendix we study two models of bilateral bargaining as an economically important example of the use of the subgame perfect Nash equilibrium concept. We begin by studying a finite horizon model of bargaining and then consider its infinite horizon counterpart.

Example 9.AA.1: *Finite Horizon Bilateral Bargaining.* Two players, 1 and 2, bargain to determine the split of v dollars. The rules are as follows: The game begins in period 1; in period 1, player 1 makes an offer of a split (a real number between 0 and v) to player 2, which player 2 may then accept or reject. If she accepts, the proposed split is immediately implemented and the game ends. If she rejects, nothing happens until period 2. In period 2, the players' roles are reversed, with player 2 making an offer to player 1 and player 1 then being able to accept or reject it. Each player has a discount factor of $\delta \in (0, 1)$, so that a dollar received in period t is worth δ^{t-1} in period 1 dollars. However, after some finite number of periods T, if an agreement has not yet been reached, the bargaining is terminated and the players each receive nothing. A portion of the extensive form of this game is depicted in Figure 9.AA.1 [this model is due to Stahl (1972)].

There is a unique subgame perfect Nash equilibrium (SPNE) in this game. To see this, suppose first that T is odd, so that player 1 makes the offer in period T if no previous agreement has been reached. Now, player 2 is willing to accept *any* offer in this period because she will get zero if she refuses and the game is terminated (she is indifferent about accepting an offer of zero). Given this fact, the unique SPNE in the subgame that begins in the final period when no agreement has been previously reached has player 1 offer player 2 zero and player 2 accept.[16] Therefore, the payoffs from equilibrium play in this subgame are $(\delta^{T-1}v, 0)$.

Now consider play in the subgame starting in period $T-1$ when no previous agreement has been reached. Player 2 makes the offer in this period. In any SPNE, player 1 will accept an offer in period $T-1$ if and only if it provides her with a payoff of at least $\delta^{T-1}v$, since otherwise she will do better rejecting it and waiting to make an offer in period T (she earns $\delta^{T-1}v$ by doing so). Given this fact, in any SPNE, player 2 must make an offer in period $T-1$ that gives player 1 a payoff of exactly $\delta^{T-1}v$, and player 1 accepts this offer (note that this is player 2's best offer

16. Note that if player 2 is unwilling to accept an offer of zero, then player 1 has no optimal strategy; she wants to make a strictly positive offer ever closer to zero (since player 1 will accept any strictly positive offer). If the reliance on player 1 accepting an offer over which she is indifferent bothers you, you can convince yourself that the analysis of the game in which offers must be in small increments (pennies) yields exactly the same outcome as that identified in the text as the size of these increments goes to zero.

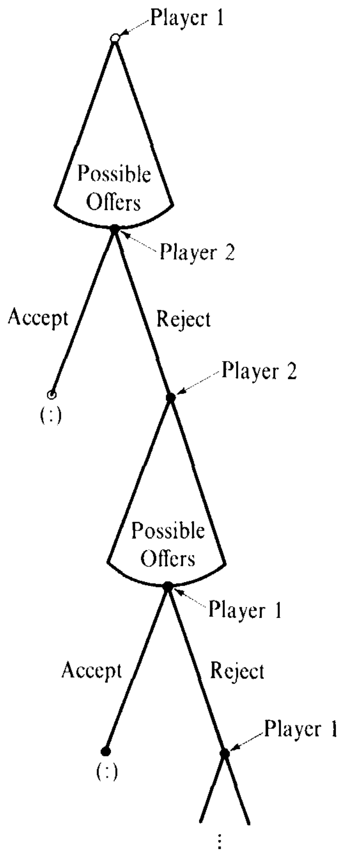


Figure 9.AA.1
The alternating-offer bilateral bargaining game.

among all those that would be accepted, and making an offer that will be rejected is worse for player 2 because it results in her receiving a payoff of zero). The payoffs arising if the game reaches period $T-1$ must therefore be $(\delta^{T-1}v, \delta^{T-2}v - \delta^{T-1}v)$.

Continuing in this fashion, we can determine that the unique SPNE when T is odd results in an agreement being reached in period 1, a payoff for player 1 of

$$\begin{aligned} v_1^*(T) &= v[1 - \delta + \delta^2 - \cdots + \delta^{T-1}] \\ &= v\left[(1-\delta)\left(\frac{1-\delta^{T-1}}{1-\delta^2}\right) + \delta^{T-1}\right], \end{aligned}$$

and a payoff to player 2 of $v_2^*(T) = v - v_1^*(T)$.

If T is instead even, then player 1 must earn $v - \delta v_1^*(T-1)$ because in any SPNE, player 2 (who will be the first offerer in the odd-number-of-periods subgame that begins in period 2 if she rejects player 1's period 1 offer) will accept an offer in period 1 if and only if it gives her at least $\delta v_1^*(T-1)$, and player 1 will offer her exactly this amount.

Finally, note that as the number of periods grows large ($T \to \infty$), player 1's payoff converges to $v/(1+\delta)$, and player 2's payoff converges to $\delta v/(1+\delta)$. ■

In Example 9.AA.1, the application of the SPNE concept was relatively straightforward; we simply needed to start at the end of the game and work backward. We now consider the infinite horizon counterpart of this game. As we noted in Section

9.B, we can no longer solve for the SPNE in this simple manner when the game has an infinite horizon. Moreover, in many games, introduction of an infinite horizon allows a broad range of behavior to emerge as subgame perfect. Nevertheless, in the infinite horizon bargaining model, the SPNE concept is quite powerful. There is a unique SPNE in this game, and it turns out to be exactly the limiting outcome of the finite horizon model as the length of the horizon T approaches ∞.

Example 9.AA.2: *Infinite Horizon Bilateral Bargaining.* Consider an extension of the finite horizon bargaining game considered in Example 9.AA.1 in which bargaining is no longer terminated after T rounds but, rather, can potentially go on forever. If this happens, the players both earn zero. This model is due to Rubinstein (1982).

We claim that this game has a unique SPNE. In this equilibrium, the players reach an immediate agreement in period 1, with player 1 earning $v/(1+\delta)$ and player 2 earning $\delta v/(1+\delta)$.

The method of analysis we use here, following Shaked and Sutton (1984), makes heavy use of the stationarity of the game (the subgame starting in period 2 looks exactly like that in period 1, but with the players' roles reversed).

To start, let $\bar{v}_1$ denote the largest payoff that player 1 gets in *any* SPNE (i.e., there may, in principle, be multiple SPNEs in this model).[17] Given the stationarity of the model, this is also the largest amount that player 2 can expect in the subgame that begins in period 2 after her rejection of player 1's period 1 offer, a subgame in which player 2 has the role of being the first player to make an offer. As a result, player 1's payoff in any SPNE cannot be lower than the amount $\underline{v}_1 = v - \delta\bar{v}_1$ because, if it was, then player 1 could do better by making a period 1 offer that gives player 2 just slightly more than $\delta\bar{v}_1$. Player 2 is certain to accept any such offer because she will earn only $\delta\bar{v}_1$ by rejecting it (note that we are using subgame perfection here, because we are requiring that the continuation of play after rejection is an SPNE in the continuation subgame and that player 2's response will be optimal given this fact).

Next, we claim that, in any SPNE, $\bar{v}_1$ cannot be larger than $v - \delta\underline{v}_1$. To see this, note that in any SPNE, player 2 is certain to reject any offer in period 1 that gives her less than $\delta\underline{v}_1$ because she can earn at least $\delta\underline{v}_1$ by rejecting it and waiting to make an offer in period 2. Thus, player 1 can do no better than $v - \delta\underline{v}_1$ by making an offer that is accepted in period 1. What about by making an offer that is rejected in period 1? Since player 2 must earn at least $\delta\underline{v}_1$ if this happens, and since agreement cannot occur before period 2, player 1 can earn no more than $\delta v - \delta\underline{v}_1$ by doing this. Hence, we have $\bar{v}_1 \le v - \delta\underline{v}_1$.

Next, note that these derivations imply that

$$\bar{v}_1 \le v - \delta\underline{v}_1 = (\underline{v}_1 + \delta\bar{v}_1) - \delta\underline{v}_1,$$

so that

$$\bar{v}_1(1-\delta) \le \underline{v}_1(1-\delta).$$

Given the definitions of $\underline{v}_1$ and $\bar{v}_1$, this implies that $\underline{v}_1 = \bar{v}_1$, and so player 1's SPNE payoff is uniquely determined. Denote this payoff by v_1°. Since $v_1^\circ = v - \delta v_1^\circ$, we find that player 1 must earn $v_1^\circ = v/(1+\delta)$ and player 2 must earn $v_2^\circ = v - v_1^\circ = \delta v/(1+\delta)$. In addition, recalling the argument in the previous paragraph, we see

17. This maximum can be shown to be well defined, but we will not do so here.

that an agreement will be reached in the first period (player 1 will find it worthwhile to make an offer that player 2 accepts). The SPNE strategies are as follows: A player who has just received an offer accepts it if and only if she is offered at least δv_1°, while a player whose turn it is to make an offer offers exactly δv_1° to the player receiving the offer.

Note that the equilibrium strategies, outcome, and payoffs are precisely the limit of those in the finite game in Example 9.AA.1 as $T \to \infty$. ■

The coincidence of the infinite horizon equilibrium with the limit of the finite horizon equilibria in this model is not a general property of infinite horizon games. The discussion of infinitely repeated games in Chapter 12 provides an illustration of this point.

We should also point out that the outcomes of game-theoretic models of bargaining can be quite sensitive to the precise specification of the bargaining process and players' preferences. Exercises 9.B.7 and 9.B.13 provide an illustration.

APPENDIX B: EXTENSIVE FORM TREMBLING-HAND PERFECT NASH EQUILIBRIUM

In this appendix we extend the analysis presented in Section 9.C by discussing another equilibrium notion that strengthens the consistency conditions on beliefs in the weak PBE concept: *extensive form trembling-hand perfect Nash equilibrium* [due to Selten (1975)]. In fact, this equilibrium concept is the strongest among those discussed in Section 9.C.

The definition of an extensive form trembling-hand perfect Nash equilibrium parallels that for the normal form (see Section 8.F) but has the trembles applied not to a player's mixed strategies, but rather to the player's choice at each of her information sets. A useful way to view this idea is with what Selten (1975) calls the *agent normal form.* This is the normal form that we would derive if we pretended that the player had a set of agents in charge of moving for her at each of her information sets (a different one for each), each acting independently to try to maximize the player's payoff.

Definition 9.BB.1: Strategy profile σ in extensive form game Γ_E is an *extensive form trembling-hand perfect Nash equilibrium* if and only if it is a normal form trembling-hand perfect Nash equilibrium of the agent normal form derived from Γ_E.

To see why it is desirable to have the trembles occurring at each information set rather than over strategies as in the normal-form concept considered in Section 8.F, consider Figure 9.BB.1, which is taken from van Damme (1983). This game has a unique subgame perfect Nash equilibrium: $(\sigma_1, \sigma_2) = ((NR, L), \ell)$. But you can check that $((NR, L), \ell)$ is not the only normal form trembling-hand perfect Nash equilibrium: so are $((R, L), r)$ and $((R, M), r)$. The reason that these two strategy profiles are normal form trembling-hand perfect is that, in the normal form, the tremble to strategy (NR, M) by player 1 can be larger than that to (NR, L) despite the fact that the latter is a better choice for player 1 at her second decision node.

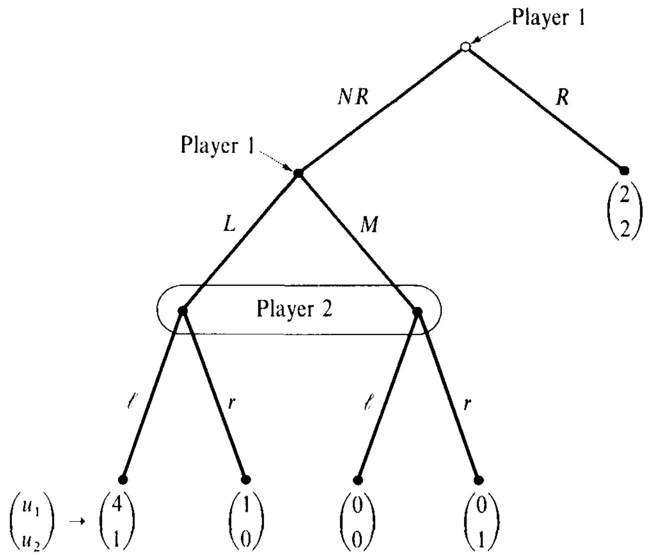


Figure 9.BB.1
Strategy profiles $((R, L), r)$ and $((R, M), r)$ are normal form trembling-hand perfect but are not subgame perfect.

With such a tremble, player 2's best response to player 1's perturbed strategy is r. It is not difficult to see, however, that the unique extensive form trembling-hand perfect Nash equilibrium of this game is $((NR, L), \ell)$ because the agent who moves at player 1's second decision node will put as high a probability as possible on L.

When we compare Definitions 9.BB.1 and 9.C.4, it is apparent that every extensive form trembling-hand perfect Nash equilibrium is a sequential equilibrium. In particular, even though the trembling-hand perfection criterion is not formulated in terms of beliefs, we can use the sequence of (strictly mixed) equilibrium strategies $\{\sigma^k\}_{k=1}^{\infty}$ in the perturbed games of the agent normal form as our strategy sequence for deriving sequential equilibrium beliefs. Because the limiting strategies σ in the extensive form trembling-hand perfect equilibrium are best responses to every element of this sequence, they are also best responses to each other with these derived beliefs. (Every extensive form trembling-hand perfect Nash equilibrium is therefore also subgame perfect.)

In essence, by introducing trembles, the extensive form trembling-hand perfect equilibrium notion makes every part of the tree be reached when strategies are perturbed, and because equilibrium strategies are required to be best responses to perturbed strategies, it insures that equilibrium strategies are sequentially rational. The primary difference between this notion and that of sequential equilibrium is that, like its normal form cousin, the extensive form trembling-hand perfect equilibrium concept can also eliminate some sequential equilibria in which weakly dominated strategies are played. Figure 9.BB.2 (a slight modification of the game in Figure 9.C.1) depicts a sequential equilibrium whose strategies are not extensive form trembling-hand perfect.

In general, however, the concepts are quite close [see Kreps and Wilson (1982) for a formal comparison]; and because it is much easier to check that strategies are best responses at the limiting beliefs than it is to check that they are best responses for a sequence of strategies, sequential equilibrium is much more commonly used. For an interesting further discussion of this concept, consult van Damme (1983).

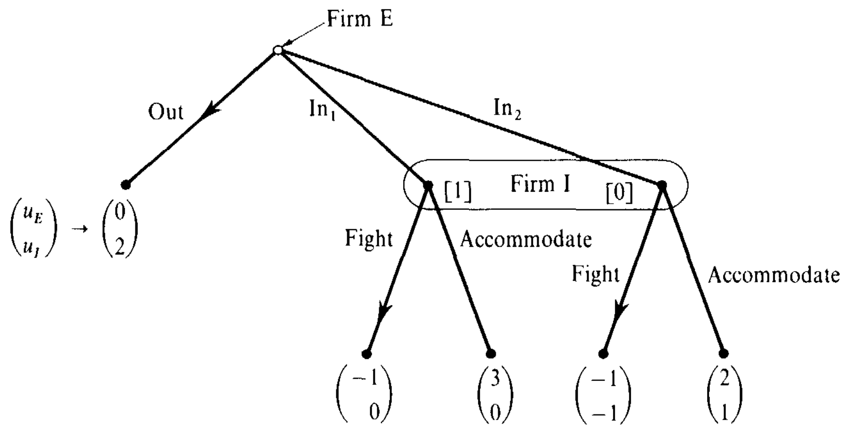


Figure 9.BB.2
A sequential equilibrium need not be extensive form trembling-hand perfect.

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EXERCISES

9.B.1^A How many subgames are there in the game of Example 9.B.2 (depicted in Figure 9.B.3)?

9.B.2^A In text.

9.B.3^B Verify that the strategies identified through backward induction in Example 9.B.2 constitute a Nash equilibrium of the game studied there. Also, identify *all other* pure strategy Nash equilibria of this game. Argue that each of these other equilibria does not satisfy the principle of sequential rationality.

9.B.4[B] Prove that in a finite *zero-sum* game of perfect information, there are unique subgame perfect Nash equilibrium payoffs.

9.B.5[B] (E. Maskin) Consider a game with two players, player 1 and player 2, in which each player i can choose an action from a finite set M_i that contains m_i actions. Player i's payoff if the action choices are (m_1, m_2) is $\phi_i(m_1, m_2)$.

(a) Suppose, first, that the two players move simultaneously. How many strategies does each player have?

(b) Now suppose that player 1 moves first and that player 2 observes player 1's move before choosing her move. How many strategies does each player have?

(c) Suppose that the game in **(b)** has multiple SPNEs. Show that if this is the case, then there exist two pairs of moves (m_1, m_2) and (m_1', m_2') (where either $m_1 \neq m_1'$ or $m_2 \neq m_2'$) such that either

$$\text{(i)} \quad \phi_1(m_1, m_2) = \phi_1(m_1', m_2')$$

or

$$\text{(ii)} \quad \phi_2(m_1, m_2) = \phi_2(m_1', m_2').$$

(d) Suppose that for any two pairs of moves (m_1, m_2) and (m_1', m_2') such that $m_1 \neq m_1'$ or $m_2 \neq m_2'$, condition (ii) is violated (i.e., player 2 is never indifferent between pairs of moves). Suppose also that there exists a pure strategy Nash equilibrium in the game in **(a)** in which π_1 is player 1's payoff. Show that in any SPNE of the game in **(b)**, player 1's payoff is at least π_1. Would this conclusion necessarily hold for any *Nash* equilibrium of the game in **(b)**?

(e) Show by example that the conclusion in **(d)** may fail either if condition (ii) holds for some strategy pairs (m_1, m_2), (m_1', m_2') with $m_1 \neq m_1'$ or $m_2 \neq m_2'$ or if we replace the phrase *pure strategy Nash equilibrium* with the phrase *mixed strategy Nash equilibrium*.

9.B.6[B] Solve for the mixed strategy equilibrium involving actual randomization in the post-entry subgame of the Niche Choice game in Example 9.B.4. Is there an SPNE that induces this behavior in the post-entry subgame? What are the SPNE strategies?

9.B.7[B] Consider the finite horizon bilateral bargaining game in Appendix A (Example 9.AA.1); but instead of assuming that players discount future payoffs, assume that it costs $c < v$ to make an offer. (Only the player making an offer incurs this cost, and players who have made offers incur this cost even if no agreement is ultimately reached.) What is the (unique) SPNE of this alternative model? What happens as T approaches ∞?

9.B.8[C] Prove that every (finite) game Γ_E has a mixed strategy subgame perfect Nash equilibrium.

9.B.9[B] Consider a game in which the following simultaneous-move game is played twice:

		Player 2		
		b_1	b_2	b_3
	a_1	10, 10	2, 12	0, 13
Player 1	a_2	12, 2	5, 5	0, 0
	a_3	13, 0	0, 0	1, 1

The players observe the actions chosen in the first play of the game prior to the second play. What are the pure strategy subgame perfect Nash equilibria of this game?

9.B.10[B] Reconsider the game in Example 9.B.3, but now change the post-entry game so that when both players choose "accommodate", instead of receiving the payoffs $(u_E, u_I) = (3, 1)$, the players now must play the following simultaneous-move game:

		Firm I: ℓ	Firm I: r
Firm E	U	3, 1	0, 0
	D	0, 0	x, 3

What are the SPNEs of this game when $x \geq 0$? When $x < 0$?

9.B.11[B] Two firms, A and B, are in a market that is declining in size. The game starts in period 0, and the firms can compete in periods 0, 1, 2, 3, ... (i.e., indefinitely) if they so choose. Duopoly profits in period t for firm A are equal to $105 - 10t$, and they are $10.5 - t$ for firm B. Monopoly profits (those if a firm is the only one left in the market) are $510 - 25t$ for firm A and $51 - 2t$ for firm B.

Suppose that at the start of each period, each firm must decide either to "stay in" or "exit" if it is still active (they do so simultaneously if both are still active). Once a firm exits, it is out of the market forever and earns zero in each period thereafter. Firms maximize their (undiscounted) sum of profits.

What is this game's subgame perfect Nash equilibrium outcome (and what are the firms' strategies in the equilibrium)?

9.B.12[C] Consider the infinite horizon bilateral bargaining model of Appendix A (Example 9.AA.2). Suppose the discount factors δ_1 and δ_2 of the two players differ. Now what is the (unique) subgame perfect Nash equilibrium?

9.B.13[B] What are the subgame perfect Nash equilibria of the infinite horizon version of Exercise 9.B.7?

9.B.14[B] At time 0, an incumbent firm (firm I) is already in the widget market, and a potential entrant (firm E) is considering entry. In order to enter, firm E must incur a cost of $K > 0$. Firm E's only opportunity to enter is at time 0. There are three production periods. In any period in which both firms are active in the market, the game in Figure 9.Ex.1 is played. Firm E moves first, deciding whether to stay in or exit the market. If it stays in, firm I decides whether to fight (the upper payoff is for firm E). Once firm E plays "out," it is out of

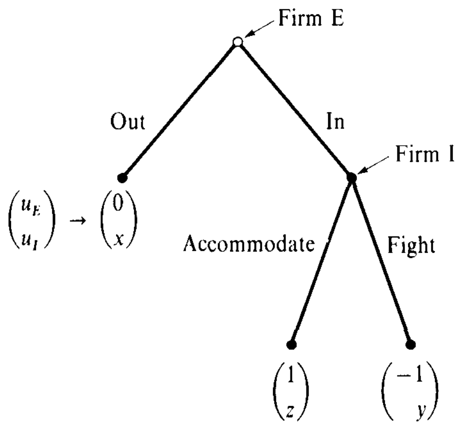

Figure 9.Ex.1

the market forever; firm E earns zero in any period during which it is out of the market, and firm I earns x. The discount factor for both firms is δ.

Assume that:

(A.1) $x > z > y$.
(A.2) $y + \delta x > (1 + \delta)z$.
(A.3) $1 + \delta > K$.

(a) What is the (unique) subgame perfect Nash equilibrium of this game?

(b) Suppose now that firm E faces a financial constraint. In particular, if firm I fights *once* against firm E (in any period), firm E will be forced out of the market from that point on. Now what is the (unique) subgame perfect Nash equilibrium of this game? (If the answer depends on the values of parameters beyond the three assumptions, indicate how.)

9.C.1^{B} Prove Proposition 9.C.1.

9.C.2^{B} What is the set of weak PBEs in the game in Example 9.C.3 when $\gamma \in (-1, 0)$?

9.C.3^{C} A buyer and a seller are bargaining. The seller owns an object for which the buyer has value $v > 0$ (the seller's value is zero). This value is known to the buyer but not to the seller. The value's prior distribution is common knowledge. There are two periods of bargaining. The seller makes a take-it-or-leave-it offer (i.e., names a price) at the start of each period that the buyer may accept or reject. The game ends when an offer is accepted or after two periods, whichever comes first. Both players discount period 2 payoffs with a discount factor of $\delta \in (0, 1)$.

Assume throughout that the buyer always accepts the seller's offer whenever she is indifferent.

(a) Characterize the (pure strategy) weak perfect Bayesian equilibria for a case in which v can take two values v_L and v_H, with $v_H > v_L > 0$, and where $\lambda = \text{Prob}\,(v_H)$.

(b) Do the same for the case in which v is uniformly distributed on $[\underline{v}, \bar{v}]$.

9.C.4^{C} A plaintiff, Ms. P, files a suit against Ms. D (the defendant). If Ms. P wins, she will collect π dollars in damages from Ms. D. Ms. D knows the likelihood that Ms. P will win, $\lambda \in [0, 1]$, but Ms. P does not (Ms. D might know if she was actually at fault). They both have strictly positive costs of going to trial of c_p and c_d. The prior distribution of λ has density $f(\lambda)$ (which is common knowledge).

Suppose pretrial settlement negotiations work as follows: Ms. P makes a take-it-or-leave-it settlement offer (a dollar amount) to Ms. D. If Ms. D accepts, she pays Ms. P and the game is over. If she does not accept, they go to trial.

(a) What are the (pure strategy) weak perfect Bayesian equilibria of this game?

(b) What effects do changes in c_p, c_d, and π have?

(c) Now allow Ms. D, after having her offer rejected, to decide not to go to court after all. What are the weak perfect Bayesian equilibria? What about the effects of the changes in **(b)**?

9.C.5^{C} Reconsider Exercise 9.C.4. Now suppose it is Ms. P who knows λ.

9.C.6^{B} What are the sequential equilibria in the games in Exercises 9.C.3 to 9.C.5?

9.C.7^{B} (Based on work by K. Bagwell and developed as an exercise by E. Maskin) Consider the extensive form game depicted in Figure 9.Ex.2.

(a) Find a subgame perfect Nash equilibrium of this game. Is it unique? Are there any other Nash equilibria?

(b) Now suppose that player 2 cannot observe player 1's move. Write down the new extensive form. What is the set of Nash equilibria?

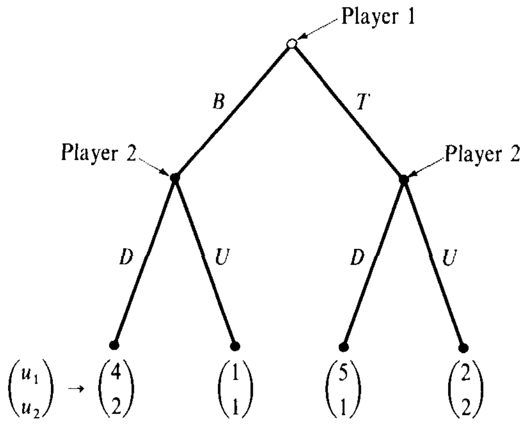

Figure 9.Ex.2

(c) Now suppose that player 2 observes player 1's move correctly with probability $p \in (0, 1)$ and incorrectly with probability $1 - p$ (e.g., if player 1 plays T, player 2 observes T with probability p and observes B with probability $1 - p$). Suppose that player 2's propensity to observe incorrectly (i.e., given by the value of p) is common knowledge to the two players. What is the extensive form now? Show that there is a unique weak perfect Bayesian equilibrium. What is it?

9.D.1[B] Show that under the condition given in Proposition 9.B.2 for existence of a unique subgame perfect Nash equilibrium in a finite game of perfect information, there is an order of iterated removal of weakly dominated strategies for which all surviving strategy profiles lead to the same outcome (i.e., have the same equilibrium path and payoffs) as the subgame perfect Nash equilibrium. [In fact, *any* order of deletion leads to this result; see Moulin (1981).]