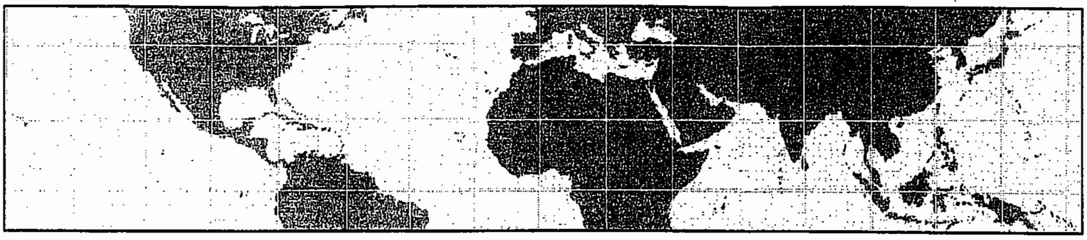


CHAPTER 1 Introduction

In the last 30 years derivatives have ~~become~~ increasingly ~~important in~~ finance. Futures and options are now traded actively on many exchanges throughout the world. Many different types of forward contracts, swaps, options, and other derivatives are regularly traded by financial institutions, fund managers, and corporate treasurers in the over-the-counter market. Derivatives are added to bond issues, used in executive compensation plans, embedded in capital investment opportunities, and so on. We have now reached the stage where anyone who works in finance needs to understand how derivatives work, how they are used, and how they are priced.

A *derivative* can be defined as a financial instrument whose value depends on (or derives from) the values of other, more basic, underlying variables. Very often the variables underlying derivatives are the prices of traded assets. A stock option, for example, is a derivative whose value is dependent on the price of a stock. However, derivatives can be dependent on almost any variable, from the price of hogs to the amount of snow falling at a certain ski resort.

Since the first edition of this book was published in 1988 there have been many developments in derivatives markets. There is now active trading in credit derivatives, electricity derivatives, weather derivatives, and insurance derivatives. Many new types of interest rate, foreign exchange, and equity derivative products have been created. There have been many new ideas in risk management and risk measurement. Analysts have also become more aware of the need to analyze what are known as *real options*. This edition of the book reflects all these developments.

In this opening chapter we take a first look at forward, futures, and options markets and provide an overview of how they are used by hedgers, speculators, and arbitrageurs. Later chapters will give more details and elaborate on many of the points made here.

1.1 EXCHANGE-TRADED MARKETS

A derivatives exchange is a market where individuals trade standardized contracts that have been defined by the exchange. Derivatives exchanges have existed for a long time. The Chicago Board of Trade (CBOT, www.cbot.com) was established in 1848 to bring farmers and merchants together. Initially its main task was to standardize the quantities and qualities of the grains that were traded. Within a few years the first futures-type contract was developed. It was known as a *to-arrive contract*. Speculators soon became interested in the contract and found trading the contract to be an attractive alternative to trading the grain itself. A rival futures exchange, the Chicago Mercantile Exchange (CME, www.cme.com), was established in 1919. Now futures exchanges exist all over the world. (See table at the end of the book.)

The Chicago Board Options Exchange (CBOE, www.cboe.com) started trading call option contracts on 16 stocks in 1973. Options had traded prior to 1973, but the CBOE succeeded in creating an orderly market with well-defined contracts. Put option contracts started trading on the exchange in 1977. The CBOE now trades options on well over 1,000 stocks and many different stock indices. Like futures, options have proved to be very popular contracts. Many other exchanges throughout the world now trade options. (See table at the end of the book.) The underlying assets include foreign currencies and futures contracts as well as stocks and stock indices.

Electronic Markets

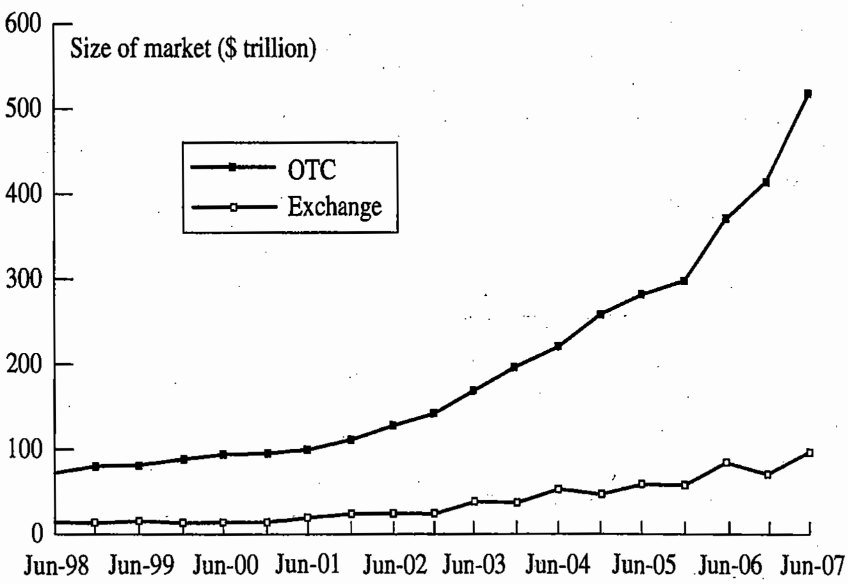
Traditionally derivatives exchanges have used what is known as the *open outcry system*. This involves traders physically meeting on the floor of the exchange, shouting, and using a complicated set of hand signals to indicate the trades they would like to carry out. Exchanges are increasingly replacing the open outcry system by *electronic trading*. This involves traders entering their desired trades at a keyboard and a computer being used to match buyers and sellers. The open outcry system has its advocates, but, as time passes, it is becoming less and less common.

1.2 OVER-THE-COUNTER MARKETS

Not all trading is done on exchanges. The *over-the-counter market* is an important alternative to exchanges and, measured in terms of the total volume of trading, has become much larger than the exchange-traded market. It is a telephone- and computer-linked network of dealers. Trades are done over the phone and are usually between two financial institutions or between a financial institution and one of its clients (typically a corporate treasurer or fund manager). Financial institutions often act as market makers for the more commonly traded instruments. This means that they are always prepared to quote both a bid price (a price at which they are prepared to buy) and an offer price (a price at which they are prepared to sell).

Telephone conversations in the over-the-counter market are usually taped. If there is a dispute about what was agreed, the tapes are replayed to resolve the issue. Trades in the over-the-counter market are typically much larger than trades in the exchange-traded market. A key advantage of the over-the-counter market is that the terms of a contract do not have to be those specified by an exchange. Market participants are free to negotiate any mutually attractive deal. A disadvantage is that there is usually some credit risk in an over-the-counter trade (i.e., there is a small risk that the contract will not be honored). As we shall see in the next chapter, exchanges have organized themselves to eliminate virtually all credit risk.

Figure 1.1 Size of over-the-counter and exchange-traded derivatives markets.



Market Size

Both the over-the-counter and the exchange-traded market for derivatives are huge. Although the statistics that are collected for the two markets are not exactly comparable, it is clear that the over-the-counter market is much larger than the exchange-traded market. The Bank for International Settlements (www.bis.org) started collecting statistics on the markets in 1998. Figure 1.1 compares (a) the estimated total principal amounts underlying transactions that were outstanding in the over-the counter markets between June 1998 and June 2007 and (b) the estimated total value of the assets underlying exchange-traded contracts during the same period. Using these measures, we see that, by June 2007, the over-the-counter market had grown to $516.4 trillion and the exchange-traded market had grown to $96.7 trillion.

In interpreting these numbers, we should bear in mind that the principal underlying an over-the-counter transaction is not the same as its value. An example of an over-the-counter contract is an agreement to buy 100 million US dollars with British pounds at a predetermined exchange rate in 1 year. The total principal amount underlying this transaction is $100 million. However, the value of the contract might be only $1 million. The Bank for International Settlements estimates the gross market value of all over-the-counter contracts outstanding in June 2007 to be about $11.1 trillion.[1]

1.3 FORWARD CONTRACTS

A relatively simple derivative is a *forward contract*. It is an agreement to buy or sell an asset at a certain future time for a certain price. It can be contrasted with a *spot*

[1] A contract that is worth $1 million to one side and −$1 million to the other side would be counted as having a gross market value of $1 million.

Table 1.1 Spot and forward quotes for the USD/GBP exchange rate, July 20, 2007 (GBP = British pound; USD = US dollar; quote is number of USD per GBP).

	Bid	*Offer*
Spot	2.0558	2.0562
1-month forward	2.0547	2.0552
3-month forward	2.0526	2.0531
6-month forward	2.0483	2.0489

contract, which is an agreement to buy or sell an asset today. A forward contract is traded in the over-the-counter market—usually between two financial institutions or between a financial institution and one of its clients.

One of the parties to a forward contract assumes a *long position* and agrees to buy the underlying asset on a certain specified future date for a certain specified price. The other party assumes a *short position* and agrees to sell the asset on the same date for the same price.

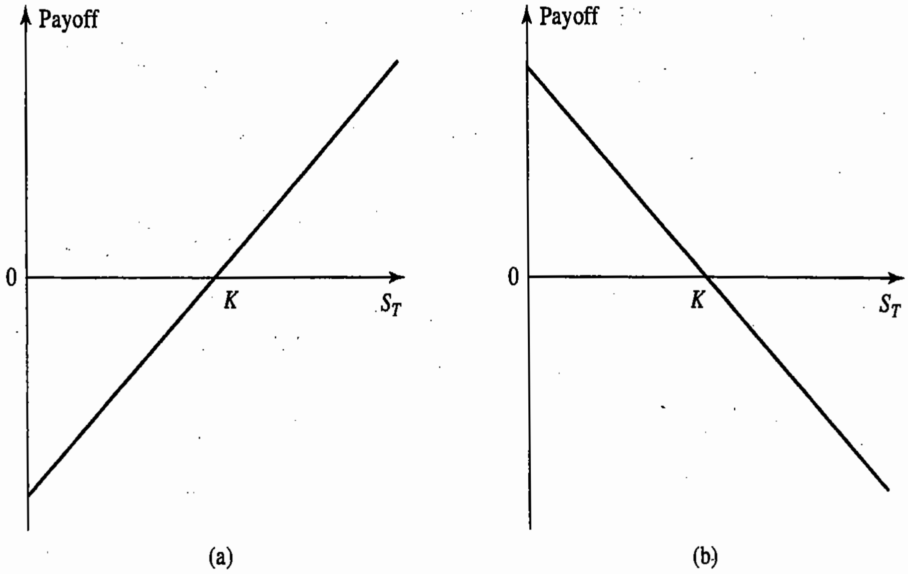
Forward contracts on foreign exchange are very popular. Most large banks employ both spot and forward foreign-exchange traders. Spot traders are trading a foreign currency for almost immediate delivery. Forward traders are trading for delivery at a future time. Table 1.1 provides the quotes on the exchange rate between the British pound (GBP) and the US dollar (USD) that might be made by a large international bank on July 20, 2007. The quote is for the number of USD per GBP. The first row indicates that the bank is prepared to buy GBP (also known as sterling) in the spot market (i.e., for virtually immediate delivery) at the rate of $2.0558 per GBP and sell sterling in the spot market at $2.0562 per GBP. The second, third, and fourth rows indicate that the bank is prepared to buy sterling in 1, 3, and 6 months at $2.0547, $2.0526, and $2.0483 per GBP, respectively, and to sell sterling in 1, 3, and 6 months at $2.0552, $2.0531, and $2.0489 per GBP, respectively.

Forward contracts can be used to hedge foreign currency risk. Suppose that, on July 20, 2007, the treasurer of a US corporation knows that the corporation will pay £1 million in 6 months (i.e., on January 20, 2008) and wants to hedge against exchange rate moves. Using the quotes in Table 1.1, the treasurer can agree to buy £1 million 6 months forward at an exchange rate of 2.0489. The corporation then has a long forward contract on GBP. It has agreed that on January 20, 2008, it will buy £1 million from the bank for $2.0489 million. The bank has a short forward contract on GBP. It has agreed that on January 20, 2008, it will sell £1 million for $2.0489 million. Both sides have made a binding commitment.

Payoffs from Forward Contracts

Consider the position of the corporation in the trade we have just described. What are the possible outcomes? The forward contract obligates the corporation to buy £1 million for $2,048,900. If the spot exchange rate rose to, say, 2.1000, at the end of the 6 months, the forward contract would be worth $51,100 (= $2,100,000 − $2,048,900) to the corporation. It would enable 1 million pounds to be purchased at an exchange rate

Figure 1.2 Payoffs from forward contracts: (a) long position, (b) short position. Delivery price = K; price of asset at contract maturity = S_T.



of 2.0489 rather than 2.1000. Similarly, if the spot exchange rate fell to 1.9000 at the end of the 6 months, the forward contract would have a negative value to the corporation of \$148,900 because it would lead to the corporation paying \$148,900 more than the market price for the sterling.

In general, the payoff from a long position in a forward contract on one unit of an asset is

$$S_T - K$$

where K is the delivery price and S_T is the spot price of the asset at maturity of the contract. This is because the holder of the contract is obligated to buy an asset worth S_T for K. Similarly, the payoff from a short position in a forward contract on one unit of an asset is

$$K - S_T$$

These payoffs can be positive or negative. They are illustrated in Figure 1.2. Because it costs nothing to enter into a forward contract, the payoff from the contract is also the trader's total gain or loss from the contract.

In the example just considered, $K = 2.0489$ and the corporation has a long contract. When $S_T = 2.1000$, the payoff is \$0.0511 per £1; when $S_T = 1.9000$, it is -0.1489 per £1.

Forward Prices and Spot Prices

We shall be discussing in some detail the relationship between spot and forward prices in Chapter 5. For a quick preview of why the two are related, consider a stock that pays no dividend and is worth \$60. You can borrow or lend money for 1 year at 5%. What should the 1-year forward price of the stock be?

The answer is \$60 grossed up at 5% for 1 year, or \$63. If the forward price is more than this, say \$67, you could borrow \$60, buy one share of the stock, and sell it forward for $67. After paying off the loan, you would net a profit of $4 in 1 year. If the forward price is less than $63, say $58, an investor owning the stock as part of a portfolio would sell the stock for $60 and enter into a forward contract to buy it back for $58 in 1 year. The proceeds of investment would be invested at 5% to earn $3. The investor would end up $5 better off than if the stock were kept in the portfolio for the year.

1.4 FUTURES CONTRACTS

Like a forward contract, a futures contract is an agreement between two parties to buy or sell an asset at a certain time in the future for a certain price. Unlike forward contracts, futures contracts are normally traded on an exchange. To make trading possible, the exchange specifies certain standardized features of the contract. As the two parties to the contract do not necessarily know each other, the exchange also provides a mechanism that gives the two parties a guarantee that the contract will be honored.

The largest exchanges on which futures contracts are traded are the Chicago Board of Trade (CBOT) and the Chicago Mercantile Exchange (CME). On these and other exchanges throughout the world, a very wide range of commodities and financial assets form the underlying assets in the various contracts. The commodities include pork bellies, live cattle, sugar, wool, lumber, copper, aluminum, gold, and tin. The financial assets include stock indices, currencies, and Treasury bonds. Futures prices are regularly reported in the financial press. Suppose that, on September 1, the December futures price of gold is quoted as $680. This is the price, exclusive of commissions, at which traders can agree to buy or sell gold for December delivery. It is determined on the floor of the exchange in the same way as other prices (i.e., by the laws of supply and demand). If more traders want to go long than to go short, the price goes up; if the reverse is true, then the price goes down.

Further details on issues such as margin requirements, daily settlement procedures, delivery procedures, bid–offer spreads, and the role of the exchange clearinghouse are given in Chapter 2.

1.5 OPTIONS

Options are traded both on exchanges and in the over-the-counter market. There are two types of option. A *call option* gives the holder the right to buy the underlying asset by a certain date for a certain price. A *put option* gives the holder the right to sell the underlying asset by a certain date for a certain price. The price in the contract is known as the *exercise price* or *strike price*; the date in the contract is known as the *expiration date* or *maturity*. *American options* can be exercised at any time up to the expiration date. *European options* can be exercised only on the expiration date itself.[2] Most of the options that are traded on exchanges are American. In the exchange-traded equity option market, one contract is usually an agreement to buy or sell 100 shares. European options are generally easier to analyze than American options, and some of the

[2] Note that the terms *American* and *European* do not refer to the location of the option or the exchange. Some options trading on North American exchanges are European.

Table 1.2 Prices of options on Intel, September 12, 2006; stock price = $19.56. (*Source*: CBOE)

	Calls			*Puts*		
Strike price ($)	*Oct. 2006*	*Jan. 2007*	*Apr. 2007*	*Oct. 2006*	*Jan. 2007*	*Apr. 2007*
15.00	4.650	4.950	5.150	0.025	0.150	0.275
17.50	2.300	2.775	3.150	0.125	0.475	0.725
20.00	0.575	1.175	1.650	0.875	1.375	1.700
22.50	0.075	0.375	0.725	2.950	3.100	3.300
25.00	0.025	0.125	0.275	5.450	5.450	5.450

properties of an American option are frequently deduced from those of its European counterpart.

It should be emphasized that an option gives the holder the right to do something. The holder does not have to exercise this right. This is what distinguishes options from forwards and futures, where the holder is obligated to buy or sell the underlying asset. Whereas it costs nothing to enter into a forward or futures contract, there is a cost to acquiring an option.

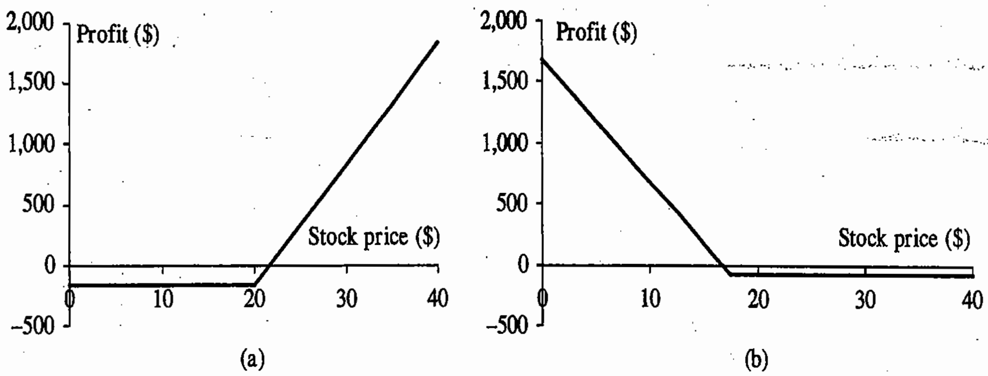
The largest exchange in the world for trading stock options is the Chicago Board Options Exchange (CBOE; www.cboe.com). Table 1.2 gives the midpoint of the bid and offer quotes for some of the American options trading on Intel (ticker symbol: INTC) on September 12, 2006. The quotes are taken from the CBOE website. The Intel stock price at the time of the quotes was $19.56. The option strike prices are $15.00, $17.50, $ 20.00, $22.50, and $25.00. The maturities are October 2006, January 2007, and April 2007. The October options have an expiration date of October 21, 2006; the January options have an expiration date of January 20, 2007; the April options have an expiration date of April 21, 2007.

Table 1.2 illustrates a number of properties of options. The price of a call option decreases as the strike price increases; the price of a put option increases as the strike price increases. Both types of options tend to become more valuable as their time to maturity increases. A put with a $25 strike price should be exercised immediately. That is why the price is the same for all maturities. These properties of options will be discussed further in Chapter 9.

Suppose an investor instructs a broker to buy one April call option contract on Intel with a strike price of $20.00. The broker will relay these instructions to a trader at the CBOE. This trader will then find another trader who wants to sell one April call contract on Intel with a strike price of $20.00, and a price will be agreed. For the purposes of our example, we ignore the bid–offer spread and assume that the price is $1.65, as indicated in Table 1.2. This is the price for an option to buy one share. In the United States, an option contract is a contract to buy or sell 100 shares. Therefore the investor must arrange for $165 to be remitted to the exchange through the broker. The exchange will then arrange for this amount to be passed on to the party on the other side of the transaction.

In our example the investor has obtained at a cost of $165 the right to buy 100

Figure 1.3 Net profit per share from (a) purchasing a contract consisting of 100 Intel April call options with a strike price of $20.00 and (b) purchasing a contract consisting of 100 Intel April put options with a strike price of $17.50.



Intel shares for $20.00 each. The party on the other side of the transaction has received $165 and has agreed to sell 100 Intel shares for $20.00 per share if the investor chooses to exercise the option. If the price of Intel does not rise above $20.00 before April 21, 2007, the option is not exercised and the investor loses $165. But if the Intel share price does well and the option is exercised when it is $30, the investor is able to buy 100 shares at $20.00 per share when they are worth $30 per share. This leads to a gain of $1000, or $835 when the initial cost of the options are taken into account.

An alternative trade for the investor would be the purchase of one April put option contract with a strike price of $17.50. From Table 1.2 we see that this would cost 100×0.725 or $72.50. The investor would obtain the right sell 100 Intel shares for $17.50 per share prior to April 21, 2007. If the Intel share price stays above $17.50, the option is not exercised and the investor loses $72.50. But if the investor exercises when the stock price is $15, the investor makes a gain of $250 by buying 100 Intel shares at $15 and selling them for $17.50. The net profit after the cost of the options is taken into account is $177.50.

The stock options trading on the CBOE are American. If we assume for simplicity that they are European, so that they can be exercised only at maturity, the investor's profit as a function of the final stock price for the two trades we have considered is shown in Figure 1.3.

Further details about the operation of options markets and how prices such as those in Table 1.2 are determined by traders are given in later chapters. At this stage we note that there are four types of participants in options markets:

1. Buyers of calls
2. Sellers of calls
3. Buyers of puts
4. Sellers of puts

Buyers are referred to as having *long positions*; sellers are referred to as having *short positions*. Selling an option is also known as *writing the option*.

Business Snapshot 1.1 Hedge Funds

Hedge funds have become major users of derivatives for hedging, speculation, and arbitrage. A hedge fund is similar to a mutual fund in that it invests funds on behalf of clients. However, unlike mutual funds, hedge funds are not required to register under US federal securities law. This is because they accept funds only from financially sophisticated individuals and do not publicly offer their securities. Mutual funds are subject to regulations requiring that shares in the funds be fairly priced, that the shares be redeemable at any time, that investment policies be disclosed, that the use of leverage be limited, that no short positions are taken, and so on. Hedge funds are relatively free of these regulations. This gives them a great deal of freedom to develop sophisticated, unconventional, and proprietary investment strategies. The fees charged by hedge fund managers are dependent on the fund's performance and are relatively high—typically 1% to 2% of the amount invested plus 20% of the profits. Hedge funds have grown in popularity with over $1 trillion being invested throughout the world for clients. "Funds of funds" have been set up to invest in a portfolio of hedge funds.

The investment strategy followed by a hedge fund manager often involves using derivatives to set up a speculative or arbitrage position. Once the strategy has been defined, the hedge fund manager must:

1. Evaluate the risks to which the fund is exposed
2. Decide which risks are acceptable and which will be hedged
3. Devise strategies (usually involving derivatives) to hedge the unacceptable risks

Here are some examples of the labels used for hedge funds together with the trading strategies followed:

Convertible arbitrage: Take a long position in a convertible bond combined with an actively managed short position in the underlying equity.

Distressed securities: Buy securities issued by companies in or close to bankruptcy.

Emerging markets: Invest in debt and equity of companies in developing or emerging countries and in the debt of the countries themselves.

Macro or global: Use leverage and derivatives to speculate on interest rate and foreign exchange rate moves.

Market neutral: Purchase securities considered to be undervalued and short securities considered to be overvalued in such a way that the exposure to the overall direction of the market is zero.

1.6 TYPES OF TRADERS

Derivatives markets have been outstandingly successful. The main reason is that they have attracted many different types of traders and have a great deal of liquidity. When an investor wants to take one side of a contract, there is usually no problem in finding someone that is prepared to take the other side.

Three broad categories of traders can be identified: hedgers, speculators, and arbitrageurs. Hedgers use derivatives to reduce the risk that they face from potential future movements in a market variable. Speculators use them to bet on the future direction of a market variable. Arbitrageurs take offsetting positions in two or more instruments to lock in a profit. As described in Business Snapshot 1.1, hedge funds have become big users of derivatives for all three purposes.

In the next few sections, we will consider the activities of each type of trader in more detail.

1.7 HEDGERS

In this section we illustrate how hedgers can reduce their risks with forward contracts and options.

Hedging Using Forward Contracts

Suppose that it is July 20, 2007, and ImportCo, a company based in the United States, knows that it will have to pay £10 million on October 20, 2007, for goods it has purchased from a British supplier. The USD–GBP exchange rate quotes made by a financial institution are shown in Table 1.1. ImportCo could hedge its foreign exchange risk by buying pounds (GBP) from the financial institution in the 3-month forward market at 2.0531. This would have the effect of fixing the price to be paid to the British exporter at $20,531,000.

Consider next another US company, which we will refer to as ExportCo, that is exporting goods to the United Kingdom and, on July 20, 2007, knows that it will receive £30 million 3 months later. ExportCo can hedge its foreign exchange risk by selling £30 million in the 3-month forward market at an exchange rate of 2.0526. This would have the effect of locking in the US dollars to be realized for the sterling at $61,578,000.

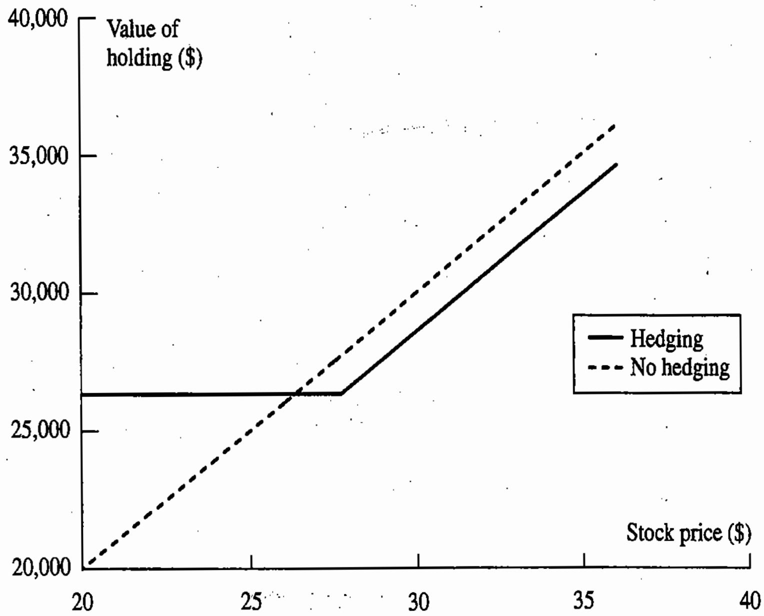
Note that a company might do better if it chooses not to hedge than if it chooses to hedge. Alternatively, it might do worse. Consider ImportCo. If the exchange rate is 1.9000 on October 20 and the company has not hedged, the £10 million that it has to pay will cost $19,000,000, which is less than $20,531,000. On the other hand, if the exchange rate is 2.1000, the £10 million will cost $21,000,000—and the company will wish that it had hedged! The position of ExportCo if it does not hedge is the reverse. If the exchange rate in October proves to be less than 2.0526, the company will wish that it had hedged; if the rate is greater than 2.0526, it will be pleased that it has not done so.

This example illustrates a key aspect of hedging. The purpose of hedging is to reduce risk. There is no guarantee that the outcome with hedging will be better than the outcome without hedging.

Hedging Using Options

Options can also be used for hedging. Consider an investor who in May of a particular year owns 1,000 Microsoft shares. The share price is $28 per share. The investor is concerned about a possible share price decline in the next 2 months and wants protection. The investor could buy ten July put option contracts on Microsoft on the Chicago Board Options Exchange with a strike price of $27.50. This would give the

Figure 1.4 Value of Microsoft holding in 2 months with and without hedging.



investor the right to sell a total of 1,000 shares for a price of $27.50. If the quoted option price is $1, then each option contract would cost 100 × $1 = $100 and the total cost of the hedging strategy would be 10 × $100 = $1,000.

The strategy costs $1,000 but guarantees that the shares can be sold for at least $27.50 per share during the life of the option. If the market price of Microsoft falls below $27.50, the options will be exercised, so that $27,500 is realized for the entire holding. When the cost of the options is taken into account, the amount realized is $26,500. If the market price stays above $27.50, the options are not exercised and expire worthless. However, in this case the value of the holding is always above $27,500 (or above $26,500 when the cost of the options is taken into account). Figure 1.4 shows the net value of the portfolio (after taking the cost of the options into account) as a function of Microsoft's stock price in 2 months. The dotted line shows the value of the portfolio assuming no hedging.

A Comparison

There is a fundamental difference between the use of forward contracts and options for hedging. Forward contracts are designed to neutralize risk by fixing the price that the hedger will pay or receive for the underlying asset. Option contracts, by contrast, provide insurance. They offer a way for investors to protect themselves against adverse price movements in the future while still allowing them to benefit from favorable price movements. Unlike forwards, options involve the payment of an up-front fee.

1.8 SPECULATORS

We now move on to consider how futures and options markets can be used by speculators. Whereas hedgers want to avoid exposure to adverse movements in the price of an asset, speculators wish to take a position in the market. Either they are betting that the price of the asset will go up or they are betting that it will go down.

Speculation Using Futures

Consider a US speculator who in February thinks that the British pound will strengthen relative to the US dollar over the next 2 months and is prepared to back that hunch to the tune of £250,000. One thing the speculator can do is purchase £250,000 in the spot market in the hope that the sterling can be sold later at a higher price. (The sterling once purchased would be kept in an interest-bearing account.) Another possibility is to take a long position in four CME April futures contracts on sterling. (Each futures contract is for the purchase of £62,500.) Table 1.3 summarizes the two alternatives on the assumption that the current exchange rate is 2.0470 dollars per pound and the April futures price is 2.0410 dollars per pound. If the exchange rate turns out to be 2.1000 dollars per pound in April, the futures contract alternative enables the speculator to realize a profit of $(2.1000 - 2.0410) \times 250{,}000 = \$14{,}750$. The spot market alternative leads to 250,000 units of an asset being purchased for \$2.0470 in February and sold for \$2.1000 in April, so that a profit of $(2.1000 - 2.0470) \times 250{,}000 = \$13{,}250$ is made. If the exchange rate falls to 2.0000 dollars per pound, the futures contract gives rise to a $(2.0410 - 2.0000) \times 250{,}000 = \$10{,}250$ loss, whereas the spot market alternative gives rise to a loss of $(2.0470 - 2.0000) \times 250{,}000 = \$11{,}750$. The alternatives appear to give rise to slightly different profits and losses. But these calculations do not reflect the interest that is earned or paid. As shown in Chapter 5, when the interest earned in sterling and the interest foregone on the dollars used to buy the sterling are taken into account, the profit or loss from the two alternatives is the same.

What then is the difference between the two alternatives? The first alternative of buying sterling requires an up-front investment of \$511,750 ($= 250{,}000 \times 2.0470$). In contrast, the second alternative requires only a small amount of cash to be deposited by the speculator in what is termed a "margin account". The operation of margin accounts is explained in Chapter 2. In Table 1.3, the initial margin requirement is assumed to be \$5,000 per contract, or \$20,000 in total, but in practice it might be even less than this. The futures market allows the speculator to obtain leverage. With a relatively small initial outlay, the investor is able to take a large speculative position.

Table 1.3 Speculation using spot and futures contracts. One futures contract is on £62,500. Initial margin on four futures contracts = \$20,000.

	Possible trades	
	Buy £250,000 *Spot price = 2.0470*	*Buy 4 futures contracts* *Futures price = 2.0410*
Investment	\$511,750	\$20,000
Profit if April spot = 2.1000	\$13,250	\$14,750
Profit if April spot = 2.0000	−\$11,750	−\$10,250

Speculation Using Options

Options can also be used for speculation. Suppose that it is October and a speculator considers that a stock is likely to increase in value over the next 2 months. The stock price is currently \$20, and a 2-month call option with a \$22.50 strike price is currently selling for \$1. Table 1.4 illustrates two possible alternatives, assuming that the speculator is willing to invest \$2,000. One alternative is to purchase 100 shares; the other involves the purchase of 2,000 call options (i.e., 20 call option contracts). Suppose that the speculator's hunch is correct and the price of the stock rises to \$27 by December. The first alternative of buying the stock yields a profit of

$$100 \times (\$27 - \$20) = \$700$$

However, the second alternative is far more profitable. A call option on the stock with a strike price of \$22.50 gives a payoff of \$4.50, because it enables something worth \$27 to be bought for \$22.50. The total payoff from the 2,000 options that are purchased under the second alternative is

$$2{,}000 \times \$4.50 = \$9{,}000$$

Subtracting the original cost of the options yields a net profit of

$$\$9{,}000 - \$2{,}000 = \$7{,}000$$

The options strategy is, therefore, 10 times more profitable than directly buying the stock.

Options also give rise to a greater potential loss. Suppose the stock price falls to \$15 by December. The first alternative of buying stock yields a loss of

$$100 \times (\$20 - \$15) = \$500$$

Because the call options expire without being exercised, the options strategy would lead to a loss of \$2,000—the original amount paid for the options. Figure 1.5 shows the profit or loss from the two strategies as a function of the stock price in 2 months.

Options like futures provide a form of leverage. For a given investment, the use of options magnifies the financial consequences. Good outcomes become very good, while bad outcomes result in the whole initial investment being lost.

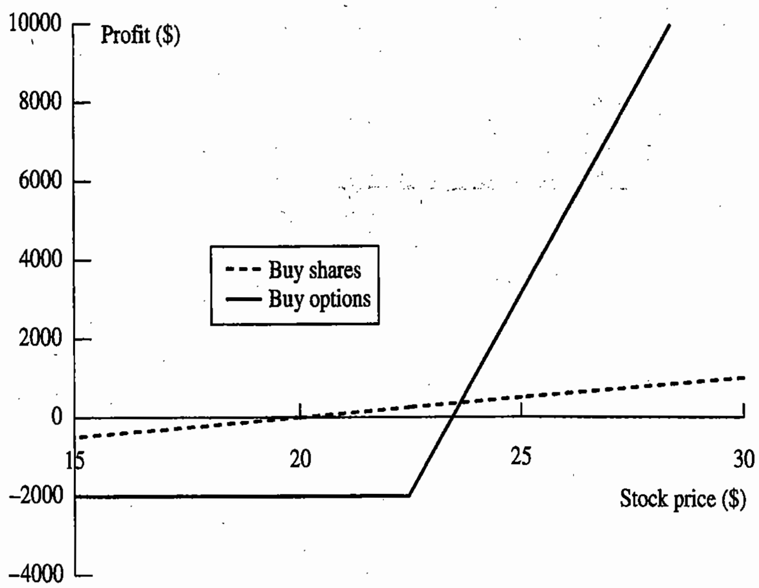
A Comparison

Futures and options are similar instruments for speculators in that they both provide a way in which a type of leverage can be obtained. However, there is an important

Table 1.4 Comparison of profits (losses) from two alternative strategies for using \$2,000 to speculate on a stock worth \$20 in October.

	December stock price	
Investor's strategy	*\$15*	*\$27*
Buy 100 shares	(\$500)	\$700
Buy 2,000 call options	(\$2,000)	\$7,000

Figure 1.5 Profit or loss from two alternative strategies for speculating on a stock currently worth $20.



difference between the two. When a speculator uses futures, the potential loss as well as the potential gain is very large. When options are used, no matter how bad things get, the speculator's loss is limited to the amount paid for the options.

1.9 ARBITRAGEURS

Arbitrageurs are a third important group of participants in futures, forward, and options markets. Arbitrage involves locking in a riskless profit by simultaneously entering into transactions in two or more markets. In later chapters we will see how arbitrage is sometimes possible when the futures price of an asset gets out of line with its spot price. We will also examine how arbitrage can be used in options markets. This section illustrates the concept of arbitrage with a very simple example.

Let us consider a stock that is traded on both the New York Stock Exchange (www.nyse.com) and the London Stock Exchange (www.stockex.co.uk). Suppose that the stock price is $200 in New York and £100 in London at a time when the exchange rate is $2.0300 per pound. An arbitrageur could simultaneously buy 100 shares of the stock in New York and sell them in London to obtain a risk-free profit of

$$100 \times [(\$2.03 \times 100) - \$200]$$

or $300 in the absence of transactions costs. Transactions costs would probably eliminate the profit for a small investor. However, a large investment bank faces very low transactions costs in both the stock market and the foreign exchange market. It would find the arbitrage opportunity very attractive and would try to take as much advantage of it as possible.

Business Snapshot 1.2 The Barings Bank Disaster

Derivatives are very versatile instruments. They can be used for hedging, speculation, and arbitrage. One of the risks faced by a company that trades derivatives is that an employee who has a mandate to hedge or to look for arbitrage opportunities may become a speculator.

Nick Leeson, an employee of Barings Bank in the Singapore office in 1995, had a mandate to look for arbitrage opportunities between the Nikkei 225 futures prices on the Singapore exchange and those on the Osaka exchange. Over time Leeson moved from being an arbitrageur to being a speculator without anyone in the Barings London head office fully understanding that he had changed the way he was using derivatives. He began to incur losses, which he was able to hide. He then began to take bigger speculative positions in an attempt to recover the losses, but only succeeded in making the losses worse.

By the time Leeson's activities were uncovered, the total loss was close to 1 billion dollars. As a result, Barings—a bank that had been in existence for 200 years—was wiped out. One of the lessons from Barings is that it is important to define unambiguous risk limits for traders and then monitor carefully what they do to make sure that these limits are adhered to.

Arbitrage opportunities such as the one just described cannot last for long. As arbitrageurs buy the stock in New York, the forces of supply and demand will cause the dollar price to rise. Similarly, as they sell the stock in London, the sterling price will be driven down. Very quickly the two prices will become equivalent at the current exchange rate. Indeed, the existence of profit-hungry arbitrageurs makes it unlikely that a major disparity between the sterling price and the dollar price could ever exist in the first place. Generalizing from this example, we can say that the very existence of arbitrageurs means that in practice only very small arbitrage opportunities are observed in the prices that are quoted in most financial markets. In this book most of the arguments concerning futures prices, forward prices, and the values of option contracts will be based on the assumption that no arbitrage opportunities exist.

1.10 DANGERS

Derivatives are very versatile instruments. As we have seen, they can be used for hedging, for speculation, and for arbitrage. It is this very versatility that can cause problems. Sometimes traders who have a mandate to hedge risks or follow an arbitrage strategy become (consciously or unconsciously) speculators. The results can be disastrous. One example of this is provided by the activities of Nick Leeson at Barings Bank (see Business Snapshot 1.2).[3]

To avoid the sort of problems Barings encountered, it is very important for both financial and nonfinancial corporations to set up controls to ensure that derivatives are being used for their intended purpose. Risk limits should be set and the activities of traders should be monitored daily to ensure that these risk limits are adhered to.

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[3] The movie *Rogue Trader* provides a good dramatization of the failure of Barings Bank.

SUMMARY

One of the exciting developments in finance over the last 30 years has been the growth of derivatives markets. In many situations, both hedgers and speculators find it more attractive to trade a derivative on an asset than to trade the asset itself. Some derivatives are traded on exchanges; others are traded by financial institutions, fund managers, and corporations in the over-the-counter market, or added to new issues of debt and equity securities. Much of this book is concerned with the valuation of derivatives. The aim is to present a unifying framework within which all derivatives—not just options or futures—can be valued.

In this chapter we have taken a first look at forward, futures, and options contracts. A forward or futures contract involves an obligation to buy or sell an asset at a certain time in the future for a certain price. There are two types of options: calls and puts. A call option gives the holder the right to buy an asset by a certain date for a certain price. A put option gives the holder the right to sell an asset by a certain date for a certain price. Forwards, futures, and options trade on a wide range of different underlying assets.

Derivatives have been very successful innovations in capital markets. Three main types of traders can be identified: hedgers, speculators, and arbitrageurs. Hedgers are in the position where they face risk associated with the price of an asset. They use derivatives to reduce or eliminate this risk. Speculators wish to bet on future movements in the price of an asset. They use derivatives to get extra leverage. Arbitrageurs are in business to take advantage of a discrepancy between prices in two different markets. If, for example, they see the futures price of an asset getting out of line with the cash price, they will take offsetting positions in the two markets to lock in a profit.

FURTHER READING

Chancellor, E. *Devil Take the Hindmost—A History of Financial Speculation*. New York: Farra Straus Giroux, 2000.

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Questions and Problems (Answers in Solutions Manual)

1.1. What is the difference between a long forward position and a short forward position?

1.2. Explain carefully the difference between hedging, speculation, and arbitrage.

1.3. What is the difference between entering into a long forward contract when the forward price is \$50 and taking a long position in a call option with a strike price of \$50?

1.4. Explain carefully the difference between selling a call option and buying a put option.

1.5. An investor enters into a short forward contract to sell 100,000 British pounds for US dollars at an exchange rate of 1.9000 US dollars per pound. How much does the investor gain or lose if the exchange rate at the end of the contract is (a) 1.8900 and (b) 1.9200?

1.6. A trader enters into a short cotton futures contract when the futures price is 50 cents per pound. The contract is for the delivery of 50,000 pounds. How much does the trader gain or lose if the cotton price at the end of the contract is (a) 48.20 cents per pound and (b) 51.30 cents per pound?

1.7. Suppose that you write a put contract with a strike price of $40 and an expiration date in 3 months. The current stock price is $41 and the contract is on 100 shares. What have you committed yourself to? How much could you gain or lose?

1.8. What is the difference between the over-the-counter market and the exchange-traded market? What are the bid and offer quotes of a market maker in the over-the-counter market?

1.9. You would like to speculate on a rise in the price of a certain stock. The current stock price is $29 and a 3-month call with a strike price of $30 costs $2.90. You have $5,800 to invest. Identify two alternative investment strategies, one in the stock and the other in an option on the stock. What are the potential gains and losses from each?

1.10. Suppose that you own 5,000 shares worth $25 each. How can put options be used to provide you with insurance against a decline in the value of your holding over the next 4 months?

1.11. When first issued, a stock provides funds for a company. Is the same true of a stock option? Discuss.

1.12. Explain why a forward contract can be used for either speculation or hedging.

1.13. Suppose that a March call option to buy a share for $50 costs $2.50 and is held until March. Under what circumstances will the holder of the option make a profit? Under what circumstances will the option be exercised? Draw a diagram illustrating how the profit from a long position in the option depends on the stock price at maturity of the option.

1.14. Suppose that a June put option to sell a share for $60 costs $4 and is held until June. Under what circumstances will the seller of the option (i.e., the party with the short position) make a profit? Under what circumstances will the option be exercised? Draw a diagram illustrating how the profit from a short position in the option depends on the stock price at maturity of the option.

1.15. It is May and a trader writes a September call option with a strike price of $20. The stock price is $18 and the option price is $2. Describe the trader's cash-flows if the option is held until September and the stock price is $25 at that time.

1.16. A trader writes a December put option with a strike price of $30. The price of the option is $4. Under what circumstances does the trader make a gain?

1.17. A company knows that it is due to receive a certain amount of a foreign currency in 4 months. What type of option contract is appropriate for hedging?

1.18. A United States company expects to have to pay 1 million Canadian dollars in 6 months. Explain how the exchange rate risk can be hedged using (a) a forward contract and (b) an option.

1.19. A trader enters into a short forward contract on 100 million yen. The forward exchange rate is \$0.0080 per yen. How much does the trader gain or lose if the exchange rate at the end of the contract is (a) \$0.0074 per yen and (b) \$0.0091 per yen?

1.20. The Chicago Board of Trade offers a futures contract on long-term Treasury bonds. Characterize the traders likely to use this contract.

1.21. "Options and futures are zero-sum games." What do you think is meant by this statement?

1.22. Describe the profit from the following portfolio: a long forward contract on an asset and a long European put option on the asset with the same maturity as the forward contract and a strike price that is equal to the forward price of the asset at the time the portfolio is set up.

1.23. In the 1980s, Bankers Trust developed *index currency option notes* (ICONs). These are bonds in which the amount received by the holder at maturity varies with a foreign exchange rate. One example was its trade with the Long Term Credit Bank of Japan. The ICON specified that if the yen–US dollar exchange rate, S_T, is greater than 169 yen per dollar at maturity (in 1995), the holder of the bond receives \$1,000. If it is less than 169 yen per dollar, the amount received by the holder of the bond is

$$1{,}000 - \max\left[0,\ 1{,}000\left(\frac{169}{S_T} - 1\right)\right]$$

When the exchange rate is below 84.5, nothing is received by the holder at maturity. Show that this ICON is a combination of a regular bond and two options.

1.24. On July 1, 2008, a company enters into a forward contract to buy 10 million Japanese yen on January 1, 2009. On September 1, 2008, it enters into a forward contract to sell 10 million Japanese yen on January 1, 2009. Describe the payoff from this strategy.

1.25. Suppose that USD/sterling spot and forward exchange rates are as follows:

Spot	2.0080
90-day forward	2.0056
180-day forward	2.0018

What opportunities are open to an arbitrageur in the following situations?
(a) A 180-day European call option to buy £1 for \$1.97 costs 2 cents.
(b) A 90-day European put option to sell £1 for \$2.04 costs 2 cents.

Assignment Questions

1.26. The price of gold is currently \$600 per ounce. The forward price for delivery in 1 year is \$800. An arbitrageur can borrow money at 10% per annum. What should the arbitrageur do? Assume that the cost of storing gold is zero and that gold provides no income.

1.27. The current price of a stock is \$94, and 3-month European call options with a strike price of \$95 currently sell for \$4.70. An investor who feels that the price of the stock will increase is trying to decide between buying 100 shares and buying 2,000 call options (= 20 contracts). Both strategies involve an investment of \$9,400. What advice would you give? How high does the stock price have to rise for the option strategy to be more profitable?

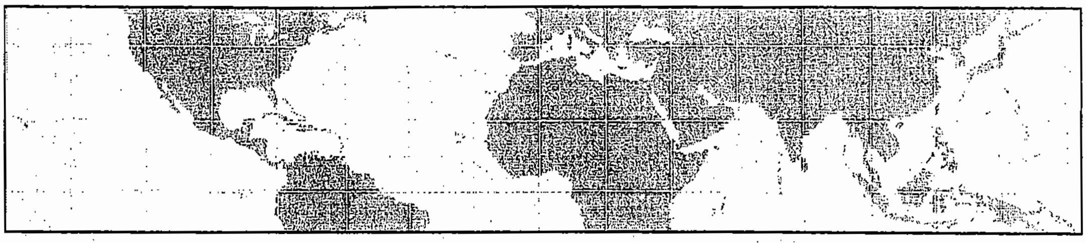
1.28. On September 12, 2006, an investor owns 100 Intel shares. As indicated in Table 1.2, the share price is \$19.56 and a January put option with a strike price of \$17.50 costs \$0.475. The investor is comparing two alternatives to limit downside risk. The first is to buy 1 January put option contract with a strike price of \$17.50. The second involves instructing a broker to sell the 100 shares as soon as Intel's price reaches \$17.50. Discuss the advantages and disadvantages of the two strategies.

1.29. A bond issued by Standard Oil some time ago worked as follows. The holder received no interest. At the bond's maturity the company promised to pay \$1,000 plus an additional amount based on the price of oil at that time. The additional amount was equal to the product of 170 and the excess (if any) of the price of a barrel of oil at maturity over \$25. The maximum additional amount paid was \$2,550 (which corresponds to a price of \$40 per barrel). Show that the bond is a combination of a regular bond, a long position in call options on oil with a strike price of \$25, and a short position in call options on oil with a strike price of \$40.

1.30. Suppose that in the situation of Table 1.1 a corporate treasurer said: "I will have £1 million to sell in 6 months. If the exchange rate is less than 2.02, I want you to give me 2.02. If it is greater than 2.09, I will accept 2.09. If the exchange rate is between 2.02 and 2.09, I will sell the sterling for the exchange rate." How could you use options to satisfy the treasurer?

1.31. Describe how foreign currency options can be used for hedging in the situation considered in Section 1.7 so that (a) ImportCo is guaranteed that its exchange rate will be less than 2.0700, and (b) ExportCo is guaranteed that its exchange rate will be at least 2.0400. Use DerivaGem to calculate the cost of setting up the hedge in each case assuming that the exchange rate volatility is 12%, interest rates in the United States are 5%, and interest rates in Britain are 5.7%. Assume that the current exchange rate is the average of the bid and offer in Table 1.1.

1.32. A trader buys a European call option and sells a European put option. The options have the same underlying asset, strike price, and maturity. Describe the trader's position. Under what circumstances does the price of the call equal the price of the put?



CHAPTER 2

Mechanics of Futures Markets

In Chapter 1 we explained that both futures and forward contracts are agreements to buy or sell an asset at a future time for a certain price. Futures contracts are traded on an organized exchange, and the contract terms are standardized by that exchange. By contrast, forward contracts are private agreements between two financial institutions or between a financial institution and one of its clients.

This chapter covers the details of how futures markets work. We examine issues such as the specification of contracts, the operation of margin accounts, the organization of exchanges, the regulation of markets, the way in which quotes are made, and the treatment of futures transactions for accounting and tax purposes. We compare futures contracts with forward contracts and explain the difference between the payoffs realized from them.

2.1 BACKGROUND

As we saw in Chapter 1, futures contracts are now traded actively all over the world. The Chicago Board of Trade (CBOT, www.cbot.com) and the Chicago Mercantile Exchange (CME, www.cme.com) are the two largest futures exchanges in the United States. (They finalized an agreement to merge in July 2007.) The two largest exchanges in Europe are Euronext (www.euronext.com), which reached an agreement to merge with the New York Stock Exchange (www.nyse.com) in 2006, and Eurex (www.eurexchange.com), which is co-owned by Deutsche Börse and the Swiss Exchange. Other large exchanges include Bolsa de Mercadorias & Futuros (www.bmf.com.br) in São Paulo, the Tokyo International Financial Futures Exchange (www.tiffe.or.jp), the Singapore International Monetary Exchange (www.sgx.com), and the Sydney Futures Exchange (www.sfe.com.au). A table at the end of this book provides a more complete list.

We examine how a futures contract comes into existence by considering the corn futures contract traded on the Chicago Board of Trade (CBOT). On March 5 a trader in New York might call a broker with instructions to buy 5,000 bushels of corn for delivery in July of the same year. The broker would immediately issue instructions to a trader to buy (i.e., take a long position in) one July corn contract. (Each corn contract on CBOT is for the delivery of exactly 5,000 bushels.) At about the same time, another

Business Snapshot 2.1 The Unanticipated Delivery of a Futures Contract

This story (which may well be apocryphal) was told to the author of this book by a senior executive of a financial institution. It concerns a new employee of the financial institution who had not previously worked in the financial sector. One of the clients of the financial institution regularly entered into a long futures contract on live cattle for hedging purposes and issued instructions to close out the position on the last day of trading. (Live cattle futures contracts trade on the Chicago Mercantile Exchange and each contract is on 40,000 pounds of cattle.) The new employee was given responsibility for handling the account.

When the time came to close out a contract the employee noted that the client was long one contract and instructed a trader at the exchange to buy (not sell) one contract. The result of this mistake was that the financial institution ended up with a long position in two live cattle futures contracts. By the time the mistake was spotted trading in the contract had ceased.

The financial institution (not the client) was responsible for the mistake. As a result, it started to look into the details of the delivery arrangements for live cattle futures contracts—something it had never done before. Under the terms of the contract, cattle could be delivered by the party with the short position to a number of different locations in the United States during the delivery month. Because it was long, the financial institution could do nothing but wait for a party with a short position to issue a *notice of intention to deliver* to the exchange and for the exchange to assign that notice to the financial institution.

It eventually received a notice from the exchange and found that it would receive live cattle at a location 2,000 miles away the following Tuesday. The new employee was sent to the location to handle things. It turned out that the location had a cattle auction every Tuesday. The party with the short position that was making delivery bought cattle at the auction and then immediately delivered them. Unfortunately the cattle could not be resold until the next cattle auction the following Tuesday. The employee was therefore faced with the problem of making arrangements for the cattle to be housed and fed for a week. This was a great start to a first job in the financial sector!

trader in Kansas might instruct a broker to sell 5,000 bushels of corn for July delivery. This broker would then issue instructions to sell (i.e., take a short position in) one corn contract. A price would be determined and the deal would be done. Under the traditional open outcry system, floor traders representing each party would physically meet to determine the price. With electronic trading, a computer would match trades and monitor prices.

The trader in New York who agreed to buy has a *long futures position* in one contract; the trader in Kansas who agreed to sell has a *short futures position* in one contract. The price agreed to is the current *futures price* for July corn, say 300 cents per bushel. This price, like any other price, is determined by the laws of supply and demand. If, at a particular time, more traders wish to sell rather than buy July corn, the price will go down. New buyers then enter the market so that a balance between buyers and sellers is maintained. If more traders wish to buy rather than sell July corn, the price goes up. New sellers then enter the market and a balance between buyers and sellers is maintained.

Closing Out Positions

The vast majority of futures contracts do not lead to delivery. The reason is that most traders choose to close out their positions prior to the delivery period specified in the contract. Closing out a position means entering into the opposite trade to the original one. For example, the New York investor who bought a July corn futures contract on March 5 can close out the position by selling (i.e., shorting) one July corn futures contract on, say, April 20. The Kansas investor who sold (i.e., shorted) a July contract on March 5 can close out the position by buying one July contract on, say, May 25. In each case, the investor's total gain or loss is determined by the change in the futures price between March 5 and the day when the contract is closed out.

Delivery is so unusual that traders sometimes forget how the delivery process works (see Business Snapshot 2.1). Nevertheless we will spend part of this chapter reviewing the delivery arrangements in futures contracts. This is because it is the possibility of final delivery that ties the futures price to the spot price.[1]

2.2 SPECIFICATION OF A FUTURES CONTRACT

When developing a new contract, the exchange must specify in some detail the exact nature of the agreement between the two parties. In particular, it must specify the asset, the contract size (exactly how much of the asset will be delivered under one contract), where delivery will be made, and when delivery will be made.

Sometimes alternatives are specified for the grade of the asset that will be delivered or for the delivery locations. As a general rule, it is the party with the short position (the party that has agreed to sell the asset) that chooses what will happen when alternatives are specified by the exchange. When the party with the short position is ready to deliver, it files a *notice of intention to deliver* with the exchange. This notice indicates selections it has made with respect to the grade of asset that will be delivered and the delivery location.

The Asset

When the asset is a commodity, there may be quite a variation in the quality of what is available in the marketplace. When the asset is specified, it is therefore important that the exchange stipulate the grade or grades of the commodity that are acceptable. The New York Board of Trade (NYBOT) has specified the asset in its frozen concentrated orange juice futures contract as orange solids from Florida and/or Brazil that are US Grade A with Brix value of not less than 62.5 degrees.

For some commodities a range of grades can be delivered, but the price received depends on the grade chosen. For example, in the Chicago Board of Trade corn futures contract, the standard grade is "No. 2 Yellow", but substitutions are allowed with the price being adjusted in a way established by the exchange. No. 1 Yellow is deliverable for 1.5 cents per bushel more than No. 2 Yellow. No. 3 Yellow is deliverable for 1.5 cents per bushel less than No. 2 Yellow.

The financial assets in futures contracts are generally well defined and unambiguous.

[1] As mentioned in Chapter 1, the spot price is the price for almost immediate delivery.

For example, there is no need to specify the grade of a Japanese yen. However, there are some interesting features of the Treasury bond and Treasury note futures contracts traded on the Chicago Board of Trade. The underlying asset in the Treasury bond contract is any long-term US Treasury bond that has a maturity of greater than 15 years and is not callable within 15 years. In the Treasury note futures contract, the underlying asset is any long-term Treasury note with a maturity of no less than 6.5 years and no more than 10 years from the date of delivery. In both cases, the exchange has a formula for adjusting the price received according to the coupon and maturity date of the bond delivered. This is discussed in Chapter 6.

The Contract Size

The contract size specifies the amount of the asset that has to be delivered under one contract. This is an important decision for the exchange. If the contract size is too large, many investors who wish to hedge relatively small exposures or who wish to take relatively small speculative positions will be unable to use the exchange. On the other hand, if the contract size is too small, trading may be expensive as there is a cost associated with each contract traded.

The correct size for a contract clearly depends on the likely user. Whereas the value of what is delivered under a futures contract on an agricultural product might be $10,000 to $20,000, it is much higher for some financial futures. For example, under the Treasury bond futures contract traded on the Chicago Board of Trade, instruments with a face value of $100,000 are delivered.

In some cases exchanges have introduced "mini" contracts to attract smaller investors. For example, the CME's Mini Nasdaq 100 contract is on 20 times the Nasdaq 100 index, whereas the regular contract is on 100 times the index.

Delivery Arrangements

The place where delivery will be made must be specified by the exchange. This is particularly important for commodities that involve significant transportation costs. In the case of the NYBOT frozen concentrate orange juice contract, delivery is to exchange-licensed warehouses in Florida, New Jersey, or Delaware.

When alternative delivery locations are specified, the price received by the party with the short position is sometimes adjusted according to the location chosen by that party. The price tends to be higher for delivery locations that are relatively far from the main sources of the commodity.

Delivery Months

A futures contract is referred to by its delivery month. The exchange must specify the precise period during the month when delivery can be made. For many futures contracts, the delivery period is the whole month.

The delivery months vary from contract to contract and are chosen by the exchange to meet the needs of market participants. For example, corn futures traded on the Chicago Board of Trade have delivery months of March, May, July, September, and December. At any given time, contracts trade for the closest delivery month and a number of subsequent delivery months. The exchange specifies when trading in a particular month's contract will begin. The exchange also specifies the last day on which trading can take place for a given contract. Trading generally ceases a few days before the last day on which delivery can be made.

Price Quotes

The exchange defines how prices will be quoted. For example, crude oil prices on the New York Mercantile Exchange are quoted in dollars and cents. Treasury bond and Treasury note futures on the Chicago Board of Trade are quoted in dollars and thirty-seconds of a dollar.

Price Limits and Position Limits

For most contracts, daily price movement limits are specified by the exchange. If in a day the price moves down from the previous day's close by an amount equal to the daily price limit, the contract is said to be *limit down*. If it moves up by the limit, it is said to be *limit up*. A *limit move* is a move in either direction equal to the daily price limit. Normally, trading ceases for the day once the contract is limit up or limit down. However, in some instances the exchange has the authority to step in and change the limits.

The purpose of daily price limits is to prevent large price movements from occurring because of speculative excesses. However, limits can become an artificial barrier to trading when the price of the underlying commodity is advancing or declining rapidly. Whether price limits are, on balance, good for futures markets is controversial.

Position limits are the maximum number of contracts that a speculator may hold. The purpose of these limits is to prevent speculators from exercising undue influence on the market.

2.3 CONVERGENCE OF FUTURES PRICE TO SPOT PRICE

As the delivery period for a futures contract is approached, the futures price converges to the spot price of the underlying asset. When the delivery period is reached, the futures price equals—or is very close to—the spot price.

To see why this is so, we first suppose that the futures price is above the spot price during the delivery period. Traders then have a clear arbitrage opportunity:

1. Sell (i.e., short) a futures contract
2. Buy the asset
3. Make delivery

These steps are certain to lead to a profit equal to the amount by which the futures price exceeds the spot price. As traders exploit this arbitrage opportunity, the futures price will fall. Suppose next that the futures price is below the spot price during the delivery period. Companies interested in acquiring the asset will find it attractive to enter into a long futures contract and then wait for delivery to be made. As they do so, the futures price will tend to rise.

The result is that the futures price is very close to the spot price during the delivery period. Figure 2.1 illustrates the convergence of the futures price to the spot price. In

Figure 2.1 Relationship between futures price and spot price as the delivery period is approached: (a) Futures price above spot price; (b) futures price below spot price.

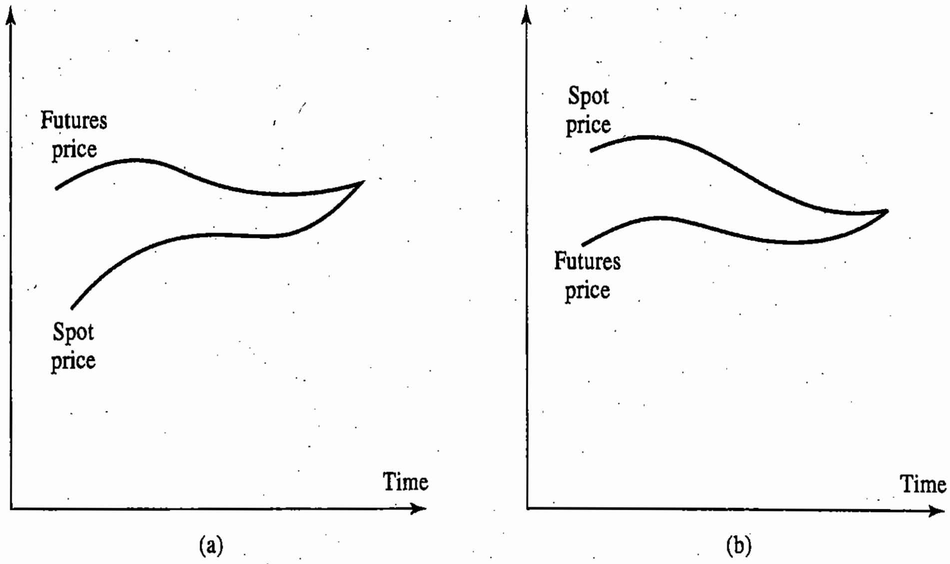


Figure 2.1(a) the futures price is above the spot price prior to the delivery period. In Figure 2.1(b) the futures price is below the spot price prior to the delivery period. The circumstances under which these two patterns are observed are discussed in Chapter 5.

2.4 DAILY SETTLEMENT AND MARGINS

If two investors get in touch with each other directly and agree to trade an asset in the future for a certain price, there are obvious risks. One of the investors may regret the deal and try to back out. Alternatively, the investor simply may not have the financial resources to honor the agreement. One of the key roles of the exchange is to organize trading so that contract defaults are avoided. This is where margins come in.

The Operation of Margins

To illustrate how margins work, we consider an investor who contacts his or her broker on Thursday, June 5, to buy two December gold futures contracts on the COMEX division of the New York Mercantile Exchange (NYMEX). We suppose that the current futures price is \$600 per ounce. Because the contract size is 100 ounces, the investor has contracted to buy a total of 200 ounces at this price. The broker will require the investor to deposit funds in a *margin account*. The amount that must be deposited at the time the contract is entered into is known as the *initial margin*. We suppose this is \$2,000 per contract, or \$4,000 in total. At the end of each trading day, the margin account is adjusted to reflect the investor's gain or loss. This practice is referred to as *marking to market* the account.

Suppose, for example, that by the end of June 5 the futures price has dropped from \$600 to \$597. The investor has a loss of \$600 (= 200 × \$3), because the 200 ounces of December gold, which the investor contracted to buy at \$600, can now be sold for only

\$597. The balance in the margin account would therefore be reduced by \$600 to \$3,400. Similarly, if the price of December gold rose to \$603 by the end of the first day, the balance in the margin account would be increased by \$600 to \$4,600. A trade is first marked to market at the close of the day on which it takes place. It is then marked to market at the close of trading on each subsequent day.

Note that marking to market is not merely an arrangement between broker and client. When there is a decrease in the futures price so that the margin account of an investor with a long position is reduced by \$600, the investor's broker has to pay the exchange \$600 and the exchange passes the money on to the broker of an investor with a short position. Similarly, when there is an increase in the futures price, brokers for parties with short positions pay money to the exchange and brokers for parties with long positions receive money from the exchange. Later we will examine in more detail the mechanism by which this happens.

The investor is entitled to withdraw any balance in the margin account in excess of the initial margin. To ensure that the balance in the margin account never becomes negative a *maintenance margin*, which is somewhat lower than the initial margin, is set. If the balance in the margin account falls below the maintenance margin, the investor receives a margin call and is expected to top up the margin account to the initial margin level the next day. The extra funds deposited are known as a *variation margin*. If the investor does not provide the variation margin, the broker closes out the position. In the case of the investor considered earlier, closing out the position would involve neutralizing the existing contract by selling 200 ounces of gold for delivery in December.

Table 2.1 illustrates the operation of the margin account for one possible sequence of futures prices in the case of the investor considered earlier. The maintenance margin is assumed for the purpose of the illustration to be \$1,500 per contract, or \$3,000 in total. On June 13 the balance in the margin account falls \$340 below the maintenance margin level. This drop triggers a margin call from the broker for an additional \$1,340. Table 2.1 assumes that the investor does in fact provide this margin by the close of trading on June 16. On June 19 the balance in the margin account again falls below the maintenance margin level, and a margin call for \$1,260 is sent out. The investor provides this margin by the close of trading on June 20. On June 26 the investor decides to close out the position by selling two contracts. The futures price on that day is \$592.30, and the investor has a cumulative loss of \$1,540. Note that the investor has excess margin on June 16, 23, 24, and 25. Table 2.1 assumes that the excess is not withdrawn.

Further Details

Many brokers allow an investor to earn interest on the balance in a margin account. The balance in the account does not, therefore, represent a true cost, provided that the interest rate is competitive with what could be earned elsewhere. To satisfy the initial margin requirements (but not subsequent margin calls), an investor can sometimes deposit securities with the broker. Treasury bills are usually accepted in lieu of cash at about 90% of their face value. Shares are also sometimes accepted in lieu of cash—but at about 50% of their market value.

The effect of the marking to market is that a futures contract is settled daily rather than all at the end of its life. At the end of each day, the investor's gain (loss) is added to (subtracted from) the margin account, bringing the value of the contract back to zero. A futures contract is in effect closed out and rewritten at a new price each day.

Table 2.1 Operation of margins for a long position in two gold futures contracts. The initial margin is $2,000 per contract, or $4,000 in total, and the maintenance margin is $1,500 per contract, or $3,000 in total. The contract is entered into on June 5 at $600 and closed out on June 26 at $592.30. The numbers in the second column, except the first and the last, represent the futures prices at the close of trading.

Day	*Futures price ($)*	*Daily gain (loss) ($)*	*Cumulative gain (loss) ($)*	*Margin account balance ($)*	*Margin call ($)*
	600.00			4,000	
June 5	597.00	(600)	(600)	3,400	
June 6	596.10	(180)	(780)	3,220	
June 9	598.20	420	(360)	3,640	
June 10	597.10	(220)	(580)	3,420	
June 11	596.70	(80)	(660)	3,340	
June 12	595.40	(260)	(920)	3,080	
June 13	593.30	(420)	(1,340)	2,660	1,340
June 16	593.60	60	(1,280)	4,060	
June 17	591.80	(360)	(1,640)	3,700	
June 18	592.70	180	(1,460)	3,880	
June 19	587.00	(1,140)	(2,600)	2,740	1,260
June 20	587.00	0	(2,600)	4,000	
June 23	588.10	220	(2,380)	4,220	
June 24	588.70	120	(2,260)	4,340	
June 25	591.00	460	(1,800)	4,800	
June 26	592.30	260	(1,540)	5,060	

Minimum levels for initial and maintenance margins are set by the exchange. Individual brokers may require greater margins from their clients than those specified by the exchange. However, they cannot require lower margins than those specified by the exchange. Margin levels are determined by the variability of the price of the underlying asset. The higher this variability, the higher the margin levels. The maintenance margin is usually about 75% of the initial margin.

Margin requirements may depend on the objectives of the trader. A bona fide hedger, such as a company that produces the commodity on which the futures contract is written, is often subject to lower margin requirements than a speculator. The reason is that there is deemed to be less risk of default. Day trades and spread transactions often give rise to lower margin requirements than do hedge transactions. In a *day trade* the trader announces to the broker an intent to close out the position in the same day. In a *spread transaction* the trader simultaneously buys (i.e., takes a long position in) a contract on an asset for one maturity month and sells (i.e., takes a short position in) a contract on the same asset for another maturity month.

Note that margin requirements are the same on short futures positions as they are on long futures positions. It is just as easy to take a short futures position as it is to take a long one. The spot market does not have this symmetry. Taking a long position in the spot market involves buying the asset for immediate delivery and presents no problems. Taking a short position involves selling an asset that you do not own. This is a more complex transaction that may or may not be possible in a particular market. It is discussed further in Chapter 5.

The Clearinghouse and Clearing Margins

A *clearinghouse* acts as an intermediary in futures transactions. It guarantees the performance of the parties to each transaction. The clearinghouse has a number of members, who must post funds with the exchange. Brokers who are not members themselves must channel their business through a member. The main task of the clearinghouse is to keep track of all the transactions that take place during a day, so that it can calculate the net position of each of its members.

Just as an investor is required to maintain a margin account with a broker, the broker is required to maintain a margin account with a clearinghouse member and the clearinghouse member is required to maintain a margin account with the clearinghouse. The latter is known as a *clearing margin*. The margin accounts for clearinghouse members are adjusted for gains and losses at the end of each trading day in the same way as are the margin accounts of investors. However, in the case of the clearinghouse member, there is an original margin, but no maintenance margin. Every day the account balance for each contract must be maintained at an amount equal to the original margin times the number of contracts outstanding. Thus, depending on transactions during the day and price movements, the clearinghouse member may have to add funds to its margin account at the end of the day. Alternatively, it may find it can remove funds from the account at this time. Brokers who are not clearinghouse members must maintain a margin account with a clearinghouse member.

In determining clearing margins, the exchange clearinghouse calculates the number of contracts outstanding on either a gross or a net basis. When the gross basis is used, the number of contracts equals the sum of the long and short positions. When the net basis is used, these are offset against each other. Suppose a clearinghouse member has two clients: one with a long position in 20 contracts, the other with a short position in 15 contracts. Gross margining would calculate the clearing margin on the basis of 35 contracts; net margining would calculate the clearing margin on the basis of 5 contracts. Most exchanges currently use net margining.

Credit Risk

The whole purpose of the margining system is to eliminate the risk that a trader who makes a profit will not be paid. Overall the system has been very successful. Traders entering into contracts at major exchanges have always had their contracts honored. Futures markets were tested on October 19, 1987, when the S&P 500 index declined by over 20% and traders with long positions in S&P 500 futures found they had negative margin balances. Traders who did not meet margin calls were closed out but still owed their brokers money. Some did not pay and as a result some brokers went bankrupt because, without their clients' money, they were unable to meet margin calls on contracts they entered into on behalf of their clients. However, the exchanges had

Business Snapshot 2.2 Long-Term Capital Management's Big Loss

Long-Term Capital Management (LTCM), a hedge fund formed in the mid-1990s, always collateralized its transactions. The hedge fund's investment strategy was known as convergence arbitrage. A very simple example of what it might do is the following. It would find two bonds, X and Y, issued by the same company that promised the same payoffs, with X being less liquid (i.e., less actively traded) than Y. The market always places a value on liquidity. As a result the price of X would be less than the price of Y. LTCM would buy X, short Y, and wait, expecting the prices of the two bonds to converge at some future time.

When interest rates increased, the company expected both bonds to move down in price by about the same amount, so that the collateral it paid on bond X would be about the same as the collateral it received on bond Y. Similarly, when interest rates decreased, LTCM expected both bonds to move up in price by about the same amount, so that the collateral it received on bond X would be about the same as the collateral it paid on bond Y. It therefore expected that there would be no significant outflow of funds as a result of its collateralization agreements.

In August 1998, Russia defaulted on its debt and this led to what is termed a "flight to quality" in capital markets. One result was that investors valued liquid instruments more highly than usual and the spreads between the prices of the liquid and illiquid instruments in LTCM's portfolio increased dramatically. The prices of the bonds LTCM had bought went down and the prices of those it had shorted increased. It was required to post collateral on both. The company was highly leveraged and unable to make the payments required under the collateralization agreements. The result was that positions had to be closed out and LTCM lost about $4 billion. If the company had been less highly leveraged it would probably have been able to survive the flight to quality and could have waited for the prices of the liquid and illiquid bonds to move closer to each other.

sufficient funds to ensure that everyone who had a short futures position on the S&P 500 got paid off.

Collateralization in OTC Markets

Credit risk has traditionally been a feature of the over-the-counter markets. There is always a chance that the party on the other side of an over-the-counter trade will default. It is interesting that, in an attempt to reduce credit risk, the over-the-counter market is now imitating the margining system adopted by exchanges with a procedure known as *collateralization.*

Consider two participants in the over-the-counter market, company A and company B, with an outstanding over-the-counter contract. They could enter into a collateralization agreement where they value the contract each day. If from one day to the next the value of the contract to company A increases, company B is required to pay company A cash equal to this increase. Similarly, if the value of the contract to company A decreases, company A is required to pay company B cash equal to the decrease. Interest is paid on outstanding cash balances.

Collateralization significantly reduces the credit risk in over-the-counter contracts and is discussed further in Section 22.8. Collateralization agreements were used by a hedge fund, Long-Term Capital Management (LTCM), in the 1990s. They allowed LTCM to be highly leveraged. The contracts did provide credit risk protection, but as described in Business Snapshot 2.2 the high leverage left the hedge fund vulnerable to other risks.

2.5 NEWSPAPER QUOTES

Many newspapers carry futures prices. Table 2.2 shows the prices for commodities as they appeared in the *Wall Street Journal* of Tuesday, January 9, 2007. The prices refer to the trading that took place on the previous day (i.e., Monday, January 8, 2007). The prices for index futures, currency futures, and interest rate futures are given in Chapters 3, 5, and 6, respectively.

The *Wall Street Journal* only shows quotes for contracts with relatively short maturities. For most commodities, contracts trade with much longer maturities than those shown. However, trading volume tends to decrease as contract maturity increases.

The asset underlying the futures contract, the exchange that the contract is traded on, the contract size, and how the price is quoted are all shown at the top of each section in Table 2.2. The first asset is copper, traded on COMEX (a division of the New York Mercantile Exchange). The contract size is 25,000 lbs, and the price is quoted in cents per lb. The maturity month of the contract is shown in the first column.

Prices

The first three numbers in each row show the opening price, the highest price achieved in trading during the day, and the lowest price achieved in trading during the day. The opening price is representative of the prices at which contracts were trading immediately after the opening bell. For March 2007 copper on January 8, 2007, the opening price was 253.50 cents per pound and, during the day, the price traded between 247.00 and 258.95 cents.

Settlement Price

The fourth number is the *settlement price*. This is the price used for calculating daily gains and losses and margin requirements. It is usually calculated as the price at which the contract traded immediately before the bell signaling the end of trading for the day. The fifth number is the change in the settlement price from the previous day. For the March 2007 copper futures contract, the settlement price was 252.80 cents on January 8, 2007, down 0.70 cents from the previous trading day.

In the case of the March 2007 futures, an investor with a long position in one contract would find his or her margin account balance reduced by \$175.00 (= 25,000 × 0.70 cents) on January 8, 2007. Similarly, an investor with a short position in one contract would find that the margin balance increased by \$175.00 on this date.

Table 2.2 Commodity futures quotes from the *Wall Street Journal*, January 9, 2007. (Columns show month, open, high, low, settle, change, and open interest, respectively.)

From platinum to orange juice: futures contracts

Commodity futures prices, including open interest, or the number of contracts outstanding. Nearby-month contracts are listed first. Most-active contracts are also listed, plus other notable months.

KEY TO EXCHANGES: CBT: Chicago Board of Trade; **CME:** Chicago Mercantile Exchange; **CMX:** Comex; **KC:** Kansas City Board of Trade; **MPLS:** Minneapolis Grain Exchange; **NYBOT:** New York Board of Trade; **NYM:** New York Mercantile Exchange, or Nymex

Metal & Petroleum Futures

	Open	High	Contract hi lo	Low	Settle	Chg	Open Interest
Copper-High (CMX)-25,000 lbs.; cents per lb.							
Jan	255.65	256.50		252.00	**251.45**	-0.80	2,527
March	253.50	258.95		247.00	**252.80**	-0.70	48,809
Gold (CMX)-100 troy oz.; $ per troy oz.							
Jan	...	...		...	**607.50**	2.60	514
Feb	609.30	612.40		605.00	**609.40**	2.50	179,246
April	616.10	617.30		611.90	**615.40**	2.50	36,615
June	622.10	623.50		615.80	**621.20**	2.50	31,616
Oct	634.80	635.00		633.50	**632.70**	2.50	19,436
Dec	638.90	640.50		638.00	**638.30**	2.50	42,082
Platinum (NYM)-50 troy oz.; $ per troy oz.							
Jan	...	...		...	**1119.40**	10.40	14
April	1114.00	1132.90		1111.50	**1126.90**	14.90	8,043
Silver (CMX)-5,000 troy oz.; cnts per troy oz.							
Jan	1225.0	1225.0		1225.0	**1226.0**	13.0	115
March	1222.0	1242.5		1209.5	**1236.0**	13.0	60,566
Crude Oil, Light Sweet (NYM)-1,000 bbls.; $ per bbl.							
Feb	56.24	57.72		55.10	**56.09**	-0.22	297,617
March	57.56	58.85		56.38	**57.36**	-0.03	189,021
April	58.30	59.81		57.40	**58.38**	0.03	63,918
June	60.20	61.33		59.08	**60.01**	0.10	78,744
Dec	62.78	64.08		62.10	**62.94**	0.28	143,083
Dec'08	65.25	65.25		63.53	**64.18**	0.33	70,370
Heating Oil No. 2 (NYM)-42,000 gal.; $ per gal.							
Feb	1.5800	1.6020		1.5475	**1.5571**	-.0087	84,979
March	1.6135	1.6385		1.5855	**1.5950**	-.0083	51,391
Gasoline-NY RBOB (NYM)-42,000 gal.; $ per gal.							
Feb	1.5072	1.5185	▼	1.4530	**1.4685**	-.0246	58,858
March	1.5345	1.5634	▼	1.5020	**1.5160**	-.0201	35,902
Natural Gas (NYM)-10,000 MMBtu.; $ per MMBtu.							
Feb	6.370	6.560		6.325	**6.378**	.194	78,114
March	6.547	6.690		6.496	**6.548**	.201	140,874
April	6.600	6.768		6.580	**6.637**	.190	103,520
May	6.750	6.857		6.680	**6.732**	.175	44,710
Oct	7.300	7.350		7.190	**7.237**	.150	39,416
March'08	8.735	8.780		8.680	**8.682**	.110	39,031

Agriculture Futures

	Open	High	Contract hi lo	Low	Settle	Chg	Open Interest
Corn (CBT)-5,000 bu.; cents per bu.							
March	369.00	369.75		361.25	**363.50**	-4.75	570,439
Dec	365.00	367.00		359.50	**364.75**	-.50	318,645
Ethanol (CBT)-29,000 gal.; $ per gal.							
Feb	2.249	2.249		2.249	**2.249**	-.011	56
Oats (CBT)-5,000 bu.; cents per bu.							
March	263.00	263.00		260.00	**260.75**	-1.00	8,412
Dec	232.50	233.00		232.50	**233.00**	...	4,039
Soybeans (CBT)-5,000 bu.; cents per bu.							
Jan	664.50	668.25		662.00	**665.00**	-3.00	5,812
March	682.75	684.00		674.50	**677.25**	-4.25	218,149
Soybean Meal (CBT)-100 tons; $ per ton.							
Jan	190.00	191.80		189.60	**190.60**	-.40	2,614
March	195.70	196.70		194.10	**195.10**	-.80	74,207
Soybean Oil (CBT)-60,000 lbs.; cents per lb.							
Jan	28.18	28.18		27.97	**28.12**	-.28	1,696
March	28.80	28.88		28.34	**28.49**	-.31	144,012
Rough Rice (CBT)-2,000 cwt.; cents per cwt.							
Jan	1024.50	1038.00	▲	1017.00	**1035.00**	14.00	248
March	1048.00	1063.00	▲	1042.00	**1059.50**	13.00	10,406
Wheat (CBT)-5,000 bu.; cents per bu.							
March	470.00	471.25		455.50	**464.00**	-6.25	238,277
July	477.25	478.75		465.00	**473.50**	-3.50	83,574
Wheat (KC)-5,000 bu.; cents per bu.							
March	481.25	481.50		475.00	**479.00**	-2.25	60,413
July	485.00	486.00		479.50	**483.50**	-2.75	32,604
Wheat (MPLS)-5,000 bu.; cents per bu.							
March	487.25	487.25		478.00	**482.25**	-4.75	22,412
Dec	502.00	506.00		501.00	**504.00**	-2.50	8,571
Cattle-Feeder (CME)-50,000 lbs.; cents per lb.							
Jan	98.800	99.500		98.625	**98.875**	.225	4,530
March	97.750	98.500		97.500	**97.850**	.200	14,509
Cattle-Live (CME)-40,000 lbs.; cents per lb.							
Feb	92.600	93.650	▲	92.600	**93.250**	.675	124,905
April	94.300	94.975	▲	94.100	**94.450**	.325	71,613
Hogs-Lean (CME)-40,000 lbs.; cents per lb.							
Feb	60.400	60.900		60.000	**60.300**	-.100	82,727
April	64.250	64.725		63.750	**63.950**	-.300	45,227
Coffee (NYBOT)-37,500 lbs.; cents per lb.							
March	120.25	120.90		119.20	**120.10**	-.35	82,758
May	123.25	123.80		122.00	**123.10**	-.40	20,611
Sugar-World (NYBOT)-112,000 lbs.; cents per lb.							
March	11.20	11.32		11.09	**11.16**	.07	263,326
May	11.23	11.32		11.14	**11.21**	.10	90,874
Sugar-Domestic (NYBOT)-112,000 lbs.; cents per lb.							
March	19.95	19.95		19.95	**19.95**	-.04	3,468
May	19.90	19.90		19.90	**19.89**	-.01	2,592
Cotton (NYBOT)-50,000 lbs.; cents per lb.							
March	54.15	54.80		54.15	**54.53**	.11	108,341
May	55.10	55.55		55.10	**55.38**	.27	24,645
Orange Juice (NYBOT)-15,000 lbs.; cents per lb.							
Jan	201.95	203.40		200.00	**201.90**	-.05	430
March	196.90	197.25		195.50	**195.80**	-.15	21,427

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Open Interest

The final column in Table 2.2 shows the *open interest* for each contract. This is the total number of contracts outstanding. The open interest is the number of long positions or, equivalently, the number of short positions. The open interest for March 2007 copper is shown as 48,809 contracts. Note that the open interest for the January 2007 contract is much lower because most traders who held long or short positions in that contract have already closed out.

Sometimes the volume of trading in a day is greater than the open interest at the end of the day for a contract. This is indicative of a large number of day trades.

Patterns of Futures Prices

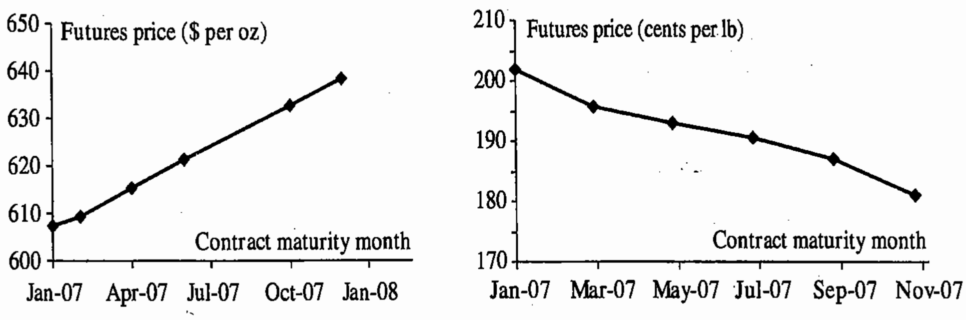
Futures prices can show a number of different patterns. In Table 2.2, gold, crude oil, and natural gas all have settlement prices that increase with the maturity of the contract. This is known as a *normal market*. Table 2.2 shows that the settlement prices for January and March orange juice futures were 201.90 and 195.80 cents, respectively. Other data show that the May 2007, July 2007, September 2007, and November 2007 contracts had settlement prices of 193.00, 190.50, 187.00, and 181.00 cents, respectively, on January 8, 2007. The orange juice futures price was therefore a decreasing function of maturity on January 8, 2007. This is known as an *inverted market*. Figure 2.2 displays the settlement price as a function of maturity for gold and orange juice on January 8, 2007.

Futures prices can show a mixture of normal and inverted markets. An example is provided by live cattle on January 8, 2007. As shown in Table 2.2, the April futures prices was higher than the February futures price. However, the June futures price (not shown) was lower than the April futures price. For later maturities, the futures price continued decreasing, then increased, then decreased again as a function of maturity.

2.6 DELIVERY

As mentioned earlier in this chapter, very few of the futures contracts that are entered into lead to delivery of the underlying asset. Most are closed out early. Nevertheless, it

Figure 2.2 Settlement futures price as a function of contract maturity on January 8, 2007, for (a) gold and (b) orange juice



(a) Gold (b) Orange juice

is the possibility of eventual delivery that determines the futures price. An understanding of delivery procedures is therefore important.

The period during which delivery can be made is defined by the exchange and varies from contract to contract. The decision on when to deliver is made by the party with the short position, whom we shall refer to as investor A. When investor A decides to deliver, investor A's broker issues a notice of intention to deliver to the exchange clearinghouse. This notice states how many contracts will be delivered and, in the case of commodities, also specifies where delivery will be made and what grade will be delivered. The exchange then chooses a party with a long position to accept delivery.

Suppose that the party on the other side of investor A's futures contract when it was entered into was investor B. It is important to realize that there is no reason to expect that it will be investor B who takes delivery. Investor B may well have closed out his or her position by trading with investor C, investor C may have closed out his or her position by trading with investor D, and so on. The usual rule chosen by the exchange is to pass the notice of intention to deliver on to the party with the oldest outstanding long position. Parties with long positions must accept delivery notices. However, if the notices are transferable, long investors have a short period of time, usually half an hour, to find another party with a long position that is prepared to accept the notice from them.

In the case of a commodity, taking delivery usually means accepting a warehouse receipt in return for immediate payment. The party taking delivery is then responsible for all warehousing costs. In the case of livestock futures, there may be costs associated with feeding and looking after the animals (see Business Snapshot 2.1). In the case of financial futures, delivery is usually made by wire transfer. For all contracts, the price paid is usually the most recent settlement price. If specified by the exchange, this price is adjusted for grade, location of delivery, and so on. The whole delivery procedure from the issuance of the notice of intention to deliver to the delivery itself generally takes about two to three days.

There are three critical days for a contract. These are the first notice day, the last notice day, and the last trading day. The *first notice day* is the first day on which a notice of intention to make delivery can be submitted to the exchange. The *last notice day* is the last such day. The *last trading day* is generally a few days before the last notice day. To avoid the risk of having to take delivery, an investor with a long position should close out his or her contracts prior to the first notice day.

Cash Settlement

Some financial futures, such as those on stock indices, are settled in cash because it is inconvenient or impossible to deliver the underlying asset. In the case of the futures contract on the S&P 500, for example, delivering the underlying asset would involve delivering a portfolio of 500 stocks. When a contract is settled in cash, all outstanding contracts are declared closed on a predetermined day. The final settlement price is set equal to the spot price of the underlying asset at either the opening or close of trading on that day. For example, in the S&P 500 futures contract trading on the Chicago Mercantile Exchange, the predetermined day is the third Friday of the delivery month and final settlement is at the opening price.

2.7 TYPES OF TRADERS AND TYPES OF ORDERS

There are two main types of traders executing trades: commission brokers and locals. *Commission brokers* are following the instructions of their clients and charge a commission for doing so; *locals* are trading on their own account.

Individuals taking positions, whether locals or the clients of commission brokers, can be categorized as hedgers, speculators, or arbitrageurs, as discussed in Chapter 1. Speculators can be classified as scalpers, day traders, or position traders. *Scalpers* are watching for very short-term trends and attempt to profit from small changes in the contract price. They usually hold their positions for only a few minutes. *Day traders* hold their positions for less than one trading day. They are unwilling to take the risk that adverse news will occur overnight. *Position traders* hold their positions for much longer periods of time. They hope to make significant profits from major movements in the markets.

Orders

The simplest type of order placed with a broker is a *market order*. It is a request that a trade be carried out immediately at the best price available in the market. However, there are many other types of orders. We will consider those that are more commonly used.

A *limit order* specifies a particular price. The order can be executed only at this price or at one more favorable to the investor. Thus, if the limit price is \$30 for an investor wanting to buy, the order will be executed only at a price of \$30 or less. There is, of course, no guarantee that the order will be executed at all, because the limit price may never be reached.

A *stop order* or *stop-loss order* also specifies a particular price. The order is executed at the best available price once a bid or offer is made at that particular price or a less-favorable price. Suppose a stop order to sell at \$30 is issued when the market price is \$35. It becomes an order to sell when and if the price falls to \$30. In effect, a stop order becomes a market order as soon as the specified price has been hit. The purpose of a stop order is usually to close out a position if unfavorable price movements take place. It limits the loss that can be incurred.

A *stop–limit order* is a combination of a stop order and a limit order. The order becomes a limit order as soon as a bid or offer is made at a price equal to or less favorable than the stop price. Two prices must be specified in a stop–limit order: the stop price and the limit price. Suppose that at the time the market price is \$35, a stop–limit order to buy is issued with a stop price of \$40 and a limit price of \$41. As soon as there is a bid or offer at \$40, the stop–limit becomes a limit order at \$41. If the stop price and the limit price are the same, the order is sometimes called a *stop-and-limit order*.

A *market-if-touched* (MIT) *order* is executed at the best available price after a trade occurs at a specified price or at a price more favorable than the specified price. In effect, an MIT becomes a market order once the specified price has been hit. An MIT is also known as a *board order*. Consider an investor who has a long position in a futures contract and is issuing instructions that would lead to closing out the contract. A stop order is designed to place a limit on the loss that can occur in the event of unfavorable price movements. By contrast, a market-if-touched order is designed to ensure that profits are taken if sufficiently favorable price movements occur.

A *discretionary order* or *market-not-held order* is traded as a market order except that execution may be delayed at the broker's discretion in an attempt to get a better price.

Some orders specify time conditions. Unless otherwise stated, an order is a day order and expires at the end of the trading day. A *time-of-day order* specifies a particular period of time during the day when the order can be executed. An *open order* or a *good-till-canceled order* is in effect until executed or until the end of trading in the particular contract. A *fill-or-kill order*, as its name implies, must be executed immediately on receipt or not at all.

2.8 REGULATION

Futures markets in the United States are currently regulated federally by the Commodity Futures Trading Commission (CFTC; www.cftc.gov), which was established in 1974. This body is responsible for licensing futures exchanges and approving contracts. All new contracts and changes to existing contracts must be approved by the CFTC. To be approved, the contract must have some useful economic purpose. Usually this means that it must serve the needs of hedgers as well as speculators.

The CFTC looks after the public interest. It is responsible for ensuring that prices are communicated to the public and that futures traders report their outstanding positions if they are above certain levels. The CFTC also licenses all individuals who offer their services to the public in futures trading. The backgrounds of these individuals are investigated, and there are minimum capital requirements. The CFTC deals with complaints brought by the public and ensures that disciplinary action is taken against individuals when appropriate. It has the authority to force exchanges to take disciplinary action against members who are in violation of exchange rules.

With the formation of the National Futures Association (NFA; www.nfa.futures.org) in 1982, some of responsibilities of the CFTC were shifted to the futures industry itself. The NFA is an organization of individuals who participate in the futures industry. Its objective is to prevent fraud and to ensure that the market operates in the best interests of the general public. It is authorized to monitor trading and take disciplinary action when appropriate. The agency has set up an efficient system for arbitrating disputes between individuals and its members.

From time to time, other bodies, such as the Securities and Exchange Commission (SEC; www.sec.gov), the Federal Reserve Board (www.federalreserve.gov), and the US Treasury Department (www.treas.gov), have claimed jurisdictional rights over some aspects of futures trading. These bodies are concerned with the effects of futures trading on the spot markets for securities such as stocks, Treasury bills, and Treasury bonds. The SEC currently has an effective veto over the approval of new stock or bond index futures contracts. However, the basic responsibility for all futures and options on futures rests with the CFTC.

Trading Irregularities

Most of the time futures markets operate efficiently and in the public interest. However, from time to time, trading irregularities do come to light. One type of trading irregularity occurs when an investor group tries to "corner the market".[2] The investor group takes a huge long futures position and also tries to exercise some control over the supply of the underlying commodity. As the maturity of the futures contracts is approached, the investor group does not close out its position, so that the number of outstanding futures contracts may exceed the amount of the commodity available for delivery. The holders of short positions realize that they will find it difficult to deliver and become desperate to close out their positions. The result is a large rise in both futures and spot prices. Regulators usually deal with this type of abuse of the market by increasing margin requirements or imposing stricter position limits or prohibiting trades that increase a speculator's open position or requiring market participants to close out their positions.

Other types of trading irregularity can involve the traders on the floor of the exchange. These received some publicity early in 1989, when it was announced that the FBI had carried out a two-year investigation, using undercover agents, of trading on the Chicago Board of Trade and the Chicago Mercantile Exchange. The investigation was initiated because of complaints filed by a large agricultural concern. The alleged offenses included overcharging customers, not paying customers the full proceeds of sales, and traders using their knowledge of customer orders to trade first for themselves (an offence known as *front running*).

2.9 ACCOUNTING AND TAX

The full details of the accounting and tax treatment of futures contracts are beyond the scope of this book. A trader who wants detailed information on this should consult experts. In this section we provide some general background information.

Accounting

Accounting standards require changes in the market value of a futures contract to be recognized when they occur unless the contract qualifies as a hedge. If the contract does qualify as a hedge, gains or losses are generally recognized for accounting purposes in the same period in which the gains or losses from the item being hedged are recognized. The latter treatment is referred to as *hedge accounting*.

Consider a company with a December year end. In September 2007 it buys a March 2008 corn futures contract and closes out the position at the end of February 2008. Suppose that the futures prices are 250 cents per bushel when the contract is entered into, 270 cents per bushel at the end of 2007, and 280 cents per bushel when the contract is closed out. The contract is for the delivery of 5,000 bushels. If the contract does not qualify as a hedge, the gains for accounting purposes are

$$5{,}000 \times (2.70 - 2.50) = \$1{,}000$$

in 2007 and

$$5{,}000 \times (2.80 - 2.70) = \$500$$

[2] Possibly the best known example of this was the attempt by the Hunt brothers to corner the silver market in 1979–80. Between the middle of 1979 and the beginning of 1980, their activities led to a price rise from \$9 per ounce to \$50 per ounce.

in 2008. If the company is hedging the purchase of 5,000 bushels of corn in February 2008 so that the contract qualifies for hedge accounting, the entire gain of $1,500 is realized in 2008 for accounting purposes.

The treatment of hedging gains and losses is sensible. If the company is hedging the purchase of 5,000 bushels of corn in February 2008, the effect of the futures contract is to ensure that the price paid is close to 250 cents per bushel. The accounting treatment reflects that this price is paid in 2008.

In June 1998, the Financial Accounting Standards Board issued FASB Statement No. 133 (FAS 133), Accounting for Derivative Instruments and Hedging Activities. FAS 133 applies to all types of derivatives (including futures, forwards, swaps, and options). It requires all derivatives to be included on the balance sheet at fair market value.[3] It increases disclosure requirements. It also gives companies far less latitude than previously in using hedge accounting. For hedge accounting to be used, the hedging instrument must be highly effective in offsetting exposures and an assessment of this effectiveness is required every three months. A similar standard IAS 39 has been issued by the International Accounting Standards Board.

Tax

Under the US tax rules, two key issues are the nature of a taxable gain or loss and the timing of the recognition of the gain or loss. Gains or losses are either classified as capital gains or losses or alternatively as part of ordinary income.

For a corporate taxpayer, capital gains are taxed at the same rate as ordinary income, and the ability to deduct losses is restricted. Capital losses are deductible only to the extent of capital gains. A corporation may carry back a capital loss for three years and carry it forward for up to five years. For a noncorporate taxpayer, short-term capital gains are taxed at the same rate as ordinary income, but long-term capital gains are subject to a maximum capital gains tax rate of 15%. (Long-term capital gains are gains from the sale of a capital asset held for longer than one year; short-term capital gains are the gains from the sale of a capital asset held one year or less.) For a noncorporate taxpayer, capital losses are deductible to the extent of capital gains plus ordinary income up to $3,000 and can be carried forward indefinitely.

Generally, positions in futures contracts are treated as if they are closed out on the last day of the tax year. For the noncorporate taxpayer, this gives rise to capital gains and losses that are treated as if they were 60% long term and 40% short term without regard to the holding period. This is referred to as the "60/40" rule. A noncorporate taxpayer may elect to carry back for three years any net losses from the 60/40 rule to offset any gains recognized under the rule in the previous three years.

Hedging transactions are exempt from this rule. The definition of a hedge transaction for tax purposes is different from that for accounting purposes. The tax regulations define a hedging transaction as a transaction entered into in the normal course of business primarily for one of the following reasons:

1. To reduce the risk of price changes or currency fluctuations with respect to property that is held or to be held by the taxpayer for the purposes of producing ordinary income

[3] Previously the attraction of derivatives in some situations was that they were "off-balance-sheet" items.

2. To reduce the risk of price or interest rate changes or currency fluctuations with respect to borrowings made by the taxpayer

The hedging transaction must be identified before the end of the day on which the taxpayer enters into the transaction. The asset being hedged must be identified within 35 days. Gains or losses from hedging transactions are treated as ordinary income. The timing of the recognition of gains or losses from hedging transactions generally matches the timing of the recognition of income or expense associated with the transaction being hedged.

2.10 FORWARD vs. FUTURES CONTRACTS

The main differences between forward and futures contracts are summarized in Table 2.3. Both contracts are agreements to buy or sell an asset for a certain price at a certain future time. A forward contract is traded in the over-the-counter market and there is no standard contract size or standard delivery arrangements. A single delivery date is usually specified and the contract is usually held to the end of its life and then settled. A futures contract is a standardized contract traded on an exchange. A range of delivery dates is usually specified. It is settled daily and usually closed out prior to maturity.

Profits from Forward and Futures Contracts

Suppose that the sterling exchange rate for a 90-day forward contract is 1.9000 and that this rate is also the futures price for a contract that will be delivered in exactly 90 days. What is the difference between the gains and losses under the two contracts?

Under the forward contract, the whole gain or loss is realized at the end of the life of the contract. Under the futures contract, the gain or loss is realized day by day because of the daily settlement procedures. Suppose that investor A is long £1 million in a 90-day forward contract and investor B is long £1 million in 90-day futures contracts. (Because each futures contract is for the purchase or sale of £62,500, investor B must purchase a total of 16 contracts.) Assume that the spot exchange rate in 90 days proves to be 2.1000 dollars per pound. Investor A makes a gain of

Table 2.3 Comparison of forward and futures contracts.

Forward	*Futures*
Private contract between two parties	Traded on an exchange
Not standardized	Standardized contract
Usually one specified delivery date	Range of delivery dates
Settled at end of contract	Settled daily
Delivery or final cash settlement usually takes place	Contract is usually closed out prior to maturity
Some credit risk	Virtually no credit risk

$200,000 on the 90th day. Investor B makes the same gain—but spread out over the 90-day period. On some days investor B may realize a loss, whereas on other days he or she makes a gain. However, in total, when losses are netted against gains, there is a gain of $200,000 over the 90-day period.

Foreign Exchange Quotes

Both forward and futures contracts trade actively on foreign currencies. However, there is sometimes a difference in the way exchange rates are quoted in the two markets. For example, futures prices where one currency is the US dollar are always quoted as the number of US dollars per unit of the foreign currency or as the number of US cents per unit of the foreign currency. Forward prices are always quoted in the same way as spot prices. This means that, for the British pound, the euro, the Australian dollar, and the New Zealand dollar, the forward quotes show the number of US dollars per unit of the foreign currency and are directly comparable with futures quotes. For other major currencies, forward quotes show the number of units of the foreign currency per US dollar (USD). Consider the Canadian dollar (CAD). A futures price quote of 0.9500 USD per CAD corresponds to a forward price quote of 1.0526 CAD per USD ($1.0526 = 1/0.9500$).

SUMMARY

A very high proportion of the futures contracts that are traded do not lead to the delivery of the underlying asset. Traders usually enter into offsetting contracts to close out their positions before the delivery period is reached. However, it is the possibility of final delivery that drives the determination of the futures price. For each futures contract, there is a range of days during which delivery can be made and a well-defined delivery procedure. Some contracts, such as those on stock indices, are settled in cash rather than by delivery of the underlying asset.

The specification of contracts is an important activity for a futures exchange. The two sides to any contract must know what can be delivered, where delivery can take place, and when delivery can take place. They also need to know details on the trading hours, how prices will be quoted, maximum daily price movements, and so on. New contracts must be approved by the Commodity Futures Trading Commission before trading starts.

Margins are an important aspect of futures markets. An investor keeps a margin account with his or her broker. The account is adjusted daily to reflect gains or losses, and from time to time the broker may require the account to be topped up if adverse price movements have taken place. The broker either must be a clearinghouse member or must maintain a margin account with a clearinghouse member. Each clearinghouse member maintains a margin account with the exchange clearinghouse. The balance in the account is adjusted daily to reflect gains and losses on the business for which the clearinghouse member is responsible.

Information on futures prices is collected in a systematic way at exchanges and relayed within a matter of seconds to investors throughout the world. Many daily newspapers such as the *Wall Street Journal* carry a summary of the previous day's trading.

Forward contracts differ from futures contracts in a number of ways. Forward contracts are private arrangements between two parties, whereas futures contracts are traded on exchanges. There is generally a single delivery date in a forward contract, whereas futures contracts frequently involve a range of such dates. Because they are not traded on exchanges, forward contracts do not need to be standardized. A forward contract is not usually settled until the end of its life, and most contracts do in fact lead to delivery of the underlying asset or a cash settlement at this time.

In the next few chapters we shall examine in more detail the ways in which forward and futures contracts can be used for hedging. We shall also look at how forward and futures prices are determined.

FURTHER READING

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Jorion, P. "Risk Management Lessons from Long-Term Capital Management," *European Financial Management*, 6, 3 (September 2000): 277–300.

Kawaller, I. G., and P. D. Koch. "Meeting the Highly Effective Expectation Criterion for Hedge Accounting," *Journal of Derivatives*, 7, 4 (Summer 2000): 79–87.

Lowenstein, R. *When Genius Failed: The Rise and Fall of Long-Term Capital Management*. New York: Random House, 2000.

Questions and Problems (Answers in Solutions Manual)

2.1. Distinguish between the terms *open interest* and *trading volume*.

2.2. What is the difference between a *local* and a *commission broker*?

2.3. Suppose that you enter into a short futures contract to sell July silver for \$10.20 per ounce on the New York Commodity Exchange. The size of the contract is 5,000 ounces. The initial margin is \$4,000, and the maintenance margin is \$3,000. What change in the futures price will lead to a margin call? What happens if you do not meet the margin call?

2.4. Suppose that in September 2009 a company takes a long position in a contract on May 2010 crude oil futures. It closes out its position in March 2010. The futures price (per barrel) is \$68.30 when it enters into the contract, \$70.50 when it closes out its position, and \$69.10 at the end of December 2009. One contract is for the delivery of 1,000 barrels. What is the company's total profit? When is it realized? How is it taxed if it is (a) a hedger and (b) a speculator? Assume that the company has a December 31 year-end.

2.5. What does a stop order to sell at \$2 mean? When might it be used? What does a limit order to sell at \$2 mean? When might it be used?

2.6. What is the difference between the operation of the margin accounts administered by a clearinghouse and those administered by a broker?

2.7. What differences exist in the way prices are quoted in the foreign exchange futures market, the foreign exchange spot market, and the foreign exchange forward market?

2.8. The party with a short position in a futures contract sometimes has options as to the precise asset that will be delivered, where delivery will take place, when delivery will take place, and so on. Do these options increase or decrease the futures price? Explain your reasoning.

2.9. What are the most important aspects of the design of a new futures contract?

2.10. Explain how margins protect investors against the possibility of default.

2.11. A trader buys two July futures contracts on orange juice. Each contract is for the delivery of 15,000 pounds. The current futures price is 160 cents per pound, the initial margin is \$6,000 per contract, and the maintenance margin is \$4,500 per contract. What price change would lead to a margin call? Under what circumstances could \$2,000 be withdrawn from the margin account?

2.12. Show that, if the futures price of a commodity is greater than the spot price during the delivery period, then there is an arbitrage opportunity. Does an arbitrage opportunity exist if the futures price is less than the spot price? Explain your answer.

2.13. Explain the difference between a market-if-touched order and a stop order.

2.14. Explain what a stop–limit order to sell at 20.30 with a limit of 20.10 means.

2.15. At the end of one day a clearinghouse member is long 100 contracts, and the settlement price is \$50,000 per contract. The original margin is \$2,000 per contract. On the following day the member becomes responsible for clearing an additional 20 long contracts, entered into at a price of \$51,000 per contract. The settlement price at the end of this day is \$50,200. How much does the member have to add to its margin account with the exchange clearinghouse?

2.16. On July 1, 2009, a Japanese company enters into a forward contract to buy \$1 million on January 1, 2010. On September 1, 2009, it enters into a forward contract to sell \$1 million on January 1, 2010. Describe the profit or loss the company will make in yen as a function of the forward exchange rates on July 1, 2009, and September 1, 2009.

2.17. The forward price of the Swiss franc for delivery in 45 days is quoted as 1.2500. The futures price for a contract that will be delivered in 45 days is 0.7980. Explain these two quotes. Which is more favorable for an investor wanting to sell Swiss francs?

2.18. Suppose you call your broker and issue instructions to sell one July hogs contract. Describe what happens.

2.19. "Speculation in futures markets is pure gambling. It is not in the public interest to allow speculators to trade on a futures exchange." Discuss this viewpoint.

2.20. Identify the commodities whose futures contracts have the highest open interest in Table 2.2.

2.21. What do you think would happen if an exchange started trading a contract in which the quality of the underlying asset was incompletely specified?

2.22. "When a futures contract is traded on the floor of the exchange, it may be the case that the open interest increases by one, stays the same, or decreases by one." Explain this statement.

2.23. Suppose that, on October 24, 2009, a company sells one April 2010 live cattle futures contract. It closes out its position on January 21, 2010. The futures price (per pound) is 91.20 cents when it enters into the contract, 88.30 cents when it closes out its position, and 88.80 cents at the end of December 2009. One contract is for the delivery of 40,000 pounds of cattle. What is the total profit? How is it taxed if the company is (a) a hedger and (b) a speculator? Assume that the company has a December 31 year-end.

2.24. A cattle farmer expects to have 120,000 pounds of live cattle to sell in 3 months. The live cattle futures contract on the Chicago Mercantile Exchange is for the delivery of 40,000 pounds of cattle. How can the farmer use the contract for hedging? From the farmer's viewpoint, what are the pros and cons of hedging?

2.25. It is July 2008. A mining company has just discovered a small deposit of gold. It will take 6 months to construct the mine. The gold will then be extracted on a more or less continuous basis for 1 year. Futures contracts on gold are available on the New York Commodity Exchange. There are delivery months every 2 months from August 2008 to December 2009. Each contract is for the delivery of 100 ounces. Discuss how the mining company might use futures markets for hedging.

Assignment Questions

2.26. A company enters into a short futures contract to sell 5,000 bushels of wheat for 450 cents per bushel. The initial margin is $3,000 and the maintenance margin is $2,000. What price change would lead to a margin call? Under what circumstances could $1,500 be withdrawn from the margin account?

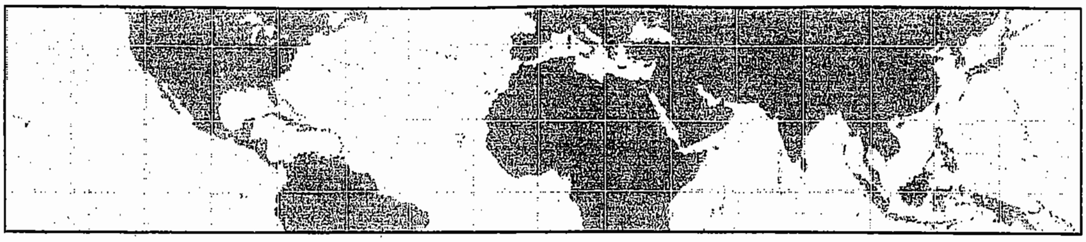
2.27. Suppose that there are no storage costs for crude oil and the interest rate for borrowing or lending is 5% per annum. How could you make money on January 8, 2007, by trading June 2007 and December 2007 contracts? Use Table 2.2.

2.28. What position is equivalent to a long forward contract to buy an asset at K on a certain date and a put option to sell it for K on that date.

2.29. The author's Web page (www.rotman.utoronto.ca/~hull/data) contains daily closing prices for crude oil and gold futures contracts. (Both contracts are traded on NYMEX.) You are required to download the data and answer the following:

(a) How high do the maintenance margin levels for oil and gold have to be set so that there is a 1% chance that an investor with a balance slightly above the maintenance margin level on a particular day has a negative balance 2 days later? How high do they have to be for a 0.1% chance? Assume daily price changes are normally distributed with mean zero. Explain why NYMEX might be interested in this calculation.

(b) Imagine an investor who starts with a long position in the oil contract at the beginning of the period covered by the data and keeps the contract for the whole of the period of time covered by the data. Margin balances in excess of the initial margin are withdrawn. Use the maintenance margin you calculated in part (a) for a 1% risk level and assume that the maintenance margin is 75% of the initial margin. Calculate the number of margin calls and the number of times the investor has a negative margin balance. Assume that all margin calls are met in your calculations. Repeat the calculations for an investor who starts with a short position in the gold contract.



CHAPTER 3

Hedging Strategies Using Futures

Many of the participants in futures markets are hedgers. Their aim is to use futures markets to reduce a particular risk that they face. This risk might relate to fluctuations in the price of oil, a foreign exchange rate, the level of the stock market, or some other variable. A *perfect hedge* is one that completely eliminates the risk. Perfect hedges are rare. For the most part, therefore, a study of hedging using futures contracts is a study of the ways in which hedges can be constructed so that they perform as close to perfect as possible.

In this chapter we consider a number of general issues associated with the way hedges are set up. When is a short futures position appropriate? When is a long futures position appropriate? Which futures contract should be used? What is the optimal size of the futures position for reducing risk? At this stage, we restrict our attention to what might be termed *hedge-and-forget* strategies. We assume that no attempt is made to adjust the hedge once it has been put in place. The hedger simply takes a futures position at the beginning of the life of the hedge and closes out the position at the end of the life of the hedge. In Chapter 17 we will examine dynamic hedging strategies in which the hedge is monitored closely and frequent adjustments are made.

The chapter initially treats futures contracts as forward contracts (that is, it ignores daily settlement). Later it explains an adjustment known as "tailing" that takes account of the difference between futures and forwards.

3.1 BASIC PRINCIPLES

When an individual or company chooses to use futures markets to hedge a risk, the objective is usually to take a position that neutralizes the risk as far as possible. Consider a company that knows it will gain \$10,000 for each 1 cent increase in the price of a commodity over the next 3 months and lose \$10,000 for each 1 cent decrease in the price during the same period. To hedge, the company's treasurer should take a short futures position that is designed to offset this risk. The futures position should lead to a loss of \$10,000 for each 1 cent increase in the price of the commodity over the 3 months and a gain of \$10,000 for each 1 cent decrease in the price during this period. If the price of the commodity goes down, the gain on the futures position offsets the loss on the rest of the company's business. If the price of the commodity goes up, the loss on the futures position is offset by the gain on the rest of the company's business.

Short Hedges

A *short hedge* is a hedge, such as the one just described, that involves a short position in futures contracts. A short hedge is appropriate when the hedger already owns an asset and expects to sell it at some time in the future. For example, a short hedge could be used by a farmer who owns some hogs and knows that they will be ready for sale at the local market in two months. A short hedge can also be used when an asset is not owned right now but will be owned at some time in the future. Consider, for example, a US exporter who knows that he or she will receive euros in 3 months. The exporter will realize a gain if the euro increases in value relative to the US dollar and will sustain a loss if the euro decreases in value relative to the US dollar. A short futures position leads to a loss if the euro increases in value and a gain if it decreases in value. It has the effect of offsetting the exporter's risk.

To provide a more detailed illustration of the operation of a short hedge in a specific situation, we assume that it is May 15 today and that an oil producer has just negotiated a contract to sell 1 million barrels of crude oil. It has been agreed that the price that will apply in the contract is the market price on August 15. The oil producer is therefore in the position where it will gain \$10,000 for each 1 cent increase in the price of oil over the next 3 months and lose \$10,000 for each 1 cent decrease in the price during this period. Suppose that on May 15 the spot price is \$60 per barrel and the crude oil futures price on the New York Mercantile Exchange (NYMEX) for August delivery is \$59 per barrel. Because each futures contract on NYMEX is for the delivery of 1,000 barrels, the company can hedge its exposure by shorting 1,000 futures contracts. If the oil producer closes out its position on August 15, the effect of the strategy should be to lock in a price close to \$59 per barrel.

To illustrate what might happen, suppose that the spot price on August 15 proves to be \$55 per barrel. The company realizes \$55 million for the oil under its sales contract. Because August is the delivery month for the futures contract, the futures price on August 15 should be very close to the spot price of \$55 on that date. The company therefore gains approximately

$$\$59 - \$55 = \$4$$

per barrel, or \$4 million in total from the short futures position. The total amount realized from both the futures position and the sales contract is therefore approximately \$59 per barrel, or \$59 million in total.

For an alternative outcome, suppose that the price of oil on August 15 proves to be \$65 per barrel. The company realizes \$65 for the oil and loses approximately

$$\$65 - \$59 = \$6$$

per barrel on the short futures position. Again, the total amount realized is approximately \$59 million. It is easy to see that in all cases the company ends up with approximately \$59 million.

Long Hedges

Hedges that involve taking a long position in a futures contract are known as *long hedges*. A long hedge is appropriate when a company knows it will have to purchase a certain asset in the future and wants to lock in a price now.

Suppose that it is now January 15. A copper fabricator knows it will require 100,000 pounds of copper on May 15 to meet a certain contract. The spot price of copper is 340 cents per pound, and the futures price for May delivery is 320 cents per pound. The fabricator can hedge its position by taking a long position in four futures contracts on the COMEX division of NYMEX and closing its position on May 15. Each contract is for the delivery of 25,000 pounds of copper. The strategy has the effect of locking in the price of the required copper at close to 320 cents per pound.

Suppose that the spot price of copper on May 15 proves to be 325 cents per pound. Because May is the delivery month for the futures contract, this should be very close to the futures price. The fabricator therefore gains approximately

$$100{,}000 \times (\$3.25 - \$3.20) = \$5{,}000$$

on the futures contracts. It pays $100{,}000 \times \$3.25 = \$325{,}000$ for the copper, making the net cost approximately $\$325{,}000 - \$5{,}000 = \$320{,}000$. For an alternative outcome, suppose that the spot price is 305 cents per pound on May 15. The fabricator then loses approximately

$$100{,}000 \times (\$3.20 - \$3.05) = \$15{,}000$$

on the futures contract and pays $100{,}000 \times \$3.05 = \$305{,}000$ for the copper. Again, the net cost is approximately \$320,000, or 320 cents per pound.

Note that it is better for the company to use futures contracts than to buy the copper on January 15 in the spot market. If it does the latter, it will pay 340 cents per pound instead of 320 cents per pound and will incur both interest costs and storage costs. For a company using copper on a regular basis, this disadvantage would be offset by the convenience of having the copper on hand.[1] However, for a company that knows it will not require the copper until May 15, the futures contract alternative is likely to be preferred.

Long hedges can be used to manage an existing short position. Consider an investor who has shorted a certain stock. (See Section 5.2 for a discussion of shorting.) Part of the risk faced by the investor is related to the performance of the whole stock market. The investor can neutralize this risk with a long position in index futures contracts. This type of hedging strategy is discussed further later in the chapter.

The examples we have looked at assume that the futures position is closed out in the delivery month. The hedge has the same basic effect if delivery is allowed to happen. However, making or taking delivery can be costly and inconvenient. For this reason, delivery is not usually made even when the hedger keeps the futures contract until the delivery month. As will be discussed later, hedgers with long positions usually avoid any possibility of having to take delivery by closing out their positions before the delivery period.

We have also assumed in the two examples that there is no daily settlement. In practice, daily settlement does have a small effect on the performance of a hedge. As

[1] See Section 5.11 for a discussion of convenience yields.

explained in Chapter 2, it means that the payoff from the futures contract is realized day by day throughout the life of the hedge rather than all at the end.

3.2 ARGUMENTS FOR AND AGAINST HEDGING

The arguments in favor of hedging are so obvious that they hardly need to be stated. Most companies are in the business of manufacturing, or retailing or wholesaling, or providing a service. They have no particular skills or expertise in predicting variables such as interest rates, exchange rates, and commodity prices. It makes sense for them to hedge the risks associated with these variables as they arise. The companies can then focus on their main activities—for which presumably they do have particular skills and expertise. By hedging, they avoid unpleasant surprises such as sharp rises in the price of a commodity that is being purchased.

In practice, many risks are left unhedged. In the rest of this section we will explore some of the reasons.

Hedging and Shareholders

One argument sometimes put forward is that the shareholders can, if they wish, do the hedging themselves. They do not need the company to do it for them. This argument is, however, open to question. It assumes that shareholders have as much information about the risks faced by a company as does the company's management. In most instances, this is not the case. The argument also ignores commissions and other transactions costs. These are less expensive per dollar of hedging for large transactions than for small transactions. Hedging is therefore likely to be less expensive when carried out by the company than when it is carried out by individual shareholders. Indeed, the size of futures contracts makes hedging by individual shareholders impossible in many situations.

One thing that shareholders can do far more easily than a corporation is diversify risk. A shareholder with a well-diversified portfolio may be immune to many of the risks faced by a corporation. For example, in addition to holding shares in a company that uses copper, a well-diversified shareholder may hold shares in a copper producer, so that there is very little overall exposure to the price of copper. If companies are acting in the best interests of well-diversified shareholders, it can be argued that hedging is unnecessary in many situations. However, the extent to which managers are in practice influenced by this type of argument is open to question.

Hedging and Competitors

If hedging is not the norm in a certain industry, it may not make sense for one particular company to choose to be different from all others. Competitive pressures within the industry may be such that the prices of the goods and services produced by the industry fluctuate to reflect raw material costs, interest rates, exchange rates, and so on. A company that does not hedge can expect its profit margins to be roughly constant. However, a company that does hedge can expect its profit margins to fluctuate!

To illustrate this point, consider two manufacturers of gold jewelry, SafeandSure

Table 3.1 Danger in hedging when competitors do not hedge.

Change in gold price	*Effect on price of gold jewelry*	*Effect on profits of TakeaChance Co.*	*Effect on profits of SafeandSure Co.*
Increase	Increase	None	Increase
Decrease	Decrease	None	Decrease

Company and TakeaChance Company. We assume that most companies in the industry do not hedge against movements in the price of gold and that TakeaChance Company is no exception. However, SafeandSure Company has decided to be different from its competitors and to use futures contracts to hedge its purchase of gold over the next 18 months. If the price of gold goes up, economic pressures will tend to lead to a corresponding increase in the wholesale price of the jewelry, so that TakeaChance Company's gross profit margin is unaffected. By contrast, SafeandSure Company's profit margin will increase after the effects of the hedge have been taken into account. If the price of gold goes down, economic pressures will tend to lead to a corresponding decrease in the wholesale price of the jewelry. Again, TakeaChance Company's profit margin is unaffected. However, SafeandSure Company's profit margin goes down. In extreme conditions, SafeandSure Company's profit margin could become negative as a result of the "hedging" carried out! The situation is summarized in Table 3.1.

This example emphasizes the importance of looking at the big picture when hedging. All the implications of price changes on a company's profitability should be taken into account in the design of a hedging strategy to protect against the price changes.

Hedging Can Lead to a Worse Outcome

It is important to realize that a hedge using futures contracts can result in a decrease or an increase in a company's profits relative to the position it would be in with no hedging. In the example involving the oil producer considered earlier, if the price of oil goes down, the company loses money on its sale of 1 million barrels of oil, and the futures position leads to an offsetting gain. The treasurer can be congratulated for having had the foresight to put the hedge in place. Clearly, the company is better off than it would be with no hedging. Other executives in the organization, it is hoped, will appreciate the contribution made by the treasurer. If the price of oil goes up, the company gains from its sale of the oil, and the futures position leads to an offsetting loss. The company is in a worse position than it would be with no hedging. Although the hedging decision was perfectly logical, the treasurer may in practice have a difficult time justifying it. Suppose that the price of oil at the end of the hedge is $69, so that the company loses $10 per barrel on the futures contract. We can imagine a conversation such as the following between the treasurer and the president:

President: This is terrible. We've lost $10 million in the futures market in the space of three months. How could it happen? I want a full explanation.

Treasurer: The purpose of the futures contracts was to hedge our exposure to the price of oil, not to make a profit. Don't forget we made $10 million from the favorable effect of the oil price increases on our business.

Business Snapshot 3.1 Hedging by Gold Mining Companies

It is natural for a gold mining company to consider hedging against changes in the price of gold. Typically it takes several years to extract all the gold from a mine. Once a gold mining company decides to go ahead with production at a particular mine, it has a big exposure to the price of gold. Indeed a mine that looks profitable at the outset could become unprofitable if the price of gold plunges.

Gold mining companies are careful to explain their hedging strategies to potential shareholders. Some gold mining companies do not hedge. They tend to attract shareholders who buy gold stocks because they want to benefit when the price of gold increases and are prepared to accept the risk of a loss from a decrease in the price of gold. Other companies choose to hedge. They estimate the number of ounces they will produce each month for the next few years and enter into short futures or forward contracts to lock in the price that will be received.

Suppose you are Goldman Sachs and have just entered into a forward contract with a hedger whereby you agree to buy a large amount of gold at a fixed price. How do you hedge your risk? The answer is that you borrow gold from a central bank and sell it at the current market price. (The central banks of many countries hold large amounts of gold.) At the end of the life of the forward contract, you buy gold from the gold mining company under the terms of the forward contract and use it to repay the central bank. The central bank charges a fee (perhaps 1.5% per annum), known as the gold lease rate for lending its gold in this way.

PRESIDENT: What's that got to do with it? That's like saying that we do not need to worry when our sales are down in California because they are up in New York.

TREASURER: If the price of oil had gone down...

PRESIDENT: I don't care what would have happened if the price of oil had gone down. The fact is that it went up. I really do not know what you were doing playing the futures markets like this. Our shareholders will expect us to have done particularly well this quarter. I'm going to have to explain to them that your actions reduced profits by $10 million. I'm afraid this is going to mean no bonus for you this year.

TREASURER: That's unfair. I was only...

PRESIDENT: Unfair! You are lucky not to be fired. You lost $10 million.

TREASURER: It all depends on how you look at it...

It is easy to see why many treasurers are reluctant to hedge! Hedging reduces risk for the company. However, it may increase risk for the treasurer if others do not fully understand what is being done. The only real solution to this problem involves ensuring that all senior executives within the organization fully understand the nature of hedging before a hedging program is put in place. Ideally, hedging strategies are set by a company's board of directors and are clearly communicated to both the company's management and the shareholders. (See Business Snapshot 3.1 for a discussion of hedging by gold mining companies.)

3.3 BASIS RISK

The hedges in the examples considered so far have been almost too good to be true. The hedger was able to identify the precise date in the future when an asset would be bought or sold. The hedger was then able to use futures contracts to remove almost all the risk arising from the price of the asset on that date. In practice, hedging is often not quite as straightforward. Some of the reasons are as follows:

1. The asset whose price is to be hedged may not be exactly the same as the asset underlying the futures contract.
2. The hedger may be uncertain as to the exact date when the asset will be bought or sold.
3. The hedge may require the futures contract to be closed out before its delivery month.

These problems give rise to what is termed *basis risk*. This concept will now be explained.

The Basis

The *basis* in a hedging situation is as follows:[2]

$$\text{Basis} = \text{Spot price of asset to be hedged} - \text{Futures price of contract used}$$

If the asset to be hedged and the asset underlying the futures contract are the same, the basis should be zero at the expiration of the futures contract. Prior to expiration, the basis may be positive or negative. From Table 2.2 and Figure 2.2, we see that on January 8, 2007, the basis was negative for gold and positive for orange juice.

As time passes, the spot price and the futures price do not necessarily change by the same amount. As a result, the basis changes. An increase in the basis is referred to as a *strengthening of the basis*; a decrease in the basis is referred to as a *weakening of the basis*. Figure 3.1 illustrates how a basis might change over time in a situation where the basis is positive prior to expiration of the futures contract.

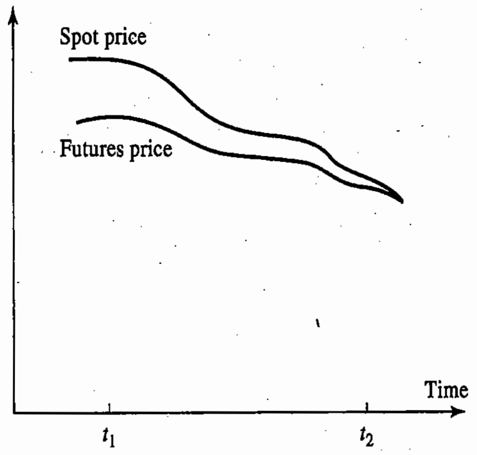
To examine the nature of basis risk, we will use the following notation:

S_1: Spot price at time t_1
S_2: Spot price at time t_2
F_1: Futures price at time t_1
F_2: Futures price at time t_2
b_1: Basis at time t_1
b_2: Basis at time t_2

We will assume that a hedge is put in place at time t_1 and closed out at time t_2. As an example, we will consider the case where the spot and futures prices at the time the hedge is initiated are \$2.50 and \$2.20, respectively, and that at the time the hedge is closed out they are \$2.00 and \$1.90, respectively. This means that $S_1 = 2.50$, $F_1 = 2.20$, $S_2 = 2.00$, and $F_2 = 1.90$.

[2] This is the usual definition. However, the alternative definition Basis = Futures price − Spot price is sometimes used, particularly when the futures contract is on a financial asset.

Figure 3.1 Variation of basis over time.



From the definition of the basis, we have

$$b_1 = S_1 - F_1 \quad \text{and} \quad b_2 = S_2 - F_2$$

so that, in our example, $b_1 = 0.30$ and $b_2 = 0.10$.

Consider first the situation of a hedger who knows that the asset will be sold at time t_2 and takes a short futures position at time t_1. The price realized for the asset is S_2 and the profit on the futures position is $F_1 - F_2$. The effective price that is obtained for the asset with hedging is therefore

$$S_2 + F_1 - F_2 = F_1 + b_2$$

In our example, this is \$2.30. The value of F_1 is known at time t_1. If b_2 were also known at this time, a perfect hedge would result. The hedging risk is the uncertainty associated with b_2 and is known as *basis risk*. Consider next a situation where a company knows it will buy the asset at time t_2 and initiates a long hedge at time t_1. The price paid for the asset is S_2 and the loss on the hedge is $F_1 - F_2$. The effective price that is paid with hedging is therefore

$$S_2 + F_1 - F_2 = F_1 + b_2$$

This is the same expression as before and is \$2.30 in the example. The value of F_1 is known at time t_1, and the term b_2 represents basis risk.

Note that basis risk can lead to an improvement or a worsening of a hedger's position. Consider a short hedge. If the basis strengthens (i.e., increases) unexpectedly, the hedger's position improves; if the basis weakens (i.e., decreases) unexpectedly, the hedger's position worsens. For a long hedge, the reverse holds. If the basis strengthens unexpectedly, the hedger's position worsens; if the basis weakens unexpectedly, the hedger's position improves.

The asset that gives rise to the hedger's exposure is sometimes different from the asset underlying the futures contract that is used for hedging. This increases the basis risk. Define S_2^* as the price of the asset underlying the futures contract at time t_2. As before, S_2 is the price of the asset being hedged at time t_2. By hedging, a company ensures that the price that will be paid (or received) for the asset is

$$S_2 + F_1 - F_2$$

This can be written as

$$F_1 + (S_2^* - F_2) + (S_2 - S_2^*)$$

The terms $S_2^* - F_2$ and $S_2 - S_2^*$ represent the two components of the basis. The $S_2^* - F_2$ term is the basis that would exist if the asset being hedged were the same as the asset underlying the futures contract. The $S_2 - S_2^*$ term is the basis arising from the difference between the two assets.

Choice of Contract

One key factor affecting basis risk is the choice of the futures contract to be used for hedging. This choice has two components:

1. The choice of the asset underlying the futures contract
2. The choice of the delivery month

If the asset being hedged exactly matches an asset underlying a futures contract, the first choice is generally fairly easy. In other circumstances, it is necessary to carry out a careful analysis to determine which of the available futures contracts has futures prices that are most closely correlated with the price of the asset being hedged.

The choice of the delivery month is likely to be influenced by several factors. In the examples given earlier in this chapter, we assumed that, when the expiration of the hedge corresponds to a delivery month, the contract with that delivery month is chosen. In fact, a contract with a later delivery month is usually chosen in these circumstances. The reason is that futures prices are in some instances quite erratic during the delivery month. Moreover, a long hedger runs the risk of having to take delivery of the physical asset if the contract is held during the delivery month. Taking delivery can be expensive and inconvenient. (Long hedgers normally prefer to close out the futures contract and buy the asset from their usual suppliers.)

In general, basis risk increases as the time difference between the hedge expiration and the delivery month increases. A good rule of thumb is therefore to choose a delivery month that is as close as possible to, but later than, the expiration of the hedge. Suppose delivery months are March, June, September, and December for a futures contract on a particular asset. For hedge expirations in December, January, and February, the March contract will be chosen; for hedge expirations in March, April, and May, the June contract will be chosen; and so on. This rule of thumb assumes that there is sufficient liquidity in all contracts to meet the hedger's requirements. In practice, liquidity tends to be greatest in short-maturity futures contracts. Therefore, in some situations, the hedger may be inclined to use short-maturity contracts and roll them forward. This strategy is discussed later in the chapter.

Example 3.1

It is March 1. A US company expects to receive 50 million Japanese yen at the end of July. Yen futures contracts on the Chicago Mercantile Exchange have delivery months of March, June, September, and December. One contract is for the delivery of 12.5 million yen. The company therefore shorts four September yen futures contracts on March 1. When the yen are received at the end of July, the company closes out its position. We suppose that the futures price on March 1 in cents per yen is 0.7800 and that the spot and futures prices when the contract is closed out are 0.7200 and 0.7250, respectively.

The gain on the futures contract is $0.7800 - 0.7250 = 0.0550$ cents per yen. The basis is $0.7200 - 0.7250 = -0.0050$ cents per yen when the contract is closed out. The effective price obtained in cents per yen is the final spot price plus the gain on the futures:

$$0.7200 + 0.0550 = 0.7750$$

This can also be written as the initial futures price plus the final basis:

$$0.7800 + (-0.0050) = 0.7750$$

The total amount received by the company for the 50 million yen is 50×0.00775 million dollars, or \$387,500.

Example 3.2

It is June 8 and a company knows that it will need to purchase 20,000 barrels of crude oil at some time in October or November. Oil futures contracts are currently traded for delivery every month on NYMEX and the contract size is 1,000 barrels. The company therefore decides to use the December contract for hedging and takes a long position in 20 December contracts. The futures price on June 8 is \$68.00 per barrel. The company finds that it is ready to purchase the crude oil on November 10. It therefore closes out its futures contract on that date. The spot price and futures price on November 10 are \$70.00 per barrel and \$69.10 per barrel.

The gain on the futures contract is $69.10 - 68.00 = \$1.10$ per barrel. The basis when the contract is closed out is $70.00 - 69.10 = \$0.90$ per barrel. The effective price paid (in dollars per barrel) is the final spot price less the gain on the futures, or

$$70.00 - 1.10 = 68.90$$

This can also be calculated as the initial futures price plus the final basis,

$$68.00 + 0.90 = 68.90$$

The total price paid is $68.90 \times 20{,}000 = \$1{,}378{,}000$.

3.4 CROSS HEDGING

In the examples considered up to now, the asset underlying the futures contract has been the same as the asset whose price is being hedged. *Cross hedging* occurs when the two assets are different. Consider, for example, an airline that is concerned about the future price of jet fuel. Because there is no futures contract on jet fuel, it might choose to use heating oil futures contracts to hedge its exposure.

The *hedge ratio* is the ratio of the size of the position taken in futures contracts to the size of the exposure. When the asset underlying the futures contract is the same as the asset being hedged, it is natural to use a hedge ratio of 1.0. This is the hedge ratio we have used in the examples considered so far. For instance, in Example 3.2, the hedger's exposure was on 20,000 barrels of oil, and futures contracts were entered into for the delivery of exactly this amount of oil.

When cross hedging is used, setting the hedge ratio equal to 1.0 is not always optimal. The hedger should choose a value for the hedge ratio that minimizes the variance of the value of the hedged position. We now consider how the hedger can do this.

Calculating the Minimum Variance Hedge Ratio

We will use the following notation:

ΔS: Change in spot price, S, during a period of time equal to the life of the hedge
ΔF: Change in futures price, F, during a period of time equal to the life of the hedge
σ_S: Standard deviation of ΔS
σ_F: Standard deviation of ΔF
ρ: Coefficient of correlation between ΔS and ΔF
h^*: Hedge ratio that minimizes the variance of the hedger's position

In the appendix at the end of this chapter, we show that

$$h^* = \rho\frac{\sigma_S}{\sigma_F} \tag{3.1}$$

The optimal hedge ratio is the product of the coefficient of correlation between ΔS and ΔF and the ratio of the standard deviation of ΔS to the standard deviation of ΔF. Figure 3.2 shows how the variance of the value of the hedger's position depends on the hedge ratio chosen.

If $\rho = 1$ and $\sigma_F = \sigma_S$, the hedge ratio, h^*, is 1.0. This result is to be expected, because in this case the futures price mirrors the spot price perfectly. If $\rho = 1$ and $\sigma_F = 2\sigma_S$, the hedge ratio h^* is 0.5. This result is also as expected, because in this case the futures price always changes by twice as much as the spot price.

The optimal hedge ratio, h^*, is the slope of the best-fit line when ΔS is regressed against ΔF, as indicated in Figure 3.3. This is intuitively reasonable, because we require h^* to correspond to the ratio of changes in ΔS to changes in ΔF. The *hedge effectiveness* can be

Figure 3.2 Dependence of variance of hedger's position on hedge ratio.

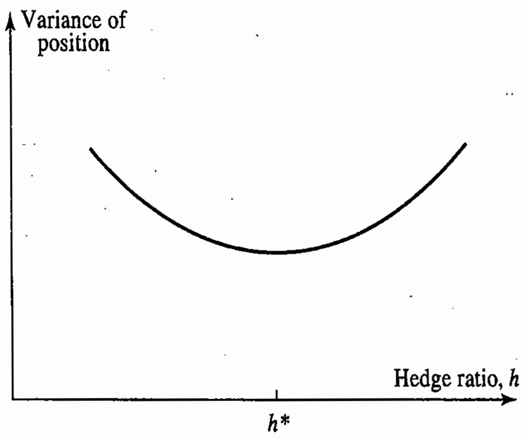
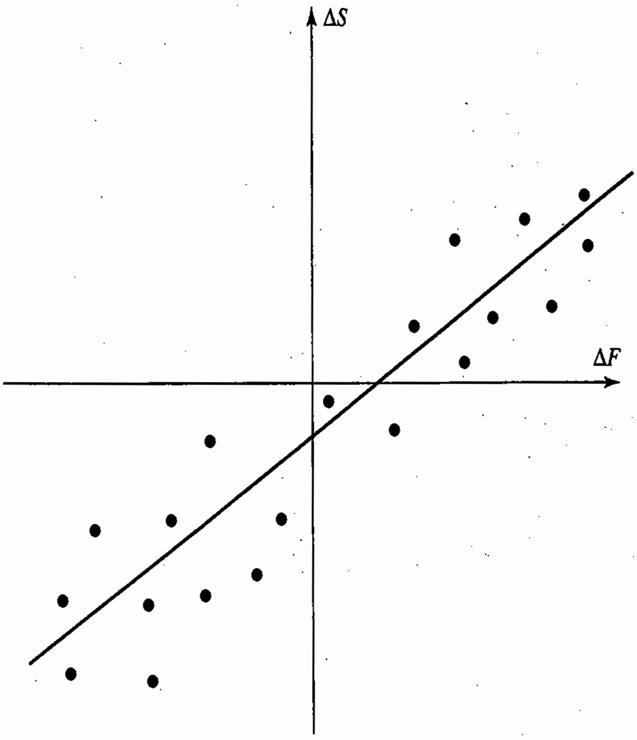


Figure 3.3 Regression of change in spot price against change in futures price.



defined as the proportion of the variance that is eliminated by hedging. This is the R^2 from the regression of ΔS against ΔF and equals ρ^2, or

$$h^{*2}\frac{\sigma_F^2}{\sigma_S^2}$$

The parameters ρ, σ_F, and σ_S in equation (3.1) are usually estimated from historical data on ΔS and ΔF. (The implicit assumption is that the future will in some sense be like the past.) A number of equal nonoverlapping time intervals are chosen, and the values of ΔS and ΔF for each of the intervals are observed. Ideally, the length of each time interval is the same as the length of the time interval for which the hedge is in effect. In practice, this sometimes severely limits the number of observations that are available, and a shorter time interval is used.

Optimal Number of Contracts

Define variables as follows:

- Q_A: Size of position being hedged (units)
- Q_F: Size of one futures contract (units)
- N^*: Optimal number of futures contracts for hedging

The futures contracts should be on h^*Q_A units of the asset. The number of futures contracts required is therefore given by

$$N^* = \frac{h^* Q_A}{Q_F} \tag{3.2}$$

Example 3.3

An airline expects to purchase 2 million gallons of jet fuel in 1 month and decides to use heating oil futures for hedging.[3] We suppose that Table 3.2 gives, for 15 successive months, data on the change, ΔS, in the jet fuel price per gallon and the corresponding change, ΔF, in the futures price for the contract on heating oil that would be used for hedging price changes during the month. The number of observations, which we will denote by n, is 15. We will denote the ith observations on ΔF and ΔS by x_i and y_i, respectively. From Table 3.2, we have

$$\sum x_i = -0.013 \qquad \sum x_i^2 = 0.0138$$
$$\sum y_i = 0.003 \qquad \sum y_i^2 = 0.0097$$
$$\sum x_i y_i = 0.0107$$

Table 3.2 Data to calculate minimum variance hedge ratio when heating oil futures contract is used to hedge purchase of jet fuel.

Month i	*Change in futures price per gallon* $(= x_i)$	*Change in fuel price per gallon* $(= y_i)$
1	0.021	0.029
2	0.035	0.020
3	−0.046	−0.044
4	0.001	0.008
5	0.044	0.026
6	−0.029	−0.019
7	−0.026	−0.010
8	−0.029	−0.007
9	0.048	0.043
10	−0.006	0.011
11	−0.036	−0.036
12	−0.011	−0.018
13	0.019	0.009
14	−0.027	−0.032
15	0.029	0.023

[3] Heating oil futures contracts are more liquid than jet fuel futures contracts. For an account of how Delta Airlines used heating oil futures to hedge its future purchases of jet fuel, see A. Ness, "Delta Wins on Fuel," *Risk*, June 2001, p. 8.

Standard formulas from statistics give the estimate of σ_F as

$$\sqrt{\frac{\sum x_i^2}{n-1} - \frac{(\sum x_i)^2}{n(n-1)}} = 0.0313$$

The estimate of σ_S is

$$\sqrt{\frac{\sum y_i^2}{n-1} - \frac{(\sum y_i)^2}{n(n-1)}} = 0.0263$$

The estimate of ρ is

$$\frac{n\sum x_i y_i - \sum x_i \sum y_i}{\sqrt{[n\sum x_i^2 - (\sum x_i)^2][n\sum y_i^2 - (\sum y_i)^2]}} = 0.928$$

From equation (3.1), the minimum variance hedge ratio, h^*, is therefore

$$0.928 \times \frac{0.0263}{0.0313} = 0.78$$

Each heating oil contract traded on NYMEX is on 42,000 gallons of heating oil. From equation (3.2), the optimal number of contracts is

$$\frac{0.78 \times 2{,}000{,}000}{42{,}000} = 37.14$$

or, rounding to the nearest whole number, 37.

Tailing the Hedge

When futures are used for hedging, a small adjustment, known as *tailing the hedge*, can be made to allow for the impact of daily settlement. In practice this means that equation (3.2) becomes[4]

$$N^* = \frac{h^* V_A}{V_F} \tag{3.3}$$

where V_A is the dollar value of the position being hedged and V_F is the dollar value of one futures contract (the futures price times Q_F). Suppose that in Example 3.3 the spot price and the futures price are 1.94 and 1.99 dollars per gallon. Then $V_A = 2{,}000{,}000 \times 1.94 = 3{,}880{,}0000$ while $V_F = 42{,}000 \times 1.99 = 83{,}580$, so that the optimal number of contracts is

$$\frac{0.78 \times 3,880,000}{83,550} = 36.22$$

If we round this to the nearest whole number, the optimal number of contracts is now 36 rather than 37. The effect of tailing the hedge is to multiply the hedge ratio in equation (3.2) by the ratio of the spot price to the futures price. Ideally the futures position used for hedging should then be adjusted as V_A and V_F change, but in practice this is not usually feasible.

[4] See Problem 5.23 for an explanation of equation (3.3).

3.5 STOCK INDEX FUTURES

We now move on to consider stock index futures and how they are used to hedge or manage exposures to equity prices.

A *stock index* tracks changes in the value of a hypothetical portfolio of stocks. The weight of a stock in the portfolio equals the proportion of the portfolio invested in the stock. The percentage increase in the stock index over a small interval of time is set equal to the percentage increase in the value of the hypothetical portfolio. Dividends are usually not included in the calculation so that the index tracks the capital gain/loss from investing in the portfolio.[5]

If the hypothetical portfolio of stocks remains fixed, the weights assigned to individual stocks in the portfolio do not remain fixed. When the price of one particular stock in the portfolio rises more sharply than others, more weight is automatically given to that stock. Some indices are constructed from a hypothetical portfolio consisting of one of each of a number of stocks. The weights assigned to the stocks are then proportional to their market prices, with adjustments being made when there are stock splits. Other indices are constructed so that weights are proportional to market capitalization (stock price × number of shares outstanding). The underlying portfolio is then automatically adjusted to reflect stock splits, stock dividends, and new equity issues.

Stock Indices

Table 3.3 shows futures prices for contracts on a number of different stock indices as they were reported in the *Wall Street Journal* of January 9, 2007. The prices refer to the close of trading on January 8, 2007.

The *Dow Jones Industrial Average* is based on a portfolio consisting of 30 blue-chip stocks in the United States. The weights given to the stocks are proportional to their prices. The Chicago Board of Trade trades two futures contracts on the index. One is on \$10 times the index. The other (the Mini DJ Industrial Average) is on \$5 times the index.

The *Standard & Poor's 500* (S&P 500) *Index* is based on a portfolio of 500 different stocks: 400 industrials, 40 utilities, 20 transportation companies, and 40 financial institutions. The weights of the stocks in the portfolio at any given time are proportional to their market capitalizations. This index accounts for 80% of the market capitalization of all the stocks listed on the New York Stock Exchange. The Chicago Mercantile Exchange (CME) trades two futures contracts on the S&P 500. One is on \$250 times the index; the other (the Mini S&P 500 contract) is on \$50 times the index.

The *Nasdaq 100* is based on 100 stocks using the National Association of Securities Dealers Automatic Quotations Service. The CME trades two contracts. One is on \$100 times the index; the other (the Mini Nasdaq 100 contract) is on \$20 times the index.

The *Russell 1000 Index* is an index of the prices of the 1000 largest capitalization stocks in the United States. The *U.S. Dollar Index* is a trade-weighted index of the values of six foreign currencies (the euro, yen, pound, Canadian dollar, Swedish krona, and Swiss franc).

[5] An exception to this is a *total return index*. This is calculated by assuming that dividends on the hypothetical portfolio are reinvested in the portfolio.

Table 3.3 Index futures quotes from *Wall Street Journal*, January 9, 2007: Columns show month, open, high, low, settle, change, and open interest, respectively.

Index Futures

DJ Industrial Average (CBT)-$10 x Index						
March	12457	12515	12405	**12492**	42	64,772
June	12530	12591	12525	**12591**	42	46
Mini DJ Industrial Average (CBT)-$5 x index						
March	12460	12514	12405	**12492**	42	106,556
June	12570	12577	12540	**12591**	42	21
S&P 500 Index (CME)-$250 x Index						
March	1417.30	1424.50	1413.00	**1422.50**	6.10	601,897
June	1426.10	1437.00	1426.10	**1435.30**	6.20	13,062
Mini S&P 500 (CME)-$50 x Index						
March	1417.25	1424.50	1413.00	**1422.50**	6.00	1,525,973
June	1430.50	1437.00	▼1425.50	**1435.25**	6.25	13,716
Nasdaq 100 (CME)-$100 x index						
March	1797.50	1812.50	1792.00	**1803.50**	6.25	45,550
Mini Nasdaq 100 (CME)-$20 x Index						
March	1798.0	1812.3	1792.3	**1803.5**	6.3	328,990
June	1819.8	1833.3	1814.3	**1825.0**	6.3	92
Russell 1000 (NYBOT)-$500 x index						
March	770.75	773.50	769.15	**773.00**	3.10	70,440
U.S. Dollar Index (NYBOT)-$1,000 x index						
March	84.43	84.62	84.27	**84.37**	-.03	24,181
June	84.10	84.30	84.01	**84.12**	-.03	2,028

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As mentioned in Chapter 2, futures contracts on stock indices are settled in cash, not by delivery of the underlying asset. All contracts are marked to market to either the opening price or the closing price of the index on the last trading day, and the positions are then deemed to be closed. For example, contracts on the S&P 500 are closed out at the opening price of the S&P 500 index on the third Friday of the delivery month.

Hedging an Equity Portfolio

Stock index futures can be used to hedge a well-diversified equity portfolio. Define:

P: Current value of the portfolio

F: Current value of one futures contract (the futures price times the contract size)

If the portfolio mirrors the index, the optimal hedge ratio, h^*, equals 1.0 and equation (3.3) shows that the number of futures contracts that should be shorted is

$$N^* = \frac{P}{F} \tag{3.4}$$

Suppose, for example, that a portfolio worth \$5,050,000 mirrors the S&P 500. The index futures price is 1,010 and each futures contract is on \$250 times the index. In this case $P = 5{,}050{,}000$ and $F = 1{,}010 \times 250 = 252{,}500$, so that 20 contracts should be shorted to hedge the portfolio.

When the portfolio does not exactly mirror the index, we can use the parameter beta (β) from the capital asset pricing model to determine the appropriate hedge ratio. Beta is the slope of the best-fit line obtained when excess return on the portfolio over the risk-free rate is regressed against the excess return of the market over the risk-free rate. When $\beta = 1.0$, the return on the portfolio tends to mirror the return on the market; when $\beta = 2.0$, the excess return on the portfolio tends to be twice as great as the excess return on the market; when $\beta = 0.5$, it tends to be half as great; and so on.

A portfolio with a β of 2.0 is twice as sensitive to market movements as a portfolio with a beta 1.0. It is therefore necessary to use twice as many contracts to hedge the portfolio. Similarly, a portfolio with a beta of 0.5 is half as sensitive to market movements as a portfolio with a beta of 1.0 and we should use half as many contracts to hedge it. In general, $h^* = \beta$, so that equation (3.3) gives

$$N^* = \beta \frac{P}{F} \tag{3.5}$$

This formula assumes that the maturity of the futures contract is close to the maturity of the hedge.

We illustrate that this formula gives good results by extending our earlier example. Suppose that a futures contract with 4 months to maturity is used to hedge the value of a portfolio over the next 3 months in the following situation:

Value of S&P 500 index = 1,000
S&P 500 futures price = 1,010
Value of portfolio = \$5,050,000
Risk-free interest rate = 4% per annum
Dividend yield on index = 1% per annum
Beta of portfolio = 1.5

One futures contract is for delivery of \$250 times the index. It follows that $F = 250 \times 1{,}010 = 252{,}500$ and from equation (3.5), the number of futures contracts that should be shorted to hedge the portfolio is

$$1.5 \times \frac{5{,}050{,}000}{252{,}500} = 30$$

Suppose the index turns out to be 900 in 3 months and the futures price is 902. The gain from the short futures position is then

$$30 \times (1010 - 902) \times 250 = \$810{,}000$$

The loss on the index is 10%. The index pays a dividend of 1% per annum, or 0.25% per 3 months. When dividends are taken into account, an investor in the index would therefore earn −9.75% in the 3-month period. Because the portfolio has a β of 1.5, the capital asset pricing model gives

Expected return on portfolio − Risk-free interest rate
= 1.5 × (Return on index − Risk-free interest rate)

The risk-free interest rate is approximately 1% per 3 months. It follows that the expected return (%) on the portfolio during the 3 months when the 3-month return on the index is −9.75% is

$$1.0 + [1.5 \times (-9.75 - 1.0)] = -15.125$$

The expected value of the portfolio (inclusive of dividends) at the end of the 3 months is therefore

$$\$5{,}050{,}000 \times (1 - 0.15125) = \$4{,}286{,}187$$

Table 3.4 Performance of stock index hedge.

Value of index in three months:	900	950	1,000	1,050	1,100
Futures price of index today:	1,010	1,010	1,010	1,010	1,010
Futures price of index in three months:	902	952	1,003	1,053	1,103
Gain on futures position ($):	810,000	435,000	52,500	−322,500	−697,500
Return on market:	−9.750%	−4.750%	0.250%	5.250%	10.250%
Expected return on portfolio:	−15.125%	−7.625%	−0.125%	7.375%	14.875%
Expected portfolio value in three months including dividends ($):	4,286,187	4,664,937	5,043,687	5,422,437	5,801,187
Total value of position in three months ($):	5,096,187	5,099,937	5,096,187	5,099,937	5,103,687

It follows that the expected value of the hedger's position, including the gain on the hedge, is

$$\$4{,}286{,}187 + \$810{,}000 = \$5{,}096{,}187$$

Table 3.4 summarizes these calculations together with similar calculations for other values of the index at maturity. It can be seen that the total expected value of the hedger's position in 3 months is almost independent of the value of the index.

The only thing we have not covered in this example is the relationship between futures prices and spot prices. We will see in Chapter 5 that the 1,010 assumed for the futures price today is roughly what we would expect given the interest rate and dividend we are assuming. The same is true of the futures prices in 3 months shown in Table 3.4.[6]

Reasons for Hedging an Equity Portfolio

Table 3.4 shows that the hedging scheme results in a value for the hedger's position at the end of the 3-month period being about 1% higher than at the beginning of the 3-month period. There is no surprise here. The risk-free rate is 4% per annum, or 1% per 3 months. The hedge results in the investor's position growing at the risk-free rate.

It is natural to ask why the hedger should go to the trouble of using futures contracts. To earn the risk-free interest rate, the hedger can simply sell the portfolio and invest the proceeds in risk-free instruments such as Treasury bills.

One answer to this question is that hedging can be justified if the hedger feels that the stocks in the portfolio have been chosen well. In these circumstances, the hedger might be very uncertain about the performance of the market as a whole, but confident that the stocks in the portfolio will outperform the market (after appropriate adjustments have been made for the beta of the portfolio). A hedge using index futures

[6] The calculations in Table 3.4 assume that the dividend yield on the index is predictable, the risk-free interest rate remains constant, and the return on the index over the 3-month period is perfectly correlated with the return on the portfolio. In practice, these assumptions do not hold perfectly, and the hedge works rather less well than is indicated by Table 3.4.

removes the risk arising from market moves and leaves the hedger exposed only to the performance of the portfolio relative to the market. Another reason for hedging may be that the hedger is planning to hold a portfolio for a long period of time and requires short-term protection in an uncertain market situation. The alternative strategy of selling the portfolio and buying it back later might involve unacceptably high transaction costs.

Changing the Beta of a Portfolio

In the example in Table 3.4, the beta of the hedger's portfolio is reduced to zero. (The hedger's expected return is independent of the performance of the index.) Sometimes futures contracts are used to change the beta of a portfolio to some value other than zero. Continuing with our earlier example:

$$\text{S\&P 500 index} = 1{,}000$$
$$\text{S\&P 500 futures price} = 1{,}010$$
$$\text{Value of portfolio} = \$5{,}050{,}000$$
$$\text{Beta of portfolio} = 1.5$$

As before, $F = 250 \times 1{,}010 = 252{,}500$ and a complete hedge requires

$$1.5 \times \frac{5{,}050{,}000}{252{,}500} = 30$$

contracts to be shorted. To reduce the beta of the portfolio from 1.5 to 0.75, the number of contracts shorted should be 15 rather than 30; to increase the beta of the portfolio to 2.0, a long position in 10 contracts should be taken; and so on. In general, to change the beta of the portfolio from β to β^*, where $\beta > \beta^*$, a short position in

$$(\beta - \beta^*)\frac{P}{F}$$

contracts is required. When $\beta < \beta^*$, a long position in

$$(\beta^* - \beta)\frac{P}{F}$$

contracts is required.

Exposure to the Price of an Individual Stock

Some exchanges do trade futures contracts on selected individual stocks, but in most cases a position in an individual stock can only be hedged using a stock index futures contract.

Hedging an exposure to the price of an individual stock using index futures contracts is similar to hedging a well-diversified stock portfolio. The number of index futures contracts that the hedger should short is given by $\beta P/F$, where β is the beta of the stock, P is the total value of the shares owned, and F is the current value of one index futures contract. Note that although the number of contracts entered into is calculated in the same way as it is when a portfolio of stocks is being hedged, the performance of the hedge is considerably worse. The hedge provides protection only against the risk arising from market movements, and this risk is a relatively small proportion of the total risk in the price movements of individual stocks. The hedge is appropriate when an investor feels that the stock will outperform the market but is unsure about the performance of the market. It can also be used by an investment bank that has underwritten a new issue of the stock and wants protection against moves in the market as a whole.

Consider an investor who in June holds 20,000 IBM shares, each worth \$100. The investor feels that the market will be very volatile over the next month but that IBM has a good chance of outperforming the market. The investor decides to use the August futures contract on the S&P 500 to hedge the position during the 1-month period. The β of IBM is estimated at 1.1. The current futures price for the August contract on the S&P 500 is 900. Each contract is for delivery of \$250 times the index. In this case, $P = 20{,}000 \times 100 = 2{,}000{,}000$ and $F = 900 \times 250 = 225{,}000$. The number of contracts that should be shorted is therefore

$$1.1 \times \frac{2{,}000{,}000}{225{,}000} = 9.78$$

Rounding to the nearest integer, the hedger shorts 10 contracts, closing out the position 1 month later. Suppose IBM rises to \$125 during the month, and the futures price of the S&P 500 rises to 1080. The investor gains $20{,}000 \times (\$125 - \$100) = \$500{,}000$ on IBM while losing $10 \times 250 \times (1080 - 900) = \$450{,}000$ on the futures contracts. .

In this example, the hedge offsets a gain on the underlying asset with a loss on the futures contracts. The offset might seem to be counterproductive. However, it cannot be emphasized often enough that the purpose of a hedge is to reduce risk. A hedge tends to make unfavorable outcomes less unfavorable but also to make favorable outcomes less favorable.

3.6 ROLLING THE HEDGE FORWARD

Sometimes the expiration date of the hedge is later than the delivery dates of all the futures contracts that can be used. The hedger must then roll the hedge forward by closing out one futures contract and taking the same position in a futures contract with a later delivery date. Hedges can be rolled forward many times. Consider a company that wishes to use a short hedge to reduce the risk associated with the price to be received for an asset at time T. If there are futures contracts 1, 2, 3, ..., n (not all necessarily in existence at the present time) with progressively later delivery dates, the company can use the following strategy:

Time t_1: Short futures contract 1

Time t_2: Close out futures contract 1
Short futures contract 2

Time t_3: Close out futures contract 2
Short futures contract 3

⋮

Time t_n: Close out futures contract $n - 1$
Short futures contract n

Time T: Close out futures contract n

Table 3.5 Data for the example on rolling oil hedge forward.

Date	*Apr. 2007*	*Sept. 2007*	*Feb. 2008*	*June 2008*
Oct. 2007 futures price	68.20	67.40		
Mar. 2008 futures price		67.00	66.50	
July 2008 futures price			66.30	65.90
Spot price	69.00			66.00

Suppose that in April 2007 a company realizes that it will have 100,000 barrels of oil to sell in June 2008 and decides to hedge its risk with a hedge ratio of 1.0. (In this example, we do not make the "tailing" adjustment described in Section 3.4.) The current spot price is \$69. Although futures contracts are traded with maturities stretching several years into the future, we suppose that only the first six delivery months have sufficient liquidity to meet the company's needs. The company therefore shorts 100 October 2007 contracts. In September 2007 it rolls the hedge forward into the March 2008 contract. In February 2008 it rolls the hedge forward again into the July 2008 contract.

One possible outcome is shown in Table 3.5. The October 2007 contract is shorted at \$68.20 per barrel and closed out at \$67.40 per barrel for a profit of \$0.80 per barrel; the March 2008 contract is shorted at \$67.00 per barrel and closed out at \$66.50 per barrel for a profit of \$0.50 per barrel. The July 2008 contract is shorted at \$66.30 per barrel and closed out at \$65.90 per barrel for a profit of \$0.40 per barrel. The final spot price is \$66.

The dollar gain per barrel of oil from the short futures contracts is

$$(68.20 - 67.40) + (67.00 - 66.50) + (66.30 - 65.90) = 1.70$$

The oil price declined from \$69 to \$66. Receiving only \$1.70 per barrel compensation for a price decline of \$3.00 may appear unsatisfactory. However, we cannot expect total compensation for a price decline when futures prices are below spot prices. The best we can hope for is to lock in the futures price that would apply to a June 2008 contract if it were actively traded.

The daily settlement of futures contracts can cause a mismatch between the timing of the cash flows on hedge and the timing of the cash flows from the position being hedged. In situations where the hedge is rolled forward so that it lasts a long time this can lead to serious problems (see Business Snapshot 3.2).

SUMMARY

This chapter has discussed various ways in which a company can take a position in futures contracts to offset an exposure to the price of an asset. If the exposure is such that the company gains when the price of the asset increases and loses when the price of the asset decreases, a short hedge is appropriate. If the exposure is the other way round (i.e., the company gains when the price of the asset decreases and loses when the price of the asset increases), a long hedge is appropriate.

Business Snapshot 3.2 Metallgesellschaft: Hedging Gone Awry

Sometimes rolling hedges forward can lead to cash flow pressures. The problem was illustrated dramatically by the activities of a German company, Metallgesellschaft (MG), in the early 1990s.

MG sold a huge volume of 5- to 10-year heating oil and gasoline fixed-price supply contracts to its customers at 6 to 8 cents above market prices. It hedged its exposure with long positions in short-dated futures contracts that were rolled forward. As it turned out, the price of oil fell and there were margin calls on the futures positions. Considerable short-term cash flow pressures were placed on MG. The members of MG who devised the hedging strategy argued that these short-term cash outflows were offset by positive cash flows that would ultimately be realized on the long-term fixed-price contracts. However, the company's senior management and its bankers became concerned about the huge cash drain. As a result, the company closed out all the hedge positions and agreed with its customers that the fixed-price contracts would be abandoned. The outcome was a loss to MG of $1.33 billion.

Hedging is a way of reducing risk. As such, it should be welcomed by most executives. In reality, there are a number of theoretical and practical reasons why companies do not hedge. On a theoretical level, we can argue that shareholders, by holding well-diversified portfolios, can eliminate many of the risks faced by a company. They do not require the company to hedge these risks. On a practical level, a company may find that it is increasing rather than decreasing risk by hedging if none of its competitors does so. Also, a treasurer may fear criticism from other executives if the company makes a gain from movements in the price of the underlying asset and a loss on the hedge.

An important concept in hedging is basis risk. The basis is the difference between the spot price of an asset and its futures price. Basis risk arises from uncertainty as to what the basis will be at maturity of the hedge.

The hedge ratio is the ratio of the size of the position taken in futures contracts to the size of the exposure. It is not always optimal to use a hedge ratio of 1.0. If the hedger wishes to minimize the variance of a position, a hedge ratio different from 1.0 may be appropriate. The optimal hedge ratio is the slope of the best-fit line obtained when changes in the spot price are regressed against changes in the futures price.

Stock index futures can be used to hedge the systematic risk in an equity portfolio. The number of futures contracts required is the beta of the portfolio multiplied by the ratio of the value of the portfolio to the value of one futures contract. Stock index futures can also be used to change the beta of a portfolio without changing the stocks that make up the portfolio.

When there is no liquid futures contract that matures later than the expiration of the hedge, a strategy known as rolling the hedge forward may be appropriate. This involves entering into a sequence of futures contracts. When the first futures contract is near expiration, it is closed out and the hedger enters into a second contract with a later delivery month. When the second contract is close to expiration, it is closed out and the hedger enters into a third contract with a later delivery month; and so on. The result of all this is the creation of a long-dated futures contract by trading a series of short-dated contracts.

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Questions and Problems (Answers in Solutions Manual)

3.1. Under what circumstances are (a) a short hedge and (b) a long hedge appropriate?

3.2. Explain what is meant by *basis risk* when futures contracts are used for hedging.

3.3. Explain what is meant by a *perfect hedge*. Does a perfect hedge always lead to a better outcome than an imperfect hedge? Explain your answer.

3.4. Under what circumstances does a minimum variance hedge portfolio lead to no hedging at all?

3.5. Give three reasons why the treasurer of a company might not hedge the company's exposure to a particular risk.

3.6. Suppose that the standard deviation of quarterly changes in the prices of a commodity is \$0.65, the standard deviation of quarterly changes in a futures price on the commodity is \$0.81, and the coefficient of correlation between the two changes is 0.8. What is the optimal hedge ratio for a 3-month contract? What does it mean?

3.7. A company has a \$20 million portfolio with a beta of 1.2. It would like to use futures contracts on the S&P 500 to hedge its risk. The index futures price is currently standing at 1080, and each contract is for delivery of \$250 times the index. What is the hedge that minimizes risk? What should the company do if it wants to reduce the beta of the portfolio to 0.6?

3.8. In the Chicago Board of Trade's corn futures contract, the following delivery months are available: March, May, July, September, and December. State the contract that should be used for hedging when the expiration of the hedge is in (a) June, (b) July, and (c) January.

3.9. Does a perfect hedge always succeed in locking in the current spot price of an asset for a future transaction? Explain your answer.

3.10. Explain why a short hedger's position improves when the basis strengthens unexpectedly and worsens when the basis weakens unexpectedly.

3.11. Imagine you are the treasurer of a Japanese company exporting electronic equipment to the United States. Discuss how you would design a foreign exchange hedging strategy and the arguments you would use to sell the strategy to your fellow executives.

3.12. Suppose that in Example 3.2 of Section 3.3 the company decides to use a hedge ratio of 0.8. How does the decision affect the way in which the hedge is implemented and the result?

3.13. "If the minimum variance hedge ratio is calculated as 1.0, the hedge must be perfect." Is this statement true? Explain your answer.

3.14. "If there is no basis risk, the minimum variance hedge ratio is always 1.0." Is this statement true? Explain your answer.

3.15. "For an asset where futures prices are usually less than spot prices, long hedges are likely to be particularly attractive." Explain this statement.

3.16. The standard deviation of monthly changes in the spot price of live cattle is (in cents per pound) 1.2. The standard deviation of monthly changes in the futures price of live cattle for the closest contract is 1.4. The correlation between the futures price changes and the spot price changes is 0.7. It is now October 15. A beef producer is committed to purchasing 200,000 pounds of live cattle on November 15. The producer wants to use the December live cattle futures contracts to hedge its risk. Each contract is for the delivery of 40,000 pounds of cattle. What strategy should the beef producer follow?

3.17. A corn farmer argues "I do not use futures contracts for hedging. My real risk is not the price of corn. It is that my whole crop gets wiped out by the weather." Discuss this viewpoint. Should the farmer estimate his or her expected production of corn and hedge to try to lock in a price for expected production?

3.18. On July 1, an investor holds 50,000 shares of a certain stock. The market price is \$30 per share. The investor is interested in hedging against movements in the market over the next month and decides to use the September Mini S&P 500 futures contract. The index futures price is currently 1,500 and one contract is for delivery of \$50 times the index. The beta of the stock is 1.3. What strategy should the investor follow?

3.19. Suppose that in Table 3.5 the company decides to use a hedge ratio of 1.5. How does the decision affect the way the hedge is implemented and the result?

3.20. A futures contract is used for hedging. Explain why the marking to market of the contract can give rise to cash flow problems.

3.21. An airline executive has argued: "There is no point in our using oil futures. There is just as much chance that the price of oil in the future will be less than the futures price as there is that it will be greater than this price." Discuss the executive's viewpoint.

3.22. Suppose that the 1-year gold lease rate is 1.5% and the 1-year risk-free rate is 5.0%. Both rates are compounded annually. Use the discussion in Business Snapshot 3.1 to calculate the maximum 1-year gold forward price Goldman Sachs should quote to the gold-mining company when the spot price is $600.

Assignment Questions

3.23. The following table gives data on monthly changes in the spot price and the futures price for a certain commodity. Use the data to calculate a minimum variance hedge ratio.

Spot price change	+0.50	+0.61	−0.22	−0.35	+0.79
Futures price change	+0.56	+0.63	−0.12	−0.44	+0.60
Spot price change	+0.04	+0.15	+0.70	−0.51	−0.41
Futures price change	−0.06	+0.01	+0.80	−0.56	−0.46

3.24. It is July 16. A company has a portfolio of stocks worth $100 million. The beta of the portfolio is 1.2. The company would like to use the CME December futures contract on the S&P 500 to change the beta of the portfolio to 0.5 during the period July 16 to November 16. The index futures price is currently 1,000, and each contract is on $250 times the index.
(a) What position should the company take?
(b) Suppose that the company changes its mind and decides to increase the beta of the portfolio from 1.2 to 1.5. What position in futures contracts should it take?

3.25. A fund manager has a portfolio worth $50 million with a beta of 0.87. The manager is concerned about the performance of the market over the next 2 months and plans to use 3-month futures contracts on the S&P 500 to hedge the risk. The current level of the index is 1,250, one contract is on 250 times the index, the risk-free rate is 6% per annum, and the dividend yield on the index is 3% per annum. The current 3-month futures price is 1259.
(a) What position should the fund manager take to hedge all exposure to the market over the next 2 months?
(b) Calculate the effect of your strategy on the fund manager's returns if the index in 2 months is 1,000, 1,100, 1,200, 1,300, and 1,400. Assume that the 1-month futures price is 0.25% higher than the index level at this time.

3.26. It is now October 2007. A company anticipates that it will purchase 1 million pounds of copper in each of February 2008, August 2008, February 2009, and August 2009. The company has decided to use the futures contracts traded in the COMEX division of the New York Mercantile Exchange to hedge its risk. One contract is for the delivery of

25,000 pounds of copper. The initial margin is $2,000 per contract and the maintenance margin is $1,500 per contract. The company's policy is to hedge 80% of its exposure. Contracts with maturities up to 13 months into the future are considered to have sufficient liquidity to meet the company's needs. Devise a hedging strategy for the company. Do not make the tailing adjustment described in Section 3.4.

Assume the market prices (in cents per pound) today and at future dates are as follows:

Date	*Oct. 2007*	*Feb. 2008*	*Aug. 2008*	*Feb. 2009*	*Aug. 2009*
Spot price	372.00	369.00	365.00	377.00	388.00
Mar. 2008 futures price	372.30	369.10			
Sept. 2008 futures price	372.80	370.20	364.80		
Mar. 2009 futures price		370.70	364.30	376.70	
Sept. 2009 futures price			364.20	376.50	388.20

What is the impact of the strategy you propose on the price the company pays for copper? What is the initial margin requirement in October 2007? Is the company subject to any margin calls?

APPENDIX

PROOF OF THE MINIMUM VARIANCE HEDGE RATIO FORMULA

Suppose we expect to sell N_A units of an asset at time t_2 and choose to hedge at time t_1 by shorting futures contracts on N_F units of a similar asset. The hedge ratio, which we will denote by h, is

$$h = \frac{N_F}{N_A} \tag{3A.1}$$

We will denote the total amount realized for the asset when the profit or loss on the hedge is taken into account by Y, so that

$$Y = S_2 N_A - (F_2 - F_1)N_F$$

or

$$Y = S_1 N_A + (S_2 - S_1)N_A - (F_2 - F_1)N_F \tag{3A.2}$$

where S_1 and S_2 are the asset prices at times t_1 and t_2, and F_1 and F_2 are the futures prices at times t_1 and t_2. From equation (3A.1), the expression for Y in equation (3A.2) can be written

$$Y = S_1 N_A + N_A(\Delta S - h\,\Delta F) \tag{3A.3}$$

where

$$\Delta S = S_2 - S_1 \quad \text{and} \quad \Delta F = F_2 - F_1$$

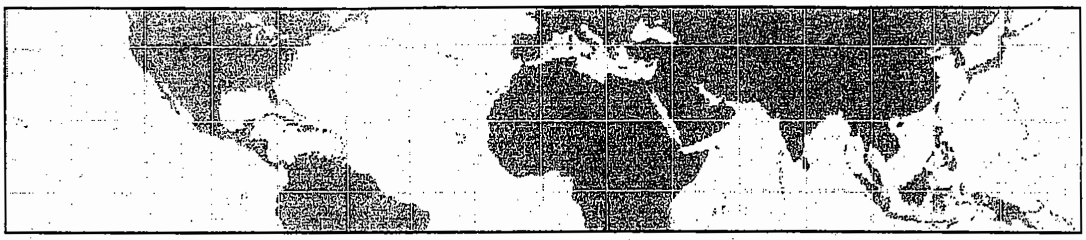
Because S_1 and N_A are known at time t_1, the variance of Y in equation (3A.3) is minimized when the variance of $\Delta S - h\,\Delta F$ is minimized. The variance of $\Delta S - h\,\Delta F$ is

$$v = \sigma_S^2 + h^2\sigma_F^2 - 2h\rho\sigma_S\sigma_F$$

where σ_S, σ_F, and ρ are as defined in Section 3.4, so that

$$\frac{dv}{dh} = 2h\sigma_F^2 - 2\rho\sigma_S\sigma_F$$

Setting this equal to zero, and noting that d^2v/dh^2 is positive, we see that the value of h that minimizes the variance is $h = \rho\sigma_S/\sigma_F$.



CHAPTER 4

Interest Rates

Interest rates are a factor in the valuation of virtually all derivatives and will feature prominently in much of the material that will be presented in the rest of this book. This chapter deals with some fundamental issues concerned with the way interest rates are measured and analyzed. It explains the compounding frequency used to define an interest rate and the meaning of continuously compounded interest rates, which are used extensively in the analysis of derivatives. It covers zero rates, par yields, and yield curves, discusses bond pricing, and outlines a procedure commonly used by a derivatives trading desk to calculate zero-coupon Treasury interest rates. It also covers forward rates and forward rate agreements and review different theories of the term structure of interest rates. Finally, it explains the use of duration and convexity measures to determine the sensitivity of bond prices to interest rate changes.

Chapter 6 will cover interest rate futures and show how the duration measure can be used when interest rate exposures are hedged. For ease of exposition, day count conventions will be ignored throughout this chapter. The nature of these conventions and their impact on calculations will be discussed in Chapters 6 and 7.

4.1 TYPES OF RATES

An interest rate in a particular situation defines the amount of money a borrower promises to pay the lender. For any given currency, many different types of interest rates are regularly quoted. These include mortgage rates, deposit rates, prime borrowing rates, and so on. The interest rate applicable in a situation depends on the credit risk. This is the risk that there will be a default by the borrower of funds, so that the interest and principal are not paid to the lender as promised. The higher the credit risk, the higher the interest rate that is promised by the borrower.

Treasury Rates

Treasury rates are the rates an investor earns on Treasury bills and Treasury bonds. These are the instruments used by a government to borrow in its own currency. Japanese Treasury rates are the rates at which the Japanese government borrows in yen; US Treasury rates are the rates at which the US government borrows in US dollars; and so on. It is usually assumed that there is no chance that a government will default on an obligation denominated in its own currency. Treasury rates are therefore totally risk-free rates in the sense that an investor who buys a Treasury bill or Treasury bond is certain that interest and principal payments will be made as promised.

Treasury rates are important because they are used to price Treasury bonds and are sometimes used to define the payoff from a derivative. However, derivatives traders (particularly those active in the over-the-counter market) do not usually use Treasury rates as risk-free rates. Instead they use LIBOR rates.

LIBOR

LIBOR is short for *London Interbank Offered Rate*. A LIBOR quote by a particular bank is the rate of interest at which the bank is prepared to make a large wholesale deposit with other banks. Large banks and other financial institutions quote LIBOR in all major currencies for maturities up to 12 months: 1-month LIBOR is the rate at which 1-month deposits are offered, 3-month LIBOR is the rate at which 3-month deposits are offered, and so on.

A deposit with a bank can be regarded as a loan to that bank. A bank must therefore satisfy certain creditworthiness criteria in order to be able to accept a LIBOR quote from another bank so that it receives deposits from that bank at LIBOR. Typically it must have a AA credit rating.[1]

AA-rated financial institutions regard LIBOR as their short-term opportunity cost of capital. They can borrow short-term funds at the LIBOR quotes of other financial institutions. Their own LIBOR quotes determine the rate at which surplus funds are lent to other financial institutions. LIBOR rates are not totally free of credit risk. There is a small chance that a AA-rated financial institution will default on a short-term LIBOR loan. However, they are close to risk-free. Derivatives traders regard LIBOR rates as a better indication of the "true" risk-free rate than Treasury rates, because a number of tax and regulatory issues cause Treasury rates to be artifically low (see Business Snapshot 4.1). To be consistent with the normal practice in derivatives markets, the term "risk-free rate" in this book should be interpreted as the LIBOR rate.[2]

In addition to quoting LIBOR rates, large banks also quote LIBID rates. This is the *London Interbank Bid Rate* and is the rate at which they will accept deposits from other banks. At any specified time, there is usually a small spread between the quoted LIBID and LIBOR rates (with LIBOR higher than LIBID). The rates themselves are determined by active trading between banks and are continually changing so that the supply of funds in the interbank market equals the demand for funds in that market. For example, if more banks want to borrow US dollars for 3 months than lend US dollars for 3 months, the 3-month US LIBID and LIBOR rates quoted by banks will increase. Similarly, if more banks want to lend 3-month funds than borrow these funds, the 3-month LIBID and LIBOR rates will decrease. LIBOR and LIBID trade in what is known as the *Eurocurrency market*. This market is outside the control of any one government.

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[1] As explained in Chapter 22, the best credit rating given to a company by the rating agency S&P is AAA. The second best is AA. The corresponding ratings from the rival rating agency Moody's are Aaa and Aa, respectively.

[2] As we shall see in Chapters 6 and 7, it is more accurate to say that the risk-free rate should be interpreted as the rate derived from LIBOR, swap, and Eurodollar futures quotes.

Business Snapshot 4.1 What Is the Risk-Free Rate?

It is natural to assume that the rates on Treasury bills and Treasury bonds are the correct benchmark risk-free rates for derivative traders working for financial institutions. In fact, these derivative traders usually use LIBOR rates as short-term risk-free rates. This is because they regard LIBOR as their opportunity cost of capital (see Section 4.1). Traders argue that Treasury rates are too low to be used as risk-free rates because:

1. Treasury bills and Treasury bonds must be purchased by financial institutions to fulfill a variety of regulatory requirements. This increases demand for these Treasury instruments driving the price up and the yield down.
2. The amount of capital a bank is required to hold to support an investment in Treasury bills and bonds is substantially smaller than the capital required to support a similar investment in other instruments with very low risk.
3. In the United States, Treasury instruments are given a favorable tax treatment compared with most other fixed-income investments because they are not taxed at the state level.

LIBOR is approximately equal to the short-term borrowing rate of a AA-rated company. It is therefore not a perfect proxy for the risk-free rate. There is a small chance that a AA borrower will default within the life of a LIBOR loan. Nevertheless, traders feel it is the best proxy for them to use. LIBOR rates are quoted out to 12 months. As Chapters 6 and 7 show, the Eurodollar futures market and the swap market are used to extend the trader's proxy for the risk-free rate beyond 12 months.

Repo Rates

Sometimes trading activities are funded with a *repo* or *repurchase agreement*. This is a contract where an investment dealer who owns securities agrees to sell them to another company now and buy them back later at a slightly higher price. The other company is providing a loan to the investment dealer. The difference between the price at which the securities are sold and the price at which they are repurchased is the interest it earns. The interest rate is referred to as the *repo rate*. If structured carefully, the loan involves very little credit risk. If the borrower does not honor the agreement, the lending company simply keeps the securities. If the lending company does not keep to its side of the agreement, the original owner of the securities keeps the cash.

The most common type of repo is an *overnight repo*, in which the agreement is renegotiated each day. However, longer-term arrangements, known as *term repos*, are sometimes used.

4.2 MEASURING INTEREST RATES

A statement by a bank that the interest rate on one-year deposits is 10% per annum sounds straightforward and unambiguous. In fact, its precise meaning depends on the way the interest rate is measured.

Table 4.1 Effect of the compounding frequency on the value of \$100 at the end of 1 year when the interest rate is 10% per annum.

Compounding frequency	*Value of \$100 at end of year (\$)*
Annually ($m = 1$)	110.00
Semiannually ($m = 2$)	110.25
Quarterly ($m = 4$)	110.38
Monthly ($m = 12$)	110.47
Weekly ($m = 52$)	110.51
Daily ($m = 365$)	110.52

If the interest rate is measured with annual compounding, the bank's statement that the interest rate is 10% means that \$100 grows to

$$\$100 \times 1.1 = \$110$$

at the end of 1 year. When the interest rate is measured with semiannual compounding, it means that 5% is earned every 6 months, with the interest being reinvested. In this case \$100 grows to

$$\$100 \times 1.05 \times 1.05 = \$110.25$$

at the end of 1 year. When the interest rate is measured with quarterly compounding, the bank's statement means that 2.5% is earned every 3 months, with the interest being reinvested. The \$100 then grows to

$$\$100 \times 1.025^4 = \$110.38$$

at the end of 1 year. Table 4.1 shows the effect of increasing the compounding frequency further.

The compounding frequency defines the units in which an interest rate is measured. A rate expressed with one compounding frequency can be converted into an equivalent rate with a different compounding frequency. For example, from Table 4.1 we see that 10.25% with annual compounding is equivalent to 10% with semiannual compounding. We can think of the difference between one compounding frequency and another to be analogous to the difference between kilometers and miles. They are two different units of measurement.

To generalize our results, suppose that an amount A is invested for n years at an interest rate of R per annum. If the rate is compounded once per annum, the terminal value of the investment is

$$A(1 + R)^n$$

If the rate is compounded m times per annum, the terminal value of the investment is

$$A\left(1 + \frac{R}{m}\right)^{mn} \tag{4.1}$$

When $m = 1$, the rate is sometimes referred to as the *equivalent annual interest rate.*

Continuous Compounding

The limit as the compounding frequency, m, tends to infinity is known as *continuous compounding*.[3] With continuous compounding, it can be shown that an amount A invested for n years at rate R grows to

$$Ae^{Rn} \tag{4.2}$$

where $e = 2.71828$. The exponential function, e^x, is built into most calculators, so the computation of the expression in equation (4.2) presents no problems. In the example in Table 4.1, $A = 100$, $n = 1$, and $R = 0.1$, so that the value to which A grows with continuous compounding is

$$100e^{0.1} = \$110.52$$

This is (to two decimal places) the same as the value with daily compounding. For most practical purposes, continuous compounding can be thought of as being equivalent to daily compounding. Compounding a sum of money at a continuously compounded rate R for n years involves multiplying it by e^{Rn}. Discounting it at a continuously compounded rate R for n years involves multiplying by e^{-Rn}.

In this book, interest rates will be measured with continuous compounding except where stated otherwise. Readers used to working with interest rates that are measured with annual, semiannual, or some other compounding frequency may find this a little strange at first. However, continuously compounded interest rates are used to such a great extent in pricing derivatives that it makes sense to get used to working with them now.

Suppose that R_c is a rate of interest with continuous compounding and R_m is the equivalent rate with compounding m times per annum. From the results in equations (4.1) and (4.2), we have

$$Ae^{R_c n} = A\left(1 + \frac{R_m}{m}\right)^{mn}$$

or

$$e^{R_c} = \left(1 + \frac{R_m}{m}\right)^{m}$$

This means that

$$R_c = m \ln\left(1 + \frac{R_m}{m}\right) \tag{4.3}$$

and

$$R_m = m(e^{R_c/m} - 1) \tag{4.4}$$

These equations can be used to convert a rate with a compounding frequency of m times per annum to a continuously compounded rate and vice versa. The natural logarithm function $\ln x$, which is built into most calculators, is the *inverse* of the exponential function, so that, if $y = \ln x$, then $x = e^y$.

Example 4.1

Consider an interest rate that is quoted as 10% per annum with semiannual compounding. From equation (4.3) with $m = 2$ and $R_m = 0.1$, the equivalent rate

[3] Actuaries sometimes refer to a continuously compounded rate as the *force of interest*.

with continuous compounding is

$$2\ln\left(1+\frac{0.1}{2}\right)=0.09758$$

or 9.758% per annum.

Example 4.2

Suppose that a lender quotes the interest rate on loans as 8% per annum with continuous compounding, and that interest is actually paid quarterly. From equation (4.4) with $m=4$ and $R_c=0.08$, the equivalent rate with quarterly compounding is

$$4\times(e^{0.08/4}-1)=0.0808$$

or 8.08% per annum. This means that on a \$1,000 loan, interest payments of \$20.20 would be required each quarter.

4.3 ZERO RATES

The n-year zero-coupon interest rate is the rate of interest earned on an investment that starts today and lasts for n years. All the interest and principal is realized at the end of n years. There are no intermediate payments. The n-year zero-coupon interest rate is sometimes also referred to as the n-year *spot rate*, the n-year *zero rate*, or just the n-year zero. Suppose a 5-year zero rate with continuous compounding is quoted as 5% per annum. This means that \$100, if invested for 5 years, grows to

$$100\times e^{0.05\times 5}=128.40$$

Most of the interest rates we observe directly in the market are not pure zero rates. Consider a 5-year government bond that provides a 6% coupon. The price of this bond does not by itself determine the 5-year Treasury zero rate because some of the return on the bond is realized in the form of coupons prior to the end of year 5. Later in this chapter we will discuss how we can determine Treasury zero rates from the market prices of coupon-bearing bonds.

4.4 BOND PRICING

Most bonds pay coupons to the holder periodically. The bond's principal (which is also known as its par value or face value) is paid at the end of its life. The theoretical price of a bond can be calculated as the present value of all the cash flows that will be received by the owner of the bond. Sometimes bond traders use the same discount rate for all the cash flows underlying a bond, but a more accurate approach is to use a different zero rate for each cash flow.

To illustrate this, consider the situation where Treasury zero rates, measured with continuous compounding, are as in Table 4.2. (We explain later how these can be calculated.) Suppose that a 2-year Treasury bond with a principal of \$100 provides coupons at the rate of 6% per annum semiannually. To calculate the present value of the first coupon of \$3, we discount it at 5.0% for 6 months; to calculate the present

Table 4.2 Treasury zero rates.

Maturity (years)	*Zero rate (%) (continuously compounded)*
0.5	5.0
1.0	5.8
1.5	6.4
2.0	6.8

value of the second coupon of \$3, we discount it at 5.8% for 1 year; and so on. Therefore the theoretical price of the bond is

$$3e^{-0.05\times0.5} + 3e^{-0.058\times1.0} + 3e^{-0.064\times1.5} + 103e^{-0.068\times2.0} = 98.39$$

or \$98.39.

Bond Yield

A bond's yield is the single discount rate that, when applied to all cash flows, gives a bond price equal to its market price. Suppose that the theoretical price of the bond we have been considering, \$98.39, is also its market value (i.e., the market's price of the bond is in exact agreement with the data in Table 4.2). If y is the yield on the bond, expressed with continuous compounding, it must be true that

$$3e^{-y\times0.5} + 3e^{-y\times1.0} + 3e^{-y\times1.5} + 103e^{-y\times2.0} = 98.39$$

This equation can be solved using an iterative ("trial and error") procedure to give $y = 6.76\%$.[4]

Par Yield

The *par yield* for a certain bond maturity is the coupon rate that causes the bond price to equal its par value. (The par value is the same as the principal value.) Usually the bond is assumed to provide semiannual coupons. Suppose that the coupon on a 2-year bond in our example is c per annum (or $\frac{1}{2}c$ per 6 months). Using the zero rates in Table 4.2, the value of the bond is equal to its par value of 100 when

$$\frac{c}{2}e^{-0.05\times0.5} + \frac{c}{2}e^{-0.058\times1.0} + \frac{c}{2}e^{-0.064\times1.5} + \left(100 + \frac{c}{2}\right)e^{-0.068\times2.0} = 100$$

This equation can be solved in a straightforward way to give $c = 6.87$. The 2-year par yield is therefore 6.87% per annum with semiannual compounding (or 6.75% with continuous compounding).

More generally, if d is the present value of \$1 received at the maturity of the bond, A is the value of an annuity that pays one dollar on each coupon payment date, and m

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[4] One way of solving nonlinear equations of the form $f(y) = 0$, such as this one, is to use the Newton–Raphson method. We start with an estimate y_0 of the solution and produce successively better estimates $y_1, y_2, y_3, \ldots$ using the formula $y_{i+1} = y_i - f(y_i)/f'(y_i)$, where $f'(y)$ denotes the derivative of f with respect to y.

is the number of coupon payments per year, then the par yield c must satisfy

$$100 = A\frac{c}{m} + 100d$$

so that

$$c = \frac{(100 - 100d)m}{A}$$

In our example, $m = 2$, $d = e^{-0.068\times 2} = 0.87284$, and

$$A = e^{-0.05\times 0.5} + e^{-0.058\times 1.0} + e^{-0.064\times 1.5} + e^{-0.068\times 2.0} = 3.70027$$

The formula confirms that the par yield is 6.87% per annum. Note that this is a rate expressed with semiannual compounding (and paid every 6 months). With continuous compounding, it would be 6.75% per annum.

4.5 DETERMINING TREASURY ZERO RATES

One way of determining Treasury zero rates such as those in Table 4.2 is to observe the yields on "strips". These are zero-coupon bonds that are synthetically created by traders when they sell coupons on a Treasury bond separately from the principal.

Another way to determine Treasury zero rates is from Treasury bills and coupon-bearing bonds. The most popular approach is known as the *bootstrap method.* To illustrate the nature of the method, consider the data in Table 4.3 on the prices of five bonds. Because the first three bonds pay no coupons, the zero rates corresponding to the maturities of these bonds can easily be calculated. The 3-month bond provides a return of 2.5 in 3 months on an initial investment of 97.5. With quarterly compounding, the 3-month zero rate is $(4 \times 2.5)/97.5 = 10.256\%$ per annum. Equation (4.3) shows that, when the rate is expressed with continuous compounding, it becomes

$$4\ln\left(1 + \frac{0.10256}{4}\right) = 0.10127$$

or 10.127% per annum. The 6-month bond provides a return of 5.1 in 6 months on an initial investment of 94.9. With semiannual compounding the 6-month rate is

Table 4.3 Data for bootstrap method.

Bond principal (*$*)	*Time to maturity* (*years*)	*Annual coupon** (*$*)	*Bond price* (*$*)
100	0.25	0	97.5
100	0.50	0	94.9
100	1.00	0	90.0
100	1.50	8	96.0
100	2.00	12	101.6

* Half the stated coupon is assumed to be paid every 6 months.

$(2 \times 5.1)/94.9 = 10.748\%$ per annum. Equation (4.3) shows that, when the rate is expressed with continuous compounding, it becomes

$$2\ln\left(1+\frac{0.10748}{2}\right) = 0.10469$$

or 10.469% per annum. Similarly, the 1-year rate with continuous compounding is

$$\ln\left(1+\frac{10}{90}\right) = 0.10536$$

or 10.536% per annum.

The fourth bond lasts 1.5 years. The payments are as follows:

6 months: \$4

1 year: \$4

1.5 years: \$104

From our earlier calculations, we know that the discount rate for the payment at the end of 6 months is 10.469% and that the discount rate for the payment at the end of 1 year is 10.536%. We also know that the bond's price, \$96, must equal the present value of all the payments received by the bondholder. Suppose the 1.5-year zero rate is denoted by R. It follows that

$$4e^{-0.10469\times0.5} + 4e^{-0.10536\times1.0} + 104e^{-R\times1.5} = 96$$

This reduces to

$$e^{-1.5R} = 0.85196$$

or

$$R = -\frac{\ln(0.85196)}{1.5} = 0.10681$$

The 1.5-year zero rate is therefore 10.681%. This is the only zero rate that is consistent with the 6-month rate, 1-year rate, and the data in Table 4.3.

The 2-year zero rate can be calculated similarly from the 6-month, 1-year, and 1.5-year zero rates, and the information on the last bond in Table 4.3. If R is the 2-year zero rate, then

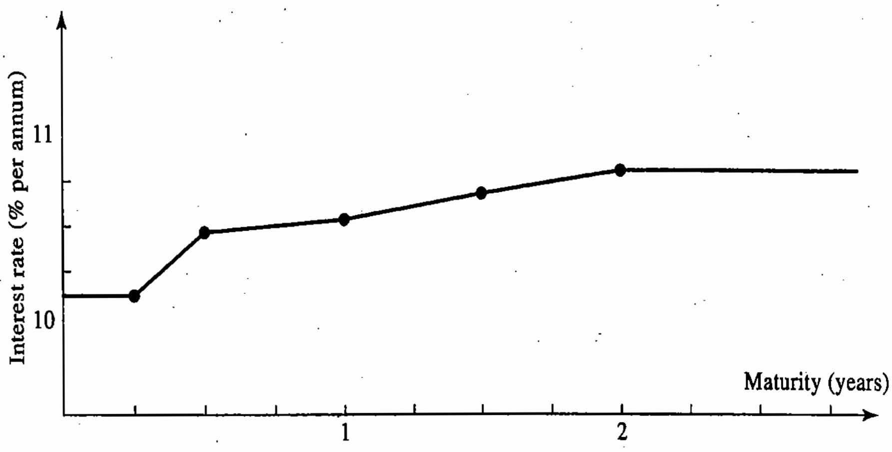
$$6e^{-0.10469\times0.5} + 6e^{-0.10536\times1.0} + 6e^{-0.10681\times1.5} + 106e^{-R\times2.0} = 101.6$$

This gives $R = 0.10808$, or 10.808%.

Table 4.4 Continuously compounded zero rates determined from data in Table 4.3.

Maturity (years)	*Zero rate (%) (continuously compounded)*
0.25	10.127
0.50	10.469
1.00	10.536
1.50	10.681
2.00	10.808

Figure 4.1 Zero rates given by the bootstrap method.



The rates we have calculated are summarized in Table 4.4. A chart showing the zero rate as a function of maturity is known as the *zero curve*. A common assumption is that the zero curve is linear between the points determined using the bootstrap method. (This means that the 1.25-year zero rate is $0.5 \times 10.536 + 0.5 \times 10.681 = 10.6085\%$ in our example.) It is also usually assumed that the zero curve is horizontal prior to the first point and horizontal beyond the last point. Figure 4.1 shows the zero curve for our data using these assumptions. By using longer maturity bonds, the zero curve would be more accurately determined beyond 2 years.

In practice, we do not usually have bonds with maturities equal to exactly 1.5 years, 2 years, 2.5 years, and so on. The approach often used by analysts is to interpolate between the bond price data before it is used to calculate the zero curve. For example, if they know that a 2.3-year bond with a coupon of 6% sells for 98 and a 2.7-year bond with a coupon of 6.5% sells for 99, it might be assumed that a 2.5-year bond with a coupon of 6.25% would sell for 98.5.

4.6 FORWARD RATES

Forward interest rates are the rates of interest implied by current zero rates for periods of time in the future. To illustrate how they are calculated, we suppose that a particular set of zero rates are as shown in the second column of Table 4.5. The rates are assumed to be continuously compounded. Thus, the 3% per annum rate for 1 year means that, in return for an investment of \$100 today, an investor receives $100e^{0.03\times 1} = \$103.05$ in 1 year; the 4% per annum rate for 2 years means that, in return for an investment of \$100 today, the investor receives $100e^{0.04\times 2} = \$108.33$ in 2 years; and so on.

The forward interest rate in Table 4.5 for year 2 is 5% per annum. This is the rate of interest that is implied by the zero rates for the period of time between the end of the first year and the end of the second year. It can be calculated from the 1-year zero interest rate of 3% per annum and the 2-year zero interest rate of 4% per annum. It is the rate of interest for year 2 that, when combined with 3% per annum for year 1, gives 4% overall for the 2 years. To show that the correct answer is 5% per annum, suppose that \$100 is invested. A rate of 3% for the first year and 5% for the second year gives

$$100e^{0.03\times 1}e^{0.05\times 1} = \$108.33$$

at the end of the second year. A rate of 4% per annum for 2 years gives

$$100e^{0.04\times 2}$$

which is also \$108.33. This example illustrates the general result that when interest rates are continuously compounded and rates in successive time periods are combined, the overall equivalent rate is simply the average rate during the whole period. In our example, 3% for the first year and 5% for the second year average to 4% over the 2 years. The result is only approximately true when the rates are not continuously compounded.

The forward rate for year 3 is the rate of interest that is implied by a 4% per annum 2-year zero rate and a 4.6% per annum 3-year zero rate. It is 5.8% per annum. The reason is that an investment for 2 years at 4% per annum combined with an investment for one year at 5.8% per annum gives an overall average return for the three years of 4.6% per annum. The other forward rates can be calculated similarly and are shown in the third column of the table. In general, if R_1 and R_2 are the zero rates for maturities T_1 and T_2, respectively, and R_F is the forward interest rate for the period of time between T_1 and T_2, then

$$R_F = \frac{R_2T_2 - R_1T_1}{T_2 - T_1} \tag{4.5}$$

To illustrate this formula, consider the calculation of the year-4 forward rate from the data in Table 4.5: $T_1 = 3$, $T_2 = 4$, $R_1 = 0.046$, and $R_2 = 0.05$, and the formula gives $R_F = 0.062$.

Equation (4.5) can be written as

$$R_F = R_2 + (R_2 - R_1)\frac{T_1}{T_2 - T_1} \tag{4.6}$$

This shows that if the zero curve is upward sloping between T_1 and T_2, so that $R_2 > R_1$, then $R_F > R_2$ (i.e., the forward rate for a period of time ending at T_2 is greater than the T_2 zero rate). Similarly, if the zero curve is downward sloping with $R_2 < R_1$, then

Table 4.5 Calculation of forward rates.

Year (*n*)	*Zero rate for an n-year investment* (*% per annum*)	*Forward rate for nth year* (*% per annum*)
1	3.0	
2	4.0	5.0
3	4.6	5.8
4	5.0	6.2
5	5.3	6.5

Business Snapshot 4.2 Orange County's Yield Curve Plays

Suppose an investor can borrow or lend at the rates given in Table 4.5 and thinks that 1-year interest rates will not change much over the next 5 years. The investor can borrow 1-year funds and invest for 5-years. The 1-year borrowings can be rolled over for further 1-year periods at the end of the first, second, third, and fourth years. If interest rates do stay about the same, this strategy will yield a profit of about 2.3% per year, because interest will be received at 5.3% and paid at 3%. This type of trading strategy is known as a *yield curve play*. The investor is speculating that rates in the future will be quite different from the forward rates observed in the market today. (In our example, forward rates observed in the market today for future 1-year periods are 5%, 5.8%, 6.2%, and 6.5%.)

Robert Citron, the Treasurer at Orange County, used yield curve plays similar to the one we have just described very successfully in 1992 and 1993. The profit from Mr. Citron's trades became an important contributor to Orange County's budget and he was re-elected. (No one listened to his opponent in the election who said his trading strategy was too risky.)

In 1994 Mr. Citron expanded his yield curve plays. He invested heavily in *inverse floaters*. These pay a rate of interest equal to a fixed rate of interest minus a floating rate. He also leveraged his position by borrowing in the repo market. If short-term interest rates had remained the same or declined he would have continued to do well. As it happened, interest rates rose sharply during 1994. On December 1, 1994, Orange County announced that its investment portfolio had lost $1.5 billion and several days later it filed for bankruptcy protection.

$R_F < R_2$ (i.e., the forward rate is less than the T_2 zero rate). Taking limits as T_2 approaches T_1 in equation (4.6) and letting the common value of the two be T, we obtain

$$R_F = R + T\frac{\partial R}{\partial T}$$

where R is the zero rate for a maturity of T. The value of R_F obtained in this way is known as the *instantaneous forward rate* for a maturity of T. This is the forward rate that is applicable to a very short future time period that begins at time T. Define $P(0, T)$ as the price of a zero-coupon bond maturing at time T. Because $P(0, T) = e^{-RT}$, the equation for the instantaneous forward rate can also be written as

$$R_F = -\frac{\partial}{\partial T}\ln P(0, T)$$

Assuming that the zero rates for borrowing and investing are the same (which is close to the truth for a large financial institution), an investor can lock in the forward rate for a future time period. Suppose, for example, that the zero rates are as in Table 4.5. If an investor borrows \$100 at 3% for 1 year and then invests the money at 4% for 2 years, the result is a cash outflow of $100e^{0.03\times 1} = \$103.05$ at the end of year 1 and an inflow of $100e^{0.04\times 2} = \$108.33$ at the end of year 2. Since $108.33 = 103.05e^{0.05}$, a return equal to the forward rate (5%) is earned on \$103.05 during the second year. Suppose next that the investor borrows \$100 for four years at 5% and invests it for three years at 4.6%. The result is a cash inflow of $100e^{0.046\times 3} = \$114.80$ at the end of the third year and a cash outflow of $100e^{0.05\times4} = \$122.14$ at the end of the fourth year. Since $122.14 = 114.80e^{0.062}$, money is being borrowed for the fourth year at the forward rate of 6.2%.

If an investor thinks that rates in the future will be different from today's forward rates there are many trading strategies that the investor will find attractive (see Business Snapshot 4.2). One of these involves entering into a contract known as a *forward rate agreement*. We will now discuss how this contract works and how it is valued.

4.7 FORWARD RATE AGREEMENTS

A forward rate agreement (FRA) is an over-the-counter agreement that a certain interest rate will apply to either borrowing or lending a certain principal during a specified future period of time. The assumption underlying the contract is that the borrowing or lending would normally be done at LIBOR.

Consider a forward rate agreement where company X is agreeing to lend money to company Y for the period of time between T_1 and T_2. Define:

R_K: The rate of interest agreed to in the FRA

R_F: The forward LIBOR interest rate for the period between times T_1 and T_2, calculated today[5]

R_M: The actual LIBOR interest rate observed in the market at time T_1 for the period between times T_1 and T_2

L: The principal underlying the contract

We will depart from our usual assumption of continuous compounding and assume that the rates R_K, R_F, and R_M are all measured with a compounding frequency reflecting the length of the period to which they apply. This means that if $T_2 - T_1 = 0.5$, they are expressed with semiannual compounding; if $T_2 - T_1 = 0.25$, they are expressed with quarterly compounding; and so on. (This assumption corresponds to the usual market practice for FRAs.)

Normally company X would earn R_M from the LIBOR loan. The FRA means that it will earn R_K. The extra interest rate (which may be negative) that it earns as a result of entering into the FRA is $R_K - R_M$. The interest rate is set at time T_1 and paid at time T_2. The extra interest rate therefore leads to a cash flow to company X at time T_2 of

$$L(R_K - R_M)(T_2 - T_1) \tag{4.7}$$

Similarly there is a cash flow to company Y at time T_2 of

$$L(R_M - R_K)(T_2 - T_1) \tag{4.8}$$

From equations (4.7) and (4.8), we see that there is another interpretation of the FRA. It is an agreement where company X will receive interest on the principal between T_1 and T_2 at the fixed rate of R_K and pay interest at the realized market rate of R_M. Company Y will pay interest on the principal between T_1 and T_2 at the fixed rate of R_K and receive interest at R_M.

Usually FRAs are settled at time T_1 rather than T_2. The payoff must then be

[5] LIBOR forward rates are calculated as described in Section 4.6 from the LIBOR/swap zero curve. The latter is determined in the way described in Section 7.6.

discounted from time T_2 to T_1. For company X, the payoff at time T_1 is

$$\frac{L(R_K - R_M)(T_2 - T_1)}{1 + R_M(T_2 - T_1)}$$

and, for company Y, the payoff at time T_1 is

$$\frac{L(R_M - R_K)(T_2 - T_1)}{1 + R_M(T_2 - T_1)}$$

Example 4.3

Suppose that a company enters into an FRA that specifies it will receive a fixed rate of 4% on a principal of \$100 million for a 3-month period starting in 3 years. If 3-month LIBOR proves to be 4.5% for the 3-month period the cash flow to the lender will be

$$100{,}000{,}000 \times (0.04 - 0.045) \times 0.25 = -\$125{,}000$$

at the 3.25-year point. This is equivalent to a cash flow of

$$-\frac{125{,}000}{1 + 0.045 \times 0.25} = -\$123{,}609$$

at the 3-year point. The cash flow to the party on the opposite side of the transaction will be +\$125,000 at the 3.25-year point or +\$123,609 at the 3-year point. (All interest rates in this example are expressed with quarterly compounding.)

Valuation

To value an FRA we first note that it is always worth zero when $R_K = R_F$.[6] This is because, as noted in Section 4.6, a large financial institution can at no cost lock in the forward rate for a future time period. For example, it can ensure that it earns the forward rate for the time period between years 2 and 3 by borrowing a certain amount of money for 2 years and investing it for 3 years. Similarly, it can ensure that it pays the forward rate for the time period between years 2 and 3 by borrowing for a certain amount of money for 3 years and investing it for 2 years.

Compare two FRAs. The first promises that the LIBOR forward rate R_F will be earned on a principal of L between times T_1 and T_2; the second promises that R_K will be earned on the same principal between the same two dates. The two contracts are the same except for the interest payments received at time T_2. The excess of the value of the second contract over the first is, therefore, the present value of the difference between these interest payments, or

$$L(R_K - R_F)(T_2 - T_1)e^{-R_2 T_2}$$

where R_2 is the continously compounded riskless zero rate for a maturity T_2.[7] Because the value of the FRA where R_F is earned is zero, the value of the FRA where R_K is earned is

$$V_{\text{FRA}} = L(R_K - R_F)(T_2 - T_1)e^{-R_2 T_2} \tag{4.9}$$

[6] It is usually the case that R_K is set equal to R_F when the FRA is first initiated.

[7] Note that R_K, R_M, and R_F are expressed with a compounding frequency corresponding to $T_2 - T_1$, whereas R_2 is expressed with continuous compounding.

Similarly, for a company receiving interest at the floating rate and paying interest at R_K, the value of the FRA is

$$V_{\text{FRA}} = L(R_F - R_K)(T_2 - T_1)e^{-R_2T_2} \tag{4.10}$$

By comparing equations (4.7) and (4.9) or equations (4.8) and (4.10), we see that an FRA can be valued if we:

1. Calculate the payoff on the assumption that forward rates are realized (that is, on the assumption that $R_M = R_F$).
2. Discount this payoff at the risk-free rate.

Example 4.4

Suppose that LIBOR zero and forward rates are as in Table 4.5. Consider an FRA where we will receive a rate of 6%, measured with annual compounding, on a principal of \$100 million between the end of year 1 and the end of year 2. In this case, the forward rate is 5% with continuous compounding or 5.127% with annual compounding. From equation (4.9), it follows that the value of the FRA is

$$100{,}000{,}000 \times (0.06 - 0.05127)e^{-0.04\times 2} = \$805{,}800$$

4.8 DURATION

The *duration* of a bond, as its name implies, is a measure of how long on average the holder of the bond has to wait before receiving cash payments. A zero-coupon bond that lasts n years has a duration of n years. However, a coupon-bearing bond lasting n years has a duration of less than n years, because the holder receives some of the cash payments prior to year n.

Suppose that a bond provides the holder with cash flows c_i at time t_i ($1 \leqslant i \leqslant n$). The bond price B and bond yield y (continuously compounded) are related by

$$B = \sum_{i=1}^{n} c_i e^{-yt_i} \tag{4.11}$$

The duration of the bond, D, is defined as

$$D = \frac{\sum_{i=1}^{n} t_i c_i e^{-yt_i}}{B} \tag{4.12}$$

This can be written

$$D = \sum_{i=1}^{n} t_i \left[\frac{c_i e^{-yt_i}}{B}\right]$$

The term in square brackets is the ratio of the present value of the cash flow at time t_i to the bond price. The bond price is the present value of all payments. The duration is therefore a weighted average of the times when payments are made, with the weight applied to time t_i being equal to the proportion of the bond's total present value provided by the cash flow at time t_i. The sum of the weights is 1.0. Note that for the purposes of the definition of duration all discounting is done at the bond yield rate of interest, y. (We do not use a different zero rate for each cash flow as described in Section 4.4.)

Table 4.6 Calculation of duration.

Time (years)	*Cash flow ($)*	*Present value*	*Weight*	*Time × weight*
0.5	5	4.709	0.050	0.025
1.0	5	4.435	0.047	0.047
1.5	5	4.176	0.044	0.066
2.0	5	3.933	0.042	0.083
2.5	5	3.704	0.039	0.098
3.0	105	73.256	0.778	2.333
Total:	130	94.213	1.000	2.653

When a small change Δy in the yield is considered, it is approximately true that

$$\Delta B = \frac{dB}{dy}\Delta y \tag{4.13}$$

From equation (4.11), this becomes

$$\Delta B = -\Delta y \sum_{i=1}^{n} c_i t_i e^{-yt_i} \tag{4.14}$$

(Note that there is a negative relationship between B and y. When bond yields increase, bond prices decrease. When bond yields decrease, bond prices increase.) From equations (4.12) and (4.14), the key duration relationship is obtained:

$$\Delta B = -BD\,\Delta y \tag{4.15}$$

This can be written

$$\frac{\Delta B}{B} = -D\,\Delta y \tag{4.16}$$

Equation (4.16) is an approximate relationship between percentage changes in a bond price and changes in its yield. It is easy to use and is the reason why duration, which was first suggested by Macaulay in 1938, has become such a popular measure.

Consider a 3-year 10% coupon bond with a face value of $100. Suppose that the yield on the bond is 12% per annum with continuous compounding. This means that $y = 0.12$. Coupon payments of $5 are made every 6 months. Table 4.6 shows the calculations necessary to determine the bond's duration. The present values of the bond's cash flows, using the yield as the discount rate, are shown in column 3 (e.g., the present value of the first cash flow is $5e^{-0.12\times 0.5} = 4.709$). The sum of the numbers in column 3 gives the bond's price as 94.213. The weights are calculated by dividing the numbers in column 3 by 94.213. The sum of the numbers in column 5 gives the duration as 2.653 years.

Small changes in interest rates are often measured in *basis points*. As mentioned earlier, a basis point is 0.01% per annum. The following example investigates the accuracy of the duration relationship in equation (4.15).

Example 4.5

For the bond in Table 4.6, the bond price, B, is 94.213 and the duration, D, is 2.653, so that equation (4.15) gives

$$\Delta B = -94.213 \times 2.653\,\Delta y$$

or

$$\Delta B = -249.95\,\Delta y$$

When the yield on the bond increases by 10 basis points (= 0.1%), $\Delta y = +0.001$. The duration relationship predicts that $\Delta B = -249.95 \times 0.001 = -0.250$, so that the bond price goes down to $94.213 - 0.250 = 93.963$. How accurate is this? Valuing the bond in terms of its yield in the usual way, we find that, when the bond yield increases by 10 basis points to 12.1%, the bond price is

$$5e^{-0.121\times 0.5} + 5e^{-0.121\times 1.0} + 5e^{-0.121\times 1.5} + 5e^{-0.121\times 2.0} + 5e^{-0.121\times 2.5} + 105e^{-0.121\times 3.0} = 93.963$$

which is (to three decimal places) the same as that predicted by the duration relationship.

Modified Duration

The preceding analysis is based on the assumption that y is expressed with continuous compounding. If y is expressed with annual compounding, it can be shown that the approximate relationship in equation (4.15) becomes

$$\Delta B = -\frac{BD\,\Delta y}{1+y}$$

More generally, if y is expressed with a compounding frequency of m times per year, then

$$\Delta B = -\frac{BD\,\Delta y}{1+y/m}$$

A variable D^*, defined by

$$D^* = \frac{D}{1+y/m}$$

is sometimes referred to as the bond's *modified duration*. It allows the duration relationship to be simplified to

$$\Delta B = -BD^*\Delta y \tag{4.17}$$

when y is expressed with a compounding frequency of m times per year. The following example investigates the accuracy of the modified duration relationship.

Example 4.6

The bond in Table 4.6 has a price of 94.213 and a duration of 2.653. The yield, expressed with semiannual compounding is 12.3673%. The modified duration, D^*, is given by

$$D^* = \frac{2.653}{1+0.123673/2} = 2.499$$

From equation (4.17),

$$\Delta B = -94.213 \times 2.4985\,\Delta y$$

or

$$\Delta B = -235.39\,\Delta y$$

When the yield (semiannually compounded) increases by 10 basis points (= 0.1%), we have $\Delta y = +0.001$. The duration relationship predicts that we expect ΔB to be $-235.39 \times 0.001 = -0.235$, so that the bond price goes down to $94.213 - 0.235 = 93.978$. How accurate is this? An exact calculation similar to that in the previous example shows that, when the bond yield (semiannually compounded) increases by 10 basis points to 12.4673%, the bond price becomes 93.978. This shows that the modified duration calculation gives good accuracy for small yield changes.

Another term that is sometimes used is *dollar duration*. This is the product of modified duration and bond price, so that $\Delta B = -D^{**}\Delta y$, where D^{**} is dollar duration.

Bond Portfolios

The duration, D, of a bond portfolio can be defined as a weighted average of the durations of the individual bonds in the portfolio, with the weights being proportional to the bond prices. Equations (4.15) to (4.17) then apply, with B being defined as the value of the bond portfolio. They estimate the change in the value of the bond portfolio for a small change Δy in the yields of all the bonds.

It is important to realize that, when duration is used for bond portfolios, there is an implicit assumption that the yields of all bonds will change by approximately the same amount. When the bonds have widely differing maturities, this happens only when there is a parallel shift in the zero-coupon yield curve. We should therefore interpret equations (4.15) to (4.17) as providing estimates of the impact on the price of a bond portfolio of a small parallel shift, Δy, in the zero curve.

By choosing a portfolio so that the duration of assets equals the duration of liabilities (i.e., the net duration is zero), a financial institution eliminates its exposure to small parallel shifts in the yield curve. It is still exposed to shifts that are either large or nonparallel.

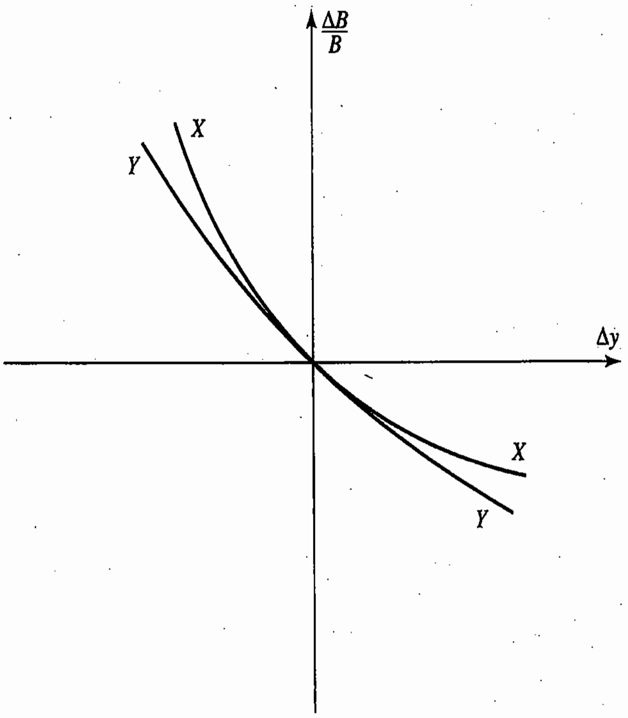
4.9 CONVEXITY

The duration relationship applies only to small changes in yields. This is illustrated in Figure 4.2, which shows the relationship between the percentage change in value and change in yield for two bond portfolios having the same duration. The gradients of the two curves are the same at the origin. This means that both bond portfolios change in value by the same percentage for small yield changes and is consistent with equation (4.16). For large yield changes, the portfolios behave differently. Portfolio X has more curvature in its relationship with yields than portfolio Y. A factor known as *convexity* measures this curvature and can be used to improve the relationship in equation (4.16).

A measure of convexity is

$$C = \frac{1}{B}\frac{d^2B}{dy^2} = \frac{\sum_{i=1}^{n} c_i t_i^2 e^{-yt_i}}{B}$$

Figure 4.2 Two bond portfolios with the same duration.



From Taylor series expansions, we obtain a more accurate expression than equation (4.13), given by

$$\Delta B = \frac{dB}{dy}\Delta y + \frac{1}{2}\frac{d^2B}{dy^2}\Delta y^2$$

This leads to

$$\frac{\Delta B}{B} = -D\,\Delta y + \tfrac{1}{2}C(\Delta y)^2$$

The convexity of a bond portfolio tends to be greatest when the portfolio provides payments evenly over a long period of time. It is least when the payments are concentrated around one particular point in time. By choosing a portfolio of assets and liabilities with a net duration of zero and a net convexity of zero, a financial institution can make itself immune to relatively large parallel shifts in the zero curve. However, it is still exposed to nonparallel shifts.

4.10 THEORIES OF THE TERM STRUCTURE OF INTEREST RATES

It is natural to ask what determines the shape of the zero curve. Why is it sometimes downward sloping, sometimes upward sloping, and sometimes partly upward sloping and partly downward sloping? A number of different theories have been proposed. The simplest is *expectations theory*, which conjectures that long-term interest rates should reflect expected future short-term interest rates. More precisely, it argues that a forward interest rate corresponding to a certain future period is equal to the expected future zero interest rate for that period. Another idea, *market segmentation theory*, conjectures that there need be no relationship between short-, medium-, and long-term interest rates. Under the theory, a major investor such as a large pension fund invests in bonds of a certain maturity and does not readily switch from one maturity to another. The short-term interest rate is determined by supply and demand in the short-term bond market; the medium-term interest rate is determined by supply and demand in the medium-term bond market; and so on.

The theory that is most appealing is *liquidity preference theory*, which argues that forward rates should always be higher than expected future zero rates. The basic assumption underlying the theory is that investors prefer to preserve their liquidity and invest funds for short periods of time. Borrowers, on the other hand, usually prefer to borrow at fixed rates for long periods of time. Liquidity preference theory leads to a situation in which forward rates are greater than expected future zero rates. It is also consistent with the empirical result that yield curves tend to be upward sloping more often than they are downward sloping.

The Management of Net Interest Income

To understand liquidity preference theory, it is useful to consider the interest rate risk faced by banks when they take deposits and make loans. The *net interest income* of the bank is the excess of the interest received over the interest paid and needs to be carefully managed.

Consider a simple situation where a bank offers consumers a one-year and a five-year deposit rate as well as a one-year and five-year mortgage rate. The rates are shown in Table 4.7. We make the simplifying assumption that the expected one-year interest rate for future time periods to equal the one-year rates prevailing in the market today. Loosely speaking this means that the market considers interest rate increases to be just as likely as interest rate decreases. As a result, the rates in Table 4.7 are "fair" in that they reflect the market's expectations (i.e., they correspond to expectations theory). Investing money for one year and reinvesting for four further one-year periods give the same expected overall return as a single five-year investment. Similarly, borrowing money for one year and refinancing each year for the next four years leads to the same expected financing costs as a single five-year loan.

Suppose you have money to deposit and agree with the prevailing view that interest rate increases are just as likely as interest rate decreases. Would you choose to deposit your money for one year at 3% per annum or for five years at 3% per annum? The chances are that you would choose one year because this gives you more financial flexibility. It ties up your funds for a shorter period of time.

Table 4.7 Example of rates offered by a bank to its customers.

Maturity (years)	*Deposit rate*	*Mortgage rate*
1	3%	6%
5	3%	6%

Now suppose that you want a mortgage. Again you agree with the prevailing view that interest rate increases are just as likely as interest rate decreases. Would you choose a one-year mortgage at 6% or a five-year mortgage at 6%? The chances are that you would choose a five-year mortgage because it fixes your borrowing rate for the next five years and subjects you to less refinancing risk.

When the bank posts the rates shown in Table 4.7, it is likely to find that the majority of its customers opt for one-year deposits and five-year mortgages. This creates an asset/liability mismatch for the bank and subjects it to risks. There is no problem if interest rates fall. The bank will find itself financing the five-year 6% loans with deposits that cost less than 3% in the future and net interest income will increase. However, if rates rise, the deposits that are financing these 6% loans will cost more than 3% in the future and net interest income will decline. A 3% rise in interest rates would reduce the net interest income to zero.

It is the job of the asset/liability management group to ensure that the maturities of the assets on which interest is earned and the maturities of the liabilities on which interest is paid are matched. One way it can do this is by increasing the five-year rate on both deposits and mortgages. For example, it could move to the situation in Table 4.8 where the five-year deposit rate is 4% and the five-year mortgage rate 7%. This would make five-year deposits relatively more attractive and one-year mortgages relatively more attractive. Some customers who chose one-year deposits when the rates were as in Table 4.7 will switch to five-year deposits in the Table 4.8 situation. Some customers who chose five-year mortgages when the rates were as in Table 4.7 will choose one-year mortgages. This may lead to the maturities of assets and liabilities being matched. If there is still an imbalance with depositors tending to choose a one-year maturity and borrowers a five-year maturity, five-year deposit and mortgage rates could be increased even further. Eventually the imbalance will disappear.

The net result of all banks behaving in the way we have just described is liquidity preference theory. Long-term rates tend to be higher than those that would be predicted by expected future short-term rates. The yield curve is upward sloping most of the time. It is downward sloping only when the market expects a really steep decline in short-term rates.

Many banks now have sophisticated systems for monitoring the decisions being made by customers so that, when they detect small differences between the maturities of the assets and liabilities being chosen by customers they can fine tune the rates they offer. Sometimes derivatives such as interest rate swaps (which will be discussed in Chapter 7) are also used to manage their exposure. The result of all this is that net interest income is very stable. As indicated in Business Snapshot 4.3, this has not always been the case.

Table 4.8 Five-year rates are increased in an attempt to match maturities of assets and liabilities.

Maturity (years)	*Deposit rate*	*Mortgage rate*
1	3%	6%
5	4%	7%

Business Snapshot 4.3 Expensive Failures of Financial Institutions in the US

Throughout the 1960s, 1970s, and 1980s, Savings and Loans (S&Ls) in the United States failed to manage interest rate risk well. They tended to take short-term deposits and offer long-term fixed-rate mortgages. As a result, they were seriously hurt by interest rate increases in 1969–70, 1974, and the killer in 1979–82. S&Ls were protected by government guarantees. Over 1,700 failed in the 1980s. A major reason for the failures was the lack of interest rate risk management. The total cost to the US tax payer of the failures has been estimated to be between \$100 and \$500 billion.

The largest bank failure in the US, Contintental Illinois, can also be attributed to a failure to manage interest rate risks well. During the period 1980 to 1983, its assets (i.e., loans) with maturities over a year totaled between \$7 and \$8 billion, whereas its liabilities (i.e., deposits) with maturities over a year were between \$1.4 and \$2.5 million. Continental failed in 1984 and was the subject of an expensive government bailout.

SUMMARY

Two important interest rates for derivative traders are Treasury rates and LIBOR rates. Treasury rates are the rates paid by a government on borrowings in its own currency. LIBOR rates are short-term lending rates offered by banks in the interbank market. Derivatives traders assume that the LIBOR rate is a risk-free rate.

The compounding frequency used for an interest rate defines the units in which it is measured. The difference between an annually compounded rate and a quarterly compounded rate is analogous to the difference between a distance measured in miles and a distance measured in kilometers. Traders frequently use continuous compounding when analyzing the value of derivatives.

Many different types of interest rates are quoted in financial markets and calculated by analysts. The n-year zero or spot rate is the rate applicable to an investment lasting for n years when all of the return is realized at the end. The par yield on a bond of a certain maturity is the coupon rate that causes the bond to sell for its par value. Forward rates are the rates applicable to future periods of time implied by today's zero rates.

The method most commonly used to calculate zero rates is known as the bootstrap method. It involves starting with short-term instruments and moving progressively to longer-term instruments, making sure that the zero rates calculated at each stage are consistent with the prices of the instruments. It is used daily by trading desks to calculate a Treasury zero-rate curve.

A forward rate agreement (FRA) is an over-the-counter agreement that a certain interest rate will apply to either borrowing or lending a certain principal at LIBOR during a specified future period of time. An FRA can be valued by assuming that forward rates are realized and discounting the resulting payoff.

An important concept in interest rate markets is *duration*. Duration measures the sensitivity of the value of a bond portfolio to a small parallel shift in the zero-coupon yield curve. Specifically,

$$\Delta B = -BD\,\Delta y$$

where B is the value of the bond portfolio, D is the duration of the portfolio, Δy is the size of a small parallel shift in the zero curve, and ΔB is the resultant effect on the value of the bond portfolio.

Liquidity preference theory can be used to explain the interest rate term structures that are observed in practice. The theory argues that most entities like to borrow long and lend short. To match the maturities of borrowers and lenders, it is necessary for financial institutions to raise long-term rates so that forward interest rates are higher than expected future spot interest rates.

FURTHER READING

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Questions and Problems (Answers in Solutions Manual)

4.1. A bank quotes an interest rate of 14% per annum with quarterly compounding. What is the equivalent rate with (a) continuous compounding and (b) annual compounding?

4.2. What is meant by LIBOR and LIBID. Which is higher?

4.3. The 6-month and 1-year zero rates are both 10% per annum. For a bond that has a life of 18 months and pays a coupon of 8% per annum (with semiannual payments and one having just been made), the yield is 10.4% per annum. What is the bond's price? What is the 18-month zero rate? All rates are quoted with semiannual compounding.

4.4. An investor receives $1,100 in one year in return for an investment of $1,000 now. Calculate the percentage return per annum with:
(a) Annual compounding
(b) Semiannual compounding
(c) Monthly compounding
(d) Continuous compounding

4.5. Suppose that zero interest rates with continuous compounding are as follows:

Maturity (months)	*Rate (% per annum)*
3	8.0
6	8.2
9	8.4
12	8.5
15	8.6
18	8.7

Calculate forward interest rates for the second, third, fourth, fifth, and sixth quarters.

4.6. Assuming that zero rates are as in Problem 4.5, what is the value of an FRA that enables the holder to earn 9.5% for a 3-month period starting in 1 year on a principal of \$1,000,000? The interest rate is expressed with quarterly compounding.

4.7. The term structure of interest rates is upward-sloping. Put the following in order of magnitude:
(a) The 5-year zero rate
(b) The yield on a 5-year coupon-bearing bond
(c) The forward rate corresponding to the period between 4.75 and 5 years in the future
What is the answer to this question when the term structure of interest rates is downward-sloping?

4.8. What does duration tell you about the sensitivity of a bond portfolio to interest rates. What are the limitations of the duration measure?

4.9. What rate of interest with continuous compounding is equivalent to 15% per annum with monthly compounding?

4.10. A deposit account pays 12% per annum with continuous compounding, but interest is actually paid quarterly. How much interest will be paid each quarter on a \$10,000 deposit?

4.11. Suppose that 6-month, 12-month, 18-month, 24-month, and 30-month zero rates are, respectively, 4%, 4.2%, 4.4%, 4.6%, and 4.8% per annum, with continuous compounding. Estimate the cash price of a bond with a face value of 100 that will mature in 30 months and pays a coupon of 4% per annum semiannually.

4.12. A 3-year bond provides a coupon of 8% semiannually and has a cash price of 104. What is the bond's yield?

4.13. Suppose that the 6-month, 12-month, 18-month, and 24-month zero rates are 5%, 6%, 6.5%, and 7%, respectively. What is the 2-year par yield?

4.14. Suppose that zero interest rates with continuous compounding are as follows:

Maturity (years)	*Rate (% per annum)*
1	2.0
2	3.0
3	3.7
4	4.2
5	4.5

Calculate forward interest rates for the second, third, fourth, and fifth years.

4.15. Use the rates in Problem 4.14 to value an FRA where you will pay 5% (compounded anually) for the third year on \$1 million.

4.16. A 10-year 8% coupon bond currently sells for \$90. A 10-year 4% coupon bond currently sells for \$80. What is the 10-year zero rate? (*Hint*: Consider taking a long position in two of the 4% coupon bonds and a short position in one of the 8% coupon bonds.)

4.17. Explain carefully why liquidity preference theory is consistent with the observation that the term structure of interest rates tends to be upward-sloping more often than it is downward-sloping.

4.18. "When the zero curve is upward-sloping, the zero rate for a particular maturity is greater than the par yield for that maturity. When the zero curve is downward-sloping the reverse is true." Explain why this is so.

4.19. Why are US Treasury rates significantly lower than other rates that are close to risk-free?

4.20. Why does a loan in the repo market involve very little credit risk?

4.21. Explain why an FRA is equivalent to the exchange of a floating rate of interest for a fixed rate of interest.

4.22. A 5-year bond with a yield of 11% (continuously compounded) pays an 8% coupon at the end of each year.
(a) What is the bond's price?
(b) What is the bond's duration?
(c) Use the duration to calculate the effect on the bond's price of a 0.2% decrease in its yield.
(d) Recalculate the bond's price on the basis of a 10.8% per annum yield and verify that the result is in agreement with your answer to (c).

4.23. The cash prices of 6-month and 1-year Treasury bills are 94.0 and 89.0. A 1.5-year bond that will pay coupons of $4 every 6 months currently sells for $94.84. A 2-year bond that will pay coupons of $5 every 6 months currently sells for $97.12. Calculate the 6-month, 1-year, 1.5-year, and 2-year zero rates.

Assignment Questions

4.24. An interest rate is quoted as 5% per annum with semiannual compounding. What is the equivalent rate with (a) annual compounding, (b) monthly compounding, and (c) continuous compounding.

4.25. The 6-month, 12-month, 18-month, and 24-month zero rates are 4%, 4.5%, 4.75%, and 5%, with semiannual compounding.
(a) What are the rates with continuous compounding?
(b) What is the forward rate for the 6-month period beginning in 18 months?
(c) What is the value of an FRA that promises to pay you 6% (compounded semiannually) on a principal of $1 million for the 6-month period starting in 18 months?

4.26. What is the 2-year par yield when the zero rates are as in Problem 4.25? What is the yield on a 2-year bond that pays a coupon equal to the par yield?

4.27. The following table gives the prices of bonds:

Bond principal ($)	*Time to maturity (years)*	*Annual coupon* ($)*	*Bond price ($)*
100	0.50	0.0	98
100	1.00	0.0	95
100	1.50	6.2	101
100	2.00	8.0	104

* Half the stated coupon is assumed to be paid every six months.

(a) Calculate zero rates for maturities of 6 months, 12 months, 18 months, and 24 months.

(b) What are the forward rates for the following periods: 6 months to 12 months, 12 months to 18 months, and 18 months to 24 months?
(c) What are the 6-month, 12-month, 18-month, and 24-month par yields for bonds that provide semiannual coupon payments?
(d) Estimate the price and yield of a 2-year bond providing a semiannual coupon of 7% per annum.

4.28. Portfolio A consists of a 1-year zero-coupon bond with a face value of $2,000 and a 10-year zero-coupon bond with a face value of $6,000. Portfolio B consists of a 5.95-year zero-coupon bond with a face value of $5,000. The current yield on all bonds is 10% per annum.
(a) Show that both portfolios have the same duration.
(b) Show that the percentage changes in the values of the two portfolios for a 0.1% per annum increase in yields are the same.
(c) What are the percentage changes in the values of the two portfolios for a 5% per annum increase in yields?

CHAPTER 5

Determination of Forward and Futures Prices

In this chapter we examine how forward prices and futures prices are related to the spot price of the underlying asset. Forward contracts are easier to analyze than futures contracts because there is no daily settlement—only a single payment at maturity. Luckily it can be shown that the forward price and futures price of an asset are usually very close when the maturities of the two contracts are the same.

In the first part of the chapter we derive some important general results on the relationship between forward (or futures) prices and spot prices. We then use the results to examine the relationship between futures prices and spot prices for contracts on stock indices, foreign exchange, and commodities. We will consider interest rate futures contracts in the next chapter.

5.1 INVESTMENT ASSETS vs. CONSUMPTION ASSETS

When considering forward and futures contracts, it is important to distinguish between investment assets and consumption assets. An *investment asset* is an asset that is held for investment purposes by significant numbers of investors. Stocks and bonds are clearly investment assets. Gold and silver are also examples of investment assets. Note that investment assets do not have to be held exclusively for investment. (Silver, for example, has a number of industrial uses.) However, they do have to satisfy the requirement that they are held by significant numbers of investors solely for investment. A *consumption asset* is an asset that is held primarily for consumption. It is not usually held for investment. Examples of consumption assets are commodities such as copper, oil, and pork bellies.

As we shall see later in this chapter, we can use arbitrage arguments to determine the forward and futures prices of an investment asset from its spot price and other observable market variables. We cannot do this for consumption assets.

5.2 SHORT SELLING

Some of the arbitrage strategies presented in this chapter involve *short selling*. This trade, usually simply referred to as "shorting", involves selling an asset that is not owned. It is something that is possible for some—but not all—investment assets. We will illustrate how it works by considering a short sale of shares of a stock.

Suppose an investor instructs a broker to short 500 IBM shares. The broker will carry out the instructions by borrowing the shares from another client and selling them in the market in the usual way. The investor can maintain the short position for as long as desired, provided there are always shares for the broker to borrow. At some stage, however, the investor will close out the position by purchasing 500 IBM shares. These are then replaced in the account of the client from which the shares were borrowed. The investor takes a profit if the stock price has declined and a loss if it has risen. If at any time while the contract is open the broker is not able to borrow shares, the investor is forced to close out the position, even if not ready to do so.

An investor with a short position must pay to the broker any income, such as dividends or interest, that would normally be received on the securities that have been shorted. The broker will transfer this income to the account of the client from whom the securities have been borrowed. Consider the position of an investor who shorts 500 shares in April when the price per share is \$120 and closes out the position by buying them back in July when the price per share is \$100. Suppose that a dividend of \$1 per share is paid in May. The investor receives $500 \times \$120 = \$60{,}000$ in April when the short position is initiated. The dividend leads to a payment by the investor of $500 \times \$1 = \$500$ in May. The investor also pays $500 \times \$100 = \$50{,}000$ for shares when the position is closed out in July. The net gain, therefore, is

$$\$60{,}000 - \$500 - \$50{,}000 = \$9{,}500$$

Table 5.1 illustrates this example and shows that the cash flows from the short sale are the mirror image of the cash flows from purchasing the shares in April and selling them in July.

The investor is required to maintain a *margin account* with the broker. The margin account consists of cash or marketable securities deposited by the investor with the broker to guarantee that the investor will not walk away from the short position if the share price increases. It is similar to the margin account discussed in Chapter 2 for

Table 5.1 Cash flows from short sale and purchase of shares.

	Purchase of shares	
April:	Purchase 500 shares for \$120	−\$60,000
May:	Receive dividend	+\$500
July:	Sell 500 shares for \$100 per share	+\$50,000
		Net profit = −\$9,500
	Short sale of shares	
April:	Borrow 500 shares and sell them for \$120	+\$60,000
May:	Pay dividend	−\$500
July:	Buy 500 shares for \$100 per share Replace borrowed shares to close short position	−\$50,000
		Net profit = +\$9,500

futures contracts. An initial margin is required and if there are adverse movements (i.e., increases) in the price of the asset that is being shorted, additional margin may be required. If the additional margin is not provided, the short position is closed out. The margin account does not represent a cost to the investor. This is because interest is usually paid on the balance in margin accounts and, if the interest rate offered is unacceptable, marketable securities such as Treasury bills can be used to meet margin requirements. The proceeds of the sale of the asset belong to the investor and normally form part of the initial margin.

The SEC abolished the "uptick rule" in the United States on July 6, 2007. This rule required the most recent movement in the price of a stock to be an increase for the shorting of a stock to be permitted.

5.3 ASSUMPTIONS AND NOTATION

In this chapter we will assume that the following are all true for some market participants:

1. The market participants are subject to no transaction costs when they trade.
2. The market participants are subject to the same tax rate on all net trading profits.
3. The market participants can borrow money at the same risk-free rate of interest as they can lend money.
4. The market participants take advantage of arbitrage opportunities as they occur.

Note that we do not require these assumptions to be true for all market participants. All that we require is that they be true—or at least approximately true—for a few key market participants such as large derivatives dealers. It is the trading activities of these key market participants and their eagerness to take advantage of arbitrage opportunities as they occur that determine the relationship between forward and spot prices.

The following notation will be used throughout this chapter:

T: Time until delivery date in a forward or futures contract (in years)

S_0: Price of the asset underlying the forward or futures contract today

F_0: Forward or futures price today

r: Zero-coupon risk-free rate of interest per annum, expressed with continuous compounding, for an investment maturing at the delivery date (i.e., in T years)

The risk-free rate, r, is the rate at which money is borrowed or lent when there is no credit risk, so that the money is certain to be repaid. As discussed in Chapter 4, financial institutions and other participants in derivatives markets assume that LIBOR rates rather than Treasury rates are the relevant risk-free rates.

5.4 FORWARD PRICE FOR AN INVESTMENT ASSET

The easiest forward contract to value is one written on an investment asset that provides the holder with no income. Non-dividend-paying stocks and zero-coupon bonds are examples of such investment assets.

Consider a long forward contract to purchase a non-dividend-paying stock in 3 months.[1] Assume the current stock price is \$40 and the 3-month risk-free interest rate is 5% per annum.

Suppose first that the forward price is relatively high at \$43. An arbitrageur can borrow \$40 at the risk-free interest rate of 5% per annum, buy one share, and short a forward contract to sell one share in 3 months. At the end of the 3 months, the arbitrageur delivers the share and receives \$43. The sum of money required to pay off the loan is

$$40e^{0.05\times 3/12} = \$40.50$$

By following this strategy, the arbitrageur locks in a profit of \$43.00 – \$40.50 = \$2.50 at the end of the 3-month period.

Suppose next that the forward price is relatively low at \$39. An arbitrageur can short one share, invest the proceeds of the short sale at 5% per annum for 3 months, and take a long position in a 3-month forward contract. The proceeds of the short sale grow to $40e^{0.05\times 3/12}$, or \$40.50 in 3 months. At the end of the 3 months, the arbitrageur pays \$39, takes delivery of the share under the terms of the forward contract, and uses it to close out the short position. A net gain of

$$\$40.50 - \$39.00 = \$1.50$$

is therefore made at the end of the 3 months. The two trading strategies we have considered are summarized in Table 5.2.

Under what circumstances do arbitrage opportunities such as those in Table 5.2 not exist? The first arbitrage works when the forward price is greater than \$40.50. The

Table 5.2 Arbitrage opportunities when forward price is out of line with spot price for asset providing no income. (Asset price = \$40; interest rate = 5%; maturity of forward contract = 3 months.)

Forward Price = \$43	*Forward Price = \$39*
Action now:	*Action now*:
Borrow \$40 at 5% for 3 months	Short 1 unit of asset to realize \$40
Buy one unit of asset	Invest \$40 at 5% for 3 months
Enter into forward contract to sell asset in 3 months for \$43	Enter into a forward contract to buy asset in 3 months for \$39
Action in 3 months:	*Action in 3 months*:
Sell asset for \$43	Buy asset for \$39
Use \$40.50 to repay loan with interest	Close short position
	Receive \$40.50 from investment
Profit realized = \$2.50	Profit realized = \$1.50

[1] Forward contracts on individual stocks do not often arise in practice. However, they form useful examples for developing our ideas. Futures on individual stocks started trading in the United States in November 2002.

Business Snapshot 5.1 Kidder Peabody's Embarrassing Mistake

Investment banks have developed a way of creating a zero-coupon bond, called a *strip*, from a coupon-bearing Treasury bond by selling each of the cash flows underlying the coupon-bearing bond as a separate security. Joseph Jett, a trader working for Kidder Peabody, had a relatively simple trading strategy. He would buy strips and sell them in the forward market. As equation (5.1) shows, the forward price of a security providing no income is always higher than the spot price. Suppose, for example, that the 3-month interest rate is 4% per annum and the spot price of a strip is \$70. The 3-month forward price of the strip is $70e^{0.04\times 3/12} = \$70.70$.

Kidder Peabody's computer system reported a profit on each of Jett's trades equal to the excess of the forward price over the spot price (\$0.70 in our example). In fact this profit was nothing more than the cost of financing the purchase of the strip. But, by rolling his contracts forward, Jett was able to prevent this cost from accruing to him.

The result was that the system reported a profit of \$100 million on Jett's trading (and Jett received a big bonus) when in fact there was a loss in the region of \$350 million. This shows that even large financial institutions can get relatively simple things wrong!

second arbitrage works when the forward price is less than \$40.50. We deduce that for there to be no arbitrage the forward price must be exactly \$40.50.

A Generalization

To generalize this example, we consider a forward contract on an investment asset with price S_0 that provides no income. Using our notation, T is the time to maturity, r is the risk-free rate, and F_0 is the forward price. The relationship between F_0 and S_0 is

$$F_0 = S_0 e^{rT} \tag{5.1}$$

If $F_0 > S_0 e^{rT}$, arbitrageurs can buy the asset and short forward contracts on the asset. If $F_0 < S_0 e^{rT}$, they can short the asset and enter into long forward contracts on it.[2] In our example, $S_0 = 40$, $r = 0.05$, and $T = 0.25$, so that equation (5.1) gives

$$F_0 = 40e^{0.05\times 0.25} = \$40.50$$

which is in agreement with our earlier calculations.

A long forward contract and a spot purchase both lead to the asset being owned at time T. The forward price is higher than the spot price because of the cost of financing the spot purchase of the asset during the life of the forward contract. This point was overlooked by Kidder Peabody in 1994, much to its cost (see Business Snapshot 5.1).

[2] For another way of seeing that equation (5.1) is correct, consider the following strategy: buy one unit of the asset and enter into a short forward contract to sell it for F_0 at time T. This costs S_0 and is certain to lead to a cash inflow of F_0 at time T. Therefore S_0 must equal the present value of F_0; that is, $S_0 = F_0 e^{-rT}$, or equivalently $F_0 = S_0 e^{rT}$.

Example 5.1

Consider a 4-month forward contract to buy a zero-coupon bond that will mature 1 year from today. (This means that the bond will have 8 months to go when the forward contract matures.) The current price of the bond is \$930. We assume that the 4-month risk-free rate of interest (continuously compounded) is 6% per annum. Because zero-coupon bonds provide no income, we can use equation (5.1) with $T = 4/12$, $r = 0.06$, and $S_0 = 930$. The forward price, F_0, is given by

$$F_0 = 930e^{0.06\times4/12} = \$948.79$$

This would be the delivery price in a contract negotiated today.

What If Short Sales Are Not Possible?

Short sales are not possible for all investment assets. As it happens, this does not matter. To derive equation (5.1), we do not need to be able to short the asset. All that we require is that there be a significant number of people who hold the asset purely for investment (and by definition this is always true of an investment asset). If the forward price is too low, they will find it attractive to sell the asset and take a long position in a forward contract.

Suppose the underlying asset is gold and assume no storage costs or income. If $F_0 > S_0e^{rT}$, an investor can adopt the following strategy:

1. Borrow S_0 dollars at an interest rate r for T years.
2. Buy 1 ounce of gold.
3. Short a forward contract on 1 ounce of gold.

At time T, 1 ounce of gold is sold for F_0. An amount S_0e^{rT} is required to repay the loan at this time and the investor makes a profit of $F_0 - S_0e^{rT}$.

Suppose next that $F_0 < S_0e^{rT}$. In this case an investor who owns 1 ounce of gold can

1. Sell the gold for S_0.
2. Invest the proceeds at interest rate r for time T.
3. Take a long position in a forward contract on 1 ounce of gold.

At time T, the cash invested has grown to S_0e^{rT}. The gold is repurchased for F_0 and the investor makes a profit of $S_0e^{rT} - F_0$ relative to the position the investor would have been in if the gold had been kept.

As in the non-dividend-paying stock example considered earlier, we can expect the forward price to adjust so that neither of the two arbitrage opportunities we have considered exists. This means that the relationship in equation (5.1) must hold.

5.5 KNOWN INCOME

In this section we consider a forward contract on an investment asset that will provide a perfectly predictable cash income to the holder. Examples are stocks paying known dividends and coupon-bearing bonds. We adopt the same approach as in the previous section. We first look at a numerical example and then review the formal arguments.

Consider a long forward contract to purchase a coupon-bearing bond whose current price is \$900. We will suppose that the forward contract matures in 9 months. We will also suppose that a coupon payment of \$40 is expected after 4 months. We assume that the 4-month and 9-month risk-free interest rates (continuously compounded) are, respectively, 3% and 4% per annum.

Suppose first that the forward price is relatively high at \$910. An arbitrageur can borrow \$900 to buy the bond and short a forward contract. The coupon payment has a present value of $40e^{-0.03\times 4/12} = \$39.60$. Of the \$900, \$39.60 is therefore borrowed at 3% per annum for 4 months so that it can be repaid with the coupon payment. The remaining \$860.40 is borrowed at 4% per annum for 9 months. The amount owing at the end of the 9-month period is $860.40e^{0.04\times 0.75} = \$886.60$. A sum of \$910 is received for the bond under the terms of the forward contract. The arbitrageur therefore makes a net profit of

$$910.00 - 886.60 = \$23.40$$

Suppose next that the forward price is relatively low at \$870. An investor can short the bond and enter into a long forward contract. Of the \$900 realized from shorting the bond, \$39.60 is invested for 4 months at 3% per annum so that it grows into an amount sufficient to pay the coupon on the bond. The remaining \$860.40 is invested for 9 months at 4% per annum and grows to \$886.60. Under the terms of the forward contract, \$870 is paid to buy the bond and the short position is closed out. The investor therefore gains

$$886.60 - 870 = \$16.60$$

The two strategies we have considered are summarized in Table 5.3.[3] The first strategy in Table 5.3 produces a profit when the forward price is greater than \$886.60, whereas the second strategy produces a profit when the forward price is less than \$886.60. It follows that if there are no arbitrage opportunities then the forward price must be \$886.60.

A Generalization

We can generalize from this example to argue that, when an investment asset will provide income with a present value of I during the life of a forward contract, we have

$$F_0 = (S_0 - I)e^{rT} \tag{5.2}$$

In our example, $S_0 = 900.00$, $I = 40e^{-0.03\times 4/12} = 39.60$, $r = 0.04$, and $T = 0.75$, so that

$$F_0 = (900.00 - 39.60)e^{0.04\times 0.75} = \$886.60$$

This is in agreement with our earlier calculation. Equation (5.2) applies to any investment asset that provides a known cash income.

If $F_0 > (S_0 - I)e^{rT}$, an arbitrageur can lock in a profit by buying the asset and shorting a forward contract on the asset; if $F_0 < (S_0 - I)e^{rT}$, an arbitrageur can lock in a profit by shorting the asset and taking a long position in a forward contract. If

[3] If shorting the bond is not possible, investors who already own the bond will sell it and buy a forward contract on the bond increasing the value of their position by \$16.60. This is similar to the strategy we described for gold in Section 5.4.

Table 5.3 Arbitrage opportunities when 9-month forward price is out of line with spot price for asset providing known cash income. (Asset price = \$900; income of \$40 occurs at 4 months; 4-month and 9-month rates are, respectively, 3% and 4% per annum.)

Forward price = \$910	*Forward price = \$870*
Action now: Borrow \$900: \$39.60 for 4 months and \$860.40 for 9 months Buy 1 unit of asset Enter into forward contract to sell asset in 9 months for \$910	*Action now*: Short 1 unit of asset to realize \$900 Invest \$39.60 for 4 months and \$860.40 for 9 months Enter into a forward contract to buy asset in 9 months for \$870
Action in 4 months: Receive \$40 of income on asset Use \$40 to repay first loan with interest	*Action in 4 months*: Receive \$40 from 4-month investment Pay income of \$40 on asset
Action in 9 months: Sell asset for \$910 Use \$886.60 to repay second loan with interest	*Action in 9 months*: Receive \$886.60 from 9-month investment Buy asset for \$870 Close out short position
Profit realized = \$23.40	Profit realized = \$16.60

short sales are not possible, investors who own the asset will find it profitable to sell the asset and enter into long forward contracts.[4]

Example 5.2

Consider a 10-month forward contract on a stock when the stock price is \$50. We assume that the risk-free rate of interest (continuously compounded) is 8% per annum for all maturities. We also assume that dividends of \$0.75 per share are expected after 3 months, 6 months, and 9 months. The present value of the dividends, I, is

$$I = 0.75e^{-0.08\times3/12} + 0.75e^{-0.08\times6/12} + 0.75e^{-0.08\times9/12} = 2.162$$

The variable T is 10 months, so that the forward price, F_0, from equation (5.2), is given by

$$F_0 = (50 - 2.162)e^{0.08\times10/12} = \$51.14$$

If the forward price were less than this, an arbitrageur would short the stock and buy forward contracts. If the forward price were greater than this, an arbitrageur would short forward contracts and buy the stock in the spot market.

[4] For another way of seeing that equation (5.2) is correct, consider the following strategy: buy one unit of the asset and enter into a short forward contract to sell it for F_0 at time T. This costs S_0 and is certain to lead to a cash inflow of F_0 at time T and an income with a present value of I. The initial outflow is S_0. The present value of the inflows is $I + F_0e^{-rT}$. Hence, $S_0 = I + F_0e^{-rT}$, or equivalently $F_0 = (S_0 - I)e^{rT}$.

5.6 KNOWN YIELD

We now consider the situation where the asset underlying a forward contract provides a known yield rather than a known cash income. This means that the income is known when expressed as a percentage of the asset's price at the time the income is paid. Suppose that an asset is expected to provide a yield of 5% per annum. This could mean that income is paid once a year and is equal to 5% of the asset price at the time it is paid, in which case the yield would be 5% with annual compounding. Alternatively, it could mean that income is paid twice a year and is equal to 2.5% of the asset price at the time it is paid, in which case the yield would be 5% per annum with semiannual compounding. In Section 4.2 we explained that we will normally measure interest rates with continuous compounding. Similarly, we will normally measure yields with continuous compounding. Formulas for translating a yield measured with one compounding frequency to a yield measured with another compounding frequency are the same as those given for interest rates in Section 4.2.

Define q as the average yield per annum on an asset during the life of a forward contract with continuous compounding. It can be shown (see Problem 5.20) that

$$F_0 = S_0 e^{(r-q)T} \tag{5.3}$$

Example 5.3

Consider a 6-month forward contract on an asset that is expected to provide income equal to 2% of the asset price once during a 6-month period. The risk-free rate of interest (with continuous compounding) is 10% per annum. The asset price is \$25. In this case, $S_0 = 25$, $r = 0.10$, and $T = 0.5$. The yield is 4% per annum with semiannual compounding. From equation (4.3), this is 3.96% per annum with continuous compounding. It follows that $q = 0.0396$, so that from equation (5.3) the forward price, F_0, is given by

$$F_0 = 25e^{(0.10-0.0396)\times 0.5} = \$25.77$$

5.7 VALUING FORWARD CONTRACTS

The value of a forward contract at the time it is first entered into is zero. At a later stage, it may prove to have a positive or negative value. It is important for banks and other financial institutions to value the contract each day. (This is referred to as marking to market the contract.) Using the notation introduced earlier, we suppose K is the delivery price for a contract that was negotiated some time ago, the delivery date is T years from today, and r is the T-year risk-free interest rate. The variable F_0 is the forward price that would be applicable if we negotiated the contract today. We also define

f: Value of forward contract today

It is important to be clear about the meaning of the variables F_0, K, and f. At the beginning of the life of the forward contract, the delivery price, K, is set equal to the forward price, F_0, and the value of the contract, f, is 0. As time passes, K stays the same (because it is part of the definition of the contract), but the forward price changes and the value of the contract becomes either positive or negative.

A general result, applicable to all long forward contracts (both those on investment assets and those on consumption assets), is

$$f = (F_0 - K)e^{-rT} \tag{5.4}$$

To see why equation (5.4) is correct, we use an argument analogous to the one we used for forward rate agreements in Section 4.7. We compare a long forward contract that has a delivery price of F_0 with an otherwise identical long forward contract that has a delivery price of K. The difference between the two is only in the amount that will be paid for the underlying asset at time T. Under the first contract, this amount is F_0; under the second contract, it is K. A cash outflow difference of $F_0 - K$ at time T translates to a difference of $(F_0 - K)e^{-rT}$ today. The contract with a delivery price F_0 is therefore less valuable than the contract with delivery price K by an amount $(F_0 - K)e^{-rT}$. The value of the contract that has a delivery price of F_0 is by definition zero. It follows that the value of the contract with a delivery price of K is $(F_0 - K)e^{-rT}$. This proves equation (5.4). Similarly, the value of a short forward contract with delivery price K is

$$(K - F_0)e^{-rT}$$

Example 5.4

A long forward contract on a non-dividend-paying stock was entered into some time ago. It currently has 6 months to maturity. The risk-free rate of interest (with continuous compounding) is 10% per annum, the stock price is \$25, and the delivery price is \$24. In this case, $S_0 = 25$, $r = 0.10$, $T = 0.5$, and $K = 24$. From equation (5.1), the 6-month forward price, F_0, is given by

$$F_0 = 25e^{0.1\times0.5} = \$26.28$$

From equation (5.4), the value of the forward contract is

$$f = (26.28 - 24)e^{-0.1\times0.5} = \$2.17$$

Equation (5.4) shows that we can value a long forward contract on an asset by making the assumption that the price of the asset at the maturity of the forward contract equals the forward price F_0. To see this, note that when we make that assumption, a long forward contract provides a payoff at time T of $F_0 - K$. This has a present value of $(F_0 - K)e^{-rT}$, which is the value of f in equation (5.4). Similarly, we can value a short forward contract on the asset by assuming that the current forward price of the asset is realized. These results are analogous to the result in Section 4.7 that we can value a forward rate agreement on the assumption that forward rates are realized.

Using equation (5.4) in conjunction with equation (5.1) gives the following expression for the value of a forward contract on an investment asset that provides no income

$$f = S_0 - Ke^{-rT} \tag{5.5}$$

Similarly, using equation (5.4) in conjunction with equation (5.2) gives the following

Business Snapshot 5.2 A Systems Error?

A foreign exchange trader working for a bank enters into a long forward contract to buy 1 million pounds sterling at an exchange rate of 1.9000 in 3 months. At the same time, another trader on the next desk takes a long position in 16 contracts for 3-month futures on sterling. The futures price is 1.9000 and each contract is on 62,500 pounds. The positions taken by the forward and futures traders are therefore the same. Within minutes of the positions being taken the forward and the futures prices both increase to 1.9040. The bank's systems show that the futures trader has made a profit of \$4,000, while the forward trader has made a profit of only \$3,900. The forward trader immediately calls the bank's systems department to complain. Does the forward trader have a valid complaint?

The answer is no! The daily settlement of futures contracts ensures that the futures trader realizes an almost immediate profit corresponding to the increase in the futures price. If the forward trader closed out the position by entering into a short contract at 1.9040, the forward trader would have contracted to buy 1 million pounds at 1.9000 in 3 months and sell 1 million pounds at 1.9040 in 3 months. This would lead to a \$4,000 profit—but in 3 months, not today. The forward trader's profit is the present value of \$4,000. This is consistent with equation (5.4).

The forward trader can gain some consolation from the fact that gains and losses are treated symmetrically. If the forward/futures prices dropped to 1.8960 instead of rising to 1.9040, then the futures trader would take a loss of \$4,000 while the forward trader would take a loss of only \$3,900.

expression for the value of a long forward contract on an investment asset that provides a known income with present value I:

$$f = S_0 - I - Ke^{-rT} \tag{5.6}$$

Finally, using equation (5.4) in conjunction with equation (5.3) gives the following expression for the value of a long forward contract on an investment asset that provides a known yield at rate q:

$$f = S_0e^{-qT} - Ke^{-rT} \tag{5.7}$$

When a futures price changes, the gain or loss on a futures contract is calculated as the change in the futures price multiplied by the size of the position. This gain is realized almost immediately because of the way futures contracts are settled daily. Equation (5.4) shows that, when a forward price changes, the gain or loss is the present value of the change in the forward price multiplied by the size of the position. The difference between the gain/loss on forward and futures contracts can cause confusion on a foreign exchange trading desk (see Business Snapshot 5.2).

5.8 ARE FORWARD PRICES AND FUTURES PRICES EQUAL?

The appendix at the end of this chapter provides an arbitrage argument to show that when the risk-free interest rate is constant and the same for all maturities, the forward price for a contract with a certain delivery date is in theory the same as the futures price for a contract with that delivery date. The argument in the appendix can be extended to cover situations where the interest rate is a known function of time.

When interest rates vary unpredictably (as they do in the real world), forward and futures prices are in theory no longer the same. We can get a sense of the nature of the relationship by considering the situation where the price of the underlying asset, S, is strongly positively correlated with interest rates. When S increases, an investor who holds a long futures position makes an immediate gain because of the daily settlement procedure. The positive correlation indicates that it is likely that interest rates have also increased. The gain will therefore tend to be invested at a higher than average rate of interest. Similarly, when S decreases, the investor will incur an immediate loss. This loss will tend to be financed at a lower than average rate of interest. An investor holding a forward contract rather than a futures contract is not affected in this way by interest rate movements. It follows that a long futures contract will be slightly more attractive than a similar long forward contract. Hence, when S is strongly positively correlated with interest rates, futures prices will tend to be slightly higher than forward prices. When S is strongly negatively correlated with interest rates, a similar argument shows that forward prices will tend to be slightly higher than futures prices.

The theoretical differences between forward and futures prices for contracts that last only a few months are in most circumstances sufficiently small to be ignored. In practice, there are a number of factors not reflected in theoretical models that may cause forward and futures prices to be different. These include taxes, transactions costs, and the treatment of margins. The risk that the counterparty will default is generally less in the case of a futures contract because of the role of the exchange clearinghouse. Also, in some instances, futures contracts are more liquid and easier to trade than forward contracts. Despite all these points, for most purposes it is reasonable to assume that forward and futures prices are the same. This is the assumption we will usually make in this book. We will use the symbol F_0 to represent both the futures price and the forward price of an asset today.

One exception to the rule that futures and forward contracts can be assumed to be the same concerns Eurodollar futures. This will be discussed in Section 6.3.

5.9 FUTURES PRICES OF STOCK INDICES

We introduced futures on stock indices in Section 3.5 and showed how a stock index futures contract is a useful tool in managing equity portfolios. Table 3.3 shows futures prices for a number of different indices. We are now in a position to consider how index futures prices are determined.

A stock index can usually be regarded as the price of an investment asset that pays dividends.[5] The investment asset is the portfolio of stocks underlying the index, and the dividends paid by the investment asset are the dividends that would be received by the holder of this portfolio. It is usually assumed that the dividends provide a known yield rather than a known cash income. If q is the dividend yield rate, equation (5.3) gives the futures price, F_0, as

$$F_0 = S_0 e^{(r-q)T} \tag{5.8}$$

[5] Occasionally this is not the case: see Business Snapshot 5.3.

Business Snapshot 5.3 The CME Nikkei 225 Futures Contract

The arguments in this chapter on how index futures prices are determined require that the index be the value of an investment asset. This means that it must be the value of a portfolio of assets that can be traded. The asset underlying the Chicago Mercantile Exchange's futures contract on the Nikkei 225 Index does not qualify, and the reason why is quite subtle. Suppose S is the value of the Nikkei 225 Index. This is the value of a portfolio of 225 Japanese stocks measured in yen. The variable underlying the CME futures contract on the Nikkei 225 has a *dollar value* of $5S$. In other words, the futures contract takes a variable that is measured in yen and treats it as though it is dollars.

We cannot invest in a portfolio whose value will always be $5S$ dollars. The best we can do is to invest in one that is always worth $5S$ yen or in one that is always worth $5QS$ dollars, where Q is the dollar value of 1 yen. The variable $5S$ dollars is not, therefore, the price of an investment asset and equation (5.8) does not apply.

CME's Nikkei 225 futures contract is an example of a *quanto*. A quanto is a derivative where the underlying asset is measured in one currency and the payoff is in another currency. Quantos will be discussed further in Chapter 29.

This shows that the futures price increases at rate $r - q$ with the maturity of the futures contract. In Table 3.3 the June futures price of the S&P 500 is about 0.9% more than the March futures price. This indicates that on January 8, 2007, the short-term risk-free rate, r, was greater than the dividend yield, q, by about 3.6% per year.

Example 5.5

Consider a 3-month futures contract on the S&P 500. Suppose that the stocks underlying the index provide a dividend yield of 1% per annum, that the current value of the index is 1,300, and that the continuously compounded risk-free interest rate is 5% per annum. In this case, $r = 0.05$, $S_0 = 1{,}300$, $T = 0.25$, and $q = 0.01$. Hence, the futures price, F_0, is given by

$$F_0 = 1{,}300e^{(0.05-0.01)\times 0.25} = \$1{,}313.07$$

In practice, the dividend yield on the portfolio underlying an index varies week by week throughout the year. For example, a large proportion of the dividends on the NYSE stocks are paid in the first week of February, May, August, and November each year. The chosen value of q should represent the average annualized dividend yield during the life of the contract. The dividends used for estimating q should be those for which the ex-dividend date is during the life of the futures contract.

Index Arbitrage

If $F_0 > S_0e^{(r-q)T}$, profits can be made by buying the stocks underlying the index at the spot price (i.e., for immediate delivery) and shorting futures contracts. If $F_0 < S_0e^{(r-q)T}$, profits can be made by doing the reverse—that is, shorting or selling the stocks underlying the index and taking a long position in futures contracts. These strategies are known as *index arbitrage*. When $F_0 < S_0e^{(r-q)T}$, index arbitrage is often done by a pension fund that owns an indexed portfolio of stocks. When $F_0 > S_0e^{(r-q)T}$, it might be

Business Snapshot 5.4 Index Arbitrage in October 1987

To do index arbitrage, a trader must be able to trade both the index futures contract and the portfolio of stocks underlying the index very quickly at the prices quoted in the market. In normal market conditions this is possible using program trading, and the relationship in equation (5.8) holds well. Examples of days when the market was anything but normal are October 19 and 20 of 1987. On what is termed "Black Monday", October 19, 1987, the market fell by more than 20%, and the 604 million shares traded on the New York Stock Exchange easily exceeded all previous records. The exchange's systems were overloaded, and orders placed to buy or sell shares on that day could be delayed by up to two hours before being executed.

For most of October 19, 1987, futures prices were at a significant discount to the underlying index. For example, at the close of trading the S&P 500 Index was at 225.06 (down 57.88 on the day), whereas the futures price for December delivery on the S&P 500 was 201.50 (down 80.75 on the day). This was largely because the delays in processing orders made index arbitrage impossible. On the next day, Tuesday, October 20, 1987, the New York Stock Exchange placed temporary restrictions on the way in which program trading could be done. This also made index arbitrage very difficult and the breakdown of the traditional linkage between stock indices and stock index futures continued. At one point the futures price for the December contract was 18% less than the S&P 500 Index. However, after a few days the market returned to normal, and the activities of arbitrageurs ensured that equation (5.8) governed the relationship between futures and spot prices of indices.

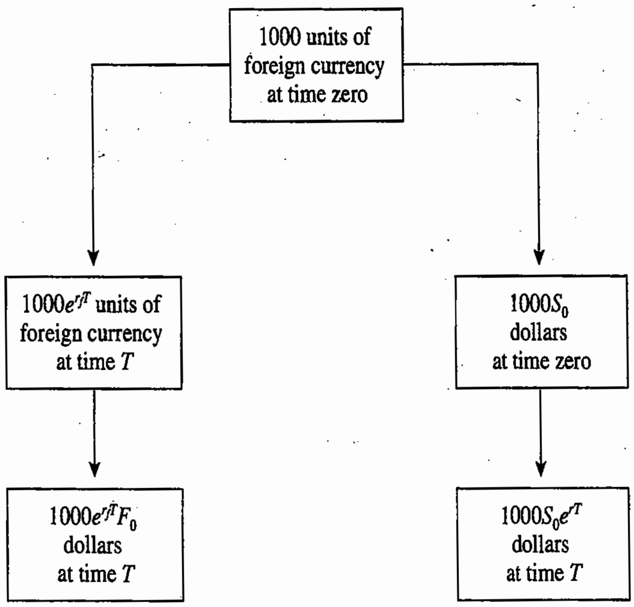
done by a corporation holding short-term money market investments. For indices involving many stocks, index arbitrage is sometimes accomplished by trading a relatively small representative sample of stocks whose movements closely mirror those of the index. Often index arbitrage is implemented through *program trading*. This involves using a computer system to generate the trades.

Most of the time the activities of arbitrageurs ensure that equation (5.8) holds, but occasionally arbitrage is impossible and the futures price does get out of line with the spot price (see Business Snapshot 5.4).

5.10 FORWARD AND FUTURES CONTRACTS ON CURRENCIES

We now move on to consider forward and futures foreign currency contracts from the perspective of a US investor. The underlying asset is one unit of the foreign currency. We will therefore define the variable S_0 as the current spot price in dollars of one unit of the foreign currency and F_0 as the forward or futures price in dollars of one unit of the foreign currency. This is consistent with the way we have defined S_0 and F_0 for other assets underlying forward and futures contracts. However, as mentioned in Section 2.10, it does not necessarily correspond to the way spot and forward exchange rates are quoted. For major exchange rates other than the British pound, euro, Australian dollar, and New Zealand dollar, a spot or forward exchange rate is normally quoted as the number of units of the currency that are equivalent to one US dollar.

Figure 5.1 Two ways of converting 1,000 units of a foreign currency to dollars at time T. Here, S_0 is spot exchange rate, F_0 is forward exchange rate, and r and r_f are the dollar and foreign risk-free rates.



A foreign currency has the property that the holder of the currency can earn interest at the risk-free interest rate prevailing in the foreign country. For example, the holder can invest the currency in a foreign-denominated bond. We define r_f as the value of the foreign risk-free interest rate when money is invested for time T. The variable r is the US dollar risk-free rate when money is invested for this period of time.

The relationship between F_0 and S_0 is

$$F_0 = S_0e^{(r-r_f)T} \tag{5.9}$$

This is the well-known interest rate parity relationship from international finance. The reason it is true is illustrated in Figure 5.1. Suppose that an individual starts with 1,000 units of the foreign currency. There are two ways it can be converted to dollars at time T. One is by investing it for T years at r_f and entering into a forward contract to sell the proceeds for dollars at time T. This generates $1{,}000e^{r_fT}F_0$ dollars. The other is by exchanging the foreign currency for dollars in the spot market and investing the proceeds for T years at rate r. This generates $1{,}000S_0e^{rT}$ dollars. In the absence of arbitrage opportunities, the two strategies must give the same result. Hence,

$$1{,}000e^{r_fT}F_0 = 1{,}000S_0e^{rT}$$

so that

$$F_0 = S_0e^{(r-r_f)T}$$

Example 5.6

Suppose that the 2-year interest rates in Australia and the United States are 5% and 7%, respectively, and the spot exchange rate between the Australian dollar

(AUD) and the US dollar (USD) is 0.6200 USD per AUD. From equation (5.9), the 2-year forward exchange rate should be

$$0.62e^{(0.07-0.05)\times 2} = 0.6453$$

Suppose first that the 2-year forward exchange rate is less than this, say 0.6300. An arbitrageur can:

1. Borrow 1,000 AUD at 5% per annum for 2 years, convert to 620 USD and invest the USD at 7% (both rates are continuously compounded).
2. Enter into a forward contract to buy 1,105.17 AUD for $1{,}105.17 \times 0.63 =$ 696.26 USD.

The 620 USD that are invested at 7% grow to $620e^{0.07\times 2} = 713.17$ USD in 2 years. Of this, 696.26 USD are used to purchase 1,105.17 AUD under the terms of the forward contract. This is exactly enough to repay principal and interest on the 1,000 AUD that are borrowed ($1{,}000e^{0.05\times 2} = 1{,}105.17$). The strategy therefore gives rise to a riskless profit of $713.17 - 696.26 = 16.91$ USD. (If this does not sound very exciting, consider following a similar strategy where you borrow 100 million AUD!)

Suppose next that the 2-year forward rate is 0.6600 (greater than the 0.6453 value given by equation (5.9)). An arbitrageur can:

1. Borrow 1,000 USD at 7% per annum for 2 years, convert to $1{,}000/0.6200 =$ 1,612.90 AUD, and invest the AUD at 5%.
2. Enter into a forward contract to sell 1,782.53 AUD for $1{,}782.53 \times 0.66 =$ 1,176.47 USD.

The 1,612.90 AUD that are invested at 5% grow to $1{,}612.90e^{0.05\times 2} = 1{,}782.53$ AUD in 2 years. The forward contract has the effect of converting this to 1,176.47 USD. The amount needed to payoff the USD borrowings is $1{,}000e^{0.07\times 2} = 1{,}150.27$ USD. The strategy therefore gives rise to a riskless profit of $1{,}176.47 - 1{,}150.27 = 26.20$ USD.

Table 5.4 shows currency futures quotes on January 8, 2007. The quotes are US dollars (or cents) per unit of the foreign currency. This is the usual quotation convention for futures contracts. Equation (5.9) applies with r equal to the US risk-free rate and r_f equal to the foreign risk-free rate.

On January 8, 2007, interest rates on the Japanese yen, Canadian dollar, British pound, Swiss franc, and euro were lower than the interest rate on the US dollar. This corresponds to the $r > r_f$ situation and explains why futures prices for these currencies increase with maturity in Table 5.4. For the Australian dollar and Mexican peso, interest rates were higher than in the United States. This corresponds to the $r_f > r$ situation and explains why the futures prices of these currencies decrease with maturity.

Example 5.7

In Table 5.4 the June settlement price for the Canadian dollar is 0.28% higher than the March settlement price. This indicates that the short-term futures prices are increasing at about 1.12% per year with maturity. From equation (5.9) this is an estimate of the amount by which short-term US interest rates exceeded short-term Canadian interest rates on January 8, 2007.

Table 5.4 Foreign exchange futures quotes from the *Wall Street Journal* on January 9, 2007. (Columns show month, open, high, low, settle, change, lifetime high, lifetime low, and open interest, respectively.)

Currency Futures

Japanese Yen (CME)-¥12,500,000; $ per 100¥						
March	.8505	.8546	.8494	**.8501**	-.0010	275,923
June	.8603	.8643	.8593	**.8599**	-.0010	5,516
Canadian Dollar (CME)-CAD 100,000; $ per CAD						
March	.8541	.8549	.8503	**.8525**	-.0018	155,395
June	.8566	.8569	.8528	**.8549**	-.0018	2,830
British Pound (CME)-£62,500; $ per £						
March	1.9308	1.9410	1.9265	**1.9382**	.0074	134,588
June	1.9326	1.9403	1.9266	**1.9377**	.0074	178
Swiss Franc (CME)-CHF 125,000; $ per CHF						
March	.8146	.8158	.8105	**.8138**	-.0007	70,774
June	.8207	.8219	.8170	**.8200**	-.0007	168
Australian Dollar (CME)-AUD 100,000; $ per AUD						
March	.7775	.7808	.7766	**.7783**	.0012	116,717
June	.7763	.7784	.7746	**.7762**	.0012	227
Mexican Peso (CME)-MXN 500,000; $ per 10MXN						
Jan	...	...	...	**.91250**	-.00100	0
March	.90700	.91150	.90700	**.91025**	-.00100	69,631
Euro (CME)-€125,000; $ per €						
March	1.3051	1.3078	1.3013	**1.3061**	.0009	174,877
June	1.3079	1.3123	1.3061	**1.3106**	.0009	1,453

A Foreign Currency as an Asset Providing a Known Yield

Equation (5.9) is identical to equation (5.3) with q replaced by r_f. This is not a coincidence. A foreign currency can be regarded as an investment asset paying a known yield. The yield is the risk-free rate of interest in the foreign currency.

To understand this, we note that the value of interest paid in a foreign currency depends on the value of the foreign currency. Suppose that the interest rate on British pounds is 5% per annum. To a US investor the British pound provides an income equal to 5% of the value of the British pound per annum. In other words it is an asset that provides a yield of 5% per annum.

5.11 FUTURES ON COMMODITIES

We now move on to consider futures contracts on commodities. First we consider the futures prices of commodities that are investment assets such as gold and silver.[6] We then move on to consider the futures prices of consumption assets.

Income and Storage Costs

As explained in Business Snapshot 3.1, the hedging strategies of gold producers leads to a requirement on the part of investment banks to borrow gold. Gold owners such as central banks charge interest in the form of what is known as the *gold lease rate* when they lend gold. The same is true of silver. Gold and silver can therefore provide income to the holder. Like other commodities they also have storage costs.

Equation (5.1) shows that, in the absence of storage costs and income, the forward

[6] Recall that, for an asset to be an investment asset, it need not be held solely for investment purposes. What is required is that some individuals hold it for investment purposes and that these individuals be prepared to sell their holdings and go long forward contracts, if the latter look more attractive. This explains why silver, although it has significant industrial uses, is an investment asset.

price of a commodity that is an investment asset is given by

$$F_0 = S_0 e^{rT} \tag{5.10}$$

Storage costs can be treated as negative income. If U is the present value of all the storage costs, net of income, during the life of a forward contract, it follows from equation (5.2) that

$$F_0 = (S_0 + U)e^{rT} \tag{5.11}$$

Example 5.8

Consider a 1-year futures contract on an investment asset that provides no income. It costs \$2 per unit to store the asset, with the payment being made at the end of the year. Assume that the spot price is \$450 per unit and the risk-free rate is 7% per annum for all maturities. This corresponds to $r = 0.07$, $S_0 = 450$, $T = 1$, and

$$U = 2e^{-0.07\times 1} = 1.865$$

From equation (5.11), the theoretical futures price, F_0, is given by

$$F_0 = (450 + 1.865)e^{0.07\times 1} = \$484.63$$

If the actual futures price is greater than 484.63, an arbitrageur can buy the asset and short 1-year futures contracts to lock in a profit. If the actual futures price is less than 484.63, an investor who already owns the asset can improve the return by selling the asset and buying futures contracts.

If the storage costs net of income incurred at any time are proportional to the price of the commodity, they can be treated as negative yield. In this case, from equation (5.3),

$$F_0 = S_0 e^{(r+u)T} \tag{5.12}$$

where u denotes the storage costs per annum as a proportion of the spot price net of any yield earned on the asset.

Consumption Commodities

Commodities that are consumption assets rather than investment assets usually provide no income, but can be subject to significant storage costs. We now review the arbitrage strategies used to determine futures prices from spot prices carefully.[7] Suppose that, instead of equation (5.11), we have

$$F_0 > (S_0 + U)e^{rT} \tag{5.13}$$

To take advantage of this opportunity, an arbitrageur can implement the following strategy:

1. Borrow an amount $S_0 + U$ at the risk-free rate and use it to purchase one unit of the commodity and to pay storage costs.
2. Short a forward contract on one unit of the commodity.

[7] For some commodities the spot price depends on the delivery location. We assume that the delivery location for spot and futures are the same.

If we regard the futures contract as a forward contract, this strategy leads to a profit of $F_0 - (S_0 + U)e^{rT}$ at time T. There is no problem in implementing the strategy for any commodity. However, as arbitrageurs do so, there will be a tendency for S_0 to increase and F_0 to decrease until equation (5.13) is no longer true. We conclude that equation (5.13) cannot hold for any significant length of time.

Suppose next that

$$F_0 < (S_0 + U)e^{rT} \tag{5.14}$$

When the commodity is an investment asset, we can argue that many investors hold the commodity solely for investment. When they observe the inequality in equation (5.14), they will find it profitable to do the following:

1. Sell the commodity, save the storage costs, and invest the proceeds at the risk-free interest rate.
2. Take a long position in a forward contract.

The result is a riskless profit at maturity of $(S_0 + U)e^{rT} - F_0$ relative to the position the investors would have been in if they had held the commodity. It follows that equation (5.14) cannot hold for long. Because neither equation (5.13) nor (5.14) can hold for long, we must have $F_0 = (S_0 + U)e^{rT}$.

This argument cannot be used for a commodity that is a consumption asset rather than an investment asset. Individuals and companies who own a consumption commodity usually plan to use it in some way. They are reluctant to sell the commodity in the spot market and buy forward or futures contracts, because forward and futures contracts cannot be consumed (for example, oil futures cannot be used to feed a refinery!). There is therefore nothing to stop equation (5.14) from holding, and all we can assert for a consumption commodity is

$$F_0 \leqslant (S_0 + U)e^{rT} \tag{5.15}$$

If storage costs are expressed as a proportion u of the spot price, the equivalent result is

$$F_0 \leqslant S_0 e^{(r+u)T} \tag{5.16}$$

Convenience Yields

We do not necessarily have equality in equations (5.15) and (5.16) because users of a consumption commodity may feel that ownership of the physical commodity provides benefits that are not obtained by holders of futures contracts. For example, an oil refiner is unlikely to regard a futures contract on crude oil in the same way as crude oil held in inventory. The crude oil in inventory can be an input to the refining process, whereas a futures contract cannot be used for this purpose. In general, ownership of the physical asset enables a manufacturer to keep a production process running and perhaps profit from temporary local shortages. A futures contract does not do the same. The benefits from holding the physical asset are sometimes referred to as the *convenience yield* provided by the commodity. If the dollar amount of storage costs is known and has a present value U, then the convenience yield y is defined such that

$$F_0 e^{yT} = (S_0 + U)e^{rT}$$

If the storage costs per unit are a constant proportion, u, of the spot price, then y is defined so that

$$F_0 e^{yT} = S_0 e^{(r+u)T}$$

or

$$F_0 = S_0 e^{(r+u-y)T} \tag{5.17}$$

The convenience yield simply measures the extent to which the left-hand side is less than the right-hand side in equation (5.15) or (5.16). For investment assets the convenience yield must be zero; otherwise, there are arbitrage opportunities. Figure 2.2 of Chapter 2 shows that the futures price of orange juice decreased as the time to maturity of the contract increased on January 8, 2007. This pattern suggests that the convenience yield, y, is greater than $r + u$ for orange juice on this date.

The convenience yield reflects the market's expectations concerning the future availability of the commodity. The greater the possibility that shortages will occur, the higher the convenience yield. If users of the commodity have high inventories, there is very little chance of shortages in the near future and the convenience yield tends to be low. If inventories are low, shortages are more likely and the convenience yield is usually higher.

5.12 THE COST OF CARRY

The relationship between futures prices and spot prices can be summarized in terms of the *cost of carry*. This measures the storage cost plus the interest that is paid to finance the asset less the income earned on the asset. For a non-dividend-paying stock, the cost of carry is r, because there are no storage costs and no income is earned; for a stock index, it is $r - q$, because income is earned at rate q on the asset. For a currency, it is $r - r_f$; for a commodity that provides income at rate q and requires storage costs at rate u, it is $r - q + u$; and so on.

Define the cost of carry as c. For an investment asset, the futures price is

$$F_0 = S_0 e^{cT} \tag{5.18}$$

For a consumption asset, it is

$$F_0 = S_0 e^{(c-y)T} \tag{5.19}$$

where y is the convenience yield.

5.13 DELIVERY OPTIONS

Whereas a forward contract normally specifies that delivery is to take place on a particular day, a futures contract often allows the party with the short position to choose to deliver at any time during a certain period. (Typically the party has to give a few days' notice of its intention to deliver.) The choice introduces a complication into the determination of futures prices. Should the maturity of the futures contract be assumed to be the beginning, middle, or end of the delivery period? Even though most futures contracts are closed out prior to maturity, it is important to know when delivery would have taken place in order to calculate the theoretical futures price.

If the futures price is an increasing function of the time to maturity, it can be seen from equation (5.19) that $c > y$, so that the benefits from holding the asset (including convenience yield and net of storage costs) are less than the risk-free rate. It is usually optimal in such a case for the party with the short position to deliver as early as possible, because the interest earned on the cash received outweighs the benefits of holding the asset. As a rule, futures prices in these circumstances should be calculated on the basis that delivery will take place at the beginning of the delivery period. If futures prices are decreasing as time to maturity increases ($c < y$), the reverse is true. It is then usually optimal for the party with the short position to deliver as late as possible, and futures prices should, as a rule, be calculated on this assumption.

5.14 FUTURES PRICES AND EXPECTED FUTURE SPOT PRICES

We refer to the market's average opinion about what the spot price of an asset will be at a certain future time as the *expected spot price* of the asset at that time. Suppose that it is now June and the September futures price of corn is 350 cents. It is interesting to ask what the expected spot price of corn in September is. Is it less than 350 cents, greater than 350 cents, or exactly equal to 350 cents? As illustrated in Figure 2.1, the futures price converges to the spot price at maturity. If the expected spot price is less than 350 cents, the market must be expecting the September futures price to decline, so that traders with short positions gain and traders with long positions lose. If the expected spot price is greater than 350 cents, the reverse must be true. The market must be expecting the September futures price to increase, so that traders with long positions gain while those with short positions lose.

Keynes and Hicks

Economists John Maynard Keynes and John Hicks argued that, if hedgers tend to hold short positions and speculators tend to hold long positions, the futures price of an asset will be below the expected spot price.[8] This is because speculators require compensation for the risks they are bearing. They will trade only if they can expect to make money on average. Hedgers will lose money on average, but they are likely to be prepared to accept this because the futures contract reduces their risks. If hedgers tend to hold long positions while speculators hold short positions, Keynes and Hicks argued that the futures price will be above the expected spot price for a similar reason.

Risk and Return

The modern approach to explaining the relationship between futures prices and expected spot prices is based on the relationship between risk and expected return in the economy. In general, the higher the risk of an investment, the higher the expected return demanded by an investor. Readers familiar with the capital asset pricing model will know that there are two types of risk in the economy: systematic and nonsystematic. Nonsystematic risk should not be important to an investor. It can be almost completely eliminated by holding a well-diversified portfolio. An investor should not therefore require a higher expected return for bearing nonsystematic risk. Systematic risk, in contrast, cannot be diversified away. It arises from a correlation between

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[8] See: J. M. Keynes, *A Treatise on Money*. London: Macmillan, 1930; and J. R. Hicks, *Value and Capital*. Oxford: Clarendon Press, 1939.

returns from the investment and returns from the whole stock market. An investor generally requires a higher expected return than the risk-free interest rate for bearing positive amounts of systematic risk. Also, an investor is prepared to accept a lower expected return than the risk-free interest rate when the systematic risk in an investment is negative.

The Risk in a Futures Position

Let us consider a speculator who takes a long position in a futures contract that lasts for T years in the hope that the spot price of the asset will be above the futures price at the end of the life of the futures contract. We ignore daily settlement and assume that the futures contract can be treated as a forward contract. We suppose that the speculator puts the present value of the futures price into a risk-free investment while simultaneously taking a long futures position. The proceeds of the risk-free investment are used to buy the asset on the delivery date. The asset is then immediately sold for its market price. The cash flows to the speculator are as follows:

Today: $-F_0e^{-rT}$

End of futures contract: $+S_T$

where F_0 is the futures price today, S_T is the price of the asset at time T at the end of the futures contract, and r is the risk-free return on funds invested for time T.

How do we value this investment? The discount rate we should use for the expected cash flow at time T equals an investor's required return on the investment. Suppose that k is an investor's required return for this investment. The present value of this investment is

$$-F_0e^{-rT} + E(S_T)e^{-kT}$$

where E denotes expected value. We can assume that all investments in securities markets are priced so that they have zero net present value. This means that

$$-F_0e^{-rT} + E(S_T)e^{-kT} = 0$$

or

$$F_0 = E(S_T)e^{(r-k)T} \tag{5.20}$$

As we have just discussed, the returns investors require on an investment depend on its systematic risk. The investment we have been considering is in essence an investment in the asset underlying the futures contract. If the returns from this asset are uncorrelated with the stock market, the correct discount rate to use is the risk-free rate r, so we should set $k = r$. Equation (5.20) then gives

$$F_0 = E(S_T)$$

This shows that the futures price is an unbiased estimate of the expected future spot price when the return from the underlying asset is uncorrelated with the stock market.

If the return from the asset is positively correlated with the stock market, $k > r$ and equation (5.20) leads to $F_0 < E(S_T)$. This shows that, when the asset underlying the futures contract has positive systematic risk, we should expect the futures price to understate the expected future spot price. An example of an asset that has positive systematic risk is a stock index. The expected return of investors on the stocks underlying an index is generally more than the risk-free rate, r. The dividends provide a return of q. The expected increase in the index must therefore be more than $r - q$. Equation (5.8) is therefore consistent with the prediction that the futures price understates the expected future stock price for a stock index.

If the return from the asset is negatively correlated with the stock market, $k < r$ and equation (5.20) gives $F_0 > E(S_T)$. This shows that, when the asset underlying the futures contract has negative systematic risk, we should expect the futures price to overstate the expected future spot price.

Normal Backwardation and Contango

When the futures price is below the expected future spot price, the situation is known as *normal backwardation*; and when the futures price is above the expected future spot price, the situation is known as *contango*. However, it should be noted that sometimes these terms are used to refer to whether the futures price is below or above the current spot price, rather than the expected future spot price.

SUMMARY

For most purposes, the futures price of a contract with a certain delivery date can be considered to be the same as the forward price for a contract with the same delivery date. It can be shown that in theory the two should be exactly the same when interest rates are perfectly predictable.

For the purposes of understanding futures (or forward) prices, it is convenient to divide futures contracts into two categories: those in which the underlying asset is held for investment by a significant number of investors and those in which the underlying asset is held primarily for consumption purposes.

In the case of investment assets, we have considered three different situations:

1. The asset provides no income.
2. The asset provides a known dollar income.
3. The asset provides a known yield.

The results are summarized in Table 5.5. They enable futures prices to be obtained for contracts on stock indices, currencies, gold, and silver. Storage costs can be treated as negative income.

In the case of consumption assets, it is not possible to obtain the futures price as a function of the spot price and other observable variables. Here the parameter known as the asset's convenience yield becomes important. It measures the extent to which users of the commodity feel that ownership of the physical asset provides benefits that are not obtained by the holders of the futures contract. These benefits may include the ability to profit from temporary local shortages or the ability to keep a production process running. We can obtain an upper bound for the futures price of consumption assets using arbitrage arguments, but we cannot nail down an equality relationship between futures and spot prices.

The concept of cost of carry is sometimes useful. The cost of carry is the storage cost

Table 5.5 Summary of results for a contract with time to maturity T on an investment asset with price S_0 when the risk-free interest rate for a T-year period is r.

Asset	*Forward/futures price*	*Value of long forward contract with delivery price K*
Provides no income:	S_0e^{rT}	$S_0 - Ke^{-rT}$
Provides known income with present value I:	$(S_0 - I)e^{rT}$	$S_0 - I - Ke^{-rT}$
Provides known yield q:	$S_0e^{(r-q)T}$	$S_0e^{-qT} - Ke^{-rT}$

of the underlying asset plus the cost of financing it minus the income received from it. In the case of investment assets, the futures price is greater than the spot price by an amount reflecting the cost of carry. In the case of consumption assets, the futures price is greater than the spot price by an amount reflecting the cost of carry net of the convenience yield.

If we assume the capital asset pricing model is true, the relationship between the futures price and the expected future spot price depends on whether the return on the asset is positively or negatively correlated with the return on the stock market. Positive correlation will tend to lead to a futures price lower than the expected future spot price, whereas negative correlation will tend to lead to a futures price higher than the expected future spot price. Only when the correlation is zero will the theoretical futures price be equal to the expected future spot price.

FURTHER READING

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Questions and Problems (Answers in Solutions Manual)

5.1. Explain what happens when an investor shorts a certain share.

5.2. What is the difference between the forward price and the value of a forward contract?

5.3. Suppose that you enter into a 6-month forward contract on a non-dividend-paying stock when the stock price is $30 and the risk-free interest rate (with continuous compounding) is 12% per annum. What is the forward price?

5.4. A stock index currently stands at 350. The risk-free interest rate is 8% per annum (with continuous compounding) and the dividend yield on the index is 4% per annum. What should the futures price for a 4-month contract be?

5.5. Explain carefully why the futures price of gold can be calculated from its spot price and other observable variables whereas the futures price of copper cannot.

5.6. Explain carefully the meaning of the terms *convenience yield* and *cost of carry*. What is the relationship between futures price, spot price, convenience yield, and cost of carry?

5.7. Explain why a foreign currency can be treated as an asset providing a known yield.

5.8. Is the futures price of a stock index greater than or less than the expected future value of the index? Explain your answer.

5.9. A 1-year long forward contract on a non-dividend-paying stock is entered into when the stock price is $40 and the risk-free rate of interest is 10% per annum with continuous compounding.

(a) What are the forward price and the initial value of the forward contract?

(b) Six months later, the price of the stock is $45 and the risk-free interest rate is still 10%. What are the forward price and the value of the forward contract?

5.10. The risk-free rate of interest is 7% per annum with continuous compounding, and the dividend yield on a stock index is 3.2% per annum. The current value of the index is 150. What is the 6-month futures price?

5.11. Assume that the risk-free interest rate is 9% per annum with continuous compounding and that the dividend yield on a stock index varies throughout the year. In February, May, August, and November, dividends are paid at a rate of 5% per annum. In other months, dividends are paid at a rate of 2% per annum. Suppose that the value of the index on July 31 is 1,300. What is the futures price for a contract deliverable in December 31 of the same year?

5.12. Suppose that the risk-free interest rate is 10% per annum with continuous compounding and that the dividend yield on a stock index is 4% per annum. The index is standing at 400, and the futures price for a contract deliverable in four months is 405. What arbitrage opportunities does this create?

5.13. Estimate the difference between short-term interest rates in Mexico and the United States on January 8, 2007, from the information in Table 5.4.

5.14. The 2-month interest rates in Switzerland and the United States are, respectively, 2% and 5% per annum with continuous compounding. The spot price of the Swiss franc is $0.8000. The futures price for a contract deliverable in 2 months is $0.8100. What arbitrage opportunities does this create?

5.15. The spot price of silver is $9 per ounce. The storage costs are $0.24 per ounce per year payable quarterly in advance. Assuming that interest rates are 10% per annum for all maturities, calculate the futures price of silver for delivery in 9 months.

5.16. Suppose that F_1 and F_2 are two futures contracts on the same commodity with times to maturity, t_1 and t_2, where $t_2 > t_1$. Prove that

$$F_2 \leqslant F_1 e^{r(t_2 - t_1)}$$

where r is the interest rate (assumed constant) and there are no storage costs. For the purposes of this problem, assume that a futures contract is the same as a forward contract.

5.17. When a known future cash outflow in a foreign currency is hedged by a company using a forward contract, there is no foreign exchange risk. When it is hedged using futures contracts, the marking-to-market process does leave the company exposed to some risk. Explain the nature of this risk. In particular, consider whether the company is better off using a futures contract or a forward contract when:
(a) The value of the foreign currency falls rapidly during the life of the contract.
(b) The value of the foreign currency rises rapidly during the life of the contract.
(c) The value of the foreign currency first rises and then falls back to its initial value.
(d) The value of the foreign currency first falls and then rises back to its initial value.
Assume that the forward price equals the futures price.

5.18. It is sometimes argued that a forward exchange rate is an unbiased predictor of future exchange rates. Under what circumstances is this so?

5.19. Show that the growth rate in an index futures price equals the excess return of the index over the risk-free rate. Assume that the risk-free interest rate and the dividend yield are constant.

5.20. Show that equation (5.3) is true by considering an investment in the asset combined with a short position in a futures contract. Assume that all income from the asset is reinvested in the asset. Use an argument similar to that in footnotes 2 and 4 and explain in detail what an arbitrageur would do if equation (5.3) did not hold.

5.21. Explain carefully what is meant by the expected price of a commodity on a particular future date. Suppose that the futures price for crude oil declines with the maturity of the contract at the rate of 2% per year. Assume that speculators tend to be short crude oil futures and hedgers tend to be long. What does the Keynes and Hicks argument imply about the expected future price of oil?

5.22. The Value Line Index is designed to reflect changes in the value of a portfolio of over 1,600 equally weighted stocks. Prior to March 9, 1988, the change in the index from one day to the next was calculated as the *geometric* average of the changes in the prices of the stocks underlying the index. In these circumstances, does equation (5.8) correctly relate the futures price of the index to its cash price? If not, does the equation overstate or understate the futures price?

5.23. A US company is interested in using the futures contracts traded on the CME to hedge its Australian dollar exposure. Define r as the interest rate (all maturities) on the US dollar and r_f as the interest rate (all maturities) on the Australian dollar. Assume that r and r_f are constant and that the company uses a contract expiring at time T to hedge an exposure at time t ($T > t$).
(a) Show that the optimal hedge ratio is $e^{(r_f-r)(T-t)}$.
(b) Show that, when t is 1 day, the optimal hedge ratio is almost exactly S_0/F_0, where S_0 is the current spot price of the currency and F_0 is the current futures price of the currency for the contract maturing at time T.
(c) Show that the company can take account of the daily settlement of futures contracts for a hedge that lasts longer than 1 day by adjusting the hedge ratio so that it always equals the spot price of the currency divided by the futures price of the currency.

Assignment Questions

5.24. A stock is expected to pay a dividend of \$1 per share in 2 months and in 5 months. The stock price is \$50, and the risk-free rate of interest is 8% per annum with continuous compounding for all maturities. An investor has just taken a short position in a 6-month forward contract on the stock.

(a) What are the forward price and the initial value of the forward contract?

(b) Three months later, the price of the stock is \$48 and the risk-free rate of interest is still 8% per annum. What are the forward price and the value of the short position in the forward contract?

5.25. A bank offers a corporate client a choice between borrowing cash at 11% per annum and borrowing gold at 2% per annum. (If gold is borrowed, interest must be repaid in gold. Thus, 100 ounces borrowed today would require 102 ounces to be repaid in 1 year.) The risk-free interest rate is 9.25% per annum, and storage costs are 0.5% per annum. Discuss whether the rate of interest on the gold loan is too high or too low in relation to the rate of interest on the cash loan. The interest rates on the two loans are expressed with annual compounding. The risk-free interest rate and storage costs are expressed with continuous compounding.

5.26. A company that is uncertain about the exact date when it will pay or receive a foreign currency may try to negotiate with its bank a forward contract that specifies a period during which delivery can be made. The company wants to reserve the right to choose the exact delivery date to fit in with its own cash flows. Put yourself in the position of the bank. How would you price the product that the company wants?

5.27. A trader owns gold as part of a long-term investment portfolio. The trader can buy gold for \$550 per ounce and sell it for \$549 per ounce. The trader can borrow funds at 6% per year and invest funds at 5.5% per year (both interest rates are expressed with annual compounding). For what range of 1-year forward prices of gold does the trader have no arbitrage opportunities? Assume there is no bid–offer spread for forward prices.

5.28. A company enters into a forward contract with a bank to sell a foreign currency for K_1 at time T_1. The exchange rate at time T_1 proves to be S_1 ($> K_1$). The company asks the bank if it can roll the contract forward until time T_2 ($> T_1$) rather than settle at time T_1. The bank agrees to a new delivery price, K_2. Explain how K_2 should be calculated.

APPENDIX

PROOF THAT FORWARD AND FUTURES PRICES ARE EQUAL WHEN INTEREST RATES ARE CONSTANT

This appendix demonstrates that forward and futures prices are equal when interest rates are constant. Suppose that a futures contract lasts for n days and that F_i is the futures price at the end of day i $(0 < i < n)$. Define δ as the risk-free rate per day (assumed constant). Consider the following strategy:[9]

1. Take a long futures position of e^{δ} at the end of day 0 (i.e., at the beginning of the contract).
2. Increase long position to $e^{2\delta}$ at the end of day 1.
3. Increase long position to $e^{3\delta}$ at the end of day 2.

   And so on.

This strategy is summarized in Table 5A.1. By the beginning of day i, the investor has a long position of $e^{\delta i}$. The profit (possibly negative) from the position on day i is

$$(F_i - F_{i-1})e^{\delta i}$$

Assume that the profit is compounded at the risk-free rate until the end of day n. Its value at the end of day n is

$$(F_i - F_{i-1})e^{\delta i}e^{(n-i)\delta} = (F_i - F_{i-1})e^{n\delta}$$

The value at the end of day n of the entire investment strategy is therefore

$$\sum_{i=1}^{n}(F_i - F_{i-1})e^{n\delta}$$

This is

$$[(F_n - F_{n-1}) + (F_{n-1} - F_{n-2}) + \cdots + (F_1 - F_0)]e^{n\delta} = (F_n - F_0)e^{n\delta}$$

Because F_n is the same as the terminal asset spot price, S_T, the terminal value of the investment strategy can be written

$$(S_T - F_0)e^{n\delta}$$

An investment of F_0 in a risk-free bond combined with the strategy involving futures just given yields

$$F_0e^{n\delta} + (S_T - F_0)e^{n\delta} = S_Te^{n\delta}$$

at time T. No investment is required for all the long futures positions described. It follows that an amount F_0 can be invested to give an amount $S_Te^{n\delta}$ at time T.

Suppose next that the forward price at the end of day 0 is G_0. Investing G_0 in a riskless bond and taking a long forward position of $e^{n\delta}$ forward contracts also guarantees an amount $S_Te^{n\delta}$ at time T. Thus, there are two investment strategies—one requiring an

[9] This strategy was proposed by J. C. Cox, J. E. Ingersoll, and S. A. Ross, "The Relation between Forward Prices and Futures Prices," *Journal of Financial Economics* 9 (December 1981): 321–46.

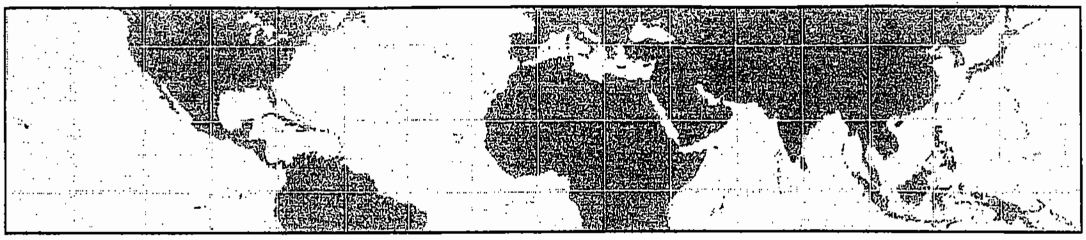
Table 5A.1 Investment strategy to show that futures and forward prices are equal.

Day	0	1	2	…	$n-1$	n
Futures price	F_0	F_1	F_2	…	F_{n-1}	F_n
Futures position	e^{δ}	$e^{2\delta}$	$e^{3\delta}$	…	$e^{n\delta}$	0
Gain/loss	0	$(F_1 - F_0)e^{\delta}$	$(F_2 - F_1)e^{2\delta}$	…	…	$(F_n - F_{n-1})e^{n\delta}$
Gain/loss compounded to day n	0	$(F_1 - F_0)e^{n\delta}$	$(F_2 - F_1)e^{n\delta}$	…	…	$(F_n - F_{n-1})e^{n\delta}$

initial outlay of F_0 and the other requiring an initial outlay of G_0—both of which yield $S_T e^{n\delta}$ at time T. It follows that, in the absence of arbitrage opportunities,

$$F_0 = G_0$$

In other words, the futures price and the forward price are identical. Note that in this proof there is nothing special about the time period of 1 day. The futures price based on a contract with weekly settlements is also the same as the forward price when corresponding assumptions are made.



CHAPTER 6

Interest Rate Futures

So far we have covered futures contracts on commodities, stock indices, and foreign currencies. We have seen how they work, how they are used for hedging, and how futures prices are determined. We now move on to consider interest rate futures.

This chapter explains the popular Treasury bond and Eurodollar futures contracts that trade in the United States. Many of the other interest rate futures contracts throughout the world have been modeled on these contracts. The chapter also shows how interest rate futures contracts, when used in conjunction with the duration measure introduced in Chapter 4, can be used to hedge a company's exposure to interest rate movements.

6.1 DAY COUNT AND QUOTATION CONVENTIONS

As a preliminary to the material in this chapter, we consider the day count and quotation conventions that apply to bonds and other interest-rate-dependent instuments.

Day Counts

The day count defines the way in which interest accrues over time. Generally, we know the interest earned over some reference period (e.g., the time between coupon payments on a bond), and we are interested in calculating the interest earned over some other period.

The day count convention is usually expressed as X/Y. When we are calculating the interest earned between two dates, X defines the way in which the number of days between the two dates is calculated, and Y defines the way in which the total number of days in the reference period is measured. The interest earned between the two dates is

$$\frac{\text{Number of days between dates}}{\text{Number of days in reference period}} \times \text{Interest earned in reference period}$$

Three day count conventions that are commonly used in the United States are:

1. Actual/actual (in period)
2. 30/360
3. Actual/360

Business Snapshot 6.1 Day Counts Can Be Deceptive

Between February 28, 2009, and March 1, 2009, you have a choice between owning a US government bond and a US corporate bond. They pay the same coupon and have the same quoted price. Which would you prefer?

It sounds as though you should be indifferent, but in fact you should have a marked preference for the corporate bond. Under the 30/360 day count convention used for corporate bonds, there are 3 days between February 28, 2009, and March 1, 2009. Under the actual/actual (in period) day count convention used for government bonds, there is only 1 day. You would earn approximately three times as much interest by holding the corporate bond!

The actual/actual (in period) day count is used for Treasury bonds in the United States. This means that the interest earned between two dates is based on the ratio of the actual days elapsed to the actual number of days in the period between coupon payments. Suppose that the bond principal is \$100, coupon payment dates are March 1 and September 1, the coupon rate is 8%, and we wish to calculate the interest earned between March 1 and July 3. The reference period is from March 1 to September 1. There are 184 (actual) days in this period, and interest of \$4 is earned during the period. There are 124 (actual) days between March 1 and July 3. The interest earned between March 1 and July 3 is therefore

$$\frac{124}{184} \times 4 = 2.6957$$

The 30/360 day count is used for corporate and municipal bonds in the United States. This means that we assume 30 days per month and 360 days per year when carrying out calculations. With the 30/360 day count, the total number of days between March 1 and September 1 is 180. The total number of days between March 1 and July 3 is $(4 \times 30) + 2 = 122$. In a corporate bond with the same terms as the Treasury bond just considered, the interest earned between March 1 and July 3 would therefore be

$$\frac{122}{180} \times 4 = 2.7111$$

As shown in Business Snapshot 6.1, sometimes the 30/360 day count convention has surprising consequences.

The actual/360 day count is used for money market instruments in the United States. This indicates that the reference period is 360 days. The interest earned during part of a year is calculated by dividing the actual number of elapsed days by 360 and multiplying by the rate. The interest earned in 90 days is therefore exactly one-fourth of the quoted rate, and the interest earned in a whole year of 365 days is 365/360 times the quoted rate.

Conventions vary from country to country and from instrument to instrument. For example, money market instruments are quoted on an actual/365 basis in Australia, Canada, and New Zealand. LIBOR is quoted on an actual/360 for all currencies except sterling, for which it is quoted on an actual/365 basis. Euro-denominated and sterling bonds are usually quoted on an actual/actual basis.

Price Quotations

The prices of money market instruments are sometimes quoted using a *discount rate*. This is the interest earned as a percentage of the final face value rather than as a percentage of the initial price paid for the instrument. An example is Treasury bills in the United States. If the price of a 91-day Treasury bill is quoted as 8, this means that the annualized rate of interest earned is 8% of the face value. Suppose that the face value is \$100. Interest of \$2.0222 (= \$100 × 0.08 × 91/360) is earned over the 91-day life. This corresponds to a true rate of interest of 2.0222/(100 − 2.0222) = 2.064% for the 91-day period. In general, the relationship between the cash price and quoted price of a Treasury bill in the United States is

$$P = \frac{360}{n}(100 - Y)$$

where P is the quoted price, Y is the cash price, and n is the remaining life of the Treasury bill measured in calendar days.

US Treasury Bonds

Treasury bond prices in the United States are quoted in dollars and thirty-seconds of a dollar. The quoted price is for a bond with a face value of \$100. Thus, a quote of 90-05 indicates that the quoted price for a bond with a face value of \$100,000 is \$90,156.25.

The quoted price, which traders refer to as the *clean price*, is not the same as the cash price paid by the purchaser of the bond, which traders refer to as the *dirty price*. In general,

Cash price = Quoted price + Accrued interest since last coupon date

To illustrate this formula, suppose that it is March 5, 2010, and the bond under consideration is an 11% coupon bond maturing on July 10, 2018, with a quoted price of 95-16 or \$95.50. Because coupons are paid semiannually on government bonds (and the final coupon is at maturity), the most recent coupon date is January 10, 2010, and the next coupon date is July 10, 2010. The number of days between January 10, 2010, and March 5, 2010, is 54, whereas the number of days between January 10, 2010, and July 10, 2010, is 181. On a bond with \$100 face value, the coupon payment is \$5.50 on January 10 and July 10. The accrued interest on March 5, 2010, is the share of the July 10 coupon accruing to the bondholder on March 5, 2010. Because actual/actual in period is used for Treasury bonds in the United States, this is

$$\frac{54}{181} \times \$5.5 = \$1.64$$

The cash price per \$100 face value for the bond is therefore

$$\$95.50 + \$1.64 = \$97.14$$

Thus, the cash price of a \$100,000 bond is \$97,140.

6.2 TREASURY BOND FUTURES

Table 6.1 shows interest rate futures quotes as they appeared in the *Wall Street Journal* on January 9, 2007. One of the most popular long-term interest rate futures contracts is the Treasury bond futures contract traded on the Chicago Board of Trade (CBOT). In this contract, any government bond that has more than 15 years to maturity on the first day of the delivery month and is not callable within 15 years from that day can be delivered. As will be explained later in this section, the CBOT has developed a procedure for adjusting the price received by the party with the short position according to the particular bond delivered.

The Treasury note and 5-year Treasury note futures contract in the United States are also very popular. With Treasury note futures, any government bond (or note) with a maturity between $6\frac{1}{2}$ and 10 years can be delivered. In the 5-year Treasury note futures contract, the bond delivered has a remaining life that is about 4 or 5 years.

The remaining discussion in this section focuses on CBOT Treasury bond futures. The Treasury note futures traded in the United States and many other futures contracts in the rest of the world are designed in a similar way to CBOT Treasury bond futures, so that many of the points we will make are applicable to these contracts as well.

Quotes

Treasury bond futures prices are quoted in the same way as the Treasury bond prices themselves (see Section 6.1). Table 6.1 shows that the settlement price on January 8, 2007, for the March 2007 contract was 112-04, or $112\frac{4}{32}$. One contract involves the delivery of \$100,000 face value of the bond. Thus, a \$1 change in the quoted futures price would lead to a \$1,000 change in the value of the futures contract. Delivery can take place at any time during the delivery month.

Table 6.1 Interest rate futures quotes from the *Wall Street Journal* on January 9, 2007. (Columns show month, open, high, low, settle, change, and open interest, respectively.)

Interest Rate Futures

Treasury Bonds (CBT)-\$100,000; pts 32nds of 100%						
March	112-05	112-07	111-27	**112-04**	-1	777,963
June	112-04	112-04	111-27	**112-02**	-1	7,450
Treasury Notes (CBT)-\$100,000; pts 32nds of 100%						
March	107-280	107-295	107-215	**107-260**	-2.0	2,296,674
June	107-260	107-270	107-240	**107-270**	-2.0	38,131
5 Yr. Treasury Notes (CBT)-\$100,000; pts 32nds of 100%						
March	105-090	105-100	105-045	**105-075**	-2.0	1,425,917
2 Yr. Treasury Notes (CBT)-\$200,000; pts 32nds of 100%						
March	102-042	102-042	102-015	**102-025**	-1.7	770,033
30 Day Federal Funds (CBT)-\$5,000,000; 100 - daily avg.						
Jan	94.750	94.760	94.750	**94.755**	...	84,247
Feb	94.755	94.760	94.755	**94.760**	...	120,416
1 Month Libor (CME)-\$3,000,000; pts of 100%						
Jan	94.6775	94.6800	94.6775	**94.6775**	-.0025	23,569
Feb	94.6775	94.6825	94.6775	**94.6825**	.0050	16,150
Eurodollar (CME)-\$1,000,000; pts of 100%						
Jan	94.6375	94.6450	94.6375	**94.6425**	...	42,487
June	94.8250	94.8300	94.7750	**94.7900**	-.0400	1,414,973
Sept	95.0000	95.0000	94.9400	**94.9550**	-.0500	1,346,082
Dec	95.1300	95.1300	95.0700	**95.0900**	-.0450	1,316,779

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Conversion Factors

As mentioned, the Treasury bond futures contract allows the party with the short position to choose to deliver any bond that has a maturity of more than 15 years and is not callable within 15 years. When a particular bond is delivered, a parameter known as its *conversion factor* defines the price received for the bond by the party with the short position. The applicable quoted price is the product of the conversion factor and the most recent settlement price for the futures contract. Taking accrued interest into account, as described in Section 6.1, the cash received for each \$100 face value of bond delivered is

$$(\text{Most recent settlement price} \times \text{Conversion factor}) + \text{Accrued interest}$$

Each contract is for the delivery of \$100,000 face value of bonds. Suppose that the most recent settlement price is 90-00, the conversion factor for the bond delivered is 1.3800, and the accrued interest on this bond at the time of delivery is \$3 per \$100 face value. The cash received by the party with the short position (and paid by the party with the long position) is then

$$(1.3800 \times 90.00) + 3.00 = \$127.20$$

per \$100 face value. A party with the short position in one contract would deliver bonds with a face value of \$100,000 and receive \$127,200.

The conversion factor for a bond is set equal to the quoted price the bond would have per dollar of principal on the first day of the delivery month on the assumption that the interest rate for all maturities equals 6% per annum (with semiannual compounding). The bond maturity and the times to the coupon payment dates are rounded down to the nearest 3 months for the purposes of the calculation. The practice enables the CBOT to produce comprehensive tables. If, after rounding, the bond lasts for an exact number of 6-month periods, the first coupon is assumed to be paid in 6 months. If, after rounding, the bond does not last for an exact number of 6-month periods (i.e., there are an extra 3 months), the first coupon is assumed to be paid after 3 months and accrued interest is subtracted.

As a first example of these rules, consider a 10% coupon bond with 20 years and 2 months to maturity. For the purposes of calculating the conversion factor, the bond is assumed to have exactly 20 years to maturity. The first coupon payment is assumed to be made after 6 months. Coupon payments are then assumed to be made at 6-month intervals until the end of the 20 years when the principal payment is made. Assume that the face value is \$100. When the discount rate is 6% per annum with semiannual compounding (or 3% per 6 months), the value of the bond is

$$\sum_{i=1}^{40} \frac{5}{1.03^i} + \frac{100}{1.03^{40}} = \$146.23$$

Dividing by the face value gives a conversion factor of 1.4623.

As a second example of the rules, consider an 8% coupon bond with 18 years and 4 months to maturity. For the purposes of calculating the conversion factor, the bond is assumed to have exactly 18 years and 3 months to maturity. Discounting all the payments back to a point in time 3 months from today at 6% per annum (compounded semiannually) gives a value of

$$4+\sum_{i=1}^{36}\frac{4}{1.03^i}+\frac{100}{1.03^{36}}=\$125.83$$

The interest rate for a 3-month period is $\sqrt{1.03}-1$, or 1.4889%. Hence, discounting back to the present gives the bond's value as 125.83/1.014889 = \$123.99. Subtracting the accrued interest of 2.0, this becomes \$121.99. The conversion factor is therefore 1.2199.

Cheapest-to-Deliver Bond

At any given time during the delivery month, there are many bonds that can be delivered in the CBOT Treasury bond futures contract. These vary widely as far as coupon and maturity are concerned. The party with the short position can choose which of the available bonds is "cheapest" to deliver. Because the party with the short position receives

(Most recent settlement price × Conversion factor) + Accrued interest

and the cost of purchasing a bond is

Quoted bond price + Accrued interest

the cheapest-to-deliver bond is the one for which

Quoted bond price − (Most recent settlement price × Conversion factor)

is least. Once the party with the short position has decided to deliver, it can determine the cheapest-to-deliver bond by examining each of the deliverable bonds in turn.

Example 6.1

The party with the short position has decided to deliver and is trying to choose between the three bonds in the table below. Assume the most recent settlement price is 93-08, or 93.25.

Bond	*Quoted bond price (\$)*	*Conversion factor*
1	99.50	1.0382
2	143.50	1.5188
3	119.75	1.2615

The cost of delivering each of the bonds is as follows:

Bond 1: 99.50 − (93.25 × 1.0382) = \$2.69

Bond 2: 143.50 − (93.25 × 1.5188) = \$1.87

Bond 3: 119.75 − (93.25 × 1.2615) = \$2.12

The cheapest-to-deliver bond is Bond 2.

Business Snapshot 6.2 The Wild Card Play

Trading in the CBOT Treasury bond futures contract ceases at 2:00 p.m. Chicago time. However, Treasury bonds themselves continue trading in the spot market until 4:00 p.m. Furthermore, a trader with a short futures position has until 8:00 p.m. to issue to the clearinghouse a notice of intention to deliver. If the notice is issued, the invoice price is calculated on the basis of the settlement price that day. This is the price at which trading was conducted just before the closing bell at 2:00 p.m.

This practice gives rise to an option known as the *wild card play*. If bond prices decline after 2:00 p.m. on the first day of the delivery month, the party with the short position can issue a notice of intention to deliver at, say, 3:45 p.m. and proceed to buy bonds for delivery at the 2:00 p.m. futures price. If the bond price does not decline, the party with the short position keeps the position open and waits until the next day when the same strategy can be used.

As with the other options open to the party with the short position, the wild card play is not free. Its value is reflected in the futures price, which is lower than it would be without the option.

A number of factors determine the cheapest-to-deliver bond. When bond yields are in excess of 6%, the conversion factor system tends to favor the delivery of low-coupon long-maturity bonds. When yields are less than 6%, the system tends to favor the delivery of high-coupon short-maturity bonds. Also, when the yield curve is upward-sloping, there is a tendency for bonds with a long time to maturity to be favored, whereas when it is downward-sloping, there is a tendency for bonds with a short time to maturity to be delivered.

In addition to the cheapest-to-deliver bond option, the party with a short position has an option known as the wild card play. This is described in Business Snapshot 6.2.

Determining the Futures Price

An exact theoretical futures price for the Treasury bond contract is difficult to determine because the short party's options concerned with the timing of delivery and choice of the bond that is delivered cannot easily be valued. However, if we assume that both the cheapest-to-deliver bond and the delivery date are known, the Treasury bond futures contract is a futures contract on a traded security (the bond) that provides the holder with known income.[1] Equation (5.2) then shows that the futures price, F_0, is related to the spot price, S_0, by

$$F_0 = (S_0 - I)e^{rT} \tag{6.1}$$

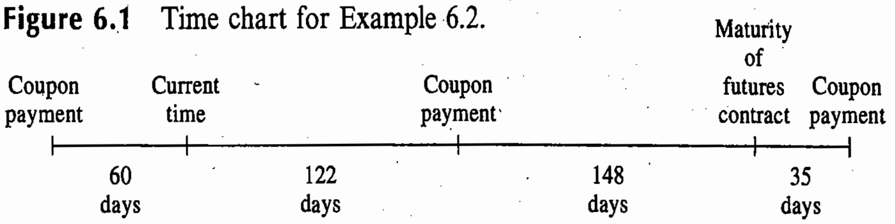
where I is the present value of the coupons during the life of the futures contract, T is the time until the futures contract matures, and r is the risk-free interest rate applicable to a time period of length T.

Example 6.2

Suppose that, in a Treasury bond futures contract, it is known that the cheapest-to-deliver bond will be a 12% coupon bond with a conversion factor of 1.4000.

[1] In practice, for the purposes of determining the cheapest-to-deliver in this calculation, analysts usually assume that zero rates at the maturity of the futures contract will equal today's forward rates.

Figure 6.1 Time chart for Example 6.2.



Suppose also that it is known that delivery will take place in 270 days. Coupons are payable semiannually on the bond. As illustrated in Figure 6.1, the last coupon date was 60 days ago, the next coupon date is in 122 days, and the coupon date thereafter is in 305 days. The term structure is flat, and the rate of interest (with continuous compounding) is 10% per annum. Assume that the current quoted bond price is \$120. The cash price of the bond is obtained by adding to this quoted price the proportion of the next coupon payment that accrues to the holder. The cash price is therefore

$$120 + \frac{60}{60+122} \times 6 = 121.978$$

A coupon of \$6 will be received after 122 days (= 0.3342 years). The present value of this is

$$6e^{-0.1\times 0.3342} = 5.803$$

The futures contract lasts for 270 days (= 0.7397 years). The cash futures price, if the contract were written on the 12% bond, would therefore be

$$(121.978 - 5.803)e^{0.1\times 0.7397} = 125.094$$

At delivery, there are 148 days of accrued interest. The quoted futures price, if the contract were written on the 12% bond, is calculated by subtracting the accrued interest

$$125.094 - 6 \times \frac{148}{148+35} = 120.242$$

From the definition of the conversion factor, 1.4000 standard bonds are considered equivalent to each 12% bond. The quoted futures price should therefore be

$$\frac{120.242}{1.4000} = 85.887$$

6.3 EURODOLLAR FUTURES

The most popular interest rate futures contract in the United States is the 3-month Eurodollar futures contract traded on the Chicago Mercantile Exchange (CME). A Eurodollar is a dollar deposited in a US or foreign bank outside the United States. The Eurodollar interest rate is the rate of interest earned on Eurodollars deposited by one bank with another bank. It is essentially the same as the London Interbank Offered Rate (LIBOR) introduced in Chapter 4.

Three-month Eurodollar futures contracts are futures contracts on the 3-month

(90-day) Eurodollar interest rate. They allow an investor to lock in an interest rate on \$1 million for a future 3-month period. The 3-month period to which the interest rate applies starts on the third Wednesday of the delivery month. The contracts have delivery months of March, June, September, and December for up to 10 years into the future. This means that in 2008 an investor can use Eurodollar futures to lock in an interest rate for 3-month periods that are as far into the future as 2018. Short-maturity contracts trade for months other than March, June, September, and December. However, these have relatively low open interest.

To understand how Eurodollar futures contracts work, consider the June 2007 contract in Table 6.1. The quoted settlement price on January 8, 2007, is 94.79. The contract ends on the third Wednesday of the delivery month. In the case of this contract, the third Wednesday of the delivery month is June 20, 2007. The contract is marked to market in the usual way until that date. However, on June 20, 2007, the settlement price is set equal to $100 - R$, where R is the actual 3-month Eurodollar interest rate on that day, expressed with quarterly compounding and an actual/360 day count convention. (Thus, if the 3-month Eurodollar interest rate on June 20, 2007, turned out to be 4%, the final settlement price would be 96.) There is a final settlement reflecting this settlement price and all contracts are declared closed.

The contract is designed so that a 1 basis point (= 0.01) move in the futures quote corresponds to a gain or loss of \$25 per contract. When a Eurodollar futures quote increases by 1 basis point, a trader who is long one contract gains \$25 and a trader who is short one contract loses \$25. Similarly, when the quote decreases by 1 basis point a trader who is long one contract loses \$25 and a trader who is short one contract gains \$25. Suppose, for example, that the settlement price changes from 94.79 to 94.90 between January 8, 2007, and January 9, 2007. Traders with long positions gain $11 \times 25 = \$275$ per contract; traders with short positions lose \$275 per contract. The \$25 per basis point rule is consistent with the point made earlier that the contract locks in an interest rate on \$1 million dollars for 3 months. When an interest rate per year changes by 1 basis point, the interest earned on 1 million dollars for 3 months changes by

$$1{,}000{,}000 \times 0.0001 \times 0.25 = 25$$

or \$25. Since the futures quote is 100 minus the futures interest rate, an investor who is long gains when interest rates fall and one who is short gains when interest rates rise.

Example 6.3

On January 8, 2007, an investor wants to lock in the interest rate that will be earned on \$5 million for 3 months starting on June 20, 2007. The investor buys five June07 Eurodollar futures contracts at 94.79. On June 20, 2007, the 3-month LIBOR interest rate is 4%, so that the final settlement price proves to be 96.00. The investor gains $5 \times 25 \times (9{,}600 - 9{,}479) = \$15{,}125$ on the long futures position. The interest earned on the \$5 million for 3 months at 4% is

$$5{,}000{,}000 \times 0.25 \times 0.04 = 50{,}000$$

or \$50,000. The gain on the futures contract brings this up to \$65,125. This is the interest that would have been earned if the interest rate had been 5.21% ($5{,}000{,}000 \times 0.25 \times 0.0521 = 65{,}125$). This illustration shows that the futures trade has the effect of locking in an interest rate equal to (100 − 94.79)%, or 5.21%.

The exchange defines the contract price as

$$10{,}000 \times [100 - 0.25 \times (100 - Q)] \tag{6.2}$$

where Q is the quote. Thus, the settlement price of 94.79 for the June 2007 contract in Table 6.1 corresponds to a contract price of

$$10{,}000 \times [100 - 0.25 \times (100 - 94.79)] = \$986{,}975$$

In Example 6.3, the final contract price is

$$10{,}000 \times [100 - 0.25 \times (100 - 96)] = \$990{,}000$$

and the difference between the initial and final contract price is \$3,025, so that an investor with a long position in five contracts gains 5 × 3,025 dollars, or \$15,125. This is consistent with the "\$25 per 1 basis point move" rule used in Example 6.3.

We can see from Table 6.1 that the interest rate term structure in the United States was downward-sloping on January 8, 2007. The futures rate for a 3-month period beginning in January 17, 2007, was 5.3575%; for a 3-month period beginning June 20, 2007, it was 5.21%; for a 3-month period beginning September 19, 2007, it was 5.045%; and for a 3-month period beginning December 19, 2007, it was 4.91%.

Other contracts similar to the CME Eurodollar futures contract trade on interest rates in other countries. The CME trades Euroyen contracts. The London International Financial Futures and Options Exchange (part of Euronext) trades 3-month Euribor contracts (i.e., contracts on the 3-month LIBOR rate for the euro) and 3-month Euroswiss futures.

Forward vs. Futures Interest Rates

The Eurodollar futures contract is similar to a forward rate agreement (FRA: see Section 4.7) in that it locks in an interest rate for a future period. For short maturities (up to a year or so), the two contracts can be assumed to be the same and the Eurodollar futures interest rate can be assumed to be the same as the corresponding forward interest rate. For longer-dated contracts, differences between the contracts become important. Compare a Eurodollar futures contract on an interest rate for the period between times T_1 and T_2 with an FRA for the same period. The Eurodollar futures contract is settled daily. The final settlement is at time T_1 and reflects the realized interest rate for the period between times T_1 and T_2. By contrast the FRA is not settled daily and the final settlement reflecting the realized interest rate between times T_1 and T_2 is made at time T_2.[2]

There are therefore two differences between a Eurodollar futures contract and an FRA. These are:

1. The difference between a Eurodollar futures contract and a similar contract where there is no daily settlement. The latter is a forward contract where a payoff equal to the difference between the forward interest rate and the realized interest rate is paid at time T_1.

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[2] As mentioned in Section 4.7, settlement may occur at time T_1, but it is then equal to the present value of the normal forward contract payoff at time T_2.

2. The difference between a forward contract where there is settlement at time T_1 and a forward contract where there is settlement at time T_2.

These two components to the difference between the contracts cause some confusion in practice. Both decrease the forward rate relative to the futures rate, but for long-dated contracts the reduction caused by the second difference is much smaller than that caused by the first. The reason why the first difference (daily settlement) decreases the forward rate follows from the arguments in Section 5.8. Suppose you have a contract where the payoff is $R_M - R_F$ at time T_1, where R_F is a predetermined rate for the period between T_1 and T_2, and R_M is the realized rate for this period, and you have the option to switch to daily settlement. In this case daily settlement tends to lead to cash inflows when rates are high and cash outflows when rates are low. You would therefore find switching to daily settlement to be attractive because you tend to have more money in your margin account when rates are high. As a result the market would therefore set R_F higher for the daily settlement alternative (reducing your cumulative expected payoff). To put this the other way round, switching from daily settlement to settlement at time T_1 reduces R_F.

To understand the reason why the second difference reduces the forward rate, suppose that the payoff of $R_M - R_F$ is at time T_2 instead of T_1 (as it is for a regular FRA). If R_M is high, the payoff is positive. Because rates are high, the cost to you of having the payoff that you receive at time T_2 rather than time T_1 is relatively high. If R_M is low, the payoff is negative. Because rates are low, the benefit to you of having the payoff you make at time T_2 rather than time T_1 is relatively low. Overall you would rather have the payoff at time T_1. If it is at time T_2 rather than T_1, you must be compensated by a reduction in R_F.[3]

Analysts make what is known as a *convexity adjustment* to account for the total difference between the two rates. One popular adjustment is[4]

$$\text{Forward rate} = \text{Futures rate} - \tfrac{1}{2}\sigma^2 T_1 T_2 \tag{6.3}$$

where, as above, T_1 is the time to maturity of the futures contract and T_2 is the time to the maturity of the rate underlying the futures contract. The variable σ is the standard deviation of the change in the short-term interest rate in 1 year. Both rates are expressed with continuous compounding.[5] A typical value for σ is 1.2% or 0.012.

Example 6.4

Consider the situation where $\sigma = 0.012$ and we wish to calculate the forward rate when the 8-year Eurodollar futures price quote is 94. In this case $T_1 = 8$, $T_2 = 8.25$, and the convexity adjustment is

$$\tfrac{1}{2} \times 0.012^2 \times 8 \times 8.25 = 0.00475$$

or 0.475% (47.5 basis points). The futures rate is 6% per annum on an actual/360

[3] Quantifying the effect of this type of timing difference on the value of a derivative is discussed further in Chapter 29.

[4] See Technical Note 1 on the author's website for a proof of this.

[5] This formula is based on the Ho–Lee interest rate model, which will be discussed in Chapter 30. See T. S. Y. Ho and S.-B. Lee, "Term structure movements and pricing interest rate contingent claims," *Journal of Finance*, 41 (December 1986), 1011–29.

basis with quarterly compounding. This corresponds to 1.5% per 90 days or an annual rate of $(365/90)\ln 1.015 = 6.038\%$ with continuous compounding and an actual/365 day count. The estimate of the forward rate given by equation (6.3), therefore, is $6.038 - 0.475 = 5.563\%$ per annum with continuous compounding. The table below shows how the size of the adjustment increases with the time to maturity.

Maturity of futures (years)	*Convexity adjustments (basis points)*
2	3.2
4	12.2
6	27.0
8	47.5
10	73.8

We can see from this table that the size of the adjustment is roughly proportional to the square of the time to maturity of the futures contract. Thus the convexity adjustment for the 8-year contract is approximately 16 times that for a 2-year contract.

Using Eurodollar Futures to Extend the LIBOR Zero Curve

The LIBOR zero curve out to 1 year is determined by the 1-month, 3-month, 6-month, and 12-month LIBOR rates. Once the convexity adjustment just described has been made, Eurodollar futures are often used to extend the zero curve. Suppose that the ith Eurodollar futures contract matures at time T_i $(i = 1, 2, \dots)$. It is usually assumed that the forward interest rate calculated from the ith futures contract applies to the period T_i to T_{i+1}. (In practice this is close to true.) This enables a bootstrap procedure to be used to determine zero rates. Suppose that F_i is the forward rate calculated from the ith Eurodollar futures contract and R_i is the zero rate for a maturity T_i. From equation (4.5),

$$F_i = \frac{R_{i+1}T_{i+1} - R_iT_i}{T_{i+1} - T_i}$$

so that

$$R_{i+1} = \frac{F_i(T_{i+1} - T_i) + R_iT_i}{T_{i+1}} \tag{6.4}$$

Other Euro rates such as Euroswiss, Euroyen, and Euribor are used in a similar way.

Example 6.5

The 400-day LIBOR zero rate has been calculated as 4.80% with continuous compounding and, from Eurodollar futures quotes, it has been calculated that (a) the forward rate for a 90-day period beginning in 400 days is 5.30% with continuous compounding, (b) the forward rate for a 90-day period beginning in 491 days is 5.50% with continuous compounding, and (c) the forward rate for a 90-day period beginning in 589 days is 5.60% with continuous compounding. We can use equation (6.4) to obtain the 491-day rate as

$$\frac{0.053 \times 91 + 0.048 \times 400}{491} = 0.04893$$

or 4.893%. Similarly we can use the second forward rate to obtain the 589-day rate as

$$\frac{0.055 \times 98 + 0.04893 \times 491}{589} = 0.04994$$

or 4.994%. The next forward rate of 5.60% would be used to determine the zero curve out to the maturity of the next Eurodollar futures contract. (Note that, even though the rate underlying the Eurodollar futures contract is a 90-day rate, it is assumed to apply to the 91 or 98 days elapsing between Eurodollar contract maturities.)

6.4 DURATION-BASED HEDGING STRATEGIES USING FUTURES

We discussed duration in Section 4.8. Consider the situation where a position in an asset that is interest rate dependent, such as a bond portfolio or a money market security, is being hedged using an interest rate futures contract. Define:

F_C: Contract price for the interest rate futures contract

D_F: Duration of the asset underlying the futures contract at the maturity of the futures contract

P: Forward value of the portfolio being hedged at the maturity of the hedge (in practice, this is usually assumed to be the same as the value of the portfolio today)

D_P: Duration of the portfolio at the maturity of the hedge

If we assume that the change in the yield, Δy, is the same for all maturities, which means that only parallel shifts in the yield curve can occur, it is approximately true that

$$\Delta P = -PD_P\,\Delta y$$

It is also approximately true that

$$\Delta F_C = -F_C D_F\,\Delta y$$

The number of contracts required to hedge against an uncertain Δy, therefore, is

$$N^* = \frac{PD_P}{F_C D_F} \tag{6.5}$$

This is the *duration-based hedge ratio*. It is sometimes also called the *price sensitivity hedge ratio*.[6] Using it has the effect of making the duration of the entire position zero.

When the hedging instrument is a Treasury bond futures contract, the hedger must base D_F on an assumption that one particular bond will be delivered. This means that the hedger must estimate which of the available bonds is likely to be cheapest to deliver

[6] For a more detailed discussion of equation (6.5), see R.J. Rendleman, "Duration-Based Hedging with Treasury Bond Futures," *Journal of Fixed Income* 9, 1 (June 1999): 84–91.

at the time the hedge is put in place. If, subsequently, the interest rate environment changes so that it looks as though a different bond will be cheapest to deliver, then the hedge has to be adjusted and its performance may be worse than anticipated.

When hedges are constructed using interest rate futures, it is important to bear in mind that interest rates and futures prices move in opposite directions. When interest rates go up, an interest rate futures price goes down. When interest rates go down, the reverse happens, and the interest rate futures price goes up. Thus, a company in a position to lose money if interest rates drop should hedge by taking a long futures position. Similarly, a company in a position to lose money if interest rates rise should hedge by taking a short futures position.

The hedger tries to choose the futures contract so that the duration of the underlying asset is as close as possible to the duration of the asset being hedged. Eurodollar futures tend to be used for exposures to short-term interest rates, whereas Treasury bond and Treasury note futures contracts are used for exposures to longer-term rates.

Example 6.6

It is August 2 and a fund manager with \$10 million invested in government bonds is concerned that interest rates are expected to be highly volatile over the next 3 months. The fund manager decides to use the December T-bond futures contract to hedge the value of the portfolio. The current futures price is 93-02, or 93.0625. Because each contract is for the delivery of \$100,000 face value of bonds, the futures contract price is \$93,062.50.

Suppose that the duration of the bond portfolio in 3 months will be 6.80 years. The cheapest-to-deliver bond in the T-bond contract is expected to be a 20-year 12% per annum coupon bond. The yield on this bond is currently 8.80% per annum, and the duration will be 9.20 years at maturity of the futures contract.

The fund manager requires a short position in T-bond futures to hedge the bond portfolio. If interest rates go up, a gain will be made on the short futures position, but a loss will be made on the bond portfolio. If interest rates decrease, a loss will be made on the short position, but there will be a gain on the bond portfolio. The number of bond futures contracts that should be shorted can be calculated from equation (6.5) as

$$\frac{10{,}000{,}000}{93{,}062.50} \times \frac{6.80}{9.20} = 79.42$$

To the nearest whole number, the portfolio manager should short 79 contracts.

6.5 HEDGING PORTFOLIOS OF ASSETS AND LIABILITIES

Financial institutions sometimes attempt to hedge themselves against interest rate risk by ensuring that the average duration of their assets equals the average duration of their liabilities. (The liabilities can be regarded as short positions in bonds.) This strategy is known as *duration matching* or *portfolio immunization*. When implemented, it ensures that a small parallel shift in interest rates will have little effect on the value of the portfolio of assets and liabilities. The gain (loss) on the assets should offset the loss (gain) on the liabilities.

> **Business Snapshot 6.3** Asset–Liability Management by Banks
>
> As described in Business Snapshot 4.3, in the 1960s, 1970s, and 1980s, many Savings and Loans and some banks in the United States made the mistake of funding long-term loans with short-term deposits. This led to some spectacular failures by these financial institutions.
>
> The asset–liability management (ALM) committees of banks now monitor their exposure to interest rates very carefully. Matching the durations of assets and liabilities is a first step, but this does not protect a bank against nonparallel shifts in the yield curve. A popular approach is known as *GAP management*. This involves dividing the zero-coupon yield curve into segments, known as *buckets*. The first bucket might be 0 to 1 month, the second 1 to 3 months, and so on. The ALM committee then investigates the effect on the values of both assets and liabilities of the zero rates corresponding to one bucket changing while those corresponding to all other buckets staying the same.
>
> If there is a mismatch, corrective action is usually taken. Luckily banks today have many more tools to manage their exposures to interest rates than they had in the 1960s. These tools include swaps, FRAs, bond futures, Eurodollar futures, and other interest rate derivatives.

Duration matching does not immunize a portfolio against nonparallel shifts in the zero curve. This is a weakness of the approach. In practice, short-term rates are usually more volatile than, and are not perfectly correlated with, long-term rates. Sometimes it even happens that short- and long-term rates move in opposite directions to each other. Duration matching is therefore only a first step and financial institutions have developed other tools to help them manage their interest rate exposure. See Business Snapshot 6.3.

SUMMARY

Two very popular interest rate contracts are the Treasury bond and Eurodollar futures contracts that trade in the United States. In the Treasury bond futures contracts, the party with the short position has a number of interesting delivery options:

1. Delivery can be made on any day during the delivery month.
2. There are a number of alternative bonds that can be delivered.
3. On any day during the delivery month, the notice of intention to deliver at the 2:00 p.m. settlement price can be made any time up to 8:00 p.m.

These options all tend to reduce the futures price.

The Eurodollar futures contract is a contract on the 3-month rate on the third Wednesday of the delivery month. Eurodollar futures are frequently used to estimate LIBOR forward rates for the purpose of constructing a LIBOR zero curve. When long-dated contracts are used in this way, it is important to make what is termed a convexity adjustment to allow for the marking to market in the futures contract.

The concept of duration is important in hedging interest rate risk. It enables a hedger to assess the sensitivity of a bond portfolio to small parallel shifts in the yield curve. It also enables the hedger to assess the sensitivity of an interest rate futures price to small changes in the yield curve. The number of futures contracts necessary to protect the bond portfolio against small parallel shifts in the yield curve can therefore be calculated.

The key assumption underlying the duration-based hedging scheme is that all interest rates change by the same amount. This means that only parallel shifts in the term structure are allowed for. In practice, short-term interest rates are generally more volatile than are long-term interest rates, and hedge performance is liable to be poor if the duration of the bond underlying the futures contract differs markedly from the duration of the asset being hedged.

FURTHER READING

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Grinblatt, M., and N. Jegadeesh. "The Relative Price of Eurodollar Futures and Forward Contracts," *Journal of Finance*, 51, 4 (September 1996): 1499–1522.

Questions and Problems (Answers in Solutions Manual)

6.1. A US Treasury bond pays a 7% coupon on January 7 and July 7. How much interest accrues per \$100 of principal to the bondholder between July 7, 2009, and August 9, 2009? How would your answer be different if it were a corporate bond?

6.2. It is January 9, 2009. The price of a Treasury bond with a 12% coupon that matures on October 12, 2020, is quoted as 102-07. What is the cash price?

6.3. How is the conversion factor of a bond calculated by the Chicago Board of Trade? How is it used?

6.4. A Eurodollar futures price changes from 96.76 to 96.82. What is the gain or loss to an investor who is long two contracts?

6.5. What is the purpose of the convexity adjustment made to Eurodollar futures rates? Why is the convexity adjustment necessary?

6.6. The 350-day LIBOR rate is 3% with continuous compounding and the forward rate calculated from a Eurodollar futures contract that matures in 350 days is 3.2% with continuous compounding. Estimate the 440-day zero rate.

6.7. It is January 30. You are managing a bond portfolio worth \$6 million. The duration of the portfolio in 6 months will be 8.2 years. The September Treasury bond futures price is currently 108-15, and the cheapest-to-deliver bond will have a duration of 7.6 years in September. How should you hedge against changes in interest rates over the next 6 months?

6.8. The price of a 90-day Treasury bill is quoted as 10.00. What continuously compounded return (on an actual/365 basis) does an investor earn on the Treasury bill for the 90-day period?

6.9. It is May 5, 2008. The quoted price of a government bond with a 12% coupon that matures on July 27, 2011, is 110-17. What is the cash price?

6.10. Suppose that the Treasury bond futures price is 101-12. Which of the following four bonds is cheapest to deliver?

Bond	*Price*	*Conversion factor*
1	125-05	1.2131
2	142-15	1.3792
3	115-31	1.1149
4	144-02	1.4026

6.11. It is July 30, 2009. The cheapest-to-deliver bond in a September 2009 Treasury bond futures contract is a 13% coupon bond, and delivery is expected to be made on September 30, 2009. Coupon payments on the bond are made on February 4 and August 4 each year. The term structure is flat, and the rate of interest with semiannual compounding is 12% per annum. The conversion factor for the bond is 1.5. The current quoted bond price is $110. Calculate the quoted futures price for the contract.

6.12. An investor is looking for arbitrage opportunities in the Treasury bond futures market. What complications are created by the fact that the party with a short position can choose to deliver any bond with a maturity of over 15 years?

6.13. Suppose that the 9-month LIBOR interest rate is 8% per annum and the 6-month LIBOR interest rate is 7.5% per annum (both with actual/365 and continuous compounding). Estimate the 3-month Eurodollar futures price quote for a contract maturing in 6 months.

6.14. Suppose that the 300-day LIBOR zero rate is 4% and Eurodollar quotes for contracts maturing in 300, 398, and 489 days are 95.83, 95.62, and 95.48. Calculate 398-day and 489-day LIBOR zero rates. Assume no difference between forward and futures rates for the purposes of your calculations.

6.15. Suppose that a bond portfolio with a duration of 12 years is hedged using a futures contract in which the underlying asset has a duration of 4 years. What is likely to be the impact on the hedge of the fact that the 12-year rate is less volatile than the 4-year rate?

6.16. Suppose that it is February 20 and a treasurer realizes that on July 17 the company will have to issue $5 million of commercial paper with a maturity of 180 days. If the paper were issued today, the company would realize $4,820,000. (In other words, the company would receive $4,820,000 for its paper and have to redeem it at $5,000,000 in 180 days' time.) The September Eurodollar futures price is quoted as 92.00. How should the treasurer hedge the company's exposure?

6.17. On August 1, a portfolio manager has a bond portfolio worth $10 million. The duration of the portfolio in October will be 7.1 years. The December Treasury bond futures price is currently 91-12 and the cheapest-to-deliver bond will have a duration of 8.8 years at maturity. How should the portfolio manager immunize the portfolio against changes in interest rates over the next 2 months?

6.18. How can the portfolio manager change the duration of the portfolio to 3.0 years in Problem 6.17?

6.19. Between October 30, 2009, and November 1, 2009, you have a choice between owning a US government bond paying a 12% coupon and a US corporate bond paying a 12%

coupon. Consider carefully the day count conventions discussed in this chapter and decide which of the two bonds you would prefer to own. Ignore the risk of default.

6.20. Suppose that a Eurodollar futures quote is 88 for a contract maturing in 60 days. What is the LIBOR forward rate for the 60- to 150-day period? Ignore the difference between futures and forwards for the purposes of this question.

6.21. The 3-month Eurodollar futures price for a contract maturing in 6 years is quoted as 95.20. The standard deviation of the change in the short-term interest rate in 1 year is 1.1%. Estimate the forward LIBOR interest rate for the period between 6.00 and 6.25 years in the future.

6.22. Explain why the forward interest rate is less than the corresponding futures interest rate calculated from a Eurodollar futures contract.

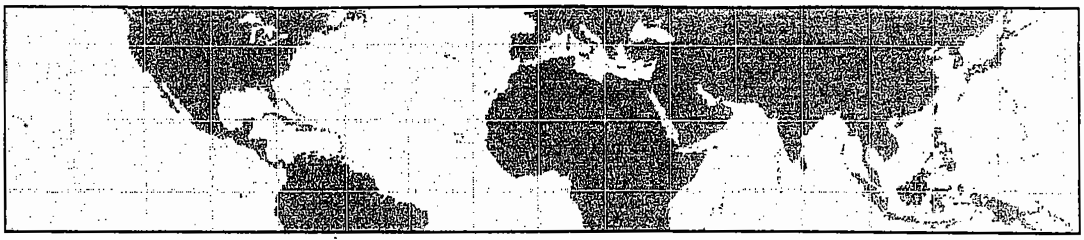
Assignment Questions

6.23. Assume that a bank can borrow or lend money at the same interest rate in the LIBOR market. The 90-day rate is 10% per annum, and the 180-day rate is 10.2% per annum, both expressed with continuous compounding and actual/actual day count. The Eurodollar futures price for a contract maturing in 91 days is quoted as 89.5. What arbitrage opportunities are open to the bank?

6.24. A Canadian company wishes to create a Canadian LIBOR futures contract from a US Eurodollar futures contract and forward contracts on foreign exchange. Using an example, explain how the company should proceed. For the purposes of this problem, assume that a futures contract is the same as a forward contract.

6.25. The futures price for the June 2009 CBOT bond futures contract is 118-23.
(a) Calculate the conversion factor for a bond maturing on January 1, 2025, paying a coupon of 10%.
(b) Calculate the conversion factor for a bond maturing on October 1, 2030, paying a coupon of 7%.
(c) Suppose that the quoted prices of the bonds in (a) and (b) are 169.00 and 136.00, respectively. Which bond is cheaper to deliver?
(d) Assuming that the cheapest-to-deliver bond is actually delivered on June 25, 2009, what is the cash price received for the bond?

6.26. A portfolio manager plans to use a Treasury bond futures contract to hedge a bond portfolio over the next 3 months. The portfolio is worth \$100 million and will have a duration of 4.0 years in 3 months. The futures price is 122, and each futures contract is on \$100,000 of bonds. The bond that is expected to be cheapest to deliver will have a duration of 9.0 years at the maturity of the futures contract. What position in futures contracts is required?
(a) What adjustments to the hedge are necessary if after 1 month the bond that is expected to be cheapest to deliver changes to one with a duration of 7 years?
(b) Suppose that all rates increase over the next 3 months, but long-term rates increase less than short-term and medium-term rates. What is the effect of this on the performance of the hedge?



CHAPTER 7 Swaps

The first swap contracts were negotiated in the early 1980s. Since then the market has seen phenomenal growth. Swaps now occupy a position of central importance in the over-the-counter derivatives market.

A swap is an agreement between two companies to exchange cash flows in the future. The agreement defines the dates when the cash flows are to be paid and the way in which they are to be calculated. Usually the calculation of the cash flows involves the future value of an interest rate, an exchange rate, or other market variable.

A forward contract can be viewed as a simple example of a swap. Suppose it is March 1, 2009, and a company enters into a forward contract to buy 100 ounces of gold for $900 per ounce in 1 year. The company can sell the gold in 1 year as soon as it is received. The forward contract is therefore equivalent to a swap where the company agrees that on March 1, 2010, it will pay $90,000 and receive $100S$, where S is the market price of 1 ounce of gold on that date.

Whereas a forward contract is equivalent to the exchange of cash flows on just one future date, swaps typically lead to cash flow exchanges on several future dates. In this chapter we examine how swaps are designed, how they are used, and how they are valued. Most of this chapter focuses on two popular swaps: plain vanilla interest rate swaps and fixed-for-fixed currency swaps. Other types of swaps are briefly reviewed at the end of the chapter and discussed in more detail in Chapter 32.

7.1 MECHANICS OF INTEREST RATE SWAPS

The most common type of swap is a "plain vanilla" interest rate swap. In this swap a company agrees to pay cash flows equal to interest at a predetermined fixed rate on a notional principal for a number of years. In return, it receives interest at a floating rate on the same notional principal for the same period of time.

LIBOR

The floating rate in most interest rate swap agreements is the London Interbank Offered Rate (LIBOR). We introduced this in Chapter 4. It is the rate of interest at which a bank is prepared to deposit money with other banks in the Eurocurrency market. Typically, 1-month, 3-month, 6-month, and 12-month LIBOR are quoted in all major currencies.

Just as prime is often the reference rate of interest for floating-rate loans in the domestic financial market, LIBOR is a reference rate of interest for loans in international financial markets. To understand how it is used, consider a 5-year bond with a rate of interest specified as 6-month LIBOR plus 0.5% per annum. The life of the bond is divided into 10 periods, each 6 months in length. For each period, the rate of interest is set at 0.5% per annum above the 6-month LIBOR rate at the beginning of the period. Interest is paid at the end of the period.

Illustration

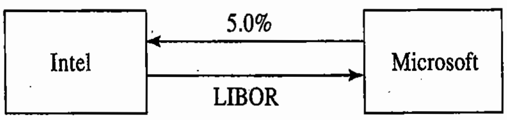
Consider a hypothetical 3-year swap initiated on March 5, 2007, between Microsoft and Intel. We suppose Microsoft agrees to pay Intel an interest rate of 5% per annum on a principal of $100 million, and in return Intel agrees to pay Microsoft the 6-month LIBOR rate on the same principal. Microsoft is the *fixed-rate payer*; Intel is the *floating-rate payer*. We assume the agreement specifies that payments are to be exchanged every 6 months and that the 5% interest rate is quoted with semiannual compounding. This swap is represented diagrammatically in Figure 7.1.

The first exchange of payments would take place on September 5, 2007, 6 months after the initiation of the agreement. Microsoft would pay Intel $2.5 million. This is the interest on the $100 million principal for 6 months at 5%. Intel would pay Microsoft interest on the $100 million principal at the 6-month LIBOR rate prevailing 6 months prior to September 5, 2007—that is, on March 5, 2007. Suppose that the 6-month LIBOR rate on March 5, 2007, is 4.2%. Intel pays Microsoft $0.5 \times 0.042 \times \$100 =$ $2.1 million.[1] Note that there is no uncertainty about this first exchange of payments because it is determined by the LIBOR rate at the time the contract is entered into.

The second exchange of payments would take place on March 5, 2008, a year after the initiation of the agreement. Microsoft would pay $2.5 million to Intel. Intel would pay interest on the $100 million principal to Microsoft at the 6-month LIBOR rate prevailing 6 months prior to March 5, 2008—that is, on September 5, 2007. Suppose that the 6-month LIBOR rate on September 5, 2007, is 4.8%. Intel pays $0.5 \times 0.048 \times \$100 =$ $2.4 million to Microsoft.

In total, there are six exchanges of payment on the swap. The fixed payments are always $2.5 million. The floating-rate payments on a payment date are calculated using the 6-month LIBOR rate prevailing 6 months before the payment date. An interest rate swap is generally structured so that one side remits the difference between the two payments to the other side. In our example, Microsoft would pay Intel $0.4 million (= $2.5 million – $2.1 million) on September 5, 2007, and $0.1 million (= $2.5 million – $2.4 million) on March 5, 2008.

Figure 7.1 Interest rate swap between Microsoft and Intel.



[1] The calculations here are simplified in that they ignore day count conventions. This point is discussed in more detail later in the chapter.

Table 7.1 Cash flows (millions of dollars) to Microsoft in a $100 million 3-year interest rate swap when a fixed rate of 5% is paid and LIBOR is received.

Date	*Six-month LIBOR rate (%)*	*Floating cash flow received*	*Fixed cash flow paid*	*Net cash flow*
Mar. 5, 2007	4.20			
Sept. 5, 2007	4.80	+2.10	−2.50	−0.40
Mar. 5, 2008	5.30	+2.40	−2.50	−0.10
Sept. 5, 2008	5.50	+2.65	−2.50	+0.15
Mar. 5, 2009	5.60	+2.75	−2.50	+0.25
Sept. 5, 2009	5.90	+2.80	−2.50	+0.30
Mar. 5, 2010		+2.95	−2.50	+0.45

Table 7.1 provides a complete example of the payments made under the swap for one particular set of 6-month LIBOR rates. The table shows the swap cash flows from the perspective of Microsoft. Note that the $100 million principal is used only for the calculation of interest payments. The principal itself is not exchanged. For this reason it is termed the *notional principal*, or just the *notional*.

If the principal were exchanged at the end of the life of the swap, the nature of the deal would not be changed in any way. The principal is the same for both the fixed and floating payments. Exchanging $100 million for $100 million at the end of the life of the swap is a transaction that would have no financial value to either Microsoft or Intel. Table 7.2 shows the cash flows in Table 7.1 with a final exchange of principal added in. This provides an interesting way of viewing the swap. The cash flows in the third column of this table are the cash flows from a long position in a floating-rate bond. The cash flows in the fourth column of the table are the cash flows from a short position in a fixed-rate bond. The table shows that the swap can be regarded as the exchange of a fixed-rate bond for a floating-rate bond. Microsoft, whose position is described by Table 7.2, is long a floating-rate bond and short a fixed-rate bond. Intel is long a fixed-rate bond and short a floating-rate bond.

Table 7.2 Cash flows (millions of dollars) from Table 7.1 when there is a final exchange of principal.

Date	*Six-month LIBOR rate (%)*	*Floating cash flow received*	*Fixed cash flow paid*	*Net cash flow*
Mar. 5, 2007	4.20			
Sept. 5, 2007	4.80	+2.10	−2.50	−0.40
Mar. 5, 2008	5.30	+2.40	−2.50	−0.10
Sept. 5, 2008	5.50	+2.65	−2.50	+0.15
Mar. 5, 2009	5.60	+2.75	−2.50	+0.25
Sept. 5, 2009	5.90	+2.80	−2.50	+0.30
Mar. 5, 2010		+102.95	−102.50	+0.45

This characterization of the cash flows in the swap helps to explain why the floating rate in the swap is set 6 months before it is paid. On a floating-rate bond, interest is generally set at the beginning of the period to which it will apply and is paid at the end of the period. The calculation of the floating-rate payments in a "plain vanilla" interest rate swap such as the one in Table 7.2 reflects this.

Using the Swap to Transform a Liability

For Microsoft, the swap could be used to transform a floating-rate loan into a fixed-rate loan. Suppose that Microsoft has arranged to borrow $100 million at LIBOR plus 10 basis points. (One basis point is one-hundredth of 1%, so the rate is LIBOR plus 0.1%.) After Microsoft has entered into the swap, it has the following three sets of cash flows:

1. It pays LIBOR plus 0.1% to its outside lenders.
2. It receives LIBOR under the terms of the swap.
3. It pays 5% under the terms of the swap.

These three sets of cash flows net out to an interest rate payment of 5.1%. Thus, for Microsoft, the swap could have the effect of transforming borrowings at a floating rate of LIBOR plus 10 basis points into borrowings at a fixed rate of 5.1%.

For Intel, the swap could have the effect of transforming a fixed-rate loan into a floating-rate loan. Suppose that Intel has a 3-year $100 million loan outstanding on which it pays 5.2%. After it has entered into the swap, it has the following three sets of cash flows:

1. It pays 5.2% to its outside lenders.
2. It pays LIBOR under the terms of the swap.
3. It receives 5% under the terms of the swap.

These three sets of cash flows net out to an interest rate payment of LIBOR plus 0.2% (or LIBOR plus 20 basis points). Thus, for Intel, the swap could have the effect of transforming borrowings at a fixed rate of 5.2% into borrowings at a floating rate of LIBOR plus 20 basis points. These potential uses of the swap by Intel and Microsoft are illustrated in Figure 7.2.

Using the Swap to Transform an Asset

Swaps can also be used to transform the nature of an asset. Consider Microsoft in our example. The swap could have the effect of transforming an asset earning a fixed rate of interest into an asset earning a floating rate of interest. Suppose that Microsoft owns $100 million in bonds that will provide interest at 4.7% per annum over the next 3 years.

Figure 7.2 Microsoft and Intel use the swap to transform a liability.

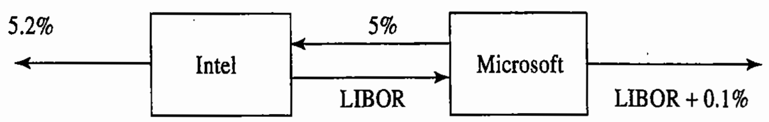
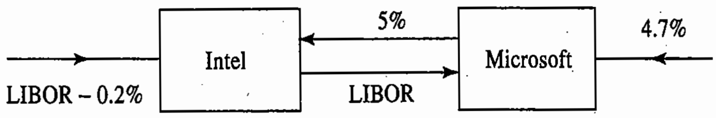


Figure 7.3 Microsoft and Intel use the swap to transform an asset.



After Microsoft has entered into the swap, it has the following three sets of cash flows:

1. It receives 4.7% on the bonds.
2. It receives LIBOR under the terms of the swap.
3. It pays 5% under the terms of the swap.

These three sets of cash flows net out to an interest rate inflow of LIBOR minus 30 basis points. Thus, one possible use of the swap for Microsoft is to transform an asset earning 4.7% into an asset earning LIBOR minus 30 basis points.

Next, consider Intel. The swap could have the effect of transforming an asset earning a floating rate of interest into an asset earning a fixed rate of interest. Suppose that Intel has an investment of $100 million that yields LIBOR minus 20 basis points. After it has entered into the swap, it has the following three sets of cash flows:

1. It receives LIBOR minus 20 basis points on its investment.
2. It pays LIBOR under the terms of the swap.
3. It receives 5% under the terms of the swap.

These three sets of cash flows net out to an interest rate inflow of 4.8%. Thus, one possible use of the swap for Intel is to transform an asset earning LIBOR minus 20 basis points into an asset earning 4.8%. These potential uses of the swap by Intel and Microsoft are illustrated in Figure 7.3.

Role of Financial Intermediary

Usually two nonfinancial companies such as Intel and Microsoft do not get in touch directly to arrange a swap in the way indicated in Figures 7.2 and 7.3. They each deal with a financial intermediary such as a bank or other financial institution. "Plain vanilla" fixed-for-floating swaps on US interest rates are usually structured so that the financial institution earns about 3 or 4 basis points (0.03% or 0.04%) on a pair of offsetting transactions.

Figure 7.4 shows what the role of the financial institution might be in the situation in Figure 7.2. The financial institution enters into two offsetting swap transactions with

Figure 7.4 Interest rate swap from Figure 7.2 when financial institution is involved.

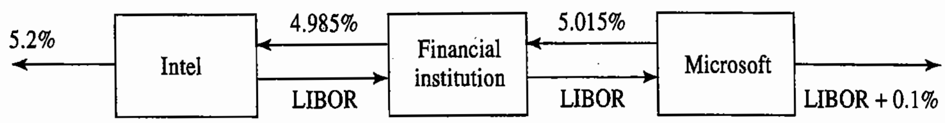
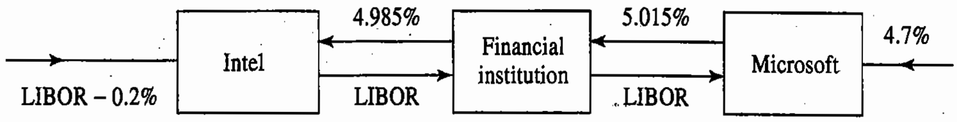


Figure 7.5 Interest rate swap from Figure 7.3 when financial institution is involved.



Intel and Microsoft. Assuming that both companies honor their obligations, the financial institution is certain to make a profit of 0.03% (3 basis points) per year multiplied by the notional principal of $100 million. This amounts to $30,000 per year for the 3-year period. Microsoft ends up borrowing at 5.115% (instead of 5.1%, as in Figure 7.2), and Intel ends up borrowing at LIBOR plus 21.5 basis points (instead of at LIBOR plus 20 basis points, as in Figure 7.2).

Figure 7.5 illustrates the role of the financial institution in the situation in Figure 7.3. The swap is the same as before and the financial institution is certain to make a profit of 3 basis points if neither company defaults. Microsoft ends up earning LIBOR minus 31.5 basis points (instead of LIBOR minus 30 basis points, as in Figure 7.3), and Intel ends up earning 4.785% (instead of 4.8%, as in Figure 7.3).

Note that in each case the financial institution has two separate contracts: one with Intel and the other with Microsoft. In most instances, Intel will not even know that the financial institution has entered into an offsetting swap with Microsoft, and vice versa. If one of the companies defaults, the financial institution still has to honor its agreement with the other company. The 3-basis-point spread earned by the financial institution is partly to compensate it for the risk that one of the two companies will default on the swap payments.

Market Makers

In practice, it is unlikely that two companies will contact a financial institution at the same time and want to take opposite positions in exactly the same swap. For this reason, many large financial institutions act as market makers for swaps. This means that they are prepared to enter into a swap without having an offsetting swap with another counterparty.[2] Market makers must carefully quantify and hedge the risks they are taking. Bonds, forward rate agreements, and interest rate futures are examples of the instruments that can be used for hedging by swap market makers. Table 7.3 shows quotes for plain vanilla US dollar swaps that might be posted by a market maker.[3] As mentioned earlier, the bid–offer spread is 3 to 4 basis points. The average of the bid and offer fixed rates is known as the *swap rate*. This is shown in the final column of Table 7.3.

Consider a new swap where the fixed rate equals the current swap rate. We can reasonably assume that the value of this swap is zero. (Why else would a market maker choose bid–offer quotes centered on the swap rate?) In Table 7.2 we saw that a swap can

[2] This is sometimes referred to as *warehousing* swaps.

[3] The standard swap in the United States is one where fixed payments made every 6 months are exchanged for floating LIBOR payments made every 3 months. In Table 7.1 we assumed that fixed and floating payments are exchanged every 6 months. The fixed rate should be almost exactly the same in both cases.

Table 7.3 Bid and offer fixed rates in the swap market and swap rates (percent per annum).

Maturity (years)	*Bid*	*Offer*	*Swap rate*
2	6.03	6.06	6.045
3	6.21	6.24	6.225
4	6.35	6.39	6.370
5	6.47	6.51	6.490
7	6.65	6.68	6.665
10	6.83	6.87	6.850

ɔe characterized as the difference between a fixed-rate bond and a floating-rate bond.)efine:

B_{fix}: Value of fixed-rate bond underlying the swap we are considering

B_{fl}: Value of floating-rate bond underlying the swap we are considering

Since the swap is worth zero, it follows that

$$B_{\text{fix}} = B_{\text{fl}} \tag{7.1}$$

We will use this result later in the chapter when discussing how the LIBOR/swap zero curve is determined.

7.2 DAY COUNT ISSUES

We discussed day count conventions in Section 6.1. The day count conventions affect payments on a swap, and some of the numbers calculated in the examples we have given do not exactly reflect these day count conventions. Consider, for example, the 6-month LIBOR payments in Table 7.1. Because it is a US money market rate, 6-month LIBOR is quoted on an actual/360 basis. The first floating payment in Table 7.1, based on the LIBOR rate of 4.2%, is shown as \$2.10 million. Because there are 184 days between March 5, 2007, and September 5, 2007, it should be

$$100 \times 0.042 \times \frac{184}{360} = \$2.1467 \text{ million}$$

In general, a LIBOR-based floating-rate cash flow on a swap payment date is calculated as $LRn/360$, where L is the principal, R is the relevant LIBOR rate, and n is the number of days since the last payment date.

The fixed rate that is paid in a swap transaction is similarly quoted with a particular day count basis being specified. As a result, the fixed payments may not be exactly equal on each payment date. The fixed rate is usually quoted as actual/365 or 30/360. It is not therefore directly comparable with LIBOR because it applies to a full year. To make the rates approximately comparable, either the 6-month LIBOR rate must be multiplied by 365/360 or the fixed rate must be multiplied by 360/365.

For clarity of exposition, we will ignore day count issues in the calculations in the rest of this chapter.

Business Snapshot 7.1 Extract from Hypothetical Swap Confirmation

Trade date:	27-February-2007
Effective date:	5-March-2007
Business day convention (all dates):	Following business day
Holiday calendar:	US
Termination date:	5-March-2010
Fixed amounts	
Fixed-rate payer:	Microsoft
Fixed-rate notional principal:	USD 100 million
Fixed rate:	5.015% per annum
Fixed-rate day count convention:	Actual/365
Fixed-rate payment dates:	Each 5-March and 5-September, commencing 5-September-2007, up to and including 5-March-2010
Floating amounts	
Floating-rate payer:	Goldman Sachs
Floating-rate notional principal:	USD 100 million
Floating rate:	USD 6-month LIBOR
Floating-rate day count convention:	Actual/360
Floating-rate payment dates:	Each 5-March and 5-September, commencing 5-September-2007, up to and including 5-March-2010

7.3 CONFIRMATIONS

A *confirmation* is the legal agreement underlying a swap and is signed by representatives of the two parties. The drafting of confirmations has been facilitated by the work of the International Swaps and Derivatives Association (ISDA; www.isda.org) in New York. This organization has produced a number of Master Agreements that consist of clauses defining in some detail the terminology used in swap agreements, what happens in the event of default by either side, and so on. In Business Snapshot 7.1, we show a possible extract from the confirmation for the swap shown in Figure 7.4 between Microsoft and a financial institution (assumed here to be Goldman Sachs). Almost certainly, the full confirmation would state that the provisions of an ISDA Master Agreement apply to the contract.

The confirmation specifies that the following business day convention is to be used and that the US calendar determines which days are business days and which days are holidays. This means that, if a payment date falls on a weekend or a US holiday, the payment is made on the next business day.[4] September 5, 2009, is a Saturday. The

[4] Another business day convention that is sometimes specified is the *modified following* business day convention, which is the same as the following business day convention except that, when the next business day falls in a different month from the specified day, the payment is made on the immediately preceding business day. *Preceding* and *modified preceding* business day conventions are defined analogously.

penultimate exchange of payments in the swap between Microsoft and Goldman Sachs is therefore on Tuesday, September 8, 2009 (Monday, September 7, is Labor day, a holiday).

7.4 THE COMPARATIVE-ADVANTAGE ARGUMENT

An explanation commonly put forward to explain the popularity of swaps concerns comparative advantages. Consider the use of an interest rate swap to transform a liability. Some companies, it is argued, have a comparative advantage when borrowing in fixed-rate markets, whereas other companies have a comparative advantage in floating-rate markets. To obtain a new loan, it makes sense for a company to go to the market where it has a comparative advantage. As a result, the company may borrow fixed when it wants floating, or borrow floating when it wants fixed. The swap is used to transform a fixed-rate loan into a floating-rate loan, and vice versa.

Suppose that two companies, AAACorp and BBBCorp, both wish to borrow $10 million for 5 years and have been offered the rates shown in Table 7.4. AAACorp has a AAA credit rating; BBBCorp has a BBB credit rating.[5] We assume that BBBCorp wants to borrow at a fixed rate of interest, whereas AAACorp wants to borrow at a floating rate of interest linked to 6-month LIBOR. Because it has a worse credit rating than AAACorp, BBBCorp pays a higher rate of interest than AAACorp in both fixed and floating markets.

A key feature of the rates offered to AAACorp and BBBCorp is that the difference between the two fixed rates is greater than the difference between the two floating rates. BBBCorp pays 1.2% more than AAACorp in fixed-rate markets and only 0.7% more than AAACorp in floating-rate markets. BBBCorp appears to have a comparative advantage in floating-rate markets, whereas AAACorp appears to have a comparative advantage in fixed-rate markets.[6] It is this apparent anomaly that can lead to a swap being negotiated. AAACorp borrows fixed-rate funds at 4% per annum. BBBCorp borrows floating-rate funds at LIBOR plus 0.6% per annum. They then enter into a swap agreement to ensure that AAACorp ends up with floating-rate funds and BBBCorp ends up with fixed-rate funds.

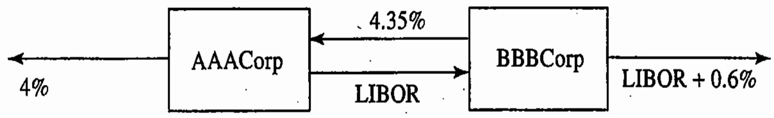
Table 7.4 Borrowing rates that provide a basis for the comparative-advantage argument.

	Fixed	*Floating*
AAACorp	4.0%	6-month LIBOR − 0.1%
BBBCorp	5.2%	6-month LIBOR + 0.6%

[5] The credit ratings assigned to companies by S&P (in order of decreasing creditworthiness) are AAA, AA, A, BBB, BB, B, CCC, CC, and C. The corresponding ratings assigned by Moody's are Aaa, Aa, A, Baa, Ba, B, Caa, Ca, and C, respectively.

[6] Note that BBBCorp's comparative advantage in floating-rate markets does not imply that BBBCorp pays less than AAACorp in this market. It means that the extra amount that BBBCorp pays over the amount paid by AAACorp is less in this market. One of my students summarized the situation as follows: "AAACorp pays more less in fixed-rate markets; BBBCorp pays less more in floating-rate markets."

Figure 7.6 Swap agreement between AAACorp and BBBCorp when rates in Table 7.4 apply.



To understand how this swap might work, we first assume that AAACorp and BBBCorp get in touch with each other directly. The sort of swap they might negotiate is shown in Figure 7.6. This is similar to our example in Figure 7.2. AAACorp agrees to pay BBBCorp interest at 6-month LIBOR on \$10 million. In return, BBBCorp agrees to pay AAACorp interest at a fixed rate of 4.35% per annum on \$10 million.

AAACorp has three sets of interest rate cash flows:

1. It pays 4% per annum to outside lenders.
2. It receives 4.35% per annum from BBBCorp.
3. It pays LIBOR to BBBCorp.

The net effect of the three cash flows is that AAACorp pays LIBOR minus 0.35% per annum. This is 0.25% per annum less than it would pay if it went directly to floating-rate markets. BBBCorp also has three sets of interest rate cash flows:

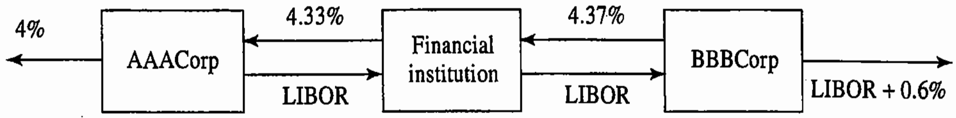
1. It pays LIBOR + 0.6% per annum to outside lenders.
2. It receives LIBOR from AAACorp.
3. It pays 4.35% per annum to AAACorp.

The net effect of the three cash flows is that BBBCorp pays 4.95% per annum. This is 0.25% per annum less than it would pay if it went directly to fixed-rate markets.

In this example, the swap has been structured so that the net gain to both sides is the same, 0.25%. This need not be the case. However, the total apparent gain from this type of interest rate swap arrangement is always $a - b$, where a is the difference between the interest rates facing the two companies in fixed-rate markets, and b is the difference between the interest rates facing the two companies in floating-rate markets. In this case, $a = 1.2\%$ and $b = 0.7\%$, so that the total gain is 0.5%.

If AAACorp and BBBCorp did not deal directly with each other and used a financial institution, an arrangement such as that shown in Figure 7.7 might result. (This is similar to the example in Figure 7.4.) In this case, AAACorp ends up borrowing at

Figure 7.7 Swap agreement between AAACorp and BBBCorp when rates in Table 7.4 apply and a financial intermediary is involved.



LIBOR minus 0.33%, BBBCorp ends up borrowing at 4.97%, and the financial institution earns a spread of 4 basis points per year. The gain to AAACorp is 0.23%; the gain to BBBCorp is 0.23%; and the gain to the financial institution is 0.04%. The total gain to all three parties is 0.50% as before.

Criticism of the Argument

The comparative-advantage argument we have just outlined for explaining the attractiveness of interest rate swaps is open to question. Why in Table 7.4 should the spreads between the rates offered to AAACorp and BBBCorp be different in fixed and floating markets? Now that the swap market has been in existence for some time, we might reasonably expect these types of differences to have been arbitraged away.

The reason that spread differentials appear to exist is due to the nature of the contracts available to companies in fixed and floating markets. The 4.0% and 5.2% rates available to AAACorp and BBBCorp in fixed-rate markets are 5-year rates (e.g., the rates at which the companies can issue 5-year fixed-rate bonds). The LIBOR − 0.1% and LIBOR + 0.6% rates available to AAACorp and BBBCorp in floating-rate markets are 6-month rates. In the floating-rate market, the lender usually has the opportunity to review the floating rates every 6 months. If the creditworthiness of AAACorp or BBBCorp has declined, the lender has the option of increasing the spread over LIBOR that is charged. In extreme circumstances, the lender can refuse to roll over the loan at all. The providers of fixed-rate financing do not have the option to change the terms of the loan in this way.[7]

The spreads between the rates offered to AAACorp and BBBCorp are a reflection of the extent to which BBBCorp is more likely than AAACorp to default. During the next 6 months, there is very little chance that either AAACorp or BBBCorp will default. As we look further ahead, the probability of a default by a company with a relatively low credit rating (such as BBBCorp) is liable to increase faster than the probability of a default by a company with a relatively high credit rating (such as AAACorp). This is why the spread between the 5-year rates is greater than the spread between the 6-month rates.

After negotiating a floating-rate loan at LIBOR + 0.6% and entering into the swap shown in Figure 7.7, BBBCorp appears to obtain a fixed-rate loan at 4.97%. The arguments just presented show that this is not really the case. In practice, the rate paid is 4.97% only if BBBCorp can continue to borrow floating-rate funds at a spread of 0.6% over LIBOR. If, for example, the credit rating of BBBCorp declines so that the floating-rate loan is rolled over at LIBOR + 1.6%, the rate paid by BBBCorp increases to 5.97%. The market expects that BBBCorp's spread over 6-month LIBOR will on average rise during the swap's life. BBBCorp's expected average borrowing rate when it enters into the swap is therefore greater than 4.97%.

The swap in Figure 7.7 locks in LIBOR − 0.33% for AAACorp for the whole of the next 5 years, not just for the next 6 months. This appears to be a good deal for AAACorp. The downside is that it is bearing the risk of a default by the financial institution. If it borrowed floating-rate funds in the usual way, it would not be bearing this risk.

[7] If the floating-rate loans are structured so that the spread over LIBOR is guaranteed in advance regardless of changes in credit rating, the spread differentials disappear.

7.5 THE NATURE OF SWAP RATES

At this stage it is appropriate to examine the nature of swap rates and the relationship between swap and LIBOR markets. We explained in Section 4.1 that LIBOR is the rate of interest at which AA-rated banks borrow for periods between 1 and 12 months from other banks. Also, as indicated in Table 7.3, a swap rate is the average of (a) the fixed rate that a swap market maker is prepared to pay in exchange for receiving LIBOR (its bid rate) and (b) the fixed rate that it is prepared to receive in return for paying LIBOR (its offer rate).

Like LIBOR rates, swap rates are not risk-free lending rates. However, they are close to risk-free. A financial institution can earn the 5-year swap rate on a certain principal by doing the following:

1. Lend the principal for the first 6 months to a AA borrower and then relend it for successive 6 month periods to other AA borrowers; and
2. Enter into a swap to exchange the LIBOR income for the 5-year swap rate.

This shows that the 5-year swap rate is an interest rate with a credit risk corresponding to the situation where 10 consecutive 6-month LIBOR loans to AA companies are made. Similarly the 7-year swap rate is an interest rate with a credit risk corresponding to the situation where 14 consecutive 6-month LIBOR loans to AA companies are made. Swap rates of other maturities can be interpreted analogously.

Note that 5-year swap rates are less than 5-year AA borrowing rates. It is much more attractive to lend money for successive 6-month periods to borrowers who are always AA at the beginning of the periods than to lend it to one borrower for the whole 5 years when all we can be sure of is that the borrower is AA at the beginning of the 5 years.

7.6 DETERMINING LIBOR/SWAP ZERO RATES

We explained in Section 4.1 that derivative traders tend to use LIBOR rates as a proxies for risk-free rates when valuing derivatives. One problem with LIBOR rates is that direct observations are possible only for maturities out to 12 months. As described in Section 6.3, one way of extending the LIBOR zero curve beyond 12 months is to use Eurodollar futures. Typically Eurodollar futures are used to produce a LIBOR zero curve out to 2 years—and sometimes out to as far as 5 years. Traders then use swap rates to extend the LIBOR zero curve further. The resulting zero curve is sometimes referred to as the LIBOR zero curve and sometimes as the swap zero curve. To avoid any confusion, we will refer to it as the LIBOR/swap zero curve. We will now describe how swap rates are used in the determination of the LIBOR/swap zero curve.

The first point to note is that the value of a newly issued floating-rate bond that pays 6-month LIBOR is always equal to its principal value (or par value) when the LIBOR/swap zero curve is used for discounting.[8] The reason is that the bond provides a rate of interest of LIBOR, and LIBOR is the discount rate. The interest on the bond exactly matches the discount rate, and as a result the bond is fairly priced at par.

In equation (7.1), we showed that for a newly issued swap where the fixed rate equals the swap rate, $B_{\text{fix}} = B_{\text{fl}}$. We have just argued that B_{fl} equals the notional principal. It

[8] The same is of course true of a newly issued bond that pays 1-month, 3-month, or 12-month LIBOR.

follows that B_{fix} also equals the swap's notional principal. Swap rates therefore define a set of par yield bonds. For example, from the swap rates in Table 7.3, we can deduce that the 2-year LIBOR/swap par yield is 6.045%, the 3-year LIBOR/swap par yield is 6.225%, and so on.[9]

Section 4.5 showed how the bootstrap method can be used to determine the Treasury zero curve from Treasury bond prices. It can be used with swap rates in a similar way to extend the LIBOR/swap zero curve.

Example 7.1

Suppose that the 6-month, 12-month, and 18-month LIBOR/swap zero rates have been determined as 4%, 4.5%, and 4.8% with continuous compounding and that the 2-year swap rate (for a swap where payments are made semiannually) is 5%. This 5% swap rate means that a bond with a principal of \$100 and a semiannual coupon of 5% per annum sells for par. It follows that, if R is the 2-year zero rate, then

$$2.5e^{-0.04\times 0.5} + 2.5e^{-0.045\times 1.0} + 2.5e^{-0.048\times 1.5} + 102.5e^{-2R} = 100$$

Solving this, we obtain $R = 4.953\%$. (Note that this calculation is simplified in that it does not take the swap's day count conventions and holiday calendars into account. See Section 7.2.)

7.7 VALUATION OF INTEREST RATE SWAPS

We now move on to discuss the valuation of interest rate swaps. An interest rate swap is worth zero, or close to zero, when it is first initiated. After it has been in existence for some time, its value may become positive or negative. There are two valuation approaches. The first regards the swap as the difference between two bonds; the second regards it as a portfolio of FRAs.

Valuation in Terms of Bond Prices

Principal payments are not exchanged in an interest rate swap. However, as illustrated in Table 7.2, we can assume that principal payments are both received and paid at the end of the swap without changing its value. By doing this, we find that, from the point of view of the floating-rate payer, a swap can be regarded as a long position in a fixed-rate bond and a short position in a floating-rate bond, so that

$$V_{\text{swap}} = B_{\text{fix}} - B_{\text{fl}}$$

where V_{swap} is the value of the swap, B_{fl} is the value of the floating-rate bond (corresponding to payments that are made), and B_{fix} is the value of the fixed-rate bond (corresponding to payments that are received). Similarly, from the point of view of the fixed-rate payer, a swap is a long position in a floating-rate bond and a short

[9] Analysts frequently interpolate between swap rates before calculating the zero curve, so that they have swap rates for maturities at 6-month intervals. For example, for the data in Table 7.3 the 2.5-year swap rate would be assumed to be 6.135%; the 7.5-year swap rate would be assumed to be 6.696%; and so on.

position in a fixed-rate bond, so that the value of the swap is

$$V_{\text{swap}} = B_{\text{fl}} - B_{\text{fix}}$$

The value of the fixed rate bond, B_{fix}, can be determined as described in Section 4.4. To value the floating-rate bond, we note that the bond is worth the notional principal immediately after an interest payment. This is because at this time the bond is a "fair deal" where the borrower pays LIBOR for each subsequent accrual period.

Suppose that the notional principal is L, the next exchange of payments is at time t^*, and the floating payment that will be made at time t^* (which was determined at the last payment date) is k^*. Immediately after the payment $B_{\text{fl}} = L$ as just explained. It follows that immediately before the payment $B_{\text{fl}} = L + k^*$. The floating-rate bond can therefore be regarded as an instrument providing a single cash flow of $L + k^*$ at time t^*. Discounting this, the value of the floating-rate bond today is

$$(L + k^*)e^{-r^*t^*}$$

where r^* is the LIBOR/swap zero rate for a maturity of t^*.

Example 7.2

Suppose that a financial institution has agreed to pay 6-month LIBOR and receive 8% per annum (with semiannual compounding) on a notional principal of \$100 million. The swap has a remaining life of 1.25 years. The LIBOR rates with continuous compounding for 3-month, 9-month, and 15-month maturities are 10%, 10.5%, and 11%, respectively. The 6-month LIBOR rate at the last payment date was 10.2% (with semiannual compounding).

The calculations for valuing the swap in terms of bonds are summarized in Table 7.5. The fixed-rate bond has cash flows of 4, 4, and 104 on the three payment dates. The discount factors for these cash flows are, respectively,

$$e^{-0.1\times 0.25}, \quad e^{-0.105\times 0.75}, \quad e^{-0.11\times 1.25}$$

and are shown in the fourth column of Table 7.5. The table shows that the value of the fixed-rate bond (in millions of dollars) is 98.238.

In this example, $L = \$100$ million, $k^* = 0.5 \times 0.102 \times 100 = \$5.1$ million, and $t^* = 0.25$, so that the floating-rate bond can be valued as though it produces a cash flow of \$105.1 million in 3 months. The table shows that the value of the floating bond (in millions of dollars) is $105.100 \times 0.9753 = 102.505$.

Table 7.5 Valuing a swap in terms of bonds (\$ millions). Here, B_{fix} is fixed-rate bond underlying the swap, and B_{fl} is floating-rate bond underlying the swap.

Time	B_{fix} *cash flow*	B_{fl} *cash flow*	*Discount factor*	*Present value* B_{fix} *cash flow*	*Present value* B_{fl} *cash flow*
0.25	4.0	105.100	0.9753	3.901	102.505
0.75	4.0		0.9243	3.697	
1.25	104.0		0.8715	90.640	
Total:				98.238	102.505

The value of the swap is the difference between the two bond prices:

$$V_{\text{swap}} = 98.238 - 102.505 = -4.267$$

or -4.267 million dollars.

If the financial institution had been in the opposite position of paying fixed and receiving floating, the value of the swap would be +\$4.267 million. Note that these calculations do not take account of day count conventions and holiday calendars.

Valuation in Terms of FRAs

A swap can be characterized as a portfolio of forward rate agreements. Consider the swap between Microsoft and Intel in Figure 7.1. The swap is a 3-year deal entered into on March 5, 2007, with semiannual payments. The first exchange of payments is known at the time the swap is negotiated. The other five exchanges can be regarded as FRAs. The exchange on March 5, 2008, is an FRA where interest at 5% is exchanged for interest at the 6-month rate observed in the market on September 5, 2007; the exchange on September 5, 2008, is an FRA where interest at 5% is exchanged for interest at the 6-month rate observed in the market on March 5, 2008; and so on.

As shown at the end of Section 4.7, an FRA can be valued by assuming that forward interest rates are realized. Because it is nothing more than a portfolio of forward rate agreements, a plain vanilla interest rate swap can also be valued by making the assumption that forward interest rates are realized. The procedure is as follows:

1. Use the LIBOR/swap zero curve to calculate forward rates for each of the LIBOR rates that will determine swap cash flows.
2. Calculate swap cash flows on the assumption that the LIBOR rates will equal the forward rates.
3. Discount these swap cash flows (using the LIBOR/swap zero curve) to obtain the swap value.

Example 7.3

Consider again the situation in Example 7.2. Under the terms of the swap, a financial institution has agreed to pay 6-month LIBOR and receive 8% per annum (with semiannual compounding) on a notional principal of \$100 million. The swap has a remaining life of 1.25 years. The LIBOR rates with continuous compounding for 3-month, 9-month, and 15-month maturities are 10%, 10.5%, and 11%, respectively. The 6-month LIBOR rate at the last payment date was 10.2% (with semiannual compounding).

The calculations are summarized in Table 7.6. The first row of the table shows the cash flows that will be exchanged in 3 months. These have already been determined. The fixed rate of 8% will lead to a cash inflow of $100 \times 0.08 \times 0.5 = \$4$ million. The floating rate of 10.2% (which was set 3 months ago) will lead to a cash outflow of $100 \times 0.102 \times 0.5 = \$5.1$ million. The second row of the table shows the cash flows that will be exchanged in 9 months assuming that forward rates are realized. The cash inflow is \$4.0 million as before. To calculate the cash outflow, we must first calculate the forward rate corresponding to the period between 3 and 9 months.

Table 7.6 Valuing swap in terms of FRAs ($ millions). Floating cash flows are calculated by assuming that forward rates will be realized.

Time	*Fixed cash flow*	*Floating cash flow*	*Net cash flow*	*Discount factor*	*Present value of net cash flow*
0.25	4.0	−5.100	−1.100	0.9753	−1.073
0.75	4.0	−5.522	−1.522	0.9243	−1.407
1.25	4.0	−6.051	−2.051	0.8715	−1.787
Total:					−4.267

From equation (4.5), this is

$$\frac{0.105 \times 0.75 - 0.10 \times 0.25}{0.5} = 0.1075,$$

or 10.75% with continuous compounding. From equation (4.4), the forward rate becomes 11.044% with semiannual compounding. The cash outflow is therefore $100 \times 0.11044 \times 0.5 = \$5.522$ million. The third row similarly shows the cash flows that will be exchanged in 15 months assuming that forward rates are realized. The discount factors for the three payment dates are, respectively,

$$e^{-0.1\times0.25}, \quad e^{-0.105\times0.75}, \quad e^{-0.11\times1.25}$$

The present value of the exchange in three months is −$1.073 million. The values of the FRAs corresponding to the exchanges in 9 months and 15 months are −$1.407 and −$1.787 million, respectively. The total value of the swap is −$4.267 million. This is in agreement with the value we calculated in Example 7.2 by decomposing the swap into bonds.

The fixed rate in an interest rate swap is chosen so that the swap is worth zero initially. This means that at the outset of a swap the sum of the values of the FRAs underlying the swap is zero. It does not mean that the value of each individual FRA is zero. In general, some FRAs will have positive values whereas others have negative values.

Consider the FRAs underlying the swap between Microsoft and Intel in Figure 7.1:

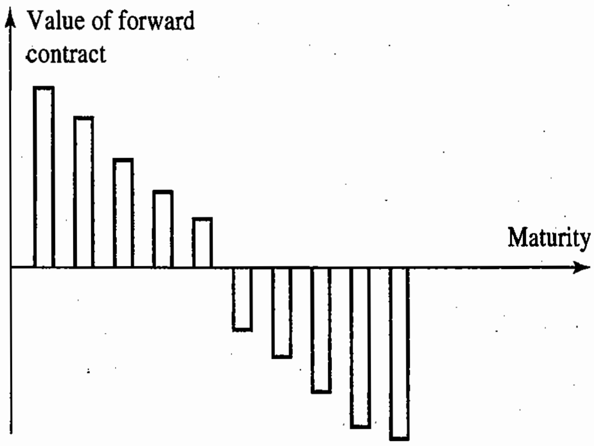
Value of FRA to Microsoft > 0 when forward interest rate > 5.0%

Value of FRA to Microsoft = 0 when forward interest rate = 5.0%

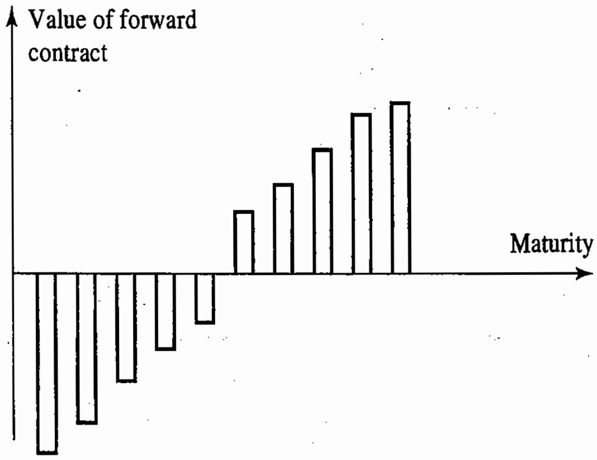
Value of FRA to Microsoft < 0 when forward interest rate < 5.0%

Suppose that the term structure of interest rates is upward-sloping at the time the swap is negotiated. This means that the forward interest rates increase as the maturity of the FRA increases. Since the sum of the values of the FRAs is zero, the forward interest rate must be less than 5.0% for the early payment dates and greater than 5.0% for the later payment dates. The value to Microsoft of the FRAs corresponding to early payment dates is therefore negative, whereas the value of the FRAs corresponding to later payment dates is positive. If the term structure of interest rates is downward-sloping at the time the swap is negotiated, the reverse is true. The impact of the shape of the term structure of interest rates on the values of the forward contracts underlying a swap is summarized in Figure 7.8.

Figure 7.8 Valuing of forward rate agreements underlying a swap as a function of maturity. In (a) the term structure of interest rates is upward-sloping and we receive fixed, or it is downward-sloping and we receive floating; in (b) the term structure of interest rates is upward-sloping and we receive floating, or it is downward-sloping and we receive fixed.



(a)



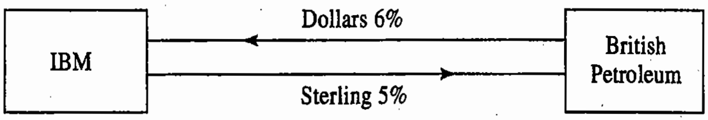
(b)

7.8 CURRENCY SWAPS

Another popular type of swap is known as a *currency swap.* In its simplest form, this involves exchanging principal and interest payments in one currency for principal and interest payments in another.

A currency swap agreement requires the principal to be specified in each of the two currencies. The principal amounts in each currency are usually exchanged at the beginning and at the end of the life of the swap. Usually the principal amounts are chosen to be approximately equivalent using the exchange rate at the swap's initiation. When they are exchanged at the end of the life of the swap, their values may be quite different.

Figure 7.9 A currency swap.



Illustration

Consider a hypothetical 5-year currency swap agreement between IBM and British Petroleum entered into on February 1, 2007. We suppose that IBM pays a fixed rate of interest of 5% in sterling and receives a fixed rate of interest of 6% in dollars from British Petroleum. Interest rate payments are made once a year and the principal amounts are \$18 million and £10 million. This is termed a *fixed-for-fixed* currency swap because the interest rate in both currencies is fixed. The swap is shown in Figure 7.9. Initially, the principal amounts flow in the opposite direction to the arrows in Figure 7.9. The interest payments during the life of the swap and the final principal payment flow in the same direction as the arrows. Thus, at the outset of the swap, IBM pays \$18 million and receives £10 million. Each year during the life of the swap contract, IBM receives \$1.08 million (= 6% of \$18 million) and pays £0.50 million (= 5% of £10 million). At the end of the life of the swap, it pays a principal of £10 million and receives a principal of \$18 million. These cash flows are shown in Table 7.7.

Use of a Currency Swap to Transform Liabilities and Assets

A swap such as the one just considered can be used to transform borrowings in one currency to borrowings in another. Suppose that IBM can issue \$18 million of US-dollar-denominated bonds at 6% interest. The swap has the effect of transforming this transaction into one where IBM has borrowed £10 million at 5% interest. The initial exchange of principal converts the proceeds of the bond issue from US dollars to sterling. The subsequent exchanges in the swap have the effect of swapping the interest and principal payments from dollars to sterling.

The swap can also be used to transform the nature of assets. Suppose that IBM can invest £10 million in the UK to yield 5% per annum for the next 5 years, but feels that

Table 7.7 Cash flows to IBM in currency swap.

Date	*Dollar cash flow (millions)*	*Sterling cash flow (millions)*
February 1, 2007	−18.00	+10.00
February 1, 2008	+1.08	−0.50
February 1, 2009	+1.08	−0.50
February 1, 2010	+1.08	−0.50
February 1, 2011	+1.08	−0.50
February 1, 2012	+19.08	−10.50

the US dollar will strengthen against sterling and prefers a US-dollar-denominated investment. The swap has the effect of transforming the UK investment into a \$18 million investment in the US yielding 6%.

Comparative Advantage

Currency swaps can be motivated by comparative advantage. To illustrate this, we consider another hypothetical example. Suppose the 5-year fixed-rate borrowing costs to General Electric and Qantas Airways in US dollars (USD) and Australian dollars (AUD) are as shown in Table 7.8. The data in the table suggest that Australian rates are higher than USD interest rates, and also that General Electric is more creditworthy than Qantas Airways, because it is offered a more favorable rate of interest in both currencies. From the viewpoint of a swap trader, the interesting aspect of Table 7.8 is that the spreads between the rates paid by General Electric and Qantas Airways in the two markets are not the same. Qantas Airways pays 2% more than General Electric in the US dollar market and only 0.4% more than General Electric in the AUD market.

This situation is analogous to that in Table 7.4. General Electric has a comparative advantage in the USD market, whereas Qantas Airways has a comparative advantage in the AUD market. In Table 7.4, where a plain vanilla interest rate swap was considered, we argued that comparative advantages are largely illusory. Here we are comparing the rates offered in two different currencies, and it is more likely that the comparative advantages are genuine. One possible source of comparative advantage is tax. General Electric's position might be such that USD borrowings lead to lower taxes on its worldwide income than AUD borrowings. Qantas Airways' position might be the reverse. (Note that we assume that the interest rates shown in Table 7.8 have been adjusted to reflect these types of tax advantages.)

We suppose that General Electric wants to borrow 20 million AUD and Qantas Airways wants to borrow 15 million USD and that the current exchange rate (USD per AUD) is 0.7500. This creates a perfect situation for a currency swap. General Electric and Qantas Airways each borrow in the market where they have a comparative advantage; that is, General Electric borrows USD whereas Qantas Airways borrows AUD. They then use a currency swap to transform General Electric's loan into an AUD loan and Qantas Airways' loan into a USD loan.

As already mentioned, the difference between the USD interest rates is 2%, whereas the difference between the AUD interest rates is 0.4%. By analogy with the interest rate swap case, we expect the total gain to all parties to be $2.0 - 0.4 = 1.6\%$ per annum.

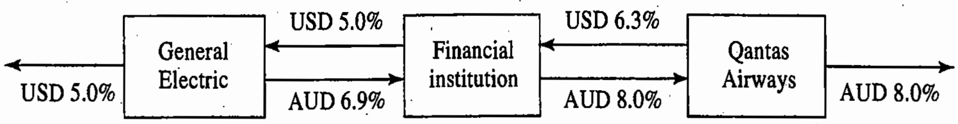
There are many ways in which the swap can be arranged. Figure 7.10 shows one way swaps might be entered into with a financial institution. General Electric borrows USD and Qantas Airways borrows AUD. The effect of the swap is to transform the USD

Table 7.8 Borrowing rates providing basis for currency swap.

	*USD**	*AUD**
General Electric	5.0%	7.6%
Qantas Airways	7.0%	8.0%

* Quoted rates have been adjusted to reflect the differential impact of taxes.

Figure 7.10 A currency swap motivated by comparative advantage.



interest rate of 5% per annum to an AUD interest rate of 6.9% per annum for General Electric. As a result, General Electric is 0.7% per annum better off than it would be if it went directly to AUD markets. Similarly, Qantas exchanges an AUD loan at 8% per annum for a USD loan at 6.3% per annum and ends up 0.7% per annum better off than it would be if it went directly to USD markets. The financial institution gains 1.3% per annum on its USD cash flows and loses 1.1% per annum on its AUD flows. If we ignore the difference between the two currencies, the financial institution makes a net gain of 0.2% per annum. As predicted, the total gain to all parties is 1.6% per annum.

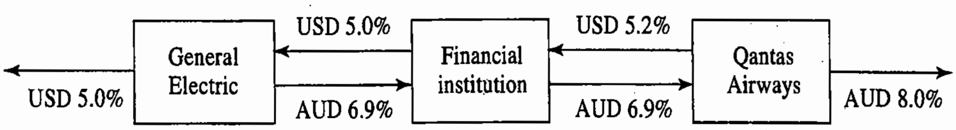
Each year the financial institution makes a gain of USD 195,000 (= 1.3% of 15 million) and incurs a loss of AUD 220,000 (= 1.1% of 20 million). The financial institution can avoid any foreign exchange risk by buying AUD 220,000 per annum in the forward market for each year of the life of the swap, thus locking in a net gain in USD.

It is possible to redesign the swap so that the financial institution makes a 0.2% spread in USD. Figures 7.11 and 7.12 present two alternatives. These alternatives are unlikely to be used in practice because they do not lead to General Electric and Qantas being free of foreign exchange risk.[10] In Figure 7.11, Qantas bears some foreign exchange risk because it pays 1.1% per annum in AUD and pays 5.2% per annum in USD. In Figure 7.12, General Electric bears some foreign exchange risk because it receives 1.1% per annum in USD and pays 8% per annum in AUD.

7.9 VALUATION OF CURRENCY SWAPS

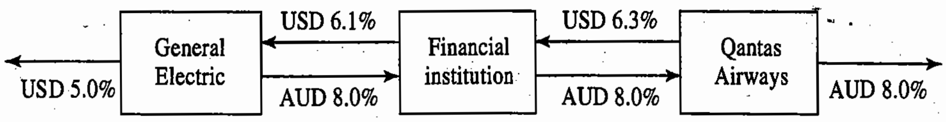
Like interest rate swaps, fixed-for-fixed currency swaps can be decomposed into either the difference between two bonds or a portfolio of forward contracts.

Figure 7.11 Alternative arrangement for currency swap: Qantas Airways bears some foreign exchange risk.



[10] Usually it makes sense for the financial institution to bear the foreign exchange risk, because it is in the best position to hedge the risk.

Figure 7.12 Alternative arrangement for currency swap: General Electric bears some foreign exchange risk.



Valuation in Terms of Bond Prices

If we define V_{swap} as the value in US dollars of an outstanding swap where dollars are received and a foreign currency is paid, then

$$V_{\text{swap}} = B_D - S_0 B_F$$

where B_F is the value, measured in the foreign currency, of the bond defined by the foreign cash flows on the swap and B_D is the value of the bond defined by the domestic cash flows on the swap, and S_0 is the spot exchange rate (expressed as number of dollars per unit of foreign currency). The value of a swap can therefore be determined from LIBOR rates in the two currencies, the term structure of interest rates in the domestic currency, and the spot exchange rate.

Similarly, the value of a swap where the foreign currency is received and dollars are paid is

$$V_{\text{swap}} = S_0 B_F - B_D$$

Example 7.4

Suppose that the term structure of LIBOR/swap interest rates is flat in both Japan and the United States. The Japanese rate is 4% per annum and the US rate is 9% per annum (both with continuous compounding). Some time ago a financial institution has entered into a currency swap in which it receives 5% per annum in yen and pays 8% per annum in dollars once a year. The principals in the two currencies are \$10 million and 1,200 million yen. The swap will last for another 3 years, and the current exchange rate is 110 yen = \$1.

The calculations are summarized in Table 7.9. In this case the cash flows from the dollar bond underlying the swap are as shown in the second column. The

Table 7.9 Valuation of currency swap in terms of bonds. (All amounts in millions.)

Time	*Cash flows on dollar bond (\$)*	*Present value (\$)*	*Cash flows on yen bond (yen)*	*Present value (yen)*
1	0.8	0.7311	60	57.65
2	0.8	0.6682	60	55.39
3	0.8	0.6107	60	53.22
3	10.0	7.6338	1,200	1,064.30
Total:		9.6439		1,230.55

present value of the cash flows using the dollar discount rate of 9% are shown in the third column. The cash flows from the yen bond underlying the swap are shown in the fourth column of the table. The present value of the cash flows using the yen discount rate of 4% are shown in the final column of the table.

The value of the dollar bond, B_D, is 9.6439 million dollars. The value of the yen bond is 1230.55 million yen. The value of the swap in dollars is therefore

$$\frac{1{,}230.55}{110} - 9.6439 = 1.5430 \text{ million}$$

Valuation as Portfolio of Forward Contracts

Each exchange of payments in a fixed-for-fixed currency swap is a forward foreign exchange contract. As shown in Section 5.7, forward foreign exchange contracts can be valued by assuming that forward exchange rates are realized. The forward exchange rates themselves can be calculated from equation (5.9).

Example 7.5

Consider again the situation in Example 7.4. The LIBOR/swap term structure of interest rates is flat in both Japan and the United States. The Japanese rate is 4% per annum and the US rate is 9% per annum (both with continuous compounding). Some time ago a financial institution has entered into a currency swap in which it receives 5% per annum in yen and pays 8% per annum in dollars once a year. The principals in the two currencies are \$10 million and 1,200 million yen. The swap will last for another 3 years, and the current exchange rate is 110 yen = \$1.

The calculations are summarized in Table 7.10. The financial institution pays $0.08 \times 10 = \$0.8$ million dollars and receives $1{,}200 \times 0.05 = 60$ million yen each year. In addition, the dollar principal of \$10 million is paid and the yen principal of 1,200 is received at the end of year 3. The current spot rate is 0.009091 dollar per yen. In this case $r = 9\%$ and $r_f = 4\%$, so that, from equation (5.9), the 1-year forward rate is

$$0.009091\, e^{(0.09-0.04)\times 1} = 0.009557$$

The 2- and 3-year forward rates in Table 7.10 are calculated similarly. The forward contracts underlying the swap can be valued by assuming that the forward rates are realized. If the 1-year forward rate is realized, the yen cash flow in year 1

Table 7.10 Valuation of currency swap as a portfolio of forward contracts. (All amounts in millions.)

Time	*Dollar cash flow*	*Yen cash flow*	*Forward exchange rate*	*Dollar value of yen cash flow*	*Net cash flow (\$)*	*Present value*
1	−0.8	60	0.009557	0.5734	−0.2266	−0.2071
2	−0.8	60	0.010047	0.6028	−0.1972	−0.1647
3	−0.8	60	0.010562	0.6337	−0.1663	−0.1269
3	−10.0	1200	0.010562	12.6746	+2.6746	2.0417
Total:						1.5430

is worth $60 \times 0.009557 = 0.5734$ million dollars and the net cash flow at the end of year 1 is $0.8 - 0.5734 = -0.2266$ million dollars. This has a present value of

$$-0.2266e^{-0.09\times 1} = -0.2071$$

million dollars. This is the value of forward contract corresponding to the exchange of cash flows at the end of year 1. The value of the other forward contracts are calculated similarly. As shown in Table 7.10, the total value of the forward contracts is \$1.5430 million. This agrees with the value calculated for the swap in Example 7.4 by decomposing it into bonds.

The value of a currency swap is normally zero when it is first negotiated. If the two principals are worth exactly the same using the exchange rate at the start of the swap, the value of the swap is also zero immediately after the initial exchange of principal. However, as in the case of interest rate swaps, this does not mean that each of the individual forward contracts underlying the swap has zero value. It can be shown that, when interest rates in two currencies are significantly different, the payer of the currency with the high interest rate is in the position where the forward contracts corresponding to the early exchanges of cash flows have negative values, and the forward contract corresponding to final exchange of principals has a positive value. The payer of the currency with the low interest rate is in the opposite position; that is, the forward contracts corresponding to the early exchanges of cash flows have positive values, while that corresponding to the final exchange has a negative value. These results are important when the credit risk in the swap is being evaluated.

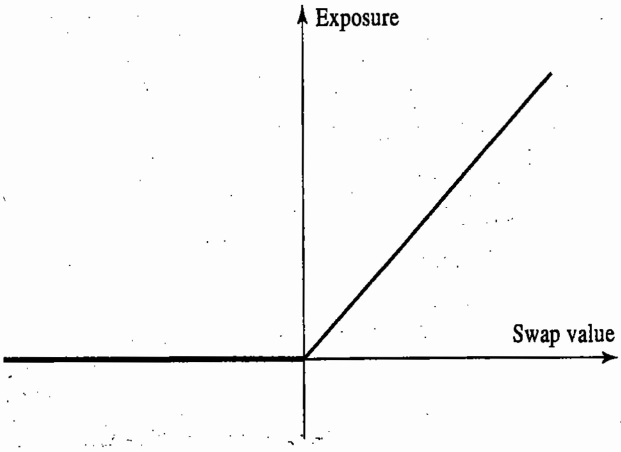
7.10 CREDIT RISK

Contracts such as swaps that are private arrangements between two companies entail credit risks. Consider a financial institution that has entered into offsetting contracts with two companies (see Figure 7.4, 7.5, or 7.7). If neither party defaults, the financial institution remains fully hedged. A decline in the value of one contract will always be offset by an increase in the value of the other contract. However, there is a chance that one party will get into financial difficulties and default. The financial institution then still has to honor the contract it has with the other party.

Suppose that, some time after the initiation of the contracts in Figure 7.4, the contract with Microsoft has a positive value to the financial institution, whereas the contract with Intel has a negative value. If Microsoft defaults, the financial institution is liable to lose the whole of the positive value it has in this contract. To maintain a hedged position, it would have to find a third party willing to take Microsoft's position. To induce the third party to take the position, the financial institution would have to pay the third party an amount roughly equal to the value of its contract with Microsoft prior to the default.

A financial institution has credit-risk exposure from a swap only when the value of the swap to the financial institution is positive. What happens when this value is negative and the counterparty gets into financial difficulties? In theory, the financial institution could realize a windfall gain, because a default would lead to it getting rid of a liability. In practice, it is likely that the counterparty would choose to sell the contract to a third party or rearrange its affairs in some way so that its positive value in the contract is not lost. The most realistic assumption for the financial institution is

Figure 7.13 The credit exposure in a swap.



therefore as follows. If the counterparty goes bankrupt, there will be a loss if the value of the swap to the financial institution is positive, and there will be no effect on the financial institution's position if the value of the swap to the financial institution is negative. This situation is summarized in Figure 7.13.

Potential losses from defaults on a swap are much less than the potential losses from defaults on a loan with the same principal. This is because the value of the swap is usually only a small fraction of the value of the loan. Potential losses from defaults on a currency swap are greater than on an interest rate swap. The reason is that, because principal amounts in two different currencies are exchanged at the end of the life of a currency swap, a currency swap is liable to have a greater value at the time of a default than an interest rate swap.

It is important to distinguish between the credit risk and market risk to a financial institution in any contract. As discussed earlier, the credit risk arises from the possibility of a default by the counterparty when the value of the contract to the financial institution is positive. The market risk arises from the possibility that market variables such as interest rates and exchange rates will move in such a way that the value of a contract to the financial institution becomes negative. Market risks can be hedged relatively easily by entering into offsetting contracts; credit risks are less easy to hedge.

One of the more bizarre stories in swap markets is outlined in Business Snapshot 7.2. It concerns the British Local Authority, Hammersmith and Fulham and shows that, in addition to bearing market risk and credit risk, banks trading swaps also sometimes bear legal risk.

7.11 OTHER TYPES OF SWAPS

In this chapter we have covered interest rate swaps where LIBOR is exchanged for a fixed rate of interest and currency swaps where a fixed rate of interest in one currency is exchanged for a fixed rate of interest in another currency. Many other types of swaps are traded. We will discuss some of them in detail in Chapters 23, 29, and 32. At this stage, we will provide an overview.

Business Snapshot 7.2 The Hammersmith and Fulham Story

Between 1987 to 1989 the London Borough of Hammersmith and Fulham in the UK entered into about 600 interest rate swaps and related instruments with a total notional principal of about 6 billion pounds. The transactions appear to have been entered into for speculative rather than hedging purposes. The two employees of Hammersmith and Fulham responsible for the trades had only a sketchy understanding of the risks they were taking and how the products they were trading worked.

By 1989, because of movements in sterling interest rates, Hammersmith and Fulham had lost several hundred million pounds on the swaps. To the banks on the other side of the transactions, the swaps were worth several hundred million pounds. The banks were concerned about credit risk. They had entered into offsetting swaps to hedge their interest rate risks. If Hammersmith and Fulham defaulted, the banks would still have to honor their obligations on the offsetting swaps and would take a huge loss.

What happened was something a little different from a default. Hammersmith and Fulham's auditor asked to have the transactions declared void because Hammersmith and Fulham did not have the authority to enter into the transactions. The British courts agreed. The case was appealed and went all the way to the House of Lords, Britain's highest court. The final decision was that Hammersmith and Fulham did not have the authority to enter into the swaps, but that they ought to have the authority to do so in the future for risk-management purposes. Needless to say, banks were furious that their contracts were overturned in this way by the courts.

Variations on the Standard Interest Rate Swap

In fixed-for-floating interest rate swaps, LIBOR is the most common reference floating interest rate. In the examples in this chapter, the tenor (i.e., payment frequency) of LIBOR has been 6 months, but swaps where the tenor of LIBOR is 1 month, 3 months, and 12 months trade regularly. The tenor on the floating side does not have to match the tenor on the fixed side. (Indeed, as pointed out in footnote 3, the standard interest rate swap in the United States is one where there are quarterly LIBOR payments and semiannual fixed payments.) LIBOR is the most common floating rate, but others such as the commercial paper (CP) rate are occasionally used. Sometimes floating-for-floating interest rates swaps are negotiated. For example, the 3-month CP rate plus 10 basis points might be exchanged for 3-month LIBOR with both being applied to the same principal. (This deal would allow a company to hedge its exposure when assets and liabilities are subject to different floating rates.)

The principal in a swap agreement can be varied throughout the term of the swap to meet the needs of a counterparty. In an *amortizing swap*, the principal reduces in a predetermined way. (This might be designed to correspond to the amortization schedule on a loan.) In a *step-up swap*, the principal increases in a predetermined way. (This might be designed to correspond to drawdowns on a loan agreement.) Deferred swaps or *forward swaps*, where the parties do not begin to exchange interest payments until some future date, can also be arranged. Sometimes swaps are negotiated where the principal to which the fixed payments are applied is different from the principal to which the floating payments are applied.

A *constant maturity swap* (CMS swap) is an agreement to exchange a LIBOR rate for a swap rate. An example would be an agreement to exchange 6-month LIBOR applied to a certain principal for the 10-year swap rate applied to the same principal every 6 months for the next 5 years. A *constant maturity Treasury swap* (CMT swap) is a similar agreement to exchange a LIBOR rate for a particular Treasury rate (e.g., the 10-year Treasury rate).

In a *compounding swap*, interest on one or both sides is compounded forward to the end of the life of the swap according to preagreed rules and there is only one payment date at the end of the life of the swap. In a *LIBOR-in arrears* swap, the LIBOR rate observed on a payment date is used to calculate the payment on that date. (As explained in Section 7.1, in a standard deal the LIBOR rate observed on one payment date is used to determine the payment on the next payment date.) In an *accrual swap*, the interest on one side of the swap accrues only when the floating reference rate is in a certain range.

Other Currency Swaps

In this chapter we have considered fixed-for-fixed currency swaps. Another type of swap is a fixed-for-floating currency swap, whereby a floating rate (usually LIBOR) in one currency is exchanged for a fixed rate in another currency. This is a combination of a fixed-for-floating interest rate swap and a fixed-for-fixed currency swap and is known as a *cross-currency interest rate swap*. A further type of currency swap is a *floating-for-floating currency swap*, where a floating rate in one currency is exchanged for a floating rate in another currency.

Sometimes a rate observed in one currency is applied to a principal amount in another currency. One such deal might be where 3-month LIBOR observed in the United States is exchanged for 3-month LIBOR in Britain, with both rates being applied to a principal of 10 million British pounds. This type of swap is referred to as a *diff swap or a quanto*.

Equity Swaps

An *equity swap* is an agreement to exchange the total return (dividends and capital gains) realized on an equity index for either a fixed or a floating rate of interest. For example, the total return on the S&P 500 in successive 6-month periods might be exchanged for LIBOR, with both being applied to the same principal. Equity swaps can be used by portfolio managers to convert returns from a fixed or floating investment to the returns from investing in an equity index, and vice versa.

Options

Sometimes there are options embedded in a swap agreement. For example, in an *extendable swap*, one party has the option to extend the life of the swap beyond the specified period. In a *puttable swap*, one party has the option to terminate the swap early. Options on swaps, or *swaptions*, are also available. These provide one party with the right at a future time to enter into a swap where a predetermined fixed rate is exchanged for floating.

Commodity Swaps, Volatility Swaps, and Other Exotic Instruments

Commodity swaps are in essence a series of forward contracts on a commodity with different maturity dates and the same delivery prices. In a *volatility swap* there are a series of time periods. At the end of each period, one side pays a preagreed volatility, while the other side pays the historical volatility realized during the period. Both volatilities are multiplied by the same notional principal in calculating payments.

Swaps are limited only by the imagination of financial engineers and the desire of corporate treasurers and fund managers for exotic structures. In Chapter 32, we will describe the famous 5/30 swap entered into between Procter and Gamble and Bankers Trust, where payments depended in a complex way on the 30-day commercial paper rate, a 30-year Treasury bond price, and the yield on a 5-year Treasury bond.

SUMMARY

The two most common types of swaps are interest rate swaps and currency swaps. In an interest rate swap, one party agrees to pay the other party interest at a fixed rate on a notional principal for a number of years. In return, it receives interest at a floating rate on the same notional principal for the same period of time. In a currency swap, one party agrees to pay interest on a principal amount in one currency. In return, it receives interest on a principal amount in another currency.

Principal amounts are not usually exchanged in an interest rate swap. In a currency swap, principal amounts are usually exchanged at both the beginning and the end of the life of the swap. For a party paying interest in the foreign currency, the foreign principal is received, and the domestic principal is paid at the beginning of the swap's life. At the end of the swap's life, the foreign principal is paid and the domestic principal is received.

An interest rate swap can be used to transform a floating-rate loan into a fixed-rate loan, or vice versa. It can also be used to transform a floating-rate investment to a fixed-rate investment, or vice versa. A currency swap can be used to transform a loan in one currency into a loan in another currency. It can also be used to transform an investment denominated in one currency into an investment denominated in another currency.

There are two ways of valuing interest rate and currency swaps. In the first, the swap is decomposed into a long position in one bond and a short position in another bond. In the second it is regarded as a portfolio of forward contracts.

When a financial institution enters into a pair of offsetting swaps with different counterparties, it is exposed to credit risk. If one of the counterparties defaults when the financial institution has positive value in its swap with that counterparty, the financial institution loses money because it still has to honor its swap agreement with the other counterparty.

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Questions and Problems (Answers in Solutions Manual)

7.1. Companies A and B have been offered the following rates per annum on a \$20 million 5-year loan:

	Fixed rate	*Floating rate*
Company A:	5.0%	LIBOR + 0.1%
Company B:	6.4%	LIBOR + 0.6%

Company A requires a floating-rate loan; company B requires a fixed-rate loan. Design a swap that will net a bank, acting as intermediary, 0.1% per annum and that will appear equally attractive to both companies.

7.2. Company X wishes to borrow US dollars at a fixed rate of interest. Company Y wishes to borrow Japanese yen at a fixed rate of interest. The amounts required by the two companies are roughly the same at the current exchange rate. The companies are subject to the following interest rates, which have been adjusted to reflect the impact of taxes:

	Yen	*Dollars*
Company X:	5.0%	9.6%
Company Y:	6.5%	10.0%

Design a swap that will net a bank, acting as intermediary, 50 basis points per annum. Make the swap equally attractive to the two companies and ensure that all foreign exchange risk is assumed by the bank.

7.3. A \$100 million interest rate swap has a remaining life of 10 months. Under the terms of the swap, 6-month LIBOR is exchanged for 7% per annum (compounded semiannually). The average of the bid–offer rate being exchanged for 6-month LIBOR in swaps of all maturities is currently 5% per annum with continuous compounding. The 6-month LIBOR rate was 4.6% per annum 2 months ago. What is the current value of the swap to the party paying floating? What is its value to the party paying fixed?

7.4. Explain what a swap rate is. What is the relationship between swap rates and par yields?

7.5. A currency swap has a remaining life of 15 months. It involves exchanging interest at 10% on £20 million for interest at 6% on \$30 million once a year. The term structure of interest rates in both the United Kingdom and the United States is currently flat, and if the swap were negotiated today the interest rates exchanged would be 4% in dollars and 7% in sterling. All interest rates are quoted with annual compounding. The current exchange rate (dollars per pound sterling) is 1.8500. What is the value of the swap to the party paying sterling? What is the value of the swap to the party paying dollars?

7.6. Explain the difference between the credit risk and the market risk in a financial contract.

7.7. A corporate treasurer tells you that he has just negotiated a 5-year loan at a competitive fixed rate of interest of 5.2%. The treasurer explains that he achieved the 5.2% rate by borrowing at 6-month LIBOR plus 150 basis points and swapping LIBOR for 3.7%. He goes on to say that this was possible because his company has a comparative advantage in the floating-rate market. What has the treasurer overlooked?

7.8. Explain why a bank is subject to credit risk when it enters into two offsetting swap contracts.

7.9. Companies X and Y have been offered the following rates per annum on a \$5 million 10-year investment:

	Fixed rate	*Floating rate*
Company X:	8.0%	LIBOR
Company Y:	8.8%	LIBOR

Company X requires a fixed-rate investment; company Y requires a floating-rate investment. Design a swap that will net a bank, acting as intermediary, 0.2% per annum and will appear equally attractive to X and Y.

7.10. A financial institution has entered into an interest rate swap with company X. Under the terms of the swap, it receives 10% per annum and pays 6-month LIBOR on a principal of \$10 million for 5 years. Payments are made every 6 months. Suppose that company X defaults on the sixth payment date (at the end of year 3) when the interest rate (with semiannual compounding) is 8% per annum for all maturities. What is the loss to the financial institution? Assume that 6-month LIBOR was 9% per annum halfway through year 3.

7.11. Companies A and B face the following interest rates (adjusted for the differential impact of taxes):

	Company A	*Company B*
US dollars (floating rate):	LIBOR + 0.5%	LIBOR + 1.0%
Canadian dollars (fixed rate):	5.0%	6.5%

Assume that A wants to borrow US dollars at a floating rate of interest and B wants to borrow Canadian dollars at a fixed rate of interest. A financial institution is planning to arrange a swap and requires a 50-basis-point spread. If the swap is to appear equally attractive to A and B, what rates of interest will A and B end up paying?

7.12. A financial institution has entered into a 10-year currency swap with company Y. Under the terms of the swap, the financial institution receives interest at 3% per annum in Swiss francs and pays interest at 8% per annum in US dollars. Interest payments are exchanged once a year. The principal amounts are 7 million dollars and 10 million francs. Suppose that company Y declares bankruptcy at the end of year 6, when the exchange rate is \$0.80 per franc. What is the cost to the financial institution? Assume that, at the end of year 6, the interest rate is 3% per annum in Swiss francs and 8% per annum in US dollars for all maturities. All interest rates are quoted with annual compounding.

7.13. After it hedges its foreign exchange risk using forward contracts, is the financial institution's average spread in Figure 7.10 likely to be greater than or less than 20 basis points? Explain your answer.

7.14. "Companies with high credit risks are the ones that cannot access fixed-rate markets directly. They are the companies that are most likely to be paying fixed and receiving floating in an interest rate swap." Assume that this statement is true. Do you think it increases or decreases the risk of a financial institution's swap portfolio? Assume that companies are most likely to default when interest rates are high.

7.15. Why is the expected loss from a default on a swap less than the expected loss from the default on a loan with the same principal?

7.16. A bank finds that its assets are not matched with its liabilities. It is taking floating-rate deposits and making fixed-rate loans. How can swaps be used to offset the risk?

7.17. Explain how you would value a swap that is the exchange of a floating rate in one currency for a fixed rate in another currency.

7.18. The LIBOR zero curve is flat at 5% (continuously compounded) out to 1.5 years. Swap rates for 2- and 3-year semiannual pay swaps are 5.4% and 5.6%, respectively. Estimate the LIBOR zero rates for maturities of 2.0, 2.5, and 3.0 years. (Assume that the 2.5-year swap rate is the average of the 2- and 3-year swap rates.)

Assignment Questions

7.19. The 1-year LIBOR rate is 10%. A bank trades swaps where a fixed rate of interest is exchanged for 12-month LIBOR with payments being exchanged annually. The 2- and 3-year swap rates (expressed with annual compounding) are 11% and 12% per annum. Estimate the 2- and 3-year LIBOR zero rates.

7.20. Company A, a British manufacturer, wishes to borrow US dollars at a fixed rate of interest. Company B, a US multinational, wishes to borrow sterling at a fixed rate of interest. They have been quoted the following rates per annum (adjusted for differential tax effects):

	Sterling	*US dollars*
Company A	11.0%	7.0%
Company B	10.6%	6.2%

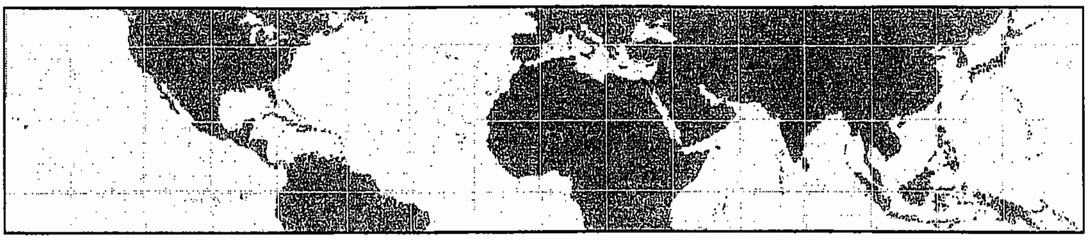
Design a swap that will net a bank, acting as intermediary, 10 basis points per annum and that will produce a gain of 15 basis points per annum for each of the two companies.

7.21. Under the terms of an interest rate swap, a financial institution has agreed to pay 10% per annum and to receive 3-month LIBOR in return on a notional principal of \$100 million with payments being exchanged every 3 months. The swap has a remaining life of 14 months. The average of the bid and offer fixed rates currently being swapped for

3-month LIBOR is 12% per annum for all maturities. The 3-month LIBOR rate 1 month ago was 11.8% per annum. All rates are compounded quarterly. What is the value of the swap?

7.22. Suppose that the term structure of interest rates is flat in the United States and Australia. The USD interest rate is 7% per annum and the AUD rate is 9% per annum. The current value of the AUD is 0.62 USD. Under the terms of a swap agreement, a financial institution pays 8% per annum in AUD and receives 4% per annum in USD. The principals in the two currencies are $12 million USD and 20 million AUD. Payments are exchanged every year, with one exchange having just taken place. The swap will last 2 more years. What is the value of the swap to the financial institution? Assume all interest rates are continuously compounded.

7.23. Company X is based in the United Kingdom and would like to borrow $50 million at a fixed rate of interest for 5 years in US funds. Because the company is not well known in the United States, this has proved to be impossible. However, the company has been quoted 12% per annum on fixed-rate 5-year sterling funds. Company Y is based in the United States and would like to borrow the equivalent of $50 million in sterling funds for 5 years at a fixed rate of interest. It has been unable to get a quote but has been offered US dollar funds at 10.5% per annum. Five-year government bonds currently yield 9.5% per annum in the United States and 10.5% in the United Kingdom. Suggest an appropriate currency swap that will net the financial intermediary 0.5% per annum.



CHAPTER 8 Mechanics of Options Markets

We introduced options in Chapter 1. This chapter explains how options markets are organized, what terminology is used, how the contracts are traded, how margin requirements are set, and so on. Later chapters will examine such topics as trading strategies involving options, the determination of option prices, and the ways in which portfolios of options can be hedged. This chapter is concerned primarily with stock options. It presents some introductory material on currency options, index options, and futures options. More details concerning these instruments can be found in Chapters 15 and 16.

Options are fundamentally different from forward and futures contracts. An option gives the holder of the option the right to do something, but the holder does not have to exercise this right. By contrast, in a forward or futures contract, the two parties have committed themselves to some action. It costs a trader nothing (except for the margin requirements) to enter into a forward or futures contract, whereas the purchase of an option requires an up-front payment.

8.1 TYPES OF OPTIONS

As mentioned in Chapter 1, there are two basic types of options. A *call option* gives the holder of the option the right to buy an asset by a certain date for a certain price. A *put option* gives the holder the right to sell an asset by a certain date for a certain price. The date specified in the contract is known as the *expiration date* or the *maturity date*. The price specified in the contract is known as the *exercise price* or the *strike price*.

Options can be either American or European, a distinction that has nothing to do with geographical location. *American options* can be exercised at any time up to the expiration date, whereas *European options* can be exercised only on the expiration date itself. Most of the options that are traded on exchanges are American. However, European options are generally easier to analyze than American options, and some of the properties of an American option are frequently deduced from those of its European counterpart.

Call Options

Consider the situation of an investor who buys a European call option with a strike price of $100 to purchase 100 shares of a certain stock. Suppose that the current stock price is \$98, the expiration date of the option is in 4 months, and the price of an option to purchase one share is \$5. The initial investment is \$500. Because the option is European, the investor can exercise only on the expiration date. If the stock price on this date is less than \$100, the investor will clearly choose not to exercise. (There is no point in buying for \$100 a share that has a market value of less than \$100.) In these circumstances, the investor loses the whole of the initial investment of \$500. If the stock price is above \$100 on the expiration date, the option will be exercised. Suppose, for example, that the stock price is \$115. By exercising the option, the investor is able to buy 100 shares for \$100 per share. If the shares are sold immediately, the investor makes a gain of \$15 per share, or \$1,500, ignoring transactions costs. When the initial cost of the option is taken into account, the net profit to the investor is \$1,000.

Figure 8.1 shows how the investor's net profit or loss on an option to purchase one share varies with the final stock price in the example. It is important to realize that an investor sometimes exercises an option and makes a loss overall. Suppose that, in the example, the stock price is \$102 at the expiration of the option. The investor would exercise the option contract for a gain of 100 × (\$102 – \$100) = \$200 and realize a loss overall of \$300 when the initial cost of the option is taken into account. It is tempting to argue that the investor should not exercise the option in these circumstances. However, not exercising would lead to an overall loss of \$500, which is worse than the \$300 loss when the investor exercises. In general, call options should always be exercised at the expiration date if the stock price is above the strike price.

Put Options

Whereas the purchaser of a call option is hoping that the stock price will increase, the purchaser of a put option is hoping that it will decrease. Consider an investor who buys a European put option with a strike price of \$70 to sell 100 shares of a certain stock. Suppose that the current stock price is \$65, the expiration date of the option is

Figure 8.1 Profit from buying a European call option on one share of a stock. Option price = \$5; strike price = \$100.

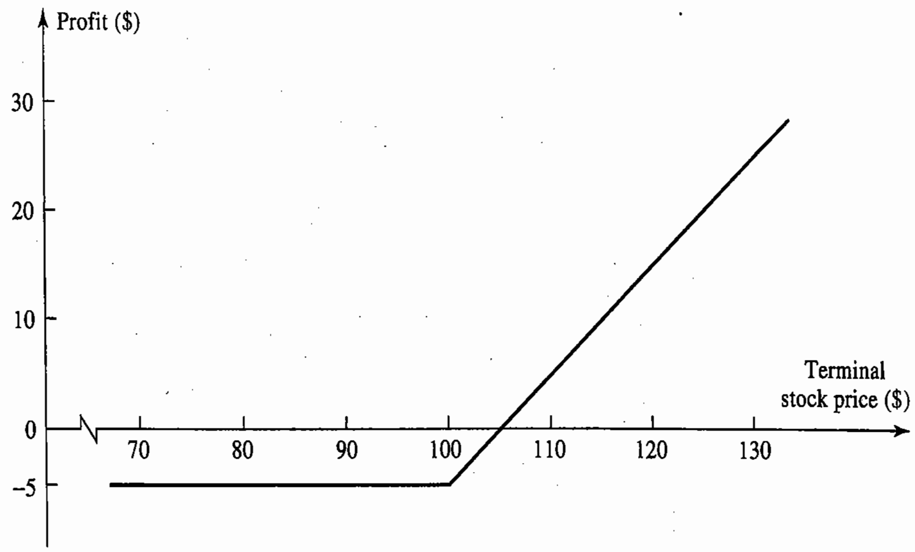
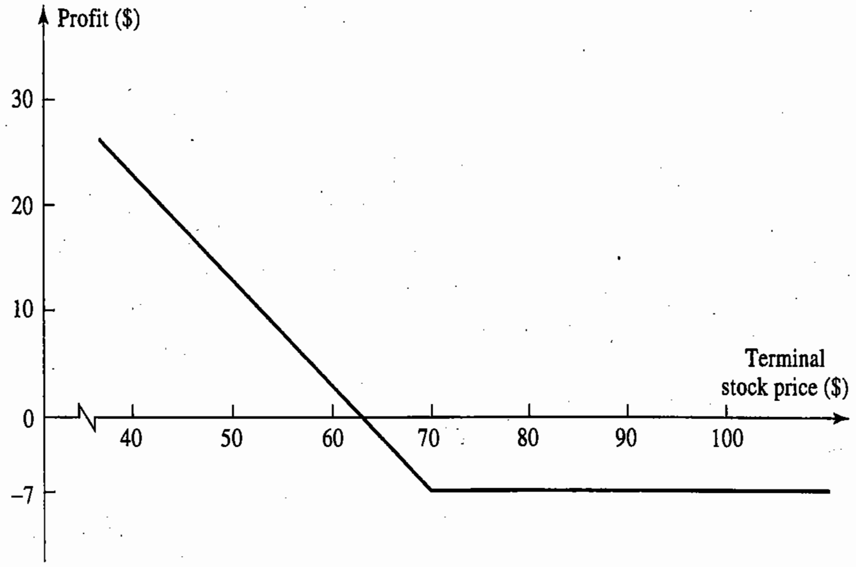


Figure 8.2 Profit from buying a European put option on one share of a stock. Option price = $7; strike price = $70.



in 3 months, and the price of an option to sell one share is $7. The initial investment is $700. Because the option is European, it will be exercised only if the stock price is below $70 on the expiration date. Suppose that the stock price is $55 on this date. The investor can buy 100 shares for $55 per share and, under the terms of the put option, sell the same shares for $70 to realize a gain of $15 per share, or $1,500. (Again, transactions costs are ignored.) When the $700 initial cost of the option is taken into account, the investor's net profit is $800. There is no guarantee that the investor will make a gain. If the final stock price is above $70, the put option expires worthless, and the investor loses $700. Figure 8.2 shows the way in which the investor's profit or loss on an option to sell one share varies with the terminal stock price in this example.

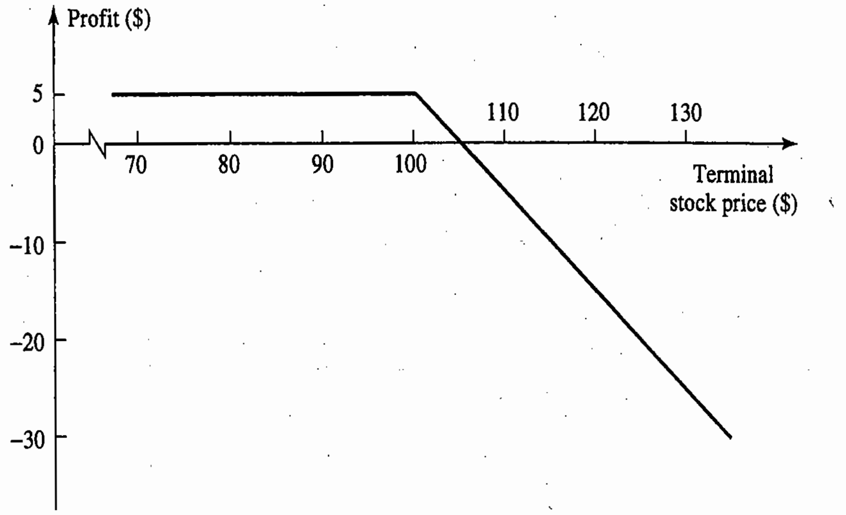
Early Exercise

As already mentioned, exchange-traded stock options are generally American rather than European. This means that the investor in the foregoing examples would not have to wait until the expiration date before exercising the option. We will see that there are some circumstances when it is optimal to exercise American options before the expiration date.

8.2 OPTION POSITIONS

There are two sides to every option contract. On one side is the investor who has taken the long position (i.e., has bought the option). On the other side is the investor who has taken a short position (i.e., has sold or *written* the option). The writer of an option receives cash up front, but has potential liabilities later. The writer's profit or loss is the reverse of that for the purchaser of the option. Figures 8.3 and 8.4 show the variation of the profit or loss with the final stock price for writers of the options considered in Figures 8.1 and 8.2.

Figure 8.3 Profit from writing a European call option on one share of a stock. Option price = \$5; strike price = \$100.



There are four types of option positions:

1. A long position in a call option
2. A long position in a put option
3. A short position in a call option
4. A short position in a put option

It is often useful to characterize a European option in terms of its payoff to the purchaser of the option. The initial cost of the option is then not included in the calculation. If K is the strike price and S_T is the final price of the underlying asset, the

Figure 8.4 Profit from writing a European put option on one share of a stock. Option price = \$7; strike price = \$70.

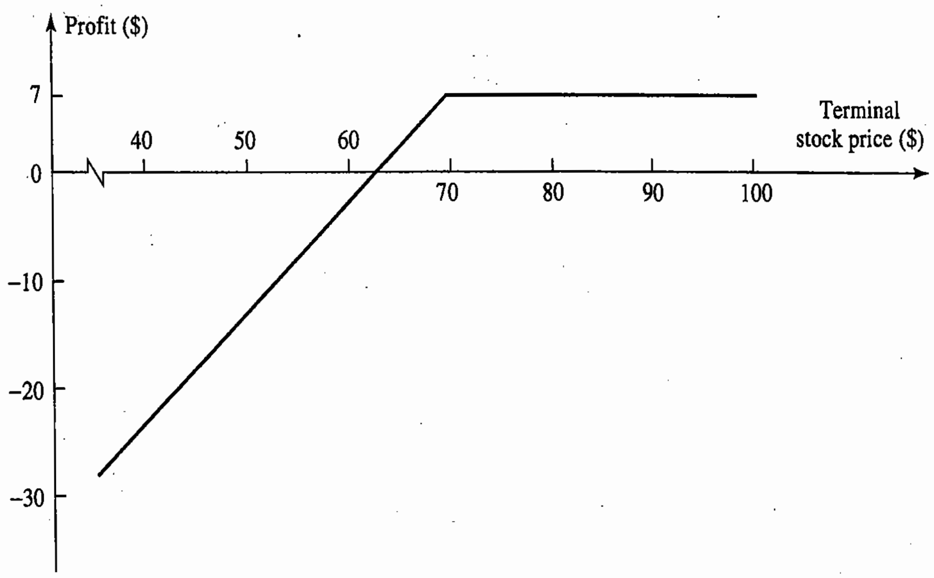
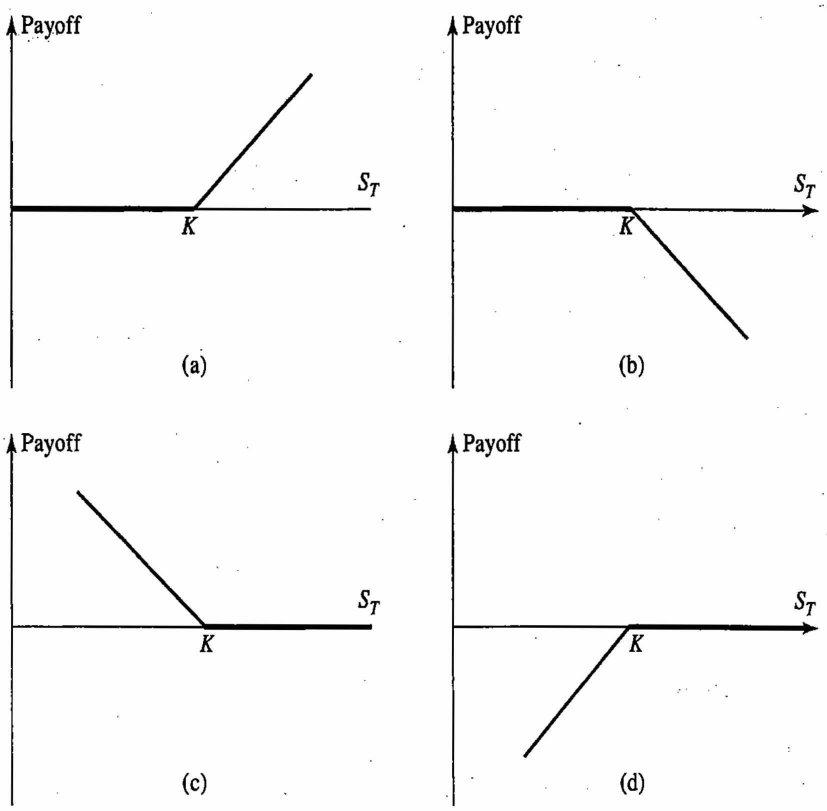


Figure 8.5 Payoffs from positions in European options: (a) long call; (b) short call; (c) long put; (d) short put. Strike price = K; price of asset at maturity = S_T.



payoff from a long position in a European call option is

$$\max(S_T - K, 0)$$

This reflects the fact that the option will be exercised if $S_T > K$ and will not be exercised if $S_T \leqslant K$. The payoff to the holder of a short position in the European call option is

$$-\max(S_T - K, 0) = \min(K - S_T, 0)$$

The payoff to the holder of a long position in a European put option is

$$\max(K - S_T, 0)$$

and the payoff from a short position in a European put option is

$$-\max(K - S_T, 0) = \min(S_T - K, 0)$$

Figure 8.5 illustrates these payoffs.

8.3 UNDERLYING ASSETS

This section provides a first look at options on stocks, currencies, stock indices, and futures.

Stock Options

Most trading in stock options is on exchanges. In the United States the main exchanges are the Chicago Board Options Exchange (www.cboe.com), the Philadelphia Stock Exchange (www.phlx.com), the American Stock Exchange (www.amex.com), the International Securities Exchange (www.iseoptions.com), and the Boston Options Exchange (www.bostonoptions.com). Options trade on more than 1,000 different stocks. One contract gives the holder the right to buy or sell 100 shares at the specified strike price. This contract size is convenient because the shares themselves are normally traded in lots of 100.

Foreign Currency Options

Most currency options trading is now in the over-the-counter market, but there is some exchange trading. The major exchange for trading foreign currency options in the United States is the Philadelphia Stock Exchange. It offers both European and American contracts on a variety of different currencies. The size of one contract depends on the currency. For example, in the case of the British pound, one contract gives the holder the right to buy or sell £31,250; in the case of the Japanese yen, one contract gives the holder the right to buy or sell 6.25 million yen. Foreign currency options contracts are discussed further in Chapter 15.

Index Options

Many different index options currently trade throughout the world in both the over-the-counter market and the exchange-traded market. The most popular exchange-traded contracts in the United States are those on the S&P 500 Index (SPX), the S&P 100 Index (OEX), the Nasdaq 100 Index (NDX), and the Dow Jones Industrial Index (DJX). All of these trade on the Chicago Board Options Exchange. Most of the contracts are European. An exception is the OEX contract on the S&P 100, which is American. One contract is usually to buy or sell 100 times the index at the specified strike price. Settlement is always in cash, rather than by delivering the portfolio underlying the index. Consider, for example, one call contract on the S&P 100 with a strike price of 980. If it is exercised when the value of the index is 992, the writer of the contract pays the holder $(992 - 980) \times 100 = \$1,200$. This cash payment is based on the index value at the end of the day on which exercise instructions are issued. (Not surprisingly, investors usually wait until the end of a day before issuing these instructions.) Index options are discussed further in Chapter 15.

Futures Options

When an exchange trades a particular futures contract it often also trades options on that contract. A futures option normally matures just before the delivery period in the futures contract. When a call option is exercised, the holder acquires from the writer a long position in the underlying futures contract plus a cash amount equal to the excess of the futures price over the strike price. When a put option is exercised, the holder acquires a short position in the underlying futures contract plus a cash amount equal to the excess of the strike price over the futures price. Futures options contracts are discussed further in Chapter 16.

8.4 SPECIFICATION OF STOCK OPTIONS

In the rest of this chapter, we will focus on stock options. As already mentioned, an exchange-traded stock option in the United States is an American-style option contract to buy or sell 100 shares of the stock. Details of the contract—the expiration date, the strike price, what happens when dividends are declared, how large a position investors can hold, and so on—are specified by the exchange.

Expiration Dates

One of the items used to describe a stock option is the month in which the expiration date occurs. Thus, a January call trading on IBM is a call option on IBM with an expiration date in January. The precise expiration date is the Saturday immediately following the third Friday of the expiration month. The last day on which options trade is the third Friday of the expiration month. An investor with a long position in an option normally has until 4:30 p.m. Central Time on that Friday to instruct a broker to exercise the option. The broker then has until 10:59 p.m. the next day to complete the paperwork notifying the exchange that exercise is to take place.

Stock options are on a January, February, or March cycle. The January cycle consists of the months of January, April, July, and October. The February cycle consists of the months of February, May, August, and November. The March cycle consists of the months of March, June, September, and December. If the expiration date for the current month has not yet been reached, options trade with expiration dates in the current month, the following month, and the next two months in the cycle. If the expiration date of the current month has passed, options trade with expiration dates in the next month, the next-but-one month, and the next two months of the expiration cycle. For example, IBM is on a January cycle. At the beginning of January, options are traded with expiration dates in January, February, April, and July; at the end of January, they are traded with expiration dates in February, March, April, and July; at the beginning of May, they are traded with expiration dates in May, June, July, and October; and so on. When one option reaches expiration, trading in another is started. Longer-term options, known as LEAPS (long-term equity anticipation securities), also trade on about 500 stocks in the United States. These have expiration dates up to 39 months into the future. The expiration dates for LEAPS on stocks are always in January.

Strike Prices

The exchange normally chooses the strike prices at which options can be written so that they are spaced \$2.50, \$5, or \$10 apart. Typically the spacing is \$2.50 when the stock price is between \$5 and \$25, \$5 when the stock price is between \$25 and \$200, and \$10 for stock prices above \$200. As will be explained shortly, stock splits and stock dividends can lead to nonstandard strike prices.

When a new expiration date is introduced, the two or three strike prices closest to the current stock price are usually selected by the exchange. If the stock price moves outside the range defined by the highest and lowest strike price, trading is usually introduced in an option with a new strike price. To illustrate these rules, suppose that the stock price is \$84 when trading begins in the October options. Call and put options would probably first be offered with strike prices of \$80, \$85, and \$90. If the stock price rose above \$90, it is likely that a strike price of \$95 would be offered; if it fell below \$80, it is likely that a strike price of \$75 would be offered; and so on.

Terminology

For any given asset at any given time, many different option contracts may be trading. Consider a stock that has four expiration dates and five strike prices. If call and put options trade with every expiration date and every strike price, there are a total of 40 different contracts. All options of the same type (calls or puts) are referred to as an *option class*. For example, IBM calls are one class, whereas IBM puts are another class. An *option series* consists of all the options of a given class with the same expiration date and strike price. In other words, an option series refers to a particular contract that is traded. For example, IBM 70 October calls would constitute an option series.

Options are referred to as *in the money*, *at the money*, or *out of the money*. If S is the stock price and K is the strike price, a call option is in the money when $S > K$, at the money when $S = K$, and out of the money when $S < K$. A put option is in the money when $S < K$, at the money when $S = K$, and out of the money when $S > K$. Clearly, an option will be exercised only when it is in the money. In the absence of transactions costs, an in-the-money option will always be exercised on the expiration date if it has not been exercised previously.

The *intrinsic value* of an option is defined as the maximum of zero and the value the option would have if it were exercised immediately. For a call option, the intrinsic value is therefore $\max(S - K, 0)$. For a put option, it is $\max(K - S, 0)$. An in-the-money American option must be worth at least as much as its intrinsic value because the holder can realize the intrinsic value by exercising immediately. Often it is optimal for the holder of an in-the-money American option to wait rather than exercise immediately. The option is then said to have *time value*. The total value of an option can be thought of as the sum of its intrinsic value and its time value.

FLEX Options

The Chicago Board Options Exchange offers FLEX (short for flexible) options on equities and equity indices. These are options where the traders on the floor of the exchange agree to nonstandard terms. These nonstandard terms can involve a strike price or an expiration date that is different from what is usually offered by the exchange. It can also involve the option being European rather than American. FLEX options are an attempt by option exchanges to regain business from the over-the-counter markets. The exchange specifies a minimum size (e.g., 100 contracts) for FLEX option trades.

Dividends and Stock Splits

The early over-the-counter options were dividend protected. If a company declared a cash dividend, the strike price for options on the company's stock was reduced on the ex-dividend day by the amount of the dividend. Exchange-traded options are not usually adjusted for cash dividends. In other words, when a cash dividend occurs, there are no adjustments to the terms of the option contract. An exception is sometimes made for large cash dividends (see Business Snapshot 8.1).

Exchange-traded options are adjusted for stock splits. A stock split occurs when the

Business Snapshot 8.1 Gucci Group's Large Dividend

When there is a large cash dividend (typically one more than 10% of the stock price), a committee of the Options Clearing Corporation (OCC) at the Chicago Board Options Exchange can decide to make adjustments to the terms of options traded on the exchange.

On May 28, 2003, Gucci Group NV (GUC) declared a cash dividend of 13.50 euros (approximately $15.88) per common share and this was approved at the GUC annual shareholders meeting on July 16, 2003. The dividend was about 16% of the share price at the time it was declared. In this case, the OCC committee decided to adjust the terms of options. The result was that the holder of a call contract paid 100 times the strike price on exercise and received $1,588 of cash in addition to 100 shares; the holder of a put contract received 100 times the strike price on exercise and delivered $1,588 of cash in addition to 100 shares. These adjustments had the effect of reducing the strike price by $15.88.

Adjustments for large dividends are not always made. For example, Deutsche Terminbörse chose not to adjust the terms of options traded on that exchange when Daimler-Benz surprised the market on March 10, 1998, with a dividend equal to about 12% of its stock price.

existing shares are "split" into more shares. For example, in a 3-for-1 stock split, three new shares are issued to replace each existing share. Because a stock split does not change the assets or the earning ability of a company, we should not expect it to have any effect on the wealth of the company's shareholders. All else being equal, the 3-for-1 stock split should cause the stock price to go down to one-third of its previous value. In general, an n-for-m stock split should cause the stock price to go down to m/n of its previous value. The terms of option contracts are adjusted to reflect expected changes in a stock price arising from a stock split. After an n-for-m stock split, the strike price is reduced to m/n of its previous value, and the number of shares covered by one contract is increased to n/m of its previous value. If the stock price declines in the way expected, the positions of both the writer and the purchaser of a contract remain unchanged.

Example 8.1

Consider a call option to buy 100 shares of a company for $30 per share. Suppose that the company makes a 2-for-1 stock split. The terms of the option contract are then changed so that it gives the holder the right to purchase 200 shares for $15 per share.

Stock options are adjusted for stock dividends. A stock dividend involves a company issuing more shares to its existing shareholders. For example, a 20% stock dividend means that investors receive one new share for each five already owned. A stock dividend, like a stock split, has no effect on either the assets or the earning power of a company. The stock price can be expected to go down as a result of a stock dividend. The 20% stock dividend referred to is essentially the same as a 6-for-5 stock split. All else being equal, it should cause the stock price to decline to 5/6 of its previous value. The terms of an option are adjusted to reflect the expected price decline arising from a stock dividend in the same way as they are for that arising from a stock split.

Example 8.2

Consider a put option to sell 100 shares of a company for $15 per share. Suppose that the company declares a 25% stock dividend. This is equivalent to a 5-for-4 stock split. The terms of the option contract are changed so that it gives the holder the right to sell 125 shares for $12.

Adjustments are also made for rights issues. The basic procedure is to calculate the theoretical price of the rights and then to reduce the strike price by this amount.

Position Limits and Exercise Limits

The Chicago Board Options Exchange often specifies a *position limit* for option contracts. This defines the maximum number of option contracts that an investor can hold on one side of the market. For this purpose, long calls and short puts are considered to be on the same side of the market. Also considered to be on the same side are short calls and long puts. The *exercise limit* usually equals the position limit. It defines the maximum number of contracts that can be exercised by any individual (or group of individuals acting together) in any period of five consecutive business days. Options on the largest and most frequently traded stocks have positions limits of 250,000 contracts. Smaller capitalization stocks have position limits of 200,000, 75,000, 50,000, or 25,000 contracts.

Position limits and exercise limits are designed to prevent the market from being unduly influenced by the activities of an individual investor or group of investors. However, whether the limits are really necessary is a controversial issue.

8.5 TRADING

Traditionally, exchanges have had to provide a large open area for individuals to meet and trade options. This is changing. Many derivatives exchanges are fully electronic, so traders do not have to physically meet. The International Securities Exchange (www.iseoptions.com) launched the first all-electronic options market for equities in the United States in May 2000. The Chicago Board Options Exchange has an electronic system that runs side by side with its floor-based open-outcry markets.

Market Makers

Most options exchanges use market makers to facilitate trading. A market maker for a certain option is an individual who, when asked to do so, will quote both a bid and an offer price on the option. The bid is the price at which the market maker is prepared to buy, and the offer or asked is the price at which the market maker is prepared to sell. At the time the bid and offer prices are quoted, the market maker does not know whether the trader who asked for the quotes wants to buy or sell the option. The offer is always higher than the bid, and the amount by which the offer exceeds the bid is referred to as the *bid–offer* spread. The exchange sets upper limits for the bid–offer spread. For example, it might specify that the spread be no more than $0.25 for options priced at less than $0.50, $0.50 for options priced between $0.50 and $10, $0.75 for options priced between $10 and $20, and $1 for options priced over $20.

The existence of the market maker ensures that buy and sell orders can always be executed at some price without any delays. Market makers therefore add liquidity to the market. The market makers themselves make their profits from the bid–offer spread. They use some of the methods discussed in Chapter 17 to hedge their risks.

Offsetting Orders

An investor who has purchased an option can close out the position by issuing an offsetting order to sell the same option. Similarly, an investor who has written an option can close out the position by issuing an offsetting order to buy the same option. (In this respect options markets are similar to futures markets.) If, when an option contract is traded, neither investor is closing an existing position, the open interest increases by one contract. If one investor is closing an existing position and the other is not, the open interest stays the same. If both investors are closing existing positions, the open interest goes down by one contract.

8.6 COMMISSIONS

The types of orders that can be placed with a broker for options trading are similar to those for futures trading (see Section 2.7). A market order is executed immediately, a limit order specifies the least favorable price at which the order can be executed, and so on.

For a retail investor, commissions vary significantly from broker to broker. Discount brokers generally charge lower commissions than full-service brokers. The actual amount charged is often calculated as a fixed cost plus a proportion of the dollar amount of the trade. Table 8.1 shows the sort of schedule that might be offered by a discount broker. Thus, the purchase of eight contracts when the option price is \$3 would cost $\$20 + (0.02 \times \$2{,}400) = \$68$ in commissions.

If an option position is closed out by entering into an offsetting trade, the commission must be paid again. If the option is exercised, the commission is the same as it would be if the investor placed an order to buy or sell the underlying stock. This could be as much as 1% to 2% of the stock's value.

Consider an investor who buys one call contract with a strike price of \$50 when the stock price is \$49. We suppose the option price is \$4.50, so that the cost of the contract is \$450. Under the schedule in Table 8.1, the purchase or sale of one contract always

Table 8.1 A typical commission schedule for a discount broker.

Dollar amount of trade	*Commission**
< \$2,500	\$20 + 2% of dollar amount
\$2,500 to \$10,000	\$45 + 1% of dollar amount
> \$10,000	\$120 + 0.25% of dollar amount

* Maximum commission is \$30 per contract for the first five contracts plus \$20 per contract for each additional contract. Minimum commission is \$30 per contract for the first contract plus \$2 per contract for each additional contract.

costs $30 (both the maximum and minimum commission is $30 for the first contract). Suppose that the stock price rises and the option is exercised when the stock reaches $60. Assuming that the investor pays 1.5% commission on stock trades, the commission payable when the option is exercised is

$$0.015 \times \$60 \times 100 = \$90$$

The total commission paid is therefore $120, and the net profit to the investor is

$$\$1{,}000 - \$450 - \$120 = \$430$$

Note that selling the option for $10 instead of exercising it would save the investor $60 in commissions. (The commission payable when an option is sold is only $30 in our example.) In general, the commission system tends to push retail investors in the direction of selling options rather than exercising them.

A hidden cost in option trading (and in stock trading) is the market maker's bid–offer spread. Suppose that, in the example just considered, the bid price was $4.00 and the offer price was $4.50 at the time the option was purchased. We can reasonably assume that a "fair" price for the option is halfway between the bid and the offer price, or $4.25. The cost to the buyer and to the seller of the market maker system is the difference between the fair price and the price paid. This is $0.25 per option, or $25 per contract.

8.7 MARGINS

When shares are purchased in the United States, an investor can borrow up to 50% of the price from the broker. This is known as *buying on margin*. If the share price declines so that the loan is substantially more than 50% of the stock's current value, there is a "margin call", where the broker requests that cash be deposited by the investor. If the margin call is not met, the broker sells the stock.

When call and put options with maturities less than 9 months are purchased, the option price must be paid in full. Investors are not allowed to buy these options on margin because options already contain substantial leverage and buying on margin would raise this leverage to an unacceptable level. For options with maturities greater than 9 months investors can buy on margin, borrowing up to 25% of the option value.

A trader who writes options is required to maintain funds in a margin account. Both the trader's broker and the exchange want to be satisfied that the investor will not default if the option is exercised. The amount of margin required depends on the trader's position.

Writing Naked Options

A *naked option* is an option that is not combined with an offsetting position in the underlying stock. The initial margin required by the CBOE for a written naked call option is the greater of the following two calculations:

1. A total of 100% of the proceeds of the sale plus 20% of the underlying share price less the amount, if any, by which the option is out of the money
2. A total of 100% of the option proceeds plus 10% of the underlying share price

For a written naked put option, it is the greater of

1. A total of 100% of the proceeds of the sale plus 20% of the underlying share price less the amount, if any, by which the option is out of the money
2. A total of 100% of the option proceeds plus 10% of the exercise price

The 20% in the preceding calculations is replaced by 15% for options on a broadly based stock index because a stock index is usually less volatile than the price of an individual stock.

Example 8.3

An investor writes four naked call option contracts on a stock. The option price is \$5, the strike price is \$40, and the stock price is \$38. Because the option is \$2 out of the money, the first calculation gives

$$400 \times (5 + 0.2 \times 38 - 2) = \$4{,}240$$

The second calculation gives

$$400 \times (5 + 0.1 \times 38) = \$3{,}520$$

The initial margin requirement is therefore \$4,240. Note that, if the option had been a put, it would be \$2 in the money and the margin requirement would be

$$400 \times (5 + 0.2 \times 38) = \$5{,}040$$

In both cases the proceeds of the sale, \$2,000, can be used to form part of the margin account.

A calculation similar to the initial margin calculation (but with the current market price replacing the proceeds of sale) is repeated every day. Funds can be withdrawn from the margin account when the calculation indicates that the margin required is less than the current balance in the margin account. When the calculation indicates that a significantly greater margin is required, a margin call will be made.

Other Rules

In Chapter 10, we will examine option trading strategies such as covered calls, protective puts, spreads, combinations, straddles, and strangles. The CBOE has special rules for determining the margin requirements when these trading strategies are used. These are described in the *CBOE Margin Manual*, which is available on the CBOE website (www.cboe.com).

As an example of the rules, consider an investor who writes a covered call. This is a written call option when the shares that might have to be delivered are already owned. Covered calls are far less risky than naked calls, because the worst that can happen is that the investor is required to sell shares already owned at below their market value. No margin is required on the written option. However, the investor can borrow an amount equal to $0.5\min(S, K)$, rather than the usual $0.5S$, on the stock position.

8.8 THE OPTIONS CLEARING CORPORATION

The Options Clearing Corporation (OCC) performs much the same function for options markets as the clearinghouse does for futures markets (see Chapter 2). It guarantees that options writers will fulfill their obligations under the terms of options contracts and keeps a record of all long and short positions. The OCC has a number of members, and all option trades must be cleared through a member. If a broker is not itself a member of an exchange's OCC, it must arrange to clear its trades with a member. Members are required to have a certain minimum amount of capital and to contribute to a special fund that can be used if any member defaults on an option obligation.

When purchasing an option, the buyer must pay for it in full by the morning of the next business day. The funds are deposited with the OCC. The writer of the option maintains a margin account with a broker, as described earlier.[1] The broker maintains a margin account with the OCC member that clears its trades. The OCC member in turn maintains a margin account with the OCC.

Exercising an Option

When an investor notifies a broker to exercise an option, the broker in turn notifies the OCC member that clears its trades. This member then places an exercise order with the OCC. The OCC randomly selects a member with an outstanding short position in the same option. The member, using a procedure established in advance, selects a particular investor who has written the option. If the option is a call, this investor is required to sell stock at the strike price. If it is a put, the investor is required to buy stock at the strike price. The investor is said to be *assigned*. When an option is exercised, the open interest goes down by one.

At the expiration of the option, all in-the-money options should be exercised unless the transactions costs are so high as to wipe out the payoff from the option. Some brokers will automatically exercise options for a client at expiration when it is in their client's interest to do so. Many exchanges also have rules for exercising options that are in the money at expiration.

8.9 REGULATION

Options markets are regulated in a number of different ways. Both the exchange and its Options Clearing Corporation have rules governing the behavior of traders. In addition, there are both federal and state regulatory authorities. In general, options markets have demonstrated a willingness to regulate themselves. There have been no major scandals or defaults by OCC members. Investors can have a high level of confidence in the way the market is run.

The Securities and Exchange Commission is responsible for regulating options markets in stocks, stock indices, currencies, and bonds at the federal level. The

[1] The margin requirements described in the previous section are the minimum requirements specified by the OCC. A brokerage house may require higher margins from its clients. However, it cannot require lower margins. Some brokerage houses do not allow their retail clients to write uncovered options at all.

Commodity Futures Trading Commission is responsible for regulating markets for options on futures. The major options markets are in the states of Illinois and New York. These states actively enforce their own laws on unacceptable trading practices.

8.10 TAXATION

Determining the tax implications of option strategies can be tricky, and an investor who is in doubt about this should consult a tax specialist. In the United States, the general rule is that (unless the taxpayer is a professional trader) gains and losses from the trading of stock options are taxed as capital gains or losses. The way that capital gains and losses are taxed in the United States was discussed in Section 2.9. For both the holder and the writer of a stock option, a gain or loss is recognized when (a) the option expires unexercised or (b) the option position is closed out. If the option is exercised, the gain or loss from the option is rolled into the position taken in the stock and recognized when the stock position is closed out. For example, when a call option is exercised, the party with a long position is deemed to have purchased the stock at the strike price plus the call price. This is then used as a basis for calculating this party's gain or loss when the stock is eventually sold. Similarly, the party with the short call position is deemed to have sold the stock at the strike price plus the call price. When a put option is exercised, the seller of the option is deemed to have bought the stock for the strike price less the original put price and the purchaser of the option is deemed to have sold the stock for the strike price less the original put price.

Wash Sale Rule

One tax consideration in option trading in the United States is the wash sale rule. To understand this rule, imagine an investor who buys a stock when the price is \$60 and plans to keep it for the long term. If the stock price drops to \$40, the investor might be tempted to sell the stock and then immediately repurchase it, so that the \$20 loss is realized for tax purposes. To prevent this sort of thing, the tax authorities have ruled that when the repurchase is within 30 days of the sale (i.e., between 30 days before the sale and 30 days after the sale), any loss on the sale is not deductible. The disallowance also applies where, within the 61-day period, the taxpayer enters into an option or similar contract to acquire the stock. Thus, selling a stock at a loss and buying a call option within a 30-day period will lead to the loss being disallowed. The wash sale rule does not apply if the taxpayer is a dealer in stocks or securities and the loss is sustained in the ordinary course of business.

Constructive Sales

Prior to 1997, if a United States taxpayer shorted a security while holding a long position in a substantially identical security, no gain or loss was recognized until the short position was closed out. This means that short positions could be used to defer recognition of a gain for tax purposes. The situation was changed by the Tax Relief Act

Business Snapshot 8.2 Tax Planning Using Options

As a simple example of a possible tax planning strategy using options, suppose that Country A has a tax regime where the tax is low on interest and dividends and high on capital gains, while Country B has a tax regime where tax is high on interest and dividends and low on capital gains. It is advantageous for a company to receive the income from a security in Country A and the capital gain, if there is one, in Country B. The company would like to keep capital losses in Country A, where they can be used to offset capital gains on other items. All of this can be accomplished by arranging for a subsidiary company in Country A to have legal ownership of the security and for a subsidiary company in Country B to buy a call option on the security from the company in Country A, with the strike price of the option equal to the current value of the security. During the life of the option, income from the security is earned in Country A. If the security price rises sharply, the option will be exercised and the capital gain will be realized in Country B. If it falls sharply, the option will not be exercised and the capital loss will be realized in Country A.

of 1997. An appreciated property is now treated as "constructively sold" when the owner does one of the following:

1. Enters into a short sale of the same or substantially identical property
2. Enters into a futures or forward contract to deliver the same or substantially identical property
3. Enters into one or more positions that eliminate substantially all of the loss and opportunity for gain

It should be noted that transactions reducing only the risk of loss or only the opportunity for gain should not result in constructive sales. Therefore an investor holding a long position in a stock can buy in-the-money put options on the stock without triggering a constructive sale.

Tax practitioners sometimes use options to minimize tax costs or maximize tax benefits (see Business Snapshot 8.2). Tax authorities in many jurisdictions have proposed legislation designed to combat the use of derivatives for tax purposes. Before entering into any tax-motivated transaction, a corporate treasurer or private individual should explore in detail how the structure could be unwound in the event of legislative change and how costly this process could be.

8.11 WARRANTS, EMPLOYEE STOCK OPTIONS, AND CONVERTIBLES

Warrants are options issued by a financial institution or nonfinancial corporation. For example, a financial institution might issue put warrants on one million ounces of gold and then proceed to create a market for the warrants. To exercise the warrant, the holder would contact the financial institution. A common use of warrants by a nonfinancial corporation is at the time of a bond issue. The corporation issues call warrants on its own stock and then attaches them to the bond issue to make it more attractive to investors.

Business Snapshot 8.3 Employee Stock Options

Stock options became an increasingly popular type of compensation for executives and other employees in the 1990s and early 2000s. In a typical arrangement, an employee is granted a certain number of call options on the stock of the company for which he or she works. The options are at the money on the grant date. They often last for ten years or even longer and there is a vesting period of up to five years. The options cannot be exercised during the vesting period, but can be exercised any time after the vesting period ends. If the employee leaves the company during the vesting period the options are forfeited. If the employee leaves the company after the end of the vesting period, in-the-money options are exercised immediately while out-of-the-money options are forfeited. Options cannot be sold to another party by the employee.

One reason why employee stock options have been so attractive has been their accounting treatment. The compensation cost charged to the income statement for an employee stock option in the United States and other countries has traditionally been its intrinsic value. Because most options were at the money when they were issued, this compensation cost was usually zero. In 1995 accounting standard FAS 123 was issued. This encouraged, but did not require, companies to expense the "fair value" of the options on their income statement. (If the fair value was not expensed on the income statement it had to be reported in a footnote to the company's accounts.)

Accounting standards have now changed to require the expensing of stock options at their fair value on the income statement. In February 2004 the International Accounting Standards Board issued IAS 2 requiring companies to start expensing stock options in 2005. In December 2004 FAS 123 was revised to require the expensing of employee stock options in the United States starting in 2005. There are signs that these rules have made stock options less popular as a method of employee compensation.

Employee stock options tend to be exercised earlier than similar exchange-traded or over-the-counter options because the employee is not allowed to sell the options. If an employee wants to realize cash from the options, he or she has to exercise the options and sell the stock. For this reason, valuing employee stock options is not as easy as valuing regular options. It requires a model of the employees' early-exercise behavior (see Section 12.10).

There have been some well publicized questionable activities concerning the granting of stock options. If the senior management of a company knows there will an announcement that will lead to an increase in the company's stock price, there is a big temptation to choose the time when they grant stock options to themselves (or to choose the time of the news announcement) so that the news announcement is shortly after the grant date. Worse still, if the company experiences a sharp rise in its stock price, there is a temptation to backdate the grant date of their at-the-money options. Some senior managers seem to have succumbed to these temptations.

Employee stock options are call options issued to executives by their company to motivate them to act in the best interests of the company's shareholders. They are usually at the money at the time of issue. They are now a cost on the income statement of the company in most countries, making them a less attractive form of compensation than they used to be. (See Business Snapshot 8.3.)

Convertible bonds, often simply referred to as *convertibles*, are bonds issued by a company that can be converted into equity at certain times using a predetermined exchange ratio. They are therefore bonds with an embedded call option on the company's stock.

One feature of warrants, employee stock options, and convertibles is that a predetermined number of options are issued. By contrast, the number of exchange-traded options outstanding is not predetermined. (As more people trade a particular option series, the number of options outstanding increases.) Warrants issued by a company on its own stock, executive stock options, and convertibles are different from exchange-traded options in another important way. When these instruments are exercised, the company issues more shares of its own stock and sells them to the option holder for the strike price. The exercise of the instruments therefore leads to an increase in the number of shares of the company's stock that are outstanding. By contrast, when an exchange-traded call option is exercised, the party with the short position buys in the market shares that have already been issued and sells them to the party with the long position for the strike price. The company whose stock underlies the option is not involved in any way.

8.12 OVER-THE-COUNTER OPTIONS MARKETS

Most of this chapter has focused on exchange-traded options markets. The over-the-counter market for options has become increasingly important since the early 1980s and is now larger than the exchange-traded market. As explained in Chapter 1, in the over-the-counter market, financial institutions, corporate treasurers, and fund managers trade over the phone. There is a wide range of assets underlying the options. Over-the-counter options on foreign exchange and interest rates are particularly popular. The chief potential disadvantage of the over-the-counter market is that option writer may default. This means that the purchaser is subject to some credit risk. In an attempt to overcome this disadvantage, market participants are increasingly requiring counterparties to post collateral. This was discussed in Section 2.4.

The instruments traded in the over-the-counter market are often structured by financial institutions to meet the precise needs of their clients. Sometimes this involves choosing exercise dates, strike prices, and contract sizes that are different from those traded by the exchange. In other cases the structure of the option is different from standard calls and puts. The option is then referred to as an *exotic option*. Chapter 24 describes a number of different types of exotic options.

SUMMARY

There are two types of options: calls and puts. A call option gives the holder the right to buy the underlying asset for a certain price by a certain date. A put option gives the holder the right to sell the underlying asset by a certain date for a certain price. There are four possible positions in options markets: a long position in a call, a short position in a call, a long position in a put, and a short position in a put. Taking a short position in an option is known as writing it. Options are currently traded on stocks, stock indices, foreign currencies, futures contracts, and other assets.

An exchange must specify the terms of the option contracts it trades. In particular, it must specify the size of the contract, the precise expiration time, and the strike price. In the United States one stock option contract gives the holder the right to buy or sell 100 shares. The expiration of a stock option contract is 10:59 p.m. Central Time on the Saturday immediately following the third Friday of the expiration month. Options with several different expiration months trade at any given time. Strike prices are at $\$2\frac{1}{2}$, \$5, or \$10 intervals, depending on the stock price. The strike price is generally fairly close to the stock price when trading in an option begins.

The terms of a stock option are not normally adjusted for cash dividends. However, they are adjusted for stock dividends, stock splits, and rights issues. The aim of the adjustment is to keep the positions of both the writer and the buyer of a contract unchanged.

Most option exchanges use market makers. A market maker is an individual who is prepared to quote both a bid price (at which he or she is prepared to buy) and an offer price (at which he or she is prepared to sell). Market makers improve the liquidity of the market and ensure that there is never any delay in executing market orders. They themselves make a profit from the difference between their bid and offer prices (known as their bid–offer spread). The exchange has rules specifying upper limits for the bid–offer spread.

Writers of options have potential liabilities and are required to maintain margins with their brokers. If it is not a member of the Options Clearing Corporation, the broker will maintain a margin account with a firm that is a member. This firm will in turn maintain a margin account with the Options Clearing Corporation. The Options Clearing Corporation is responsible for keeping a record of all outstanding contracts, handling exercise orders, and so on.

Not all options are traded on exchanges. Many options are traded by phone in the over-the-counter market. An advantage of over-the-counter options is that they can be tailored by a financial institution to meet the particular needs of a corporate treasurer or fund manager.

FURTHER READING

Arzac, E. R. "PERCs, DECs, and Other Mandatory Convertibles," *Journal of Applied Corporate Finance*, 10, 1 (1997): 54–63.

McMillan, L. G. *McMillan on Options*, 2nd edn. New Jersey: Wiley, 2004.

Questions and Problems (Answers in Solutions Manual)

8.1. An investor buys a European put on a share for \$3. The stock price is \$42 and the strike price is \$40. Under what circumstances does the investor make a profit? Under what circumstances will the option be exercised? Draw a diagram showing the variation of the investor's profit with the stock price at the maturity of the option.

8.2. An investor sells a European call on a share for \$4. The stock price is \$47 and the strike price is \$50. Under what circumstances does the investor make a profit? Under what circumstances will the option be exercised? Draw a diagram showing the variation of the investor's profit with the stock price at the maturity of the option.

8.3. An investor sells a European call option with strike price of K and maturity T and buys a put with the same strike price and maturity. Describe the investor's position.

8.4. Explain why brokers require margins when clients write options but not when they buy options.

8.5. A stock option is on a February, May, August, and November cycle. What options trade on (a) April 1 and (b) May 30?

8.6. A company declares a 2-for-1 stock split. Explain how the terms change for a call option with a strike price of \$60.

8.7. "Employee stock options issued by a company are different from regular exchange-traded call options on the company's stock because they can affect the capital structure of the company." Explain this statement.

8.8. A corporate treasurer is designing a hedging program involving foreign currency options. What are the pros and cons of using (a) the Philadelphia Stock Exchange and (b) the over-the-counter market for trading?

8.9. Suppose that a European call option to buy a share for \$100.00 costs \$5.00 and is held until maturity. Under what circumstances will the holder of the option make a profit? Under what circumstances will the option be exercised? Draw a diagram illustrating how the profit from a long position in the option depends on the stock price at maturity of the option.

8.10. Suppose that a European put option to sell a share for \$60 costs \$8 and is held until maturity. Under what circumstances will the seller of the option (the party with the short position) make a profit? Under what circumstances will the option be exercised? Draw a diagram illustrating how the profit from a short position in the option depends on the stock price at maturity of the option.

8.11. Describe the terminal value of the following portfolio: a newly entered-into long forward contract on an asset and a long position in a European put option on the asset with the same maturity as the forward contract and a strike price that is equal to the forward price of the asset at the time the portfolio is set up. Show that the European put option has the same value as a European call option with the same strike price and maturity.

8.12. A trader buys a call option with a strike price of \$45 and a put option with a strike price of \$40. Both options have the same maturity. The call costs \$3 and the put costs \$4. Draw a diagram showing the variation of the trader's profit with the asset price.

8.13. Explain why an American option is always worth at least as much as a European option on the same asset with the same strike price and exercise date.

8.14. Explain why an American option is always worth at least as much as its intrinsic value.

8.15. Explain carefully the difference between writing a put option and buying a call option.

8.16. The treasurer of a corporation is trying to choose between options and forward contracts to hedge the corporation's foreign exchange risk. Discuss the advantages and disadvantages of each.

8.17. Consider an exchange-traded call option contract to buy 500 shares with a strike price of \$40 and maturity in 4 months. Explain how the terms of the option contract change when there is: (a) a 10% stock dividend; (b) a 10% cash dividend; and (c) a 4-for-1 stock split.

8.18. "If most of the call options on a stock are in the money, it is likely that the stock price has risen rapidly in the last few months." Discuss this statement.

8.19. What is the effect of an unexpected cash dividend on (a) a call option price and (b) a put option price?

8.20. Options on General Motors stock are on a March, June, September, and December cycle. What options trade on (a) March 1, (b) June 30, and (c) August 5?

8.21. Explain why the market maker's bid–offer spread represents a real cost to options investors.

8.22. A United States investor writes five naked call option contracts. The option price is \$3.50, the strike price is \$60.00, and the stock price is \$57.00. What is the initial margin requirement?

Assignment Questions

8.23. The price of a stock is \$40. The price of a 1-year European put option on the stock with a strike price of \$30 is quoted as \$7 and the price of a 1-year European call option on the stock with a strike price of \$50 is quoted as \$5. Suppose that an investor buys 100 shares, shorts 100 call options, and buys 100 put options. Draw a diagram illustrating how the investor's profit or loss varies with the stock price over the next year. How does your answer change if the investor buys 100 shares, shorts 200 call options, and buys 200 put options?

8.24. "If a company does not do better than its competitors but the stock market goes up, executives do very well from their stock options. This makes no sense." Discuss this viewpoint. Can you think of alternatives to the usual employee stock option plan that take the viewpoint into account.

8.25. Use DerivaGem to calculate the value of an American put option on a non-dividend-paying stock when the stock price is \$30, the strike price is \$32, the risk-free rate is 5%, the volatility is 30%, and the time to maturity is 1.5 years. (Choose "Binomial American" for the "option type" and 50 time steps.)
(a) What is the option's intrinsic value?
(b) What is the option's time value?
(c) What would a time value of zero indicate? What is the value of an option with zero time value?
(d) Using a trial and error approach, calculate how low the stock price would have to be for the time value of the option to be zero.

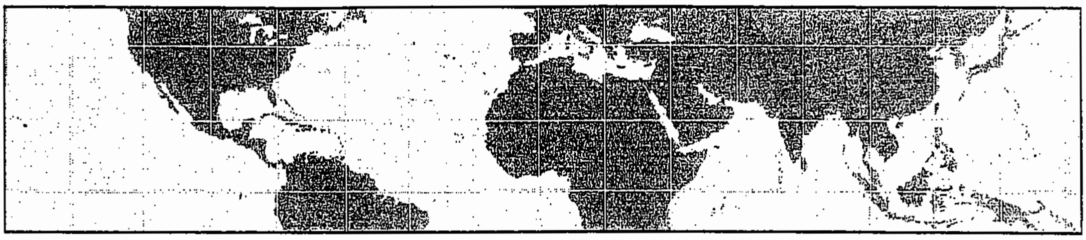
8.26. On July 20, 2004, Microsoft surprised the market by announcing a \$3 dividend. The ex-dividend date was November 17, 2004, and the payment date was December 2, 2004. Its stock price at the time was about \$28. It also changed the terms of its employee stock options so that each exercise price was adjusted downward to

$$\text{Predividend exercise price} \times \frac{\text{Closing price} - \$3.00}{\text{Closing price}}$$

The number of shares covered by each stock option outstanding was adjusted upward to

$$\text{Number of shares predividend} \times \frac{\text{Closing price}}{\text{Closing price} - \$3.00}$$

"Closing Price" means the official NASDAQ closing price of a share of Microsoft common stock on the last trading day before the ex-dividend date. Evaluate this adjustment. Compare it with the system used by exchanges to adjust for extraordinary dividends (see Business Snapshot 8.1).



CHAPTER 9 Properties of Stock Options

In this chapter we look at the factors affecting stock option prices. We use a number of different arbitrage arguments to explore the relationships between European option prices, American option prices, and the underlying stock price. The most important of these relationships is put–call parity, which is a relationship between the price of a European call option, the price of a European put option, and the underlying stock price.

The chapter examines whether American options should be exercised early. It shows that it is never optimal to exercise an American call option on a non-dividend-paying stock prior to the option's expiration, but that under some circumstances the early exercise of an American put option on such a stock is optimal.

9.1 FACTORS AFFECTING OPTION PRICES

There are six factors affecting the price of a stock option:

1. The current stock price, S_0
2. The strike price, K
3. The time to expiration, T
4. The volatility of the stock price, σ
5. The risk-free interest rate, r
6. The dividends expected during the life of the option

In this section we consider what happens to option prices when one of these factors changes, with all the others remaining fixed. The results are summarized in Table 9.1.

Figures 9.1 and 9.2 show how European call and put prices depend on the first five factors in the situation where $S_0 = 50$, $K = 50$, $r = 5\%$ per annum, $\sigma = 30\%$ per annum, $T = 1$ year, and there are no dividends. In this case the call price is 7.116 and the put price is 4.677.

Stock Price and Strike Price

If a call option is exercised at some future time, the payoff will be the amount by which the stock price exceeds the strike price. Call options therefore become more valuable as

Table 9.1 Summary of the effect on the price of a stock option of increasing one variable while keeping all others fixed.*

Variable	*European call*	*European put*	*American call*	*American put*
Current stock price	+	−	+	−
Strike price	−	+	−	+
Time to expiration	?	?	+	+
Volatility	+	+	+	+
Risk-free rate	+	−	+	−
Amount of future dividends	−	+	−	+

* + indicates that an increase in the variable causes the option price to increase; − indicates that an increase in the variable causes the option price to decrease; ? indicates that the relationship is uncertain.

the stock price increases and less valuable as the strike price increases. For a put option, the payoff on exercise is the amount by which the strike price exceeds the stock price. Put options therefore behave in the opposite way from call options: they become less valuable as the stock price increases and more valuable as the strike price increases. Figures 9.1(a–d) illustrate the way in which put and call prices depend on the stock price and strike price.

Time to Expiration

Now consider the effect of the expiration date. Both put and call American options become more valuable (or at least do not decrease in value) as the time to expiration increases. Consider two American options that differ only as far as the expiration date is concerned. The owner of the long-life option has all the exercise opportunities open to the owner of the short-life option—and more. The long-life option must therefore always be worth at least as much as the short-life option.

Although European put and call options usually become more valuable as the time to expiration increases (see, e.g., Figures 9.1(e, f)), this is not always the case. Consider two European call options on a stock: one with an expiration date in 1 month, the other with an expiration date in 2 months. Suppose that a very large dividend is expected in 6 weeks. The dividend will cause the stock price to decline, so that the short-life option could be worth more than the long-life option.

Volatility

The precise way in which volatility is defined is discussed in Chapter 13. Roughly speaking, the *volatility* of a stock price is a measure of how uncertain we are about future stock price movements. As volatility increases, the chance that the stock will do very well or very poorly increases. For the owner of a stock, these two outcomes tend to offset each other. However, this is not so for the owner of a call or put. The owner of a call benefits from price increases but has limited downside risk in the event of price decreases because the most the owner can lose is the price of the option. Similarly, the owner of a put benefits from price decreases, but has limited downside risk in the event of price increases. The values of both calls and puts therefore increase as volatility increases (see Figures 9.2(a, b)).

Risk-Free Interest Rate

The risk-free interest rate affects the price of an option in a less clear-cut way. As interest rates in the economy increase, the expected return required by investors from the stock

Figure 9.1 Effect of changes in stock price, strike price, and expiration date on option prices when $S_0 = 50$, $K = 50$, $r = 5\%$, $\sigma = 30\%$, and $T = 1$.

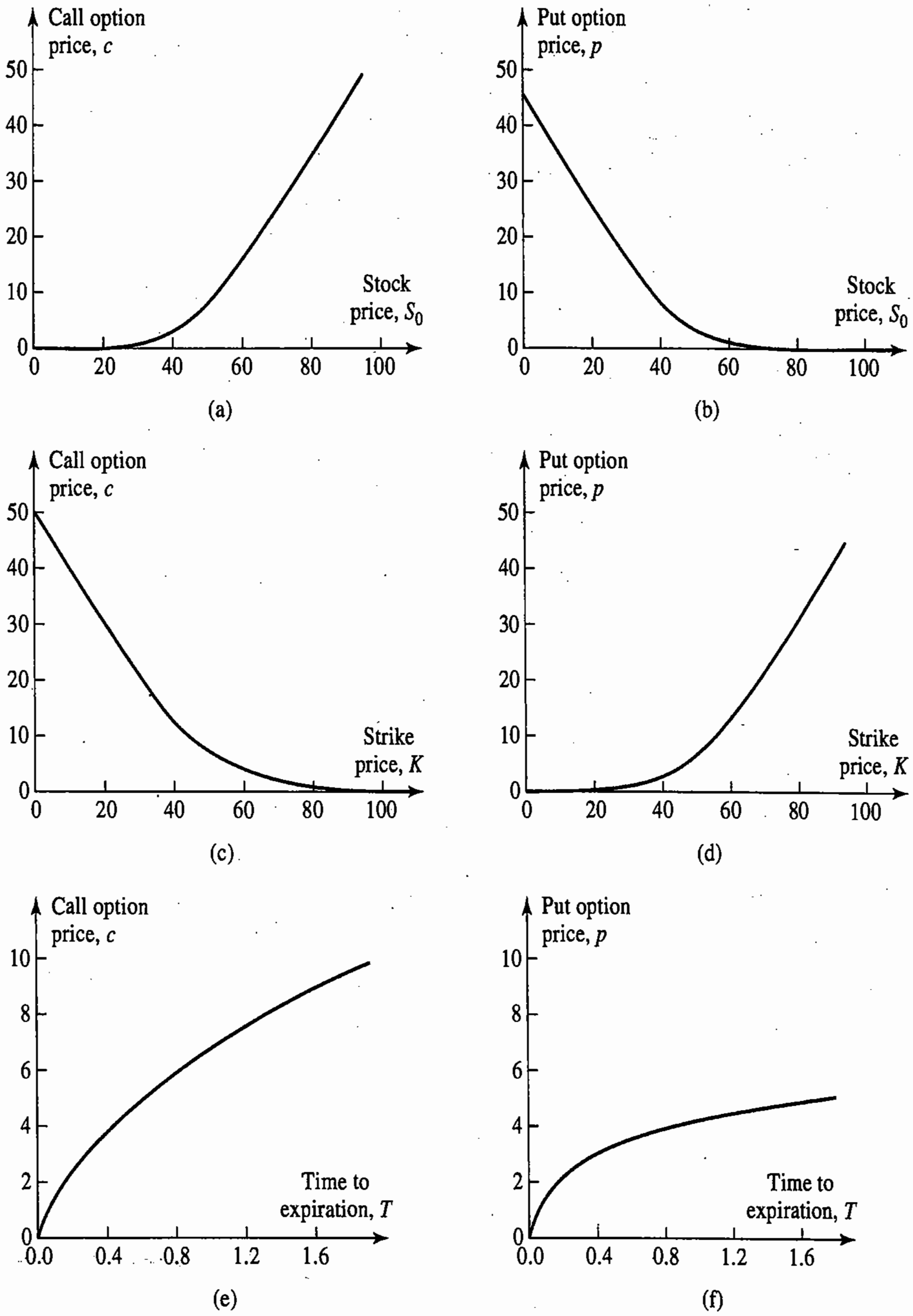
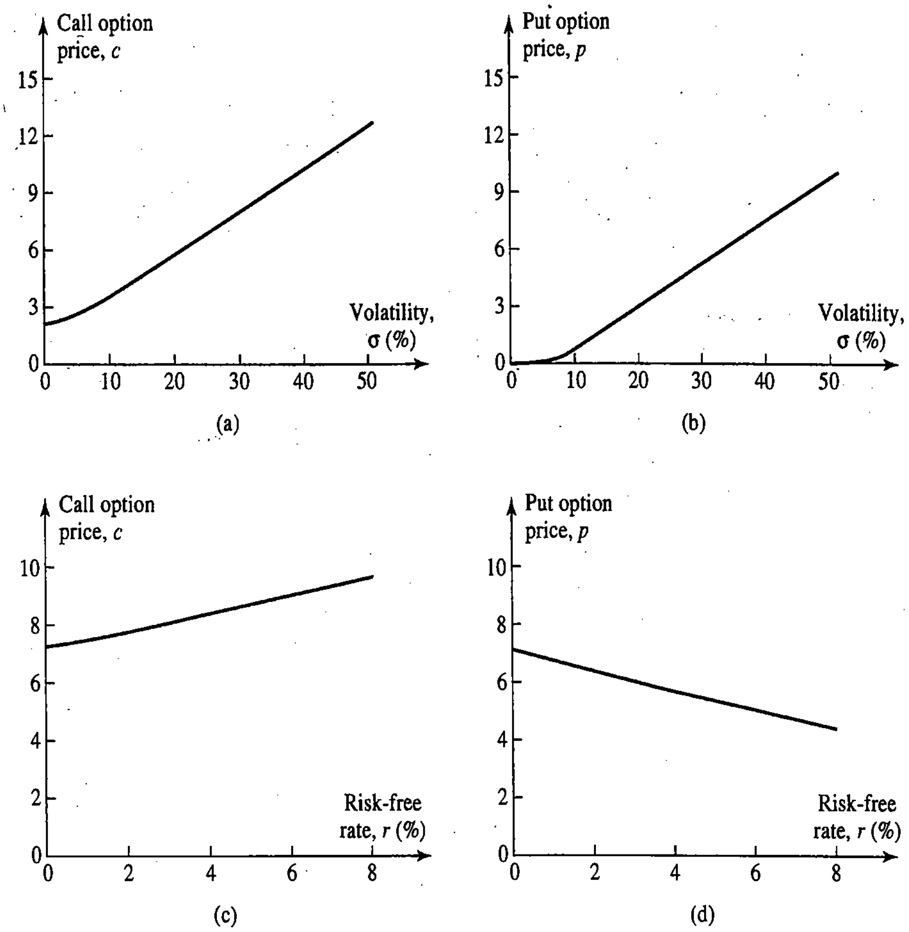


Figure 9.2 Effect of changes in volatility and risk-free interest rate on option prices when $S_0 = 50$, $K = 50$, $r = 5\%$, $\sigma = 30\%$, and $T = 1$.



tends to increase. In addition, the present value of any future cash flow received by the holder of the option decreases. The combined impact of these two effects is to increase the value of call options and decrease the value of put options (see Figures 9.2(c, d)).

It is important to emphasize that we are assuming that interest rates change while all other variables stay the same. In particular we are assuming in Table 9.1 that interest rates change while the stock price remains the same. In practice, when interest rates rise (fall), stock prices tend to fall (rise). The net effect of an interest rate increase and the accompanying stock price decrease can be to decrease the value of a call option and increase the value of a put option. Similarly, the net effect of an interest rate decrease and the accompanying stock price increase can be to increase the value of a call option and decrease the value of a put option.

Amount of Future Dividends

Dividends have the effect of reducing the stock price on the ex-dividend date. This is bad news for the value of call options and good news for the value of put options. The value of a call option is therefore negatively related to the size of an anticipated future dividend, and the value of a put option is positively related to the size of an anticipated future dividend.

9.2 ASSUMPTIONS AND NOTATION

In this chapter we will make assumptions similar to those made for deriving forward and futures prices in Chapter 5. We assume that there are some market participants, such as large investment banks, for which the following statements are true:

1. There are no transactions costs.
2. All trading profits (net of trading losses) are subject to the same tax rate.
3. Borrowing and lending are possible at the risk-free interest rate.

We assume that these market participants are prepared to take advantage of arbitrage opportunities as they arise. As discussed in Chapters 1 and 5, this means that any available arbitrage opportunities disappear very quickly. For the purposes of our analysis, it is therefore reasonable to assume that there are no arbitrage opportunities.

We will use the following notation:

S_0: Current stock price
K: Strike price of option
T: Time to expiration of option
S_T: Stock price on the expiration date
r: Continuously compounded risk-free rate of interest for an investment maturing in time T
C: Value of American call option to buy one share
P: Value of American put option to sell one share
c: Value of European call option to buy one share
p: Value of European put option to sell one share

It should be noted that r is the nominal rate of interest, not the real rate of interest. We can assume that $r > 0$. Otherwise, a risk-free investment would provide no advantages over cash. (Indeed, if $r < 0$, cash would be preferable to a risk-free investment.)

9.3 UPPER AND LOWER BOUNDS FOR OPTION PRICES

In this section we derive upper and lower bounds for option prices. These bounds do not depend on any particular assumptions about the factors mentioned in Section 9.1 (except $r > 0$). If an option price is above the upper bound or below the lower bound, then there are profitable opportunities for arbitrageurs.

Upper Bounds

An American or European call option gives the holder the right to buy one share of a stock for a certain price. No matter what happens, the option can never be worth more than the stock. Hence, the stock price is an upper bound to the option price:

$$c \leqslant S_0 \quad \text{and} \quad C \leqslant S_0$$

If these relationships were not true, an arbitrageur could easily make a riskless profit by buying the stock and selling the call option.

An American or European put option gives the holder the right to sell one share of a stock for K. No matter how low the stock price becomes, the option can never be worth more than K. Hence,

$$p \leqslant K \quad \text{and} \quad P \leqslant K$$

For European options, we know that at maturity the option cannot be worth more than K. It follows that it cannot be worth more than the present value of K today:

$$p \leqslant Ke^{-rT}$$

If this were not true, an arbitrageur could make a riskless profit by writing the option and investing the proceeds of the sale at the risk-free interest rate.

Lower Bound for Calls on Non-Dividend-Paying Stocks

A lower bound for the price of a European call option on a non-dividend-paying stock is

$$S_0 - Ke^{-rT}$$

We first look at a numerical example and then consider a more formal argument.

Suppose that $S_0 = \$20$, $K = \$18$, $r = 10\%$ per annum, and $T = 1$ year. In this case,

$$S_0 - Ke^{-rT} = 20 - 18e^{-0.1} = 3.71$$

or \$3.71. Consider the situation where the European call price is \$3.00, which is less than the theoretical minimum of \$3.71. An arbitrageur can short the stock and buy the call to provide a cash inflow of \$20.00 − \$3.00 = \$17.00. If invested for 1 year at 10% per annum, the \$17.00 grows to $17e^{0.1} = \$18.79$. At the end of the year, the option expires. If the stock price is greater than \$18.00, the arbitrageur exercises the option for \$18.00, closes out the short position, and makes a profit of

$$\$18.79 - \$18.00 = \$0.79$$

If the stock price is less than \$18.00, the stock is bought in the market and the short position is closed out. The arbitrageur then makes an even greater profit. For example, if the stock price is \$17.00, the arbitrageur's profit is

$$\$18.79 - \$17.00 = \$1.79$$

For a more formal argument, we consider the following two portfolios:

Portfolio A: one European call option plus an amount of cash equal to Ke^{-rT}

Portfolio B: one share

In portfolio A, the cash, if it is invested at the risk-free interest rate, will grow to K in time T. If $S_T > K$, the call option is exercised at maturity and portfolio A is worth S_T. If $S_T < K$, the call option expires worthless and the portfolio is worth K. Hence, at time T, portfolio A is worth

$$\max(S_T, K)$$

Portfolio B is worth S_T at time T. Hence, portfolio A is always worth as much as, and can be worth more than, portfolio B at the option's maturity. It follows that in the absence of arbitrage opportunities this must also be true today. Hence,

$$c + Ke^{-rT} \geqslant S_0$$

or

$$c \geqslant S_0 - Ke^{-rT}$$

Because the worst that can happen to a call option is that it expires worthless, its value cannot be negative. This means that $c \geqslant 0$ and therefore

$$c \geqslant \max(S_0 - Ke^{-rT}, 0) \tag{9.1}$$

Example 9.1

Consider a European call option on a non-dividend-paying stock when the stock price is \$51, the strike price is \$50, the time to maturity is 6 months, and the risk-free rate of interest is 12% per annum. In this case, $S_0 = 51$, $K = 50$, $T = 0.5$, and $r = 0.12$. From equation (9.1), a lower bound for the option price is $S_0 - Ke^{-rT}$, or

$$51 - 50e^{-0.12\times 0.5} = \$3.91$$

Lower Bound for European Puts on Non-Dividend-Paying Stocks

For a European put option on a non-dividend-paying stock, a lower bound for the price is

$$Ke^{-rT} - S_0$$

Again, we first consider a numerical example and then look at a more formal argument.

Suppose that $S_0 = \$37$, $K = \$40$, $r = 5\%$ per annum, and $T = 0.5$ years. In this case,

$$Ke^{-rT} - S_0 = 40e^{-0.05\times 0.5} - 37 = \$2.01$$

Consider the situation where the European put price is \$1.00, which is less than the theoretical minimum of \$2.01. An arbitrageur can borrow \$38.00 for 6 months to buy both the put and the stock. At the end of the 6 months, the arbitrageur will be required to repay $38e^{0.05\times 0.5} = \$38.96$. If the stock price is below \$40.00, the arbitrageur exercises the option to sell the stock for \$40.00, repays the loan, and makes a profit of

$$\$40.00 - \$38.96 = \$1.04$$

If the stock price is greater than \$40.00, the arbitrageur discards the option, sells the stock, and repays the loan for an even greater profit. For example, if the stock price is \$42.00, the arbitrageur's profit is

$$\$42.00 - \$38.96 = \$3.04$$

For a more formal argument, we consider the following two portfolios:

Portfolio C: one European put option plus one share

Portfolio D: an amount of cash equal to Ke^{-rT}

If $S_T < K$, then the option in portfolio C is exercised at option maturity, and the portfolio becomes worth K. If $S_T > K$, then the put option expires worthless, and the portfolio is worth S_T at this time. Hence, portfolio C is worth

$$\max(S_T, K)$$

in time T. Assuming the cash is invested at the risk-free interest rate, portfolio D is worth K in time T. Hence, portfolio C is always worth as much as, and can sometimes be worth more than, portfolio D in time T. It follows that in the absence of arbitrage opportunities portfolio C must be worth at least as much as portfolio D today. Hence,

$$p + S_0 \geqslant Ke^{-rT}$$

or

$$p \geqslant Ke^{-rT} - S_0$$

Because the worst that can happen to a put option is that it expires worthless, its value cannot be negative. This means that

$$p \geqslant \max(Ke^{-rT} - S_0, 0) \tag{9.2}$$

Example 9.2

Consider a European put option on a non-dividend-paying stock when the stock price is \$38, the strike price is \$40, the time to maturity is 3 months, and the risk-free rate of interest is 10% per annum. In this case $S_0 = 38$, $K = 40$, $T = 0.25$, and $r = 0.10$. From equation (9.2), a lower bound for the option price is $Ke^{-rT} - S_0$, or

$$40e^{-0.1\times0.25} - 38 = \$1.01$$

9.4 PUT–CALL PARITY

We now derive an important relationship between p and c. Consider the following two portfolios that were used in the previous section:

Portfolio A: one European call option plus an amount of cash equal to Ke^{-rT}

Portfolio C: one European put option plus one share

Both are worth

$$\max(S_T, K)$$

at expiration of the options. Because the options are European, they cannot be exercised prior to the expiration date. The portfolios must therefore have identical values today. This means that

$$c + Ke^{-rT} = p + S_0 \tag{9.3}$$

This relationship is known as *put–call parity*. It shows that the value of a European call with a certain strike price and exercise date can be deduced from the value of a European put with the same strike price and exercise date, and vice versa.

If equation (9.3) does not hold, there are arbitrage opportunities. Suppose that the stock price is \$31, the strike price is \$30, the risk-free interest rate is 10% per annum, the price of a 3-month European call option is \$3, and the price of a three-month European put option is \$2.25. In this case,

$$c + Ke^{-rT} = 3 + 30e^{-0.1\times3/12} = \$32.26$$

and

$$p + S_0 = 2.25 + 31 = \$33.25$$

Portfolio C is overpriced relative to portfolio A. An arbitrageur can buy the securities in portfolio A and short the securities in portfolio C. The strategy involves buying the call and shorting both the put and the stock, generating a positive cash flow of

$$-3 + 2.25 + 31 = \$30.25$$

up front. When invested at the risk-free interest rate, this amount grows to

$$30.25e^{0.1\times 0.25} = \$31.02$$

in 3 months. If the stock price at expiration of the option is greater than \$30, the call will be exercised; and if it is less than \$30, the put will be exercised. In either case, the arbitrageur ends up buying one share for \$30. This share can be used to close out the short position. The net profit is therefore

$$\$31.02 - \$30.00 = \$1.02$$

For an alternative situation, suppose that the call price is \$3 and the put price is \$1. In this case,

$$c + Ke^{-rT} = 3 + 30e^{-0.1\times 3/12} = \$32.26$$

and

$$p + S_0 = 1 + 31 = \$32.00$$

Portfolio A is overpriced relative to portfolio C. An arbitrageur can short the securities in portfolio A and buy the securities in portfolio C to lock in a profit. The strategy involves

Table 9.2 Arbitrage opportunities when put–call parity does not hold. Stock price = \$31; interest rate = 10%; call price = \$3. Both put and call have a strike price of \$30 and 3 months to maturity.

Three-month put price = \$2.25	***Three-month put price = \$1***
Action now:	*Action now*:
Buy call for \$3	Borrow \$29 for 3 months
Short put to realize \$2.25	Short call to realize \$3
Short the stock to realize \$31	Buy put for \$1
Invest \$30.25 for 3 months	Buy the stock for \$31
Action in 3 months if $S_T > 30$:	*Action in 3 months if* $S_T > 30$:
Receive \$31.02 from investment	Call exercised: sell stock for \$30
Exercise call to buy stock for \$30	Use \$29.73 to repay loan
Net profit = \$1.02	Net profit = \$0.27
Action in 3 months if $S_T < 30$:	*Action in 3 months if* $S_T < 30$:
Receive \$31.02 from investment	Exercise put to sell stock for \$30
Put exercised: buy stock for \$30	Use \$29.73 to repay loan
Net profit = \$1.02	Net profit = \$0.27

Business Snapshot 9.1 Put–Call Parity and Capital Structure

The pioneers of option pricing were Fischer Black, Myron Scholes, and Robert Merton. In the early 1970s, they showed that options can be used to characterize the capital structure of a company. Today their analysis is widely used by financial institutions to assess a company's credit risk.

To illustrate the analysis, consider a company that has assets that are financed with zero-coupon bonds and equity. Suppose that the bonds mature in 5 years at which time a principal payment of K is required. The company pays no dividends. If the assets are worth more than K in 5 years, the equity holders choose to repay the bondholders. If the assets are worth less than K, the equity holders choose to declare bankruptcy and the bondholders end up owning the company.

The value of the equity in 5 years is therefore $\max(A_T - K, 0)$, where A_T is the value of the company's assets at that time. This shows that the equity holders have a 5-year European call option on the assets of the company with a strike price of K. What about the bondholders? They get $\min(A_T, K)$ in 5 years. This is the same as $K - \max(K - A_T, 0)$. This shows that today the bonds are worth the present value of K minus the value of a 5-year European put option on the assets with a strike price of K.

To summarize, if c and p are the value of the call and put options on the company's assets at time T, then

$$\text{Value of equity} = c$$

$$\text{Value of debt} = PV(K) - p$$

Denote the value of the assets of the company today by A_0. The value of the assets must equal the total value of the instruments used to finance the assets. This means that it must equal the sum of the value of the equity and the value of the debt, so that

$$A_0 = c + [PV(K) - p]$$

Rearranging this equation, we have

$$c + PV(K) = p + A_0$$

This is the put–call parity result in equation (9.3) for call and put options on the assets of the company.

shorting the call and buying both the put and the stock with an initial investment of

$$\$31 + \$1 - \$3 = \$29$$

When the investment is financed at the risk-free interest rate, a repayment of $29e^{0.1\times 0.25} =$ \$29.73 is required at the end of the 3 months. As in the previous case, either the call or the put will be exercised. The short call and long put option position therefore leads to the stock being sold for \$30.00. The net profit is therefore

$$\$30.00 - \$29.73 = \$0.27$$

These examples are illustrated in Table 9.2. Business Snapshot 9.1 shows how options and put–call parity can help us understand the positions of the debt and equity holders in a company.

American Options

Put–call parity holds only for European options. However, it is possible to derive some results for American option prices. It can be shown (see Problem 9.18) that, when there are no dividends,

$$S_0 - K \leqslant C - P \leqslant S_0 - Ke^{-rT} \tag{9.4}$$

Example 9.3

An American call option on a non-dividend-paying stock with strike price \$20.00 and maturity in 5 months is worth \$1.50. Suppose that the current stock price is \$19.00 and the risk-free interest rate is 10% per annum. From equation (9.4), we have

$$19 - 20 \leqslant C - P \leqslant 19 - 20e^{-0.1\times 5/12}$$

or

$$1 \geqslant P - C \geqslant 0.18$$

showing that $P - C$ lies between \$1.00 and \$0.18. With C at \$1.50, P must lie between \$1.68 and \$2.50. In other words, upper and lower bounds for the price of an American put with the same strike price and expiration date as the American call are \$2.50 and \$1.68.

9.5 EARLY EXERCISE: CALLS ON A NON-DIVIDEND-PAYING STOCK

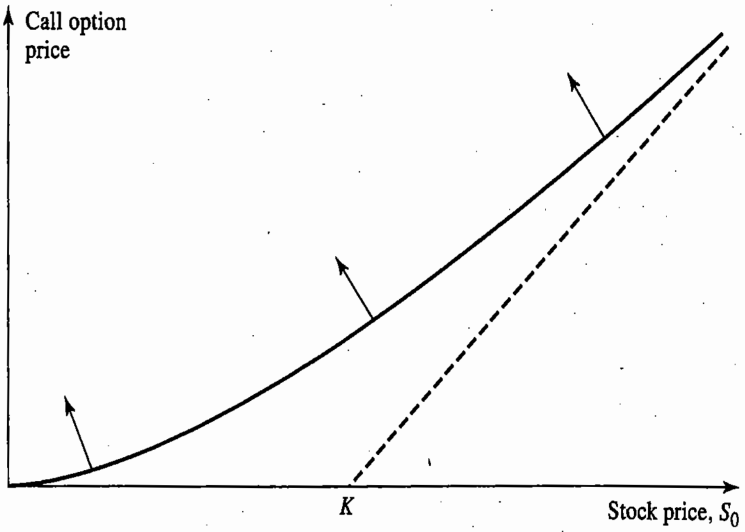
This section demonstrates that it is never optimal to exercise an American call option on a non-dividend-paying stock before the expiration date.

To illustrate the general nature of the argument, consider an American call option on a non-dividend-paying stock with 1 month to expiration when the stock price is \$50 and the strike price is \$40. The option is deep in the money, and the investor who owns the option might well be tempted to exercise it immediately. However, if the investor plans to hold the stock obtained by exercising the option for more than 1 month, this is not the best strategy. A better course of action is to keep the option and exercise it at the end of the month. The \$40 strike price is then paid out 1 month later than it would be if the option were exercised immediately, so that interest is earned on the \$40 for 1 month. Because the stock pays no dividends, no income from the stock is sacrificed. A further advantage of waiting rather than exercising immediately is that there is some chance (however remote) that the stock price will fall below \$40 in 1 month. In this case, the investor will not exercise in 1 month and will be glad that the decision to exercise early was not taken!

This argument shows that there are no advantages to exercising early if the investor plans to keep the stock for the remaining life of the option (1 month, in this case). What if the investor thinks the stock is currently overpriced and is wondering whether to exercise the option and sell the stock? In this case, the investor is better off selling the option than exercising it.[1] The option will be bought by another investor who does want to hold the stock. Such investors must exist: otherwise the current stock price would not be \$50. The price obtained for the option will be greater than its intrinsic value of \$10, for the reasons mentioned earlier.

[1] As an alternative strategy, the investor can keep the option and short the stock to lock in a better profit than \$10.

Figure 9.3 Variation of price of an American or European call option on a non-dividend-paying stock with the stock price, S_0.



For a more formal argument, we can use equation (9.1):

$$c \geqslant S_0 - Ke^{-rT}$$

Because the owner of an American call has all the exercise opportunities open to the owner of the corresponding European call, we must have

$$C \geqslant c$$

Hence,

$$C \geqslant S_0 - Ke^{-rT}$$

Given $r > 0$, it follows that $C > S_0 - K$. If it were optimal to exercise early, C would equal $S_0 - K$. We deduce that it can never be optimal to exercise early.

Figure 9.3 shows the general way in which the call price varies with S_0. It indicates that the call price is always above its intrinsic value of $\max(S_0 - K, 0)$. As r or T or the volatility increases, the line relating the call price to the stock price moves in the direction indicated by the arrows (i.e., farther away from the intrinsic value).

To summarize, there are two reasons an American call on a non-dividend-paying stock should not be exercised early. One relates to the insurance that it provides. A call option, when held instead of the stock itself, in effect insures the holder against the stock price falling below the strike price. Once the option has been exercised and the strike price has been exchanged for the stock price, this insurance vanishes. The other reason concerns the time value of money. From the perspective of the option holder, the later the strike price is paid out, the better.

9.6 EARLY EXERCISE: PUTS ON A NON-DIVIDEND-PAYING STOCK

It can be optimal to exercise an American put option on a non-dividend-paying stock early. Indeed, at any given time during its life, a put option should always be exercised early if it is sufficiently deep in the money.

To illustrate this, consider an extreme situation. Suppose that the strike price is \$10 and the stock price is virtually zero. By exercising immediately, an investor makes an immediate gain of \$10. If the investor waits, the gain from exercise might be less than \$10, but it cannot be more than \$10 because negative stock prices are impossible. Furthermore, receiving \$10 now is preferable to receiving \$10 in the future. It follows that the option should be exercised immediately.

Like a call option, a put option can be viewed as providing insurance. A put option, when held in conjunction with the stock, insures the holder against the stock price falling below a certain level. However, a put option is different from a call option in that it may be optimal for an investor to forgo this insurance and exercise early in order to realize the strike price immediately. In general, the early exercise of a put option becomes more attractive as S_0 decreases, as r increases, and as the volatility decreases.

It will be recalled from equation (9.2) that

$$p \geqslant Ke^{-rT} - S_0$$

For an American put with price P, the stronger condition

$$P \geqslant K - S_0$$

must always hold because immediate exercise is always possible.

Figure 9.4 shows the general way in which the price of an American put varies with S_0. Provided that $r > 0$, it is always optimal to exercise an American put immediately when the stock price is sufficiently low. When early exercise is optimal, the value of the option is $K - S_0$. The curve representing the value of the put therefore merges into the put's intrinsic value, $K - S_0$, for a sufficiently small value of S_0. In Figure 9.4, this value of S_0 is shown as point A. The line relating the put price to the stock price moves in the direction indicated by the arrows when r decreases, when the volatility increases, and when T increases.

Because there are some circumstances when it is desirable to exercise an American put option early, it follows that an American put option is always worth more than the

Figure 9.4 Variation of price of an American put option with stock price, S_0.

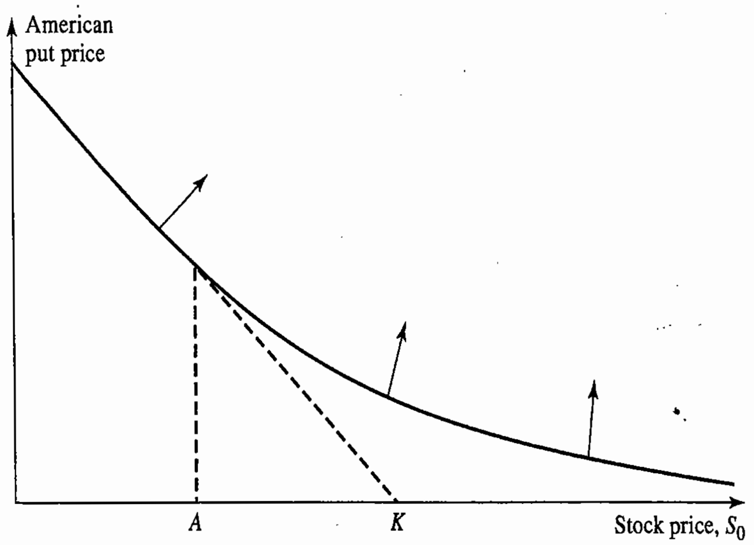
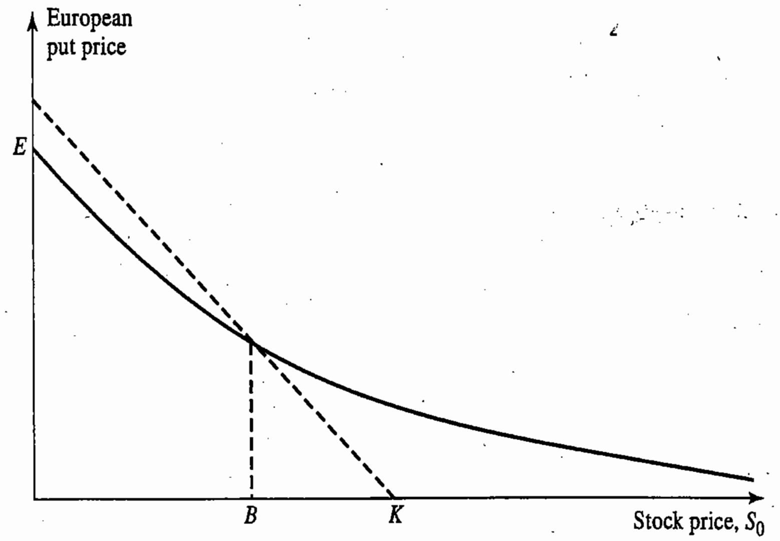


Figure 9.5 Variation of price of a European put option with the stock price, S_0.



corresponding European put option. Furthermore, because an American put is sometimes worth its intrinsic value (see Figure 9.4), it follows that a European put option must sometimes be worth less than its intrinsic value. Figure 9.5 shows the variation of the European put price with the stock price. Note that point B in Figure 9.5, at which the price of the option is equal to its intrinsic value, must represent a higher value of the stock price than point A in Figure 9.4. Point E in Figure 9.5 is where $S_0 = 0$ and the European put price is Ke^{-rT}.

9.7 EFFECT OF DIVIDENDS

The results produced so far in this chapter have assumed that we are dealing with options on a non-dividend-paying stock. In this section we examine the impact of dividends. In the United States most exchange-traded stock options have a life of less than 1 year and dividends payable during the life of the option can usually be predicted with reasonable accuracy. We will use D to denote the present value of the dividends during the life of the option. In the calculation of D, a dividend is assumed to occur at the time of its ex-dividend date.

Lower Bound for Calls and Puts

We can redefine portfolios A and B as follows:

Portfolio A: one European call option plus an amount of cash equal to $D + Ke^{-rT}$

Portfolio B: one share

A similar argument to the one used to derive equation (9.1) shows that

$$c \geqslant S_0 - D - Ke^{-rT} \tag{9.5}$$

We can also redefine portfolios C and D as follows:

Portfolio C: one European put option plus one share

Portfolio D: an amount of cash equal to $D + Ke^{-rT}$

A similar argument to the one used to derive equation (9.2) shows that

$$p \geqslant D + Ke^{-rT} - S_0 \tag{9.6}$$

Early Exercise

When dividends are expected, we can no longer assert that an American call option will not be exercised early. Sometimes it is optimal to exercise an American call immediately prior to an ex-dividend date. It is never optimal to exercise a call at other times. This point is discussed further in Section 13.12.

Put–Call Parity

Comparing the value at option maturity of the redefined portfolios A and C shows that, with dividends, the put–call parity result in equation (9.3) becomes

$$c + D + Ke^{-rT} = p + S_0 \tag{9.7}$$

Dividends cause equation (9.4) to be modified (see Problem 9.19) to

$$S_0 - D - K \leqslant C - P \leqslant S_0 - Ke^{-rT} \tag{9.8}$$

SUMMARY

There are six factors affecting the value of a stock option: the current stock price, the strike price, the expiration date, the stock price volatility, the risk-free interest rate, and the dividends expected during the life of the option. The value of a call generally increases as the current stock price, the time to expiration, the volatility, and the risk-free interest rate increase. The value of a call decreases as the strike price and expected dividends increase. The value of a put generally increases as the strike price, the time to expiration, the volatility, and the expected dividends increase. The value of a put decreases as the current stock price and the risk-free interest rate increase.

It is possible to reach some conclusions about the value of stock options without making any assumptions about the volatility of stock prices. For example, the price of a call option on a stock must always be worth less than the price of the stock itself. Similarly, the price of a put option on a stock must always be worth less than the option's strike price.

A European call option on a non-dividend-paying stock must be worth more than

$$\max(S_0 - Ke^{-rT}, 0)$$

where S_0 is the stock price, K is the strike price, r is the risk-free interest rate, and T is the time to expiration. A European put option on a non-dividend-paying stock must be worth more than

$$\max(Ke^{-rT} - S_0, 0)$$

When dividends with present value D will be paid, the lower bound for a European call option becomes

$$\max(S_0 - D - Ke^{-rT}, 0)$$

and the lower bound for a European put option becomes

$$\max(Ke^{-rT} + D - S_0, 0)$$

Put–call parity is a relationship between the price, c, of a European call option on a stock and the price, p, of a European put option on a stock. For a non-dividend-paying stock, it is

$$c + Ke^{-rT} = p + S_0$$

For a dividend-paying stock, the put–call parity relationship is

$$c + D + Ke^{-rT} = p + S_0$$

Put–call parity does not hold for American options. However, it is possible to use arbitrage arguments to obtain upper and lower bounds for the difference between the price of an American call and the price of an American put.

In Chapter 13, we will carry the analyses in this chapter further by making specific assumptions about the probabilistic behavior of stock prices. The analysis will enable us to derive exact pricing formulas for European stock options. In Chapters 11 and 19, we will see how numerical procedures can be used to price American options.

FURTHER READING

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Questions and Problems (Answers in Solutions Manual)

9.1. List the six factors that affect stock option prices.

9.2. What is a lower bound for the price of a 4-month call option on a non-dividend-paying stock when the stock price is \$28, the strike price is \$25, and the risk-free interest rate is 8% per annum?

9.3. What is a lower bound for the price of a 1-month European put option on a non-dividend-paying stock when the stock price is \$12, the strike price is \$15, and the risk-free interest rate is 6% per annum?

9.4. Give two reasons why the early exercise of an American call option on a non-dividend-paying stock is not optimal. The first reason should involve the time value of money. The second should apply even if interest rates are zero.

9.5. "The early exercise of an American put is a trade-off between the time value of money and the insurance value of a put." Explain this statement.

9.6. Explain why an American call option on a dividend-paying stock is always worth at least as much as its intrinsic value. Is the same true of a European call option? Explain your answer.

9.7. The price of a non-dividend-paying stock is \$19 and the price of a 3-month European call option on the stock with a strike price of \$20 is \$1. The risk-free rate is 4% per annum. What is the price of a 3-month European put option with a strike price of \$20?

9.8. Explain why the arguments leading to put–call parity for European options cannot be used to give a similar result for American options.

9.9. What is a lower bound for the price of a 6-month call option on a non-dividend-paying stock when the stock price is \$80, the strike price is \$75, and the risk-free interest rate is 10% per annum?

9.10. What is a lower bound for the price of a 2-month European put option on a non-dividend-paying stock when the stock price is \$58, the strike price is \$65, and the risk-free interest rate is 5% per annum?

9.11. A 4-month European call option on a dividend-paying stock is currently selling for \$5. The stock price is \$64, the strike price is \$60, and a dividend of \$0.80 is expected in 1 month. The risk-free interest rate is 12% per annum for all maturities. What opportunities are there for an arbitrageur?

9.12. A 1-month European put option on a non-dividend-paying stock is currently selling for \$2.50. The stock price is \$47, the strike price is \$50, and the risk-free interest rate is 6% per annum. What opportunities are there for an arbitrageur?

9.13. Give an intuitive explanation of why the early exercise of an American put becomes more attractive as the risk-free rate increases and volatility decreases.

9.14. The price of a European call that expires in 6 months and has a strike price of \$30 is \$2. The underlying stock price is \$29, and a dividend of \$0.50 is expected in 2 months and again in 5 months. The term structure is flat, with all risk-free interest rates being 10%. What is the price of a European put option that expires in 6 months and has a strike price of \$30?

9.15. Explain carefully the arbitrage opportunities in Problem 9.14 if the European put price is \$3.

9.16. The price of an American call on a non-dividend-paying stock is \$4. The stock price is \$31, the strike price is \$30, and the expiration date is in 3 months. The risk-free interest rate is 8%. Derive upper and lower bounds for the price of an American put on the same stock with the same strike price and expiration date.

9.17. Explain carefully the arbitrage opportunities in Problem 9.16 if the American put price is greater than the calculated upper bound.

9.18. Prove the result in equation (9.4). (*Hint*: For the first part of the relationship, consider (a) a portfolio consisting of a European call plus an amount of cash equal to K, and (b) a portfolio consisting of an American put option plus one share.)

9.19. Prove the result in equation (9.8). (*Hint*: For the first part of the relationship, consider (a) a portfolio consisting of a European call plus an amount of cash equal to $D+K$, and (b) a portfolio consisting of an American put option plus one share.)

9.20. Consider a 5-year employee stock option on a non-dividend-paying stock. The option can be exercised at any time after the end of the first year. Unlike a regular exchange-traded call option, the employee stock option cannot be sold. What is the likely impact of this restriction on the early-exercise decision?

9.21. Use the software DerivaGem to verify that Figures 9.1 and 9.2 are correct.

Assignment Questions

9.22. A European call option and put option on a stock both have a strike price of \$20 and an expiration date in 3 months. Both sell for \$3. The risk-free interest rate is 10% per annum, the current stock price is \$19, and a \$1 dividend is expected in 1 month. Identify the arbitrage opportunity open to a trader.

9.23. Suppose that c_1, c_2, and c_3 are the prices of European call options with strike prices K_1, K_2, and K_3, respectively, where $K_3 > K_2 > K_1$ and $K_3 - K_2 = K_2 - K_1$. All options have the same maturity. Show that

$$c_2 \leqslant 0.5(c_1 + c_3)$$

(*Hint*: Consider a portfolio that is long one option with strike price K_1, long one option with strike price K_3, and short two options with strike price K_2.)

9.24. What is the result corresponding to that in Problem 9.23 for European put options?

9.25. Suppose that you are the manager and sole owner of a highly leveraged company. All the debt will mature in 1 year. If at that time the value of the company is greater than the face value of the debt, you will pay off the debt. If the value of the company is less than the face value of the debt, you will declare bankruptcy and the debt holders will own the company.
(a) Express your position as an option on the value of the company.
(b) Express the position of the debt holders in terms of options on the value of the company.
(c) What can you do to increase the value of your position?

9.26. Consider an option on a stock when the stock price is \$41, the strike price is \$40, the risk-free rate is 6%, the volatility is 35%, and the time to maturity is 1 year. Assume that a dividend of \$0.50 is expected after 6 months.
(a) Use DerivaGem to value the option assuming it is a European call.
(b) Use DerivaGem to value the option assuming it is a European put.
(c) Verify that put–call parity holds.
(d) Explore using DerivaGem what happens to the price of the options as the time to maturity becomes very large. For this purpose, assume there are no dividends. Explain the results you get.

CHAPTER 10

Trading Strategies Involving Options

The profit pattern from an investment in a single option was discussed in Chapter 8. In this chapter we cover more fully the range of profit patterns obtainable using options. We assume that the underlying asset is a stock. Similar results can be obtained for other underlying assets, such as foreign currencies, stock indices, and futures contracts. The options used in the strategies we discuss are European. American options may lead to slightly different outcomes because of the possibility of early exercise.

In the first section we consider what happens when a position in a stock option is combined with a position in the stock itself. We then move on to examine the profit patterns obtained when an investment is made in two or more different options on the same stock. One of the attractions of options is that they can be used to create a wide range of different payoff functions. (A payoff function is the payoff as a function of the stock price.) If European options were available with every single possible strike price, any payoff function could in theory be created.

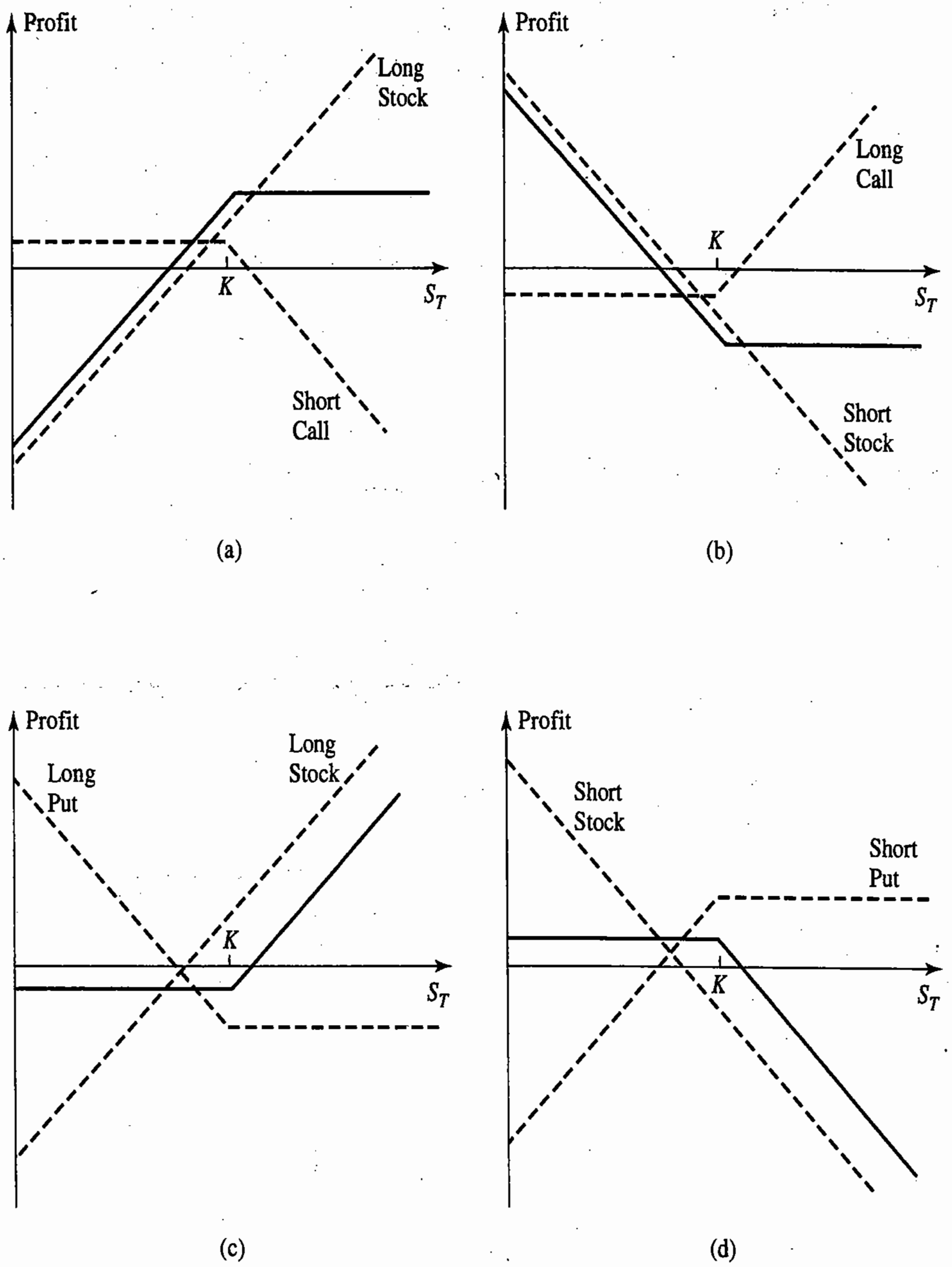
For ease of exposition the figures and tables showing the profit from a trading strategy will ignore the time value of money. The profit will be shown as the final payoff minus the initial cost. (In theory, it should be calculated as the present value of the final payoff minus the initial cost.)

10.1 STRATEGIES INVOLVING A SINGLE OPTION AND A STOCK

There are a number of different trading strategies involving a single option on a stock and the stock itself. The profits from these are illustrated in Figure 10.1. In this figure and in other figures throughout this chapter, the dashed line shows the relationship between profit and the stock price for the individual securities constituting the portfolio, whereas the solid line shows the relationship between profit and the stock price for the whole portfolio.

In Figure 10.1(a), the portfolio consists of a long position in a stock plus a short position in a call option. This is known as *writing a covered call.* The long stock position "covers" or protects the investor from the payoff on the short call that becomes necessary if there is a sharp rise in the stock price. In Figure 10.1(b), a short position in a stock is combined with a long position in a call option. This is the reverse of writing

Figure 10.1 Profit patterns (a) long position in a stock combined with short position in a call; (b) short position in a stock combined with long position in a call; (c) long position in a put combined with long position in a stock; (d) short position in a put combined with short position in a stock.



a covered call. In Figure 10.1(c), the investment strategy involves buying a put option on a stock and the stock itself. The approach is sometimes referred to as a *protective put* strategy. In Figure 10.1(d), a short position in a put option is combined with a short position in the stock. This is the reverse of a protective put.

The profit patterns in Figures 10.1 have the same general shape as the profit patterns discussed in Chapter 8 for short put, long put, long call, and short call, respectively. Put–call parity provides a way of understanding why this is so. From Chapter 9, the put–call parity relationship is

$$p + S_0 = c + Ke^{-rT} + D \tag{10.1}$$

where p is the price of a European put, S_0 is the stock price, c is the price of a European call, K is the strike price of both call and put, r is the risk-free interest rate, T is the time to maturity of both call and put, and D is the present value of the dividends anticipated during the life of the options.

Equation (10.1) shows that a long position in a put combined with a long position in the stock is equivalent to a long call position plus a certain amount ($= Ke^{-rT} + D$) of cash. This explains why the profit pattern in Figure 10.1(c) is similar to the profit pattern from a long call position. The position in Figure 10.1(d) is the reverse of that in Figure 10.1(c) and therefore leads to a profit pattern similar to that from a short call position.

Equation (10.1) can be rearranged to become

$$S_0 - c = Ke^{-rT} + D - p$$

This shows that a long position in a stock combined with a short position in a call is equivalent to a short put position plus a certain amount ($= Ke^{-rT} + D$) of cash. This equality explains why the profit pattern in Figure 10.1(a) is similar to the profit pattern from a short put position. The position in Figure 10.1(b) is the reverse of that in Figure 10.1(a) and therefore leads to a profit pattern similar to that from a long put position.

10.2 SPREADS

A spread trading strategy involves taking a position in two or more options of the same type (i.e., two or more calls or two or more puts).

Bull Spreads

One of the most popular types of spreads is a *bull spread*. This can be created by buying a call option on a stock with a certain strike price and selling a call option on the same

Figure 10.2 Profit from bull spread created using call options.

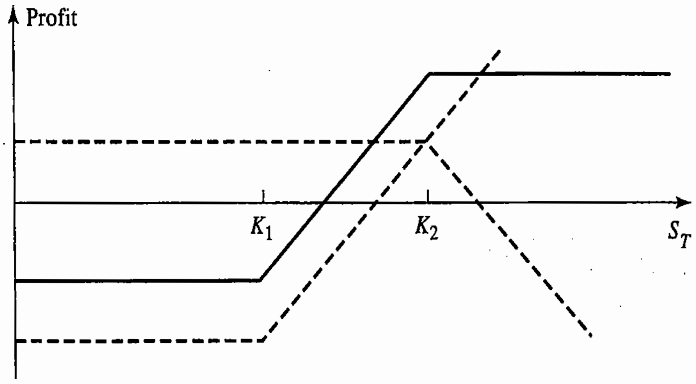


Table 10.1 Payoff from a bull spread created using calls.

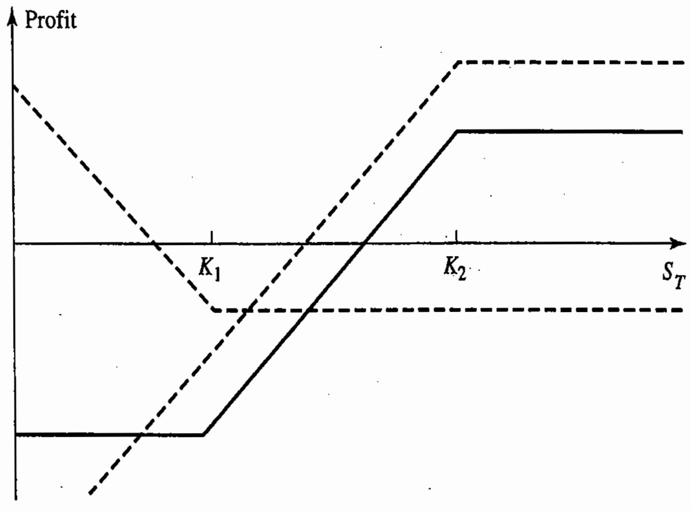
Stock price range	*Payoff from long call option*	*Payoff from short call option*	*Total payoff*
$S_T \leqslant K_1$	0	0	0
$K_1 < S_T < K_2$	$S_T - K_1$	0	$S_T - K_1$
$S_T \geqslant K_2$	$S_T - K_1$	$-(S_T - K_2)$	$K_2 - K_1$

stock with a higher strike price. Both options have the same expiration date. The strategy is illustrated in Figure 10.2. The profits from the two option positions taken separately are shown by the dashed lines. The profit from the whole strategy is the sum of the profits given by the dashed lines and is indicated by the solid line. Because a call price always decreases as the strike price increases, the value of the option sold is always less than the value of the option bought. A bull spread, when created from calls, therefore requires an initial investment.

Suppose that K_1 is the strike price of the call option bought, K_2 is the strike price of the call option sold, and S_T is the stock price on the expiration date of the options. Table 10.1 shows the total payoff that will be realized from a bull spread in different circumstances. If the stock price does well and is greater than the higher strike price, the payoff is the difference between the two strike prices, or $K_2 - K_1$. If the stock price on the expiration date lies between the two strike prices, the payoff is $S_T - K_1$. If the stock price on the expiration date is below the lower strike price, the payoff is zero. The profit in Figure 10.2 is calculated by subtracting the initial investment from the payoff.

A bull spread strategy limits the investor's upside as well as downside risk. The strategy can be described by saying that the investor has a call option with a strike price equal to K_1 and has chosen to give up some upside potential by selling a call option with strike price K_2 ($K_2 > K_1$). In return for giving up the upside potential, the investor gets the

Figure 10.3 Profit from bull spread created using put options.



price of the option with strike price K_2. Three types of bull spreads can be distinguished:

1. Both calls are initially out of the money.
2. One call is initially in the money; the other call is initially out of the money.
3. Both calls are initially in the money.

The most aggressive bull spreads are those of type 1. They cost very little to set up and have a small probability of giving a relatively high payoff ($= K_2 - K_1$). As we move from type 1 to type 2 and from type 2 to type 3, the spreads become more conservative.

Example 10.1

An investor buys for \$3 a call with a strike price of \$30 and sells for \$1 a call with a strike price of \$35. The payoff from this bull spread strategy is \$5 if the stock price is above \$35, and zero if it is below \$30. If the stock price is between \$30 and \$35, the payoff is the amount by which the stock price exceeds \$30. The cost of the strategy is \$3 – \$1 = \$2. The profit is therefore as follows:

Stock price range	*Profit*
$S_T \leqslant 30$	-2
$30 < S_T < 35$	$S_T - 32$
$S_T \geqslant 35$	3

Bull spreads can also be created by buying a put with a low strike price and selling a put with a high strike price, as illustrated in Figure 10.3. Unlike the bull spread created from calls, bull spreads created from puts involve a positive up-front cash flow to the investor (ignoring margin requirements) and a payoff that is either negative or zero.

Bear Spreads

An investor who enters into a bull spread is hoping that the stock price will increase. By contrast, an investor who enters into a *bear spread* is hoping that the stock price will decline. Bear spreads can be created by buying a put with one strike price and selling a put with another strike price. The strike price of the option purchased is greater than the strike price of the option sold. (This is in contrast to a bull spread, where the strike

Figure 10.4 Profit from bear spread created using put options.

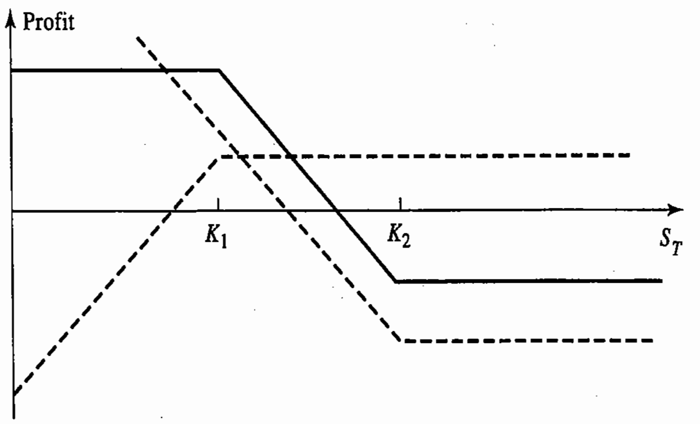


Table 10.2 Payoff from a bear spread created with put options.

Stock price range	*Payoff from long put option*	*Payoff from short put option*	*Total payoff*
$S_T \leqslant K_1$	$K_2 - S_T$	$-(K_1 - S_T)$	$K_2 - K_1$
$K_1 < S_T < K_2$	$K_2 - S_T$	0	$K_2 - S_T$
$S_T \geqslant K_2$	0	0	0

price of the option purchased is always less than the strike price of the option sold.) In Figure 10.4, the profit from the spread is shown by the solid line. A bear spread created from puts involves an initial cash outflow because the price of the put sold is less than the price of the put purchased. In essence, the investor has bought a put with a certain strike price and chosen to give up some of the profit potential by selling a put with a lower strike price. In return for the profit given up, the investor gets the price of the option sold.

Assume that the strike prices are K_1 and K_2, with $K_1 < K_2$. Table 10.2 shows the payoff that will be realized from a bear spread in different circumstances. If the stock price is greater than K_2, the payoff is zero. If the stock price is less than K_1, the payoff is $K_2 - K_1$. If the stock price is between K_1 and K_2, the payoff is $K_2 - S_T$. The profit is calculated by subtracting the initial cost from the payoff.

Example 10.2

An investor buys for \$3 a put with a strike price of \$35 and sells for \$1 a put with a strike price of \$30. The payoff from this bear spread strategy is zero if the stock price is above \$35, and \$5 if it is below \$30. If the stock price is between \$30 and \$35, the payoff is $35 - S_T$. The options cost $\$3 - \$1 = \$2$ up front. The profit is therefore as follows:

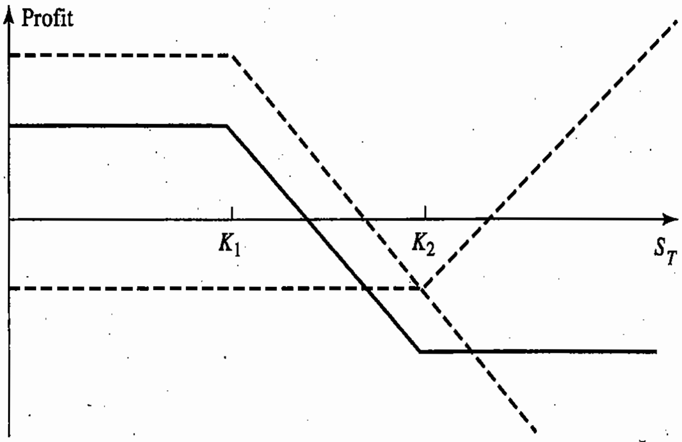
Stock price range	*Profit*
$S_T \leqslant 30$	+3
$30 < S_T < 35$	$33 - S_T$
$S_T \geqslant 35$	−2

Like bull spreads, bear spreads limit both the upside profit potential and the downside risk. Bear spreads can be created using calls instead of puts. The investor buys a call with a high strike price and sells a call with a low strike price, as illustrated in Figure 10.5. Bear spreads created with calls involve an initial cash inflow (ignoring margin requirements).

Box Spreads

A box spread is a combination of a bull call spread with strike prices K_1 and K_2 and a bear put spread with the same two strike prices. As shown in Table 10.3 the payoff from a box spread is always $K_2 - K_1$. The value of a box spread is therefore always the present value of this payoff or $(K_2 - K_1)e^{-rT}$. If it has a different value there is an arbitrage opportunity. If the market price of the box spread is too low, it is profitable to

Figure 10.5 Profit from bear spread created using call options.



buy the box. This involves buying a call with strike price K_1, buying a put with strike price K_2, selling a call with strike price K_2, and selling a put with strike price K_1. If the market price of the box spread is too high, it is profitable to sell the box. This involves buying a call with strike price K_2, buying a put with strike price K_1, selling a call with strike price K_1, and selling a put with strike price K_2.

It is important to realize that a box-spread arbitrage only works with European options. Most of the options that trade on exchanges are American. As shown in Business Snapshot 10.1, inexperienced traders who treat American options as European are liable to lose money.

Butterfly Spreads

A *butterfly spread* involves positions in options with three different strike prices. It can be created by buying a call option with a relatively low strike price, K_1, buying a call option with a relatively high strike price, K_3, and selling two call options with a strike price, K_2, halfway between K_1 and K_3. Generally K_2 is close to the current stock price. The pattern of profits from the strategy is shown in Figure 10.6. A butterfly spread leads to a profit if the stock price stays close to K_2, but gives rise to a small loss if there is a significant stock price move in either direction. It is therefore an appropriate strategy for an investor who feels that large stock price moves are unlikely. The strategy requires a small investment initially. The payoff from a butterfly spread is shown in Table 10.4.

Table 10.3 Payoff from a box spread.

Stock price range	*Payoff from bull call spread*	*Payoff from bear put spread*	*Total payoff*
$S_T \leqslant K_1$	0	$K_2 - K_1$	$K_2 - K_1$
$K_1 < S_T < K_2$	$S_T - K_1$	$K_2 - S_T$	$K_2 - K_1$
$S_T \geqslant K_2$	$K_2 - K_1$	0	$K_2 - K_1$

Business Snapshot 10.1 Losing Money with Box Spreads

Suppose that a stock has a price of \$50 and a volatility of 30%. No dividends are expected and the risk-free rate is 8%. A trader offers you the chance to sell on the CBOE a 2-month box spread where the strike prices are \$55 and \$60 for \$5.10. Should you do the trade?

The trade certainly sounds attractive. In this case $K_1 = 55$, $K_2 = 60$, and the payoff is certain to be \$5 in 2 months. By selling the box spread for \$5.10 and investing the funds for 2 months you would have more than enough funds to meet the \$5 payoff in 2 months. The theoretical value of the box spread today is $5 \times e^{-0.08\times 2/12} = \$4.93$.

Unfortunately there is a snag. CBOE stock options are American and the \$5 payoff from the box spread is calculated on the assumption that the options comprising the box are European. Option prices for this example (calculated using DerivaGem) are shown in the table below. A bull call spread where the strike prices are \$55 and \$60 costs $0.96 - 0.26 = \$0.70$. (This is the same for both European and American options because, as we saw in Chapter 9, the price of a European call is the same as the price of an American call when there are no dividends.) A bear put spread with the same strike prices costs $9.46 - 5.23 = \$4.23$ if the options are European and $10.00 - 5.44 = \$4.56$ if they are American. The combined value of both spreads if they are created with European options is $0.70 + 4.23 = \$4.93$. This is the theoretical box spread price calculated above. The combined value of buying both spreads if they are American is $0.70 + 4.56 = \$5.26$. Selling a box spread created with American options for \$5.10 would not be a good trade. You would realize this almost immediately as the trade involves selling a \$60 strike put and this would be exercised against you almost as soon as you sold it!

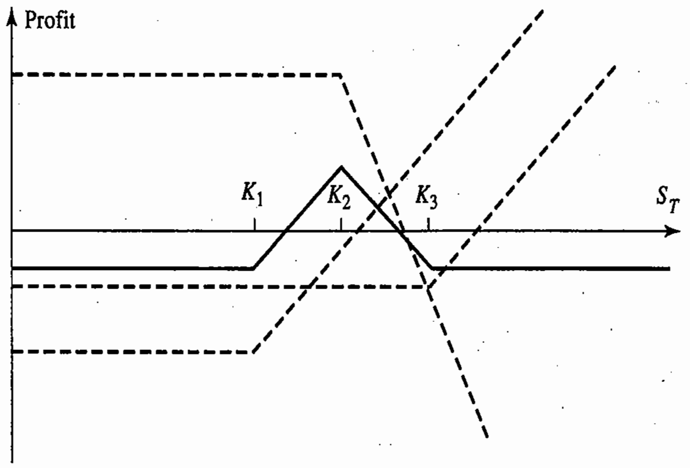
Option type	*Strike price*	*European option price*	*American option price*
Call	60	0.26	0.26
Call	55	0.96	0.96
Put	60	9.46	10.00
Put	55	5.23	5.44

Suppose that a certain stock is currently worth \$61. Consider an investor who feels that a significant price move in the next 6 months is unlikely. Suppose that the market prices of 6-month calls are as follows:

Strike price (\$)	Call price (\$)
55	10
60	7
65	5

The investor could create a butterfly spread by buying one call with a \$55 strike price, buying one call with a \$65 strike price, and selling two calls with a \$60 strike price. It costs $\$10 + \$5 - (2 \times \$7) = \$1$ to create the spread. If the stock price in 6 months is greater than \$65 or less than \$55, the total payoff is zero, and the investor incurs a net

Figure 10.6 Profit from butterfly spread using call options.



loss of \$1. If the stock price is between \$56 and \$64, a profit is made. The maximum profit, \$4, occurs when the stock price in 6 months is \$60.

Butterfly spreads can be created using put options. The investor buys one put with a low strike price, another with a high strike price, and sells two puts with an intermediate strike price, as illustrated in Figure 10.7. The butterfly spread in the example considered above would be created by buying one put with a strike price of \$55, another with a strike price of \$65, and selling two puts with a strike price of \$60. If all options are European, the use of put options results in exactly the same spread as the use of call options. Put–call parity can be used to show that the initial investment is the same in both cases.

A butterfly spread can be sold or shorted by following the reverse strategy. Options are sold with strike prices of K_1 and K_3, and two options with the middle strike price K_2 are purchased. This strategy produces a modest profit if there is a significant movement in the stock price.

Calendar Spreads

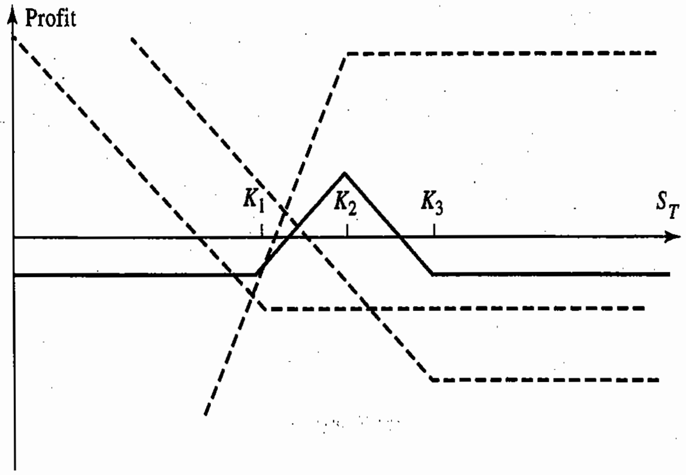
Up to now we have assumed that the options used to create a spread all expire at the same time. We now move on to *calendar spreads* in which the options have the same strike price and different expiration dates.

Table 10.4 Payoff from a butterfly spread.

Stock price range	*Payoff from first long call*	*Payoff from second long call*	*Payoff from short calls*	*Total payoff**
$S_T \leqslant K_1$	0	0	0	0
$K_1 < S_T < K_2$	$S_T - K_1$	0	0	$S_T - K_1$
$K_2 < S_T < K_3$	$S_T - K_1$	0	$-2(S_T - K_2)$	$K_3 - S_T$
$S_T \geqslant K_3$	$S_T - K_1$	$S_T - K_3$	$-2(S_T - K_2)$	0

* These payoffs are calculated using the relationship $K_2 = 0.5(K_1 + K_3)$.

Figure 10.7 Profit from butterfly spread using put options.



A calendar spread can be created by selling a call option with a certain strike price and buying a longer-maturity call option with the same strike price. The longer the maturity of an option, the more expensive it usually is. A calendar spread therefore usually requires an initial investment. Profit diagrams for calendar spreads are usually produced so that they show the profit when the short-maturity option expires on the assumption that the long-maturity option is sold at that time. The profit pattern for a calendar spread produced from call options is shown in Figure 10.8. The pattern is similar to the profit from the butterfly spread in Figure 10.6. The investor makes a profit if the stock price at the expiration of the short-maturity option is close to the strike price of the short-maturity option. However, a loss is incurred when the stock price is significantly above or significantly below this strike price.

Figure 10.8 Profit from calendar spread created using two calls.

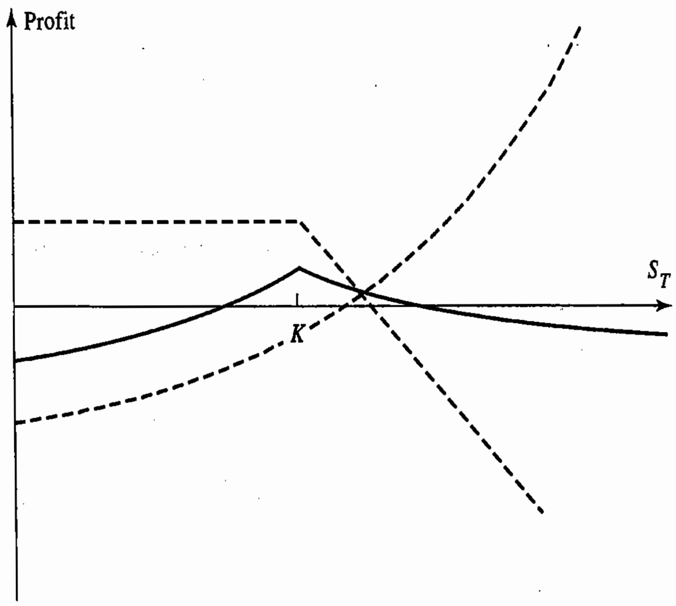
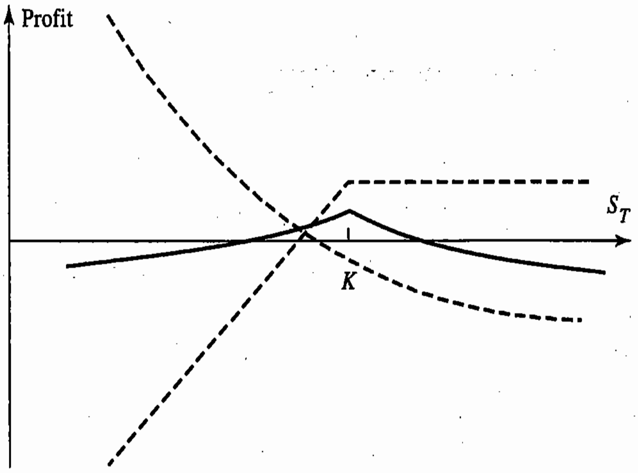


Figure 10.9 Profit from a calendar spread created using two puts.



To understand the profit pattern from a calendar spread, first consider what happens if the stock price is very low when the short-maturity option expires. The short-maturity option is worthless and the value of the long-maturity option is close to zero. The investor therefore incurs a loss that is close to the cost of setting up the spread initially. Consider next what happens if the stock price, S_T, is very high when the short-maturity option expires. The short-maturity option costs the investor $S_T - K$, and the long-maturity option is worth close to $S_T - K$, where K is the strike price of the options. Again, the investor makes a net loss that is close to the cost of setting up the spread initially. If S_T is close to K, the short-maturity option costs the investor either a small amount or nothing at all. However, the long-maturity option is still quite valuable. In this case a significant net profit is made.

In a *neutral calendar spread*, a strike price close to the current stock price is chosen. A *bullish calendar spread* involves a higher strike price, whereas a *bearish calendar spread* involves a lower strike price.

Calendar spreads can be created with put options as well as call options. The investor buys a long-maturity put option and sells a short-maturity put option. As shown in Figure 10.9, the profit pattern is similar to that obtained from using calls.

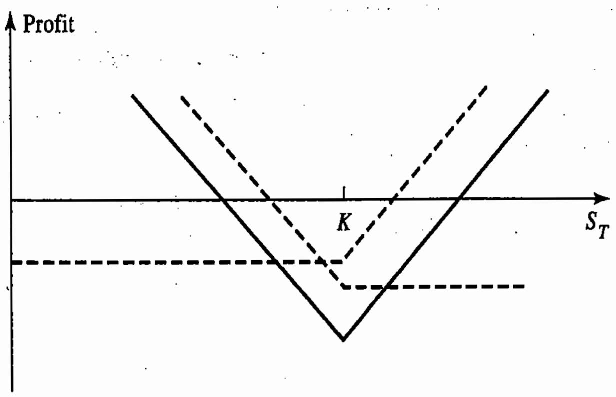
A *reverse calendar spread* is the opposite to that in Figures 10.8 and 10.9. The investor buys a short-maturity option and sells a long-maturity option. A small profit arises if the stock price at the expiration of the short-maturity option is well above or well below the strike price of the short-maturity option. However, a significant loss results if it is close to the strike price.

Diagonal Spreads

Bull, bear, and calendar spreads can all be created from a long position in one call and a short position in another call. In the case of bull and bear spreads, the calls have different strike prices and the same expiration date. In the case of calendar spreads, the calls have the same strike price and different expiration dates.

In a *diagonal spread* both the expiration date and the strike price of the calls are different. This increases the range of profit patterns that are possible.

Figure 10.10 Profit from a straddle.



10.3 COMBINATIONS

A *combination* is an option trading strategy that involves taking a position in both calls and puts on the same stock. We will consider straddles, strips, straps, and strangles.

Straddle

One popular combination is a *straddle*, which involves buying a call and put with the same strike price and expiration date. The profit pattern is shown in Figure 10.10. The strike price is denoted by K. If the stock price is close to this strike price at expiration of the options, the straddle leads to a loss. However, if there is a sufficiently large move in either direction, a significant profit will result. The payoff from a straddle is calculated in Table 10.5.

A straddle is appropriate when an investor is expecting a large move in a stock price but does not know in which direction the move will be. Consider an investor who feels that the price of a certain stock, currently valued at \$69 by the market, will move significantly in the next 3 months. The investor could create a straddle by buying both a put and a call with a strike price of \$70 and an expiration date in 3 months. Suppose that the call costs \$4 and the put costs \$3. If the stock price stays at \$69, it is easy to see that the strategy costs the investor \$6. (An up-front investment of \$7 is required, the call expires worthless, and the put expires worth \$1.) If the stock price moves to \$70, a loss of \$7 is experienced. (This is the worst that can happen.) However, if the stock price jumps up to \$90, a profit of \$13 is made; if the stock moves down to \$55, a profit of \$8 is made; and so on. As discussed in Business Snapshot 10.2 an investor should carefully

Table 10.5 Payoff from a straddle.

Range of stock price	*Payoff from call*	*Payoff from put*	*Total payoff*
$S_T \leqslant K$	0	$K - S_T$	$K - S_T$
$S_T > K$	$S_T - K$	0	$S_T - K$

Business Snapshot 10.2 How to Make Money from Trading Straddles

Suppose that a big move is expected in a company's stock price because there is a takeover bid for the company or the outcome of a major lawsuit involving the company is about to be announced. Should you trade a straddle?

A straddle seems a natural trading strategy in this case. However, if your view of the company's situation is much the same as that of other market participants, this view will be reflected in the prices of options. Options on the stock will be significantly more expensive than options on a similar stock for which no jump is expected. The V-shaped profit pattern from the straddle in Figure 10.10 will have moved downward, so that a bigger move in the stock price is necessary for you to make a profit.

For a straddle to be an effective strategy, you must believe that there are likely to be big movements in the stock price and these beliefs must be different from those of most other investors. Market prices incorporate the beliefs of market participants. To make money from any investment strategy, you must take a view that is different from most of the rest of the market—and you must be right!

consider whether the jump that he or she anticipates is already reflected in option prices before putting on a straddle trade.

The straddle in Figure 10.10 is sometimes referred to as a *bottom straddle* or *straddle purchase*. A *top straddle* or *straddle write* is the reverse position. It is created by selling a call and a put with the same exercise price and expiration date. It is a highly risky strategy. If the stock price on the expiration date is close to the strike price, a significant profit results. However, the loss arising from a large move is unlimited.

Strips and Straps

A *strip* consists of a long position in one call and two puts with the same strike price and expiration date. A *strap* consists of a long position in two calls and one put with the

Figure 10.11 Profit from a strip and a strap.

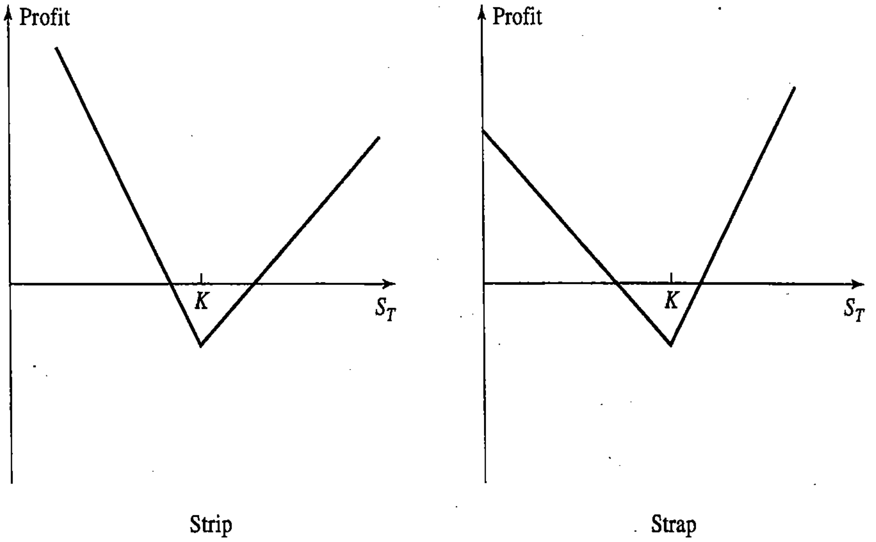
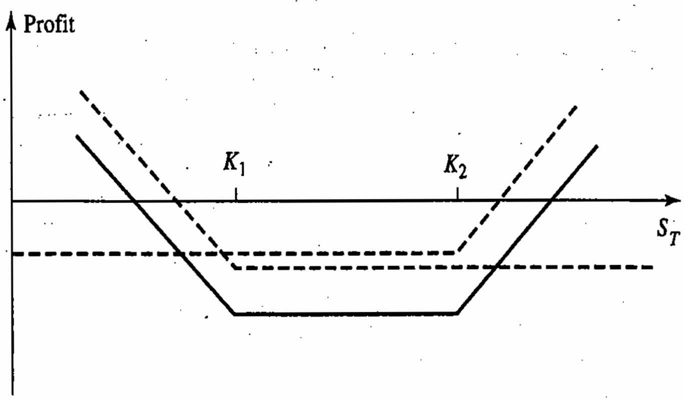


Figure 10.12 Profit from a strangle.



same strike price and expiration date. The profit patterns from strips and straps are shown in Figure 10.11. In a strip the investor is betting that there will be a big stock price move and considers a decrease in the stock price to be more likely than an increase. In a strap the investor is also betting that there will be a big stock price move. However, in this case, an increase in the stock price is considered to be more likely than a decrease.

Strangles

In a *strangle*, sometimes called a *bottom vertical combination*, an investor buys a put and a call with the same expiration date and different strike prices. The profit pattern that is obtained is shown in Figure 10.12. The call strike price, K_2, is higher than the put strike price, K_1. The payoff function for a strangle is calculated in Table 10.6.

A strangle is a similar strategy to a straddle. The investor is betting that there will be a large price move, but is uncertain whether it will be an increase or a decrease. Comparing Figures 10.12 and 10.10, we see that the stock price has to move farther in a strangle than in a straddle for the investor to make a profit. However, the downside risk if the stock price ends up at a central value is less with a strangle.

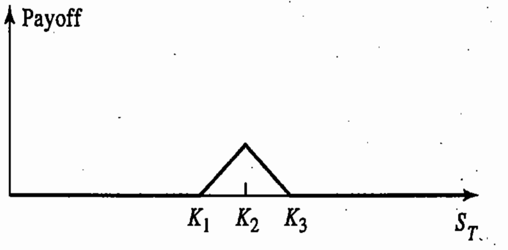
The profit pattern obtained with a strangle depends on how close together the strike prices are. The farther they are apart, the less the downside risk and the farther the stock price has to move for a profit to be realized.

The sale of a strangle is sometimes referred to as a *top vertical combination*. It can be appropriate for an investor who feels that large stock price moves are unlikely. However, as with sale of a straddle, it is a risky strategy involving unlimited potential loss to the investor.

Table 10.6 Payoff from a strangle.

Range of stock price	*Payoff from call*	*Payoff from put*	*Total payoff*
$S_T \leqslant K_1$	0	$K_1 - S_T$	$K_1 - S_T$
$K_1 < S_T < K_2$	0	0	0
$S_T \geqslant K_2$	$S_T - K_2$	0	$S_T - K_2$

Figure 10.13 "Spike payoff" from a butterfly spread that can be used as a building block to create other payoffs.



10.4 OTHER PAYOFFS

This chapter has demonstrated just a few of the ways in which options can be used to produce an interesting relationship between profit and stock price. If European options expiring at time T were available with every single possible strike price, any payoff function at time T could in theory be obtained. The easiest illustration of this involves butterfly spreads. Recall that a butterfly spread is created by buying options with strike prices K_1 and K_3 and selling two options with strike price K_2, where $K_1 < K_2 < K_3$ and $K_3 - K_2 = K_2 - K_1$. Figure 10.13 shows the payoff from a butterfly spread. The pattern could be described as a spike. As K_1 and K_3 move closer together, the spike becomes smaller. Through the judicious combination of a large number of very small spikes, any payoff function can be approximated.

SUMMARY

A number of common trading strategies involve a single option and the underlying stock. For example, writing a covered call involves buying the stock and selling a call option on the stock; a protective put involves buying a put option and buying the stock. The former is similar to selling a put option; the latter is similar to buying a call option.

Spreads involve either taking a position in two or more calls or taking a position in two or more puts. A bull spread can be created by buying a call (put) with a low strike price and selling a call (put) with a high strike price. A bear spread can be created by buying a put (call) with a high strike price and selling a put (call) with a low strike price. A butterfly spread involves buying calls (puts) with a low and high strike price and selling two calls (puts) with some intermediate strike price. A calendar spread involves selling a call (put) with a short time to expiration and buying a call (put) with a longer time to expiration. A diagonal spread involves a long position in one option and a short position in another option such that both the strike price and the expiration date are different.

Combinations involve taking a position in both calls and puts on the same stock. A straddle combination involves taking a long position in a call and a long position in a put with the same strike price and expiration date. A strip consists of a long position in one call and two puts with the same strike price and expiration date. A strap consists of a long position in two calls and one put with the same strike price and expiration date. A strangle consists of a long position in a call and a put with different strike prices and the same expiration date. There are many other ways in which options can be used to produce interesting payoffs. It is not surprising that option trading has steadily increased in popularity and continues to fascinate investors.

FURTHER READING

Bharadwaj, A. and J.B. Wiggins. "Box Spread and Put–Call Parity Tests for the S&P Index LEAPS Markets," *Journal of Derivatives*, 8, 4 (Summer 2001): 62–71.

Chaput, J. S., and L. H. Ederington, "Option Spread and Combination Trading," *Journal of Derivatives*, 10, 4 (Summer 2003): 70–88.

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Questions And Problems (Answers in Solutions Manual)

10.1. What is meant by a protective put? What position in call options is equivalent to a protective put?

10.2. Explain two ways in which a bear spread can be created.

10.3. When is it appropriate for an investor to purchase a butterfly spread?

10.4. Call options on a stock are available with strike prices of \$15, \$17$\frac{1}{2}$, and \$20, and expiration dates in 3 months. Their prices are \$4, \$2, and \$$\frac{1}{2}$, respectively. Explain how the options can be used to create a butterfly spread. Construct a table showing how profit varies with stock price for the butterfly spread.

10.5. What trading strategy creates a reverse calendar spread?

10.6. What is the difference between a strangle and a straddle?

10.7. A call option with a strike price of \$50 costs \$2. A put option with a strike price of \$45 costs \$3. Explain how a strangle can be created from these two options. What is the pattern of profits from the strangle?

10.8. Use put–call parity to relate the initial investment for a bull spread created using calls to the initial investment for a bull spread created using puts.

10.9. Explain how an aggressive bear spread can be created using put options.

10.10. Suppose that put options on a stock with strike prices \$30 and \$35 cost \$4 and \$7, respectively. How can the options be used to create (a) a bull spread and (b) a bear spread? Construct a table that shows the profit and payoff for both spreads.

10.11. Use put–call parity to show that the cost of a butterfly spread created from European puts is identical to the cost of a butterfly spread created from European calls.

10.12. A call with a strike price of \$60 costs \$6. A put with the same strike price and expiration date costs \$4. Construct a table that shows the profit from a straddle. For what range of stock prices would the straddle lead to a loss?

10.13. Construct a table showing the payoff from a bull spread when puts with strike prices K_1 and K_2, with $K_2 > K_1$, are used.

10.14. An investor believes that there will be a big jump in a stock price, but is uncertain as to the direction. Identify six different strategies the investor can follow and explain the differences among them.

10.15. How can a forward contract on a stock with a particular delivery price and delivery date be created from options?

10.16. "A box spread comprises four options. Two can be combined to create a long forward position and two can be combined to create a short forward position." Explain this statement.

10.17. What is the result if the strike price of the put is higher than the strike price of the call in a strangle?

10.18. One Australian dollar is currently worth \$0.64. A 1-year butterfly spread is set up using European call options with strike prices of \$0.60, \$0.65, and \$0.70. The risk-free interest rates in the United States and Australia are 5% and 4% respectively, and the volatility of the exchange rate is 15%. Use the DerivaGem software to calculate the cost of setting up the butterfly spread position. Show that the cost is the same if European put options are used instead of European call options.

Assignment Questions

10.19. Three put options on a stock have the same expiration date and strike prices of \$55, \$60, and \$65. The market prices are \$3, \$5, and \$8, respectively. Explain how a butterfly spread can be created. Construct a table showing the profit from the strategy. For what range of stock prices would the butterfly spread lead to a loss?

10.20. A diagonal spread is created by buying a call with strike price K_2 and exercise date T_2 and selling a call with strike price K_1 and exercise date T_1, where $T_2 > T_1$. Draw a diagram showing the profit when (a) $K_2 > K_1$ and (b) $K_2 < K_1$.

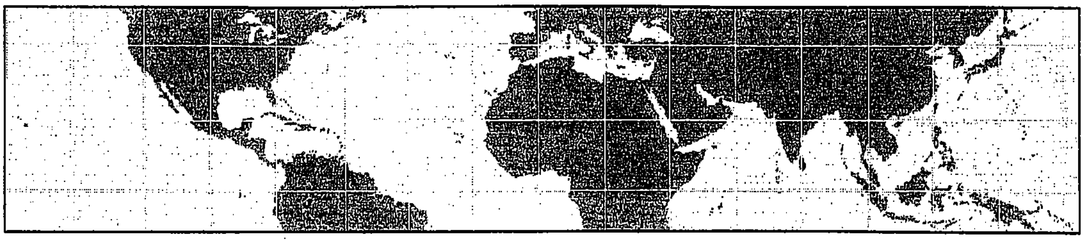
10.21. Draw a diagram showing the variation of an investor's profit and loss with the terminal stock price for a portfolio consisting of:
(a) One share and a short position in one call option
(b) Two shares and a short position in one call option
(c) One share and a short position in two call options
(d) One share and a short position in four call options
In each case, assume that the call option has an exercise price equal to the current stock price.

10.22. Suppose that the price of a non-dividend-paying stock is \$32, its volatility is 30%, and the risk-free rate for all maturities is 5% per annum. Use DerivaGem to calculate the cost of setting up the following positions:
(a) A bull spread using European call options with strike prices of \$25 and \$30 and a maturity of 6 months
(b) A bear spread using European put options with strike prices of \$25 and \$30 and a maturity of 6 months
(c) A butterfly spread using European call options with strike prices of \$25, \$30, and \$35 and a maturity of 1 year

(d) A butterfly spread using European put options with strike prices of \$25, \$30, and \$35 and a maturity of 1 year
(e) A straddle using options with a strike price of \$30 and a 6-month maturity
(f) A strangle using options with strike prices of \$25 and \$35 and a 6-month maturity

In each case provide a table showing the relationship between profit and final stock price. Ignore the impact of discounting.

10.23. What trading position is created from a long strangle and a short straddle when both have the same time to maturity? Assume that the strike price in the straddle is halfway between the two strike prices of the strangle.



CHAPTER 11

Binomial Trees

A useful and very popular technique for pricing an option involves constructing a *binomial tree*. This is a diagram representing different possible paths that might be followed by the stock price over the life of an option. The underlying assumption is that the stock price follows a *random walk*. In each time step, it has a certain probability of moving up by a certain percentage amount and a certain probability of moving down by a certain percentage amount. In the limit, as the time step becomes smaller, this model leads to the lognormal assumption for stock prices that underlies the Black–Scholes model we will be discussing in Chapter 13.

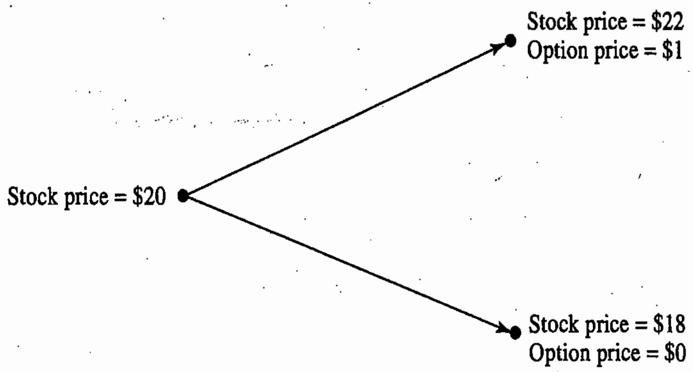
In this chapter we will take a first look at binomial trees. We show how they can be used to value options using both no-arbitrage arguments and a principle known as risk-neutral valuation. The general approach adopted here is similar to that in an important paper published by Cox, Ross, and Rubinstein in 1979. More details on numerical procedures involving binomial and trinomial trees are given in Chapter 19.

11.1 A ONE-STEP BINOMIAL MODEL AND A NO-ARBITRAGE ARGUMENT

We start by considering a very simple situation. A stock price is currently \$20, and it is known that at the end of 3 months it will be either \$22 or \$18. We are interested in valuing a European call option to buy the stock for \$21 in 3 months. This option will have one of two values at the end of the 3 months. If the stock price turns out to be \$22, the value of the option will be \$1; if the stock price turns out to be \$18, the value of the option will be zero. The situation is illustrated in Figure 11.1.

It turns out that a relatively simple argument can be used to price the option in this example. The only assumption needed is that arbitrage opportunities do not exist. We set up a portfolio of the stock and the option in such a way that there is no uncertainty about the value of the portfolio at the end of the 3 months. We then argue that, because the portfolio has no risk, the return it earns must equal the risk-free interest rate. This enables us to work out the cost of setting up the portfolio and therefore the option's price. Because there are two securities (the stock and the stock option) and only two possible outcomes, it is always possible to set up the riskless portfolio.

Figure 11.1 Stock price movements for numerical example in Section 11.1.



Consider a portfolio consisting of a long position in Δ shares of the stock and a short position in one call option. We calculate the value of Δ that makes the portfolio riskless. If the stock price moves up from $20 to $22, the value of the shares is 22Δ and the value of the option is 1, so that the total value of the portfolio is $22\Delta - 1$. If the stock price moves down from $20 to $18, the value of the shares is 18Δ and the value of the option is zero, so that the total value of the portfolio is 18Δ. The portfolio is riskless if the value of Δ is chosen so that the final value of the portfolio is the same for both alternatives. This means that

$$22\Delta - 1 = 18\Delta$$

or

$$\Delta = 0.25$$

A riskless portfolio is therefore

Long: 0.25 shares
Short: 1 option

If the stock price moves up to $22, the value of the portfolio is

$$22 \times 0.25 - 1 = 4.5$$

If the stock price moves down to $18, the value of the portfolio is

$$18 \times 0.25 = 4.5$$

Regardless of whether the stock price moves up or down, the value of the portfolio is always 4.5 at the end of the life of the option.

Riskless portfolios must, in the absence of arbitrage opportunities, earn the risk-free rate of interest. Suppose that in this case the risk-free rate is 12% per annum. It follows that the value of the portfolio today must be the present value of 4.5, or

$$4.5e^{-0.12\times3/12} = 4.367$$

The value of the stock price today is known to be $20. Suppose the option price is denoted by f. The value of the portfolio today is

$$20 \times 0.25 - f = 5 - f$$

It follows that

$$5 - f = 4.367$$

or

$$f = 0.633$$

This shows that, in the absence of arbitrage opportunities, the current value of the option must be 0.633. If the value of the option were more than 0.633, the portfolio would cost less than 4.367 to set up and would earn more than the risk-free rate. If the value of the option were less than 0.633, shorting the portfolio would provide a way of borrowing money at less than the risk-free rate.

A Generalization

We can generalize the no-arbitrage argument just presented by considering a stock whose price is S_0 and an option on the stock whose current price is f. We suppose that the option lasts for time T and that during the life of the option the stock price can either move up from S_0 to a new level, S_0u, where $u > 1$, or down from S_0 to a new level, S_0d, where $d < 1$. The percentage increase in the stock price when there is an up movement is $u - 1$; the percentage decrease when there is a down movement is $1 - d$. If the stock price moves up to S_0u, we suppose that the payoff from the option is f_u; if the stock price moves down to S_0d, we suppose the payoff from the option is f_d. The situation is illustrated in Figure 11.2.

As before, we imagine a portfolio consisting of a long position in Δ shares and a short position in one option. We calculate the value of Δ that makes the portfolio riskless. If there is an up movement in the stock price, the value of the portfolio at the end of the life of the option is

$$S_0u\Delta - f_u$$

If there is a down movement in the stock price, the value becomes

$$S_0d\Delta - f_d$$

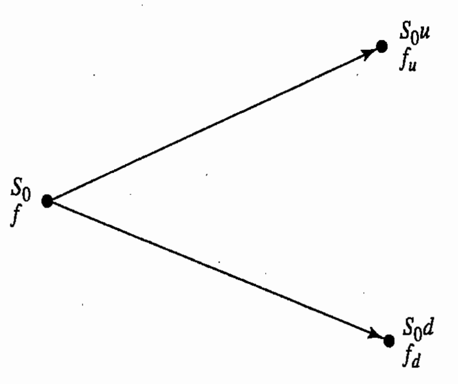
The two are equal when

$$S_0u\Delta - f_u = S_0d\Delta - f_d$$

or

$$\Delta = \frac{f_u - f_d}{S_0u - S_0d} \tag{11.1}$$

Figure 11.2 Stock and option prices in a general one-step tree.



In this case, the portfolio is riskless and, for there to be no arbitrage opportunities, it must earn the risk-free interest rate. Equation (11.1) shows that Δ is the ratio of the change in the option price to the change in the stock price as we move between the nodes at time T.

If we denote the risk-free interest rate by r, the present value of the portfolio is

$$(S_0 u\Delta - f_u)e^{-rT}$$

The cost of setting up the portfolio is

$$S_0\Delta - f$$

It follows that

$$S_0\Delta - f = (S_0 u\Delta - f_u)e^{-rT}$$

or

$$f = S_0\Delta(1 - ue^{-rT}) + f_u e^{-rT}$$

Substituting from equation (11.1) for Δ and simplifying, we can reduce this equation to

$$f = e^{-rT}[pf_u + (1-p)f_d] \tag{11.2}$$

where

$$p = \frac{e^{rT} - d}{u - d} \tag{11.3}$$

Equations (11.2) and (11.3) enable an option to be priced when stock price movements are given by a one-step binomial tree. The only assumption needed for the equation is that there are no arbitrage opportunities.

In the numerical example considered previously (see Figure 11.1), $u = 1.1$, $d = 0.9$, $r = 0.12$, $T = 0.25$, $f_u = 1$, and $f_d = 0$. From equation (11.3), we have

$$p = \frac{e^{0.12\times 3/12} - 0.9}{1.1 - 0.9} = 0.6523$$

and, from equation (11.2), we have

$$f = e^{-0.12\times 0.25}(0.6523 \times 1 + 0.3477 \times 0) = 0.633$$

The result agrees with the answer obtained earlier in this section.

Irrelevance of the Stock's Expected Return

The option pricing formula in equation (11.2) does not involve the probabilities of the stock price moving up or down. For example, we get the same option price when the probability of an upward movement is 0.5 as we do when it is 0.9. This is surprising and seems counterintuitive. It is natural to assume that, as the probability of an upward movement in the stock price increases, the value of a call option on the stock increases and the value of a put option on the stock decreases. This is not the case.

The key reason is that we are not valuing the option in absolute terms. We are calculating its value in terms of the price of the underlying stock. The probabilities of future up or down movements are already incorporated into the stock price: we do not need to take them into account again when valuing the option in terms of the stock price.

11.2 RISK-NEUTRAL VALUATION

We do not need to make any assumptions about the probabilities of up and down movements in order to derive equation (11.2). All we require is the absence of arbitrage opportunities. However, it is natural to interpret the variable p in equation (11.2) as the probability of an up movement in the stock price. The variable $1-p$ is then the probability of a down movement, and the expression

$$pf_u + (1-p)f_d$$

is the expected payoff from the option. With this interpretation of p, equation (11.2) then states that the value of the option today is its expected future payoff discounted at the risk-free rate.

We now investigate the expected return from the stock when the probability of an up movement is p. The expected stock price at time T, $E(S_T)$, is given by

$$E(S_T) = pS_0u + (1-p)S_0d$$

or

$$E(S_T) = pS_0(u-d) + S_0d$$

Substituting from equation (11.3) for p, we obtain

$$E(S_T) = S_0e^{rT} \tag{11.4}$$

showing that the stock price grows on average at the risk-free rate. Setting the probability of the up movement equal to p is therefore equivalent to assuming that the return on the stock equals the risk-free rate.

In a *risk-neutral world* all individuals are indifferent to risk. In such a world, investors require no compensation for risk, and the expected return on all securities is the risk-free interest rate. Equation (11.4) shows that we are assuming a risk-neutral world when we set the probability of an up movement to p. Equation (11.2) shows that the value of the option is its expected payoff in a risk-neutral world discounted at the risk-free rate.

This result is an example of an important general principle in option pricing known as *risk-neutral valuation*. The principle states that we can with complete impunity assume the world is risk neutral when pricing options. The resulting prices are correct not just in a risk-neutral world, but in other worlds as well.

The One-Step Binomial Example Revisited

We now return to the example in Figure 11.1 and illustrate that risk-neutral valuation gives the same answer as no-arbitrage arguments. In Figure 11.1, the stock price is currently \$20 and will move either up to \$22 or down to \$18 at the end of 3 months. The option considered is a European call option with a strike price of \$21 and an expiration date in 3 months. The risk-free interest rate is 12% per annum.

We define p as the probability of an upward movement in the stock price in a risk-neutral world. We can calculate p from equation (11.3). Alternatively, we can argue that the expected return on the stock in a risk-neutral world must be the risk-free rate of 12%. This means that p must satisfy

$$22p + 18(1-p) = 20e^{0.12\times 3/12}$$

or

$$4p = 20e^{0.12\times3/12} - 18$$

That is, p must be 0.6523.

At the end of the 3 months, the call option has a 0.6523 probability of being worth 1 and a 0.3477 probability of being worth zero. Its expected value is therefore

$$0.6523 \times 1 + 0.3477 \times 0 = 0.6523$$

In a risk-neutral world this should be discounted at the risk-free rate. The value of the option today is therefore

$$0.6523e^{-0.12\times3/12}$$

or \$0.633. This is the same as the value obtained earlier, demonstrating that no-arbitrage arguments and risk-neutral valuation give the same answer.

Real World vs. Risk-Neutral World

It should be emphasized that p is the probability of an up movement in a risk-neutral world. In general this is not the same as the probability of an up movement in the real world. In our example $p = 0.6523$. When the probability of an up movement is 0.6523, the expected return on both the stock and the option is the risk-free rate of 12%. Suppose that, in the real world, the expected return on the stock is 16% and p^* is the probability of an up movement. It follows that

$$22p^* + 18(1 - p^*) = 20e^{0.16\times3/12}$$

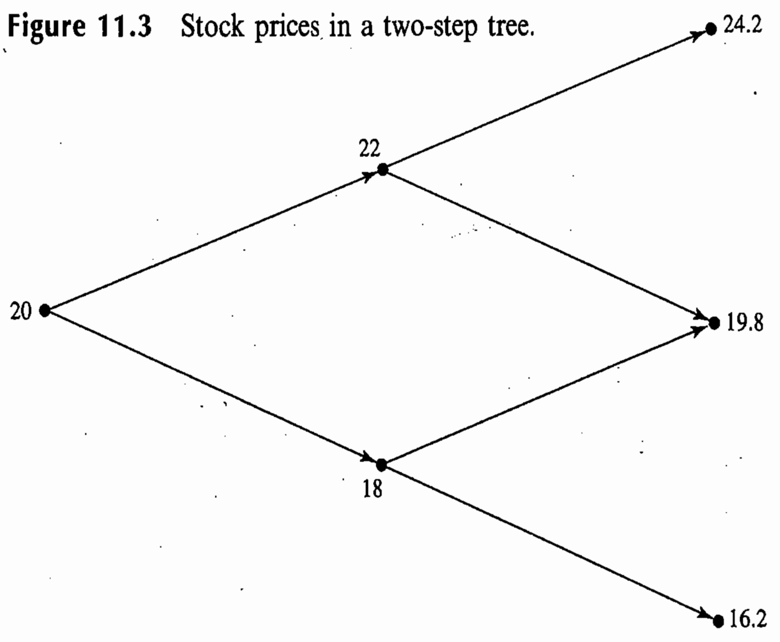
so that $p^* = 0.7041$.

The expected payoff from the option in the real world is then given by

$$p^* \times 1 + (1 - p^*) \times 0$$

This is 0.7041. Unfortunately it is not easy to know the correct discount rate to apply to

Figure 11.3 Stock prices in a two-step tree.



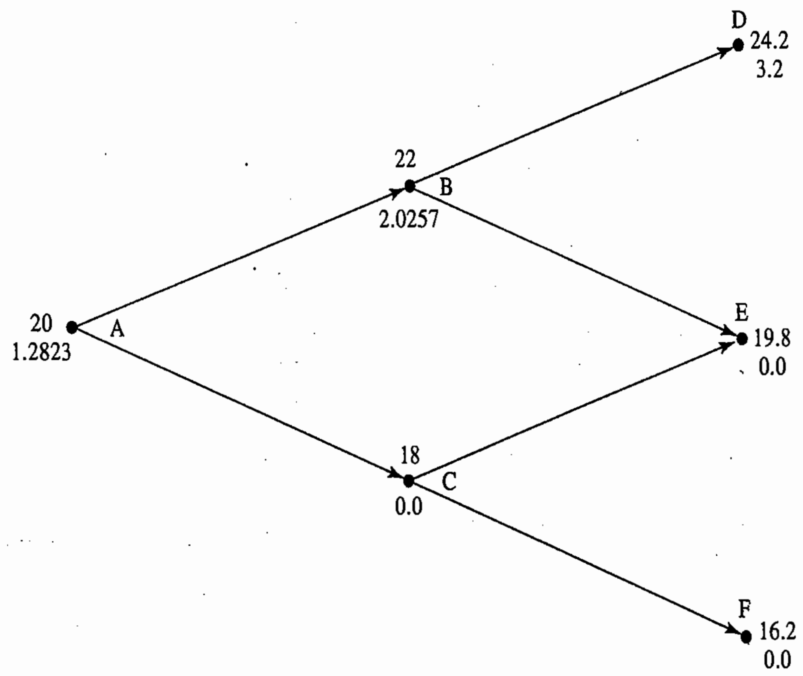
the expected payoff in the real world. A position in a call option is riskier than a position in the stock. As a result the discount rate to be applied to the payoff from a call option is greater than 16%. Without knowing the option's value, we do not know how much greater than 16% it should be.[1] Using risk-neutral valuation is convenient because we know that in a risk-neutral world the expected return on all assets (and therefore the discount rate to use for all expected payoffs) is the risk-free rate.

11.3 TWO-STEP BINOMIAL TREES

We can extend the analysis to a two-step binomial tree such as that shown in Figure 11.3. Here the stock price starts at $20 and in each of two time steps may go up by 10% or down by 10%. We suppose that each time step is 3 months long and the risk-free interest rate is 12% per annum. As before, we consider an option with a strike price of $21.

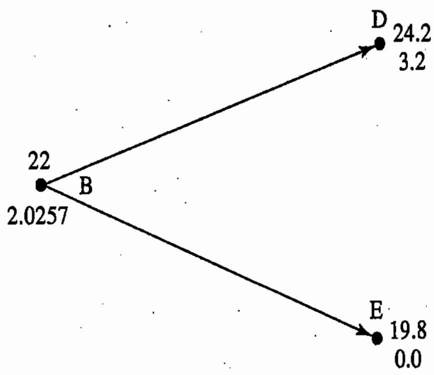
The objective of the analysis is to calculate the option price at the initial node of the tree. This can be done by repeatedly applying the principles established earlier in the chapter. Figure 11.4 shows the same tree as Figure 11.3, but with both the stock price and the option price at each node. (The stock price is the upper number and the option price is the lower number.) The option prices at the final nodes of the tree are easily calculated. They are the payoffs from the option. At node D the stock price is 24.2 and the option price is $24.2 - 21 = 3.2$; at nodes E and F the option is out of the money and its value is zero.

Figure 11.4 Stock and option prices in a two-step tree. The upper number at each node is the stock price and the lower number is the option price.



[1] Because the correct value of the option is 0.633, we can deduce that the correct discount rate is 42.58%. This is because $0.633 = 0.7041e^{-0.4258\times3/12}$.

Figure 11.5 Evaluation of option price at node B.



At node C the option price is zero, because node C leads to either node E or node F and at both of those nodes the option price is zero. We calculate the option price at node B by focusing our attention on the part of the tree shown in Figure 11.5. Using the notation introduced earlier in the chapter, $u = 1.1$, $d = 0.9$, $r = 0.12$, and $T = 0.25$, so that $p = 0.6523$, and equation (11.2) gives the value of the option at node B as

$$e^{-0.12\times 3/12}(0.6523 \times 3.2 + 0.3477 \times 0) = 2.0257$$

It remains for us to calculate the option price at the initial node A. We do so by focusing on the first step of the tree. We know that the value of the option at node B is 2.0257 and that at node C it is zero. Equation (11.2) therefore gives the value at node A as

$$e^{-0.12\times 3/12}(0.6523 \times 2.0257 + 0.3477 \times 0) = 1.2823$$

The value of the option is \$1.2823.

Note that this example was constructed so that u and d (the proportional up and down movements) were the same at each node of the tree and so that the time steps were of the same length. As a result, the risk-neutral probability, p, as calculated by equation (11.3) is the same at each node.

A Generalization

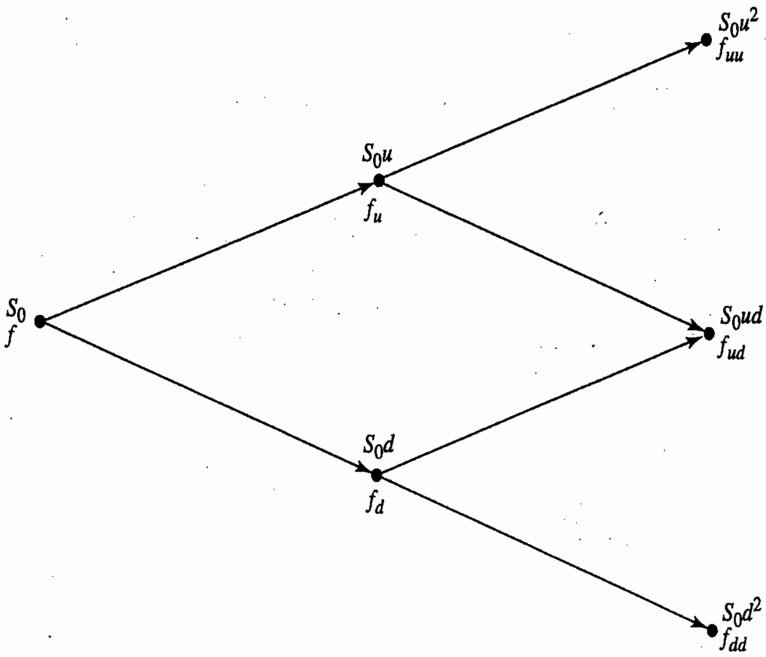
We can generalize the case of two time steps by considering the situation in Figure 11.6. The stock price is initially S_0. During each time step, it either moves up to u times its initial value or moves down to d times its initial value. The notation for the value of the option is shown on the tree. (For example, after two up movements the value of the option is f_{uu}.) We suppose that the risk-free interest rate is r and the length of the time step is Δt years.

Because the length of a time step is now Δt rather than T, equations (11.2) and (11.3) become

$$f = e^{-r\Delta t}[pf_u + (1-p)f_d] \tag{11.5}$$

$$p = \frac{e^{r\Delta t} - d}{u - d} \tag{11.6}$$

Figure 11.6 Stock and option prices in general two-step tree.



Repeated application of equation (11.5) gives

$$f_u = e^{-r\Delta t}[pf_{uu} + (1-p)f_{ud}] \tag{11.7}$$

$$f_d = e^{-r\Delta t}[pf_{ud} + (1-p)f_{dd}] \tag{11.8}$$

$$f = e^{-r\Delta t}[pf_u + (1-p)f_d] \tag{11.9}$$

Substituting from equations (11.7) and (11.8) into (11.9), we get

$$f = e^{-2r\Delta t}[p^2 f_{uu} + 2p(1-p)f_{ud} + (1-p)^2 f_{dd}] \tag{11.10}$$

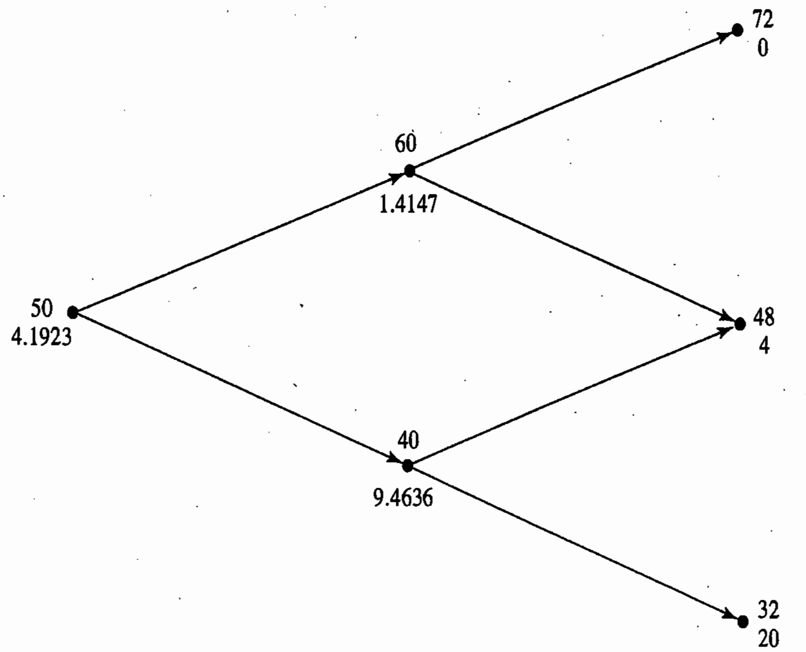
This is consistent with the principle of risk-neutral valuation mentioned earlier. The variables p^2, $2p(1-p)$, and $(1-p)^2$ are the probabilities that the upper, middle, and lower final nodes will be reached. The option price is equal to its expected payoff in a risk-neutral world discounted at the risk-free interest rate.

As we add more steps to the binomial tree, the risk-neutral valuation principle continues to hold. The option price is always equal to its expected payoff in a risk-neutral world discounted at the risk-free interest rate.

11.4 A PUT EXAMPLE

The procedures described in this chapter can be used to price puts as well as calls. Consider a 2-year European put with a strike price of \$52 on a stock whose current price is \$50. We suppose that there are two time steps of 1 year, and in each time step the stock price either moves up by 20% or moves down by 20%. We also suppose that the risk-free interest rate is 5%.

Figure 11.7 Using a two-step tree to value a European put option. At each node, the upper number is the stock price and the lower number is the option price.



The tree is shown in Figure 11.7. In this case $u = 1.2$, $d = 0.8$, $\Delta t = 1$, and $r = 0.05$. From equation (11.6) the value of the risk-neutral probability, p, is given by

$$p = \frac{e^{0.05\times 1} - 0.8}{1.2 - 0.8} = 0.6282$$

The possible final stock prices are: \$72, \$48, and \$32. In this case, $f_{uu} = 0$, $f_{ud} = 4$, and $f_{dd} = 20$. From equation (11.10),

$$f = e^{-2\times 0.05\times 1}(0.6282^2 \times 0 + 2 \times 0.6282 \times 0.3718 \times 4 + 0.3718^2 \times 20) = 4.1923$$

The value of the put is \$4.1923. This result can also be obtained using equation (11.5) and working back through the tree one step at a time. Figure 11.7 shows the intermediate option prices that are calculated.

11.5 AMERICAN OPTIONS

Up to now all the options we have considered have been European. We now move on to consider how American options can be valued using a binomial tree such as that in Figure 11.4 or 11.7. The procedure is to work back through the tree from the end to the beginning, testing at each node to see whether early exercise is optimal. The value of the option at the final nodes is the same as for the European option. At earlier nodes the value of the option is the greater of

1. The value given by equation (11.5)
2. The payoff from early exercise

Figure 11.8 Using a two-step tree to value an American put option. At each node, the upper number is the stock price and the lower number is the option price.

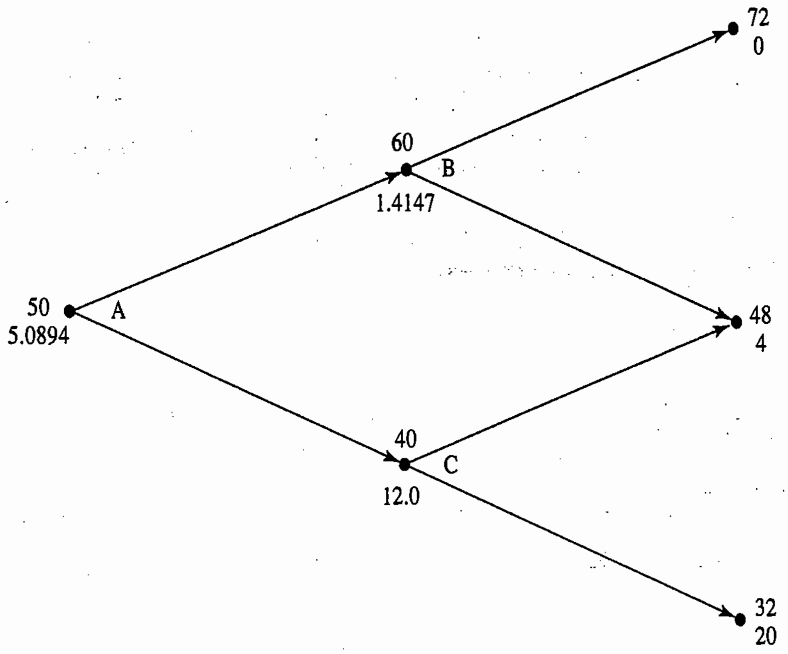


Figure 11.8 shows how Figure 11.7 is affected if the option under consideration is American rather than European. The stock prices and their probabilities are unchanged. The values for the option at the final nodes are also unchanged. At node B, equation (11.5) gives the value of the option as 1.4147, whereas the payoff from early exercise is negative (= −8). Clearly early exercise is not optimal at node B, and the value of the option at this node is 1.4147. At node C, equation (11.5) gives the value of the option as 9.4636, whereas the payoff from early exercise is 12. In this case, early exercise is optimal and the value of the option at the node is 12. At the initial node A, the value given by equation (11.5) is

$$e^{-0.05\times 1}(0.6282 \times 1.4147 + 0.3718 \times 12.0) = 5.0894$$

and the payoff from early exercise is 2. In this case early exercise is not optimal. The value of the option is therefore \$5.0894.

11.6 DELTA

At this stage it is appropriate to introduce *delta*, an important parameter in the pricing and hedging of options.

The delta of a stock option is the ratio of the change in the price of the stock option to the change in the price of the underlying stock. It is the number of units of the stock we should hold for each option shorted in order to create a riskless portfolio. It is the same as the Δ introduced earlier in this chapter. The construction of a riskless hedge is sometimes referred to as *delta hedging*. The delta of a call option is positive, whereas the delta of a put option is negative.

From Figure 11.1, we can calculate the value of the delta of the call option being considered as

$$\frac{1-0}{22-18} = 0.25$$

This is because when the stock price changes from \$18 to \$22, the option price changes from \$0 to \$1.

In Figure 11.4 the delta corresponding to stock price movements over the first time step is

$$\frac{2.0257-0}{22-18} = 0.5064$$

The delta for stock price movements over the second time step is

$$\frac{3.2-0}{24.2-19.8} = 0.7273$$

if there is an upward movement over the first time step, and

$$\frac{0-0}{19.8-16.2} = 0$$

if there is a downward movement over the first time step.

From Figure 11.7, delta is

$$\frac{1.4147-9.4636}{60-40} = -0.4024$$

at the end of the first time step, and either

$$\frac{0-4}{72-48} = -0.1667 \quad \text{or} \quad \frac{4-20}{48-32} = -1.0000$$

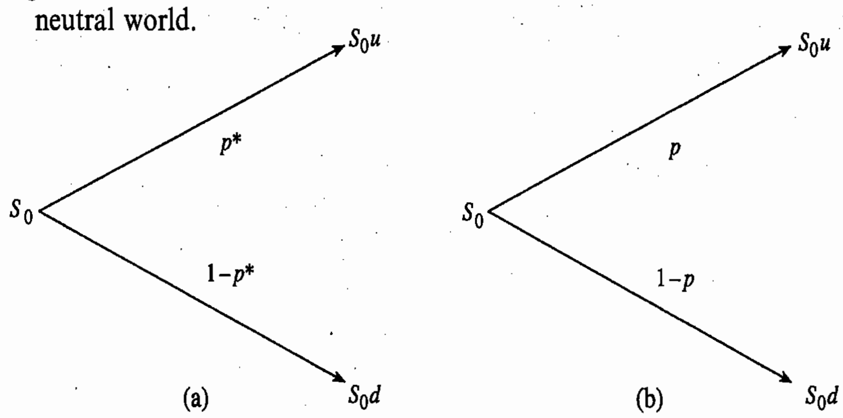
at the end of the second time step.

The two-step examples show that delta changes over time. (In Figure 11.4, delta changes from 0.5064 to either 0.7273 or 0; and, in Figure 11.7, it changes from −0.4024 to either −0.1667 or −1.0000.) Thus, in order to maintain a riskless hedge using an option and the underlying stock, we need to adjust our holdings in the stock periodically. This is a feature of options that we will return to in Chapter 17.

11.7 MATCHING VOLATILITY WITH u AND d

In practice, when constructing a binomial tree to represent the movements in a stock price, we choose the parameters u and d to match the volatility of the stock price. To see how this is done, we suppose that the expected return on a stock (in the real world) is μ and its volatility is σ. Figure 11.9 shows stock price movements over the first step of a binomial tree. The step is of length Δt. The stock price starts at S_0 and moves either up to S_0u or down to S_0d. These are the only two possible outcomes in both the real world and the risk-neutral world. The probability of an up movement in the real world is denoted by p^* and, consistent with our earlier notation, in the risk-neutral world this probability is p.

Figure 11.9 Change in stock price in time Δt in (a) the real world and (b) the risk-neutral world.



The expected stock price at the end of the first time step in the real world is $S_0e^{\mu\Delta t}$. On the tree the expected stock price at this time is

$$p^*S_0u + (1-p^*)S_0d$$

In order to match the expected return on the stock with the tree's parameters, we must therefore have

$$p^*S_0u + (1-p^*)S_0d = S_0e^{\mu\Delta t}$$

or

$$p^* = \frac{e^{\mu\Delta t} - d}{u - d} \tag{11.11}$$

As we will explain in Chapter 13, the volatility σ of a stock price is defined so that $\sigma\sqrt{\Delta t}$ is the standard deviation of the return on the stock price in a short period of time of length Δt. Equivalently, the variance of the return is $\sigma^2\Delta t$. On the tree in Figure 11.9(a), the variance of the stock price return is[2]

$$p^*u^2 + (1-p^*)d^2 - [p^*u + (1-p^*)d]^2$$

In order to match the stock price volatility with the tree's parameters, we must therefore have

$$p^*u^2 + (1-p^*)d^2 - [p^*u + (1-p^*)d]^2 = \sigma^2\Delta t \tag{11.12}$$

Substituting from equation (11.11) into equation (11.12) gives

$$e^{\mu\Delta t}(u+d) - ud - e^{2\mu\Delta t} = \sigma^2\Delta t$$

When terms in Δt^2 and higher powers of Δt are ignored, one solution to this equation is[3]

$$u = e^{\sigma\sqrt{\Delta t}} \tag{11.13}$$

$$d = e^{-\sigma\sqrt{\Delta t}} \tag{11.14}$$

---

[2] This uses the result that the variance of a variable X equals $E(X^2) - [E(X)]^2$, where E denotes expected value.

[3] We are here using the series expansion

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

These are the values of u and d proposed by Cox, Ross, and Rubinstein (1979) for matching volatility.

The analysis in Section 11.2 shows that we can replace the tree in Figure 11.9(a) by the tree in Figure 11.9(b), where the probability of an up movement is p, and then behave as though the world is risk neutral. The variable p is given by equation (11.6) as

$$p = \frac{a-d}{u-d} \tag{11.15}$$

where

$$a = e^{r\Delta t} \tag{11.16}$$

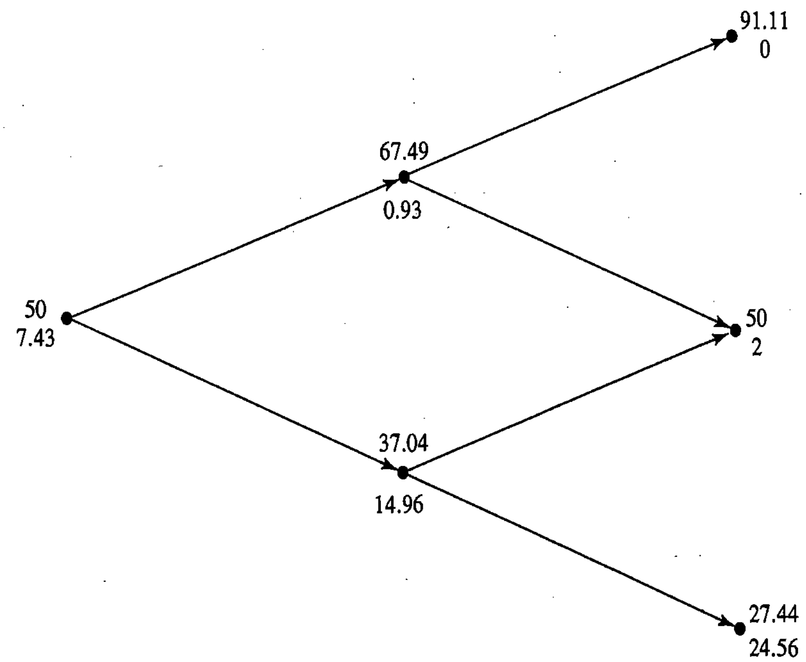
In Figure 11.9(b), the expected stock price at the end of the time step is $S_0 e^{r\Delta t}$, as shown in equation (11.4). The variance of the stock price return in the risk-neutral world is

$$pu^2 + (1-p)d^2 - [pu + (1-p)d]^2 = [e^{r\Delta t}(u+d) - ud - e^{2r\Delta t}]$$

Substituting for u and d from equations (11.13) and (11.14), we find this equals $\sigma^2 \Delta t$ when terms in Δt^2 and higher powers of Δt are ignored.

This analysis shows that when we move from the real world to the risk-neutral world the expected return on the stock changes, but its volatility remains the same (at least in the limit as Δt tends to zero). This is an illustration of an important general result known as *Girsanov's theorem*. When we move from a world with one set of risk preferences to a world with another set of risk preferences, the expected growth rates in variables change, but their volatilities remain the same. We will examine the impact of risk preferences on the behavior of market variables in more detail in Chapter 27. Moving from one set of risk preferences to another is sometimes referred to as *changing*

Figure 11.10 Two-step tree to value a 2-year American put option when the stock price is 50, strike price is 52, risk-free rate is 5%, and volatility is 30%.



the measure. The real-world measure is sometimes referred to as the *P-measure*, while the risk-neutral world measure is referred to as the *Q-measure*.[4]

Consider again the American put option in Figure 11.8, where the stock price is \$50, the strike price is \$52, the risk-free rate is 5%, the life of the option is 2 years, and there are two time steps. In this case, $\Delta t = 1$. Suppose that the volatility σ is 30%. Then, from equations (11.13) to (11.16), we have

$$u = e^{0.3\times 1} = 1.3499, \qquad d = \frac{1}{1.3499} = 0.7408, \qquad a = e^{0.05\times 1} = 1.0513$$

and

$$p = \frac{1.053 - 0.7408}{1.3499 - 0.7408} = 0.5097$$

The tree is shown in Figure 11.10. The value of the put option is 7.43. (This is different from the value obtained in Figure 11.8 by assuming $u = 1.2$ and $d = 0.8$.)

11.8 INCREASING THE NUMBER OF STEPS

The binomial model presented above is unrealistically simple. Clearly, an analyst can expect to obtain only a very rough approximation to an option price by assuming that stock price movements during the life of the option consist of one or two binomial steps.

When binomial trees are used in practice, the life of the option is typically divided into 30 or more time steps. In each time step there is a binomial stock price movement. With 30 time steps there are 31 terminal stock prices and 2^{30}, or about 1 billion, possible stock price paths are implicitly considered.

The equations defining the tree are equations (11.13) to (11.16), regardless of the number of time steps. Suppose, for example, that there are five steps instead of two in the example we considered in Figure 11.10. The parameters would be $\Delta t = 2/5 = 0.4$, $r = 0.05$, and $\sigma = 0.3$. These values give $u = e^{0.3\times\sqrt{0.4}} = 1.2089$, $d = 1/1.2089 = 0.8272$, $a = e^{0.05\times 0.4} = 1.0202$, and $p = (1.0202 - 0.8272)/(1.2089 - 0.8272) = 0.5056$.

Using DerivaGem

The software accompanying this book, DerivaGem, is a useful tool for becoming comfortable with binomial trees. After loading the software in the way described at the end of this book, go to the Equity_FX_Index_Futures_Options worksheet. Choose Equity as the Underlying Type and select Binomial American as the Option Type. Enter the stock price, volatility, risk-free rate, time to expiration, exercise price, and tree steps, as 50, 30%, 5%, 2, 52, and 2, respectively. Click on the *Put* button and then on *Calculate*. The price of the option is shown as 7.428 in the box labeled Price. Now click on *Display Tree* and you will see the equivalent of Figure 11.10. (The red numbers in the software indicate the nodes where the option is exercised.)

Return to the Equity_FX_Index_Futures_Options worksheet and change the number of time steps to 5. Hit *Enter* and click on *Calculate*. You will find that the value of the option changes to 7.671. By clicking on *Display Tree* the five-step tree is displayed, together with the values of u, d, a, and p calculated above.

[4] With the notation we have been using, p is the probability under the Q-measure, while p^* is the probability under the P-measure.

DerivaGem can display trees that have up to 10 steps, but the calculations can be done for up to 500 steps. In our example, 500 steps gives the option price (to two decimal places) as 7.47. This is an accurate answer. By changing the Option Type to Binomial European we can use the tree to value a European option. Using 500 time steps the value of a European option with the same parameters as the American option is 6.76. (By changing the option type to Analytic European we can display the value of the option using the Black–Scholes formula that will be presented in Chapter 13. This is also 6.76.)

By changing the Underlying Type, we can consider options on assets other than stocks. These will now be discussed.

11.9 OPTIONS ON OTHER ASSETS

We introduced options on indices, currencies, and futures contracts in Chapter 8 and will cover them in more detail in Chapters 15 and 16. It turns out that we can construct and use binomial trees for these options in exactly the same way as for options on stocks except that the equations for p change. As in the case of options on stocks, equation (11.2) applies so that the value at a node (before the possibility of early exercise is considered) is p times the value if there is an up movement plus $1-p$ times the value if there is a down movement, discounted at the risk-free rate.

Options on Stocks Paying a Continuous Dividend Yield

Consider a stock paying a known dividend yield at rate q. The total return from dividends and capital gains in a risk-neutral world is r. The dividends provide a return of q. Capital gains must therefore provide a return of $r-q$. If the stock starts at S_0, its expected value after one time step of length Δt must be $S_0e^{(r-q)\Delta t}$. This means that

$$pS_0u + (1-p)S_0d = S_0e^{(r-q)\Delta t}$$

so that

$$p = \frac{e^{(r-q)\Delta t} - d}{u-d}$$

As in the case of options on non-dividend-paying stocks, we match volatility by setting $u = e^{\sigma\sqrt{\Delta t}}$ and $d = 1/u$. This means that we can use equations (11.13) to (11.16), except that we set $a = e^{(r-q)\Delta t}$ instead of $a = e^{r\Delta t}$.

Options on Stock Indices

When calculating a futures price for a stock index in Chapter 5 we assumed that the stocks underlying the index provided a dividend yield at rate q. We make a similar assumption here. The valuation of an option on a stock index is therefore very similar to the valuation of an option on a stock paying a known dividend yield.

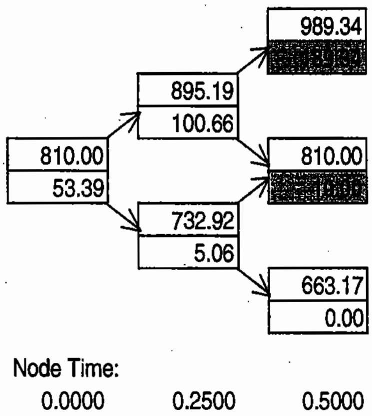
Example 11.1

A stock index is currently 810 and has a volatility of 20% and a dividend yield of 2%. The risk-free rate is 5%. Figure 11.11 shows the output from DerivaGem for valuing a European 6-month call option with a strike price of 800 using a two-step tree.

Figure 11.11 Two-step tree to value a European 6-month call option on an index when the index level is 810, strike price is 800, risk-free rate is 5%, volatility is 20%, and dividend yield is 2% (DerivaGem output).

At each node:
Upper value = Underlying Asset Price
Lower value = Option Price
Shading indicates where option is exercised

Strike price = 800
Discount factor per step = 0.9876
Time step, dt = 0.2500 years, 91.25 days
Growth factor per step, a = 1.0075
Probability of up move, p = 0.5126
Up step size, u = 1.1052
Down step size, d = 0.9048



In this case,

$$\Delta t = 0.25, \qquad u = e^{0.20\times\sqrt{0.25}} = 1.1052,$$
$$d = 1/u = 0.9048, \qquad a = e^{(0.05-0.02)\times 0.25} = 1.0075$$
$$p = (1.0075 - 0.9048)/(1.1052 - 0.9048) = 0.5126$$

The value of the option is 53.39.

Options on Currencies

As pointed out in Section 5.10, a foreign currency can be regarded as an asset providing a yield at the foreign risk-free rate of interest, r_f. By analogy with the stock index case we can construct a tree for options on a currency by using equations (11.13) to (11.16) and setting $a = e^{(r-r_f)\Delta t}$.

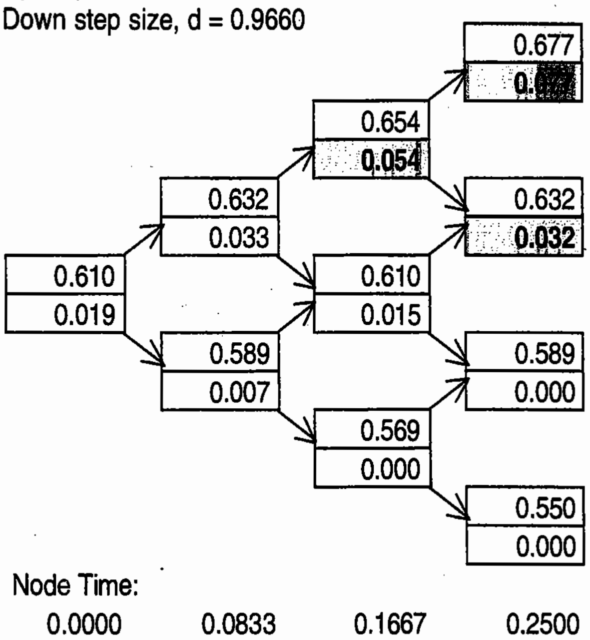
Example 11.2

The Australian dollar is currently worth 0.6100 U.S. dollars and this exchange rate has a volatility of 12%. The Australian risk-free rate is 7% and the U.S. risk-free rate is 5%. Figure 11.12 shows the output from DerivaGem for valuing a 3-month American call option with a strike price of 0.6000 using a three-step tree.

Figure 11.12 Three-step tree to value an American 3-month call option on a currency when the value of the currency is 0.6100, strike price is 0.6000, risk-free rate is 5%, volatility is 12%, and foreign risk-free rate is 7% (DerivaGem output).

At each node:
Upper value = Underlying Asset Price
Lower value = Option Price
Shading indicates where option is exercised

Strike price = 0.6
Discount factor per step = 0.9958
Time step, dt = 0.0833 years, 30.42 days
Growth factor per step, a = 0.9983
Probability of up move, p = 0.4673
Up step size, u = 1.0352
Down step size, d = 0.9660



In this case,

$$\Delta t = 0.08333, \quad u = e^{0.12\times\sqrt{0.08333}} = 1.0352$$
$$d = 1/u = 0.9660, \quad a = e^{(0.05-0.07)\times 0.08333} = 0.9983$$
$$p = (0.9983 - 0.9660)/(1.0352 - 0.9660) = 0.4673$$

The value of the option is 0.019.

Options on Futures

It costs nothing to take a long or a short position in a futures contract. It follows that in a risk-neutral world a futures price should have an expected growth rate of zero. (We discuss this point in more detail in Section 16.7.) As above, we define p as the probability of an up movement in the futures price, u as the percentage up movement, and d as the percentage down movement. If F_0 is the initial futures price, the expected futures price at the end of one time step of length Δt should also be F_0. This means that

$$pF_0u + (1-p)F_0d = F_0$$

so that

$$p = \frac{1-d}{u-d}$$

and we can use equations (11.13) to (11.16) with $a = 1$.

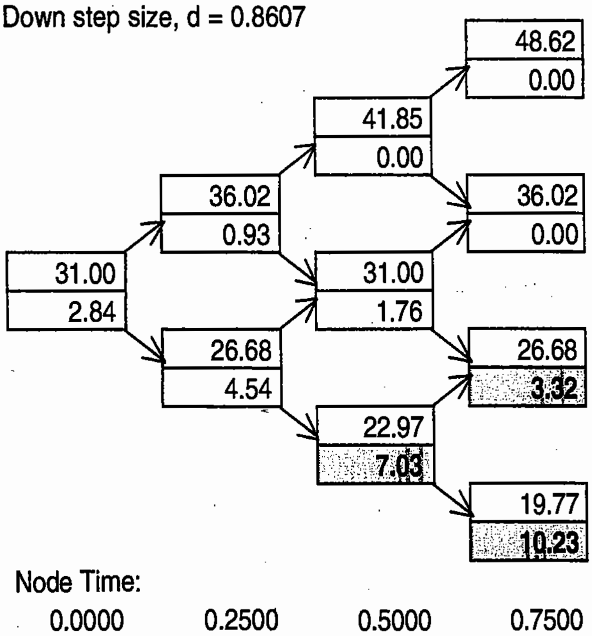
Example 11.3

A futures price is currently 31 and has a volatility of 30%. The risk-free rate is 5%. Figure 11.13 shows the output from DerivaGem for valuing a 9-month American put option with a strike price of 30 using a three-step tree.

Figure 11.13 Three-step tree to value an American 9-month put option on a futures contract when the futures price is 31, strike price is 30, risk-free rate is 5%, and volatility is 30% (DerivaGem output).

At each node:
Upper value = Underlying Asset Price
Lower value = Option Price
Shading indicates where option is exercised

Strike price = 30
Discount factor per step = 0.9876
Time step, dt = 0.2500 years, 91.25 days
Growth factor per step, a = 1.000
Probability of up move, p = 0.4626
Up step size, u = 1.1618
Down step size, d = 0.8607



In this case,

$$\Delta t = 0.25, \quad u = e^{0.3\sqrt{0.25}} = 1.1618$$
$$d = 1/u = 1/1.1618 = 0.8607, \quad a = 1,$$
$$p = (1 - 0.8607)/(1.1618 - 0.8607) = 0.4626$$

The value of the option is 2.84.

SUMMARY

This chapter has provided a first look at the valuation of options on stocks and other assets. In the simple situation where movements in the price of a stock during the life of an option are governed by a one-step binomial tree, it is possible to set up a riskless portfolio consisting of a stock option and a certain amount of the stock. In a world with no arbitrage opportunities, riskless portfolios must earn the risk-free interest. This enables the stock option to be priced in terms of the stock. It is interesting to note that no assumptions are required about the probabilities of up and down movements in the stock price at each node of the tree.

When stock price movements are governed by a multistep binomial tree, we can treat each binomial step separately and work back from the end of the life of the option to the beginning to obtain the current value of the option. Again only no-arbitrage arguments are used, and no assumptions are required about the probabilities of up and down movements in the stock price at each node.

A very important principle states that we can assume the world is risk-neutral when valuing an option. This chapter has shown, through both numerical examples and algebra, that no-arbitrage arguments and risk-neutral valuation are equivalent and lead to the same option prices.

The delta of a stock option, Δ, considers the effect of a small change in the underlying stock price on the change in the option price. It is the ratio of the change in the option price to the change in the stock price. For a riskless position, an investor should buy Δ shares for each option sold. An inspection of a typical binomial tree shows that delta changes during the life of an option. This means that to hedge a particular option position, we must change our holding in the underlying stock periodically.

Constructing binomial trees for valuing options on stock indices, currencies, and futures contracts is very similar to doing so for valuing options on stocks. In Chapter 19, we will return to binomial trees and provide more details on how they can be used in practice.

FURTHER READING

Coval, J. E. and T. Shumway. "Expected Option Returns," *Journal of Finance*, 56, 3 (2001): 983–1009.

Cox, J. C., S. A. Ross, and M. Rubinstein. "Option Pricing: A Simplified Approach." *Journal of Financial Economics* 7 (October 1979): 229–64.

Rendleman, R., and B. Bartter. "Two State Option Pricing." *Journal of Finance* 34 (1979): 1092–1110.

Questions and Problems (Answers in Solutions Manual)

11.1. A stock price is currently \$40. It is known that at the end of 1 month it will be either \$42 or \$38. The risk-free interest rate is 8% per annum with continuous compounding. What is the value of a 1-month European call option with a strike price of \$39?

11.2. Explain the no-arbitrage and risk-neutral valuation approaches to valuing a European option using a one-step binomial tree.

11.3. What is meant by the "delta" of a stock option?

11.4. A stock price is currently \$50. It is known that at the end of 6 months it will be either \$45 or \$55. The risk-free interest rate is 10% per annum with continuous compounding. What is the value of a 6-month European put option with a strike price of \$50?

11.5. A stock price is currently \$100. Over each of the next two 6-month periods it is expected to go up by 10% or down by 10%. The risk-free interest rate is 8% per annum with continuous compounding. What is the value of a 1-year European call option with a strike price of \$100?

11.6. For the situation considered in Problem 11.5, what is the value of a 1-year European put option with a strike price of \$100? Verify that the European call and European put prices satisfy put–call parity.

11.7. What are the formulas for u and d in terms of volatility?

11.8. Consider the situation in which stock price movements during the life of a European option are governed by a two-step binomial tree. Explain why it is not possible to set up a position in the stock and the option that remains riskless for the whole of the life of the option.

11.9. A stock price is currently \$50. It is known that at the end of 2 months it will be either \$53 or \$48. The risk-free interest rate is 10% per annum with continuous compounding. What is the value of a 2-month European call option with a strike price of \$49? Use no-arbitrage arguments.

11.10. A stock price is currently \$80. It is known that at the end of 4 months it will be either \$75 or \$85. The risk-free interest rate is 5% per annum with continuous compounding. What is the value of a 4-month European put option with a strike price of \$80? Use no-arbitrage arguments.

11.11. A stock price is currently \$40. It is known that at the end of 3 months it will be either \$45 or \$35. The risk-free rate of interest with quarterly compounding is 8% per annum. Calculate the value of a 3-month European put option on the stock with an exercise price of \$40. Verify that no-arbitrage arguments and risk-neutral valuation arguments give the same answers.

11.12. A stock price is currently \$50. Over each of the next two 3-month periods it is expected to go up by 6% or down by 5%. The risk-free interest rate is 5% per annum with continuous compounding. What is the value of a 6-month European call option with a strike price of \$51?

11.13. For the situation considered in Problem 11.12, what is the value of a 6-month European put option with a strike price of \$51? Verify that the European call and European put prices satisfy put–call parity. If the put option were American, would it ever be optimal to exercise it early at any of the nodes on the tree?

11.14. A stock price is currently \$25. It is known that at the end of 2 months it will be either \$23 or \$27. The risk-free interest rate is 10% per annum with continuous compounding. Suppose S_T is the stock price at the end of 2 months. What is the value of a derivative that pays off S_T^2 at this time?

11.15. Calculate u, d, and p when a binomial tree is constructed to value an option on a foreign currency. The tree step size is 1 month, the domestic interest rate is 5% per annum, the foreign interest rate is 8% per annum, and the volatility is 12% per annum.

Assignment Questions

11.16. A stock price is currently \$50. It is known that at the end of 6 months it will be either \$60 or \$42. The risk-free rate of interest with continuous compounding is 12% per annum. Calculate the value of a 6-month European call option on the stock with an exercise price of \$48. Verify that no-arbitrage arguments and risk-neutral valuation arguments give the same answers.

11.17. A stock price is currently \$40. Over each of the next two 3-month periods it is expected to go up by 10% or down by 10%. The risk-free interest rate is 12% per annum with continuous compounding.

(a) What is the value of a 6-month European put option with a strike price of \$42?

(b) What is the value of a 6-month American put option with a strike price of \$42?

11.18. Using a "trial-and-error" approach, estimate how high the strike price has to be in Problem 11.17 for it to be optimal to exercise the option immediately.

11.19. A stock price is currently \$30. During each 2-month period for the next 4 months it will increase by 8% or reduce by 10%. The risk-free interest rate is 5%. Use a two-step tree to calculate the value of a derivative that pays off $[\max(30 - S_T, 0)]^2$, where S_T is the stock price in 4 months. If the derivative is American-style, should it be exercised early?

11.20. Consider a European call option on a non-dividend-paying stock where the stock price is \$40, the strike price is \$40, the risk-free rate is 4% per annum, the volatility is 30% per annum, and the time to maturity is 6 months.

(a) Calculate u, d, and p for a two-step tree.

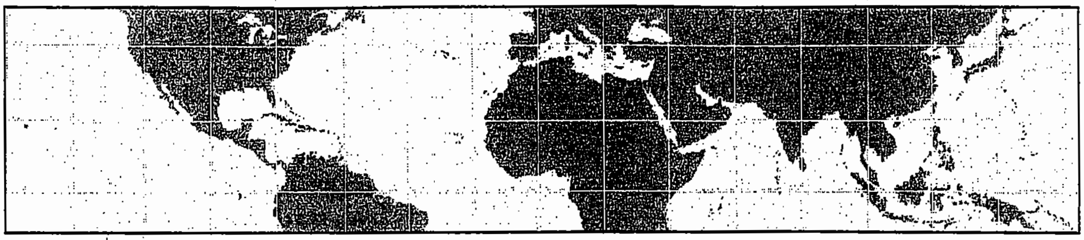
(b) Value the option using a two-step tree.

(c) Verify that DerivaGem gives the same answer.

(d) Use DerivaGem to value the option with 5, 50, 100, and 500 time steps.

11.21. Repeat Problem 11.20 for an American put option on a futures contract. The strike price and the futures price are \$50, the risk-free rate is 10%, the time to maturity is 6 months, and the volatility is 40% per annum.

11.22. Footnote 1 shows that the correct discount rate to use for the real-world expected payoff in the case of the call option considered in Figure 11.1 is 42.6%. Show that if the option is a put rather than a call the discount rate is −52.5%. Explain why the two real-world discount rates are so different.



CHAPTER 12

Wiener Processes and Itô's Lemma

Any variable whose value changes over time in an uncertain way is said to follow a *stochastic process*. Stochastic processes can be classified as *discrete time* or *continuous time*. A discrete-time stochastic process is one where the value of the variable can change only at certain fixed points in time, whereas a continuous-time stochastic process is one where changes can take place at any time. Stochastic processes can also be classified as *continuous variable* or *discrete variable*. In a continuous-variable process, the underlying variable can take any value within a certain range, whereas in a discrete-variable process, only certain discrete values are possible.

This chapter develops a continuous-variable, continuous-time stochastic process for stock prices. Learning about this process is the first step to understanding the pricing of options and other more complicated derivatives. It should be noted that, in practice, we do not observe stock prices following continuous-variable, continuous-time processes. Stock prices are restricted to discrete values (e.g., multiples of a cent) and changes can be observed only when the exchange is open. Nevertheless, the continuous-variable, continuous-time process proves to be a useful model for many purposes.

Many people feel that continuous-time stochastic processes are so complicated that they should be left entirely to "rocket scientists". This is not so. The biggest hurdle to understanding these processes is the notation. Here we present a step-by-step approach aimed at getting the reader over this hurdle. We also explain an important result known as *Itô's lemma* that is central to the pricing of derivatives.

12.1 THE MARKOV PROPERTY

A *Markov process* is a particular type of stochastic process where only the present value of a variable is relevant for predicting the future. The past history of the variable and the way that the present has emerged from the past are irrelevant.

Stock prices are usually assumed to follow a Markov process. Suppose that the price of IBM stock is \$100 now. If the stock price follows a Markov process, our predictions for the future should be unaffected by the price one week ago, one month ago, or one year ago. The only relevant piece of information is that the price is now \$100.[1] Predictions for the future are uncertain and must be expressed in terms of probability distributions. The Markov property implies that the probability distribution of the price at any particular future time is not dependent on the particular path followed by the price in the past.

The Markov property of stock prices is consistent with the weak form of market efficiency. This states that the present price of a stock impounds all the information contained in a record of past prices. If the weak form of market efficiency were not true, technical analysts could make above-average returns by interpreting charts of the past history of stock prices. There is very little evidence that they are in fact able to do this.

It is competition in the marketplace that tends to ensure that weak-form market efficiency holds. There are many investors watching the stock market closely. Trying to make a profit from it leads to a situation where a stock price, at any given time, reflects the information in past prices. Suppose that it was discovered that a particular pattern in stock prices always gave a 65% chance of subsequent steep price rises. Investors would attempt to buy a stock as soon as the pattern was observed, and demand for the stock would immediately rise. This would lead to an immediate rise in its price and the observed effect would be eliminated, as would any profitable trading opportunities.

12.2 CONTINUOUS-TIME STOCHASTIC PROCESSES

Consider a variable that follows a Markov stochastic process. Suppose that its current value is 10 and that the change in its value during 1 year is $\phi(0, 1)$, where $\phi(m, v)$ denotes a probability distribution that is normally distributed with mean m and variance v.[2] What is the probability distribution of the change in the value of the variable during 2 years?

The change in 2 years is the sum of two normal distributions, each of which has a mean of zero and variance of 1.0. Because the variable is Markov, the two probability distributions are independent. When we add two independent normal distributions, the result is a normal distribution where the mean is the sum of the means and the variance is the sum of the variances. The mean of the change during 2 years in the variable we are considering is, therefore, zero and the variance of this change is 2.0. Hence, the change in the variable over 2 years has the distribution $\phi(0, 2)$. The standard deviation of the distribution is $\sqrt{2}$.

Consider next the change in the variable during 6 months. The variance of the change in the value of the variable during 1 year equals the variance of the change during the first 6 months plus the variance of the change during the second 6 months. We assume these are the same. It follows that the variance of the change during a 6-month period must be 0.5. Equivalently, the standard deviation of the change is $\sqrt{0.5}$. The probability distribution for the change in the value of the variable during 6 months is $\phi(0, 0.5)$.

[1] Statistical properties of the stock price history of IBM may be useful in determining the characteristics of the stochastic process followed by the stock price (e.g., its volatility). The point being made here is that the particular path followed by the stock in the past is irrelevant.

[2] Variance is the square of standard deviation. The variance of a 1-year change in the value of the variable we are considering is therefore 1.0.

A similar argument shows that the probability distribution for the change in the value of the variable during 3 months is $\phi(0, 0.25)$. More generally, the change during any time period of length T is $\phi(0, T)$. In particular, the change during a very short time period of length Δt is $\phi(0, \Delta t)$.

Note that, when Markov processes are considered, the variances of the changes in successive time periods are additive. The standard deviations of the changes in successive time periods are not additive. The variance of the change in the variable in our example is 1.0 per year, so that the variance of the change in 2 years is 2.0 and the variance of the change in 3 years is 3.0. The standard deviations of the changes in 2 and 3 years are $\sqrt{2}$ and $\sqrt{3}$, respectively. Strictly speaking, we should not refer to the standard deviation of the variable as 1.0 per year. The results explain why uncertainty is sometimes referred to as being proportional to the square root of time.

Wiener Processes

The process followed by the variable we have been considering is known as a *Wiener process*. It is a particular type of Markov stochastic process with a mean change of zero and a variance rate of 1.0 per year. It has been used in physics to describe the motion of a particle that is subject to a large number of small molecular shocks and is sometimes referred to as *Brownian motion*.

Expressed formally, a variable z follows a Wiener process if it has the following two properties:

PROPERTY 1. *The change Δz during a small period of time Δt is*

$$\Delta z = \epsilon\sqrt{\Delta t} \tag{12.1}$$

where ϵ has a standardized normal distribution $\phi(0, 1)$.

PROPERTY 2. *The values of Δz for any two different short intervals of time, Δt, are independent.*

It follows from the first property that Δz itself has a normal distribution with

$$\text{mean of } \Delta z = 0$$
$$\text{standard deviation of } \Delta z = \sqrt{\Delta t}$$
$$\text{variance of } \Delta z = \Delta t$$

The second property implies that z follows a Markov process.

Consider the change in the value of z during a relatively long period of time, T. This can be denoted by $z(T) - z(0)$. It can be regarded as the sum of the changes in z in N small time intervals of length Δt, where

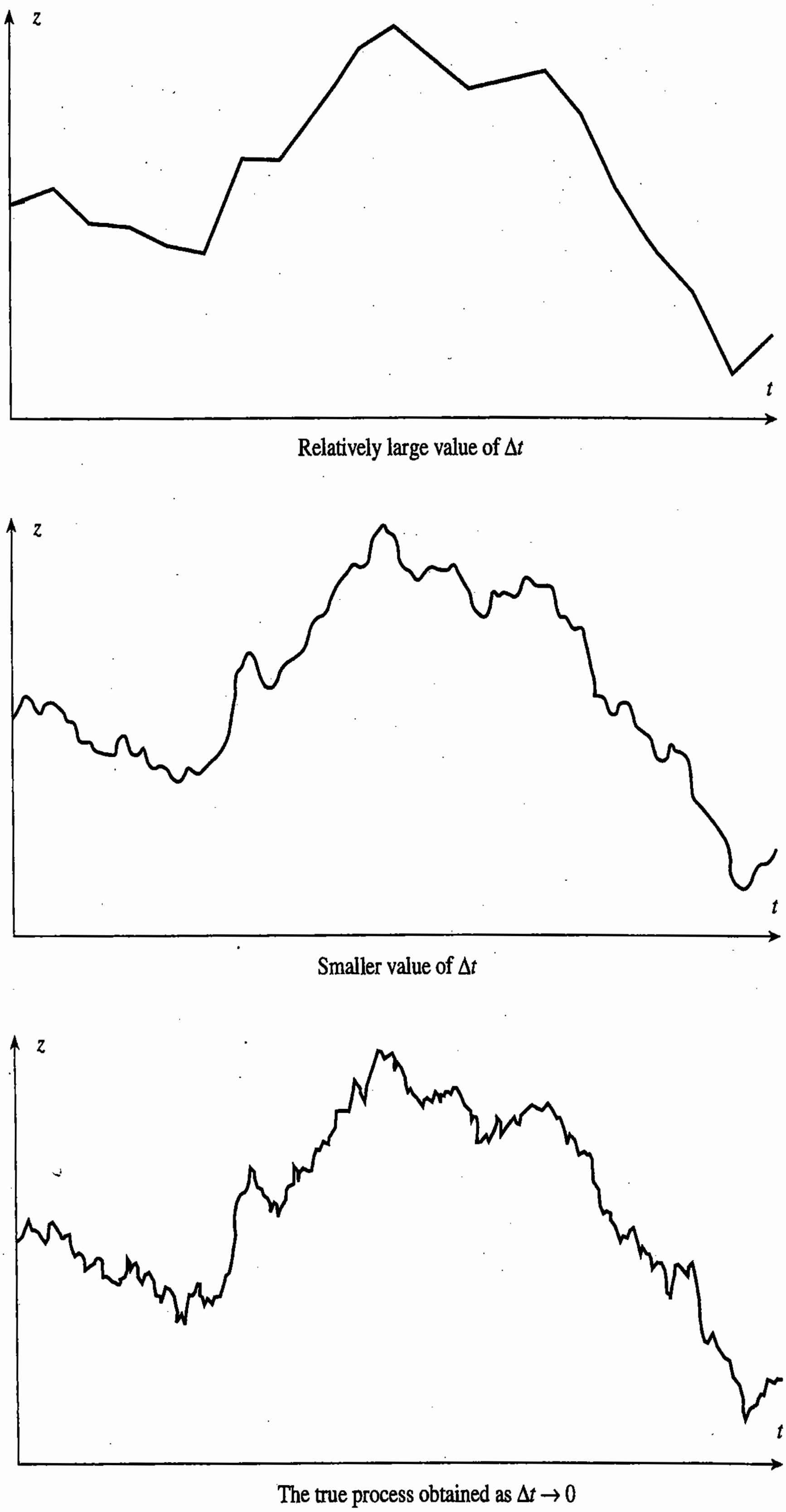
$$N = \frac{T}{\Delta t}$$

Thus,

$$z(T) - z(0) = \sum_{i=1}^{N} \epsilon_i \sqrt{\Delta t} \tag{12.2}$$

where the ϵ_i $(i = 1, 2, \ldots, N)$ are distributed $\phi(0, 1)$. We know from the second property of Wiener processes that the ϵ_i are independent of each other. It follows

Figure 12.1 How a Wiener process is obtained when $\Delta t \to 0$ in equation (12.1).



from equation (12.2) that $z(T) - z(0)$ is normally distributed, with

$$\text{mean of } [z(T) - z(0)] = 0$$
$$\text{variance of}[z(T) - z(0)] = N\,\Delta t = T$$
$$\text{standard deviation of } [z(T) - z(0)] = \sqrt{T}$$

This is consistent with the discussion earlier in this section.

Example 12.1

Suppose that the value, z, of a variable that follows a Wiener process is initially 25 and that time is measured in years. At the end of 1 year, the value of the variable is normally distributed with a mean of 25 and a standard deviation of 1.0. At the end of 5 years, it is normally distributed with a mean of 25 and a standard deviation of $\sqrt{5}$, or 2.236. Our uncertainty about the value of the variable at a certain time in the future, as measured by its standard deviation, increases as the square root of how far we are looking ahead.

In ordinary calculus, it is usual to proceed from small changes to the limit as the small changes become closer to zero. Thus, $dx = a\,dt$ is the notation used to indicate that $\Delta x = a\,\Delta t$ in the limit as $\Delta t \to 0$. We use similar notational conventions in stochastic calculus. So, when we refer to dz as a Wiener process, we mean that it has the properties for Δz given above in the limit as $\Delta t \to 0$.

Figure 12.1 illustrates what happens to the path followed by z as the limit $\Delta t \to 0$ is approached. Note that the path is quite "jagged". This is because the size of a movement in z in time Δt is proportional to $\sqrt{\Delta t}$ and, when Δt is small, $\sqrt{\Delta t}$ is much bigger than Δt. Two intriguing properties of Wiener processes, related to this $\sqrt{\Delta t}$ property, are as follows:

1. The expected length of the path followed by z in any time interval is infinite.
2. The expected number of times z equals any particular value in any time interval is infinite.

Generalized Wiener Process

The mean change per unit time for a stochastic process is known as the *drift rate* and the variance per unit time is known as the *variance rate*. The basic Wiener process, dz, that has been developed so far has a drift rate of zero and a variance rate of 1.0. The drift rate of zero means that the expected value of z at any future time is equal to its current value. The variance rate of 1.0 means that the variance of the change in z in a time interval of length T equals T. A *generalized Wiener process* for a variable x can be defined in terms of dz as

$$dx = a\,dt + b\,dz \tag{12.3}$$

where a and b are constants.

To understand equation (12.3), it is useful to consider the two components on the right-hand side separately. The $a\,dt$ term implies that x has an expected drift rate of a per unit of time. Without the $b\,dz$ term, the equation is $dx = a\,dt$, which implies that $dx/dt = a$. Integrating with respect to time, we get

$$x = x_0 + at$$

where x_0 is the value of x at time 0. In a period of time of length T, the variable x increases by an amount aT. The $b\,dz$ term on the right-hand side of equation (12.3) can be regarded as adding noise or variability to the path followed by x. The amount of this noise or variability is b times a Wiener process. A Wiener process has a standard deviation of 1.0. It follows that b times a Wiener process has a standard deviation of b. In a small time interval Δt, the change Δx in the value of x is given by equations (12.1) and (12.3) as

$$\Delta x = a\,\Delta t + b\epsilon\sqrt{\Delta t}$$

where, as before, ϵ has a standard normal distribution. Thus Δx has a normal distribution with

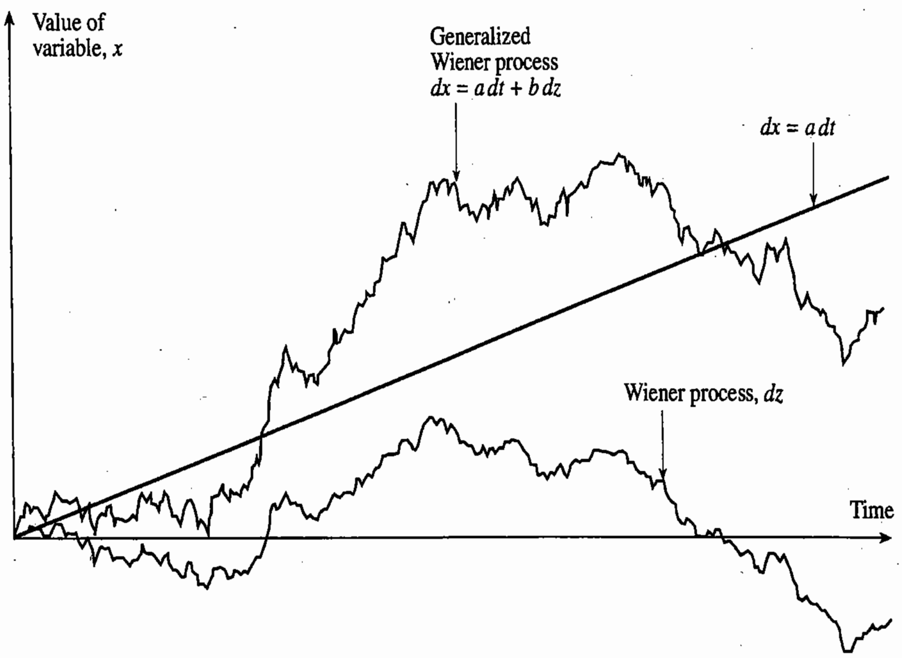
$$\text{mean of } \Delta x = a\,\Delta t$$
$$\text{standard deviation of } \Delta x = b\sqrt{\Delta t}$$
$$\text{variance of } \Delta x = b^2\Delta t$$

Similar arguments to those given for a Wiener process show that the change in the value of x in any time interval T is normally distributed with

$$\text{mean of change in } x = aT$$
$$\text{standard deviation of change in } x = b\sqrt{T}$$
$$\text{variance of change in } x = b^2T$$

Thus, the generalized Wiener process given in equation (12.3) has an expected drift rate (i.e., average drift per unit of time) of a and a variance rate (i.e., variance per unit of time) of b^2. It is illustrated in Figure 12.2.

Figure 12.2 Generalized Wiener process with $a = 0.3$ and $b = 1.5$.



Example 12.2

Consider the situation where the cash position of a company, measured in thousands of dollars, follows a generalized Wiener process with a drift of 20 per year and a variance rate of 900 per year. Initially, the cash position is 50. At the end of 1 year the cash position will have a normal distribution with a mean of 70 and a standard deviation of $\sqrt{900}$, or 30. At the end of 6 months it will have a normal distribution with a mean of 60 and a standard deviation of $30\sqrt{0.5} = 21.21$. Our uncertainty about the cash position at some time in the future, as measured by its standard deviation, increases as the square root of how far ahead we are looking. Note that the cash position can become negative. (We can interpret this as a situation where the company is borrowing funds.)

Itô Process

A further type of stochastic process, known as an *Itô process*, can be defined. This is a generalized Wiener process in which the parameters a and b are functions of the value of the underlying variable x and time t. An Itô process can be written algebraically as

$$dx = a(x, t)\,dt + b(x, t)\,dz \tag{12.4}$$

Both the expected drift rate and variance rate of an Itô process are liable to change over time. In a small time interval between t and $t + \Delta t$, the variable changes from x to $x + \Delta x$, where

$$\Delta x = a(x, t)\Delta t + b(x, t)\epsilon\sqrt{\Delta t}$$

This relationship involves a small approximation. It assumes that the drift and variance rate of x remain constant, equal to $a(x, t)$ and $b(x, t)^2$, respectively, during the time interval between t and $t + \Delta t$.

12.3 THE PROCESS FOR A STOCK PRICE

In this section we discuss the stochastic process usually assumed for the price of a non-dividend-paying stock.

It is tempting to suggest that a stock price follows a generalized Wiener process; that is, that it has a constant expected drift rate and a constant variance rate. However, this model fails to capture a key aspect of stock prices. This is that the expected percentage return required by investors from a stock is independent of the stock's price. If investors require a 14% per annum expected return when the stock price is \$10, then, *ceteris paribus*, they will also require a 14% per annum expected return when it is \$50.

Clearly, the assumption of constant expected drift rate is inappropriate and needs to be replaced by the assumption that the expected return (i.e., expected drift divided by the stock price) is constant. If S is the stock price at time t, then the expected drift rate in S should be assumed to be μS for some constant parameter μ. This means that in a short interval of time, Δt, the expected increase in S is $\mu S\,\Delta t$. The parameter μ is the expected rate of return on the stock, expressed in decimal form.

If the volatility of the stock price is always zero, then this model implies that

$$\Delta S = \mu S\,\Delta t$$

In the limit, as $\Delta t \to 0$,

$$dS = \mu S\,dt$$

or

$$\frac{dS}{S} = \mu\,dt$$

Integrating between time 0 and time T, we get

$$S_T = S_0 e^{\mu T} \tag{12.5}$$

where S_0 and S_T are the stock price at time 0 and time T. Equation (12.5) shows that, when the variance rate is zero, the stock price grows at a continuously compounded rate of μ per unit of time.

In practice, of course, a stock price does exhibit volatility. A reasonable assumption is that the variability of the percentage return in a short period of time, Δt, is the same regardless of the stock price. In other words, an investor is just as uncertain of the percentage return when the stock price is \$50 as when it is \$10. This suggests that the standard deviation of the change in a short period of time Δt should be proportional to the stock price and leads to the model

$$dS = \mu S\,dt + \sigma S\,dz$$

or

$$\frac{dS}{S} = \mu\,dt + \sigma\,dz \tag{12.6}$$

Equation (12.6) is the most widely used model of stock price behavior. The variable σ is the volatility of the stock price. The variable μ is its expected rate of return. The model in equation (12.6) can be regarded as the limiting case of the random walk represented by the binomial trees in Chapter 11 as the time step becomes smaller.

Discrete-Time Model

The model of stock price behavior we have developed is known as *geometric Brownian motion*. The discrete-time version of the model is

$$\frac{\Delta S}{S} = \mu\,\Delta t + \sigma\epsilon\sqrt{\Delta t} \tag{12.7}$$

or

$$\Delta S = \mu S\,\Delta t + \sigma S\epsilon\sqrt{\Delta t} \tag{12.8}$$

The variable ΔS is the change in the stock price, S, in a small time interval Δt, and ϵ has a standard normal distribution (i.e., a normal distribution with a mean of zero and standard deviation of 1.0). The parameter μ is the expected rate of return per unit of time from the stock and the parameter σ is the volatility of the stock price. In this chapter we will assume these parameters are constant.

The left-hand side of equation (12.7) is the return provided by the stock in a short period of time, Δt. The term $\mu\,\Delta t$ is the expected value of this return, and the term $\sigma\epsilon\sqrt{\Delta t}$ is the stochastic component of the return. The variance of the stochastic component (and, therefore, of the whole return) is $\sigma^2\Delta t$. This is consistent with the definition of the volatility σ given in Section 11.7; that is, σ is such that $\sigma\sqrt{\Delta t}$ is the standard deviation of the return in a short time period Δt.

Equation (12.7) shows that $\Delta S/S$ is normally distributed with mean $\mu\,\Delta t$ and standard deviation $\sigma\sqrt{\Delta t}$. In other words,

$$\frac{\Delta S}{S} \sim \phi(\mu\,\Delta t,\ \sigma^2\Delta t) \tag{12.9}$$

Example 12.3

Consider a stock that pays no dividends, has a volatility of 30% per annum, and provides an expected return of 15% per annum with continuous compounding. In this case, $\mu = 0.15$ and $\sigma = 0.30$. The process for the stock price is

$$\frac{dS}{S} = 0.15\,dt + 0.30\,dz$$

If S is the stock price at a particular time and ΔS is the increase in the stock price in the next small interval of time,

$$\frac{\Delta S}{S} = 0.15\Delta t + 0.30\epsilon\sqrt{\Delta t}$$

where ϵ has a standard normal distribution. Consider a time interval of 1 week, or 0.0192 year, so that $\Delta t = 0.0192$. Then

$$\frac{\Delta S}{S} = 0.00288 + 0.0416\epsilon$$

or

$$\Delta S = 0.00288S + 0.0416S\epsilon$$

Monte Carlo Simulation

A Monte Carlo simulation of a stochastic process is a procedure for sampling random outcomes for the process. We will use it as a way of developing some understanding of the nature of the stock price process in equation (12.6).

Consider the situation in Example 12.3 where the expected return from a stock is 15% per annum and the volatility is 30% per annum. The stock price change over 1 week was shown to be

$$\Delta S = 0.00288S + 0.0416S\epsilon \tag{12.10}$$

A path for the stock price over 10 weeks can be simulated by sampling repeatedly for ϵ from $\phi(0, 1)$ and substituting into equation (12.10). The expression =RAND( ) in Excel produces a random sample between 0 and 1. The inverse cumulative normal distribution is NORMSINV. The instruction to produce a random sample from a standard normal distribution in Excel is therefore =NORMSINV(RAND( )). Table 12.1 shows one path for a stock price that was sampled in this way. The initial stock price is assumed to be \$100. For the first period, ϵ is sampled as 0.52. From equation (12.10), the change during the first time period is

$$\Delta S = 0.00288 \times 100 + 0.0416 \times 100 \times 0.52 = 2.45$$

Therefore, at the beginning of the second time period, the stock price is \$102.45. The value of ϵ sampled for the next period is 1.44. From equation (12.10), the change during the second time period is

$$\Delta S = 0.00288 \times 102.45 + 0.0416 \times 102.45 \times 1.44 = 6.43$$

Table 12.1 Simulation of stock price when $\mu = 0.15$ and $\sigma = 0.30$ during 1-week periods.

Stock price at start of period	*Random sample for ϵ*	*Change in stock price during period*
100.00	0.52	2.45
102.45	1.44	6.43
108.88	−0.86	−3.58
105.30	1.46	6.70
112.00	−0.69	−2.89
109.11	−0.74	−3.04
106.06	0.21	1.23
107.30	−1.10	−4.60
102.69	0.73	3.41
106.11	1.16	5.43
111.54	2.56	12.20

So, at the beginning of the next period, the stock price is \$108.88, and so on. Note that, because the process we are simulating is Markov, the samples for ϵ should be independent of each other.[3]

Table 12.1 assumes that stock prices are measured to the nearest cent. It is important to realize that the table shows only one possible pattern of stock price movements. Different random samples would lead to different price movements. Any small time interval Δt can be used in the simulation. In the limit as $\Delta t \to 0$, a perfect description of the stochastic process is obtained. The final stock price of 111.54 in Table 12.1 can be regarded as a random sample from the distribution of stock prices at the end of 10 weeks. By repeatedly simulating movements in the stock price, a complete probability distribution of the stock price at the end of this time is obtained. Monte Carlo simulation is discussed in more detail in Chapter 19.

12.4 THE PARAMETERS

The process for a stock price developed in this chapter involves two parameters, μ and σ. The parameter μ is the expected return (annualized) earned by an investor in a short period of time. Most investors require higher expected returns to induce them to take higher risks. It follows that the value of μ should depend on the risk of the return from the stock.[4] It should also depend on the level of interest rates in the economy. The higher the level of interest rates, the higher the expected return required on any given stock.

Fortunately, we do not have to concern ourselves with the determinants of μ in any detail because the value of a derivative dependent on a stock is, in general, independent

[3] In practice, it is more efficient to sample $\ln S$ rather than S, as will be discussed in Section 19.6.

[4] More precisely, μ depends on that part of the risk that cannot be diversified away by the investor.

of μ. The parameter σ, the stock price volatility, is, by contrast, critically important to the determination of the value of many derivatives. We will discuss procedures for estimating σ in Chapter 13. Typical values of σ for a stock are in the range 0.15 to 0.60 (i.e., 15% to 60%).

The standard deviation of the proportional change in the stock price in a small interval of time Δt is $\sigma\sqrt{\Delta t}$. As a rough approximation, the standard deviation of the proportional change in the stock price over a relatively long period of time T is $\sigma\sqrt{T}$. This means that, as an approximation, volatility can be interpreted as the standard deviation of the change in the stock price in 1 year. In Chapter 13, we will show that the volatility of a stock price is exactly equal to the standard deviation of the continuously compounded return provided by the stock in 1 year.

12.5 ITÔ'S LEMMA

The price of a stock option is a function of the underlying stock's price and time. More generally, we can say that the price of any derivative is a function of the stochastic variables underlying the derivative and time. A serious student of derivatives must, therefore, acquire some understanding of the behavior of functions of stochastic variables. An important result in this area was discovered by the mathematician K. Itô in 1951,[5] and is known as *Itô's lemma.*

Suppose that the value of a variable x follows the Itô process

$$dx = a(x,t)\,dt + b(x,t)\,dz \tag{12.11}$$

where dz is a Wiener process and a and b are functions of x and t. The variable x has a drift rate of a and a variance rate of b^2. Itô's lemma shows that a function G of x and t follows the process

$$dG = \left(\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \tfrac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\right)dt + \frac{\partial G}{\partial x}b\,dz \tag{12.12}$$

where the dz is the same Wiener process as in equation (12.11). Thus, G also follows an Itô process, with a drift rate of

$$\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \tfrac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2$$

and a variance rate of

$$\left(\frac{\partial G}{\partial x}\right)^2 b^2$$

A completely rigorous proof of Itô's lemma is beyond the scope of this book. In the appendix to this chapter, we show that the lemma can be viewed as an extension of well-known results in differential calculus.

Earlier, we argued that

$$dS = \mu S\,dt + \sigma S\,dz \tag{12.13}$$

[5] See K. Itô, "On Stochastic Differential Equations," *Memoirs of the American Mathematical Society,* 4 (1951): 1–51.

with μ and σ constant, is a reasonable model of stock price movements. From Itô's lemma, it follows that the process followed by a function G of S and t is

$$dG = \left(\frac{\partial G}{\partial S}\mu S + \frac{\partial G}{\partial t} + \tfrac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2\right) dt + \frac{\partial G}{\partial S}\sigma S\, dz \tag{12.14}$$

Note that both S and G are affected by the same underlying source of uncertainty, dz. This proves to be very important in the derivation of the Black–Scholes results.

Application to Forward Contracts

To illustrate Itô's lemma, consider a forward contract on a non-dividend-paying stock. Assume that the risk-free rate of interest is constant and equal to r for all maturities. From equation (5.1),

$$F_0 = S_0 e^{rT}$$

where F_0 is the forward price at time zero, S_0 is the spot price at time zero, and T is the time to maturity of the forward contract.

We are interested in what happens to the forward price as time passes. We define F as the forward price at a general time t, and S as the stock price at time t, with $t < T$. The relationship between F and S is given by

$$F = Se^{r(T-t)} \tag{12.15}$$

Assuming that the process for S is given by equation (12.13), we can use Itô's lemma to determine the process for F. From equation (12.15),

$$\frac{\partial F}{\partial S} = e^{r(T-t)}, \qquad \frac{\partial^2 F}{\partial S^2} = 0, \qquad \frac{\partial F}{\partial t} = -rSe^{r(T-t)}$$

From equation (12.14), the process for F is given by

$$dF = \left[e^{r(T-t)}\mu S - rSe^{r(T-t)}\right] dt + e^{r(T-t)}\sigma S\, dz$$

Substituting F for $Se^{r(T-t)}$ gives

$$dF = (\mu - r)F\, dt + \sigma F\, dz \tag{12.16}$$

Like S, the forward price F follows geometric Brownian motion. It has an expected growth rate of $\mu - r$ rather than μ. The growth rate in F is the excess return of S over the risk-free rate.

12.6 THE LOGNORMAL PROPERTY

We now use Itô's lemma to derive the process followed by $\ln S$ when S follows the process in equation (12.13). We define

$$G = \ln S$$

Since

$$\frac{\partial G}{\partial S} = \frac{1}{S}, \qquad \frac{\partial^2 G}{\partial S^2} = -\frac{1}{S^2}, \qquad \frac{\partial G}{\partial t} = 0$$

it follows from equation (12.14) that the process followed by G is

$$dG = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma\, dz \tag{12.17}$$

Since μ and σ are constant, this equation indicates that $G = \ln S$ follows a generalized Wiener process. It has constant drift rate $\mu - \sigma^2/2$ and constant variance rate σ^2. The change in $\ln S$ between time 0 and some future time T is therefore normally distributed, with mean $(\mu - \sigma^2/2)T$ and variance $\sigma^2 T$. This means that

$$\ln S_T - \ln S_0 \sim \phi\left[\left(\mu - \frac{\sigma^2}{2}\right)T,\ \sigma^2 T\right] \tag{12.18}$$

or

$$\ln S_T \sim \phi\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T,\ \sigma^2 T\right] \tag{12.19}$$

where S_T is the stock price at a future time T, S_0 is the stock price at time 0, and as before $\phi(m, v)$ denotes a normal distribution with mean m and variance v.

Equation (12.19) shows that $\ln S_T$ is normally distributed. A variable has a lognormal distribution if the natural logarithm of the variable is normally distributed. The model of stock price behavior we have developed in this chapter therefore implies that a stock's price at time T, given its price today, is lognormally distributed. The standard deviation of the logarithm of the stock price is $\sigma\sqrt{T}$. It is proportional to the square root of how far ahead we are looking.

SUMMARY

Stochastic processes describe the probabilistic evolution of the value of a variable through time. A Markov process is one where only the present value of the variable is relevant for predicting the future. The past history of the variable and the way in which the present has emerged from the past is irrelevant.

A Wiener process dz is a process describing the evolution of a normally distributed variable. The drift of the process is zero and the variance rate is 1.0 per unit time. This means that, if the value of the variable is x_0 at time 0, then at time T it is normally distributed with mean x_0 and standard deviation $\sqrt{T}$.

A generalized Wiener process describes the evolution of a normally distributed variable with a drift of a per unit time and a variance rate of b^2 per unit time, where a and b are constants. This means that if, as before, the value of the variable is x_0 at time 0, it is normally distributed with a mean of $x_0 + aT$ and a standard deviation of $b\sqrt{T}$ at time T.

An Itô process is a process where the drift and variance rate of x can be a function of both x itself and time. The change in x in a very short period of time is, to a good approximation, normally distributed, but its change over longer periods of time is liable to be nonnormal.

One way of gaining an intuitive understanding of a stochastic process for a variable is to simulate the behavior of the variable. This involves dividing a time interval into many small time steps and randomly sampling possible paths for the variable. The future probability distribution for the variable can then be calculated. Monte Carlo simulation is discussed further in Chapter 19.

Itô's lemma is a way of calculating the stochastic process followed by a function of a variable from the stochastic process followed by the variable itself. As we shall see in Chapter 13, Itô's lemma plays a very important part in the pricing of derivatives. A key point is that the Wiener process *dz* underlying the stochastic process for the variable is exactly the same as the Wiener process underlying the stochastic process for the function of the variable. Both are subject to the same underlying source of uncertainty.

The stochastic process usually assumed for a stock price is geometric Brownian motion. Under this process the return to the holder of the stock in a small period of time is normally distributed and the returns in two nonoverlapping periods are independent. The value of the stock price at a future time has a lognormal distribution. The Black–Scholes model, which we cover in the next chapter, is based on the geometric Brownian motion assumption.

FURTHER READING

On Efficient Markets and the Markov Property of Stock Prices

Brealey, R. A. *An Introduction to Risk and Return from Common Stock*, 2nd edn. Cambridge, MA: MIT Press, 1986.

Cootner, P. H. (ed.) *The Random Character of Stock Market Prices*. Cambridge, MA: MIT Press, 1964.

On Stochastic Processes

Cox, D. R., and H. D. Miller. *The Theory of Stochastic Processes*. London: Chapman & Hall, 1977.

Feller, W. *Introduction to Probability Theory and Its Applications*. New York: Wiley, 1968.

Karlin, S., and H. M. Taylor. *A First Course in Stochastic Processes*, 2nd edn. New York: Academic Press, 1975.

Neftci, S. *Introduction to Mathematics of Financial Derivatives*, 2nd edn. New York: Academic Press, 2000.

Questions and Problems (Answers in Solutions Manual)

12.1. What would it mean to assert that the temperature at a certain place follows a Markov process? Do you think that temperatures do, in fact, follow a Markov process?

12.2. Can a trading rule based on the past history of a stock's price ever produce returns that are consistently above average? Discuss.

12.3. A company's cash position, measured in millions of dollars, follows a generalized Wiener process with a drift rate of 0.5 per quarter and a variance rate of 4.0 per quarter. How high does the company's initial cash position have to be for the company to have a less than 5% chance of a negative cash position by the end of 1 year?

12.4. Variables X_1 and X_2 follow generalized Wiener processes, with drift rates μ_1 and μ_2 and variances σ_1^2 and σ_2^2. What process does $X_1 + X_2$ follow if:

(a) The changes in X_1 and X_2 in any short interval of time are uncorrelated?

(b) There is a correlation ρ between the changes in X_1 and X_2 in any short time interval?

12.5. Consider a variable S that follows the process

$$dS = \mu\, dt + \sigma\, dz$$

For the first three years, $\mu = 2$ and $\sigma = 3$; for the next three years, $\mu = 3$ and $\sigma = 4$. If the initial value of the variable is 5, what is the probability distribution of the value of the variable at the end of year 6?

12.6. Suppose that G is a function of a stock price S and time. Suppose that σ_S and σ_G are the volatilities of S and G. Show that, when the expected return of S increases by $\lambda\sigma_S$, the growth rate of G increases by $\lambda\sigma_G$, where λ is a constant.

12.7. Stock A and stock B both follow geometric Brownian motion. Changes in any short interval of time are uncorrelated with each other. Does the value of a portfolio consisting of one of stock A and one of stock B follow geometric Brownian motion? Explain your answer.

12.8. The process for the stock price in equation (12.8) is

$$\Delta S = \mu S\,\Delta t + \sigma S\epsilon\sqrt{\Delta t}$$

where μ and σ are constant. Explain carefully the difference between this model and each of the following:

$$\Delta S = \mu\,\Delta t + \sigma\epsilon\sqrt{\Delta t}$$
$$\Delta S = \mu S\,\Delta t + \sigma\epsilon\sqrt{\Delta t}$$
$$\Delta S = \mu\,\Delta t + \sigma S\epsilon\sqrt{\Delta t}$$

Why is the model in equation (12.8) a more appropriate model of stock price behavior than any of these three alternatives?

12.9. It has been suggested that the short-term interest rate r follows the stochastic process

$$dr = a(b - r)\,dt + rc\,dz$$

where a, b, c are positive constants and dz is a Wiener process. Describe the nature of this process.

12.10. Suppose that a stock price S follows geometric Brownian motion with expected return μ and volatility σ:

$$dS = \mu S\,dt + \sigma S\,dz$$

What is the process followed by the variable S^n? Show that S^n also follows geometric Brownian motion.

12.11. Suppose that x is the yield to maturity with continuous compounding on a zero-coupon bond that pays off \$1 at time T. Assume that x follows the process

$$dx = a(x_0 - x)\,dt + sx\,dz$$

where a, x_0, and s are positive constants and dz is a Wiener process. What is the process followed by the bond price?

Assignment Questions

12.12. Suppose that a stock price has an expected return of 16% per annum and a volatility of 30% per annum. When the stock price at the end of a certain day is \$50, calculate the following:

(a) The expected stock price at the end of the next day.
(b) The standard deviation of the stock price at the end of the next day.
(c) The 95% confidence limits for the stock price at the end of the next day.

12.13. A company's cash position, measured in millions of dollars, follows a generalized Wiener process with a drift rate of 0.1 per month and a variance rate of 0.16 per month. The initial cash position is 2.0.

(a) What are the probability distributions of the cash position after 1 month, 6 months, and 1 year?

(b) What are the probabilities of a negative cash position at the end of 6 months and 1 year?

(c) At what time in the future is the probability of a negative cash position greatest?

12.14. Suppose that x is the yield on a perpetual government bond that pays interest at the rate of \$1 per annum. Assume that x is expressed with continuous compounding, that interest is paid continuously on the bond, and that x follows the process

$$dx = a(x_0 - x)\,dt + sx\,dz$$

where a, x_0, and s are positive constants, and dz is a Wiener process. What is the process followed by the bond price? What is the expected instantaneous return (including interest and capital gains) to the holder of the bond?

12.15. If S follows the geometric Brownian motion process in equation (12.6), what is the process followed by

(a) $y = 2S$

(b) $y = S^2$

(c) $y = e^S$

(d) $y = e^{r(T-t)}/S$

In each case express the coefficients of dt and dz in terms of y rather than S.

12.16. A stock price is currently 50. Its expected return and volatility are 12% and 30%, respectively. What is the probability that the stock price will be greater than 80 in 2 years? (*Hint*: $S_T > 80$ when $\ln S_T > \ln 80$.)

APPENDIX

DERIVATION OF ITÔ'S LEMMA

In this appendix, we show how Itô's lemma can be regarded as a natural extension of other, simpler results. Consider a continuous and differentiable function G of a variable x. If Δx is a small change in x and ΔG is the resulting small change in G, a well-known result from ordinary calculus is

$$\Delta G \approx \frac{dG}{dx}\Delta x \tag{12A.1}$$

In other words, ΔG is approximately equal to the rate of change of G with respect to x multiplied by Δx. The error involves terms of order Δx^2. If more precision is required, a Taylor series expansion of ΔG can be used:

$$\Delta G = \frac{dG}{dx}\Delta x + \tfrac{1}{2}\frac{d^2G}{dx^2}\Delta x^2 + \tfrac{1}{6}\frac{d^3G}{dx^3}\Delta x^3 + \cdots$$

For a continuous and differentiable function G of two variables x and y, the result analogous to equation (12A.1) is

$$\Delta G \approx \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial y}\Delta y \tag{12A.2}$$

and the Taylor series expansion of ΔG is

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial y}\Delta y + \tfrac{1}{2}\frac{\partial^2 G}{\partial x^2}\Delta x^2 + \frac{\partial^2 G}{\partial x\,\partial y}\Delta x\,\Delta y + \tfrac{1}{2}\frac{\partial^2 G}{\partial y^2}\Delta y^2 + \cdots \tag{12A.3}$$

In the limit, as Δx and Δy tend to zero, equation (12A.3) becomes

$$dG = \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial y}dy \tag{12A.4}$$

We now extend equation (12A.4) to cover functions of variables following Itô processes. Suppose that a variable x follows the Itô process

$$dx = a(x, t)\,dt + b(x, t)\,dz \tag{12A.5}$$

and that G is some function of x and of time t. By analogy with equation (12A.3), we can write

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial t}\Delta t + \tfrac{1}{2}\frac{\partial^2 G}{\partial x^2}\Delta x^2 + \frac{\partial^2 G}{\partial x\,\partial t}\Delta x\,\Delta t + \tfrac{1}{2}\frac{\partial^2 G}{\partial t^2}\Delta t^2 + \cdots \tag{12A.6}$$

Equation (12A.5) can be discretized to

$$\Delta x = a(x, t)\,\Delta t + b(x, t)\epsilon\sqrt{\Delta t}$$

or, if arguments are dropped,

$$\Delta x = a\,\Delta t + b\epsilon\sqrt{\Delta t} \tag{12A.7}$$

This equation reveals an important difference between the situation in equation (12A.6) and the situation in equation (12A.3). When limiting arguments were used to move from equation (12A.3) to equation (12A.4), terms in Δx^2 were ignored because they were second-order terms. From equation (12A.7), we have

$$\Delta x^2 = b^2\epsilon^2\Delta t + \text{terms of higher order in } \Delta t \qquad \textbf{(12A.8)}$$

This shows that the term involving Δx^2 in equation (12A.6) has a component that is of order Δt and cannot be ignored.

The variance of a standardized normal distribution is 1.0. This means that

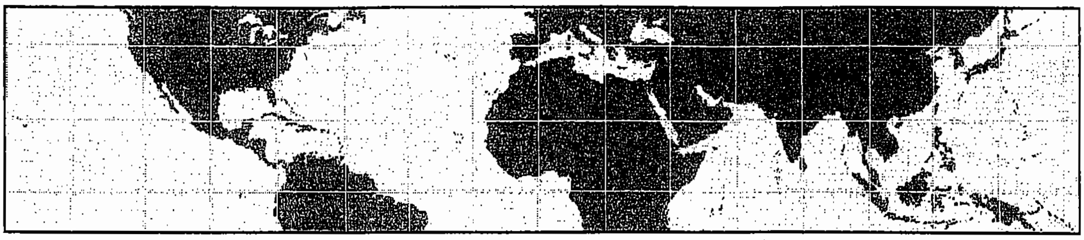
$$E(\epsilon^2) - [E(\epsilon)]^2 = 1$$

where E denotes expected value. Since $E(\epsilon) = 0$, it follows that $E(\epsilon^2) = 1$. The expected value of $\epsilon^2\Delta t$, therefore, is Δt. It can be shown that the variance of $\epsilon^2\Delta t$ is of order Δt^2 and that, as a result, we can treat $\epsilon^2\Delta t$ as nonstochastic and equal to its expected value, Δt, as Δt tends to zero. It follows from equation (12A.8) that Δx^2 becomes nonstochastic and equal to b^2dt as Δt tends to zero. Taking limits as Δx and Δt tend to zero in equation (12A.6), and using this last result, we obtain

$$dG = \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial t}dt + \tfrac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2dt \qquad \textbf{(12A.9)}$$

This is Itô's lemma. If we substitute for dx from equation (12A.5), equation (12A.9) becomes

$$dG = \left(\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \tfrac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\right)dt + \frac{\partial G}{\partial x}b\,dz$$



CHAPTER 13 The Black–Scholes–Merton Model

In the early 1970s, Fischer Black, Myron Scholes, and Robert Merton achieved a major breakthrough in the pricing of stock options.[1] This involved the development of what has become known as the Black–Scholes (or Black–Scholes–Merton) model. The model has had a huge influence on the way that traders price and hedge options. It has also been pivotal to the growth and success of financial engineering in the last 30 years. In 1997, the importance of the model was recognized when Robert Merton and Myron Scholes were awarded the Nobel prize for economics. Sadly, Fischer Black died in 1995, otherwise he too would undoubtedly have been one of the recipients of this prize.

This chapter shows how the Black–Scholes model for valuing European call and put options on a non-dividend-paying stock is derived. It explains how volatility can be either estimated from historical data or implied from option prices using the model. It shows how the risk-neutral valuation argument introduced in Chapter 11 can be used. It also shows how the Black–Scholes model can be extended to deal with European call and put options on dividend-paying stocks and presents some results on the pricing of American call options on dividend-paying stocks.

13.1 LOGNORMAL PROPERTY OF STOCK PRICES

The model of stock price behavior used by Black, Scholes, and Merton is the model we developed in Chapter 12. It assumes that percentage changes in the stock price in a short period of time are normally distributed. Define

μ: Expected return on stock per year
σ: Volatility of the stock price per year

The mean of the return in time Δt is $\mu\,\Delta t$ and the standard deviation of the return is

[1] See F. Black and M. Scholes, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, 81 (May/June 1973): 637–59; R.C. Merton, "Theory of Rational Option Pricing," *Bell Journal of Economics and Management Science*, 4 (Spring 1973): 141–83.

$\sigma\sqrt{\Delta t}$, so that

$$\frac{\Delta S}{S} \sim \phi(\mu\,\Delta t,\ \sigma^2 \Delta t) \tag{13.1}$$

where ΔS is the change in the stock price S in time Δt, and $\phi(m, v)$ denotes a normal distribution with mean m and variance v.

As shown in Section 12.6, the model implies that

$$\ln S_T - \ln S_0 \sim \phi\left[\left(\mu - \frac{\sigma^2}{2}\right)T,\ \sigma^2 T\right]$$

From this, it follows that

$$\ln\frac{S_T}{S_0} \sim \phi\left[\left(\mu - \frac{\sigma^2}{2}\right)T,\ \sigma^2 T\right] \tag{13.2}$$

and

$$\ln S_T \sim \phi\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T,\ \sigma^2 T\right] \tag{13.3}$$

where S_T is the stock price at a future time T and S_0 is the stock price at time 0. Equation (13.3) shows that $\ln S_T$ is normally distributed, so that S_T has a lognormal distribution. The mean of $\ln S_T$ is $\ln S_0 + (\mu - \sigma^2/2)T$ and the standard deviation is $\sigma\sqrt{T}$.

Example 13.1

Consider a stock with an initial price of \$40, an expected return of 16% per annum, and a volatility of 20% per annum. From equation (13.3), the probability distribution of the stock price S_T in 6 months' time is given by

$$\ln S_T \sim \phi[\ln 40 + (0.16 - 0.2^2/2) \times 0.5,\ 0.2^2 \times 0.5]$$

$$\ln S_T \sim \phi(3.759,\ 0.02)$$

There is a 95% probability that a normally distributed variable has a value within 1.96 standard deviations of its mean. In this case, the standard deviation is $\sqrt{0.02} = 0.141$. Hence, with 95% confidence,

$$3.759 - 1.96 \times 0.141 < \ln S_T < 3.759 + 1.96 \times 0.141$$

This can be written

$$e^{3.759 - 1.96\times 0.141} < S_T < e^{3.759 + 1.96 \times 0.141}$$

or

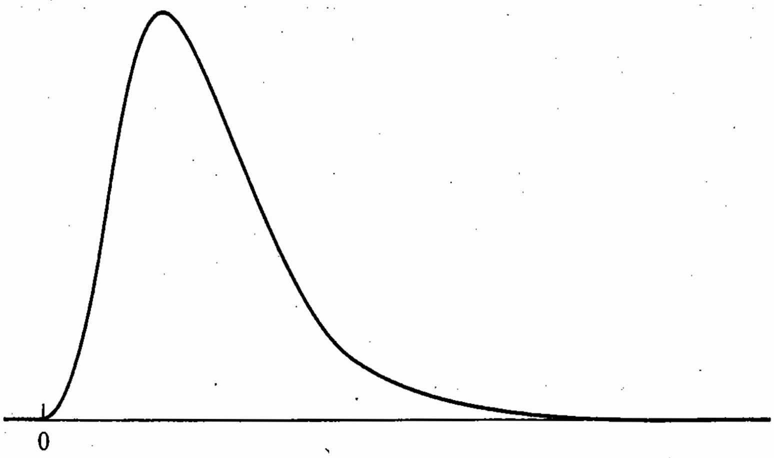
$$32.55 < S_T < 56.56$$

Thus, there is a 95% probability that the stock price in 6 months will lie between 32.55 and 56.56.

A variable that has a lognormal distribution can take any value between zero and infinity. Figure 13.1 illustrates the shape of a lognormal distribution. Unlike the normal distribution, it is skewed so that the mean, median, and mode are all different. From equation (13.3) and the properties of the lognormal distribution, it can be shown that the expected value $E(S_T)$ of S_T is given by

$$E(S_T) = S_0 e^{\mu T} \tag{13.4}$$

Figure 13.1 Lognormal distribution.



This fits in with the definition of μ as the expected rate of return. The variance $\text{var}(S_T)$ of S_T, can be shown to be given by[2]

$$\text{var}(S_T) = S_0^2 e^{2\mu T}(e^{\sigma^2 T} - 1) \tag{13.5}$$

Example 13.2

Consider a stock where the current price is \$20, the expected return is 20% per annum, and the volatility is 40% per annum. The expected stock price, $E(S_T)$, and the variance of the stock price, $\text{var}(S_T)$, in 1 year are given by

$$E(S_T) = 20e^{0.2\times 1} = 24.43 \quad \text{and} \quad \text{var}(S_T) = 400e^{2\times 0.2\times 1}(e^{0.4^2\times 1} - 1) = 103.54$$

The standard deviation of the stock price in 1 year is $\sqrt{103.54}$, or 10.18.

13.2 THE DISTRIBUTION OF THE RATE OF RETURN

The lognormal property of stock prices can be used to provide information on the probability distribution of the continuously compounded rate of return earned on a stock between times 0 and T. If we define the continuously compounded rate of return per annum realized between times 0 and T as x, then

$$S_T = S_0 e^{xT}$$

so that

$$x = \frac{1}{T}\ln\frac{S_T}{S_0} \tag{13.6}$$

From equation (13.2), it follows that

$$x \sim \phi\left(\mu - \frac{\sigma^2}{2}, \frac{\sigma^2}{T}\right) \tag{13.7}$$

[2] See Technical Note 2 on the author's website for a proof of the results in equations (13.4) and (13.5). For a more extensive discussion of the properties of the lognormal distribution, see J. Aitchison and J. A. C. Brown, *The Lognormal Distribution*. Cambridge University Press, 1966.

Thus, the continuously compounded rate of return per annum is normally distributed with mean $\mu - \sigma^2/2$ and standard deviation $\sigma/\sqrt{T}$. As T increases, the standard deviation of x declines. To understand the reason for this, consider two cases: $T = 1$ and $T = 20$. We are more certain about the average return per year over 20 years than we are about the return in any one year.

Example 13.3

Consider a stock with an expected return of 17% per annum and a volatility of 20% per annum. The probability distribution for the average rate of return (continuously compounded) realized over 3 years is normal, with mean

$$0.17 - \frac{0.2^2}{2} = 0.15$$

or 15% per annum, and standard deviation

$$\sqrt{\frac{0.2^2}{3}} = 0.1155$$

or 11.55% per annum. Because there is a 95% chance that a normally distributed variable will lie within 1.96 standard deviations of its mean, we can be 95% confident that the average return realized over 3 years will be between −7.6% and +37.6% per annum.

13.3 THE EXPECTED RETURN

The expected return, μ, required by investors from a stock depends on the riskiness of the stock. The higher the risk, the higher the expected return. It also depends on the level of interest rates in the economy. The higher the level of interest rates, the higher the expected return required on any given stock. Fortunately, we do not have to concern ourselves with the determinants of μ in any detail. It turns out that the value of a stock option, when expressed in terms of the value of the underlying stock, does not depend on μ at all. Nevertheless, there is one aspect of the expected return from a stock that frequently causes confusion and should be explained.

Equation (13.1) shows that $\mu\,\Delta t$ is the expected percentage change in the stock price in a very short period of time, Δt. It is natural to assume from this that μ is the expected continuously compounded return on the stock. However, this is not the case. The continuously compounded return, x, actually realized over a period of time of length T is given by equation (13.6) as

$$x = \frac{1}{T}\ln\frac{S_T}{S_0}$$

and, as indicated in equation (13.7), the expected value $E(x)$ of x is $\mu - \sigma^2/2$.

The reason why the expected continuously compounded return is different from μ is subtle, but important. Suppose we consider a very large number of very short periods of time of length Δt. Define S_i as the stock price at the end of the ith interval and ΔS_i as $S_{i+1} - S_i$. Under the assumptions we are making for stock price behavior, the average of the returns on the stock in each interval is close to μ. In other words, $\mu\,\Delta t$ is close to the arithmetic mean of the $\Delta S_i/S_i$. However, the expected return over the whole period

Business Snapshot 13.1 Mutual Fund Returns Can Be Misleading

The difference between μ and $\mu - \sigma^2/2$ is closely related to an issue in the reporting of mutual fund returns. Suppose that the following is a sequence of returns per annum reported by a mutual fund manager over the last five years (measured using annual compounding):

15%, 20%, 30%, −20%, 25%

The arithmetic mean of the returns, calculated by taking the sum of the returns and dividing by 5, is 14%. However, an investor would actually earn less than 14% per annum by leaving the money invested in the fund for 5 years. The dollar value of \$100 at the end of the 5 years would be

$$100 \times 1.15 \times 1.20 \times 1.30 \times 0.80 \times 1.25 = \$179.40$$

By contrast, a 14% return with annual compounding would give

$$100 \times 1.14^5 = \$192.54$$

The return that gives \$179.40 at the end of five years is 12.4%. This is because

$$100 \times (1.124)^5 = 179.40$$

What average return should the fund manager report? It is tempting for the manager to make a statement such as: "The average of the returns per year that we have realized in the last 5 years is 14%." Although true, this is misleading. It is much less misleading to say: "The average return realized by someone who invested with us for the last 5 years is 12.4% per year." In some jurisdictions, regulations require fund managers to report returns the second way.

This phenomenon is an example of a result that is well known by mathematicians. The geometric mean of a set of numbers (not all the same) is always less than the arithmetic mean. In our example, the return multipliers each year are 1.15, 1.20, 1.30, 0.80, and 1.25. The arithmetic mean of these numbers is 1.140, but the geometric mean is only 1.124.

covered by the data, expressed with a compounding interval of Δt, is close to $\mu - \sigma^2/2$, not μ.[3] Business Snapshot 13.1 provides a numerical example concerning the mutual fund industry to illustrate why this is so. For a mathematical explanation of what is going on, we start with equation (13.4):

$$E(S_T) = S_0 e^{\mu T}$$

Taking logarithms, we get

$$\ln[E(S_T)] = \ln(S_0) + \mu T$$

It is now tempting to set $\ln[E(S_T)] = E[\ln(S_T)]$, so that $E[\ln(S_T)] - \ln(S_0) = \mu T$, or $E[\ln(S_T/S_0)] = \mu T$, which leads to $E(x) = \mu$. However, we cannot do this because ln

[3] The arguments in this section show that the term "expected return" is ambiguous. It can refer either to μ or to $\mu - \sigma^2/2$. Unless otherwise stated, it will be used to refer to μ throughout this book.

is a nonlinear function. In fact, $\ln[E(S_T)] > E[\ln(S_T)]$, so that $E[\ln(S_T/S_0)] < \mu T$, which leads to $E(x) < \mu$. (As pointed out above, $E(x) = \mu - \sigma^2/2$.)

13.4 VOLATILITY

The volatility σ of a stock is a measure of our uncertainty about the returns provided by the stock. Stocks typically have a volatility between 15% and 60%.

From equation (13.7), the volatility of a stock price can be defined as the standard deviation of the return provided by the stock in 1 year when the return is expressed using continuous compounding.

When Δt is small, equation (13.1) shows that $\sigma^2 \Delta t$ is approximately equal to the variance of the percentage change in the stock price in time Δt. This means that $\sigma\sqrt{\Delta t}$ is approximately equal to the standard deviation of the percentage change in the stock price in time Δt. Suppose that $\sigma = 0.3$, or 30%, per annum and the current stock price is \$50. The standard deviation of the percentage change in the stock price in 1 week is approximately

$$30 \times \sqrt{\frac{1}{52}} = 4.16\%$$

A one-standard-deviation move in the stock price in 1 week is therefore 50×0.0416, or \$2.08.

Uncertainty about a future stock price, as measured by its standard deviation, increases—at least approximately—with the square root of how far ahead we are looking. For example, the standard deviation of the stock price in 4 weeks is approximately twice the standard deviation in 1 week.

Estimating Volatility from Historical Data

To estimate the volatility of a stock price empirically, the stock price is usually observed at fixed intervals of time (e.g., every day, week, or month). Define:

$n+1$: Number of observations
S_i: Stock price at end of ith interval, with $i = 0, 1, \dots, n$
τ: Length of time interval in years

and let

$$u_i = \ln\left(\frac{S_i}{S_{i-1}}\right) \quad \text{for } i = 1, 2, \dots, n$$

The usual estimate, s, of the standard deviation of the u_i is given by

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(u_i - \bar{u})^2}$$

or

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n} u_i^2 - \frac{1}{n(n-1)}\left(\sum_{i=1}^{n} u_i\right)^2}$$

where $\bar{u}$ is the mean of the u_i.[4]

[4] The mean $\bar{u}$ is often assumed to be zero when estimates of historical volatilities are made.

From equation (13.2), the standard deviation of the u_i is $\sigma\sqrt{\tau}$. The variable s is therefore an estimate of $\sigma\sqrt{\tau}$. It follows that σ itself can be estimated as $\hat{\sigma}$, where

$$\hat{\sigma} = \frac{s}{\sqrt{\tau}}$$

The standard error of this estimate can be shown to be approximately $\hat{\sigma}/\sqrt{2n}$.

Choosing an appropriate value for n is not easy. More data generally lead to more accuracy, but σ does change over time and data that are too old may not be relevant for predicting the future volatility. A compromise that seems to work reasonably well is to use closing prices from daily data over the most recent 90 to 180 days. Alternatively, as a rule of thumb, n can be set equal to the number of days to which the volatility is to be applied. Thus, if the volatility estimate is to be used to value a 2-year option, daily data for the last 2 years are used. More sophisticated approaches to estimating volatility involving GARCH models are discussed in Chapter 21.

Example 13.4

Table 13.1 shows a possible sequence of stock prices during 21 consecutive trading days. In this case,

$$\sum u_i = 0.09531 \text{ and } \sum u_i^2 = 0.00326$$

Table 13.1 Computation of volatility.

Day	*Closing stock price (dollars)*	*Price relative* S_i/S_{i-1}	*Daily return* $u_i = \ln(S_i/S_{i-1})$
0	20.00		
1	20.10	1.00500	0.00499
2	19.90	0.99005	−0.01000
3	20.00	1.00503	0.00501
4	20.50	1.02500	0.02469
5	20.25	0.98780	−0.01227
6	20.90	1.03210	0.03159
7	20.90	1.00000	0.00000
8	20.90	1.00000	0.00000
9	20.75	0.99282	−0.00720
10	20.75	1.00000	0.00000
11	21.00	1.01205	0.01198
12	21.10	1.00476	0.00475
13	20.90	0.99052	−0.00952
14	20.90	1.00000	0.00000
15	21.25	1.01675	0.01661
16	21.40	1.00706	0.00703
17	21.40	1.00000	0.00000
18	21.25	0.99299	−0.00703
19	21.75	1.02353	0.02326
20	22.00	1.01149	0.01143

and the estimate of the standard deviation of the daily return is

$$\sqrt{\frac{0.00326}{19} - \frac{0.09531^2}{380}} = 0.01216$$

or 1.216%. Assuming that there are 252 trading days per year, $\tau = 1/252$ and the data give an estimate for the volatility per annum of $0.01216\sqrt{252} = 0.193$, or 19.3%. The standard error of this estimate is

$$\frac{0.193}{\sqrt{2 \times 20}} = 0.031$$

or 3.1% per annum.

The foregoing analysis assumes that the stock pays no dividends, but it can be adapted to accommodate dividend-paying stocks. The return, u_i, during a time interval that includes an ex-dividend day is given by

$$u_i = \ln \frac{S_i + D}{S_{i-1}}$$

where D is the amount of the dividend. The return in other time intervals is still

$$u_i = \ln \frac{S_i}{S_{i-1}}$$

However, as tax factors play a part in determining returns around an ex-dividend date, it is probably best to discard altogether data for intervals that include an ex-dividend date.

Trading Days vs. Calendar Days

An important issue is whether time should be measured in calendar days or trading days when volatility parameters are being estimated and used. As shown in Business Snapshot 13.2, research shows that volatility is much higher when the exchange is open for trading than when it is closed. As a result, practitioners tend to ignore days when the exchange is closed when estimating volatility from historical data and when calculating the life of an option. The volatility per annum is calculated from the volatility per trading day using the formula

$$\begin{matrix}\text{Volatility} \\ \text{per annum}\end{matrix} = \begin{matrix}\text{Volatility} \\ \text{per trading day}\end{matrix} \times \sqrt{\begin{matrix}\text{Number of trading days} \\ \text{per annum}\end{matrix}}$$

This is what we did in Example 13.4 when calculating volatility from the data in Table 13.1. The number of trading days in a year is usually assumed to be 252 for stocks.

The life of an option is also usually measured using trading days rather than calendar days. It is calculated as T years, where

$$T = \frac{\text{Number of trading days until option maturity}}{252}$$

Business Snapshot 13.2 What Causes Volatility?

It is natural to assume that the volatility of a stock is caused by new information reaching the market. This new information causes people to revise their opinions about the value of the stock. The price of the stock changes and volatility results. This view of what causes volatility is not supported by research. With several years of daily stock price data, researchers can calculate:

1. The variance of stock price returns between the close of trading on one day and the close of trading on the next day when there are no intervening nontrading days
2. The variance of the stock price returns between the close of trading on Friday and the close of trading on Monday

The second of these is the variance of returns over a 3-day period. The first is a variance over a 1-day period. We might reasonably expect the second variance to be three times as great as the first variance. Fama (1965), French (1980), and French and Roll (1986) show that this is not the case. These three research studies estimate the second variance to be, respectively, 22%, 19%, and 10.7% higher than the first variance.

At this stage one might be tempted to argue that these results are explained by more news reaching the market when the market is open for trading. But research by Roll (1984) does not support this explanation. Roll looked at the prices of orange juice futures. By far the most important news for orange juice futures prices is news about the weather and this is equally likely to arrive at any time. When Roll did a similar analysis to that just described for stocks, he found that the second (Friday-to-Monday) variance for orange juice futures is only 1.54 times the first variance.

The only reasonable conclusion from all this is that volatility is to a large extent caused by trading itself. (Traders usually have no difficulty accepting this conclusion!)

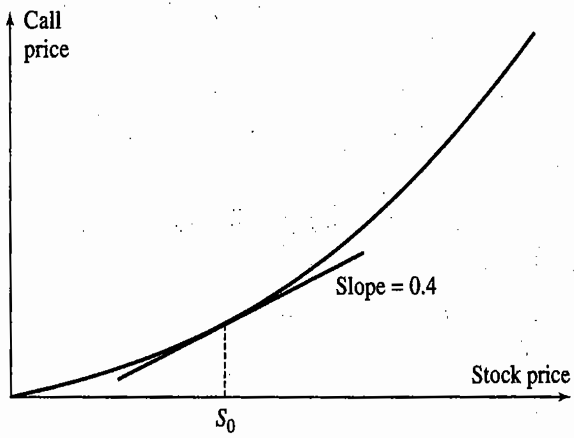
13.5 THE IDEA UNDERLYING THE BLACK–SCHOLES–MERTON DIFFERENTIAL EQUATION

The Black–Scholes–Merton differential equation is an equation that must be satisfied by the price of any derivative dependent on a non-dividend-paying stock. The equation is derived in the next section. Here we consider the nature of the arguments we will use.

These are similar to the no-arbitrage arguments we used to value stock options in Chapter 11 for the situation where stock price movements are binomial. They involve setting up a riskless portfolio consisting of a position in the derivative and a position in the stock. In the absence of arbitrage opportunities, the return from the portfolio must be the risk-free interest rate, r. This leads to the Black-Scholes-Merton differential equation.

The reason a riskless portfolio can be set up is that the stock price and the derivative price are both affected by the same underlying source of uncertainty: stock price movements. In any short period of time, the price of the derivative is perfectly correlated with the price of the underlying stock. When an appropriate portfolio of the stock and the derivative is established, the gain or loss from the stock position always offsets the gain or loss from the derivative position so that the overall value of the portfolio at the end of the short period of time is known with certainty.

Figure 13.2 Relationship between call price and stock price. Current stock price is S_0.



Suppose, for example, that at a particular point in time the relationship between a small change ΔS in the stock price and the resultant small change Δc in the price of a European call option is given by

$$\Delta c = 0.4\,\Delta S$$

This means that the slope of the line representing the relationship between c and S is 0.4, as indicated in Figure 13.2. The riskless portfolio would consist of:

1. A long position in 0.4 shares
2. A short position in one call option

Suppose, for example, that the stock price increases by 10 cents. The option price will increase by 4 cents and the $40 \times 0.10 = \$4$ gain on the shares is equal to the $100 \times 0.04 = \$4$ loss on the short option position.

There is one important difference between the Black–Scholes–Merton analysis and our analysis using a binomial model in Chapter 11. In Black–Scholes–Merton, the position in the stock and the derivative is riskless for only a very short period of time. (Theoretically, it remains riskless only for an instantaneously short period of time.) To remain riskless, it must be adjusted, or *rebalanced*, frequently.[5] For example, the relationship between Δc and ΔS in our example might change from $\Delta c = 0.4\,\Delta S$ today to $\Delta c = 0.5\,\Delta S$ in 2 weeks. This would mean that, in order to maintain the riskless position, an extra 0.1 share would have to be purchased for each call option sold. It is nevertheless true that the return from the riskless portfolio in any very short period of time must be the risk-free interest rate. This is the key element in the Black–Scholes analysis and leads to their pricing formulas.

Assumptions

The assumptions we use to derive the Black–Scholes–Merton differential equation are as follows:

1. The stock price follows the process developed in Chapter 12 with μ and σ constant.
2. The short selling of securities with full use of proceeds is permitted.

[5] We discuss the rebalancing of portfolios in more detail in Chapter 17.

3. There are no transactions costs or taxes. All securities are perfectly divisible.
4. There are no dividends during the life of the derivative.
5. There are no riskless arbitrage opportunities.
6. Security trading is continuous.
7. The risk-free rate of interest, r, is constant and the same for all maturities.

As we discuss in later chapters, some of these assumptions can be relaxed. For example, σ and r can be known functions of t. We can even allow interest rates to be stochastic provided that the stock price distribution at maturity of the option is still lognormal.

13.6 DERIVATION OF THE BLACK–SCHOLES–MERTON DIFFERENTIAL EQUATION

The stock price process we are assuming is the one we developed in Section 12.3:

$$dS = \mu S\,dt + \sigma S\,dz \tag{13.8}$$

Suppose that f is the price of a call option or other derivative contingent on S. The variable f must be some function of S and t. Hence, from equation (12.14),

$$df = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \tfrac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial f}{\partial S}\sigma S\,dz \tag{13.9}$$

The discrete versions of equations (13.8) and (13.9) are

$$\Delta S = \mu S\,\Delta t + \sigma S\,\Delta z \tag{13.10}$$

and

$$\Delta f = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \tfrac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t + \frac{\partial f}{\partial S}\sigma S\,\Delta z \tag{13.11}$$

where Δf and ΔS are the changes in f and S in a small time interval Δt. Recall from the discussion of Itô's lemma in Section 12.5 that the Wiener processes underlying f and S are the same. In other words, the Δz $(= \epsilon\sqrt{\Delta t})$ in equations (13.10) and (13.11) are the same. It follows that a portfolio of the stock and the derivative can be constructed so that the Wiener process is eliminated.

The portfolio is

-1: derivative

$+\partial f/\partial S$: shares

The holder of this portfolio is short one derivative and long an amount $\partial f/\partial S$ of shares. Define Π as the value of the portfolio. By definition

$$\Pi = -f + \frac{\partial f}{\partial S}S \tag{13.12}$$

The change $\Delta\Pi$ in the value of the portfolio in the time interval Δt is given by

$$\Delta\Pi = -\Delta f + \frac{\partial f}{\partial S}\Delta S \tag{13.13}$$

Substituting equations (13.10) and (13.11) into equation (13.13) yields

$$\Delta\Pi = \left(-\frac{\partial f}{\partial t} - \tfrac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t \tag{13.14}$$

Because this equation does not involve Δz, the portfolio must be riskless during time Δt. The assumptions listed in the preceding section imply that the portfolio must instantaneously earn the same rate of return as other short-term risk-free securities. If it earned more than this return, arbitrageurs could make a riskless profit by borrowing money to buy the portfolio; if it earned less, they could make a riskless profit by shorting the portfolio and buying risk-free securities. It follows that

$$\Delta\Pi = r\Pi\,\Delta t \tag{13.15}$$

where r is the risk-free interest rate. Substituting from equations (13.12) and (13.14) into (13.15), we obtain

$$\left(\frac{\partial f}{\partial t} + \tfrac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t = r\left(f - \frac{\partial f}{\partial S}S\right)\Delta t$$

so that

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \tfrac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2} = rf \tag{13.16}$$

Equation (13.16) is the Black–Scholes–Merton differential equation. It has many solutions, corresponding to all the different derivatives that can be defined with S as the underlying variable. The particular derivative that is obtained when the equation is solved depends on the *boundary conditions* that are used. These specify the values of the derivative at the boundaries of possible values of S and t. In the case of a European call option, the key boundary condition is

$$f = \max(S - K,\ 0) \quad \text{when } t = T$$

In the case of a European put option, it is

$$f = \max(K - S,\ 0) \quad \text{when } t = T$$

One point that should be emphasized about the portfolio used in the derivation of equation (13.16) is that it is not permanently riskless. It is riskless only for an infinitesimally short period of time. As S and t change, $\partial f/\partial S$ also changes. To keep the portfolio riskless, it is therefore necessary to frequently change the relative proportions of the derivative and the stock in the portfolio.

Example 13.5

A forward contract on a non-dividend-paying stock is a derivative dependent on the stock. As such, it should satisfy equation (13.16). From equation (5.5), we know that the value of the forward contract, f, at a general time t is given in terms of the stock price S at this time by

$$f = S - Ke^{-r(T-t)}$$

where K is the delivery price. This means that

$$\frac{\partial f}{\partial t} = -rKe^{-r(T-t)}, \qquad \frac{\partial f}{\partial S} = 1, \qquad \frac{\partial^2 f}{\partial S^2} = 0$$

When these are substituted into the left-hand side of equation (13.16), we obtain

$$-rKe^{-r(T-t)} + rS$$

This equals rf, showing that equation (13.16) is indeed satisfied.

The Prices of Tradeable Derivatives

Any function $f(S, t)$ that is a solution of the differential equation (13.16) is the theoretical price of a derivative that could be traded. If a derivative with that price existed, it would not create any arbitrage opportunities. Conversely, if a function $f(S, t)$ does not satisfy the differential equation (13.16), it cannot be the price of a derivative without creating arbitrage opportunities for traders.

To illustrate this point, consider first the function e^S. This does not satisfy the differential equation (13.16). It is therefore not a candidate for being the price of a derivative dependent on the stock price. If an instrument whose price was always e^S existed, there would be an arbitrage opportunity. As a second example, consider the function

$$\frac{e^{(\sigma^2-2r)(T-t)}}{S}$$

This does satisfy the differential equation, and so is, in theory, the price of a tradeable security. (It is the price of a derivative that pays off $1/S_T$ at time T.) For other examples of tradeable derivatives, see Problems 13.11, 13.12, 13.23, and 13.28.

13.7 RISK-NEUTRAL VALUATION

We introduced risk-neutral valuation in connection with the binomial model in Chapter 11. It is without doubt the single most important tool for the analysis of derivatives. It arises from one key property of the Black–Scholes–Merton differential equation (13.16). This property is that the equation does not involve any variables that are affected by the risk preferences of investors. The variables that do appear in the equation are the current stock price, time, stock price volatility, and the risk-free rate of interest. All are independent of risk preferences.

The Black–Scholes–Merton differential equation would not be independent of risk preferences if it involved the expected return, μ, on the stock. This is because the value of μ does depend on risk preferences. The higher the level of risk aversion by investors, the higher μ will be for any given stock. It is fortunate that μ happens to drop out in the derivation of the differential equation.

Because the Black–Scholes–Merton differential equation is independent of risk preferences, an ingenious argument can be used. If risk preferences do not enter the equation, they cannot affect its solution. Any set of risk preferences can, therefore, be used when evaluating f. In particular, the very simple assumption that all investors are risk neutral can be made.

In a world where investors are risk neutral, the expected return on all investment assets is the risk-free rate of interest, r. The reason is that risk-neutral investors do not require a premium to induce them to take risks. It is also true that the present value of any cash flow in a risk-neutral world can be obtained by discounting its expected value at the risk-free rate. The assumption that the world is risk neutral does, therefore, considerably simplify the analysis of derivatives.

Consider a derivative that provides a payoff at one particular time. It can be valued using risk-neutral valuation by using the following procedure:

1. Assume that the expected return from the underlying asset is the risk-free interest rate, r (i.e., assume $\mu = r$).
2. Calculate the expected payoff from the derivative.
3. Discount the expected payoff at the risk-free interest rate.

It is important to appreciate that risk-neutral valuation (or the assumption that all investors are risk neutral) is merely an artificial device for obtaining solutions to the Black–Scholes differential equation. The solutions that are obtained are valid in all worlds, not just those where investors are risk neutral. When we move from a risk-neutral world to a risk-averse world, two things happen. The expected growth rate in the stock price changes and the discount rate that must be used for any payoffs from the derivative changes. It happens that these two changes always offset each other exactly.

Application to Forward Contracts on a Stock

We valued forward contracts on a non-dividend-paying stock in Section 5.7. In Example 13.5, we verified that the pricing formula satisfies the Black–Scholes differential equation. In this section we derive the pricing formula from risk-neutral valuation. We make the assumption that interest rates are constant and equal to r. This is somewhat more restrictive than the assumption in Chapter 5.

Consider a long forward contract that matures at time T with delivery price, K. As indicated in Figure 1.2, the value of the contract at maturity is

$$S_T - K$$

where S_T is the stock price at time T. From the risk-neutral valuation argument, the value of the forward contract at time 0 is its expected value at time T in a risk-neutral world discounted at the risk-free rate of interest. Denoting the value of the forward contract at time zero by f, this means that

$$f = e^{-rT}\hat{E}(S_T - K)$$

where $\hat{E}$ denotes the expected value in a risk-neutral world. Since K is a constant, this equation becomes

$$f = e^{-rT}\hat{E}(S_T) - Ke^{-rT} \tag{13.17}$$

The expected return μ on the stock becomes r in a risk-neutral world. Hence, from equation (13.4), we have

$$\hat{E}(S_T) = S_0 e^{rT} \tag{13.18}$$

Substituting equation (13.18) into equation (13.17) gives

$$f = S_0 - Ke^{-rT} \tag{13.19}$$

This is in agreement with equation (5.5).

13.8 BLACK–SCHOLES PRICING FORMULAS

The Black–Scholes formulas for the prices at time 0 of a European call option on a non-dividend-paying stock and a European put option on a non-dividend-paying stock are

$$c = S_0 N(d_1) - Ke^{-rT} N(d_2) \tag{13.20}$$

and

$$p = Ke^{-rT} N(-d_2) - S_0 N(-d_1) \tag{13.21}$$

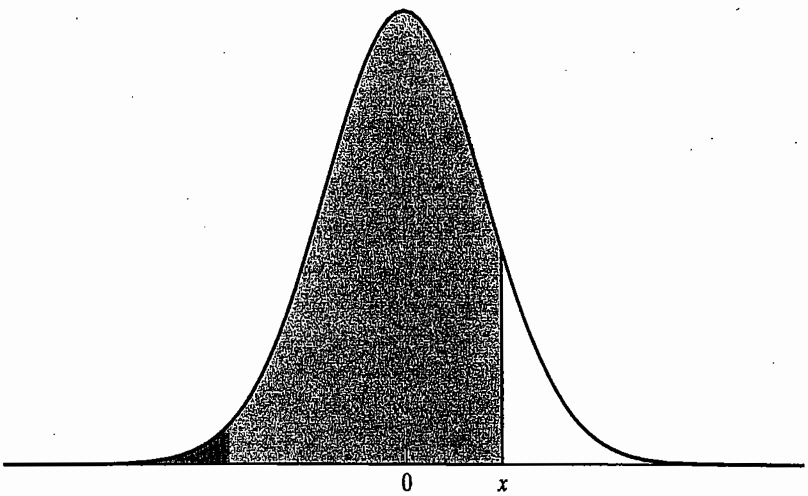
where

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

The function $N(x)$ is the cumulative probability distribution function for a standardized normal distribution. In other words, it is the probability that a variable with a standard normal distribution, $\phi(0, 1)$, will be less than x. It is illustrated in Figure 13.3. The remaining variables should be familiar. The variables c and p are the European call and European put price, S_0 is the stock price at time zero, K is the strike price, r is the

Figure 13.3 Shaded area represents $N(x)$.



continuously compounded risk-free rate, σ is the stock price volatility, and T is the time to maturity of the option.

One way of deriving the Black–Scholes formulas is by solving the differential equation (13.16) subject to the boundary condition mentioned in Section 13.6.[6] Another approach is to use risk-neutral valuation. Consider a European call option. The expected value of the option at maturity in a risk-neutral world is

$$\hat{E}[\max(S_T - K, 0)]$$

where, as before, $\hat{E}$ denotes the expected value in a risk-neutral world. From the risk-neutral valuation argument, the European call option price c is this expected value discounted at the risk-free rate of interest, that is,

$$c = e^{-rT}\hat{E}[\max(S_T - K, 0)] \tag{13.22}$$

The appendix at the end of this chapter shows that this equation leads to the result in equation (13.20).

To provide an interpretation of the terms in equation (13.20), we note that it can be written

$$c = e^{-rT}[S_0N(d_1)e^{rT} - KN(d_2)] \tag{13.23}$$

The expression $N(d_2)$ is the probability that the option will be exercised in a risk-neutral world, so that $KN(d_2)$ is the strike price times the probability that the strike price will be paid. The expression $S_0N(d_1)e^{rT}$ is the expected value in a risk-neutral world of a variable that is equal to S_T if $S_T > K$ and to zero otherwise.

Since it is never optimal to exercise early an American call option on a non-dividend-paying stock (see Section 9.5), equation (13.20) is the value of an American call option on a non-dividend-paying stock. Unfortunately, no exact analytic formula for the value of an American put option on a non-dividend-paying stock has been produced. Numerical procedures for calculating American put values are discussed in Chapter 19.

When the Black–Scholes formula is used in practice the interest rate r is set equal to the zero-coupon risk-free interest rate for a maturity T. As we show in later chapters, this is theoretically correct when r is a known function of time. It is also theoretically correct when the interest rate is stochastic provided that the stock price at time T is lognormal and the volatility parameter is chosen appropriately. As mentioned earlier, time is normally measured as the number of trading days left in the life of the option divided by the number of trading days in 1 year.

Properties of the Black–Scholes Formulas

We now show that the Black–Scholes formulas have the right general properties by considering what happens when some of the parameters take extreme values.

When the stock price, S_0, becomes very large, a call option is almost certain to be exercised. It then becomes very similar to a forward contract with delivery price K.

[6] The differential equation gives the call and put prices at a general time t. For example, the call price that satisfies the differential equation is $c = SN(d_1) - Ke^{-r(T-t)}N(d_2)$, where

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

and $d_2 = d_1 - \sigma\sqrt{T-t}$. See Problem 13.17 to prove that the differential equation is satisfied.

From equation (5.5), we expect the call price to be

$$S_0 - Ke^{-rT}$$

This is, in fact, the call price given by equation (13.20) because, when S_0 becomes very large, both d_1 and d_2 become very large, and $N(d_1)$ and $N(d_2)$ become close to 1.0. When the stock price becomes very large, the price of a European put option, p, approaches zero. This is consistent with equation (13.21) because $N(-d_1)$ and $N(-d_2)$ are both close to zero in this case.

Consider next what happens when the volatility σ approaches zero. Because the stock is virtually riskless, its price will grow at rate r to S_0e^{rT} at time T and the payoff from a call option is

$$\max(S_0e^{rT} - K, 0)$$

Discounting at rate r, the value of the call today is

$$e^{-rT}\max(S_0e^{rT} - K, 0) = \max(S_0 - Ke^{-rT}, 0)$$

To show that this is consistent with equation (13.20), consider first the case where $S_0 > Ke^{-rT}$. This implies that $\ln(S_0/K) + rT > 0$. As σ tends to zero, d_1 and d_2 tend to $+\infty$, so that $N(d_1)$ and $N(d_2)$ tend to 1.0 and equation (13.20) becomes

$$c = S_0 - Ke^{-rT}$$

When $S_0 < Ke^{-rT}$, it follows that $\ln(S_0/K) + rT < 0$. As σ tends to zero, d_1 and d_2 tend to $-\infty$, so that $N(d_1)$ and $N(d_2)$ tend to zero and equation (13.20) gives a call price of zero. The call price is therefore always $\max(S_0 - Ke^{-rT}, 0)$ as σ tends to zero. Similarly, it can be shown that the put price is always $\max(Ke^{-rT} - S_0, 0)$ as σ tends to zero.

13.9 CUMULATIVE NORMAL DISTRIBUTION FUNCTION

The only problem in implementing equations (13.20) and (13.21) is in calculating the cumulative normal distribution function, $N(x)$. Tables for $N(x)$ are provided at the end of this book. The NORMSDIST function calculates $N(x)$ in Excel. A polynomial approximation that gives six-decimal-place accuracy is[7]

$$N(x) = \begin{cases} 1 - N'(x)(a_1k + a_2k^2 + a_3k^3 + a_4k^4 + a_5k^5) & \text{when } x \geqslant 0 \\ 1 - N(-x) & \text{when } x < 0 \end{cases}$$

where

$$k = \frac{1}{1 + \gamma x}, \quad \gamma = 0.2316419$$

$$a_1 = 0.319381530, \quad a_2 = -0.356563782$$

$$a_3 = 1.781477937, \quad a_4 = -1.821255978, \quad a_5 = 1.330274429$$

[7] See M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions*. New York: Dover Publications, 1972.

and

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

Example 13.6

The stock price 6 months from the expiration of an option is \$42, the exercise price of the option is \$40, the risk-free interest rate is 10% per annum, and the volatility is 20% per annum. This means that $S_0 = 42$, $K = 40$, $r = 0.1$, $\sigma = 0.2$, $T = 0.5$,

$$d_1 = \frac{\ln(42/40) + (0.1 + 0.2^2/2) \times 0.5}{0.2\sqrt{0.5}} = 0.7693$$

$$d_2 = \frac{\ln(42/40) + (0.1 - 0.2^2/2) \times 0.5}{0.2\sqrt{0.5}} = 0.6278$$

and

$$Ke^{-rT} = 40e^{-0.05} = 38.049$$

Hence, if the option is a European call, its value c is given by

$$c = 42N(0.7693) - 38.049N(0.6278)$$

If the option is a European put, its value p is given by

$$p = 38.049N(-0.6278) - 42N(-0.7693)$$

Using the polynomial approximation just given or the NORMSDIST function in Excel,

$$N(0.7693) = 0.7791, \qquad N(-0.7693) = 0.2209$$

$$N(0.6278) = 0.7349, \qquad N(-0.6278) = 0.2651$$

so that

$$c = 4.76, \qquad p = 0.81$$

Ignoring the time value of money, the stock price has to rise by \$2.76 for the purchaser of the call to break even. Similarly, the stock price has to fall by \$2.81 for the purchaser of the put to break even.

13.10 WARRANTS AND EMPLOYEE STOCK OPTIONS

The exercise of a regular call option on a company has no effect on the number of the company's shares outstanding. If the writer of the option does not own the company's shares, he or she must buy them in the market in the usual way and then sell them to the option holder for the strike price. As explained in Chapter 8, warrants and employee stock options are different from regular call options in that exercise leads to the company issuing more shares and then selling them to the option holder for the strike price. As the strike price is less than the market price, this dilutes the interest of the existing shareholders.

How should potential dilution affect the way we value outstanding warrants and employee stock options? The answer is that it should not! Assuming markets are

Business Snapshot 13.3 Warrants, Employee Stock Options, and Dilution

Consider a company with 100,000 shares each worth \$50. It surprises the market with an announcement that it is granting 100,000 stock options to its employees with a strike price of \$50. If the market sees little benefit to the shareholders from the employee stock options in the form of reduced salaries and more highly motivated managers, the stock price will decline immediately after the announcement of the employee stock options. If the stock price declines to \$45, the dilution cost to the current shareholders is \$5 per share or \$500,000 in total.

Suppose that the company does well so that by the end of three years the share price is \$100. Suppose further that all the options are exercised at this point. The payoff to the employees is \$50 per option. It is tempting to argue that there will be further dilution in that 100,000 shares worth \$100 per share are now merged with 100,000 shares for which only \$50 is paid, so that (a) the share price reduces to \$75 and (b) the payoff to the option holders is only \$25 per option. However, this argument is flawed. The exercise of the options is anticipated by the market and already reflected in the share price. The payoff from each option exercised is \$50.

This example illustrates the general point that when markets are efficient the impact of dilution from executive stock options or warrants is reflected in the stock price as soon as they are announced and does not need to be taken into account again when the options are valued.

efficient the stock price will reflect potential dilution from all outstanding warrants and employee stock options. This is explained in Business Snapshot 13.3.[8]

Consider next the situation a company is in when it is contemplating a new issue of warrants (or employee stock options). We suppose that the company is interested in calculating the cost of the issue assuming that there are no compensating benefits. We assume that the company has N shares worth S_0 each and the number of new options contemplated is M, with each option giving the holder the right to buy one share for K. The value of the company today is NS_0. This value does not change as a result of the warrant issue. Suppose that without the warrant issue the share price will be S_T at the warrant's maturity. This means that (with or without the warrant issue) the total value of the equity and the warrants at time T will NS_T. If the warrants are exercised, there is a cash inflow from the strike price increasing this to $NS_T + MK$. This value is distributed among $N + M$ shares, so that the share price immediately after exercise becomes

$$\frac{NS_T + MK}{N + M}$$

Therefore the payoff to an option holder if the option is exercised is

$$\frac{NS_T + MK}{N + M} - K$$

[8] Analysts sometimes assume that the sum of the values of the warrants and the equity (rather than just the value of the equity) is lognormal. The result is a Black–Scholes type of equation for the value of the warrant in terms of the value of the warrant. See Technical Note 3 on the author's website for an explanation of this model.

or

$$\frac{N}{N+M}(S_T - K)$$

This shows that the value of each option is the value of

$$\frac{N}{N+M}$$

regular call options on the company's stock. Therefore the total cost of the options is M times this.

Example 13.7

A company with 1 million shares worth \$40 each is considering issuing 200,000 warrants each giving the holder the right to buy one share with a strike price of \$60 in 5 years. It wants to know the cost of this. The interest rate is 3% per annum, and the volatility is 30% per annum. The company pays no dividends. From equation (13.20), the value of a 5-year European call option on the stock is \$7.04. In this case, $N = 1{,}000{,}000$ and $M = 200{,}000$, so that the value of each warrant is

$$\frac{1{,}000{,}000}{1{,}000{,}000 + 200{,}000} \times 7.04 = 5.87$$

or \$5.87. The total cost of the warrant issue is $200{,}000 \times 5.87 = \$1.17$ million. Assuming the market perceives no benefits from the warrant issue, we expect the stock price to decline by \$1.17 to \$38.83.

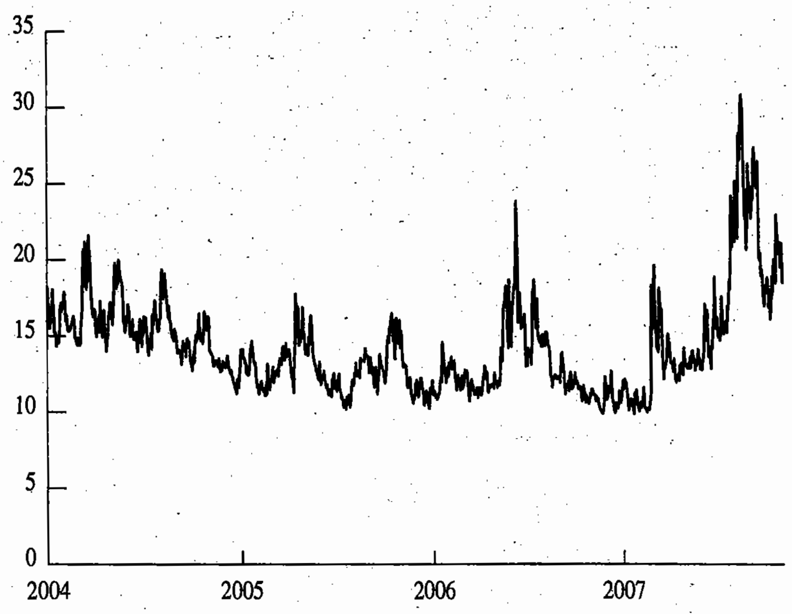
13.11 IMPLIED VOLATILITIES

The one parameter in the Black–Scholes pricing formulas that cannot be directly observed is the volatility of the stock price. In Section 13.4, we discussed how this can be estimated from a history of the stock price. In practice, traders usually work with what are known as *implied volatilities*. These are the volatilities implied by option prices observed in the market.

To illustrate how implied volatilities are calculated, suppose that the value of a European call option on a non-dividend-paying stock is 1.875 when $S_0 = 21$, $K = 20$, $r = 0.1$, and $T = 0.25$. The implied volatility is the value of σ that, when substituted into equation (13.20), gives $c = 1.875$. Unfortunately, it is not possible to invert equation (13.20) so that σ is expressed as a function of S_0, K, r, T, and c. However, an iterative search procedure can be used to find the implied σ. For example, we can start by trying $\sigma = 0.20$. This gives a value of c equal to 1.76, which is too low. Because c is an increasing function of σ, a higher value of σ is required. We can next try a value of 0.30 for σ. This gives a value of c equal to 2.10, which is too high and means that σ must lie between 0.20 and 0.30. Next, a value of 0.25 can be tried for σ. This also proves to be too high, showing that σ lies between 0.20 and 0.25. Proceeding in this way, we can halve the range for σ at each iteration and the correct value of σ can be calculated to any required accuracy.[9] In this example, the implied volatility is 0.235, or 23.5%, per

[9] This method is presented for illustration. Other more powerful methods, such as the Newton–Raphson method, are often used in practice (see footnote 4 of Chapter 4). DerivaGem can be used to calculate implied volatilities.

Figure 13.4 The VIX index, January 2004 to October 2007.



annum. A similar procedure can be used in conjunction with binomial trees to find implied volatilities for American options.

Implied volatilities are used to monitor the market's opinion about the volatility of a particular stock. Whereas historical volatilities (see Section 13.4) are backward looking, implied volatilities are forward looking. Traders often quote the implied volatility of an option rather than its price. This is convenient because the implied volatility tends to be less variable than the option price. As will be explained in Chapter 18, the implied volatilities of actively traded options are used by traders to estimate appropriate implied volatilities for other options.

The VIX Index

The CBOE publishes indices of implied volatility. The most popular index, the SPX VIX, is an index of the implied volatility of 30-day options on the S&P 500 calculated from a wide range of calls and puts.[10] Information on the way the index is calculated is in Section 24.13. Trading in futures on the VIX started in 2004 and trading in options on the VIX started in 2006. A trade involving futures or options on the S&P 500 is a bet on both the future level of the S&P 500 and the volatility of the S&P 500. By contrast, a futures or options contract on the VIX is a bet only on volatility. One contract is on 1,000 times the index. Figure 13.4 shows the VIX index between 2004 and 2007.

Example 13.8

Suppose that a trader buys an April futures contract on the VIX when the futures price is 18.5 (corresponding to a 30-day S&P 500 volatility of 18.5%) and closes out the contract when the futures price is 19.3 (corresponding to an S&P 500 volatility of 19.3%). The trader makes a gain of $800.

[10] Similarly, the VXN is an index of the volatility of the NASDAQ 100 index and the VXD is an index of the volatility of the Dow Jones Industrial Average.

13.12 DIVIDENDS

Up to now, we have assumed that the stock upon which the option is written pays no dividends. In this section, we modify the Black–Scholes model to take account of dividends. We assume that the amount and timing of the dividends during the life of an option can be predicted with certainty. For short-life options this is not an unreasonable assumption. (For long-life options it is usual to assume that the dividend yield rather the cash dividend payments are known. Options can then be valued as will be described in the next chapter.) The date on which the dividend is paid should be assumed to be the ex-dividend date. On this date the stock price declines by the amount of the dividend.[11]

European Options

European options can be analyzed by assuming that the stock price is the sum of two components: a riskless component that corresponds to the known dividends during the life of the option and a risky component. The riskless component, at any given time, is the present value of all the dividends during the life of the option discounted from the ex-dividend dates to the present at the risk-free rate. By the time the option matures, the dividends will have been paid and the riskless component will no longer exist. The Black–Scholes formula is therefore correct if S_0 is equal to the risky component of the stock price and σ is the volatility of the process followed by the risky component.[12] Operationally, this means that the Black–Scholes formula can be used provided that the stock price is reduced by the present value of all the dividends during the life of the option, the discounting being done from the ex-dividend dates at the risk-free rate. As already mentioned, a dividend is counted as being during the life of the option only if its ex-dividend date occurs during the life of the option.

Example 13.9

Consider a European call option on a stock when there are ex-dividend dates in two months and five months. The dividend on each ex-dividend date is expected to be \$0.50. The current share price is \$40, the exercise price is \$40, the stock price volatility is 30% per annum, the risk-free rate of interest is 9% per annum, and the time to maturity is six months. The present value of the dividends is

$$0.5e^{-0.1667\times0.09} + 0.5e^{-0.4167\times0.09} = 0.9741$$

The option price can therefore be calculated from the Black–Scholes formula,

[11] For tax reasons the stock price may go down by somewhat less than the cash amount of the dividend. To take account of this phenomenon, we need to interpret the word 'dividend' in the context of option pricing as the reduction in the stock price on the ex-dividend date caused by the dividend. Thus, if a dividend of \$1 per share is anticipated and the share price normally goes down by 80% of the dividend on the ex-dividend date, the dividend should be assumed to be \$0.80 for the purposes of the analysis.

[12] In theory, this is not quite the same as the volatility of the stochastic process followed by the whole stock price. The volatility of the risky component is approximately equal to the volatility of the whole stock price multiplied by $S_0/(S_0 - D)$, where D is the present value of the dividends. However, an adjustment is only necessary when volatilities are estimated using historical data. An implied volatility is calculated after the present value of dividends have been subtracted from the stock price and is the volatility of the risky component.

with $S_0 = 40 - 0.9741 = 39.0259$, $K = 40$, $r = 0.09$, $\sigma = 0.3$, and $T = 0.5$:

$$d_1 = \frac{\ln(39.0259/40) + (0.09 + 0.3^2/2) \times 0.5}{0.3\sqrt{0.5}} = 0.2017$$

$$d_2 = \frac{\ln(39.0259/40) + (0.09 - 0.3^2/2) \times 0.5}{0.3\sqrt{0.5}} = -0.0104$$

Using the polynomial approximation in Section 13.9 or the NORMSDIST function in Excel gives

$$N(d_1) = 0.5800, \qquad N(d_2) = 0.4959$$

and, from equation (13.20), the call price is

$$39.0259 \times 0.5800 - 40e^{-0.09\times0.5} \times 0.4959 = 3.67$$

or \$3.67.

American Options

Consider next American call options. Section 9.5 showed that in the absence of dividends American options should never be exercised early. An extension to the argument shows that, when there are dividends, it can only be optimal to exercise at a time immediately before the stock goes ex-dividend. We assume that n ex-dividend dates are anticipated and that they are at times $t_1, t_2, \dots, t_n$, with $t_1 < t_2 < \dots < t_n$. The dividends corresponding to these times will be denoted by $D_1, D_2, \dots, D_n$, respectively.

We start by considering the possibility of early exercise just prior to the final ex-dividend date (i.e., at time t_n). If the option is exercised at time t_n, the investor receives

$$S(t_n) - K$$

where $S(t)$ denotes the stock price at time t. If the option is not exercised, the stock price drops to $S(t_n) - D_n$. As shown by equation (9.5), the value of the option is then greater than

$$S(t_n) - D_n - Ke^{-r(T-t_n)}$$

It follows that, if

$$S(t_n) - D_n - Ke^{-r(T-t_n)} \geqslant S(t_n) - K$$

that is,

$$D_n \leqslant K\left[1 - e^{-r(T-t_n)}\right] \tag{13.24}$$

it cannot be optimal to exercise at time t_n. On the other hand, if

$$D_n > K\left[1 - e^{-r(T-t_n)}\right] \tag{13.25}$$

for any reasonable assumption about the stochastic process followed by the stock price, it can be shown that it is always optimal to exercise at time t_n for a sufficiently high value of $S(t_n)$. The inequality in (13.25) will tend to be satisfied when the final ex-dividend date is fairly close to the maturity of the option (i.e., $T - t_n$ is small) and the dividend is large.

Consider next time t_{n-1}, the penultimate ex-dividend date. If the option is exercised immediately prior to time t_{n-1}, the investor receives $S(t_{n-1}) - K$. If the option is not exercised at time t_{n-1}, the stock price drops to $S(t_{n-1}) - D_{n-1}$ and the earliest subsequent time at which exercise could take place is t_n. Hence, from equation (9.5), a lower bound to the option price if it is not exercised at time t_{n-1} is

$$S(t_{n-1}) - D_{n-1} - Ke^{-r(t_n - t_{n-1})}$$

It follows that if

$$S(t_{n-1}) - D_{n-1} - Ke^{-r(t_n - t_{n-1})} \geqslant S(t_{n-1}) - K$$

or

$$D_{n-1} \leqslant K\left[1 - e^{-r(t_n - t_{n-1})}\right]$$

it is not optimal to exercise immediately prior to time t_{n-1}. Similarly, for any $i < n$, if

$$D_i \leqslant K\left[1 - e^{-r(t_{i+1} - t_i)}\right] \tag{13.26}$$

it is not optimal to exercise immediately prior to time t_i.

The inequality in (13.26) is approximately equivalent to

$$D_i \leqslant Kr(t_{i+1} - t_i)$$

Assuming that K is fairly close to the current stock price, this inequality is satisfied when the dividend yield on the stock is less than the risk-free rate of interest. This is often the case.

We can conclude from this analysis that, in many circumstances, the most likely time for the early exercise of an American call is immediately before the final ex-dividend date, t_n. Furthermore, if inequality (13.26) holds for $i = 1, 2, \ldots, n - 1$ and inequality (13.24) holds, we can be certain that early exercise is never optimal.

Black's Approximation

Black suggests an approximate procedure for taking account of early exercise in call options.[13] This involves calculating, as described earlier in this section, the prices of European options that mature at times T and t_n, and then setting the American price equal to the greater of the two. This approximation seems to work well in most cases.[14]

Example 13.10

Consider the situation in Example 13.9, but suppose that the option is American rather than European. In this case $D_1 = D_2 = 0.5$, $S_0 = 40$, $K = 40$, $r = 0.09$, $t_1 = 2/12$, and $t_2 = 5/12$. Since

$$K\left[1 - e^{-r(t_2 - t_1)}\right] = 40(1 - e^{-0.09 \times 0.25}) = 0.89$$

is greater than 0.5, it follows (see inequality (13.26)) that the option should never be exercised immediately before the first ex-dividend date. In addition, since

$$K\left[1 - e^{-r(T - t_2)}\right] = 40(1 - e^{-0.09 \times 0.0833}) = 0.30$$

[13] See F. Black, "Fact and Fantasy in the Use of Options," *Financial Analysts Journal*, 31 (July/August 1975): 36–41, 61–72.

[14] For an exact formula, suggested by Roll, Geske, and Whaley, for valuing American calls when there is only one ex-dividend date, see Technical Note 4 on the author's website. This involves the cumulative bivariate normal distribution function. A procedure for calculating this function is given in Technical Note 5 also on the author's website.