1 Games in Strategic Form and Nash Equilibrium

We begin with a simple, informal example of a game. Rousseau, in his *Discourse on the Origin and Basis of Equality among Men*, comments:

> If a group of hunters set out to take a stag, they are fully aware that they would all have to remain faithfully at their posts in order to succeed; but if a hare happens to pass near one of them, there can be no doubt that he pursued it without qualm, and that once he had caught his prey, he cared very little whether or not he had made his companions miss theirs.[1]

To make this into a game, we need to fill in a few details. Suppose that there are only two hunters, and that they must decide simultaneously whether to hunt for stag or for hare. If both hunt for stag, they will catch one stag and share it equally. If both hunt for hare, they each will catch one hare. If one hunts for hare while the other tries to take a stag, the former will catch a hare and the latter will catch nothing. Each hunter prefers half a stag to one hare.

This is a simple example of a game. The hunters are the players. Each player has the choice between two strategies: hunt stag and hunt hare. The payoff to their choice is the prey. If, for instance, a stag is worth 4 "utils" and a hare is worth 1, then when both players hunt stag each has a payoff of 2 utils. A player who hunts hare has payoff 1, and a player who hunts stag by himself has payoff 0.

What prediction should one make about the outcome of Rousseau's game? Cooperation—both hunting stag—is an equilibrium, or more precisely a "Nash equilibrium," in that neither player has a unilateral incentive to change his strategy. Therefore, stag hunting seems like a possible outcome of the game. However, Rousseau (and later Waltz (1959)) also warns us that cooperation is by no means a foregone conclusion. If each player believes the other will hunt hare, each is better off hunting hare himself. Thus, the noncooperative outcome—both hunting hare—is also a Nash equilibrium, and without more information about the context of the game and the hunters' expectations it is difficult to know which outcome to predict.

This chapter will give precise definitions of a "game" and a "Nash equilibrium," among other concepts, and explore their properties. There are two nearly equivalent ways of describing games: the *strategic* (or *normal*) form and the *extensive* form.[2] Section 1.1 develops the idea of the strategic form and of dominated strategies. Section 1.2 defines the solution concept of Nash equilibrium, which is the starting point of most applications of game theory. Section 1.3 offers a first look at the question of when Nash equilibria exist; it is the one place in this chapter where powerful mathematics is used.

1. Quoted by Ordeshook (1986).
2. Historically, the term "normal form" has been standard, but many game theorists now prefer to use "strategic form," as this formulation treats the players' strategies as primitives of the model.

It may appear at first that the strategic form can model only those games in which the players act simultaneously and once and for all, but this is not the case. Chapter 3 develops the extensive-form description of a game, which explicitly models the timing of the players' decisions. We will then explain how the strategic form can be used to analyze extensive-form games.

1.1 Introduction to Games in Strategic Form and Iterated Strict Dominance⁺

1.1.1 Strategic-Form Games

A game in strategic (or normal) form has three elements: the set of players $i \in \mathscr{I}$, which we take to be the finite set $\{1, 2, \ldots, I\}$, the *pure-strategy space* S_i for each player i, and *payoff functions* u_i that give player i's von Neumann-Morgenstern utility $u_i(s)$ for each profile $s = (s_1, \ldots, s_I)$ of strategies. We will frequently refer to all players other than some given player i as "player i's opponents" and denote them by "$-i$." To avoid misunderstanding, let us emphasize that this terminology does not mean that the other players are trying to "beat" player i. Rather, each player's objective is to maximize his own payoff function, and this may involve "helping" or "hurting" the other players. For economists, the most familiar interpretations of strategies may be as choices of prices or output levels, which correspond to Bertrand and Cournot competition, respectively. For political scientists, strategies might be votes or choices of electoral platforms.

A two-player zero-sum game is a game such that $\sum_{i=1}^{2} u_i(s) = 0$ for all s. (The key feature of these games is that the sum of the utilities is a constant; setting the constant to equal 0 is a normalization.) In a two-player zero-sum game, whatever one player wins the other loses. This is the extreme case where the players are indeed pure "opponents" in the colloquial sense. Although such games are amenable to elegant analysis and have been widely studied in game theory, most games of interest in the social sciences are non-zero-sum.

It is helpful to think of players' strategies as corresponding to various "buttons" on a computer keyboard. The players are thought of as being in separate rooms, and being asked to choose a button without communicating with each other. Usually we also assume that all players know the structure of the strategic form, and know that their opponents know it, and know that their opponents know that they know, and so on *ad infinitum*. That is, the structure of the game is *common knowledge*, a concept examined more formally in chapter 14. This chapter uses common knowledge informally, to motivate the solution concept of Nash equilibrium and iterated strict dominance. As will be seen, common knowledge of payoffs on its own is in fact neither necessary nor sufficient to justify Nash equilibrium. In

	L	M	R
U	4,3	5,1	6,2
M	2,1	8,4	3,6
D	3,0	9,6	2,8

Figure 1.1

particular, for some justifications it suffices that the players simply know their *own* payoffs.

We focus our attention on finite games, that is, games where $S = \times_i S_i$ is finite; finiteness should be assumed wherever we do not explicitly note otherwise. Strategic forms for finite two-player games are often depicted as matrices, as in figure 1.1. In this matrix, players 1 and 2 have three pure strategies each: U, M, D (up, middle, and down) and L, M, R (left, middle, and right), respectively. The first entry in each box is player 1's payoff for the corresponding strategy profile; the second is player 2's.

A *mixed strategy* σ_i is a probability distribution over pure strategies. (We postpone the motivation for mixed strategies until later in this chapter.) Each player's randomization is statistically independent of those of his opponents, and the payoffs to a profile of mixed strategies are the expected values of the corresponding pure-strategy payoffs. (One reason we assume that the space of pure strategies is finite is to avoid measure-theoretic complications.) We will denote the space of player i's mixed strategies by Σ_i, where $\sigma_i(s_i)$ is the probability that σ_i assigns to s_i. The space of mixed-strategy profiles is denoted $\Sigma = \times_i \Sigma_i$, with element σ. The *support* of a mixed strategy σ_i is the set of pure strategies to which σ_i assigns positive probability. Player i's payoff to profile σ is

$$\sum_{s \in S} \left(\prod_{j=1}^{I} \sigma_j(s_j) \right) u_i(s),$$

which we denote $u_i(\sigma)$ in a slight abuse of notation. Note that player i's payoff to a mixed-strategy profile is a linear function of player i's mixing probability σ_i, a fact which has many important implications. Note also that player i's payoff is a polynomial function of the strategy profile, and so in particular is continuous. Last, note that the set of mixed strategies contains the pure strategies, as degenerate probability distributions are included. (We will speak of nondegenerate mixed strategies when we want to exclude pure strategies from consideration.)

For instance, in figure 1.1 a mixed strategy for player 1 is a vector $(\sigma_1(\text{U}), \sigma_1(\text{M}), \sigma_1(\text{D}))$ such that $\sigma_1(\text{U})$, $\sigma_1(\text{M})$, and $\sigma_1(\text{D})$ are nonnegative and $\sigma_1(\text{U}) + \sigma_1(\text{M}) + \sigma_1(\text{D}) = 1$. The payoffs to profiles $\sigma_1 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $\sigma_2 = (0, \frac{1}{2}, \frac{1}{2})$ are

$$u_1(\sigma_1, \sigma_2) = \tfrac{1}{3}(0 \cdot 4 + \tfrac{1}{2} \cdot 5 + \tfrac{1}{2} \cdot 6) + \tfrac{1}{3}(0 \cdot 2 + \tfrac{1}{2} \cdot 8 + \tfrac{1}{2} \cdot 3)$$
$$+ \tfrac{1}{3}(0 \cdot 3 + \tfrac{1}{2} \cdot 9 + \tfrac{1}{2} \cdot 2)$$
$$= \tfrac{11}{2}.$$

Similarly, $u_2(\sigma_1, \sigma_2) = \frac{27}{6}$.

1.1.2 Dominated Strategies

Is there an obvious prediction of how the game described in figure 1.1 should be played? Note that, no matter how player 1 plays, R gives player 2 a strictly higher payoff than M does. In formal language, strategy M is *strictly dominated*. Thus, a "rational" player 2 should not play M. Furthermore, if player 1 knows that player 2 will not play M, then U is a better choice than M or D. Finally, if player 2 knows that player 1 knows that player 2 will not play M, then player 2 knows that player 1 will play U, and so player 2 should play L.

The process of elimination described above is called *iterated dominance*, or, more precisely, *iterated strict dominance*.[3] In section 2.1 we give a formal definition of iterated strict dominance, as well as an application to an economic example. The reader may worry at this stage that the set of strategies that survive iterated strict dominance depends on the order in which strategies are eliminated, but this is not the case. (The key is that, if strategy s_i is strictly worse than strategy s_i' against all opponents' strategies in some set D, then strategy s_i is strictly worse than strategy s_i' against all opponents' strategies in any subset of D. Exercise 2.1 asks for a formal proof.)

Next, consider the game illustrated in figure 1.2. Here player 1's strategy M is not dominated by U, because M is better than U if player 2 moves R; and M is not dominated by D, because M is better than D when 2 moves L. However, if player 1 plays U with probability $\frac{1}{2}$ and D with probability $\frac{1}{2}$, he is guaranteed an expected payoff of $\frac{1}{2}$ regardless of how player 2 plays, which exceeds the payoff of 0 he receives from M. Hence, a pure strategy

	L	R
U	2,0	−1,0
M	0,0	0,0
D	−1,0	2,0

Figure 1.2

3. Iterated elimination of weakly dominated strategies has been studied by Luce and Raiffa (1957), Fahrquarson (1969), and Moulin (1979).

may be strictly dominated by a mixed strategy even if it is not strictly dominated by any pure strategy.

We will frequently wish to discuss varying the strategy of a single player i while holding the strategies of his opponents fixed. To do so, we let

$$s_{-i} \in S_{-i}$$

denote a strategy selection for all players but i, and write

$$(s_i', s_{-i})$$

for the profile

$$(s_1, \ldots, s_{i-1}, s_i', s_{i+1}, \ldots, s_I).$$

Similarly, for mixed strategies we let

$$(\sigma_i', \sigma_{-i}) = (\sigma_1, \ldots, \sigma_{i-1}, \sigma_i', \sigma_{i+1}, \ldots, \sigma_I).$$

Definition 1.1 Pure strategy s_i is *strictly dominated for player i* if there exists $\sigma_i' \in \Sigma_i$ such that

$$u_i(\sigma_i', s_{-i}) > u_i(s_i, s_{-i}) \text{ for all } s_{-i} \in S_{-i}. \tag{1.1}$$

The strategy s_i is *weakly dominated* if there exists a σ_i' such that inequality 1.1 holds with weak inequality, and the inequality is strict for at least one s_{-i}.

Note that, for a given s_i, strategy σ_i' satisfies inequality 1.1 for all pure strategies s_{-i} of the opponents if and only if it satisfies inequality 1.1 for all mixed strategies σ_{-i} as well, because player i's payoff when his opponents play mixed strategies is a convex combination of his payoffs when his opponents play pure strategies.

So far we have considered dominated pure strategies. It is easy to see that a mixed strategy that assigns positive probability to a dominated pure strategy is dominated. However, a mixed strategy may be strictly dominated even though it assigns positive probability only to pure strategies that are not even weakly dominated. Figure 1.3 gives an example. Playing U with probability $\frac{1}{2}$ and M with probability $\frac{1}{2}$ gives expected payoff

	L	R
U	1,3	−2,0
M	−2,0	1,3
D	0,1	0,1

Figure 1.3

	L	R
U	8,10	−100,9
D	7,6	6,5

Figure 1.4

$-\frac{1}{2}$ regardless of player 2's play and so is strictly dominated by playing D, even though neither U nor M is dominated.

When a game is solvable by iterated strict dominance in the sense that each player is left with a single strategy, as in figure 1.1, the unique strategy profile obtained is an obvious candidate for the prediction of how the game will be played. Although this candidate is often a good prediction, this need not be the case, especially when the payoffs can take on extreme values. When our students have been asked how they would play the game illustrated in figure 1.4, about half have chosen D even though iterated dominance yields (U, L) as the unique solution. The point is that although U is better than D when player 2 is certain not to use the dominated strategy R, D is better than U when there is a 1-percent chance that player 2 plays R. (The same casual empiricism shows that our students in fact do always play L.) If the loss to (U, R) is less extreme, say only -1, then almost all players 1 choose U, as small fears about R matter less. This example illustrates the role of the assumptions that payoffs and the strategy spaces are common knowledge (as they were in this experiment) and that "rationality," in the sense of not playing a strictly dominated strategy, is common knowledge (as apparently was not the case in this experiment). The point is that the analysis of some games, such as the one illustrated in figure 1.4, is very sensitive to small uncertainties about the behavioral assumptions players make about each other. This kind of "robustness" test—testing how the theory's predictions change with small changes in the model—is an idea that will return in chapters 3, 8, and 11.

At this point we can illustrate a major difference between the analysis of games and the analysis of single-player decisions: In a decision, there is a single decision maker, whose only uncertainty is about the possible moves of "nature," and the decision maker is assumed to have fixed, exogenous beliefs about the probabilities of nature's moves. In a game, there are several decision makers, and the expectations players have about their opponents' play are not exogenous. One implication is that many familiar comparative-statics conclusions from decision theory do not extend once we take into account the way a change in the game may change the actions of *all* players.

Consider for example the game illustrated in figure 1.5. Here player 1's dominant strategy is U, and iterated strict dominance predicts that the

	L	R
U	1,3	4,1
D	0,2	3,4

Figure 1.5

	L	R
U	−1,3	2,1
D	0,2	3,4

Figure 1.6

solution is (U, L). Could it help player 1 to change the game and *reduce* his payoffs if U occurs by 2 utils, which would result in the game shown in figure 1.6? Decision theory teaches that such a change would not help, and indeed it would not *if we held player 2's action fixed at L.* Thus, player 1 would not benefit from this reduction in payoff if it were done without player 2's knowledge. However, if player 1 could arrange for this reduction to occur, and to become known to player 2 before player 2 chose his action, player 1 would indeed benefit, for then player 2 would realize that D is player 1's dominant choice, and player 2 would play R, giving player 1 a payoff of 3 instead of 1.

As we will see, similar observations apply to changes such as decreasing a player's choice set or reducing the quality of his information: Such changes cannot help a player in a fixed decision problem, but in a game they may have beneficial effects on the play of opponents. This is true both when one is making predictions using iterated dominance and when one is studying the equilibria of a game.

1.1.3 Applications of the Elimination of Dominated Strategies

In this subsection we present two classic games in which a *single* round of elimination of dominated strategies reduces the strategy set of each player to a single pure strategy. The first example uses the elimination of strictly dominated strategies, and the second uses the elimination of weakly dominated strategies.

Example 1.1: Prisoner's Dilemma
One round of the elimination of strictly dominated strategies gives a unique answer in the famous "prisoner's dilemma" game, depicted in figure 1.7. The story behind the game is that two people are arrested for a crime. The police lack sufficient evidence to convict either suspect and consequently

	C	D
C	1,1	−1,2
D	2,−1	0,0

Figure 1.7

need them to give testimony against each other. The police put each suspect in a different cell to prevent the two suspects from communicating with each other. The police tell each suspect that if he testifies against (doesn't cooperate with) the other, he will be released and will receive a reward for testifying, provided the other suspect does not testify against him. If neither suspect testifies, both will be released on account of insufficient evidence, and no rewards will be paid. If one testifies, the other will go to prison; if both testify, both will go to prison, but they will still collect rewards for testifying. In this game, both players simultaneously choose between two actions. If both players cooperate (C) (do not testify), they get 1 each. If they both play noncooperatively (D, for defect), they obtain 0. If one cooperates and the other does not, the latter is rewarded (gets 2) and the former is punished (gets -1). Although cooperating would give each player a payoff of 1, self-interest leads to an inefficient outcome with payoffs 0. (To readers who feel this outcome is not reasonable, our response is that their intuition probably concerns a different game—perhaps one where players "feel guilty" if they defect, or where they fear that defecting will have bad consequences in the future. If the game is played repeatedly, other outcomes can be equilibria; this is discussed in chapters 4, 5, and 9.)

Many versions of the prisoner's dilemma have appeared in the social sciences. One example is moral hazard in teams. Suppose that there are two workers, $i = 1, 2$, and that each can "work" ($s_i = 1$) or "shirk" ($s_i = 0$). The total output of the team is $4(s_1 + s_2)$ and is shared equally between the two workers. Each worker incurs private cost 3 when working and 0 when shirking. With "work" identified with C and "shirk" with D, the payoff matrix for this moral-hazard-in-teams game is that of figure 1.7, and "work" is a strictly dominated strategy for each worker.

Exercise 1.7 gives another example where strict dominance leads to a unique solution: that of a mechanism for deciding how to pay for a public good.

Example 1.2: Second-Price Auction

A seller has one indivisible unit of an object for sale. There are I potential buyers, or bidders, with valuations $0 \leq v_1 \leq \cdots \leq v_I$ for the object, and these valuations are common knowledge. The bidders simultaneously submit bids $s_i \in [0, +\infty)$. The highest bidder wins the object and pays the second bid (i.e., if he wins ($s_i > \max_{j \neq i} s_j$), bidder i has utility $u_i =$

$v_i - \max_{j \neq i} s_j$), and the other bidders pay nothing (and therefore have utility 0). If several bidders bid the highest price, the good is allocated randomly among them. (The exact probability determining the allocation is irrelevant because the winner and the losers have the same surplus, i.e., 0.)

For each player i the strategy of bidding his valuation ($s_i = v_i$) weakly dominates all other strategies. Let $r_i \equiv \max_{j \neq i} s_j$. Suppose first that $s_i > v_i$. If $r_i \geq s_i$, bidder i obtains utility 0, which he would get by bidding v_i. If $r_i \leq v_i$, bidder i obtains utility $v_i - r_i$, which again is what he would get by bidding v_i. If $v_i < r_i < s_i$, then bidder i has utility $v_i - r_i < 0$; if he were to bid v_i, his utility would be 0. The reasoning is similar for $s_i < v_i$: When $r_i \leq s_i$ or $r_i \geq v_i$, the bidder's utility is unchanged when he bids v_i instead of s_i. However, if $s_i < r_i < v_i$, the bidder forgoes a positive utility by underbidding.

Thus, it is reasonable to predict that bidders bid their valuation in the second-price auction. Therefore, bidder I wins and has utility $v_I - v_{I-1}$. Note also that because bidding one's valuation is a dominant strategy, it does not matter whether the bidders have information about one another's valuations. Hence, if bidders know their own valuation but do not know the other bidders' valuations (see chapter 6), it is still a dominant strategy for each bidder to bid his valuation.

1.2 Nash Equilibrium†

Unfortunately, many if not most games of economic interest are not solvable by iterated strict dominance. In contrast, the concept of a Nash-equilibrium solution has the advantage of existing in a broad class of games.

1.2.1 Definition of Nash Equilibrium

A Nash equilibrium is a profile of strategies such that each player's strategy is an optimal response to the other players' strategies.

Definition 1.2 A mixed-strategy profile σ^* is a *Nash equilibrium* if, for all players i,

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(s_i, \sigma_{-i}^*) \text{ for all } s_i \in S_i. \tag{1.2}$$

A pure-strategy Nash equilibrium is a pure-strategy profile that satisfies the same conditions. Since expected utilities are "linear in the probabilities," if a player uses a nondegenerate mixed strategy in a Nash equilibrium (one that puts positive weight on more than one pure strategy) he must be indifferent between all pure strategies to which he assigns positive probability. (This linearity is why, in equation 1.2, it suffices to check that no player has a profitable pure-strategy deviation.)

A Nash equilibrium is *strict* (Harsanyi 1973b) if each player has a unique best response to his rivals' strategies. That is, s^* is a strict equilibrium if and only if it is a Nash equilibrium and, for all i and all $s_i \neq s_i^*$,

$$u_i(s_i^*, s_{-i}^*) > u_i(s_i, s_{-i}^*).$$

By definition, a strict equilibrium is necessarily a pure-strategy equilibrium. Strict equilibria remain strict when the payoff functions are slightly perturbed, as the strict inequalities remain satisfied.[4,5]

Strict equilibria may seem more compelling than equilibria where players are indifferent between their equilibrium strategy and a nonequilibrium response, as in the latter case we may wonder why players choose to conform to the equilibrium. Also, strict equilibria are robust to various small changes in the nature of the game, as is discussed in chapters 11 and 14. However, strict equilibria need not exist, as is shown by the "matching pennies" game of example 1.6 below: The unique equilibrium of that game is in (nondegenerate) mixed strategies, and no (nondegenerate) mixed-strategy equilibrium can be strict.[6] (Even pure-strategy equilibria need not be strict; an example is the profile (D, R) in figure 1.18 when $\lambda = 0$.)

To put the idea of Nash equilibrium in perspective, observe that it was implicit in two of the first games to have been studied, namely the Cournot (1838) and Bertrand (1883) models of oligopoly. In the Cournot model, firms simultaneously choose the quantities they will produce, which they then sell at the market-clearing price. (The model does not specify how this price is determined, but it is helpful to think of it being chosen by a Walrasian auctioneer so as to equate total output and demand.) In the Bertrand model, firms simultaneously choose prices and then must produce enough output to meet demand after the price choices become known. In each model, equilibrium is determined by the condition that all firms choose the action that is a best response to the anticipated play of their opponents. It is common practice to speak of the equilibria of these two models as "Cournot equilibrium" and "Bertrand equilibrium," respectively, but it is more helpful to think of them as the *Nash* equilibria of the two different games. We show below that the concepts of "Stackelberg equi-

4. Harsanyi called this "strong" equilibrium; we use the term "strict" to avoid confusion with "strong equilibrium" of Aumann 1959—see note 11.

5. An equilibrium is *quasi-strict* if each pure-strategy best response to one's rivals' strategies belongs to the support of the equilibrium strategy: $\{\sigma_i^*\}_{i \in \mathscr{I}}$ is a quasi-strict equilibrium if it is a Nash equilibrium and if, for all i and s_i,

$$u_i(s_i, \sigma_{-i}^*) = u_i(\sigma_i^*, \sigma_{-i}^*) \Rightarrow \sigma_i^*(s_i) > 0.$$

The equilibrium in matching pennies is quasi-strict, but some games have equilibria that are not quasi-strict. The game in figure 1.18b for $\lambda = 0$ has two Nash equilibria, (U, L) and (D, R). The equilibrium (U, L) is strict, but the equilibrium (D, R) is not even quasi-strict. Harsanyi (1973b) has shown that, for "almost all games," all equilibria are quasi-strict (that is, the set of all games that possess an equilibrium that is not quasi-strict is a closed set of measure 0 in the Euclidean space of strategic-form payoff vectors).

6. Remember that in a mixed-strategy equilibrium a player must receive the same expected payoff from every pure strategy he assigns positive probability.

librium" and "open-loop equilibrium" are also best thought of as shorthand ways of referring to the equilibria of different *games*.

Nash equilibria are "consistent" predictions of how the game will be played, in the sense that if all players predict that a particular Nash equilibrium will occur then no player has an incentive to play differently. Thus, a Nash equilibrium, and only a Nash equilibrium, can have the property that the players can predict it, predict that their opponents predict it, and so on. In contrast, a prediction that any fixed non-Nash profile will occur implies that at least one player will make a "mistake," either in his prediction of his opponents' play or (given that prediction) in his optimization of his payoff.

We do not maintain that such mistakes never occur. In fact, they may be likely in some special situations. But predicting them requires that the game theorist know more about the outcome of the game than the participants know. This is why most economic applications of game theory restrict attention to Nash equilibria.

The fact that Nash equilibria pass the test of being consistent predictions does not make them good predictions, and in situations it seems rash to think that a precise prediction is available. By "situations" we mean to draw attention to the fact that the likely outcome of a game depends on more information than is provided by the strategic form. For example, one would like to know how much experience the players have with games of this sort, whether they come from a common culture and thus might share certain expectations about how the game will be played, and so on.

When one round of elimination of strictly dominated strategies yields a unique strategy profile $s^* = (s_1^*, \ldots, s_I^*)$, this strategy profile is necessarily a Nash equilibrium (actually the unique Nash equilibrium). This is because any strategy $s_i \neq s_i^*$ is necessarily strictly dominated by s_i^*. In particular,

$$u_i(s_i, s^*_{-i}) < u_i(s_i^*, s^*_{-i}).$$

Thus, s^* is a pure-strategy Nash equilibrium (indeed a strict equilibrium). In particular, not cooperating is the unique Nash equilibrium in the prisoner's dilemma of example 1.1.[7]

We show in section 2.1 that the same property holds for iterated dominance. That is, if a single strategy profile survives iterated deletion of strictly dominated strategies, then it is the unique Nash equilibrium of the game.

Conversely, any Nash-equilibrium strategy profile must put weight only on strategies that are not strictly dominated (or, more generally, do not survive iterated deletion of strictly dominated strategies), because a player

7. The same reasoning shows that if there exists a single strategy profile surviving one round of deletion of weakly dominated strategies, this strategy profile is a Nash equilibrium. So, bidding one's valuation in the second-price auction (example 1.2) is a Nash equilibrium.

could increase his payoff by replacing a dominated strategy with one that dominates it. However, Nash equilibria may assign positive probability to weakly dominated strategies.

1.2.2 Examples of Pure-Strategy Equilibria

Example 1.3: Cournot Competition

We remind the reader of the Cournot model of a duopoly producing a homogeneous good. The strategies are quantities. Firm 1 and firm 2 simultaneously choose their respective output levels, q_i, from feasible sets $Q_i = [0, \infty)$, say. They sell their output at the market-clearing price $p(q)$, where $q = q_1 + q_2$. Firm i's cost of production is $c_i(q_i)$, and firm i's total profit is then

$$u_i(q_1, q_2) = q_i p(q) - c_i(q_i).$$

The feasible sets Q_i and the payoff functions u_i determine the strategic form of the game. The "Cournot reaction functions" $r_1: Q_2 \to Q_1$ and $r_2: Q_1 \to Q_2$ specify each firm's optimal output for each fixed output level of its opponent. If the u_i are differentiable and strictly concave, and the appropriate boundary conditions are satisfied,[8] we can solve for these reaction functions using the first-order conditions. For example, $r_2(\cdot)$ satisfies

$$p(q_1 + r_2(q_1)) + p'(q_1 + r_2(q_1))r_2(q_1) - c_2'(r_2(q_1)) = 0. \tag{1.3}$$

The intersections (if any exist) of the two reaction functions r_1 and r_2 are the Nash equilibria of the Cournot game: Neither firm can gain by a change in output, given the output level of its opponent.

For instance, for linear demand ($p(q) = \max(0, 1 - q)$) and symmetric, linear cost ($c_i(q_i) = cq_i$ where $0 \le c \le 1$), firm 2's reaction function, given by equation 1.3, is (over the relevant range)

$$r_2(q_1) = (1 - q_1 - c)/2.$$

By symmetry, firm 1's reaction function is

$$r_1(q_2) = (1 - q_2 - c)/2.$$

The Nash equilibrium satisfies $q_2^* = r_2(q_1^*)$ and $q_1^* = r_1(q_2^*)$ or $q_1^* = q_2^* = (1 - c)/3$.

Example 1.4: Hotelling Competition

Consider Hotelling's (1929) model of differentiation on the line. A linear city of length 1 lies on the abscissa of a line, and consumers are uniformly

8. The "appropriate boundary conditions" refer to sufficient conditions for the optimal reaction of each firm to be in the interior of the feasible set Q_i. For example, if all positive outputs are feasible ($Q_i = [0, +\infty)$), it suffices that $p(q) - c_2'(0) > 0$ for all q (which, in general, implies that $c_2'(0) = 0$) for $r_2(q_1)$ to be strictly positive for all q_1, and $\lim_{q \to \infty} p(q) + p'(q)q - c_2'(q) < 0$ for $r_2(q_1)$ to be finite for all q_1.

distributed with density 1 along this interval. There are two stores (firms) located at the two extremes of the city, which sell the same physical product. Firm 1 is at $x = 0$, firm 2 at $x = 1$. The unit cost of each store is c. Consumers incur a transportation cost t per unit of distance. They have unit demands and buy one unit if and only if the minimum generalized price (price plus transportation cost) for the two stores does not exceed some large number $\bar{s}$. If prices are "not too high," the demand for firm 1 is equal to the number of consumers who find it cheaper to buy from firm 1. Letting p_i denote the price of firm i, the demand for firm 1 is given by

$$D_1(p_1, p_2) = x,$$

where

$$p_1 + tx = p_2 + t(1 - x)$$

or

$$D_1(p_1, p_2) = \frac{p_2 - p_1 + t}{2t}$$

and

$$D_2(p_1, p_2) = 1 - D_1(p_1, p_2).$$

Suppose that prices are chosen simultaneously. A Nash equilibrium is a profile (p_1^*, p_2^*) such that, for each player i,

$$p_i^* \in \arg\max_{p_i} \{(p_i - c)D_i(p_i, p_{-i}^*)\}.$$

For instance, firm 2's reaction curve, $r_2(p_1)$, is given (in the relevant range) by

$$D_2(p_1, r_2(p_1)) + [r_2(p_1) - c]\frac{\partial D_2}{\partial p_2}(p_1, r_2(p_1)) = 0.$$

In our example, the Nash equilibrium is given by $p_1^* = p_2^* = c + t$ (and the above analysis is valid as long as $c + 3t/2 \leq \bar{s}$).

Example 1.5: Majority Voting

There are three players, 1, 2, and 3, and three alternatives, A, B, and C. Players vote simultaneously for an alternative; abstaining is not allowed. Thus, the strategy spaces are $S_i = \{A, B, C\}$. The alternative with the most votes wins; if no alternative receives a majority, then alternative A is selected. The payoff functions are

$$u_1(A) = u_2(B) = u_3(C) = 2,$$

$$u_1(B) = u_2(C) = u_3(A) = 1,$$

and

$$u_1(C) = u_2(A) = u_3(B) = 0.$$

This game has three pure-strategy equilibrium *outcomes*: A, B, and C. There are more *equilibria* than this: If players 1 and 3 vote for outcome A, then player 2's vote does not change the outcome, and player 3 is indifferent about how he votes. Hence, the profiles (A, A, A) and (A, B, A) are both Nash equilibria whose outcome is A. (The profile (A, A, B) is not a Nash equilibrium, since if player 3 votes for B then player 2 would prefer to vote for B as well.)

1.2.3 Nonexistence of a Pure-Strategy Equilibrium

Not all games have pure-strategy Nash equilibria. Two examples of games whose only Nash equilibrium is in (nondegenerate) mixed strategies follow.

Example 1.6: Matching Pennies

A simple example of nonexistence is "matching pennies" (figure 1.8). Players 1 and 2 simultaneously announce heads (H) or tails (T). If the announcements match, then player 1 gains a util and player 2 loses a util. If the announcements differ, it is player 2 who wins the util and player 1 who loses. If the predicted outcome is that the announcements will match, then player 2 has an incentive to deviate, while player 1 would prefer to deviate from any prediction in which announcements do not match. The only "stable" situation is one in which each player randomizes between his two pure strategies, assigning equal probability to each. To see this, note that if player 2 randomizes $\frac{1}{2}$-$\frac{1}{2}$ between H and T, player 1's payoff is $\frac{1}{2}\cdot 1 + \frac{1}{2}\cdot(-1) = 0$ when playing H and $\frac{1}{2}\cdot(-1) + \frac{1}{2}\cdot 1 = 0$ when playing T. In this case player 1 is completely indifferent between his possible choices and is willing to randomize himself.

This raises the question of why a player should bother to play a mixed strategy when he knows that any of the pure strategies in its support would do equally well. In matching pennies, if player 1 knows that player 2 will randomize between H and T with equal probabilities, player 1 has expected value 0 from all possible choices. As far as his payoff goes, he could just as well play "heads" with certainty, but if this is anticipated by player 2 the equilibrium disintegrates. Subsection 1.2.5 mentions one defense of mixed strategies, which is that it represents a large population of players

	H	T
H	1, −1	−1, 1
T	−1, 1	1, −1

Figure 1.8

who use different pure strategies. If we insist that there is only one "player 1," though, this interpretation does not apply. Harsanyi (1973a) offered the alternative defense that the "mixing" should be interpreted as the result of small, unobservable variations in a player's payoffs. Thus, in our example, sometimes player 1 might prefer matching on T to matching on H, and conversely. Then, for each value of his payoff, player 1 would play a pure strategy. This "purification" of mixed-strategy equilibria is discussed in chapter 6.

Example 1.7: Inspection Game

A popular variant of the "matching pennies" game is the "inspection game," which has been applied to arms control, crime deterrence, and worker incentives. The simplest version of this game is depicted in figure 1.9. An agent (player 1) works for a principal (player 2). The agent can either shirk (S) or work (W). Working costs the agent g and produces output of value v for the principal. The principal can either inspect (I) or not inspect (NI). An inspection costs h to the principal but provides evidence of whether the worker shirks. The principal pays the agent a wage w unless he has evidence that the agent has shirked. (The principal is not allowed to condition the wage on the observed level of output.) If the agent is caught shirking, he gets 0 (because of limited liability). The two players choose their strategies simultaneously (in particular, the principal does not know whether the worker has chosen to shirk when he decides whether to inspect). To limit the number of cases to consider, assume that $g > h > 0$. To make things interesting we also assume that $w > g$ (otherwise working would be a weakly or strictly dominated strategy for the agent).

There is no pure-strategy equilibrium in the inspection game: If the principal does not inspect, the agent strictly prefers shirking, and therefore the principal is better off inspecting as $w > h$. On the other hand, if the principal inspects with probability 1 in equilibrium, the agent prefers working (as $w > g$), which implies that the principal is better off not inspecting. Thus, the principal must play a mixed strategy in equilibrium. Similarly, the agent must also randomize. Let x and y denote the probabilities that the agent shirks and the principal inspects, respectively. For the agent to be indifferent between shirking and working, it must be the case that the gain from shirking (g) equals the expected loss in income (yw). For the principal to be indifferent between inspecting and not inspecting, the

	I	NI
S	$0, -h$	$w, -w$
W	$w-g, v-w-h$	$w-g, v-w$

Figure 1.9

cost of inspection (h) must equal the expected wage savings (xw). Hence, $y = g/w$ and $x = h/w$ (both x and y belong to $(0, 1)$).[9]

1.2.4 Multiple Nash Equilibria, Focal Points, and Pareto Optimality

Many games have several Nash equilibria. When this is the case, the assumption that a Nash equilibrium is played relies on there being some mechanism or process that leads all the players to expect the same equilibrium.

One well-known example of a game with multiple equilibria is the "battle of the sexes," illustrated by figure 1.10a. The story that goes with the name "battle of the sexes" is that the two players wish to go to an event together, but disagree about whether to go to a football game or the ballet. Each player gets a utility of 2 if both go to his or her preferred event, a utility of 1 if both go to the other's preferred event, and 0 if the two are unable to agree and stay home or go out individually. Figure 1.10b displays a closely related game that goes by the names of "chicken" and "hawk-dove." (Chapter 4 discusses a related dynamic game that is also called "chicken.") One version of the story here is that the two players meet at a one-lane bridge and each must choose whether to cross or to wait for the other. If both play T (for "tough"), they crash in the middle of the bridge and get -1 each; if both play W (for "weak"), they wait and get 0; if one player chooses T and the other chooses W, then the tough player crosses first, receiving 2, and the weak one receives 1. In the bridge-crossing story, the term "chicken" is used in the colloquial sense of "coward." (Evolutionary biologists call this game "hawk-dove," because they interpret strategy T as "hawk-like" and strategy W as "dove-like.")

Though the different payoff matrices in figures 1.10a and 1.10b describe different sorts of situations, the two games are very similar. Each of them has three equilibria: two in pure strategies, with payoffs (2, 1) and (1, 2), and

9. Building on this result, one can compute the optimal contract, i.e., the w that maximizes the principal's expected payoff

$$v(1-x) - w(1-xy) - hy = v(1-h/w) - w.$$

The optimal wage is thus $w = \sqrt{hv}$ (assuming $\sqrt{hv} > g$). Note that the principal would be better off if he could "commit" to an inspection level. To see this, consider the different game in which the principal plays first and chooses a probability y of inspection, and the agent, after observing y, chooses whether to shirk. For a given w ($>g$), the principal can choose $y = g/w + \varepsilon$, where ε is positive and arbitrarily small. The agent then works with probability 1, and the principal has (approximately) payoff

$$v - w - hg/w > v(1-h/w) - w.$$

Technically, commitment eliminates the constraint $xw \geq h$, (i.e., that it is *ex post* worthwhile to inspect). (It is crucial that the principal is committed to inspecting with probability y. If the "toss of the coin" determining inspection is not public, the principal has an *ex post* incentive not to inspect, as he knows that the agent works.) This reasoning will become familiar in chapter 3. See chapters 5 and 10 for discussions of how repeated play might make the commitment credible whereas it would not be if the game was played only once.

	B	F
F	0,0	2,1
B	1,2	0,0

a

	T	W
T	−1,−1	2,1
W	1,2	0,0

b

Figure 1.10

one that is mixed. In the battle of the sexes, the mixed equilibrium is that player 1 plays F with probability $\frac{2}{3}$ (and B with probability $\frac{1}{3}$) and player 2 plays B with probability $\frac{2}{3}$ (and F with probability $\frac{1}{3}$). To obtain these probabilities, we solve out the conditions that the players be indifferent between their two pure strategies. So, if x and y denote the probabilities that player 1 plays F and player 2 plays B, respectively, player 1's indifference between F and B is equivalent to

$$0 \cdot y + 2 \cdot (1 - y) = 1 \cdot y + 0 \cdot (1 - y),$$

or

$$y = \tfrac{2}{3}.$$

Similarly, for player 2 to be indifferent between B and F it must be the case that

$$0 \cdot x + 2 \cdot (1 - x) = 1 \cdot x + 0 \cdot (1 - x),$$

or

$$x = \tfrac{2}{3}.$$

In the chicken game of figure 1.10b, the mixed-strategy equilibrium has players 1 and 2 play tough with probability $\frac{1}{2}$.

If the two players have not played the battle of the sexes before, it is hard to see just what the right prediction might be, because there is no obvious way for the players to coordinate their expectations. In this case we would not be surprised to see the outcome (B, F). (We would still be surprised if (B, F) turned out to be the "right" prediction, i.e., if it occurred almost every time.) However, Schelling's (1960) theory of "focal points" suggests that in some "real-life" situations players may be able to coordinate on a particular equilibrium by using information that is abstracted away by the strategic form. For example, the *names* of the strategies may have some commonly understood "focal" power. For example, suppose two players are asked to name an exact time, with the promise of a reward if their choices match. Here "12 noon" is focal; "1:43 P.M." is not. One reason that game theory abstracts away from such considerations is that the "focalness" of various strategies depends on the players' culture and past experiences. Thus, the focal point when choosing between "Left" and "Right" may vary across countries with the direction of flow of auto traffic.

Another example of multiple equilibria is the stag-hunt game we used to begin this chapter, where each player has to choose whether to hunt hare by himself or to join a group that hunts stag. Suppose now that there are I players, that choosing hare gives payoff 1 regardless of the other players' actions, and that choosing stag gives payoff 2 if all players choose stag and gives payoff 0 otherwise. This game has two pure-strategy equilibria: "all stag" and "all hare." Nevertheless, it is not clear which equilibrium should be expected. In particular, which equilibrium is more plausible may depend on the number of players. With only two players, stag is better than hare provided that the single opponent plays stag with probability $\frac{1}{2}$ or more, and given that "both stag" is efficient the opponent might be judged this likely to play stag. However, with nine players stag is optimal only if there is a probability of at least $\frac{1}{2}$ that all eight opponents play stag; if each opponent plays stag with probability p independent of the others, then this requires $p^8 \geq \frac{1}{2}$, or $p \gtrsim 0.93$. In the language of Harsanyi and Selten (1988), "all hare" *risk-dominates* "all stag."[10] (See Harsanyi and Selten 1988 for a formal definition. In a symmetric 2×2 game—that is, a symmetric two-player game with two strategies per player—if both players strictly prefer the same action when their prediction is that the opponent randomizes $\frac{1}{2}$-$\frac{1}{2}$, then the profile where both players play that action is the risk-dominant equilibrium.)

Although risk dominance then suggests that a Pareto-dominant equilibrium need not always be played, it is sometimes argued that players will

10. Very similar games have been discussed in the economics literature, where they are called "coordination failures." For example, Diamond (1982) considered a game where two players have to decide whether to produce one unit of a good that they cannot consume themselves in the hope of trading it for a good produced by the other player. Consumption yields 2 units of utility, and production costs 1 unit. Trade takes place only if both players have produced. Not producing yields 0; producing yields 1 if the opponent produces and -1 otherwise. This game is exactly "stag hunt" in the two-player case. With more players the two games can differ, as the payoff to producing might not equal 2 but might instead be

$$2(\text{no. of opponents who produce})/(\text{total no. of opponents}) - 1,$$

assuming that a trader is matched randomly to another trader, who may or may not have produced. The literature on network externalities in adopting a new technology (e.g. Farrell and Saloner 1985) is a more recent study of coordination problems in economics. For example, all players gain if all switch to the new technology; but if less than half of the population is going to switch, each individual is better off staying with the old technology.

	L	R
U	9,9	0,8
D	8,0	7,7

Figure 1.11

in fact coordinate on the Pareto-dominant equilibrium (provided one exists) if they are able to talk to one another before the game is played. The intuition for this is that, even though the players cannot commit themselves to play the way they claim they will, the preplay communication lets the players reas ure one another about the low risk of playing the strategy of the Pareto-dominant equilibrium. Although preplay communication may indeed make the Pareto-dominant equilibrium more likely in the stag-hunt game, it is not clear that it does so in general.

Consider the game illustrated in figure 1.11 (from Harsanyi and Selten 1988). This game has two pure-strategy equilibria ((U, L) with payoffs (9, 9) and (D, R) with payoffs (7, 7)) and a mixed equilibrium with even lower payoffs. Equilibrium (U, L) Pareto-dominates the others. Is this the most reasonable prediction of how the game will be played?

Suppose first that the players do not communicate before play. Then, while the Pareto efficiency of (U, L) may tend to make it a focal point, playing D is much safer for player 1, as it guarantees 7 regardless of how player 2 plays, and player 1 should play D if he assesses the probability of R to be greater than $\frac{1}{8}$ (so (D, R) is risk dominant). Moreover, player 1 knows that player 2 should play R if player 2 believes the probability of D is more than $\frac{1}{8}$. In this situation we are not certain what outcome to predict.

Does (U, L) become compelling if we suppose that the players are able to meet and communicate before they play? Aumann (1990) argues that the answer is no. Suppose that the players meet and assure each other that they plan to play (U, L). Should player 1 take player 2's assurances at face value? As Aumann observes, regardless of his own play, player 2 gains if player 1 plays U; thus, no matter how player 2 intends to play, he should tell player 1 that he intends to play L. Thus, it is not clear that the players should expect their assurances to be believed, which means that (D, R) might be the outcome after all. Thus, even with preplay communication, (U, L) does not seem like the necessary outcome, although it may seem more likely than when communication is not possible.

Another difficulty with the idea that the Pareto-dominant equilibrium is the natural prediction arises in games with more than two players. Consider the game illustrated in figure 1.12 (taken from Bernheim, Peleg, and Whinston 1987), where player 1 chooses rows, player 2 chooses columns, and

	L	R
U	0,0,10	−5,−5,0
D	−5,−5,0	1,1,−5

A

	L	R
U	−2,−2,0	−5,−5,0
D	−5,−5,0	−1,−1,5

B

Figure 1.12

player 3 chooses matrices. (Harsanyi and Selten (1988) give a closely related example where player 3 moves before players 1 and 2.) This game has two pure-strategy Nash equilibria, (U, L, A) and (D, R, B), and an equilibrium in mixed strategies. Bernheim, Peleg, and Whinston do not consider mixed strategies, so we will temporarily restrict our attention to pure ones. The equilibrium (U, L, A) Pareto-dominates (D, R, B). Is (U, L, A) then the obvious focal point? Imagine that this was the expected solution, and hold player 3's choice fixed. This induces a two-player game between players 1 and 2. In this two-player game, (D, R) is the Pareto-dominant equilibrium! Thus, if players 1 and 2 expect that player 3 will play A, and if they can coordinate their play on their Pareto-preferred equilibrium in matrix A, they should do so, upseting the "good" equilibrium (U, L, A).

In response to this example, Bernheim, Peleg, and Whinston propose the idea of a coalition-proof equilibrium, as a way of extending the idea of coordinating on the Pareto-dominant equilibrium to games with more than two players.[11]

To summarize our remarks on multiple equilibria: Although some games have focal points that are natural predictions, game theory lacks a general and convincing argument that a Nash outcome will occur.[12] However, equilibrium analysis has proved useful to economists, and we will focus attention on equilibrium in this book. (Chapter 2 discusses the "rationalizability" notion of Bernheim and Pearce, which investigates the predictions

11. The definition of a coalition-proof equilibrium proceeds by induction on coalition size. First one requires that no one-player coalition can deviate, i.e., that the given strategies are a Nash equilibrium. Then one requires that no two-player coalition can deviate, given that once such a deviation has "occurred" either of the deviating players (but none of the others) is free to deviate again. That is, the two-player deviations must be Nash equilibria of the two-player game induced by holding the strategies of the others fixed. And one proceeds in this way up to the coalition of all players. Clearly (U, L, A) in figure 1.12 is not coalition-proof; brief inspection shows that (D, R, B) is.

Coalition-proof equilibrium is a weakening of Aumann's (1959) "strong equilibrium," which requires that no subset of players, taking the actions of others as given, can jointly deviate in a way that increases the payoffs of all its members. Since this requirement applies to the grand coalition of all players, strong equilibria must be Pareto efficient, unlike coalition-proof equilibria. No strong equilibrium exists in the game of figure 1.12.

12. Aumann (1987) argues that the "Harsanyi doctrine," according to which all players' beliefs must be consistent with Bayesian updating from a common prior, implies that Bayesian rational players must predict a "correlated equilibrium" (a generalization of Nash equilibrium defined in section 2.2).

one can make without invoking equilibrium. As we will see, rationalizability is closely linked to the notion of iterated strict dominance.)

1.2.5 Nash Equilibrium as the Result of Learning or Evolution

To this point we have motivated the solution concepts of dominance, iterated dominance, and Nash equilibrium by supposing that players make their predictions of their opponents' play by introspection and deduction, using their knowledge of the opponents' payoffs, the knowledge that the opponents are rational, the knowledge that each player knows that the others know these things, and so on through the infinite regress implied by "common knowledge."

An alternative approach to introspection for explaining how players predict the behavior of their opponents is to suppose that players extrapolate from their past observations of play in "similar games," either with their current opponents or with "similar"[13] ones. At the end of this subsection we will discuss how introspection and extrapolation differ in the nature of their assumptions about the players' information about one another.

The idea of using learning-type adjustment processes to explain equilibrium goes back to Cournot, who proposed a process that might lead the players to play the Cournot-Nash equilibrium outputs. In the Cournot adjustment process, players take turns setting their outputs, and each player's chosen output is a best response to the output his opponent chose the period before. Thus, if player 1 moves first in period 0, and chooses q_1^0, then player 2's output in period 1 is $q_2^1 = r_2(q_1^0)$, where r_2 is the Cournot reaction function defined in example 1.3. Continuing to iterate the process,

$$q_1^2 = r_1(q_2^1) = r_1(r_2(q_1^0)),$$

and so on. This process may settle down to a steady state where the output levels are constant, but it need not do so. *If* the process does converge to (q_1^*, q_2^*), then $q_2^* = r_2(q_1^*)$ and $q_1^* = r_1(q_2^*)$, so the steady state is a Nash equilibrium.

If the process converges to a particular steady state for all initial quantities sufficiently close to it, we say that the steady state is *asymptotically stable*. As an example of an asymptotically stable equilibrium, consider the Cournot game where $p(q) = 1 - q$, $c_i(q_i) = 0$, and the feasible sets are $Q_i = [0, 1]$. The reaction curves for this game are $r_i(q_j) = (1 - q_j)/2$, and the unique Nash equilibrium is at the intersection of the reaction curves, which is the point $A = (\frac{1}{3}, \frac{1}{3})$. Figure 1.13 displays the path of the Cournot adjust-

13. Of course the distinction between introspection and extrapolation is not absolute. One might suppose that introspection leads to the idea that extrapolation is likely to work, or conversely that past experience has shown that introspection is likely to make the correct prediction.

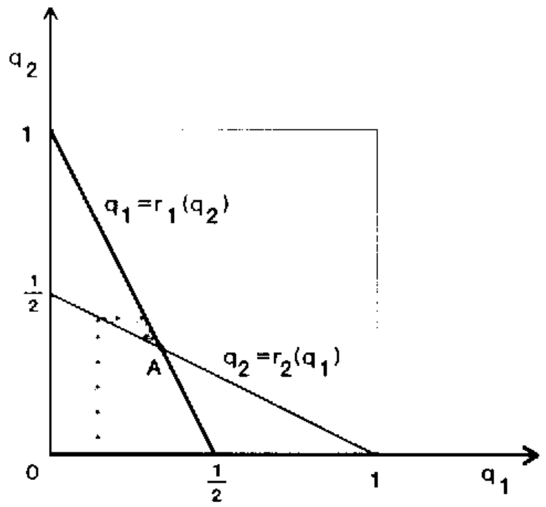

Figure 1.13

ment or *tâtonnement* process for the initial condition $q_1^0 = \frac{1}{6}$. The process converges to the Nash equilibrium from every starting point; that is, the Nash equilibrium is globally stable.

Now suppose that the cost and demand functions yield reaction curves as in figure 1.14 (we spare the reader the derivation of such reaction functions from a specification of cost and demand functions). The reaction functions in figure 1.14 intersect at three points, B, C, and D, all of which are Nash equilibria. Now, however, the intermediate Nash equilibrium, C, is not stable, as the adjustment process converges either to B or to D unless it starts at exactly C.

Comparing figures 1.13 and 1.14 may suggest that the question of asymptotic stability is related to the relative slopes of the reaction functions, and this is indeed the case. If the payoff functions are twice continuously differentiable, the slope of firm i's reaction function is

$$\frac{dr_i}{dq_j} = -\frac{\partial^2 u_i}{\partial q_i \partial q_j} \bigg/ \frac{\partial^2 u_i}{\partial q_i^2},$$

and a sufficient condition for an equilibrium to be asymptotically stable is that

$$\left|\frac{dr_1}{dq_2}\right| \left|\frac{dr_2}{dq_1}\right| < 1$$

or

$$\frac{\partial^2 u_1}{\partial q_1 \partial q_2} \frac{\partial^2 u_2}{\partial q_1 \partial q_2} < \frac{\partial^2 u_1}{\partial q_1^2} \frac{\partial^2 u_2}{\partial q_2^2}$$

in an open neighborhood of the Nash equilibrium.

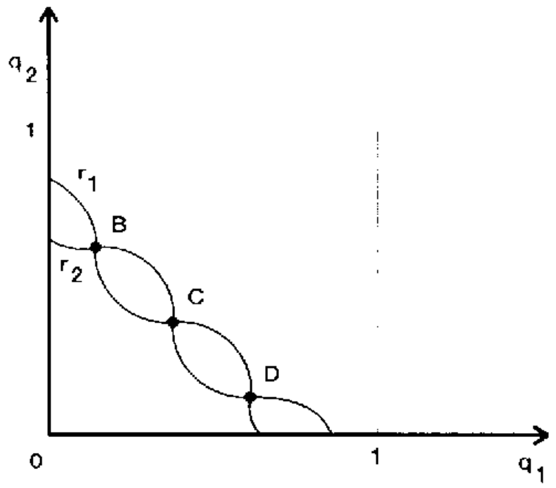

Figure 1.14

Technical aside The condition for asymptotic stability when firms react *simultaneously*, instead of alternatively, to their opponent's most recent outputs is the same as the one just described. To see this, suppose that both players simultaneously adjust their quantities each period by choosing a best response to their opponent's output in the previous period. View this as a dynamic process

$$q^t = (q_1^t, q_2^t) = (r_1(q_2^{t-1}), r_2(q_1^{t-1})) \equiv f(q^{t-1}).$$

From the study of dynamical systems (Hirsch and Smale 1974), we know that a fixed point q^* of f is asymptotically stable in this process if all the eigenvalues of $\partial f(q^*)$ have real parts whose absolute value is less than 1. The condition on the slopes of the reaction functions is exactly sufficient to imply that this eigenvalue condition is satisfied. Classic references on the stability of the Cournot adjustment process include Fisher 1961, Hahn 1962, Seade 1980, and Dixit 1986; see Moulin 1986 for a discussion of more recent work and of subtleties that arise with more than two players.

One way to interpret Cournot's adjustment process with either alternating or simultaneous adjustment is that in each period the player making a move expects that his opponent's output in the future will be the same as it is now. Since output in fact changes every period, it may seem more plausible that players base their forecasts on the average value of their opponent's past play, which suggests the alternative dynamic process

$$q_i^t = r_i\left(\sum_{\tau=0}^{t-1} q_j^\tau / t\right).$$

This alternative has the added value of converging under a broader set

	L	M	R
U	0,0	4,5	5,4
M	5,4	0,0	4,5
D	4,5	5,4	0,0

Figure 1.15

of assumptions, which makes it more useful as a tool for computing equilibria.[14]

However, even when players do respond to the past averages of their opponents' play, the adjustment process need not converge, especially once we move away from games with one-dimensional strategy spaces and concave payoffs. The first example of cycles in this context is due to Shapley (1964), who considered the game illustrated here in figure 1.15.

Suppose first that, in each period, each player chooses a best response to the action his opponent played the period before. If play starts at the point (M, L), it will proceed to trace out the cycle (M, L), (M, R), (U, R), (U, M), (D, M), (D, L), (M, L). If instead players take turns reacting to one another's previous action, then once again play switches from one point to the next each period. If players respond to their opponents' average play, the play cycles increasingly (in fact, geometrically) slowly but never converges: Once (M, L) is played, (M, R) occurs for the next two periods, then player 1 switches to U; (U, R) occurs for the next four periods, then player 2 switches to M; after eight periods of (U, M), player 1 switches to D; and so on.

Thus, even assuming that behavior follows an adjustment process does not imply that play must converge to a Nash equilibrium. And the adjustment processes are not compelling as a description of players' behavior. One problem with all the processes we have discussed so far is that the players ignore the way that their current action will influence their opponent's action in the next period. That is, the adjustment process itself may not be an equilibrium of the "repeated game," where players know they face one another repeatedly.[15] It might seem natural that if the same two players face each other repeatedly they would come to recognize the dynamic effect of their choices. (Note that the effect is smaller if players react to past averages.)

14. For a detailed study of convergence when Cournot oligopolists respond to averages, see Thorlund-Petersen 1990.

15. If firms have perfect foresight, they choose their output taking into account its effect on their rival's future reaction. On this, see exercise 13.2. The Cournot tâtonnement process can be viewed as a special case of the perfect-foresight model where the firms have discount factor 0.

A related defense of Nash equilibrium supposes that there is a large group of players who are matched at random and asked to play a specific game. The players are not allowed to communicate or even to know who their opponents are. At each round, each player chooses a strategy, observes the strategy chosen by his opponent, and receives the corresponding payoff. If there are a great many players then a pair of players who are matched today are unlikely to meet again, and players have no reason to worry about how their current choice will affect the play of their future opponents. Thus, in each period the players should tend to play the strategy that maximizes that period's expected payoff. (We say "tend to play" to allow for the possibility that players may occasionally "experiment" with other choices.)

The next step is to specify how players adjust their expectations about their opponents' play in light of their experience. Many different specifications are possible, and, as with the Cournot process, the adjustment process need not converge to a stable distribution. However, if players observe their opponents' strategies at the end of each round, and players eventually receive a great many observations, then one natural specification is that each player's expectations about the play of his opponents converges to the probability distribution corresponding to the sample average of play he has observed in the past. In this case, *if* the system converges to a steady state, the steady state must be a Nash equilibrium.[16]

Caution The assumption that players observe one another's strategies at the end of each round makes sense in games like the Cournot competition where strategies correspond to uncontingent choices of actions. In the general extensive-form games we introduce in chapter 3, strategies are contingent plans, and the observed outcome of play need not reveal the action a player would have used in a contingency that did not arise (Fudenberg and Kreps 1988).

The idea of a large population of players can also be used to provide an alternative interpretation of mixed strategies and mixed-strategy equilibria. Instead of supposing that individual players randomize among several strategies, a mixed strategy can be viewed as describing a situation in which different fractions of the population play different pure strategies. Once again a Nash equilibrium in mixed strategies requires that all pure strategies that receive positive probability are equally good responses, since if one pure strategy did better than the other we would expect more and more of the players to learn this and switch their play to the strategy with the higher payoff.

16. Recent papers on the explanation of Nash equilibrium as the result of learning include Gul 1989, Milgrom and Roberts 1989, and Nyarko 1989.

The large-population model of adjustment to Nash equilibrium has yet another application: It can be used to discuss the adjustment of population fractions by *evolution* as opposed to learning. In theoretical biology, Maynard Smith and Price (1973) pioneered the idea that animals are genetically programmed to play different pure strategies, and that the genes whose strategies are more successful will have higher reproductive fitness. Thus, the population fractions of strategies whose payoff against the current distribution of opponents' play is relatively high will tend to grow at a faster rate, and, any stable steady state must be a Nash equilibrium. (Non-Nash profiles can be unstable steady states, and not all Nash equilibria are locally stable.) It is interesting to note that there is an extensive literature applying game theory to questions of animal behavior and of the determination of the relative frequency of male and female offspring. (Maynard Smith 1982 is the classic reference.)

More recently, some economists and political scientists have argued that evolution can be taken as a metaphor for learning, and that evolutionary stability should be used more broadly in economics. Work in this area includes Axelrod's (1984) study of evolutionary stability in the repeated prisoner's dilemma game we discuss in chapter 4 and Sugden's (1986) study of how evolutionary stability can be used to ask which equilibria are more likely to become focal points in Schelling's sense.

To conclude this section we compare the informational assumptions of deductive and extrapolative explanations of Nash equilibrium and iterated strict dominance. The deductive justification of the iterated deletion of strictly dominated strategies requires that players are rational and know the payoff functions of all players, that they know their opponents are rational and know the payoff functions, that they know the opponents know, and so on for as many steps as it takes for the iterative process to terminate. In contrast, if players play one another repeatedly, then, even if players do not know their opponents' payoffs, they will eventually learn that the opponents do not play certain strategies, and the dynamics of the learning system will replicate the iterative deletion process. And for an extrapolative justification of Nash equilibrium, it suffices that players know their own payoffs, that play eventually converges to a steady state, and that if play does converge all players eventually learn their opponents' steady-state strategies. Players need not have *any* information about the payoff functions or information of their opponents.

Of course, the reduction in the informational requirements is made possible by the additional hypotheses of the learning story: Players must have enough experience to learn how their opponents play, and play must converge to a steady state. Moreover, we must suppose either that there is a large population of players who are randomly matched, or that, even though the same players meet one another repeatedly, they ignore any dynamic links between their play today and their opponents' play tomorrow.

1.3 Existence and Properties of Nash Equilibria (technical)††

We now tackle the question of the existence of Nash equilibria. Although some of the material in this section is technical, it is quite important for those who wish to read the formal game-theory literature. However, the section can be skipped in a first reading by those who are pressed for time and have little interest in technical detail.

1.3.1 Existence of a Mixed-Strategy Equilibrium

Theorem 1.1 (Nash 1950b) Every finite strategic-form game has a mixed-strategy equilibrium.

Remark Remember that a pure-strategy equilibrium is an equilibrium in degenerate mixed strategies. The theorem does not assert the existence of an equilibrium with nondegenerate mixing.

Proof Since this is the archetypal existence proof in game theory, we will go through it in detail. The idea of the proof is to apply Kakutani's fixed-point theorem to the players' "reaction correspondences." Player i's *reaction correspondence*, r_i, maps each strategy profile σ to the set of mixed strategies that maximize player i's payoff when his opponents play σ_{-i}. (Although r_i depends only on σ_{-i} and not on σ_i, we write it as a function of the strategies of all players, because later we will look for a fixed point in the space Σ of strategy profiles.) This is the natural generalization of the Cournot reaction function we defined above. Define the correspondence $r: \Sigma \rightrightarrows \Sigma$ to be the Cartesian product of the r_i. A *fixed point* of r is a σ such that $\sigma \in r(\sigma)$, so that, for each player, $\sigma_i \in r_i(\sigma)$. Thus, a fixed point of r is a Nash equilibrium.

From Kakutani's theorem, the following are sufficient conditions for $r: \Sigma \rightrightarrows \Sigma$ to have a fixed point:

(1) Σ is a compact,[17] convex,[18] nonempty subset of a (finite-dimensional) Euclidean space.

(2) $r(\sigma)$ is nonempty for all σ.

(3) $r(\sigma)$ is convex for all σ.

17. A subset X of a Euclidean space is compact if any sequence in X has a subsequence that converges to a limit point in X. The definition of compactness for more general topological spaces uses the notion of "cover," which is a collection of open sets whose union includes the set X. X is compact if any cover has a finite subcover.

18. A set X in a linear vector space is convex if, for any x and x' belonging to X and any $\lambda \in [0, 1]$, $\lambda x + (1 - \lambda)x'$ belongs to X.

(4) $r(\cdot)$ has a closed graph: If $(\sigma^n, \hat{\sigma}^n) \to (\sigma, \hat{\sigma})$ with $\hat{\sigma}^n \in r(\sigma^n)$, then $\hat{\sigma} \in r(\sigma)$. (This property is also often referred to as *upper hemi-continuity*.[19])

Let us check that these conditions are satisfied.

Condition 1 is easy—each Σ_i is a simplex of dimension $(\# S_i - 1)$. Each player's payoff function is linear, and therefore continuous in his own mixed strategy, and since continuous functions on compact sets attain maxima, condition 2 is satisfied. If $r(\sigma)$ were not convex, there would be a $\sigma' \in r(\sigma)$, a $\sigma'' \in r(\sigma)$, and a $\lambda \in (0,1)$ such that $\lambda\sigma' + (1-\lambda)\sigma'' \notin r(\sigma)$. But for each player i,

$$u_i(\lambda\sigma_i' + (1-\lambda)\sigma_i'', \sigma_{-i}) = \lambda u_i(\sigma_i', \sigma_{-i}) + (1-\lambda)u_i(\sigma_i'', \sigma_{-i}),$$

so that if both σ_i' and σ_i'' are best responses to σ_{-i}, then so is their weighted average. This verifies condition 3.

Finally, assume that condition 4 is violated so there is a sequence $(\sigma^n, \hat{\sigma}^n) \to (\sigma, \hat{\sigma})$, $\hat{\sigma}^n \in r(\sigma^n)$, but $\hat{\sigma} \notin r(\sigma)$. Then $\hat{\sigma}_i \notin r_i(\sigma)$ for some player i. Thus, there is an $\varepsilon > 0$ and a σ_i' such that $u_i(\sigma_i', \sigma_{-i}) > u_i(\hat{\sigma}_i, \sigma_{-i}) + 3\varepsilon$. Since u_i is continuous and $(\sigma^n, \hat{\sigma}^n) \to (\sigma, \hat{\sigma})$, for n sufficiently large we have

$$u_i(\sigma_i', \sigma_{-i}^n) > u_i(\sigma_i', \sigma_{-i}) - \varepsilon > u_i(\hat{\sigma}_i, \sigma_{-i}) + 2\varepsilon > u_i(\hat{\sigma}_i^n, \sigma_{-i}^n) + \varepsilon.$$

Thus, σ_i' does *strictly* better against $\sigma^n{}_{-i}$ than $\hat{\sigma}_i^n$ does, which contradicts $\hat{\sigma}_i^n \in r_i(\sigma^n)$. This verifies condition 4. ■

Once existence has been established, it is natural to consider the characterization of the equilibrium set. Ideally one would prefer there to be a unique equilibrium, but this is true only under very strong conditions. When several equilibria exist, one must see which, if any, seem to be reasonable predictions, but this requires examination of the entire Nash set. The reasonableness of one equilibrium may depend on whether there are others with competing claims. Unfortunately, in many interesting games the set of equilibria is difficult to characterize.

1.3.2 The Nash-Equilibrium Correspondence Has a Closed Graph

We now analyze how the set of Nash equilibria changes when the payoff functions change continuously with some parameters. The intuition for the results can be gleaned from the case of a single decision maker (see figure 1.16). Suppose that the decision maker gets payoff $1 + \lambda$ when playing L and $1 - \lambda$ when playing R. Let x denote the probability that the decision maker plays L, and consider the optimal x for each λ in $[-1, 1]$. This

19. The graph of a correspondence $f: X \rightrightarrows Y$ is the set of (x, y) such that $y \in f(x)$. Upper hemi-continuity requires that, for any x_0, and for any open set V that contains $f(x_0)$, there exists a neighborhood U of x_0 such that $f(x) \subseteq V$ if $x \in U$. In general this differs from the closed-graph notion, but the two concepts coincide if the range of f is compact and $f(x)$ is closed for each x—conditions which are generally satisfied when applying fixed-point theorems. See Green and Heller 1981.

L	R
$1+\lambda$	$1-\lambda$

Figure 1.16

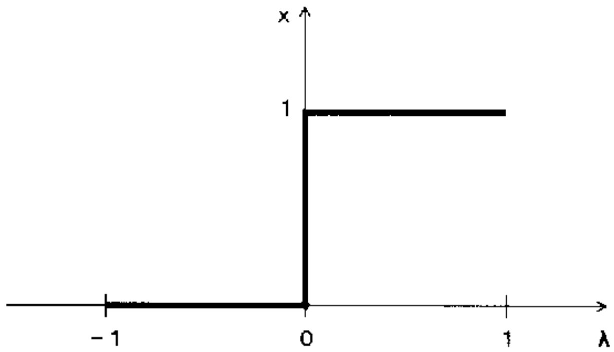

Figure 1.17

defines the Nash-equilibrium correspondence for this one-player game. In particular, for $\lambda = 0$, any $x \in [0,1]$ is optimal. Figure 1.17, which exhibits the graph of the Nash correspondence (in bold), suggests its main properties. First, the correspondence has a closed graph (is upper hemi-continuous). For any sequence (λ^n, x^n) belonging to the graph of the correspondence and converging to some (λ, x), the limit (λ, x) belongs to the graph of correspondence.[20] Second, the correspondence may not be "lower hemi-continuous." That is, there may exist (λ, x) belonging to the graph of the correspondence and a sequence $\lambda^n \to \lambda$ such that there exists no x^n such that (λ^n, x^n) belongs to the graph of the correspondence and $x^n \to x$. Here, take $\lambda = 0$ and $x \in (0,1)$. These two properties generalize to multi-player situations.[21]

One key step in the proof of existence of subsection 1.3.1 is verifying that when payoffs are continuous the reaction correspondences have closed graphs. The same argument applies to the set of Nash equilibria: Consider a family of strategic-form games with the same finite pure-strategy space S and payoffs $u_i(s, \lambda)$ that are continuous functions of λ. Let $G(\lambda)$ denote the game associated with λ and let $E(\cdot)$ be the Nash correspondence that associates with each λ the set of (mixed-strategy) Nash equilibria of $G(\lambda)$. Then, if the set of possible values Λ of λ is compact, the Nash correspondence has a closed graph and, in particular, $E(\lambda)$ is closed for each λ. The proof is as in the verification of condition (4) in the existence proof. Con-

20. This result is part of the "theorem of the maximum" (Berge 1963).
21. A correspondence $f: X \rightrightarrows Y$ is lower hemi-continuous if, for any $(x, y) \in X \times Y$ such that $y \in f(x)$, and any sequence $x^n \in X$ such that $x^n \to x$, there exists a sequence y^n in Y such that $y^n \to y$ and $y^n \in f(x^n)$ for each x^n.

	L	R
U	1,1	0,0
D	0,0	λ,2

a

	L	R
U	1,1	0,0
D	0,0	λ,λ

b

Figure 1.18

sider two sequences $\lambda^n \to \lambda$ and $\sigma^n \to \sigma$ such that $\sigma^n \in r(\sigma^n)$ and $\sigma \notin r(\sigma)$. That is, σ^n is a Nash equilibrium of $G(\lambda^n)$, but σ is not a Nash equilibrium of $G(\lambda)$. Then there is a player i and a $\hat{\sigma}_i$ that does strictly better than σ_i against σ_{-i}. Since payoffs are continuous in λ, for any λ^n near λ and any σ^n_{-i} near σ_{-i}, $\hat{\sigma}_i$ is a strictly better response to σ^n_{-i} than σ_i^n is—a contradiction.

It is important to note that this does not mean that the correspondence $E(\cdot)$ is continuous. Loosely speaking, a closed graph (plus compactness) implies that the set of equilibria cannot shrink in passing to the limit. If σ^n are Nash equilibria of $G(\lambda^n)$ and $\lambda^n \to \lambda$, then σ^n has a limit point $\sigma \in E(\lambda)$. However, $E(\lambda)$ can contain additional equilibria that are not limits of equilibria of "nearby" games. Thus, $E(\cdot)$ is not lower hemi-continuous, and hence is not continuous. We illustrate this with the two games in figure 1.18. In both of these games, (U, L) is the unique Nash equilibrium if $\lambda < 0$, while for $\lambda > 0$ there are three equilibria (U, L), (D, R), and an equilibrium in mixed strategies. While the equilibrium correspondence has a closed graph in both games, the two games have very different sets of equilibria at the point $\lambda = 0$.

First consider the game illustrated in figure 1.18a. For $\lambda > 0$, there are two pure-strategy equilibria and a unique equilibrium with nondegenerate mixing, as each player can be indifferent between his two choices only if the other player randomizes. If we let p denote the probability of U and q denote the probability of L, a simple computation shows that the unique mixed-strategy equilibrium is

$$(p, q) = \left(\frac{2}{3}, \frac{\lambda}{1+\lambda}\right).$$

As required by a closed graph, the profiles $(p, q) = (1, 1)$, $(0, 0)$, and $(\frac{2}{3}, 0)$ are all Nash equilibria at $\lambda = 0$. There are also additional equilibria for $\lambda = 0$ that are not limits of equilibria for any sequence $\lambda^n \to 0$, namely $(p, 0)$ for any $p \in [0, \frac{2}{3}]$. When $\lambda = 0$, player 1 is willing to randomize even if player 2 plays R with probability 1, and so long as the probability of U is not too large player 2 is still willing to play R. This illustrates how the equilibrium correspondence can fail to be lower hemi-continuous.

In the game of figure 1.18b, the equilibria for $\lambda > 0$ are $(1, 1)$, $(0, 0)$, and $(\lambda/(1+\lambda), \lambda/(1+\lambda))$, whereas for $\lambda = 0$ there are only *two* equilibria:

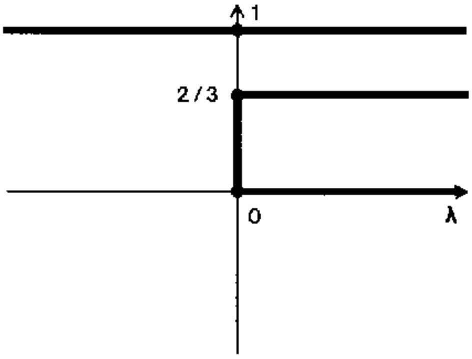

Figure 1.19

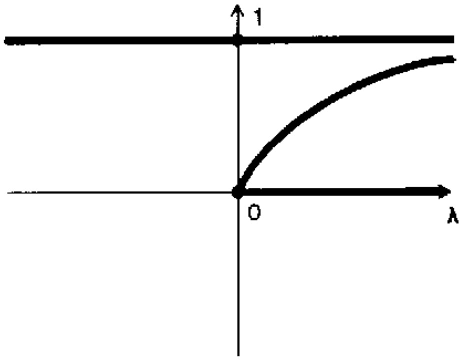

Figure 1.20

$(1, 1)$ and $(0, 0)$. (To see this, note that if p is greater than 0 then player 2 will set $q = 1$, and so p must equal 1, and $(1, 1)$ is the only equilibrium with $q > 0$.)

At first sight a decrease in the number of equilibria might appear to violate the closed-graph property, but this is not the case: For λ positive but small, the mixed-strategy equilibrium $(\lambda/(1 + \lambda), \lambda/(1 + \lambda))$ is very close to the pure-strategy equilibrium (0,0). Figures 1.19 and 1.20 display the equilibrium correspondences of these two games. More precisely, for each λ we display the set of p such that (p, q) is an equilibrium of $N(\lambda)$ for some q; this allows us to give a two-dimensional diagram.

Inspection of the diagrams reveals that each of these games has an odd number of Nash equilibria everywhere except $\lambda = 0$. Chapter 12 explains that this observation is generally true: If the strategy spaces are held fixed, there is an odd number of Nash equilibria for "almost all" payoff functions.

Finally, note that in figures 1.18a and 1.18b, although (D, R) is not a Nash equilibrium for $\lambda < 0$, it is an "ε-Nash equilibrium" in the sense of Radner (1980) if $\varepsilon \geq |\lambda|$: Each player's maximum gain to deviation is less than ε. More generally, an equilibrium of a given game will be an ε-Nash equilibrium for games "nearby"—a point developed and exploited by Fudenberg and Levine (1983, 1986), whose results are discussed in chapter 4.

1.3.3 Existence of Nash Equilibrium in Infinite Games with Continuous Payoffs

Economists often use models of games with an uncountable number of actions (as in the Cournot game of example 1.3 and the Hotelling game of example 1.4). Some might argue that prices or quantities are "really" infinitely divisible, while others might argue that "reality" is discrete and the continuum is a mathematical abstraction, but it is often easier to work with a continuum of actions rather than a large finite grid. Moreover, as Dasgupta and Maskin (1986) argue, when the continuum game does not have a Nash equilibrium, the equilibria corresponding to fine, discrete grids (whose existence was proved in subsection 1.3.1) could be very sensitive to exactly which finite grid is specified: If there were equilibria of the finite-grid version of the game that were fairly insensitive to the choice of the grid, one could take a sequence of finer and finer grids "converging" to the continuum, and the limit of a convergent subsequence of the discrete-action-space equilibria would be a continuum equilibrium under appropriate continuity assumptions. (To put it another way, one can pick equilibria of the discrete-grid version of the game that do not fluctuate with the grid if the continuum game has an equilibrium.)

Theorem 1.2 (Debreu 1952; Glicksberg 1952; Fan 1952) Consider a strategic-form game whose strategy spaces S_i are nonempty compact convex subsets of an Euclidean space. If the payoff functions u_i are continuous in s and quasi-concave in s_i, there exists a pure-strategy Nash equilibrium.[22]

Proof The proof is very similar to that of Nash's theorem: We verify that continuous payoffs imply nonempty, closed-graph reaction correspondences, and that quasi-concavity in players' own actions implies that the reaction correspondences are convex-valued. ■

Note that Nash's theorem is a special case of this theorem. The set of mixed strategies over a finite set of actions, being a simplex, is a compact, convex subset of an Euclidean space; the payoffs are polynomial, and therefore quasi-concave, in the player's own mixed strategy.

If the payoff functions are not continuous, the reaction correspondences can fail to have a closed graph and/or fail to be nonempty. The latter problem arises because discontinuous functions need not attain a maximum, as for example the function $f(x) = -|x|$, $x \neq 0$, $f(0) = -1$. To see how the reaction correspondence may fail to have a closed graph even when optimal reactions always exist, consider the following two-player game:

$$S_1 = S_2 = [0, 1],$$

$$u_1(s_1, s_2) = -(s_1 - s_2)^2,$$

22. It is interesting to note that Debreu (1952) used a generalization of theorem 1.2 to prove that competitive equilibria exist when consumers have quasi-convex preferences.

$$u_2(s_1, s_2) = \begin{cases} -(s_1 - s_2 - \frac{1}{3})^2, & s_1 \geq \frac{1}{3} \\ -(s_1 - s_2 + \frac{1}{3})^2, & s_1 < \frac{1}{3}. \end{cases}$$

Here each player's payoff is strictly concave in his own strategy, and a best response exists (and is unique) for each strategy of the opponent. However, the game does not have a pure-strategy equilibrium: Player 1's reaction function is $r_1(s_2) = s_2$, while player 2's reaction function is $r_2(s_1) = s_1 - \frac{1}{3}$ for $s_1 \geq \frac{1}{3}, r_2(s_1) = s_1 + \frac{1}{3}$ for $s_1 < \frac{1}{3}$, and these reaction functions do not intersect.

Quasi-concavity is hard to satisfy in some contexts. For example, in the Cournot game the quasi-concavity of payoffs requires strong conditions on the second derivatives of the price and cost functions. Of course, Nash equilibria can exist even when the conditions of the existence theorems are not satisfied, as these conditions are sufficient but not necessary. However, in the Cournot case Roberts and Sonnenschein (1976) show that pure-strategy Cournot equilibria can fail to exist with "nice" preferences and technologies.

The absence of a pure-strategy equilibrium in some games should not be surprising, since pure-strategy equilibria need not exist in finite games, and these games can be approximated by games with real-valued action spaces but nonconcave payoffs. Figure 1.21 depicts the payoffs of player 1, who chooses an action s_1 in the interval $[\underline{s}_1, \bar{s}_1]$. Payoff function u_1 is continuous in s but not quasi-concave in s_1. This game is "almost" a game where player 1 has two actions, s_1' and s_1''. Suppose the same holds for player 2. Then the game is similar to a game with two actions per player, and we know (from "matching pennies," for instance) that such games may have no pure-strategy equilibrium.

When payoffs are continuous (but not necessarily quasi-concave), mixed strategies can be used to obtain convex-valued reactions, as in the following theorem.

Theorem 1.3 (Glicksberg 1952) Consider a strategic-form game whose strategy spaces S_i are nonempty compact subsets of a metric space. If the payoff functions u_i are continuous then there exists a Nash equilibrium in mixed strategies.

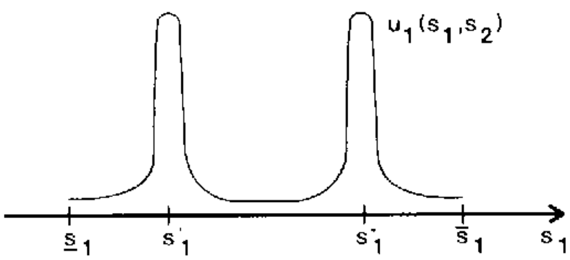

Figure 1.21

Here the mixed strategies are the (Borel) probability measures over the pure strategies, which we endow with the topology of weak convergence.[23] Once more, the proof applies a fixed-point theorem to the reaction correspondences. As we remarked above, the introduction of mixed strategies again makes the strategy spaces convex, the payoffs linear in own strategy and continuous in all strategies (when payoffs are continuous functions of the pure strategies, they are continuous in the mixed strategies as well[24]), and the reaction correspondences convex-valued. With infinitely many pure strategies, the space of mixed strategies is infinite-dimensional, so a more powerful fixed-point theorem than Kakutani's is required. Alternatively, one can approximate the strategy spaces by a sequence of finite grids. From Nash's theorem, each grid has a mixed-strategy equilibrium. One then argues that, since the space of probability measures is weakly compact, the sequence of these discrete equilibria has an accumulation point. Since the payoffs are continuous, it is easy to verify that the limit point is an equilibrium.

We have already seen that pure-strategy equilibria need not exist when payoffs are discontinuous. There are many examples to show that in this case mixed-strategy equilibria may fail to exist as well. (The oldest such example we know of is given in Sion and Wolfe 1957—see exercise 2.2 below.) Note: The Glicksberg theorem used above fails because when the pure-strategy payoffs are discontinuous the mixed-strategy payoffs are discontinuous too. Thus, as before, best responses may fail to exist for some of the opponents' strategies. Section 12.2 discusses the existence of mixed-strategy equilibria in discontinuous games and conditions that guarantee the existence of pure-strategy equilibria.

Exercises

Exercise 1.1* This exercise asks you to work through the characterization of all the Nash equilibria of general two-player games in which each player has two actions (i.e., 2×2 matrix games). This process is time consuming but straightforward and is recommended to the student who is unfamiliar with the mechanics of determining Nash equilibria.

Let the game be as illustrated in figure 1.22.

The pure-strategy Nash equilibria are easily found by testing each cell of the matrix; e.g., (U, L) is a Nash equilibrium if and only if $a \geq e$ and $b \geq d$.

23. Fix a compact metric space A. A sequence of measures μ^n on A converges "weakly" to a limit μ if $\int f\, d\mu^n \to \int f\, d\mu$ for every real-valued continuous function f on A. The set of probability measures on A endowed with the topology of weak convergence is compact.

24. This is an immediate consequence of the definition of convergence we gave in note 23.

	L	R
U	a,b	c,d
D	e,f	g,h

Figure 1.22

	L	R
U	1,−1	3,0
D	4,2	0,−1

Figure 1.23

To determine the mixed-strategy equilibria requires more work. Let x be the probability player 1 plays U and let y be the probability player 2 plays L. We provide an outline, which the student should complete:

(i) Compute each player's reaction correspondence as a function of his opponent's randomizing probability.

(ii) For which parameters is player i indifferent between his two strategies regardless of the play of his opponent?

(iii) For which parameters does player i have a strictly dominant strategy?

(iv) Show that if neither player has a strictly dominant strategy, and the game has a unique equilibrium, the equilibrium must be in mixed strategies.

(v) Consider the particular example illustrated in figure 1.23.

(a) Derive the best-response correspondences graphically by plotting player i's payoff to his two pure strategies as a function of his opponent's mixed strategy.

(b) Plot the two reaction correspondences in the (x, y) space. What are the Nash equilibria?

Exercise 1.2* Find all the equilibria of the voting game of example 1.5.

Exercise 1.3 (Nash demand game)* Consider the problem of dividing a pie between two players. If we let x and y denote player 1's and player 2's payoffs, the vector (x, y) is feasible if and only if $x \geq x_0$, $y \geq y_0$, and $g(x, y) \leq 1$, where g is a differentiable function with $\partial g/\partial x > 0$ and $\partial g/\partial y > 0$ (for instance, $g(x, y) = x + y$). Assume that the feasible set is convex. The point (x_0, y_0) will be called the *status quo*. Nash (1950a) proposed axioms which implied that the "right" way to divide the pie is the allocation (x^*, y^*) that maximizes the product of the differences from the status quo $(x - x_0)(y - y_0)$ subject to the feasibility constraint $g(x, y) \leq 1$. In his 1953 paper, Nash looked for a game that would give this axiomatic bargaining solution as a Nash equilibrium.

(a) Suppose that both players simultaneously formulate demands x and y. If (x, y) is feasible, each player gets what he demanded. If (x, y) is infeasible, player 1 gets x_0 and player 2 gets y_0. Show that there exists a continuum of pure-strategy equilibria, and, more precisely, that any efficient division (x, y) (i.e., feasible and satisfying $g(x, y) = 1$) is a pure-strategy-equilibrium outcome.

(b)** Consider Binmore's (1981) version of the Nash "modified demand game." The feasible set is defined by $x \geq x_0$, $y \geq y_0$, and $g(x, y) \leq z$, where z has cumulative distribution F on $[\underline{z}, \bar{z}]$ (suppose that $\forall z$, the feasible set is nonempty). The players do not know the realization of z before making demands. The allocation is made as previously, after the demands are made and z is realized. Derive the Nash-equilibrium conditions. Show that when F converges to a mass point at 1, any Nash equilibrium converges to the axiomatic bargaining solution.

Exercise 1.4 (Edgeworth duopoly)** There are two identical firms producing a homogeneous good whose demand curve is $q = 100 - p$. Firms simultaneously choose prices. Each firm has a capacity constraint of K. If the firms choose the same price they share the market equally. If the prices are unequal, $p_i < p_j$, the low-price firm, i, sells $\min(100 - p_i, K)$ and the high-price firm, j, sells $\min[\max(0, 100 - p_i - K), K]$. (There are many possible rationing rules, depending on the distribution of consumers' preferences and on how consumers are allocated to firms. If the aggregate demand represents a group of consumers each of whom buys one unit if the price p_i is less than his reservation price of r, and buys no units otherwise, and the consumer's reservation prices are uniformly distributed on $[0, 100]$, the above rationing rule says that the high-value consumers are allowed to purchase at price p_i before lower-value consumers are.) The cost of production is 10 per unit.

(a) Show that firm 1's payoff function is

$$u_1(p_1, p_2) = \begin{cases} (p_1 - 10)\min(100 - p_1, K), & p_1 < p_2 \\ (p_1 - 10)\min(50 - p_1/2, K), & p_1 = p_2 \\ (p_1 - 10)\min(100 - K - p_1, K), & p_1 > p_2, p_1 < 100 - K \\ 0, & \text{otherwise.} \end{cases}$$

(b) Suppose $30 < K < 45$. (Note that these inequalities are strict.) Show that this game does not have a pure-strategy Nash equilibrium by proving the following sequence of claims:

(i) If (p_1, p_2) is a pure-strategy Nash equilibrium, then $p_1 = p_2$. (Hint: If $p_1 \neq p_2$, then the higher-price firm has customers (Why?) and so the lower-price firm's capacity constraint is strictly binding. What happens if this firm charges a slightly higher price?)

(ii) If (p, p) is a pure-strategy Nash equilibrium, then $p > 10$.

(iii) If (p, p) is a pure-strategy Nash equilibrium, then p satisfies $p \leq 100 - 2K$.

(iv) If (p, p) is a pure-strategy Nash equilibrium, then $p = 100 - 2K$. (Hint: If $p < 100 - 2K$, is a deviation to a price between p and $100 - 2K$ profitable for either firm?)

(v) Since $K > 30$, there exists $\delta > 0$ such that a price of $100 - 2K + \delta$ earns a firm a higher profit than $100 - 2K$ when the other firm charges $100 - 2K$.

Note: The Edgeworth duopoly game does satisfy the assumptions of theorem 1.3 (restrict prices to the set $[0, 100]$) and so has a mixed-strategy equilibrium.

Exercise 1.5 (final-offer arbitration)* Farber (1980) proposes the following model of final-offer arbitration. There are three players: a management ($i = 1$), a union ($i = 2$), and an arbitrator ($i = 3$). The arbitrator must choose a settlement $t \in \mathbb{R}$ from the two offers, $s_1 \in \mathbb{R}$ and $s_2 \in \mathbb{R}$, made by the management and the union respectively. The arbitrator has exogenously given preferences $v_0 = -(t - s_0)^2$. That is, he would like to be as close to his "bliss point," s_0, as possible. The management and the union don't know the arbitrator's bliss point; they know only that it is drawn from the distribution P with continuous, positive density p on $[\underline{s}_0, \bar{s}_0]$. The management and the union choose their offers simultaneously. Their objective functions are $u_1 = -t$ and $u_2 = +t$, respectively.

Derive and interpret the first-order conditions for a Nash equilibrium. Show that the two offers are equally likely to be chosen by the arbitrator.

Exercise 1.6** Show that the two-player game illustrated in figure 1.24 has a unique equilibrium. (Hint: Show that it has a unique pure-strategy equilibrium; then show that player 1, say, cannot put positive weight on both U and M; then show that player 1, say, cannot put positive weight on both U and D, but not on M, for instance.)

	L	M	R
U	1, −2	−2, 1	0, 0
M	−2, 1	1, −2	0, 0
D	0, 0	0, 0	1, 1

Figure 1.24

Exercise 1.7 (public good)* Consider an economy with I consumers with "quasi-linear" utility functions,

$$u_i = V_i(x, \theta_i) + t_i,$$

where t_i is consumer i's income, x is a public decision (for instance, the quantity of a public good), $V_i(x, \theta_i)$ is consumer i's gross surplus for decision x, and θ_i is a utility parameter. The monetary cost of decision x is $C(x)$.

The socially efficient decision is

$$x^*(\theta_1, \ldots, \theta_I) \in \arg\max_x \left\{ \sum_{i=1}^{I} V_i(x, \theta_i) - C(x) \right\}.$$

Assume (i) that the maximand in this program is strictly concave and (ii) that for all θ_{-i}, θ_i, and θ_i',

$$\theta_i' \neq \theta_i \Rightarrow x^*(\theta_{-i}, \theta_i') \neq x^*(\theta_{-i}, \theta_i).$$

Condition ii says that the optimal decision is responsive to the utility parameter of each consumer. (Condition i is satisfied if x belongs to $\mathbb{R}$, V_i is strictly concave in x, and C is strictly convex in x. Furthermore, if θ_i belongs to an interval of $\mathbb{R}$, V_i and C are twice differentiable, $\partial V_i/\partial x \partial \theta_i > 0$ or < 0, and x^* is an interior solution, then x^* is strictly increasing or strictly decreasing in θ_i, so that condition (ii) is satisfied as well.)

Now consider the following "demand-revelation game": Consumers are asked to announce their utility parameters simultaneously. A pure strategy for consumer i is thus an announcement $\hat{\theta}_i$ of his parameter ($\hat{\theta}_i$ may differ from the true parameter θ_i). The realized decision is the optimal one for the announced parameters $x^*(\hat{\theta}_1, \ldots, \hat{\theta}_I)$, and consumer i receives a transfer from a "social planner" equal to

$$t_i(\hat{\theta}_1, \ldots, \hat{\theta}_I) = K_i + \sum_{j \neq i} V_j(x^*(\hat{\theta}_1, \ldots, \hat{\theta}_I), \hat{\theta}_j) - C(x^*(\hat{\theta}_1, \ldots, \hat{\theta}_I)),$$

when K_i is a constant.

Show that telling the truth is dominant, in that any report $\hat{\theta}_i \neq \theta_i$ is strictly dominated by the truthful report $\hat{\theta}_i = \theta_i$.

Because each player has a dominant strategy, it does not matter whether he knows the other players' utility parameters. Hence, even if the players do not know one another's payoffs (see chapter 6), it is still rational for them to tell the truth. This property of the dominant-strategy demand-revelation mechanism (called the *Groves mechanism*) makes it particularly interesting in a situation in which a consumer's utility parameter is known only to that consumer.

Exercise 1.8* Consider the following model of bank runs, which is due to Diamond and Dybvig (1983). There are three periods ($t = 0, 1, 2$). There are many consumers—a continuum of them, for simplicity. All consumers are *ex ante* identical. At date 0, they deposit their entire wealth, \$1, in a bank.

The bank invests in projects that yield \$$R$ each if the money is invested for two periods, where $R > 1$. However, if a project is interrupted after one period, it yields only \$1 (it breaks even). Each consumer "dies" (or "needs money immediately") at the end of date 1 with probability x, and lives for two periods with probability $1 - x$. He learns which one obtains at the beginning of date 1. A consumer's utility is $u(c_1)$ if he dies in period 1 and $u(c_1 + c_2)$ if he dies in period 2, where $u' > 0$, $u'' < 0$, and c_1 and c_2 are the consumptions in periods 1 and 2.

An optimal insurance contract (c_1^*, c_2^*) maximizes a consumer's *ex ante* or expected utility. The consumer receives c_1^* if he dies at date 1, and otherwise consumes nothing at date 1 and receives c_2^* at date 2. The contract satisfies $xc_1^* + (1 - x)c_2^*/R = 1$ (the bank breaks even) and $u'(c_1^*) = R\,u'(c_2^*)$ (equality between the marginal rates of substitution). Note that $1 < c_1^* < c_2^*$. The issue is whether the bank can implement this optimal insurance scheme if it is unable to observe who needs money at the end of the first period. Suppose that the bank offers to pay $r_1 = c_1^*$ to consumers who want to withdraw their money in period 1. If $f \in [0, 1]$ is the fraction of consumers who withdraw at date 1, each withdrawing consumer gets r_1 if $fr_1 \le 1$, and gets $1/f$ if $fr_1 > 1$. Similarly, consumers who do not withdraw at date 1 receive $\max\{0, R(1 - r_1 f)/(1 - f)\}$ in period 2.

(a) Show that it is a Nash equilibrium for each consumer to withdraw at date 1 if and only if he "dies" at that date.

(b) Show that another Nash equilibrium exhibits a bank run ($f = 1$).

(c) Compare with the stag hunt.

Exercise 1.9* Suppose $p(q) = a - bq$ in the Cournot duopoly game of example 1.3.

(a) Check that the second-order and boundary conditions for equation (1.3) are satisfied. Compute the Nash equilibrium.

(b) Now suppose there are I identical firms, which all have cost function $c_i(q_i) = cq_i$. Compute the limit of the Nash equilibria as $I \to \infty$. Comment.

Exercise 1.10* Suppose there are I farmers, each of whom has the right to graze cows on the village common. The amount of milk a cow produces depends on the total number of cows, N, grazing on the green. The revenue produced by n_i cows is $n_i v(N)$ for $N < \bar{N}$, and $v(N) \equiv 0$ for $N \ge \bar{N}$, where $v(0) > 0$, $v' < 0$, and $v'' \le 0$. Each cow costs c, and cows are perfectly divisible. Suppose $v(0) > c$. Farmers simultaneously decide how many cows to purchase; all purchased cows will graze on the common.

(a) Write this as a game in strategic form.

(b) Find the Nash equilibrium, and compare it against the social optimum.

(c) Discuss the relationship between this game and the Cournot oligopoly model.

(This exercise, constructed by R. Gibbons, is based on a discussion in Hume 1739.)

Exercise 1.11** We mentioned that theorem 1.3, which concerns the existence of a mixed-strategy Nash equilibrium when strategy spaces are nonempty, compact subsets of a metric space ($\mathbb{R}^n$, say) and when the payoff functions are continuous, can also be proved by taking a sequence of discrete approximations of the strategy spaces that "converge" to it. Go through the steps of the proof as carefully as you can.

Here is a sketch of the proof: Each discrete grid has a mixed-strategy equilibrium. By compactness, the sequence of discrete-grid equilibria has an accumulation point. Argue that this limit must be an equilibrium of the limit game with a continuum of actions. (This relies on the discrete grids becoming increasingly good approximations and the payoffs being continuous.)

Exercise 1.12* Consider a simultaneous-move auction in which two players simultaneously choose bids, which must be in nonnegative integer multiples of one cent. The higher bidder wins a dollar bill. If the bids are equal, neither player receives the dollar. Each player must pay his own bid, whether or not he wins the dollar. (The loser pays too.) Each player's utility is simply his net winnings; that is, the players are risk neutral. Construct a symmetric mixed-strategy equilibrium in which every bid less than 1.00 has a positive probability.

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2 Iterated Strict Dominance, Rationalizability, and Correlated Equilibrium

Most economic applications of game theory use the concept of Nash equilibrium or one of the more restrictive "equilibrium refinements" we introduce in later chapters. However, as we warned in chapter 1, in some situations the Nash concept seems too demanding. Thus, it is interesting to know what predictions one can make without assuming that a Nash equilibrium will occur. Section 2.1 presents the notions of iterated strict dominance and rationalizability, which derive predictions using only the assumptions that the structure of the game (i.e., the strategy spaces and the payoffs) and the rationality of the players are common knowledge. As we will see, these two notions are closely related, as rationalizability is essentially the contrapositive of iterated strict dominance.

Section 2.2 introduces the idea of a correlated equilibrium, which extends the Nash concept by supposing that players can build a "correlating device" that sends each of them a private signal before they choose their strategy.

2.1 Iterated Strict Dominance and Rationalizability††

We introduced iterated strict dominance informally at the beginning of chapter 1. We will now define it formally, derive some of its properties, and apply it to the Cournot model. We will then define rationalizability and relate the two concepts. As throughout, we restrict our attention to finite games except where we explicitly indicate otherwise.

2.1.1 Iterated Strict Dominance: Definition and Properties

Definition 2.1 The process of iterated deletion of strictly dominated strategies proceeds as follows: Set $S_i^0 \equiv S_i$ and $\Sigma_i^0 \equiv \Sigma_i$. Now define S_i^n recursively by

$$S_i^n = \{s_i \in S_i^{n-1} \mid \text{there is no } \sigma_i \in \Sigma_i^{n-1} \text{ such that}$$
$$u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i}) \text{ for all } s_{-i} \in S_{-i}^{n-1}\}$$

and define

$$\Sigma_i^n = \{\sigma_i \in \Sigma_i \mid \sigma_i(s_i) > 0 \text{ only if } s_i \in S_i^n\}.$$

Set

$$S_i^\infty = \bigcap_{n=0}^{\infty} S_i^n.$$

S_i^∞ is the set of player i's pure strategies that survive iterated deletion of strictly dominated strategies. Set Σ_i^∞ to be all mixed strategies σ_i such that there is no σ_i' with $u_i(\sigma_i', s_{-i}) > u_i(\sigma_i, s_{-i})$ for all $s_{-i} \in S_{-i}^\infty$. This is the set of player i's mixed strategies that survive iterated strict dominance.

In words, S_i^n is the set of player i's strategies that are not strictly dominated when players $j \neq i$ are constrained to play strategies in S_j^{n-1} and Σ_i^n

is the set of mixed strategies over S_i^n. Note, however, that Σ_i^∞ may be smaller than the set of mixed strategies over S_i^∞. The reason for this, as was shown in figure 1.3, is that some mixed strategies with support S_i^∞ can be dominated. (In that example, $S_i^\infty = S_i$ for both players i because no pure strategy is eliminated in the first round of the process.)

Note that in a finite game the sequence of iterations defined above must cease to delete further strategies after a finite number of steps. The intersection S_i^∞ is simply the final set of surviving strategies. Note also that each step of the iteration requires one more level of the assumption "I know that you know... that I know the payoffs." For this reason, conclusions based on a large number of iterations tend to be less robust to small changes in the information players have about one another.

The reader may wonder whether the limit set $S^\infty = S_1^\infty \times \cdots \times S_I^\infty$ depends on the particular way that we have specified the process of deletion proceeds: We assumed that at each iteration all dominated strategies of each player are deleted simultaneously. Alternatively, we could have eliminated player 1's dominated strategies, than player 2's,..., then player I's, and started again with player 1,..., *ad infinitum*. Clearly there are many other iterative procedures that can be defined to eliminate strictly dominated strategies. Fortunately *all these procedures yield the same surviving strategies S^∞ and Σ^∞*, as is shown by exercise 2.1. (We will show in chapter 11 that this property does not hold for weakly dominated strategies; that is, which strategies survive in the limit may depend on the order of deletion.)

The reader may also wonder whether one could not delete all the dominated (pure and mixed) strategies at each round of the iterative process instead of first deleting only dominated pure strategies and then deleting mixed strategies at the end. The two ways to proceed actually yield the same sets Σ_i^∞. The reason is that a strategy is strictly dominated against all pure strategies of the opponents if and only if it is dominated against all of their mixed strategies, as we saw in subsection 1.1.2. Thus, whether a nondegenerate mixed strategy σ_i for player i is deleted at round n doesn't alter which strategies of player i's opponents are deleted at the next round. Thus, at each round, the sets of remaining *pure* strategies are the same under the two alternative definitions. Therefore, the undominated mixed strategies Σ_i^∞ are the same.

Definition 2.2 A game is solvable by iterated (strict) dominance if, for each player i, S_i^∞ is a singleton (i.e., a one-element set).

When the iterated deletion of strictly dominated strategies yields a unique strategy profile (as is the case in figure 1.1 or in the prisoner's dilemma of figure 1.7), this strategy profile is necessarily a Nash equilibrium (indeed, it is the unique Nash equilibrium). The proof goes as follows: Let $(s_1^*, \ldots, s_I^*)$ denote this strategy profile, and suppose that there exist i and $s_i \in S_i$ such that $u_i(s_i, s_{-i}^*) > u_i(s_i^*, s_{-i}^*)$. Then if one round of elimination of strictly dominated strategies has sufficed to yield this unique profile, s_i^* must dominate all other strategies in S_i, which is impossible as s_i is a better response to s_{-i}^* than s_i^*. More generally, suppose that in the iterated deletion s_i is strictly dominated at some round by s_i', which in turn is eliminated at a later round because it becomes strictly dominated by $s_i'', \ldots,$ which is finally eliminated by s_i^*. Because s_{-i}^* belongs to the undominated strategies of player i's opponents at each round, by transitivity s_i^* must be a better response to s_{-i}^* than s_i—a contradiction. Conversely, it is easy to see that in any Nash equilibrium the players must play strategies that are not eliminated by iterated strict dominance.

It is also easy to see that if players repeatedly play the same game, and infer their opponents' behavior from past observations, eventually only strategies that survive iterated deletion of strictly dominated strategies will be played. First, because opponents won't play dominated strategies, players will learn that such strategies are not used. They will then use only strategies that are not strictly dominated, given that the dominated strategies of their opponents are not used. After more learning, this will be learned by the opponents, and so on.

2.1.2 An Application of Iterated Strict Dominance

Example 2.1: Iterated Deletion in the Cournot Model[1]

We now make stronger assumptions on the (infinite-action) Cournot model introduced in example 1.3: Suppose that u_i is strictly concave in q_i ($\partial^2 u_i/\partial q_i^2 < 0$), that the cross-partial derivative is negative ($\partial^2 u_i/\partial q_i \partial q_j < 0$, which is the case if $p' < 0$ and $p'' \leq 0$), and that the reaction curves r_1 and r_2 (which are continuous and downward-sloping from the previous two assumptions) intersect only once at a point N, at which r_1 is strictly steeper than r_2. This situation is depicted in figure 2.1. (Note that N is stable, in the terminology introduced in subsection 1.2.5.)

Let q_1^m and q_2^m denote the monopoly outputs: $q_1^m = r_1(0)$ and $q_2^m = r_2(0)$. The first round of deletion of strictly dominated strategies yields $S_i^1 = [0, q_i^m]$. The second round of deletion yields $S_i^2 = [r_i(q_j^m), q_i^m] \equiv [\underline{q}_i^2, q_i^m]$, as indicated in figure 2.1. Consider, for instance, firm 2. Knowing that firm 1 won't pick output greater than q_i^m, choosing output q_2 under $r_2(q_1^m) \equiv \underline{q}_2^2$ is strictly dominated by playing $\underline{q}_2^2$ by strict concavity of firm 2's payoff in its own output. And similarly for firm 1. The third round of deletion yields $S_i^3 = [\underline{q}_i^2, r_i(\underline{q}_j^2)] \equiv [\underline{q}_i^2, q_i^3]$, and so on. More generally, iterated deletion yields a sequence of shrinking intervals around the outputs (q_1^*, q_2^*) corresponding to the intersection N of the reaction curves. For $n = 2k + 1$,

$$\underline{q}_i^{2k+1} = \underline{q}_i^{2k} \qquad \text{and} \qquad q_i^{2k+1} = r_i(\underline{q}_j^{2k});$$

1. This example is inspired by Gabay and Moulin 1980. See also Moulin 1984.

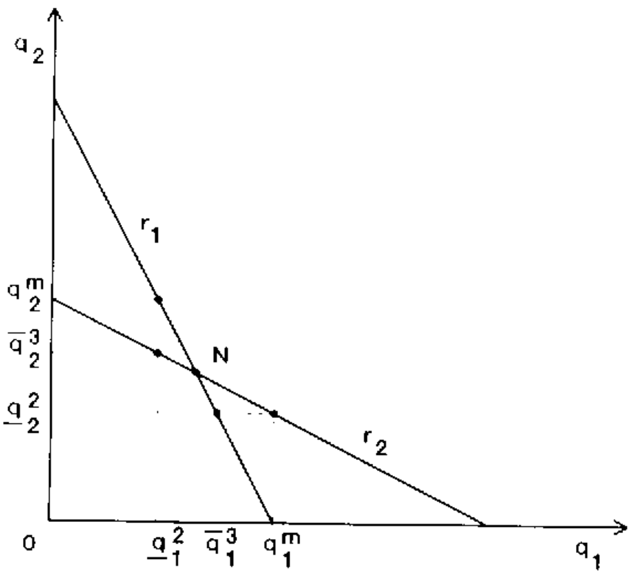


Figure 2.1

for $n = 2k$,

$$q_i^{2k} = r_i(q_j^{2k-1}) \qquad \text{and} \qquad q_i^{2k} = \overline{q}_i^{2k-1}.$$

A difference between this process and the case of finite strategy spaces is that the process of deletion does not stop after a finite number of steps. Nevertheless, the process does converge, because the sequences $\underline{q}_i^n$ and $\overline{q}_i^n$ both converge to q_i^*, so that the process of iterated deletion of strictly dominated strategies yields N as the unique "reasonable" prediction. (Let $\underline{q}_i^\infty \equiv \lim \underline{q}_i^n \leq q_i^*$ and $\overline{q}_i^\infty \equiv \lim \overline{q}_i^n \geq q_i^*$. From the definition of $\underline{q}_i^n$ and $\overline{q}_i^n$ and by continuity of the reaction curves, one has $\overline{q}_i^\infty = r_i(\underline{q}_j^\infty)$ and $\underline{q}_j^\infty = r_j(\overline{q}_i^\infty)$. Hence, $\overline{q}_i^\infty = r_i(r_j(\overline{q}_i^\infty))$, which is possible only if $\overline{q}_i^\infty = q_i^*$; and similarly for $\underline{q}_i^\infty$.)

We conclude that this Cournot game is solvable by iterated strict dominance. This need not be the case for other specifications of the payoff functions; see exercise 2.4.

2.1.3 Rationalizability

The concept of rationalizability was introduced independently by Bernheim (1984) and Pearce (1984), and was used by Aumann (1987) and by Brandenberger and Dekel (1987) in their papers on the "Bayesian approach" to the choice of strategies.

Like iterated strict dominance, rationalizability derives restrictions on play from the assumptions that the payoffs and the "rationality" of the players are common knowledge. The starting point of iterated strict dominance is the observation that a rational player will never play a strictly dominated strategy. The starting point of rationalizability is the complementary question: What are *all* the strategies that a rational player could play? The answer is that a rational player will use only those strategies that are best responses to some beliefs he might have about the strategies of his opponents. Or, to use the contrapositive, a player cannot reasonably play a strategy that is not a best response to some beliefs about his opponents' strategies. Moreover, since the player knows his opponents' payoffs, and knows they are rational, he should not have arbitrary beliefs about their strategies. He should expect his opponents to use only strategies that are best responses to some beliefs that they might have. And these opponents' beliefs, in turn, should also not be arbitrary, which leads to an infinite regress. In the two-player case, the infinite regress has the form "I'm playing strategy σ_1 because I think player 2 is using σ_2, which is a reasonable belief because I would play it if I were player 2 and I thought player 1 was using σ_1', which is a reasonable thing for player 2 to expect because σ_1' is a best response to $\sigma_2', \ldots$."

Formally, rationalizability is defined by the following iterative process.

Definition 2.3 Set $\tilde{\Sigma}_i^0 \equiv \Sigma_i$, and for each i recursively define

$$\tilde{\Sigma}_i^n = \left\{ \sigma_i \in \tilde{\Sigma}_i^{n-1} \mid \exists\, \sigma_{-i} \in \underset{j \neq i}{\times} \text{convex hull}\,(\tilde{\Sigma}_j^{n-1}) \text{ such that } u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i}) \text{ for all } \sigma_i' \in \tilde{\Sigma}_i^{n-1} \right\}.$$

The *rationalizable strategies for player i* are $R_i = \bigcap_{n=0}^{\infty} \tilde{\Sigma}_i^n$.

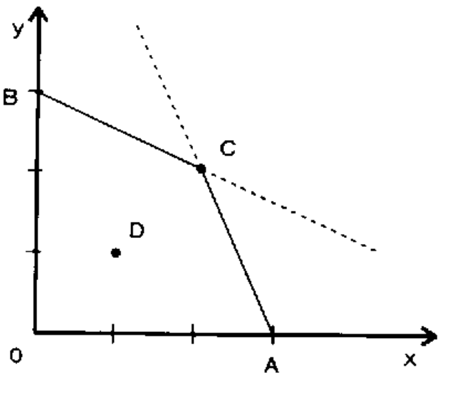
In words, $\tilde{\Sigma}_{-i}^{n-1}$ are the strategies for player i's opponents that "survive" through round $(n-1)$, and $\tilde{\Sigma}_i^n$ is the set of i's surviving strategies that are best responses to some strategy in $\tilde{\Sigma}_{-i}^{n-1}$. The reason the convex hull operator appears in the definition is that player i might not be certain which of several strategies $\sigma_j \in \tilde{\Sigma}_j^{n-1}$ player j will use.[2] And it may be that, although both σ_j' and σ_j'' are in $\tilde{\Sigma}_j^{n-1}$, the mixture $(\frac{1}{2}\sigma_j', \frac{1}{2}\sigma_j'')$ is not. This is illustrated in figure 2.2. In the game of figure 2.2, player 2 has only two pure strategies: L and R. Then any pure strategy s_1 of player 1 is associated with two potential payoffs: $x \equiv u_1(s_1, \mathrm{L})$ and $y \equiv u_1(s_1, \mathrm{R})$. Figure 2.2a describes x and y for player 1's four pure strategies. Strategy A is a best response for player 1 to L and strategy B is a best response to R, but the mixed strategy $(\frac{1}{2}\mathrm{A}, \frac{1}{2}\mathrm{B})$ is dominated by C and hence is not a best response to any strategy of player 2.

A strategy profile σ is *rationalizable* if σ_i is rationalizable for each player i. Note that every Nash equilibrium is rationalizable, since if σ^* is a Nash

2. The convex hull of a set X is the smallest convex set that contains it.

s_1	x	y
A	3	0
B	0	3
C	2	2
D	1	1

a



b

Figure 2.2

equilibrium then $\sigma_i^* \in \tilde{\Sigma}_i^n$ for each n. Thus, the set of rationalizable strategies is nonempty.

Theorem 2.1 (Bernheim 1984; Pearce 1984) The set of rationalizable strategies is nonempty and contains at least one pure strategy for each player. Further, each $\sigma_i \in R_i$ is (in Σ_i) a best response to an element of

$\times_{j \neq i}$ convex hull (R_j).

Sketch of Proof The proof shows inductively that the $\tilde{\Sigma}_i^n$ in the definition of rationalizability are closed, nonempty, and nested and that they contain a pure strategy. Their infinite intersection is thus nonempty and contains a pure strategy. The existence of an element of $\times_{j \neq i}$ convex hull (R_j) to which $\sigma_i \in R_i$ is a best response is obtained by induction on n. ■

2.1.4 Rationalizability and Iterated Strict Dominance (technical)

The condition of not being a best response, which is used in defining rationalizability, looks very close to that of being strictly dominated. In fact these two conditions are equivalent in two-player games.

It is clear that, with any number of players, a strictly dominated strategy is never a best response: If σ_i' strictly dominates σ_i relative to Σ_{-i}, then σ_i' is a strictly better response than σ_i to every σ_{-i} in Σ_{-i}. Thus, *in general games, the set of rationalizable strategies is contained in the set that survives*

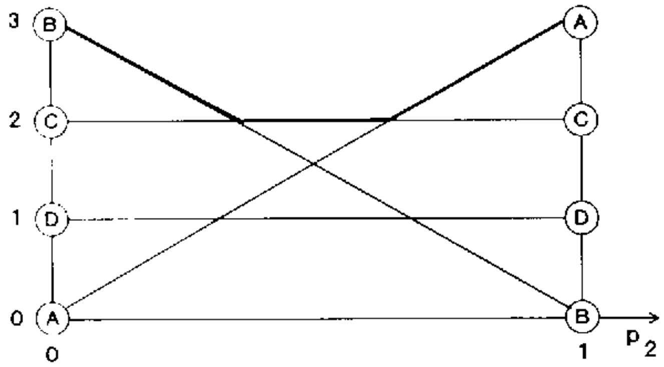


Figure 2.3

iterated strict dominance. The converse in two-player games is a consequence of the separating hyperplane theorem.

To gain some intuition, consider the game in figure 2.2. Figure 2.2b plots player 1's payoff possibilities (x, y) corresponding to different strategies of player 1. A strategy σ_1 is strictly dominated if there is another strategy that gives player 1 a strictly higher payoff no matter how player 2 plays—i.e., if σ_1 yields payoff possibility (x, y) and there is a strategy that yields (x', y'), with $x' > x$ and $y' > y$. It is clear that the pure strategy D is dominated by C and that the only undominated strategies correspond to payoffs on the line segments AC and BC. (Note that mixtures of A and B are dominated, even though neither A nor B is dominated as a pure strategy.)

It is also easy to see from the diagram which strategies are not best responses to any strategy of player 2: A strategy $(p_2, 1 - p_2)$ for player 2 corresponds to weights on L and R, and thus to a family of lines which we can interpret as player 1's "indifference curves." Player 1's best responses to $(p_2, 1 - p_2)$ are those strategies yielding the maximal payoffs with these weights, which are exactly the points of the efficient set AC $\cup$ BC with which the indifference curves are "tangent" (subtangent) (see figure 2.3). Any strategy of player 1 that is a best response to some strategy of player 2 thus corresponds to a point on the efficient frontier (the bold line in the figure), so any strategy that is not a best response must lie in the interior and hence be dominated. This is the intuition for the following theorem.

Theorem 2.2 (Pearce 1984) Rationalizability and iterated strict dominance coincide in two-player games.

Proof Let S^n denote the set of pure strategies remaining after n rounds of the deletion of strictly dominated strategies, let Σ^n be the corresponding mixed strategies, and let $\tilde{\Sigma}^n$ be the set of mixed strategies that survive n rounds of the iteration in the definition of rationalizability. Clearly the set Σ^0 of mixed strategies corresponding to S^0 equals $\tilde{\Sigma}^0$. Assume that $\Sigma^n = \tilde{\Sigma}^n$. For any finite set A, let $\Delta(A)$ denote the space of probability distributions over A. Any s_i in S_i^{n+1} is undominated in $\Delta(S_i^{n+1})$ given that σ_j belongs to

Σ_j^n; otherwise it would have been eliminated. Now consider the vectors

$$\vec{u}_i(\sigma_i) = \{u_i(\sigma_i, s_j)\}_{s_j \in S_j^n}$$

for each $\sigma_i \in \Sigma_i^n$. The set of such vectors is convex, and, by the definition of iterated dominance, S_i^{n+1} contains exactly the s_i such that $\vec{u}_i(s_i)$ is undominated in this set. Fix $\tilde{s}_i$ in S_i^{n+1}. By the separating hyperplane theorem, there exists

$$\sigma_j = \{\sigma_j(s_j)\}_{s_j \in S_j^n}$$

such that, for all $\sigma_i \in \Sigma_i^n$,

$$\sigma_j \cdot (\vec{u}_i(\tilde{s}_i) - \vec{u}_i(\sigma_i)) \geq 0$$

(where a dot denotes the inner product), or

$$u_i(\tilde{s}_i, \sigma_j) \geq u_i(\sigma_i, \sigma_j) \; \forall \; \sigma_i \in \Sigma_i^n = \tilde{\Sigma}_i^n.$$

This means that $\tilde{s}_i$ is a best response in $\tilde{\Sigma}_i^n$ to a strategy σ_j in convex hull $(\tilde{\Sigma}_j^n)$. Thus, $\tilde{s}_i \in \tilde{\Sigma}_i^{n+1}$, and we conclude that $\Sigma^{n+1} = \tilde{\Sigma}^{n+1}$. ■

Remark Pearce gives a different proof based on the existence of the minmax value in finite two-player zero-sum games.[3] The minmax theorem, in turn, is usually proved with the separating hyperplane theorem.

The equivalence between being strictly dominated and not being a best response breaks down in games with three or more players (see exercise 2.7). The point is that, since mixed strategies assume independent mixing, the set of mixed strategies is not convex. In figure 2.2, the problem becomes that the mixed strategies no longer correspond to the set of all tangents to the efficient surface, so a strategy might be on the efficient surface without being a best response to a mixed strategy. However, allowing for correlation in the definition of rationalizability restores equivalence: A strategy is strictly dominated if and only if it is not a best response to a correlated mixed strategy of the opponents. (A correlated mixed strategy for player i's opponents is a general probability distribution on S_{-i}, i.e., an element of $\Delta(S_{-i})$, while a mixed-strategy profile for player i's opponents is an element of $\times_{j \neq i} \Delta(S_j)$.) This gives rise to the notion of *correlated rationalizability*, which is equivalent to iterated strict dominance.

To see this, modify the proof above, replacing the subscript j with the subscript $-i$. The separating hyperplane theorem shows that if $\tilde{s}_i \in S_i^{n+1}$,

3. A two-person, zero-sum game with strategy spaces S_1 and S_2 has a (minmax) value if

$$\sup_{s_1 \in S_1} \inf_{s_2 \in S_2} u_1(s_1, s_2) = \inf_{s_2 \in S_2} \sup_{s_1 \in S_1} u_1(s_1, s_2).$$

If a game has a value u_1^* and if there exists (s_1^*, s_2^*) such that $u_1(s_1^*, s_2^*) = u_1^*$, then (s_1^*, s_2^*) is called a *saddle point*. Von Neumann (1928) and Fan (1952, 1953) have given sufficient conditions for the existence of a saddle point.

there is a vector

$$\sigma_{-i} = \{\sigma(s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_I)\}_{s_{-i} \in \tilde{S}^n_{-i}}$$

such that $u_i(\tilde{s}_i, \sigma_{-i}) \geq u_i(\sigma_i, \sigma_{-i})$ for all $\sigma_i \in \tilde{\Sigma}^n_i$. However, σ_{-i} is an arbitrary probability distribution over S^n_{-i}, and in general it cannot be interpreted as a mixed strategy, as it may involve player i's rivals' correlating their randomizations.

2.1.5 Discussion

Rationalizability, by design, makes very weak predictions; it does not distinguish between any outcomes that cannot be excluded on the basis of common knowledge of rationality. For example, in the battle of the sexes (figure 1.10a), rationalizability allows the prediction that the players are certain to end up at (F, B), where both get 0. (F, B) is not a Nash equilibrium; in fact, both players can gain by deviating. We can see how it might nevertheless occur: Player 1 plays F, expecting player 2 to play F, and 2 plays B expecting 1 to play B. Thus, we might be unwilling to say (F, B) *wouldn't* happen, especially if these players haven't played each other before. In some special cases, such as if we know that player 2's past play with other opponents has led him to expect (B, B) while player 1's has led him to expect (F, F), (F, B) might even be the most likely outcome. However, such situations seem rare; most often we might hesitate to predict that (F, B) has high probability. Rabin (1989) formalizes this idea by asking how likely each player can consider a given outcome. If player 1 is choosing a best response to his subjective beliefs $\hat{\sigma}_2$ about player 2's strategy, then for any value of $\hat{\sigma}_2$ player 1 must assign (F, B) a probability no greater than $\frac{1}{3}$: If he assigns a probability greater than $\frac{1}{3}$ to player 2's playing B, player 1 will play B. Similarly, player 2 cannot assign (F, B) a probability greater than $\frac{2}{3}$. Thus, Rabin argues that we ought to be hesitant to assign (F, B) a probability greater than the maximum of the two probabilities (that is, $\frac{2}{3}$).

2.2 Correlated Equilibrium††

The concept of Nash equilibrium is intended to be a minimal necessary condition for "reasonable" predictions in situations where the players must choose their strategies independently. Now consider players who may engage in preplay discussion, but then go off to isolated rooms to choose their strategies. In some situations, both players might gain if they could build a "signaling device" that sent signals to the separate rooms. Aumann's (1974) notion of a correlated equilibrium captures what could be achieved with any such signals. (See Myerson 1986 for a fuller introduction to this concept, and for a discussion of its relationship to the theory of mechanism design.)

	L	R
U	5,1	0,0
D	4,4	1,5

Figure 2.4

To motivate this concept, consider Aumann's example, presented in figure 2.4. This game has three equilibria: (U, L),(D, R), and a mixed-strategy equilibrium in which each player puts equal weight on each of his pure strategies and that gives each player 2.5. If they can jointly observe a "coin flip" (or sunspots, or any other publicly observable random variable) before play, they can achieve payoffs (3, 3) by a joint randomization between the two pure-strategy equilibria. (For example, flip a fair coin, and use the strategies "player 1 plays U if heads and D if tails; player 2 plays L if heads and R if tails"). More generally, by using a publicly observable random variable, the players can obtain any payoff vector in the convex hull of the set of Nash-equilibrium payoffs. Conversely, the players cannot obtain any payoff vector outside the convex hull of Nash payoffs by using publicly observable random variables.

However, the players can do even better (still without binding contracts) if they can build a device that sends *different but correlated signals* to each of them. This device will have three equally likely states: A, B, and C. Suppose that if A occurs player 1 is perfectly informed, but if the state is B or C player 1 does not know which of the two prevails. Player 2, conversely, is perfectly informed if the state is C, but he cannot distinguish between A and B. In this transformed game, the following is a Nash equilibrium: Player 1 plays U when told A, and D when told (B, C); player 2 plays R when told C, and L when told (A, B). Let's check that player 1 does not want to deviate. When he observes A, he knows that player 2 observes (A, B), and thus that player 2 will play L; in this case U is player 1's best response. If player 1 observes (B, C), then conditional on his information he expects player 2 to play L and R with equal probability. In this case player 1 will average 2.5 from either of his choices, so he is willing to choose D. So player 1 is choosing a best response; the same is easily seen to be true for player 2. Thus, we have constructed an equilibrium in which the players' choices are correlated: The outcomes (U, L),(D, L), and (D, R) are chosen with probability $\frac{1}{3}$ each, and the "bad" outcome (U, R) never occurs. In this new equilibrium the expected payoffs are $3\frac{1}{3}$ each, which is outside the convex hull of the equilibrium payoffs of the original game without the signaling device. (Note that adding the signaling device does not remove the "old" equilibria: Since the signals do not influence payoffs, if player 1 ignores his signal, player 2 may as well ignore hers.)

	L	R
U	0,1,3	0,0,0
D	1,1,1	1,0,0

A

	L	R
U	2,2,2	0,0,0
D	2,2,0	2,2,2

B

	L	R
U	0,1,0	0,0,0
D	1,1,0	1,0,3

C

Figure 2.5

The next example of a correlated equilibrium illustrates the familiar game-theoretic point that a player may *gain* from limiting his own information *if the opponents know he has done so*, because this may induce the opponents to play in a desirable fashion.

In the game illustrated in figure 2.5, player 1 chooses rows, player 2 chooses columns, and player 3 chooses matrices. In this game the unique Nash equilibrium is (D, L, A), with payoffs (1, 1, 1).

Now imagine that the players build a correlating device with two equally likely outcomes, H ("heads") and T ("tails"), and that they arrange for the outcome to be perfectly revealed to players 1 and 2, while player 3 receives no information at all. In this game, a Nash equilibrium is for player 1 to play U if H and D if T, player 2 to play L if H and R if T, and player 3 to play B. Player 3 now faces a distribution of $\frac{1}{2}$(U, L) and $\frac{1}{2}$(D, R), which makes B a best response. Note the importance of players 1 and 2 knowing that player 3 does not know whether heads or tails prevailed when choosing the matrix. If the random variable were publicly observable and players 1 and 2 played the above strategies, then player 3 would choose matrix A if H and matrix C if T, and thus players 1 and 2 would deviate as well. As we observed, the equilibrium would then give player 3 a payoff of 1.

With these examples as an introduction, we turn to a formal definition of correlated equilibrium. There are two equivalent ways to formulate the definition.

The first definition explicitly defines strategies for the "expanded game" with a correlating device and then applies the definition of Nash equilibrium to the expanded game. Formally, we identify a correlating device with a triple $(\Omega, \{H_i\}, p)$. Here Ω is a (finite) state space corresponding to the outcomes of the device (e.g., H or T in our discussion of figure 2.5), and p is a probability measure on the state space Ω.

Player i's information about which $\omega \in \Omega$ occurred is represented by the *information partition* H_i; if the true state is ω, player i is told that the state lies in $h_i(\omega)$. In our discussion of figure 2.4, player 1's information partition is ((A), (B, C)) and player 2's partition is ((A, B), (C)). In the discussion of figure 2.5, players 1 and 2 have the partition ((H), (T)); player 3's partition is the one-element set (H, T).

More generally, a *partition* of a finite set Ω is a collection of disjoint subsets of Ω whose union is Ω. An *information partition* H_i assigns an $h_i(\omega)$

to each ω in such a way that $\omega \in h_i(\omega)$ for all ω. The set $h_i(\omega)$ consists of those states that player i regards as possible when the truth is ω; the requirement that $\omega \in h_i(\omega)$ means that player i is never "wrong" in the weak sense that he never regards the true state as impossible. However, player i may be poorly informed. If his partition is the one-element set $h_i(\omega) = \Omega$ for all ω, he has no information at all beyond his prior. (This is called the "trivial partition.")

For all h_i with positive prior probability, player i's posterior beliefs about Ω are given by Bayes' law: $p(\omega \mid h_i) = p(\omega)/p(h_i)$ for ω in h_i, and $p(\omega \mid h_i) = 0$ for ω not in h_i.

Given a correlating device $(\Omega, \{H_i\}, p)$, the next step is to define strategies for the expanded game where players can condition their play on the signal the correlating device sends them. A pure strategy for the expanded game can be viewed as a function $\jmath_i$ that maps elements h_i of H_i—the possible signals that player i receives—to pure strategies $s_i \in S_i$ of the game without the correlating device. Note that if $\omega' \in h_i(\omega)$, then necessarily $\jmath_i$ prescribes the same actions in states ω and ω'. Instead of defining strategies in this way as maps from information sets to elements of S_i, it will be more convenient for our analysis to use an equivalent formulation: We will define pure strategies $\jmath_i$ as maps from Ω to S_i with the additional property that $\jmath_i(\omega) = \jmath_i(\omega')$ if $\omega' \in h_i(\omega)$. The formal term for this is that the strategies are *adapted* to the information structure. (Mixed strategies can be defined in the obvious way, but they will be irrelevant if we take the state space Ω to be sufficiently large. For example, instead of player 1 playing $(\frac{1}{2}\text{U}, \frac{1}{2}\text{D})$ when given signal h_i, we could construct an expanded state space $\hat{\Omega}$ where each $\omega \in h_i$ is replaced by two equally likely states, ω' and ω'', and player 1 is told both "h_i" and whether the state is of the single-prime or the double-prime kind. Then player i can use the pure strategy "play U if told h_i and single-prime, play D if told h_i and double-prime." This will be equivalent to the original mixed strategy.)

Definition 2.4A A *correlated equilibrium* $\jmath$ relative to information structure $(\Omega, \{H_i\}, p)$ is a Nash equilibrium in strategies that are adapted to this information structure. That is, $(\jmath_1, \ldots, \jmath_I)$ is a correlated equilibrium if, for every i and every adapted strategy $\tilde{\jmath}_i$,

$$\sum_{\omega \in \Omega} p(\omega) u_i(\jmath_i(\omega), \jmath_{-i}(\omega)) \geq \sum_{\omega \in \Omega} p(\omega) u_i(\tilde{\jmath}_i(\omega), \jmath_{-i}(\omega)). \tag{2.1}$$

This definition, where the distribution p over Ω is the same for all players, is sometimes called an "objective correlated equilibrium" to distinguish it from "subjective correlated equilibria" where players may disagree on prior beliefs and each player i is allowed to have different beliefs p_i. We say more about subjective correlated equilibrium in section 2.3.

Definition 2.4A, which requires that $\jmath_i$ maximize player i's "*ex ante*" payoff her expected payoff before knowing which h_i contains the true state implies that s_i maximizes player i's payoff *conditional on* h_i for each h_i that player i assigns positive prior probability (this conditional payoff is often called an "interim" payoff). That is, (2.1) is equivalent to the condition that, for all players i, information sets h_i with $p(h_i) > 0$, and all s_i,

$$\sum_{\{\omega | h_i(\omega) = h_i\}} p(\omega | h_i) u_i(s_i(\omega), s_{-i}(\omega)) \geq \sum_{\{\omega | h_i(\omega) = h_i\}} p(\omega | h_i) u_i(s_i, s_{-i}(\omega)). \quad (2.2)$$

When all players have the same prior, any h_i with $p(h_i) = 0$ is irrelevant, and all states $\omega \in h_i$ can be omitted from the specification of Ω. New issues arise when the priors are different, as we will see when we discuss Brandenburger and Dekel 1987.

An awkward feature of this definition is that it depends on the particular information structure specified, yet there are an infinite number of possible state spaces Ω and many information structures possible for each. Fortunately there is a more concise way to define correlated equilibrium. This alternative definition is based on the realization that any joint distribution over actions that forms a correlated equilibrium for some correlating device can be attained as an equilibrium with the "universal device" whose signals to each player constitute a recommendation of how that player should play. In the example of figure 2.4, player 1 would be told "play D" instead of "the state is (B, C)," and player 1 would be willing to follow this recommendation so long as, when he is told to play D, the conditional probability of player 2 being instructed to play R is $\frac{1}{2}$. (Those familiar with the literature on mechanism design will recognize this observation as a version of the "revelation principle"; see chapter 7.)

Definition 2.4B A *correlated equilibrium* is any probability distribution $p(\cdot)$ over the pure strategies $S_1 \times \cdots \times S_I$ such that, for every player i and every function $d_i(\cdot)$ that maps S_i to S_i,

$$\sum_{s \in S} p(s) u_i(s_i, s_{-i}) \geq \sum_{s \in S} p(s) u_i(d_i(s_i), s_{-i}).$$

Just as with definition 2.4A, there is an equivalent version of the definition stated in terms of maximization conditional on each recommendation: $p(\cdot)$ is a correlated equilibrium if, for every player i and every s_i with $p(s_i) > 0$,

$$\sum_{s_{-i} \in S_{-i}} p(s_{-i} | s_i) u_i(s_i, s_{-i}) \geq \sum_{s_{-i} \in S_{-i}} p(s_{-i} | s_i) u_i(s_i', s_{-i}) \quad \forall s_i' \in S_i.$$

That is, player i should not be able to gain by disobeying the recommendation to play s_i if every other player obeys his recommendation.

Let us explain why the two definitions of correlated equilibrium are equivalent. Clearly an equilibrium in the sense of definition 2.4B is an equilibrium according to definition 2.4A—just take $\Omega = S$, and $h_i(s) = \{s' | s_i' = s_i\}$.

Conversely, if σ is an equilibrium relative to some $(\Omega, \{H_i\}, \tilde{p})$ as in definition 2.4A, set $p(s)$ to be the sum of $\tilde{p}(\omega)$ over all $\omega \in \Omega$ such that $\sigma_i(\omega) = s_i$ for all players i. Let us check that no player i can gain by disobeying any recommendation $s_i \in S_i$. (The only reason this isn't completely obvious is that there may have been several information sets h_i where player i played s_i, in which case his information has been reduced to s_i alone.) Set

$$J_i(s_i) = \{\omega \mid \sigma_i(\omega) = s_i\},$$

so that $\tilde{p}(J_i(s_i)) = p(s_i)$ is the probability that player i is told to play s_i. If we view each pure-strategy profile $\sigma_{-i}(\omega)$ as a degenerate mixed strategy that places probability 1 on $s_{-i} = \sigma_{-i}(\omega)$, then the probability distribution on opponents' strategies that player i believes he faces, conditional on being told to play s_i, is

$$\sum_{\omega \in J_i(s_i)} \frac{\tilde{p}(\omega)\sigma_{-i}(\omega)}{\tilde{p}(J_i(s_i))},$$

which is a convex combination of the distributions conditional on each h_i such that $\sigma_i(h_i) = s_i$. Since player i could not gain by deviating from σ_i at any such h_i, he cannot gain by deviating when this finer information structure is replaced by the one that simply tells him his recommended strategy.

A pure-strategy Nash equilibrium is a correlated equilibrium in which the distribution $p(\cdot)$ is degenerate. Mixed-strategy Nash equilibria are also correlated equilibria: Just take $p(\cdot)$ to be the joint distribution implied by the equilibrium strategies, so that the recommendations made to each player convey no information about the play of his opponents.

Inspection of the definition shows that the set of correlated equilibria is convex, so the set of correlated equilibria is at least as large as the convex hull of the Nash equilibria. This convexification could be attained by using only public correlating devices. But, as we have seen, nonpublic (imperfect) correlation can lead to equilibria outside the convex hull of the Nash set.

Since Nash equilibria exist in finite games, correlated equilibria do too. Actually, the existence of correlated equilibria would seem to be a simpler problem than the existence of Nash equilibria, because the set of correlated equilibria is defined by a system of linear inequalities and is therefore convex; indeed, Hart and Schmeidler (1989) have provided an existence proof that uses only linear methods (as opposed to fixed-point theorems). One might also like to know when the set of correlated equilibria differs "greatly" from the convex hull of the Nash equilibria, but this question has not yet been answered.

One may take the view that the correlation in correlated equilibria should be thought of as the result of the players receiving "endogenous" correlated signals, so that the notion of correlated equilibrium is particularly appropriate in situations with preplay communication, for then the players might be able to design and implement a procedure for obtaining correlated, private signals.[4] When players do not meet and design particular correlated devices, it is plausible that they may still observe exogenous random signals (i.e., "sunspots" or "moonspots") on which they can condition their play. If the signals are publicly observed they can only serve to convexify the set of Nash equilibrium payoffs. But if the signals are observed privately and yet are correlated, they also allow imperfectly correlated equilibria, which may have payoffs outside the convex hull of Nash equilibria, such as $(3\frac{1}{3}, 3\frac{1}{3})$ in figure 2.4. (Aumann (1987) argues that Bayesian rationality, broadly construed, implies that play *must* correspond to a correlated equilibrium, though not necessarily to a Nash equilibrium.)

2.3 Rationalizability and Subjective Correlated Equilibria[†††]

In matching pennies (figure 1.10a), rationalizability allows player 1 to be sure he will outguess player 2, and player 2 to be sure he'll outguess player 1; the players' strategic beliefs need not be consistent. It is interesting to note that this kind of inconsistency in beliefs can be modeled as a kind of correlated equilibrium with inconsistent beliefs. We mentioned the possibility of inconsistent beliefs when we defined subjective correlated equilibrium, which generalizes objective correlated equilibrium by allowing each player i to have different beliefs $p_i(\cdot)$ over the joint recommendation $s \in S$. That notion is weaker than rationalizability, as is shown by figure 2.6 (which is drawn from Brandenburger and Dekel 1987). One subjective correlated equilibrium for this game has player 1's beliefs assign probability 1 to (U, L) and player 2's beliefs assign probability $\frac{1}{2}$ each to (U, L) and (D, L). Given his beliefs, player 2 is correct to play L. However, that

	L	R
U	2,0	1,1
D	1,1	0,0

Figure 2.6

4. Barany (1988) shows that if there are at least four players ($I \geq 4$), any *correlated* equilibrium of a strategic-form game coincides with a *Nash* equilibrium of an extended game in which the players engage in costless conversations (cheap talk) before they play the strategic-form game in question. If there are only two players, then the set of Nash equilibria with cheap talk coincides with the subset of correlated equilibria induced by perfectly correlated signals (i.e., publicly observed randomizing devices.)

strategy is deleted by iterated dominance, and so we see that subjective correlated equilibrium is less restrictive than rationalizability.

The point is that subjective correlated equilibrium allows each player's beliefs about his opponents to be completely arbitrary, and thus cannot capture the restrictions implied by common knowledge of the payoffs. Brandenburger and Dekel introduce the idea of an *a posteriori* equilibrium, which does capture these restrictions.

Although this equilibrium concept, like correlated equilibrium, can be defined either with reference to explicit correlating devices or in a "direct version," it is somewhat simpler here to make the correlating device explicit.

Given state space Ω, partition H_i, and priors $p_i(\cdot)$, we now require, for each ω (even those with $p_i(\omega) = 0$),[5] that player i have well-defined conditional beliefs $p_i(\omega' \mid h_i(\omega))$, satisfying $p_i(h_i(\omega) \mid h_i(\omega)) = 1$.

Definition 2.5 The adapted strategies $(\mathfrak{s}_1, \ldots, \mathfrak{s}_I)$ are an *a posteriori* equilibrium if, for all $\omega \in \Omega$, all players i, and all s_i,

$$\sum_{\omega' \in h_i(\omega)} p_i(\omega' \mid h_i(\omega)) u_i(\mathfrak{s}_i(\omega), \mathfrak{s}_{-i}(\omega'))$$

$$\geq \sum_{\omega' \in h_i(\omega)} p_i(\omega' \mid h_i(\omega)) u_i(s_i, \mathfrak{s}_{-i}(\omega')).$$

Thus, player i's strategy is required to be optimal for all ω, even those to which he assigns prior probability 0.

Brandenburger and Dekel show that the set of correlated rationalizable payoffs is precisely the set of interim payoffs to *a posteriori* equilibria; that is, they are the payoffs player i can expect to receive conditional on a particular $\omega \in \Omega$.

Exercises

Exercise 2.1**

(a) Consider an alternative definition of iterated strict dominance that proceeds as in section 2.1 except that, at each state n, only the strictly dominated pure strategies of players $I(n) \subseteq I$ are deleted. Suppose that, for each player i, there exists an infinite number of steps n such that $i \in I(n)$. If the game is finite, show that the resulting limit set is S^∞ (as given in definition 2.1), so that there is no loss of generality in taking $I(n) = I$ for all n. Hint: The intuition is that if a strategy s_i is strictly dominated at step n but is not eliminated because $i \notin I(n)$, then it will be eliminated at the next step $n' > n$ such that $i \in I(n')$, as (i) the set of strategies s_{-i} remaining at step n' is no larger than the set of strategies s_{-i} remaining at step n and (ii) if

5. Note that we do not require priors to be absolutely continuous with respect to each other that is, they may disagree on which ω's have positive probability.

strategy s_i is strictly dominated relative to a set Σ'_{-i} of opponents' mixed strategies it is strictly dominated relative to any subset $\Sigma_{-i} \subseteq \Sigma'_{-i}$. Show by induction on n that any strategy that is deleted at stage n under the maximal-deletion process $I(k) = I$ for all k is deleted in a finite number of steps (no fewer than n) when deletion is not required to be maximal.

(b) Verify that, in a finite game, the two definitions of iterated deletion of dominated mixed strategies given in section 2.1 are equivalent.

Exercise 2.2* Prove that if a game is solvable by iterated strict dominance, it has a unique Nash equilibrium.

Exercise 2.3** Consider an arbitrary two-player game with action spaces $A_1 = A_2 = [0, 1]$ and payoff functions that are twice continuously differentiable and concave in own action. Say that the game is *locally solvable by iterated strict dominance at* a^* if there is a rectangle N containing a^* such that when players are restricted to choosing actions in N, the successive elimination of strictly dominated strategies yields the unique point a^*. Relate the conditions for local solvability by iterated strict dominance of the simultaneous-move process to those for local stability of the alternating-move Cournot adjustment process. (The answer is in Gabay and Moulin 1980.)

Exercise 2.4* Show that in the Cournot game with three Nash equilibria with the reaction curves depicted in figure 1.14, the strategies that survive iterated deletion of strictly dominated strategies are the outputs that belong to the interval whose boundaries are the projections of B and D.

Exercise 2.5** A competitive economy may be described as a game with a continuum of players. Concepts such as iterated dominance, rationalizability, and Nash equilibrium can, with minor adjustments, be applied to such situations. Consider the following "wheat market": There is a continuum of farmers indexed by a parameter i distributed with a density $f(i)$ on $[\underline{i}, \bar{i}]$, where $\underline{i} > 0$. They must choose the size of their crop $q(i)$ before the market for wheat opens. The cost function of farmer i is $C(q, i) = q^2/2i$. The farmer's utility function is thus $u_i = p\,q(i) - q(i)^2/2i$, where p is the price of wheat. Let $O(p)$ denote the aggregate supply function when farmers perfectly predict p:

$$O(p) = \left(\int_{\underline{i}}^{\bar{i}} i\, f(i)\, di\right) p \equiv kp.$$

The demand curve is $D(p) = a - bp$ for $0 \le p \le a/b$ and 0 otherwise. The timing is such that the farmers simultaneously choose the size of their crop, then the price clears the market:

$$\int_{\underline{i}}^{\bar{i}} q(i) f(i)\, di = D(p).$$

A perfect foresight (or Nash equilibrium) is a price p^* such that $O(p^*) = D(p^*)$ (more correctly, it is a strategy profile $q^*(\cdot)$ such that $q^*(i) = ip^*$.) Note that $p^* = a/(b + k)$.

Apply iterated strict dominance in this game among farmers. Show that if $b > k$, the game is solvable by iterated strict dominance, which yields the perfect-foresight equilibrium. When $b \leq k$, determine the interval of prices that correspond to outputs that survive iterated strict dominance. Draw the link with the stability of the "cobweb" tatônnement in which the market is repeated over time, and farmers have point price expectations equal to the last period's price. (This exercise is drawn from Guesnerie 1989, which also addresses production and demand uncertainty, price floors and ceilings, and sequential timing of crop planting.)

Exercise 2.6** Consider the two-player game in figure 2.7. This is matching pennies with an outside option α for player 1. Suppose that $\alpha \in (0, 1)$.

(a) Show that the set of mixed strategies for player 1 surviving iterated deletion of strictly dominated strategies consists of two "edges of the strategy simplex": the set of mixed strategies with support (H, α) and the set of mixed strategies with support (T, α).

(b) Show directly (that is, without applying theorem 2.2) that the set of rationalizable strategies for player 1 is also composed of these two edges. (Hint: Use a diagram similar to figure 2.3.)

	H	T
H	1, −1	−1, 1
T	−1, 1	1, −1
α	α, 0	α, 0

Figure 2.7

	L	R
U	9	0
D	0	0

A

	L	R
U	0	9
D	9	0

B

	L	R
U	0	0
D	0	9

C

	L	R
U	6	0
D	0	6

D

Figure 2.8

Exercise 2.7** Consider the game in figure 2.8, where player 1 chooses rows, player 2 chooses columns, and player 3 chooses matrices. Player 3's payoffs are given in the figure. Show that action D is not a best response to any mixed strategies of players 1 and 2, but that D is *not* dominated. Comment.

Exercise 2.8*** Find *all* the correlated equilibria of the games illustrated in figures 2.4 and 2.5.

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II DYNAMIC GAMES OF COMPLETE INFORMATION

3 Extensive-Form Games

3.1 Introduction†

In the examples we examined in part I, such as the stag hunt, the prisoner's dilemma, and the battle of the sexes, the players choose their actions simultaneously. Much of the recent interest in the economic applications of game theory has been in situations with an important dynamic structure, such as entry and entry deterrence in industrial organization and the "time-consistency" problem in macroeconomics. Game theorists use the concept of a *game in extensive form* to model such dynamic situations. The extensive form makes explicit the order in which players move, and what each player knows when making each of his decisions. In this setting, strategies correspond to contingent plans instead of uncontingent actions. As we will see, the extensive form can be viewed as a multi-player generalization of a decision tree. Not surprisingly, many results and intuitions from decision theory have game-theoretic analogs. We will also see how to build up the strategic-form representation of a game from its extensive form. Thus, we will be able to apply the concepts and results of part I to dynamic games.

As a simple example of an extensive-form game, consider the idea of a "Stackelberg equilibrium" in a duopoly. As in the Cournot model, the actions of the firms are choices of output levels, q_1 for player 1 and q_2 for player 2. The difference is that we now suppose that player 1, the "Stackelberg leader," chooses her output level q_1 first, and that player 2 observes q_1 before choosing his own output level. To make things concrete, we suppose that production is costless, and that demand is linear, with $p(q) = 12 - q$, so that player i's payoff is $u_i(q_1, q_2) = [12 - (q_1 + q_2)]q_i$. How should we extend the idea of Nash equilibrium to this setting? And how should we expect the players to play?

Since player 2 observes player 1's choice of output q_1 before choosing q_2, in principle player 2 could condition his choice of q_2 on the observed level of q_1. And since player 1 moves first, she cannot condition her output on player 2's. Thus, it is natural that player 2's *strategies* in this game should be maps of the form $s_2: Q_1 \to Q_2$ (where Q_1 is the space of feasible q_1's and Q_2 is the space of feasible q_2's), while player 1's strategies are simply choices of q_1. Given a (pure) strategy profile of this form, the outcome is the output vector $(q_1, s_2(q_1))$, with payoffs $u_i(q_1, s_2(q_1))$.

Now that we have identified strategy spaces and the payoff functions, we can define a Nash equilibrium of this game in the obvious way: as a strategy profile such that neither player can gain by switching to a different strategy. Let's consider two particular Nash equilibria of this game.

The first equilibrium gives rise to the Stackelberg output levels normally associated with this game. In this equilibrium, player 2's strategy s_2 is to choose, for each q_1, the level of q_2 that solves $\max_{q_2'} u_2(q_1, q_2')$, so that s_2 is identically equal to the Cournot reaction function r_2 defined in chapter 1. With the payoffs we have specified, $r_2(q_1) = 6 - q_1/2$.

Nash equilibrium requires that player 1's strategy maximize her payoff given that $s_2 = r_2$, so that player 1's output level q_1^* is the solution to $\max_{q_1} u_1(q_1, r_2(q_1))$, which with the payoffs we specified gives $q_1^* = 6$.

The output levels $(q_1^*, r_2(q_1^*))$ (here equal to $(6, 3)$) are called the *Stackelberg outcome* of the game; this is the outcome economics students are taught to expect. In the usual case, r_2 is a decreasing function, and so player 1 can decrease player 2's output by increasing her own. As a result, player 1's Stackelberg output level and payoff are typically higher than in the Cournot equilibrium where both players move simultaneously, and player 2's output and payoff are typically lower. (In our case the unique Cournot equilibrium is $q_1^C = q_2^C = 4$, with payoffs of 16 each; in the Stackelberg equilibrium the leader's payoff is 18 and the follower's is 9.)

Though the Stackelberg outcome may seem the natural prediction in this game, there are many other Nash equilibria. One of them is the profile "$q_1 = q_1^C; s_2(q_1) = q_2^C$ for all q_1." These strategies really are a Nash equilibrium: Given that player 2's output will be q_2^C independent of q_1, player 1's problem is to maximize $u_1(q_1, q_2^C)$, and by definition this maximization is solved by the Cournot output q_1^C. And given that $q_1 = q_1^C$, player 2's payoff will be $u_2(q_1^C, s_2(q_1^C))$, which is maximized by *any* strategy s_2 such that $s_2(q_1^C) = q_2^C$, including the constant strategy $s_2(\cdot) \equiv q_2^C$. Note, though, that this strategy is *not* a best response to other output levels that player 1 might have chosen but did not; i.e., q_2^C is not in general a best response to q_1 for $q_1 \neq q_1^C$.

So we have identified two Nash equilibria for the game where player 1 chooses her output first: one equilibrium with the "Stackelberg outputs" and one where the output levels are the same as if the players moved simultaneously. Why is the first equilibrium more reasonable, and what is wrong with the second one? Most game theorists would answer that the second equilibrium is "not credible," as it relies on an "empty threat" by player 2 to hold his output at q_2^C regardless of player 1's choice. This threat is empty because if player 1 were to present player 2 with the *fait accompli* of choosing the Stackelberg output q_1^*, player 2 would do better to choose a different level of q_2—in particular, $q_2 = r_2(q_1^*)$. Thus, if player 1 knows player 2's payoffs, the argument goes, she should not believe that player 2 would play q_2^C no matter what player 1's output. Rather, player 1 should predict that player 2 will play an optimal response to whatever q_1 player 1 actually chooses, so that player 1 should predict that whatever level of q_1 she chooses, player 2 will choose the optimal response $r_2(q_1)$. This argument picks out the "Stackelberg equilibrium" as the unique credible outcome. A more formal way of putting this is that the Stackelberg equilibrium is consistent with *backward induction*, so called because the idea is to start by solving for the optimal choice of the last mover for each possible situation he might face, and then work backward to compute the optimal choice for the player before. The ideas of credibility and backward induction are clearly present in the textbook analysis of the Stackelberg game; they were informally applied by Schelling (1960) to the analysis of commitment in a number of settings. Selten (1965) formalized the intuition with his concept of a *subgame-perfect equilibrium*, which extends the idea of backward induction to extensive games where players move simultaneously in several periods, so the backward-induction algorithm is not applicable because there are several "last movers" and each of them must know the moves of the others to compute his own optimal choice.

This chapter will develop the formalism for modeling extensive games and develop the solution concepts of backward induction and subgame perfection. Although the extensive form is a fundamental concept in game theory, its definition may be a bit detailed for readers who are more interested in applications of games than in mastering the general theory. With such readers in mind, section 3.2 presents a first look at dynamic games by treating a class of games with a particularly simple structure: the class of "multi-stage games with observed actions." These games have "stages" such that (1) in each stage every player knows all the actions taken by any player, including "Nature," at any previous stage, and (2) players move "simultaneously" within each stage.

Though very special, this class of games includes the Stackelberg example we have just discussed, as well as many other examples from the economics literature. We use multi-stage games to illustrate the idea that strategies can be contingent plans, and to give a first definition of subgame perfection. As an illustration of the concepts, subsection 3.2.3 discusses how to model the idea of commitment, and addresses the particular example called the "time-consistency problem" in macroeconomics. Readers who lack the time or interest for the general extensive-game model are advised to skip from the end of section 3.2 to section 3.6, which gives a few cautions about the potential drawbacks of the ideas of backward induction and subgame perfection.

Section 3.3 introduces the concepts involved in defining an extensive form. Section 3.4 discusses strategies in the extensive form, called "behavior strategies," and shows how to relate them to the strategic-form strategies discussed in chapters 1 and 2. Section 3.5 gives the general definition of subgame perfection. We postpone discussion of more powerful equilibrium refinements to chapters 8 and 11 in order to first study several interesting classes of games which can be fruitfully analyzed with the tools we develop in this chapter.

Readers who already have some informal understanding of dynamic games and subgame perfection probably already know the material of section 3.2, and are invited to skip directly to section 3.3. (Teaching note: When planning to cover all of this chapter, it is probably not worth taking the time to teach section 3.2 in class; you may or may not want to ask the students to read it on their own.)

3.2 Commitment and Perfection in Multi-Stage Games with Observed Actions[+]

3.2.1 What Is a Multi-Stage Game?

Our first step is to give a more precise definition of a "multi-stage game with observed actions." Recall that we said that this meant that (1) all players knew the actions chosen at all previous stages $0, 1, 2, \ldots, k-1$ when choosing their actions at stage k, and that (2) all players move "simultaneously" in each stage k. (We adopt the convention that the first stage is "stage 0" in order to simplify the notation concerning discounting when stages are interpreted as periods.) Players move simultaneously in stage k if each player chooses his or her action at stage k without knowing the stage-k action of any other player. Common usage to the contrary, "simultaneous moves" does not exclude games where players move in alternation, as we allow for the possibility that some of the players have the one-element choice set "do nothing." For example, the Stackelberg game has two stages: In the first stage, the leader chooses an output level (and the follower "does nothing"). In the second stage, the follower knows the leader's output and chooses an output level of his own (and the leader "does nothing"). Cournot and Bertrand games are one-stage games: All players choose their actions at once and the game ends. Dixit's (1979) model of entry and entry deterrence (based on work by Spence (1977)) is a more complex example: In the first stage of this game, an incumbent invests in capacity; in the second stage, an entrant observes the capacity choice and decides whether to enter. If there is no entry, the incumbent chooses output as a monopolist in the third stage; if entry occurs, the two firms choose output simultaneously as in Cournot competition.

Often it is natural to identify the "stages" of the game with time periods, but this is not always the case. A counterexample is the Rubinstein-Ståhl model of bargaining (discussed in chapter 4), where each "time period" has two stages. In the first stage of each period, one player proposes an agreement; in the second stage, the other player either accepts or rejects the proposal. The distinction is that time periods refer to some physical measure of the passing of time, such as the accumulation of delay costs in the bargaining model, whereas the stages need not have a direct temporal interpretation.

In the first stage of a multi-stage game (stage 0), all players $i \in \mathscr{I}$ simultaneously choose actions from choice sets $A_i(h^0)$. (Remember that some of the choice sets may be the singleton "do nothing." We let $h^0 = \varnothing$ be the "history" at the start of play.) At the end of each stage, all players observe the stage's action profile. Let $a^0 \equiv (a_1^0, \ldots, a_I^0)$ be the stage-0 action profile. At the beginning of stage 1, players know history h^1, which can be identified with a^0 given that h^0 is trivial. In general, the actions player i has available in stage 1 may depend on what has happened previously, so we let $A_i(h^1)$ denote the possible second-stage actions when the history is h^1. Continuing iteratively, we define h^{k+1}, the history at the end of stage k, to be the sequence of actions in the previous periods,

$$h^{k+1} = (a^0, a^1, \ldots, a^k),$$

and we let $A_i(h^{k+1})$ denote player i's feasible actions in stage $k+1$ when the history is h^{k+1}. We let $K+1$ denote the total number of stages in the game, with the understanding that in some applications $K = +\infty$, corresponding to an infinite number of stages; in this case the "outcome" when the game is played will be an infinite history, h^∞. Since each h^{K+1} by definition describes an entire sequence of actions from the beginning of the game on, the set H^{K+1} of all "terminal histories" is the same as the set of possible outcomes when the game is played.

In this setting, a *pure strategy for player i* is simply a contingent plan of how to play in each stage k for possible history h^k. (We will postpone discussion of mixed strategies until section 3.3, as they will not be used in the examples we discuss here.) If we let H^k denote the set of all stage-k histories, and let

$$A_i(H^k) = \bigcup_{h^k \in H^k} A_i(h^k),$$

a pure strategy for player i is a sequence of maps $\{s_i^k\}_{k=0}^K$, where each s_i^k maps H^k to the set of player i's feasible actions $A_i(H^k)$ (i.e., satisfies $s_i^k(h^k) \in A_i(h^k)$ for all h^k). It should be clear how to find the sequence of actions generated by a profile of such strategies: The stage-0 actions are $a^0 = s^0(h^0)$, the stage-1 actions are $a^1 = s^1(a^0)$, the stage-2 actions are $a^2 = s^2(a^0, a^1)$, and so on. This is called the *path* of the strategy profile. Since the terminal histories represent an entire sequence of play, we can represent each player i's payoff as a function $u_i\colon H^{K+1} \to \mathbb{R}$. In most applications the payoff functions are additively separable over stages (i.e., each player's overall payoff is some weighted average of single-stage payoffs $g_i(a^k)$, $k = 0, \ldots, K$), but this restriction is not necessary.

Since we can assign an outcome in H^{K+1} to each strategy profile, and a payoff vector to each outcome, we can now compute the payoff to any strategy profile; in an abuse of notation, we will represent the payoff vector to profile s as $u(s)$. A (pure-strategy) *Nash equilibrium* in this context is simply a strategy profile s such that no player i can do better with a different strategy, which is the familiar condition that $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$ for all s_i'.

The Cournot and Bertrand "equilibria" discussed in chapter 1 are trivial examples of Nash equilibria of multi-stage (actually one-stage) games. We saw two other examples of Nash equilibria when we discussed the Stackelberg game at the beginning of this chapter. We also saw that some of these Nash equilibria may rely on "empty threats" of suboptimal play at histories that are not expected to occur—that is, at histories off the path of the equilibrium.

3.2.2 Backward Induction and Subgame Perfection

In the Stackelberg game, it was easy to see how player 2 "ought" to play, because once q_1 was fixed player 2 faced a simple decision problem. This allowed us to solve for player 2's optimal second-stage choice for each q_1 and then work backward to find the optimal choice for player 1. This algorithm can be extended to other games where only one player moves at each stage. We say that a multi-stage game has *perfect information* if, for every stage k and history h^k, exactly one player has a nontrivial choice set — a choice set with more than one element — and all the others have the one-element choice set "do nothing." A simple example of such a game has player 1 moving in stages 0, 2, 4, etc. and player 2 moving in stages 1, 3, 5, and so on. More generally, some players could move several times in a row, and which player gets to move in stage k could depend on the previous history. The key thing is that only one player moves at each stage k. Since we have assumed that each player knows the past choices of all rivals, this implies that the single player on move at k is "perfectly informed" of all aspects of the game except those which will occur in the future.

Backward induction can be applied to any finite game of perfect information, where *finite* means that the number of stages is finite and the number of feasible actions at any stage is finite, too.[1] The algorithm begins by determining the optimal choices in the final stage K for each history h^K — that is, the action for the player on move, given history h^K, that maximizes that player's payoff conditional on h^K being reached. (There may be more than one maximizing choice; in this case backward induction allows the player to choose any of the maximizers.) Then we work back to stage $K - 1$, and determine the optimal action for the player on move there, given that the player on move at stage K with history h^K will play the action we determined previously. The algorithm proceeds to "roll back," just as in solving decision problems, until the initial stage is reached. At this point we have constructed a strategy profile, and it is easy to verify that this profile is a Nash equilibrium. Moreover, it has the nice property that each player's actions are optimal at every possible history.

The argument for the backward-induction solution in the two-stage Stackelberg game—that player 1 should be able to forecast player 2's second-stage play—strikes us as quite compelling. In a three-stage game,

1. Section 4.6 extends backward induction to infinite games of perfect information, where there is no last period from which to work backward.

the argument is a bit more complex: The player on move at stage 0 must forecast that the player on move at stage 1 will correctly forecast the play of the player on move at stage 2, which clearly is a more demanding hypothesis. And the arguments for backward induction in longer games require correspondingly more involved hypotheses. For this reason, backward-induction arguments may not be compelling in "long" games. For the moment, though, we will pass over the arguments against backward induction; section 3.6 discusses its limitations in more detail.

As defined above, backward induction applies only to games of perfect information. It can be extended to a slightly larger class of games. For instance, in a multi-stage game, if all players have a dominant strategy in the last stage, given the history of the game (or, more generally, if the last stage is solvable by iterated strict dominance), one can replace the last-stage strategies by the dominant strategies, then consider the penultimate stage and apply the same reasoning, and so on. However, this doesn't define backward induction for games that cannot be solved by this backward-induction version of dominance solvability. Yet one would think that the backward-induction idea of predicting what the players are likely to choose in the future ought to carry over to more general games. Suppose that a firm—call it firm 1—has to decide whether or not to invest in a new cost-reducing technology. Its choice will be observed by its only competitor, firm 2. Once the choice is made and observed, the two firms will choose output levels simultaneously, as in Cournot competition. This is a two-stage game, but not one of perfect information. How should firm 1 forecast the second-period output choice of its opponent? In the spirit of equilibrium analysis, a natural conjecture is that the second-period output choices will be those of a Cournot equilibrium for the prevailing cost structure of the industry. That is, each history h^1 generates a simultaneous-move game between the two firms, and firm 1 forecasts that play in this game will correspond to an equilibrium for the payoffs prevailing under h^1. This is exactly the idea of Selten's (1965) *subgame-perfect equilibrium.*

Defining subgame perfection requires a few preliminary steps. First, since all players know the history h^k of moves before stage k, we can view the game from stage k on with history h^k as a game in its own right, which we will denote $G(h^k)$. To define the payoff functions in this game, note that if the actions in stages k through K are a^k though a^K, the final history will be $h^{K+1} = (h^k, a^k, a^{k+1}, \ldots, a^K)$, and so the payoffs will be $u_i(h^{K+1})$. Strategies in $G(h^k)$ are defined in the obvious way: as maps from histories to actions, where the only histories we need consider are those consistent with h^k. So now we can speak of the Nash equilibria of $G(h^k)$.

Next, any strategy profile s of the whole game induces a strategy profile $s|h^k$ on any $G(h^k)$ in the obvious way: For each player i, $s_i|h^k$ is simply the restriction of s_i to the histories consistent with h^k.

Definition 3.1 A strategy profile s of a multi-stage game with observed actions is a *subgame-perfect equilibrium* if, for every h^k, the restriction $s|h^k$ to $G(h^k)$ is a Nash equilibrium of $G(h^k)$.

This definition reduces to backward induction in finite games of perfect information, for the only Nash equilibrium in game $G(h^K)$ at the final stage is for the player on move to choose (one of) his preferred action(s) as in backward induction, the only Nash-equilibrium choice in the next-to-last stage given Nash play at the last stage is as in backward induction, and so on.

Example 3.1

To illustrate the ideas of this section, consider the following model of strategic investment in a duopoly: Firm 1 and firm 2 currently both have a constant average cost of 2 per unit. Firm 1 can install a new technology with an average cost of 0 per unit; installing the technology costs f. Firm 2 will observe whether or not firm 1 invests in the new technology. Once firm 1's investment decision is observed, the two firms will simultaneously choose output levels q_1 and q_2 as in Cournot competition. Thus, this is a two-stage game.

To define the payoffs, we suppose that the demand is $p(q) = 14 - q$ and that each firm's goal is to maximize its net revenue minus costs. Firm 1's payoff is then $[12 - (q_1 + q_2)]q_1$ if it does not invest, and $[14 - (q_1 + q_2)]q_1 - f$ if it does; firm 2's payoff is $[12 - (q_1 + q_2)]q_2$.

To find the subgame-perfect equilibria, we work backward. If firm 1 does not invest, both firms have unit cost 2, and hence their reaction functions are $r_i(q_j) = 6 - q_j/2$. These reaction functions intersect at the point (4,4), with payoffs of 16 each. If firm 1 does invest, its reaction becomes $\tilde{r}_1(q_2) = 7 - q_2/2$, the second-stage equilibrium is $(\frac{16}{3}, \frac{10}{3})$, and firm 1's total payoff is $256/9 - f$. Thus, firm 1 should make the investment if $256/9 - f > 16$, or $f < 112/9$.

Note that making the investment increases firm 1's second-stage profit in two ways. First, firm 1's profit is higher at any fixed pair of outputs, because its cost of production has gone down. Second, firm 1 gains because firm 2's second-stage output is decreased. The reason firm 2's output is lower is because by lowering its cost firm 1 altered its own second-period incentives, and in particular made itself "more aggressive" in the sense that $\tilde{r}_1(q_2) > r_1(q_2)$ for all q_2. We say more about this kind of "self-commitment" in the next subsection. Note that firm 2's output would not decrease if it continued to believe that firm 1's cost equaled 2.

3.2.3 The Value of Commitment and "Time Consistency"

One of the recurring themes in the analysis of dynamic games has been that in many situations players can benefit from the opportunity to make a binding commitment to play in a certain way. In a one-player game—i.e., a decision problem —such commitments cannot be of value, as any payoff that a player could attain while playing according to the commitment could be attained by playing in exactly the same way without being committed to do so. With more than one player, though, commitments can be of value, since by committing himself to a given sequence of actions a player may be able to alter the play of his opponents. This "paradoxical" value of commitment is closely related to our observation in chapter 1 that a player can gain by reducing his action set or decreasing his payoff to some outcomes, provided that his opponents are aware of the change. Indeed, some forms of commitment can be represented in exactly this way.

The way to model the possibility of commitments (and related moves like "promises") is to explicitly include them as actions the players can take. (Schelling (1960) was an early proponent of this view.) We have already seen one example of the value of commitment in our study of the Stackelberg game, which describes a situation where one firm (the "leader") can commit itself to an output level that the follower is forced to take as given when making its own output decision. Under the typical assumption that each firm's optimal reaction $r_i(q_j)$ is a decreasing function of its opponent's output, the Stackelberg leader's payoff is higher than in the "Cournot equilibrium" outcome where the two firms choose their output levels simultaneously.

In the Stackelberg example, commitment is achieved simply by moving earlier than the opponent. Although this corresponds to a different extensive form than the simultaneous moves of Cournot competition, the set of "physical actions" is in some sense the same. The search for a way to commit oneself can also lead to the use of actions that would not otherwise have been considered. Classic examples include a general burning his bridges behind him as a commitment not to retreat and Odysseus having himself lashed to the mast and ordering his sailors to plug their ears with wax as a commitment not to go to the Sirens' island. (Note that the natural way to model the Odysseus story is with two "players," corresponding to Odysseus before and Odysseus after he is exposed to the Sirens.) Both of these cases correspond to a "total commitment": Once the bridge is burned, or Odysseus is lashed to the mast and the sailors' ears are filled with wax, the cost of turning back or escaping from the mast is taken to be infinite. One can also consider partial commitments, which increase the cost of, e.g., turning back without making it infinite.

As a final example of the value of commitment, we consider what is known as the "time-consistency problem" in macroeconomics. This problem was first noted by Kydland and Prescott (1977); our discussion draws on the survey by Mankiw (1988). Suppose that the government sets the inflation rate π, and has preferences over inflation and output y represented by $u_g(\pi, y) = y - \pi^2$, so that it is prepared to tolerate inflation if doing so increases the output level. The working of the macroeconomy is such that only unexpected inflation changes output:

$$y = y^* + (\pi - \hat{\pi}), \tag{3.1}$$

where y^* is the "natural level" of output and $\hat{\pi}$ is the expected inflation.[2]

Regardless of the timing of moves, the agents' expectations of inflation are correct in any pure-strategy equilibrium, and so output is at its natural level. (In a mixed-strategy equilibrium the expectations need only be correct on average.) The variable of interest is thus the level of inflation. Suppose first that the government can commit itself to an inflation rate, i.e., the government moves first and chooses a level of π that is observed by the agents. Then output will equal y^* regardless of the chosen level of π, so the government should choose $\pi = 0$.

As Kydland and Prescott point out, this solution to the commitment game is not "time consistent," meaning that if the agents mistakenly believe that π is set equal to 0 when in fact the government is free to choose any level of π it wishes, then the government would prefer to choose a different level of π. That is, the commitment solution is not an equilibrium of the game without commitment.

If the government cannot commit itself, it will choose the level of inflation that equates the marginal benefit from increased output to the marginal cost of increased inflation. The government's utility function is such that this tradeoff is independent of the level of output or the level of expected inflation, and the government will choose $\pi = \frac{1}{2}$. Since output is the same in the two cases, the government does strictly worse without commitment. In the context of monetary policy, the "commitment path" can be interpreted as a "money growth rule," and noncommitment corresponds to a "discretionary policy"; hence the conclusion that "rules can be better than discretion."[3]

As a gloss on the time-consistency problem, let us consider the analogous questions in relation to Stackelberg and Cournot equilibria. If we think of the government and the agents as both choosing output levels, the commitment solution corresponds to the Stackelberg outcome (q_1^*, q_2^*). This outcome is not an equilibrium of the game where the government cannot commit itself, because in general q_1^* is not a best response to q_2^* when q_2^*

2. Equation 3.1 is a reduced form that incorporates the way that the agents' expectations influence their production decisions and in turn influence output. Since the actions of the agents have been suppressed, the model does not directly correspond to an extensive-form game, but the same intuitions apply. Here is an artificial extensive-form game with the same qualitative properties: The government chooses the money supply m, and a single agent chooses a nominal price p. Aggregate demand is $y = \max(0, m - p)$, and the agent is constrained to supply all demanders. The agent's utility is $p - p^2/2m$, and the government's utility is $y - (m - 1)^2$. This does not quite give equation 3.1, but the resulting model has very similar properties.

3. In the extensive-game model where the agent chooses prices (see note 2), the agent chooses $p = m$ and the commitment solution is to set $m = 1$. Without commitment this is not an equilibrium, since for fixed p the government could gain by choosing a larger value of m.

is held fixed. The no-commitment solution $\pi = \frac{1}{2}$ derived above corresponds to a situation of simultaneous moves that is, to the Cournot outcome.

Whether and when a commitment to a monetary rule is credible have been important topics of theoretical and applied research in macroeconomics. This research has started from the observation that decisions about the money supply are not made once and for all, but rather are made repeatedly. Chapter 5, on repeated games, and chapter 9, on reputation effects, discuss game-theoretic analyses of the question of when repeated play makes commitments credible.

Finally, note that a player does not always do better when he moves first (and his choice of action is observed) than when players move simultaneously: In "matching pennies" (example 1.6) each player's equilibrium payoff is 0, whereas if one player moves first his equilibrium payoff is -1.

3.3 The Extensive Form††

This section gives a formal development of the idea of an extensive-form game. The extensive form is a fundamental concept in game theory and one to which we will refer frequently, particularly in chapters 8 and 11, but the details of the definitions are not essential for much of the material in the rest of the book. Thus, readers who are primarily interested in applications of the theory should not be discouraged if they do not master all the fine points of the extensive-form methodology. Instead of dwelling on this section, they should proceed along, remembering to review this material before beginning section 8.3.

3.3.1 Definition

The extensive form of a game contains the following information:

(1) the set of players
(2) the order of moves—i.e., who moves when
(3) the players' payoffs as a function of the moves that were made
(4) what the players' choices are when they move
(5) what each player knows when he makes his choices
(6) the probability distributions over any exogenous events.

The set of players is denoted by $i \in \mathscr{I}$; the probability distributions over exogenous events (point 6) are represented as moves by "Nature," which is denoted by N. The order of play (point 2) is represented by a *game tree*, T, such as the one shown in figure 3.1.[4] A tree is a finite collection of ordered

4. Our development of the extensive form follows that of Kreps and Wilson 1982 with a simplification suggested by Jim Ratliff. Their assumptions (and ours) are equivalent to those of Kuhn 1953.

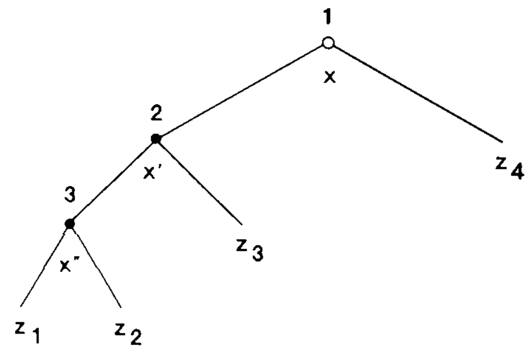


Figure 3.1

nodes $x \in X$ endowed with a precedence relation denoted by $\succ$; $x \succ x'$ means "x is before x'." We assume that the precedence relation is transitive (if x is before x' and x' is before x'', then x is before x'') and asymmetric (if x is before x', then x' is not before x). These assumptions imply that the precedence relation is a *partial order*. (It is not a complete order, because two nodes may not be comparable: In figure 3.1, z_3 is not before x'', and x'' is not before z_3.) We include a single initial node $o \in X$ that is before all other nodes in X; this node will correspond to a move by nature if any. Figure 3.1 describes a situation where "nature's move" is trivial, as nature simply gives the move to player 1. As in this figure, we will suppress nature's move whenever it is trivial, and begin the tree with the first "real" choice. The initial node will be depicted with o to distinguish it from the others. In figure 3.1, the precedence order is from the top of the diagram down. Given the assumptions we will impose, the precedence ordering will be clear in most diagrams; when the intended precedence is not clear we will use arrows ($\rightarrow$) to connect a node to its immediate successors.

The assumption that precedence is a partial order rules out cycles of the kind shown in figure 3.2a: If $x \succ x' \succ x'' \succ x$, then by transitivity $x'' \succ x'$. Since we already have $x' \succ x''$, this would violate the asymmetry condition. However, the partial ordering does not rule out the situation shown in figure 3.2b, where both x and x' are immediate predecessors of node x''.

We wish to rule out the situation in figure 3.2b, because each node of the tree is meant to be a complete description of all events that preceded it, and not just of the "physical situation" at a given point in time. For example, in figure 3.2c, a firm in each of two markets, A and B, might have entered A and then B (node x and then x'') or B and then A (node x' and then x''), but we want our formalism to distinguish between these two sequences of events instead of describing them by a single node x''. (Of course, we are free to specify that both sequences lead to the same payoff for the firm.) In order to ensure that there is only one path through the tree to a given node, so that each node is a complete description of the path

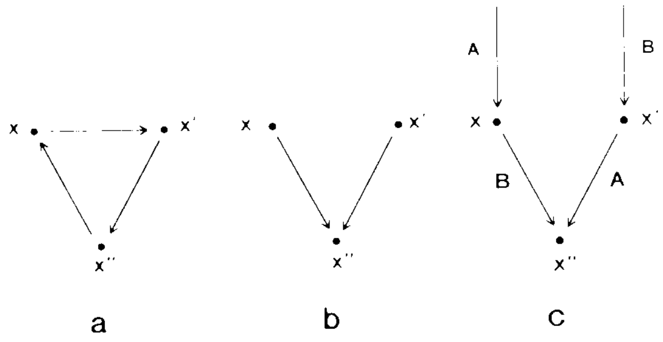


Figure 3.2

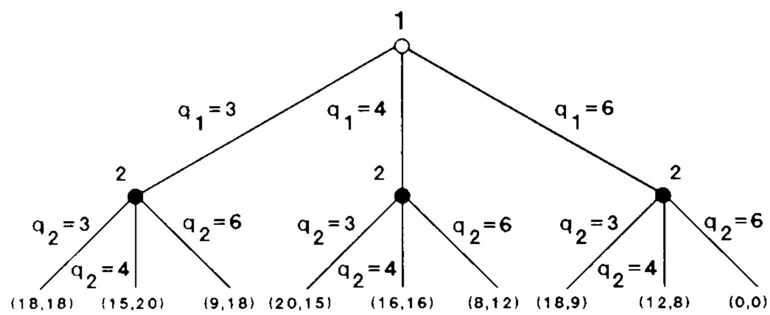


Figure 3.3

preceding it, we require that each node x (except the initial node o) have exactly one immediate predecessor—that is, one node $x' \succ x$ such that $x'' \succ x$ and $x'' \neq x'$ implies $x'' \succ x'$. Thus, if x' and x'' are both predecessors of x, then either x' is before x'' or x'' is before x'. (This makes the pair $(X, \succ)$ an *arborescence*.)

The nodes that are not predecessors of any other node are called "terminal nodes" and denoted by $z \in Z$. Because each z completely determines a path through the tree, we can assign payoffs to sequences of moves using functions $u_i\colon Z \to \mathbb{R}$, with $u_i(z)$ being player i's payoff if terminal node z is reached. In drawing extensive forms, the payoff vectors (point 3 in the list above) are displayed next to the corresponding terminal nodes, as in figures 3.3 and 3.4. To complete the specification of point 2 (who moves when), we introduce a map $\iota\colon X \to \mathcal{I}$ with the interpretation that player $\iota(x)$ moves at node x. Next we must describe what player $\iota(x)$'s choices are, which was point 4 of our list. To do so, we introduce a finite set A of actions and a function ℓ that labels each noninitial node x with the last action taken to reach it. We require that ℓ be one-to-one on the set of immediate successors of each node x, so that different successors correspond to different actions, and let $A(x)$ denote the set of feasible actions at x. (Thus $A(x)$ is the range of ℓ on the set of immediate successors of x.)

Point 5, the information players have when choosing their actions, is the most subtle of the six points. This information is represented using *information sets* $h \in H$, which *partition* the nodes of the tree—that is, every node is in exactly one information set.[5] The interpretation of the information set $h(x)$ containing node x is that the player who is choosing an action at x is uncertain if he is at x or at some other $x' \in h(x)$. We require that if $x' \in h(x)$ the same player move at x and x'. Without this requirement, the players might disagree about who was supposed to move. Also, we require that if $x' \in h(x)$ then $A(x') = A(x)$, so the player on move has the same set of choices at each node of this information set. (Otherwise he might "play" an infeasible action.) Thus, we can let $A(h)$ denote the action set at information set h.

A special case of interest is that of *games of perfect information*, in which all the information sets are singletons. In a game of perfect information, players move one at a time, and each player knows all previous moves when making his decision. The Stackelberg game we discussed at the start of this chapter is a game of perfect information. Figure 3.3 displays a tree for this

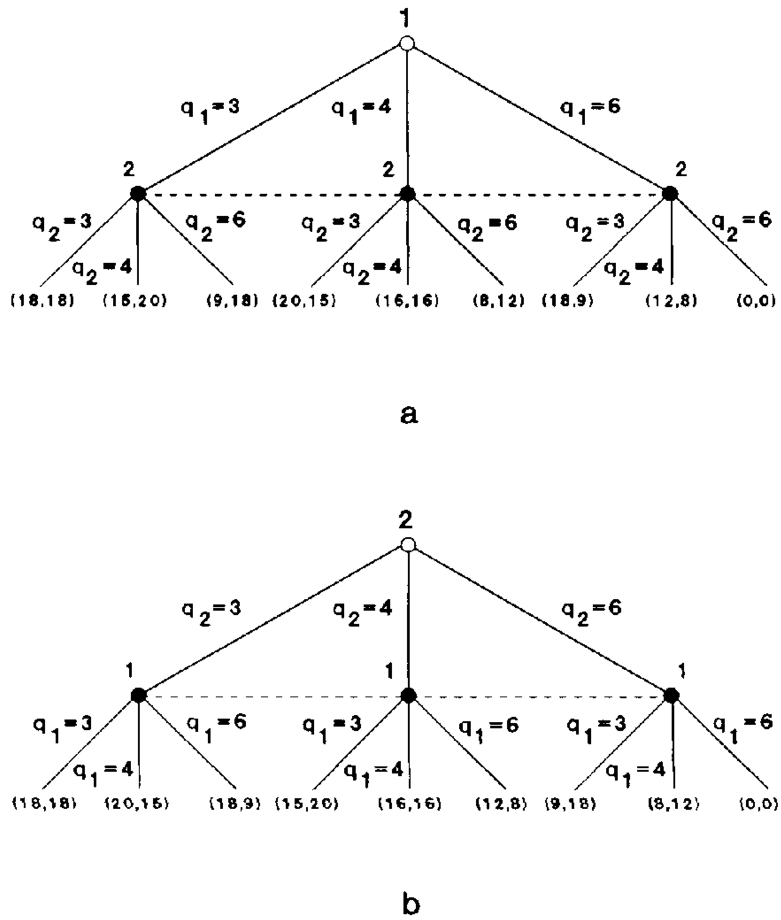


Figure 3.4

5. Note that we use the same notation, h, for information sets and for histories in multi-stage games. This should not cause too much confusion, especially as information sets can be viewed as a generalization of the idea of a history.

game on the assumption that each player has only three possible output levels: 3, 4, and 6. The vectors at the end of each branch of the tree are the payoffs of players 1 and 2, respectively.

Figure 3.4a displays an extensive form for the Cournot game, where players 1 and 2 choose their output levels simultaneously. Here player 2 does not know player 1's output level when choosing his own output. We model this by placing the nodes corresponding to player 1's three possible actions in the same information set for player 2. This is indicated in the figure by the broken line connecting the three nodes. (Some authors use "loops" around the nodes instead.) Note well the way simultaneous moves are represented: As in figure 3.3, player 1's decision comes "before" player 2's in terms of the precedence ordering of the tree; the difference is in player 2's information set. As this shows, the precedence ordering in the tree need not correspond to calendar time. To emphasize this point, consider the extensive form in figure 3.4b, which begins with a move by player 2. Figures 3.4a and 3.4b describe exactly the same strategic situation: Each player chooses his action not knowing the choice of his opponent. However, the situation represented in figure 3.3, where player 2 observed player 1's move before choosing his own, can only be described by an extensive form in which player 1 moves first.

Almost all games in the economics literature are games of *perfect recall*: No player ever forgets any information he once knew, and all players know the actions they have chosen previously. To impose this formally, we first require that if x and x' are in the same information set then neither is a predecessor of the other. This is not enough to ensure that a player never forgets, as figure 3.5 shows. To rule out this situation, we require that if $x'' \in h(x')$, if x is a predecessor of x', and if the same player i moves at x and at x' (and thus at x''), then there is a node $\hat{x}$ (possibly x itself) that is in the same information set as x, that $\hat{x}$ is a predecessor of x'', and that the action taken at x along the path to x' is the same as the action taken at $\hat{x}$ along the path to x''. Intuitively, the nodes x' and x'' are distinguished by information the player doesn't have, so he can't have had it when he was at information set $h(x)$; x' and x'' must be consistent with the same action at $h(x)$, since the player remembers his action there.

When a game involves moves by Nature, the exogenous probabilities are displayed in *brackets*, as in the two-player extensive form of figure 3.6. In

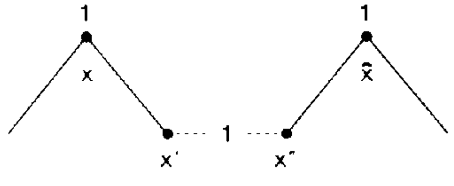


Figure 3.5

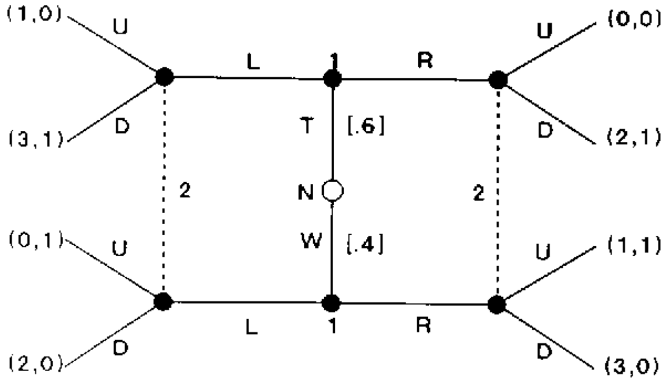


Figure 3.6

figure 3.6, Nature moves first and chooses a "type" or "private information" for player 1. With probability 0.6 player 1 learns that his type is "tough" (T), and with probability 0.4 he learns that his type is "weak" (W). Player 1 then plays left (L) or right (R). Player 2 observes player 1's action but not his type, and chooses between up (U) and down (D). Note that we have allowed both players' payoffs to depend on the choice by Nature even though this choice is initially observed only by player 1. (Player 2 will be able to infer Nature's move from his payoffs.) Figure 3.6 is an example of a "signaling game," as player 1's action may reveal information about his type to player 2. Signaling games, the simplest games of incomplete information, will be studied in detail in chapters 8 and 11.

3.3.2 Multi-Stage Games with Observed Actions

Many of the applications of game theory to economics, political science, and biology have used the special class of extensive forms that we discussed in section 3.2: the class of "*multi-stage games with observed actions.*"[6] These games have "stages" such that (1) in each stage k every player knows all the actions, including those by Nature, that were taken at any previous stage; (2) each player moves at most once within a given stage; and (3) no information set contained in stage k provides any knowledge of play in that stage. (Exercise 3.4 asks you to give a formal definition of these conditions in terms of a game tree and information sets.)

In a multi-stage game, all past actions are common knowledge at the beginning of stage k, so there is a well-defined "history" h^k at the start of each stage k. Here a pure strategy for player i is a function s_i that specifies an action $a_i \in A_i(h^k)$ for each k and each history h^k; mixed strategies specify probability mixtures over the actions in each stage.

Caution Although the idea of a multi-stage game seems natural and intrinsic, it suffers from the following drawback: There may be two exten-

6. Such games are also often called "games of almost-perfect information."

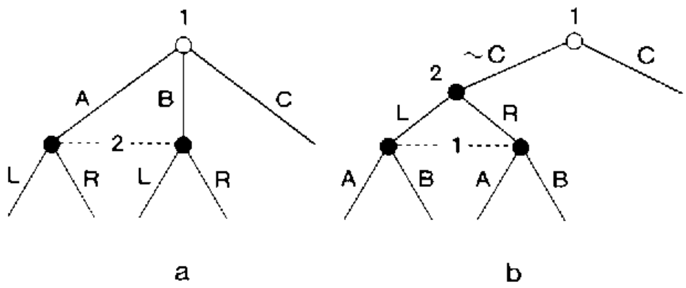


Figure 3.7

sive forms that seem to represent the same real game, with one of them a multi-stage game and the other one not. Consider for example figure 3.7. The extensive form on the left is not a multi-stage game: Player 2's information set is not a singleton, and so it must belong to the first stage and not to a second one. However, player 2 does have some information about player 1's first move (if player 2's information set is reached, then player 1 did not play C), so player 2's information set cannot belong to the first stage either. However, the extensive form on the right is a two-stage game, and the two extensive forms seem to depict the same situation: When player 2 moves, he knows that player 1 is choosing A or B but not C; player 1 chooses A or B without knowing player 2's choice of L or R. The question as to which extensive forms are "equivalent" is still a topic of research—see Elmes and Reny 1988. We will have more to say about this topic when we discuss recent work on equilibrium refinements in chapter 11.

Before proceeding to the next section, we should point out that in applications the extensive form is usually described without using the apparatus of the formal definition, and game trees are virtually never drawn except for very simple "toy" examples. The test of a good informal description is whether it provides enough information to construct the associated extensive form; if the extensive form is not clear, the model has not been well specified.

3.4 Strategies and Equilibria in Extensive-Form Games††

3.4.1 Behavior Strategies

This section defines strategies and equilibria in extensive-form games and relates them to strategies and equilibria of the strategic-form model. Let H_i be the set of player i's information sets, and let $A_i \equiv \bigcup_{h_i \in H_i} A(h_i)$ be the set of all actions for player i. A *pure strategy* for player i is a map s_i: $H_i \to A_i$, with $s_i(h_i) \in A(h_i)$ for all $h_i \in H_i$. Player i's space of pure strategies, S_i, is simply the space of all such s_i. Since each pure strategy is a map from information sets to actions, we can write S_i as the Cartesian product of the action spaces at each h_i:

$$S_i = \underset{h_i \in H_i}{\times} A(h_i).$$

In the Stackelberg example of figure 3.3, player 1 has a single information set and three actions, so that he has three pure strategies. Player 2 has three information sets, corresponding to the three possible choices of player 1, and player 2 has three possible actions at each information set, so player 2 has 27 pure strategies in all. More generally, the number of player i's pure strategies, $\# S_i$, equals

$$\prod_{h_i \in H_i} \#(A(h_i)).$$

Given a pure strategy for each player i and the probability distribution over Nature's moves, we can compute a probability distribution over outcomes and thus assign expected payoffs $u_i(s)$ to each strategy profile s. The information sets that are reached with positive probability under profile s are called the *path* of s.

Now that we have defined the payoffs to each pure strategy, we can proceed to define a pure-strategy Nash equilibrium for an extensive-form game as a strategy profile s^* such that each player i's strategy s_i^* maximizes his expected payoff given the strategies s_{-i}^* of his opponents. Note that since the definition of Nash equilibrium holds the strategies of player i's opponents fixed in testing whether player i wishes to deviate, it is as if the players choose their strategies simultaneously. This does *not* mean that in Nash equilibrium players necessarily choose their *actions* simultaneously. For example, if player 2's fixed strategy in the Stackelberg game of figure 3.3 is the Cournot reaction function $\hat{s}_2 = (4,4,3)$, then when player 1 treats player 2's strategy as fixed he does not presume that player 2's action is unaffected by his own, but rather that player 2 will respond to player 1's action in the way specified by $\hat{s}_2$.

To fill in the details missing from our discussion of the Stackelberg game in the introduction: The "Stackelberg equilibrium" of this game is the outcome $q_1 = 6$, $q_2 = 3$. This outcome corresponds to the Nash-equilibrium strategy profile $s_1 = 6$, $s_2 = \hat{s}_2$. The Cournot outcome is (4,4); this is the outcome of the Nash equilibrium $s_1 = 4$, $s_2 = (4,4,4)$.

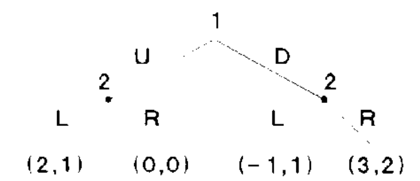
The next order of business is to define mixed strategies and mixed-strategy equilibria for extensive-form games. Such strategies are called *behavior strategies* to distinguish them from the strategic-form mixed strategies we introduced in chapter 1. Let $\Delta(A(h_i))$ be the probability distributions on $A(h_i)$. A *behavior strategy for player i*, denoted b_i, is an element of the Cartesian product $\times_{h_i \in H_i} \Delta(A(h_i))$. That is, a behavior strategy specifies a probability distribution over actions at each h_i, and the probability distributions at different information sets are independent. (Note that a pure strategy is a special kind of behavior strategy in which the distribution at each information set is degenerate.) A profile $b = (b_1, \ldots, b_I)$ of behavior strategies generates a probability distribution over outcomes in the obvious way, and hence gives rise to an expected payoff for each player. A *Nash equilibrium in behavior strategies* is a profile such that no player can increase his expected payoff by using a different behavior strategy.

3.4.2 The Strategic-Form Representation of Extensive-Form Games

Our next step is to relate extensive-form games and equilibria to the strategic-form model. To define a strategic form from an extensive form, we simply let the pure strategies $s \in S$ and the payoffs $u_i(s)$ be exactly those we defined in the extensive form. A different way of saying this is that the same pure strategies can be interpreted as either extensive-form or strategic-form objects. With the extensive-form interpretation, player i "waits" until h_i is reached before deciding how to play there; with the strategic-form interpretation, he makes a complete contingent plan in advance.

Figure 3.8 illustrates this passage from the extensive form to the strategic form in a simple example. We order player 2's information sets from left to right, so that, for example, the strategy $s_2 = (\mathrm{L}, \mathrm{R})$ means that he plays L after U and R after D.

As another example, consider the Stackelberg game illustrated in figure 3.3. We will again order player 2's information sets from left to right, so that player 2's strategy $\hat{s}_2 = (4,4,3)$ means that he plays 4 in response to $q_1 = 3$, plays 4 in response to 4, and plays 3 in response to 6. (This strategy happens to be player 2's Cournot reaction function.) Since player 2 has three information sets and three possible actions at each of these sets, he has 27 pure strategies. We trust that the reader will forgive our not displaying the strategic form in a matrix diagram!



a. Extensive Form

	(L,L)	(L,R)	(R,L)	(R,R)
U	2,1	2,1	0,0	0,0
D	−1,1	3,2	−1,1	3,2

b. Strategic Form

Figure 3.8

There can be several extensive forms with the same strategic form, as the example of simultaneous moves shows: Figures 3.4a and 3.4b both correspond to the same strategic form for the Cournot game.

At this point we should note that the strategy space as we have defined it may be unnecessarily large, as it may contain pairs of strategies that are "equivalent" in the sense of having the same consequences regardless of how the opponents play.

Definition 3.2 Two pure strategies s_i and s_i' are *equivalent* if they lead to the same probability distribution over outcomes for all pure strategies of the opponents.

Consider the example in figure 3.9. Here player 1 has four pure strategies: (a, c), (a, d), (b, c), and (b, d). However, if player 1 plays b, his second information set is never reached, and the strategies (b, c) and (b, d) are equivalent.

Definition 3.3 The *reduced strategic form* (or reduced normal form) of an extensive-form game is obtained by identifying equivalent pure strategies (i.e., eliminating all but one member of each equivalence class).

Once we have derived the strategic form from the extensive form, we can (as in chapter 1) define mixed strategies to be probability distributions over pure strategies in the reduced strategic form. Although the extensive form and the strategic form have exactly the same pure strategies, the sets of mixed and behavior strategies are different. With behavior strategies, player i performs a different randomization at each information set. Luce and Raiffa (1957) use the following analogy to explain the relationship between mixed and behavior strategies: A pure strategy is a book of instructions, where each page tells how to play at a particular information set. The strategy space S_i is like a library of these books, and a mixed strategy is a probability measure over books—i.e., a random way of making a selection from the library. A given behavior strategy, in contrast, is a single book, but it prescribes a random choice of action on each page.

The reader should suspect that these two kinds of strategies are closely related. Indeed, they are equivalent in games of perfect recall, as was proved

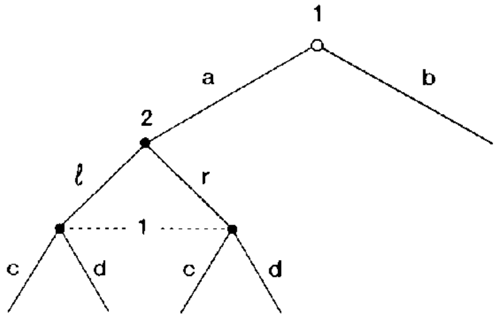


Figure 3.9

by Kuhn (1953). (Here we use "equivalence" as in our earlier definition: Two strategies are equivalent if they give rise to the same distributions over outcomes for all strategies of the opponents.)

3.4.3 The Equivalence between Mixed and Behavior Strategies in Games of Perfect Recall

The equivalence between mixed and behavior strategies under perfect recall is worth explaining in some detail, as it also helps to clarify the workings of the extensive-form model. Any mixed strategy σ_i of the strategic form (not of the reduced strategic form) generates a unique behavior strategy b_i as follows: Let $R_i(h_i)$ be the set of player i's pure strategies that do not preclude h_i, so that for all $s_i \in R_i(h_i)$ there is a profile s_{-i} for player i's opponents that reaches h_i. If σ_i assigns positive probability to some s_i in $R_i(h_i)$, define the probability that b_i assigns to $a_i \in A(h_i)$ as

$$b_i(a_i|h_i) = \sum_{\{s_i \in R_i(h_i) \text{ and } s_i(h_i)=a_i\}} \sigma_i(s_i) \Big/ \sum_{\{s_i \in R_i(h_i)\}} \sigma_i(s_i).$$

If σ_i assigns probability 0 to all $s_i \in R_i(h_i)$, then set

$$b_i(a_i|h_i) = \sum_{\{s_i(h_i)=a_i\}} \sigma_i(s_i).^7$$

In either case, the $b_i(\cdot|\cdot)$ are nonnegative, and

$$\sum_{a_i \in A(h_i)} b_i(a_i|h_i) = 1,$$

because each s_i specifies an action for player i at h_i.

Note that in the notation $b_i(a_i|h_i)$, the variable h_i is redundant, as $a_i \in A(h_i)$, but the conditioning helps emphasize that a_i is an action that is feasible at information set h_i.

It is useful to work through some examples to illustrate the construction of behavior strategies from mixed strategies. In figure 3.10, a single player (player 1) moves twice. Consider the mixed strategy $\sigma_1 = (\frac{1}{2}(\mathrm{L}, \ell), \frac{1}{2}(\mathrm{R}, r))$.

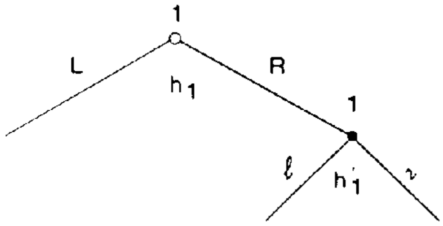


Figure 3.10

7. Since h_i cannot be reached under σ_i, the behavior strategies at h_i are arbitrary in the same sense that Bayes' rule does not determine posterior probabilities after probability-0 events. Our formula is one of many possible specifications.

The strategy plays r with probability 1 at information set h'_1, as only $(\mathrm{R}, r) \in R_1(h'_1)$.

Figure 3.11 gives another example. Player 2's strategy σ_2 assigns probability $\frac{1}{2}$ each to $s_2 = (\mathrm{L}, \mathrm{L}', \mathrm{R}'')$ and $\tilde{s}_2 = (\mathrm{R}, \mathrm{R}', \mathrm{L}'')$. The equivalent behavior strategy is

$$b_2(\mathrm{L} \mid h_2) = b_2(\mathrm{R} \mid h_2) = \tfrac{1}{2};$$

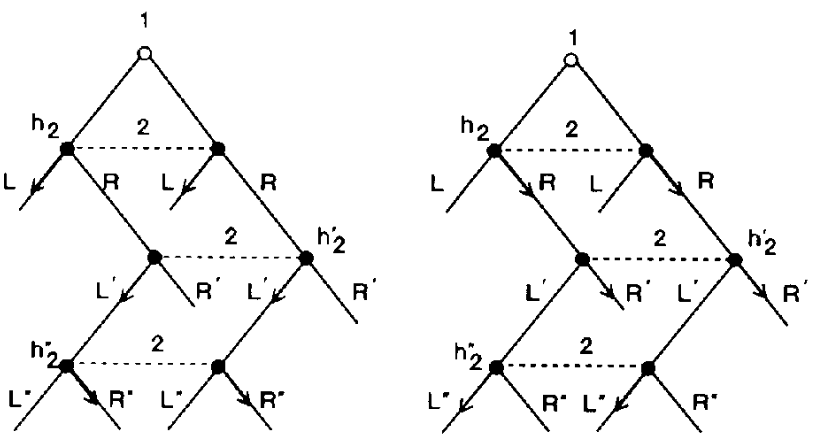
$$b_2(\mathrm{L}' \mid h'_2) = 0 \text{ and } b_2(\mathrm{R}' \mid h'_2) = 1,$$

and

$$b_2(\mathrm{L}'' \mid h''_2) = b_2(\mathrm{R}'' \mid h''_2) = \tfrac{1}{2}.$$

Many different mixed strategies can generate the same behavior strategy. This can be seen from figure 3.12, where player 2 has four pure strategies: $s_2 = (\mathrm{A}, \mathrm{C})$, $s'_2 = (\mathrm{A}, \mathrm{D})$, $s''_2 = (\mathrm{B}, \mathrm{C})$, and $s'''_2 = (\mathrm{B}, \mathrm{D})$.

Now consider two mixed strategies: $\sigma_2 = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$, which assigns probability $\frac{1}{4}$ to each pure strategy, and $\hat{\sigma}_2 = (\frac{1}{2}, 0, 0, \frac{1}{2})$, which assigns probability $\frac{1}{2}$ to s_2 and $\frac{1}{2}$ to s'''_2. Both of these mixed strategies generate the behavior strategy b_2, where $b_2(\mathrm{A} \mid h) = b_2(\mathrm{B} \mid h) = \frac{1}{2}$ and $b_2(\mathrm{C} \mid h') = b_2(\mathrm{D} \mid h') = \frac{1}{2}$.



Pure strategy s_2 Pure strategy $\tilde{s}_2$

Figure 3.11

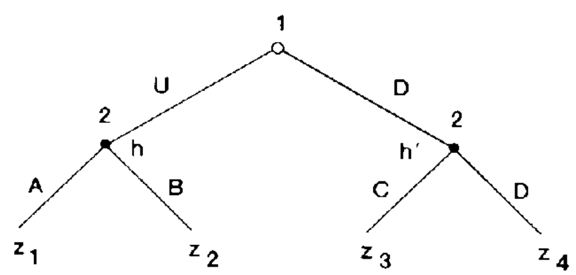


Figure 3.12

Moreover, for any strategy σ_1 of player 1, σ_2, $\hat{\sigma}_2$, and b_2 all lead to the same probability distribution over terminal nodes; for example, the probability of reaching node z_1 equals the probability of player 1's playing U times $b_2(\mathrm{A}|h)$.

The relationship between mixed and behavior strategies is different in the game illustrated in figure 3.13, which is not a game of perfect recall. (Exercise 3.2 asks you to verify this using the formal definition.) Here, player 1 has four strategies in the strategic form:

$$s_1 = (\mathrm{A}, \mathrm{C}),\ s_1' = (\mathrm{A}, \mathrm{D}),\ s_1'' = (\mathrm{B}, \mathrm{C}),\ s_1''' = (\mathrm{B}, \mathrm{D}).$$

Now consider the mixed strategy $\sigma_1 = (\frac{1}{2}, 0, 0, \frac{1}{2})$. As in the last example, this generates the behavior strategy $b_1 = \{(\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2})\}$, which says that player 1 mixes $\frac{1}{2}$-$\frac{1}{2}$ at each information set. But b_1 is *not* equivalent to the σ_1 that generated it. Consider the strategy $s_2 = \mathrm{L}$ for player 2. Then (σ_1, L) generates a $\frac{1}{2}$ probability of the terminal node corresponding to (A, L, C), and a $\frac{1}{2}$ probability of (B, L, D). However, since behavior strategies describe independent randomizations at each information set, (b_1, L) assigns probability $\frac{1}{4}$ to each of the four paths (A, L, C), (A, L, D), (B, L, C), and (B, L, D). Since both A vs. B and C vs. D are choices made by player 1, the strategic-form strategy σ_1 can have the property that both A and B have positive probability but C is played wherever A is. Put differently, the strategic-form strategies, where player 1 makes all his decisions at once, allow the decisions at different information sets to be *correlated*. Behavior strategies can't produce this correlation in the example, because when it comes time to choose between C and D player 1 has forgotten whether he chose A or B. This forgetfulness means that there is not perfect recall in this game. If we change the extensive form so that there is perfect recall (by partitioning player 1's second information set into two, corresponding to his choice of A or B), it is easy to see that every mixed strategy is indeed equivalent to the behavior strategy it generates.

Theorem 3.1 (Kuhn 1953) In a game of perfect recall, mixed and behavior strategies are equivalent. (More precisely: Every mixed strategy is equiv-

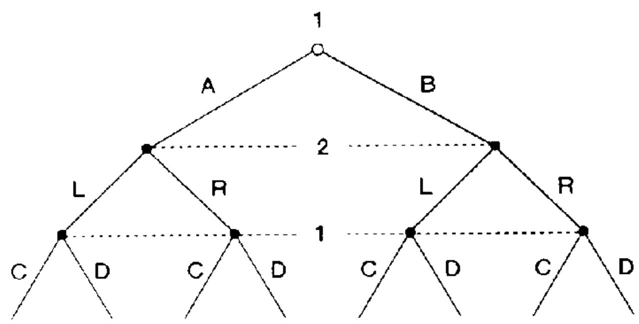


Figure 3.13

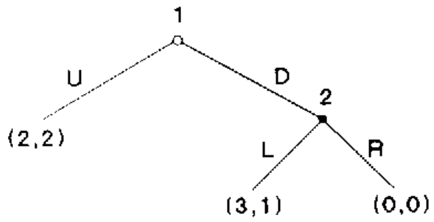
alent to the unique behavior strategy it generates, and each behavior strategy is equivalent to every mixed strategy that generates it.)

We will restrict our attention to games of perfect recall throughout this book, and will use the terms "mixed strategy" and "Nash equilibrium" to refer to the mixed and behavior formulations interchangeably. This leads us to the following important *notational convention*: In the rest of part II and in most of part IV (except in sections 8.3 and 8.4), we will be studying behavioral strategies. Thus, when we speak of a mixed strategy of an extensive form, we will mean a behavior strategy unless we state otherwise. Although the distinction between the mixed strategy σ_i and the behavior strategy b_i was necessary to establish their equivalence, we will follow standard usage by denoting both objects by σ_i (thus, the notation b_i is not used in the rest of the book). In a multi-stage game with observed actions, we will let $\sigma_i(a_i^k|h^k)$ denote player i's probability of playing action $a_i^k \in A_i(h^k)$ given the history of play h^k at stage k. In general extensive forms (with perfect recall), we let $\sigma_i(a_i|h_i)$ denote player i's probability of playing action a_i at information set h_i.

3.4.4 Iterated Strict Dominance and Nash Equilibrium

If the extensive form is finite, so is the corresponding strategic form, and the Nash existence theorem yields the existence of a mixed-strategy equilibrium. The notion of iterated strict dominance extends to extensive-form games as well; however, as we mentioned above, this concept turns out to have little force in most extensive forms. The point is that a player cannot strictly prefer one action over another at an information set that is not reached given his opponents' play.

Consider figure 3.14. Here, player 2's strategy R is not strictly dominated, as it is as good as L when player 1 plays U. Moreover, this fact is not "pathological." It obtains for all strategic forms whose payoffs are derived from an extensive form with the tree on the left-hand side of the figure. That is, for any assignment of payoffs to the terminal nodes of the tree, the payoffs to (U, L) and (U, R) must be the same, as both strategy profiles lead to the same terminal node. This shows that the set of strategic-form payoffs of a fixed game tree is of lower dimension than the set of all payoffs of the corresponding strategic form, so theorems based on generic strategic-form payoffs (see chapter 12) do not apply. In particular, there can be an even number of Nash equilibria for an open set of extensive-form payoffs. The game illustrated in figure 3.14 has two Nash equilibria, (U, R) and (D, L), and this number is not changed if the extensive-form payoffs are slightly perturbed. The one case where the odd-number theorem of chapter 12 applies is to a simultaneous-move game such as that of figure 3.4; in such a game, each terminal node corresponds to a unique strategy profile. Put



	L	R
U	2,2	2,2
D	3,1	0,0

Figure 3.14

differently: In simultaneous-move games, every strategy profile reaches every information set, and so no player's strategy can involve a choice that is not implemented given his opponents' play.

Recall that a game of perfect information has all its information sets as singletons, as in the games illustrated in figures 3.3 and 3.14.

Theorem 3.2 (Zermelo 1913; Kuhn 1953) A finite game of perfect information has a pure-strategy Nash equilibrium.

The proof of this theorem constructs the equilibrium strategies using "Zermelo's algorithm," which is a many-player generalization of backward induction in dynamic programming. Since the game is finite, it has a set of penultimate nodes i.e., nodes whose immediate successors are terminal nodes. Specify that the player who can move at each such node chooses whichever strategy leads to the successive terminal node with the highest payoff for him. (In case of a tie, make an arbitrary selection.) Now specify that each player at nodes whose immediate successors are penultimate nodes chooses the action that maximizes her payoff over the feasible successors, given that players at the penultimate nodes play as we have just specified. We can now roll back through the tree, specifying actions at each node. When we are done, we will have specified a strategy for each player, and it is easy to check that these strategies form a Nash equilibrium. (In fact, the strategies satisfy the more restrictive concept of subgame perfection, which we will introduce in the next section.)

Zermelo's algorithm is not well defined if the hypotheses of the theorem are weakened. First consider infinite games. An infinite game necessarily has either a single node with an infinite number of successors (as do games with a continuum of actions) or a path consisting of an infinite number of nodes (as do multi-stage games with an infinite number of stages). In the first case, an optimal choice need not exist without further restrictions on the payoff functions[8]; in the second, there need not be a penultimate node on a given path from which to work backward. Finally, consider a game of imperfect information in which some of the information sets are not singletons, as in figure 3.4a. Here there is no way to define an optimal choice for player 2 at his information set without first specifying player 2's belief about the previous choice of player 1; the algorithm fails because it presumes that such an optimal choice exists at every information set given a specification of play at its successors.

We will have much more to say about this issue when we treat equilibrium refinements in detail. We conclude this section with one caveat about the assertion that the Nash equilibrium is a minimal requirement for a "reasonable" point prediction: Although the Nash concept can be applied to any game, the assumption that each player correctly forecasts his opponents' strategy may be less plausible when the strategies correspond to choices of contingent plans than when the strategies are simply choices of actions. The issue here is that when some information sets may not be reached in the equilibrium, Nash equilibrium requires that players correctly forecast their opponents' play at information sets that have 0 probability according to the equilibrium strategies. This may not be a problem if the forecasts are derived from introspection, but if the forecasts are derived from observations of previous play it is less obvious why forecasts should be correct at the information sets that are not reached. This point is examined in detail in Fudenberg and Kreps 1988 and in Fudenberg and Levine 1990.

3.5 Backward Induction and Subgame Perfection††

As we have seen, the strategic form can be used to represent arbitrarily complex extensive-form games, with the strategies of the strategic form being complete contingent plans of action in the extensive form. Thus, the concept of Nash equilibrium can be applied to all games, not only to games where players choose their actions simultaneously. However, many game theorists doubt that Nash equilibrium is the right solution concept for

8. The existence of an optimal choice from a compact set of actions requires that payoffs be upper semi-continuous in the choice made. (A real-valued function $f(x)$ is upper semi-continuous if $x^n \to x$ implies $\lim_{n\to\infty} f(x^n) \le f(x)$.)

Assuming that payoffs u_i are continuous in s does not guarantee that an optimal action exists at each node. Although the last mover's payoff is continuous and therefore an optimum exists if his action set is compact, the last mover's optimal action need not be a continuous function of the action chosen by the previous player. In this case, when we replace that last mover by an arbitrary specification of an optimal action on each path, the next-to-last mover's derived payoff function need not be upper semi-continuous, even though that player's payoff is a continuous function of the actions chosen at each node. Thus, the simple backward-induction algorithm defined above cannot be applied. However, subgame-perfect equilibria do exist in infinite-action games of perfect information, as shown by Harris (1985) and by Hellwig and Leininger (1987).

general games. In this section we will present a first look at "equilibrium refinements," which are designed to separate the "reasonable" Nash equilibria from the "unreasonable" ones. In particular, we will discuss the ideas of backward induction and "subgame perfection." Chapters 4, 5 and 13 apply these ideas to some classes of games of interest to economists.

Selten (1965) was the first to argue that in general extensive games some of the Nash equilibria are "more reasonable" than others. He began with the example illustrated here in figure 3.14. This is a finite game of perfect information, and the backward-induction solution (that is, the one obtained using Kuhn's algorithm) is that player 2 should play L if his information set is reached, and so player 1 should play D. Inspection of the strategic form corresponding to this game shows that there is another Nash equilibrium, where player 1 plays U and player 2 plays R. The profile (U, R) *is* a Nash equilibrium because, given that player 1 plays U, player 2's information set is not reached, and player 2 loses nothing by playing R. But Selten argued, and we agree, that this equilibrium is suspect. After all, if player 2's information set is reached, then, as long as player 2 is convinced that his payoffs are as specified in the figure, player 2 should play L. And if we were player 2, this is how we would play. Moreover, if we were player 1, we would expect player 2 to play L, and so we would play D.

In the now-familiar language, the equilibrium (U, R) is not "credible," because it relies on an "empty threat" by player 2 to play R. The threat is "empty" because player 2 would never wish to carry it out.

The idea that backward induction gives the right answer in simple games like that of figure 3.14 was implicit in the economics literature before Selten's paper. In particular, it is embodied in the idea of Stackelberg equilibrium: The requirement that player 2's strategy be the Cournot reaction function is exactly the idea of backward induction, and all other Nash equilibria of the game are inconsistent with backward induction. So we see that the expression "Stackelberg equilibrium" does not simply refer to the extensive form of the Stackelberg game, but instead is shorthand for "the backward-induction solution to the sequential quantity-choice game." Just as with "Cournot equilibrium," this shorthand terminology can be convenient when no confusion can arise. However, our experience suggests that the terminology can indeed lead to confusion, so we advise the student to use the more precise language instead.

Consider the game illustrated in figure 3.15. Here neither of player 2's choices is dominated at his last information set, and so backward induction does not apply. However, given that one accepts the logic of backward induction, the following argument seems compelling as well: "The game beginning at player 1's second information set is a zero-sum simultaneous-move game ('matching pennies') whose unique Nash equilibrium has expected payoffs (0, 0). Player 2 should choose R only if he expects that there is

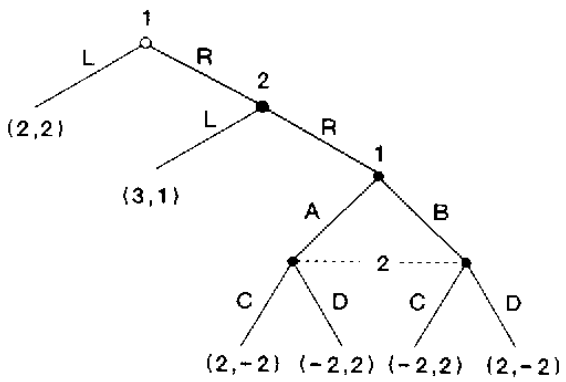


Figure 3.15

probability $\frac{3}{4}$ or better that he will outguess player 1 in the simultaneous-move subgame and end up with $+2$ instead of -2. Since player 2 assumes that player 1 is as rational as he is, it would be very rash of player 2 to expect to get the better of player 1, especially to such an extent. Thus, player 2 should go L, and so player 1 should go R." This is the logic of subgame perfection: Replace any "proper subgame" of the tree with one of its Nash-equilibrium payoffs, and perform backward induction on the reduced tree. (If the subgame has multiple Nash equilibria, this requires that all players agree on which of them would occur; we will come back to this point in subsection 3.6.1.) Once the subgame starting at player 1's second information set is replaced by its Nash-equilibrium outcome, the games described in figures 3.14 and 3.15 coincide.

To define subgame perfection formally we must first define the idea of a proper subgame. Informally, a proper subgame is a portion of a game that can be analyzed as a game in its own right, like the simultaneous-move game embedded in figure 3.15. The formal definition is not much more complicated:

Definition 3.4 A *proper subgame* G of an extensive-form game T consists of a *single* node and all its successors in T, with the property that if $x' \in G$ and $x'' \in h(x')$ then $x'' \in G$. The information sets and payoffs of the subgame are inherited from the original game. That is, x' and x'' are in the same information set in the subgame if and only if they are in the same information set in the original game, and the payoff function on the subgame is just the restriction of the original payoff function to the terminal nodes of the subgame.

Here the word "proper" here does not mean strict inclusion, as it does in the term "proper subset." Any game is always a proper subgame of itself. Proper subgames are particularly easy to identify in the class of deterministic multi-stage games with observed actions. In these games, all

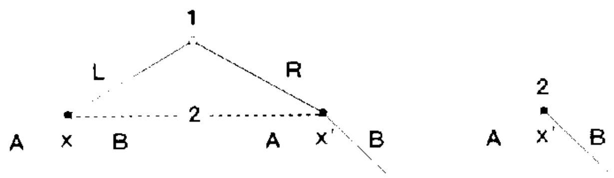


Figure 3.16

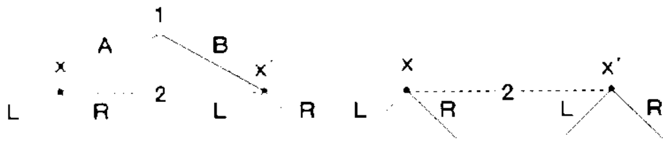


Figure 3.17

previous actions are known to all players at the start of each stage, so each stage begins a new proper subgame. (Checking this is part of exercise 3.4.)

The requirements that all the successors of x be in the subgame and that the subgame not "chop up" any information set ensure that the subgame corresponds to a situation that could arise in the original game. In figure 3.16, the game on the right isn't a subgame of the game on the left, because on the right player 2 knows that player 1 didn't play L, which he did not know in the original game.

Together, the requirements that the subgame begin with a single node x and that the subgame respect information sets imply that in the original game x must be a singleton information set, i.e., $h(x) = \{x\}$. This ensures that the payoffs in the subgame, conditional on the subgame being reached, are well defined. In figure 3.17, the "game" on the right has the problem that player 2's optimal choice depends on the relative probabilities of nodes x and x', but the specification of the game does not provide these probabilities. In other words, the diagram on the right cannot be analyzed as an independent game; it makes sense only as a component of the game on the left, which is needed to provide the missing probabilities.

Since payoffs conditional on reaching a proper subgame are well defined, we can test whether strategies yield a Nash equilibrium when restricted to the subgame in the obvious way. That is, if σ_i is a behavior strategy for player i in the original game, and $\hat{H}_i$ is the collection of player i's information sets in the proper subgame, then the restriction of σ_i to the subgame is the map $\hat{\sigma}_i$ such that $\hat{\sigma}_i(\cdot\,|h_i) = \sigma_i(\cdot\,|h_i)$ for every $h_i \in \hat{H}_i$.

We have now developed the machinery needed to define subgame perfection.

Definition 3.5 A behavior-strategy profile σ of an extensive-form game is a *subgame-perfect equilibrium* if the restriction of σ to G is a Nash equilibrium of G for every proper subgame G.

Because any game is a proper subgame of itself, a subgame-perfect equilibrium profile is necessarily a Nash equilibrium. If the only proper subgame is the whole game, the sets of Nash and subgame-perfect equilibria coincide. If there are other proper subgames, some Nash equilibria may fail to be subgame perfect.

It is easy to see that subgame perfection coincides with backward induction in finite games of perfect information. Consider the penultimate nodes of the tree, where the last choices are made. Each of these nodes begins a trivial one-player proper subgame, and Nash equilibrium in these subgames requires that the player now make a choice that maximizes his payoff; thus, any subgame-perfect equilibrium must coincide with a backward-induction solution at every penultimate node, and we can continue up the tree by induction. But subgame perfection is more general than backward induction; for example, it gives the suggested answer in the game of figure 3.15.

We remarked above that in multi-stage games with observed actions every stage begins a new proper subgame. Thus, in these games, subgame perfection is simply the requirement that the restrictions of the strategy profile yield a Nash equilibrium from the start of each stage k for each history h^k. If the game has a fixed finite number of stages $(K + 1)$, then we can characterize the subgame-perfect equilibria using backward induction: The strategies in the last stage must be a Nash equilibrium of the corresponding one-shot simultaneous-move game, and for each history h^K we replace the last stage by one of its Nash-equilibrium payoffs. For each such assignment of Nash equilibria to the last stage, we then consider the set of Nash equilibria beginning from each stage h^{K-1}. (With the last stage replaced by a payoff vector, the game from h^{K-1} on is a one-shot simultaneous-move game.) The characterization proceeds to "roll back the tree" in the manner of the Kuhn-Zermelo algorithm. Note that even if two different stage-K histories lead to the "same game" in the last stage (that is, if there is a way of identifying strategies in the two games that preserves payoffs), the two histories still correspond to different subgames, and subgame perfection allows us to specify a different Nash equilibrium for each history. This has important consequences, as we will see in section 4.3 and in chapter 5.

3.6 Critiques of Backward Induction and Subgame Perfection††

This section discusses some of the limitations of the arguments for backwards induction and subgame perfection as necessary conditions for reasonable play. Although these concepts seem compelling in simple two-stage games of perfect information, such as the Stackelberg game we discussed at the start of the chapter, things are more complicated if there are many players or if each player moves several times; in these games, equilibrium refinements are less compelling.

3.6.1 Critiques of Backward Induction

Consider the I-player game illustrated in figure 3.18, where each player $i < I$ can either end the game by playing "D" or play "A" and give the move to player $i + 1$. (To readers who skipped sections 3.3–3.5: Figure 3.18 depicts a "game tree." Though you have not seen a formal definition of such trees, we trust that the particular trees we use in this subsection will be clear.) If player i plays D, each player gets $1/i$; if all players play A, each gets 2.

Since only one player moves at a time, this is a game of perfect information, and we can apply the backward-induction algorithm, which predicts that all players should play A. If I is small, this seems like a reasonable prediction. If I is very large, then, as player 1, we ourselves would play D and not A on the basis of a "robustness" argument similar to the one that suggested the inefficient equilibrium in the stag-hunt game of subsection 1.2.4.

First, the payoff 2 requires that all $I - 1$ other players play A. If the probability that a given player plays A is $p < 1$, independent of the others, the probability that all $I - 1$ other players play A is p^{I-1}, which can be quite small even if p is very large. Second, we would worry that player 2 might have these same concerns; that is, player 2 might play D to safeguard against either "mistakes" by future players or the possibility that player 3 might intentionally play D.

A related observation is that longer chains of backward induction presume longer chains of the hypothesis that "player 1 knows that player 2 knows that player 3 knows... the payoffs." If $I = 2$ in figure 3.18, backward induction supposes that player 1 knows player 2's payoff, or at least that player 1 is fairly sure that player 2's optimal choice is A. If $I = 3$, not only must players 1 and 2 know player 3's payoff, in addition, player 1 must know that player 2 knows player 3's payoff, so that player 1 can forecast player 2's forecast of player 3's play. If player 1 thinks that player 2 will forecast player 3's play incorrectly, then player 1 may choose to play D. Traditionally, equilibrium analysis is motivated by the assumption that

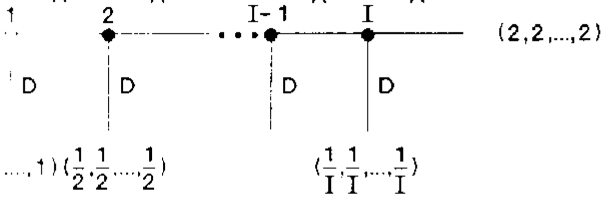


Figure 3.18

payoffs are "common knowledge," so that arbitrarily long chains of "i knows that j knows that k knows" are valid, but conclusions that require very long chains of this form are less compelling than conclusions that require less of the power of the common-knowledge assumption. (In part this is because longer chains of backward induction are more sensitive to small changes in the information structure of the game, as we will see in chapter 9.)

The example in figure 3.18 is most troubling if I is very large. A second complication with backward induction arises whenever the same player can move several times in succession. Consider the game illustrated in figure 3.19. Here the backward-induction solution is that at every information set the player who has the move plays D. Is this solution compelling? Imagine that it is, that you are player 2, and that, contrary to expectation, player 1 plays A_1 at his first move. How should you play? Backward induction says to play D_2 because player 1 will choose D_3 if given a chance, but backward induction also says that player 1 should have played D_1. In this game, unlike the simple examples we started with, player 2's best action if player 1 deviates from the predicted play A_1 depends on how player 2 expects player 1 to play in the future: If player 2 thinks there is at least a 25 percent chance that player 1 will play A_3, then player 2 should play A_2. How should player 2 form these beliefs, and what beliefs are reasonable? In particular, how should player 2 predict how player 1 will play if, contrary to backward induction, player 1 decides to play A_1? In some contexts, playing A_2 may seem like a good gamble.

Most analyses of dynamic games in the economics literature continue to use backward induction and its refinements without reservations, but recently the skeptics have become more numerous. The game depicted in figure 3.19 is based on an example provided by Rosenthal (1981), who was one of the first to question the logic of backward induction. Basu (1988, 1990), Bonanno (1988), Binmore (1987, 1988), and Reny (1986) have argued that reasonable theories of play should not try to rule out any behavior once an event to which the theory assigns probability 0 has occurred, because the theory provides no way for players to form their predictions conditional on these events. Chapter 11 discusses the work of Fudenberg, Kreps, and Levine (1988), who propose that players interpret unexpected deviations as being due to the payoffs' differing from those that were

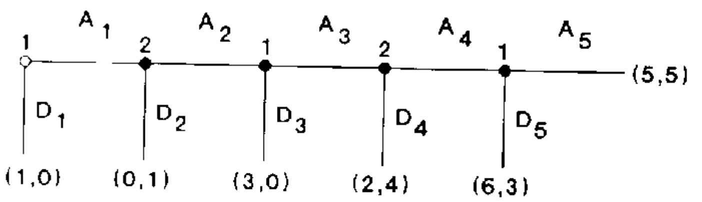


Figure 3.19

originally thought to be most likely. Since any observation of play can be explained by some specification of the opponents' payoffs, this approach sidesteps the difficulty of forming beliefs conditional on probability-0 events, and it recasts the question of how to predict play after a "deviation" as a question of which alternative payoffs are most likely given the observed play. Fudenberg and Kreps (1988) extend this to a methodological principle: They argue that any theory of play should be "complete" in the sense of assigning positive probability to any possible sequence of play, so that, using the theory, the players' conditional forecasts of subsequent play are always well defined.

Payoff uncertainty is not the only way to obtain a complete theory. A second family of complete theories is obtained by interpreting any extensive-form game as implicitly including the fact that players sometimes make small "mistakes" or "trembles" in the sense of Selten 1975. If, as Selten assumes, the probabilities of "trembling" at different information sets are independent, then no matter how often past play has failed to conform to the predictions of backward induction, a player is justified in continuing to use backward induction to predict play in the current subgame. Thus, interpreting "trembles" as deviations is a way to defend backward induction. The relevant question is how likely players view this "trembles" explanation of deviations as opposed to others. In figure 3.19, if player 2 observes A_1, should she (or will she) interpret this as a "tremble," or as a signal that player 1 is likely to play A_3?

3.6.2 Critiques of Subgame Perfection

Since subgame perfection is an extension of backward induction, it is vulnerable to the critiques just discussed. Moreover, subgame perfection requires that players all agree on the play in a subgame even if that play cannot be predicted from backward-induction arguments. This point is emphasized by Rabin (1988), who proposes alternative, weaker equilibrium refinements that allow players to disagree about which Nash equilibrium will occur in a subgame off the equilibrium path.

To see the difference this makes, consider the following three-player game. In the first stage, player 1 can either play L, ending the game with payoffs (6, 0, 6), or play R, which gives the move to player 2. Player 2 can then either play R, ending the game with payoffs (8, 6, 8), or play L, in which case players 1 and 3 (but not player 2) play a simultaneous-move "coordination game" in which they each choose F or G. If their choices differ, they each receive 7 and player 2 gets 10; if the choices match, all three players receive 0. This game is depicted in figure 3.20.

The coordination game between players 1 and 3 at the third stage has three Nash equilibria: two in pure strategies with payoffs (7, 10, 7) and a mixed-strategy equilibrium with payoffs $(3\frac{1}{2}, 5, 3\frac{1}{2})$. If we specify an equilibrium in which players 1 and 3 successfully coordinate, then player 2 plays

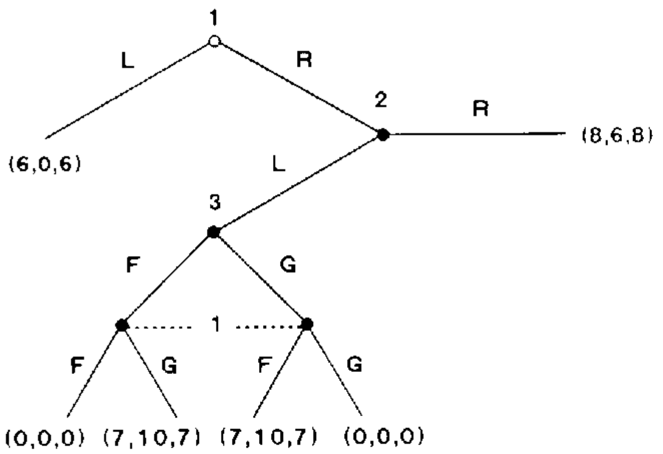


Figure 3.20

L and so player 1 plays R, expecting a payoff of 7. If we specify the inefficient mixed equilibrium in the third stage, then player 2 will play R and again player 1 plays R, this time expecting a payoff of 8. Thus, in all subgame-perfect equilibria of this game, player 1 plays R.

As Rabin argues, it may nevertheless be reasonable for player 1 to play L. He would do so if he saw no way to coordinate in the third stage, and hence expected a payoff of $3\frac{1}{2}$ conditional on that stage being reached, but feared that player 2 would believe that play in the third stage would result in coordination on an efficient equilibrium.

The point is that subgame perfection supposes not only that the players expect Nash equilibria in all subgames but also that all players expect the *same* equilibria. Whether this is plausible depends on the reason one thinks an equilibrium might arise in the first place.

Exercises

Exercise 3.1* Players 1 and 2 must decide whether or not to carry an umbrella when leaving home. They know that there is a 50-50 chance of rain. Each player's payoff is -5 if he doesn't carry an umbrella and it rains, -2 if he carries an umbrella and it rains, -1 if he carries an umbrella and it is sunny, and 1 if he doesn't carry an umbrella and it is sunny. Player 1 learns the weather before leaving home; player 2 does not, but he can observe player 1's action before choosing his own. Give the extensive and strategic forms of the game. Is it dominance solvable?

Exercise 3.2.* Verify that the game in figure 3.13 does not meet the formal definition of a game of perfect recall.

Exercise 3.3* Player 1, the "government," wishes to influence the choice of player 2. Player 2 chooses an action $a_2 \in A_2 = \{0, 1\}$ and receives a transfer $t \in T = \{0, 1\}$ from the government, which observes a_2. Player 2's objective is to maximize the expected value of his transfer, minus the cost of his action, which is 0 for $a_2 = 0$ and $\frac{1}{2}$ for $a_2 = 1$. Player 1's objective is to minimize the sum $2(a_2 - 1)^2 + t$. Before player 2 chooses his action, the government can announce a transfer rule $t(a_2)$.

(a) Draw the extensive form for the case where the government's announcement is not binding and has no effect on payoffs.

(b) Draw the extensive form for the case where the government is constrained to implement the transfer rule it announced.

(c) Give the strategic forms for both games.

(d) Characterize the subgame-perfect equilibria of the two games.

Exercise 3.4** Define a deterministic multi-stage game with observed actions using conditions on the information sets of an extensive form. Show that in these games the start of each stage begins a proper subgame.

Exercise 3.5** Show that subgame-perfect equilibria exist in finite multi-stage games.

Exercise 3.6* There are two players, a seller and a buyer, and two dates. At date 1, the seller chooses his investment level $I \geq 0$ at cost I. At date 2, the seller may sell one unit of a good and the seller has cost $c(I)$ of supplying it, where $c'(0) = -\infty$, $c' < 0$, $c'' > 0$, and $c(0)$ is less than the buyer's valuation. There is no discounting, so the socially optimal level of investment, I^*, is given by $1 + c'(I^*) = 0$.

(a) Suppose that at date 2 the buyer observes the investment I and makes a take-it-or-leave-it offer to the seller. What is this offer? What is the perfect equilibrium of the game?

(b) Can you think of a contractual way of avoiding the inefficient outcome of (a)? (Assume that contracts cannot be written on the level of I.)

Exercise 3.7* Consider a voting game in which three players, 1, 2, and 3, are deciding among three alternatives, A, B, and C. Alternative B is the "status quo" and alternatives A and C are "challengers." At the first stage, players choose which of the two challengers should be considered by casting votes for either A or C, with the majority choice being the winner and abstentions not allowed. At the second stage, players vote between the status quo B and whichever alternative was victorious in the first round, with majority rule again determining the winner. Players vote simultaneously in each round. The players care only about the alternative that is finally selected, and are indifferent as to the sequence of votes that leads to a given selection. The payoff functions are $u_1(A) = 2$, $u_1(B) = 0$, $u_1(C) = 1$; $u_2(A) = 1$, $u_2(B) = 2$, $u_2(C) = 0$; $u_3(A) = 0$, $u_3(B) = 1$, $u_3(C) = 2$.

(a) What would happen if at each stage the players voted for the alternative they would most prefer as the final outcome?

(b) Find the subgame-perfect equilibrium outcome that satisfies the additional condition that no strategy can be eliminated by iterated weak dominance. Indicate what happens if dominated strategies are allowed.

(c) Discuss whether different "agendas" for arriving at a final decision by voting between two alternatives at a time would lead to a different equilibrium outcome.
(This exercise is based on Eckel and Holt 1989, in which the play of this game in experiments is reported.)

Exercise 3.8* Subsection 3.2.3 discussed a player's "strategic incentive" to alter his first-period actions in order to change his own second-period incentives and thus alter the second-period equilibrium. A player may also have a strategic incentive to alter the second-period incentives of others. One application of this idea is the literature on strategic trade policies (e.g. Brander and Spencer 1985; Eaton and Grossman 1986—see Helpman and Krugman 1989, chapters 5 and 6, for a clear review of the arguments). Consider two countries, A and B, and a single good which is consumed only in country B. The inverse demand function is $p = P(Q)$, where Q is the total output produced by firms in countries A and B. Let c denote the constant marginal cost of production and Q_m the monopoly output (Q_m maximizes $Q(P(Q) - c)$).

(a) Suppose that country B does not produce the good. The I (≥ 1) firms in country A are Cournot competitors. Find conditions under which an optimal policy for the government of country A is to levy a unit export *tax* equal to $-P'(Q_m)(I - 1)Q_m/I$. (The objective of country A's government is to maximize the sum of its own receipts and the profit of its firm.) Give an externality interpretation.

(b) Suppose now that there are two producers, one in each country. The game has two periods. In period 1, the government of country A chooses an export tax or subsidy (per unit of exports); in period 2, the two firms, which have observed the government's choice, simultaneously choose quantities. Suppose that the Cournot reaction curves are downward sloping and intersect only once, at a point at which country A's firm's reaction curve is steeper than country B's firm's reaction curve in the (q_A, q_B) space. Show that an export *subsidy* is optimal.

(c) What would happen in question (b) if there were more than one firm in country A? If the strategic variables of period 2 gave rise to upward-sloping reaction curves? Caution: The answer to the latter depends on a "stability condition" of the kind discussed in subsection 1.2.5.

Exercise 3.9** Consider the three-player extensive-form game depicted in figure 3.21.

(a) Show that (A, A) is not the outcome of a Nash equilibrium.

(b) Consider the nonequilibrium situation where player 1 expects player 3 to play R, player 2 expects player 3 to play L, and consequently players

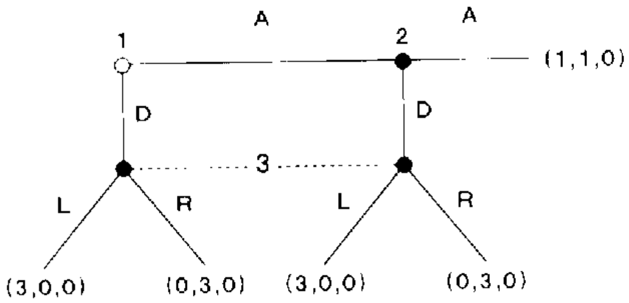


Figure 3.21

1 and 2 both play A. When might this be a fixed point of a learning process like those discussed in chapter 1? When might learning be expected to lead players 1 and 2 to have the same beliefs about player 3's action, as required for Nash equilibrium? (Give an informal answer.) For more on this question see Fudenberg and Kreps 1988 and Fudenberg and Levine 1990.

Exercise 3.10*** In the class of zero-sum games, the sets of outcomes of Nash and subgame-perfect equilibria are the same. That is, for every outcome (probability distribution over terminal nodes) of a Nash-equilibrium strategy profile, there is a perfect equilibrium profile with the same outcome. This result has limited interest, because most games in the social sciences are not zero-sum; however, its proof, which we give in the context of a multi-stage game with observed actions, is a nice way to get acquainted with the logic of perfect equilibrium. Consider a two-person game and let $u_1(\sigma_1, \sigma_2)$ denote player 1's expected payoff (by definition of a zero-sum game, $u_2 = -u_1$). Let $u_1(\sigma_1, \sigma_2 \mid h^t)$ denote player 1's expected payoff conditional on history h^t having been reached at date t (for simplicity, we identify "stages" with "dates"). Last, let $\sigma_i/\hat{\sigma}_i^{h^t}$ denote player i's strategy σ_i, except that if h^t is reached at date t, player i adopts strategy $\hat{\sigma}_i^{h^t}$ in the subgame associated with history h^t (henceforth called "the subgame").

(a) Let (σ_1, σ_2) be a Nash equilibrium. If (σ_1, σ_2) is not perfect, there is a date t, a history h^t, and a player (say player 1) such that this player does not maximize his payoff conditional on history h^t being reached. (Of course, this history h^t must have probability 0 of being reached according to strategies (σ_1, σ_2); otherwise player 1 will not be maximizing his unconditional payoff $u_1(\sigma_1, \sigma_2)$ given σ_2.)

Let $\hat{\sigma}_1^{h^t}$ denote the strategy that maximizes $u_1(\sigma_1/\hat{\sigma}_1^{h^t}, \sigma_2 \mid h^t)$. Last, let $(\sigma_1^{*h^t}, \sigma_2^{*h^t})$ denote a Nash equilibrium of the subgame. Show that for *any* $\tilde{\sigma}_1$

$$u_1(\tilde{\sigma}_1/\hat{\sigma}_1^{h^t}, \sigma_2 \mid h^t) \geq u_1(\tilde{\sigma}_1/\sigma_1^{*h^t}, \sigma_2/\sigma_2^{*h^t} \mid h^t).$$

(Hint: Use the facts that $\sigma_2^{*h^t}$ is a best response to $\sigma_1^{*h^t}$ in subgame h^t, that the game is a zero-sum game, and that $\hat{\sigma}_1^{h^t}$ is an optimal response to σ_2 in the subgame.)

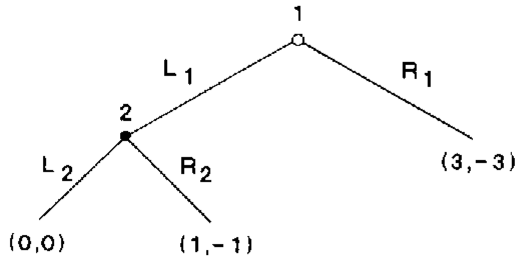


Figure 3.22

(b) Show that the strategy profile $(\sigma_1/\sigma_1^{*h'}, \sigma_2/\sigma_2^{*h'})$ is also a Nash equilibrium. (Hint: Use the fact that subgame h' is not reached under (σ_1, σ_2) and the definition of Nash behavior in the subgame.)

(c) Conclude that the Nash-equilibrium outcome (the probability distribution on terminal nodes generated by σ_1 and σ_2) is also a perfect-equilibrium outcome.

Note that although *outcomes* coincide, the Nash-equilibrium *strategies* need not be perfect-equilibrium strategies—as is demonstrated in figure 3.22, where (R_1, R_2) is a Nash, but not a perfect, equilibrium.

Exercise 3.11* Consider the agenda-setter model of Romer and Rosenthal (1978) (see also Shepsle 1981). The object of the game is to make a one-dimensional decision. There are two players. The "agenda-setter" (player 1, who may stand for a committee in a closed-rule voting system) offers a point $s_1 \in \mathbb{R}$. The "voter" (player 2, who may stand for the median voter in the legislature) can then accept s_1 or refuse it; in the latter case, the decision is the *status quo* or reversion point s_0. Thus, $s_2 \in \{s_0, s_1\}$. The adopted policy is thus s_2. The voter has quadratic preferences $-(s_2 - \hat{s}_2)^2$, where $\hat{s}_2$ is his bliss point.

(a) Suppose that the agenda setter's objective is s_2 (she prefers higher policy levels). Show that, in perfect equilibrium, the setter offers $s_1 = s_0$ if $s_0 \geq \hat{s}_2$ and $s_1 = 2\hat{s}_2 - s_0$ if $s_0 < \hat{s}_2$.

(b) Suppose that the agenda setter's objective function is quadratic as well: $-(s_2 - \hat{s}_1)^2$. Fixing $\hat{s}_1$ and $\hat{s}_2$ ($\hat{s}_1 \gtrless \hat{s}_2$), depict how the perfect-equilibrium policy varies with the reversion s_0.

Exercise 3.12** Consider the twice-repeated version of the agenda-setter model developed in the previous exercise. The new status quo in period 2 is whatever policy (agenda setter's proposal or initial status quo) was adopted in period 1. Suppose that the objective function of the agenda setter is the sum of the two periods' policies, and that the voter's preferences are $-(s_2^1 - 4)^2 - (s_2^2 - 12)^2$ (that is, his bliss point is 4 for the first-period policy and 12 for the second-period one). The initial status quo is 2.

(a) Suppose first that the voter is myopic (acts as if his discount factor were 0 instead of 1), but that the agenda setter is not. Show that the agenda