TABLE 8.2b Swap to Company A

Operation	Fixed	Floating
Issue debt	Pay 10.00%	
Enter swap	Receive 10.00%	Pay LIBOR + 0.05%
Net		Pay LIBOR + 0.05%
Direct cost		Pay LIBOR + 0.30%
Savings		0.25%

TABLE 8.2c Swap to Company B

Operation	Floating	Fixed
Issue debt	Pay LIBOR + 1.00%	
Enter swap	Receive LIBOR + 0.05%	Pay 10.00%
Net		Pay 10.95%
Direct cost		Pay 11.20%
Savings		0.25%

A, however, has a **comparative advantage** in raising fixed as the cost is 1.2% lower than for B. In contrast, the cost of raising floating is only 0.70% lower than for B. Conversely, Company B must have a comparative advantage in raising floating.

This provides a rationale for a swap that will be to the mutual advantage of both parties. If both companies directly issue funds in their final desired market, the total cost will be LIBOR + 0.30% (for A) and 11.20% (for B), for a total of LIBOR + 11.50%. In contrast, the total cost of raising capital where each has a comparative advantage is 10.00% (for A) and LIBOR + 1.00% (for B), for a total of LIBOR + 11.00%. The gain to both parties from entering a swap is 11.50% − 11.00% = 0.50%. For instance, the swap described in Tables 8.2b and 8.2c splits the benefit equally between the two parties.

Company A issues fixed debt at 10.00%, and then enters a swap whereby it promises to pay LIBOR + 0.05% in exchange for receiving 10.00% fixed payments. Its net, effective funding cost is therefore LIBOR + 0.05%, which is less than the direct cost by 25bp.

Similarly, Company B issues floating debt at LIBOR + 1.00%, and then enters a swap whereby it receives LIBOR + 0.05% in exchange for paying 10.00% fixed. Its net, effective funding cost is therefore 11.00%−0.05% = 10.95%, which is less than the direct cost by 25bp. Both parties benefit from the swap.

In terms of actual cash flows, swap payments are typically *netted* against each other. For instance, if the first LIBOR rate is at 9% assuming annual payments, company A would be owed 10% × \$100 = \$1 million, and would have to pay LIBOR +0.05%, or 9.05% × \$100 = \$0.905 million. This gives a net receipt of

\$95,000. There is no need to exchange principals since both involve the same amount.

8.3.2 Quotations

Swaps can be quoted in terms of spreads relative to the yield of similar-maturity Treasury notes. For instance, a dealer may quote 10-year swap spreads as 31/34bp against LIBOR. If the current note yield is 6.72, this means that the dealer is willing to pay $6.72 + 0.31 = 7.03\%$ against receiving LIBOR, or that the dealer is willing to receive $6.72 + 0.34 = 7.06\%$ against paying LIBOR. Of course, the dealer makes a profit from the spread, which is rather small, at 3bp only. Equivalently, the outright quote is 7.03/7.06 for the swap.

Note that the swap should trade at a positive credit spread to Treasuries. This is because the other leg is quoted in relation to LIBOR, which also has credit risk. More precisely, swap rates correspond to the credit risk of AA-rated counterparties.

Table 5.1 has shown that the interest rate swap market is by far the largest derivative market in terms of notional. Because the market is very liquid, market quotations for the fixed rate leg have become benchmark interest rates. Thus, swap rates form the basis for the **swap curve**, which is also called the par curve, because it is equivalent to yields on bonds selling at par. Because the floating-rate leg is indexed to LIBOR, which carries credit risk, the swap curve is normally higher than the par curve for government bonds in the same currency.

8.3.3 Pricing

We now discuss the pricing of interest rate swaps. Consider, for instance, a three-year \$100 million swap, where we receive a fixed coupon of 5.50% against LIBOR. Payments are annual and we ignore credit spreads. We can price the swap using either of two approaches, taking the difference between two bond prices or valuing a sequence of forward contracts. This is illustrated in Figure 8.2.

The top part of the figure shows that this swap is equivalent to a long position in a fixed-rate, 5.5% three-year bond and a short position in a three-year floating-rate note (FRN). If B_F is the value of the fixed-rate bond and B_f is the value of the FRN, the value of the swap is $V = B_F - B_f$.

The value of the FRN should be close to par. Just before a reset, B_f will behave exactly like a cash investment, as the coupon for the next period will be set to the prevailing interest rate. Therefore, its market value should be close to the face value. Just after a reset, the FRN will behave like a bond with a six-month maturity. But overall, fluctuations in the market value of B_f should be small.

Consider now the swap value. If at initiation the swap coupon is set to the prevailing par yield, B_F is equal to the face value, $B_F = 100$. Because $B_f = 100$ just before the reset on the floating leg, the value of the swap is zero, $V = B_F - B_f = 0$. This is like a forward contract at initiation.

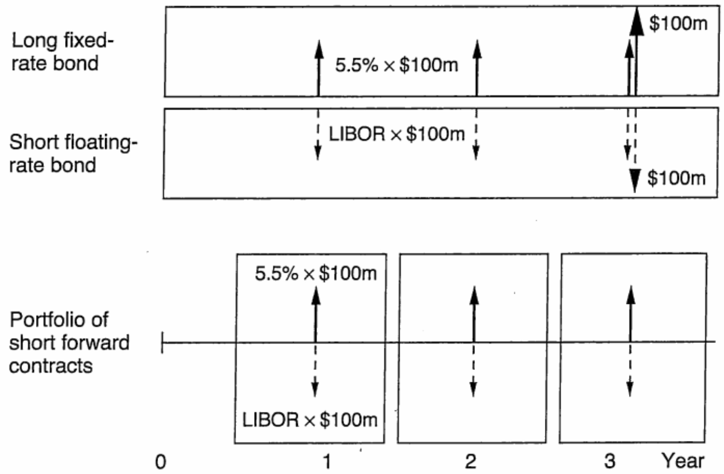

FIGURE 8.2 Alternative Decompositions for Swap Cash Flows

After the swap is consummated, its value will be affected by interest rates. If rates fall, the swap will move in-the-money, since it receives higher coupons than prevailing market yields. B_F will increase whereas B_f will barely change.

Thus the duration of a receive-fixed swap is similar to that of a fixed-rate bond, including the fixed coupons and principal at maturity. This is because the duration of the floating leg is close to zero. The fact that the principals are not exchanged does not mean that the duration computation should not include the principal. Duration should be viewed as an interest rate sensitivity.

KEY CONCEPT

A position in a receive-fixed swap is equivalent to a long position in a bond with similar coupon characteristics and maturity offset by a short position in a floating-rate note. Its duration is close to that of the fixed-rate note.

We now value the three-year swap using term-structure data from the preceding chapter. The time is just before a reset, so $B_f = \$100$ million. We compute B_F (in millions) as

$$B_F = \frac{\$5.5}{(1+4.000\%)} + \frac{\$5.5}{(1+4.618\%)^2} + \frac{\$105.5}{(1+5.192\%)^3} = \$100.95$$

The outstanding value of the swap is therefore $V = \$100.95 - \$100 = \$0.95$ million.

Alternatively, the swap can be valued as a sequence of forward contracts, as shown in the bottom part of Figure 8.2. Recall from Chapter 5 that the value of a unit position in a long forward contract is given by

$$V_i = (F_i - K)\exp(-r_i\tau_i) \tag{8.8}$$

where F_i is the current forward rate, K the prespecified rate, and r_i the spot rate for time τ_i. Extending this to multiple maturities, and to discrete compounding using R_i, the swap can be valued as

$$V = \sum_i n_i(F_i - K)/(1 + R_i)^{\tau_i} \tag{8.9}$$

where n_i is the notional amount for maturity i.

A long forward rate agreement benefits if rates go up. Indeed Equation (8.8) shows that the value increases if F_i goes up. In the case of our swap, we *receive* a fixed rate K. So, the position loses money if rates go up, as we could have received a higher rate. Hence, the sign on Equation (8.9) must be reversed.

Using the forward rates listed in Table 7.4, we find

$$V = -\frac{\$100(4.000\% - 5.50\%)}{(1 + 4.000\%)} - \frac{\$100(5.240\% - 5.50\%)}{(1 + 4.618\%)^2}$$

$$-\frac{\$100(6.350\% - 5.50\%)}{(1 + 5.192\%)^3}$$

$$V = +1.4423 + 0.2376 - 0.7302 = \$0.95 \text{ million}$$

This is identical to the previous result, as it should be. The swap is in-the-money primarily because of the first payment, which pays a rate of 5.5% whereas the forward rate is only 4.00%.

Thus, interest rate swaps can be priced and hedged using a sequence of forward rates, such as those implicit in Eurodollar contracts. In practice, the practice of daily marking-to-market futures induces a slight convexity bias in futures rates, which have to be adjusted downward to get forward rates.

Figure 8.3 compares a sequence of quarterly forward rates with the five-year swap rate prevailing at the same time. Because short-term forward rates are less than the swap rate, the near payments are in-the-money. In contrast, the more distant payments are out-of-the-money. The current market value of this swap is zero, which implies that all the near-term positive values must be offset by distant negative values.

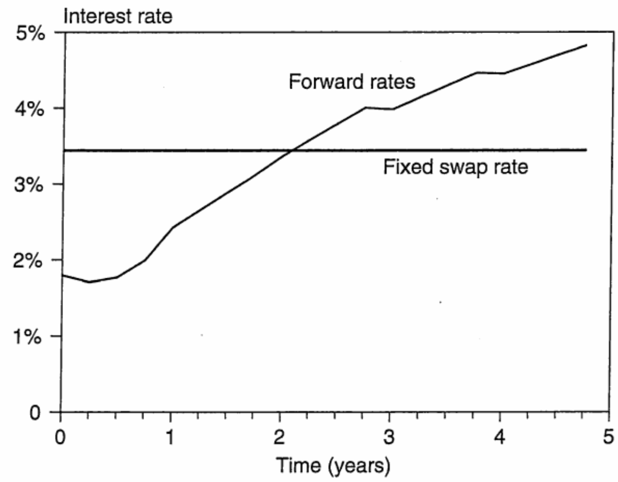

FIGURE 8.3 Sequence of Forward Rates and Swap Rate

EXAMPLE 8.6: FRM EXAM 2005—QUESTION 51

You are given the following information about an interest rate swap: two-year term, semiannual payment, fixed rate = 6%, floating rate = LIBOR + 50 basis points, notional USD 10 million. Calculate the net coupon exchange for the first period if LIBOR is 5% at the beginning of the period and 5.5% at the end of the period.

a. Fixed-rate payer pays USD 0
b. Fixed-rate payer pays USD 25,000
c. Fixed-rate payer pays USD 50,000
d. Fixed-rate payer receives USD 25,000

EXAMPLE 8.7: FRM EXAM 2000—QUESTION 55

Bank One enters into a five-year swap contract with Mervin Co. to pay LIBOR in return for a fixed 8% rate on a principal of $100 million. Two years from now, the market rate on three-year swaps at LIBOR is 7%. At this time Mervin Co. declares bankruptcy and defaults on its swap obligation. Assume that the net payment is made only at the end of each year for the swap contract period. What is the market value of the loss incurred by Bank One as a result of the default?

a. $1.927 million
b. $2.245 million
c. $2.624 million
d. $3.011 million

8.4 OPTIONS

There is a large variety of fixed-income options. We will briefly describe here caps and floors, swaptions, and exchange-traded options. In addition to these standalone instruments, fixed-income options are embedded in many securities. For instance, a callable bond can be viewed as a regular bond plus a short position in an option.

When considering fixed-income options, the underlying can be a yield or a price. Due to the negative price–yield relationship, a call option on a bond can also be viewed as a put option on the underlying yield.

8.4.1 Caps and Floors

A **cap** is a call option on interest rates with unit value

$$C_T = \text{Max}[i_T - K, 0] \tag{8.10}$$

where $K = i_C$ is the cap rate and i_T is the rate prevailing at maturity.

In practice, caps are purchased jointly with the issuance of floating-rate notes that pay LIBOR plus a spread on a periodic basis for the term of the note. By purchasing the cap, the issuer ensures that the cost of capital will not exceed the capped rate. Such caps are really a combination of individual options, called **caplets**.

The payment on each caplet is determined by C_T, the notional, and an accrual factor. Payments are made in **arrears**, that is, at the end of the period. For instance, take a one-year cap on a notional of \$1 million and a six-month LIBOR cap rate of 5%. The agreement period is from January 15 to the next January with a reset on July 15. Suppose that on July 15, LIBOR is at 5.5%. On the following January, the payment is

$$\$1 \text{ million} \times (0.055 - 0.05)(184/360) = \$2{,}555.56$$

using *Actual*/360 interest accrual. If the cap is used to hedge a FRN, this would help to offset the higher coupon payment, which is now 5.5%.

A **floor** is a put option on interest rates with value

$$P_T = \text{Max}[K - i_T, 0] \tag{8.11}$$

where $K = i_F$ is the floor rate. A **collar** is a combination of buying a cap and selling a floor. This combination decreases the net cost of purchasing the cap protection. Figure 8.4 shows an example of price path, with a cap rate of 3.5% and a floor rate of 2%. There are three instances where the cap is exercised, leading to a receipt of payment. There is one instance where the rate is below the floor, requiring a payment.

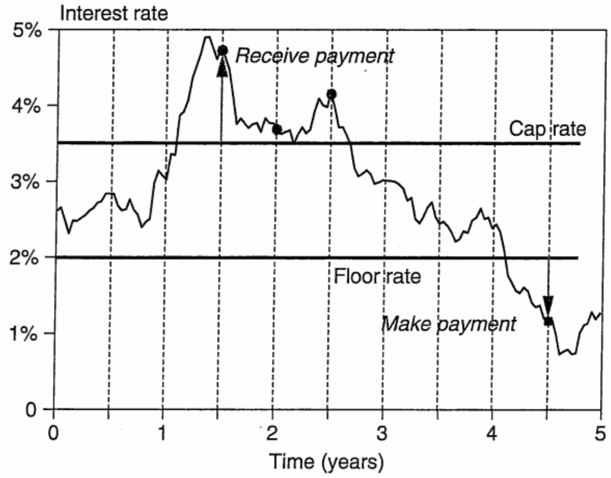

FIGURE 8.4 Exercise of Cap and Floor

When the cap and floor rates converge to the same value $K = i_C = i_F$, the overall debt cost becomes fixed instead of floating. The collar is then the same as a pay-fixed swap, which is the equivalent of put–call parity,

$$\text{Long Cap}(i_C = K) + \text{Short Floor}(i_F = K) = \text{Long Pay-Fixed Swap} \tag{8.12}$$

Caps are typically priced using a variant of the Black model, assuming that interest rate changes are lognormal. The value of the cap is set equal to a portfolio of N caplets, which are European-style individual options on different interest rates with regularly spaced maturities

$$c = \sum_{j=1}^{N} c_j \tag{8.13}$$

For each caplet, the unit price is

$$c_j = [F\,N(d_1) - K\,N(d_2)]\text{PV}(\$1) \tag{8.14}$$

where F is the current forward rate for the period t_j to t_{j+1}, K is the cap rate, and PV($1) is the discount factor to time t_{j+1}. To obtain a dollar amount, we must adjust for the notional amount as well as the length of the accrual period.

The volatility entering the pricing model, σ, is that of the forward rate between now and the expiration of the option contract, that is, at t_j. Generally, volatilities are quoted as one number for all caplets within a cap, which is called **flat volatilities.**

$$\sigma_j = \sigma$$

Alternatively, volatilities can be quoted separately for each forward rate in the caplet, which is called **spot volatilities.**

Example: Computing the Value of a Cap

Consider the previous cap on $1 million at the capped rate of 5%. Assume a flat term structure at 5.5% and a volatility of 20% pa. The reset is on July 15, in 181 days. The accrual period is 184 days.

Since the term structure is flat, the six-month forward rate starting in six months is also 5.5%. First, we compute the present value factor, which is $PV(\$1) = 1/(1 + 0.055 \times 365/360) = 0.9472$, and the volatility, which is $\sigma\sqrt{\tau} = 0.20\sqrt{181/360} = 0.1418$.

We then compute the value of $d_1 = \ln[F/K]/\sigma\sqrt{\tau} + \sigma\sqrt{\tau}/2 = \ln[0.055/0.05]/0.1418 + 0.1418/2 = 0.7430$ and $d_2 = d_1 - \sigma\sqrt{\tau} = 0.7430 - 0.1418 = 0.6012$. We find $N(d_1) = 0.7713$ and $N(d_2) = 0.7261$. The unit value of the call is $c = [FN(d_1) - KN(d_2)]PV(\$1) = 0.5789\%$. Finally, the total price of the call is $1 million $\times$ 0.5789% $\times$ (184/360)= $2,959.

Figure 8.3 can be taken as an illustration of the sequence of forward rates. If the cap rate is the same as the prevailing swap rate, the cap is said to be *at-the-money*. In the figure, the near caplets are out-of-the-money because $F_i < K$. The distant caplets, however, are in-the-money.

EXAMPLE 8.8: FRM EXAM 2002—QUESTION 22

An interest rate cap runs for 12 months based on three-month Libor with a strike price of 4%. Which of the following is generally *true*?

a. The cap consists of three caplet options with maturities of three months, the first one starting today based on three-month LIBOR set in advance and paid in arrears.
b. The cap consists of four caplets starting today, based on LIBOR set in advance and paid in arrears.
c. The implied volatility of each caplet will be identical no matter how the yield curve moves.
d. Rate caps have only a single option based on the maturity of the structure.

EXAMPLE 8.9: FRM EXAM 2004—QUESTION 10

The payoff to a swap where the investor receives fixed and pays floating can be replicated by all of the following *except*

a. A short position in a portfolio of FRAs
b. A long position in a fixed rate bond and a short position in a floating rate bond
c. A short position in an interest rate cap and a long position in a floor
d. A long position in a floating rate note and a short position in a floor

EXAMPLE 8.10: FRM EXAM 2003—QUESTION 27

A portfolio management firm manages the fixed-rate corporate bond portfolio owned by a defined-benefit pension fund. The duration of the bond portfolio is five years; the duration of the pension fund's liabilities is seven years. Assume that the fund sponsor strongly believes that rates will decline over the next six months and is concerned about the duration mismatch between portfolio assets and pension liabilities. Which of the following strategies would be the best way to eliminate the duration mismatch?

a. Enter into a swap transaction in which the firm pays fixed and receives floating.
b. Enter into a swap transaction in which the firm receives fixed and pays floating.
c. Purchase an interest rate cap expiring in six months
d. Sell Eurodollar futures contracts.

8.4.2 Swaptions

Swaptions are OTC options that give the buyer the right to enter a swap at a fixed point in time at specified terms, including a fixed coupon rate.

These contracts take many forms. A **European swaption** is exercisable on a single date at some point in the future. On that date, the owner has the right to enter a swap with a specific rate and term. Consider for example a "1Y × 5Y" swaption. This gives the owner the right to enter in one year a long or short position in a five-year swap.

A fixed-term **American swaption** is exercisable on any date during the exercise period. In our example, this would be during the next year. If, for instance, exercise occurs after six months, the swap would terminate in five years and six months from now. So, the termination date of the swap depends on the exercise date. In contrast, a **contingent American swaption** has a prespecified termination date, for instance exactly six years from now. Finally, a **Bermudan option** gives the holder the right to exercise on a specific set of dates during the life of the option.

As an example, consider a company that, in one year, will issue five-year floating-rate debt. The company wishes to have the option to swap the floating payments into fixed payments. The company can purchase a swaption that will give it the right to create a five-year pay-fixed swap at the rate of 8%. If the prevailing swap rate in one year is higher than 8%, the company will exercise the swaption, otherwise not. The value of the option at expiration will be

$$P_T = \text{Max}[V(i_T) - V(K), 0] \tag{8.15}$$

where $V(i)$ is the value of a swap to pay a fixed rate i, i_T is the prevailing swap rate for the swap maturity, and K is the locked-in swap rate. This contract is called a European 6/1 put swaption, or one into five-year payer option.

Such a swap is equivalent to an option on a bond. As this swaption creates a profit if rates rise, it is akin to a one-year put option on a six-year bond. A

TABLE 8.3 Summary of Terminology for OTC Swaps and Options

Product	Buy (long)	Sell (short)
Fixed/Floating Swap	Pay fixed wReceive floating	Pay floating Receive fixed
Cap	Pay premium Receive Max$(i - i_C, 0)$	Receive premium Pay Max$(i - i_C, 0)$
Floor	Pay premium Receive Max$(i_F - i, 0)$	Receive premium Pay Max$(i_F - i, 0)$
Put Swaption (payer option)	Pay premium Option to pay fixed and receive floating	Receive premium If exercised, receive fixed and pay floating
Call Swaption (receiver option)	Pay premium Option to pay floating and receive fixed	Receive premium If exercised, receive floating and pay fixed

put option benefits when the bond value falls, which happens when rates rise. Conversely, a swaption that gives the right to receive fixed is akin to a call option on a bond. Table 8.3 summarizes the terminology for swaps, caps and floors, and swaptions.

Swaptions can be used for a variety of purposes. Consider an investor in a mortgage-backed security (MBS). If long-term rates fall, prepayment will increase, leading to a shortfall in the price appreciation in the bond. This risk can be hedged by buying receiver swaptions. If rates fall, the buyer will exercise the option, which creates a profit to offset the loss on the MBS. Alternatively, this risk can also be hedged by issuing callable debt. This creates a long position in an option that generates a profit if rates fall. As an example, Fannie Mae, a government-sponsored enterprise that invests heavily in mortgages, uses these techniques to hedge its prepayment risk.

Finally, swaptions are typically priced using a variant of the Black model, assuming that interest rates are lognormal. The value of the swaption is then equal to a portfolio of options on different interest rates, all with the same maturity. In practice, swaptions are traded in terms of volatilities instead of option premiums. The applicable forward rate starts at the same time as the option, with a term equal to that of the option.

EXAMPLE 8.11: FRM EXAM 2003—QUESTION 56

As your company's risk manager, you are looking for protection against adverse interest rate changes in five years. Using Black's model for options on futures to price a European swap option (swaption) which gives the option holder the right to cancel a seven-year swap after five years, which of the following would you use in the model?

a. The two-year forward par swap rate starting in five years time
b. The five-year forward par swap rate starting in two years time
c. The two-year par swap rate
d. The five-year par swap rate

8.4.3 Exchange-Traded Options

Among exchange-traded fixed-income options, we describe options on Eurodollar futures and on T-bond futures.

Options on Eurodollar futures give the owner the right to enter a long or short position in Eurodollar futures at a fixed price. The payoff on a put option, for example, is

$$P_T = \text{Notional} \times \text{Max}[K - \text{FQ}_T, 0] \times (90/360) \tag{8.16}$$

where K is the strike price and FQ_T the prevailing futures price quote at maturity. In addition to the cash payoff, the option holder enters a position in the underlying futures. Since this is a put, it creates a short position after exercise, with the counterparty taking the opposing position. Note that, since futures are settled daily, the value of the contract is zero.

Since the futures price can also be written as $\text{FQ}_T = 100 - i_T$ and the strike price as $K = 100 - i_C$, the payoff is also

$$P_T = \text{Notional} \times \text{Max}[i_T - i_C, 0] \times (90/360) \tag{8.17}$$

which is equivalent to that of a cap on rates. Thus, a put on Eurodollar futures is equivalent to a caplet on LIBOR.

In practice, there are minor differences in the contracts. Options on Eurodollar futures are American style instead of European style. Also, payments are made at the expiration date of Eurodollar futures options instead of in arrears.

Options on T-bond futures give the owner the right to enter a long or short position in futures at a fixed price. The payoff on a call option, for example, is

$$C_T = \text{Notional} \times \text{Max}[F_T - K, 0] \tag{8.18}$$

An investor who thinks that rates will fall, or that the bond market will rally, could buy a call on T-bond futures. In this manner, he or she will participate in the upside, without downside risk.

EXAMPLE 8.12: FRM EXAM 2007—QUESTION 95

To hedge against future, unanticipated, and significant increases in borrowing rates, which of the following alternatives offers the greatest flexibility for the borrower?

- **a.** Interest rate collar
- **b.** Fixed for floating swap
- **c.** Call swaption
- **d.** Interest rate floor

8.5 IMPORTANT FORMULAS

Long FRA 1×4 = Invest for one period, borrow for four

Payment on FRA: $V_T = (S_T - F) \times \tau \times \text{Notional} \times \text{PV}(\$1)$

Valuation of Eurodollar contract:
$P_t = 10{,}000 \times [100 - 0.25(100 - \text{FQ}_t)] = 10{,}000 \times [100 - 0.25F_t]$

Eurodollar contract risk: DV01 = \$25

Futures convexity adjustment: Futures Rate = Forward Rate + $\left(\frac{1}{2}\right)\sigma^2 t_1 t_2$
(negative relationship between contract value and rates)

T-bond futures net delivery cost: Cost = Price − Futures Quote × CF

T-bond futures conversion factor: CF = NPV of bond at 6%

Valuation of interest rate swap: $V = B_F(\text{fixed} - \text{rate}) - B_f(\text{floating} - \text{rate})$
Long receive-fixed = long fixed-coupon bond + short FRN

Valuation of interest rate swap as forward contracts:
$V = \sum_i n_i (F_i - K)/(1 + R_i)^{\tau_i}$

Interest-rate cap: $C_T = \text{Max}[i_T - K, 0]$

Interest-rate floor: $P_T = \text{Max}[K - i_T, 0]$

Collar: Long cap plus short floor

Cap valuation: $c = \sum_{j=1}^{N} c_j, \quad c_j = [F N(d_1) - K N(d_2)]\text{PV}(\$1)$

Put swaption (1Y × 5Y): (right to pay fixed, starting in one year for five years)
$P_T = \text{Max}[V(i_T) - V(K), 0]$

Call option on Eurodollar futures = cap on rates

8.6 ANSWERS TO CHAPTER EXAMPLES

Example 8.1: FRM Exam 2002—Question 27

b. An FRA defined as $t_1 \times t_2$ involves a forward rate starting at time t_1 and ending at time t_2. The buyer of this FRA locks in a borrowing rate for months 3 to 5. This is equivalent to borrowing for five months and reinvesting the funds for the first two months.

Example 8.2: FRM Exam 2005—Question 57

d. The market-implied forward rate is given by $\exp(-R_2 \times 2) = \exp(-R_1 \times 1 - F_{1,2} \times 1)$, or $F_{1,2} = 2 \times 3.50 - 1 \times 3.25 = 3.75\%$. Given that this is exactly equal to the quoted rate, the value must be zero. If instead this rate was 3.50%, for example, the value would be $V = \$1{,}000{,}000 \times (3.75\% - 3.50\%) \times (2 - 1)\exp(-3.50\% \times 2) = 2{,}331$.

Example 8.3: FRM Exam 2001—Question 70

b. The seller of an FRA agrees to receive fixed. Since rates are now higher than the contract rate, this contract must show a loss for the seller. The loss is $10,000,000 $\times$ (6.85% $-$ 6.35%) $\times$ (90/360) = $12,500 when paid in arrears, i.e., in nine months. On the settlement date, i.e., brought forward by three months, the loss is $12,500/(1 + 6.85% $\times$ 0.25) = $12,290.

Example 8.4: FRM Exam 2005—Question 49

c. Compute the ratio of the price to the CF. This gives, respectively, 125.69/1.1979 = 11.29, then 12.87, 8.09, and 11.58. Hence, bond C is the cheapest-to-deliver.

Example 8.5: FRM Exam 2007—Question 80

c. Equation (8.4) shows that the futures rate exceeds the forward rate.

Example 8.6: FRM Exam 2005—Question 51

b. The floating leg uses LIBOR at the beginning of the period, plus 50bp, or 5.5%. The payment is given by $10,000,000 $\times$ (0.06 $-$ 0.055) $\times$ 0.5 = $25,000.

Example 8.7: FRM Exam 2000—Question 55

c. Using Equation (8.9) for three remaining periods, we have the discounted value of the net interest payment, or (8% $-$ 7%)100m$ = 1m$, discounted at 7%, which is $934,579 + $873,439 + $816,298 = $2,624,316.

Example 8.8: FRM Exam 2002—Question 22

a. Interest rate caps involve multiple options, or caplets. The first one has terms that are set in three months. It locks in $\mathrm{Max}[R(t+3) - 4\%, 0]$. Payment occurs in arrears in six months. The second one is a function of $\mathrm{Max}[R(t+6) - 4\%, 0]$. The third is a function of $\mathrm{Max}[R(t+9) - 4\%, 0]$ and is paid at $t + 12$. The sequence then stops because the cap has a term of 12 months only. This means there are three caplets.

Example 8.9: FRM Exam 2004—Question 10

d. A receive-fixed swap position is equivalent to being long a fixed-rate bond, or being short a portfolio of FRAs (which gain if rates go down), or selling a cap and buying a floor with the same strike price (which gains if rates go up). A short position in a floor does not generate a gain if rates drop. It is asymmetric anyway.

Example 8.10: FRM Exam 2003—Question 27

b. The manager should increase the duration of assets, or buy coupon-paying bonds. This can be achieved by entering a receive-fixed swap, so b. is correct and a. is wrong. Buying a cap will not provide protection if rates drop. Selling Eurodollar futures will lose money if rates drop.

Example 8.11: FRM Exam 2003—Question 56

a. The forward rate should start at the beginning of the option in five years, with a maturity equal to the duration of the option, or two years.

Example 8.12: FRM Exam 2007—Question 95

c. A swaption gives the borrower the flexibility to lock in a low rate. On the other hand, a regular swap does not offer flexibility as an option. A collar fixes a range of rates, but not much flexibility. A floor involves protection if rates go down, not up. (Note that buying a cap would have been another good choice.)

CHAPTER 9

Equity, Currency, and Commodity Markets

Having covered fixed-income instruments, we now turn to equity, currency, and commodity markets. Equities, or common stocks, represent ownership shares in a corporation. Due to the uncertainty in their cash flows, as well as in the appropriate discount rate, equities are much more difficult to value than fixed-income securities. They are also less amenable to the quantitative analysis that is used in fixed-income markets. Equity derivatives, however, can be priced reasonably precisely in relation to underlying stock prices.

Next, the foreign currency markets include spot, forward, options, futures, and swap markets. The foreign exchange markets are by far the largest financial markets in the world, with daily turnover estimated at $1,880 billion in 2004.

Commodity markets consist of agricultural products, metals, energy, and other products. Commodities differ from financial assets as their holding provides an implied benefit known as convenience yield but also incurs storage costs.

Section 9.1 introduces equity markets and presents valuation methods. Section 9.2 briefly discusses convertible bonds and warrants. These differ from the usual equity options in that exercising them creates new shares. Section 9.3 then provides an overview of important equity derivatives, including stock index futures, stock options, stock index options, and equity swaps. Section 9.4 presents a brief introduction to currency markets. Contracts such as futures, forwards, and options have been developed in previous chapters and do not require special treatment. In contrast, currency swaps are analyzed in some detail in Section 9.5 due to their unique features and importance. Finally, Section 9.6 discusses commodity markets.

9.1 EQUITIES

9.1.1 Overview

Common stocks, also called **equities**, are securities that represent ownership in a corporation. Bonds are *senior* to equities, that is, have a prior claim on the firm's assets in case of bankruptcy. Hence equities represent **residual claims** to what is left of the value of the firm after bonds, loans, and other contractual obligations have been paid off.

TABLE 9.1 Global Equity Markets—2007 (Billions of U.S. Dollars)

United States	19,922
Japan	4,543
Eurozone	10,047
United Kingdom	3,852
Other Europe	2,867
Other Pacific	4,949
Canada	2,187
Developed	48,366
Emerging	12,508
World	60,874

Source: World Federation of Exchanges

Another important feature of common stocks is their **limited liability**, which means that the most shareholders can lose is their original investment. This is unlike owners of unincorporated businesses, whose creditors have a claim on the personal assets of the owner should the business turn bad.

Table 9.1 describes the global equity markets. The total market value of common stocks was worth approximately $61 trillion at the end of 2007. The United States accounts for the largest share, followed by Japan, the Eurozone, and the United Kingdom. In 2008, global stocks fell by 42%, which implies a loss of market value of about $26 trillion.

Preferred stocks differ from common stock because they promise to pay a specific stream of dividends. So, they behave like a perpetual bond, or consol. Unlike bonds, however, failure to pay these dividends does not result in default. Instead, the corporation must withhold dividends to common stock holders until the preferred dividends have been paid out. In other words, preferred stocks are junior to bonds, but senior to common stocks.

With **cumulative preferred dividends**, all current and previously postponed dividends must be paid before any dividends on common stock shares can be paid. Preferred stocks usually have no voting rights.

Unlike interest payments, preferred stocks dividends are not tax-deductible expenses. Preferred stocks, however, have an offsetting tax advantage. Corporations that receive preferred dividends only pay taxes on 30% of the amount received, which lowers their income tax burden. As a result, most preferred stocks are held by corporations. The market capitalization of preferred stocks is much lower than that of common stocks, as seen from the IBM example below. Trading volumes are also much lower.

Example: IBM Preferred Stock

IBM issued 11.25 million preferred shares in June 1993. These are traded as 45 million "depositary" shares, each representing one-fourth of the preferred, under the ticker "IBM-A" on the NYSE. Dividends accrue at the rate of $7.50 per annum, or $1.875 per depositary share.

As of April 2001, the depositary shares were trading at \$25.4, within a narrow 52-week trading range of [\$25.00, \$26.25]. Using the valuation formula for a consol, the shares trade at an implied yield of 7.38%. The total market capitalization of the IBM-A shares amounts to approximately \$260 million. In comparison, the market value of the common stock is \$214,602 million, which is more than 800 times larger.

9.1.2 Valuation

Common stocks are extremely difficult to value. Like any other asset, their value derives from their future benefits, that is, from their stream of future cash flows (i.e., dividend payments) or future stock price.

We have seen that valuing Treasury bonds is relatively straightforward, as the stream of cash flows, coupon, and principal payments, can be easily laid out and discounted into the present.

This is an entirely different affair for common stocks. Consider for illustration a "simple" case where a firm pays out a dividend D over the next year that grows at the constant rate of g. We ignore the final stock value and discount at the constant rate of r, such that $r > g$. The firm's value, P, can be assessed using the net present value formula, like a bond

$$
\begin{aligned}
P &= \textstyle\sum_{t=1}^{\infty} C_t/(1+r)^t \\
&= \textstyle\sum_{t=1}^{\infty} D(1+g)^{(t-1)}/(1+r)^t \\
&= [D/(1+r)] \textstyle\sum_{t=0}^{\infty} [(1+g)/(1+r)]^t \\
&= [D/(1+r)] \times \left[\frac{1}{1-(1+g)/(1+r)}\right] \\
&= [D/(1+r)] \times [(1+r)/(r-g)]
\end{aligned}
$$

This is also the so-called "Gordon-growth" model,

$$P = \frac{D}{r-g} \tag{9.1}$$

as long as the discount rate exceeds the growth rate of dividends, $r > g$.

The problem with equities is that the growth rate of dividends is uncertain and that, in addition, it is not clear what the required discount rate should be. To make things even harder, some companies simply do not pay any dividend and instead create value from the appreciation of their share price.

Still, this valuation formula indicates that large variations in equity prices can arise from small changes in the discount rate or in the growth rate of dividends, explaining the large volatility of equities. More generally, the risk and expected return of the equity depends on the underlying business fundamentals as well as on the amount of leverage, or debt in the capital structure.

For financial intermediaries for which the value of underlying assets can be measured precisely, we can value the equity from the underlying assets and the cost of borrowing. This situation, however, is more akin to the pricing of a derivative from the price of the underlying than pricing the asset directly.

EXAMPLE 9.1: LEVERAGE AND RETURN ON EQUITY

A hedge fund leverages its $100 million of investor capital by a factor of three and invests it into a portfolio of junk bonds yielding 14%. If its borrowing costs are 8%, what is the yield on investor capital?

a. 14%
b. 18%
c. 26%
d. 42%

9.2 CONVERTIBLE BONDS AND WARRANTS

9.2.1 Definitions

We now turn to convertible bonds and warrants. While these instruments have option-like features, they differ from regular options. When a call option is exercised, for instance, the "long" purchases an outstanding share from the "short." There is no net creation of shares. In contrast, the exercise of convertible bonds, of warrants, (and of executive stock options) entails the creation of new shares, as the option is sold by the corporation itself. Because the number of shares goes up, the existing shares are said to be **diluted** by the creation of new shares.

Warrants are long-term call options issued by a corporation on its own stock. They are typically created at the time of a bond issue, but they trade separately from the bond to which they were originally attached. When a warrant is exercised, it results in a cash inflow to the firm which issues more shares.

Convertible bonds are bonds issued by a corporation that can be converted into equity at certain times using a predetermined exchange ratio. They are equivalent to a regular bond plus a warrant. This allows the company to issue debt with a lower coupon than otherwise.

For example, a bond with a **conversion ratio** of 10 allows its holder to convert one bond with par value of $1,000 into 10 shares of the common stock. The **conversion price**, which is really the strike price of the option, is $1,000/10 = $100. The corporation will typically issue the convertible deep out-of-the-money, for example when the stock price is at $50. When the stock price moves, for instance to $120, the bond can be converted into stock for an immediate option profit of ($120 − $100) × 10 = $200.

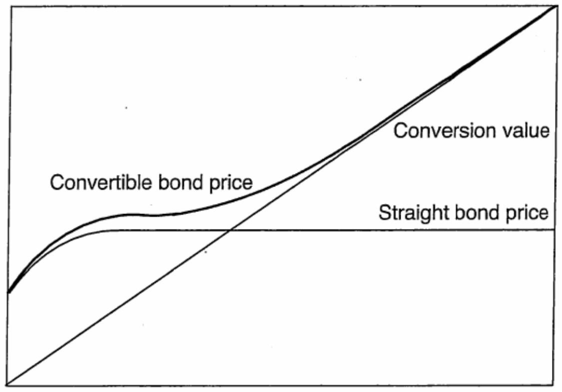

FIGURE 9.1 Convertible Bond Price and Conversion Value

Figure 9.1 describes the relationship between the value of the convertible bond and the **conversion value**, defined as the current stock price times the conversion ratio. The convertible bond value must be greater than the price of an otherwise identical straight bond and the conversion value.

For high values of the stock price, the firm is unlikely to default and the straight bond price is constant, reflecting the discounting of cash flows at the risk-free rate. In this situation, it is almost certain the option will be exercised and the convertible value is close to the conversion value. For low values of the stock price, the firm is likely to default and the straight bond price drops, reflecting the likely loss upon default. In this situation, it is almost certain the option will not be exercised, and the convertible value is close to the straight bond value. In the intermediate region, the convertible value depends on both the conversion and straight bond values. The convertible is also sensitive to interest rate risk.

Example: A Convertible Bond

Consider an 8% annual coupon, 10-year convertible bond with a face value of $1,000. The yield on similar maturity straight debt issued by the company is currently 8.50%, which gives a current value of straight debt of $967. The bond can be converted into common stock at a ratio of 10-to-1.

Assume first that the stock price is $50. The conversion value is then $500, much less than the straight debt value of $967. This corresponds to the left area of Figure 9.1. If the convertible trades at $972, its promised yield is 8.42%. This is close to the yield of straight debt, as the option has little value.

Assume next that the stock price is $150. The conversion value is then $1,500, much higher than the straight debt value of $967. This corresponds to the right area of Figure 9.1. If the convertible trades at $1,505, its promised yield is 2.29%. In this case, the conversion option is in-the-money, which explains why the yield is so low.

9.2.2 Valuation

Warrants can be valued by adapting standard option pricing models to the dilution effect of new shares. Consider a company with N outstanding shares and M outstanding warrants, each allowing the holder to purchase γ shares at the fixed price of K. At origination, the value of the firm includes the warrant, or

$$V_0 = NS_0 + MW_0 \tag{9.2}$$

where S_0 is the initial stock price just before issuing the warrant, and W_0 is the up-front value of the warrant.

After dilution, the total value of the firm includes the value of the firm before exercise (including the original value of the warrants) plus the proceeds from exercise, i.e., $V_T + M\gamma K$. The number of shares then increases to $N + \gamma M$. The total payoff to the warrant holder is

$$W_T = \gamma \text{Max}(S_T - K, 0) = \gamma(S_T - K) = \gamma \left(\frac{V_T + M\gamma K}{N + \gamma M} - K \right) \tag{9.3}$$

which must be positive. After simplification, this is also

$$W_T = \gamma \left(\frac{V_T - NK}{N + \gamma M} \right) = \frac{\gamma}{N + \gamma M}(V_T - NK) = \frac{\gamma N}{N + \gamma M} \left(\frac{V_T}{N} - K \right) \tag{9.4}$$

which is equivalent to $n = \gamma N/(N + \gamma M)$ options on the stock price. The warrant can be valued by standard option models with the asset value equal to the stock price plus the warrant proceeds, multiplied by the factor n,

$$W_0 = n \times c\left(S_0 + \frac{M}{N} W_0, K, \tau, \sigma, r, d \right) \tag{9.5}$$

with the usual parameters. Here, the unit asset value is $\frac{V_0}{N} = S_0 + \frac{M}{N} W_0$. This must be solved iteratively since W_0 appears on both sides of the equation. If, however, M is small relative to the current float, or number of outstanding shares N, the formula reduces to a simple call option in the amount γ

$$W_0 = \gamma\ c(S_0, K, \tau, \sigma, r, d) \tag{9.6}$$

Example: Pricing a Convertible Bond

Consider a zero-coupon, 10-year convertible bond with face value of \$1,000. The yield on similar maturity straight debt issued by the company is currently 8.158%, using continuous compounding, which gives a straight debt value of \$442.29.

The bond can be converted into common stock at a ratio of 10-to-1 at expiration only. This gives a strike price of $K = \$100$. The current stock price is \$60. The stock pays no dividend and has annual volatility of 30%. The risk-free rate is 5%, also continuously compounded.

Ignoring dilution effects, the Black–Scholes model gives an option value of \$216.79. So, the theoretical value for the convertible bond is $P = \$442.29 + \$216.79 = \$659.08$. If the market price is lower than \$659, the convertible is said to be cheap. This, of course, assumes that the pricing model and input assumptions are correct.

One complication is that most convertibles are also callable at the discretion of the firm. Convertible securities can be called for several reasons. First, an issue can be called to force conversion into common stock when the stock price is high enough. Bondholders have typically a month during which they can still convert, in which case this is a **forced conversion**. This call feature gives the corporation more control over conversion. It also allows the company to raise equity capital by forcing the bondholders to pay the exercise price.

Second, the call may be exercised when the option value is worthless and the firm can refinance its debt at a lower coupon. This is similar to the call of a nonconvertible bond, except that the convertible must be *busted*, which occurs when the stock price is much lower than the conversion price.

In terms of risk factors, a long position in a convertible bond is exposed to increasing interest rates and credit spreads, like regular corporate bonds, but also to factors that decrease the value of the embedded call, such as a decreasing stock price and decreasing implied volatility.

EXAMPLE 9.2: FRM EXAM 2001—QUESTION 119

A corporate bond with face value of \$100 is convertible at \$40 and the corporation has called it for redemption at \$106. The bond is currently selling at \$115 and the stock's current market price is \$45. Which of the following would a bondholder most likely do?

a. Sell the bond
b. Convert the bond into common stock
c. Allow the corporation to call the bond at 106
d. None of the above

EXAMPLE 9.3: FRM EXAM 2001—QUESTION 117

What is the main reason why convertible bonds are generally issued with a call?

a. To make their analysis less easy for investors
b. To protect against unwanted takeover bids
c. To reduce duration
d. To force conversion if in-the-money

9.3 EQUITY DERIVATIVES

Equity derivatives can be traded on over-the-counter markets as well as organized exchanges. We only consider the most popular instruments.

9.3.1 Stock Index Futures

Stock index futures are actively traded all over the world. In fact, the turnover corresponding to the notional amount is often greater than the total amount of trading in physical stocks in the same market. The success of these contracts can be explained by their versatility for risk management. Stock index futures allow investors to manage efficiently their exposure to broad stock market movements. Speculators can take easily directional bets with futures, on the upside or downside. Hedgers also find that futures provide a cost-efficient method to protect against price risk.

Perhaps the most active contract is the S&P 500 futures contract on the Chicago Mercantile Exchange (CME). The contract notional is defined as \$250 times the index level. Table 9.2 displays quotations as of December 31, 1999.

The table shows that most of the volume was concentrated in the "near" contract, that is, March in this case. Translating the trading volume in number of contracts into a dollar equivalent, we find \$250 × 1484.2 × 34,897, which gives \$13 billion. So, these markets are very liquid. As a comparison, the average daily volume was \$35 billion in 2001. This was close to the trading volume of \$42 billion for stocks on the New York Stock Exchange (NYSE).

We can also compute the daily profit on a long position, which would have been \$250 × (+3.40), or \$850 on that day. In relative terms, this daily move was +3.4/1480.8, which is only 0.23%. The typical daily standard deviation is about 1%, which gives a typical profit or loss of \$3,710.50.

These contracts are cash settled. They do not involve delivery of the underlying stocks at expiration. In terms of valuation, the futures contract is priced according to the usual cash-and-carry relationship,

$$F_t e^{-r\tau} = S_t e^{-y\tau} \tag{9.7}$$

where y is now the dividend yield defined per unit time. For instance, the yield on the S&P was $y = 0.94$ percent per annum on that day.

Here, we assume that the dividend yield is known in advance and paid on a continuous basis. In general, this is not necessarily the case but can be viewed as

TABLE 9.2 Sample S&P Futures Quotations

Maturity	Open	Settle	Change	Volume	Open Interest
March	1480.80	1484.20	+3.40	34,897	356,791
June	1498.00	1503.10	+3.60	410	8,431

a good approximation. With a large number of firms in the index, dividends will be spread reasonably evenly over the quarter.

To check if the futures contract was fairly valued, we need the spot price, $S = 1469.25$, the short-term interest rate, $r = 5.3\%$, and the number of days to maturity, which was 76 (to March 16). Note that rates are not continuously compounded. The present value factor is $\text{PV}(\$1) = 1/(1 + r\tau) = 1/(1 + 5.3\%(76/365)) = 0.9891$. Similarly, the present value of the dividend stream is $1/(1 + y\tau) = 1/(1 + 0.94\%(76/365)) = 0.9980$. The fair price is then

$$F = [S/(1 + y\tau)]\,(1 + r\tau) = [1469.25 \times 0.9980]/0.9891 = 1482.6$$

This is rather close to the settlement value of $F = 1484.2$. The discrepancy is probably because the quotes were not measured simultaneously. Because the yield is less than the interest rate, the forward price is greater than the spot price.

Figure 9.2 displays the convergence of futures and cash prices for the December 1999 S&P 500 futures contract traded on the CME. The futures price is always above the spot price. The correlation between the two prices is very high, reflecting the cash-and-carry relationship in Equation (9.7).

Because financial institutions engage in stock index arbitrage, we would expect the cash-and-carry relationship to hold very well, One notable exception was during the market crash of October 19, 1987. The market lost more than 20% in a single day. Throughout the day, however, futures prices were more up-to-date than cash prices because of execution delays in cash markets. As a result, the S&P stock index futures value was very cheap compared with the underlying cash market. Arbitrage, however, was made difficult due to chaotic market conditions.

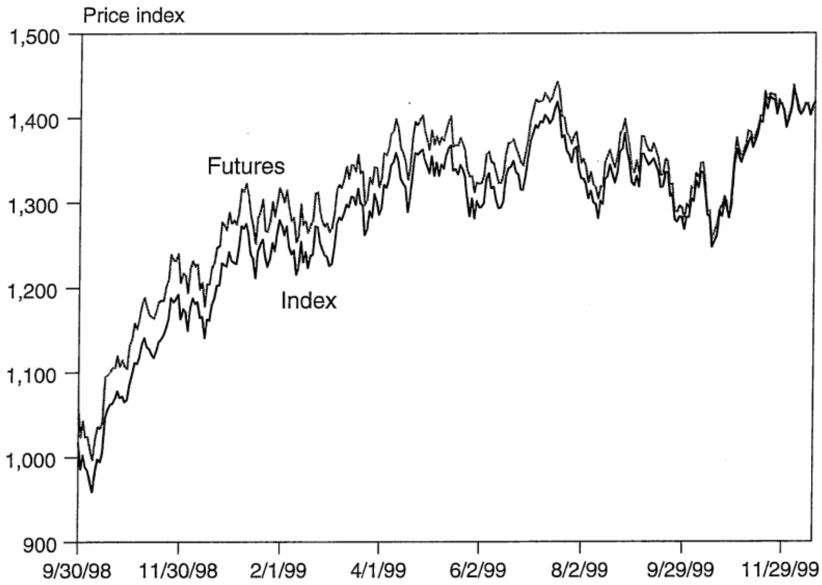

FIGURE 9.2 Futures and Cash Prices for S&P 500 Futures

EXAMPLE 9.4: FRM EXAM 2000—QUESTION 12

Suppose the price for a six-month S&P index futures contract is 552.3. If the risk-free interest rate is 7.5% per year and the dividend yield on the stock index is 4.2% per year, and the market is complete and there is no arbitrage, what is the price of the index today?

a. 543.26
b. 552.11
c. 555.78
d. 560.02

9.3.2 Single Stock Futures

In late 2000, the United States passed legislation authorizing trading in **single stock futures**, which are futures contracts on individual stocks. Such contracts were already trading in Europe and elsewhere. In the United States, electronic trading started in November 2002 and now takes place on "OneChicago," a joint venture of Chicago exchanges.

Each contract gives the obligation to buy or sell 100 shares of the underlying stock. Settlement usually involves physical delivery, that is, the exchange of the underlying stock. Relative to trading in the underlying stocks, single stock futures have many advantages. Positions can be established more efficiently due to their low margin requirements, which are generally 20% of the cash value. In contrast, margin for stocks are higher. Also, short selling eliminates the costs and inefficiencies associated with the stock loan process. Other than physical settlement, these contracts trade like stock index futures.

9.3.3 Equity Options

Options can be traded on individual stocks, on stock indices, or on stock index futures. In the United States, stock options trade, for example, on the Chicago Board Options Exchange (CBOE). Each option gives the right to buy or sell a round lot of 100 shares. Settlement involves physical delivery.

Traded options are typically American-style, so their valuation should include the possibility of early exercise. In practice, however, their values do not differ much from those of European options, which can be priced by the Black–Scholes model. When the stock pays no dividend, the values are the same. For more precision, we can use numerical models such as binomial trees to take into account dividend payments.

The most active *index* options in the United States are options on the S&P 100 and S&P 500 index traded on the CBOE. The former are American-style, while the latter are European-style. These options are cash settled, as it would be too complicated to deliver a basket of 100 or 500 underlying stocks. Each contract is for \$100 times the value of the index. European options on stock indices can be priced using the Black–Scholes formula, using y as the dividend yield on the index as we have done in the previous section for stock index futures.

Finally, options on S&P 500 stock index futures are also popular. These give the right to enter a long or short futures position at a fixed price. Exercise is cash settled.

9.3.4 Equity Swaps

An **equity swap** is an agreement to exchange cash flows tied to the return on a stock market index in exchange for a fixed or floating rate of interest. An example is a swap that provides the return on the S&P 500 index every six months in exchange for payment of LIBOR plus a spread. The swap will be typically priced so as to have zero value at initiation. Equity swaps can be valued as portfolios of forward contracts, as in the case of interest rate swaps. We will later see how to price currency swaps. The same method can be used for equity swaps.

These swaps are used by investment managers to acquire exposure to, for example, an emerging stock market, without having to invest in the market itself. In some cases, these swaps can also be used to skirt restrictions on foreign investments.

9.3.5 Variance Swaps

A **variance swap** is a forward contract on the variance. The payoff is computed as

$$V_T = (\sigma^2_{t_0,T} - K_V)N \tag{9.8}$$

where N is the notional amount, σ^2 is the realized variance over the life of the contract, usually measured as

$$\sigma^2 = \frac{252}{\tau}\sum_{i=1}^{\tau}[\ln(S_i/S_{i-1})]^2 \tag{9.9}$$

and K_V is the strike price, or forward price. Variance swaps can be written on any underlying asset, but are most common for equities or equity indices. They allow trades based on direct views on variance. Long positions are bets on high volatility.

For example, suppose a dealer quotes a one-year contract on the S&P 500 index, with $K_V = (15\%)^2$ and notional of $N = \$100{,}000/(\text{one volatility point})^2$. If at expiration the realized volatility is 17%, the payoff to the long position is $[\$100{,}000/(1^2)][(17)^2 - (15)^2] = \$100{,}000(289 - 225) = \$6{,}400{,}000$. Therefore, the payoff is a quadratic function of the volatility. In theory, it is unlimited.[1] Like any forward contract, K_V is determined so that the initial value of the contract is zero. In fact, the widely quoted **VIX index** is the fair strike price for a variance swap on the S&P 500 index, quoted as volatility.

The market value of an outstanding variance swap with $\tau = T - t$ days remaining to maturity is

$$V_t = Ne^{-r\tau}[w(\sigma^2_{t_0,t} - K_V) + (1-w)(K_t - K_V)] \tag{9.10}$$

where $\sigma^2_{t_0,t}$ is the elapsed variance between the initial time t_0 and the current time, t, w is the fraction of days elapsed since t_0, and K_t is the current forward price. We can verify that at the initial time, $w = 0$, and V_0 is simply proportional to $K_t - K_V$, which is zero if the contract starts at-the-market. At expiration, this converges to Equation (9.8).

Such contracts also allow **correlation trading**. Consider for example an index of two stocks. A variance swap is available for each constituent stock as well as for the index. The realized variance of the index depends on the two variances as well as the correlation coefficient. All else equal, a higher correlation translates into a higher portfolio variance. A **long correlation** trade would buy a variance swap on the index and short variance swaps on the components.[2] If the correlation increases, the long position should gain more than the short positions, thereby generating a gain.

9.4 CURRENCY MARKETS

9.4.1 Overview

The **forex**, or currency markets have enormous trading activity, with daily turnover estimated at \$3,210 billion in 2007. Their size and growth is described in Table 9.3. This trading activity dwarfs that of bond or stock markets. In comparison, the daily trading volume on the New York Stock Exchange (NYSE) is approximately \$80 billion. Even though the largest share of these transactions is between dealers, or with other financial institutions, the volume of trading with other, nonfinancial institutions is still quite large, at \$549 billion daily.

Spot transactions are exchanges of two currencies for settlement as soon as it is practical, typically in two business days. They account for about 35% of trading volume. Other transactions are outright forward contracts and forex swaps. **Outright forward contracts** are agreements to exchange two currencies at a future date, and account for about 12% of the total market. **Forex swaps** involve two transactions, an exchange of currencies on a given date and a reversal at a later date, and account for 53% of the total market. Note that forex swaps are typically

[1] In practice, most contracts are capped to a maximum value for the variance equal to $m^2 K_V$. Volatility swaps are also available but are much less common. This is because variance swaps can be hedged relatively easily, using a combination of options. This is not the case for volatility swaps.

[2] Note that keeping the position variance-neutral requires a greater notional amount for the index swaps than for the sum of component swaps.

TABLE 9.3 Average Daily Trading Volume in Currency Markets (Billions of U.S. Dollars)

Year	Spot	Forwards & Forex Swaps	Total
1989	350	240	590
1992	416	404	820
1995	517	673	1,190
1998	592	898	1,490
2001	399	811	1,210
2004	656	1,224	1,880
2007	1,005	2,076	3,210
Of which, between:			
Dealers			1,374
Financials			1,287
Others			549

Source: Bank for International Settlements surveys.

of a short-term nature and should not be confused with long-term currency swaps, which involve a stream of payments over longer horizons.

In addition to these contracts, the market also includes OTC forex options ($212 billion daily) and exchange-traded derivatives ($72 billion daily). The most active currency futures are traded on the Chicago Mercantile Exchange (CME) and settled by physical delivery. The CME also trades options on currency futures.

As we have seen before, currency forwards, futures, and options can be priced according to standard valuation models, specifying the income payment to be a continuous flow defined by the foreign interest rate, r^*.

Currencies are generally quoted in **European terms**, that is, in units of the foreign currency per dollar. The yen, for example, is quoted as 120 yen per U.S. dollar. Two notable exceptions are the British pound (sterling) and the euro, which are quoted in **American terms**, that is, in dollars per unit of the foreign currency, The pound, for example, is quoted as 1.6 dollar per pound.

9.4.2 Currency Products

Thus, currency markets offer the full range of financial instruments. Because of their importance, currency swaps will be examined in more detail in the following section.

One type of instrument, however, is specific to the currency markets. This is the **quanto**, or quantity-adjusted derivative. The payoff is defined by variables associated with one currency but is paid in another currency.

As an example, suppose a U.S. investor considers buying Japanese stocks, represented by the Nikkei 225 index. Normally, buying the stocks involves taking a position in stocks and in the Japanese yen currency. We could have a situation where the Nikkei increases in value but this gain is wiped out by a fall in the value of the yen. To avoid this currency risk, the investor could buy a quanto forward contract on the index with expiry in one year. The contract has a notional in dollars, N^{USD} and a forward price $F^{\mathcal{Q}}$ in yen. The payout at expiration depends on the yen value of the index, S_T, and is given by

$$V_T = N^{USD}(S_T - F^{\mathcal{Q}}) \tag{9.11}$$

The forward price on the quanto depends on the usual forward price in the foreign currency but also the covariance between movements in the yen stock price and in the dollar price of the yen

$$F^{\mathcal{Q}} = F \exp(\rho\sigma_{ST}\sigma_{FX}\,\tau) \tag{9.12}$$

As an example, suppose we consider a one-year contract. The Nikkei index is at $S_0 = 10{,}000$, the dividend yield is at 2%, and the interest rate at 1%. The usual forward price in yen is $F = S\exp(-[y-r]\tau) = 10{,}000\exp(-[0.01 - 0.02] \times 1) = 10{,}100$. Next, we estimate that the volatility of the Nikkei is $\sigma_{ST} = 20\%$, the volatility of the yen is $\sigma_{FX} = 10\%$, and their correlation 0.3. Based on this information, the quanto forward price should be $F^{\mathcal{Q}} = 10{,}100\exp(0.3 \times 0.2 \times 0.1 \times 1) = 10{,}161$.

Now assume that after one year, the Nikkei has gone up to 11,000 but that the yen has depreciated from \$(1/100) to \$(1/120). Normally, this fall would have wiped out the gain on the index. An initial investment of $V_0 = 10{,}000/100 = \$100$ would have turned into $V_1 = (11{,}000 + 2\% \times 10{,}000)/120 = \$93.33$, which is a loss of \$6.67.

If instead the investor had entered a quanto forward with notional amount of \$0.01, the payoff on the contract would be $V_T = N^{USD}(S_T - F^{\mathcal{Q}}) = 0.01(11{,}000 - 10{,}161) = \$8.39$. Thus, the quanto has succeeded in extracting the stock price appreciation only.

An example of such contracts is the dollar-denominated Nikkei futures contract traded on the CME, where the payoff is given by \$5 times the change in the Nikkei futures. The CME also has traditional yen-denominated Nikkei futures contracts, where the value is 500 yen times the Nikkei futures. Unlike the quanto, profits on such contracts are affected by currency swings. Effectively, the quanto has an embedded currency forward contract with a variable notional amount.

EXAMPLE 9.5: FRM EXAM 2003—QUESTION 2

The current spot CHF/USD rate is 1.3680CHF. The three-month USD interest rate is 1.05%, the three-month Swiss interest rate is 0.35%, both continuously compounded and per annum. A currency trader notices that the three-month forward price is USD 0.7350. In order to arbitrage, the trader should

a. Borrow CHF, buy USD spot, go long Swiss franc forward
b. Borrow CHF, sell Swiss franc spot, go short Swiss franc forward
c. Borrow USD, buy Swiss franc spot, go short Swiss franc forward
d. Borrow USD, sell USD spot, go long Swiss franc forward

9.5 CURRENCY SWAPS

Currency swaps are agreements by two parties to exchange a stream of cash flows in different currencies according to a prearranged formula.

9.5.1 Instruments

Consider two counterparties, company A and company B that can raise funds either in dollars or in yen, $100 million or Y10 billion at the current rate of 100Y/$, over 10 years. Company A wants to raise dollars, and company B wants to raise yen. Table 9.4a displays borrowing costs. This example is similar to that of interest rate swaps, except that rates are now in different currencies.

Company A has an **absolute advantage** in the two markets as it can raise funds at rates systematically lower than company B. Company B, however, has a **comparative advantage** in raising dollars as the cost is only 0.50% higher than for company A, compared to the cost difference of 1.50% in yen. Conversely, company A must have a comparative advantage in raising yen.

This provides the basis for a swap which will be to the mutual advantage of both parties. If both institutions directly issue funds in their final desired market, the total cost will be 9.50% (for A) and 6.50% (for B), for a total of 16.00%. In contrast, the total cost of raising capital where each has a comparative advantage is 5.00% (for A) and 10.00% (for B), for a total of 15.00%. The gain to both parties from entering a swap is 16.00 − 15.00 = 1.00%. For instance, the swap described in Tables 9.4b and 9.4c splits the benefit equally between the two parties.

Company A issues yen debt at 5.00%, then enters a swap whereby it promises to pay 9.00% in dollars in exchange for receiving 5.00% yen payments. Its effective funding cost is therefore 9.00%, which is less than the direct cost by 50bp.

TABLE 9.4a Cost of Capital Comparison

Company	Yen	Dollar
A	5.00%	9.50%
B	6.50%	10.00%

TABLE 9.4b Swap to Company A

Operation	Yen	Dollar
Issue debt	Pay yen 5.00%	
Enter swap	Receive yen 5.00%	Pay dollar 9.00%
Net		Pay dollar 9.00%
Direct cost		Pay dollar 9.50%
Savings		0.50%

TABLE 9.4c Swap to Company B

Operation	Dollar	Yen
Issue debt	Pay dollar 10.00%	
Enter swap	Receive dollar 9.00%	Pay yen 5.00%
Net		Pay yen 6.00%
Direct cost		Pay yen 6.50%
Savings		0.50%

Similarly, company B issues dollar debt at 10.00%, then enters a swap whereby it receives 9.00% in dollars in exchange for paying 5.00% yen. If we add up the difference in dollar funding cost of 1.00% to the 5.00% yen funding costs, the effective funding cost is therefore 6.00%, which is less than the direct cost by 50bp.[3] Both parties benefit from the swap.

While payments are typically netted for an interest rate swap, because they are in the same currency, this is not the case for currency swaps. Full interest payments are made in different currencies. In addition, at initiation and termination, there is exchange of principal in different currencies. For instance, assuming annual payments, company A will receive 5.0% on a notional of Y10b, which is Y500 million in exchange for paying 9.0% on a notional of \$100 million, or \$9 million every year.

9.5.2 Pricing

Consider now the pricing of the swap to company A. This involves receiving 5.00% yen in exchange for paying 9.00% dollars. As with interest rate swaps, we can price the swap using either of two approaches, taking the difference between two bond prices or valuing a sequence of forward contracts.

This swap is equivalent to a long position in a fixed-rate, a 5% 10-year yen denominated bond and a short position in a 10-year 9% dollar denominated bond. The value of the swap is that of a long yen bond minus a dollar bond. Defining S as the dollar price of the yen and P and P^* as the dollar and yen bond, we have:

$$V = S(\$/Y)P^*(Y) - P(\$) \tag{9.13}$$

Here, we indicate the value of the yen bond by an asterisk, P^*.

[3] Note that B is somewhat exposed to currency risk, as funding costs cannot be simply added when they are denominated in different currencies. The error, however, is of a second-order magnitude.

In general, the bond value can be written as $P(c, r, F)$ where the coupon is c, the yield is r and the face value is F. Our swap is initially worth (in millions)

$$V = \frac{1}{100}P(5\%, 5\%, Y10{,}000) - P(9\%, 9\%, \$100)$$
$$= \frac{\$1}{Y100}Y10{,}000 - \$100 = \$0$$

Thus, the initial value of the swap is zero, assuming a flat term structure for both countries and no credit risk.

We can identify three conditions under which the swap will be in-the-money. This will happen if the value of the yen S appreciates, or if the yen interest rate r^* falls, or if the dollar interest rate r goes up.

Thus the swap is exposed to three risk factors, the spot rate, and two interest rates. The latter exposures are given by the duration of the equivalent bond.

KEY CONCEPT

A position in a receive-foreign currency swap is equivalent to a long position in a foreign currency bond offset by a short position in a dollar bond.

The swap can be alternatively valued as a sequence of forward contracts. Recall that the valuation of a forward contract on one yen is given by

$$V_i = (F_i - K)\exp(-r_i \tau_i) \tag{9.14}$$

using continuous compounding. Here, r_i is the dollar interest rate, F_i is the prevailing forward rate (in \$/yen), and K is the locked-in rate of exchange defined as the ratio of the dollar to yen payment on this maturity. Extending this to multiple maturities, the swap is valued as

$$V = \sum_i n_i(F_i - K)\exp(-r_i \tau_i) \tag{9.15}$$

where $n_i F_i$ is the dollar value of the yen payments translated at the forward rate and the other term $n_i K$ is the dollar payment in exchange.

Table 9.5 compares the two approaches for a three-year swap with annual payments. Market rates have now changed and $r = 8\%$ for U.S. yields and $r^* = 4\%$ for yen yields. We assume annual compounding. The spot exchange rate has moved from 100Y/\$ to 95Y/\$, reflecting a depreciation of the dollar (or appreciation of the yen).

TABLE 9.5 Pricing a Currency Swap

	Specifications		Market Data
	Notional Amount (millions)	Contract Rates	Market Rates
Dollar	\$100	9%	8%
Yen	Y10,000	5%	4%
Exchange rate		100Y/\$	95Y/\$

Valuation Using Bond Approach (millions)

	Dollar Bond			Yen Bond		
Time (year)	Dollar Payment	PV(\$1)	PV(CF)	Yen Payment	PV(Y1)	PV(CF)
1	9	0.9259	8.333	500	0.9615	480.769
2	9	0.8573	7.716	500	0.9246	462.278
3	109	0.7938	86.528	10,500	0.8890	9,334.462
Total			\$102.58			Y10,277.51
Swap (\$)			−\$102.58			\$108.18
Value						\$5.61

Valuation Using Forward Contract Approach (millions)

Time (year)	Forward Rate (Y/\$)	Yen Receipt (Y)	Yen Receipt (Y)	Dollar Payment (\$)	Difference CF (\$)	PV(CF) (\$)
1	91.48	500	5.47	−9.00	−3.534	−3.273
2	88.09	500	5.68	−9.00	−3.324	−2.850
3	84.83	10,500	123.78	−109.00	14.776	11.730
Value						\$5.61

The middle panel shows the valuation using the difference between the two bonds. First, we discount the cash flows in each currency at the newly prevailing yield. This gives $P = \$102.58$ for the dollar bond and Y10,277.51 for the yen bond. Translating the latter at the new spot rate of Y95, we get \$108.18. The swap is now valued at \$108.18 − \$102.58, which is a positive value of $V = \$5.61$ million. The appreciation of the swap is principally driven by the appreciation of the yen.

The bottom panel shows how the swap can be valued by a sequence of forward contracts. First, we compute the forward rates for the three maturities. For example, the one-year rate is $95 \times (1 + 4\%)/(1 + 8\%) = 91.48$ Y/\$, by interest rate parity. Next, we convert each yen receipt into dollars at the forward rate, for example Y500 million in one year, which is \$5.47 million. This is offset against a payment of \$9 million, for a net planned cash outflow of −\$3.53 million. Discounting and adding up the planned cash flows, we get $V = \$5.61$ million, which must be exactly equal to the value found using the alternative approach.

EXAMPLE 9.6: FRM EXAM 2004—QUESTION 54

Which of the following statements is correct when comparing the differences between an interest rate swap and a currency swap?

a. At maturity, there is no exchange of principal between the counterparties in interest rate swaps and there is an exchange of principal in currency swaps.
b. At maturity, there is no exchange of principal between the counterparties in currency swaps and there is an exchange of principle in interest rate swaps.
c. The counterparties in a interest rate swap need to consider fluctuations in exchange rates, while currency swap counterparties are only exposed to fluctuations in interest rates.
d. Currency swap counterparties are exposed to less counterparty credit risk due to the offsetting effect of currency and interest rate risk in the transaction.

EXAMPLE 9.7: FRM EXAM 2006—QUESTION 88

You have entered into a currency swap in which you receive 4%pa in yen and pay 6%pa in dollars once a year. The principals are 1,000 million yen and 10 million dollars. The swap will last for another two years, and the current exchange rate is 115 yen/$. The annualized spot rates (with continuous compounding) are 2.00% and 2.50% in yen for one- and two-year maturities, and 4.50% and 4.75% in dollars. What is the value of the swap to you in million dollars?

a. −1.270
b. −0.447
c. 0.447
d. 1.270

EXAMPLE 9.8: FRM EXAM 2007—QUESTION 87

Your company is expecting a major export order from a London-based client. The receivables under the contract are to be billed in GBP, while your reporting currency is USD. Since the order is a large sum, your company does not want to bear the exchange risk and wishes to hedge it using derivatives. To minimize the cost of hedging, which of the following is the most suitable contract?

a. A chooser option for GBP/USD pair
b. A currency swap where you pay fixed in USD and receive floating in GBP
c. A barrier put option to sell GBP against USD
d. An Asian call option on GBP against USD

9.6 COMMODITIES

9.6.1 Products

Commodities are typically traded on exchanges. Contracts include spot, futures, and options on futures. There is also an OTC market for long-term commodity swaps, where payments are tied to the price of a commodity against a fixed or floating rate.

Commodity contracts can be classified into:

- **Agricultural products**, including grains and oilseeds (corn, wheat, soybean) food and fiber (cocoa, coffee, sugar, orange juice)
- **Livestock and meat** (cattle, hogs)
- **Base metals** (aluminum, copper, nickel, and zinc)
- **Precious metals** (gold, silver, platinum)
- **Energy products** (natural gas, heating oil, unleaded gasoline, crude oil)

The **Goldman Sachs Commodity Index** (GSCI) is a broad production-weighted index of commodity price performance, which is composed of 24 liquid exchange-traded futures contracts as of 2008. The index contains 72% energy products, 7% industrial metals, 2% precious metals, 14% agricultural products, and 5% livestock products. The CME trades futures and options contracts on the GSCI.

In the last few years, active markets have developed for **electricity products**, electricity futures for delivery at specific locations, for instance California/Oregon border (COB), Palo Verde, and so on. These markets have mushroomed following the deregulation of electricity prices, which has led to more variability in electricity prices.

More recently, OTC markets and exchanges have introduced **weather derivatives**, where the payout is indexed to temperature or precipitation. On the CME, for instance, contract payouts are based on the Degree Day Index over a calendar month. This index measures the extent to which the daily temperature deviates from the average. These contracts allow users to hedge situations where their income is negatively affected by extreme weather. Markets are also evolving in newer products, such as indices of consumer bankruptcy and catastrophe insurance contracts.

Such commodity markets allow participants to exchange risks. Farmers, for instance, can sell their crops at a fixed price on a future date, insuring themselves against variations in crop prices. Likewise, consumers can buy these crops at a fixed price.

9.6.2 Pricing of Futures

Commodities differ from financial assets in two notable dimensions: they may be expensive, even impossible, to store and they may generate a flow of benefits that are not directly measurable.

The first dimension involves the cost of carrying a physical inventory of commodities. For most financial instruments, this cost is negligible. For bulky commodities, this cost may be high. Other commodities, like electricity cannot be stored easily.

The second dimension involves the benefit from holding the physical commodity. For instance, a company that manufactures copper pipes benefits from an inventory of copper which is used up in its production process. This flow is also called **convenience yield** for the holder. For a financial asset, this flow would be a monetary income payment for the investor. When an asset such as gold can be lent out for a profit, the yield represents the **lease rate**, which is the return to lending gold short-term.

Consider the first factor, storage cost only. The cash-and-carry relationship should be modified as follows. We compare two positions. In the first, we buy the commodity spot plus pay up-front the present value of storage costs PV(C). In the second, we enter a forward contract and invest the present value of the forward price. Since the two positions are identical at expiration, they must have the same initial value:

$$F_t e^{-r\tau} = S_t + \mathrm{PV}(C) \tag{9.16}$$

where $e^{-r\tau}$ is the present value factor. Alternatively, if storage costs are incurred per unit time and defined as c, we can restate this relationship as

$$F_t e^{-r\tau} = S_t e^{c\tau} \tag{9.17}$$

Due to these costs, the forward rate should be much greater than the spot rate, as the holder of a forward contract benefits not only from the time value of money but also from avoiding storage costs.

Example: Computing the Forward Price of Gold

Let us use data from December 1999. The spot price of gold is $S = \$288$, the one-year interest rate is $r = 5.73\%$ (continuously compounded), and storage costs are \$2 per ounce per year, paid up front. The fair price for a one-year forward contract should be $F = [S + \mathrm{PV}(C)]e^{r\tau} = [\$288 + \$2]e^{5.73\%} = \$307.1$.

Let us now turn to the convenience yield, which can be expressed as y per unit time. In fact, y represents the net benefit from holding the commodity, after storage costs. Following the same reasoning as before, the forward price on a commodity should be given by

$$F_t e^{-r\tau} = S_t e^{-y\tau} \tag{9.18}$$

where $e^{-y\tau}$ is an actualization factor. This factor may have an economically identifiable meaning, reflecting demand and supply conditions in the cash and futures markets. Alternatively, it can be viewed as a *plug-in* that, given F, S, and $e^{-r\tau}$, will make Equation (9.18) balance.

Figure 9.3, for example, displays the shape of the term structure of spot and futures prices for the New York Mercantile Exchange (NYMEX) crude oil contract. On December 1997, the term structure is relatively flat. On December 1998, the term structure becomes strongly upward-sloping. Part of this slope can be explained by the time value of money (the term $e^{-r\tau}$ in the equation). In contrast, the term structure is downward-sloping on December 1999. This can be interpreted in terms of a large convenience yield from holding the physical asset (in other words, the term $e^{-y\tau}$ in the equation dominates).

Let us focus for example on the one-year contract. Using $S = \$25.60$, $F = \$20.47$, $r = 5.73\%$ and solving for y,

$$y = r - \frac{1}{\tau}\ln(F/S) \tag{9.19}$$

we find $y = 28.10\%$, which is quite large. In fact, variations in y can be substantial. Just one year before, a similar calculation would have given $y = -9\%$, which implies a negative convenience yield, or a storage cost.

Table 9.6 displays futures prices for selected contracts. Futures prices are generally increasing with maturity, reflecting the time value of money, storage cost, and low convenience yields. There are some irregularities, however, reflecting anticipated imbalances between demand and supply. For instance, gasoline futures prices increase in the summer due to increased automobile driving. Natural gas displays the opposite pattern, where prices increase during the winter due to the demand for heating. Agricultural products can also be highly seasonal. In contrast, futures prices for gold are going up monotonically with time, since this is a perfectly storable good.

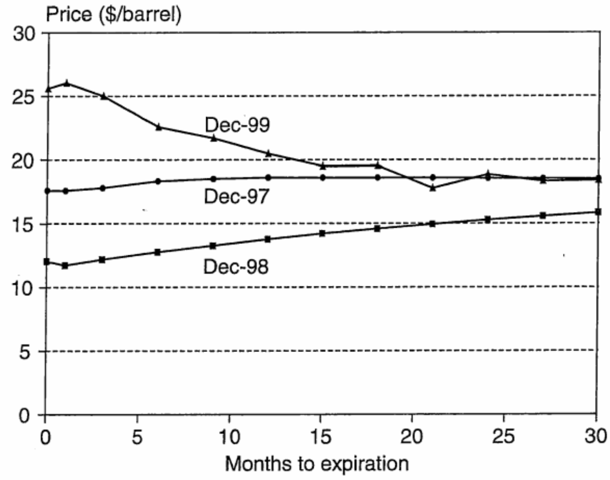

FIGURE 9.3 Spot and Futures Prices for Crude Oil

TABLE 9.6 Futures Prices as of December 30, 1999

Maturity	Corn	Sugar	Copper	Gold	Nat.Gas	Gasoline
Jan			85.25	288.5		0.6910
Mar	204.5	18.24	86.30	290.6	2.328	0.6750
July	218.0	19.00	87.10	294.9	2.377	0.6675
Sept	224.0	19.85	87.90	297.0	2.418	0.6245
Dec	233.8	18.91	88.45	300.1	2.689	
Mar 01	241.5	18.90	88.75	303.2	2.494	
...						
Dec 01	253.5			312.9	2.688	

9.6.3 Futures and Expected Spot Prices

An interesting issue is whether today's futures price gives the best forecast of the future spot price. If so, it satisfies the **expectations hypothesis**, which can be written as:

$$F_t = E_t[S_T] \tag{9.20}$$

The reason this relationship may hold is as follows. Say that the one-year oil futures price is $F = \$20.47$. If the market forecasts that oil prices in one year will be at \$25, one could make a profit by going long a futures contract at the cheap futures price of $F = \$20.47$, waiting a year, then buying oil at \$20.47, and reselling it at the higher price of \$25. In other words, deviations from this relationship imply **speculative profits**.

To be sure, these profits are not risk-free. Hence, they may represent some compensation for risk. For instance, if the market is dominated by producers who want to hedge by selling oil futures, F will be abnormally low compared with expectations. Thus the relationship between futures prices and expected spot prices can be complex.

For financial assets for which the arbitrage between cash and futures is easy, the futures or forward rate is solely determined by the cash-and-carry relationship, i.e., the interest rate and income on the asset. For commodities, however, the arbitrage may not be so easy. As a result, the futures price may deviate from the cash-and-carry relationship through this convenience yield factor. Such prices may reflect expectations of futures spot prices, as well as speculative and hedging pressures.

A market is said to be in **contango** when the futures price trades at a premium relative to the spot price, as shown in Figure 9.4. Using Equation (9.19), this implies that the convenience yield is smaller than the interest rate $y < r$.

Normally, the size of the premium should be limited by arbitrage opportunities. If this became too large, traders could buy the commodity spot, put it in storage, and simultaneously sell it for future delivery at the higher forward price. In December 2008, however, the premium for one-year oil contracts reached an

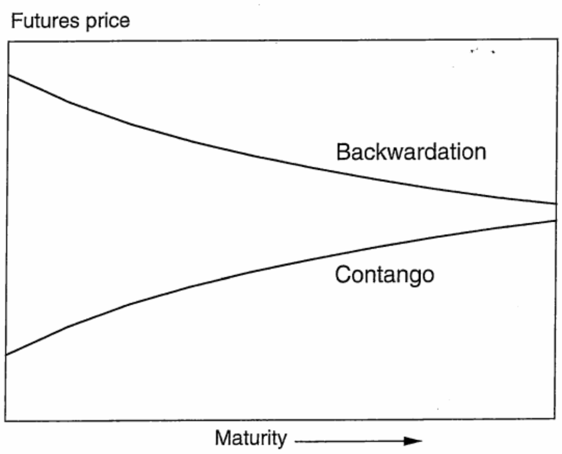

FIGURE 9.4 Patterns of Contango and Backwardation

all-time high of $13 per barrel. This was explained by the credit crunch, which prevented oil traders to secure loans to finance oil storage.

A market is said to be in **backwardation** (or inverted) when forward prices trade at a discount relative to spot prices. This implies that the convenience yield is greater than the interest rate $y > r$. In other words, a high convenience yields puts a higher price on the cash market, as there is great demand for immediate consumption of the commodity.

With backwardation, the futures price tends to increase as the contract nears maturity. In such a situation, a **roll-over strategy** should be profitable, provided that prices do not move too much. This involves buying a long maturity contract, waiting, and then selling it at a higher price in exchange for buying a cheaper, longer-term contract.

This strategy is comparable to **riding the yield curve** when upward-sloping. This involves buying long maturities and waiting to have yields fall due to the passage of time. If the shape of the yield curve does not change too much, this will generate a capital gain from bond price appreciation. Because of the negative price-yield relationship, a positively sloped yield curve is equivalent to backwardation in bond prices.

This was basically the strategy followed by Metallgesellschaft Refining & Marketing (MGRM), the U.S. subsidiary of **Metallgesellschaft,** which had made large sales of long-term oil to clients on the OTC market. These were hedged by rolling over long positions in West Texas Intermediate (WTI) crude oil futures. This made money as long as the market was in backwardation. When the market turned to contango, however, the long positions started to lose money as they got closer to maturity. In addition, the positions were so large that they moved markets against MG. These losses caused cash-flow, or liquidity problems. MGRM ended up liquidating the positions, which led to a realized loss of $1.3 billion.

A similar problem afflicted **Amaranth,** a hedge fund that lost $6.6 billion as a result of bad bets against natural gas futures. In September 2006, the price of natural gas fell sharply. In addition, the spread between prices in winter and summer months collapsed. As the size of the positions were huge, this led to large losses that worsened when the fund attempted to liquidate the contracts.

KEY CONCEPT

Markets are in contango if spot prices are lower than forward prices. Markets are in backwardation if spot prices are higher than forward prices. Backwardation occurs when there is high current demand for the commodity, which implies high convenience yields.

EXAMPLE 9.9: FRM EXAM 2007—QUESTION 29

On January 1, a risk manager observes that the one-year continuously compounded interest rate is 5% and storage costs of a commodity product A is USD 0.05 per quarter (payable at each quarter end). He further observes the following forward prices for product A: March, 5.35; June, 5.90; September, 5.30; December, 5.22. Given the following explanation of supply and demand for this product, how would you best describe its forward price curve from June to December?

a. Backwardation as the supply of product A is expected to decline after summer
b. Contango as the supply of product A is expected to decline after summer
c. Contango as there is excess demand for product A in early summer
d. Backwardation as there is excess demand for product A in early summer

EXAMPLE 9.10: FRM EXAM 2007—QUESTION 30

Continuing with the previous question, what is the annualized rate of return earned on a cash-and-carry trade entered into in March and closed out in June?

a. 9.8%
b. 8.9%
c. 39.1%
d. 35.7%

EXAMPLE 9.11: FRM EXAM 2004—QUESTION 5

Which of the following causes led MGRM into severe financial distress?

I. There was a mismatch of cash flows from hedge and physical transactions.
II. MGRM failed to consider hedging market risk from fixed price physical sales contracts.
III. MGRM held a great percentage of the total open interest on the NYMEX.
IV. The futures market went from backwardation to contango.

a. I and III
b. I and IV
c. I, III, and IV
d. II, III and IV

9.7 IMPORTANT FORMULAS

Gordon-growth model for valuation of stocks: $P = \frac{D}{r-g}$

Warrant valuation: $W_0 = n \times c(S_0 + \frac{M}{N} W_0, K, \tau, \sigma, r, d)$

Stock index futures: $F_t e^{-r\tau} = S_t e^{-y\tau}$

Payoff on a variance swap: $V_T = (\sigma^2 - K_V)N$

Valuation of an outstanding variance swap:

$$V_t = Ne^{-r\tau}[w(\sigma^2_{t_0,t} - K_V) + (1 - w)(K_t - K_V)]$$

Pricing a currency swap as two bond positions: $V = S(\$/Y)P^*(Y) - P(\$)$

Pricing a currency swap as a sequence of forwards:

$$V = \sum_i n_i(F_i - K)\exp(-r_i\tau_i)$$

Pricing of commodity futures with storage costs:

$$F_t e^{-r\tau} = S_t + \mathrm{PV}(C), \text{ or } F_t e^{-r\tau} = S_t e^{c\tau}$$

Expectations hypothesis: $F_t = E_t[S_T]$

9.8 ANSWERS TO CHAPTER EXAMPLES

Example 9.1: Leverage and Return on Equity

c. The fund borrows \$200 million and invests \$300 million, which creates a yield of \$300 $\times$ 14% = \$42 million. Borrowing costs are \$200 $\times$ 8% = \$16 million, for a difference of \$26 million on equity of \$100 million, or 26%. Note that this is a yield, not expected rate of return if we expect some losses from default. This higher yield also implies higher risk.

Example 9.2: FRM Exam 2001—Question 119

a. The conversion rate is expressed here in terms of the conversion price. The conversion rate for this bond is \$100 into \$40, or one bond into 2.5 shares. Immediate conversion will yield $2.5 \times \$45 = \$112.5$. The call price is \$106. Since the market price is higher than the call price and the conversion value, and the bond is being called, the best value is achieved by selling the bond.

Example 9.3: FRM Exam 2001—Question 117

d. Companies issue convertible bonds because the coupon is lower than for regular bonds. In addition, these bonds are callable in order to force conversion into the stock at a favorable ratio. In the previous question, for instance, conversion would provide equity capital to the firm at the price of \$40, while the market price is at \$45.

Example 9.4: FRM Exam 2000—Question 12

a. This is the cash-and-carry relationship, solved for S. We have $Se^{-y\tau} = Fe^{-r\tau}$, or $S = 552.3 \times \exp(-7.5/200)/\exp(-4.2/200) = 543.26$. We verify that the forward price is greater than the spot price since the dividend yield is less than the risk-free rate.

Example 9.5: FRM Exam 2003—Question 2

c. For consistency, translate the spot rate in dollars, $S = 0.7310$. The CHF interest rate is lower than the USD rate, so the CHF must be selling at a forward premium. The fair forward price is $F = S\exp((r - r^*)\tau) = 0.7310\exp((0.0105 - 0.0035)\,0.25) = 0.7323$. Because this is less than the observed price of 0.7350, we sell at the expensive forward price and borrow USD, buy CHF spot, invest in CHF. At maturity, we liquidate the CHF investment to satisfy the forward sale into dollars, repay the loan, and make a tidy profit.

Example 9.6: FRM Exam 2004—Question 54

a. Because payments on currency swaps are in different currencies, they cannot be netted.

Example 9.7: FRM Exam 2006—Question 88

a. The net present values of the payoffs in two currencies are described in the next table. As a result, the value of the currency swap is given by the dollar value of a long position in the yen bond minus a position in the dollar bond, or $(1/115)\,1{,}000(102.85/100) - 10(102.13/100) = \$8.943 - \$10.213 = -\$1.270$.

	Yen			USD		
T	Rate	CF	NPV	Rate	CF	NPV
1	2.00%	4	3.92	4.50%	6	5.74
2	2.50%	104	98.93	4.75%	106	96.39
Sum			102.85			102.13

Example 9.8: FRM Exam 2007—Question 87

c. A cross-currency swap is inappropriate because there is no stream of payment but just one. Also, one would want to pay GBP, not receive it. An Asian option is generally cheap, but this should be a put option, not a call. Among the two remaining choices, the chooser option is more expensive because it involves a call and put.

Example 9.9: FRM Exam 2007—Question 29

d. From June to December, prices go down, which is backwardation. June prices are abnormally high because of excess demand, which pushes prices up.

Example 9.10: FRM Exam 2007—Question 30

d. The trade involves now going long a March contract and short a June contract. In practice, this means taking delivery of the commodity and holding it for three months until resale in June. The final payout is $5.90 - 0.05$ on a base of 5.35. This gives an annualized rate of return of $r = 4\ln(5.85/5.35) = 35.7\%$.

Example 9.11: FRM Exam 2003—Question 5

c. MGRM did consider hedging its OTC contracts with futures but was hit with liquidity risk as the long futures positions lost money due to the move into contango. In addition, the positions were very large, which led to losses on the unwinding of the hedges.

PART Three

Market Risk Management

CHAPTER 10

Introduction to Market Risk

This chapter provides an introduction to market risk. Market risk is primarily measured with **position-based risk measures** such as **value at risk** (VAR). VAR is a statistical measure of *total* portfolio risk, based on the most current positions, which takes into account portfolio diversification and leverage.

In theory, risk managers should consider the entire distribution of profits and losses over the specified horizon. In practice, this distribution is summarized by one number, the worst loss at a specified confidence level, such as 99%. VAR, however, is only one of the measures that risk managers focus on. It should be complemented by **stress-testing**, which identifies potential losses under extreme market conditions.

Section 10.1 gives a brief overview of financial market risks and the history of risk measurement systems. Section 10.2 then introduces measures of downside risk. It shows how to compute VAR for a very simple portfolio. It also discusses caveats, or pitfalls to be aware of when interpreting VAR numbers. Section 10.3 extends VAR methods to cash flow at risk. Section 10.4 turns to the choice of VAR parameters, that is, the confidence level and horizon. Next, Section 10.5 describes the broad components of a VAR system. Finally, Section 10.6 shows how to complement VAR by stress tests.

10.1 INTRODUCTION TO FINANCIAL MARKET RISKS

10.1.1 Types of Financial Risks

Financial risks include market risk, credit risk, and operational risk. **Market risk** is the risk of losses due to movements in financial market prices or volatilities. This usually includes **liquidity risk**, which is the risk of losses due to the need to liquidate positions to meet funding requirements. Liquidity risk, unfortunately, is not amenable to formal quantification. Because of its importance, it will be covered in Chapter 25. **Credit risk** is the risk of losses due to the fact that counterparties may be unwilling or unable to fulfill their contractual obligations. **Operational risk** is the risk of loss resulting from failed or inadequate internal processes, systems, and people, or from external events. Oftentimes, however, these three categories interact with each other, so that any classification is, to some extent, arbitrary.

For example, credit risk can interact with other types of risks. At the most basic level, it involves the risk of default on the asset, such as a loan or bond. When the asset is traded, however, market risk also reflects credit risk. Take a corporate bond, for example. Some of the price movement may be due to movements in risk-free interest rates, which is pure market risk. The remainder will reflect the market's changing perception of the likelihood of default. Thus, for traded assets, there is no clear-cut delineation of market and credit risk. Some arbitrary classification must take place. Furthermore, operational risk is often involved as well.

Consider a simple transaction whereby a trader purchases 1 million worth of a British pound (BP) spot from Bank A. The current rate is \$1.5/BP, for settlement in two business days. So, our bank will have to deliver \$1.5 million in two days in exchange for receiving BP 1 million. This simple transaction involves a series of risks.

- *Market risk:* During the day, the spot rate could change. Say that after a few hours the rate moves to \$1.4/BP. The trader cuts the position and enters a spot sale with another bank, Bank B. The million pounds is now worth only \$1.4 million, for a loss of \$100,000 to be realized in two days. The loss is the change in the market value of the investment.
- *Credit risk:* The next day, Bank B goes bankrupt. The trader must now enter a new, replacement trade with Bank C. If the spot rate has dropped further from \$1.4/BP to \$1.35/BP, the gain of \$50,000 on the spot sale with Bank B is now at risk. The loss is the change in the market value of the investment, if positive. Thus there is interaction between market and credit risk.
- *Settlement risk:* The next day, our bank wires the \$1.5 million to Bank A in the morning, which defaults at noon and does not deliver the promised BP 1 million. This is also known as **Herstatt risk** because this German bank defaulted on such obligations in 1974, potentially destabilizing the whole financial system. The loss is now potentially the whole principal in dollars.
- *Operational risk:* Suppose that our bank wired the \$1.5 million to a wrong bank, Bank D. After two days, our back office gets the money back, which is then wired to Bank A plus compensatory interest. The loss is the interest on the amount due.

10.1.2 Risk Management Tools

In the past, risks were measured using a variety of ad hoc tools, none of which was satisfactory. These included **notional amounts**, **sensitivity measures**, and **scenarios**. While these measures provide some intuition of risk, they do not measure what matters, that is, the potential for downside loss for the total portfolio. They fail to take into account differences in volatilities across markets, correlations across risk factors, as well as the probability of adverse moves in the risk factors.

Consider for instance a five-year **inverse floater**, which pays a coupon equal to 16 percent minus twice current LIBOR, if positive, on a notional principal of \$100 million. The initial market value of the note is \$100 million. This investment is extremely sensitive to movements in interest rates. If rates go up, the present value of the cash flows will drop sharply. In addition, the discount rate also increases. The combination of a decrease in the numerator terms and an increase in the denominator terms will push the price down sharply.

The question is, how much could an investor lose on this investment over a specified horizon? The *notional amount* only provides an indication of the potential loss. The worst case scenario is one where interest rates rise above 8%. In this situation, the coupon will drop to $16 - 2 \times 8 =$ zero. The bond becomes a zero-coupon bond, whose value is \$68 million, discounted at 8%. This gives a loss of \$100 − \$68 = \$32 million. While sizable, this is still less than the notional.

A *sensitivity measure* such as duration is more helpful. As we have seen in Chapter 7, the bond has three times the duration of a similar five-year note. Assume the latter is 4.5 years. This gives a modified duration of $D = 3 \times 4.5 = 13.5$ years. This duration measure reveals the extreme sensitivity of the bond to interest rates but does not answer the question of whether such a disastrous movement in interest rates is likely. It also ignores the nonlinearity between the note price and yields.

Scenario analysis provides some improvement, as it allows the investor to investigate nonlinear, extreme effects in price. But again, the method does not associate the loss with a probability.

Another general problem is that these sensitivity or scenario measures do not allow the investor to aggregate risk across different markets. Let us say that this investor also holds a position in a bond denominated in another currency, the euro. Do the risks add up, or diversify each other?

The great beauty of VAR is that it provides a neat answer to all these questions. One number aggregates the risks across the whole portfolio, taking into account leverage and diversification, and providing a risk measure with an associated probability.

If the worst increase in yield at the 95% level is 1.65%, we can compute VAR as

$$\text{VAR} = \text{Market value} \times \text{Modified Duration} \times \text{Worst yield increase} \qquad (10.1)$$

In this case, VAR = \$100 × 13.5 × 0.0165 = \$22 million. The investor can now make a statement such as: The worst loss at the 95% confidence level is approximately \$22 million. With appropriate caveats, this is a huge improvement over traditional risk measurement methods, as it expresses risk in an intuitive fashion. Once the VAR apparatus is in place, it is easy to implement stress-tests, which provide information about extreme losses that is complementary to VAR.

Measures such as notional amounts and exposures have been, and are still used to set limits, in an attempt to control risk before it occurs, or "ex ante." These measures should be supplemented by VAR, which is an ex ante measure of the potential dollar loss. Other risk management tools include **stop losses**, which are rules enforcing position cuts after losses occur, that is "ex post." While stop losses are useful, especially in trending markets, they only provide partial protection because they are applied *after* a loss.

The VAR revolution started in 1993 when it was endorsed by the Group of Thirty (G-30) as part of "best practices" for dealing with derivatives. The methodology behind VAR, however, is not new. It results from a merging of finance theory, which focuses on the pricing and sensitivity of financial instruments, and statistics, which studies the behavior of the risk factors. The idea behind VAR, or total portfolio risk, can be traced to the pioneering work of Markowitz in 1952.

EXAMPLE 10.1: FRM EXAM 2005—QUESTION 32

Which of the following statements about trader limits are *correct*?

I. Stop-loss limits are useful if markets are trending.
II. Exposure limits do not allow for diversification.
III. VAR limits are not susceptible to arbitrage.
IV. Stop-loss limits are effective in preventing losses.

a. I and II
b. III and IV
c. I and III
d. II and IV

10.2 DOWNSIDE RISK MEASURES

10.2.1 VAR: Definition

Value at Risk VAR is a summary measure of downside risk expressed in dollars, or in the reference currency. A general definition is

> *VAR is the maximum loss over a target horizon such that there is a low, prespecified probability that the actual loss will be larger.*

Consider for instance a position of $4 billion short the yen, long the dollar. This position corresponds to a well-known hedge fund that took a bet that the yen would fall in value against the dollar. How much could this position lose over a day?

To answer this question, we could use 10 years of historical daily data on the yen/dollar rate and simulate a daily return. The simulated daily return in dollars is then

$$R_t(\$) = Q_0(\$)[S_t - S_{t-1}]/S_{t-1} \tag{10.2}$$

where Q_0 is the current dollar value of the position and S is the spot rate in yen per dollar measured over two consecutive days.

For instance, for two hypothetical days $S_1 = 112.0$ and $S_2 = 111.8$. The simulated return is

$$R_2(\$) = \$4{,}000 \text{ million} \times [111.8 - 112.0]/112.0 = -\$7.2 \text{ million}$$

Repeating this operation over the whole sample, or 2,527 trading days, creates a time-series of fictitious returns, which is plotted in Figure 10.1.

We can now construct a frequency distribution of daily returns. For instance, there are four losses below \$160 million, three losses between \$160 million and \$120 million, and so on. The histogram, or frequency distribution, is graphed in Figure 10.2. We can also order the losses from worst to best return.

Define x as the dollar profit or loss. VAR is typically reported as a positive number, even if it is a loss. It is defined implicitly by

$$c = \int_{-\mathrm{VAR}}^{\infty} f(x)dx \tag{10.3}$$

When the outcomes are discrete, VAR is the smallest loss such that the right-tail probability is at least c.

Sometimes, VAR is reported as the deviation between the mean and the quantile. This second definition is more consistent than the usual one. Because it considers the deviation between two values on the target date, it takes into account the time value of money. In most applications, however, the time horizon is very

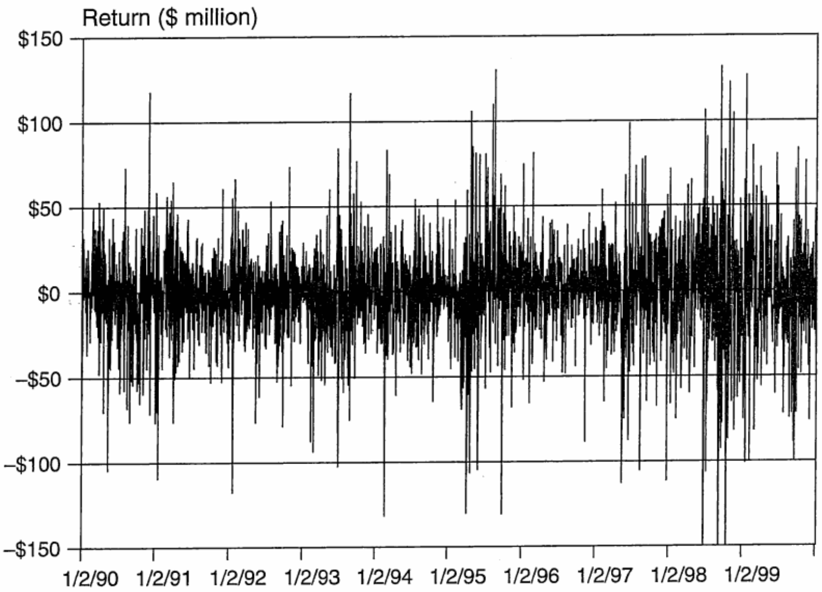

FIGURE 10.1 Simulated Daily Returns

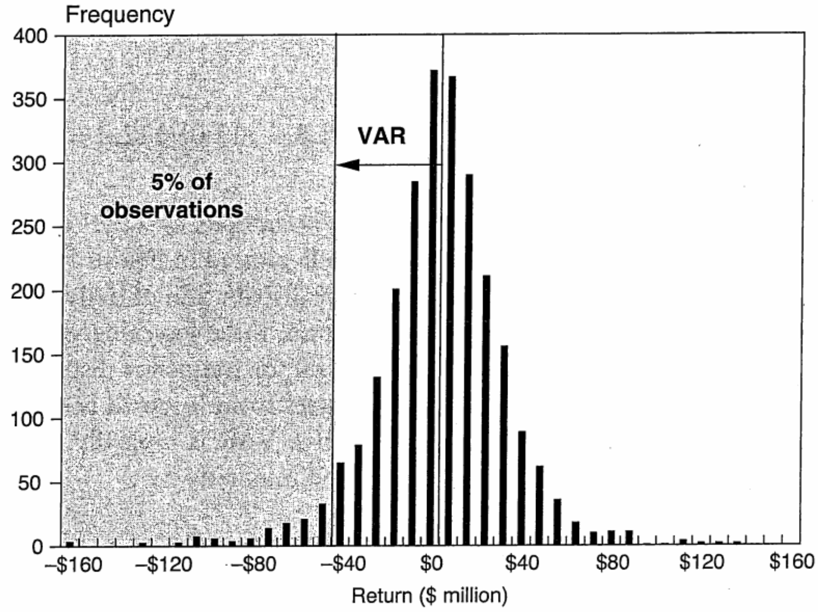

FIGURE 10.2 Distribution of Daily Return

short in which case the average return on financial series is close to zero. As a result, the two definitions usually give similar values.

In this hedge fund example, we want to find the cutoff value R^* such that the probability of a loss worse than $-R^*$ is $p = 1 - c = 5\%$. With a total of $T = 2{,}527$ observations, this corresponds to a total of $pT = 0.05 \times 2{,}527 = 126$ observations in the left tail. We pick from the ordered distribution the cutoff value, which is $R^* = \$47.1$ million. We can now make a statement such as: The maximum loss over one day is about \$47 million at the 95% confidence level. This describes risk in a way that notional amounts or exposures cannot convey.

We now wish to summarize the distribution by one number. We could describe the quantile, that is, the level of loss that will not be exceeded at some high **confidence level**. Select for instance this confidence level as $c = 95\%$. This corresponds to a **right-tail probability**. We could as well define VAR in terms of a **left-tail probability**, which we write as $p = 1 - c$.

It is essential to verify the quality of VAR forecasts by checking whether the number of losses worse than VAR, also called **exceptions**, is in line with expectations. From the confidence level, we can determine the number of expected exceedences n over a period of N days:

$$n = p \times N \tag{10.4}$$

Say that we want to backtest VAR measured at the 99% confidence level over the past 250 days. The expected number of exceptions is $n = 0.01 \times 250 = 2.5$.

Under the null hypothesis that the model is correctly calibrated, or that the true probability of exceptions is p, the distribution of number of exceptions x follows a binomial distribution

$$f(x) = \binom{N}{x} p^x (1-p)^{N-x}, \quad x = 0, 1, \ldots, n \tag{10.5}$$

For example, the probability of having exactly zero exceptions is $f(x=0) = (0.99)^{250} = 0.081$. The probability of having five exceptions is $f(x=5) = \binom{250}{5} 0.01^5 (0.99)^{245} = 0.0666$. As shown in Chapter 2, this distribution can be used to compute a cutoff value for the number of exceptions, beyond which we would have to conclude that the model is flawed. Overall, there is only a probability of 10.8% of observing five or more exceptions if the VAR model were correctly specified.

Finally, we should note that exception tests only focus on the frequency of occurrences of exceptions. They do not take into account the size of losses.

EXAMPLE 10.2: FRM EXAM 2005—QUESTION 43

The 10-Q report of ABC Bank states that the monthly VAR of ABC Bank is USD 10 million at 95% confidence level. What is the proper interpretation of this statement?

a. If we collect 100 monthly gain/loss data of ABC Bank, we will always see five months with losses larger than \$10m.
b. There is a 95% probability that the bank will lose less than \$10m over a month.
c. There is a 5% probability that the bank will gain less than \$10m each month.
d. There is a 5% probability that the bank will lose less than \$10m over a month.

EXAMPLE 10.3: FRM EXAM 2003—QUESTION 11

Based on a 90% confidence level, how many exceptions in backtesting a VAR would be expected over a 250-day trading year?

a. 10
b. 15
c. 25
d. 50

EXAMPLE 10.4: FRM EXAM 2007—QUESTION 101

A large, international bank has a trading book whose size depends on the opportunities perceived by its traders. The market risk manager estimates the one-day VAR, at the 95% confidence level, to be USD 50 million. You are asked to evaluate how good of a job the manager is doing in estimating the one-day VAR. Which of the following would be the most convincing evidence that the manager is doing a poor job, assuming that losses are identically independently distributed?

a. Over the last 250 days, there are eight exceedences.
b. Over the last 250 days, the largest loss is USD 500 million.
c. Over the last 250 days, the mean loss is USD 60 million.
d. Over the last 250 days, there is no exceedence.

10.2.2 VAR: Caveats

VAR is a useful summary measure of risk but is subject to caveats:

- *VAR does not describe the worst loss.* This is not what VAR is designed to measure. Indeed we would expect the VAR number to be exceeded with a frequency of p, that is, five days out of a hundred for a 95% confidence level. This is perfectly normal. In fact, backtesting procedures are designed to check whether the frequency of exceedences is in line with p.
- *VAR does not describe the losses in the left tail.* VAR does not say anything about the distribution of losses in its left tail. It just indicates the probability of such a value occurring. For the same VAR number, however, we can have very different distribution shapes. In the case of Figure 10.2, the average value of the losses worse than \$47 million is around \$74 million, which is 60% worse than the VAR. So, it would be unusual to sustain many losses beyond \$200 million.

  Other distributions are possible, however, while maintaining the same VAR. Figure 10.3 illustrates a distribution with 125 occurrences of large losses of \$160 million. Because there is still one observation left at \$47 million, the VAR is unchanged at \$47 million. Yet this distribution implies a high probability of sustaining very large losses, unlike the original one.

  This can create other strange results. For instance, one can construct examples, albeit stretched, where the VAR of a portfolio is greater than the sum of the VARs for its components. As a result, VAR fails to qualify as a *subadditive* risk measure, which is one of the desirable properties listed in the appendix. Subadditivity implies that the risk of a portfolio must be less than the sum of risks for portfolio components.
- *VAR is measured with some error.* The VAR number itself is subject to normal sampling variation. In our example, we used 10 years of daily data. Another

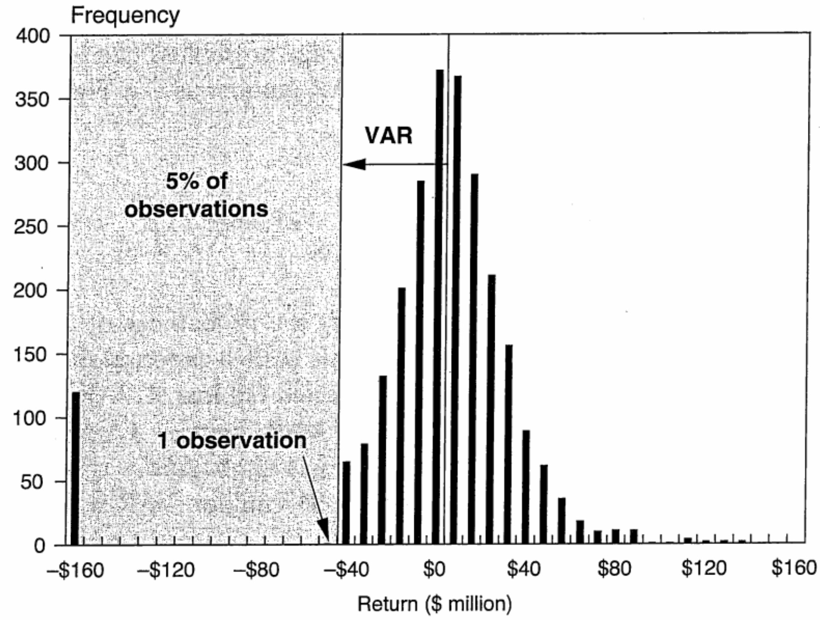

FIGURE 10.3 Altered Distribution with Same VAR

sample period, or a period of different length, will lead to a different VAR number. Different statistical methodologies or simplifications can also lead to different VAR numbers. One can experiment with sample periods and methodologies to get a sense of the precision in VAR. Hence, it is useful to remember that there is limited precision in VAR numbers. What matters is the first-order magnitude.

In addition, VAR measures are subject to the same problems that affect all risk measures based on a "window" of recent historical data. Ideally, the past window should reflect the range of future outcomes. If not, all risk measures based on recent historical data may be misleading.

10.2.3 Alternative Measures of Risk

The conventional VAR measure is the *quantile* of the distribution measured in dollars. This single number is a convenient summary, but its very simplicity may be dangerous. We have seen in Figure 10.3 that the same VAR can hide very different distribution patterns. The appendix reviews desirable properties for risk measures and shows that VAR may display some undesirable properties under some conditions. In particular, the VAR of a portfolio can be greater than the sum of subportfolios VARs. If so, merging portfolios can increase risk, which is an unexpected result. Alternative measures of risk are described below.

The Entire Distribution In our example, VAR is simply one quantile in the distribution. The risk manager, however, has access to the whole distribution and could report a range of VAR numbers for increasing confidence levels.

The Conditional VAR A related concept is the expected value of the loss when it exceeds VAR. This measures the average of the loss conditional on the fact that it is greater than VAR. Define the VAR number as $-q$. Formally, the **conditional VAR** (CVAR) is the negative of

$$E[X \mid X < q] = \int_{-\infty}^{q} xf(x)dx \Big/ \int_{-\infty}^{q} f(x)dx \tag{10.6}$$

Note that the denominator represents the probability of a loss exceeding VAR, which is also $p = 1 - c$. This ratio is also called **expected shortfall, tail conditional expectation, conditional loss,** or **expected tail loss.** CVAR indicates the potential loss if the portfolio is "hit" beyond VAR. Because CVAR is an average of the tail loss, one can show that it qualifies as a *subadditive* risk measure. For our yen position, the average loss beyond the $47 million VAR is CVAR = $74 million.

The Standard Deviation A simple summary measure of the distribution is the usual standard deviation (SD)

$$\mathrm{SD}(X) = \sqrt{\frac{1}{(N-1)} \sum_{i=1}^{N} [x_i - E(X)]^2} \tag{10.7}$$

The advantage of this measure is that it takes into account all observations, not just the few around the quantile. Any large negative value, for example, will affect the computation of the variance, increasing SD(X). If we are willing to take a stand on the shape of the distribution, say normal or Student's t, we do know that the standard deviation is the most efficient measure of dispersion. For example, for our yen position, this value is SD = $29.7 million.

Furthermore, we can translate this standard deviation into a VAR measure, using a multiplier $\alpha(c)$ that depends on the distribution and the selected confidence level c

$$\mathrm{VAR} = \alpha\sigma W \tag{10.8}$$

where σ is the volatility of the rate of return, which is unitless, and W is the amount invested, measured in the reference currency. Here SD $= \sigma W$.

With a normal distribution and $c = 95\%$, we have $\alpha = 1.645$. This gives a VAR estimate of $1.645 \times 29.7 =$ $49 million, which is not far from the empirical quantile of $47 million.

Note that Equation (10.8) measures VAR relative to the mean, because the standard deviation is a measure of dispersion around the mean. If it is important to measure the loss relative to the initial value, VAR is then

$$\mathrm{VAR} = (\alpha\sigma - \mu)W \tag{10.9}$$

where μ is the expected rate of return over the horizon. In this case, the mean is very small, at $-\$0.4$ million, which leads to very close VAR measures.

Under these conditions, VAR inherits all properties of the standard deviation. In particular, the SD of a portfolio must be smaller than the sum of the SDs of subportfolios, so it is subadditive.

The disadvantage of the standard deviation is that it is symmetrical and cannot distinguish between large losses or gains. Also, computing VAR from SD requires a distributional assumption, which may not be valid.

The Semistandard Deviation This is a simple extension of the usual standard deviation that considers only data points that represent a loss. Define N_L as the number of such points. The measure is

$$\mathrm{SD}_L(X) = \sqrt{\frac{1}{(N_L)} \sum_{i=1}^{N} [\mathrm{Min}(x_i, 0)]^2} \tag{10.10}$$

The advantage of this measure is that it accounts for asymmetries in the distribution, e.g., negative skewness which is especially dangerous. The semistandard deviation is sometimes used to report downside risk, but is much less intuitive and less popular than VAR.

The Drawdown Drawdown is the decline from peak over a fixed time interval. Define x^{MAX} as the local maximum over this period $[0, T]$, which occurs at time $t_{MAX} \in [0, T]$. Relative to this value, the drawdown at time t is

$$\mathrm{DD}(X) = \frac{(x^{MAX} - x_t)}{x^{MAX}} \tag{10.11}$$

The maximum drawdown is the largest such value over the period, or decline from peak to trough (local maximum to local minimum).

This measure is useful if returns are not independent from period to period. When a market trends, for example, the cumulative loss over a longer period is greater than the loss extrapolated from a shorter period. Alternatively, drawdowns are useful measures of risk if the portfolio is actively managed. A portfolio insurance program, for example, should have lower drawdowns relative to a fixed position in the risky asset because it cuts the position as losses accumulate.

The disadvantage of this measure is that it is backward-looking. It cannot be constructed from the current position, as in the case of VAR. In addition, the maximum drawdown corresponds to different time intervals, i.e., $t_{MAX} - t_{MIN}$. As a result, maximum drawdown measures are not directly comparable across portfolios, in contrast with VAR or the standard deviation, which are defined over a fixed horizon or in annual terms.

EXAMPLE 10.5: FRM EXAM 2003—QUESTION 5

Given the following 30 ordered percentage returns of an asset, calculate the VAR and expected shortfall at a 90% confidence level: $-16, -14, -10, -7, -7, -5, -4, -4, -4, -3, -1, -1, 0, 0, 0, 1, 2, 2, 4, 6, 7, 8, 9, 11, 12, 12, 14, 18, 21, 23$.

a. VAR (90%) = 10, expected shortfall = 14
b. VAR (90%) = 10, expected shortfall = 15
c. VAR (90%) = 14, expected shortfall = 15
d. VAR (90%) = 18, expected shortfall = 22

10.3 CASH FLOW AT RISK

VAR methods have been developed to measure the mark-to-market risk of commercial bank portfolios. By now, these methods have spread to other financial institutions (e.g., investment banks, savings and loans), and the investment management industry.

In each case, the objective function is the market value of the portfolio, assuming fixed positions. VAR methods, however, are now also spreading to other sectors (e.g., nonfinancial corporations), where the emphasis is on periodic earnings. **Cash flow at risk** (CFAR) measures the worst shortfall in cash flows due to unfavorable movements in market risk factors. This involves quantities, Q, unit revenues, P, and unit costs, C. Simplifying, we can write

$$\mathrm{CF} = Q \times (P - C) \tag{10.12}$$

Suppose we focus on the exchange rate, S, as the market risk factor. Each of these variables can be affected by S. Revenues and costs can be denominated in the foreign currency, partially or wholly. Quantities can also be affected by the exchange rate through foreign competition effects. Because quantities are random, this creates **quantity uncertainty**. The risk manager needs to model the relationship between quantities and risk factors. Once this is done, simulations can be used to project the cash-flow distribution and identify the worst loss at some confidence level. If this is unacceptably high, the company should hedge its exposure. The role of the risk manager is then to evaluate the effect of different instruments (e.g., forwards, options) and help the company decide on a risk management program.

Over the long-term, the firm may have **strategic options**. For example, the firm may decide to withdraw from a foreign market if it becomes unprofitable due to a depreciation of the country's currency. Alternatively, the firm may decide to open up an assembly operation in the foreign country to take advantage of lower local costs. Such strategic options can be incorporated in the risk analysis.

A classic example is the value of a farmer's harvest, say corn. At the beginning of the year, costs are fixed. The price of corn and the size of the harvest in the fall, however, are unknown. Suppose price movements are primarily driven by supply shocks, such as the weather. If there is a drought during the summer, quantities will fall and prices will increase, conversely if there is an exceptionally abundant harvest. Because of the negative correlation between Q and P, total revenues will fluctuate less than if quantities were fixed. Such relationships need to be factored into the risk measurement system because they will affect the hedging program.

EXAMPLE 10.6: FRM EXAM 2007—QUESTION 112

The Chief Risk Officer (CRO) of an exporting firm is attempting to estimate the firm's one-year cash flow at risk. Which of the following issues describes an approach that is *irrelevant* to the task of the CRO?

a. Because cash flow at risk is generally estimated over a quarter or over a year, it is necessary to forecast the future values of risk factors.
b. To the extent that the firm's income from exports is best approximated by a real option because the firm does not have to export when the price of the foreign currency is unexpectedly low, the CRO can use option analysis and does not have to worry about forecasting exchange rates.
c. A parametric approach can be used if exposures to foreign exchange risk factors are linear, if there are no other risk factors, and if exchange rate changes are normally distributed.
d. Using a Monte Carlo approach will help the CRO if the firm's foreign currency income is a nonlinear function of exchange rates.

10.4 VAR PARAMETERS

To measure VAR, we first need to define two quantitative parameters, the confidence level and the horizon.

10.4.1 Confidence Level

The higher the confidence level c, the greater the VAR measure. Varying the confidence level provides useful information about the return distribution and potential extreme losses. It is not clear, however, whether one should stop at 99%, 99.9%, 99.99%, and so on. Each of these values will create an increasingly larger loss, but less likely.

Another problem is that as c increases, the number of occurrences below VAR shrinks, leading to poor measures of high quantiles. With 1,000 observations, for example, VAR can be taken as the tenth lowest observation for a 99% confidence level. If the confidence level increases to 99.9%, VAR is taken from the lowest observation only. Finally, there is no simple way to estimate a 99.99% VAR from this sample because it has too few observations.

The choice of the confidence level depends on the use of VAR. For most applications, VAR is simply a benchmark measure of downside risk. If so, what really matters is *consistency* of the VAR confidence level across trading desks or time.

In contrast, if the VAR number is being used to decide how much capital to set aside to avoid bankruptcy, then a high confidence level is advisable. Obviously, institutions would prefer to go bankrupt very infrequently. This **capital adequacy** use, however, applies to the overall institution and not to trading desks.

Another important point is that VAR models are only useful insofar as they can be verified. This is the purpose of backtesting, which systematically checks whether the frequency of losses exceeding VAR is in line with $p = 1 - c$. For this purpose, the risk manager should choose a value of c that is not too high. Picking, for instance, $c = 99.99\%$ should lead, on average, to one exceedence out of 10,000 trading days, or 40 years. In other words, it is going to be impossible to verify if the true probability associated with VAR is indeed 99.99%. For all these reasons, the usual recommendation is to pick a confidence level that is not too high, such as 95% to 99%.

10.4.2 Horizon

The longer the horizon T, the greater the VAR measure. This extrapolation is driven by two factors, the behavior of the risk factors, and the portfolio positions.

To extrapolate from a one-day horizon to a longer horizon, we need to assume that returns are independently and identically distributed. If so, the daily volatility can be transformed into to a multiple-day volatility by multiplication by the square root of time. We also need to assume that the distribution of daily returns is unchanged for longer horizons, which restricts the class of distribution to the so-called "stable" family, of which the normal is a member. If so, we have

$$\text{VAR}(T \text{ days}) = \text{VAR}(1 \text{ day}) \times \sqrt{T} \tag{10.13}$$

This requires (1) the distribution to be invariant to the horizon (i.e., the same α, as for the normal), (2) the distribution to be the same for various horizons (i.e., no time decay in variances), and (3) innovations to be independent across days.

KEY CONCEPT

VAR can be extended from a one-day horizon to T days by multiplication by the square root of time. This adjustment is valid with independent and identically distributed (i.i.d.) returns that have a normal distribution.

The choice of the horizon also depends on the characteristics of the portfolio. If the positions change quickly, or if exposures (e.g., option deltas) change as underlying prices change, increasing the horizon will create "slippage" in the VAR measure.

Again, the choice of the horizon depends on the use of VAR. If the purpose is to provide an accurate benchmark measure of downside risk, the horizon should be relatively short, ideally less than the average period for major portfolio rebalancing.

In contrast, if the VAR number is being used to decide how much capital to set aside to avoid bankruptcy, then a long horizon is advisable. Institutions will want to have enough time for corrective action as problems start to develop.

In practice, the horizon cannot be less than the frequency of reporting of profits and losses. Typically, banks measure P&L on a daily basis, and corporates on a longer interval (ranging from daily to monthly). This interval is the minimum horizon for VAR.

Another criterion relates to the backtesting issue. Shorter time intervals create more data points matching the forecast VAR with the actual, subsequent P&L. As the power of the statistical tests increases with the number of observations, it is advisable to have a horizon as short as possible.

For all these reasons, the usual recommendation is to pick a horizon that is as short as feasible, for instance one day for trading desks. The horizon needs to be appropriate to the asset classes and the purpose of risk management. For institutions such as pension funds, for instance, a one-month horizon may be more appropriate.

For **capital adequacy purposes**, institutions should select a high confidence level and a long horizon. There is a trade-off, however, between these two parameters. Increasing one *or* the other will increase VAR.

EXAMPLE 10.7: FRM EXAM 2003—QUESTION 19

The VAR on a portfolio using a 1-day horizon is USD 100 million. The VAR using a 10-day horizon is

a. USD 316 million if returns are not independently and identically distributed
b. USD 316 million if returns are independently and identically distributed
c. USD 100 million since VAR does not depend on any day horizon
d. USD 31.6 million irrespective of any other factors

10.4.3 Application: The Basel Rules

The Basel Committee on Banking Supervision has laid out minimum capital requirements for commercial banks to cover market risk. This **Market Risk Charge** (MRC) is based on VAR models computed with the following parameters:

- A horizon of 10 trading days, or two calendar weeks
- A 99% confidence interval
- An observation period based on at least a year of historical data and updated at least once a quarter

Under the **Internal Models Approach** (IMA), the MRC is computed as the sum of a **general market risk charge** (GMRC) plus a **specific market risk charge** (SRC):

$$\mathrm{MRC}_t^{IMA} = \mathrm{Max}\left(k\frac{1}{60}\sum_{i=1}^{60}\mathrm{VAR}_{t-i},\ \mathrm{VAR}_{t-1}\right) + \mathrm{SRC}_t \tag{10.14}$$

The GMRC involves the average of the trading VAR over the last 60 days, times a supervisor-determined multiplier k (with a minimum value of 3), as well as yesterday's VAR, The Basel Committee allows the 10-day VAR to be obtained from an extrapolation of one-day VAR figures. Thus, VAR is really

$$\mathrm{VAR}_t(10, 99\%) = \sqrt{10} \times \mathrm{VAR}_t(1, 99\%)$$

Presumably, the 10-day period corresponds to the time required for corrective action by bank regulators, should an institution start to run into trouble. Presumably as well, the 99% confidence level corresponds to a low probability of bank failure due to market risk. Even so, one occurrence every 100 periods implies a high frequency of failure. There are $52/2 = 26$ two-week periods in one year. Thus, one failure should be expected to happen every $100/26 = 3.8$ years, which is still much too frequent. This explains why the Basel Committee has applied a multiplier factor, $k \geq 3$ to guarantee further safety. In addition, this factor is supposed to protect against fat tails, unstable parameters, changing positions, and more generally, model risk.

The **specific risk charge** is designed to provide a buffer against losses due to idiosyncractic factors related to the individual issuer of the security. It includes the risk that an individual debt or equity moves by more or less than the general market factors, reflecting basis risk or event risk such as a downgrade or default. Chapter 30 gives more details on the market risk framework, which is being updated following the credit crisis that started in 2007.

EXAMPLE 10.8: MARKET RISK CHARGE

The 95%, one-day RiskMetrics VAR for a bank trading portfolio is $1,000,000. What is the approximate general market risk charge?

a. $3,000,000
b. $9,500,000
c. $4,200,000
d. $13,400,000

10.5 COMPONENTS OF RISK MEASUREMENT SYSTEMS

As described in Figure 10.4, a risk measurement system combines the following steps:

1. From market data, construct the distribution of risk factors (e.g., normal, empirical, or other).
2. Collect the portfolio positions and map them onto the risk factors.
3. Construct the portfolio VAR using one of the three methods (delta-normal, historical, Monte Carlo), which will be explained in Chapter 15.

10.5.1 Portfolio Positions

We start with portfolio positions. The assumption is that all positions are constant over the horizon. This, of course, cannot be true in an environment where traders turn over their portfolio actively. Rather, it is a simplification.

The true risk can be greater or lower than the VAR measure. It can be greater if VAR is based on close-to-close positions that reflect lower trader limits. If traders take more risks during the day, the true risk will be greater than indicated by VAR. Conversely, the true risk can be lower if management enforces loss limits, in other words, cuts down the risk that traders can take if losses develop.

10.5.2 Risk Factors

The **risk factors** represent a subset of all market variables that adequately span the risks of the current, or allowed, portfolio. There are literally tens of thousands of securities available, but a much more restricted set of useful risk factors.

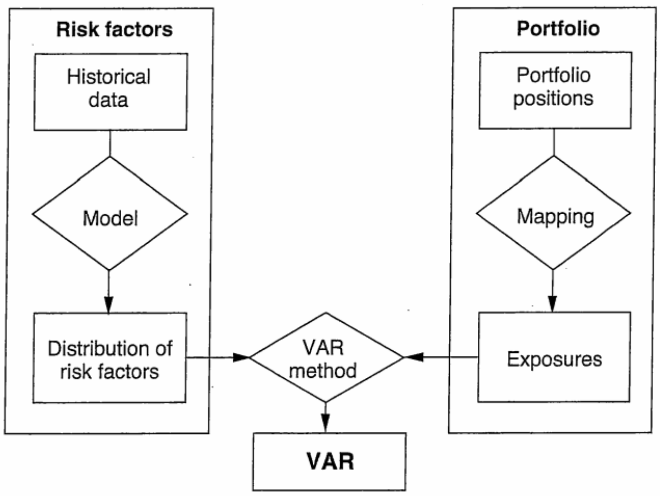

FIGURE 10.4 Elements of a VAR System

The key is to choose market factors that are adequate for the portfolio. For a simple fixed-income portfolio, one bond market risk factor may be enough. In contrast, for a highly leveraged portfolio, multiple risk factors are needed. For an option portfolio, volatilities should be added as risk factors. In general, the more complex the strategies, the greater the number of risk factors that should be used.

10.5.3 VAR Methods

Similarly, the choice of the VAR method depends on the nature of the portfolio. A simple method may be sufficient for simple portfolios. For a fixed-income portfolio, a linear method may be adequate. In contrast, if the portfolio contains options, we need to include nonlinear effects. For simple, plain-vanilla options, we may be able to approximate their price behavior with a first and second derivative (delta and gamma). For more complex options, such as digital or barrier options, this may not be sufficient.

This is why risk management is as much an art as it is a science. Risk managers need to make reasonable approximations to come up with a cost-efficient measure of risk. They also need to be aware of the fact that traders could be induced to find "holes" in the risk management system.

Once this risk measurement system is in place, it can also be used to perform stress-tests. The risk manager can easily submit the current portfolio to various scenarios, which are simply predefined movements in the risk factors. Therefore such systems are essential to evaluate the risks of the current portfolio.

EXAMPLE 10.9: POSITION-BASED RISK MEASURES

The standard VAR calculation for extension to multiple periods also assumes that positions are fixed. If risk management enforces loss limits, the true VAR will be

a. The same
b. Greater than calculated
c. Less than calculated
d. Unable to be determined

10.6 STRESS-TESTING

The Counterparty Risk Management Policy Group (2008) said that "Risk monitoring and risk management cannot be left to quantitative risk metrics, which by nature are backward looking."[1] This is why VAR should be complemented by **stress-testing**, which aims at identifying situations that could create extraordinary losses for the institution. As shown in the yen example in Figure 10.2, VAR does not purport to account for extreme losses.

Stress-testing is a key risk management process, which includes (1) scenario analysis, (2) stressing models, volatilities and correlations, and (3) developing policy responses. **Scenario analysis** submits the portfolio to large movements in financial market variables. These scenarios can be created using a number of methods.

- *Moving key variables one at a time*, which is a simple and intuitive method. Unfortunately, it is difficult to assess realistic comovements in financial variables. It is unlikely that all variables will move in the worst possible direction at the same time.
- *Using historical scenarios*, for instance the 1987 stock market crash, the devaluation of the British pound in 1992, the bond market debacle of 1984, and so on.
- *Creating prospective scenarios*, for instance working through the effects, direct and indirect, of a U.S. stock market crash. Ideally, the scenario should be tailored to the portfolio at hand, assessing the worst thing that could happen to current positions.
- *Reverse stress-tests* start from assuming a large loss and then explore the conditions that would lead to this loss. This type of analysis forces institutions to think of other scenarios and to address issues not normally covered in regular stress-tests, such as financial contagion.

Stress-testing is useful to guard against **event risk**, which is the risk of loss due to an observable political or economic event. The problem (from the viewpoint of stress-testing) is that such events are relatively rare and may be difficult to anticipate. These include

- *Changes in governments* leading to changes in economic policies
- *Changes in economic policies*, such as default, capital controls, inconvertibility, changes in tax laws, expropriations, and so on
- *Coups, civil wars, invasions*, or other signs of political instability
- *Currency devaluations*, which are usually accompanied by other drastic changes in market variables

These risks often arise in **emerging markets**,[2] perhaps due to their lack of relative political stability. To guard against event risk, risk managers should

[1] Counterparty Risk Management Policy Group (2008), *Containing Systemic Risk: The Road to Reform*, New York: CRMPG.

[2] The term "emerging stock market" was coined by the International Finance Corporation (IFC), in 1981. IFC defines an emerging stock market as one located in a developing country. Using the World Bank's definition, this includes all countries with a GNP per capita less than $8,625 in 1993.

construct prospective events and analyze their impact on portfolio values. Even so, this is not an easy matter. Recent years have demonstrated that markets seem to be systematically taken by surprise. Few people seem to have anticipated the Russian default, for instance. The Argentinian default was also unique in many respects.

Example: Turmoil in Argentina

Argentina is a good example of political risk in emerging markets. Up to 2001, the Argentine peso was fixed to the U.S. dollar at a one-to-one exchange rate. The government had promised it would defend the currency at all costs. Argentina, however, suffered from the worst economic crisis in decades, compounded by the cost of excessive borrowing.

In December 2001, Argentina announced it would stop paying interest on its $135 billion foreign debt. This was the largest sovereign default recorded so far. Economy Minister Cavallo also announced sweeping restrictions on withdrawals from bank deposits to avoid capital flight. On December 20, President Fernando de la Rua resigned after 25 people died in street protest and rioting. President Duhalde took office on January 2 and devalued the currency on January 6. The exchange rate promptly moved from 1 peso/dollar to more than 3 pesos.

Such moves could have been factored into risk management systems by scenario analysis. What was totally unexpected, however, was the government's announcement that it would treat differentially bank loans and deposits. Dollar-denominated bank deposits were converted into devalued pesos, but dollar-denominated bank loans were converted into pesos at a one-to-one rate. This mismatch rendered much of the banking system technically insolvent, because loans (bank assets) overnight became less valuable than deposits (bank liabilities). Whereas risk managers had contemplated the market risk effect of a devaluation, few had considered this possibility of such political actions.

By 2005, the Argentinian government proposed to pay back about 30% of the face value of its debt. This recovery rate was very low by historical standards.

The goal of stress-testing is to identify areas of potential vulnerability. This is not to say that the institution should be totally protected against every possible contingency, as this would make it impossible to take any risk. Rather, the objective of stress-testing and management response should be to ensure that the institution can withstand likely scenarios without going bankrupt. Stress-testing can be easily implemented once the VAR structure is in place. In Figure 10.4, all that is needed is to enter the scenario values into the risk factor inputs.

EXAMPLE 10.10: FRM EXAM 2000—QUESTION 105

Value-at-risk analysis should be complemented by stress-testing because stress-testing

a. Provides a maximum loss, expressed in dollars
b. Summarizes the expected loss over a target horizon within a minimum confidence interval
c. Assesses the behavior of portfolio at a 99% confidence level
d. Identifies losses that go beyond the normal losses measured by VAR

EXAMPLE 10.11: FRM EXAM 2006—QUESTION 87

Which of the following is true about stress-testing?

a. It is used to evaluate the potential impact on portfolio values of unlikely, although plausible, events or movements in a set of financial variables.
b. It is a risk management tool that directly compares predicted results to observed actual results. Predicted values are also compared with historical data.
c. Both a. and b. above are true.
d. None of the above are true.

10.7 IMPORTANT FORMULAS

VAR: $c = \int_{-\text{VAR}}^{\infty} f(x)dx$

CVAR: $E[X \mid X < q] = \int_{-\infty}^{q} xf(x)dx / \int_{-\infty}^{q} f(x)dx$

Drawdown: $\text{DD}(X) = \frac{(x^{MAX} - x_t)}{x^{MAX}}$

Square root of time adjustment: $\text{VAR}(T \text{ days}) = \text{VAR}(1 \text{ day}) \times \sqrt{T}$

Market Risk Charge: $\text{MRC}_t^{IMA} = \text{Max}\left(k\frac{1}{60}\sum_{i=1}^{60} \text{VAR}_{t-i},\ \text{VAR}_{t-1}\right) + \text{SRC}_t$

10.8 ANSWERS TO CHAPTER EXAMPLES

Example 10.1: FRM Exam 2005—Question 32

a. Stop-loss limits cut down the positions after a loss is incurred, which is useful if market are trending. Exposure limits do not allow for diversification because correlations are not considered. VAR limits can be arbitraged, especially with weak

VAR models. Finally, stop-loss limits are put in place after losses are incurred, so cannot prevent losses. As a result, statement I. and II. are correct.

Example 10.2: FRM Exam 2005—Question 43

b. VAR is the worst loss, such that there is a 95% probability that the losses will be less severe. Alternatively, there is a 5% probability that the loss will be worse. So b. is correct. Answer d. says "lose less" and therefore is incorrect.

Example 10.3: FRM Exam 2003—Question 11

b. Based on Equation (10.4), this is $10\% \times 250 = 25$.

Example 10.4: FRM Exam 2007—Question 101

d. We should expect $(1 - 95\%)250 = 12.5$ exceptions on average. Having eight exceptions is too few, but the difference could be due to luck. Having zero exceptions, however, would be very unusual, with a probability of $1 - (1 - 5\%)^{250}$, which is very low. This means that the risk manager is providing VAR estimates that are much too high. Otherwise, the largest or mean losses are not directly useful without more information on the distribution of profits.

Example 10.5: FRM Exam 2003—Question 5

b. The 10% lower cutoff point is the third lowest observation, which is VAR = 10. The expected shortfall is then the average of the observations in the tails, which is 15.

Example 10.6: FRM Exam 2007—Question 27

b. Conditional VAR is coherent, so statement a. is correct. A low conditional VAR means that when a loss exceeds VAR, it should be small on average, so statement c. is correct. Conditional VAR must be greater than VAR, so statement d. is correct. Economic capital is a VAR measure, not a conditional VAR measure.

Example 10.6: FRM Exam 2007—Question 112

b. A risk analysis requires constructing the distribution of risk factors, so statement a. is correct. Statement c. suggests that the delta-normal VAR method may be appropriate in some contexts, which is correct. Statement d. explains that a Monte Carlo can be used to measure risk, which is correct. Statement b. is false because the distribution of cash flows should be evaluated even in the presence of strategic options.

Example 10.7: FRM Exam 2003—Question 19

b. The square root of time $\sqrt{10} = 3.16$ adjustment applies if the distribution is i.i.d. (and normal).

Example 10.8: Market Risk Charge

d. First, we have to convert the 95% VAR to a 99% measure, assuming a normal distribution in the absence of other information. The GMRC is then $3 \times VAR \times \sqrt{10} = 3 \times \$1,000,000(2.33)/1.65) \times \sqrt{10} = \$13,396,000$.

Example 10.9: Position-Based Risk Measures

c. Less than calculated. Loss limits cut down the positions as losses accumulate. This is similar to a long position in an option, where the delta increases as the price increases, and vice versa. Long positions in options have shortened left tails, and hence involve less risk than an unprotected position.

Example 10.10: FRM Exam 2000—Question 105

d. Stress-testing identifies low-probability losses beyond the usual VAR measures. It does not, however, provide a maximum loss.

Example 10.11: FRM Exam 2006—Question 87

a. Stress-testing is indeed used to evaluate the effect of extreme events. Answer b. is about backtesting, not stress-testing.

APPENDIX: DESIRABLE PROPERTIES FOR RISK MEASURES

The purpose of a risk measure is to summarize the entire distribution of dollar returns X by one number, $\rho(X)$. Artzner et al. (1999) list four desirable properties of risk measures for capital adequacy purposes.[3]

1. **Monotonicity:** if $X_1 \leq X_2$, $\rho(X_1) \geq \rho(X_2)$.
   In other words, if a portfolio has systematically lower values than another (in each state of the world), it must have greater risk.
2. **Translation Invariance:** $\rho(X + k) = \rho(X) - k$.
   In other words, adding cash k to a portfolio should reduce its risk by k. This reduces the lowest portfolio value. As with X, k is measured in dollars.

[3] See Artzner, P., Delbaen F., Eber J.-M., and Heath D. (1999), Coherent Measures of Risk. *Mathematical Finance*, 9 (July), 203–228.

3. **Homogeneity:** $\rho(bX) = b\rho(X)$.
   In other words, increasing the size of a portfolio by a factor b should scale its risk measure by the same factor b. This property applies to the standard deviation.[4]
4. **Subadditivity:** $\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$.
   In other words, the risk of a portfolio must be less than the sum of separate risks. Merging portfolios cannot increase risk.

The usefulness of these criteria is that they force us to think about ideal properties and, more importantly, potential problems with simplified risk measures. Indeed, Artzner et al. show that the quantile-based VAR measure fails to satisfy the last property. They give some pathological examples of positions that combine to create portfolios with larger VAR. They also show that the conditional VAR, $E[-X \mid X \leq -\text{VAR}]$, satisfies all these desirable coherence properties.

Assuming a normal distribution, however, the standard deviation-based VAR satisfies the subadditivity property. This is because the volatility of a portfolio is less than the sum of volatilities: $\sigma(X_1 + X_2) \leq \sigma(X_1) + \sigma(X_2)$. We only have a strict equality when the correlation is perfect (positive for long positions). More generally, this property holds for **elliptical distributions**, for which contours of equal density are ellipsoids.

Example: Why VAR Is Not Necessarily Subadditive

Consider a trader with an investment in a corporate bond with face value of \$100,000 and default probability of 0.5%. Over the next period, we can either have no default, with a return of zero, or default with a loss of \$100,000. The payoffs are thus −\$100,000 with probability of 0.5% and +\$0 with probability of 99.5%. Since the probability of getting \$0 is greater than 99%, the VAR at the 99% confidence level is \$0, without taking the mean into account. This is consistent with the definition that VAR is the smallest loss, such that the right-tail probability is at least 99%.

Now, consider a portfolio invested in three bonds (A,B,C) with the same characteristics and independent payoffs. The VAR numbers add up to $\sum_i \text{VAR}_i =$ \$0. To compute the portfolio VAR, we tabulate the payoffs and probabilities:

State	Bonds	Probability	Payoff
No default		$0.995 \times 0.995 \times 0.995 = 0.9850749$	\$0
1 default	A,B,C	$3 \times 0.005 \times 0.995 \times 0.995 = 0.0148504$	−\$100,000
2 defaults	AB,AC,BC	$3 \times 0.005 \times 0.005 \times 0.995 = 0.0000746$	−\$200,000
3 defaults	ABC	$0.005 \times 0.005 \times 0.005 = 0.0000001$	−\$300,000

[4] This assumption, however, may be questionable in the case of huge portfolios that could not be liquidated without substantial market impact. Thus, it ignores liquidity risk.

Here, the probability of zero or one default is $0.9851 + 0.0148 = 99.99\%$. The portfolio VAR is therefore \$100,000, which is the lowest number, such that the probability exceeds 99%. Thus, the portfolio VAR is greater than the sum of individual VARs. In this example, VAR is not subadditive. This is an undesirable property because it creates disincentives to aggregate the portfolio, since it appears to have higher risk.

Admittedly, this example is a bit contrived. Nevertheless, it illustrates the danger of focusing on VAR as a sole measure of risk. The portfolio may be structured to display a low VAR. When a loss occurs, however, this may be a huge loss. This is an issue with asymmetrical positions, such as short positions in options or undiversified portfolios exposed to credit risk.

CHAPTER 11

Sources of Market Risk

We now turn to a systematic analysis of the major financial market risk factors. Section 11.1 presents a general overview of financial market risks. Downside risk can be decomposed into two types of drivers: exposures and risk factors. This decomposition is useful because it separates risk into a component over which the risk manager has control (exposure) and another component that is exogenous (the risk factors). We illustrate this decomposition in the context of a simple asset, a fixed-coupon bond.

The next four sections then turn to the major categories of market risk. Currency, fixed-income, equity, and commodities risk are analyzed in Sections 11.2, 11.3, 11.4, and 11.5, respectively. Currency risk refers to the volatility of floating exchange rates and devaluation risk, for fixed currencies. Fixed-income risk relates to term-structure risk, global interest rate risk, real yield risk, credit spread risk, and prepayment risk. Equity risk can be described in terms of country risk, industry risk, and stock-specific risk. Commodity risk includes volatility risk, convenience yield risk, delivery and liquidity risk. This chapter primarily focuses on volatility and correlation measures as drivers of risk. Also important is the shape of the distribution, however.

Finally, Section 11.6 discusses simplifications in risk models. We explain how the multitude of risk factors can be summarized into a few essential drivers. Such factor models include the diagonal model, which decomposes returns into a market-wide factor and residual risk.

11.1 SOURCES OF LOSS: A DECOMPOSITION

To examine sources of losses, consider a plain fixed-coupon bond. The potential for loss can be decomposed into the effect of *dollar duration* $D^* P$ and the changes in the yield dy. The bond's value change is given by

$$dP = -(D^* P) \times dy \tag{11.1}$$

This illustrates the general principle that losses can occur because of a combination of two factors:

1. The exposure to the factor, or dollar duration (a choice variable)
2. The movement in the factor itself (which is external to the portfolio)

This linear characterization also applies to *systematic risk* and option *delta*. We can, for instance, decompose the return on stock i, R_i into a component due to the market R_M and some residual risk, which we ignore for now because its effect washes out in a large portfolio:

$$R_i = \alpha_i + \beta_i \times R_M + \epsilon_i \approx \beta_i \times R_M \tag{11.2}$$

We ignore the constant α_i because it does not contribute to risk, as well as the residual ϵ_i, which is diversified. **Specific risk** can be defined as risk that is due to issuer-specific price movements, after accounting for general market factors.

Note that R_i is expressed here in terms of **rate of return** and, hence, has no dimension. To get a change in a dollar price, we write

$$dP_i = R_i P_i \approx (\beta P_i) \times R_M \tag{11.3}$$

Similarly, the change in the value of a derivative f can be expressed in terms of the change in the price of the underlying asset S,

$$df = \Delta \times dS \tag{11.4}$$

To avoid confusion, we use the conventional notations of Δ for the first partial derivative of the option. Changes are expressed in infinitesimal amounts df and dS.

Equations (11.1), (11.2), and (11.4) all reveal that the change in value is linked to an **exposure** coefficient and a change in market variable:

$$\text{Market Loss} = \text{Exposure} \times \text{Adverse Movement in Financial Variable}$$

To have a loss, we need to have some exposure *and* an unfavorable move in the risk factor. This decomposition is also useful to understand the drivers of discontinuities in the portfolio value. These can come from either discontinuous payoffs, or from jumps in the risk factors. Discontinuous payoffs arise with some instruments, such as binary options, which pay a fixed amount if the options ends up in the money and none otherwise. Discontinuities can also arise if there are jumps in the risk factors, such as the 1987 stock market crash, or due to event risk.

11.2 CURRENCY RISK

Currency risk arises from potential movements in the value of foreign currencies. This includes currency-specific volatility, correlations across currencies, and devaluation risk. Currency risk arises in the following environments.

- In a *pure currency float*, the external value of a currency is free to move, to depreciate or appreciate, as pushed by market forces. An example is the dollar/euro exchange rate.

- In a *fixed currency system*, a currency's external value is fixed (or pegged) to another currency. An example is the Hong Kong dollar, which is fixed against the U.S. dollar. This does not mean there is no risk, however, due to possible readjustments in the **parity value**, called devaluations or revaluations.
- In a *change in currency regime*, a currency that was previously fixed becomes flexible, or vice versa. For instance, the Argentinian peso was fixed against the dollar until 2001, and floated thereafter. Changes in regime can also lower currency risk, as in the recent case of the euro.[1]

11.2.1 Currency Volatility

Table 11.1 compares the RiskMetrics volatility forecasts for a group of 21 currencies.[2] Ten of these correspond to "industrialized" countries, the others to "emerging" markets.

These numbers are standard deviations, adapted from value-at-risk (VAR) forecasts at the 95% confidence level divided by 1.645. The table reports daily, monthly, and annualized standard deviations at the end of 2006 and 1996. Annual volatilities are obtained from monthly volatilities multiplied by the square root of 12.

TABLE 11.1 Currency Volatility Against U.S. Dollar (Percent)

		End 2006			End 1996
Country	Currency Code	Daily	Monthly	Annual	Annual
Argentina	ARS	0.269	1.073	3.72	0.42
Australia	AUD	0.374	1.995	6.91	8.50
Canada	CAD	0.338	1.840	6.37	3.60
Switzerland	CHF	0.403	2.223	7.70	10.16
Denmark	DKK	0.361	1.933	6.70	7.78
United Kingdom	GBP	0.383	2.072	7.18	9.14
Hong Kong	HKD	0.035	0.161	0.56	0.26
Indonesia	IDR	0.286	1.443	5.00	1.61
Japan	JPY	0.363	2.040	7.07	6.63
South Korea	KRW	0.325	1.675	5.80	4.49
Mexico	MXN	0.324	1.856	6.43	6.94
Malaysia	MYR	0.311	1.430	4.95	1.60
Norway	NOK	0.520	2.760	9.56	7.60
New Zealand	NZD	0.455	2.642	9.15	7.89
Philippines	PHP	0.197	1.087	3.76	0.57
Sweden	SEK	0.498	2.535	8.78	6.38
Singapore	SGD	0.183	0.991	3.43	1.79
Thailand	THB	0.645	2.647	9.17	1.23
Taiwan	TWD	0.217	1.093	3.79	0.94
South Africa	ZAR	0.666	4.064	14.08	8.37
Euro	EUR	0.360	1.934	6.70	8.26

[1] As of 2007, the Eurozone includes a block of 13 countries, Austria, Belgium/Luxembourg, Finland, France, Germany, Ireland, Italy, Netherlands, Portugal, and Spain. Greece joined on January 1, 2001. Slovenia joined on January 1, 2007. Cyprus joined on January 1, 2008. Currency risk is not totally eliminated, however, as there is always a possibility that the currency union could dissolve.

[2] For updates, see www.riskmetrics.com.

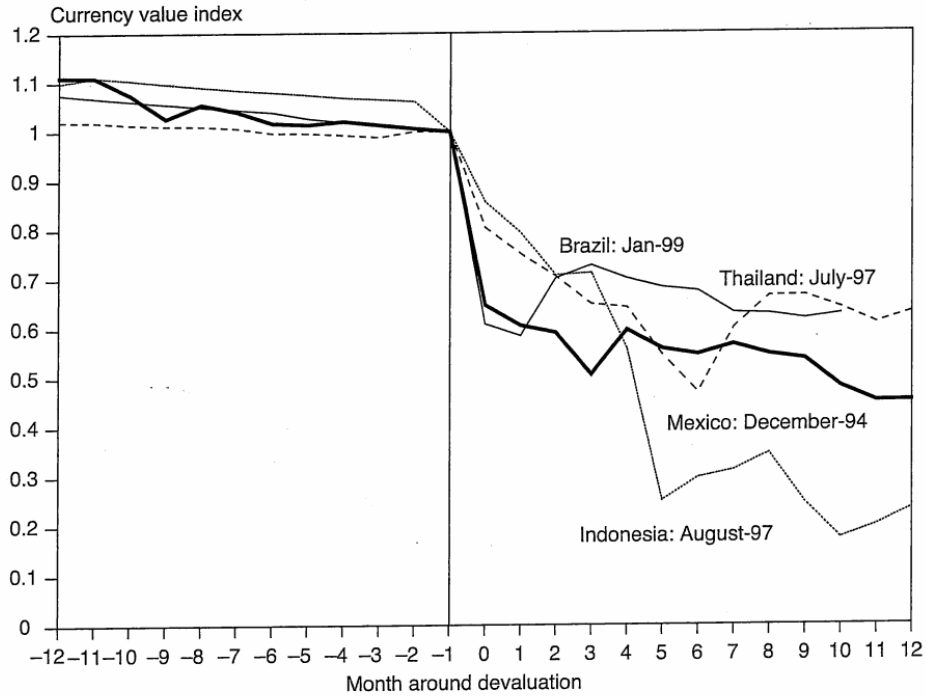

FIGURE 11.1 Effect of Currency Devaluation

Across developed markets, volatility typically ranges from 6% to 10% per annum. The Canadian dollar is notably lower at 4% to 6% volatility. Some currencies, such as the Hong Kong dollar have very low volatility, reflecting their pegging to the dollar. This does not mean that they have low risk, however. They are subject to **devaluation risk**, which is the risk that the currency peg could fail. This has happened to Thailand and Indonesia, which in 1996 had low volatility but converted to a floating exchange rate regime, with much higher volatility in the latter period.

The typical impact of a currency devaluation is illustrated in Figure 11.1. Each currency has been scaled to a unit value at the end of the month just before the devaluation. In previous months, we observe only small variations in exchange rates. In contrast, the devaluation itself leads to a dramatic drop in value ranging from 20% to an extreme 80% in the case of the rupiah.

Currency risk is also related to other financial risks, in particular interest rate risk. Often, interest rates are raised in an effort to stem the depreciation of a currency, resulting in a positive correlation between the currency and the bond market. These interactions should be taken into account when designing scenarios for stress-tests.

11.2.2 Correlations

Next, we briefly describe the correlations between these currencies against the U.S. dollar. Generally, correlations are low, mostly in the range of −0.10 to 0.20. This indicates substantial benefits from holding a well-diversified currency portfolio.

There are, however, blocks of currencies with very high correlations. European currencies, such as the DKK, SEK, NOK, CHF, have high correlation with each other and the Euro, on the order of 0.90. The GBP also has high correlations with European currencies, around 0.60–0.70. As a result, investing across European currencies does little to diversify risk, from the viewpoint of a U.S. dollar-based investor.

11.2.3 Cross-Rate Volatility

Exchange rates are expressed relative to a base currency, usually the dollar. The **cross rate** is the exchange rate between two currencies other than the reference currency. For instance, say that S_1 represents the dollar/pound rate and that S_2 represents the dollar/euro (EUR) rate. Then the euro/pound rate is given by the ratio

$$S_3(\mathrm{EUR/BP}) = \frac{S_1(\$/\mathrm{BP})}{S_2(\$/\mathrm{EUR})} \tag{11.5}$$

Using logs, we can write

$$\ln[S_3] = \ln[S_1] - \ln[S_2] \tag{11.6}$$

The volatility of the cross rate is

$$\sigma_3^2 = \sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2 \tag{11.7}$$

Alternatively, this shows that we could infer the correlation coefficient ρ_{12} from the triplet of variances. Note that this assumes both the numerator and denominator are in the same currency. Otherwise, the log of the cross rate is the sum of the logs, and the negative sign in Equation (11.7) must be changed to a positive sign.

EXAMPLE 11.1: IMPLIED CORRELATION

What is the implied correlation between JPY/EUR and EUR/USD when given the following volatilities for foreign exchange rates? JPY/USD at 8%; JPY/EUR at 10%; EUR/USD at 6%.

a. 60%
b. 30%
c. −30%
d. −60%

11.3 FIXED-INCOME RISK

Fixed-income risk arises from potential movements in the level and volatility of bond yields. Yield curves move up and down and in various other ways. For the risk manager, this will creates **yield curve risk** for fixed-income portfolios.

11.3.1 Factors Affecting Yields

Movements in yields reflects economic fundamentals. The primary determinant of movements in interest rates is **inflationary expectations.** Any perceived increase in the forecast rate of inflation will make bonds with fixed nominal coupons less attractive, thereby increasing their yield.

Figure 11.2 compares the level of short-term U.S. interest rates with the concurrent level of inflation. The graphs show that most of the long-term movements in nominal rates can be explained by inflation. In more recent years, however, inflation has been subdued. Rates have fallen accordingly.

The **real interest rate** is defined as the nominal rate minus the rate of inflation over the same period. This is generally positive but in recent years has been negative as the Federal Reserve has kept nominal rates to very low levels in order to stimulate economic activity.

Movements in the term structure of interest rates can be summarized by two maturities. In practice, market observers focus on a long-term rate (e.g., from the 10-year Treasury note) and a short-term rate (e.g., from the three-month Treasury bill), as shown in Figure 11.3.

Shaded areas indicate periods of U.S. economic recessions. Recession periods are officially defined by the National Bureau of Economic Research (NBER),

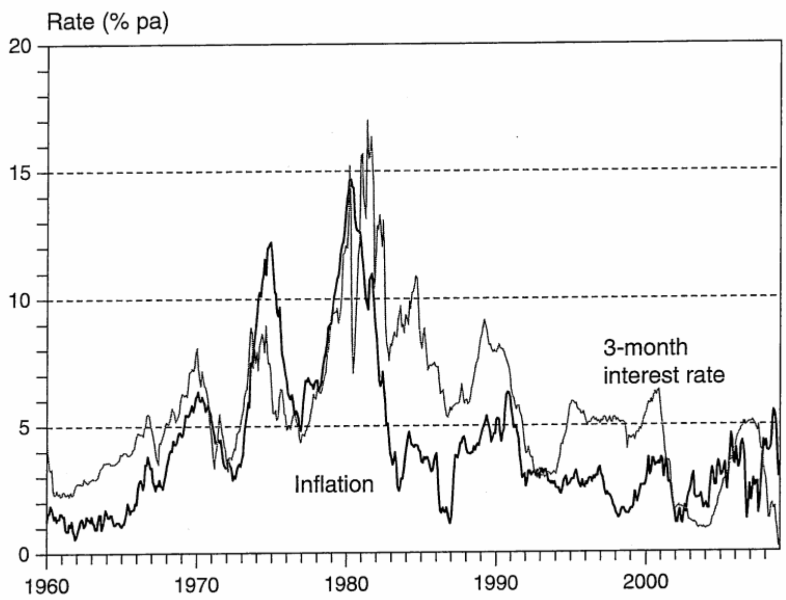

FIGURE 11.2 Inflation and Interest Rates

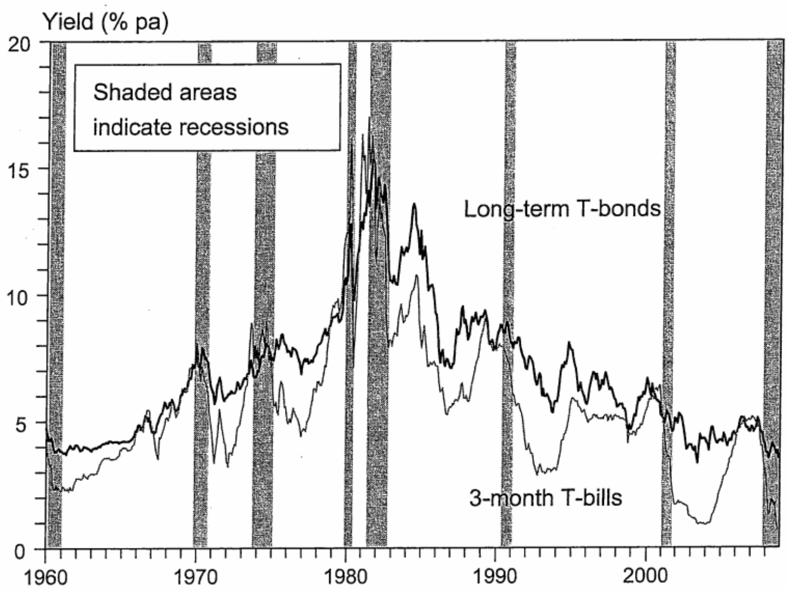

FIGURE 11.3 Movements in the Term Structure

always with a substantial delay. This explains why the 2008 recession is not yet recorded in Figure 11.3.

Generally, the two rates move in tandem, although the short-term rate displays more variability. The **term spread** is defined as the difference between the long rate and the short rate. Figure 11.4 relates the term spread to economic activity. As the graph shows, periods of recessions usually witness an increase in the term spread. Slow economic activity decreases the demand for capital, which in turn

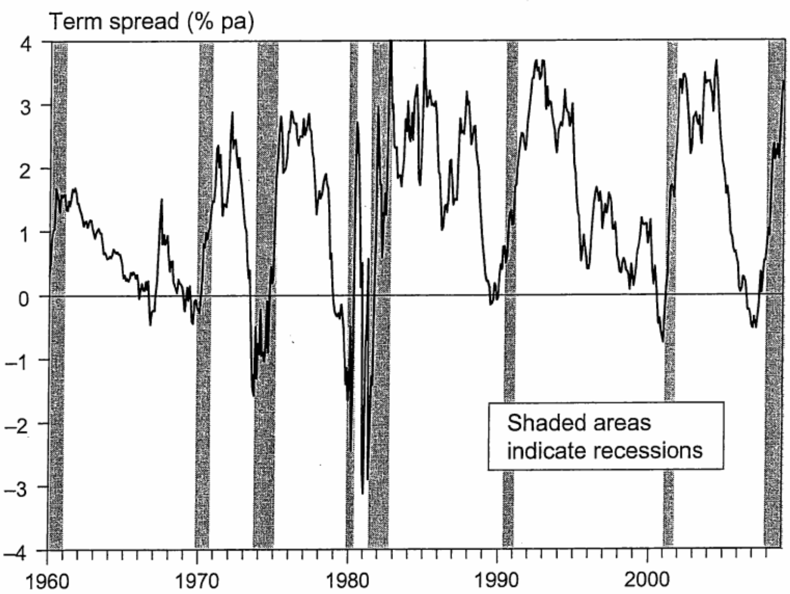

FIGURE 11.4 Term Structure Spread

decreases short-term rates and increases the term spread. Equivalently, the central banks lowers short-term rates to stimulate the economy.

11.3.2 Bond Price and Yield Volatility

Table 11.2 compares the RiskMetrics volatility forecasts for U.S. bond prices as of 2006 and 1996. The table includes Eurodeposits, fixed swap rates, and zero-coupon Treasury rates, for maturities ranging from 30 day to 30 years.

Risk can be measured as either return volatility or yield volatility. Using the duration approximation, the volatility of the rate of return in the bond price is

$$\sigma\left(\frac{dP}{P}\right) = \mid D^* \mid \times \sigma(dy) \tag{11.8}$$

The table shows that short-term investments have very little price risk, as expected, due to their short maturity and duration. The price risk of 10-year bonds is around 6% to 10% annually, which is similar to that of floating currencies. The risk of 30-year bonds is higher, around 20%, which is similar to that of equities.

Instead of measuring price volatility, it is more intuitive to consider yield volatility, $\sigma(dy)$. This is because yield volatility remains more constant over time

TABLE 11.2 U.S. Fixed-Income Return Volatility (Percent)

Type/		Yield	End 2006			End 1996
Maturity	Code	Level	Daily	Monthly	Annual	Annual
Euro-30d	R030	5.325	0.001	0.004	0.01	0.05
Euro-90d	R090	5.365	0.002	0.010	0.04	0.08
Euro-180d	R180	5.375	0.005	0.030	0.11	0.19
Euro-360d	R360	5.338	0.028	0.148	0.51	0.58
Swap-2Y	S02	5.158	0.081	0.420	1.45	1.57
Swap-3Y	S03	5.100	0.127	0.657	2.27	2.59
Swap-4Y	S04	5.062	0.172	0.890	3.08	3.59
Swap-5Y	S05	5.075	0.219	1.120	3.88	4.70
Swap-7Y	S07	5.116	0.283	1.460	5.06	6.69
Swap-10Y	S10	5.177	0.383	1.965	6.81	9.82
Zero-2Y	Z02	4.811	0.088	0.444	1.54	1.64
Zero-3Y	Z03	4.716	0.130	0.663	2.30	2.64
Zero-4Y	Z04	4.698	0.173	0.871	3.02	3.69
Zero-5Y	Z05	4.688	0.216	1.084	3.76	4.67
Zero-7Y	Z07	4.692	0.279	1.395	4.83	6.81
Zero-9Y	Z09	4.695	0.343	1.714	5.94	8.64
Zero-10Y	Z10	4.698	0.375	1.874	6.49	9.31
Zero-15Y	Z15	4.772	0.531	2.647	9.17	13.82
Zero-20Y	Z20	4.810	0.690	3.441	11.92	17.48
Zero-30Y	Z30	4.847	1.014	5.049	17.49	23.53

TABLE 11.3 U.S. Fixed-Income Yield Volatility, 2006 (Percent)

Type/ Maturity	Code	Yield Level	End 2006 Daily	Monthly	Annual
Euro-30d	R030	5.325	0.008	0.048	0.17
Euro-90d	R090	5.365	0.006	0.041	0.14
Euro-180d	R180	5.375	0.010	0.062	0.22
Euro-360d	R360	5.338	0.030	0.157	0.54
Swap-2Y	S02	5.158	0.033	0.181	0.63
Swap-3Y	S03	5.100	0.020	0.115	0.40
Swap-4Y	S04	5.062	0.023	0.131	0.46
Swap-5Y	S05	5.075	0.022	0.132	0.46
Swap-7Y	S07	5.116	0.020	0.096	0.33
Swap-10Y	S10	5.177	0.008	0.048	0.17
Zero-2Y	Z02	4.811	0.012	0.064	0.22
Zero-3Y	Z03	4.716	0.020	0.113	0.39
Zero-4Y	Z04	4.698	0.020	0.110	0.38
Zero-5Y	Z05	4.688	0.024	0.126	0.44
Zero-7Y	Z07	4.692	0.023	0.126	0.44
Zero-9Y	Z09	4.695	0.024	0.130	0.45
Zero-10Y	Z10	4.698	0.026	0.139	0.48
Zero-15Y	Z15	4.772	0.027	0.143	0.49
Zero-20Y	Z20	4.810	0.036	0.183	0.63
Zero-30Y	Z30	4.847	0.036	0.184	0.64

than price volatility, which must approach zero as the bond approaches maturity. Yield volatilities are displayed in Table 11.3. Yield volatility averages around 0.50 percent per annum for swaps and zeros. As shown from Table 11.2, however, volatility was fairly low at the end of 2006 compared to 1996, and the historical average. Typical yield volatility is around 1%.

EXAMPLE 11.2: FRM EXAM 2007—QUESTION 50

A portfolio consists of two zero coupon bonds, each with a current value of \$10. The first bond has a modified duration of one year and the second has a modified duration of nine years. The yield curve is flat, and all yields are 5%. Assume all moves of the yield curve are parallel shifts. Given that the daily volatility of the yield is 1%, which of the following is the best estimate of the portfolio daily VAR at the 95% confidence level?

a. USD 1.65
b. USD 2.33
c. USD 1.16
d. USD 0.82

11.3.3 Correlations

Table 11.4 displays correlation coefficients for all maturity pairs at a one-day horizon. Correlations are generally very high, suggesting that yields are affected by a common factor.

If the yield curve were to move in strict parallel fashion, all correlations should be equal to 1.000. In practice, the yield curve displays more complex patterns but still satisfies some smoothness conditions. This implies that movements in adjoining maturities are highly correlated. For instance, the correlation between the nine-year zero and 10-year zero is 0.9998, which is very high. Correlations are the lowest for maturities further apart, for instance 0.848 between the two-year and 30-year zero.

These high correlations give risk managers an opportunity to simplify the number of risk factors they have to deal with. Suppose, for instance, that the portfolio consists of global bonds in 17 different currencies. Initially, the risk manager decides to keep 14 risk factors in each market. This leads to a very large number of correlations within, but also across all markets. With 17 currencies, and 14 maturities, the total number of risk factors is $N = 17 \times 14 = 238$. The correlation matrix has $N \times (N - 1) = 238 \times 237 = 56{,}406$ elements off the diagonal. Surely some of this information is superfluous.

The matrix in Table 11.4 can be simplified using principal components. **Principal components** is a statistical technique that extracts linear combinations of the original variables that explain the highest proportion of diagonal components of the matrix. For this matrix, the first principal component explains 94% of the total variance and has similar weights on all maturities. Hence, it could be called a **level risk factor**. The second principal component explains 4% of the total variance. As it is associated with opposite positions on short and long maturities, it could be called a **slope risk factor** (or *twist*). Sometimes a third factor is found that represents **curvature risk factor**, or a **bend risk factor** (also called a *butterfly*).

Previous research has indeed found that, in the United States and other fixed-income markets, movements in yields could be usefully summarized by two to three factors that typically explain over 95% of the total variance.

TABLE 11.4 U.S. Fixed-Income Return Correlations, 2006 (Daily)

	Z02	Z03	Z04	Z05	Z07	Z09	Z10	Z15	Z20	Z30
Z02	1.000									
Z03	0.991	1.000								
Z04	0.980	0.994	1.000							
Z05	0.968	0.985	0.998	1.000						
Z07	0.949	0.972	0.991	0.996	1.000					
Z09	0.934	0.960	0.982	0.990	0.998	1.000				
Z10	0.927	0.954	0.978	0.987	0.997	0.9998	1.000			
Z15	0.896	0.931	0.960	0.971	0.986	0.992	0.994	1.000		
Z20	0.874	0.913	0.944	0.958	0.976	0.983	0.985	0.998	1.000	
Z30	0.848	0.891	0.925	0.940	0.960	0.969	0.972	0.992	0.998	1.000

EXAMPLE 11.3: FRM EXAM 2005—QUESTION 109

In using principal components analysis (PCA) to analyze movements in yields for fixed-income investments, which of the following is *correct*?

a. Changes in long-term yields tend to be larger than changes in short-term yields.
b. The "twist" factor explains more of the variance than the "butterfly" factor.
c. Movements in yields can be usefully summarized by two to three factors that typically explain approximately 50% of the variance in developed markets.
d. PCA extracts nonlinear combinations of the original variables that explain the highest proportion of diagonal components of the matrix.

11.3.4 Global Interest Rate Risk

Different fixed-income markets are exposed to their own sources of risk. The Japanese government bond market, for example is exposed to yen interest rates. Yet in all markets, we observe similar patterns. To illustrate, Table 11.5 shows price and yield volatilities for 17 different fixed-income markets, focusing only on 10-year zeros.

TABLE 11.5 Global Fixed-Income Volatility, 2006 (Percent)

Country	Code	Yield Level	Return Vol. Daily	Return Vol. Monthly	Return Vol. Annual	Yield Vol. $\sigma(dy)$ Daily	Yield Vol. $\sigma(dy)$ Monthly	Yield Vol. $\sigma(dy)$ Annual
Austrl.	AUD	5.847	0.355	1.859	6.44	0.038	0.198	0.691
Belgium	BEF	4.018	0.230	1.257	4.35	0.025	0.137	0.471
Canada	CAD	4.096	0.288	1.485	5.14	0.030	0.153	0.532
Germany	DEM	3.963	0.225	1.246	4.32	0.024	0.135	0.471
Denmark	DKK	3.952	0.244	1.318	4.56	0.026	0.141	0.492
Spain	ESP	4.029	0.239	1.288	4.46	0.026	0.140	0.482
France	FRF	4.006	0.229	1.250	4.33	0.025	0.136	0.471
U.K.	GBP	4.695	0.287	1.458	5.05	0.030	0.153	0.531
Ireland	IEP	3.980	0.244	1.300	4.50	0.026	0.141	0.492
Italy	ITL	4.251	0.253	1.358	4.70	0.027	0.144	0.501
Japan	JPY	1.715	0.287	1.410	4.89	0.030	0.143	0.498
Nether.	NLG	4.004	0.226	1.240	4.29	0.024	0.134	0.471
New Zl.	NZD	5.894	0.267	1.474	5.11	0.028	0.155	0.549
Sweden	SEK	3.793	0.253	1.345	4.66	0.027	0.143	0.503
U.S.	USD	4.698	0.375	1.874	6.49	0.039	0.195	0.684
S.Afr.	ZAR	7.727	0.459	2.881	9.98	0.046	0.285	0.992
Euro	EUR	3.962	0.222	1.230	4.26	0.024	0.133	0.461

The level of yields falls within a remarkably narrow range, 4% to 6%. This reflects the fact that yields are primarily driven by **inflationary expectations**, which have become similar across all these markets. Indeed central banks across all these countries have proved their common determination to keep inflation in check. Two notable exceptions are South Africa, where yields are higher and Japan, where yields are lower. These two countries are experiencing much higher and lower inflation, respectively, than the rest of the group.

The table also shows that most countries have an annual volatility of yield changes around 0.60%. In fact, we would expect this volatility to decrease as yields drop toward zero and to be higher when yields are higher. This can be modeled by relating the volatility of yield changes to a function of the yield level, as explained in Chapter 4. One such function is the Cox, Ingersoll, and Ross (1985) model, which posits that movements in yields should be proportional to the square root of the yield level. Thus neither the normal nor the lognormal model is totally appropriate.

Finally, correlations are very high across continental European bond markets that are part of the euro. For example, the correlation between French and German bonds is above 0.986. These markets are now moving in synchronization, as monetary policy is dictated by the **European Central Bank** (ECB). Eurozone bonds only differ in terms of credit risk.

11.3.5 Real Yield Risk

So far, the analysis has only considered **nominal interest rate risk**, as most bonds represent obligations in nominal terms, i.e., in dollars for the coupon and principal payment. Recently, however, many countries have issued inflation-protected bonds, which make payments that are fixed in real terms but indexed to the rate of inflation.

In this case, the source of risk is **real interest rate risk**. This real yield can be viewed as the internal rate of return that will make the discounted value of promised real bond payments equal to the current real price. This is a new source of risk, as movements in real interest rates may not correlate perfectly with movements in nominal yields.

These two markets can be used to infer inflationary expectations. The implied rate of inflation can be measured as the nominal yield minus the real yield.

Example: Real and Nominal Yields

Consider for example the 10-year Treasury Inflation Protected (TIP) note paying a 3% coupon in real terms. The actual coupon and principal payments are indexed to the increase in Consumer Price Index (CPI).

The TIP is now trading at a clean real price of 108-23+. Discounting the coupon payments and the principal gives a real yield of $r = 1.98\%$. Note that since the bond is trading at a premium, the real yield must be lower than the coupon.

Projecting the rate of inflation at $\pi = 2\%$, semiannually compounded, we infer the projected nominal yield as $(1 + y/200) = (1 + r/200)(1 + \pi/200)$, which gives 4.00%. This is the same order of magnitude as the current nominal yield on the 10-year Treasury note, which is 3.95%. The two bonds have a very different risk profile, however. If the rate of inflation moves to 5%, payments on the TIP will grow at 5% plus 2%, while the coupon on the regular note will stay fixed.

11.3.6 Credit Spread Risk

Credit spread risk is the risk that yields on duration-matched credit-sensitive bond and Treasury bonds could move differently. The topic of credit risk will be analyzed in more detail in another part of this book. Suffice to say that the credit spread represent a compensation for the loss due to default, plus perhaps a risk premium that reflects investor risk aversion.

A position in a credit spread can be established by investing in credit-sensitive bonds, such as corporates, agencies, mortgage-backed securities (MBSs), and shorting Treasuries with the appropriate duration. This type of position benefits from a stable or shrinking credit spread, but loses from a widening of spreads. Because credit spreads cannot turn negative, their distribution is asymmetric, however. When spreads are tight, large moves imply increases in spreads rather than decreases. Thus positions in credit spreads can be exposed to large losses.

Figure 11.5 displays the time-series of credit spreads since 1960, taken from the Baa-Treasury spread. The graph shows that credit spreads display cyclical patterns, increasing during a recession and decreasing during economic expansions. Greater spreads during recessions reflect the greater number of defaults during difficult times. In 2008, the spread reached 5%, a record over this period.

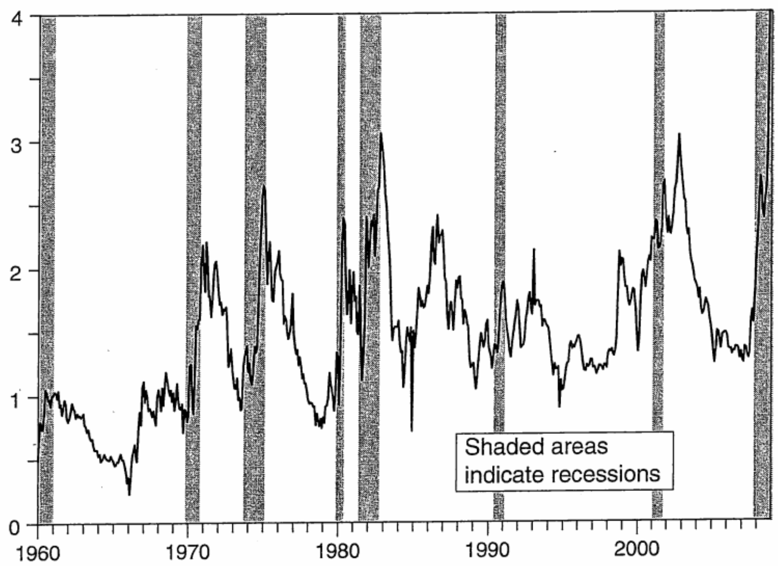

FIGURE 11.5 Credit Spreads

11.3.7 Prepayment Risk

Prepayment risk arises in the context of home mortgages when there is uncertainty about whether the homeowner will refinance his loan early. It is a prominent feature of **mortgage-backed securities**, where the investor has granted the borrower an option to repay the debt early.

This option, however, is much more complex than an ordinary option, due to the multiplicity of factors involved. We have seen in Chapter 7 that it depends on the age of the loan (seasoning), the current level of interest rates, the previous path of interest rates (burnout), economic activity, and seasonal patterns. Assuming that the prepayment model adequately captures all these features, investors can evaluate the attractiveness of MBSs by calculating their **option-adjusted spread** (OAS). This represents the spread over the equivalent Treasury minus the cost of the option component.

EXAMPLE 11.4: FRM EXAM 2002—QUESTION 128

During 2002, an Argentinean pension fund with 80% of its assets in dollar-denominated debt lost more than 40% of its value. Which of the following reasons could explain all of the 40% loss:

a. The assets were invested in a diversified portfolio of AAA firms in the U.S.
b. The assets invested in local currency in Argentina lost all their value and the value of the dollar-denominated assets stayed constant.
c. The dollar-denominated assets were invested in U.S. Treasury debt, but the fund had bought credit protection on sovereign debt from Argentina.
d. The fund had invested 80% of its funds in dollar-denominated sovereign debt from Argentina.

11.4 EQUITY RISK

Equity risk arises from potential movements in the value of stock prices. We will show that we can usefully decompose the total risk into a marketwide risk and stock-specific risk.

11.4.1 Stock Market Volatility

Table 11.6 compares the RiskMetrics volatility forecasts for a group of 31 stock markets. The selected indices are those most recognized in each market, for example the S&P 500 in the United States, Nikkei 225 in Japan, and FTSE-100 in Britain. Most of these have an associated futures contract, so positions can be

TABLE 11.6 Equity Volatility (Percent)

Stock Market Country	Code	End 2006 Daily	End 2006 Monthly	End 2006 Annual	End 1996 Annual
Australia	AUD	0.598	3.302	11.4	13.4
Canada	CAD	0.586	3.380	11.7	13.8
Switzerland	CHF	0.606	3.212	11.1	11.1
Germany	DEM	0.701	3.616	12.5	18.6
France	FRF	0.719	3.690	12.8	16.1
U.K.	GBP	0.474	2.691	9.3	11.1
Hong Kong	HKD	0.983	4.604	15.9	17.3
Japan	JPY	0.686	4.046	14.0	19.9
South Korea	KRW	0.731	3.873	13.4	25.5
U.S.	USD	0.444	2.380	8.2	12.9
South Africa	ZAR	0.769	4.268	14.8	11.9

taken or hedged in futures. Nearly all of these indices are weighted by market capitalization, although there is now a trend toward weighting by **market float**, which is the market value of freely traded shares. For some companies, a large fraction of outstanding shares may be kept by strategic investors (company management, or a government, for example).

We immediately note that risk is much greater than for currencies, typically ranging from 12% to 20%. Markets that are less diversified or are exposed to greater fluctuations in economic fundamentals are more volatile. **Concentration** refers to the proportion of the index due to the biggest stocks. In Finland, for instance, half of the index represents one firm only, Nokia, which makes the index more volatile than otherwise.

11.5 COMMODITY RISK

Commodity risk arises from potential movements in the value of commodity contracts, which include agricultural products, metals, and energy products.

11.5.1 Commodity Volatility

Table 11.7 displays the volatility of the commodity contracts currently covered by the RiskMetrics system. These can be grouped into *precious metals* (gold, platinum, silver), *base metals* (aluminum, copper, nickel, zinc), and *energy products* (natural gas, heating oil, unleaded gasoline, crude oil–West Texas Intermediate).

Precious metals have an annual volatility ranging from 20% to 30% in 2006, on the same order of magnitude as equity markets. Among base metals, spot volatility are similarly volatile. Energy products, in contrast, are much more volatile with numbers ranging from 20% to 70% in 2006. This is because energy products are less storable than metals and as a result, are much more affected by variations in demand and supply.

TABLE 11.7 Commodity Volatility (Percent)

Commodity Term	Code	End 2006 Daily	Monthly	Annual	End 1996 Annual
Gold, spot	GLD.C00	0.841	5.03	17.4	5.5
Platinum, spot	PLA.C00	1.508	8.45	29.3	6.5
Silver, spot	SLV.C00	1.722	9.40	32.6	18.1
Aluminium, spot	ALU.C00	1.409	7.99	27.7	16.8
15-month	ALU.C15	0.966	5.65	19.6	13.9
Copper, spot	COP.C00	1.479	8.75	30.3	35.4
15-month	COP.C15	1.300	7.82	27.1	21.5
Nickel, spot	NIC.C00	2.060	11.73	40.7	22.7
15-month	NIC.C15	2.327	12.93	44.8	22.7
Zinc, spot	ZNC.C00	1.751	10.24	35.5	12.4
15-month	ZNC.C15	1.282	8.12	28.1	11.6
Natural gas, 1m	GAS.C01	3.910	20.74	71.9	95.8
15-month	GAS.C06	2.396	11.86	41.1	34.4
Heating oil, 1m	HTO.C01	1.824	9.52	33.0	34.4
12-month	HTO.C12	1.075	5.86	20.3	22.7
Unleaded gas, 1m	UNL.C01	1.910	10.25	35.5	31.0
6-month	UNL.C06	1.220	7.08	24.5	23.5
Crude oil, 1m	WTI.C01	1.467	8.21	28.4	32.8
12-month	WTI.C12	1.045	5.60	19.4	28.9

11.5.2 Futures Risk

The forward or futures price on a commodity can be expressed as

$$F_t e^{-r\tau} = S_t e^{-y\tau} \tag{11.9}$$

where $e^{-r\tau}$ is the present value factor in the base currency and $e^{-y\tau}$ includes a **convenience yield** y. Any storage cost should be deducted from y. This represents an implicit flow benefit from holding the commodity, as was explained in Chapter 9. For precious metals, this convenience yield is close to zero.

While this convenience yield is conceptually similar to that of a dividend yield on a stock index, it cannot be measured as regular income. Rather, it should be viewed as a "plug-in" that, given F, S, and $e^{-r\tau}$, will make Equation (11.9) balance. Further, it can be quite volatile.

As Table 11.7 shows, futures prices are less volatile for longer maturities. This decreasing term structure of volatility is more marked for energy products and less so for base metals. In addition, movements in futures prices are much less tightly related to spot prices than for financial contracts.

This is illustrated in Table 11.8, which displays correlations for copper contracts as well as for natural gas contracts. The correlations for natural gas are much lower than for copper. Thus variations in the basis are much more important for energy products than for financial products, or even metals.

TABLE 11.8 Correlations across Maturities, 2006 (Daily)

Copper	COP.C00	COP.C03	COP.C15	COP.C27
COP.C00	1			
COP.C03	.998	1		
COP.C15	.979	.985	1	
COP.C27	.923	.935	.974	1
Nat.Gas	GAS.C01	GAS.C03	GAS.C06	GAS.C12
GAS.C01	1			
GAS.C03	.916	1		
GAS.C06	.897	.968	1	
GAS.C12	.884	.824	.878	1

This is confirmed by Figure 11.6, which compares the spot and futures prices for crude oil. There is much more variation in the basis between the spot and futures prices for crude oil. The market switches from backwardation ($S > F$) to contango ($S < F$). As a result, the futures contract represents a separate risk factor. Energy risk measurement systems require separate risk factors for each maturity.

In addition to traditional market sources of risk, positions in commodity futures are also exposed to delivery and liquidity risks. Asset liquidity risk is due to the relative low volume in some of these markets, relative to other financial products.

Also, taking delivery or having to deliver on a futures contract that is carried to expiration is costly. Transportation, storage and insurance costs can be quite high. Futures delivery also requires complying with the type and location of the commodity that is to be delivered.

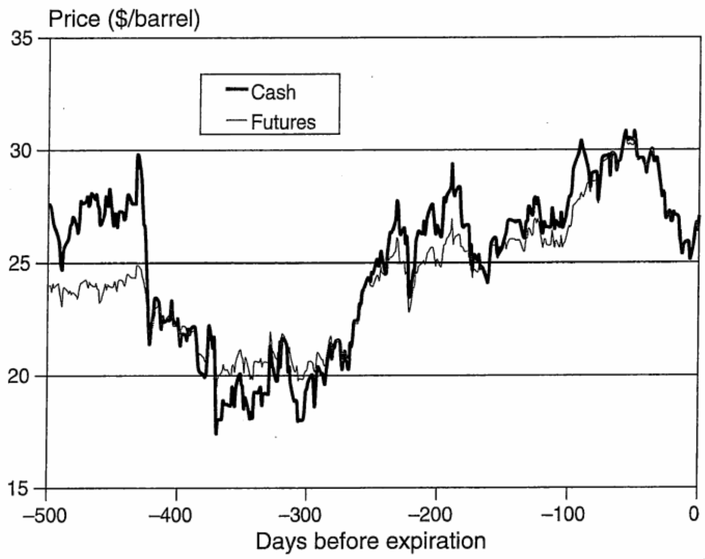

FIGURE 11.6 Futures and Spot for Crude Oil

EXAMPLE 11.5: FRM EXAM 2006—QUESTION 115

Assume the risk-free rate is 5% per annum, the cost of storing oil for a year is 1% per annum, the convenience yield for owning oil for a year is 2% per annum, and the current price of oil is USD 50 per barrel. All rates are continuously compounded. What is the forward price of oil in a year?

a. USD 49.01
b. USD 52.04
c. USD 47.56
d. USD 49.50

EXAMPLE 11.6: FRM EXAM 2006—QUESTION 138

Imagine a stack-and-roll hedge of monthly commodity deliveries that you continue for the next five years. Assume the hedge ratio is adjusted to take into effect the mistiming of cash flows but is not adjusted for the basis risk of the hedge. In which of the following situations is your calendar basis risk likely to be greatest?

a. Stack-and-roll in the front month in oil futures
b. Stack-and-roll in the 12-month contract in natural gas futures
c. Stack-and-roll in the three-year contract in gold futures
d. All four situations will have the same basis risk

11.6 RISK SIMPLIFICATION

The fundamental idea behind modern risk measurement methods is to aggregate the portfolio risk at the highest level. In practice, it would be too complex to model each risk factor individually. Instead, some simplification is required. We have seen, for example, that movements in the terms structure of interest rates could be simplified to a few major risk factors. This approach expands on the **diagonal model** proposed by Professor William Sharpe. This was initially applied to stocks, but the methodology can be used in any market.

11.6.1 Diagonal Model

The diagonal model starts with a statistical decomposition of the return on stock i into a marketwide return and an idiosyncratic risk. The diagonal model adds the assumption that all specific risks are uncorrelated. Hence, any correlation across two stocks must come from the joint effect of the market.

We decompose the return on stock i, R_i, into a constant, a component due to the market, R_M, and some residual risk:

$$R_i = \alpha_i + \beta_i \times R_M + \epsilon_i \tag{11.10}$$

where β_i is called systematic risk of stock i. It is also the regression slope ratio:

$$\beta_i = \frac{\mathrm{Cov}[R_i, R_M]}{V[R_M]} = \rho_{iM}\frac{\sigma(R_i)}{\sigma(R_M)} \tag{11.11}$$

Note that the residual is uncorrelated with R_M by assumption. The contribution of William Sharpe was to show that equilibrium in capital markets imposes restrictions on the α_i. For risk managers, who primarily focus on risk, however, the diagonal model allows considerable simplifications in the risk model, so we ignore the intercept in what follows.

Consider a portfolio that consists of positions w_i on the various assets. We have

$$R_p = \sum_{i=1}^{N} w_i R_i \tag{11.12}$$

Using Equation (11.10), the portfolio return is also

$$R_p = \sum_{i=1}^{N} (w_i \beta_i R_M + w_i \epsilon_i) = \beta_p R_M + \sum_{i=1}^{N} (w_i \epsilon_i) \tag{11.13}$$

The portfolio variance is

$$V[R_p] = \beta_p^2 V[R_M] + \sum_{i=1}^{N} \left(w_i^2 V[\epsilon_i]\right) \tag{11.14}$$

since all the residual terms are uncorrelated. Suppose that, for simplicity, the portfolio is equally weighted and that the residual variances are all the same $V[\epsilon_i] = V$. This implies $w_i = w = 1/N$. As the number of assets, N, increases, the second term will tend to

$$\sum_{i=1}^{N} \left(w_i^2 V[\epsilon_i]\right) \rightarrow N \times [(1/N)^2 V] = (V/N)$$

which should vanish as N increases. In this situation, the only remaining risk is the general market risk, consisting of the beta squared times the variance of the market:

$$V[R_p] \rightarrow \beta_p^2 V[R_M]$$

So, this justifies ignoring specific risk in large, well-diversified portfolios. For portfolio with a small number of stocks concentrated in one sector, this approach will underestimate risk.

More generally, this approach can be expanded to multiple factors. The appendix shows how this approach can be used to build a covariance matrix from general market factors. Each security is "mapped" on the selected risk factors. Exposures are then added up across the entire portfolio, for which risk is aggregated at the top level. This mapping approach is particularly useful when there is no history of returns for some positions. Instead, these positions can be mapped on the risk factors.

11.6.2 Fixed-Income Portfolio Risk

As an example of portfolio simplification, we turn to the analysis of a corporate bond portfolio with N individual bonds. Each "name" is potentially a source of risk. Instead of modeling all securities, the risk manager should attempt to simplify the risk profile of the portfolio. Potential major risk factors are movements in a set of J Treasury zero-coupon rates, z_j, and in K credit spreads, s_k, sorted by credit rating. The goal is to provide a good approximation to the risk of the portfolio.

In addition, it is not practical to model the risk of all bonds. The bonds may not have a sufficient history. Even if they do, the history may not be relevant if it does not account for the probability of default.

We model the movement in each corporate bond yield y_i by a movement in the Treasury factor z_j at the closest maturity and in the credit rating s_k class to which it belongs. The remaining component is ϵ_i, which is assumed to be independent across i. We have $y_i = z_j + s_k + \epsilon_i$. This decomposition is illustrated in Figure 11.7 for a corporate bond rated BBB with a 20-year maturity.

The movement in the bond price is

$$\Delta P_i = -\mathrm{DVBP}_i \Delta y_i = -\mathrm{DVBP}_i \; \Delta z_j - \mathrm{DVBP}_i \; \Delta s_k - \mathrm{DVBP}_i \; \Delta \epsilon_i \qquad (11.15)$$

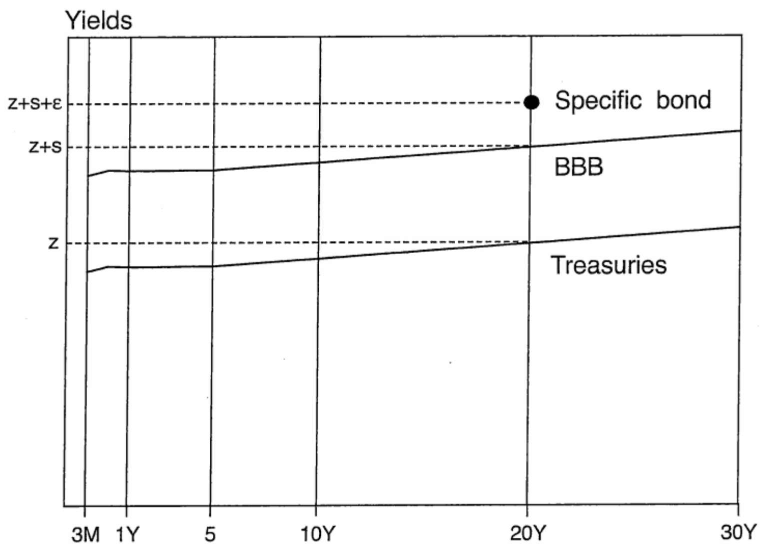

FIGURE 11.7 Yield Decomposition

where DVBP is the total dollar value of a basis point for the associated risk factor. We hold n_i units of this bond, so that its value is

$$P = \sum_{i=1}^{N} n_i P_i \tag{11.16}$$

Expanding the portfolio into its components, we have

$$\Delta P = -\sum_{i=1}^{N} n_i \Delta P_i = -\sum_{i=1}^{N} n_i \text{DVBP}_i \Delta y_i \tag{11.17}$$

Using the risk factor decomposition, the portfolio price movement is

$$\Delta P = -\sum_{j=1}^{J} \text{DVBP}_j^z \, \Delta z_j - \sum_{k=1}^{K} \text{DVBP}_k^s \, \Delta s_k - \sum_{i=1}^{N} n_i \text{DVBP}_i \, \Delta \epsilon_i \tag{11.18}$$

where DVBP_j^z results from the summation of $n_i\text{DVBP}_i$ for all bonds that are exposed to the jth maturity. As in Equation (11.14), The total variance can be decomposed into

$$V[\Delta P] = \text{General Risk} + \sum_{i=1}^{N} n_i^2 \text{DVBP}_i^2 \, V[\Delta \epsilon_i] \tag{11.19}$$

If the portfolio is well diversified, the general risk term should dominate. So, we could simply ignore the second term. Ignoring specific risk, a portfolio composed of thousands of securities can be characterized by its exposure to just a few government maturities and credit spreads. This is a considerable simplification.

EXAMPLE 11.7: FRM EXAM 2002—QUESTION 44

The historical simulation (HS) approach is based on the empirical distributions and a large number of risk factors. The RiskMetrics approach assumes normal distributions and uses mapping on equity indices. The HS approach is more likely to provide an accurate estimate of VAR than the RiskMetrics approach for a portfolio that consists of

a. A small number of emerging market securities
b. A small number of broad market indexes
c. A large number of emerging market securities
d. A large number of broad market indexes

EXAMPLE 11.8: FRM EXAM 2007—QUESTION 11

A hedge fund manager has to choose a risk model for a large "equity market neutral" portfolio, which has zero beta. Many of the stocks held are recent IPOs. Among the following alternatives, the best is

a. A single index model with no specific risk, estimated over the last year
b. A diagonal index model with idiosyncratic risk, estimated over the last year
c. A model that maps positions on industry and style factors
d. A full covariance matrix model using a very short window

11.7 IMPORTANT FORMULAS

Cross-exchange rate: $S_3(\text{EUR/BP}) = \frac{S_1(\$/\text{BP})}{S_2(\$/\text{EUR})}$, $\sigma_3^2 = \sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2$

Volatility of the rate of return in the bond price: $\sigma\left(\frac{dP}{P}\right) = |D^*| \times \sigma(dy)$

Futures price on a commodity: $F_t e^{-r\tau} = S_t e^{-y\tau}$

Diagonal Model: $R_i = \alpha_i + \beta_i \times R_M + \epsilon_i$

11.8 ANSWERS TO CHAPTER EXAMPLES

Example 11.1: Implied Correlation

d. The logs of JPY/EUR and EUR/USD add up to that of JPY/USD: $\ln[\text{JPY/USD}] = \ln[\text{JPY/EUR}] + \ln[\text{EUR/USD}]$. So, $\sigma^2(\text{JPY/USD}) = \sigma^2(\text{JPY/EUR}) + \sigma^2(\text{EUR/USD}) + 2\rho\sigma(\text{JPY/EUR})\sigma(\text{EUR/USD})$, or $8^2 = 10^2 + 6^2 + 2\rho 10 \times 6$, or $2\rho 10 \times 6 = -72$, or $\rho = -0.60$.

Example 11.2: FRM Exam 2007—Question 50

a. The dollar duration of the portfolio is $1 \times \$10 + 9 \times \$10 = \$100$. Multiplied by 0.01 and 1.65, this gives \$1.65.

Example 11.3: FRM Exam 2005—Question 109

b. Answer a. is incorrect, as short-term rates tend to move more than long-term rates. Answer c. understates the explanatory power, which is 95%, not 50%. Answer d. is incorrect, because PCA is a linear combination of the factors.

Example 11.4: FRM Exam 2002—Question 128

d. In 2001, Argentina defaulted on its debt, both in the local currency and in dollars. Answer a. is wrong because a diversified portfolio could not have lost so much. The funds were invested at 80% in dollar-denominated assets, so b. is wrong. Even a total wipeout of the local-currency portion could not explain a loss of 40% on the portfolio. If the fund had bought credit protection, it would have not lost as much, so c. is wrong. The fund must have had credit exposure to Argentina, so answer d. is correct.

Example 11.5: FRM Exam 2006—Question 115

b. Using $F_t e^{-r\tau} = S_t e^{-y\tau}$, we have $F = S\exp(-(y-c)\tau + r\tau) = 50\exp(-(0.02 - 0.01) + 0.05) = 52.04$.

Example 11.6: FRM Exam 2006—Question 138

a. For gold, forward rates closely follow spot rates, so there is little basis risk. For oil and natural gas, there is most movement at the short end of the term structure of futures prices. So using short maturities, or the front month, has the greatest basis risk.

Example 11.7: FRM Exam 2002—Question 44

a. The question deals with the distribution of the assets and the effect of diversification. Emerging market securities are more volatile and less likely to be normally distributed than broad market indices. In addition, a small portfolio is less likely to be well represented by a mapping approach, and is less likely to be normal. The RiskMetrics approach assumes that the conditional distribution is normal and simplifies risk by mapping. This will be acceptable with a large number of securities with distributions close to the normal, which is answer d. Answer a. describes the least diversified portfolio, for which the HS method is best.

Example 11.8: FRM Exam 2007—Question 11

c. Answer a. is incorrect because it only considers the portfolio beta, which is zero by construction. So, it would erroneously conclude that there is no risk. Answer b. is better but would miss the risk of the IPO positions because they have no history. Answer c. will produce unreliable numbers because of the short window. The best solution is to replace the IPO positions by exposures on industry and style factors.

APPENDIX: SIMPLIFICATION OF THE COVARIANCE MATRIX

This appendix shows how the diagonal model can be used to construct a simplified covariance matrix, which is useful for some VAR approaches. Say that we have $N = 100$ assets, which implies a covariance matrix with $N(N+1)/2 = 5{,}050$ entries.

First, we derive the covariance between any two stocks under the one-factor model

$$\mathrm{Cov}[R_i, R_j] = \mathrm{Cov}[\beta_i R_M + \epsilon_i, \beta_j R_M + \epsilon_j] = \beta_i \beta_j \sigma_M^2 \qquad (11.20)$$

using the assumption that the residual components are uncorrelated with each other and with the market. Also, the variance of a stock is

$$\mathrm{Cov}[R_i, R_i] = \beta_i^2 \sigma_M^2 + \sigma_{\epsilon,i}^2 \qquad (11.21)$$

The covariance matrix is then

$$\Sigma = \begin{bmatrix} \beta_1^2\sigma_M^2 + \sigma_{\epsilon,1}^2 & \beta_1\beta_2\sigma_M^2 & \dots & \beta_1\beta_N\sigma_M^2 \\ \vdots & & & \\ \beta_N\beta_1\sigma_M^2 & \beta_N\beta_2\sigma_M^2 & \dots & \beta_N^2\sigma_M^2 + \sigma_{\epsilon,N}^2 \end{bmatrix}$$

which can also be written as

$$\Sigma = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_N \end{bmatrix} [\beta_1 \dots \beta_N]\sigma_M^2 + \begin{bmatrix} \sigma_{\epsilon,1}^2 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & \sigma_{\epsilon,N}^2 \end{bmatrix}$$

Using matrix notation, we have

$$\Sigma = \beta\beta'\sigma_M^2 + D_\epsilon \qquad (11.22)$$

This consists of N elements in the vector β, of N elements on the diagonal of the matrix D_ϵ, plus the variance of the market itself. The diagonal model reduces the number of parameters from $N \times (N+1)/2$ to $2N+1$, a considerable improvement. For example, with 100 assets the number is reduced from 5,050 to 201.

In summary, this diagonal model substantially simplifies the risk structure of an equity portfolio. Risk managers can proceed in two steps: first, managing the overall market risk of the portfolios, and second, managing the concentration risk of individual securities.

Still, this one-factor model could miss common effects among groups of stocks, such as industry effects. To account for these, Equation (11.10) can be generalized to K factors

$$R_i = \alpha_i + \beta_{i1} y_1 + \dots + \beta_{iK} y_K + \epsilon_i \qquad (11.23)$$

where $y_1, \dots, y_K$ are the factors, which are assumed independent of each other for simplification. The covariance matrix generalizes Equation (11.22) to

$$\Sigma = \beta_1\beta_1'\sigma_1^2 + \dots + \beta_K\beta_K'\sigma_K^2 + D_\epsilon \qquad (11.24)$$

The number of parameters is now $(N \times K + K + N)$. For example, with 100 assets and five factors, this number is 605, which is still much lower than 5,050 for the unrestricted model.

CHAPTER 12
Hedging Linear Risk

Risk that has been measured can be managed. This chapter turns to the active management of market risks.

The traditional approach to market risk management includes **hedging.** Hedging consists of taking positions that lower the risk profile of the portfolio. The techniques for hedging have been developed in the futures markets, where farmers, for instance, use financial instruments to hedge the price risk of their products.

This implementation of hedging is quite narrow, however. Its objective is to find the optimal position in a futures contract that minimizes the variance, or more generally the VAR, of the total position. The portfolio consists of two position, a fixed inventory to be hedged and a "hedging" instrument. In this chapter, the value of the hedging instrument is linearly related to the underlying risk factor.

More generally, we can distinguish between

- **Static hedging**, which consists of putting on, and leaving, a position until the hedging horizon.
- **Dynamic hedging**, which consists of continuously rebalancing the portfolio to the horizon. This can create a risk profile similar to positions in options.

Dynamic hedging is associated with options, which will be examined in the next chapter. Since options have nonlinear payoffs in the underlying asset, the hedge ratio, which can be viewed as the slope of the tangent to the payoff function, must be readjusted as the price moves.

Even with static hedging, hedging will create **hedge slippage**, or **basis risk**. Basis risk arises when changes in payoffs on the hedging instrument do not perfectly offset changes in the value of the inventory position.

A final note on hedging is in order. Obviously, if the objective of hedging is to lower volatility, hedging will eliminate downside risk but also any upside potential. The objective of hedging is to lower risk, not to make profits, so this is a double-edged sword. Whether hedging is beneficial should be examined in the context of the trade-off between risk and return.

This chapter discusses linear hedging. A particularly important application is hedging with futures. Section 12.1 presents an introduction to futures hedging with a unit hedge ratio. Section 12.2 then turns to a general method for finding the optimal hedge ratio. This method is applied in Section 12.3 for hedging bonds and equities.

12.1 INTRODUCTION TO FUTURES HEDGING

12.1.1 Unitary Hedging

Consider the situation of a U.S. exporter who has been promised a payment of 125 million Japanese yen in seven months. This is cash position, or anticipated inventory. The perfect hedge would be to enter a seven-month forward contract over-the-counter (OTC). Assume for this illustration that this OTC contract is not convenient. Instead, the exporter decides to turn to an exchange-traded futures contract, which can be bought or sold easily.

The Chicago Mercantile Exchange (CME) lists yen contracts with face amount of Y12,500,000 that expire in nine months. The exporter places an order to sell 10 contracts, with the intention of reversing the position in seven months, when the contract will still have two months to maturity.[1] Because the amount sold is the same as the underlying, this is called a **unitary hedge**.

Table 12.1 describes the initial and final conditions for the contract. At each date, the futures price is determined by interest parity. Suppose that the yen depreciates sharply, or that the dollar goes up from Y125 to Y150. This lead to a loss on the anticipated cash position of Y125,000,000 $\times$ (0.006667 $-$ 0.00800) $= -$\$166,667. This loss, however, is offset by a gain on the futures, which is ($-$10) $\times$ Y12,500,000 $\times$ (0.006711 $-$ 0.00806) $=$ \$168,621. The net is a small gain of \$1,954. Overall, the exporter has been hedged.

This example shows that futures hedging can be quite effective, removing the effect of fluctuations in the risk factor. Define Q as the amount of yen transacted and S and F as the spot and futures rates, indexed by 1 at the initial time and by 2

TABLE 12.1 A Futures Hedge

Item	Initial Time	Exit Time	Gain or Loss
Market Data:			
Maturity (months)	9	2	
US interest rate	6%	6%	
Yen interest rate	5%	2%	
Spot (Y/\$)	Y125.00	Y150.00	
Futures (Y/\$)	Y124.07	Y149.00	
Contract Data:			
Spot (\$/Y)	0.008000	0.006667	−\$166,667
Futures (\$/Y)	0.008060	0.006711	\$168,621
Basis (\$/Y)	−0.000060	−0.000045	\$1,954

[1] In practice, if the liquidity of long-dated contracts is not adequate, the exporter could use nearby contracts and roll them over prior to expiration into the next contracts. When there are multiple exposures, this practice is known as a **stack hedge**. Another type of hedge is the **strip hedge**, which involves hedging the exposures with a number of different contracts. While a stack hedge has superior liquidity, it also entails greater basis risk than a strip hedge. Hedgers must decide whether the greater liquidity of a stack hedge is worth the additional basis risk.

at the exit time. The P&L on the unhedged transaction is

$$Q[S_2 - S_1] \tag{12.1}$$

Instead, the hedged profit is

$$Q[(S_2 - S_1) - (F_2 - F_1)] = Q[(S_2 - F_2) - (S_1 - F_1)] = Q[b_2 - b_1] \tag{12.2}$$

where $b = S - F$ is the **basis**. The profit only depends on the movement in the basis. Hence the effect of hedging is to transform price risk into basis risk. A short hedge position is said to be *long the basis*, since it benefits from an increase in the basis.

In this case, the basis risk is minimal for a number of reasons. First, the cash and futures correspond to the same asset. Second, the cash-and-carry relationship holds very well for currencies. Third, the remaining maturity at exit is rather short. This is not always the case, however.

12.1.2 Basis Risk

Basis risk arises when the characteristics of the futures contract differ from those of the underlying position. Futures contracts are standardized to a particular grade, say West Texas Intermediate (WTI) for oil futures traded on the NYMEX. This defines the grade of crude oil deliverable against the contract. A hedger, however, may have a position in a different grade, which may not be perfectly correlated with WTI. Thus basis risk is the uncertainty whether the cash-futures spread will widen or narrow during the hedging period. Hedging can be effective, however, if movements in the basis are dominated by movements in cash markets.

For most commodities, basis risk is inevitable. Organized exchanges strive to create enough trading and liquidity in their listed contracts, which requires standardization. Speculators also help to increase trading volumes and provide market liquidity. Thus there is a trade-off between liquidity and basis risk.

Basis risk is higher with **cross-hedging**, which involves using a futures on a totally different asset or commodity than the cash position. For instance, a U.S. exporter who is due to receive a payment in Norwegian Kroner (NK) could hedge using a futures contract on the $/euro exchange rate. Relative to the dollar, the euro and the NK should behave similarly, but there is still some basis risk.

Basis risk is lowest when the underlying position and the futures correspond to the same asset. Even so, some basis risk remains because of differing maturities. As we have seen in the yen hedging example, the maturity of the futures contract is 9 instead of 7 months. As a result, the liquidation price of the futures is uncertain.

Figure 12.1 describes the various time components for a hedge using T-bond futures. The first component is the *maturity of the underlying bond*, say 20 years. The second component is the *time to futures expiration*, say nine months. The third component is the *hedge horizon*, say seven months. Basis risk occurs when the hedge horizon does not match the time to futures expiration.

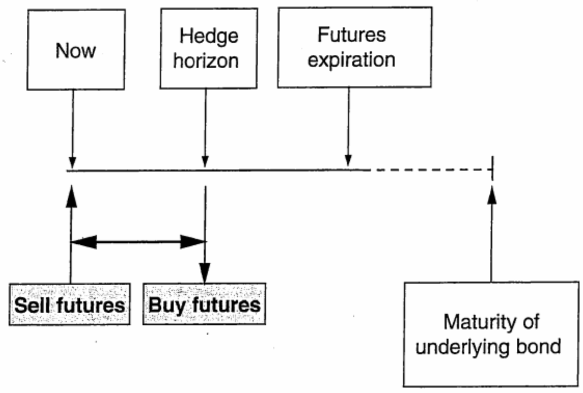

FIGURE 12.1 Hedging Horizon and Contract Maturity

EXAMPLE 12.1: FRM EXAM 2000—QUESTION 79

Under which scenario is basis risk likely to exist?

a. A hedge (which was initially matched to the maturity of the underlying) is lifted before expiration.
b. The correlation of the underlying and the hedge vehicle is less than one and their volatilities are unequal.
c. The underlying instrument and the hedge vehicle are dissimilar.
d. All of the above are correct.

EXAMPLE 12.2: FRM EXAM 2007—QUESTION 99

Which of the following trade(s) contain mainly basis risk?

I. Long 1,000 lots Nov 07 ICE Brent Oil contracts and short 1,000 lots Nov 07 NYMEX WTI Crude Oil contracts
II. Long 1,000 lots Nov 07 ICE Brent Oil contracts and long 2,000 lots Nov 07 ICE Brent Oil at-the-money put
III. Long 1,000 lots Nov 07 ICE Brent Oil contracts and short 1,000 lots Dec 07 ICE Brent Oil contracts
IV. Long 1,000 lots Nov 07 ICE Brent Oil contracts and short 1,000 lots Dec 07 NYMEX WTI Crude Oil contracts

a. II and IV only
b. I and III only
c. I, III, and IV only
d. III and IV only

12.2 OPTIMAL HEDGING

The previous section gave an example of a unit hedge, where the amounts transacted are identical in the two markets. In general, this is not appropriate. We have to decide how much of the hedging instrument to transact.

Consider a situation where a portfolio manager has an inventory of carefully selected corporate bonds that should do better than their benchmark. The manager wants to guard against interest rate increases, however, over the next three months. In this situation, it would be too costly to sell the entire portfolio only to buy it back later. Instead, the manager can implement a temporary hedge using derivative contracts, for instance T-bond futures.

Here, we note that the only risk is **price risk**, as the quantity of the inventory is known. This may not always be the case, however. Farmers, for instance, have uncertainty over both prices and the size of their crop. If so, the hedging problem is substantially more complex as it involves hedging *revenues*, which involves analyzing demand and supply conditions.

12.2.1 The Optimal Hedge Ratio

Define ΔS as the change in the dollar value of the inventory and ΔF as the change in the dollar value of one futures contract. The inventory, or position to be hedged, can be existing or **anticipatory**, that is, to be received in the future with a great degree of certainty. The manager is worried about potential movements in the value of the inventory ΔS.

If the manager goes long N futures contracts, the total change in the value of the portfolio is

$$\Delta V = \Delta S + N\Delta F \tag{12.3}$$

One should try to find the hedge that reduces risk to the minimum level. The variance of total profits is equal to

$$\sigma^2_{\Delta V} = \sigma^2_{\Delta S} + N^2\sigma^2_{\Delta F} + 2N\sigma_{\Delta S,\Delta F} \tag{12.4}$$

Note that volatilities are initially expressed in dollars, not in rates of return, as we attempt to stabilize dollar values.

Taking the derivative with respect to N

$$\frac{\partial \sigma^2_{\Delta V}}{\partial N} = 2N\sigma^2_{\Delta F} + 2\sigma_{\Delta S,\Delta F} \tag{12.5}$$

For simplicity, drop the Δ in the subscripts. Setting Equation (12.5) equal to zero and solving for N, we get

$$N^* = -\frac{\sigma_{\Delta S,\Delta F}}{\sigma^2_{\Delta F}} = -\frac{\sigma_{SF}}{\sigma^2_F} = -\rho_{SF}\frac{\sigma_S}{\sigma_F} \tag{12.6}$$

where σ_{SF} is the covariance between futures and spot price changes. Here, N^* is the **minimum variance hedge ratio.**

In practice, there is often confusion about the definition of the portfolio value and unit prices. Here S consists of the number of units (shares, bonds, bushels, gallons) times the unit price (stock price, bond price, wheat price, fuel price).

It is sometimes easier to deal with unit prices and to express volatilities in terms of *rates of changes in unit prices*, which are unitless. Defining quantities Q and unit prices s, we have $S = Qs$. Similarly, the notional amount of one futures contract is $F = Q_f f$. We can then write

$$\sigma_{\Delta S} = Q\sigma(\Delta s) = Qs\sigma(\Delta s/s)$$
$$\sigma_{\Delta F} = Q_f\sigma(\Delta f) = Q_f f\sigma(\Delta f/f)$$
$$\sigma_{\Delta S,\Delta F} = \rho_{sf}[Qs\sigma(\Delta s/s)][Q_f f\sigma(\Delta f/f)]$$

Using Equation (12.6), the optimal hedge ratio N^* can also be expressed as

$$N^* = -\rho_{SF}\frac{Qs\sigma(\Delta s/s)}{Q_f f\sigma(\Delta f/f)} = -\rho_{SF}\frac{\sigma(\Delta s/s)}{\sigma(\Delta f/f)}\frac{Qs}{Q_f f} = -\beta_{sf}\frac{Q \times s}{Q_f \times f} \tag{12.7}$$

where β_{sf} is the coefficient in the regression of $\Delta s/s$ over $\Delta f/f$. The second term represents an adjustment factor for the size of the cash position and of the futures contract.

The optimal amount N^* can be derived from the slope coefficient of a regression of $\Delta s/s$ on $\Delta f/f$:

$$\frac{\Delta s}{s} = \alpha + \beta_{sf}\frac{\Delta f}{f} + \epsilon \tag{12.8}$$

As seen in Chapter 3, standard regression theory shows that

$$\beta_{sf} = \frac{\sigma_{sf}}{\sigma_f^2} = \rho_{sf}\frac{\sigma_s}{\sigma_f} \tag{12.9}$$

Thus, the **best hedge** is obtained from a regression of the (change in the) value of the inventory on the value of the hedge instrument.

KEY CONCEPT

The optimal hedge is given by the negative of the beta coefficient of a regression of changes in the cash value on changes in the payoff on the hedging instrument.