PART THREE

Market Equilibrium and Market Failure

In Part III, our focus shifts to the fundamental issue of economics: *the organization of production and the allocation of the resulting commodities among consumers.* This fundamental issue can be addressed from two perspectives, one *positive* and the other *normative*.

From a positive (or *descriptive*) perspective, we can investigate the determination of production and consumption under various institutional mechanisms. The institutional arrangement that is our central focus is that of a *market* (or *private ownership*) *economy*. In a market economy, individual consumers have ownership rights to various assets (such as their labor) and are free to trade these assets in the marketplace for other assets or goods. Likewise, firms, which are themselves owned by consumers, decide on their production plan and trade in the market to secure necessary inputs and sell the resulting outputs. Roughly speaking, we can identify a *market equilibrium* as an outcome of a market economy in which each agent in the economy (i.e., each consumer and firm) is doing as well as he can given the actions of all other agents.

In contrast, from a normative (or *prescriptive*) perspective, we can ask what constitutes a *socially optimal* plan of production and consumption (of course, we will need to be more specific about what "socially optimal" means), and we can then examine the extent to which specific institutions, such as a market economy, perform well in this regard.

In Chapter 10, we study *competitive* (or *perfectly competitive*) *market economies* for the first time. These are market economies in which every relevant good is traded in a market at publicly known prices and all agents act as price takers (recall that much of the analysis of individual behavior in Part I was geared to this case). We begin by defining, in a general way, two key concepts: *competitive* (or *Walrasian*) *equilibrium* and *Pareto optimality* (or *Pareto efficiency*). The concept of competitive equilibrium provides us with an appropriate notion of market equilibrium for competitive market economies. The concept of Pareto optimality offers a minimal and uncontroversial test that any social optimal economic outcome should pass. An economic outcome is said to be Pareto optimal if it is impossible to make some individuals better off without making some other individuals worse off. This concept is a formalization of the idea that there is no waste in society, and it conveniently separates the issue of economic efficiency from more controversial (and political) questions regarding the ideal *distribution* of well-being across individuals.

Chapter 10 then explores these two concepts and the relationships between them in the special context of the *partial equilibrium model*. The partial equilibrium model, which forms the basis for our analysis throughout Part III, offers a considerable analytical simplification; in it, our analysis can be conducted by analyzing a single market (or a small group of related markets) at a time. In this special context, we establish two central results regarding the optimality properties of competitive equilibria, known as the *fundamental theorems of welfare economics*. These can be roughly paraphrased as follows:

The First Fundamental Welfare Theorem. If every relevant good is traded in a market at publicly known prices (i.e., if there is a complete set of markets), and if households and firms act perfectly competitively (i.e., as price takers), then the market outcome is Pareto optimal. That is, when markets are complete, *any competitive equilibrium is necessarily Pareto optimal.*

The Second Fundamental Welfare Theorem. If household preferences and firm production sets are convex, there is a complete set of markets with publicly known prices, and every agent acts as a price taker, then *any Pareto optimal outcome can be achieved as a competitive equilibrium if appropriate lump-sum transfers of wealth are arranged.*

The first welfare theorem provides a set of conditions under which we can be assured that a market economy will achieve a Pareto optimal result; it is, in a sense, the formal expression of Adam Smith's claim about the "invisible hand" of the market. The second welfare theorem goes even further. It states that under the same set of assumptions as the first welfare theorem plus convexity conditions, *all* Pareto optimal outcomes can in principle be implemented through the market mechanism. That is, a public authority who wishes to implement a particular Pareto optimal outcome (reflecting, say, some political consensus on proper distributional goals) may always do so by appropriately redistributing wealth and then "letting the market work."

In an important sense, the first fundamental welfare theorem establishes the perfectly competitive case as a benchmark for thinking about outcomes in market economies. In particular, any inefficiencies that arise in a market economy, and hence any role for Pareto-improving market intervention, *must* be traceable to a violation of at least one of the assumptions of this theorem.

The remainder of Part III, Chapters 11 to 14, can be viewed as a development of this theme. In these chapters, we study a number of ways in which actual markets may depart from this perfectly competitive ideal and where, as a result, market equilibria fail to be Pareto optimal, a situation known as *market failure*.

In Chapter 11, we study *externalities* and *public goods*. In both cases, the actions of one agent directly affect the utility functions or production sets of other agents in the economy. We see there that the presence of these nonmarketed "goods" or "bads" (which violates the complete markets assumption of the first welfare theorem) undermines the Pareto optimality of market equilibrium.

In Chapter 12, we turn to the study of settings in which some agents in the economy have *market power* and, as a result, fail to act as price takers. Once again, an assumption of the first fundamental welfare theorem fails to hold, and market equilibria fail to be Pareto optimal as a result.

In Chapters 13 and 14, we consider situations in which an *asymmetry of information* exists among market participants. The complete markets assumption of the first welfare theorem implicitly requires that the characteristics of traded commodities be observable by all market participants because, without this observability, distinct markets cannot exist for commodities that have different characteristics. Chapter 13 focuses on the case in which asymmetric information exists between agents at the time of contracting. Our discussion highlights several phenomena—*adverse selection, signaling*, and *screening*—that can arise as a result of this informational imperfection, and the welfare loss that it causes. Chapter 14 in contrast, investigates the case of postcontractual asymmetric information, a problem that leads us to the study of the *principal-agent model*. Here, too, the presence of asymmetric information prevents trade of all relevant commodities and can lead market outcomes to be Pareto inefficient.

We rely extensively in some places in Part III on the tools that we developed in Parts I and II. This is particularly true in Chapter 10, where we use material developed in Part I, and Chapters 12 and 13, where we use the game-theoretic tools developed in Part II.

A much more complete and general study of competitive market economies and the fundamental welfare theorems is reserved for Part IV.

CHAPTER

10 Competitive Markets

10.A Introduction

In this chapter, we consider, for the first time, an entire economy in which consumers and firms interact through markets. The chapter has two principal goals: first, to formally introduce and study two key concepts, the notions of *Pareto optimality* and *competitive equilibrium*, and second, to develop a somewhat special but analytically very tractable context for the study of market equilibrium, the *partial equilibrium model*.

We begin in Section 10.B by presenting the notions of a *Pareto optimal* (or *Pareto efficient*) *allocation* and of a *competitive* (or *Walrasian*) *equilibrium* in a general setting.

Starting in Section 10.C, we narrow our focus to the partial equilibrium context. The partial equilibrium approach, which originated in Marshall (1920), envisions the market for a single good (or group of goods) for which each consumer's expenditure constitutes only a small portion of his overall budget. When this is so, it is reasonable to assume that changes in the market for this good will leave the prices of all other commodities approximately unaffected and that there will be, in addition, negligible wealth effects in the market under study. We capture these features in the simplest possible way by considering a two-good model in which the expenditure on all commodities other than that under consideration is treated as a single composite commodity (called the *numeraire* commodity), and in which consumers' utility functions take a quasilinear form with respect to this numeraire. Our study of the competitive equilibria of this simple model lends itself to extensive demand-and-supply graphical analysis. We also discuss how to determine the comparative statics effects that arise from exogenous changes in the market environment. As an illustration, we consider the effects on market equilibrium arising from the introduction of a distortionary commodity tax.

In Section 10.D, we analyze the properties of Pareto optimal allocations in the partial equilibrium model. Most significantly, we establish for this special context the validity of the *fundamental theorems of welfare economics*: Competitive equilibrium allocations are necessarily Pareto optimal, and any Pareto optimal allocation can be achieved as a competitive equilibrium if appropriate lump-sum transfers are made.

As we noted in the introduction to Part III, these results identify an important benchmark case in which market equilibria yield desirable economic outcomes. At the same time, they provide a framework for identifying situations of market failure, such as those we study in Chapters 11 to 14.

In Section 10.E, we consider the measurement of welfare changes in the partial equilibrium context. We show that these can be represented by areas between properly defined demand and supply curves. As an application, we examine the deadweight loss of distortionary taxation.

Section 10.F contemplates settings characterized by *free entry*, that is, settings in which all potential firms have access to the most efficient technology and may enter and exit markets in response to the profit opportunities they present. We define a notion of *long-run competitive equilibrium* and then use it to distinguish between long-run and short-run comparative static effects in response to changes in market conditions.

In Section 10.G, we provide a more extended discussion of the use of partial equilibrium analysis in economic modeling.

The material covered in this chapter traces its roots far back in economic thought. An excellent source for further reading is Stigler (1987). We should emphasize that the analysis of competitive equilibrium and Pareto optimality presented here is very much a first pass. In Part IV we return to the topic for a more complete and general investigation; many additional references will be given there.

10.B Pareto Optimality and Competitive Equilibria

In this section, we introduce and discuss the concepts of *Pareto optimality* (or *Pareto efficiency*) and *competitive* (or *Walrasian*) *equilibrium* in a general setting.

Consider an economy consisting of I consumers (indexed by $i = 1, \dots, I$), J firms (indexed by $j = 1, \dots, J$), and L goods (indexed by $\ell = 1, \dots, L$). Consumer i's preferences over consumption bundles $x_i = (x_{1i}, \dots, x_{Li})$ in his consumption set $X_i \subset \mathbb{R}^L$ are represented by the utility function $u_i(\cdot)$. The total amount of each good $\ell = 1, \dots, L$ initially available in the economy, called the total *endowment* of good ℓ, is denoted by $\omega_\ell \geq 0$ for $\ell = 1, \dots, L$. It is also possible, using the production technologies of the firms, to transform some of the initial endowment of a good into additional amounts of other goods. Each firm j has available to it the production possibilities summarized by the production set $Y_j \subset \mathbb{R}^L$. An element of Y_j is a production vector $y_j = (y_{1j}, \dots, y_{Lj}) \in \mathbb{R}^L$. Thus, if $(y_1, \dots, y_J) \in \mathbb{R}^{LJ}$ are the production vectors of the J firms, the total (net) amount of good ℓ available to the economy is $\omega_\ell + \sum_j y_{\ell j}$ (recall that negative entries in a production vector denote input usage; see Section 5.B).

We begin with Definition 10.B.1, which identifies the set of possible outcomes in this economy:

Definition 10.B.1: An *economic allocation* $(x_1, \dots, x_I, y_1, \dots, y_J)$ is a specification of a consumption vector $x_i \in X_i$ for each consumer $i = 1, \dots, I$ and a production vector $y_j \in Y_j$ for each firm $j = 1, \dots, J$. The allocation $(x_1, \dots, x_I, y_1, \dots, y_J)$ is *feasible* if

$$\sum_{i=1}^{I} x_{\ell i} \leq \omega_\ell + \sum_{j=1}^{J} y_{\ell j} \quad \text{for } \ell = 1, \dots, L.$$

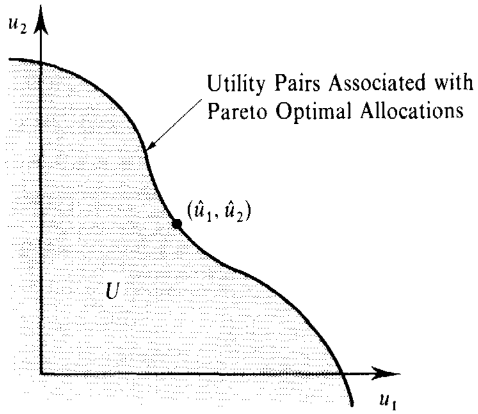


Figure 10.B.1
A utility possibility set.

Thus, an economic allocation is feasible if the total amount of each good consumed does not exceed the total amount available from both the initial endowment and production.

Pareto Optimality

It is often of interest to ask whether an economic system is producing an "optimal" economic outcome. An essential requirement for any optimal economic allocation is that it possess the property of *Pareto optimality* (or *Pareto efficiency*).

Definition 10.B.2: A feasible allocation $(x_1, \ldots, x_I, y_1, \ldots, y_J)$ is *Pareto optimal* (or *Pareto efficient*) if there is no other feasible allocation $(x'_1, \ldots, x'_I, y'_1, \ldots, y'_J)$ such that $u_i(x'_i) \geq u_i(x_i)$ for all $i = 1, \ldots, I$ and $u_i(x'_i) > u_i(x_i)$ for some i.

An allocation that is Pareto optimal uses society's initial resources and technological possibilities efficiently in the sense that there is no alternative way to organize the production and distribution of goods that makes some consumer better off without making some other consumer worse off.

Figure 10.B.1 illustrates the concept of Pareto optimality. There we depict the set of attainable utility levels in a two-consumer economy. This set is known as a *utility possibility set* and is defined in this two-consumer case by

$$U = \{(u_1, u_2) \in \mathbb{R}^2\text{: there exists a feasible allocation } (x_1, x_2, y_1, \ldots, y_J) \text{ such that } u_i \leq u_i(x_i) \text{ for } i = 1, 2\}.$$

The set of Pareto optimal allocations corresponds to those allocations that generate utility pairs lying in the utility possibility set's northeast boundary, such as point $(\hat{u}_1, \hat{u}_2)$. At any such point, it is impossible to make one consumer better off without making the other worse off.

It is important to note that the criterion of Pareto optimality does not insure that an allocation is in any sense equitable. For example, using all of society's resources and technological capabilities to make a single consumer as well off as possible, subject to all other consumers receiving a subsistence level of utility, results in an allocation that is Pareto optimal but not in one that is very desirable on distributional grounds. Nevertheless, Pareto optimality serves as an important minimal test for the desirability of an allocation; it does, at the very least, say that there is no waste in the allocation of resources in society.

Competitive Equilibria

Throughout this chapter, we are concerned with the analysis of competitive market economies. In such an economy, society's initial endowments and technological possibilities (i.e., the firms) are owned by consumers. We suppose that consumer i initially owns $\omega_{\ell i}$ of good ℓ, where $\sum_i \omega_{\ell i} = \omega_\ell$. We denote consumer i's vector of endowments by $\omega_i = (\omega_{1i}, \ldots, \omega_{Li})$. In addition, we suppose that consumer i owns a share θ_{ij} of firm j (where $\sum_i \theta_{ij} = 1$), giving him a claim to fraction θ_{ij} of firm j's profits.

In a competitive economy, a market exists for each of the L goods, and all consumers and producers act as price takers. The idea behind the price-taking assumption is that if consumers and producers are small relative to the size of the market, they will regard market prices as unaffected by their own actions.[1]

Denote the vector of market prices for goods $1, \ldots, L$ by $p = (p_1, \ldots, p_L)$. Definition 10.B.3 introduces the notion of a competitive (or Walrasian) equilibrium.

Definition 10.B.3: The allocation $(x_1^*, \ldots, x_I^*, y_1^*, \ldots, y_J^*)$ and price vector $p^* \in \mathbb{R}^L$ constitute a *competitive* (or *Walrasian*) *equilibrium* if the following conditions are satisfied:

(i) *Profit maximization*: For each firm j, y_j^* solves

$$\max_{y_j \in Y_j} \; p^* \cdot y_j. \tag{10.B.1}$$

(ii) *Utility maximization*: For each consumer i, x_i^* solves

$$\max_{x_i \in X_i} \; u_i(x_i) \tag{10.B.2}$$

$$\text{s.t. } p^* \cdot x_i \le p^* \cdot \omega_i + \sum_{j=1}^{J} \theta_{ij}(p^* \cdot y_j^*).$$

(iii) *Market clearing*: For each good $\ell = 1, \ldots, L$,

$$\sum_{i=1}^{I} x_{\ell i}^* = \omega_\ell + \sum_{j=1}^{J} y_{\ell j}^*. \tag{10.B.3}$$

Definition 10.B.3 delineates three sorts of conditions that must be met for a competitive economy to be considered to be in equilibrium. Conditions (i) and (ii) reflect the underlying assumption, common to nearly all economic models, that agents in the economy seek to do as well as they can for themselves. Condition (i) states that each firm must choose a production plan that maximizes its profits, taking as given the equilibrium vector of prices of its outputs and inputs (for the justification of the profit-maximization assumption, see Section 5.G). We studied this competitive behavior of the firm extensively in Chapter 5.

Condition (ii) requires that each consumer chooses a consumption bundle that maximizes his utility given the budget constraint imposed by the equilibrium prices and by his wealth. We studied this competitive behavior of the consumer extensively in Chapter 3. One difference here, however, is that the consumer's wealth is now a function of prices. This dependence of wealth on prices arises in

1. Strictly speaking, it is *equilibrium* market prices that they will regard as unaffected by their actions. For more on this point, see the small-type discussion later in this section.

two ways: First, prices determine the value of the consumer's initial endowments; for example, an individual who initially owns real estate is poorer if the price of real estate falls. Second, the equilibrium prices affect firms' profits and hence the value of the consumer's shareholdings.

Condition (iii) is somewhat different. It requires that, at the equilibrium prices, the desired consumption and production levels identified in conditions (i) and (ii) are in fact mutually compatible; that is, the aggregate supply of each commodity (its total endowment plus its net production) equals the aggregate demand for it. If excess supply or demand existed for a good at the going prices, the economy could not be at a point of equilibrium. For example, if there is excess demand for a particular commodity at the existing prices, some consumer who is not receiving as much of the commodity as he desires could do better by offering to pay just slightly more than the going market price and thereby get sellers to offer the commodity to him first. Similarly, if there is excess supply, some seller will find it worthwhile to offer his product at a slight discount from the going market price.[2]

Note that in justifying why an equilibrium must involve no excess demand or supply, we have actually made use of the fact that consumers and producers *might not* simply take market prices as given. How are we to reconcile this argument with the underlying price-taking assumption?

An answer to this apparent paradox comes from recognizing that consumers and producers *always* have the ability to alter their offered prices (in the absence of any institutional constraints preventing this). For the price-taking assumption to be appropriate, what we want is that they have no *incentive* to alter prices that, if taken as given, equate demand and supply (we have already seen that they *do* have an incentive to alter prices that do not equate demand and supply).

Notice that as long as consumers can make their desired trades at the going market prices, they will not wish to offer more than the market price to entice sellers to sell to them first. Similarly, if producers are able to make their desired sales, they will have no incentive to undercut the market price. Thus, at a price that equates demand and supply, consumers do not wish to raise prices, and firms do not wish to lower them.

More troublesome is the possibility that a buyer might try to lower the price he pays or that a seller might try to raise the price he charges. A seller, for example, may possess the ability to raise profitably prices of the goods he sells above their competitive level (see Chapter 12). In this case, there is no reason to believe that this market power will not be exercised. To rescue the price-taking assumption, one needs to argue that under appropriate (competitive) conditions such market power does not exist. This we do in Sections 12.F and 18.C, where we formalize the idea that if market participants' desired trades are small relative to the size of the market, then they will have little incentive to depart from market prices. Thus, in a suitably defined equilibrium, they will act approximately like price takers.

Note from Definition 10.B.3 that if the allocation $(x_1^*, \dots, x_I^*, y_1^*, \dots, y_J^*)$ and price vector $p^* \gg 0$ constitute a competitive equilibrium, then so do the allocation

2. Strictly speaking, this second part of the argument requires the price to be positive; indeed, if the price is zero (i.e., if the good is free), then excess supply should be permissible at equilibrium. In the remainder of this chapter, however, consumer preferences will be such as to preclude this possibility (goods will be assumed to be desirable). Hence, we neglect this possibility here.

$(x_1^*, \ldots, x_I^*, y_1^*, \ldots, y_J^*)$ and price vector $\alpha p^* = (\alpha p_1^*, \ldots, \alpha p_L^*)$ for any scalar $\alpha > 0$ (see Exercise 10.B.2). As a result, we can normalize prices without loss of generality. In this chapter, we always normalize by setting one good's price equal to 1.

Lemma 10.B.1 will also prove useful in identifying competitive equilibria.

Lemma 10.B.1: If the allocation $(x_1, \ldots, x_I, y_1, \ldots, y_J)$ and price vector $p \gg 0$ satisfy the market clearing condition (10.B.3) for all goods $\ell \neq k$, and if every consumer's budget constraint is satisfied with equality, so that $p \cdot x_i = p \cdot \omega_i + \sum_j \theta_{ij} p \cdot y_j$ for all i, then the market for good k also clears.

Proof: Adding up the consumers' budget constraints over the I consumers and rearranging terms, we get

$$\sum_{\ell \neq k} p_\ell \left(\sum_{i=1}^{I} x_{\ell i} - \omega_\ell - \sum_{j=1}^{J} y_{\ell j} \right) = -p_k \left(\sum_{i=1}^{I} x_{ki} - \omega_k - \sum_{j=1}^{J} y_{kj} \right).$$

By market clearing in goods $\ell \neq k$, the left-hand side of this equation is equal to zero. Thus, the right-hand side must be equal to zero as well. Because $p_k > 0$, this implies that we have market clearing in good k. ■

In the models studied in this chapter, Lemma 10.B.1 will allow us to identify competitive equilibria by checking for market clearing in only $L - 1$ markets. Lemma 10.B.1 is really just a matter of double-entry accountancy. If consumers' budget constraints hold with equality, the dollar value of each consumer's planned purchases equals the dollar value of what he plans to sell plus the dollar value of his share (θ_{ij}) of the firms' (net) supply, and so the total value of planned purchases in the economy must equal the total value of planned sales. If those values are equal to each other in all markets but one, then equality must hold in the remaining market as well.

10.C Partial Equilibrium Competitive Analysis

Marshallian partial equilibrium analysis envisions the market for one good (or several goods, as discussed in Section 10.G) that constitutes a small part of the overall economy. The small size of the market facilitates two important simplifications for the analysis of market equilibrium:[3] First, as Marshall (1920) emphasized, when the expenditure on the good under study is a small portion of a consumer's total expenditure, only a small fraction of any additional dollar of wealth will be spent on this good; consequently, we can expect wealth effects for it to be small. Second, with similarly dispersed substitution effects, the small size of the market under study should lead the prices of other goods to be approximately unaffected by changes in this market.[4] Because of this fixity of other prices, we are justified in treating the expenditure on these other goods as a single composite commodity, which we call the *numeraire* (see Exercise 3.G.5).

3. The following points have been formalized by Vives (1987). (See Exercise 10.C.1 for an illustration.)

4. This is not the only possible justification for taking other goods' prices as being unaffected by the market under study; see Section 10.G.

With this partial equilibrium interpretation as our motivation, we proceed to study a simple two-good quasilinear model. There are two commodities: good ℓ and the numeraire. We let x_i and m_i denote consumer i's consumption of good ℓ and the numeraire, respectively. Each consumer $i = 1, \ldots, I$ has a utility function that takes the quasilinear form (see Sections 3.B and 3.C):

$$u_i(m_i, x_i) = m_i + \phi_i(x_i).$$

We let each consumer's consumption set be $\mathbb{R} \times \mathbb{R}_+$, and so we assume for convenience that consumption of the numeraire commodity m can take negative values. This is to avoid dealing with boundary problems. We assume that $\phi_i(\cdot)$ is bounded above and twice differentiable, with $\phi_i'(x_i) > 0$ and $\phi_i''(x_i) < 0$ at all $x_i \geq 0$. We normalize $\phi_i(0) = 0$.

In terms of our partial equilibrium interpretation, we think of good ℓ as the good whose market is under study and of the numeraire as representing the composite of all other goods (m stands for the total money expenditure on these other goods). Recall that with quasilinear utility functions, wealth effects for non-numeraire commodities are null.

In the discussion that follows, we normalize the price of the numeraire to equal 1, and we let p denote the price of good ℓ.

Each firm $j = 1, \ldots, J$ in this two-good economy is able to produce good ℓ from good m. The amount of the numeraire required by firm j to produce $q_j \geq 0$ units of good ℓ is given by the cost function $c_j(q_j)$ (recall that the price of the numeraire is 1). Letting z_j denote firm j's use of good m as an input, its production set is therefore

$$Y_j = \{(-z_j, q_j)\colon\ q_j \geq 0 \text{ and } z_j \geq c_j(q_j)\}.$$

In what follows, we assume that $c_j(\cdot)$ is twice differentiable, with $c_j'(q_j) > 0$ and $c_j''(q_j) \geq 0$ at all $q_j \geq 0$. [In terms of our partial equilibrium interpretation, we can think of $c_j(q_j)$ as actually arising from some multiple-input cost function $c_j(\bar{w}, q_j)$, given the fixed vector of factor prices $\bar{w}$.[5]]

For simplicity, we shall assume that there is no initial endowment of good ℓ, so that all amounts consumed must be produced by the firms. Consumer i's initial endowment of the numeraire is the scalar $\omega_{mi} > 0$, and we let $\omega_m = \sum_i \omega_{mi}$.

We now proceed to identify the competitive equilibria for this two-good quasilinear model. Applying Definition 10.B.3, we consider first the implications of profit and utility maximization.

Given the price p^* for good ℓ, firm j's equilibrium output level q_j^* must solve

$$\underset{q_j \geq 0}{\text{Max}} \quad p^* q_j - c_j(q_j),$$

which has the necessary and sufficient first-order condition

$$p^* \leq c_j'(q_j^*), \quad \text{with equality if } q_j^* > 0.$$

On the other hand, consumer i's equilibrium consumption vector (m_i^*, x_i^*) must

5. Some of the exercises at the end of the chapter investigate the effects of exogenous changes in these factor prices.

solve

$$\underset{m_i \in \mathbb{R}, x_i \in \mathbb{R}_+}{\text{Max}} \quad m_i + \phi_i(x_i)$$

$$\text{s.t.} \quad m_i + p^* x_i \le \omega_{mi} + \sum_{j=1}^{J} \theta_{ij}(p^* q_j^* - c_j(q_j^*)).$$

In any solution to this problem, the budget constraint holds with equality. Substituting for m_i from this constraint, we can rewrite consumer i's problem solely in terms of choosing his optimal consumption of good ℓ. Doing so, we see that x_i^* must solve

$$\underset{x_i \ge 0}{\text{Max}} \quad \phi_i(x_i) - p^* x_i + \left[\omega_{mi} + \sum_{j=1}^{J} \theta_{ij}(p^* q_j^* - c_j(q_j^*))\right],$$

which has the necessary and sufficient first-order condition

$$\phi_i'(x_i^*) \le p^*, \quad \text{with equality if } x_i^* > 0.$$

In what follows, it will be convenient to adopt the convention of identifying an equilibrium allocation by the levels of good ℓ consumed and produced, $(x_1^*, \ldots, x_I^*, q_1^*, \ldots, q_J^*)$, with the understanding that consumer i's equilibrium consumption of the numeraire is then $m_i^* = [\omega_{mi} + \sum_j \theta_{ij}(p^* q_j^* - c_j(q_j^*))] - p^* x_i^*$ and that firm j's equilibrium usage of the numeraire as an input is $z_j^* = c_j(q_j^*)$.

To complete the development of the equilibrium conditions for this model, recall that by Lemma 10.B.1, we need only check that the market for good ℓ clears.[6] Hence, we conclude that the allocation $(x_1^*, \ldots, x_I^*, q_1^*, \ldots, q_J^*)$ and the price p^* constitute a competitive equilibrium if and only if

$$p^* \le c_j'(q_j^*), \quad \text{with equality if } q_j^* > 0 \qquad j = 1, \ldots, J. \tag{10.C.1}$$

$$\phi_i'(x_i^*) \le p^*, \quad \text{with equality if } x_i^* > 0 \qquad i = 1, \ldots, I. \tag{10.C.2}$$

$$\sum_{i=1}^{I} x_i^* = \sum_{j=1}^{J} q_j^*. \tag{10.C.3}$$

At any interior solution, condition (10.C.1) says that firm j's marginal benefit from selling an additional unit of good ℓ, p^*, exactly equals its marginal cost $c_j'(q_j^*)$. Condition (10.C.2) says that consumer i's marginal benefit from consuming an additional unit of good ℓ, $\phi_i'(x_i^*)$, exactly equals its marginal cost p^*. Condition (10.C.3) is the market-clearing equation. Together, these $I + J + 1$ conditions characterize the $(I + J + 1)$ equilibrium values $(x_1^*, \ldots, x_I^*, q_1^*, \ldots, q_J^*)$ and p^*. Note that as long as $\text{Max}_i\, \phi_i'(0) > \text{Min}_j\, c_j'(0)$, the aggregate consumption and production of good ℓ must be strictly positive in a competitive equilibrium [this follows from conditions (10.C.1) and (10.C.2)]. For simplicity, we assume that this is the case in the discussion that follows.

Conditions (10.C.1) to (10.C.3) have a very important property: They do not involve, in any manner, the endowments or the ownership shares of the consumers. As a result, we see that *the equilibrium allocation and price are independent of the*

6. Note that we must have $p^* > 0$ in any competitive equilibrium; otherwise, consumers would demand an infinite amount of good ℓ [recall that $\phi_i'(\cdot) > 0$].

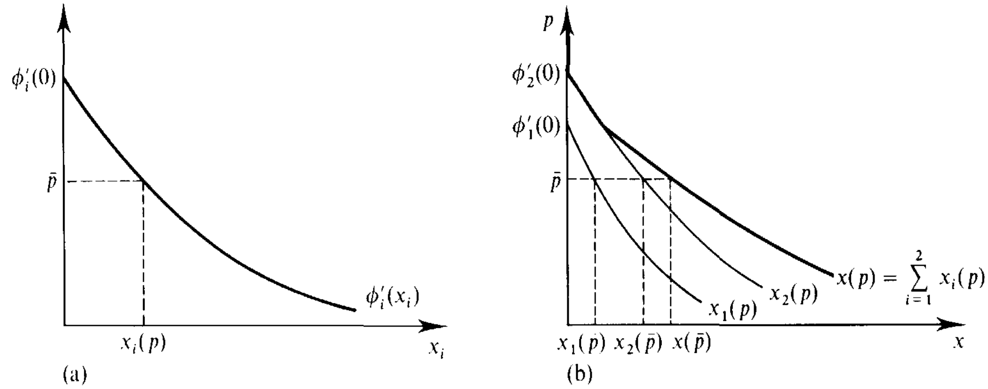


Figure 10.C.1
Construction of the aggregate demand function.
(a) Determination of consumer i's demand.
(b) Construction of the aggregate demand function ($I = 2$).

distribution of endowments and ownership shares. This important simplification arises from the quasilinear form of consumer preferences.[7]

The competitive equilibrium of this model can be nicely represented using the traditional Marshallian graphical technique that identifies the equilibrium price as the point of intersection of aggregate demand and aggregate supply curves.

We can derive the aggregate demand function for good ℓ from condition (10.C.2). Because $\phi_i''(\cdot) < 0$ and $\phi_i(\cdot)$ is bounded, $\phi_i'(\cdot)$ is a strictly decreasing function of x_i taking all values in the set $(0, \phi_i'(0)]$. Therefore, for each possible level of $p > 0$, we can solve for a unique level of x_i, denoted $x_i(p)$, that satisfies condition (10.C.2). Note that if $p \geq \phi_i'(0)$, then $x_i(p) = 0$. Figure 10.C.1(a) depicts this construction for a price $\bar{p} > 0$. The function $x_i(\cdot)$ is consumer i's *Walrasian demand function* for good ℓ (see Section 3.D) which, because of quasilinearity, does not depend on the consumer's wealth. It is continuous and nonincreasing in p at all $p > 0$, and is strictly decreasing at any $p < \phi_i'(0)$ [at any such p, we have $x_i'(p) = 1/\phi_i''(x_i(p)) < 0$].

The *aggregate demand function for good ℓ* is then the function $x(p) = \sum_i x_i(p)$, which is continuous and nonincreasing at all $p > 0$, and is strictly decreasing at any $p < \text{Max}_i\, \phi_i'(0)$. Its construction is depicted in Figure 10.C.1(b) for the case in which $I = 2$; it is simply the horizontal summation of the individual demand functions and is drawn in the figure with a heavy trace. Note that $x(p) = 0$ whenever $p \geq \text{Max}_i\, \phi_i'(0)$.

The aggregate supply function can be similarly derived from condition (10.C.1).[8] Suppose, first, that every $c_j(\cdot)$ is strictly convex and that $c_j'(q_j) \to \infty$ as $q_j \to \infty$. Then, for any $p > 0$, we can let $q_j(p)$ denote the unique level of q_j that satisfies condition (10.C.1). Note that for $p \leq c_j'(0)$, we have $q_j(p) = 0$. Figure 10.C.2(a) illustrates this construction for a price $\bar{p} > 0$. The function $q_j(\cdot)$ is firm j's *supply function* for good ℓ (see Sections 5.C and 5.D). It is continuous and nondecreasing at all $p > 0$, and is strictly increasing at any $p > c_j'(0)$ [for any such p, $q_j'(p) = 1/c_j''(q_j(p)) > 0$].

The *aggregate* (or *industry*) *supply function* for good ℓ is then the function $q(p) = \sum_j q_j(p)$, which is continuous and nondecreasing at all $p > 0$, and is strictly increasing at any $p > \text{Min}_j\, c_j'(0)$. Its construction is depicted in Figure 10.C.2(b) for

7. See Section 10.G for a further discussion of this general feature of equilibrium in economies with quasilinear utility functions.

8. See Section 5.D for an extensive discussion of individual supply in the one-input, one-output case.

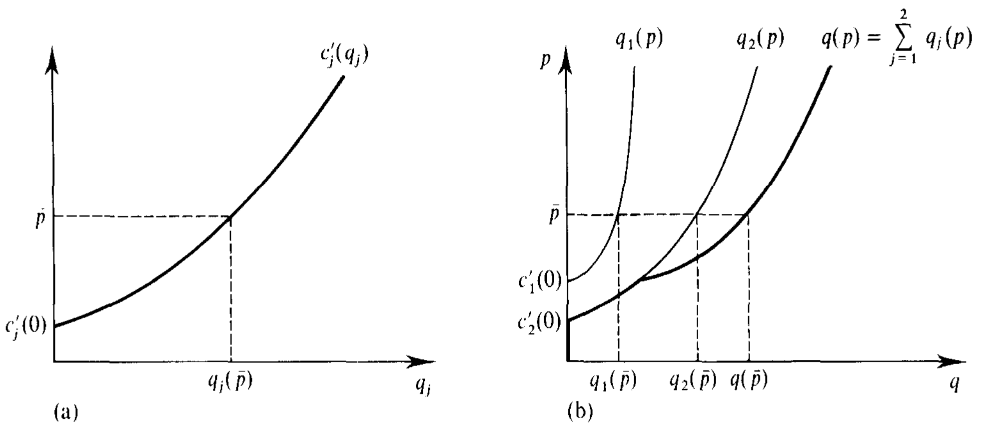


Figure 10.C.2
Construction of the aggregate supply function.
(a) Determination of firm j's supply.
(b) Construction of the aggregate supply function ($J = 2$).

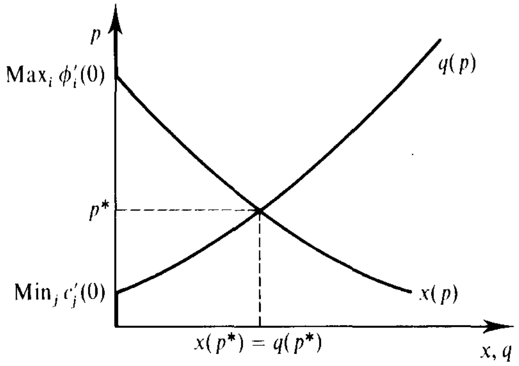


Figure 10.C.3
The equilibrium price equates demand and supply.

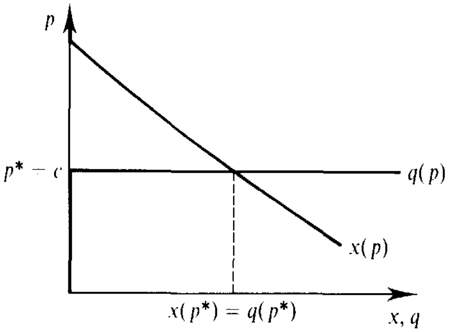
the case in which $J = 2$; it is equal to the horizontal sum of the individual firms' supply functions and is drawn in the figure with a heavy trace. Note that $q(p) = 0$ whenever $p \leq \text{Min}_j\, c_j'(0)$.

To find the equilibrium price of good ℓ, we need only find the price p^* at which aggregate demand equals aggregate supply, that is, at which $x(p^*) = q(p^*)$. When $\text{Max}_i\, \phi_i'(0) > \text{Min}_j\, c_j'(0)$ as we have assumed, at any $p \geq \text{Max}_i\, \phi_i'(0)$ we have $x(p) = 0$ and $q(p) > 0$. Likewise, at any $p \leq \text{Min}_j\, c_j'(0)$ we have $x(p) > 0$ and $q(p) = 0$. The existence of an equilibrium price $p^* \in (\text{Min}_j\, c_j'(0), \text{Max}_i\, \phi_i'(0))$ then follows from the continuity properties of $x(\cdot)$ and $q(\cdot)$. The solution is depicted in Figure 10.C.3. Note also that because $x(\cdot)$ is strictly decreasing at all $p < \text{Max}_i\, \phi_i'(0)$ and $q(\cdot)$ is strictly increasing at all $p > \text{Min}_j\, c_j'(0)$, this equilibrium price is uniquely defined.[9] The individual consumption and production levels of good ℓ in this equilibrium are then given by $x_i^* = x_i(p^*)$ for $i = 1, \ldots, I$ and $q_j^* = q_j(p^*)$ for $j = 1, \ldots, J$.

More generally, if some $c_j(\cdot)$ is merely convex [e.g., if $c_j(\cdot)$ is linear, as in the constant returns case], then $q_j(\cdot)$ is a convex-valued correspondence rather than a function and it may be well defined only on a subset of prices.[10] Nevertheless, the

9. Be warned, however, that the uniqueness of equilibrium is a property that need not hold in more general settings in which wealth effects are present. (See Chapter 17.)

10. For example, if firm j has $c_j(q_j) = c_j q_j$ for some scalar $c_j > 0$, then when $p > c_j$, we have $q_j(p) = \infty$. As a result, if $p > c_j$, the aggregate supply is $q(p) = \sum_j q_j(p) = \infty$; consequently $q(\cdot)$ is not well defined for this p.



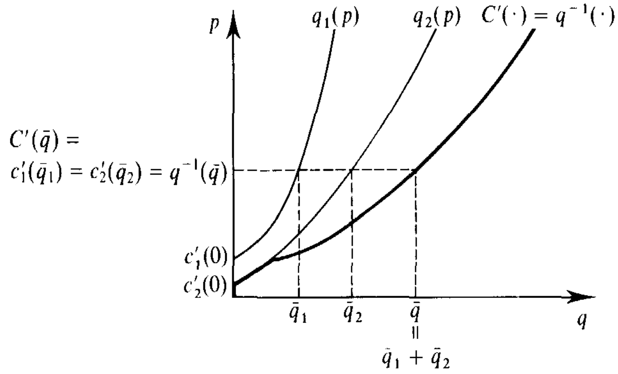


Figure 10.C.4 (left)
Equilibrium when $c_j(q_j) = cq_j$ for all $j = 1, \dots, J$.

Figure 10.C.5 (right)
The industry marginal cost function.

basic features of the analysis do not change. Figure 10.C.4 depicts the determination of the equilibrium value of p in the case where, for all j, $c_j(q_j) = cq_j$ for some scalar $c > 0$. The only difference from the strictly convex case is that, when $J > 1$, individual firms' equilibrium production levels are not uniquely determined.

The inverses of the aggregate demand and supply functions also have interpretations that are of interest. At any given level of aggregate output of good ℓ, say $\bar{q}$, the inverse of the industry supply function, $q^{-1}(\bar{q})$, gives the price that brings forth aggregate supply $\bar{q}$. That is, when each firm chooses its optimal output level facing the price $p = q^{-1}(\bar{q})$, aggregate supply is exactly $\bar{q}$. Figure 10.C.5 illustrates this point. Note that in selecting these output levels, all active firms set their marginal cost equal to $q^{-1}(\bar{q})$. As a result, the marginal cost of producing an additional unit of good ℓ at $\bar{q}$ is precisely $q^{-1}(\bar{q})$, regardless of which active firm produces it. Thus $q^{-1}(\cdot)$, the inverse of the industry supply function, can be viewed as the *industry marginal cost function*, which we now denote by $C'(\cdot) = q^{-1}(\cdot)$.[11]

The derivation of $C'(\cdot)$ just given accords fully with our discussion in Section 5.E. We saw there that the aggregate supply of the J firms, $q(p)$, maximizes aggregate profits given p; therefore, we can relate $q(\cdot)$ to the industry marginal cost function $C'(\cdot)$ in exactly the same manner as we did in Section 5.D for the case of a single firm's marginal cost function and supply behavior. With convex technologies, the aggregate supply locus for good ℓ therefore coincides with the graph of the industry marginal cost function $C'(\cdot)$, and so $q^{-1}(\cdot) = C'(\cdot)$.[12]

Likewise, at any given level of aggregate demand $\bar{x}$, the *inverse demand function* $P(\bar{x}) = x^{-1}(\bar{x})$ gives the price that results in aggregate demand of $\bar{x}$. That is, when each consumer optimally chooses his demand for good ℓ at this price, total demand exactly equals $\bar{x}$. Note that at these individual demand levels (assuming that they are positive), each consumer's marginal benefit in terms of the numeraire from an additional unit of good ℓ, $\phi_i'(x_i)$, is exactly equal to $P(\bar{x})$. This is illustrated in Figure

11. Formally, the industry marginal cost function $C'(\cdot)$ is the derivative of the aggregate cost function $C(\cdot)$ that gives the total production cost that would be incurred by a central authority who operates all J firms and seeks to produce any given aggregate level of good ℓ at minimum total cost. (See Exercise 10.C.3.)

12. More formally, by Proposition 5.E.1, aggregate supply behavior can be determined by maximizing profit given the aggregate cost function $C(\cdot)$. This yields first-order condition $p = C'(q(p))$. Hence, $q(\cdot) = C'^{-1}(\cdot)$, or equivalently $q^{-1}(\cdot) = C'(\cdot)$.

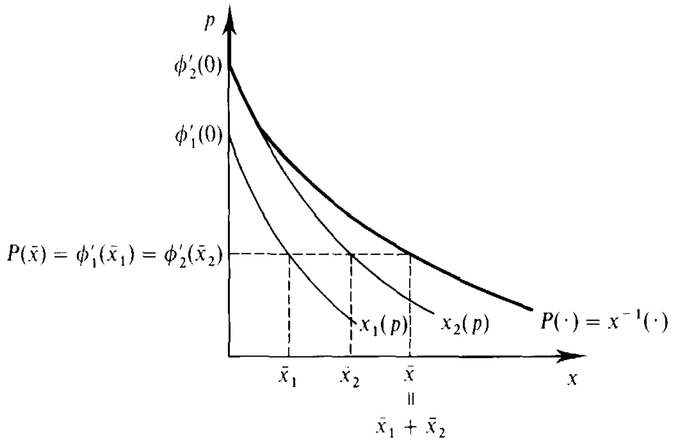


Figure 10.C.6
The inverse demand function.

10.C.6. The value of the inverse demand function at quantity $\bar{x}$, $P(\bar{x})$, can thus be viewed as giving the *marginal social benefit of good* ℓ given that the aggregate quantity $\bar{x}$ is efficiently distributed among the I consumers (see Exercise 10.C.4 for a precise statement of this fact).

Given these interpretations, we can view the competitive equilibrium as involving an aggregate output level at which the marginal social benefit of good ℓ is exactly equal to its marginal cost. This suggests a social optimality property of the competitive allocation, a topic that we investigate further in Section 10.D.

Comparative Statics

It is often of interest to determine how a change in underlying market conditions affects the equilibrium outcome of a competitive market. Such questions may arise, for example, because we may be interested in comparing market outcomes across several similar markets that differ in some measurable way (e.g., we might compare the price of ice cream in a number of cities whose average temperatures differ) or because we want to know how a change in market conditions will alter the outcome in a particular market. The analysis of these sorts of questions is known as *comparative statics analysis*.

As a general matter, we might imagine that each consumer's preferences are affected by a vector of exogenous parameters $\alpha \in \mathbb{R}^M$, so that the utility function $\phi_i(\cdot)$ can be written as $\phi_i(x_i, \alpha)$. Similarly, each firm's technology may be affected by a vector of exogenous parameters $\beta \in \mathbb{R}^S$, so that the cost function $c_j(\cdot)$ can be written as $c_j(q_j, \beta)$. In addition, in some circumstances, consumers and firms face taxes or subsidies that may make the effective (i.e., net of taxes and subsidies) price paid or received differ from the market price p. We let $\hat{p}_i(p, t)$ and $\hat{p}_j(p, t)$ denote, respectively, the effective price paid by consumer i and the effective price received by firm j given tax and subsidy parameters $t \in \mathbb{R}^K$. For example, if consumer i must pay a tax of t_i (in units of the numeraire) per unit of good i purchased, then $\hat{p}_i(p, t) = p + t_i$. If consumer i instead faces a tax that is a percentage t_i of the sales price, then $\hat{p}_i(p, t) = p(1 + t_i)$.

For given values (α, β, t) of the parameters, the $I + J$ equilibrium quantities $(x_1^*, \ldots, x_I^*, q_1^*, \ldots, q_J^*)$ and the equilibrium price p^* are determined as the solution to the following $I + J + 1$ equations (we assume, for simplicity, that $x_i^* > 0$ for all

i and $q_j^* > 0$ for all j):

$$\phi_i'(x_i^*, \alpha) = \hat{p}_i(p^*, t) \qquad i = 1, \ldots, I. \tag{10.C.4}$$

$$c_j'(q_j^*, \beta) = \hat{p}_j(p^*, t) \qquad j = 1, \ldots, J. \tag{10.C.5}$$

$$\sum_{i=1}^{I} x_i^* = \sum_{j=1}^{J} q_j^*. \tag{10.C.6}$$

These $I + J + 1$ equations implicitly define the equilibrium allocation and price as functions of the exogenous parameters (α, β, t). If all the relevant functions are differentiable, we can use the implicit function theorem to derive the marginal change in the equilibrium allocation and price in response to a differential change in the values of these parameters (see Section M.E of the Mathematical Appendix). In Example 10.C.1, we consider one such comparative statics exercise; it is only one among a large number of possibilities that arise naturally in economic applications. (The exercises at the end of this chapter include additional examples.)

Example 10.C.1: *Comparative Statics Effects of a Sales Tax.* Suppose that a new sales tax is proposed under which consumers must pay an amount $t \geq 0$ (in units of the numeraire) for each unit of good ℓ consumed. We wish to determine the effect of this tax on the market price. Let $x(p)$ and $q(p)$ denote the aggregate demand and supply functions, respectively, for good ℓ in the absence of the tax (we maintain all our previous assumptions regarding these functions).

In terms of our previous notation, the $\phi_i(\cdot)$ and $c_j(\cdot)$ functions do not depend on any exogenous parameters, $\hat{p}_i(p, t) = p + t$ for all i, and $\hat{p}_j(p, t) = p$ for all j. In principle, by substituting these expressions into the system of equilbrium equations (10.C.4) to (10.C.6), we can derive the effect of a marginal increase in the tax on the price by direct use of the implicit function theorem (see Exercise 10.C.5). Here, however, we pursue a more instructive way to get the answer. In particular, note that aggregate demand with a tax of t and price p is exactly $x(p + t)$ because the tax is equivalent for consumers to the price being increased by t. Thus, the equilibrium market price when the tax is t, which we denote by $p^*(t)$, must satisfy

$$x(p^*(t) + t) = q(p^*(t)). \tag{10.C.7}$$

Suppose that we now want to determine the effect on prices paid and received of a marginal increase in the tax. Assuming that $x(\cdot)$ and $q(\cdot)$ are differentiable at $p = p^*(t)$, differentiating (10.C.7) yields

$$p^{*\prime}(t) = -\frac{x'(p^*(t) + t)}{x'(p^*(t) + t) - q'(p^*(t))}. \tag{10.C.8}$$

It is immediate from (10.C.8) and our assumptions on $x'(\cdot)$ and $q'(\cdot)$ that $-1 \leq p^{*\prime}(t) < 0$ at any t. Therefore, the price $p^*(t)$ received by producers falls as t increases while the overall cost of the good to consumers $p^*(t) + t$ rises (weakly). The total quantities produced and consumed fall (again weakly). See Figure 10.C.7(a), where the equilibrium level of aggregate consumption at tax rate t is denoted by $x^*(t)$. Notice from (10.C.8) that when $q'(p^*(t))$ is large we have $p^{*\prime}(t) \approx 0$, and so the price received by the firms is hardly affected by the tax; nearly all the impact of the tax is felt by consumers. In contrast, when $q'(p^*(t)) = 0$, we have $p^{*\prime}(t) = -1$, and so the impact of the tax is felt entirely by the firms. Figures 10.C.7(b) and (c) depict these two cases.

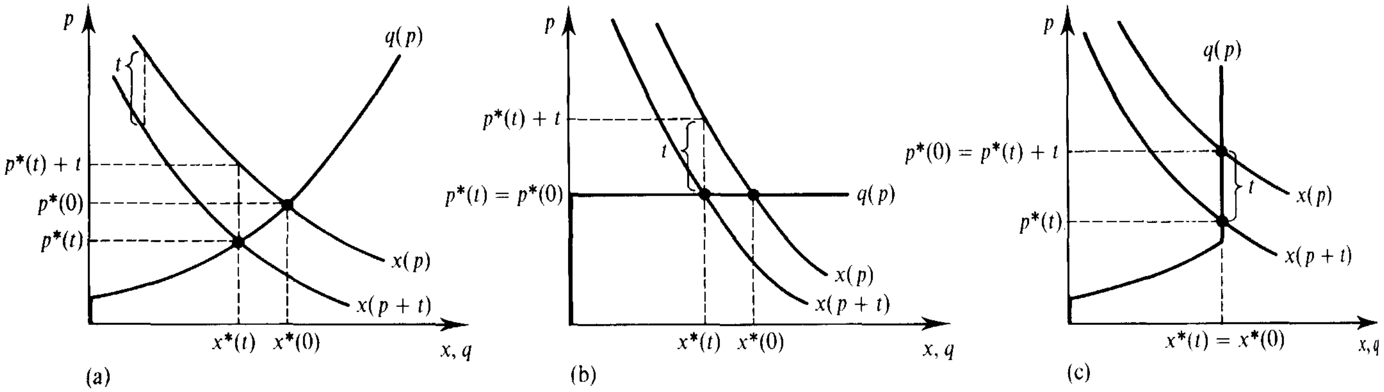


Figure 10.C.7 Comparative statics effects of a sales tax.

By substituting into (10.C.8) for $x'(\cdot)$ and $q'(\cdot)$, the marginal change in p^* can be expressed in terms of derivatives of the underlying individual utility and cost functions. For example, if we let $p^* = p^*(0)$ be the pretax price, we see that

$$p^{*\prime}(0) = -\frac{\sum_{i=1}^{I} [\phi_i''(x_i(p^*))]^{-1}}{\sum_{i=1}^{I} [\phi_i''(x_i(p^*))]^{-1} - \sum_{j=1}^{J} [c_j''(q_j(p^*))]^{-1}}.$$

■

We have assumed throughout this section that consumers' preferences and firms' technologies are convex (and strictly so in the case of consumer preferences). What if this is not the case? Figure 10.C.8 illustrates one problem that can then arise; it shows the demand function and

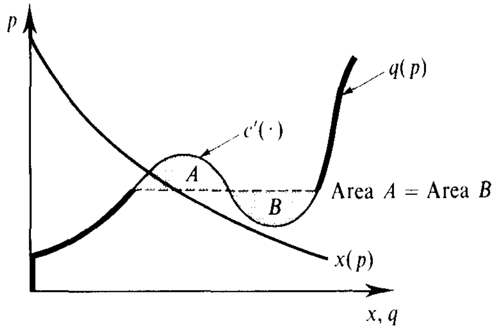


Figure 10.C.8 Nonexistence of competitive equilibrium with a nonconvex technology.

supply correspondence for an economy in which there is a single firm (so $J = 1$).[13] This firm's cost function $c(\cdot)$ is continuous and differentiable but not convex. In the figure, the light curve is the graph of the firm's marginal cost function $c'(\cdot)$. As the figure illustrates, $c'(\cdot)$ fails to be nondecreasing. The heavier curve is the firm's actual supply correspondence $q(\cdot)$ (you should verify that it is determined as indicated in the figure).[14] The graph of the supply correspondence no longer coincides with the marginal cost curve and, as is evident in the figure, no intersection exists between the graph of the supply correspondence and the demand curve. Thus, in this case, *no competitive equilibrium exists.*

13. We set $J = 1$ here solely for expositional purposes.

14. See Section 5.D for a more detailed discussion of the relation between a firm's supply correspondence and its marginal cost function when its technology is nonconvex.

This observation suggests that convexity assumptions are key to the existence of a competitive equilibrium. We shall confirm this in Chapter 17, where we provide a more general discussion of the conditions under which existence of a competitive equilibrium is assured.

10.D The Fundamental Welfare Theorems in a Partial Equilibrium Context

In this section, we study the properties of Pareto optimal allocations in the framework of the two-good quasilinear economy introduced in Section 10.C, and we establish a fundamental link between the set of Pareto optimal allocations and the set of competitive equilibria.

The identification of Pareto optimal allocations is considerably facilitated by the quasilinear specification. In particular, *when consumer preferences are quasilinear, the boundary of the economy's utility possibility set is linear* (see Section 10.B for the definition of this set) *and all points in this boundary are associated with consumption allocations that differ only in the distribution of the numeraire among consumers.*

To see this important fact, suppose that we fix the consumption and production levels of good ℓ at $(\bar{x}_1, \ldots, \bar{x}_I, \bar{q}_1, \ldots, \bar{q}_J)$. With these production levels, the total amount of the numeraire available for distribution among consumers is $\omega_m - \sum_j c_j(\bar{q}_j)$. Because the quasilinear form of the utility functions allows for an unlimited unit-for-unit transfer of utility across consumers through transfers of the numeraire, the set of utilities that can be attained for the I consumers by appropriately distributing the available amounts of the numeraire is given by

$$\left\{(u_1, \ldots, u_I)\colon \sum_{i=1}^{I} u_i \leq \sum_{i=1}^{I} \phi_i(\bar{x}_i) + \omega_m - \sum_{j=1}^{J} c_j(\bar{q}_j)\right\}. \tag{10.D.1}$$

The boundary of this set is a hyperplane with normal vector $(1, \ldots, 1)$. The set is depicted for the case $I = 2$ by the hatched set in Figure 10.D.1.

Note that by altering the consumption and production levels of good ℓ, we necessarily shift the boundary of this set in a parallel manner. Thus, every Pareto optimal allocation must involve the quantities $(x_1^*, \ldots, x_I^*, q_1^*, \ldots, q_J^*)$ that extend this boundary as far out as possible, as illustrated by the heavily drawn boundary of the shaded utility possibility set in Figure 10.D.1. We call these quantities the *optimal consumption and production levels* for good ℓ. As long as these optimal consumption and production levels for good ℓ are uniquely determined, Pareto optimal allocations can differ only in the distribution of the numeraire among consumers.[15]

15. The optimal individual production levels need not be unique if firms' cost functions are convex but not strictly so. Indeterminacy of optimal individual production levels arises, for example, when all firms have identical constant returns to scale technologies. However, under our assumptions that the $\phi_i(\cdot)$ functions are strictly concave and that the $c_j(\cdot)$ functions are convex, the optimal individual consumption levels of good ℓ are necessarily unique and, hence, so is the optimal aggregate production level $\sum_j q_j^*$ of good ℓ. This implies that, under our assumptions, the consumption allocations in two different Pareto optimal allocations can differ only in the distribution of numeraire among consumers. If, moreover, the $c_j(\cdot)$ functions are strictly convex, then the optimal individual production levels are also uniquely determined. (See Exercise 10.D.1.)

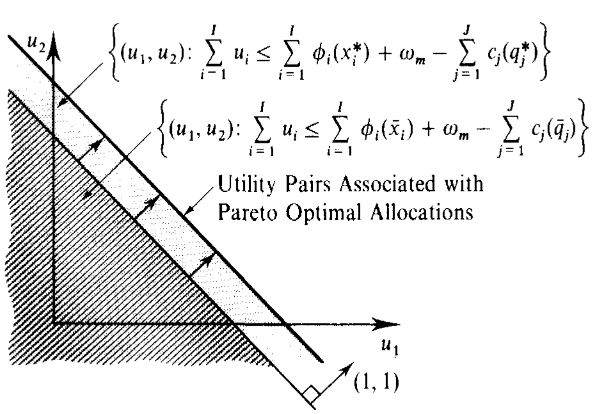


Figure 10.D.1
The utility possibility set in a quasilinear economy.

It follows from expression (10.D.1) that the optimal consumption and production levels of good ℓ can be obtained as the solution to

$$\underset{\substack{(x_1,\ldots,x_I)\ge 0\\(q_1,\ldots,q_J)\ge 0}}{\text{Max}} \quad \sum_{i=1}^{I} \phi_i(x_i) - \sum_{j=1}^{J} c_j(q_j) + \omega_m \tag{10.D.2}$$

$$\text{s.t.} \quad \sum_{i=1}^{I} x_i - \sum_{j=1}^{J} q_j = 0.$$

The value of the term $\sum_i \phi_i(x_i) - \sum_j c_j(q_j)$ in the objective function of problem (10.D.2) is known as the *Marshallian aggregate surplus* (or, simply, the *aggregate surplus*). It can be thought of as the total utility generated from consumption of good ℓ less its costs of production (in terms of the numeraire). The optimal consumption and production levels for good ℓ maximize this aggregate surplus measure.

Given our convexity assumptions, the first-order conditions of problem (10.D.2) yield necessary and sufficient conditions that characterize the optimal quantities. If we let μ be the multiplier on the constraint in problem (10.D.2), the $I + J$ optimal values $(x_1^*, \ldots, x_I^*, q_1^*, \ldots, q_J^*)$ and the multiplier μ satisfy the following $I + J + 1$ conditions:

$$\mu \le c_j'(q_j^*), \quad \text{with equality if } q_j^* > 0 \qquad j = 1, \ldots, J. \tag{10.D.3}$$

$$\phi_i'(x_i^*) \le \mu, \quad \text{with equality if } x_i^* > 0 \qquad i = 1, \ldots, I. \tag{10.D.4}$$

$$\sum_{i=1}^{I} x_i^* = \sum_{j=1}^{J} q_j^*. \tag{10.D.5}$$

These conditions should look familiar: They exactly parallel conditions (10.C.1) to (10.C.3) in Section 10.C, with μ replacing p^*. This observation has an important implication. We can immediately infer from it that any competitive equilibrium outcome in this model is Pareto optimal because any competitive equilibrium allocation has consumption and production levels of good ℓ, $(x_1^*, \ldots, x_I^*, q_1^*, \ldots, q_J^*)$, that satisfy conditions (10.D.3) to (10.D.5) when we set $\mu = p^*$. Thus, we have established the *first fundamental theorem of welfare economics* (Proposition 10.D.1) in the context of this quasilinear two-good model.

Proposition 10.D.1: (*The First Fundamental Theorem of Welfare Economics*) If the price p^* and allocation $(x_1^*, \ldots, x_I^*, q_1^*, \ldots, q_J^*)$ constitute a competitive equilibrium, then this allocation is Pareto optimal.

The first fundamental welfare theorem establishes conditions under which market equilibria are necessarily Pareto optimal. It is a formal expression of Adam Smith's "invisible hand" and is a result that holds with considerable generality (see Section 16.C for a much more extensive discussion). Equally important, however, are the conditions under which it fails to hold. In the models for which we establish the first fundamental welfare theorem here and in Section 16.C, markets are "complete" in the sense that there is a market for every relevant commodity and all market participants act as price takers. In Chapters 11 to 14, we study situations in which at least one of these conditions fails, and market outcomes fail to be Pareto optimal as a result.

We can also develop a converse to Proposition 10.D.1, known as the *second fundamental theorem of welfare economics.* In Section 10.C, we saw that good ℓ's equilibrium price p^*, its equilibrium consumption and production levels $(x_1^*, \ldots, x_I^*, q_1^*, \ldots, q_J^*)$, and firms' profits are unaffected by changes in consumers' wealth levels. As a result, a transfer of one unit of the numeraire from consumer i to consumer i' will cause each of these consumers' equilibrium consumption of the numeraire to change by exactly the amount of the transfer and will cause no other changes. Thus, by appropriately transferring endowments of the numeraire commodity, the resulting competitive equilibrium allocation can be made to yield any utility vector in the boundary of the utility possibility set. The second welfare theorem therefore tells us that, in this two-good quasilinear economy, a central authority interested in achieving a particular Pareto optimal allocation can always implement this outcome by transferring the numeraire among consumers and then "allowing the market to work." This is stated formally in Proposition 10.D.2.

Proposition 10.D.2: (*The Second Fundamental Theorem of Welfare Economics*) For any Pareto optimal levels of utility $(u_1^*, \ldots, u_I^*)$, there are transfers of the numeraire commodity $(T_1, \ldots, T_I)$ satisfying $\sum_i T_i = 0$, such that a competitive equilibrium reached from the endowments $(\omega_{m1} + T_1, \ldots, \omega_{mI} + T_I)$ yields precisely the utilities $(u_1^*, \ldots, u_I^*)$.

In Section 16.D, we study the conditions under which the second welfare theorem holds in more general competitive economies. A critical requirement, in addition to those needed for the first welfare theorem, turns out to be convexity of preferences and production sets, an assumption we have made in the model under consideration here. In contrast, we shall see in Chapter 16 that no such convexity assumptions are needed for the first welfare theorem.

The correspondence between p and μ in the equilibrium conditions (10.C.1) to (10.C.3) and the Pareto optimality conditions (10.D.3) to (10.D.5) is worthy of emphasis: The competitive price is exactly equal to the shadow price on the resource constraint for good ℓ in the Pareto optimality problem (10.D.2). In this sense, then, we can say that a good's price in a competitive equilibrium reflects precisely its marginal social value. In a competitive equilibrium, each firm, by operating at a point where price equals marginal cost, equates its marginal production cost to the marginal social value of its output. Similarly, each consumer, by consuming up to the point where marginal utility from a good equals its price, is at a point where the marginal benefit from consumption of the good exactly equals its marginal cost. This correspondence between equilibrium market prices and optimal shadow prices holds quite generally in competitive economies (see Section 16.F for further discussion of this point).

An alternative way to characterize the set of Pareto optimal allocations is to solve

$$\begin{aligned}
\underset{\{x_i, m_i\}_{i=1}^I, \{z_j, q_j\}_{j=1}^J}{\text{Max}} \quad & m_1 + \phi_1(x_1) & \text{(10.D.6)} \\
\text{s.t.} \quad (1) \quad & m_i + \phi_i(x_i) \geq \bar{u}_i \qquad i = 2, \ldots, I \\
(2\ell) \quad & \sum_{i=1}^I x_i - \sum_{j=1}^J q_j \leq 0 \\
(2m) \quad & \sum_{i=1}^I m_i + \sum_{j=1}^J z_j \leq \omega_m \\
(3) \quad & z_j \geq c_j(q_j) \qquad j = 1, \ldots, J.
\end{aligned}$$

Problem (10.D.6) expresses the Pareto optimality problem as one of trying to maximize the well-being of individual 1 subject to meeting certain required utility levels for the other individuals in the economy [constraints (1)], resource constraints [constraints (2ℓ) and ($2m$)], and technological constraints [constraints (3)]. By solving problem (10.D.6) for various required levels of utility for these other individuals, $(\bar{u}_2, \ldots, \bar{u}_I)$, we can identify all the Pareto optimal outcomes for this economy (see Exercise 10.D.3; more generally, we can do this whenever consumer preferences are strongly monotone). Exercise 10.D.4 asks you to derive conditions (10.D.3) to (10.D.5) in this alternative manner.

10.E Welfare Analysis in the Partial Equilibrium Model

It is often of interest to measure the change in the level of social welfare that would be generated by a change in market conditions such as an improvement in technology, a new government tax policy, or the elimination of some existing market imperfection. In the partial equilibrium model, it is particularly simple to carry out this welfare analysis. This fact accounts to a large extent for the popularity of the model.

In the discussion that follows, we assume that the welfare judgments of society are embodied in a social welfare function $W(u_1, \ldots, u_I)$ assigning a social welfare value to every utility vector $(u_1, \ldots, u_I)$ (see Chapters 4, 16, and 22 for more on this concept). In addition, we suppose that (as in the theory of the normative representative consumer discussed in Section 4.D) there is some central authority who redistributes wealth by means of transfers of the numeraire commodity in order to maximize social welfare.[16] The critical simplification offered by the quasilinear specification of individual utility functions is that when there is a central authority who redistributes wealth in this manner, *changes in social welfare can be measured by changes in the Marshallian aggregate surplus* (introduced in Section 10.D) *for any social welfare function that society may have.*

To see this point (which we have in fact already examined in Example 4.D.2), consider some given consumption and production levels of good ℓ, $(x_1, \ldots, x_I, q_1, \ldots, q_J)$,

16. As in Section 4.D, we assume that consumers treat these transfers as independent of their own actions; that is, in the standard terminology, they are *lump-sum* transfers. You should think of the central authority as making the transfers prior to the opening of markets.

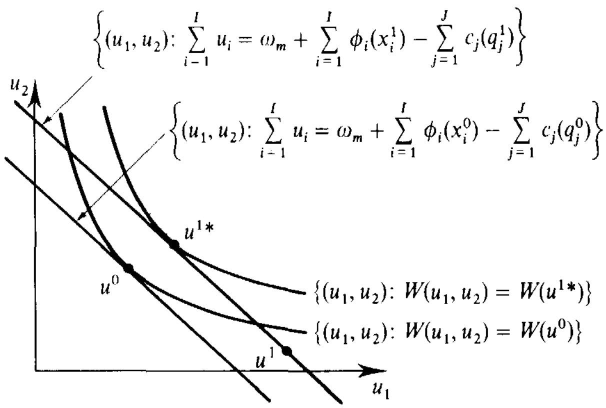


Figure 10.E.1
With lump-sum redistribution occurring to maximize social welfare, changes in welfare correspond to changes in aggregate surplus in a quasilinear model.

having $\sum_i x_i = \sum_j q_j$. From Section 10.D and Figure 10.D.1 we know that the utility vectors $(u_1, \ldots, u_I)$ that are achievable through reallocation of the numeraire given these consumption and production levels of good ℓ are

$$\left\{(u_1, \ldots, u_I)\colon \sum_{i=1}^{I} u_i \le \omega_m + \sum_{i=1}^{I} \phi_i(x_i) - \sum_{j=1}^{J} c_j(q_j)\right\}.$$

Now, if a central authority is redistributing the numeraire to maximize $W(u_1, \ldots, u_I)$, the ultimate maximized value of welfare must be greater the larger this set is (i.e., the farther out the boundary of the set is). Hence, we see that a change in the consumption and production levels of good ℓ leads to an increase in welfare (given optimal redistribution of the numeraire) if and only if it increases the Marshallian aggregate surplus

$$S(x_1, \ldots, x_I, q_1, \ldots, q_J) = \sum_{i=1}^{I} \phi_i(x_i) - \sum_{j=1}^{J} c_j(q_j). \qquad (10.E.1)$$

Figure 10.E.1 provides an illustration. It shows three utility vectors for the case $I = 2$: An initial utility vector $u^0 = (u_1^0, u_2^0)$ associated with an allocation in which the consumption and production levels of good ℓ are $(x_1^0, \ldots, x_I^0, q_1^0, \ldots, q_J^0)$ and in which the wealth distribution has been optimized, a utility vector $u^1 = (u_1^1, u_2^1)$ that results from a change in the consumption and production levels of good ℓ to $(x_1^1, \ldots, x_I^1, q_1^1, \ldots, q_J^1)$ in the absence of any transfers of the numeraire, and a utility vector $u^{1*} = (u_1^{1*}, u_2^{1*})$ that results from this change once redistribution of the numeraire occurs to optimize social welfare. As can be seen in the figure, the change increases aggregate surplus and also increases welfare once optimal transfers of the numeraire occur, even though welfare would decrease in the absence of the transfers. Thus, as long as redistribution of wealth is occurring to maximize a social welfare function, changes in welfare can be measured by changes in Marshallian aggregate surplus (to repeat: for *any* social welfare function).[17]

In many circumstances of interest, the Marshallian surplus has a convenient and

17. Notice that no transfers would be necessary in the special case in which the social welfare function is in fact the "utilitarian" social welfare function $\sum_i u_i$; in this case, it is sufficient that all available units of the numeraire go to consumers (i.e., none goes to waste or is otherwise withheld).

historically important formulation in terms of areas lying vertically between the aggregate demand and supply functions for good ℓ.

To expand on this point, we begin by making two key assumptions. Denoting by $x = \sum_i x_i$ the aggregate consumption of good ℓ, we assume, first, that for any x, the individual consumptions of good ℓ are distributed optimally across consumers. That is, recalling our discussion of the inverse demand function $P(\cdot)$ in Section 10.C (see Figure 10.C.6), that we have $\phi_i'(x_i) = P(x)$ for every i. This condition will be satisfied if, for example, consumers act as price-takers and all consumers face the same price. Similarly, denoting by $q = \sum_j q_j$ the aggregate output of good ℓ, we assume that the production of any total amount q is distributed optimally across firms. That is, recalling our discussion of the industry marginal cost curve $C'(\cdot)$ in Section 10.C (see Figure 10.C.5), that we have $c_j'(q_j) = C'(q)$ for every j. This will be satisfied if, for example, firms act as price takers and all firms face the same price. Observe that we do not require that the price faced by consumers and firms be the same.[18]

Consider now a differential change $(dx_1, \ldots, dx_I, dq_1, \ldots, dq_J)$ in the quantities of good ℓ consumed and produced satisfying $\sum_i dx_i = \sum_j dq_j$, and denote $dx = \sum_i dx_i$. The change in aggregate Marshallian surplus is then

$$dS = \sum_{i=1}^{I} \phi_i'(x_i)\, dx_i - \sum_{j=1}^{J} c_j'(q_j)\, dq_j. \tag{10.E.2}$$

Since $\phi_i'(x_i) = P(x)$ for all i, and $c_j'(q_j) = C'(q)$ for all j, we get

$$dS = P(x) \sum_{i=1}^{I} dx_i - C'(q) \sum_{j=1}^{J} dq_j. \tag{10.E.3}$$

Finally, since $x = q$ (by market feasibility) and $\sum_j dq_j = \sum_i dx_i = dx$, this becomes

$$dS = [P(x) - C'(x)]\, dx. \tag{10.E.4}$$

This differential change in Marshallian surplus is depicted in Figure 10.E.2(a). Expression (10.E.4) is quite intuitive; it tells us that starting at aggregate consumption level x the marginal effect on social welfare of an increase in the aggregate quantity consumed, dx, is equal to consumers' marginal benefit from this consumption, $P(x)\, dx$, less the marginal cost of this extra production, $C'(x)\, dx$ (both in terms of the numeraire).

We can also integrate (10.E.4) to express the total value of the aggregate Marshallian surplus at the aggregate consumption level x, denoted $S(x)$, in terms of an integral of the difference between the inverse demand function and the industry marginal cost function,

$$S(x) = S_0 + \int_0^x [P(s) - C'(s)]\, ds, \tag{10.E.5}$$

18. For example, consumers may face a tax per unit purchased that makes the price they pay differ from the price received by the firms (see Example 10.C.1). The assumptions made here also hold in the monopoly model to be studied in Section 12.B. In that model, there is a single firm (and so there is no issue of optimal allocation of production), and all consumers act as price takers facing the same price. An example where the assumption of an optimal allocation of production is not valid is the Cournot duopoly model of Chapter 12 when firms have different efficiencies. There, firms with different costs have different levels of marginal cost in an equilibrium.

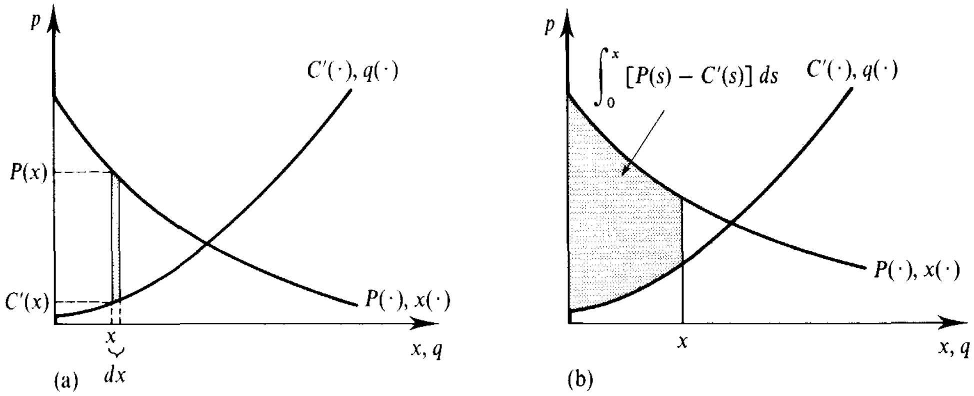


Figure 10.E.2
(a) A differential change in Marshallian surplus. (b) The Marshallian surplus at aggregate consumption level x.

where S_0 is a constant of integration equal to the value of the aggregate surplus when there is no consumption or production of good ℓ [it is equal to zero if $c_j(0) = 0$ for all j]. The integral in (10.E.5) is depicted in Figure 10.E.2(b); it is exactly equal to the area lying vertically between the aggregate demand and supply curves for good ℓ up to quantity x.

Note from (10.E.5) that the value of the aggregate Marshallian surplus is maximized at the aggregate consumption level x^* such that $P(x^*) = C'(x^*)$, which is exactly the competitive equilibrium aggregate consumption level.[19] This accords with Proposition 10.D.1, the first fundamental welfare theorem, which states that the competitive allocation is Pareto optimal.

Example 10.E.1: *The Welfare Effects of a Distortionary Tax.* Consider again the commodity tax problem studied in Example 10.C.1. Suppose now that the welfare authority keeps a balanced budget and returns the tax revenue raised to consumers by means of lump-sum transfers. What impact does this tax-and-transfer scheme have on welfare?[20]

To answer this question, it is convenient to let $(x_1^*(t), \dots, x_I^*(t), q_1^*(t), \dots, q_J^*(t))$ and $p^*(t)$ denote the equilibrium consumption, production, and price levels of good ℓ when the tax rate is t. Note that $\phi_i'(x_i^*(t)) = p^*(t) + t$ for all i and that $c_j'(q_j^*(t)) = p^*(t)$ for all j. Thus, letting $x^*(t) = \sum_i x_i^*(t)$ and $S^*(t) = S(x^*(t))$, we can use (10.E.5) to express the change in aggregate Marshallian surplus resulting from

19. To see this, check first that $S''(x) \leq 0$ at all x. Hence, $S(\cdot)$ is a concave function and therefore $x^* > 0$ maximizes aggregate surplus if and only if $S'(x^*) = 0$. Then verify that $S'(x) = P(x) - C'(x)$ at all $x > 0$.

20. This problem is closely related to that studied in Example 3.I.1 (we could equally well motivate the analysis here by asking, as we did there, about the welfare cost of the distortionary tax relative to the use of a lump-sum tax that raises the same revenue; the measure of deadweight loss that emerges would be the same as that developed here). The discussion that follows amounts to an extension, in the quasilinear context, of the analysis of Example 3.I.1 to situations with many consumers and the presence of firms. For an approach that uses the theory of a normative representative consumer presented in Section 4.D, see the small-type discussion at the end of this section.

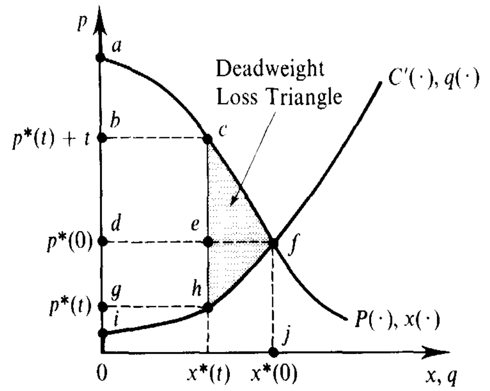


Figure 10.E.3
The deadweight welfare loss from distortionary taxation.

the introduction of the tax as

$$S^*(t) - S^*(0) = \int_{x^*(0)}^{x^*(t)} [P(s) - C'(s)]\, ds. \tag{10.E.6}$$

Expression (10.E.6) is negative because $x^*(t) < x^*(0)$ (recall the analysis of Example 10.C.1) and $P(x) \geq C'(x)$ for all $x \leq x^*(0)$, with strict inequality for $x < x^*(0)$. Hence, social welfare is optimized by setting $t = 0$. The loss in welfare from $t > 0$ is known as the *deadweight loss of distortionary taxation* and is equal to the area of the shaded region in Figure 10.E.3, called the *deadweight loss triangle.*

Notice that since $S^{*\prime}(t) = [P(x^*(t)) - C'(x^*(t))]x^{*\prime}(t)$, we have $S^{*\prime}(0) = 0$. That is, starting from a position without any tax, the first-order welfare effect of an infinitesimal tax is zero. Only as the tax rate increases above zero does the marginal effect become strictly negative. This is as it should be: if we start at an (interior) welfare maximum, then a small displacement from the optimum cannot have a first-order effect on welfare.

It is sometimes of interest to distinguish between the various components of aggregate Marshallian surplus that accrue directly to consumers, firms, and the tax authority.[21] The *aggregate consumer surplus* when consumers' effective price is $\hat{p}$ and therefore aggregate consumption is $x(\hat{p})$ is defined as the gross consumer benefits from consumption of good ℓ minus the consumers' total expenditure on this good (the latter is the cost to consumers in terms of forgone consumption of the numeraire):

$$CS(\hat{p}) = \sum_{i=1}^{I} \phi_i(x_i(\hat{p})) - \hat{p}x(\hat{p}).$$

Using again the fact that consumption is distributed optimally, we have

$$\begin{aligned} CS(\hat{p}) &= \int_0^{x(\hat{p})} P(s)\, ds - \hat{p}x(\hat{p}) \\ &= \int_0^{x(\hat{p})} [P(s) - \hat{p}]\, ds. \end{aligned} \tag{10.E.7}$$

21. For example, if the set of active consumers of good ℓ is distinct from the set of owners of the firms producing the good, then this distinction tells us something about the distributional effects of the tax in the absence of transfers between owners and consumers.

Finally, the integral in (10.E.7) is equal to[22]

$$CS(\hat{p}) = \int_{\hat{p}}^{\infty} x(s)\,ds. \tag{10.E.8}$$

Thus, because consumers face an effective price of $p^*(t) + t$ when the tax is t, the change in consumer surplus from imposition of the tax is

$$CS(p^*(t) + t) - CS(p^*(0)) = -\int_{p^*(0)}^{p^*(t)+t} x(s)\,ds. \tag{10.E.9}$$

In Figure 10.E.3, the reduction in consumer surplus is depicted by area $(dbcf)$.

The aggregate profit, or *aggregate producer surplus*, when firms face effective price $\hat{p}$ is

$$\Pi(\hat{p}) = \hat{p}q(\hat{p}) - \sum_{j=1}^{J} c_j(q_j(\hat{p})).$$

Again, using the optimality of the allocation of production across firms, we have[23]

$$\Pi(\hat{p}) = \Pi_0 + \int_0^{q(\hat{p})} [\hat{p} - C'(s)]\,ds \tag{10.E.10}$$

$$= \Pi_0 + \int_0^{\hat{p}} q(s)\,ds, \tag{10.E.11}$$

where Π_0 is a constant of integration equal to profits when $q_j = 0$ for all j [$\Pi_0 = 0$ if $c_j(0) = 0$ for all j]. Since producers pay no tax, they face price $p^*(t)$ when the tax rate is t. The change in producer surplus is therefore

$$\Pi(p^*(t)) - \Pi(p^*(0)) = -\int_{p^*(t)}^{p^*(0)} q(s)\,ds. \tag{10.E.12}$$

The reduction in producer surplus is depicted by area $(gdfh)$ in Figure 10.E.3.

Finally, the *tax revenue* is $tx^*(t)$; it is depicted in Figure 10.E.3 by area $(gbch)$.

The total deadweight welfare loss from the tax is then equal to the sum of the reductions in consumer and producer surplus less the tax revenue. ■

The welfare measure developed here is closely related to our discussion of normative representative consumers in Section 4.D. We showed there that if a central authority is redistributing wealth to maximize a social welfare function given prices p, leading to a wealth distribution rule $(w_1(p, w), \ldots, w_I(p, w))$, then there is a normative representative consumer with indirect utility function $v(p, w)$ whose demand $x(p, w)$ is exactly equal to aggregate demand [i.e., $x(p, w) = \sum_i x_i(p, w_i(p, w))$] and whose utility can be used as a measure of social welfare. Recalling our discussion in Section 3.I, this means that we can measure the change in welfare resulting from a price–wealth change by adding the representative consumer's

22. This can be seen geometrically. For example, when $\hat{p} = p^*(0)$, the integrals in both (10.E.7) and (10.E.8) are equal to area (daf) in Figure 10.E.3. Formally, the equivalence follows from a change of variables and integration by parts (see Exercise 10.E.2).

23. When $\hat{p} = p^*(0)$, the integrals in both (10.E.10) and (10.E.11) are equal to area (idf) in Figure 10.E.3. The equivalence of these two integrals again follows formally by a change of variables and integration by parts.

compensating or equivalent variation for the price change to the change in the representative consumer's wealth (see Exercise 3.I.12). But in the quasilinear case, the representative consumer's compensating and equivalent variations are the same and can be calculated by direct integration of the representative consumer's Walrasian demand function, that is, by integration of the aggregate demand function. Hence, in Example 10.E.1, the representative consumer's compensating variation for the price change is exactly equal to the change in aggregate consumer surplus, expression (10.E.9). The change in the representative consumer's wealth, on the other hand, is equal to the change in aggregate profits plus the tax revenue rebated to consumers. Thus, the total welfare change arising from the introduction of the tax-and-transfer scheme, as measured using the normative representative consumer, is exactly equal to the deadweight loss calculated in Example 10.E.1.[24]

Another way to justify the use of aggregate surplus as a welfare measure in the quasilinear model is as a measure of *potential Pareto improvement*. Consider the tax example. We could say that a change in the tax represents a *potential* Pareto improvement if there is a set of lump-sum transfers of the numeraire that would make all consumers better off than they were before the tax change. In the present quasilinear context, this is true if and only if aggregate surplus increases with the change in the tax. This approach is sometimes referred to as the *compensation principle* because it asks whether, in principle, it is possible given the change for the winners to compensate the losers so that all are better off than before. (See also the discussion in Example 4.D.2 and especially Section 22.C.)

We conclude this section with a warning: When the numeraire represents many goods, the welfare analysis we have performed is justified only if the prices of goods other than good ℓ are undistorted in the sense that they equal these goods' true marginal utilities and production costs. Hence, these other markets must be competitive, and all market participants must face the same price. If this condition does not hold, then the costs of production faced by producers of good ℓ do not reflect the true social costs incurred from their use of these goods as inputs. Exercise 10.G.3 provides an illustration of this problem.

10.F Free-Entry and Long-Run Competitive Equilibria

Up to this point, we have taken the set of firms and their technological capabilities as fixed. In this section, we consider the case in which an infinite number of firms can potentially be formed, each with access to the most efficient production technology. Moreover, firms may enter or exit the market in response to profit opportunities. This scenario, known as a situation of *free entry*, is often a reasonable approximation when we think of long-run outcomes in a market. In the discussion that follows, we introduce and study a notion of *long-run competitive equilibrium* and then discuss how this concept can be used to analyze long-run and short-run comparative statics effects.

To begin, suppose that each of an infinite number of potential firms has access to a technology for producing good ℓ with cost function $c(q)$, where q is the *individual* firm's output of good ℓ. We assume that $c(0) = 0$; that is, a firm can earn zero profits by simply deciding to be inactive and setting $q = 0$. In the terminology of Section

24. This deadweight loss measure corresponds also to the measure developed for the one-consumer case in Example 3.I.1, where we implicitly limited ourselves to the case in which the taxed good has a constant unit cost.

5.B, there are no sunk costs in the long run. The aggregate demand function is $x(\cdot)$, with inverse demand function $P(\cdot)$.

In a long-run competitive equilibrium, we would like to determine not only the price and output levels for the firms but also the number of firms that are active in the industry. Given our assumption of identical firms, we focus on equilibria in which all active firms produce the same output level, so that a long-run competitive equilibrium can be described by a triple (p, q, J) formed by a price p, an output per firm q, and an integer number of active firms J (hence the total industry output is $Q = Jq$).[25] The central assumption determining the number of active firms is one of free entry and exit: A firm will enter the market if it can earn positive profits at the going market price and will exit if it can make only negative profits at any positive production level given this price. If all firms, active and potential, take prices as unaffected by their own actions, this implies that active firms must earn exactly zero profits in any long-run competitive equilibrium; otherwise, we would have either no firms willing to be active in the market (if profits were negative) or an infinite number of firms entering the market (if profits were positive). This leads us to the formulation given in Definition 10.F.1.

Definition 10.F.1: Given an aggregate demand function $x(p)$ and a cost function $c(q)$ for each potentially active firm having $c(0) = 0$, a triple (p^*, q^*, J^*) is a *long-run competitive equilibrium* if

(i) q^* solves $\underset{q \geq 0}{\text{Max}}\; p^*q - c(q)$ (Profit maximization)

(ii) $x(p^*) = J^*q^*$ (Demand = supply)

(iii) $p^*q^* - c(q^*) = 0$ (Free Entry Condition).

The long-run equilibrium price can be thought of as equating demand with long-run supply, where the long-run supply takes into account firms' entry and exit decisions. In particular, if $q(\cdot)$ is the supply correspondence of an individual firm with cost function $c(\cdot)$ and $\pi(\cdot)$ is its profit function, we can define a *long-run aggregate supply correspondence* by[26]

$$Q(p) = \begin{cases} \infty & \text{if } \pi(p) > 0, \\ \{Q \geq 0: Q = Jq \text{ for some integer } J \geq 0 \text{ and } q \in q(p)\} & \text{if } \pi(p) = 0. \end{cases}$$

If $\pi(p) > 0$, then every firm wants to supply an amount strictly bounded away from zero. Hence, the aggregate supply is infinite. If $\pi(p) = 0$ and $Q = Jq$ for some $q \in q(p)$, then we can have J firms each supply q and have the rest remain inactive [since $c(0) = 0$, this is a profit-maximizing choice for the inactive firms as well]. With this

25. The assumption that all active firms produce the same output level is without loss of generality whenever $c(\cdot)$ is strictly convex on the set $(0, \infty]$. A firm's supply correspondence can then include at most one positive output level at any given price p.

26. In terms of the basic properties of production sets presented in Section 5.B, the long-run supply correspondence is the supply correspondence of the production set Y^+, where Y is the production set associated with the individual firm [i.e., with $c(\cdot)$], and Y^+ is its "additive closure" (i.e., the smallest set that contains Y and is additive: $Y^+ + Y^+ \subset Y^+$; see Exercise 5.B.4).

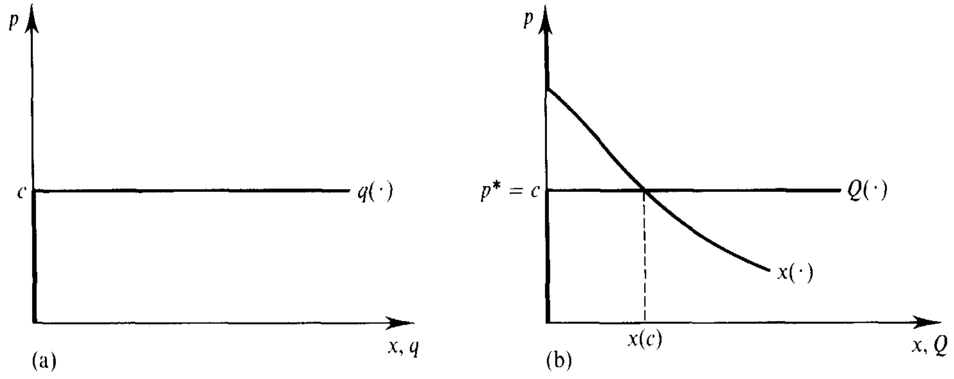


Figure 10.F.1
Long-run competitive equilibrium with constant returns to scale. (a) A firm's supply correspondence. (b) Long-run equilibrium.

notion of a long-run supply correspondence, p^* is a long-run competitive equilibrium price if and only if $x(p^*) \in Q(p^*)$.[27]

We now investigate this long-run competitive equilibrium notion. Consider first the case in which the cost function $c(\cdot)$ exhibits constant returns to scale, so that $c(q) = cq$ for some $c > 0$, and assume that $x(c) > 0$. In this case, condition (i) of Definition 10.F.1 tells us that in any long-run competitive equilbrium we have $p^* \leq c$ (otherwise, there is no profit-maximizing production). However, at any such price, aggregate consumption is strictly positive since $x(c) > 0$, so condition (ii) requires that $q^* > 0$, By condition (iii), we must have $(p^* - c)q^* = 0$. Hence, we conclude that $p^* = c$ and aggregate consumption is $x(c)$. Note, however, that J^* and q^* are *indeterminate*: any J^* and q^* such that $J^*q^* = x(c)$ satisfies conditions (i) and (ii).

Figure 10.F.1 depicts this long-run equilibrium. The supply correspondence of an individual firm $q(\cdot)$ is illustrated in Figure 10.F.1(a); Figure 10.F.1(b) shows the long-run equilibrium price and aggregate output as the intersection of the graph of the aggregate demand function $x(\cdot)$ with the graph of the long-run aggregate supply correspondence

$$Q(p) = \begin{cases} \infty & \text{if } p > c \\ [0, \infty) & \text{if } p = c \\ 0 & \text{if } p < c. \end{cases}$$

We move next to the case in which $c(\cdot)$ is increasing and strictly convex (i.e., the production technology of an individual firm displays strictly decreasing returns to scale). We assume also that $x(c'(0)) > 0$. With this type of cost function, *no long-run competitive equilibrium can exist*. To see why this is so, note that if $p > c'(0)$, then $\pi(p) > 0$ and therefore the long-run supply is infinite. On the other hand, if $p \leq c'(0)$, then the long-run supply is zero while $x(p) > 0$. The problem is illustrated in Figure 10.F.2, where the graph of the demand function $x(\cdot)$ has no intersection with the

27. In particular, if (p^*, q^*, J^*) is a long-run equilibrium, then condition (i) of Definition 10.F.1 implies that $q^* \in q(p^*)$ and condition (iii) implies that $\pi(p^*) = 0$. Hence, by condition (ii), $x(p^*) \in Q(p^*)$. In the other direction, if $x(p^*) \in Q(p^*)$, then $\pi(p^*) = 0$ and there exists $q^* \in q(p^*)$ and J^* with $x(p^*) = J^*q^*$. Therefore, the three conditions of Definition 10.F.1 are satisfied.

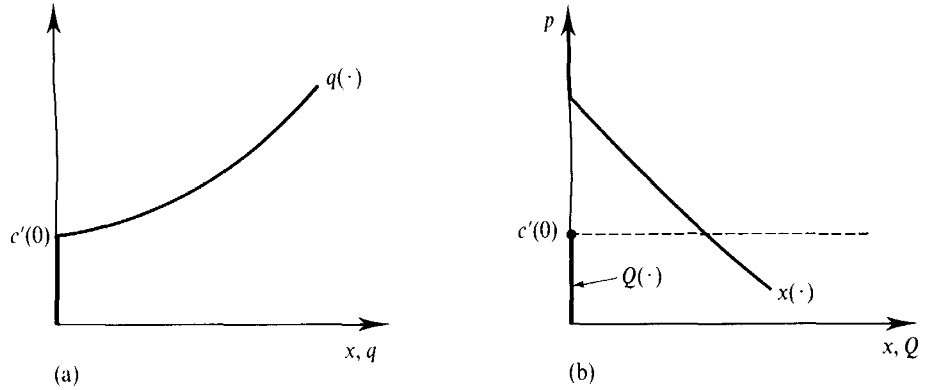


Figure 10.F.2
Nonexistence of long-run competitive equilibrium with strictly convex costs. (a) A firm's supply correspondence. (b) No intersection of long-run supply and demand.

graph of the long-run aggregate supply correspondence

$$Q(p) = \begin{cases} \infty & \text{if } p > c'(0) \\ 0 & \text{if } p \le c'(0). \end{cases}$$

The difficulty can be understood in a related way. As discussed in Exercise 5.B.4, the long-run aggregate production set in the situation just described is convex but not closed. This can be seen in Figure 10.F.3, where the industry marginal cost function with J firms,

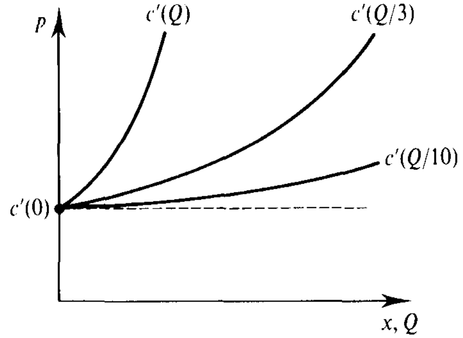


Figure 10.F.3
The limiting behavior of industry marginal cost as $J \to \infty$ with strictly convex costs.

$c'(Q/J)$, is shown for various values of J (in particular, for $J = 1$, $J = 3$, and $J = 10$). Note that as J increases, this marginal cost function approaches *but never reaches* the marginal cost function corresponding to a constant marginal cost of $c'(0)$.

Perhaps not surprisingly, to generate the existence of an equilibrium with a determinate number of firms, the long-run cost function must exhibit a strictly positive efficient scale; that is, *there must exist a strictly positive output level $\bar{q}$ at which a firm's average costs of production are minimized* (see Section 5.D for a further discussion of the efficient scale concept).

Suppose, in particular, that $c(\cdot)$ has a unique efficient scale $\bar{q} > 0$, and let the minimized level of average cost be $\bar{c} = c(\bar{q})/\bar{q}$. Assume, moreover, that $x(\bar{c}) > 0$. If at a long-run equilibrium (p^*, q^*, J^*) we had $p^* > \bar{c}$, then $p^*\bar{q} > \bar{c}\bar{q}$, and so we would have $\pi(p^*) > 0$. Thus, at any long-run equilibrium we must have $p^* \le \bar{c}$. In contrast, if $p^* < \bar{c}$, then $x(p^*) > 0$; but since $p^*q - c(q) = p^*q - (c(q)/q)q \le (p^* - \bar{c})q < 0$

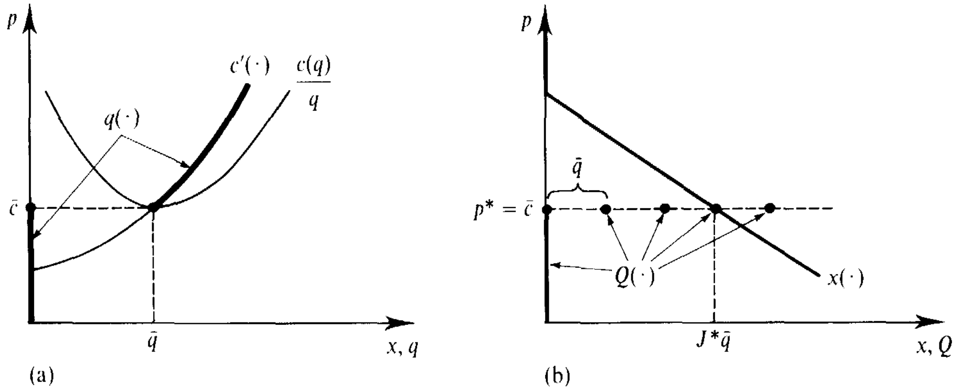


Figure 10.F.4
Long-run competitive equilibrium when average costs exhibit a strictly positive efficient scale. (a) A firm's supply correspondence. (b) Long-run equilibrium.

for all $q > 0$, a firm would earn strictly negative profits at any positive level of output. So $p^* < \bar{c}$ also cannot be a long-run equilibrium price. Thus, at any long-run equilibrium we must have $p^* = \bar{c}$. Moreover, if $p^* = \bar{c}$, then each active firm's supply must be $q^* = \bar{q}$ (this is the only strictly positive output level at which the firm earns nonnegative profits), and the equilibrium number of active firms is therefore $J^* = x(\bar{c})/\bar{q}$.[28] In conclusion, the number of active firms is a well-determined quantity at long-run equilibrium. Figure 10.F.4 depicts such an equilibrium. The long-run aggregate supply correspondence is

$$Q(p) = \begin{cases} \infty & \text{if } p > \bar{c} \\ \{Q \geq 0 \colon Q = J\bar{q} \text{ for some integer } J \geq 0\} & \text{if } p = \bar{c} \\ 0 & \text{if } p < \bar{c}. \end{cases}$$

Observe that the equilibrium price and aggregate output are exactly the same as if the firms had a constant returns to scale technology with unit cost $\bar{c}$.

Several points should be noted about the equilibrium depicted in Figure 10.F.4. First, if the efficient scale of operation is large relative to the size of market demand, it could well turn out that the equilibrium number of active firms is small. In these cases, we may reasonably question the appropriateness of the price-taking assumption (e.g., what if $J^* = 1$?). Indeed, we are then likely to be in the realm of the situations with market power studied in Chapter 12.

Second, we have conveniently shown the demand at price $\bar{c}$, $x(\bar{c})$, to be an integer multiple of $\bar{q}$. Were this not so, no long-run equilibrium would exist because the graphs of the demand function and the long-run supply correspondence would

28. Note that when $c(\cdot)$ is differentiable, condition (i) of Definition 10.F.1 implies that $c'(q^*) = p^*$, while condition (iii) implies $p^* = c(q^*)/q^*$. Thus, a *necessary* condition for an equilibrium is that $c'(q^*) = c(q^*)/q^*$. This is the condition for q^* to be a critical point of average costs [differentiate $c(q)/q$ and see Exercise 5.D.1]. In the case where average cost $c(q)/q$ is U-shaped (i.e., with no critical point other than the global minimum, as shown in Figure 10.F.4), this implies that $q^* = \bar{q}$, and so $p^* = \bar{c}$ and $J^* = x(\bar{c})/\bar{q}$. Note, however, that the argument in the text does not require this assumption about the shape of average costs.

not intersect.[29] The nonexistence of competitive equilibrium can occur here for the same reason that we have already alluded to in small type in Section 10.C: The long-run production technologies we are considering exhibit nonconvexities.

It seems plausible, however, that when the efficient scale of a firm is small relative to the size of the market, this "integer problem" should not be too much of a concern. In fact, when we study oligopolistic markets in Chapter 12, we shall see that when firms' efficient scales are small in this sense, the oligopolistic equilibrium price is close to $\bar{c}$, the equilibrium price we would derive if we simply ignored the integer constraint on the number of firms J^*. Intuitively, when the efficient scale is small, we will have many firms in the industry and the equilibrium, although not strictly competitive, will involve a price close to $\bar{c}$. Thus, if the efficient scale is small relative to the size of the market [as measured by $x(\bar{c})$], then ignoring the integer problem and treating firms as price takers gives approximately the correct answer.

Third, when an equilibrium exists, as in Figure 10.F.4, the equilibrium outcome maximizes Marshallian aggregate surplus and therefore is Pareto optimal. To see this, note from Figure 10.F.4 that aggregate surplus at the considered equilibrium is equal to

$$\operatorname*{Max}_{x \geq 0} \int_0^x P(s)\, ds - \bar{c}x,$$

the maximized value of aggregate surplus when firms' cost functions are $\bar{c}q$. But because $c(q) \geq \bar{c}q$ for all q, this must be the largest attainable value of aggregate surplus given the actual cost function $c(\cdot)$; that is,

$$\operatorname*{Max}_{x > 0} \int_0^x P(s)\, ds - \bar{c}x \geq \int_0^{\hat{x}} P(s)\, ds - Jc(\hat{x}/J),$$

for all $\hat{x}$ and J. This fact provides an example of a point we raised at the end of Section 10.D (and will substantiate with considerable generality in Chapter 16): The first welfare theorem continues to be valid even in the absence of convexity of individual production sets.

Short-Run and Long-Run Comparative Statics

Although firms may enter and exit the market in response to profit opportunities in the long run, these changes may take time. For example, factories may need to be shut down, the workforce reduced, and machinery sold when a firm exits an industry. It may even pay a firm to continue operating until a suitable buyer for its plant and equipment can be found. When examining the comparative statics effects of a shock to a market, it is therefore important to distinguish between long-run and short-run effects.

Suppose, for example, that we are at a long-run equilibrium with J^* active firms

29. An intermediate case between constant returns (where any scale is efficient) and the case of a unique efficient scale occurs when there is a range $[\underline{q}, \bar{q}]$ of efficient scales (the average cost curve has a flat bottom). In this case, the integer problem is mitigated. For a long-run competitive equilibrium to exist, we now only need there to be some $q \in [\underline{q}, \bar{q}]$ such that $x(\bar{c})/q$ is an integer. Of course, as the interval $[\underline{q}, \bar{q}]$ grows larger, not only are the chances of a long-run equilibrium existing greater, but so are the chances of indeterminacy of the equilibrium number of firms (i.e., of multiple equilibria involving differing numbers of firms).

each producing q^* units of output and that there is some shock to demand (similar points can be made for supply shocks). In the short run, it may be impossible for any new firms to organize and enter the industry, and so we will continue to have J^* firms for at least some period of time. Moreover, these J^* firms may face a short-run cost function $c_s(\cdot)$ that differs from the long-run cost function $c(\cdot)$ because various input levels may be fixed in the short run. For example, firms may have the long-run cost function

$$c(q) = \begin{cases} K + \psi(q) & \text{if } q > 0 \\ 0 & \text{if } q = 0, \end{cases} \tag{10.F.1}$$

where $\psi(0) = 0$, $\psi'(q) > 0$, and $\psi''(q) > 0$. But in the short run, it may be impossible for an active firm to recover its fixed costs if it exits and sets $q = 0$. Hence, in the short run the firm has the cost function

$$c_s(q) = K + \psi(q) \qquad \text{for all } q \geq 0. \tag{10.F.2}$$

Another possibility is that $c(q)$ might be the cost function of some multiple-input production process, and in the short run an active firm may be unable to vary its level of some inputs. (See the discussion in Section 5.B on this point and also Exercises 10.F.5 and 10.F.6 for illustrations.)

Whenever the distinction between short run and long run is significant, the *short-run comparative statics effects* of a demand shock may best be determined by solving for the competitive equilibrium given J^* firms, each with cost function $c_s(\cdot)$, and the new demand function. This is just the equilibrium notion studied in Section 10.C, where we take firms' cost functions to be $c_s(\cdot)$. The *long-run comparative statics effects* can then be determined by solving for the long-run (i.e., free entry) equilibrium given the new demand function and long-run cost function $c(\cdot)$.

Example 10.F.1: *Short-Run and Long-Run Comparative Statics with Lumpy Fixed Costs that Are Sunk in the Short Run.* Suppose that the long-run cost function $c(\cdot)$ is given by (10.F.1) but that in the short run the fixed cost K is sunk so that $c_s(\cdot)$ is given by (10.F.2). The aggregate demand function is initially $x(\cdot, \alpha_0)$, and the industry is at a long-run equilibrium with J_0 firms, each producing $\bar{q}$ units of output [the efficient scale for cost function $c(\cdot)$], and a price of $p^* = \bar{c} = c(\bar{q})/\bar{q}$. This equilibrium position is depicted in Figure 10.F.5.

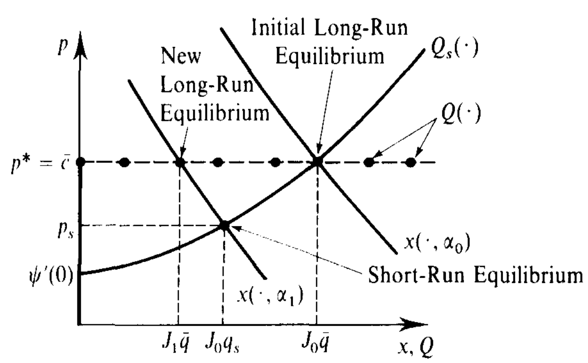


Figure 10.F.5 Short-run and long-run comparative statics in Example 10.F.1.

Now suppose that we have a shift to the demand function $x(\cdot, \alpha_1)$ shown in Figure 10.F.5. The short-run equilibrium is determined by the intersection of the graph of this demand function with the graph of the industry supply correspondence of the J_0 firms, each of which has short-run cost function $c_s(\cdot)$. The short-run aggregate supply correspondence is depicted as $Q_s(\cdot)$ in the figure. Thus, in the short run, the shock to demand causes price to fall to p_s and output per firm to fall to q_s. Firms' profits also fall; since $p_s < \bar{c}$, active firms lose money in the short run.

In the long run, however, firms exit in response to the decrease in demand, with the number of firms falling to $J_1 < J_0$, each producing output $\bar{q}$. Price returns to $p^* = \bar{c}$, aggregate consumption is $x(\bar{c}, \alpha_1)$, and all active firms once again earn zero profits. This new long-run equilibrium is also shown in Figure 10.F.5. ■

This division of dynamic adjustment into two periods, although useful as a first approximation, is admittedly crude. It may often be reasonable to think that there are several distinct short-run stages corresponding to different levels of adjustment costs associated with different decisions: in the very short run, production may be completely fixed; in the medium run, some inputs may be adjusted while others may not be; perhaps entry and exit take place only in the "very long run." Moreover, the methodology that we have discussed treats the two periods in isolation from each other. This approach ignores, for example, the possibility of intertemporal substitution by consumers when tomorow's price is expected to differ from today's (intertemporal substitution might be particularly important for very short-run periods when the fact that many production decisions are fixed can make prices very sensitive to demand shocks).

These weaknesses are not flaws in the competitive model per se, but rather only in the somewhat extreme methodological simplification adopted here. A fully satisfactory treatment of these issues requires an explicitly dynamic model that places expectations at center stage. In Chapter 20 we study dynamic models of competitive markets in greater depth. Nevertheless, this simple dichotomization into long-run and short-run periods of adjustment is often a useful starting point for analysis.

10.G Concluding Remarks on Partial Equilibrium Analysis

In principle, the analysis of Pareto optimal outcomes and competitive equilibria requires the simultaneous consideration of the entire economy (a task we undertake in Part IV). Partial equilibrium analysis can be thought of as facilitating matters on two accounts. On the positive side, it allows us to determine the equilibrium outcome in the particular market under study in isolation from all other markets. On the normative side, it allows us to use Marshallian aggregate surplus as a welfare measure that, in many cases of interest, has a very convenient representation in terms of the area lying vertically between the aggregate demand and supply curves.

In the model considered in Sections 10.C to 10.F, the validity of both of these simplifications rested, implicitly, on two premises: first, that the prices of all commodities other than the one under consideration remain fixed; second, that there are no wealth effects in the market under study. We devote this section to a few additional interpretative comments regarding these assumptions. (See also Section 15.E for an example illustrating the limits of partial equilibrium analysis.)

The assumption that the prices of goods other than the good under consideration (say, good ℓ) remain fixed is essential for limiting our positive and normative analysis to a single market. In Section 10.B, we justified this assumption in terms of the market for good ℓ being small and having a diffuse influence over the remaining markets. However, this is not its only possible justification. For example, the nonsubstitution theorem (see Appendix A of Chapter 5) implies that the prices of all other goods will remain fixed if the numeraire is the only primary (i.e., nonproduced) factor, all produced goods other than ℓ are produced under conditions of constant returns using the numeraire and produced commodities other than ℓ as inputs, and there is no joint production.[30]

Even when we cannot assume that all other prices are fixed, however, a generalization of our single-market partial equilibrium analysis is sometimes possible. Often we are interested not in a single market but in a group of commodities that are strongly interrelated either in consumers' tastes (tea and coffee are the classic examples) or in firms' technologies. In this case, studying one market at a time while keeping other prices fixed is no longer a useful approach because what matters is the *simultaneous* determination of *all* prices in the group. However, if the prices of goods outside the group may be regarded as unaffected by changes within the markets for this group of commodities, and if there are no wealth effects for commodities in the group, then we can extend much of the analysis presented in Sections 10.C to 10.F.

To this effect, suppose that the group is composed of M goods, and let $x_i \in \mathbb{R}^M_+$ and $q_j \in \mathbb{R}^M$ be vectors of consumptions and productions for these M goods. Each consumer has a utility function of the form

$$u_i(m_i, x_i) = m_i + \phi_i(x_i),$$

where m_i is the consumption of the numeraire commodity (i.e., the total expenditure on commodities outside the group). Firms' cost functions are $c_j(q_j)$. With this specification, many of the basic results of the previous sections go through unmodified (often it is just a matter of reinterpreting x_i and q_j as vectors). In particular, the results discussed in Section 10.C on the uniqueness of equilibrium and its independence from initial endowments still hold (see Exercise 10.G.1), as do the welfare theorems of Section 10.D. However, our ability to conduct welfare analysis using the areas lying vertically between demand and supply curves becomes much more limited. The cross-effects among markets with changing and interrelated prices cannot be

30. A simple example of this result arises when all produced goods other than ℓ are produced directly from the numeraire with constant returns to scale. In this case, the equilibrium price of each of these goods is equal to the amount of the numeraire that must be used as an input in its production per unit of output produced. More generally, prices for produced goods other than ℓ will remain fixed under the conditions of the nonsubstitution theorem because all efficient production vectors can be generated using a single set of techniques. In any equilibrium, the price of each produced good other than ℓ must be equal to the amount of the numeraire embodied in a unit of the good in the efficient production technique, either directly through the use of the numeraire as an input or indirectly through the use as inputs of produced goods other than ℓ that are in turn produced using the numeraire (or using other produced goods that are themselves produced using the numeraire, and so on).

ignored.[31] (Exercises 10.G.3 to 10.G.5 ask you to consider some issues related to this point.)

The assumption of no wealth effects for good ℓ, on the other hand, is critical for the validity of the style of welfare analysis that we have carried out in this chapter. Without it, as we shall see in Part IV, Pareto optimality cannot be determined independently from the particular distribution of welfare sought, and we already know from Section 3.I that area measures calculated from Walrasian demand functions are not generally correct measures of compensating or equivalent variations (for which the Hicksian demand functions should be used). However, the assumption of no wealth effects is much less critical for positive analysis (determination of equilibrium, comparative statics effects, and so on). Even with wealth effects, the demand-and-supply apparatus can still be quite helpful for the positive part of the theory. The behavior of firms, for example, is not changed in any way. Consumers, on the other hand, have a demand function that, with prices of the other goods kept fixed, now depends only on the price for good ℓ and wealth. If wealth is determined from initial endowments and shareholdings, then we can view wealth as itself a function of the price of good ℓ (recall that other prices are fixed), and so we can again express demand as a function of this good's price alone. Formally, the analysis reduces to that presented in Section 10.C: The equilibrium in market ℓ can be identified as an intersection point of demand and supply curves.[32]

31. A case in which the single-market analysis for good ℓ is still fully justified is when utility and cost functions have the form

$$u_i(m_i, x_i) = m_i + \phi_{\ell i}(x_{\ell i}) + \phi_{-\ell,i}(x_{-\ell,i}),$$

and

$$c_j(q_j) = c_{\ell j}(q_{\ell j}) + c_{-\ell,j}(q_{-\ell,j}),$$

where $x_{-\ell,i}$ and $q_{-\ell,j}$ are consumption and production vectors for goods in the group other than ℓ. With this additive separability in good ℓ, the markets for goods in the group other than ℓ do not influence the equilibrium price in market ℓ. Good ℓ is effectively independent of the group, and we can treat it in isolation, as we have done in the previous sections. (In point of fact, we do not even need to assume that the remaining markets in the group keep their prices fixed. What happens in them is simply irrelevant for equilibrium and welfare analysis in the market for good ℓ.) See Exercise 10.G.2.

32. The presence of wealth effects can lead, however, to some interesting new phenomena on the consumer's side. One is the *backward-bending* demand curve, where demand for a good is *increasing* in its price over some range. This can happen if consumers have endowments of good ℓ, because then an increase in its price increases consumers' wealth and could lead to a net increase in their demands for good ℓ, even if it is a normal good.

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EXERCISES

10.B.1^B The concept defined in Definition 10.B.2 is sometimes known as *strong Pareto efficiency*. An outcome is *weakly Pareto efficient* if there is no alternative feasible allocation that makes *all* individuals *strictly* better off.

(a) Argue that if an outcome is strongly Pareto efficient, then it is weakly Pareto efficient as well.

(b) Show that if all consumers' preferences are continuous and strongly monotone, then these two notions of Pareto efficiency are equivalent for any *interior* outcome (i.e., an outcome in which each consumer's consumption lies in the interior of his consumption set). Assume for simplicity that $X_i = \mathbb{R}^L_+$ for all i.

(c) Construct an example where the two notions are not equivalent. Why is the strong monotonicity assumption important in **(b)**? What about interiority?

10.B.2^A Show that if allocation $(x_1^*, \dots, x_I^*, y_1^*, \dots, y_J^*)$ and price vector $p^* \gg 0$ constitute a competitive equilibrium, then allocation $(x_1^*, \dots, x_I^*, y_1^*, \dots, y_J^*)$ and price vector αp^* also constitute a competitive equilibrium for any scalar $\alpha > 0$.

10.C.1^B Suppose that consumer i's preferences can be represented by the utility function $u_i(x_{1i}, \dots, x_{Li}) = \sum_\ell \log(x_{\ell i})$ (these are Cobb–Douglas preferences).

(a) Derive his demand for good ℓ. What is the wealth effect?

(b) Now consider a sequence of situations in which we proportionately increase both the number of goods and the consumer's wealth. What happens to the wealth effect in the limit?

10.C.2^B Consider the two-good quasilinear model presented in Section 10.C with one consumer and one firm (so that $I = 1$ and $J = 1$). The initial endowment of the numeraire is $\omega_m > 0$, and the initial endowment of good ℓ is 0. Let the consumer's quasilinear utility function be $\phi(x) + m$, where $\phi(x) = \alpha + \beta \ln x$ for some $(\alpha, \beta) \gg 0$. Also, let the firm's cost function be $c(q) = \sigma q$ for some scalar $\sigma > 0$. Assume that the consumer receives all the profits of the firm. Both the firm and the consumer act as price takers. Normalize the price of good m to equal 1, and denote the price of good ℓ by p.

(a) Derive the consumer's and the firm's first-order conditions.

(b) Derive the competitive equilibrium price and output of good ℓ. How do these vary with α, β, and σ?

10.C.3^B Consider a central authority who operates J firms with differentiable convex cost functions $c_j(q_j)$ for producing good ℓ from the numeraire. Define $C(q)$ to be the central authority's minimized cost level for producing aggregate quantity q; that is

$$C(q) = \underset{(q_1,\dots,q_J)\geq 0}{\text{Min}} \quad \sum_{j=1}^{J} c_j(q_j)$$
$$\text{s.t.} \quad \sum_{j=1}^{J} q_j \geq q.$$

(a) Derive the first-order conditions for this cost-minimization problem.

(b) Show that at the cost-minimizing production allocation $(q_1^*, \dots, q_J^*)$, $C'(q) = c_j'(q_j^*)$ for all j with $q_j^* > 0$ (i.e., the central authority's marginal cost at aggregate output level q equals each firm's marginal cost level at the optimal production allocation for producing q).

(c) Show that if firms all maximize profit facing output price $p = C'(q)$ (with the price of the numeraire equal to 1), then the consequent output choices result in an aggregate output of q. Conclude that $C'(\cdot)$ is the inverse of the industry supply function $q(\cdot)$.

10.C.4$^{\mathrm{B}}$ Consider a central authority who has x units of good ℓ to allocate among I consumers, each of whom has a quasilinear utility function of the form $\phi_i(x_i) + m_i$, with $\phi_i(\cdot)$ a differentiable, increasing, and strictly concave function. The central authority allocates good ℓ to maximize the sum of consumers' utilities $\sum_i u_i$.

(a) Set up the central authority's problem and derive its first-order condition.

(b) Let $\gamma(x)$ be the value function of the central authority's problem, and let $P(x) = \gamma'(x)$ be its derivative. Show that if $(x_1^*, \ldots, x_I^*)$ is the optimal allocation of good ℓ given available quantity x, then $P(x) = \phi_i'(x_i^*)$ for all i with $x_i^* > 0$.

(c) Argue that if all consumers maximize utility facing a price for good ℓ of $P(x)$ (with the price of the numeraire equal to 1), then the aggregate demand for good ℓ is exactly x. Conclude that $P(\cdot)$ is, in fact, the inverse of the aggregate demand function $x(\cdot)$.

10.C.5$^{\mathrm{B}}$ Derive the differential change in the equilibrium price in response to a differential change in the tax in Example 10.C.1 by applying the implicit function theorem to the system of equations (10.C.4) to (10.C.6).

10.C.6$^{\mathrm{B}}$ A tax is to be levied on a commodity bought and sold in a competitive market. Two possible forms of tax may be used: In one case, a *specific* tax is levied, where an amount t is paid per unit bought or sold (this is the case considered in the text); in the other case, an *ad valorem* tax is levied, where the government collects a tax equal to τ times the amount the seller receives from the buyer. Assume that a partial equilibrium approach is valid.

(a) Show that, with a specific tax, the ultimate cost of the good to consumers and the amounts purchased are independent of whether the consumers or the producers pay the tax.

(b) Show that this is not generally true with an ad valorem tax. In this case, which collection method leads to a higher cost to consumers? Are there special cases in which the collection method is irrelevant with an ad valorem tax?

10.C.7$^{\mathrm{B}}$ An ad valorem tax of τ (see Exercise 10.C.6 for a definition) is to be levied on consumers in a competitive market with aggregate demand curve $x(p) = Ap^{\varepsilon}$, where $A > 0$ and $\varepsilon < 0$, and aggregate supply curve $q(p) = \alpha p^{\gamma}$, where $\alpha > 0$ and $\gamma > 0$. Calculate the percentage change in consumer cost and producer receipts per unit sold for a small ("marginal") tax. Denote $\kappa = (1 + \tau)$. Assume that a partial equilibrium approach is valid.

Compute the elasticity of the equilibrium price with respect to κ. Argue that when $\gamma = 0$ producers bear the full effect of the tax while consumers' total costs of purchase are unaffected, and that when $\varepsilon = 0$ it is consumers who bear the full burden of the tax. What happens when each of these elasticities approaches ∞ in absolute value?

10.C.8$^{\mathrm{B}}$ Suppose that there are J firms producing good ℓ, each with a differentiable cost function $c(q, \alpha)$ that is strictly convex in q, where α is an exogenous parameter that affects costs (it could be a technological parameter or an input price). Assume that $\partial c(q, \alpha)/\partial \alpha > 0$. The differentiable aggregate demand function for good ℓ is $x(p)$, with $x'(\cdot) \leq 0$. Assume that partial equilibrium analysis is justified.

Let $q^*(\alpha)$ be the *per firm* output and $p^*(\alpha)$ be the equilibrium price in the competitive equilibrium given α.

(a) Derive the marginal change in a firm's profits with respect to α.

(b) Give the weakest possible sufficient condition, stated in terms of marginal and average costs and/or their derivatives, that guarantees that if α increases marginally, then firms' equilibrium profits decline for any demand function $x(\cdot)$ having $x'(\cdot) \leq 0$. Show that if this condition is not satisfied, then there are demand functions such that profits increase when α increases.

(c) In the case where α is the price of factor input k, interpret the condition in **(b)** in terms of the conditional factor demand for input k.

10.C.9$^{\mathbf{B}}$ Suppose that in a partial equilibrium context there are J identical firms that produce good ℓ with cost function $c(w, q)$, where w is a vector of factor input prices. Show that an increase in the price of factor k, w_k, lowers the equilibrium price of good ℓ if and only if factor k is an *inferior* factor, that is, if at fixed input prices, the use of factor k is decreasing in a firm's output level.

10.C.10$^{\mathbf{B}}$ Consider a market with demand curve $x(p) = \alpha p^{\varepsilon}$ and with J firms, each of which has marginal cost function $c'(q) = \beta q^{\eta}$, where $(\alpha, \beta, \eta) \gg 0$ and $\varepsilon < 0$. Calculate the competitive equilibrium price and output levels. Examine the comparative statics change in these variables as a result of changes in α and β. How are these changes affected by ε and η?

10.C.11$^{\mathbf{B}}$ Assume that partial equilibrium analysis is valid. Suppose that firms 1 and 2 are producing a positive level of output in a competitive equilibrium. The cost function for firm j is given by $c(q, \alpha_j)$, where α_j is an exogenous technological parameter. If α_1 differs from α_2 marginally, what is the difference in the two firms' profits?

10.D.1$^{\mathbf{B}}$ Prove that under the assumptions that the $\phi_i(\cdot)$ functions are strictly concave and the cost functions $c_j(\cdot)$ are convex, the optimal individual consumption levels of good ℓ in problem (10.D.2) are uniquely defined. Conclude that the optimal aggregate production level of good ℓ is therefore also uniquely defined. Show that if the cost functions $c_j(\cdot)$ are *strictly* convex, then the optimal individual production levels of good ℓ in problem (10.D.2) are also uniquely defined.

10.D.2$^{\mathbf{B}}$ Determine the optimal consumption and production levels of good ℓ for the economy described in Exercise 10.C.2. Compare these with the equilibrium levels you identified in that exercise.

10.D.3$^{\mathbf{B}}$ In the context of the two-good quasilinear economy studied in Section 10.D, show that any allocation that is a solution to problem (10.D.6) is Pareto optimal and that any Pareto optimal allocation is a solution to problem (10.D.6) for *some* choice of utility levels $(\bar{u}_2, \dots, \bar{u}_I)$.

10.D.4$^{\mathbf{B}}$ Derive the first-order conditions for problem (10.D.6) and compare them with conditions (10.D.3) to (10.D.5).

10.E.1$^{\mathbf{C}}$ Suppose that $J_d > 0$ of the firms that produce good ℓ are domestic firms, and $J_f > 0$ are foreign firms. All domestic firms have the same convex cost function for producing good ℓ, $c_d(q_j)$. All foreign firms have the same convex cost function $c_f(q_j)$. Assume that partial equilibrium analysis is valid.

The government of the domestic country is considering imposing a per-unit tariff of τ on imports of good ℓ. The government wants to maximize domestic welfare as measured by the *domestic* Marshallian surplus (i.e., the sum of domestic consumers' utilities less domestic firms' costs).

(a) Show that if $c_f(\cdot)$ is strictly convex, then imposition of a small tariff raises domestic welfare.

(b) Show that if $c_f(\cdot)$ exhibits constant returns to scale, then imposition of a small tariff lowers domestic welfare.

10.E.2$^{\mathbf{B}}$ Consumer surplus when consumers face effective price $\hat{p}$ can be written as

$$CS(\hat{p}) = \int_0^{x(\hat{p})} [P(s) - \hat{p}]\, ds.$$

Prove by means of a change of variables and integration by parts that this integral is equal to $\int_{\bar{p}}^{\alpha} x(s)\,ds$.

10.E.3[C] (*Ramsey tax problem*) Consider a fully separable quasilinear model with L goods in which each consumer has preferences of the form $u_i(x_i) = x_{1i} + \sum_{\ell=2}^{L} \phi_{\ell i}(x_{\ell i})$ and each good $2, \ldots, L$ is produced with constant returns to scale from good 1, using c_ℓ units of good 1 per unit of good ℓ produced. Assume that consumers initially hold endowments only of the numeraire, good 1. Hence, consumers are net sellers of good 1 to the firms and net purchasers of goods $2, \ldots, L$.

In this setting, consumer i's demand for each good $\ell \neq 1$ can be written in the form $x_{\ell i}(p_\ell)$, so that demand for good ℓ is independent of the consumer's wealth and all other prices, and welfare can be measured by the sum of the Marshallian aggregate surpluses in the $L - 1$ markets for nonnumeraire commodities (see Section 10.G and Exercise 10.G.2 for more on this).

Suppose that the government must raise R units of good 1 through (specific) commodity taxes. Note, in particular, that such taxes involve taxing a *transaction* of a good, *not* an individual's consumption level of that good.

Let t_ℓ denote the tax to be paid by a consumer in units of good 1 for each unit of good $\ell \neq 1$ purchased, and let t_1 be the tax in units of good 1 to be paid by consumers for each unit of good 1 sold to a firm. Normalize the price paid by firms for good 1 to equal 1. Under our assumptions, each choice of $t = (t_1, \ldots, t_L)$ results in a consumer paying a total of $c_\ell + t_\ell$ per unit of good $\ell \neq 1$ purchased and having to part with $(1 + t_1)$ units of good 1 for each unit of good 1 sold to a firm.

(a) Consider two possible tax vectors t and t'. Show that if t' is such that $(c_\ell + t'_\ell) = \alpha(c_\ell + t_\ell)$ and $(1 + t'_1) = (1/\alpha)(1 + t_1)$ for some scalar $\alpha > 0$, then the two sets of taxes raise the same revenue. Conclude from this fact that the government can restrict attention to tax vectors that leave one good untaxed.

(b) Let good 1 be the untaxed good (i.e., set $t_1 = 0$). Derive conditions describing the taxes that should be set on goods $2, \ldots, L$ if the government wishes to minimize the welfare loss arising from this taxation. Express this formula in terms of the elasticity of demand for each good.

(c) Under what circumstances should the tax rate on all goods be equal? In general, which goods should have higher tax rates? When would taxing only good 1 be optimal?

10.F.1[A] Show that if $c(q)$ is strictly convex in q and $c(0) = 0$, then $\pi(p) > 0$ if and only if $p > c'(0)$.

10.F.2[B] Consider a market with demand function $x(p) = A - Bp$ in which every potential firm has cost function $c(q) = K + \alpha q + \beta q^2$, where $\alpha > 0$ and $\beta > 0$.

(a) Calculate the long-run competitive equilibrium price, output per firm, aggregate output, and number of firms. Ignore the integer constraint on the number of firms. How does each of these vary with A?

(b) Now examine the short-run competitive equilibrium response to a change in A starting from the long-run equilibrium you identified in (a). How does the change in price depend on the level of A in the initial equilibrium? What happens as $A \to \infty$? What accounts for this effect of market size?

10.F.3[B] (D. Pearce) Consider a partial equilibrium setting in which each (potential) firm has a long-run cost function $c(\cdot)$, where $c(q) = K + \phi(q)$ for $q > 0$ and $c(0) = 0$. Assume that $\phi'(q) > 0$ and $\phi''(q) < 0$, and denote the firm's efficient scale by $\bar{q}$. Suppose that there is initially a long-run equilibrium with J^* firms. The government considers imposing two different types of taxes: The first is an ad valorem tax of τ (see Exercise 10.C.6) on sales of the good. The second is a tax T that must be paid by any operating firm (where a firm is considered to be "operating" if it sells a positive amount). If the two taxes would raise an equal amount of revenue with the initial level of sales and number of firms, which will raise more after the industry adjusts to a new long-run equilibrium? (You should ignore the integer constraint on the number of firms.)

10.F.4^B (J. Panzar) Assume that partial equilibrium analysis is valid. The single-output, many-input technology for producing good ℓ has a differentiable cost function $c(w, q)$, where $w = (w_1, \dots, w_K)$ is a vector of factor input prices and q is the firm's output of good ℓ. Given factor prices w, let $\bar{q}(w)$ denote the firm's efficient scale. Assume that $\bar{q}(w) > 0$ for all w. Also let $p_\ell^*(w)$ denote the long-run equilibrium price of good ℓ when factor prices are w. Show that the function $p_\ell^*(w)$ is nondecreasing, homogeneous of degree one, and concave. (You should ignore the integer constraint on the number of firms.)

10.F.5^C Suppose that there are J firms that can produce good ℓ from K factor inputs with differentiable cost function $c(w, q)$. Assume that this function is strictly convex in q. The differentiable aggregate demand function for good ℓ is $x(p, \alpha)$, where $\partial x(p, \alpha)/\partial p < 0$ and $\partial x(p, \alpha)/\partial \alpha > 0$ (α is an exogenous parameter affecting demand). However, although $c(w, q)$ is the cost function when all factors can be freely adjusted, factor k cannot be adjusted in the short run.

Suppose that we are initially at an equilibrium in which all inputs are optimally adjusted to the equilibrium level of output q^* and factor prices w so that, letting $z_k(w, q)$ denote a firm's conditional factor demand for input k when all inputs can be adjusted, $z_k^* = z_k(w, q^*)$.

(a) Show that a firm's equilibrium response to an increase in the price of good ℓ is larger in the long run than in the short run.

(b) Show that this implies that the long-run equilibrium response of p_ℓ to a marginal increase in α is smaller than the short-run response. Show that the reverse is true for the response of the equilibrium aggregate consumption of good ℓ (hold the number of firms equal to J in both the short run and long run).

10.F.6^B Suppose that the technology for producing a good uses capital (z_1) and labor (z_2) and takes the Cobb Douglas form $f(z_1, z_2) = z_1^\alpha z_2^{1-\alpha}$, where $\alpha \in (0, 1)$. In the long run, both factors can be adjusted; but in the short run, the use of capital is fixed. The industry demand function takes the form $x(p) = a - bp$. The vector of input prices is (w_1, w_2). Find the long-run equilibrium price and aggregate quantity. Holding the number of firms and the level of capital fixed at their long-run equilibrium levels, what is the short-run industry supply function?

10.F.7^B Consider a case where in the short run active firms can increase their use of a factor but cannot decrease it. Show that the short-run cost curve will exhibit a kink (i.e., be nondifferentiable) at the current (long-run) equilibrium. Analyze the implications of this fact for the relative variability of short-run prices and quantities.

10.G.1^B Consider the case of an interrelated group of M commodities. Let consumer i's utility function take the form $u_i(x_{1i}, \dots, x_{Mi}) = m_i + \phi_i(x_{1i}, \dots, x_{Mi})$. Assume that $\phi_i(\cdot)$ is differentiable and strictly concave. Let firm j's cost function be the differentiable convex function $c_j(q_{1j}, \dots, q_{Mj})$.

Normalize the price of the numeraire to be 1. Derive $(I + J + 1)M$ equations characterizing the $(I + J + 1)M$ equilibrium quantities $(x_{1i}^*, \dots, x_{Mi}^*)$ for $i = 1, \dots, I$, $(q_{1j}^*, \dots, q_{Mj}^*)$ for $j = 1, \dots, J$, and $(p_1^*, \dots, p_M^*)$. [*Hint*: Derive consumers' and firms' first-order conditions and the $M - 1$ market-clearing conditions in parallel to our analysis of the single-market case.] Argue that the equilibrium prices and quantities of these M goods are independent of consumers' wealths, that equilibrium individual consumptions and aggregate production levels are unique, and that if the $c_j(\cdot)$ functions are strictly convex, then equilibrium individual production levels are also unique.

10.G.2^B Consider the case in which the functions $\phi_i(\cdot)$ and $c_j(\cdot)$ in Exercise 10.G.1 are separable in good ℓ (one of the goods in the group): $\phi_i(\cdot) = \phi_{\ell i}(x_{\ell i}) + \phi_{-\ell,i}(x_{-\ell,i})$ and $c_j(\cdot) = c_{\ell j}(q_{\ell j}) + c_{-\ell,j}(q_{-\ell,j})$. Argue that in this case, the equilibrium price, consumption, and production of good ℓ can be determined independently of other goods in the group. Also argue that under the same assumptions as in the single-market case studied in Section 10.E, changes in welfare caused by changes in the market for this good can be captured by the Marshallian aggregate surplus for this good, $\sum_i \phi_{\ell i}(x_{\ell i}) - \sum_j c_{\ell j}(q_{\ell j})$, which can be represented in terms of the areas lying vertically between the demand and supply curves for good ℓ. Note the implication of these results for the case in which we have separability of all goods: $\phi_i(\cdot) = \sum_\ell \phi_{\ell i}(x_{\ell i})$ and $c_j(\cdot) = \sum_\ell c_{\ell j}(q_{\ell j})$.

10.G.3^B Consider a three-good economy ($\ell = 1, 2, 3$) in which every consumer has preferences that can be described by the utility function $u(x) = x_1 + \phi(x_2, x_3)$ and there is a single production process that produces goods 2 and 3 from good 1 having $c(q_2, q_3) = c_2 q_2 + c_3 q_3$. Suppose that we are considering a tax change in only a single market, say market 2.

(a) Show that if the price in market 3 is undistorted (i.e., if $t_3 = 0$), then the change in aggregate surplus caused by the tax change can be captured solely through the change in the area lying vertically between market 2's demand and supply curves holding the price of good 3 at its initial level.

(b) Show that if market 3 is initially distorted because $t_3 > 0$, then by using only the single-market measure in **(a)**, we would overstate the decrease in aggregate surplus if good 3 is a substitute for good 2 and would understate it if good 3 is a complement. Provide an intuitive explanation of this result. What is the correct measure of welfare change?

10.G.4^B Consider a three-good economy ($\ell = 1, 2, 3$) in which every consumer has preferences that can be described by the utility function $u(x) = x_1 + \phi(x_2, x_3)$ and there is a single production process that produces goods 2 and 3 from good 1 having $c(q_2, q_3) = c_2 q_2 + c_3 q_3$. Derive an expression for the welfare loss from an increase in the tax rates on both goods.

10.G.5^B Consider a three-good economy ($\ell = 1, 2, 3$) in which every consumer has preferences that can be described by the utility function $u(x) = x_1 + \phi(x_2, x_3)$ and there is a single production process that produces goods 2 and 3 from good 1 having $c(q_2, q_3) = c_2(q_2) + c_3(q_3)$, where $c_2(\cdot)$ and $c_3(\cdot)$ are strictly increasing and strictly convex.

(a) If goods 2 and 3 are substitutes, what effect does an increase in the tax on good 2 have on the price paid by consumers for good 3? What if they are complements?

(b) What is the bias from applying the formula for welfare loss you derived in part (b) of Exercise 10.G.3 using the price paid by consumers for good 3 prior to the tax change in both the case of substitutes and that of complements?

CHAPTER

Externalities and Public Goods 11

11.A Introduction

In Chapter 10, we saw a close connection between competitive, price-taking equilibria and Pareto optimality (or, Pareto efficiency).[1] The first welfare theorem tells us that competitive equilibria are necessarily Pareto optimal. From the second welfare theorem, we know that under suitable convexity hypotheses, any Pareto optimal allocation can be achieved as a competitive allocation after an appropriate lump-sum redistribution of wealth. Under the assumptions of these theorems, the possibilities for welfare-enhancing intervention in the marketplace are strictly limited to the carrying out of wealth transfers for the purposes of achieving distributional aims.

With this chapter, we begin our study of *market failures*: situations in which some of the assumptions of the welfare theorems do *not* hold and in which, as a consequence, market equilibria cannot be relied on to yield Pareto optimal outcomes. In this chapter, we study two types of market failure, known as *externalities* and *public goods*.

In Chapter 10, we assumed that the preferences of a consumer were defined solely over the set of goods that she might herself decide to consume. Similarly, the production of a firm depended only on its own input choices. In reality, however, a consumer or firm may in some circumstances be directly affected by the actions of other agents in the economy; that is, there may be *external effects* from the activities of other consumers or firms. For example, the consumption by consumer i's neighbor of loud music at three in the morning may prevent her from sleeping. Likewise, a fishery's catch may be impaired by the discharges of an upstream chemical plant. Incorporating these concerns into our preference and technology formalism is, in principle, a simple matter: We need only define an agent's preferences or production set over both her own actions and those of the agent creating the external effect. But the effect on market equilibrium is significant: In general, when external effects are present, competitive equilibria are not Pareto optimal.

Public goods, as the name suggests, are commodities that have an inherently "public" character, in that consumption of a unit of the good by one agent does not preclude its consumption by another. Examples abound: Roadways, national defense,

1. See also Chapter 16.

flood-control projects, and knowledge all share this characteristic. The private provision of public goods generates a special type of externality: if one individual provides a unit of a public good, all individuals benefit. As a result, private provision of public goods is typically Pareto inefficient.

We begin our investigation of externalities and public goods in Section 11.B by considering the simplest possible externality: one that involves only two agents in the economy, where one of the agents engages in an activity that directly affects the other. In this setting, we illustrate the inefficiency of competitive equilibria when an externality is present. We then go on to consider three traditional solutions to this problem: quotas, taxes, and the fostering of decentralized bargaining over the extent of the externality. The last of these possibilities also suggests a connection between the presence of externalities and the nonexistence of certain commodity markets, a topic that we explore in some detail.

In Section 11.C, we study public goods. We first derive a condition that characterizes the optimal level of a public good and we then illustrate the inefficiency resulting from private provision. This Pareto inefficiency can be seen as arising from an externality among the consumers of the good, which in this context is known as the *free-rider problem*. We also discuss possible solutions to this free-rider problem. Both quantity-based intervention (here, direct governmental provision) and price-based intervention (taxes and subsidies) can, in principle, correct it. In contrast, decentralized bargaining and competitive market-based solutions are unlikely to be viable in the context of public goods.

In Section 11.D, we return to the analysis of externalities. We study cases in which many agents both produce and are affected by the externality. Multilateral externalities can be classified according to whether the externality is *depletable* (or *private* or *rivalrous*) or *nondepletable* (or *public* or *nonrivalrous*). We argue that market solutions are likely to work well in the former set of cases but poorly in the latter, where the externality possesses the characteristics of a public good (or bad). Indeed, this may well explain why most externalities that are regarded as serious social problems (e.g., water pollution, acid rain, congestion) take the form of nondepletable multilateral externalities.

In Section 11.E, we examine another problem that may arise in these settings: Individuals may have privately held information about the effects of externalities on their well-being. We see there that this type of informational asymmetry may confound both private and government efforts to achieve optimal outcomes.

In Appendix A, we study the connection between externalities and the presence of technological nonconvexities, and we examine the implications of these nonconvexities for our analysis.

The literature on externalities and public goods is voluminous. Useful introductions and further references to these subjects may be found in Baumol and Oates (1988) and Laffont (1988).

11.B A Simple Bilateral Externality

Surprisingly, perhaps, a fully satisfying definition of an externality has proved somewhat elusive. Nevertheless, informal Definition 11.B.1 provides a serviceable point of departure.

Definition 11.B.1: An *externality* is present whenever the well-being of a consumer or the production possibilities of a firm are directly affected by the actions of another agent in the economy.

Simple as Definition 11.B.1 sounds, it contains a subtle point that has been a source of some confusion. When we say "directly," we mean to exclude any effects that are mediated by prices. That is, an externality is present if, say, a fishery's productivity is affected by the emissions from a nearby oil refinery, but not simply because the fishery's profitability is affected by the price of oil (which, in turn, is to some degree affected by the oil refinery's output of oil). The latter type of effect [referred to as a *pecuniary externality* by Viner (1931)] is present in any competitive market but, as we saw in Chapter 10, creates no inefficiency. Indeed, with price-taking behavior, the market is precisely the mechanism that guarantees a Pareto optimal outcome. This suggests that the presence of an externality is not merely a technological phenomenon but also a function of the set of markets in existence. We return to this point later in the section.

In the remainder of this section, we explore the implications of external effects for competitive equilibria and public policy in the context of a very simple two-agent, partial equilibrium model. We consider two consumers, indexed by $i = 1, 2$, who constitute a small part of the overall economy. In line with this interpretation, we suppose that the actions of these consumers do not affect the prices $p \in \mathbb{R}^L$ of the L traded goods in the economy. At these prices, consumer i's wealth is w_i.

In contrast with the standard competitive model, however, we assume that each consumer has preferences not only over her consumption of the L traded goods $(x_{1i}, \dots, x_{Li})$ but also over some action $h \in \mathbb{R}_+$ taken by consumer 1. Thus, consumer i's (differentiable) utility function takes the form $u_i(x_{1i}, \dots, x_{Li}, h)$, and we assume that $\partial u_2(x_{12}, \dots, x_{L2}, h)/\partial h \neq 0$. Because consumer 1's choice of h affects consumer 2's well-being, it generates an externality. For example, the two consumers may live next door to each other, and h may be a measure of how loudly consumer 1 plays music. Or the consumers may live on a river, with consumer 1 further upstream. In this case, h could represent the amount of pollution put into the river by consumer 1; more pollution lowers consumer 2's enjoyment of the river. We should hasten to add that external effects need not be detrimental to those affected by them. Action h could, for example, be consumer 1's beautification of her property, which her neighbor, consumer 2, also gets to enjoy.[2]

In what follows, it will be convenient to define for each consumer i a derived utility function over the level of h, assuming optimal commodity purchases by consumer i at prices $p \in \mathbb{R}^L$ and wealth w_i:

$$v_i(p, w_i, h) = \underset{x_i \geq 0}{\text{Max}} \quad u_i(x_i, h)$$
$$\text{s.t. } p \cdot x_i \leq w_i.$$

For expositional purposes, we shall also assume that the consumers' utility functions

2. An externality favorable to the recipient is usually called a *positive externality*, and conversely for a *negative externality*.

take a quasilinear form with respect to a numeraire commodity (we comment below, in small type, on the simplifications afforded by this assumption). In this case, we can write the derived utility function $v_i(\cdot)$ as $v_i(p, w_i, h) = \phi_i(p, h) + w_i$.[3] Since prices of the L traded goods are assumed to be unaffected by any of the changes we are considering, we shall suppress the price vector p and simply write $\phi_i(h)$. We assume that $\phi_i(\cdot)$ is twice differentiable with $\phi_i''(\cdot) < 0$. Be warned, however, that the concavity assumption is less innocent than it looks: see Appendix A for further discussion of this point.

Although we shall speak in terms of this consumer interpretation, everything we do here applies equally well to the case in which the two agents are firms (or, for that matter, one firm and one consumer). For example, we could consider a firm j that has a derived profit function $\pi_j(p, h)$ over h given prices p. Suppressing the price vector p, the firm's profit can be written as $\pi_j(h)$, which plays the same role as the function $\phi_i(h)$ in the analysis that follows.

Nonoptimality of the Competitive Outcome

Suppose that we are at a competitive equilibrium in which commodity prices are p. That is, at the equilibrium position, each of the two consumers maximizes her utility limited only by her wealth and the prices p of the traded goods. It must therefore be the case that consumer 1 chooses her level of $h \geq 0$ to maximize $\phi_1(h)$. Thus, the equilibrium level of h, h^*, satisfies the necessary and sufficient first-order condition

$$\phi_1'(h^*) \leq 0, \quad \text{with equality if } h^* > 0. \tag{11.B.1}$$

For an interior solution, we therefore have $\phi_1'(h^*) = 0$.

In contrast, in any Pareto optimal allocation, the optimal level of h, h°, must maximize the *joint surplus* of the two consumers, and so must solve[4]

$$\underset{h \geq 0}{\text{Max}} \quad \phi_1(h) + \phi_2(h).$$

This problem gives us the necessary and sufficient first-order condition for h° of

$$\phi_1'(h^\circ) \leq -\phi_2'(h^\circ), \text{ with equality if } h^\circ > 0. \tag{11.B.2}$$

Hence, for an interior solution to the Pareto optimality problem, $\phi_1'(h^\circ) = -\phi_2'(h^\circ)$.

When external effects are present, so that $\phi_2'(h) \neq 0$ at all h, the equilibrium level of h is not optimal unless $h^\circ = h^* = 0$. Consider, for example, the case in which we have interior solutions, that is, where $(h^*, h^\circ) \gg 0$. If $\phi_2'(\cdot) < 0$, so that h generates

3. Indeed, suppose that $u_i(x_i, h) = g_i(x_{-1i}, h) + x_{1i}$, where x_{-1i} is consumer i's consumption of traded goods other than good 1. Then, the consumer's Walrasian demand function for these $L-1$ traded goods, $x_{-1i}(\cdot)$, is independent of her wealth, and $v_i(p, w_i, h) = g_i(x_{-1i}(p, h), h) - p \cdot x_{-1i}(p, h) + w_i$. Thus, denoting $\phi_i(p, h) = g_i(x_{-1i}(p, h), h) - p \cdot x_{-1i}(p, h)$, we have obtained the desired form.

4. Recall the reasoning of Sections 10.D and 10.E, or note that at any Pareto optimal allocation in which h° is the level of h and w_i is consumer i's wealth level for $i = 1, 2$, it must be impossible to change h and reallocate wealth so as to make one consumer better off without making the other worse off. Thus, $(h^\circ, 0)$ must solve $\text{Max}_{h,T}\ \phi_1(h) + w_1 - T$ subject to $\phi_2(h) + w_2 + T \geq \bar{u}_2$, for some $\bar{u}_2$. Because the constraint holds with equality in any solution to this problem, substituting from the constraint for T in the objective function shows that h° must maximize the joint surplus of the two consumers $\phi_1(h) + \phi_2(h)$.

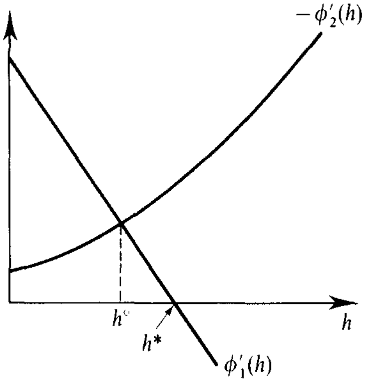

Figure 11.B.1
The equilibrium (h^*) and optimal (h°) levels of a negative externality.

a negative externality, then we have $\phi_1'(h^\circ) = -\phi_2'(h^\circ) > 0$; because $\phi_1'(\cdot)$ is decreasing and $\phi_1'(h^*) = 0$, this implies that $h^* > h^\circ$. In contrast, when $\phi_2'(\cdot) > 0$, h represents a positive externality, and $\phi_1'(h^\circ) = -\phi_2'(h^\circ) < 0$ implies that $h^* < h^\circ$.

Figure 11.B.1 depicts the solution for a case in which h constitutes a negative external effect, so that $\phi_2'(h) < 0$ at all h. In the figure, we graph $\phi_1'(\cdot)$ and $-\phi_2'(\cdot)$. The competitive equilibrium level of the externality h^* occurs at the point where the graph of $\phi_1'(\cdot)$ crosses the horizontal axis. In contrast, the optimal externality level h° corresponds to the point of intersection between the graphs of the two functions.

Note that optimality does not usually entail the complete elimination of a negative externality. Rather, the externality's level is adjusted to the point where the marginal benefit to consumer 1 of an additional unit of the externality-generating activity, $\phi_1'(h^\circ)$, equals its marginal cost to consumer 2, $-\phi_2'(h^\circ)$.

In the current example, quasilinear utilities lead the optimal level of the externality to be independent of the consumers' wealth levels. In the absence of quasilinearity, however, wealth effects for the consumption of the externality make its optimal level depend on the consumers' wealths. See Exercise 11.B.2 for an illustration. Note, however, that when the agents under consideration are firms, wealth effects are always absent.

Traditional Solutions to the Externality Problem

Having identified the inefficiency of the competitive market outcome in the presence of an externality, we now consider three possible solutions to the problem. We first look at government-implemented quotas and taxes, and then analyze the possibility that an efficient outcome can be achieved in a much less intrusive manner by simply fostering bargaining between the consumers over the extent of the externality.

Quotas and taxes

To fix ideas, suppose that h generates a negative external effect, so that $h^\circ < h^*$. The most direct sort of government intervention to achieve efficiency is direct control of the externality-generating activity itself. The government can simply mandate that h be no larger than h°, its optimal level. With this constraint, consumer 1 will indeed fix the level of the externality at h°.

A second option is for the government to attempt to restore optimality by imposing a tax on the externality-generating activity. This solution is known as

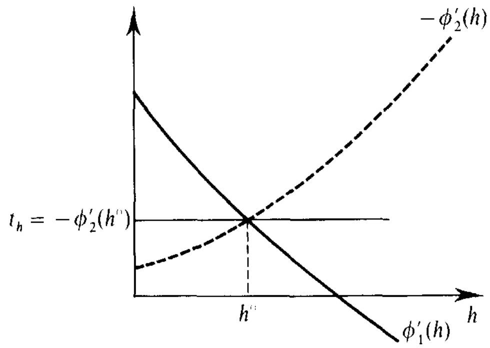

Figure 11.B.2
The optimality-restoring Pigouvian tax.

Pigouvian taxation, after Pigou (1932). To this effect, suppose that consumer 1 is made to pay a tax of t_h per unit of h. It is then not difficult to see that a tax of

$$t_h = -\phi_2'(h^\circ) > 0$$

will implement the optimal level of the externality. Indeed, consumer 1 will then choose the level of h that solves

$$\underset{h \geq 0}{\text{Max}} \quad \phi_1(h) - t_h h, \tag{11.B.3}$$

which has the necessary and sufficient first-order condition

$$\phi_1'(h) \leq t_h, \quad \text{with equality if } h > 0. \tag{11.B.4}$$

Given $t_h = -\phi_2'(h^\circ)$, $h = h^\circ$ satisfies condition (11.B.4) [recall that h° is defined by the condition: $\phi_1'(h^\circ) \leq -\phi_2'(h^\circ)$, with equality if $h^\circ > 0$]. Moreover, given $\phi_1''(\cdot) < 0$, h° must be the unique solution to problem (11.B.3). Figure 11.B.2 illustrates this solution for a case in which $h^\circ > 0$.

Note that the optimality-restoring tax is exactly equal to the *marginal externality* at the optimal solution.[5] That is, it is exactly equal to the amount that consumer 2 would be willing to pay to reduce h slightly from its optimal level h°. When faced with this tax, consumer 1 is effectively led to carry out an individual cost benefit computation that *internalizes* the externality that she imposes on consumer 2.

The principles for the case of a positive externality are exactly the same, only now when we set $t_h = -\phi_2'(h^\circ) < 0$, t_h takes the form of a per-unit *subsidy* (i.e., consumer 1 *receives* a payment for each unit of the externality she generates).

Several additional points are worth noting about this Pigouvian solution. First, we can actually achieve optimality either by taxing the externality or by subsidizing its reduction. Consider, for example, the case of a negative externality. Suppose the government pays a subsidy of $s_h = -\phi_2'(h^\circ) > 0$ for every unit that consumer 1's choice of h is below h^*, its level in the competitive equilibrium. If so, then consumer 1 will maximize $\phi_1(h) + s_h(h^* - h) = \phi_1(h) - t_h h + t_h h^*$. But this is equivalent to a tax of t_h per unit on h combined with a lump-sum payment of $t_h h^*$. Hence, a subsidy for the reduction of the externality combined with a lump-sum transfer can exactly replicate the outcome of the tax.

Second, a point implicit in the derivation above is that, in general, it is essential

5. In the case where $h^\circ = 0$, any tax greater than $-\phi_2'(0)$ also implements the optimal outcome.

to tax the externality-producing activity directly. For instance, suppose that, in the example of consumer 1 playing loud music, we tax purchases of music equipment instead of taxing the playing of loud music itself. In general, this will not restore optimality. Consumer 1 will be led to lower her consumption of music equipment (perhaps she will purchase only a CD player, rather than a CD player and a tape player) but may nevertheless play whatever equipment she does purchase too loudly. A common example of this sort arises when a firm pollutes in the process of producing output. A tax on its output leads the firm to reduce its output level but may not have any effect (or, more generally, may have too little effect) on its pollution emissions. Taxing output achieves optimality only in the special case in which emissions bear a fixed monotonic relationship to the level of output. In this special case, emissions can be measured by the level of output, and a tax on output is essentially equivalent to a tax on emissions. (See Exercise 11.B.5 for an illustration.)

Third, note that the tax/subsidy and the quota approaches are equally effective in achieving an optimal outcome. However, the government must have a great deal of information about the benefits and costs of the externality for the two consumers to set the optimal levels of either the quota or the tax. In Section 11.E we will see that when the government does not possess this information the two approaches typically are not equivalent.

Fostering bargaining over externalities: enforceable property rights

Another approach to the externality problem aims at a less intrusive form of intervention, merely seeking to insure that conditions are met for the parties to themselves reach an optimal agreement on the level of the externality.

Suppose that we establish enforceable property rights with regard to the externality-generating activity. Say, for example, that we assign the right to an "externality-free" environment to consumer 2. In this case, consumer 1 is unable to engage in the externality-producing activity without consumer 2's permission. For simplicity, imagine that the bargaining between the parties takes a form in which consumer 2 makes consumer 1 a take-it-or-leave-it offer, demanding a payment of T in return for permission to generate externality level h.[6] Consumer 1 will agree to this demand if and only if she will be at least as well off as she would be by rejecting it, that is, if and only if $\phi_1(h) - T \geq \phi_1(0)$. Hence, consumer 2 will choose her offer (h, T) to solve

$$\begin{aligned} \underset{h \geq 0,\, T}{\text{Max}} \quad & \phi_2(h) + T \\ \text{s.t.} \quad & \phi_1(h) - T \geq \phi_1(0). \end{aligned}$$

Since the constraint is binding in any solution to this problem, $T = \phi_1(h) - \phi_1(0)$. Therefore, consumer 2's optimal offer involves the level of h that solves

$$\underset{h \geq 0}{\text{Max}} \quad \phi_2(h) + \phi_1(h) - \phi_1(0). \tag{11.B.5}$$

But this is precisely h^o, the socially optimal level.

Note, moreover, that the precise allocation of these rights between the two

6. Either of the bargaining processes discussed in Appendix A of Chapter 9 would yield the same conclusions.

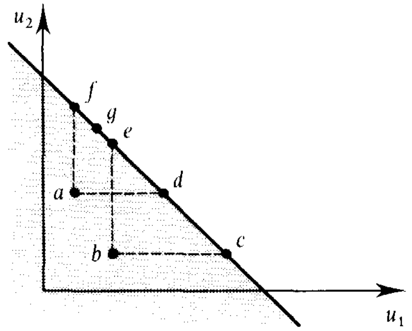

Figure 11.B.3
The final distribution of utilities under different property rights institutions and different bargaining procedures.

consumers is inessential to the achievement of optimality. Suppose, for example, that consumer 1 instead has the right to generate as much of the externality as she wants. In this case, in the absence of any agreement, consumer 1 will generate externality level h^*. Now consumer 2 will need to offer a $T < 0$ (i.e., to pay consumer 1) to have $h < h^*$. In particular, consumer 1 will agree to externality level h if and only if $\phi_1(h) - T \geq \phi_1(h^*)$. As a consequence, consumer 2 will offer to set h at the level that solves $\text{Max}_{h \geq 0}\, (\phi_2(h) + \phi_1(h) - \phi_1(h^*))$. Once again, the optimal externality level h° results. The allocation of rights affects only the final wealth of the two consumers by altering the payment made by consumer 1 to consumer 2. In the first case, consumer 1 pays $\phi_1(h^\circ) - \phi_1(0) > 0$ to be allowed to set $h^\circ > 0$; in the second, she "pays" $\phi_1(h^\circ) - \phi_1(h^*) < 0$ in return for setting $h^\circ < h^*$.

We have here an instance of what is known as the *Coase theorem* [for Coase (1960)]: If trade of the externality can occur, then bargaining will lead to an efficient outcome no matter how property rights are allocated.

All this is illustrated in Figure 11.B.3, in which we represent the utility possibility set for the two consumers. Every point in the boundary of this set corresponds to an allocation with externality level h°. The points a and b correspond to the utility levels arising, respectively, from externality levels 0 and h^* in the absence of any transfers. They constitute the initial situation after the assignment of property rights (to consumers 2 and 1, respectively) but before bargaining. In the particular bargaining procedure we have adopted (which gives the power to make a take-it-or-leave-it offer to consumer 2), the utility levels after bargaining are points f and e, respectively. If the bargaining power (i.e., the power to make the take-it-or-leave-it offer) had been instead in the hands of consumer 1, the post-bargaining utility levels would have been points d and c, respectively. Other bargaining procedures (such as the ones studied in Appendix A of Chapter 9) may yield other points in the segments $[f, d]$ and $[e, c]$, respectively.

Note that the existence of both well-defined and enforceable property rights is essential for this type of bargaining to occur. If property rights are not well defined, it will be unclear whether consumer 1 must gain consumer 2's permission to generate the externality. If property rights cannot be enforced (perhaps the level of h is not easily measured), then consumer 1 has no need to purchase the right to engage in the externality-generating activity from consumer 2. For this reason, proponents of this type of approach focus on the absence of these legal institutions as a central impediment to optimality.

This solution to the externality problem has a significant advantage over the tax and quota schemes in terms of the level of knowledge required of the government. The consumers must know each other's preferences, but the government need not. We should emphasize, however, that for bargaining over the externality to lead to efficiency, it is important that the consumers know this information. In Section 11.E, we will see that when the agents are to some extent ignorant of each others' preferences, bargaining need *not* lead to an efficient outcome.

Two further points regarding these three types of solutions to the externality problem are worthy of note. First, in the case in which the two agents are firms, one form that an efficient bargain might take is the sale of one of the firms to the other. The resulting merged firm would then fully internalize the externality in the process of maximizing its profits.[7]

Second, note that all three approaches require that the externality-generating activity be measureable. This is not a trivial requirement; in many cases, such measurement may be either technologically infeasible or very costly (consider the cost of measuring air pollution or noise). A proper computation of costs and benefits should take these costs into account. If measurement is very costly, then it may be optimal to simply allow the externality to persist.

Externalities and Missing Markets

The observation that bargaining can generate an optimal outcome suggests a connection between externalities and missing markets. After all, a market system can be viewed as a particular type of trading procedure.

Suppose that property rights are well defined and enforceable and that a competitive market for the right to engage in the externality-generating activity exists. For simplicity, we assume that consumer 2 has the right to an externality-free environment. Let p_h denote the price of the right to engage in one unit of the activity. In choosing how many of these rights to purchase, say h_1, consumer 1 will solve

$$\underset{h_1 \geq 0}{\text{Max}}\ \phi_1(h_1) - p_h h_1,$$

which has the first-order condition

$$\phi_1'(h_1) \leq p_h, \quad \text{with equality if } h_1 > 0. \tag{11.B.6}$$

In deciding how many rights to sell, h_2, consumer 2 will solve

$$\underset{h_2 > 0}{\text{Max}}\ \ \phi_2(h_2) + p_h h_2,$$

which has the first-order condition

$$\phi_2'(h_2) \leq -p_h, \quad \text{with equality if } h_2 > 0. \tag{11.B.7}$$

7. Note, however, that this conclusion presumes that the owner of a firm has full control over all its functions. In more complicated (but realistic) settings in which this is not true, say because owners must hire managers whose actions cannot be perfectly controlled, the results of a merger and of an agreement over the level of the externality need not be the same. Chapters 14 and 23 provide an introduction to the topic of incentive design. See Holmstrom and Tirole (1989) for a discussion of these issues in the theory of the firm.

In a competitive equilibrium, the market for these rights must clear; that is, we must have $h_1 = h_2$. Hence, (11.B.6) and (11.B.7) imply that the level of rights traded in this competitive rights market, say h^{**}, satisfies

$$\phi_1'(h^{**}) \leq -\phi_2'(h^{**}), \quad \text{with equality if } h^{**} > 0.$$

Comparing this expression with (11.B.2), we see that h^{**} equals the optimal level h°. The equilibrium price of the externality is $p_h^* = \phi_1'(h^\circ) = -\phi_2'(h^\circ)$.

Consumer 1 and 2's equilibrium utilities are then $\phi_1(h^\circ) - p_h^* h^\circ$ and $\phi_2(h^\circ) + p_h^* h^\circ$, respectively. The market therefore works as a particular bargaining procedure for splitting the gains from trade; for example, point g in Figure 11.B.3 could represent the utilities in the competitive equilibrium.

We see that if a competitive market exists for the externality, then optimality results. Thus, externalities can be seen as being inherently tied to the absence of certain competitive markets, a point originally noted by Meade (1952) and substantially extended by Arrow (1969). Indeed, recall that our original definition of an externality, Definition 11.B.1, explicitly required that an action chosen by one agent must *directly* affect the well-being or production capabilities of another. Once a market exists for an externality, however, each consumer decides for herself how much of the externality to consume at the going prices.

Unfortunately, the idea of a competitive market for the externality in the present example is rather unrealistic; in a market with only one seller and one buyer, price taking would be unlikely.[8] However, most important externalities are produced and felt by *many* agents. Thus, we might hope that in these multilateral settings, price taking would be a more reasonable assumption and, as a result, that a competitive market for the externality would lead to an efficient outcome. In Section 11.D, where we study multilateral externalities, we see that the correctness of this conclusion depends on whether the externality is "private" or "public" in nature. Before coming to this, however, we first study the nature of public goods.

11.C Public Goods

In this section, we study commodities that, in contrast with those considered so far, have a feature of "publicness" to their consumption. These commodities are known as *public goods*.

Definition 11.C.1: A *public good* is a commodity for which use of a unit of the good by one agent does not preclude its use by other agents.

Put somewhat differently, public goods possess the feature that they are *nondepletable*: Consumption by one individual does not affect the supply available for other individuals. Knowledge provides a good illustration. The use of a piece of knowledge for one purpose does not preclude its use for others. In contrast, the commodities studied up to this point have been assumed to be of a *private*, or *depletable*, nature;

8. For that matter, the idea that the externality rights are all sold at the same price lacks justification here, because there is no natural unit of measurement for the externality.

that is, for each additional unit consumed by individual i, there is one unit less available for individuals $j \neq i$.[9]

A distinction can also be made according to whether *exclusion* of an individual from the benefits of a public good is possible. Every private good is automatically excludable, but public goods may or may not be. The patent system, for example, is a mechanism for excluding individuals (although imperfectly) from the use of knowledge developed by others. On the other hand, it might be technologically impossible, or at the least very costly, to exclude some consumers from the benefits of national defense or of a project to improve air quality. For simplicity, our discussion here will focus primarily on the case in which exclusion is not possible.

Note that a public "good" need not necessarily be desirable; that is, we may have public *bads* (e.g., foul air). In this case, we should read the phrase "does not preclude" in Definition 11.C.1 to mean "does not decrease."

Conditions for Pareto Optimality

Consider a setting with I consumers and one public good, in addition to L traded goods of the usual, private, kind. We again adopt a partial equilibrium perspective by assuming that the quantity of the public good has no effect on the prices of the L traded goods and that each consumer's utility function is quasilinear with respect to the same numeraire, traded commodity. As in Section 11.B, we can therefore define, for each consumer i, a derived utility function over the level of the public good. Letting x denote the quantity of the public good, we denote consumer i's utility from the public good by $\phi_i(x)$. We assume that this function is twice differentiable, with $\phi_i''(x) < 0$ at all $x \geq 0$. Note that precisely because we are dealing with a public good, the argument x does not have an i subscript.

The cost of supplying q units of the public good is $c(q)$. We assume that $c(\cdot)$ is twice differentiable, with $c''(q) > 0$ at all $q \geq 0$.

To describe the case of a desirable public good whose production is costly, we take $\phi_i'(\cdot) > 0$ for all i and $c'(\cdot) > 0$. Except where otherwise noted, however, the analysis applies equally well to the case of a public bad whose reduction is costly, where $\phi_i'(\cdot) < 0$ for all i and $c'(\cdot) < 0$.

In this quasilinear model, any Pareto optimal allocation must maximize aggregate surplus (see Section 10.D) and therefore must involve a level of the public good that solves

$$\underset{q \geq 0}{\text{Max}} \quad \sum_{i=1}^{I} \phi_i(q) - c(q).$$

The necessary and sufficient first-order condition for the optimal quantity q° is then

$$\sum_{i=1}^{I} \phi_i'(q^\circ) \leq c'(q^\circ), \quad \text{with equality if } q^\circ > 0. \tag{11.C.1}$$

Condition (11.C.1) is the classic optimality condition for a public good first derived by Samuelson (1954; 1955). (Here it is specialized to the partial equilibrium setting;

9. Intermediate cases are also possible in which the consumption of the good by one individual affects to some degree its availability to others. A classic example is the presence of congestion effects. For this reason, goods for which there is no depletability whatsoever are sometimes referred to as *pure* public goods.

see Section 16.G for a more general treatment.) At an interior solution, we have $\sum_i \phi_i'(q^\circ) = c'(q^\circ)$, so that at the optimal level of the public good *the sum of consumers' marginal benefits from the public good is set equal to its marginal cost.* This condition should be contrasted with conditions (10.D.3) to (10.D.5) for a private good, where *each* consumer's marginal benefit from the good is equated to its marginal cost.

Inefficiency of Private Provision of Public Goods

Consider the circumstance in which the public good is provided by means of private purchases by consumers. We imagine that a market exists for the public good and that each consumer i chooses how much of the public good to buy, denoted by $x_i \geq 0$, taking as given its market price p. The total amount of the public good purchased by consumers is then $x = \sum_i x_i$. Formally, we treat the supply side as consisting of a single profit-maximizing firm with cost function $c(\cdot)$ that chooses its production level taking the market price as given. Note, however, that by the analysis of Section 5.E, we can actually think of the supply behavior of this firm as representing the industry supply of J price-taking firms whose aggregate cost function is $c(\cdot)$.

At a competitive equilibrium involving price p^*, each consumer i's purchase of the public good x_i^* must maximize her utility and so must solve

$$\underset{x_i \geq 0}{\text{Max}} \quad \phi_i(x_i + \sum_{k \neq i} x_k^*) - p^* x_i. \tag{11.C.2}$$

In determining her optimal purchases, consumer i takes as given the amount of the private good being purchased by each other consumer (as in the Nash equilibrium concept studied in Section 8.D). Consumer i's purchases x_i^* must therefore satisfy the necessary and sufficient first-order condition

$$\phi_i'(x_i^* + \sum_{k \neq i} x_k^*) \leq p^*, \quad \text{with equality if } x_i^* > 0.$$

Letting $x^* = \sum_i x_i^*$ denote the equilibrium level of the public good, for each consumer i we must therefore have

$$\phi_i'(x^*) \leq p^*, \quad \text{with equality if } x_i^* > 0. \tag{11.C.3}$$

The firm's supply q^*, on the other hand, must solve $\text{Max}_{q \geq 0}\,(p^* q - c(q))$ and therefore must satisfy the standard necessary and sufficient first-order condition

$$p^* \leq c'(q^*), \quad \text{with equality if } q^* > 0. \tag{11.C.4}$$

At a competitive equilibrium, $q^* = x^*$. Thus, letting $\delta_i = 1$ if $x_i^* > 0$ and $\delta_i = 0$ if $x_i^* = 0$, (11.C.3) and (11.C.4) tell us that $\sum_i \delta_i[\phi_i'(q^*) - c'(q^*)] = 0$. Recalling that $\phi_i'(\cdot) > 0$ and $c'(\cdot) > 0$, this implies that whenever $I > 1$ and $q^* > 0$ (so that $\delta_i = 1$ for some i) we have

$$\sum_{i=1}^{I} \phi_i'(q^*) > c'(q^*). \tag{11.C.5}$$

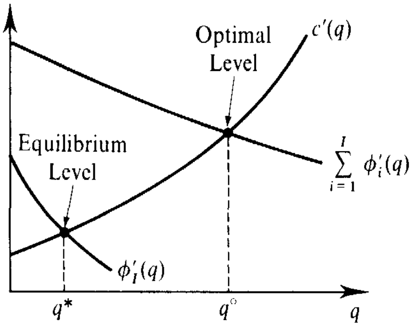

Figure 11.C.1
Private provision leads to an insufficient level of a desirable public good.

Comparing (11.C.5) with (11.C.1), we see that whenever $q^\circ > 0$ and $I > 1$, the level of the public good provided is too low; that is, $q^* < q^\circ$.[10]

The cause of this inefficiency can be understood in terms of our discussion of externalities in Section 11.B. Here each consumer's purchase of the public good provides a direct benefit not only to the consumer herself but also to every other consumer. Hence, private provision creates a situation in which externalities are present. The failure of each consumer to consider the benefits for others of her public good provision is often referred to as the *free-rider problem*: Each consumer has an incentive to enjoy the benefits of the public good provided by others while providing it insufficiently herself.

In fact, in the present model, the free-rider problem takes a very stark form. To see this most simply, suppose that we can order the consumers according to their marginal benefits, in the sense that $\phi_1'(x) < \cdots < \phi_I'(x)$ at all $x \geq 0$. Then condition (11.C.3) can hold with equality only for a *single* consumer and, moreover, this must be the consumer labeled I. Therefore, only the consumer who derives the largest (marginal) benefit from the public good will provide it; all others will set their purchases equal to zero in the equilibrium. The equilibrium level of the public good is then the level q^* that satisfies $\phi_I'(q^*) = c'(q^*)$. Figure 11.C.1 depicts both this equilibrium and the Pareto optimal level. Note that the curve representing $\sum_i \phi_i'(q)$ geometrically corresponds to a *vertical* summation of the individual curves representing $\phi_i(q)$ for $i = 1, \ldots, I$ (whereas in the case of a private good, the market demand curve is identified by adding the individual demand curves *horizontally*).

The inefficiency of private provision is often remedied by governmental intervention in the provision of public goods. Just as with externalities, this can happen not only through quantity-based intervention (such as direct governmental provision) but also through "price-based" intervention in the form of taxes or subsidies. For example, suppose that there are two consumers with benefit functions $\phi_1(x_1 + x_2)$ and $\phi_2(x_1 + x_2)$, where x_i is the amount of the public good purchased by consumer i, and that $q^\circ > 0$. By analogy with the analysis in Section 11.B, a subsidy to each consumer i per unit purchased of $s_i = \phi'_{-i}(q^\circ)$ [or, equivalently, a tax of $-\phi'_{-i}(q^\circ)$ per unit that consumer i's purchases of the public good fall below some specified

10. The conclusion follows immediately if $q^* = 0$. So suppose instead that $q^* > 0$. Then since $\sum_i \phi_i'(q^*) - c'(q^*) > 0$ and $\sum_i \phi_i'(\cdot) - c'(\cdot)$ is decreasing, any solution to (11.C.1) must have a larger value than q^*. Note that, in contrast, if we are dealing with a public bad, so that $\phi_i'(\cdot) < 0$ and $c'(\cdot) < 0$, then the inequalities reverse and $q^\circ < q^*$.

level] faces each consumer with the marginal external effect of her actions and so generates an optimal level of public good provision by consumer i. Formally, if $(\tilde{x}_1, \tilde{x}_2)$ are the competitive equilibrium levels of the public good purchased by the two consumers given these subsidies, and if $\tilde{p}$ is the equilbrium price, then consumer i's purchases of the public good, $\tilde{x}_i$, must solve $\text{Max}_{x_i \geq 0}\ \phi_i(x_i + \tilde{x}_j) + s_i x_i - \tilde{p} x_i$, and so $\tilde{x}_i$ must satisfy the necessary and sufficient first-order condition

$$\phi_i'(\tilde{x}_1 + \tilde{x}_2) + s_i \leq \tilde{p}, \text{ with equality of } \tilde{x}_i > 0.$$

Substituting for s_i, and using both condition (11.C.4) and the market-clearing condition that $\tilde{x}_1 + \tilde{x}_2 = \tilde{q}$, we conclude that $\tilde{q}$ is the total amount of the public good in the competitive equilibrium given these subsidies if and only if

$$\phi_i'(\tilde{q}) + \phi_{-i}'(q^\circ) \leq c'(\tilde{q}),$$

with equality for some i if $\tilde{q} > 0$. Recalling (11.C.1) we see that $\tilde{q} = q^\circ$. (Exercise 11.C.1 asks you to extend this argument to the case where $I > 2$; formally, we then have a multilateral externality of the sort studied in Section 11.D.)

Note that both optimal direct public provision and this subsidy scheme require that the government know the benefits derived by consumers from the public good (i.e., their willingness to pay in terms of private goods). In Section 11.E, we study the case in which this is not so.

Lindahl Equilibria

Although private provision of the sort studied above results in an inefficient level of the public good, there is *in principle* a market institution that can achieve optimality. Suppose that, for each consumer i, we have a market for the public good "as experienced by consumer i." That is, we think of each consumer's consumption of the public good as a distinct commodity with its own market. We denote the price of this personalized good by p_i. Note that p_i may differ across consumers. Suppose also that, given the equilibrium price p_i^{**}, each consumer i sees herself as deciding on the *total amount of the public good she will consume*, x_i, so as to solve

$$\underset{x_i > 0}{\text{Max}} \quad \phi_i(x_i) - p_i^{**} x_i.$$

Her equilibrium consumption level x_i^{**} must therefore satisfy the necessary and sufficient first-order condition

$$\phi_i'(x_i^{**}) \leq p_i^{**}, \quad \text{with equality if } x_i^{**} > 0. \tag{11.C.6}$$

The firm is now viewed as producing a bundle of I goods with a fixed-proportions technology (i.e., the level of production of each personalized good is necessarily the same). Thus, the firm solves

$$\underset{q \geq 0}{\text{Max}} \quad \left(\sum_{i=1}^{I} p_i^{**} q \right) - c(q).$$

The firm's equilibrium level of output q^{**} therefore satisfies the necessary and sufficient first-order condition

$$\sum_{i=1}^{I} p_i^{**} \leq c'(q^{**}), \quad \text{with equality if } q^{**} > 0. \tag{11.C.7}$$

Together, (11.C.6), (11.C.7), and the market-clearing condition that $x_i^{**} = q^{**}$ for all i imply that

$$\sum_{i=1}^{I} \phi_i'(q^{**}) \leq c'(q^{**}), \quad \text{with equality if } q^{**} > 0. \tag{11.C.8}$$

Comparing (11.C.8) with (11.C.1), we see that the equilibrium level of the public good consumed by each consumer is exactly the efficient level: $q^{**} = q^{\circ}$.

This type of equilibrium in personalized markets for the public good is known as a *Lindahl equilibrium*, after Lindahl (1919). [See also Milleron (1972) for a further discussion.] To understand why we obtain efficiency, note that once we have defined personalized markets for the public good, each consumer, taking the price in her personalized market as given, fully determines her own level of consumption of the public good; externalities are eliminated.

Yet, despite the attractive properties of Lindahl equilibria, their realism is questionable. Note, first, that the ability to exclude a consumer from use of the public good is essential if this equilibrium concept is to make sense; otherwise a consumer would have no reason to believe that in the absence of making any purchases of the public good she would get to consume none of it.[11] Moreover, even if exclusion is possible, these are markets with only a single agent on the demand side. As a result, price-taking behavior of the sort presumed is unlikely to occur.

The idea that inefficiencies can in principle be corrected by introducing the right kind of markets, encountered here and in Section 11.B, is a very general one. In particular cases, however, this "solution" may or may not be a realistic possibility. We encounter this issue again in our study of multilateral externalities in Section 11.D. As we shall see, these types of externalities often share many of the features of public goods.

11.D Multilateral Externalities

In most cases, externalities are felt and generated by numerous parties. This is particularly true of those externalities, such as industrial pollution, smog caused by automobile use, or congestion, that are widely considered to be "important" policy problems. In this section, we extend our analysis of externalities to these multilateral settings.

An important distinction can be made in the case of multilateral externalities according to whether the externality is *depletable* (or *private*, or *rivalrous*) or *nondepletable* (or *public*, or *nonrivalrous*). Depletable externalities have the feature that experience of the externality by one agent reduces the amount that will be felt by other agents. For example, if the externality takes the form of the dumping of garbage on people's property, if an additional unit of garbage is dumped on one

11. Thus, the possibility of exclusion can be important for efficient supply of the public good, even though the use of an exclusion technology is itself inefficient (a Pareto optimal allocation cannot involve any exclusion).

piece of property, that much less is left to be dumped on others.[12] Depletable externalities therefore share the characteristics of our usual (private) sort of commodity. In contrast, air pollution is a nondepletable externality; the amount of air pollution experienced by one agent is not affected by the fact that others are also experiencing it. Nondepletable externalities therefore have the characteristics of public goods (or bads).

In this section we argue that a decentralized market solution can be expected to work well for multilateral depletable externalities as long as well-defined and enforceable property rights can be created. In contrast, market-based solutions are unlikely to work in the nondepletable case, in parallel to our conclusions regarding public goods in Section 11.C.

We shall assume throughout this section that the agents who generate externalities are distinct from those who experience them. This simplification is inessential but eases the exposition and facilitates comparison with the previous sections (Exercise 11.D.2 asks you to consider the general case). For ease of reference, we assume here that the generators of the externality are firms and that those experiencing the externality are consumers. We also focus on the special, but central, case in which the externality generated by the firms is homogeneous (i.e., consumers are indifferent to the source of the externality). (Exercise 11.D.4 asks you to consider the case in which the source matters.)

We again adopt a partial equilibrium approach and assume that agents take as given the price vector p of L traded goods. There are J firms that generate the externality in the process of production. As discussed in Section 11.B, given price vector p, we can determine firm j's derived profit function over the level of the externality it generates, $h_j \geq 0$, which we denote by $\pi_j(h_j)$. There are also I consumers, who have quasilinear utility functions with respect to a numeraire, traded commodity. Given price vector p, we denote by $\phi_i(\tilde{h}_i)$ consumer i's derived utility function over the amount of the externality $\tilde{h}_i$ she experiences. We assume that $\pi_j(\cdot)$ and $\phi_i(\cdot)$ are twice differentiable with $\pi_j''(\cdot) < 0$ and $\phi_i''(\cdot) < 0$. To fix ideas, we shall focus on the case where $\phi_i'(\cdot) < 0$ for all i, so that we are dealing with a negative externality.

Depletable Externalities

We begin by examining the case of depletable externalities. As in Section 11.B, it is easy to see that the level of the (negative) externality is excessive at an unfettered competitive equilibrium. Indeed, at any competitive equilibrium, each firm j will wish to set the externality-generating activity at the level h_j^* satisfying the condition

$$\pi_j'(h_j^*) \leq 0, \quad \text{with equality if } h_j^* > 0.\text{[13]} \tag{11.D.1}$$

In contrast, any Pareto optimal allocation involves the levels $(\tilde{h}_1^\circ, \ldots, \tilde{h}_I^\circ, h_1^\circ, \ldots, h_J^\circ)$

12. A distinction can also be made as to whether a depletable externality is *allocable*. For example, acid rain is depletable in the sense that the total amount of chemicals put into the air will fall somewhere, but it is not readily allocable because where it falls is determined by weather patterns. Throughout this section, we take depletable externalities to be allocable. The analytical implications of nonallocable depletable externalities parallel those of nondepletable ones.

13. The firms are indifferent about which consumer is affected by their externality. Therefore, apart from the fact that $\sum_i \tilde{h}_i = \sum_j h_j^*$, the particular values of the individual $\tilde{h}_i$'s are indeterminate.

that solve[14]

$$\begin{aligned}\underset{\substack{(h_1,\ldots,h_J)\geq 0\\(\tilde{h}_1,\ldots,\tilde{h}_I)\geq 0}}{\text{Max}} \quad & \sum_{i=1}^{I} \phi_i(\tilde{h}_i) + \sum_{j=1}^{J} \pi_j(h_j) \\ \text{s.t.} \quad & \sum_{j=1}^{J} h_j = \sum_{i=1}^{I} \tilde{h}_i.\end{aligned} \tag{11.D.2}$$

The constraint in (11.D.2) reflects the depletability of the externality: If $\tilde{h}_i$ is increased by one unit, there is one unit less of the externality that needs to be experienced by others. Letting μ be the multiplier on this constraint, the necessary and sufficient first-order conditions to problem (11.D.2) are

$$\phi_i'(\tilde{h}_i^\circ) \leq \mu, \quad \text{with equality if } \tilde{h}_i^\circ > 0,\ i = 1,\ldots,I, \tag{11.D.3}$$

and

$$\mu \leq -\pi_j'(h_j^\circ), \quad \text{with equality if } h_j^\circ > 0,\ j = 1,\ldots,J. \tag{11.D.4}$$

Conditions (11.D.3) and (11.D.4), along with the constraint in problem (11.D.2), characterize the optimal levels of externality generation and consumption. Note that they exactly parallel the efficiency conditions for a private good derived in Chapter 10, conditions (10.D.3) to (10.D.5), where we interpret $-\pi_j'(\cdot)$ as firm j's marginal cost of producing more of the externality. If well-defined and enforceable property rights can be specified over the externality, and if I and J are large numbers so that price taking is a reasonable hypothesis, then by analogy with the analysis of competitive markets for private goods in Chapter 10, a market for the externality can be expected to lead to the optimal levels of externality production and consumption in the depletable case.

Nondepletable Externalities

We now move to the case in which the externality is nondepletable. To be specific, assume that the level of the externality experienced by *each* consumer is $\sum_j h_j$, the total amount of the externality produced by the firms.

In an unfettered competitive equilibrium, each firm j's externality generation h_j^* again satisfies condition (11.D.1). In contrast, any Pareto optimal allocation involves externality generation levels $(h_1^\circ,\ldots,h_J^\circ)$ that solve

$$\underset{(h_1,\ldots,h_J)\geq 0}{\text{Max}} \quad \sum_{i=1}^{I} \phi_i(\textstyle\sum_j h_j) + \displaystyle\sum_{j=1}^{J} \pi_j(h_j). \tag{11.D.5}$$

This problem has necessary and sufficient first-order conditions for each firm j's optimal level of externality generation, h_j°, of

$$\sum_{i=1}^{I} \phi_i'(\textstyle\sum_j h_j^\circ) \leq -\pi_j'(h_j^\circ), \quad \text{with equality if } h_j^\circ > 0. \tag{11.D.6}$$

14. The objective function in (11.D.2) amounts to the usual difference between benefits and costs arising in the aggregate surplus measure. Note, to this effect, that $-\pi_j(\cdot)$ can be viewed as firm j's cost function for producing the externality.

Condition (11.D.6) is exactly analogous to the optimality condition for a public good, condition (11.C.1), where $-\pi_j'(\cdot)$ is firm j's marginal cost of externality production.[15]

By analogy with our discussion of private provision of public goods in Section 11.C, the introduction of a standard sort of market for the externality will *not* lead here, as it did in the bilateral case of Section 11.B, to an optimal outcome. The free-rider problem reappears, and the equilibrium level of the (negative) externality will exceed its optimal level. Instead, in the case of a multilateral nondepletable externality, a market-based solution would require personalized markets for the externality, as in the Lindahl equilibrium concept. However, all the problems with Lindahl equilibrium discussed in Section 11.C will similarly afflict these markets. As a result, purely market-based solutions, personalized or not, are unlikely to work in the case of a depletable externality.[16]

In contrast, given adequate information (a strong assumption!), the government *can* achieve optimality using quotas or taxes. With quotas, the government simply sets an upper bound on each firm j's level of externality generation equal to its optimal level h_j°. On the other hand, as in Section 11.B, optimality-restoring taxes face each firm with the marginal social cost of their externality. Here the optimal tax is identical for each firm and is equal to $t_h = -\sum_i \phi_i'(\sum_j h_j^\circ)$ per unit of the externality generated. Given this tax, each firm j solves

$$\underset{h_j \geq 0}{\text{Max}} \quad \pi_j(h_j) - t_h h_j,$$

which has the necessary and sufficient first-order condition

$$\pi_j'(h_j) \leq t_h, \quad \text{with equality if } h_j > 0.$$

Given $t_h = -\sum_i \phi_i'(\sum_j h_j^\circ)$, firm j's optimal choice is $h_j = h_j^\circ$.

A partial market-based approach that can achieve optimality with a nondepletable multilateral externality involves specification of a quota on the *total* level of the externality and distribution of that number of *tradeable externality permits* (each permit grants a firm the right to generate one unit of the externality). Suppose that $h^\circ = \sum_j h_j^\circ$ permits are given to the firms, with firm j receiving $\bar{h}_j$ of them. Let p_h^* denote the equilibrium price of these permits. Then each firm j's demand for permits, h_j, solves $\text{Max}_{h_j \geq 0}\,(\pi_j(h_j) + p_h^*(\bar{h}_j - h_j))$ and so satisfies the necessary and sufficient first-order condition $\pi_j'(h_j) \leq p_h^*$, with equality if $h_j > 0$. In addition, market clearing in the permits market requires that $\sum_j h_j = h^\circ$. The competitive equilibrium in the market for permits then has price $p_h^* = -\sum_i \phi_i'(h^\circ)$ and each firm j using h_j° permits and so yields an optimal allocation. The advantage of this scheme relative to a strict quota method arises when the government has limited information about the $\pi_j(\cdot)$ functions and cannot tell which particular firms can efficiently bear the burden of externality reduction, although it has enough information, perhaps of a statistical sort, to allow the computation of the optimal aggregate level of the externality, h°.

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15. Recall that the single firm's cost function $c(\cdot)$ in Section 11.C could be viewed as the aggregate cost function of J separate profit-maximizing firms. Were we to explicitly model these J firms in Section 11.C, the optimality conditions for public good production would take exactly the form in (11.D.6) with $c_j'(h_j^\circ)$ replacing $-\pi_j'(h_j^\circ)$.

16. The public nature of the externality leads to similar free-rider problems in any bargaining solution. (See Exercise 11.D.6 for an illustration.)

11.E Private Information and Second-Best Solutions

In practice, the degree to which an agent is affected by an externality or benefits from a public good will often be known only to her. The presence of *privately held* (or *asymmetrically held*) information can confound both centralized (e.g., quotas and taxes) and decentralized (e.g., bargaining) attempts to achieve optimality. In this section, we provide an introduction to these issues, focusing for the sake of specificity on the case of a bilateral externality such as that studied in Section 11.B. Following the convention adopted in Section 11.D, we shall assume here that the externality-generating agent is a firm and the affected agent is a consumer. (For a more general treatment of some of the topics covered in this section, see Chapter 23.)

Suppose, then, that we can write the consumer's derived utility function from externality level h (see Section 11.B for more on this construction) as $\phi(h, \eta)$, where $\eta \in \mathbb{R}$ is a parameter, to be called the consumer's *type*, that affects the consumer's costs from the externality. Similarly, we let $\pi(h, \theta)$ denote the firm's derived profit given its type $\theta \in \mathbb{R}$. The actual values of θ and η are *privately observed*: Only the consumer knows her type η, and only the firm observes its type θ. The ex ante likelihoods (probability distributions) of various values of θ and η are, however, publicly known. For convenience, we assume that θ and η are independently distributed. As previously, we assume that $\pi(h, \theta)$ and $\phi(h, \eta)$ are strictly concave in h for any given values of θ and η.

Decentralized Bargaining

Consider the decentralized approach to the externality problem first. In general, bargaining in the presence of bilateral asymmetric information will *not* lead to an efficient level of the externality. To see this, consider again the case in which the consumer has the right to an externality-free environment, and the simple bargaining process in which the consumer makes a take-it-or-leave-it offer to the firm. For simplicity, we assume that there are only two possible levels of the externality, 0 and $\bar{h} > 0$, and we focus on the case of a negative externality in which externality level $\bar{h}$, relative to the level 0, is detrimental for the consumer and beneficial for the firm (the analysis is readily applied to the case of a positive externality).

It is convenient to define $b(\theta) = \pi(\bar{h}, \theta) - \pi(0, \theta) > 0$ as the measure of the firm's benefit from the externality-generating activity when its type is θ. Similarly, we let $c(\eta) = \phi(0, \eta) - \phi(\bar{h}, \eta) > 0$ give the consumer's cost from externality level $\bar{h}$. In this simplified setting, the only aspects of the consumer's and firm's types that matter are the values of b and c that these types generate. Hence, we can focus directly on the various possible values of b and c that the two agents might have. Denote by $G(b)$ and $F(c)$ the distribution functions of these two variables induced by the underlying probability distributions of θ and η (note that, given the independence of θ and η, b and c are independent). For simplicity, we assume that these distributions have associated density functions $g(b)$ and $f(c)$, with $g(b) > 0$ and $f(c) > 0$ for all $b > 0$ and $c > 0$.

Since the consumer has the right to an externality-free environment, in the absence of any agreement with the firm she will always insist that the firm set $h = 0$ (recall that $c > 0$). However, in any arrangement that guarantees Pareto optimal outcomes for all values of b and c, the firm should be allowed to set $h = \bar{h}$ whenever $b > c$.

Now consider the amount that the consumer will demand from the firm when her cost is c in exchange for permission to engage in the externality-generating activity. Since the firm knows that the consumer will insist on $h = 0$ if there is no agreement, the firm will agree to pay the amount T if and only if $b \geq T$. Hence, the consumer knows that if she demands a payment of T, the probability that the firm will accept her offer equals the probability that $b \geq T$; that is, it is equal to $1 - G(T)$. Given her cost $c > 0$ (and assuming risk neutrality), the consumer optimally chooses the value of T she demands to solve

$$\operatorname*{Max}_{T} \quad (1 - G(T))(T - c). \tag{11.E.1}$$

The objective function of problem (11.E.1) is the probability that the firm accepts the demand, multiplied by the net gain to the consumer when this happens $(T - c)$. Under our assumptions, the objective function in (11.E.1) is strictly positive for all $T > c$ and equal to zero when $T = c$. Therefore, the solution, say T_c^*, is such that $T_c^* > c$. But this implies that this bargaining process must result in a strictly positive probability of an inefficient outcome, since whenever the firm's benefit b satisfies $c < b < T_c^*$, the firm will reject the consumer's offer, resulting in an externality level of zero, even though optimality requires that $h = \bar{h}$.[17, 18]

Quotas and Taxes

Just as decentralized bargaining will involve inefficiencies in the presence of privately held information, so too will the use of quotas and taxes. Moreover, as originally noted by Weitzman (1974), the presence of asymmetrically held information causes these two policy instruments to no longer be perfect substitutes for one another, as they were in the model of Section 11.B.[19]

To begin, note that given θ and η, the aggregate surplus resulting from externality level h (we return to a continuum of possible externality levels here) is $\phi(h, \eta) + \pi(h, \theta)$. Thus, the externality level that maximizes aggregate surplus depends in general on the realized values of (θ, η). We denote this optimal value by the function $h^\circ(\theta, \eta)$. Figure 11.E.1 depicts this optimum value for two different pairs of parameters, (θ', η') and (θ'', η'').

Suppose, first, that a quota level of $\hat{h}$ is fixed. The firm will then choose the level of the externality to solve

$$\begin{aligned} \operatorname*{Max}_{h \geq 0} \quad & \pi(h, \theta) \\ \text{s.t.} \quad & h \leq \hat{h}. \end{aligned}$$

Denote its optimal choice by $h^q(\hat{h}, \theta)$. The typical effect of the quota will be to make

17. Note the similarity between problem (11.E.1) and the monopolist's problem studied in Section 12.B. Here the consumer's inability to discriminate among firms of different types leads her optimal offer to be one that yields an inefficient outcome.

18. We could, of course, also consider the outcomes from other, perhaps more elaborate, bargaining procedures. In Chapter 23, however, we shall study a result due to Myerson and Satterthwaite (1983) that implies that *no* bargaining procedure can lead to an efficient outcome for all values of b and c in this setting.

19. The discussion that follows also has implications for the relative advantages of quantity- versus price-based control mechanisms in organizations.

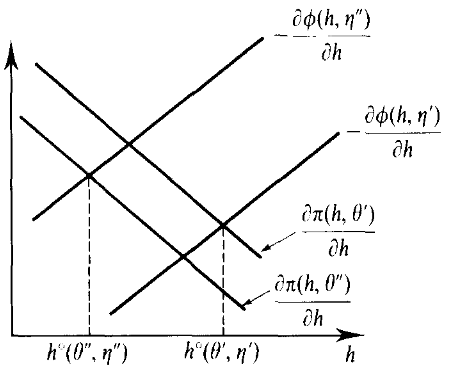


Figure 11.E.1 The surplus-maximizing aggregate externality level for two different pairs of parameters, (θ', η') and (θ'', η'').

the actual level of the externality much less sensitive to the values of θ and η than is required by optimality. The firm's externality level will be completely insensitive to η. Moreover, if the level of the quota $\hat{h}$ is such that $\partial\pi(\hat{h}, \theta)/\partial h > 0$ for all θ, we will have $h^q(\hat{h}, \theta) = \hat{h}$ for every θ. The loss in aggregate surplus arising under the quota for types (θ, η) is given by

$$\phi(h^q(\hat{h}, \theta), \eta) + \pi(h^q(\hat{h}, \theta), \theta) - \phi(h^\circ(\theta, \eta), \eta) - \pi(h^\circ(\theta, \eta), \theta)$$
$$= \int_{h(\theta,\eta)}^{h^q(\hat{h},\theta)} \left(\frac{\partial\pi(h,\theta)}{\partial h} + \frac{\partial\phi(h,\eta)}{\partial h}\right) dh.$$

This loss is represented by the shaded region in Figure 11.E.2 for a case in which the quota is set equal to $\hat{h} = h^\circ(\bar{\theta}, \bar{\eta})$, the externality level that maximizes social surplus when θ and η each take their mean values, $\bar{\theta}$ and $\bar{\eta}$ [the dashed lines in the figure are the graphs of $\partial\pi(h, \bar{\theta})/\partial h$ and $-\partial\phi(h, \bar{\eta})/\partial h$ and the solid lines are the graphs of $\partial\pi(h, \theta)/\partial h$ and $-\partial\phi(h, \eta)/\partial h$; note that in the case depicted, the firm wishes to produce the externality up to the allowed quota $\hat{h}$].

Consider next the use of a tax on the firm of t units of the numeraire per unit of the externality. For any given value of θ, the firm will then choose the level of externality to solve

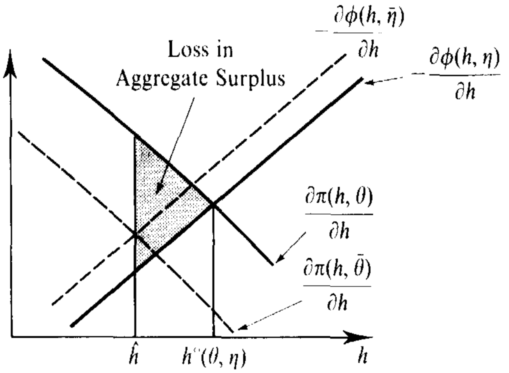
$$\underset{h \geq 0}{\text{Max}} \quad \pi(h, \theta) - th.$$

Denote its optimal choice by $h^t(t, \theta)$. The loss in aggregate surplus from the tax relative to the optimal outcome for types (θ, η) is therefore given by

$$\phi(h^t(t, \theta), \eta) + \pi(h^t(t, \theta), \theta) - \phi(h^\circ(\theta, \eta), \eta) - \pi(h^\circ(\theta, \eta), \theta)$$
$$= \int_{h(\theta,\eta)}^{h^t(t,\theta)} \left(\frac{\partial\pi(h,\theta)}{\partial h} + \frac{\partial\phi(h,\eta)}{\partial h}\right) dh.$$

Its value is depicted by the shaded region in Figure 11.E.3 for the same situation as in Figure 11.E.2, but now assuming that a tax is set at $t = -\partial\phi(h^\circ(\bar{\theta}, \bar{\eta}), \bar{\eta})/\partial h$, the value that results in the maximization of aggregate surplus when $(\theta, \eta) = (\bar{\theta}, \bar{\eta})$. Note that under a tax, as under a quota, the level of the externality is responsive to changes in the marginal benefits of the firm but not to changes in the marginal costs of the consumer.

Which of these instruments, quota or tax, performs better? The answer is that it depends. Imagine, for example, that η is a constant, say equal to $\bar{\eta}$. Then, for θ such that the benefits of the externality's use to the firm are high, a quota will typically miss the optimal externality level by not allowing the externality to increase above the quota level. On the other hand, because a fixed tax rate t does not reflect any



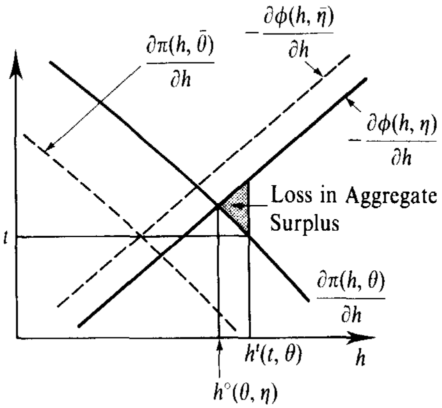


Figure 11.E.2 (left)
The loss in aggregate surplus under a quota for types (θ, η).

Figure 11.E.3 (right)
The loss in aggregate surplus under a tax for types (θ, η).

increasing marginal costs of the externality to the consumer at higher externality levels, for such values of θ the tax may result in excess production of the externality.

Intuitively, when the optimal externality level varies little with θ, we expect a quota to be better. Figure 11.E.4(a), for example, depicts a case in which the marginal cost to the consumer of the externality is zero up to some point h^* and infinite thereafter. In this case, by setting a quota of $\hat{h} = h^*$, we can maximize aggregate surplus for any value of (θ, η), but no tax can accomplish this. A tax would have to be very high to guarantee that with probability one the externality level fixed by the firm is not larger than h^*. But if so, the resulting externality level would be too low most of the time.

In contrast, in Figure 11.E.4(b) we depict a case in which the marginal cost to the consumer of the externality is independent of the level of h. In this case, a tax equal to this marginal cost ($t = t^*$) achieves the surplus-maximizing externality level for all (θ, η), but no quota can do so.

If we take the expected value of aggregate surplus as our welfare measure, we therefore see from these two examples that either policy instrument may be preferable, depending on the circumstances.[20] (Exercise 11.E.1 asks you to provide a full analysis for a linear-quadratic example.) Note also that the bargaining procedure we have discussed will not result in optimality in *either* case depicted in Figure 11.E.4.[21] Thus, we have here two cases in which either a quota or a tax performs better than a particular decentralized outcome.[22]

20. In Chapter 13, we discuss in greater detail some of the issues that arise in making welfare comparisons in settings with privately held information. There we shall justify the maximization of expected aggregate surplus in this partial equilibrium setting as a requirement of a notion of ex ante Pareto optimality for the two agents. See also the discussion in Section 23.F.

21. Strictly speaking, our previous discussion of bargaining assumed only two possible levels of the externality, while here we have a continuum of levels. This difference is not important. The inefficiency of the bargaining procedure previously studied would hold in this continuous environment as well.

22. We should emphasize that in these two examples other bargaining procedures will perform better than the procedure involving a take-it-or-leave-it offer by the consumer. For example, if a take-or-leave offer is made by the firm, then full optimality results in *both* of these cases because the type of the consumer is known with certainty. The conclusion of our discussion is therefore a qualitative one: With asymmetric information, it is difficult to make very general assertions about the relative performance of centralized versus decentralized approaches.

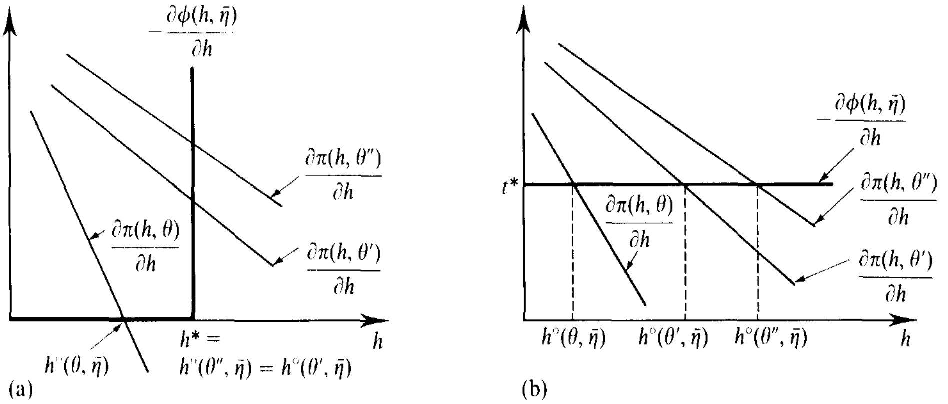


Figure 11.E.4
Two cases in which a quota or tax maximizes aggregate surplus for every realization of θ.
(a) Quota $\hat{h} = h^*$ maximizes aggregate surplus for all θ.
(b) Tax $t = t^*$ maximizes aggregate surplus for all θ.

In Exercise 11.E.2, you are asked to extend the analysis just given to a case with two firms ($j = 1, 2$) generating an externality, where the two firms are identical except possibly for their realized levels of θ_j. The exercise illustrates the importance of the degree of correlation between the θ_j's for the relative performance of quotas versus taxes. In comparing a uniform quota policy versus a uniform tax policy ("uniform" here means that the two firms face the same quota or tax rate), the less correlated the shocks across the firms, the better the tax looks. The reason is not difficult to discern. With imperfect correlation, a uniform tax has a benefit that is not achieved with a uniform quota: It allows for the individual levels of externalities generated to be responsive to the realized values of the θ_j's. Indeed, with a uniform tax, the production of the total amount of externality generated is always efficiently distributed across the two firms.

The presence of multiple generators of an externality also raises the possibility that a market for tradeable emissions permits could be created, as discussed at the end of Section 11.D. This simple addition to the quota policy can potentially eliminate the inefficient distribution of externality generation across different generators that is often a feature of a quota policy. In particular, suppose that instead of simply giving each firm a quota level, we now give them tradeable externality permits entitling them to generate the same number of units of the externality as in the quota. Suppose also that each firm would always fully use its quota if no trade was possible. Then trade must result in *at least* as large a value of aggregate surplus as the simple quota scheme for any realization of the firms' and consumer's types, because we still get the same total level of emissions and we can never get a trade between firms that lowers aggregate profits.[23] Of course, the same bargaining problems that we studied above can prevent a fully efficient distribution of externality generation from arising; but if the firms know each others' values of θ_j or are numerous enough to act competitively in the market for these rights, then we can expect a distribution of the total externality generation that is efficient across generators. In fact, in the case where the statistical distribution of costs among the firms is known but the particular realizations for individual firms are not known, this type of policy can achieve a fully optimal outcome.

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23. Note, however, that the assumption that the externalities generated by the different firms are perfect substitutes to the consumer is crucial to this conclusion. If this is not true, then the reallocation of externality generation can reduce aggregate surplus by lowering the well-being of the agents affected by the externality.

More General Policy Mechanisms

The tax and quota schemes considered above are, as we have seen, completely unresponsive to changes in the marginal costs of the externality to the affected agent (the consumer in this case). It is natural to wonder whether any other sorts of schemes can do better, perhaps by making the level of the externality responsive to the consumer's costs. The problem in doing so is that these benefits and costs are unobservable, and the parties involved may not have incentives to reveal them truthfully if asked. For example, suppose that the government simply asks the consumer and the firm to report their benefits and costs from the externality and then enforces whatever appears to be the optimal outcome based on these reports. In this case, the consumer will have an incentive to exaggerate her costs when asked in order to prevent the firm from being allowed to generate the externality. The question, then, is how to design mechanisms that control these incentives for misreporting and, as a consequence, enable the government to achieve an efficient outcome. This problem is studied in a very general form in Chapter 23; here we confine ourselves to a brief examination of one well-known scheme.

Return to the case in which there are only two possible levels of the externality, 0 and $\bar{h}$. Can we design a scheme that achieves the optimal level of externality generation for every realization of b (the firm's benefit from the externality) and c (the consumer's cost)? We now verify that the answer is "yes."

Imagine the government setting up the following *revelation mechanism*: The firm and the consumer are each asked to report their values of b and c, respectively. Let $\hat{b}$ and $\hat{c}$ denote these announcements. For each possible pair of announcements $(\hat{b}, \hat{c})$, the government sets an allowed level of the externality as well as a tax or subsidy payment for each of the two agents. Suppose, in particular, that the government declares that it will set the allowed externality level h to maximize aggregate surplus given the announcements. That is, $h = \bar{h}$ if and only if $\hat{b} > \hat{c}$. In addition, if externality generation is allowed (i.e., if $h = \bar{h}$), the government will tax the firm an amount equal to $\hat{c}$ and will subsidize the consumer with a payment equal to $\hat{b}$. That is, if the firm wants to generate the externality (which it indicates by reporting a large value of b), it is asked to pay the externality's cost as declared by the consumer; and if the consumer allows the externality (by reporting a low value of c) she receives a payment equal to the externality's benefit as declared by the firm.

In fact, *under this scheme both the firm and the consumer will tell the truth*, so that an optimal level of externality generation will, indeed, result for every possible (b, c) pair. To see this, consider the consumer's optimal announcement when her cost level is c. If the firm announces some $\hat{b} > c$, then the consumer prefers to have the externality-generating activity allowed (she does $\hat{b} - c$ better than if it is prevented). Hence, her optimal announcement satisfies $\hat{c} < \hat{b}$; moreover, because any such announcement will give her the same payoff, she might as well announce the truth, that is, $\hat{c} = c < \hat{b}$. On the other hand, if the firm announces $\hat{b} \leq c$, the consumer prefers to have the externality level set to zero. Hence, she would like announce $\hat{c} \geq \hat{b}$; and again, because any of these announcements will give her the same payoff, she may as well announce the truth, that is, $\hat{c} = c \geq \hat{b}$. Thus, whatever the firm's announcement, truth-telling is an optimal strategy for the consumer. (Formally, telling the truth is a *weakly dominant strategy* for the consumer in the sense studied in Section 8.B. In fact, it is the consumer's *only* weakly dominant strategy; see Exercise 11.E.3.) A parallel analysis yields the same conclusion for the firm.

Exercise 11.E.4: Show that in the tax-subsidy part of the mechanism above we could add, without affecting the mechanism's truth-telling or optimality properties, an additional payment to each agent that depends in an arbitrary way on the other agent's announcement.

The scheme we have described here is an example of the *Groves Clarke mechanism* [due to Groves (1973) and Clarke (1971); see also Section 23.C] and was originally proposed as a mechanism for deciding whether to carry out public good projects. Some examples for the public goods context are contained in the exercises at the end of the chapter.

The Groves Clarke mechanism has two very attractive features: it implements the optimal level of the externality for every (b, c) pair, and it induces truth-telling in a very strong (i.e., dominant strategy) sense. But the mechanism has some unattractive features as well. In particular, it does not result in a balanced budget for the government: The government has a deficit equal to $(b - c)$ whenever $b > c$. We could use the flexibility offered by Exercise 11.E.4 to eliminate this deficit for all possible (b, c), but then we would necessarily create a budget surplus and therefore a Pareto inefficient outcome for some values of (b, c) (not all units of the numeraire will be left in the hands of the firm or the consumer).

In fact, this problem is unavoidable with this type of mechanism: If we want to preserve the properties that, for every (b, c), truth-telling is a dominant strategy and the optimal level of externality is implemented, then we generally cannot achieve budget balance for every (b, c). In Chapter 23 we discuss this issue in greater detail and also consider other mechanisms that can, under certain circumstances, get around the problem. (See also Exercise 11.E.5 for an analysis in which budget balance is required only on average.)

APPENDIX A: NONCONVEXITIES AND THE THEORY OF EXTERNALITIES

Throughout this chapter, we have maintained the assumption that preferences and production sets are convex, leading the derived utility and profit functions we have considered to be concave. With these assumptions, all the decision problems we have studied have been well behaved; they had unique solutions (or, more generally, convex-valued solutions) that varied continously with the underlying parameters of the problems (e.g., the prices of the L traded commodities or the price of the externality if a market existed for it). Yet, this is not a completely innocent assumption. In this appendix, we present some simple examples designed to illustrate that externalities may themselves generate nonconvexities, and we comment on some of the implications of this fact.

We consider here a bilateral externality situation involving two firms. We suppose that firm 1 may engage in an externality-generating activity that affects firm 2's production. The level of externality generated by firm 1 is denoted by h, and firm j's profits conditional on the production of externality level h are $\pi_j(h)$ for $j = 1, 2$. It is perfectly natural to assume that $\pi_1(\cdot)$ is concave: The level h could, for example,

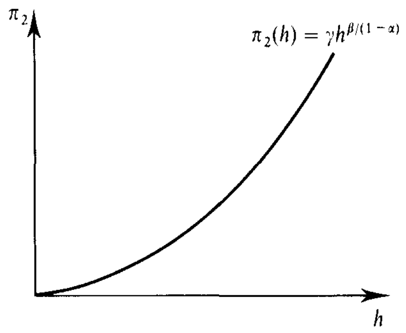

Figure 11.AA.1
The derived profit function of firm 2 (the externality recipient) in Example 11.AA.1 when $\alpha + \beta > 1$.

be equal to firm 1's output.[24] As Examples 11.AA.1 and 11.AA.2 illustrate, however, this may not be true of firm 2's profit function.

Example 11.AA.1: *Positive Externalities as a Source of Increasing Returns.* Suppose that firm 2 produces an output whose price is 1, using an input whose price, for simplicity, we also take to equal 1. Firm 2's production function is $q = h^{\beta}z^{\alpha}$, where $\alpha, \beta \in [0, 1]$. Thus, the externality is a positive one.[25] Note that, for fixed h, the problem of firm 2 is concave and perfectly well behaved. Given a level of h, the maximized profits of firm 2 can be calculated to be $\pi_2(h) = \gamma h^{\beta/(1-\alpha)}$, where $\gamma > 0$ is a constant. In Figure 11.AA.1, we represent $\pi_2(h)$ for $\beta > 1 - \alpha$. We see there that firm 2's derived profit function is *not* concave in h; in fact, it is convex. This reflects the fact that if we think of the externality h as an input to firm 2's production process, then firm 2's overall production function exhibits increasing returns to scale because $\alpha + \beta > 1$. ■

Example 11.AA.2: *Negative Externalities as a Source of Nonconvexities.* In Example 11.AA.1, the nonconvexity in firm 2's production set, and the resulting failure of concavity in its derived profit function, were caused by a positive externality. In this example the failure of concavity of firm 2's derived profit function is the result of a negative externality.

Suppose, in particular, that $\pi_2'(h) \leq 0$ for all h, with strict inequality for some h, and that firm 2 has the option of shutting down when experiencing externality level h and receiving profits of zero.[26] In this case, the function $\pi_2(\cdot)$ can *never* be concave

24. Note also that we may well have $\pi_1(h) < 0$ for some levels $h \geq 0$ because $\pi_1(h)$ is firm 1's maximized profit *conditional* on producing externality level h (and so shutting down is not possible if $h > 0$).

25. More generally, we could think that there is an industry composed of many firms and that the externality is produced and felt by all firms in the industry (e.g., h could be an index, correlated with output, of accumulated know-how in the industry). Externalities were first studied by Marshall (1920) in this context. See also Chipman (1970) and Romer (1986).

26. In the more typical case of a multilateral externality, the ability of affected parties to shut down in this manner often depends on whether the externality is depletable. In the case of a nondepletable externality, such as air pollution, affected firms can always shut down and receive zero profits. In contrast, in the case of a depletable externality (such as garbage), where $\pi_j(h)$ reflects firm j's profits when it individually absorbs h units of the externality, the absorption of the externality may itself require the use of some inputs (e.g., land to absorb garbage). Indeed, were this not the case for a depletable externality, the externality could always be absorbed in a manner that creates no social costs by allocating all of the externality to a firm that shuts down.

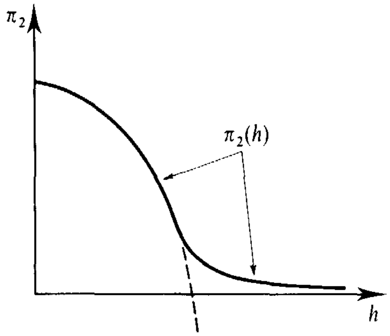

Figure 11.AA.2
If the recipient of a negative externality can shut down and earn zero profits for any level of the externality, then its derived profit function $\pi_2(h)$ cannot be concave over $h \in [0, \infty]$.

over all $h \in [0, \infty)$, a point originally noted by Starrett (1972). The reason can be seen in Figure 11.AA.2: If $\pi_2(\cdot)$ were a strictly decreasing concave function, then it would have to become negative at some level of h (see the dashed curve), but $\pi_2(\cdot)$ must be nonnegative if firm 2 can always choose to shut down. ■

The failure of $\pi_2(\cdot)$ to be concave can create problems for both centralized and decentralized solutions to the externality problem. For example, if property rights over the externality are defined and a market for the externality is introduced in either Example 11.AA.1 or Example 11.AA.2, a competitive equilibrium may fail to exist (even assuming that the two agents act as price takers). Firm 2's objective function will not be concave, and so its optimal demand may fail to be well defined and continuous (recall our discussion in Section 10.C of the equilibrium existence problems caused by nonconvexities in firms' cost functions).

In contrast, taxes and quotas can, in principle, still implement the optimal outcome despite the failure of firm 2's profit function to be concave because their use depends only on the profit function of the externality generator (here, firm 1) being well behaved. In practice, however, nonconvexities in firm 2's profit function may create problems for these centralized solutions as well. Example 11.AA.3 illustrates this point.

Example 11.AA.3: *Externalities as a Source of Multiple Local Social Optima.* It is, in principle, true that if the decision problem of the generator of an externality is concave, then the optimum can be sustained by means of quotas or taxes. But if $\pi_2(\cdot)$ is not concave, then the aggregate surplus function $\pi_1(h) + \pi_2(h)$ may not be concave and, as a result, the first-order conditions for aggregate surplus maximization may suffice only for determining local optima. In fact, as emphasized by Baumol and Oates (1988), the nonconvexities created by externalities may easily generate situations with multiple local social optima, so that identifying a global optimum may be a formidable task.

Suppose, for example, that the profit functions of the two firms are

$$\pi_1(h) = \begin{cases} h & \text{for } h \le 1 \\ 1 & \text{for } h > 1 \end{cases}$$

and

$$\pi_2(h) = \begin{cases} 2(1-h)^2 & \text{for } h \le 1 \\ 0 & \text{for } h > 1. \end{cases}$$

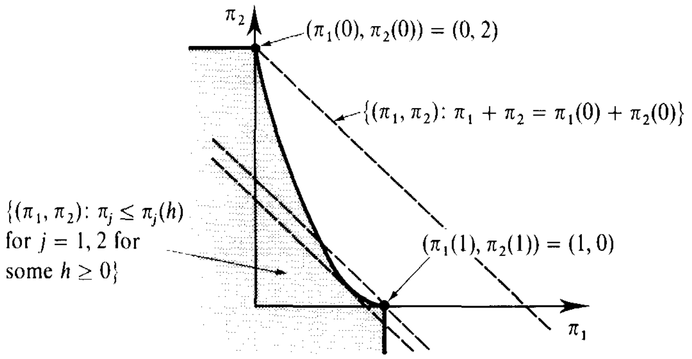

Figure 11.AA.3
The set of possible profit pairs (π_1, π_2) in Example 11.AA.3 exhibits multiple local maxima of aggregate surplus $\pi_1(h) + \pi_2(h)$.

The function $\pi_2(\cdot)$ is not concave, something that the two previous examples have shown us can easily happen with externalities. The profit levels for the two firms that are attainable for different levels of h are depicted in Figure 11.AA.3 by the shaded set $\{(\pi_1, \pi_2)\colon \pi_j \le \pi_j(h)$ for $j = 1, 2$ for some $h > 0\}$ (note that this definition allows for free disposal of profits). The social optimum has $h = 0$ (joint profits are then equal to 2), in which case firm 2 is able to operate in an environment free from the externality. This can be implemented by setting a tax rate on firm 1 of $t > 1$ per unit of the externality. But note that the outcome $h = 1$ (implemented by setting a tax rate on firm 1 of $t = 0$) is a local social optimum: As we decrease h, it is not until $h < \frac{1}{2}$ that we get an aggregate surplus level higher than that at $h = 1$. Hence, this latter outcome satisfies both the first-order and second-order conditions for the maximization of aggregate surplus (e.g., at this point, the marginal benefits of the externality exactly equal its marginal costs), and it will be easy for a social planner to be misled into thinking that she is at a welfare maximum. ■

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EXERCISES

11.B.1[B] (M. Weitzman) On Farmer Jones' farm, only honey is produced. There are two ways to make honey: with and without bees. A bucket full of artificial honey, absolutely indistinguishable from the real thing, is made out of 1 gallon of maple syrup with one unit of labor. If the same honey is made the old-fashioned way (with bees), k total units of labor are required (including bee-keeping) and b bees are required per bucket. Either way, Farmer Jones has the capacity to produce up to H buckets of honey on his farm.

The neighboring farm, belonging to Smith, produces apples. If bees are present, less labor is needed because bees pollinate the blossoms instead of workers doing it. For this reason, c bees replace one worker in the task of pollinating. Up to A bushels of apples can be grown on Smith's farm.

Suppose that the market wage rate is w, bees cost p_b per bee, and maple syrup costs p_m per gallon. If each farmer produces her maximal output at the cheapest cost to her (assume the output prices they face make maximal production efficient), is the resulting outcome efficient? How does the answer depend on k, b, c, w, p_b, and p_m? Give an intuitive explanation of your result. Up to how much would Smith be willing to bribe Jones to produce honey with bees? What would happen to efficiency if both farms belonged to the same owner? How could the government achieve efficient production through taxes?

11.B.2[C] Consider the two-consumer externality problem studied in Section 11.B, but now assume that consumer 2's derived utility function over the externality level h and her wealth available for commodity purchases w_2 takes the form $\phi_2(h, w_2)$. Assume that $\phi_2(h, w_2)$ is a twice-differentiable, strictly quasiconcave function with $\partial\phi_2(h, w_2)/\partial w_2 > 0$ and, for simplicity, that we have a positive externality so that $\partial\phi_2(h, w_2)/\partial h > 0$.

(a) Set up the Pareto optimality problem as one of choosing h and a wealth transfer T to maximize consumer 1's welfare subject to giving consumer 2 a utility level of at least $\bar{u}_2$. Derive the (necessary and sufficient) first-order condition characterizing the optimal levels of h and T, say h° and T°.

(b) Imagine that consumer 1 could purchase h on an externality market. Let p_h be the price per unit, and let $h(p_h, w_2)$ be consumer 2's demand function for h. Express the wealth effect $\partial h(p_h, w_2)/\partial w_2$ in terms of first-order and second-order partial derivatives of consumer 2's utility function.

(c) Derive the comparative statics change in the Pareto optimal level of the externality h° (for a given u_2) with respect to a differential increase $dw_2 > 0$ in consumer 2's wealth. Show that if consumer 2's demand for the externality, derived in **(b)**, is normal at price $p_h = [\partial\phi_2(h^\circ, w_2 - T^\circ)/\partial h]/\,[\partial\phi_2(h^\circ, w_2 - T^\circ)/\partial w_2]$ and wealth level $\bar{w}_2 = w_2 - T^\circ$ [i.e., if $\partial h(\bar{p}_h, \bar{w}_2)/\partial w_2 > 0$], then a marginal increase in consumer 2's wealth w_2 causes the Pareto optimal level of the externality h° to increase. (Similarly, in the case of a negative externality, if consumer 2's demand for reductions in the externality is a normal good, then when consumer 2 becomes wealthier, the Pareto optimal level of the externality declines.)

11.B.3^B Consider the optimal Pigouvian tax identified in Section 11.B for the two-consumer externality problem studied there. What happens if, given this tax, the two consumers are able to bargain with each other? Will the efficient level of the externality still result? What about with the optimal quota?

11.B.4^B Consider again the two-consumer externality problem studied in Section 11.B. Suppose that consumer 2 can take some action, say $e \in \mathbb{R}$, that affects the degree to which she is affected by the externality, so that we now write her derived utility function as $\phi_2(h, e) + w_2$. To fix ideas, let h be a negative externality, and suppose that $\partial^2\phi_2(h, e)/\partial h \partial e > 0$, so that increases in e reduce the negative effect of the externality on the margin. Suppose that both h and e can in principle be taxed or subsidized. Should e be taxed or subsidized in the optimal tax scheme? Why or why not?

11.B.5^B Suppose that at fixed input prices of $\bar{w}$ a firm produces output with the differentiable and strictly convex cost function $c(q, h)$, where $q \geq 0$ is its output level (whose price is $p > 0$) and h is the level of a negative externality generated by the firm. The externality affects a single consumer, whose derived utility function takes the form $\phi(\text{h}) + w$. The actions of the firm and consumer do not affect any market prices.

(a) Derive the first-order condition for the firm's choice of q and h.

(b) Derive the first-order conditions characterizing the Pareto optimal levels of q and h.

(c) Suppose that the government taxes the firm's output level. Show that this cannot restore efficiency. Show that a direct tax on the externality *can* restore efficiency.

(d) Show, however, that in the limiting case where h is necessarily produced in fixed proportions with q, so that $h(q) = \alpha q$ for some $\alpha > 0$, a tax on the firm's output *can* restore efficiency. What is the efficiency-restoring tax?

11.C.1^A Consider the model discussed in Section 11.C, in which I consumers privately purchase a public good. Identify per-unit subsidies $s_1, \ldots, s_I$, such that when each consumer i faces subsidy rate s_i, the total level of the public good provided is optimal.

11.C.2^A Consider the model discussed in Section 11.C, in which I consumers privately purchase a public good. Show that a per-unit subsidy on the firm's output (paid to the firm) can also restore efficiency.

11.C.3^C Reconsider the Ramsey tax problem from Exercise 10.E.3, but now suppose that the government can also provide a public good x_0 that can be produced from good 1 at cost $c(x_0)$. However, the government must still balance its budget (including any expenditures on the public good). Consumer i's utility function now takes the form $x_{1i} + \sum_{\ell=2}^{L} \phi_{\ell i}(x_{\ell i}, x_0)$. Derive and interpret the conditions characterizing the optimal commodity taxes and the optimal level of the public good. How do the two problems of Ramsey taxation and provision of the public good interact?

11.D.1^B (M. Weitzman) First-year graduate students are a hard-working group. Consider a typical class of I students. Suppose that each student i puts in h_i hours of work on her classes. This effort involves a disutility of $h_i^2/2$. Her benefits depend on how she performs relative to her peers and take the form $\phi(h_i/\bar{h})$ for all i, where $\bar{h} = (1/I)\sum_i h_i$ is the average number of hours put in by all students in the class and $\phi(\cdot)$ is a differentiable concave function, with

$\phi'(\cdot) > 0$ and $\lim_{h \to 0} \phi'(h) = \infty$. Characterize the symmetric (Nash) equilibrium. Compare it with the Pareto optimal symmetric outcome. Interpret.

11.D.2[B] Consider a setting with I consumers. Each consumer i chooses an action $h_i \in \mathbb{R}_+$. Consumer i's derived utility function over her choice of h and the choices of other consumers takes the form $\phi_i(h_i, \sum_i h_i) + w_i$, where $\phi_i(\cdot)$ is strictly concave. Characterize the optimal levels of $h_1, \ldots, h_I$. Compare these with the equilibrium levels. What tax/subsidy scheme induces the optimal outcome?

11.D.3[B] Consider an industry composed of $J > 1$ identical firms that act as price takers. The price of their output is p, and the prices of their inputs are unaffected by their actions. Suppose that partial equilibrium analysis is valid and that the aggregate demand for their product is given by the function $x(p)$. The industry is characterized by "learning by doing," in that each firm's total cost of producing a given level of output is declining in the level of total industry output; that is, each firm j has a twice-differentiable cost function of the form $c(q_j, Q)$ for $Q = \sum_j q_j$, where $c(\cdot)$ is strictly increasing in its first argument and strictly decreasing in its second. Letting subscripts denote partial derivatives, assume that $c_q + Jc_Q > 0$ and $(1/n)c_{qq} + 2c_{qQ} + nc_{QQ} > 0$ for $n = 1$ and J. Compare the equilibrium and optimal industry output levels. Interpret. What tax or subsidy restores efficiency?

11.D.4[B] Reconsider the nondepletable externality example discussed in Section 11.D, but now assume that the externalities produced by the J firms are not homogeneous. In particular, suppose that if $h_1, \ldots, h_J$ are the firms' externality levels, then consumer i's derived utility is given by $\phi_i(h_1, \ldots, h_J) + w_i$ for each $i = 1, \ldots, I$. Compare the equilibrium and efficient levels of $h_1, \ldots, h_J$. What tax/subsidy scheme can restore efficiency? Under what condition should each firm face the same tax/subsidy rate?

11.D.5[B] (*The problem of the commons*) Lake Ec can be freely accessed by fishermen. The cost of sending a boat out on the lake is $r > 0$. When b boats are sent out onto the lake, $f(b)$ fish are caught in total [so each boat catches $f(b)/b$ fish], where $f'(b) > 0$ and $f''(b) < 0$ at all $b \geq 0$. The price of fish is $p > 0$, which is unaffected by the level of the catch from Lake Ec.

(a) Characterize the equilibrium number of boats that are sent out on the lake.

(b) Characterize the optimal number of boats that should be sent out on the lake. Compare this with your answer to **(a)**.

(c) What per-boat fishing tax would restore efficiency?

(d) Suppose that the lake is instead owned by a single individual who can choose how many boats to send out. What level would this owner choose?

11.D.6[B] Suppose that there is a piece of land that is affected adversely by an externality produced by a single firm. The firm's derived profit function for the externality is $\pi(h) = \alpha + \beta h - \mu h^2$, where h is the level of the externality and $(\alpha, \beta, \mu) \gg 0$. There are I consumers who farm the land, each owning a fraction $1/I$ of it. The total yield of the land is $\phi(h) = \gamma - \eta h$, where $(\gamma, \eta) \gg 0$. Each of the I consumers then has a derived utility function of $\phi(h)/I + w$.

Bargaining among the consumers and the firm works as follows: Each consumer simultaneously decides whether to be in or out of a bargaining coalition. After this, the bargaining coalition makes the firm a take-it-or-leave-it offer specifying a level of h and a transfer. The firm then accepts or rejects this offer. In the absence of any agreement, the firm can generate any level of the externality it wishes.

(a) Let θ denote the fraction of the I consumers who join the bargaining coalition. Characterize the subgame perfect Nash equilibrium level of θ (for simplicity, treat θ as a continuous variable). Show that when $I = 1$ the optimal level of the externality results, but that when $I > 1$ we have $\theta < 1$ in equilibrium and too much of the externality is generated.

(b) Show that as I increases, the equilibrium level of θ declines. Also show that $\lim_{I \to \infty} \theta = 0$.

11.D.7[C] Individuals can build their houses in one of two neighborhoods, A or B. It costs c_A to build a house in neighborhood A and $c_B < c_A$ to build in neighborhood B. Individuals care about the prestige of the people living in their neighborhood. Individuals have varying levels of prestige, denoted by the parameter θ. Prestige varies between 0 and 1 and is uniformly distributed across the population. The prestige of neighborhood k ($k = A, B$) is a function of the average value of θ in that neighborhood, denoted by $\bar{\theta}_k$. If individual i has prestige parameter θ and builds her house in neighborhood k, her derived utility net of building costs is $(1 + \theta)(1 + \bar{\theta}_k) - c_k$. Thus, individuals with more prestige value a prestigous neighborhood more. Assume that c_A and c_B are less than 1 and that $c_A - c_B \in (\frac{1}{2}, 1)$.

(a) Show that in any building-choice equilibrium (technically, the Nash equilibrium of the simultaneous-move game in which individuals simultaneously choose where to build their house) both neighborhoods must be occupied.

(b) Show that in any equilibrium in which the prestige levels of the two neighborhoods differ, every resident of neighborhood A must have at least as high a prestige level as every resident of neighborhood B; that is, there is a cutoff level of θ, say $\hat{\theta}$, such that all types $\theta \geq \hat{\theta}$ build in neighborhood A and all $\theta < \hat{\theta}$ build in neighborhood B. Characterize this cutoff level.

(c) Show that in any equilibrium of the type identified in (b), a Pareto improvement can be achieved by altering the cutoff value of θ slightly.

11.E.1[B] Consider the setting studied in Section 11.E, and suppose that $\partial\pi(h, \theta)/\partial h = \beta - bh + \theta$ and $\partial\phi(h, \eta)/\partial h = \gamma - ch + \eta$, where θ and η are random variables with $E[\theta] = E[\eta] = E[\theta\eta] = 0$, $(\beta, b, c) \gg 0$, and $\gamma < 0$. Denote $E[\theta^2] = \sigma_\theta^2$ and $E[\eta^2] = \sigma_\eta^2$.

(a) Identify the best quota $\hat{h}^*$ for a planner who wants to maximize the expected value of aggregate surplus. (Assume the firm must produce an amount exactly equal to the quota.)

(b) Identify the best tax t^* for this same planner.

(c) Compare the two instruments: Which is better and when?

11.E.2[C] Extend the model in Exercise 11.E.1 to the case of two producers. Now let $\partial\pi_i(h_i, \theta_i)/\partial h = \beta - bh_i + \theta_i$ for $i = 1, 2$. Let $\sigma_{12} = E[\theta_1\theta_2]$. Calculate and compare the optimal quotas and taxes. How does the choice depend on σ_{12}?

11.E.3[B] Show that truth-telling is the consumer's only weakly dominant strategy in the (Groves–Clarke) revelation mechanism studied in Section 11.E.

11.E.4[A] In text.

11.E.5[B] Suppose that the government is considering building a public project. The cost is K. There are I individuals indexed by i. Individual i's privately known benefit from the project is b_i. The government's objective is to maximize the expected value of aggregate surplus. Derive the extension of the Groves–Clarke mechanism discussed in Section 11.E for this case. Can you construct your scheme so that the government balances its budget on average (over all realizations of the b_i's)?

11.E.6[B] Extend Exercise 11.E.5 to the case in which there are N possible projects, $n = 1, \ldots, N$, with individual i deriving a (privately known) benefit of $b_i(n)$ from project n.

11.E.7[B] Suppose that in the model of Section 11.E the consumer's type η takes only one possible value, $\bar{\eta}$. We have seen in the text that in this case neither a quota nor a tax will maximize aggregate surplus for all realizations of θ when the derived utility function $\phi(h, \bar{\eta})$ for the consumer has $\partial\phi(h, \bar{\eta})/\partial h \in (0, -\infty)$. Show, however, that a variable tax per unit in which the total tax collected from the firm is $\phi(h, \bar{\eta})$ when the level of the externality is h will maximize aggregate surplus for all values of θ for *any* derived utility function $\phi(h, \bar{\eta})$.

CHAPTER

12 Market Power

12.A Introduction

In the competitive model, all consumers and producers are assumed to act as price takers, in effect behaving as if the demand or supply functions that they face are infinitely elastic at going market prices. However, this assumption may not be a good one when there are only a few agents on one side of a market, for these agents will often possess *market power*—the ability to alter profitably prices away from competitive levels.

The simplest example of market power arises when there is only a single seller, a *monopolist*, of some good. If this good's market demand is a continuous decreasing function of price, then the monopolist, recognizing that a small increase in its price above the competitive level leads to only a small reduction in its sales, will find it worthwhile to raise its price above the competitive level.

Similar effects can occur when there is more than one agent, but still not many, on one side of a market. Most often, these agents with market power are firms, whose fewness arises from nonconvexities in production technologies (recall the discussion of entry in Section 10.F).

In this chapter, we study the functioning of markets in which market power is present. We begin, in Section 12.B, by considering the case in which there is a monopolist seller of some good. We review the theory of monopoly pricing and identify the welfare loss that it creates.

The remaining sections focus on situations of *oligopoly*, in which a number of firms compete in a market. In Sections 12.C and 12.D, we discuss several models of oligopolistic pricing. Each incorporates different assumptions about the underlying structure of the market and behavior of firms. The discussion highlights the implicatons of these differing assumptions for market outcomes. In Section 12.C, we focus on static models of oligopolistic pricing, where competition is viewed as a one-shot, simultaneous event. In contrast, in Section 12.D, we study how repeated interaction among firms may affect pricing in oligopolistic markets. This discussion constitutes an application of the theory of repeated games, a subject that we discuss in greater generality in Appendix A.

The analysis in Sections 12.B to 12.D treats the number of firms in the market as exogeneously given. In reality, however, the number of active firms in a market is likely to be affected by factors such as the size of market demand and the nature of competition within the market. Sections 12.E and 12.F consider issues that arise when the number of active firms in a market is determined endogenously.

Section 12.E specifies a simple model of entry into an oligopolistic market and studies the determinants of the number of active firms. It offers an analysis that parallels that considered in Section 10.F for competitive markets.

Section 12.F returns to a theme raised in Chapter 10. We illustrate how the competitive (price-taking) model can be viewed as a limiting case of oligopoly in which the size of the market, and hence the number of firms that can profitably operate in it, grows large. In the model we study, an active firm's market power diminishes as the market size expands; in the limit, the equilibrium market price comes to approximate the competitive level.

In Section 12.G, we briefly consider how firms in oligopolistic markets can make strategic precommitments to affect the conditions of future competition in a manner favorable to themselves. This issue nicely illustrates the importance of credible commitments in strategic settings, an issue we studied extensively in Chapter 9. In Appendix B, we consider in greater detail a particularly striking example of strategic precommitment to affect future market conditions, the case of entry deterrence through capacity choice.

If you have not done so already, you should review the game theory chapters in Part II before studying Sections 12.C to 12.G (in particular, review all of Chapter 7, Sections 8.A to 8.D, and Sections 9.A and 9.B).

An excellent source for further study of the topics covered in this chapter is Tirole (1988).[1]

12.B Monopoly Pricing

In this section, we study the pricing behavior of a profit-maximizing *monopolist*, a firm that is the only producer of a good. The demand for this good at price p is given by the function $x(p)$, which we take to be continuous and strictly decreasing at all p such that $x(p) > 0$.[2] For convenience, we also assume that there exists a price $\bar{p} < \infty$ such that $x(p) = 0$ for all $p \geq \bar{p}$.[3] Throughout, we suppose that the monopolist knows the demand function for its product and can produce output level q at a cost of $c(q)$.

The monopolist's decision problem consists of choosing its price p so as to maximize its profits (in terms of the numeraire), or formally, of solving

$$\underset{p}{\text{Max}} \quad px(p) - c(x(p)). \tag{12.B.1}$$

1. See also the survey by Shapiro (1989) for the topics covered in Sections 12.C, 12.D, and 12.G.

2. Throughout this chapter we take a partial equilibrium approach; see Chapter 10 for a discussion of this approach.

3. This assumption helps to insure that an optimal solution to the monopolist's problem exists. (See Exercise 12.B.2 for an example in which the failure of this condition leads to nonexistence.)

An equivalent formulation in terms of quantity choices can be derived by thinking instead of the monopolist as deciding on the level of output that it desires to sell, $q \geq 0$, letting the price at which it can sell this output be given by the *inverse demand function* $p(\cdot) = x^{-1}(\cdot)$.[4] Using this inverse demand function, the monopolist's problem can then be stated as

$$\underset{q \geq 0}{\text{Max}} \quad p(q)q - c(q). \tag{12.B.2}$$

We shall focus our analysis on this quantity formulation of the monopolist's problem [identical conclusions could equally well be developed from problem (12.B.1)]. We assume throughout that $p(\cdot)$ and $c(\cdot)$ are continuous and twice differentiable at all $q \geq 0$, that $p(0) > c'(0)$, and that there exists a unique output level $q^\circ \in (0, \infty)$ such that $p(q^\circ) = c'(q^\circ)$. Thus, q° is the unique socially optimal (competitive) output level in this market (see Chapter 10).

Under these assumptions, a solution to problem (12.B.2) can be shown to exist.[5] Given the differentiability assumed, the monopolist's optimal quantity, which we denote by q^m, must satisfy the first-order condition[6]

$$p'(q^m)\, q^m + p(q^m) \leq c'(q^m), \quad \text{with equality if } q^m > 0. \tag{12.B.3}$$

The left-hand side of (12.B.3) is the *marginal revenue* from a differential increase in q at q^m, which is equal to the derivative of revenue $d[p(q)q]/dq$, while the right-hand side is the corresponding marginal cost at q^m. Since $p(0) > c'(0)$, condition (12.B.3) can be satisfied only at $q^m > 0$. Hence, under our assumptions, *marginal revenue must equal marginal cost* at the monopolist's optimal output level:

$$p'(q^m)q^m + p(q^m) = c'(q^m). \tag{12.B.4}$$

For the typical case in which $p'(q) < 0$ at all $q \geq 0$, condition (12.B.4) implies that we must have $p(q^m) > c'(q^m)$, and so *the price under monopoly exceeds marginal cost*. Correspondingly, the monopolist's optimal output q^m must be below the socially optimal (competitive) output level q°. The cause of this quantity distortion is the monopolist's recognition that a reduction in the quantity it sells allows it to increase the price charged on its remaining sales, an increase whose effect on profits is captured by the term $p'(q^m)q^m$ in condition (12.B.4).

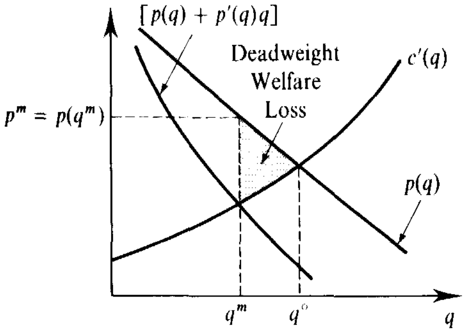
The welfare loss from this quantity distortion, known as the *deadweight loss of monopoly*, can be measured using the change in Marshallian aggregate surplus

?

4. More precisely, to take account of the fact that $x(p) = 0$ for more than one value of p, we take $p(q) = \text{Min}\,\{p: x(p) = q\}$ at all $q \geq 0$. Thus, $p(0) = \bar{p}$, the lowest price at which $x(p) = 0$.

5. In particular, it follows from condition (12.B.3) and from the facts that $p'(q) \leq 0$ for all $q \geq 0$ and $p(q) < c'(q)$ for all $q > q^\circ$, that the monopolist's optimal choice must lie in the compact set $[0, q^\circ]$. Because the objective function in problem (12.B.2) is continuous, a solution must therefore exist (see Section M.F of the Mathematical Appendix).

6. Satisfaction of first-order condition (12.B.3) is sufficient for q^m to be an optimal choice if the objective function of problem (12.B.2) is concave on $[0, q^\circ]$. Note, however, that concavity of this objective function depends not only on the technology of the firm, as in the competitive model, but also on the shape of the inverse demand function. In particular, even with a convex cost function, the monopolist's profit function can violate this concavity condition if demand is a convex function of price.



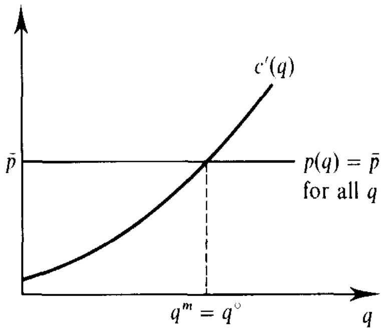


Figure 12.B.1 (left) The monopoly solution and welfare loss when $p'(\cdot) < 0$.

Figure 12.B.2 (right) The monopoly solution when $p'(q) = 0$ for all q.

(see Section 10.E),

$$\int_{q^m}^{q^o} [p(s) - c'(s)]\, ds > 0,$$

where q^o is the socially optimal (competitive) output level.

Figure 12.B.1 illustrates the monopoly outcome in this case. The monopolist's quantity q^m is determined by the intersection of the graphs of marginal revenue $p'(q)q + p(q)$ and marginal cost $c'(q)$. The monopoly price $p(q^m)$ can then be determined from the inverse demand curve. The deadweight welfare loss is equal to the area of the shaded region.

Note from condition (12.B.4) that the monopoly quantity distortion is absent in the special case in which $p'(q) = 0$ for all q. In this case, where $p(q)$ equals some constant $\bar{p}$ at all $q > 0$, the monopolist sells the same quantity as a price-taking competitive firm because it perceives that any increase in its price above the competitive price $\bar{p}$ causes it to lose all its sales.[7] Figure 12.B.2 depicts this special case.

Example 12.B.1: *Monopoly Pricing with a Linear Inverse Demand Function and Constant Returns to Scale.* Suppose that the inverse demand function in a monopolized market is $p(q) = a - bq$ and that the monopolist's cost function is $c(q) = cq$, where $a > c \geq 0$ [so that $p(0) > c'(0)$] and $b > 0$. In this case, the objective function of the monopolist's problem (12.B.2) is concave, and so condition (12.B.4) is both necessary and sufficient for a solution to the monopolist's problem. From condition (12.B.4), we can calculate the monopolist's optimal quantity and price to be $q^m = (a - c)/2b$ and $p^m = (a + c)/2$. In contrast, the socially optimal (competitive) output level and price are $q^o = (a - c)/b$ and $p^o = p(q^o) = c$. ■

Although we do not discuss these issues here, we point out that the behavioral distortions arising under monopoly are not limited to pricing decisions. (Exercises 12.B.9 and 12.B.10 ask you to investigate two examples.)

The monopoly quantity distortion is fundamentally linked to the fact that if the monopolist wants to increase the quantity it sells, it must lower its price on *all* its existing sales. In fact,

7. This inverse demand function arises, for example, when each consumer i has quasilinear preferences of the form $u_i(q_i) + m_i$ with $u_i(q_i) = \bar{p}q_i$, where q_i is consumer i's consumption of the good under study and m_i is his consumption of the numeraire commodity. [Strictly speaking, with these preferences we now have a multivalued demand correspondence rather than a demand function, but $p(\cdot)$ is nevertheless a function as before.]

if the monopolist were able to *perfectly discriminate* among its customers in the sense that it could make a distinct offer to each consumer, knowing the consumer's preferences for its product, then the monopoly quantity distortion would disappear.

To see this formally, let each consumer i have a quasilinear utility function of the form $u_i(q_i) + m_i$ over the amount q_i of the monopolist's good that he consumes and the amount m_i that he consumes of the numeraire good, and normalize $u_i(0) = 0$. Suppose that the monopolist makes a take-it-or-leave-it offer to each consumer i of the form (q_i, T_i), where q_i is the quantity offered to consumer i and T_i is the total payment that the consumer must make in return. Given offer (q_i, T_i), consumer i will accept the monopolist's offer if and only if $u_i(q_i) - T_i \geq 0$. As a result, the monopolist can extract a payment of exactly $u_i(q_i)$ from consumer i in return for q_i units of its product, leaving the consumer with a surplus of exactly zero from consumption of the good. Given this fact, the monopolist will choose the quantities it sells to the I consumers $(q_1, \ldots, q_I)$ to solve

$$\underset{(q_1,\ldots,q_I)\geq 0}{\text{Max}} \quad \sum_{i=1}^{I} u_i(q_i) - c(\textstyle\sum_i q_i). \tag{12.B.5}$$

Note, however, that any solution to problem (12.B.5) maximizes the aggregate surplus in the market, and so the monopolist will sell each consumer exactly the socially optimal (competitive) quantity. Of course, the distributional properties of this outcome would not be terribly attractive in the absence of wealth redistribution: The monopolist would get all the aggregate surplus generated by its product, and each consumer i would receive a surplus of zero (i.e., each consumer i's welfare would be exactly equal to the level he would achieve if he consumed none of the monopolist's product). But in principle, these distributional problems can be corrected through lump-sum redistribution of the numeraire.

Thus, the welfare loss from monopoly pricing can be seen as arising from constraints that prevent the monopolist from charging fully discriminatory prices. In practice, however, these constraints can be significant. They may include the costs of assessing separate charges for different consumers, the monopolist's lack of information about consumer preferences, and the possibility of consumer resale. Exercise 12.B.5 explores some of these factors. It provides conditions under which the best the monopolist can do is to name a single per-unit price, as we assumed at the beginning of this section.

12.C Static Models of Oligopoly

We now turn to cases in which more than one, but still not many, firms compete in a market. These are known as situations of *oligopoly*. Competition among firms in an oligopolistic market is inherently a setting of strategic interaction. For this reason, the appropriate tool for its analysis is game theory. Because this discussion constitutes our first application of the theory of games, we focus on relatively simple *static* models of oligopoly, in which there is only one period of competitive interaction and firms take their actions simultaneously.

We begin by studying a model of simultaneous price choices by firms with constant returns to scale technologies, known as the *Bertrand model*. This model displays a striking feature: With just two firms in a market, we obtain a perfectly competitive outcome. Motivated by this finding, we then consider three alterations of this model that weaken its strong and often implausible conclusion: a change in the firm's strategy from choosing its price to choosing its quantity of output (the *Cournot model*); the introduction of capacity constraints (or, more generally, decreasing returns to scale); and the presence of product differentiation.[8]

One lesson of this analysis is that a critical part of game-theoretic modeling goes into choosing the strategies and payoff functions of the players. In the context of oligopolistic markets, this choice requires that considerable thought be given both to the demand and technological features of the market and to the underlying processes of competition.

Unless otherwise noted, we restrict our attention to pure strategy equilibria of the models we study.

The Bertrand Model of Price Competition

We begin by considering the model of oligopolistic competition proposed by Bertrand (1883). There are two profit-maximizing firms, firms 1 and 2 (a *duopoly*), in a market whose demand function is given by $x(p)$. As in Section 10.B, we assume that $x(\cdot)$ is continuous and strictly decreasing at all p such that $x(p) > 0$ and that there exists a $\bar{p} < \infty$ such that $x(p) = 0$ for all $p \geq \bar{p}$. The two firms have constant returns to scale technologies with the same cost, $c > 0$, per unit produced. We assume that $x(c) \in (0, \infty)$, which implies that the socially optimal (competitive) output level in this market is strictly positive and finite (see Chapter 10).

Competition takes place as follows: The two firms simultaneously name their prices p_1 and p_2. Sales for firm j are then given by

$$x_j(p_j, p_k) = \begin{cases} x(p_j) & \text{if } p_j < p_k \\ \frac{1}{2}x(p_j) & \text{if } p_j = p_k \\ 0 & \text{if } p_j > p_k. \end{cases}$$

The firms produce to order and so they incur production costs only for an output level equal to their actual sales. Given prices p_j and p_k, firm j's profits are therefore equal to $(p_j - c)x_j(p_j, p_k)$.

The Bertrand model constitutes a well-defined simultaneous-move game to which we can apply the concepts developed in Chapter 8. In fact, the Nash equilibrium outcome of this model, presented in Proposition 12.C.1, is relatively simple to discern.

Proposition 12.C.1: There is a unique Nash equilibrium (p_1^*, p_2^*) in the Bertrand duopoly model. In this equilibrium, both firms set their prices equal to cost: $p_1^* = p_2^* = c$.

Proof: To begin, note that both firms setting their prices equal to c is indeed a Nash equilibrium. At these prices, both firms earn zero profits. Neither firm can gain by raising its price because it will then make no sales (thereby still earning zero); and by lowering its price below c a firm increases its sales but incurs losses. What remains is to show that there can be no other Nash equilibrium.[9] Suppose, first, that the lower of the two prices named is less than c. In this case, the firm naming this price

8. Section 12.D studies a fourth variation that involves repeated interaction among firms.

9. Recall that we restrict attention to pure strategy equilibria here. See Exercise 12.C.2 for a consideration of mixed strategy equilibria. There you are asked to show that under the conditions assumed here, Proposition 12.C.1 continues to hold: $p_1^* = p_2^* = c$ is the unique Nash equilibrium, pure or mixed, of the Bertrand model.

incurs losses. But by raising its price above c, the worst it can do is earn zero. Thus, these price choices could not constitute a Nash equilibrium.

Now suppose that one firm's price is equal to c and that the other's price is strictly greater than c: $p_j = c$, $p_k > c$. In this case, firm j is selling to the entire market but making zero profits. By raising its price a little, say to $\hat{p}_j = c + (p_k - c)/2$, firm j would still make all the sales in the market, but at a strictly positive profit. Thus, these price choices also could not constitute an equilibrium.

Finally, suppose that both price choices are strictly greater than c: $p_j > c$, $p_k > c$. Without loss of generality, assume that $p_j \le p_k$. In this case, firm k can be earning at most $\frac{1}{2}(p_j - c)x(p_j)$. But by setting its price equal to $p_j - \varepsilon$ for $\varepsilon > 0$, that is, by undercutting firm j's price, firm k will get the entire market and earn $(p_j - \varepsilon - c)x(p_j - \varepsilon)$. Since $(p_j - \varepsilon - c)x(p_j - \varepsilon) > \frac{1}{2}(p_j - c)x(p_j)$ for small-enough $\varepsilon > 0$, firm k can strictly increase its profits by doing so. Thus, these price choices are also not an equilibrium.

The three types of price configurations that we have just ruled out constitute all the possible price configurations other than $p_1 = p_2 = c$, and so we are done. ■

The striking implication of Proposition 12.C.1 is that with only two firms we get the perfectly competitive outcome. In effect, competition between the two firms makes each firm face an infinitely elastic demand curve at the price charged by its rival.

The basic idea of Proposition 12.C.1 can also be readily extended to any number of firms greater than two. [In this case, if firm j names the lowest price in the market, say $\bar{p}$, along with $\tilde{J} - 1$ other firms, it earns $(1/\tilde{J})x(\bar{p})$.] You are asked to show this in Exercise 12.C.1.

Exercise 12.C.1: Show that in any Nash equilibrium of the Bertrand model with $J > 2$ firms, all sales take place at a price equal to cost.

Thus, the Bertrand model predicts that the distortions arising from the exercise of market power are limited to the special case of monopoly. Notable as this result is, it also seems an unrealistic conclusion in many (although not all) settings. In the remainder of this section, we examine three changes in the Bertrand model that considerably weaken this strong conclusion: First, we make *quantity* the firms' strategic variable. Second, we introduce *capacity constraints* (or, more generally, decreasing returns to scale). Third, we allow for *product differentiation.*

Quantity Competition (The Cournot Model)

Suppose now that competition between the two firms takes a somewhat different form: The two firms simultaneously decide how much to produce, q_1 and q_2. Given these quantity choices, price adjusts to the level that clears the market, $p(q_1 + q_2)$, where $p(\cdot) = x^{-1}(\cdot)$ is the inverse demand function. This model is known as the *Cournot model*, after Cournot (1838). You can imagine farmers deciding how much of a perishable crop to pick each morning and send to a market. Once they have done so, the price at the market ends up being the level at which all the crops that have been sent are sold.[10] In this discussion, we assume that $p(\cdot)$ is differentiable

10. One scenario that will lead to this outcome arises when buyers bid for the crops sent that day (very much like sellers in the Bertrand model; see Exercise 12.C.5).

with $p'(q) < 0$ at all $q \geq 0$. As before, both firms produce output at a cost of $c > 0$ per unit. We also assume that $p(0) > c$ and that there exists a unique output level $q^\circ \in (0, \infty)$ such that $p(q^\circ) = c$ [in terms of the demand function $x(\cdot)$, $q^\circ = x(c)$]. Quantity q° is therefore the socially optimal (competitive) output level in this market.

To find a (pure strategy) Nash equilibrium of this model, consider firm j's maximization problem given an output level $\bar{q}_k$ of the other firm, $k \neq j$:

$$\underset{q_j \geq 0}{\text{Max}} \quad p(q_j + \bar{q}_k)q_j - cq_j. \tag{12.C.1}$$

In solving problem (12.C.1), firm j acts exactly like a monopolist who faces inverse demand function $\tilde{p}(q_j) = p(q_j + \bar{q}_k)$. An optimal quantity choice for firm j given its rival's output $\bar{q}_k$ must therefore satisfy the first-order condition

$$p'(q_j + \bar{q}_k)\, q_j + p(q_j + \bar{q}_k) \leq c, \quad \text{with equality if } q_j > 0. \tag{12.C.2}$$

For each $\bar{q}_k$, we let $b_j(\bar{q}_k)$ denote firm j's set of optimal quantity choices; $b_j(\cdot)$ is firm j's *best-response correspondence* (or *function* if it is single-valued).

A pair of quantity choices (q_1^*, q_2^*) is a Nash equilibrium if and only if $q_j^* \in b_j(q_k^*)$ for $k \neq j$ and $j = 1, 2$. Hence, if (q_1^*, q_2^*) is a Nash equilibrium, these quantities must satisfy[11]

$$p'(q_1^* + q_2^*)q_1^* + p(q_1^* + q_2^*) \leq c, \quad \text{with equality if } q_1^* > 0 \tag{12.C.3}$$

and

$$p'(q_1^* + q_2^*)q_2^* + p(q_1^* + q_2^*) \leq c, \quad \text{with equality if } q_2^* > 0. \tag{12.C.4}$$

It can be shown that under our assumptions we must have $(q_1^*, q_2^*) \gg 0$, and so conditions (12.C.3) and (12.C.4) must both hold with equality in any Nash equilibrium.[12] Adding these two equalities tells us that in any Nash equilibrium we must have

$$p'(q_1^* + q_2^*)\left(\frac{q_1^* + q_2^*}{2}\right) + p(q_1^* + q_2^*) = c. \tag{12.C.5}$$

Condition (12.C.5) allows us to reach the conclusion presented in Proposition 12.C.2.

Proposition 12.C.2: In any Nash equilibrium of the Cournot duopoly model with cost $c > 0$ per unit for the two firms and an inverse demand function $p(\cdot)$ satisfying $p'(q) < 0$ for all $q \geq 0$ and $p(0) > c$, the market price is greater than c (the competitive price) and smaller than the monopoly price.

11. Note that this method of analysis, which relies on the use of first-order conditions to calculate best responses, differs from the method used in the analysis of the Bertrand model. The reason is that in the Bertrand model each firm's objective function is discontinuous in its decision variable, so that differential optimization techniques cannot be used. Fortunately, the determination of the Nash equilibrium in the Bertrand model turned out, nevertheless, to be quite simple.

12. To see this, suppose that $q_1^* = 0$. Condition (12.C.3) then implies that $p(q_2^*) \leq c$. By condition (12.C.4) and the fact that $p'(\cdot) < 0$, this implies that were $q_2^* > 0$ we would have $p'(q_2^*)q_2^* + p(q_2^*) < c$, and so $q_2^* = 0$. But this means that $p(0) \leq c$, contradicting the assumption that $p(0) > c$. Hence, we must have $q_1^* > 0$. A similar argument shows that $q_2^* > 0$. Note, however, that this conclusion depends on our assumption of equal costs for the two firms. For example, a firm might set its output equal to zero if it is much less efficient than its rival. Exercise 12.C.9 considers some of the issues that arise when firms have differing costs.

Proof: That the equilibrium price is above c (the competitive price) follows immediately from condition (12.C.5) and the facts that $q_1^* + q_2^* > 0$ and $p'(q) < 0$ at all $q \geq 0$. We next argue that $(q_1^* + q_2^*) > q^m$, that is, that the equilibrium duopoly price $p(q_1^* + q_2^*)$ is strictly less than the monopoly price $p(q^m)$. The argument is in two parts.

First, we argue that $(q_1^* + q_2^*) \geq q^m$. To see this, suppose that $q^m > (q_1^* + q_2^*)$. By increasing its quantity to $\hat{q}_j = q^m - q_k^*$, firm j would (weakly) increase the joint profit of the two firms (the firms' joint profit then equals the monopoly profit level, its largest possible level). In addition, because aggregate quantity increases, price must fall, and so firm k is strictly worse off. This implies that firm j is strictly better off, and so firm j would have a profitable deviation if $q^m > (q_1^* + q_2^*)$. We conclude that we must have $(q_1^* + q_2^*) \geq q^m$.

Second, condition (12.C.5) implies that we cannot have $(q_1^* + q_2^*) = q^m$ because then

$$p'(q^m)\frac{q^m}{2} + p(q^m) = c,$$

in violation of the monopoly first-order condition (12.B.4). Thus, we must in fact have $(q_1^* + q_2^*) > q^m$. ■

Proposition 12.C.2 tells us that the presence of two firms is *not* sufficient to obtain a competitive outcome in the Cournot model, in contrast with the prediction of the Bertrand model. The reason is straightforward. In this model, a firm no longer sees itself as facing an infinitely elastic demand. Rather, if the firm reduces its quantity by a (differential) unit, it increases the market price by $-p'(q_1 + q_2)$. If the firms found themselves jointly producing the competitive quantity and consequently earning zero profits, either one could do strictly better by reducing its output slightly.

At the same time, competition does lower the price below the monopoly level, the price that would maximize the firms' joint profit. This occurs because when each firm determines the profitability of selling an additional unit it fails to consider the reduction in its rival's profit that is caused by the ensuing decrease in the market price [note that in firm j's first-order condition (12.C.2), only q_j multiplies the term $p'(\cdot)$, whereas in the first-order condition for joint profit maximization $(q_1 + q_2)$ does].

Example 12.C.1: *Cournot Duopoly with a Linear Inverse Demand Function and Constant Returns to Scale.* Consider a Cournot duopoly in which the firms have a cost per unit produced of c and the inverse demand function is $p(q) = a - bq$, with $a > c \geq 0$ and $b > 0$. Recall that the monopoly quantity and price are $q^m = (a - c)/2b$ and $p^m = (a + c)/2$ and that the socially optimal (competitive) output and price are $q^\circ = (a - c)/b$ and $p^\circ = p(q^\circ) = c$. Using the first-order condition (12.C.2), we find that firm j's best-response function in this Cournot model is given by $b_j(q_k) = \text{Max}\,\{0, (a - c - bq_k)/2b\}$.

Firm 1's best-response function $b_1(q_2)$ is depicted graphically in Figure 12.C.1. Since $b_1(0) = (a - c)/2b$, its graph hits the q_1 axis at the monopoly output level $(a - c)/2b$. This makes sense: Firm 1's best response to firm 2 producing no output is to produce exactly its monopoly output level. Similarly, since $b_1(q_2) = 0$ for all

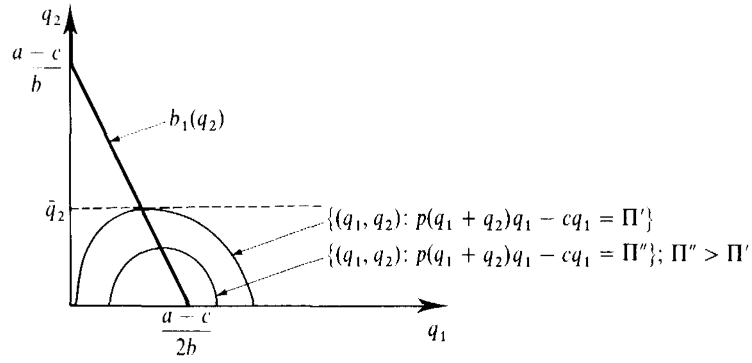


Figure 12.C.1
Firm 1's best-response function in the Cournot duopoly model of Example 12.C.1.

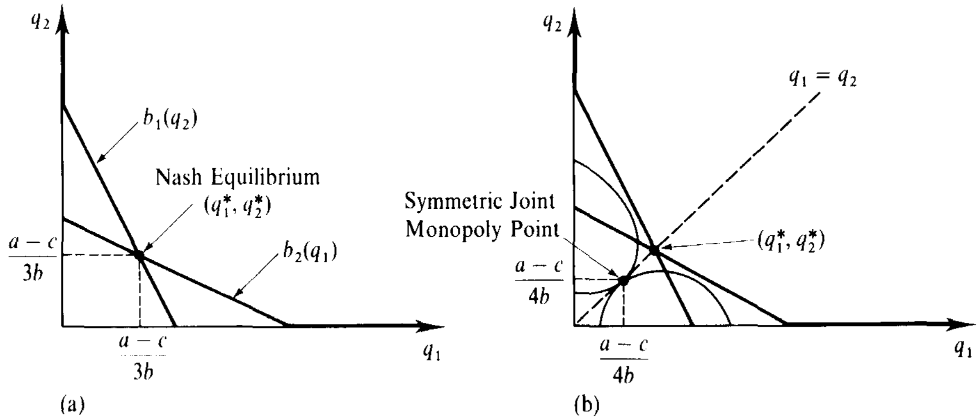


Figure 12.C.2
Nash equilibrium in the Cournot duopoly model of Example 12.C.1.

$q_2 \geq (a-c)/b$, the graph of firm 1's best-response function hits the q_2 axis at the socially optimal (competitive) output level $(a-c)/b$. Again, this makes sense: If firm 2 chooses an output level of at least $(a-c)/b$, any attempt by firm 1 to make sales results in a price below c. Two isoprofit loci of firm 1 are also drawn in the figure; these are sets of the form $\{(q_1, q_2)\colon p(q_1+q_2)q_1 - cq_1 = \Pi\}$ for some profit level Π. The profit levels associated with these loci increase as we move toward firm 1's monopoly point $(q_1, q_2) = ((a-c)/2b, 0)$. Observe that firm 1's isoprofit loci have a zero slope where they cross the graph of firm 1's best-response function. This is because the best response $b_1(\bar{q}_2)$ identifies firm 1's maximal profit point on the line $q_2 = \bar{q}_2$ and must therefore correspond to a point of tangency between this line and an isoprofit locus. Firm 2's best-response function can be depicted similarly; given the symmetry of the firms, it is located symmetrically with respect to firm 1's best-response function in (q_1, q_2)-space [i.e., it hits the q_2 axis at $(a-c)/2b$ and hits the q_1 axis at $(a-c)/b$].

The Nash equilibrium, which in this example is unique, can be computed by finding the output pair (q_1^*, q_2^*) at which the graphs of the two best-response functions intersect, that is, at which $q_1^* = b_1(q_2^*)$ and $q_2^* = b_2(q_1^*)$. It is depicted in Figure 12.C.2(a) and corresponds to individual outputs of $q_1^* = q_2^* = \frac{1}{3}[(a-c)/b]$, total output of $\frac{2}{3}[(a-c)/b]$, and a market price of $p(q_1^* + q_2^*) = \frac{1}{3}(a+2c) \in (c, p^m)$.

Also shown in Figure 12.C.2(b) is the symmetric joint monopoly point $(q^m/2, q^m/2) = ((a-c)/4b, (a-c)/4b)$. It can be seen that this point, at which each

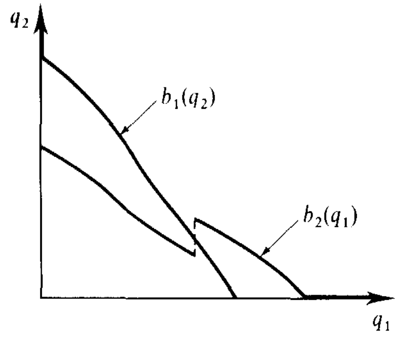


Figure 12.C.3
Nonexistence of (pure strategy) Nash equilibrium in the Cournot model.

firm produces half of the monopoly output of $(a - c)/2b$, is each firm's most profitable point on the $q_1 = q_2$ ray. ■

Exercise 12.C.6: Verify the computations and other claims in Example 12.C.1.

Up to this point we have not made any assumptions about the quasiconcavity in q_j of each firm j's objective function in problem (12.C.1). Without quasiconcavity of these functions, however, a pure strategy Nash equilibrium of this quantity game may not exist. For example, as happens in Figure 12.C.3, the best-response function of a firm lacking a quasiconcave objective function may "jump," leading to the possibility of nonexistence. (Strictly speaking, for a situation like the one depicted in Figure 12.C.3 to arise, the two firms must have different cost functions; see Exercise 12.C.8.) With quasiconcavity, we can use Proposition 8.D.3 to show that a pure strategy Nash equilibrium necessarily exists.

Suppose now that we have $J > 2$ identical firms facing the same cost and demand functions as above. Letting Q_J^* be aggregate output at equilibrium, an argument parallel to that above leads to the following generalization of condition (12.C.5):

$$p'(Q_J^*)\frac{Q_J^*}{J} + p(Q_J^*) = c. \tag{12.C.6}$$

At one extreme, when $J = 1$, condition (12.C.6) coincides with the monopoly first-order condition that we have seen in Section 12.B. At the other extreme, we must have $p(Q_J^*) \to c$ as $J \to \infty$. To see this, note that since Q_J^* is always less than the socially optimal (competitive) quantity q^o, it must be the case that $p'(Q_J^*)(Q_J^*/J) \to 0$ as $J \to \infty$. Hence, condition (12.C.6) implies that price must approach marginal cost as the number of firms grows infinitely large. This provides us with our first taste of a "competitive limit" result, a topic we shall return to in Section 12.F. Exercise 12.C.7 asks you to verify these claims for the model of Example 12.C.1.

Exercise 12.C.7: Derive the Nash equilibrium price and quantity levels in the Cournot model with J firms where each firm has a constant unit production cost of c and the inverse demand function in the market is $p(q) = a - bq$, with $a > c \geq 0$ and $b > 0$. Verify that when $J = 1$, we get the monopoly outcome; that output rises and price falls as J increases; and that as $J \to \infty$ the price and aggregate output in the market approach their competitive levels.

In contrast with the Bertrand model, the Cournot model displays a gradual reduction in market power as the number of firms increases. Yet, the "farmer sending crops to market" scenario may not seem relevant to a wide class of situations. After all, most firms seem to choose their prices, not their quantities. For this reason, many economists have thought that the Cournot model gives the right answer for the wrong reason. Fortunately, the departure from the Bertrand model that we study next offers an alternative interpretation of the Cournot model. The basic idea is that we can think of the quantity choices in the Cournot model as long-run choices of *capacity*, with the determination of price from the inverse demand function being a proxy for the outcome of short-run price competition given these capacity choices.

Capacity Constraints and Decreasing Returns to Scale

In many settings, it is natural to suppose that firms operate under conditions of eventual decreasing returns to scale, at least in the short run when capital is fixed. One special case of decreasing returns occurs when a firm has a capacity constraint that prevents it from producing more than some maximal amount, say $\bar{q}$. Here we consider, somewhat informally, how the introduction of capacity constraints affects the prediction of the Bertrand model.

With capacity constraints (or, for that matter, costs that exhibit decreasing returns to scale in a smoother way), it is no longer sensible to assume that a price announcement represents a commitment to provide *any* demanded quantity, since the costs of an order larger than capacity are infinite. We therefore make a minimal adjustment to the rules of the Bertrand model by taking price announcements to be a commitment to supply demand only up to capacity. We also assume that capacities are commonly known among the firms.

To see how capacity constraints can affect the outcome of the duopoly pricing game, suppose that each of the two firms has a constant marginal cost of $c > 0$ and a capacity constraint of $\bar{q} = \frac{3}{4}x(c)$. As before, the market demand function $x(\cdot)$ is continuous, is strictly decreasing at all p such that $x(p) > 0$, and has $x(c) > 0$.

In this case, the Bertrand outcome $p_1^* = p_2^* = c$ is no longer an equilibrium. To see this, note that because firm 2 cannot supply all demand at price $p_2^* = c$, firm 1 can anticipate making a strictly positive level of sales if it raises p_1 slightly above c. As a result, it has an incentive to deviate from $p_1^* = c$.

In fact, whenever the capacity level $\bar{q}$ satisfies $\bar{q} < x(c)$, each firm can *assure* itself of a strictly positive level of sales at a strictly positive profit margin by setting its price below $p(\bar{q})$ but above c. This is illustrated in Figure 12.C.4. In the figure, we assume that the lower-priced firm 2 fills the highest-valuation demands. By charging

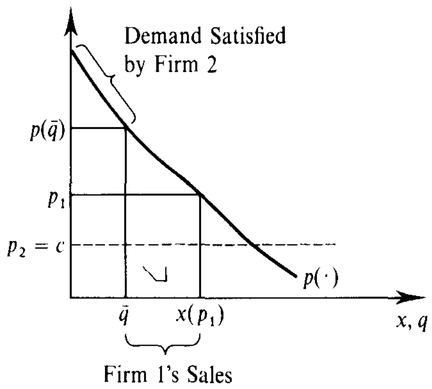


Figure 12.C.4
Calculation of demand in the presence of capacity constraints when the low-priced firm satisfies high-valuation demands first.

a price $p_1 \in (c, p(\bar{q}))$, firm 1 sells to the remaining demand at price p_1, making sales of $x(p_1) - \bar{q} > 0$. Hence, with capacity constraints, competition will not generally drive price down to cost, a point originally noted by Edgeworth (1897).

Determining the equilibrium outcome in situations in which capacity constraints are present can be tricky because knowledge of prices is no longer enough to determine each firm's sales. When the prices quoted are such that the low-priced firm cannot supply all demand at its quoted price, the demand for the higher-priced firm will generally depend on precisely *who* manages to buy from the low-priced firm. The high-priced firm will typically have greater sales if consumers with low valuations buy from the low-priced firm (in contrast with the assumption made in Figure 12.C.4) than if high-valuation consumers do. Thus, to determine demand functions for the firms, we now need to state a *rationing rule* specifying which consumers manage to buy from the low-priced firm when demand exceeds its capacity. In fact, the choice of a rationing rule can have important effects on equilibrium behavior. Exercise 12.C.11 asks you to explore some of the features of the equilibrium outcome when the highest valuation demands are served first, as in Figure 12.C.4. This is the rationing rule that tends to give the nicest results. Yet, it is neither more nor less plausible than other rules, such as a queue system or a random allocation of available units among possible buyers.

Up to this point in our discussion, we have taken a firm's capacity level as exogenous. Typically, however, we think of firms as *choosing* their capacity levels. This raises a natural question: What is the outcome in a model in which firms first choose their capacity levels and then compete in prices? Kreps and Scheinkman (1983) address this question and show that under certain conditions (among these is the assumption that high-valuation demands get served first when demand for a low-priced firm outstrips its capacity), the unique subgame perfect Nash equilibrium in this two-stage model is the *Cournot outcome*. This result is natural: the computation of price from the inverse demand curve in the Cournot model can be thought of as a proxy for this second-stage price competition. Indeed, for a wide range of capacity choices $(\bar{q}_1, \bar{q}_2)$, the unique equilibrium of the pricing subgame involves both firms setting their prices equal to $p(\bar{q}_1 + \bar{q}_2)$ (see Exercise 12.C.11). Thus, this two-stage model of capacity choice/price competition gives us the promised reinterpretation of the Cournot model: We can think of Cournot quantity competition as capturing long-run competition through capacity choice, with price competition occurring in the short run given the chosen levels of capacity.

Product Differentiation

In the Bertrand model, firms faced an infinitely elastic demand curve in equilibrium: With an arbitrarily small price differential, every consumer would prefer to buy from the lowest-priced firm. Often, however, consumers perceive differences among the products of different firms. When product differentiation exists, each firm will possess some market power as a result of the uniqueness of its product. Suppose, for example, that there are $J > 1$ firms. Each firm produces at a constant marginal cost of $c > 0$. The demand for firm j's product is given by the continuous function $x_j(p_j, p_{-j})$, where p_{-j} is a vector of prices of firm j's rivals.[13] In a setting of simultaneous price

13. Note the departure from the Bertrand model: In the Bertrand model, $x_j(p_j, p_{-j})$ is discontinuous at $p_j = \text{Min}_{k \neq j}\, p_k$.

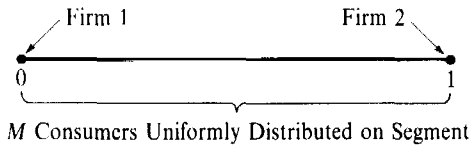


Figure 12.C.5 The linear city.

choices, each firm j takes its rivals' price choices $\bar{p}_{-j}$ as given and chooses p_j to solve

$$\underset{p_j}{\text{Max}} \quad (p_j - c)x_j(p_j, \bar{p}_{-j}).$$

Note that as long as $x_j(c, \bar{p}_{-j}) > 0$, firm j's best response necessarily involves a price in excess of its costs ($p_j > c$) because it can assure itself of strictly positive profits by setting its price slightly above c. Thus, in the presence of product differentiation, equilibrium prices will be above the competitive level. As with quantity competition and capacity constraints, the presence of product differentiation softens the strongly competitive result of the Bertrand model.

A number of models of product differentiation are popular in the applied literature. Example 12.C.2 describes one in some detail.

Example 12.C.2: *The Linear City Model of Product Differentiation.* Consider a city that can be represented as lying on a line segment of length 1, as shown in Figure 12.C.5. There is a continuum of consumers whose total number (or, more precisely, measure) is M and who are assumed to be located uniformly along this line segment. A consumer's location is indexed by $z \in [0, 1]$, the distance from the left end of the city. At each end of the city is located one supplier of widgets: Firm 1 is at the left end; firm 2, at the right. Widgets are produced at a constant unit cost of $c > 0$. Every consumer wants at most 1 widget and derives a gross benefit of v from its consumption. The total cost of buying from firm j for a consumer located a distance d from firm j is $p_j + td$, where $t/2 > 0$ can be thought of as the cost or disutility per unit of distance traveled by the consumer in going to and from firm j's location. The presence of travel costs introduces differentiation between the two firms' products because various consumers may now strictly prefer purchasing from one of the two firms even when the goods sell at the same price.

Figure 12.C.6(a) illustrates the purchase decisions of consumers located at various points in the city for a given pair of prices p_1 and p_2. Consumers at locations $[0, z_1)$

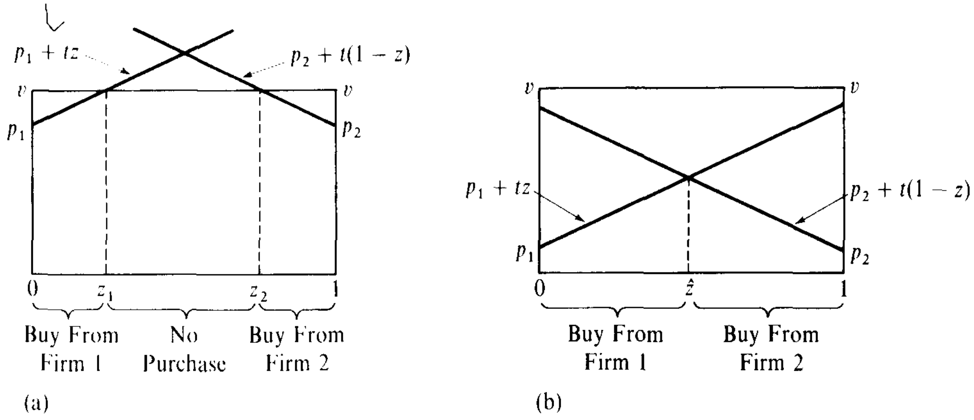


Figure 12.C.6 Consumer purchase decisions given p_1 and p_2. (a) Some consumers do not buy. (b) All consumers buy.

buy from firm 1. At these locations, $p_1 + tz < p_2 + t(1 - z)$ (purchasing from firm 1 is better than purchasing from firm 2), and $v - p_1 - tz > 0$ (purchasing from firm 1 is better than not purchasing at all). At location z_1, a consumer is indifferent between purchasing from firm 1 and not purchasing at all; that is, z_1 satisfies $v - p_1 - tz_1 = 0$. In Figure 12.C.6(a), consumers in the interval (z_1, z_2) do not purchase from either firm, while those in the interval $(z_2, 1]$ buy from firm 2.

Figure 12.C.6(b), by contrast, depicts a case in which, given prices p_1 and p_2, all consumers can obtain a strictly positive surplus by purchasing the good from one of the firms. The location of the consumer who is indifferent between the two firms is the point $\hat{z}$ such that

$$p_1 + t\hat{z} = p_2 + t(1 - \hat{z})$$

or

$$\hat{z} = \frac{t + p_2 - p_1}{2t}. \tag{12.C.7}$$

In general, the analysis of this model is complicated by the fact that depending on the parameters (v, c, t), the equilibria may involve market areas for the firms that do not touch [as in Figure 12.C.6(a)], or may have the firms battling for consumers in the middle of the market [as in Figure 12.C.6(b)]. To keep things as simple as possible here, we shall assume that consumers' value from a widget is large relative to production and travel costs, or more precisely, that $v > c + 3t$. In this case, it can be shown that a firm never wants to set its price at a level that causes some consumers not to purchase from either firm (see Exercise 12.C.13). In what follows, we shall therefore ignore the possibility of nonpurchase.

Given p_1 and p_2, let $\hat{z}$ be defined as in (12.C.7). Then firm 1's demand, given a pair of prices (p_1, p_2), equals $M\hat{z}$ when $\hat{z} \in [0, 1]$, M when $\hat{z} > 1$, and 0 when $\hat{z} < 0$.[14] Substituting for $\hat{z}$ from (12.C.7), we have

$$x_1(p_1, p_2) = \begin{cases} 0 & \text{if } p_1 > p_2 + t \\ (t + p_2 - p_1)M/2t & \text{if } p_1 \in [p_2 - t, p_2 + t] \\ M & \text{if } p_1 < p_2 - t. \end{cases} \tag{12.C.8}$$

By the symmetry of the two firms, the demand function of firm 2, $x_2(p_1, p_2)$, is

$$x_2(p_1, p_2) = \begin{cases} 0 & \text{if } p_2 > p_1 + t \\ (t + p_1 - p_2)M/2t & \text{if } p_2 \in [p_1 - t, p_1 + t] \\ M & \text{if } p_2 < p_1 - t. \end{cases} \tag{12.C.9}$$

Note from (12.C.8) and (12.C.9) that each firm j, in searching for its best response to any price choice $\bar{p}_{-j}$ by its rival, can restrict itself to prices in the interval $[\bar{p}_{-j} - t, \bar{p}_{-j} + t]$. Any price $p_j > \bar{p}_{-j} + t$ yields the same profits as setting $p_j = \bar{p}_{-j} + t$ (namely, zero), and any price $p_j < \bar{p}_{-j} - t$ yields lower profits than setting $p_j = \bar{p}_{-j} - t$ (all such prices result in sales of M units). Thus, firm j's best

14. Recall that the M consumers are uniformly distributed on the line segment, so $\hat{z}$ is the fraction who buy from firm 1.

response to $\bar{p}_{-j}$ solves

$$\begin{aligned} \underset{p_j}{\text{Max}} \quad & (p_j - c)(t + \bar{p}_{-j} - p_j)\frac{M}{2t} \\ \text{s.t. } & p_j \in [\bar{p}_{-j} - t, \bar{p}_{-j} + t]. \end{aligned} \tag{12.C.10}$$

The necessary and sufficient (Kuhn–Tucker) first-order condition for this problem is

$$t + \bar{p}_{-j} + c - 2p_j \begin{cases} \leq 0 & \text{if } p_j = \bar{p}_{-j} - t \\ = 0 & \text{if } p_j \in (\bar{p}_{-j} - t, \bar{p}_{-j} + t) \\ \geq 0 & \text{if } p_j = \bar{p}_{-j} + t. \end{cases} \tag{12.C.11}$$

Solving (12.C.11), we find that each firm j's best-response function is

$$b(\bar{p}_{-j}) = \begin{cases} \bar{p}_{-j} + t & \text{if } \bar{p}_{-j} \leq c - t \\ (t + \bar{p}_{-j} + c)/2 & \text{if } \bar{p}_{-j} \in (c - t, c + 3t) \\ \bar{p}_{-j} - t & \text{if } \bar{p}_{-j} \geq c + 3t. \end{cases} \tag{12.C.12}$$

When $\bar{p}_{-j} < c - t$, firm j prices in a manner that leads its sales to equal zero (it cannot make profits because it cannot make sales at any price above c). When $\bar{p}_{-j} > c + 3t$, firm j prices in a manner that captures the entire market. In the intermediate case, firm j's best response to $\bar{p}_{-j}$ leaves both firms with strictly positive sales levels.

Given the symmetry of the model, we look for a symmetric equilibrium, that is, an equilibrium in which $p_1^* = p_2^* = p^*$. In any symmetric equilibrium, $p^* = b(p^*)$. Examining (12.C.12), we see that this condition can be satisfied only in the middle case (note also that this is the only case in which both firms can have strictly positive sales, as they must in any symmetric equilibrium). Thus, p^* must satisfy

$$p^* = \tfrac{1}{2}(t + p^* + c),$$

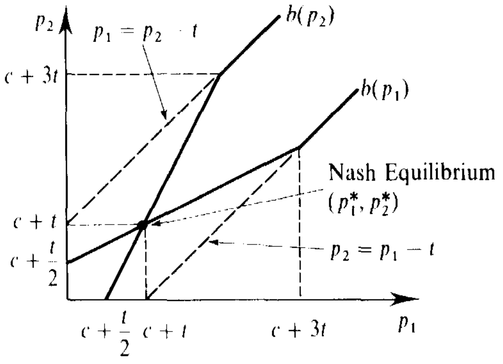
and so

$$p^* = c + t.$$

In this Nash equilibrium, each firm has sales of $M/2$ and a profit of $tM/2$. Note that as t approaches zero, the firms' products become completely undifferentiated and the equilibrium prices approach c, as in the Bertrand model. In the other direction, as the travel cost t becomes greater, thereby increasing the differentiation between the firms' products, equilibrium prices and profits increase.

Figure 12.C.7 depicts the best-response functions for the two firms (for prices greater than or equal to c) and the Nash equilibrium. As usual, the Nash equilibrium lies at the intersection of the graphs of these best-response functions. Note that there are no asymmetric equilibria here. ■

Matters become more complicated when $v < c + 3t$ because firms may wish to set prices at which some consumers do not want to purchase from either firm. One can show, however, that the equilibrium just derived remains valid as long as $v \geq c + \frac{3}{2}t$. In contrast, when $v < c + t$, in equilibrium the firms' market areas do not touch (the firms are like "local monopolists"). In the intermediate case where $v \in [c + t, c + \frac{3}{2}t]$, firms are at a "kink" in their demand functions and the consumer at the indifferent location $\hat{z}$ receives no surplus from his purchase in the equilibrium. Exercise 12.C.14 asks you to investigate these cases.



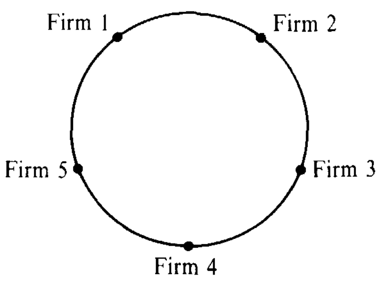


Figure 12.C.7 (left) Best-response functions and Nash equilibrium in the linear city model when $v > c + 3t$.

Figure 12.C.8 (right) The circular city model when $J = 5$.

The essential features of the linear city model can be extended to the case in which $J > 2$. In doing so, it is often most convenient for analytical purposes to consider instead a model of a *circular city*, so that firms can be kept in symmetric positions.[15] In this model, which is due to Salop (1979), consumers are uniformly distributed along a circle of circumference 1, and the firms are positioned at equal intervals from one another. Figure 12.C.8 depicts a case where $J = 5$.

Models like the linear and circular city models are known as *spatial* models of product differentiation because each firm is identified with an "address" in product space. More generally, we can imagine firms' products located in some N-dimensional characteristics space, with consumers' "addresses" (their ideal points of consumption) distributed over this space.

Spatial models share the characteristic that each firm competes for customers only locally, that is, solely with the firms offering similar products. A commonly used alternative to spatial formulations, in which each product competes instead for sales with *all* other products, is the *representative consumer model* introduced by Spence (1976) and Dixit and Stiglitz (1977). In this model, a representative consumer is postulated whose preferences over consumption of the J products $(x_1, \ldots, x_J)$ and a numeraire good m take the quasilinear form

$$u(m, x_1, \ldots, x_J) = G\left(\sum_{j=1}^{J} f(x_j)\right) + m,$$

where both $G(\cdot)$ and $f(\cdot)$ are concave.[16] Normalizing the price of the numeraire to be 1, the first-order conditions for the representative consumer's maximization problem are

$$G'\left(\sum_{j=1}^{J} f(x_j)\right) f'(x_j) = p_j \qquad \text{for } j = 1, \ldots, J. \tag{12.C.13}$$

These first-order conditions can be inverted to yield demand functions $x_j(p_1, \ldots, p_J)$ for $j = 1, \ldots, J$, which can then be used to specify a game of simultaneous price choices.[17]

An important variant of this representative consumer model arises in the limiting case where we have many products, each of which constitutes a small fraction of the sales in the overall market. In the limit, we can write the representative consumer's utility function as $G(\int f(x_j)\,dj) + m$, where x_j is now viewed as a function of the continuous index variable j.

15. In the segment $[0, 1]$, only with two firms can we have symmetric positioning. With more than two firms, the two firms closest to the endpoints of the segment would have only one nearest neighbor but the firms in the interior would have two.

16. Dixit and Stiglitz (1977) actually consider more general utility functions of the form $u(G(\sum_j f(x_j)), m)$.

17. It is also common in the literature to study games of simultaneous quantity choices, using the expression in (12.C.13) directly as the inverse demand functions for the firms.

This leads to a considerable simplification because each firm j, in deciding on its price choice, can take the value of $\bar{x} = \int f(x_j)\, dj$, called the *index of aggregate output*, as given; its own production has no effect on the value of this index. Given the value of $\bar{x}$, firm j faces the demand function

$$x_j(p_j, \bar{x}) = \psi\left(\frac{p_j}{G'(\bar{x})}\right),$$

where $\psi(\cdot) = f'^{-1}(\cdot)$. Its optimal choice can then be viewed as a function $p_j^*(\bar{x})$ of the index $\bar{x}$. Thus, the equilibrium value of the aggregate output index, say $\bar{x}^*$, satisfies $\bar{x}^* = \int f(x_j(p_j^*(\bar{x}^*), \bar{x}^*))\, dj$.

This limiting case is known as the *monopolistic competition model*. It originates in Chamberlin (1933); see Hart (1985) for a modern treatment. In markets characterized by monopolistic competition, market power is accompanied by a low level of strategic interaction, in that the strategies of any particular firm do not affect the payoff of any other firm.[18]

12.D Repeated Interaction

One unrealistic assumption in the models presented in Section 12.C was their static, one-shot nature. In these models, a firm never had to consider the reaction of its competitors to its price or quantity choice. In the Bertrand model, for instance, a firm could undercut its rival's price by a penny and steal all the rival's customers. In practice, however, a firm in this circumstance may well worry that if it does undercut its rival in this manner, the rival will respond by cutting its own price, ultimately leading to only a short-run gain in sales but a long-run reduction in the price level in the market.

In this section, we consider the simplest type of dynamic model in which these concerns arise. Two identical firms complete for sales repeatedly, with competition in each period t described by the Bertrand model. When they do so, the two firms know all the prices that have been chosen (by *both* firms) previously. There is a discount factor $\delta < 1$, and each firm j attempts to maximize the discounted value of profits, $\sum_{t=1}^{\infty} \delta^{t-1} \pi_{jt}$, where π_{jt} is firm j's profit in period t. The game that this situation gives rise to is a dynamic game (see Chapter 9) of a special kind: it is obtained by repeated play of the same static simultaneous-move game and is known as a *repeated game*.

In this repeated Bertrand game, firm j's strategy specifies what price p_{jt} it will charge in each period t as a function of the history of all past price choices by the two firms, $H_{t-1} = \{p_{1\tau}, p_{2\tau}\}_{\tau=1}^{t-1}$. Strategies of this form allow for a range of interesting behaviors. For example, a firm's strategy could call for retaliation if the firm's rival ever lowers its price below some "threshold price." This retaliation could be brief, calling for the firm to lower its price for only a few periods after the rival "crosses the line," or it could be unrelenting. The retaliation could be tailored to the amount by which the firm's rival undercut it, or it could be severe no matter how minor the rival's transgression. The firm could also respond with increasingly cooperative

18. In contrast, in spatial models, even in the limit of a continuum of firms, strategic interaction remains. In that case, firms interact locally, and neighbors count, no matter how large the economy is.

behavior in return for its rival acting cooperatively in the past. And, of course, the firm's strategy could also make the firm's behavior in any period t independent of past history (a strategy involving no retaliation or rewards).

Of particular interest to us is the possibility that these types of behavioral responses could allow firms, in settings of repeated interaction, to sustain behavior more cooperative than the outcome predicted by the simple one-shot Bertrand model. We explore this possibility in the remainder of this section.

We begin by considering the case in which the firms compete only a finite number of times T (this is known as a *finitely repeated game*). Can the rich set of possible behaviors just described actually arise in a subgame perfect Nash equilibrium of this model? Recalling Proposition 9.B.4, we see the answer is "no." The unique subgame perfect Nash equilibrium of the finitely repeated Bertrand game simply involves T repetitions of the static Bertrand equilibrium in which prices equal cost. This is a simple consequence of backward induction: In the last period, T, we must be at the Bertrand solution, and therefore profits are zero in that period *regardless of what has happened earlier*. But, then, in period $T-1$ we are, strategically speaking, at the last period, and the Bertrand solution must arise again. And so on, until we get to the first period. In summary, backward induction rules out the possibility of more cooperative behavior in the finitely repeated Bertrand game.

Things can change dramatically, however, when the horizon is extended to an infinite number of periods (this is known as an *infinitely repeated game*). To see this, consider the following strategies for firms $j = 1, 2$:

$$p_{jt}(H_{t-1}) = \begin{cases} p^m & \text{if all elements of } H_{t-1} \text{ equal } (p^m, p^m) \text{ or } t = 1 \\ c & \text{otherwise.} \end{cases} \qquad (12.D.1)$$

In words, firm j's strategy calls for it to initially play the monopoly price p^m in period 1. Then, in each period $t > 1$, firm j plays p^m if in every previous period both firms have charged price p^m and otherwise charges a price equal to cost. This type of strategy is called a *Nash reversion strategy*: Firms cooperate until someone deviates, and any deviation triggers a permanent retaliation in which both firms thereafter set their prices equal to cost, the one-period Nash strategy. Note that if both firms follow the strategies in (12.D.1), then both firms will end up charging the monopoly price in every period. They start by charging p^m, and therefore no deviation from p^m will ever be triggered.

For the strategies in (12.D.1), we have the result presented in Proposition 12.D.1.

Proposition 12.D.1: *The strategies described in (12.D.1) constitute a subgame perfect Nash equilibrium (SPNE) of the infinitely repeated Bertrand duopoly game if and only if $\delta \geq \frac{1}{2}$.*

Proof: Recall that a set of strategies is an SPNE of an infinite horizon game if and only if it specifies Nash equilibrium play in every subgame (see Section 9.B). To start, note that although each subgame of this repeated game has a distinct history of play leading to it, all of these subgames have an identical structure: Each is an infinitely repeated Bertrand duopoly game exactly like the game as a whole. Thus, to establish that the strategies in (12.D.1) constitute an SPNE, we need to show that after any previous history of play, the strategies specified for the remainder of the game constitute a Nash equilibrium of an infinitely repeated Bertrand game.

In fact, given the form of the strategies in (12.D.1), we need to be concerned with only two types of previous histories: those in which there has been a previous deviation (a price not equal to p^m) and those in which there has been no deviation.

Consider, first, a subgame arising after a deviation has occurred. The strategies call for each firm to set its price equal to c in every future period regardless of its rival's behavior. This pair of strategies is a Nash equilibrium of an infinitely repeated Bertrand game because each firm j can earn at most zero when its opponent always sets its price equal to c, and it earns exactly this amount by itself setting its price equal to c in every remaining period.

Now consider a subgame starting in, say, period t after no previous deviation has occurred. Each firm j knows that its rival's strategy calls for it to charge p^m until it encounters a deviation from p^m and to charge c thereafter. Is it in firm j's interest to use this strategy itself given that its rival does? That is, do these strategies constitute a Nash equilibrium in this subgame?

Suppose that firm j contemplates deviating from price p^m in period $\tau \geq t$ of the subgame if no deviation has occurred prior to period τ.[19] From period t through period $\tau - 1$, firm j will earn $\frac{1}{2}(p^m - c)x(p^m)$ in each period, exactly as it does if it never deviates. Starting in period τ, however, its payoffs will differ from those that would arise if it does not deviate. In periods after it deviates (periods $\tau + 1, \tau + 2, \ldots$), firm j's rival charges a price of c regardless of the form of firm j's deviation in period τ, and so firm j can earn at most zero in each of these periods. In period τ, firm j optimally deviates in a manner that maximizes its payoff in that period (note that the payoffs firm j receives in later periods are the same for any deviation from p^m that it makes). It will therefore charge $p^m - \varepsilon$ for some arbitrarily small $\varepsilon > 0$, make all sales in the market, and earn a one-period payoff of $(p^m - c - \varepsilon)x(p^m)$. Thus, its overall discounted payoff from period τ onward as a result of following this deviation strategy, discounted to period τ, can be made arbitrarily close to $(p^m - c)x(p^m)$.

On the other hand, if firm j never deviates, it earns a discounted payoff from period τ onward, discounted to period τ, of $[\frac{1}{2}(p^m - c)x(p^m)]/(1 - \delta)$. Hence, for any t and $\tau \geq t$, firm j will prefer no deviation to deviation in period τ if and only if

$$\frac{1}{1 - \delta}[\tfrac{1}{2}(p^m - c)x(p^m)] \geq (p^m - c)x(p^m),$$

or

$$\delta \geq \tfrac{1}{2}. \qquad (12.D.2)$$

Thus, the strategies in (12.D.1) constitute an SPNE if and only if $\delta \geq \frac{1}{2}$. ■

The implication of Proposition 12.D.1 is that the perfectly competitive outcome of the static Bertrand game may be avoided if the firms foresee infinitely repeated interaction. The reason is that, in contemplating a deviation, each firm takes into account not only the one-period gain it earns from undercutting its rival but also the profits forgone by triggering retaliation. The size of the discount factor δ is

19. From our previous argument, we know that once a deviation has occurred within this subgame, firm j can do no better than to play c in every period given that its rival will do so. Hence, to check whether these strategies form a Nash equilibrium in this subgame, we need only check whether firm j will wish to deviate from p^m if no such deviation has yet occurred.

important here because it affects the relative weights put on the future losses versus the present gain from a deviation. The monopoly price is sustainable if and only if the present value of these future losses is large enough relative to the possible current gain from deviation to keep the firms from going for short-run profits.

The discount factor need not be interpreted literally. For example, in a model in which market demand is growing at rate γ [i.e., $x_t(p) = \gamma^t x(p)$], larger values of γ make the model behave as if there is a larger discount factor because demand growth increases the size of any future losses caused by a current deviation. Alternatively, we can imagine that in each period there is a probability γ that the firms' interaction might end. The larger γ is, the more firms will effectively discount the future. (This interpretation makes clear that the infinitely repeated game framework can be relevant even when the firms may cease their interaction within some finite amount of time; what is needed to fit the analysis into the framework above is a strictly positive probability of continuing upon having reached any period.) Finally, the value of δ can reflect how long it takes to detect a deviation. These interpretations are developed in Exercise 12.D.1.

Although the strategies in (12.D.1) constitute an SPNE when $\delta \geq \frac{1}{2}$, they are *not* the only SPNE of the repeated Bertrand model. In particular, we can obtain the result presented in Proposition 12.D.2.

Proposition 12.D.2: In the infinitely repeated Bertand duopoly game, when $\delta \geq \frac{1}{2}$ repeated choice of any price $p \in [c, p^m]$ can be supported as a subgame perfect Nash equilibrium outcome path using Nash reversion strategies. By contrast, when $\delta < \frac{1}{2}$, any subgame perfect Nash equilibrium outcome path must have all sales occurring at a price equal to c in every period.

Proof: For the first part of the result, we have already shown in Proposition 12.D.1 that repeated choice of price p^m can be sustained as an SPNE outcome when $\delta \geq \frac{1}{2}$. The proof for any price $p \in [c, p^m)$ follows exactly the same lines; simply change price p^m in the strategies of (12.D.1) to $p \in [c, p^m)$.

The proof of the second part of the result is presented in small type.

We now show that all sales must occur at a price equal to c when $\delta < \frac{1}{2}$. To begin, let $v_{jt} = \sum_{\tau \geq t} \delta^{\tau - t} \pi_{jt}$ denote firm j's profits, discounted to period t, when the equilibrium strategies are played from period t onward. Also define $\pi_t = \pi_{1t} + \pi_{2t}$.

Observe that, because every firm j finds it optimal to conform to the equilibrium strategies in every period t, it must be that

$$\pi_t \leq v_{jt} \quad \text{for } j = 1, 2 \text{ and every } t, \tag{12.D.3}$$

since each firm j can obtain a payoff arbitrarily close to π_t in period t by deviating and undercutting the lowest price in the market by an arbitrarily small amount and can assure itself a nonnegative payoff in any period thereafter.

Suppose that there exists at least one period t in which $\pi_t > 0$. We will derive a contradiction. There are two cases to consider:

(i) Suppose, first, that there is a period τ with $\pi_\tau > 0$ such that $\pi_\tau \geq \pi_t$ for all t. If so, then adding (12.D.3) for $t = \tau$ over $j = 1, 2$, we have

$$2\pi_\tau \leq (v_{1\tau} + v_{2\tau}).$$

But $(v_{1\tau} + v_{2\tau}) \leq [1/(1 - \delta)]\pi_\tau$, and so this is impossible if $\delta < \frac{1}{2}$.

(ii) Suppose, instead, that no such period exists; that is, for any period t, there is a period $\tau > t$ such that $\pi_\tau > \pi_t$. Define $\tau(t)$ for $t \geq 1$ recursively as follows: Let $\tau(1) = 1$ and for $t \geq 2$ define $\tau(t) = \text{Min}\,\{\tau > \tau(t-1)\colon \pi_\tau > \pi_{\tau(t-1)}\}$. Note that, for all t, π_t is bounded above by the monopoly profit level $\pi^m = (p^m - c)x(p^m)$ and that the sequence $\{\pi_{\tau(t)}\}_{t=1}^{\infty}$ is monotonically increasing. Hence, as $t \to \infty$, $\pi_{\tau(t)}$ must converge to some $\bar{\pi} \in (0, \pi^m]$ such that $\pi_t < \bar{\pi}$ for all t. Now, adding (12.D.3) over $j = 1, 2$, we see that we must have

$$2\pi_{\tau(t)} \leq v_{1\tau(t)} + v_{2\tau(t)} \tag{12.D.4}$$

for all t. Moreover, $v_{1\tau(t)} + v_{2\tau(t)} \leq [1/(1-\delta)]\bar{\pi}$ for all t, and so we must have

$$2\pi_{\tau(t)} \leq \frac{1}{1-\delta}\,\bar{\pi} \tag{12.D.5}$$

for all t. But when $\delta < \frac{1}{2}$, condition (12.D.5) must be violated for t sufficiently large.

This completes the proof of the proposition. ■

The presence of multiple equilibria identified in Proposition 12.D.2 for $\delta \geq \frac{1}{2}$ is common in infinitely repeated games. Typically, a range of cooperative equilibria is possible for a given level of δ, as is a complete lack of cooperation in the form of the static Nash equilibrium outcome repeated forever.

Proposition 12.D.2 also tells us that the set of SPNE of the repeated Bertrand game grows as δ gets larger.[20] The discontinuous behavior as a function of δ of the set of SPNE displayed in Proposition 12.D.2 is, however, a special feature of the repeated Bertrand model. The repeated Cournot model and models of repeated price competition with differentiated products generally display a smoother increase in the maximal level of joint profits that can be sustained as δ increases (see Exercise 12.D.3).

In fact, a general result in the theory of repeated games, known as the *folk theorem*, tells us the following: In an infinitely repeated game, *any feasible discounted payoffs that give each player, on a per-period basis, more than the lowest payoff that he could guarantee himself in a single play of the simultaneous-move component game can be sustained as the payoffs of an SPNE if players discount the future to a sufficiently small degree.* In Appendix A, we provide a more precise statement and extended discussion of the folk theorem for general repeated games. Its message is clear: Although infinitely repeated games allow for cooperative behavior, they also allow 1
for an *extremely wide range* of possible behavior.

The wide range of equilibria in repeated game models of oligopoly is somewhat disconcerting. From a practical point of view, how do we know which equilibrium behavior will arise? Can "anything happen" in oligopolistic markets? To get around this problem, researchers often assume that symmetrically placed firms will find the symmetric profit-maximizing equilibrium focal (see Section 8.D). However, even restricting attention to the case of symmetric firms, the validity of this assumption is likely to depend on the setting. For example, the history of an industry could make other equilibria focal: An industry that has historically been very noncooperative (maybe because δ has always been low) may find noncooperative outcomes more focal. The assumption that the symmetric profit-maximizing equilibrium arises is

20. Strictly speaking, Proposition 12.D.2 shows this only for the class of stationary, symmetric equilibria (i.e., equilibria in which the firms adopt identical strategies and in which, on the equilibrium path, the actions taken are the same in every period).

more natural when the self-enforcing agreement interpretation of these equilibria is relevant, as when oligopolists secretly meet to discuss their pricing plans. Because antitrust laws preclude oligopolists from writing a formal contract specifying their behavior, any secret collusive agreement among them must be self-enforcing and so must constitute an SPNE. It seems reasonable to think that, in such circumstances, identical firms will therefore agree to the most profitable symmetric SPNE. (If the firms are not identical, similar logic suggests that the firms would agree to an SPNE corresponding to a point on the frontier of their set of SPNE payoffs.)

Finally, just as with the static models discussed in Section 12.C, it is of interest to investigate how the number of firms in a market affects its competitiveness. You are asked to do so in Exercise 12.D.2.

Exercise 12.D.2: Show that with J firms, repeated choice of any price $p \in (c, p^m]$ can be sustained as a stationary SPNE outcome path of the infinitely repeated Bertrand game using Nash reversion strategies *if and only if* $\delta \geq (J-1)/J$. What does this say about the effect of having more firms in a market on the difficulty of sustaining collusion?

In practice, an important feature of many settings of oligopolistic collusion (as well as other settings of cooperation) is that firms are likely to be able to observe their rivals' behavior only imperfectly. For example, as emphasized by Stigler (1960), an oligopolist's rivals may make secret price cuts to consumers. If the market demand is stochastic, a firm will be unable to tell with certainty whether there have been any deviations from collusive pricing simply from observation of its own demand. This possibility leads formally to study of *repeated games with imperfect observability*; see, for example, Green and Porter (1984) and Abreu, Pearce, and Stachetti (1990). A feature of this class of models is that they are able to explain observed breakdowns of cooperation as being an inevitable result of attempts to cooperate in environments characterized by imperfect observability. This is so because equilibrium strategies must be such that some negative realizations of demand result in a breakdown of cooperation if firms are to be prevented from secretly deviating from a collusive scheme.

12.E Entry

In Sections 12.B to 12.D, we analyzed monopolistic and oligopolistic market outcomes, keeping the number of active firms exogenously fixed. In most cases, however, we wish to view the number of firms that will be operating in an industry as an endogeneous variable. Doing so also raises a new question regarding the welfare properties of situations in which market power is present: Is the equilibrium number of firms that enter the market socially efficient? In Section 10.F, we saw that the answer to this question is "yes" in the case of competitive markets as long as an equilibrium exists. In this section, however, we shall see that this is no longer true when market power is present.

We now take the view that there is an infinite (or finite but very large) number of potential firms, each of which could enter and produce the good under consideration if it were profitable to do so. As in Section 10.F, we focus on the case in which all potential firms are identical. (See Exercise 12.E.1 for a case in which they are not.)

A natural way to conceptualize entry in oligopolistic settings is as a two-stage process in which a firm first incurs some setup cost $K > 0$ in entering the industry and then, once this cost is sunk, competes for business. The simplest sort of model that captures this idea has the following structure:

Stage 1: All potential firms simultaneously decide "in" or "out." If a firm decides "in," it pays a setup cost $K > 0$.

Stage 2: All firms that have entered play some oligopolistic game.

The oligopoly game in stage 2 could be any of those considered in Sections 12.C and 12.D.

Formally, this two-stage entry model defines a dynamic game (see Chapter 9). Note that its stage 2 subgames are exactly like the games we have analyzed in the previous sections because, at that stage, the number of firms is fixed. Throughout our discussion we shall assume that for each possible number of active firms, there is a unique, symmetric (across firms) equilibrium in stage 2, and we let π_J denote the profits of a firm in this stage 2 equilibrium when J firms have entered (π_J does not include the entry cost K).

This two-stage entry model provides a very simple representation of the entry process. There is very little dynamic structure, and no firm has any "first-mover" advantage that enables it to deter entry or lessen competition from other firms (see Section 12.G and Appendix B for a discussion of these possibilities).

Consider now the (pure strategy) subgame perfect Nash equilibria (SPNEs) of this model. In any SPNE of this game, no firm must want to change its entry decision given the entry decisions of the other firms. For expositional purposes, we shall also adopt the convention that a firm chooses to enter the market when it is indifferent. With this assumption, there is an equilibrium with J^* firms choosing to enter the market if and only if

$$\pi_{J^*} \geq K \tag{12.E.1}$$

and

$$\pi_{J^*+1} < K. \tag{12.E.2}$$

Condition (12.E.1) says that a firm that has chosen to enter does at least as well by doing so as it would do if it were to change its decision to "out," given the anticipated result of competition with J^* firms. Condition (12.E.2) says that a firm that has decided to remain out of the market does strictly worse by changing its decision to "in," given the anticipated result of competition with $J^* + 1$ firms.

Typically, we expect that π_J is decreasing in J and that $\pi_J \to 0$ and $J \to \infty$. In this case, there is a unique integer $\hat{J}$ such that $\pi_J \geq K$ for all $J \leq \hat{J}$ and $\pi_J < K$ for all $J > \hat{J}$, and so $J^* = \hat{J}$ is the unique equilibrium number of firms.[21,22]

21. Note, however, that although there is a unique number of entrants, there are many equilibria, in each of which the particular firms choosing to enter differ.

22. Without the assumption that firms enter when indifferent, condition (12.E.2) would be a weak inequality. This change in (12.E.2) matters for the identification of the equilibrium number of firms only in the case in which there is an integer number of firms $\tilde{J}$ such that $\pi_{\tilde{J}} = K$ (so that with $\tilde{J}$ firms in the market each firm earns exactly zero net of its entry cost K). When this is so, this change allows both $\tilde{J}$ and $\tilde{J} - 1$ to be equilibria. With minor adaptations but some loss of expositional simplicity, all the points made in this section can be extended to cover this case.

We illustrate the determination of the equilibrium number of firms with two examples in which the stage 2 oligopoly games correspond, respectively, to the Cournot and Bertrand models discussed in Section 12.C.

Example 12.E.1: *Equilibrium Entry with Cournot Competition.* Suppose that competition in stage 2 of the two-stage entry game corresponds to the Cournot model studied in Section 12.C, with $c(q) = cq$, $p(q) = a - bq$, $a > c \geq 0$, and $b > 0$. The stage 2 output per firm, q_J, and profit per firm, π_J, are given (see Exercise 12.C.7) by

$$q_J = \left(\frac{a-c}{b}\right)\left(\frac{1}{J+1}\right), \tag{12.E.3}$$

$$\pi_J = \left(\frac{a-c}{J+1}\right)^2\left(\frac{1}{b}\right). \tag{12.E.4}$$

Note that π_J is strictly decreasing in J and that $\pi_J \to 0$ as $J \to \infty$. Also, $Jq_J \to (a-c)/b$ as $J \to \infty$, so that aggregate quantity approaches the competitive level. Solving for the real number $\tilde{J} \in \mathbb{R}$ at which $\pi_{\tilde{J}} = K$ gives

$$(\tilde{J}+1)^2 = \frac{(a-c)^2}{bK}$$

or

$$\tilde{J} = \frac{(a-c)}{\sqrt{bK}} - 1.$$

The equilibrium number of entrants J^* is the largest integer that is less than or equal to $\tilde{J}$. Note that as K decreases, the number of firms active in the market (weakly) increases, and that as more firms become active, aggregate output increases and price decreases. Indeed, $J^* \to \infty$ as $K \to 0$, and output and price approach their competitive levels. Note also that a proportional increase in demand at every price, captured by a reduction in b, changes the equilibrium number of firms and price in a manner that is identical to a decrease in K. ■

Example 12.E.2: *Equilibrium Entry with Bertrand Competition.* Suppose now that competition in stage 2 of the two-stage entry game takes the form of the Bertrand model studied in Section 10.C. Once again, $c(q) = cq$, $p(q) = a - bq$, $a > c \geq 0$, and $b > 0$. Now $\pi_1 = \pi^m$, the monopoly profit level, and $\pi_J = 0$ for all $J \geq 2$. Thus, assuming that $\pi^m > K$, the SPNE must have $J^* = 1$ and result in the monopoly price and quantity levels. Comparing this result with the result in Example 12.E.1 for the Cournot model, we see that the presence of more intense stage 2 competition here actually *lowers* the ultimate level of competition in the market! ■

Entry and Welfare

Consider now how the number of firms entering an oligopolistic market compares with the number that would maximize social welfare given the presence of oligopolistic competition in the market. We begin by considering this issue for the case of a homogeneous-good industry.

Let q_J be the symmetric equilibrium output per firm when there are J firms in the market. As usual, the inverse demand function is denoted by $p(\cdot)$. Thus, $p(Jq_J)$ is the price when there are J active firms; and so $\pi_J = p(Jq_J)q_J - c(q_J)$, where $c(\cdot)$ is the cost function of a firm after entry. We assume that $c(0) = 0$.