Table 5: Team output

	Excellent **($\boldsymbol{\theta_1}$)**	**Good** **($\boldsymbol{\theta_2}$)**	**Bad** **($\boldsymbol{\theta_3}$)**
$High, High$	100	100	60
$High, Low$	100	50	20
Low, Low	50	20	0

a Suppose $\theta_1 = \theta_2 = \theta_3$. Why is $\{(w(100) = 30, w(not\ 100) = 0), (High, High)\}$ not an equilibrium?

b Suppose $\theta_1 = \theta_2 = \theta_3$. Is it optimal to induce high effort? What is an optimal contract with nonnegative wages?

c Suppose $\theta_1 = 0.5$, $\theta_2 = 0.5$, and $\theta_3 = 0$. Is it optimal to induce high effort? What is an optimal contract (possibly with negative wages)?

d Should the principal stop the agents from talking to each other?

8.5: Efficiency Wages and Risk Aversion (see Rasmusen [1992c])
In each of two periods of work, a worker decides whether to steal amount v, and is detected with probability α and suffers legal penalty p if he, in fact, did steal. A worker who is caught stealing can also be fired, after which he earns the reservation wage w_0. If the worker does not steal, his utility in the period is $U(w)$; if he steals, it is $U(w+v) - \alpha p$, where $U(w_0+v) - \alpha p > U(w_0)$. The worker's marginal utility of income is diminishing: $U' > 0$, $U'' < 0$, and $lim_{x\rightarrow\infty} U'(x) = 0$. There is no discounting. The firm definitely wants to deter stealing in each period, if at all possible.

(a) Show that the firm can indeed deter theft, even in the second period, and, in fact, do so with a second-period wage w_2^* that is higher than the reservation wage w_0.

(b) Show that the equilibrium second-period wage w_2^* is higher than the first-period wage w_1^*.

8.6. The Revelation Principle
If you apply the Revelation Principle, that
(a) Increases the welfare of all the players in the model.
(b) Increases the welfare of just the player offering the contract.
(c) Increases the welfare of just the player accepting the contract.
(d) Makes the problem easier to model, but does not raise the welfare of the players.
(e) Makes the problem easier to model and raises the welfare of some players, but not all.

8.7 Machinery
Mr. Smith is thinking of buying a custom- designed machine from either Mr. Jones or Mr. Brown. This machine costs 5000 dollars to build, and it is useless to anyone but Smith. It is common knowledge that with 90 percent probability the machine will be worth 10,000 dollars to Smith at the time of delivery, one year from today, and with 10 percent probability it will only be worth 2,000 dollars. Smith owns assets of 1,000 dollars. At the time of contracting, Jones and Brown

believe there is there is a 20 percent chance that Smith is actually acting as an "undisclosed agent" for Anderson, who has assets of 50,000 dollars.

Find the price be under the following two legal regimes: (a) An undisclosed principal is not responsible for the debts of his agent; and (b) even an undisclosed principal is responsible for the debts of his agent. Also, explain (as part [c]) which rule a moral hazard model like this would tend to support.

September 11, 1999. January 18, 2000. August 5, 2003. November 30, 2003. 25 March 2005. Eric Rasmusen, Erasmuse@indiana.edu. Http://www.rasmusen.org. Footnotes starting with xxx are the author's notes to himself. Comments welcomed.

9 Adverse Selection

9.1 Introduction: Production Game VI

In Chapter 7, games of asymmetric information were divided between games with moral hazard, in which agents are identical, and games with adverse selection, in which agents differ. In moral hazard with hidden knowledge and adverse selection, the principal tries to sort out agents of different types. In moral hazard with hidden knowledge, the emphasis is on the agent's action rather than his choice of contract, and agents accept contracts before acquiring information. Under adverse selection, the agent has private information about his type or the state of the world before he agrees to a contract, which means that the emphasis is on which contract he will accept.

For comparison with moral hazard, let us consider still another version of the Production Game of Chapters 7 and 8.

Production Game VI: Adverse Selection

Players
The principal and the agent.

The Order of Play

(0) Nature chooses the agent's ability a, unobserved by the principal, according to distribution $F(a)$.

(1) The principal offers the agent one or more wage contracts $w_1(q), w_2(q), \ldots$

(2) The agent accepts one contract or rejects them all.

(3) Nature chooses a value for the state of the world, θ, according to distribution $G(\theta)$. Output is then $q = q(a, \theta)$.

Payoffs
If the agent rejects all contracts, then $\pi_{agent} = \overline{U}$ and $\pi_{principal} = 0$.
Otherwise, $\pi_{agent} = U(w)$ and $\pi_{principal} = V(q - w)$.

Under adverse selection, it is not the worker's effort but his ability that is noncontractible. Without uncertainty (move (3)), the principal would provide a single contract specifying high wages for high output and low wages for low output, but unlike under moral hazard, either high or low output might be observed in equilibrium if both types of agent accepted the contract. Also, in adverse selection, unlike moral hazard, offering multiple contracts can be an improvement over offering a single contract. The principal might, for example, provide a contract with a flat wage for the low-ability agents and an incentive contract for the high-ability agents. Production Game VIa puts specific functional forms into the game to illustrate equilibium.

Production Game VIa: Adverse Selection, with Particular Parameters '

Players
The principal and the agent.

The Order of Play

(0) Nature chooses the agent's ability a, unobserved by the principal, according to distribution $F(a)$, which puts probability 0.9 on low ability, $a = 0$, and probability 0.1 on high ability, $a = 10$.

(1) The principal offers the agent one or more wage contracts $W_1 = \{w_1(q = 0), w_1(q = 10)\}, W_2 = \{w_2(q = 0), w_2(q = 10)\} \ldots$

(2) The agent accepts one contract or rejects them all.

(3) Nature chooses a value for the state of the world, θ, according to distribution $G(\theta)$, which puts equal weight on 0 and 10. Output is then $q = Min(a + \theta, 10)$. (Thus, output is 0 or 10 for the low-ability agent, and always 10 for hte high-ability.)

Payoffs
If the agent rejects all contracts, then depending on his type his reservation payoff is either $\pi_L = 3$ or $\pi_H = 4$ and the principal's payoff is $\pi_{principal} = 0$.
Otherwise, $\pi_{agent} = w$ and $\pi_{principal} = q - w$.

An equilibrium is

$$\begin{array}{l} Principal: Offer\ W_1 = \{w_1(q = 0) = 3, w_1(q = 10) = 3\}, W_2 = \{w_2(q = 0) = 0, w_2(q = 10) = 4\} \\ Low\ agent: \ \ Accept\ W_1. \\ High\ agent: \ \ Accept\ W_2. \end{array}$$

As usual, this is a weak equilibrium. Both Low and High agents are indifferent about whether they accept or reject a contract. But the equilibrium indifference of the agents arises from the open-set problem; if the principal were to specify a wage of 2.01 for W_1, for example, the low-ability agent would no longer be indifferent about accepting it.

This equilibrium can be obtained by what is a standard method for hidden-knowledge models. In hidden-action models, the principal tries to construct a contract which will induce the agent to take the single appropriate action. In hidden-knowledge models, the principal tries to make different actions attractive under different states of the world, so the agent's choice depends on the hidden information. The principal's problem, as in Production Game V, is to maximize his profits subject to

(1) **Incentive compatibility** (the agent picks the desired contract and actions).

(2) **Participation** (the agent prefers the contract to his reservation utility).

In a model with hidden knowledge, the incentive compatibility constraint is customarily called the **self-selection constraint**, because it induces the different types of agents to pick different contracts. The big difference is that there will be an entire set of self-selection constraints, one for each type of agent or each state of the world, since the appropriate contract depends on the hidden information.

First, what action does the principal desire from each type of agent? The agents do not choose effort, but they do choose whether or not to work for the principal, and which contract to accept. The low-ability agent's expected output is 0.5(0) + 0.5(10)= 5, compared to a reservation payoff of 3, so the principal will want to hire the low- ability agent if he can do it at an expected wage of 5 or less. The high ability agent's expected output is 0.5(10) + 0.5(10)= 10, compared to a reservation payoff of 4, so the principal will want to hire the high- ability agent, if he can do it at an expected wage of 10 or less. The principal will want to induce the low-ability agent to choose a cheaper contract and not to choose the necessarily more expensive contract needed to attract the high-ability agent.

The participation constraints are

$$\begin{aligned} U_L(W_1) \geq \overline{\pi_L};\ \ 0.5w_1(0) + 0.5w_1(10) \geq 3 \\ U_H(W_2) \geq \overline{\pi_H};\ \ 0.5w_2(10) + 0.5w_2(10) \geq 4 \end{aligned} \tag{1}$$

Clearly the contracts $W_1 = \{3, 3\}$ and $W_2 = \{0, 10\}$ satisfy the participation constraints. The constraints show that both the low- output wage and the high-output wage matter to the low-ability agent, but only the high-output wage matters to the high-ability agent, so it makes sense to make W_2 as risky as possible.

The self selection constraints are

$$\begin{aligned} U_L(W_1) \geq U_L(W_2);\ \ 0.5w_1(0) + 0.5w_1(10) \geq 0.5w_2(0) + 0.5w_2(10) \\ U_H(W_2) \geq U_H(W_1);\ \ 0.5w_2(10) + 0.5w_2(10) \geq 0.5w_1(10) + 0.5w_1(10) \end{aligned} \tag{2}$$

The risky wage contract W_2 has to have a low enough expected return for the low-ability agent to deter him from accepting it; but the safe wage contract W_1 must be less attractive than W_1 to the high-ability agent. The contracts $W_1 = \{3, 3\}$ and $W_2 = \{0, 10\}$ do this, as can be seen by substituting their values into the constraints:

$$\begin{aligned} U_L(W_1) \geq U_L(W_2);\ \ 0.5(3) + 0.5(3) \geq 0.5(0) + 0.5(4) \\ U_H(W_2) \geq U_H(W_1);\ \ 0.5(4) + 0.5(4) \geq 0.5(3)) + 0.5(3) \end{aligned} \tag{3}$$

Since the self selection and participation constraints are satisfied, the agents will not deviate from their equilibrium actions. All that remains to check is whether the principal could increase his payoff. He cannot, because he makes a profit from either contract, and having driven the agents down to their reservation utilities, he cannot further reduce their pay.

As with hidden actions, if principals compete in offering contracts under hidden information, a **competition constraint** is added: the equilibrium contract must be as attractive as possible to the agent, since otherwise another principal could profitably lure him away. An equilibrium may also need to satisfy a part of the competition constraint not found in hidden actions models: either a **nonpooling constraint** or a **nonseparating constraint**. If one of several competing principals wishes to construct a pair of separating contracts C_1 and C_2, he must construct it so that not only do agents choose C_1 and C_2 depending on the state of the world (to satisfy incentive compatibility), but also they prefer (C_1, C_2) to a pooling contract C_3 (to satisfy nonpooling). We only have one principal in Production Game VI, though, so competition constraints are irrelevant.

It is always true that the self selection and participation constraints must be satisfied for agents who accept the contracts, but it is not always the case that they accept different contracts.

If all types of agents choose the same strategy in all states, the equilibrium is **pooling**. *Otherwise, it is* **separating**.

The distinction between pooling and separating is different from the distinction between equilibrium concepts. A model might have multiple Nash equilibria, some pooling and some separating. Moreover, a single equilibrium— even a pooling one— can include several contracts, but if it is pooling the agent always uses the same strategy, regardless of type. If the agent's equilibrium strategy is mixed, the equilibrium is pooling if the agent always picks the same mixed strategy, even though the messages and efforts would differ across realizations of the game.

These two terms came up in Section 6.2 in the game of PhD Admissions. Neither type of student applied in the pooling equilibrium, but one type did in the separating equilibrium. In a principal-agent model, the principal tries to design the contract to achieve separation unless the incentives turn out to be too costly. In Production Game VI, the equilibrium was separating, since the two types of agents choose different contracts.

A separating contract need not be fully separating. If agents who observe $\theta \leq 4$ accept contract C_1 but other agents accept C_2, then the equilibrium is separating but it does not separate out every type. We say that the equilibrium is **fully revealing** if the agent's choice of contract always conveys his private information to the principal. Between pooling and fully revealing equilibria are the **imperfectly separating** equilibria synonymously called **semi-separating**, **partially separating**, **partially revealing**, or **partially pooling** equilibria.

Production Game VI is a fairly complicated game, so let us start in Sections 9.2 and 9.3 with a certainty game, although we will return to uncertainty in Section 9.4. The first game will model a used car market in which the quality of the car is known to the seller but not the buyer, and the various versions of the game will differ in the types and numbers of the buyers and sellers. Section 9.4 will return to models with uncertainty, in a model of adverse selection in insurance. One result there will be that a Nash equilibrium in pure strategies fails to exist for certain parameter values. Section 9.5 applies the idea of adverse selection to explain the magnitude of the bid-ask spread in financial markets, and Section 9.6 touches on a variety of other applications.

9.2 Adverse Selection under Certainty: Lemons I and II

Akerlof stimulated an entire field of research with his 1970 model of the market for shoddy used cars ("lemons"), in which adverse selection arises because car quality is better known to the seller than to the buyer. In agency terms, the principal contracts to buy from the agent a car whose quality, which might be high or low, is noncontractible despite the lack of uncertainty. Such a model may sound like moral hazard with hidden knowledge, but the difference is that in the used car market the seller has private information about his own type before making any kind of agreement. If, instead, the seller agreed to resell his car when he first bought it, the model would be moral hazard with hidden knowledge, because there would be no asymmetric information at the time of contracting, just an expectation of future asymmetry.

We will spend considerable time adding twists to a model of the market in used cars. The game will have one buyer and one seller, but this will simulate competition between buyers, as discussed in Section 7.2, because the seller moves first. If the model had symmetric information there would be no consumer surplus. It will often be convenient to discuss the game as if it had many sellers, interpreting a seller whom Nature randomly assigns a type as a population of sellers of different types, one of whom is drawn by Nature to participate in the game.

The Basic Lemons Model

Players
A buyer and a seller.

The Order of Play

(0) Nature chooses quality type θ for the seller according to the distribution $F(\theta)$. The seller knows θ, but while the buyer knows F, he does not know the θ of the particular seller he faces.

(1) The buyer offers a price P.

(2) The seller accepts or rejects.

Payoffs
If the buyer rejects the offer, both players receive payoffs of zero.
Otherwise, $\pi_{buyer} = V(\theta) - P$ and $\pi_{seller} = P - U(\theta)$, where V and U will be defined later.

The payoffs of both players are normalized to zero if no transaction takes place. A normalization is part of the notation of the model rather than a substantive assumption.

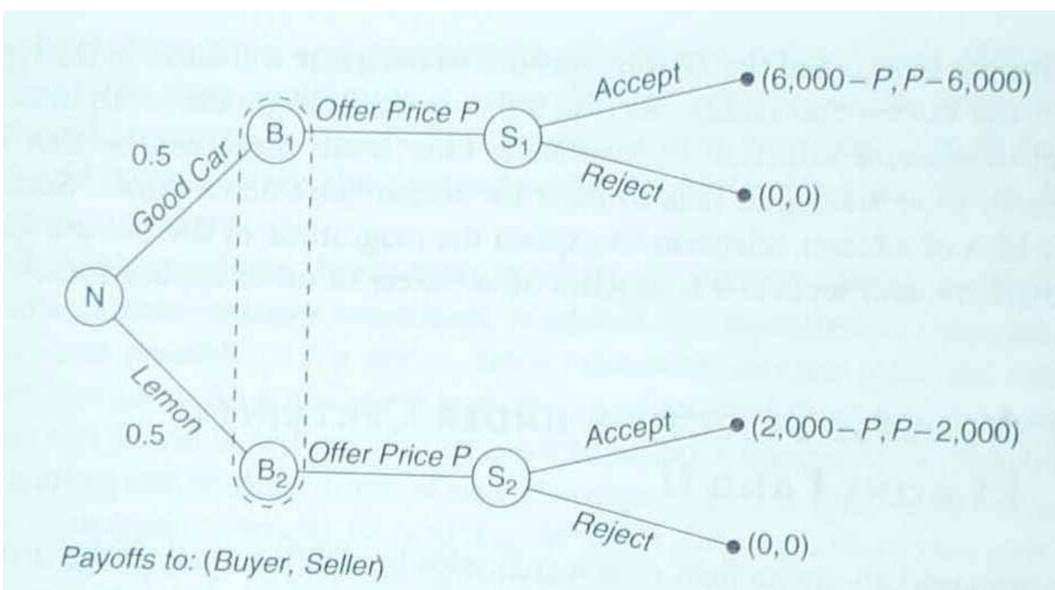
Here, the model assigns the players' utility a base value of zero when no transaction takes place, and the payoff functions show changes from that base. The seller, for instance, gains P if the sale takes place but loses $U(\theta)$ from giving up the car.

There are various ways to specify $F(\theta)$, $U(\theta)$, and $V(\theta)$. We start with identical tastes and two types (Lemons I), and generalize to a continuum of types (Lemons II). Section 9.3 specifies first that the sellers are identical and value cars more than buyers (Lemons III), next that the sellers have heterogeneous tastes (Lemons IV). We will look less formally at other modifications involving risk aversion and the relative numbers of buyers and sellers.

Lemons I: Identical Tastes, Two Types of Sellers

Let good cars have quality 6,000 and bad cars (lemons) quality 2,000, so $\theta \in \{2,000,\ 6,000\}$, and suppose that half the cars in the world are of the first type and the other half of the second type. A payoff combination of (0,0) will represent the status quo, in which the buyer has \$50,000 and the seller has the car. Assume that both players are risk neutral and they value quality at one dollar per unit, so after a trade the payoffs are $\pi_{buyer} = \theta - P$ and $\pi_{seller} = P - \theta$. The extensive form is shown in Figure 1.

Figure 1: An Extensive Form for Lemons I



If he could observe quality at the time of his purchase, the buyer would be willing to accept a contract to pay \$6,000 for a good car and \$2,000 for a lemon. He cannot observe quality, and we assume that he cannot enforce a contract based on his discoveries once the purchase is made. Given these restrictions, if the seller offers \$4,000, a price equal to the average quality, the buyer will deduce that the seller does not have a good car. The very fact that the car is for sale demonstrates its low quality. Knowing that for \$4,000 he would be sold only lemons, the buyer would refuse to pay more than \$2,000. Let us assume that an indifferent seller sells his car, in which case half of the cars are traded in equilibrium, all of them lemons.

A friendly advisor might suggest to the owner of a good car that he wait until all the lemons have been sold and then sell his own car, since everyone knows that only good cars have remained unsold. But allowing for such behavior changes the model by adding a new action. If it were anticipated, the owners of lemons would also hold back and wait for the price to rise. Such a game could be formally analyzed as a War of Attrition (Section 3.2).

The outcome that half the cars are held off the market is interesting, though not startling, since half the cars do have genuinely higher quality. It is a formalization of Groucho Marx's wisecrack that he would refuse to join any club that would accept him as a member. Lemons II will have a more dramatic outcome.

Lemons II: Identical Tastes, a Continuum of Types of Sellers

One might wonder whether the outcome of Lemons I was an artifact of the assumption of just two types. Lemons II generalizes the game by allowing the seller to be any of a continuum of types. We will assume that the quality types are uniformly distributed between $2,000$ and $6,000$. The average quality is $\overline{\theta} = 4,000$, which is therefore the price the buyer would be willing to pay for a car of unknown quality if all cars were on the market. The probability density is zero except on the support [2,000, 6,000], where it is $f(\theta) = 1/(6,000 - 2,000)$, and the cumulative density is

$$F(\theta) = \int_{2,000}^{\theta} f(x)dx. \tag{4}$$

After substituting the uniform density for $f(\theta)$ and integrating (1) we obtain

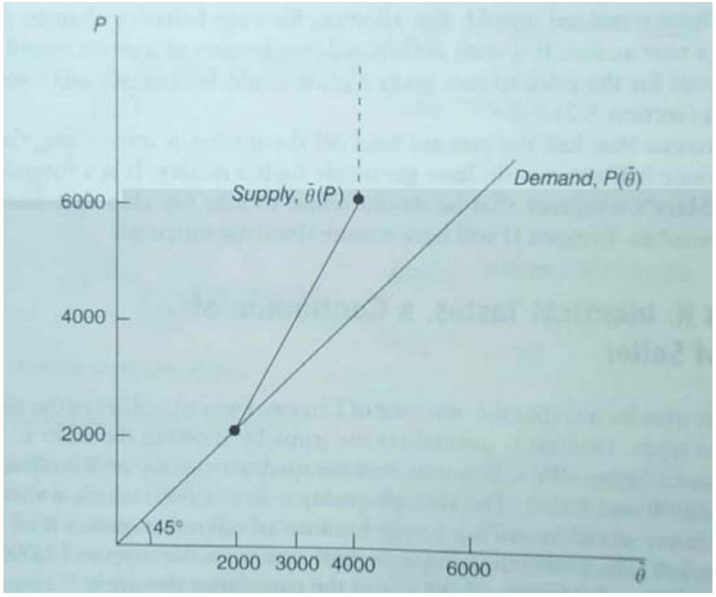
$$F(\theta) = \frac{\theta}{4,000} - 0.5. \tag{5}$$

The payoff functions are the same as in Lemons I.

The equilibrium price must be less than \$4,000 in Lemons II because, as in Lemons I, not all cars are put on the market at that price. Owners are willing to sell only if the quality of their cars is less than 4,000, so while the average quality of all used cars is 4,000, the average quality offered for sale is 3,000. The price cannot be \$4,000 when the average quality is 3,000, so the price must drop at least to \$3,000. If that happens, the owners of cars with values from 3,000 to 4,000 pull their cars off the market and the average of those remaining is 2,500. The acceptable price falls to \$2,500, and the unravelling continues until the price reaches its equilibrium level of \$2,000. But at $P = 2,000$ the number of cars on the market is infinitesimal. The market has completely collapsed!

Figure 2 puts the price of used cars on one axis and the average quality of cars offered for sale on the other. Each price leads to a different average quality, $\overline{\theta}(P)$, and the slope of $\overline{\theta}(P)$ is greater than one because average quality does not rise proportionately with price. If the price rises, the quality of the *marginal* car offered for sale equals the new price, but the quality of the *average* car offered for sale is much lower. In equilibrium, the average quality must equal the price, so the equilibrium lies on the 45° line through the origin. That line is a demand schedule of sorts, just as $\overline{\theta}(P)$ is a supply schedule. The only intersection is the point (\$2,000, 2,000).

Figure 2: Lemons II: Identical Tastes



9.3 Heterogeneous Tastes: Lemons III and IV

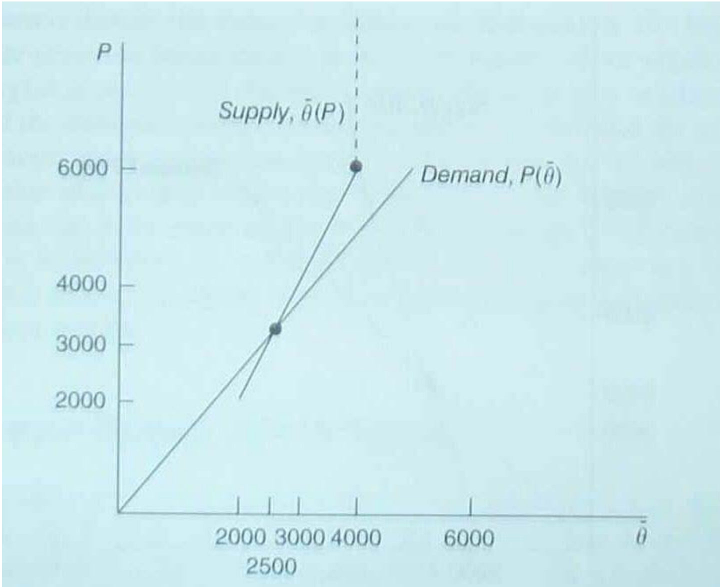
The outcome that no cars are traded is extreme, but there is no efficiency loss in either Lemons I or Lemons II. Since all the players have identical tastes, it does not matter who ends up owning the cars. But the players of this section, whose tastes differ, have real need of a market.

Lemons III : Buyers Value Cars More than Sellers

Assume that sellers value their cars at exactly their qualities θ, but that buyers have valuations 20 percent greater, and, moreover, outnumber the sellers. The payoffs if a trade occurs are $\pi_{buyer} = 1.2\theta - P$ and $\pi_{seller} = P - \theta$. In equilibrium, the sellers will capture the gains from trade.

In Figure 3, the curve $\overline{\theta}(P)$ is much the same as in Lemons II, but the equilibrium condition is no longer that price and average quality lie on the 45° line, but that they lie on the demand schedule $P(\overline{\theta})$, which has a slope of 1.2 instead of 1.0. The demand and supply schedules intersect only at $(P = \$3,000, \overline{\theta}(P) = 2,500)$. Because buyers are willing to pay a premium, we only see **partial adverse selection;** the equilibrium is partially pooling. The outcome is inefficient, because in a world of perfect information all the cars would be owned by the "buyers," who value them more, but under adverse selection they only end up owning the low-quality cars.

Figure 3: Adverse Selection When Buyers Value Cars More Than Sellers: Lemons III



Lemons IV : Sellers' Valuations Differ

In Lemons IV, we dig a little deeper to explain why trade occurs, and we model sellers as consumers whose valuations of quality have changed since they bought their cars. For a particular seller, the valuation of one unit of quality is $1+\varepsilon$, where the random disturbance ε can be either positive or negative and has an expected value of zero. The disturbance could arise because of the seller's mistake— he did not realize how much he would enjoy driving when he bought the car— or because conditions changed— he switched to a job closer to home. Payoffs if a trade occurs are $\pi_{buyer} = \theta - P$ and $\pi_{seller} = P - (1+\varepsilon)\theta$.

If $\varepsilon = -0.15$ and $\theta = 2,000$, then \$1,700 is the lowest price at which the player would resell his car. The average quality of cars offered for sale at price P is the expected quality of cars valued by their owners at less than P, i.e.,

$$\overline{\theta}(P) = E\left(\theta \mid (1+\varepsilon)\theta \leq P\right). \tag{6}$$

Suppose that a large number of new buyers, greater in number than the sellers, appear in the market, and let their valuation of one unit of quality be \$1. The demand schedule, shown in Figure 4, is the 45° line through the origin. Figure 4 shows one possible shape for the supply schedule $\overline{\theta}(P)$, although to specify it precisely we would have to specify the distribution of the disturbances.

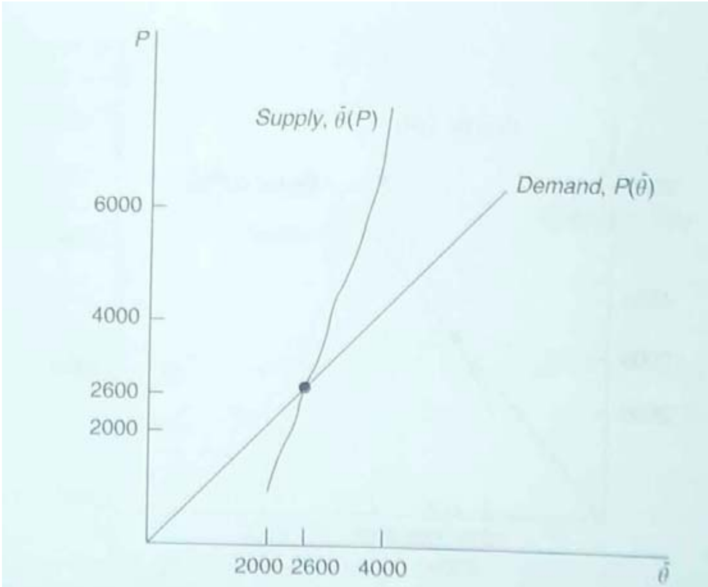


Figure 4: Lemons IV: Sellers' Valuations Differ

In contrast to Lemons I, II, and III, here if $P \geq \$6,000$ some car owners would be reluctant to sell, because they received positive disturbances to their valuations. The average quality of cars on the market is less than 4,000 even at $P = \$6,000$. On the other hand, even if $P = \$2,000$ some sellers with low quality cars *and* negative realizations of the disturbance still sell, so the average quality remains above 2,000. Under some distributions of ε, a few sellers hate their cars so much they would pay to have them taken away.

The equilibrium drawn in Figure 4 is $(P = \$2,600, \overline{\theta} = 2,600)$. Some used cars are sold, but the number is inefficiently low. Some of the sellers have high quality cars but negative disturbances, and although they would like to sell their cars to someone who values them more, they will not sell at a price of $2,600.

A theme running through all four Lemons models is that when quality is unknown to the buyer, less trade occurs. Lemons I and II show how trade diminishes, while Lemons III and IV show that the disappearance can be inefficient because some sellers value cars less than some buyers. Next we will use Lemons III, the simplest model with gains from trade, to look at various markets with more sellers than buyers, excess supply, and risk-averse buyers.

More Sellers than Buyers

In analyzing *Lemons III*, we assumed that buyers outnumbered sellers. As a result, the sellers earned producer surplus. In the original equilibrium, all the sellers with quality less than $3,000$ offered a price of $3,000 and earned a surplus of up to $1000. There were more buyers than sellers, so every seller who wished to sell was able to do so, but the price equalled the buyers' expected utility, so no buyer who failed to purchase was dissatisfied. The market cleared.

If, instead, sellers outnumber buyers, what price should a seller offer? At $3,000, not all would-be sellers can find buyers. A seller who proposed a lower price would find willing buyers despite the somewhat lower expected quality. The buyer's tradeoff between lower price and lower quality is shown in Figure 3, in which the expected consumer surplus is the vertical distance between the price (the height of the supply schedule) and the demand schedule. When the price is $3,000 and the average quality is 2,500, the buyer expects a consumer surplus of zero, which is $\$3,000 - \$1.2 \cdot 2,500$. The combination of price and quality that buyers like best is ($2,000, 2,000), because if there were enough sellers with quality $\theta = 2,000$ to satisfy the demand, each buyer would pay $P = \$2,000$ for a car worth $2,400 to him, acquiring a surplus of $400. If there were fewer sellers, the equilibrium price would be higher and some sellers would receive producer surplus.

Heterogeneous Buyers: Excess Supply

If buyers have different valuations for quality, the market might not clear, as Charles Wilson (1980) points out. Assume that the number of buyers willing to pay $1.2 per unit of quality exceeds the number of sellers, but that buyer Smith is an eccentric whose demand for high quality is unusually strong. He would pay $100,000 for a car of quality 5,000 or greater, and $0 for a car of any lower quality.

In Lemons III without Smith, the outcome is a price of $3,000, an average market quality of 2,500, and a market quality range between 2,000 and 3,000. Smith would be unhappy with this, since he has zero probability of finding a car he likes. In fact, he would be willing to accept a price of $6,000, so that all the cars, from quality 2,000 to 6,000, would be offered for sale and the probability that he buys a satisfactory car would rise from 0 to 0.25. But Smith would not want to buy all the cars offered to him, so the equilibrium has two prices, $3,000 and $6,000, with excess supply at the higher price.

Strangely enough, Smith's demand function is upward sloping. At a price of $3,000, he is unwilling to buy; at a price of $6,000, he is willing, because expected quality rises with price. This does not contradict basic price theory, for the standard assumption of *ceteris paribus* is violated. As the price increases, the quantity demanded would fall if all else stayed the same, but all else does not— quality rises.

Risk Aversion

We have implicitly assumed, by the choice of payoff functions, that the buyers and sellers are both risk neutral. What happens if they are risk averse— that is, if the marginal utilities of wealth and car quality are diminishing? Again we will use *Lemons III* and the assumption of many buyers.

On the seller's side, risk aversion changes nothing. The seller runs no risk because he knows exactly the price he receives and the quality he surrenders. But the buyer does bear risk, because he buys a car of uncertain quality. Although he would pay $3,600 for a car he knows has quality 3,000, if he is risk averse he will not pay that much for a car with expected quality 3,000 but actual quality of possibly 2,500 or 3,500: he would obtain less utility from adding 500 quality units than from subtracting 500. The buyer would pay perhaps $2,900 for a car whose expected quality is 3,000 where the demand schedule is nonlinear, lying everywhere below the demand schedule of the risk-neutral buyer. As a result, the equilibrium has a lower price and average quality.

9.4 Adverse Selection under Uncertainty: *Insurance Game III*

The term "adverse selection," like "moral hazard," comes from insurance. Insurance pays more if there is an accident than otherwise, so it benefits accident-prone customers more than safe ones and a firm's customers are "adversely selected" to be accident-prone. The classic article on adverse selection in insurance markets is Rothschild & Stiglitz (1976), which begins, "Economic theorists traditionally banish discussions of information to footnotes." How things have changed! Within ten years, information problems came to dominate research in both microeconomics and macroeconomics.

We will follow Rothschild & Stiglitz in using state-space diagrams, and we will use a version of *The Insurance Game* of Section 8.5. Under moral hazard, Smith chose whether to be $Careful$ or $Careless$. Under adverse selection, Smith cannot affect the probability of a theft, which is chosen by Nature. Rather, Smith is either $Safe$ or $Unsafe$, and while he cannot affect the probability that his car will be stolen, he does know what the probability is.

Insurance Game III

Players
Smith and two insurance companies.

The Order of Play

(0) Nature chooses Smith to be either $Safe$, with probability 0.6, or $Unsafe$, with probability 0.4. Smith knows his type, but the insurance companies do not.

(1) Each insurance company offers its own contract (x, y) under which Smith pays premium x unconditionally and receives compensation y if there is a theft.

(2) Smith picks a contract.

(3) Nature chooses whether there is a theft, using probability 0.5 if Smith is $Safe$ and 0.75 if he is $Unsafe$.

Payoffs.
Smith's payoff depends on his type and the contract (x, y) that he accepts. Let $U' > 0$ and $U'' < 0$.

$$\pi_{Smith}(Safe) = 0.5U(12 - x) + 0.5U(0 + y - x).$$

$$\pi_{Smith}(Unsafe) = 0.25U(12 - x) + 0.75(0 + y - x).$$

The companies' payoffs depend on what types of customers accept their contracts, as shown in Table 1.

Table 1: *Insurance Game III*: **Payoffs**

Company payoff	Types of customers
0	no customers
$0.5x + 0.5(x - y)$	just $Safe$
$0.25x + 0.75(x - y)$	just $Unsafe$
$0.6[0.5x + 0.5(x - y)] + 0.4[0.25x + 0.75(x - y)]$	$Unsafe$ and $Safe$

Smith is $Safe$ with probability 0.6 and $Unsafe$ with probability 0.4. Without insurance, Smith's dollar wealth is 12 if there is no theft and 0 if there is, depicted in Figure 5 as his endowment in state space, $\omega = (12, 0)$. If Smith is $Safe$, a theft occurs with probability 0.5, but if he is $Unsafe$ the probability is 0.75. Smith is risk averse (because $U'' < 0$) and the insurance companies are risk neutral.

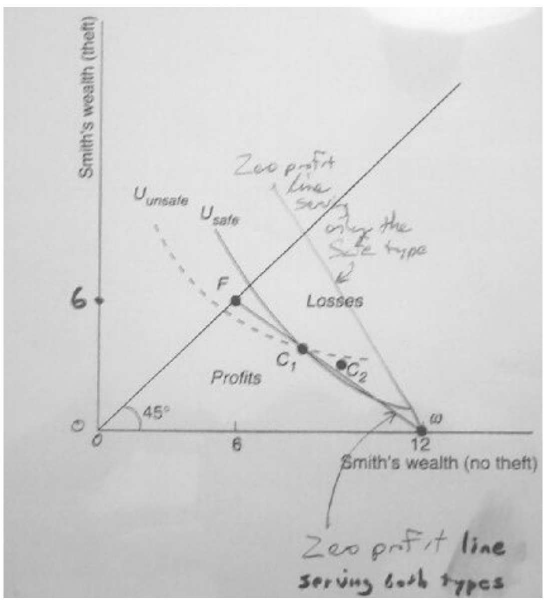


Figure 5: Insurance Game III: Nonexistence of a Pooling Equilibrium

If an insurance company knew that Smith was $Safe$, it could offer him insurance at a premium of 6 with a payout of 12 after a theft, leaving Smith with an allocation of $(6, 6)$. This is the most attractive contract that is not unprofitable, because it fully insures Smith. Whatever the state, his allocation is 6.

Figure 5 shows the indifference curves of Smith and an insurance company. The insurance company is risk neutral, so its indifference curve is a straight line. If Smith will be a customer regardless of his type, the company's indifference curve based on its expected profits is ωF (although if the company knew that Smith was Safe, the indifference curve would be steeper, and if it knew he was Unsafe, the curve would be less steep).'The insurance company is indifferent between ω and C_1, at both of which its expected profits are zero. Smith is risk averse, so his indifference curves are convex, and closest to the origin along the 45 degree if the probability of $Theft$ is 0.5. He has two sets of indifference curves, solid if he is $Safe$ and dotted if he is $Unsafe$.

Figure 5 shows why no Nash pooling equilibrium exists. To make zero profits, the equilibrium must lie on the line ωF. It is easiest to think about these problems by imagining an entire population of Smiths, whom we will call "customers." Pick a contract C_1 anywhere on ωF and think about drawing the indifference curves for the $Unsafe$ and $Safe$ customers that pass through C_1. $Safe$ customers are always willing to trade $Theft$ wealth for *No Theft* wealth at a higher rate than $Unsafe$ customers. At any point, therefore, the slope of the solid ($Safe$) indifference curve is steeper than that of the dashed ($Unsafe$) curve. Since the slopes of the dashed and solid indifference curves differ, we can insert another contract, C_2, between them and just barely to the right of ωF. The $Safe$ customers prefer contract C_2 to C_1, but the $Unsafe$ customers stay with C_1, so C_2 is profitable— since C_2 only attracts $Safe$s, it need not be to the left of ωF to avoid losses. But then the original contract C_1 was not a Nash equilibrium, and since our argument holds for any pooling contract, no pooling equilibrium exists.

The attraction of the $Safe$ customers away from pooling is referred to as **cream skimming**, although profits are still zero when there is competition for the cream. We next consider whether a separating equilibrium exists, using Figure 6. The zero profit condition requires that the $Safe$ customers take contracts on ωC_4 and the $Unsafe$'s on ωC_3.

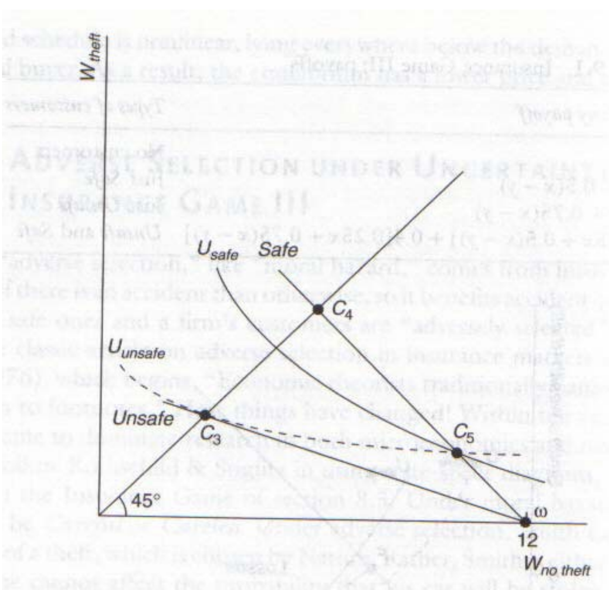


Figure 6: A Separating Equilibrium for *Insurance Game III*

The $Unsafe$s will be completely insured in any equilibrium, albeit at a high price. On the zero-profit line ωC_3, the contract they like best is C_3, which the $Safe$'s are not tempted to take. The $Safe$'s would prefer contract C_4, but C_4 uniformly dominates C_3, so it would attract $Unsafe$s too, and generate losses. To avoid attracting $Unsafe$s, the $Safe$ contract must be below the $Unsafe$ indifference curve. Contract C_5 is the fullest insurance the $Safe$s can get without attracting $Unsafe$s: it satisfies the self-selection and competition constraints.

Contract C_5, however, might not be an equilibrium either. Figure 7 is the same as Figure 6 with a few additional points marked. If one firm offered C_6, it would attract both types, $Unsafe$ and $Safe$, away from C_3 and C_5, because it is to the right of the indifference curves passing through those points. Would C_6 be profitable? That depends on the proportions of the different types. The assumption on which the equilibrium of Figure 6 is based is that the proportion of $Safe$'s is 0.6, so that the zero-profit line for pooling contracts is ωF and C_6 would be unprofitable. In Figure 7 it is assumed that the proportion of $Safe$s is higher, so the zero-profit line for pooling contracts would be $\omega F'$ and C_6, lying to its left, is profitable. But we already showed that no pooling contract is Nash, so C_6 cannot be an equilibrium. Since neither a separating pair like (C_3, C_5) nor a pooling contract like C_6 is an equilibrium, no equilibrium whatsoever exists.

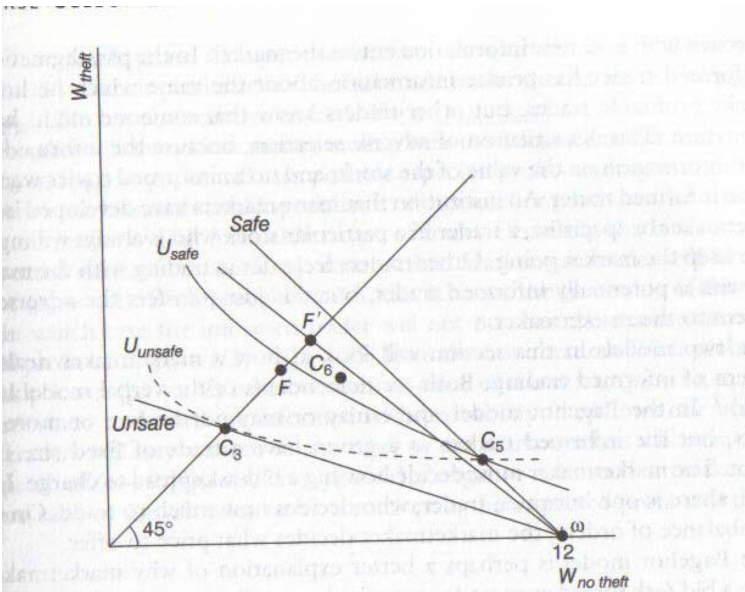


Figure 7: Curves for Which There is No Equilibrium in *Insurance Game III*

The essence of nonexistence here is that if separating contracts are offered, some company is willing to offer a superior pooling contract; but if a pooling contract is offered, some company is willing to offer a separating contract that makes it unprofitable. A monopoly would have a pure-strategy equilibrium, but in a competitive market only a mixed-strategy Nash equilibrium exists (see Dasgupta & Maskin [1986b]).

*9.5 Market Microstructure

The prices of securities such as stocks depend on what investors believe is the value of the assets that underly them. The value are highly uncertain, and new information about them is constantly being generated. The market microstructure literature is concerned with how new information enters the market. In the paradigmatic situation, an informed trader has private information about the value which he hopes to use to make profitable trades, but other traders know that someone might have private information. This is a situation of adverse selection, because the informed trader has better information on the value of the stock, and no uninformed trader wants to trade with an informed trader. An institution that many markets have developed is that of the marketmaker or specialist, a trader in a particular stock who is always willing to buy or sell to keep the market going. Other traders feel safer in trading with the marketmaker than with a potentially informed trader, but this just transfers the adverse selection problem to the marketmaker.

The two models in this section will look at how a marketmaker deals with the problem of informed trading. Both are descendants of the verbal model in Bagehot (1971).("Bagehot", pronounced "badget", is a pseudonym for Jack Treynor. See Glosten & Milgrom (1985) for a formalization.) In the Bagehot model, there may or may not be one or more informed traders, but the informed traders as a group have a trade of fixed size if they are present. The marketmaker must decide how big a bid-ask spread to charge. In the Kyle model, there is one informed trader, who decides how much to trade. On observing the imbalance of orders, the marketmaker decides what price to offer.

The Bagehot model is perhaps a better explanation of why marketmakers might charge a bid/ask spread even under competitive conditions and with zero transactions costs. Its assumption is that the marketmaker cannot change the price depending on volume, but must instead offer a price, and then accept whatever order comes along—a buy order, or a sell order.

The Bagehot Model

Players
The informed trader and two competing marketmakers.

The Order of Play

(0) Nature chooses the asset value v to be either $p_0 - \delta$ or $p_0 + \delta$ with equal probability. The marketmakers never observe the asset value, nor do they observe whether anyone else observes it, but the "informed" trader observes v with probability θ.

(1) The marketmakers choose their spreads s, offering prices $p_{bid} = p_0 - \frac{s}{2}$ at which they will buy the security and $p_{ask} = p_0 + \frac{s}{2}$ for which they will sell it.

(2) The informed trader decides whether to buy one unit, sell one unit, or do nothing.

(3) Noise traders buy n units and sell n units.

Payoffs
Everyone is risk neutral. The informed trader's payoff is $v - p_{ask}$ if he buys, $p_{bid} - v$ if he

sells, and zero if he does nothing. The marketmaker who offers the highest p_{bid} trades with all the customers who wish to sell, and the marketmaker who offers the lowest p_{ask} trades with all the customers who wish to buy. If the marketmakers set equal prices, they split the market evenly. A marketmaker who sells x units gets a payoff of $x(p_{ask} - v)$, and a marketmaker who buys x units gets a payoff of $x(v - p_{bid})$.

This is a very simple game. Competition between the marketmakers will make their prices identical and their profits zero. The informed trader should buy if $v > p_{ask}$ and sell if $v < p_{bid}$. He has no incentive to trade if $[p_{bid}, p_{ask}]$.

A marketmaker will always lose money trading with the informed trader, but if $s > 0$, so $p_{ask} > p_0$ and $p_{bid} < p_0$, he will earn positive expected profits trading with the noise traders. Since a marketmaker could specialize in either sales or purchases, he must earn zero expected profits overall from either type of trade. Centering the bid-ask spread on the expected value of the stock, p_0, ensures this. Marketmaker sales will be at the ask price of $(p_0 + s/2)$. With probability 0.5, this is above the true value of the stock, $(p_0 - \delta)$, in which case the informed trader will not buy but the marketmakers will earn a total profit of $n[(p_0+s/2)-(p_0-\delta)]$ from the noise traders. With probability 0.5, the ask price of $(p_0+s/2)$ is below the true value of the stock, $(p_0 + \delta)$, in which case the informed trader will be informed with probability θ and buy one unit and the noise traders will buy n more in any case, so the marketmakers will earn a total expected profit of $(n+\theta)[(p_0+s/2)-(p_0+\delta)]$, a negative number. For marketmaker profits from sales at the ask price to be zero overall, this expected profit must be set to zero:

$$.5n[(p_0 + s/2) - (p_0 - \delta)] + .5(n + \theta)[(p_0 + s/2) - (p_0 + \delta)] = 0 \tag{7}$$

This equation implies that $n[s/2 + \delta] + (n + \theta)[s/2 - \delta] = 0$, so

$$s^* = \frac{2\delta\theta}{2n + \theta}. \tag{8}$$

The profit from marketmaker purchases must similarly equal zero, and will for the same spread s^*, though we will not go through the algebra here.

Equation (8) has a number of implications. First, the spread s^* is positive. Even though marketmakers compete and have zero transactions costs, they charge a different price to buy and to sell. They make money dealing with the noise traders but lose money with the informed trader, if he is present. The comparative statics reflect this. s^* rises in δ, the variance of the true value, because divergent true values increase losses from trading with the informed trader, and s^* falls in n, which reflects the number of noise traders relative to informed traders, because when there are more noise traders, the profits from trading with them are greater. The spread s^* rises in θ, the probability that the informed trader really has inside information, which is also intuitive but requires a little calculus to demonstrate starting from equation (8):

$$\frac{\partial s^*}{\partial \theta} = \frac{2\delta}{2n + \theta} - \frac{2\delta\theta}{(2n + \theta)^2} = \left(\frac{1}{(2n + \theta)^2}\right)(4\delta n + 2\delta\theta - 2\delta\theta) > 0. \tag{9}$$

The second model of market microstructure, important because it is commonly used as a foundation for more complicated models, is the Kyle model, which focuses on the decision of the informed trader, not the marketmaker. The Kyle model is set up so that marketmaker observes the trade volume before he chooses the price.

The Kyle Model (Kyle [1985])

Players
The informed trader and two competing marketmakers.

The Order of Play

(0) Nature chooses the asset value v from a normal distribution with mean p_0 and variance σ_v^2, observed by the informed trader but not by the marketmakers.

(1) The informed trader offers a trade of size $x(v)$, which is a purchase if positive and a sale if negative, unobserved by the marketmaker.

(2) Nature chooses a trade of size u by noise traders, unobserved by the marketmaker, where u is distributed normally with mean zero and variance σ_u^2.

(3) The marketmakers observe the total market trade offer $y = x + u$, and choose prices $p(y)$.

(4) Trades are executed. If y is positive (the market wants to purchase, in net), whichever marketmaker offers the lowest price executes the trades; if y is negative (the market wants to sell, in net), whichever marketmaker offers the highest price executes the trades. v is then revealed to everyone.

Payoffs
All players are risk neutral. The informed trader's payoff is $(v - p)x$. The marketmaker's payoff is zero if he does not trade and $(p - v)y$ if he does.

An equilibrium for this game is the strategy combination

$$x(v) = (v - p_0)\left(\frac{\sigma_u}{\sigma_v}\right) \tag{10}$$

and

$$p(y) = p_0 + \left(\frac{\sigma_v}{2\sigma_u}\right) y. \tag{11}$$

This is reasonable. It says that the informed trader will increase the size of his trade as v gets bigger relative to p_0 (and he will sell, not buy, if $v - p_0 < 0$), and the marketmaker will increase the price he charges for selling if y is bigger, meaning that more people want to sell, which is an indicator that the informed trader might be trading heavily. The variances of the asset value (σ_v^2) and the noise trading (σ_u^2) enter as one would expect, and they matter only in their relation to each other. If $\frac{\sigma_v^2}{\sigma_u^2}$ is large, then the asset value fluctuates more than the amount of noise trading, and it is difficult for the informed trader to conceal his trades under the noise. The informed trader will trade less, and a given amount of trading will cause a greater response from the marketmaker. One might say that the market is less "liquid": a trade of given size will have a greater impact on the price.

I will not (and cannot) prove uniqueness of the equilibrium, since it is very hard to check all possible combinations of nonlinear strategies, but I will show that $\{(10), (11)\}$ is Nash and is the unique linear equilibrium. To start, hypothesize that the informed trader uses a linear strategy, so

$$x(v) = \alpha + \beta v \tag{12}$$

for some constants α and β. Competition between the marketmakers means that their expected profits will be zero, which requires that the price they offer be the expected value of v. Thus, their equilibrium strategy $p(y)$ will be an unbiased estimate of v given their data y, where they know that y is normally distributed and that

$$\begin{aligned} y &= x + u \\ &= \alpha + \beta v + u. \end{aligned} \tag{13}$$

This means that their best estimate of v given the data y is, following the usual regression rule (which readers unfamiliar with statistics must accept on faith),

$$\begin{aligned} E(v|y) &= E(v) + \left(\tfrac{cov(v,y)}{var(y)}\right) y \\ &= p_0 + \left(\tfrac{\beta\sigma_v^2}{\beta^2\sigma_v^2+\sigma_u^2}\right) y \\ &= p_0 + \lambda y, \end{aligned} \tag{14}$$

where λ is a new shorthand variable to save writing out the term in parentheses in what follows.

The function $p(y)$ will be a linear function of y under our assumption that x is a linear function of v. Given that $p(y) = p_0 + \lambda y$, what must next be shown is that x will indeed be a linear function of v. Start by writing the informed trader's expected payoff, which is

$$\begin{aligned} E\pi_i &= E([v - p(y)]x) \\ &= E([v - p_0 - \lambda(x + u)]x) \\ &= [v - p_0 - \lambda(x + 0)]x, \end{aligned} \tag{15}$$

since $E(u) = 0$. Maximizing the expected payoff with respect to x gives the first order condition

$$v - p_0 - 2\lambda x = 0, \tag{16}$$

which on rearranging becomes

$$x = -\frac{p_0}{2\lambda} + \left(\frac{1}{2\lambda}\right) v. \tag{17}$$

Equation (17) establishes that $x(v)$ is linear, given that $p(y)$ is linear. All that is left is to find the value of λ. See by comparing (17) and (12) that $\beta = \frac{1}{2\lambda}$. Substituting this β into the value of λ from (14) gives

$$\lambda = \frac{\beta\sigma_v^2}{\beta^2\sigma_v^2 + \sigma_u^2} = \frac{\frac{\sigma_v^2}{2\lambda}}{\frac{\sigma_v^2}{(4\lambda^2)} + \sigma_u^2}, \tag{18}$$

which upon solving for λ yields $\lambda = \frac{\sigma_v}{2\sigma_u}$. Since $\beta = \frac{1}{2\lambda}$, it follows that $\beta = \frac{\sigma_u}{\sigma_v}$. These values of λ and β together with equation (17) give the strategies asserted at the start in equations (10) and (11).

The two main divisions of the field of finance are corporate finance and asset pricing. Corporate finance, the study of such things as project choice, capital structure, and mergers has the most obvious applications of game theory, but the Bagehot and Kyle models show that the same techniques are also important in asset pricing. For more information, I recommend Harris & Raviv (1995).

*9.6 A Variety of Applications

Price Dispersion

Usually the best model for explaining price dispersion is a search model— Salop & Stiglitz (1977), for example, which is based on buyers whose search costs differ. But although we passed over it quickly in Section 9.3, the Lemons model with Smith, the quality-conscious consumer, generated not only excess supply, but price dispersion as well. Cars of the same average quality were sold for \$3,000 and \$6,000.

Similarly, while the most obvious explanation for why brands of stereo amplifiers sell at different prices is that customers are willing to pay more for higher quality, adverse selection contributes another explanation. Consumers might be willing to pay high prices because they know that high-priced brands could include both high-quality and low-quality amplifiers, whereas low-priced brands are invariably low quality. The low-quality amplifier ends up selling at two prices: a high price in competition with high-quality amplifiers, and, in different stores or under a different name, a low price aimed at customers less willing to trade dollars for quality.

This explanation does depend on sellers of amplifiers incurring a large enough fixed set-up or operating cost. Otherwise, too many low-quality brands would crowd into the market, and the proportion of high-quality brands would be too small for consumers to be willing to pay the high price. The low-quality brands would benefit as a group from entry restrictions: too many of them spoil the market, not through price competition but through degrading the average quality.

Health Insurance

Medical insurance is subject to adverse selection because some people are healthier than others. The variance in health is particularly high among old people, who sometimes have difficulty in obtaining insurance at all. Under basic economic theory this is a puzzle: the price should rise until supply equals demand. The problem is pooling: when the price of insurance is appropriate for the average old person, healthier ones stop buying. The price must rise to keep profits nonnegative, and the market disappears, just as in Lemons II.

If the facts indeed fit this story, adverse selection is an argument for government-enforced pooling. If all old people are required to purchase government insurance, then while the healthier of them may be worse off, the vast majority could be helped.

Using adverse selection to justify medicare, however, points out how dangerous many of the models in this book can be. For policy questions, the best default opinion is that markets are efficient. On closer examination, we have found that many markets are inefficient because of strategic behavior or information asymmetry. It is dangerous, however, to immediately conclude that the government should intervene, because the same arguments applied to government show that the cure might be worse than the disease. The analyst of health care needs to take seriously the moral hazard and rent-seeking that arise from government insurance. Doctors and hospitals will increase the cost and amount of treatment if the government pays for it, and the transfer of wealth from young people to the elderly, which is likely to swamp the gains in efficiency, might distort the shape of the government program from the economist's ideal.

Henry Ford's Five-Dollar Day

In 1914 Henry Ford made a much-publicized decision to raise the wage of his auto workers to $5 a day, considerably above the market wage. This pay hike occurred without pressure from the workers, who were non-unionized. Why did Ford do it?

The pay hike could be explained by either moral hazard or adverse selection. In accordance with the idea of efficiency wages (Section 8.1), Ford might have wanted workers who worried about losing their premium job at his factory, because they would work harder and refrain from shirking. Adverse selection could also explain the pay hike: by raising his wage Ford attracted a mixture of low -and high- quality workers, rather than low-quality alone (see Raff & Summers [1987]).

Bank Loans

Suppose that two people come to you for an unsecured loan of $10,000. One offers to pay an interest rate of 10 percent and the other offers 200 percent. Who do you accept? Like the car buyer who chooses to buy at a high price, you may choose to lend at a low interest rate.

If a lender raises his interest rate, both his pool of loan applicants and their behavior change because adverse selection and moral hazard contribute to a rise in default rates. Borrowers who expect to default are less concerned about the high interest rate than dependable borrowers, so the number of loans shrinks and the default rate rises (see Stiglitz & Weiss [1981]). In addition, some borrowers shift to higher-risk projects with greater chance of default but higher yields when they are successful. In Section 6.6 we will go through the model of D. Diamond (1989) which looks at the evolution of this problem as firms age.

Whether because of moral hazard or adverse selection, asymmetric information can also result in excess demand for bank loans. The savers who own the bank do not save enough at the equilibrium interest rate to provide loans to all the borrowers who want loans. Thus, the bank makes a loan to John Smith, while denying one to Joe, his observationally equivalent twin. Policymakers should carefully consider any laws that rule out arbitrary loan criteria or require banks to treat all customers equally. A bank might wish to restrict its loans to left-handed people, neither from prejudice nor because it is useful to ration loans according to some criterion arbitrary enough to avoid the moral hazard of favoritism by loan officers.

Bernanke (1983) suggests adverse selection in bank loans as an explanation for the Great Depression in the United States. The difficulty in explaining the Depression is not so much the initial stock market crash as the persistence of the unemployment that followed. Bernanke notes that the crash wiped out local banks and dispersed the expertise of the loan officers. After the loss of this expertise, the remaining banks were less willing to lend because of adverse selection, and it was difficult for the economy to recover.

Solutions to Adverse Selection

Even in markets where it apparently does not occur, the threat of adverse selection, like the threat of moral hazard, can be an important influence on market institutions. Adverse selection can be circumvented in a number of ways besides the contractual solutions we have been analyzing. I will mention some of them in the context of the used car market.

One set of solutions consists of ways to make car quality contractible. Buyers who find that their car is defective may have recourse to the legal system if the sellers were fraudulent, although in the United States the courts are too slow and costly to be fully effective. Other government bodies such as the Federal Trade Commission may do better by issuing regulations particular to the industry. Even without regulation, private warranties— promises to repair the car if it breaks down— may be easier to enforce than oral claims, by disspelling ambiguity about what level of quality is guaranteed.

Testing (the equivalent of moral hazard's monitoring) is always used to some extent. The prospective driver tries the car on the road, inspects the body, and otherwise tries to reduce information asymmetry. At a cost, he could even reverse the asymmetry by hiring mechanics, learning more about the car than the owner himself. The rule is not always *caveat emptor*; what should one's response be to an antique dealer who offers to pay $500 for an apparently worthless old chair?

Reputation can solve adverse selection, just as it can solve moral hazard, but only if the transaction is repeated and the other conditions of the models in Chapters 5 and 6 are met. An almost opposite solution is to show that there are innocent motives for a sale; that the owner of the car has gone bankrupt, for example, and his creditor is selling the car cheaply to avoid the holding cost.

Penalties not strictly economic are also important. One example is the social ostracism inflicted by the friend to whom a lemon has been sold; the seller is no longer invited to dinner. Or, the seller might have moral principles that prevent him from defrauding buyers. Such principles, provided they are common knowledge, would help him obtain a higher price in the used-car market. Akerlof himself has worked on the interaction between social custom and markets in his 1980 and 1983 articles. The second of these looks directly at the value of inculcating moral principles, using theoretical examples to show that parents might wish to teach their children principles, and that society might wish to give hiring preference to students from elite schools.

It is by violating the assumptions needed for perfect competition that asymmetric information enables government and social institutions to raise efficiency. This points to a major reason for studying asymmetric information: where it is important, noneconomic interference can be helpful instead of harmful. I find the social solutions particularly interesting since, as mentioned earlier in connection with health care, government solutions introduce agency problems as severe as the information problems they solve. Noneconomic behavior is important under adverse selection, in contrast to perfect competition, which allows an "Invisible Hand" to guide the market to efficiency, regardless of the moral beliefs of the traders. If everyone were honest, the lemons problem would disappear because the sellers would truthfully disclose quality. If some fraction of the sellers were honest, but buyers could not distinguish them from the dishonest sellers, the outcome would presumably be somewhere between the outcomes of complete honesty and complete dishonesty. The subject of market ethics is important, and would profit from investigation by scholars trained in economic analysis.

Notes

N9.1 Introduction: Production Game VI

- For an example of an adverse selection model in which workers also choose effort level, see Akerlof (1976) on the "rat race." The model is not moral hazard, because while the employer observes effort, the worker's types— their utility costs of hard work— are known only to themselves.

- In moral hazard with hidden knowledge, the contract must ordinarily satisfy only one participation constraint, whereas in adverse selection problems there is a different participation constraint for each type of agent. An exception is if there are constraints limiting how much an agent can be punished in different states of the world. If, for example, there are bankruptcy constraints, then, if the agent has different wealths across the N possible states of the world, there will be N constraints for how negative his wage can be, in addition to the single participation constraint. These can be looked at as **interim** participation constraints, since they represent the idea that the agent wants to get out of the contract once he observes the state of the world midway through the game.

- Gresham's Law ("Bad money drives out good") is a statement of adverse selection. Only debased money will be circulated if the payer knows the quality of his money better than the receiver. The same result occurs if quality is common knowledge, but for legal reasons the receiver is obligated to take the money, whatever its quality. An example of the first is Roman coins with low silver content; and of the second, Zambian currency with an overvalued exchange rate.

- Most adverse selection models have types that could be called "good" and "bad," because one type of agent would like to pool with the other, who would rather be separate. It is also possible to have a model in which both types would rather separate— types of workers who prefer night shifts versus those who prefer day shifts, for example— or two types who both prefer pooling— male and female college students.

- Two curious features of labor markets is that workers of widely differing outputs seem to be paid identical wages and that tests are not used more in hiring decisions. Schmidt and Judiesch (as cited in Seligman [1992], p. 145) have found that in jobs requiring only unskilled and semi-skilled blue-collar workers, the top 1 percent of workers, as defined by performance on ability tests not directly related to output, were 50 percent more productive than the average. In jobs defined as "high complexity" the difference was 127 percent.

  At about the same time as Akerlof (1970), another seminal paper appeared on adverse selection, Mirrlees (1971), although the relation only became clear later. Mirrlees looked at optimal taxation and the problem of how the government chooses a tax schedule given that it cannot observe the abilities of its citizens to earn income, and this began the literature on mechanism design. Used cars and income taxes do not appear similar, but in both situations an uninformed player must decide how to behave to another player whose type he does not know. Section 10.4 sets out a descendant of Mirrlees (1971) in a model of government procurement: much of government policy is motivated by the desire to create incentives for efficiency at minimum cost while eliciting information from individuals with superior information.

N9.2 Adverse Selection under Certainty: Lemons I and II

- Dealers in new cars and other durables have begun offering "extended-service contracts" in recent years. These contracts, offered either by the manufacturers or by independent companies, pay for repairs after the initial warranty expires. For reasons of moral hazard or adverse selection, the contracts usually do not cover damage from accidents. Oddly enough, they also do not cover items like oil changes despite their usefulness in prolonging engine life. Such contracts have their own problems, as shown by the fact that several of the independent companies went bankrupt in the late 1970s and early 1980s, making their contracts worthless.See "Extended-Service Contracts for New Cars Shed Bad Reputation as Repair Bills Grow," *Wall Street Journal*, June 10, 1985, p. 25.

- Suppose that the cars of Lemons II lasted two periods and did not physically depreciate. A naive economist looking at the market would see new cars selling for \$6,000 (twice \$3,000) and old cars selling for \$2,000 and conclude that the service stream had depreciated by 33 percent. Depreciation and adverse selection are hard to untangle using market data.

- Lemons II uses a uniform distribution. For a general distribution F, the average quality $\overline{\theta}(P)$ of cars with quality P or less is

$$\overline{\theta}(P) = E(\theta|\theta \leq P) = \frac{\int_{-\infty}^{P} xF'(x)dx}{F(P)}. \tag{19}$$

  Equation (19) also arises in physics (equation for a center of gravity) and nonlinear econometrics (the likelihood equation). Think of $\overline{\theta}(P)$ as a weighted average of the values of θ up to P, the weights being densities. Having multiplied by all these weights in the numerator, we have to divide by their "sum," $F(P) = \int_{-\infty}^{P} F'(x)dx$, in the denominator, giving rise to equation (19).

N9.3 Heterogeneous Tastes: Lemons III and IV

- You might object to a model in which the buyers of used cars value quality more than the sellers, since the sellers are often richer people. Remember that quality here is "quality of used cars," which is different from "quality of cars." The utility functions could be made more complicated without abandoning the basic model. We could specify something like $\pi_{buyer} = \theta + k/\theta - P$, where $\theta^2 > k$. Such a specification implies that the lower is the quality of the car, the greater the difference between the valuations of buyer and seller.

- In Akerlof (1970) the quality of new cars is uniformly distributed between zero and 2, and the model is set up differently, with the demand and supply curves offered by different types of traders and net supply and gross supply presented rather confusingly. Usually the best way to model a situation in which traders sell some of their endowment and consume the rest is to use only gross supplies and demands. Each old owner supplies his car to the market, but in equilibrium he might buy it back, having better information about his car than the other consumers. Otherwise, it is easy to count a given unit of demand twice, once in the demand curve and once in the net supply curve.

- See Stiglitz (1987) for a good survey of the relation between price and quality. Leibenstein (1950) uses diagrams to analyze the implications of individual demand being linked to the market price of quantity in markets for "bandwagon," "snob," and "Veblen" goods. See also "Pricing of Products is Still an Art, Often Having Little Link to Costs," *Wall Street Journal*, p. 29 (25 November 1981).

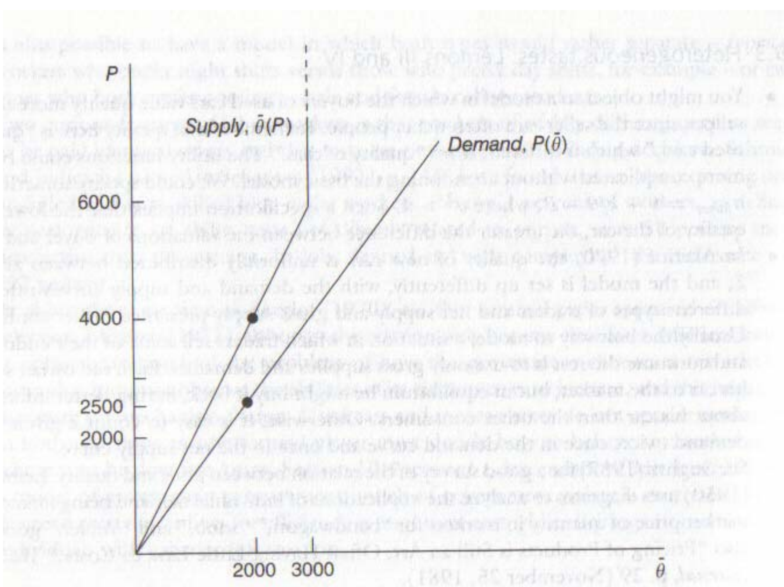
- Risk aversion is concerned only with variability of outcomes, not their level. If the quality of used cars ranges from 2,000 to 6,000, buying a used car is risky. If all used cars are of quality 2,000, buying a used car is riskless, because the buyer knows exactly what he is getting.

  In Insurance Game III in Section 9.4, the separating contract for the $Unsafe$ consumer fully insures him: he bears no risk. But in constructing the equilibrium, we had to be very careful to keep the $Unsafe$s from being tempted by the risky contract designed for the $Safe$s. Risk is a bad thing, but as with old age, the alternative is worse. If Smith were certain his car would be stolen, he would bear no risk, because he would be certain to have low utility.

- To the buyers in Lemons IV, the average quality of cars for a given price is stochastic because they do not know which values of ε were realized. To them, the curve $\overline{\theta}(P)$ is only the *expectation* of the average quality.

- **Lemons III′: Minimum Quality of Zero.** If the minimum quality of car in Lemons III were 0, not 2,000, the resulting game (Lemons III′) would be close to the original Akerlof (1970) specification. As Figure 8 shows, the supply schedule and the demand schedule intersect at the origin, so that the equilibrium price is zero and no cars are traded. The market has shut down entirely because of the unravelling effect described in Lemons II. Even though the buyers are willing to accept a quality lower than the dollar price, the price that buyers are willing to pay does not rise with quality as fast as the price needed to extract that average quality from the sellers, and a car of minimum quality is valued exactly the same by buyers and sellers. A 20 percent premium on zero is still zero. The efficiency implications are even stronger than before, because at the optimum all the old cars are sold to new buyers, but in equilibrium, none are.

Figure 8: Lemons III′ When Buyers Value Cars More and the Minimum Quality is Zero



N9.4 Adverse Selection under Uncertainty: Insurance Game III

- Markets with two types of customers are very common in insurance, because it is easy to distinguish male from female, both those types are numerous, and the difference between them is important. Males under age 25 pay almost twice the auto insurance premiums of females, and females pay 10 to 30 percent less for life insurance. The difference goes both ways, however: Aetna charges a 35-year old woman 30 to 50 percent more than a man for medical insurance. One market in which rates do not differ much is disability insurance. Women do make more claims, but the rates are the same because relatively few women buy the product (*Wall Street Journal*, p. 21, 27 August 1987).

N9.6 A Variety of Applications

- Economics professors sometimes make use of self-selection for student exams. One of my colleagues put the following instructions on an MBA exam, after stating that either Question 5 or 6 must be answered.

> "The value of Question 5 is less than that of Question 6. Question 5, however, is straightforward and the average student may expect to answer it correctly. Question 6 is more tricky: only those who have understood and absorbed the content of the course well will be able to answer it correctly... For a candidate to earn a final course grade of A or higher, it will be *necessary* for him to answer Question 6 successfully."

  Making the question even more self-referential, he asked the students for an explanation of its purpose.

  Another of my colleagues tried asking who in his class would be willing to skip the exam and settle for an A−. Those students who were willing, received an A−. The others got A's. But nobody had to take the exam (this method did upset a few people). More formally, Guasch & Weiss (1980) have looked at adverse selection and the willingess of workers with different abilities to take tests.

- Nalebuff & Scharfstein (1987) have written on testing, generalizing Mirrlees (1974), who showed how a forcing contract in which output is costlessly observed might attain efficiency by punishing only for very low output. In Nalebuff & Scharfstein, testing is costly and agents are risk averse. They develop an equilibrium in which the employer tests workers with small probability, using high-quality tests and heavy punishments to attain almost the first-best. Under a condition which implies that large expenditures on each test can eliminate false accusations, they show that the principal will test workers with small probability, but use expensive, accurate tests when he does test a worker, and impose a heavy punishment for lying.

Problems

9.1. Insurance with Equations and Diagrams
The text analyzes Insurance Game III using diagrams. Here, let us use equations too. Let $U(t) = log(t)$.

(a) Give the numeric values (x, y) for the full-information separating contracts C_3 and C_4 from Figure 6. What are the coordinates for C_3 and C_4?

(b) Why is it not necessary to use the $U(t) = log(t)$ function to find the values?

(c) At the separating contract under incomplete information, C_5, $x = 2.01$. What is y? Justify the value 2.01 for x. What are the coordinates of C_5?

(d) What is a contract C_6 that might be profitable and that would lure both types away from C_3 and C_5?

9.2: Testing and Commitment. Fraction β of workers are talented, with output $a_t = 5$, and fraction $(1 - \beta)$ are untalented, with output $a_u = 0$. Both types have a reservation wage of 1 and are risk neutral. At a cost of 2 to itself and 1 to the job applicant, employer Apex can test a job applicant and discover his true ability with probability θ, which takes a value of something over 0.5. There is just one period of work. Let $\beta = 0.001$. Suppose that Apex can commit itself to a wage schedule before the workers take the test, and that Apex must test all applicants and pay all the workers it hires the same wage, to avoid grumbling among workers and corruption in the personnel division.

(a) What is the lowest wage, w_t, that will induce talented workers to apply? What is the lowest wage, w_u, that will induce untalented workers to apply? Which is greater?

(b) What is the minimum accuracy value θ that will induce Apex to use the test? What are the firm's expected profits per worker who applies?

(c) Now suppose that Apex can pay w_p to workers who pass the test and w_f to workers who flunk. What are w_p and w_f? What is the minimum accuracy value θ that will induce Apex to use the test? What are the firm's expected profits per worker who applies?

(d) What happens if Apex cannot commit to paying the advertised wage, and can decide each applicant's wage individually?

(e) If Apex cannot commit to testing every applicant, why is there no equilibrium in which either untalented workers do not apply or the firm tests every applicant?

9.3. Finding the Mixed-Strategy Equilibrium in a Testing Game
Half of high school graduates are talented, producing output $a = x$, and half are untalented, producing output $a = 0$. Both types have a reservation wage of 1 and are risk neutral. At a cost of 2 to himself and 1 to the job applicant, an employer can test a graduate and discover his true ability. Employers compete with each other in offering wages but they cooperate in revealing test results, so an employer knows if an applicant has already been tested and failed. There is just one period of work. The employer cannot commit to testing every applicant or any fixed percentage of them.

(a) Why is there no equilibrium in which either untalented workers do not apply or the employer tests every applicant?

(b) In equilibrium, the employer tests workers with probability γ and pays those who pass the test w, the talented workers all present themselves for testing, and the untalented workers present themselves with probability α, where possibly $\gamma = 1$ or $\alpha = 1$. Find an expression for the equilibrium value of α in terms of w. Explain why α is not directly a function of x in this expression, even though the employer's main concern is that some workers have a productivity advantage of x.

(c) If $x = 9$, what are the equilibrium values of α, γ, and w?

(d) If $x = 8$, what are the equilibrium values of α, γ, and w?

9.4: Two-Time Losers. Some people are strictly principled and will commit no robberies, even if there is no penalty. Others are incorrigible criminals and will commit two robberies, regardless of the penalty. Society wishes to inflict a certain penalty on criminals as retribution. Retribution requires an expected penalty of 15 per crime (15 if detection is sure, 150 if it has probability 0.1, etc.). Innocent people are sometimes falsely convicted, as shown in Table 2.

Table 2: Two-Time Losers

	Convictions		
Robberies	0	1	2
0	0.81	0.18	0.01
1	0.60	0.34	0.06
2	0.49	0.42	0.09

Two systems are proposed: (i) a penalty of X for each conviction, and (ii) a penalty of 0 for the first conviction, and some amount P for the second conviction.

(a) What must X and P be to achieve the desired amount of retribution?

(b) Which system inflicts the smaller cost on innocent people? How much is the cost in each case?

(c) Compare this with Problem 8.2. How are they different?

9.5. Insurance and State-Space Diagrams

Two types of risk-averse people, clean-living and dissolute, would like to buy health insurance. Clean-living people become sick with probability 0.3, and dissolute people with probability 0.9. In state-space diagrams with the person's wealth if he is healthy on the vertical axis and if he is sick on the horizontal, every person's initial endowment is (5,10), because his initial wealth is 10 and the cost of medical treatment is 5.

(a) What is the expected wealth of each type of person?

(b) Draw a state-space diagram with the indifference curves for a risk-neutral insurance company that insures each type of person separately. Draw in the post-insurance allocations C_1 for the dissolute and C_2 for the clean-living under the assumption that a person's type is contractible.

(c) Draw a new state-space diagram with the initial endowment and the indifference curves for the two types of people that go through that point.

(d) Explain why, under asymmetric information, no pooling contract C_3 can be part of a Nash equilibrium.

(e) If the insurance company is a monopoly, can a pooling contract be part of a Nash equilibrium?

August 28a, 1999. December 29, 2003. March 7, 2005. 25 March 2005. Eric Rasmusen, Erasmuse@indiana.edu. Http://www.rasmusen/org/GI/chap10_mechanisms.pdf. Footnotes starting with xxx are the author's notes to himself. Comments welcomed.

10 Mechanism Design [1]

10.1 The Revelation Principle and Moral Hazard with Hidden Knowledge

In Chapter 10 we will look at mechanism design. A mechanism is a set of rules that one player constructs and another freely accepts in order to convey information from the second player to the first. Thus, a mechanism consists of an information report by the second player and a mapping from each possible report to some action by the first.

Adverse selection models can be viewed as mechanism design. *Insurance Game III* was about an insurance company which wanted to know whether a customer was safe or not. In equilibrium it offers two contracts, an expensive full insurance contract preferred by the safe customers and a cheap partial insurance contract preferred by the unsafe. A mechanism design view is that the insurance company sets up a game in which a customer reports his type as Safe or Unsafe, whichever he prefers to report, and the company then assigns him either partial or full insurance as a consequence. The contract is a mechanism for getting the agents to truthfully report their types.

Mechanism design goes beyond simple adverse selection. It can be useful even when players begin a game with symmetric information or when both players have hidden information that they would like to exchange.

Section 10.1 introduces moral hazard with hidden knowledge and discusses a modelling simplification called the Revelation Principle and a paradox known as Unravelling. Section 10.2 uses diagrams to apply the model to sales quotas, and Section 10.3 uses a product quality game of Roger Myerson's to compare moral hazard with hidden information with adverse selection. Section 10.4 applies the principles of mechanism design to price discrimination. Section 10.4 contains a more complicated model of rate-of-return regulation by a government that constructs a mechanism to induce a regulated company to reveal how high its costs are. Section 10.4 introduces a multilateral mechanism, the Groves Mechanism, for use when the problem is to elicit truthful reports from not one but N agents who need to decide whether to invest in a public good.

Moral Hazard with Hidden Knowledge

[1] xxx THis chapter is unsatisfactory. The explanations are too difficult, and perhaps repetitive. Things need to be unified more. The order might be rearranged too. The notation shoudl be unified.

Information is complete in moral hazard games, but in moral hazard with hidden knowledge the agent, but not the principal, observes a move of Nature after the game begins. Information is symmetric at the time of contracting but becomes asymmetric later. From the principal's point of view, agents are identical at the beginning of the game but develop private types midway through, depending on what they have seen. His chief concern is to give them incentives to disclose their types later, which gives games with hidden knowledge a flavor close to that of adverse selection. (In fact, an alternative name for this might be **post-contractual adverse selection.**) The agent might exert effort, but effort's contractibility is less important when the principal does not know which effort is appropriate because he is ignorant of the state of the world chosen by Nature. The main difference technically is that if information is symmetric at the start and only becomes asymmetric after a contract is signed, the participation constraint is based on the agent's expected payoffs across the different types of agent he might become. Thus, there is just one participation constraint even if there are eventually n possible types of agents in the model, rather than the n participation constraints that would be required in a standard adverse selection model.

There is more hope for obtaining efficient outcomes in moral hazard with hidden knowledge than in adverse selection or simple moral hazard. The advantage over adverse selection is that information is symmetric at the time of contracting, so neither player can use private information to extract surplus from the other by choosing inefficient contract terms. The advantage over simple moral hazard is that the post-contractual asymmetry is with respect to knowledge only, which is neutral in itself, rather than over whether the agent exerted high effort, which causes direct disutility to him.

For a comparison between the two types of moral hazard, let us modify *Production Game V* from Section 7.2 and turn it into a game of hidden knowledge.

Production Game VII: Hidden Knowledge

Players
The principal and the agent.

The Order of Play
1 The principal offers the agent a wage contract of the form $w(q, m)$, where q is output and m is a message to be sent by the agent.
2 The agent accepts or rejects the principal's offer.
3 Nature chooses the state of the world θ, according to probability distribution $F(\theta)$. The agent observes θ, but the principal does not.
4 If the agent accepts, he exerts effort e and sends a message m, both observed by the principal.
5 Output is $q(e, \theta)$.

Payoffs
If the agent rejects the contract, $\pi_{agent} = \bar{U}$ and $\pi_{principal} = 0$.
If the agent accepts the contract, $\pi_{agent} = U(e, w, \theta)$ and $\pi_{principal} = V(q - w)$.

The principal would like to know θ so he can tell which effort level is appropriate. In an ideal world he would employ an honest agent who always chose $m = \theta$, but in noncooperative games, talk is cheap. Since the agent's words are worthless, the principal must try to design a contract that either provides incentive for truth-telling or takes lying into account – he **implements** a **mechanism** to extract the agent's information.

Unravelling the Truth when Silence is the Only Alternative

Before going on to look at a self-selection contract, let us look at a special case in which hidden knowledge paradoxically makes no difference. The usual hidden knowledge model has no penalty for lying, but let us briefly consider what happens if the agent cannot lie, though he can be silent. Suppose that Nature uses the uniform distribution to assign the variable θ some value in the interval $[0, 10]$ and the agent's payoff is increasing in the principal's estimate of θ. Usually we assume that the agent can lie freely, sending a message m taking any value in $[0, 10]$, but let us assume instead that he cannot lie but he can conceal information. Thus, if $\theta = 2$, he can send the uninformative message $m \geq 0$ (equivalent to no message), or the message $m \geq 1$, or $m = 2$, but not the lie that $m \geq 4$.

When $\theta = 2$ the agent might as well send a message that is the exact truth: "$m = 2$." If he were to choose the message "$m \geq 1$" instead, the principal's first thought might be to estimate θ as the average value of the interval $[1, 10]$, which is 5.5. But the principal would realize that no agent with a value of θ greater than 5.5 would want to send that message "$m \geq 1$" if that was the resulting deduction. This realization restricts the possible interval to [1, 5.5], which in turn has an average of 3.25. But then no agent with $\theta > 3.25$ would send the message "$m \geq 1$." The principal would continue this process of logical **unravelling** to conclude that $\theta = 1$. The message "$m \geq 0$" would be even worse, making the principal believe that $\theta = 0$. In this model, "No news is bad news." The agent would therefore not send the message "$m \geq 1$" and he would be indifferent between "$m = 2$" and "$m \geq 2$" because the principal would make the same deduction from either message.

Perfect unravelling is paradoxical, but that is because the assumptions behind the reasoning in the last paragraph are rarely satisfied in the real world. In particular, unpunishable lying and genuine ignorance allow information to be concealed. If the seller is free to lie without punishment then in the absence of other incentives he always pretends that his information is extremely favorable, so nothing he says conveys any information, good or bad. If he really is ignorant in some states of the world, then his silence could mean either that he has nothing to say or that he has nothing he wants to say. The unravelling argument fails because if he sends an uninformative message the buyers will attach some probability to "no news" instead of "bad news." Problem 10.1 explores unravelling further.

The Revelation Principle

A principal might choose to offer a contract that induces his agent to lie in equilibrium, since he can take lying into account when he designs the contract, but this complicates the analysis. Each state of the world has a single truth, but a continuum of lies. Generically speaking, almost everything is false. The following principle helps us simplify contract design.

The Revelation Principle. *For every contract $w(q,m)$ that leads to lying (that is, to $m \neq \theta$), there is a contract $w^*(q,m)$ with the same outcome for every θ but no incentive for the agent to lie.*

Many possible contracts make false messages profitable for the agent because when the state of the world is a he receives a reward of x_1 for the true report of a and $x_2 > x_1$ for the false report of b. A contract which gives the agent the same reward of x_2 regardless of whether he reports a or b would lead to exactly the same payoffs for each player while giving the agent no incentive to lie. The revelation principle notes that a truth- telling contract like this can always be found by imitating the relation between states of the world and payoffs in the equilibrium of a contract with lying. The idea can also be applied to games in which two players must make reports to each other.

Applied to concrete examples, the revelation principle is obvious. Suppose we are concerned with the effect on the moral climate of cheating on income taxes, but anyone who makes \$70,000 a year can claim he makes \$50,000 and we do not have the resources to catch him. The revelation principle says that we can rewrite the tax code to set the tax to be the same for taxpayers earning \$70,000 and for those earning \$50,000, and the same amount of taxes will be collected without anyone having incentive to lie. Applied to moral education, the principle says that the mother who agrees never to punish her daughter if she tells her all her escapades will never hear any untruths. Clearly, the principle's usefulness is not so much to improve outcomes as to simplify contracts. The principal (and the modeller) need only look at contracts which induce truth-telling, so the relevant strategy space is shrunk and we can add a third constraint to the incentive compatibility and participation constraints to help calculate the equilibrium:

(3) **Truth-telling.** The equilibrium contract makes the agent willing to choose $m = \theta$.

The revelation principle says that a truth-telling equilibrium exists, but not that it is unique. It may well happen that the equilibrium is a weak Nash equilibrium in which the optimal contract gives the agent no incentive to lie but also no incentive to tell the truth. This is similar to the open-set problem discussed in Section 4.3; the optimal contract may satisfy the agent's participation constraint but makes him indifferent between accepting and rejecting the contract. If agents derive the slightest utility from telling the truth, of course, then truthtelling becomes a strong equilibrium, but if their utility from telling the truth is really significant, it should be made an explicit part of the model. If the utility of truth- telling is strong enough, in fact, agency problems and the costs associated with them disappear. This is one reason why morality is useful to business.

10.2: An Example of Moral Hazard with Hidden Knowledge: *The Salesman Game*

Suppose the manager of a company has told his salesman to investigate a potential customer, who is either a *Pushover* or a *Bonanza*. If he is a *Pushover*, the efficient sales effort is low and sales should be moderate. If he is a *Bonanza*, the effort and sales should be higher.

The Salesman Game

Players
A manager and a salesman.

The Order of Play
1 The manager offers the salesman a contract of the form $w(q, m)$, where q is sales and m is a message.
2 The salesman decides whether or not to accept the contract.
3 Nature chooses whether the customer is a *Bonanza* or a *Pushover* with probabilities 0.2 and 0.8. Denote the state variable "customer status" by θ. The salesman observes the state, but the manager does not.
4 If the salesman has accepted the contract, he chooses his sales level q, which implicitly measures his effort.

Payoffs
The manager is risk neutral and the salesman is risk averse. If the salesman rejects the contract, his payoff is $\bar{U} = 8$ and the manager's is zero. If he accepts the contract, then

$$\begin{aligned} \pi_{manager} &= q - w \\ \pi_{salesman} &= U(q, w, \theta), \text{ where } \frac{\partial U}{\partial q} < 0, \frac{\partial^2 U}{\partial q^2} < 0, \frac{\partial U}{\partial w} > 0, \frac{\partial^2 U}{\partial w^2} < 0 \end{aligned}$$

Figure 1 shows the indifference curves of manager and salesman, labelled with numerical values for exposition. The manager's indifference curves are straight lines with slope 1 because he is acting on behalf of a risk-neutral company. If the wage and the quantity both rise by a dollar, profits are unchanged, and the profits do not depend directly on whether θ takes the value *Pushover* or *Bonanza*.

The salesman's indifference curves also slope upwards, because he must receive a higher wage to compensate for the extra effort that makes q greater. They are convex because the marginal utility of dollars is decreasing and the marginal disutility of effort is increasing. As Figure 1 shows, the salesman has two sets of indifference curves, solid for *Pushovers* and dashed for *Bonanzas*, since the effort that secures a given level of sales depends on the state.

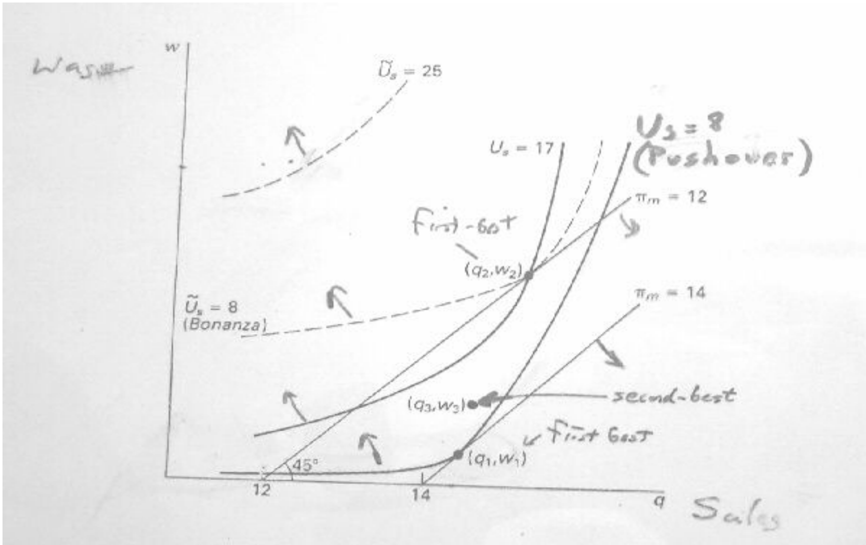


Figure 1: The Salesman Game with curves for pooling equilibrium

Because of the participation constraint, the manager must provide the salesman with a contract giving him at least his reservation utility of 8, which is the same in both states. If the true state is that the customer is a *Bonanza*, the manager would like to offer a contract that leaves the salesman on the dashed indifference curve $\tilde{U_S} = 8$, and the efficient outcome is (q_2,w_2), the point at which the salesman's indifference curve is tangent to one of the manager's indifference curves. At that point, if the salesman sells an extra dollar he requires an extra dollar of compensation.

If it were common knowledge that the customer was a *Bonanza*, the principal could choose w_2 so that $U(q_2, w_2, Bonanza) = 8$ and offer the forcing contract

$$w = \begin{cases} 0 & \text{if } q < q_2. \\ w_2 & \text{if } q \geq q_2. \end{cases} \tag{1}$$

The salesman would accept the contract and choose $q = q_2$. But if the customer were actually a *Pushover*, the salesman would still choose $q = q_2$, an inefficient outcome that does not maximize profits. High sales would be inefficient because the salesman would be willing to give up more than a dollar of wages to escape having to make his last dollar of sales. Profits would not be maximized, because the salesman achieves a utility of 17, and he would have been willing to work for less.

The revelation principle says that in searching for the optimal contract we need only look at contracts that induce the agent to truthfully reveal what kind of customer he faces. If it required more effort to sell any quantity to the *Bonanza*, as shown in Figure 1, the salesman would always want the manager to believe that he faced a *Bonanza*, so he could extract the extra pay necessary to achieve a utility of 8 selling to *Bonanzas*. The only optimal truth-telling contract is the pooling contract that pays the intermediate wage of w_3 for the intermediate quantity of q_3, and zero for any other quantity, regardless of the message. The pooling contract is a second-best contract, a compromise between the optimum for *Pushovers* and the optimum for *Bonanzas*. The point (q_3, w_3) is closer to (q_1, w_1) than to (q_2, w_2), because the probability of a *Pushover* is higher and the contract must satisfy the participation constraint,

$$0.8U(q_3, w_3, Pushover) + 0.2U(q_3, w_3, Bonanza) \geq 8. \quad (2)$$

The nature of the equilibrium depends on the shapes of the indifference curves. If they are shaped as in Figure 2, the equilibrium is separating, not pooling, and there does exist a first-best, fully revealing contract.

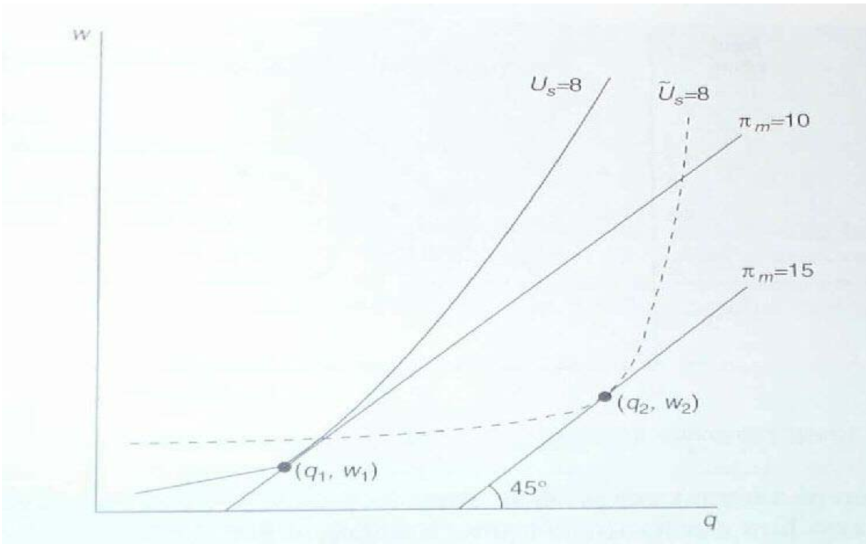


Figure 2: Indifference Curves for a Separating Equilibrium

$$\text{Separating Contract} \begin{cases} \text{Agent announces } Pushover: & w = \begin{cases} 0 \text{ if } q < q_1 \\ w_1 \text{ if } q \geq q_1 \end{cases} \\ \\ \text{Agent announces } Bonanza: & w = \begin{cases} 0 \text{ if } q < q_2 \\ w_2 \text{ if } q \geq q_2 \end{cases} \end{cases} \quad (3)$$

Again, we know from the revelation principle that we can narrow attention to contracts that induce the salesman to tell the truth. With the indifference curves of Figure 2, contract (3) induces the salesman to be truthful and the incentive compatibility constraint is satisfied. If the customer is a *Bonanza*, but the salesman claims to observe a *Pushover* and chooses q_1, his utility is less than 8 because the point (q_1, w_1) lies below the $\tilde{U_S} = 8$ indifference curve. If the customer is a *Pushover* and the salesman claims to observe a *Bonanza*, then although (q_2, w_2) does yield the salesman a higher wage than (q_1, w_1), the extra income is not worth the extra effort, because (q_2, w_2) is far below the indifference curve $U_S = 8$.

Another way to look at a separating equilibrium is to think of it as a choice of contracts rather than as one contract with different wages for different outputs. The salesman agrees to work for the manager, and after he discovers what type the customer is he chooses either the contract (q_1, w_1) or the contract (q_2, w_2), where each is a forcing contract that pays him 0 if after choosing the contract (q_i, w_i) he produces output of $q \neq q_i$. In this interpretation, the manager offers a **menu of contracts** and the salesman selects one of them after learning his type.

Sales contracts in the real world are often complicated because it is easy to measure sales and hard to measure efforts when workers who are out in the field away from direct supervision. The Salesman Game is a real problem. Gonik (1978) describes hidden knowledge contracts used by IBM's subsidiary in Brazil. Salesmen were first assigned quotas. They then announced their own sales forecast as a percentage of quota and chose from among a set of contracts, one for each possible forecast. Inventing some numbers for illustration, if Smith were assigned a quota of 400 and he announced 100 percent, he might get $w = 70$ if he sold 400 and $w = 80$ if he sold 450; but if he had announced 120 percent, he would have gotten $w = 60$ for 400 and $w = 90$ for 450. The contract encourages extra effort when the extra effort is worth the extra sales. The idea here, as in the Salesman Game, is to reward salesmen not just for high effort, but for appropriate effort.

The Salesman Game illustrates a number of ideas. It can have either a pooling or a separating equilibrium, depending on the utility function of the salesman. The revelation principle can be applied to avoid having to consider contracts in which the manager must interpret the salesman's lies. It also shows how to use diagrams when the algebraic functions are intractable or unspecified, a problem that does not arise in most of the two-valued numerical examples in this book.

10.3: Myerson Mechanism Design Example

Myerson (1991) uses a trading example in Sections 6.4 and 10.3 of his book to illustrate mechanism design. A seller has 100 units of a good. If it is high quality, he values it at 40 dollars per unit; if it is low quality, at 20 dollars. The buyer, who cannot observe quality before purchase, values high quality at 50 dollars per unit and low quality at 30 dollars. For efficiency, all of the good should be transferred from the seller to the buyer. The only way to get the seller to truthfully reveal the quality of the good, however, is for the buyer to say that if the seller admits the quality is bad, he will buy more units than if the seller claims it is good. Let us see how this works out.

Depending on who offers the contract and when it is offered, various games result. We will start with one in which the seller makes the offer, and does so before he knows whether his quality is high or low.

Myerson Trading Game I

Players

A buyer and a seller.

The Order of Play
1 The seller offers the buyer a contract $(Q_H, P_H, T_H, Q_L, P_L, T_L)$ under which the seller will later declare his quality to be high or low, and the buyer will first pay the lump sum T to the seller (perhaps with $T < 0$) and then buy Q units of the 100 the seller has available, at price P.
2 The buyer accepts or rejects the contract.
3 Nature chooses whether the seller's good is High quality (probability 0.2) or low quality (probability 0.8), unobserved by the buyer.
4. If the contract was accepted by both sides, the seller declares his type to be L or H and sells at the appropriate quantity and price as stated in the contract.

Payoffs
If the buyer rejects the contract, $\pi_{buyer} = 0$, $\pi_{seller\ H} = 40 * 100$, and $\pi_{seller\ L} = 20 * 100$.
If the buyer accepts the contract and the seller declares a type that has price P, quantity Q, and transfer T, then

$$\pi_{buyer|seller\ H} = -T + (50 - P)Q \quad and \quad \pi_{buyer|seller\ L} = -T + (30 - P)Q \tag{4}$$

and

$$\pi_{seller\ H} = T + 40(100 - Q) + PQ \quad and \quad \pi_{seller\ L} = T + 20(100 - Q) + PQ. \tag{5}$$

The seller wants to design a contract subject to two sets of constraints. First, the buyer must accept the contract. Thus, the participation constraint is[2]

$$\begin{aligned} 0.8\pi_{buyer|seller\ H}(Q_L, P_L, T_L) + 0.2\pi_{buyer|seller\ L}(Q_H, P_H, T_H) &\geq 0 \\ 0.8[-T_L + (30 - P_L)Q_L] + 0.2[-T_H + (30 - P_H)Q_H] &\geq 0 \end{aligned} \tag{6}$$

There might also be a participation constraint for the seller himself, because it might be that even when he designs the contract that maximizes his payoff, his payoff is no higher than when he refuses to offer a contract. He can always offer the acceptable if vacuous null contract $(Q_L = 0, P_L = 0, T_L = 0, Q_H = 0, P_H = 0, T_H = 0)$, however, so we do not need to write out the seller's participation constraint separately.

Second, the seller must design a contract that will induce himself to tell the truth later once he discovers his type. This is, of course a bit unusual— the seller is like a principal designing a contract for himself as agent. That is why things will be different in this chapter than in the chapters on basic moral hazard. What is happening is that the seller is trying to sell not just a good, but a contract, and so he must make the contract attractive to the

[2] Another kind of participation constraint would apply if the buyer had the option to reject purchasing anything, after accepting the contract and hearing the seller's type announcement. That would not make a difference here.

buyer. Thus, he faces incentive compatibility constraints: one for when he is low quality,

$$\begin{aligned} \pi_{seller\ L}(Q_L, P_L, T_L) &\geq \pi_{seller\ L}(Q_H, P_H, T_H) \\ 20(100 - Q_L) + P_L Q_L + T_L &\geq 20(100 - Q_H) + P_H Q_H + T_H, \end{aligned} \tag{7}$$

and one for when he has high quality,

$$\begin{aligned} \pi_{seller\ H}(Q_H, P_H, T_H) &\geq \pi_{seller\ H}(Q_L, P_L, T_L) \\ 40(100 - Q_H) + P_H Q_H + T_H &\geq 40(100 - Q_L) + P_L Q_L + T_L. \end{aligned} \tag{8}$$

To make the contract incentive compatible, the seller needs to set P_H greater than P_L, but if he does that it will be necessary to set Q_H less than Q_L. If he does that, then the low-quality seller will not be irresistably tempted to pretend his quality is high: he would be able to sell at a higher price, but not as great a quantity.

Since Q_H is being set below 100 only to make pretending to be high-quality unattractive, there is no reason to set Q_L below 100, so $Q_L = 100$. The buyer will accept the contract if $P_L \leq 30$, so the seller should set $P_L = 30$. The low-quality seller's incentive compatibility constraint, inequality (7), will be binding, and thus becomes

$$\begin{aligned} \pi_{seller\ L}(Q_L, P_L, T_L) &\geq \pi_{seller\ L}(Q_H, P_H, T_H) \\ 20(100 - 100) + 30 * 100 + 0 &= 20(100 - Q_H) + P_H Q_H + 0. \end{aligned} \tag{9}$$

Solving for Q_H gives us $Q_H = \frac{1000}{P_H - 20}$, which when substituted into the seller's payoff function yields

$$\begin{aligned} \pi_s &= 0.8\pi_{seller\ L}(Q_L, P_L, T_L) + 0.2\pi_{seller\ H}(Q_H, P_H, T_H) \\ &= 0.8[(20)(100 - Q_L) + P_L Q_L + T_L] + 0.2[(40)(100 - Q_H) + P_H Q_H + T_H] \\ &= 0.8[(20)(100 - 100) + 30 * 100 + 0] + 0.2[(40)(100 - \tfrac{1000}{P_H - 20}) + P_H(\tfrac{1000}{P_H - 20}) + 0] \end{aligned} \tag{10}$$

Maximizing with respect to P_H subject to the constraint that $P_H \leq 50$ (or else the buyer will turn down the contract) yields the corner solution of $P_H = 50$, which allows for $Q_H = 33\frac{1}{3}$.

The participation constraint for the buyer is already binding, so we do not need the transfers T_L and T_H to take away any remaining surplus, as we might in other situations.[3] Thus, the equilibrium contract is

$$(Q_L = 100, P_L = 30, T_L = 0, Q_H = 33\frac{1}{3}, P_H = 50, T_H = 0). \tag{11}$$

Note that this mechanism will not work if further offers can be made after the end of the game. The mechanism is not first-best efficient; if the seller is high- quality, then he

[3] The transfers could be used to adjust the prices, too. We could have $Q_L = 20$ and $T_L = 1000$ in equation (10) without changing anything important.

only sells $33\frac{1}{3}$ units to the buyer instead of all 100, even though both realize that the buyer's value is 50 and the seller's is only 40. If they could agree to sell the remaining $66\frac{2}{3}$ units, then the mechanism would not be incentive compatible in the first place, though, because then the low-quality seller would pretend to be high-quality, first selling $33\frac{1}{3}$ units and then selling the rest. The importance of commitment is a general feature of mechanisms.

What if it is the buyer who makes the offer?

Myerson Trading Game II

The Order of Play
The same as in Myerson Trading Game I except that the buyer makes the contract offer in move (1) and the seller accepts or rejects in move (2).

Payoffs
The same as in Myerson Trading Game I.

The participation constraint in the buyer's mechanism design problem is

$$0.8\pi_{seller\ L}(Q_L, P_L, T_L) + 0.2\pi_{seller\ H}(Q_H, P_H, T_H) \quad \geq 0. \tag{12}$$

The incentive compatibility constraints are just as they were before, since the buyer has to design a mechanism which makes the seller truthfully reveal his type.

As before, the mechanism will set $Q_L = 100$, but it will have to make $Q_H < 100$ to deter the low-quality seller from pretending he is high- quality. Also, $P_H \geq 40$, or the high-quality seller will pretend to be low-quality.

Suppose $P_H = 40$. The low-quality seller's incentive compatibility constraint, inequality (7), will be binding, and thus becomes

$$\begin{aligned} \pi_{seller\ L}(Q_L, P_L, T_L) \qquad\qquad &\geq \pi_{seller\ H}(Q_H, P_H, T_H) \\ 20(100-100) + P_L * 100 + 0 \quad &= 20(100 - Q_H) + 40Q_H + 0. \end{aligned} \tag{13}$$

Solving for Q_H gives us $Q_H = 5P_L - 100$, which when substituted into the buyer's payoff function yields

$$\begin{aligned} \pi_b \quad &= 0.8\pi_{b|L}(Q_L, P_L, T_L) + 0.2\pi_{b|H}(Q_H, P_H, T_H) \\ &= 0.8[(30 - P_L)Q_L] + 0.2[(50 - P_H)Q_H] \\ &= 0.8[(30 - P_L)100] + 0.2[(50 - 40)(5P_L - 100)] \\ &= 2400 - 80P_L + 10P_L - 200 = 2200 - 70P_L \end{aligned} \tag{14}$$

Maximizing with respect to P_L subject to the constraint that $P_L \geq 20$ (or else we would come out with $Q_H < 0$ to satisfy incentive compatibility constraint (9)) yields the corner solution of $P_L = 20$, which requires that $Q_H = 0$.

Would setting $P_H > 40$ help? No, because that just makes it harder to satisfy the low-quality seller's incentive compatibility constraint. We would continue to have $Q_H = 0$, and, of course, P_H does not matter if nothing is sold. And as before, we do not need to make use of transfers to make the participation constraint binding. Thus, the equilibrium contract has P_H take any possible value and

$$(Q_L = 100, P_L = 20, T_L = 0, Q_H = 0, T_H = 0). \tag{15}$$

In the next version of the game, we will continue to let the buyer make the offer, but he makes it at a time when the seller already knows his type. Thus, this will be an adverse selection model.

Myerson Trading Game III

The Order of Play

0. Nature chooses whether the seller's good is high quality (probability 0.2) or low quality (probability 0.8), unobserved by the buyer.
1 The buyer offers the seller a contract $(Q_H, P_H, T_H, Q_L, P_L, T_L)$ under which the seller will later declare his quality to be high or low, and the buyer will first pays the lump sum T to the seller (perhaps with $T < 0$) and then buy Q units of the 100 the seller has available, at price P.
2 The seller accepts or rejects the contract.
3. If the contract was accepted by both sides, the seller declares his type to be L or H and sells at the appropriate quantity and price as stated in the contract.

Payoffs

The same as in Myerson Trading Games I and II.

The incentive compatibility constraints are unchanged from the previous two versions of the game, but now the participation constraints are different for the two types of seller.

$$\pi_L(Q_L, P_L, T_L) \quad \geq 0 \tag{16}$$

and

$$\pi_H(Q_H, P_H, T_H) \quad \geq 0. \tag{17}$$

Any mechanism which satisfies these two constraints would also satisfy the single participation constraint in MTG II, since it says that a weighted average of the payoffs of the two sellers must be positive. Thus, any mechanism which maximized the buyer's payoff in MTG II would also maximize his payoff in MTG III, if it satisfied the tougher bifurcated participation constraints. The mechanism we found for the game does satisfy the tougher constraints, so it is the optimal mechanism here too.

This is not a general feature of mechanisms. More generally the optimal mechanism will not have as high a payoff when one player starts the game with superior information, because of the extra constraints on the mechanism.

In the last of our versions of this game, the seller makes the offer, but after he knows his type.

Myerson Trading Game IV

The Order of Play
The same as in Myerson Trading Game III except that in (1) the seller makes the offer and in (2) the buyer accepts or rejects.

Payoffs
The same as in Myerson Trading Games I, II, and III.

The incentive compatibility constraints are the same as in the previous games, and the participation constraint is inequality (6), just as in *Myerson Trading Game I.* The big difference now is that unlike in the first three versions, MTG IV has an informed player making the contract offer. As a result, the form of the offer can convey information, and we have to consider out-of- equilibrium beliefs, as in the dynamic games of incomplete information in Chapter 6 (and we will see more of this in the signalling models of Chapter 11). Surprisingly, however, the importance of out-of-equilibrium beliefs does not lead to multiple equilibria. Instead, the equilibrium contract is

M1: $(Q_L = 100, P_L = 30, T_L = 0, Q_H = 33\frac{1}{3}, P_H = 50, T_H = 0)$,

This is part of equilibrium under the out-of-equilibrium belief that if the seller offers any other contract, the buyer believes the quality is low.

This is the same equilibrium mechanism as in MTG I. It is interesting to compare it to two other mechanisms, M2 and M3, which satisfy the two incentive compatibility constraints and the participation constraint, but which are not equilibrium choices here:[4]

M2: $(Q_L = 100, P_L = 28, T_L = 0, Q_H = 0, P_H = 40, T_H = 800)$.

M3: $(Q_L = 100, P_L = 31\frac{3}{7}, T_L = 0, Q_H = 57\frac{1}{7}, P_H = 40, T_H = 0)$.

Mechanism M2 is interesting because the buyer expects a positive payoff of (30- 28) (100) = 200 if the seller is low-quality and a negative payoff of 800 if the seller is high-quality, for an overall expected payoff of zero. The contract is incentive compatible because a low-quality seller could not increase his payoff of 28*100 by pretending to be high-quality (he would get 20*100 + 800 instead), and a high- quality seller would reduce his payoff

[4]xxx M2– hwo about an oo belief that a deviator is a HIGH type? Then no deviation is profitable.

of (40*100 + 800) if he pretended to have low quality. Here, for the first time, we see a positive value for the transfer T_H.

Under mechanism M3, the buyer expects a negative payoff of (30-31 $\frac{1}{7}$) (100) = -11 $\frac{3}{7}$ if the seller is low- quality and a positive payoff of $(57\frac{1}{7})(50-40)$ =11 $\frac{3}{7}$ if the seller is high-quality, for an overall expected payoff of zero. The contract is incentive compatible because a low-quality seller could not increase his payoff of 3,142 $\frac{6}{7}$ (31 $\frac{3}{7}$) (100) by pretending to be high-quality (he would get (57 $\frac{1}{7}$) (40) + (42 $\frac{6}{7}$) (20) instead, which comes to the same figure), and a high-quality seller would reduce his payoff if he pretended to have low quality and sold something he valued at 40 at a price of 31 $\frac{1}{7}$.

In MTG IV, unlike the previous versions of the game, the particular mechanism chosen in equilibrium is not necessarily the one that the player who offers the contract likes best. Instead, an informed offeror— here, the seller— must worry that his offer might make the uninformed receiver believe the offeror's type is undesirable.

Mechanism M1 maximizes the payoff of the average seller, as we found in MTG I, yielding the low-quality seller a payoff of 3,000 and the high- quality seller a payoff of 4,333 $(= (33\frac{1}{3})(50)+66\frac{2}{3}(40))$, for an average payoff of 3,867. If the seller is high-quality, however, he would prefer mechanism M2, which has payoffs of 2800 and 4800 (=800+ 40(100)), for an average payoff of 3200. If the seller is low-quality, he would prefer mechanism M3, which has payoffs of $3,142\frac{6}{7}$ and 4000, for an average payoff of $3,314\frac{6}{7}$.

Suppose that the seller chose M2, regardless of his type. This could not be an equilibrium, because a low-quality seller would want to deviate. Suppose he deviated and offered a contract almost like M1, except that $P_L = 29.99$ instead of 30 and $P_H = 49.99$ instead of 50. This new contract would yield positive expected payoff to the buyer whether the buyer believes the seller is low-quality or high-quality, and so it would be accepted. It would yield higher payoff to the low- quality seller than M2, and so the deviation would have been profitable. Similarly, if the seller chose M3 regardless of his type, a high- quality seller could profitably deviate in the same way.

The *Myerson Trading Game* is a good introduction to the flavor of the algebra in mechanism design problems. For more on this game, in a very different style of presentation, see Sections 6.4 and Chapter 10 of Myerson (1991). We will next go on to particular economic applications of mechanism design.[5]

10.4: Price Discrimination

When a firm has market power – most simply when it is a monopolist– it would like to charge different prices to different consumers. To the consumer who would pay up to

[5]xxx Think abvout risk sharing too, with risk aversion. Then it makes a big difference when the seller learns his type.

$45,000 for a car, the firm would like to charge $45,000; to the consumer who would pay up to $36,000, the profit-maximizing price is $36, 000. But how does the car dealer know how much each consumer is willing to pay?

He does not, and that is what makes this a problem of mechanism design under adverse selection. The consumer who would be willing to pay $45,000 can hide under the guise of being a less intense consumer, and despite facing a monopolist he can end up retaining consumer surplus – an **informational rent**, a return to the consumer's private information about his own type.[6]

Pigou was a contemporary of Keynes at Cambridge who usefully divided price discrimination into three types in 1920 but named them so obscurely that I relegate his names to the endnotes and use better ones here:

1 Interbuyer price discrimination. This is when the seller can charge different prices to different buyers. Smith's price for a hamburger is $4 per burger, but Jones's is $6.

2 Interquantity price discrimination or **Nonlinear pricing**. This is when the seller can charge different unit prices for different quantities. A consumer can buy a first sausage for $9, a second sausage for $4, and a third sausage for $3. Rather than paying the "linear" total price of $9 for one sausage, $18 for two, and $27 for three, he thus pays the nonlinear price of $9 for one sausage, $13 for two, and $16 for three, the concave price path shown in Figure 3.

3 Perfect price discrimination. This combines interbuyer and interquantity price discrimination. When the seller does have perfect information and can charge each buyer that buyer's reservation price for each unit bought, Smith might end up paying $50 for his first hot dog and $20 for his second, while next to him Jones pays $4 for his first and $3 for his second.

[6]In real life, a standard opening ploy of car salesman is simply to ask. "So, how much are you able to spend on a car today?" My recommendation: don't tell him. This may sound obvious, but remember it the next time your department chairman asks you how high a salary it would take to keep you from leaving for another university.

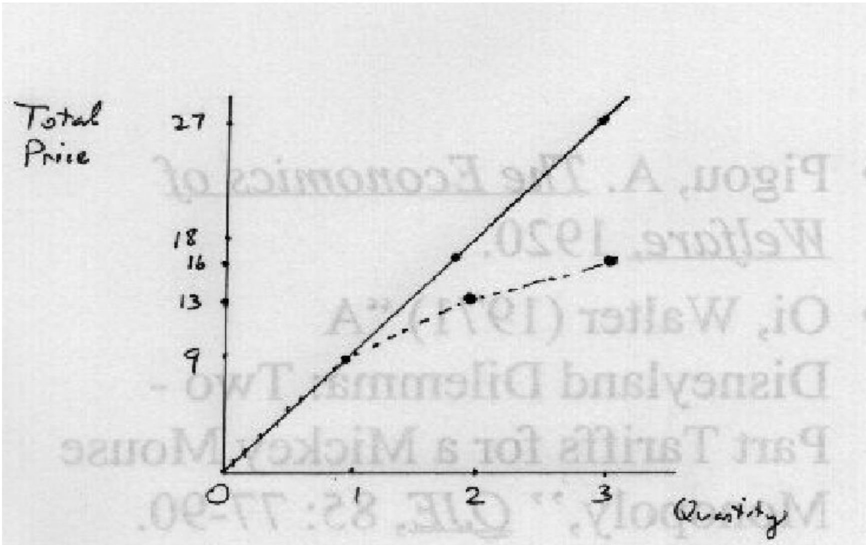


Figure 3: Linear and Nonlinear Pricing

To illustrate price discrimination as mechanism design we will use a modified version of an example in Chapter 14 of Hal Varian's third edition (Varian, 1992).

Varian's Nonlinear Pricing Game

Players
One seller and two buyers, Smith and Jones.

The Order of Play
0 Nature assigns one of the two buyers to be Unenthusiastic with utility function u and the other to be Valuing with utility function v, Smith and Jones having equal probabilities of filling each role. The seller does not observe Nature's move.
1 The seller offers a price mechanism $r(x)$ under which a buyer can buy amount x for total price $r(x)$.
2 The buyers simultaneously choose to buy quantities x_u and x_v.

Payoffs
The seller has a constant marginal cost of c, so his payoff is $r(x_u) + r(x_v) - c \cdot (x_u + x_v)$. The buyers' payoffs are $u(x_u) - r(x_u)$ and $v(x_v) - r(x_v)$ if x is positive, and 0 if $x = 0$, with $u', v' > 0$ and $u'', v'' < 0$. The total and marginal willingnesses to pay are greater for the Valuing buyer. For all x,

$$\begin{array}{ll} (a) & u(x) < v(x) \text{ and} \\ (b) & u'(x) < v'(x) \end{array} \tag{18}$$

Condition (18b) is known as the **single-crossing property**, since it implies that the indifference curves of the two agents cross at most one time (see also Section 11.1).

Combined with Condition (18a), it means they *never* cross – the Valuing buyer always has stronger demand. Figure 4 illustrates the single-crossing property in two different ways. Figure 4a on the next page directly illustrates assumptions (18a) and (18b) – the utility function of the Valuing player starts higher and rises more steeply. Since utility units can be rescaled, though, this assumption should make you uncomfortable – do we really want to assume that the Valuing player is a happier person at zero consumption than the Unenthusiastic player? I personally am not bothered, but many economists prefer to restrict themselves to models in which utility is only ordinal, not cardinal, and is in units that cannot be compared between people. We can do that here. Figure 4b illustrates the single-crossing property in goods-space, where it says that if we pick one indifference curve for each player, the two curves only cross once. In words, this says that the Valuing player always requires more extra money as compensation for reducing his consumption of the commodity we are studying than the Unenthusiastic player would. This approach avoids the issue of which player is happier, but the cost is that Figure 4b is harder to understand than Figure 4a.

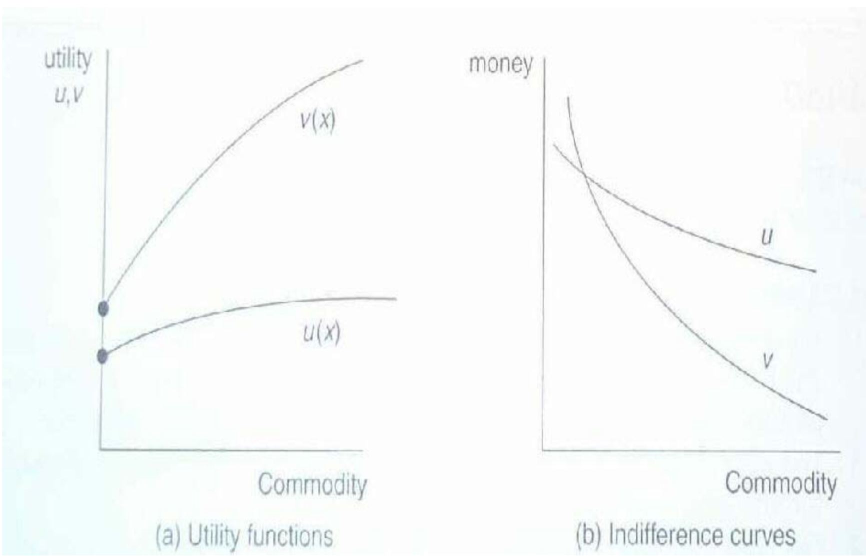


Figure 4: The Single- Crossing Property

To ease into the difficult problem of solving for the equilibrium mechanism, let us solve for the equilibrium of two simpler versions of the game that limit it to (a) perfect price discrimination and (b) interbuyer discrimination.

Perfect Price Discrimination

The game would allow perfect price discrimination if the seller did know which buyer had which utility function. He can then just maximize profit subject to the participation constraints for the two buyers:

$$\underset{r(x_u), r(x_v), x_u, x_v}{Maximize} \quad r(x_u) + r(x_v) - c \cdot (x_u + x_v). \tag{19}$$

subject to

$$\begin{array}{ll} (a) & u(x_u) - r(x_u) \geq 0 \text{ and} \\ (b) & v(x_v) - r(x_v) \geq 0. \end{array} \tag{20}$$

The constraints will be satisfied as equalities, since the seller will charge all that the buyers will pay. Substituting for $r(x_u)$ and $r(x_v)$ into the maximand, the first order conditions become

$$\begin{array}{ll} (a) & u'(x_u^*) - c = 0 \quad \text{and} \\ (b) & v'(x_v^*) - c = 0. \end{array} \tag{21}$$

Thus, the seller will choose quantities so that each buyer's marginal utility equals the marginal cost of production, and will choose prices so that the entire consumer surpluses are eaten up: $r^*(x_u^*) = u(x_u^*)$ and $r^*(x_v^*) = v(x_v^*)$. Figure 5 shows this for the unenthusiastic buyer.

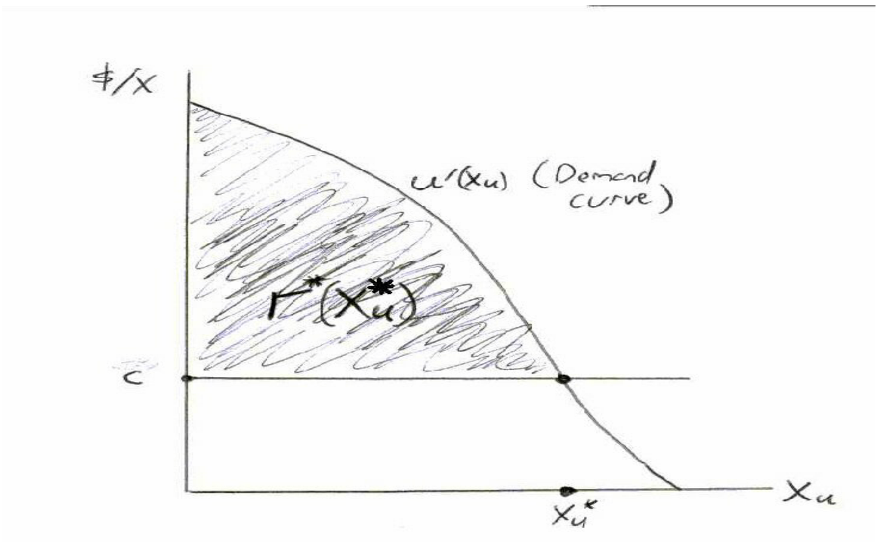


Figure 5: Perfect Price Discrimination

Interbuyer Price Discrimination

The interbuyer price discrimination problem arises when the seller knows which utility functions Smith and Jones have and can sell to them separately but he must charge each buyer a single price per unit and let the buyer choose the quantity. The seller's problem is

$$\underset{x_u, x_v, p_u, p_v}{Maximize} \quad p_u x_u + p_v x_v - c \cdot (x_u + x_v), \tag{22}$$

subject to

$$\begin{array}{ll} (a) & u(x_u) - p_u x_u \geq 0 \quad \text{and} \\ (b) & v(x_v) - p_v x_v \geq 0 \end{array} \tag{23}$$

and

$$\begin{array}{ll} (a) & x_u = argmax[u(x_u) - p_u x_u] \quad \text{and} \\ (b) & x_v = argmax[v(x_v) - p_v x_v]. \end{array} \tag{24}$$

This should remind you of moral hazard. It is very like the problem of a principal designing two incentive contracts for two agents to induce appropriate effort levels given their different disutilities of effort.

The agents will solve their quantity choice problems in (24), yielding

$$\begin{array}{ll} (a) & u'(x_u) - p_u = 0 \quad \text{and} \\ (b) & v'(x_v) - p_v = 0. \end{array} \tag{25}$$

Thus, we can simplify the original problem in (22) to

$$\underset{x_u, x_v}{Maximize} \quad u'(x_u)x_u + v'(x_v)x_v - c \cdot (x_u + x_v), \tag{26}$$

subject to

$$\begin{array}{ll} (a) & u(x_u) - p_u x_u \geq 0 \quad \text{and} \\ (b) & v(x_v) - p_v x_v \geq 0. \end{array} \tag{27}$$

The first order conditions are

$$\begin{array}{ll} (a) & u''(x_u)x_u + u' = c \quad \text{and} \\ (b) & v''(x_v)x_v + v' = c. \end{array} \tag{28}$$

This is just the 'marginal revenue equals marginal cost' condition that any monopolist uses, but one for each buyer instead of one for the entire market.

Back to Nonlinear Pricing

Neither the perfect price discrimination nor the interbuyer problems are mechanism design problems, since the seller is perfectly informed about the types of the buyers and has no need to worry about designing incentives to separate them. In the original game, however, separation is the seller's main concern. He must satisfy not just the participation constraints, but self-selection constraints. The seller's problem is

$$\underset{x_u, x_v, r(x_u), r(x_v)}{Maximize} \quad r(x_u) + r(x_v) - c \cdot (x_u + x_v), \tag{29}$$

subject to the participation constraints,

$$\begin{array}{ll} (a) & u(x_u) - r(x_u) \geq 0 \quad \text{and} \\ (b) & v(x_v) - r(x_v) \geq 0, \end{array} \tag{30}$$

and the self-selection constraints,

$$\begin{array}{ll} (a) & u(x_u) - r(x_u) \geq u(x_v) - r(x_v) \\ (b) & v(x_v) - r(x_v) \geq v(x_u) - r(x_u). \end{array} \tag{31}$$

Not all of these constraints will be binding. If neither type had a binding participation constraint, the principal would be losing a chance to increase his profits, unless there were moral hazard in the model too and some kind of efficiency wage was at work. In a mechanism design problem like this, what always happens is that the contracts are designed so that one type of agent is pushed down to his reservation utility.

Suppose the optimal contract is in fact separating, and also that both types of agent accept a contract. I have shown that at least one type will have a binding participation constraint. The second type could accept that same contract and receive more than his reservation utility, so to separate the two types the principal must offer the second type a contract which also yields more than his reservation utility. The principal will not want to be overly generous to the second type, however, so he makes sure the second type gets no more utility from his assigned contract than from accepting the first type's contract. Thus, one type of agent will have a binding participation constraint, and the other will have a binding self-selection constraint, and the other two constraints will be nonbinding. The question is: which type of buyer has which constraint binding in Varian's Nonlinear Pricing Game?

Let us start with the premise that a given constraint is binding and see if we can use our data to find a contradiction. Assume that the Valuing participation constraint, (30b), is binding. Then $v(x_v) = r(x_v)$. Substituting for $v(x_v)$ in the self-selection constraint (31b) then yields

$$r(x_v) - r(x_v) \geq v(x_u) - r(x_u), \tag{32}$$

so $r(x_u) \geq v(x_u)$. It follows from assumption (18a), which says that $u(x) < v(x)$, that $r(x_u) \geq u(x_u)$. But the Unenthusiastic participation constraint, (30a), says that $r(x_u) \leq u(x_u)$, and since these are compatible only when $r(x_u) = u(x_u)$ and we have assumed that (30b) is the binding participation constraint, we have arrived at a contradiction. Our starting point must be false, and it is in fact (30a), not (30b), that is the binding participation constraint.

We could next start with the premise that self-selection constraint (31a) is binding and derive a contradiction using assumption (18b). But the reasoning above showed that if the participation constraint is binding for one type of agent then the self-selection constraint will be binding for the other, so we can jump to the conclusion that it is in fact self-selection constraint (31b) that is binding.

Rearranging our two binding constraints and setting them out as equalities yields:

$$(30a') \quad r(x_u) = u(x_u)$$

and

$$(31b') \quad r(x_v) = r(x_u) - v(x_u) + v(x_v)$$

This allows us to reformulate the seller's problem from (29) as

$$\underset{x_u,\, x_v}{Maximize} \quad u(x_u) + u(x_u) - v(x_u) - v(x_v) - c \cdot (x_u + x_v), \tag{33}$$

which has the first-order conditions

$$\begin{aligned} &(a) \quad u'(x_u) - c + [u'(x_u) - v'(x_u)] = 0 \\ &(b) \quad v'(x_v) - c = 0 \end{aligned} \tag{34}$$

These first-order conditions could be solved for exact values of x_u and x_v if we chose particular functional forms, but they are illuminating even if we do not. Equation (34b) tells us that the Valuing type of buyer buys a quantity such that his last unit's marginal utility exactly equals the marginal cost of production; his consumption is at the efficient level. The Unenthusiastic type, however, buys less than his first-best amount, something we can deduce using the single-crossing property, assumption (18 b), that $u'(x) < v'(x)$, which implies from (34a) that $u'(x_u) - c > 0$ and the Unenthusiastic type has not bought enough to drive his marginal utility down to marginal cost. The intuition is that the seller must sell less than first-best optimal to the Unenthusiastic type so as not to make that contract too attractive to the Valuing type. On the other hand, making the Valuing type's contract more valuable to him actually helps separation, so x_v is chosen to maximize social surplus.

The single-crossing property has another important implication. Substituting from first-order condition (34b) into first-order condition (34a) yields

$$[u'(x_u) - v'(x_v)] + [u'(x_u) - v'(x_u)] = 0 \tag{35}$$

The second term in square brackets is negative by the single- crossing property. Thus, the first term must be positive. But since the single-crossing property tells us that $[u'(x_u) - v'(x_u)] < 0$, it must be true, since $v'' < 0$, that if $x_u \geq x_v$ then $[u'(x_u) - v'(x_v)] < 0$ – that is, that the first term is negative. We cannot have that without contradiction, so it must be that $x_u < x_v$. The Unenthusiastic buyer buys strictly less than the Valuing buyer. This accords with our intuition, and also lets us know that the equilibrium is separating, not pooling (though we still have not proven that the equilibrium involves both players buying a positive amount, something hard to prove elegantly since one player buying zero would be a corner solution to our maximization problem).

A Graphical Approach to the Same Problem

Under perfect price discrimination, the seller would charge $r_u = A + B$ and $r_v = A+B+J+K+L$ to the two buyers for quantities x_u^* and x_v^*, as shown in Figure 10.6a. An attempt to charge $r(x_u^*) = A + B$ and $r(x_v^*) = A + B + J + K + L$, however, would simply lead to both buyers choosing to buy x_u^*, which would yield the Valuing buyer a payoff of $J + K$ rather than the 0 he would get as a payoff from buying x_v^*.

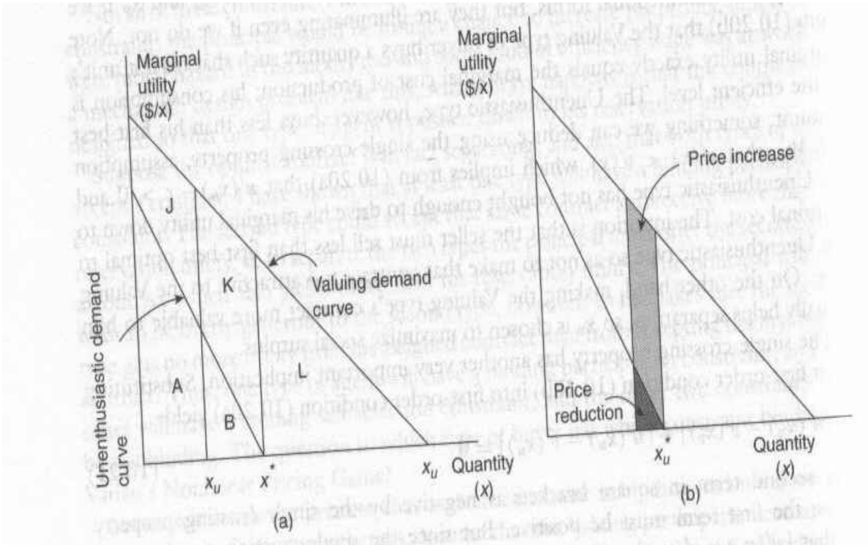


Figure 6: The Varian Nonlinear Pricing Game

The seller could separate the two buyers by charging $r(x_u^*) = A+B$ and $r(x_v^*) = A+B$, since the Unenthusiastic buyer would have no reason to switch to the greater quantity, but that would not increase his profits any over pooling. Figure 6b shows that the seller would do better to slightly reduce the quantity sold to the Unenthusiastic buyer and reduce the price by the amount of the dark shading, while selling x_u^* to the Valuing buyer and raising the price to him by the light shaded area. The Valuing buyer will still not be tempted to buy the smaller quantity at the lower price.

The profit-maximizing mechanism found earlier is shown in Figure 10.6a by $r(x_u') = A$ and $r(x_v^*) = A + B + K + L$. The Unenthusiastic buyer is left with a binding participation constraint, because $r(x_u') = A = u(x_u')$. The Valuing buyer has a nonbinding participation constraint, because $r(x_v^*) = A + B + K + L < v(x_v^*) = A + B + J + K + L$. But the Valuing buyer does have a binding self selection constraint, because he is exactly indifferent between buying x_u' and x_v^* — $v(x_u')$, because $r(x_u') = (A + J) - A$ and $v(x_v^*) - r(x_v^*) = (A + B + J + K + L) - (A + B + K + L)$. Thus, the diagram replicates the algebraic conclusions.

*10.5 Rate-of-Return Regulation and Government Procurement

The central idea in both government procurement and regulation of natural monopolies is that the government is trying to induce a private firm to efficiently provide a good to the public while covering the cost of production. If information is symmetric, this is an easy problem; the government simply pays the firm the cost of producing the good efficiently, whether the good be a missile or electricity. Usually, however, the firm has better information about costs and demand than the government does.

The variety of ways the firm might have better information and the government might extract it has given rise to a large literature in which moral hazard with hidden actions, moral hazard with hidden knowledge, adverse selection, and signalling all put in appearances. Suppose the government wants a firm to provide cable television service to a city. The firm knows more about its costs before agreeing to accept the franchise (adverse selection), discovers more after accepting it and beginning operations (moral hazard with hidden knowledge), and exerts greater or smaller effort to keep costs low (moral hazard with hidden actions). The government's problem is to acquire cable service at the lowest cost. It wants to be generous enough to induce the firm to accept the franchise in the first place but no more generous than necessary. It cannot simply agree to cover the firm's costs, because the firm would always claim high costs and exert low effort. Instead, the government might auction off the right to provide the service, might allow the firm a maximum price (a **price cap**), or might agree to compensate the firm to varying degrees for different levels of cost (**rate-of- return regulation**).

The problems of regulatory franchises and government procurement are the same in many ways. If the government wants to purchase a cruise missile, it also has the problem of how much to offer the firm. Roughly speaking, the equivalent of a price cap is a flat price, and the equivalent of rate-of-return regulation is a cost-plus contract, although the details differ in interesting ways. (A price cap allows downwards flexibility in prices, and rate-of-return regulation allows an expected but not guaranteed profit, for example.)

Many of these situations are problems of moral hazard with hidden information, because one player is trying to design a contract that the other will accept that will then induce him to use his private information properly.

Although the literature on mechanism design can be traced back to Mirrlees (1971), its true blossoming has occurred since Baron & Myerson's 1982 article, "Regulating a Monopolist with Unknown Costs." McAfee & McMillan (1988), Spulber (1989) and Laffont & Tirole (1993) provide 168-page, 690-page, and 702-page treatments of the confusing array of possible models and policies in their books on government regulation. Here, we will look at a version of the model Laffont and Tirole use to introduce their book on pages 55 to 62. This is a two-type model in which a special cost characteristic and the effort of a firm is its private information but its realized cost is public and nonstochastic. The model combines moral hazard and adverse selection, but it will behave more like an adverse selection model. The government will reimburse the firm's costs, but also fixes a price (which if negative becomes a tax) that depend on the level of the firm's costs. The questions the model hopes to answer are (a) whether effort will be too high or too low and (b) whether the price is positive and rises with costs.

Procurement I: Perfect Information [7]

[7] I have changed the notation from the 3rd edition of this book. The special problem variable x replaces the ability variable a; p replaces s; the type L firm becomes a special-cost firm.

Players
The government and the firm.

The Order of Play
0 Nature determines whether the firm has special problems that add costs of x, which has probability θ, or no special problems, which has probability $(1-\theta)$. We will call these "special" and "normal" firms, with the understanding that "special" problems may be the norm in engineering projects. The government and the firm both observe this move.
1 The government offers a contract agreeing to cover the firm's cost c of producing a cruise missile and specifying an additional price $p(c)$ for each cost level that the firm might report.
2 The firm accepts or rejects the contract.
3 If the firm accepts, it chooses effort level e, unobserved by the government.
4 The firm finishes the cruise missile at a cost of $c = c_0 + x - e$ or $c = c_0 - e$ which is observed by the government, plus an additional cost $f(e - c_0)$ that the government does not observe. The government reimburses c and pays $p(c)$.

Payoffs
Both firm and government are risk neutral and both receive payoffs of zero if the firm rejects the contract. If the firm accepts, its payoff is

$$\pi_{firm} = p - f(e - c_0), \tag{36}$$

where $f(e - c_0)$, the cost of effort, is increasing and convex, so $f' > 0$ and $f'' > 0$. Assume, too, for technical convenience, that f is increasingly convex, so $f''' > 0$.[8] The government's payoff is

$$\pi_{government} = B - (1+\lambda)c - \lambda p - f, \tag{37}$$

where B is the benefit of the cruise missile and λ is the deadweight loss from the taxation needed for government spending.[9]

The model differs from other principal-agent models in this book because the principal cares about the welfare of the agent. If the government cared only about the value of the cruise missile and the cost to taxpayers, its payoff would be $[B-(1+\lambda)c-(1+\lambda)p]$. Instead, the payoff function maximizes social welfare, the sum of the welfares of the taxpayers and the firm. The welfare of the firm is $(p - f)$, and summing the two welfares yields equation (37). Either kind of government payoff function may be realistic, depending on the political balance in the country being modelled, and the model will have similar properties whichever one is used.

Assume for the moment that B is large enough that the government definitely wishes to build the missile (how large will become apparent later). Cost, not output, is the focus of this model. The optimal output is one cruise missile regardless of agency problems, but the government wants to minimize the cost of producing the missile.

[8] The argument of f is normalized to be $(c_0 - e)$ rather than just e to avoid clutter in the algebra later. The assumption that $f''' > 0$ allows the use of first-order conditions by making concave the maximand in (48), which is a difference of two concave functions. It will also make deterministic contracts superior to stochastic ones. See p. 58 of Laffont & Tirole (1993).

[9] Hausman & Poterba (1987) estimate this loss to be around \$0.30 for each \$1 of tax revenue raised at the margin for the United States.

In this first variant of the game, whether the firm has special problems is observed by the government, which can therefore specify a contract conditioned on the type of the firm. The government pays prices of p_N to a normal firm with the cost $\underline{c}$, p_S to a special firm with the cost $\overline{c}$, and a price of $p = 0$ to a firm that does not achieve its appropriate cost level.

The special firm exerts effort $e = c_0 + x - \overline{c}$, achieves $c = \overline{c}$, generating unobserved effort disutility $f(e - c_0) = f(x - \overline{c})$, so its participation constraint is:

$$\begin{aligned} \pi_S(S) &\geq 0 \\ p_S - f(x - \overline{c}) &\geq 0. \end{aligned} \tag{38}$$

Similarly, in equilibrium the normal firm exerts effort $e = c_0 - \underline{c}$, so its participation constraint is

$$\begin{aligned} \pi_N(N) &\geq 0 \\ p_N - f(-\underline{c}) &\geq 0 \end{aligned} \tag{39}$$

To make a firm's payoff zero and reduce the deadweight loss from taxation, the government will provide prices that exactly cover the firm's disutility of effort. Since there is no uncertainty we can invert the cost equation and write it as $e = c_0 + x - c$ or $e = c_0 - c$. The prices will be $p_S = f(e - c_0) = f(x - \overline{c})$ and $p_N = f(e - c_0) = f(-\underline{c})$.

Suppose the government knows the firm has special problems. Substituting the price P_S into the government's payoff function, equation (37), yields

$$\pi_{government} = B - (1 + \lambda)\overline{c} - \lambda f(c_0 + x - \overline{c}) - f((x - \overline{c}) - c_0). \tag{40}$$

Since $f'' > 0$, the government's payoff function is concave, and standard optimization techniques can be used. The first-order condition for $\overline{c}$ is

$$\frac{\partial \pi_{government}}{\partial \overline{c}} = -(1 + \lambda) + (1 + \lambda) f'(x - \overline{c}) = 0, \tag{41}$$

so

$$f'(x - \overline{c}) = 1. \tag{42}$$

Since $f'(x - \overline{c}) = f'([c_0 + x - \overline{c}] - c_0)$ and $c_0 + x - \overline{c} = e$, equation (42) says that $\overline{c}$ should be chosen so that $f'(e - c_0) = 1$; at the optimal effort level, the marginal disutility of effort equals the marginal reduction in cost because of effort. This is the first-best efficient effort level, which we will denote by $e^* \equiv e : \{f'(e - c_0) = 1\}$.

Exactly the same is true for the normal firm, so $f'(x - \overline{c}) = f'(-\underline{c}) = 1$ and $\underline{c} = \overline{c} - x$. The cost targets assigned to each firm are $\overline{c} = c_0 + x - e^*$ and $\underline{c} = c_0 - e^*$. Since both types must exert the same effort, e^*, to achieve their different targets, $p_S = f(e^* - c_0) = p_N$. The two firms exert the same efficient effort level and are paid the same price to compensate for the disutility of effort. Let us call this price level p^*.

The assumption that B is sufficiently large can now be made more specific: it is that $B - (1 + \lambda)\overline{c} - \lambda f(e^* - c_0) - f(e^* - c_0) \geq 0$, which requires that $B - (1 + \lambda)(c_0 + x - e^*) - (1 + \lambda)p^* \geq 0$.

Procurement II: Incomplete Information

In the second variant of the game, the existence of special problems is not observed by the government, which must therefore provide incentives for the firm to volunteer its type if the normal firm is to produce at lower cost than the firm with special problems.

The government could use a pooling contract, simply providing a price of p^* for a cost of $c = c_0 + x - e^*$, enough to compensate the firm with special problems for its effort, with $p = 0$ for any other cost. Both types would accept this, but the normal firm could exert effort less than e^* and still get costs down enough to receive the price. (Notice that this is the cheapest possible pooling contract; any cheaper contract would be rejected by the firm with special problems.) Thus, if the government would build the cruise missile under full information knowing that the firm has special problems, it would also build it under incomplete information, when the firm might have special problems.

The pooling contract, however, is not optimal. Instead, the government could offer a choice between the contract $(p^*, c = c_0 + x - e^*)$ and a new contract that offers a higher price but requires reimbursable costs to be lower. By definition of e^*, $f'(c_0 + x - e^* - c_0) = 1$, so $f'(c_0 - e^* - c_0) < 1$, which is to say that the normal firm's marginal disutility of effort when it exerts just enough effort to get costs down to $c = c_0 + x - e^*$ is less than 1. This means that if the government can offer a new contract with slightly lower c but slightly higher p that will be acceptable to the normal firm but will have a lower combined expense of $(p + c)$. This tell us that a separating contract exists that is superior to the pooling contract.

Let us therefore find the optimal contract with values $(\underline{c}, p_N)$ and $(\overline{c}, p_S)$ and $p = 0$ for other cost levels. It will turn out that the $(\overline{c}, p_S)$ part of the optimal separating contract will not be the same as the pooling contract in the previous paragraph, because to find the optimal separating contract it is not enough to find the optimal "new contract;" we need to find the optimal *pair* of contracts, and by finding a new contract for the special-problems firm too, we will be able to reduce the government's expense from the normal firm's contract.

A separating contract must satisfy participation constraints and incentive compatibility constraints for each type of firm. The participation constraints are the same as in Procurement I, inequalities (38) and (39).

The incentive compatibility constraint for the special firm is

$$\begin{aligned} \pi_S(S) &\geq \pi_S(N) \\ p_S - f(x - \overline{c}) &\geq p_N - f(x - \underline{c}), \end{aligned} \tag{43}$$

and for the normal firm it is

$$\begin{aligned} \pi_N(N) &\geq \pi_N(S) \\ p_N - f(-\underline{c}) &\geq p_S - f(-\overline{c}). \end{aligned} \tag{44}$$

Since the normal firm can achieve the same cost level as the special firm with less effort, inequality (44) tells us that if we are to have $\underline{c} < \overline{c}$, as is necessary for us to have a separating equilibrium, we need $P_N > P_S$. The second half of inequality (44) must be positive, If the special firm's participation constraint, inequality (38), is satisfied, then $p_S - f(-\overline{c}) > 0$. This, in turn implies that (39) is a strong inequality; the normal firm's participation constraint is nonbinding.

The special firm's participation constraint, (38), will be binding (and therefore satisfied as an equality), because the government will reduce the price as much as possible in order to avoid the deadweight loss of taxation. The normal firm's incentive compatibility constraint must also be binding, because if the pair $(\underline{c}, p_N)$ were strictly more attractive for the normal firm, the government could reduce the price p_N. Constraint (44) is therefore satisfied as an equality.[10] Knowing that constraints (38) and (44) are binding, we can write from constraint (38),

$$p_S = f(x - \overline{c}) \tag{45}$$

and, making use of both (38) and (44),

$$p_N = f(-\underline{c}) + f(x - \overline{c}) - f(-\overline{c}). \tag{46}$$

From (37), the government's maximization problem under incomplete information is

$$\underset{\underline{c}, \overline{c}, p_N, p_S}{Maximize} \quad \theta \left[B - (1+\lambda)\overline{c} - \lambda p_S - f(x - \overline{c}) \right] + [1 - \theta] \left[B - (1+\lambda)\underline{c} - \lambda p_N - f(-\underline{c}) \right]. \tag{47}$$

Substituting for p_S and p_N from (45) and (46) reduces the problem to

$$\begin{aligned} \underset{\underline{c}, \overline{c}}{Maximize} \quad & \theta[B - (1+\lambda)\overline{c} - \lambda(f(x - \overline{c}) - f(x - \overline{c})] + [1 - \theta][B - (1+\lambda)\underline{c} \\ & -\lambda f(-\underline{c}) - \lambda f(x - \overline{c}) + \lambda f(-\overline{c}) - f(-\underline{c})]. \end{aligned} \tag{48}$$

(1) The first-order condition with respect to $\underline{c}$ is

$$(1 - \theta)[-(1+\lambda) + \lambda f'(-\underline{c}) + f'(-\underline{c})] = 0, \tag{49}$$

which simplifies to

$$f'(-\underline{c}) = 1. \tag{50}$$

Thus, as earlier, $f'_N = 1$. The normal firm chooses the efficient effort level e^* in equilibrium, and $\underline{c}$ takes the same value as it did in Procurement I. Equation (46) can be rewritten as

$$p_N = p^* + f(x - \overline{c}) - f(-\overline{c}). \tag{51}$$

Because $f(x - \overline{c}) > f(-\overline{c})$, equation (51) shows that $p_N > p^*$. Incomplete information increases the price for the normal firm, which earns more than its reservation utility in the game with incomplete information. Since the firm with special problems will earn exactly zero, this means that the government is on average providing its supplier with an above-market rate of return, not because of corruption or political influence, but because that

[10]The same argument does not hold for the special firm, because if p_S were reduced, the participation constraint would be violated.

is the way to induce normal suppliers to reveal that they do not have special costs. This should be kept in mind as an alternative to the product quality model of Chapter 5 and the efficiency wage model of Section 8.1 for why above-average rates of return persist.

(2) The first-order condition with respect to $\overline{c}$ is

$$\theta \left[-(1+\lambda) + \lambda f'(x-\overline{c}) + f'(x-\overline{c})\right] + [1-\theta] \left[\lambda f'(x-\overline{c}) + f'(-\overline{c})\right] = 0. \tag{52}$$

This can be rewritten as

$$f'(x-\overline{c}) = 1 - \left(\frac{1-\theta}{\theta(1+\lambda)}\right) \left[\lambda f'(x-\overline{c}) + f'(-\overline{c})\right]. \tag{53}$$

Since the right-hand-side of equation (53) is less than one, the special firm has a lower level of f' than the normal firm, and must be exerting effort less than e^* since $f'' > 0$. Perhaps this explains the expression "good enough for government work". Also since the special firm's participation constraint, (38), is satisfied as an equality, it must also be true that $p_S < p^*$. The special firm's price is lower than under full information, although since its effort is also lower, its payoff stays the same.

We must also see that the incentive compatibility constraint for the firm with special problems is satisfied as a weak inequality; the firm with special problems is not near being tempted to pick the normal firm's contract. This is a bit subtle. Setting the left-hand-side of the incentive compatibility constraint (43) equal to zero because the participation constraint is binding for the firm with special problems, substituting in for p_N from equation (46) and rearranging yields

$$f(x-\underline{c}) - f(-\underline{c}) \geq f(x-\overline{c}) - f(-\overline{c}). \tag{54}$$

This is true, and true as a strict inequality, because $f'' > 0$ and the arguments of f on the left-hand-side of equation (54) take larger values than on the right-hand side, as shown in Figure 10.7.

Figure 7: The Disutility of Effort

To summarize, the government's optimal contract will induce the normal firm to exert the first-best efficient effort level and achieve the first-best cost level, but will yield that firm a positive profit. The contract will induce the firm with special costs to exert something less than the first-best effort level and result in a cost level higher than the first-best, but its profit will be zero.

There is a tradeoff between the government's two objectives of inducing the correct amount of effort and minimizing the subsidy to the firm. Even under complete information, the government cannot provide a subsidy of zero, or the firms will refuse to build the cruise missile. Under incomplete information, not only must the subsidies be positive but the normal firm earns **informational rents**; the government offers a contract that pays the normal firm with more then under complete information to prevent it from mimicking a firm with special problems by choosing an inefficiently low effort. The firm with special problems, however, does choose an inefficiently low effort, because if it were assigned greater effort it would have to be paid a greater subsidy, which would tempt the normal firm to imitate it. In equilibrium, the government has compromised by having some probability of an inefficiently high subsidy ex post, and some probability of inefficiently low effort.

In the last version of the game, the firm's type is not known to either player until after the contract is agreed upon. The firm, however, learns its type before it must choose its effort level.

Procurement III: Moral Hazard with Hidden Information

The Order of Play
1 The government offers a contract agreeing to cover the firm's cost c of producing a cruise missile and specifying an additional price $p(c)$ for each cost level that the firm might report.
2 The firm accepts or rejects the contract.
3 Nature determines whether the firm has special problems that add costs of x, which has probability θ, or no special problems, which has probability $(1-\theta)$. We will call these "special" and "normal" firms, with the understanding that "special" problems may be the norm in engineering projects. The government and the firm both observe this move.
4 If the firm accepts, it chooses effort level e, unobserved by the government.
5 The firm finishes the cruise missile at a cost of $c = c_0 + x - e$ or $c = c_0 - e$ which is observed by the government, plus an additional cost $f(e - c_0)$ that the government does not observe. The government reimburses c and pays $p(c)$.

The contract must satisfy one overall participation constraint and and two incentive compatibility constraints, one for each type of firm. The participation constraint is

$$\theta[p_S - f(x - \overline{c})] + [1-\theta][p_N - f(-\underline{c})] \geq 0. \tag{55}$$

The incentive compatibility constraints are the same as before: for the special firm,

$$p_S - f(x - \overline{c}) \geq p_N - f(-x - \underline{c}), \tag{56}$$

and for the normal firm,

$$p_N - f(-\underline{c}) \geq p_S - f(-\overline{c}). \tag{57}$$

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As before, constraint (55) will be binding (and therefore satisfied as an equality), because the government will reduce the price as much as possible in order to avoid the deadweight loss of taxation. The normal firm's incentive compatibility constraint must also be binding, because if the pair $(\overline{c}, p_N)$ were strictly more attractive for the normal firm, the government could reduce the price p_N. Constraint (57) is therefore satisfied as an equality.[11] Knowing that constraints (55) and (57) are binding, we can write from constraint (55),

$$p_S = f(x-\overline{c}) - \frac{[1-\theta][p_N - f(-\underline{c})]}{\theta}. \tag{58}$$

Substituting from (58) for p_S into (57), we get

$$p_N - f(-\underline{c}) = f(x-\overline{c}) - \frac{[1-\theta][p_N - f(-\underline{c})]}{\theta} - f(-\overline{c}). \tag{59}$$

This can be solved for p_N to yield

$$p_N = \theta[f(x-\overline{c}) - f(-\overline{c})] + f(-\underline{c}), \tag{60}$$

which when substituted into (58) yields

$$p_S = [1-\theta][f(x-\overline{c}) - f(-\overline{c})]. \tag{61}$$

From (37), the government's maximization problem under incomplete information is

$$\underset{\underline{c},\overline{c},p_N,p_S}{Maximize} \quad \theta\left[B - (1+\lambda)\overline{c} - \lambda p_S - f(x-\overline{c})\right] + [1-\theta]\left[B - (1+\lambda)\underline{c} - \lambda p_N - f(-\underline{c})\right]. \tag{62}$$

Substituting for p_N and p_S from (60) and (61) reduces the problem to

$$\begin{aligned}\underset{\underline{c},\overline{c},p_N,p_S}{Maximize} \quad & \theta\left\{B - (1+\lambda)\overline{c} - \lambda[1-\theta][f(x-\overline{c}) - f(-\overline{c})] - f(x-\overline{c})\right\} \\ & + [1-\theta]\left[B - (1+\lambda)\underline{c} - \lambda\{\theta[f(x-\overline{c}) - f(-\overline{c})] + f(-\underline{c})\} - f(-\underline{c})\right].\end{aligned} \tag{63}$$

(1) The first-order condition with respect to $\underline{c}$ is

$$(1-\theta)[-(1+\lambda) + \lambda f'(-\underline{c}) + f'(-\underline{c})] = 0, \tag{64}$$

just as under adverse selection, which simplifies to

$$f'(-\underline{c}) = 1. \tag{65}$$

Thus, as earlier, $f'_N = 1$. The normal firm chooses the efficient effort level e^* in equilibrium, and $\underline{c}$ takes the same value as it did in Procurement I. Equation (59) can be rewritten as

$$p_N = p^* + f(x-\overline{c}) - f(-\overline{c}). \tag{66}$$

[11]The same argument does not hold for the firm with special costs, because if p_S were reduced, the participation constraint would be violated.xxx check this

Because $f(x-\overline{c}) > f(-\overline{c})$, equation (66) shows that $p_N > p^*$. The normal firm earns more than its reservation utility, even under complete information. The special firm must therefore earn less than its reservation utility, so that the overall participation constraint will be satisfied as an equality.

(2) The first-order condition with respect to $\overline{c}$ is

$$\theta\left\{-(1+\lambda)-\lambda(1-\theta)[-f'(x-\overline{c})+f'(-\overline{c})]+f'(x-\overline{c})\right\}+\lambda\left[1-\theta\right]\left[f'(x-\overline{c})-f'(-\overline{c})\right]=0. \tag{67}$$

This can be rewritten as

$$xcxcvxcvcxf'(x-\overline{c}) = 1 - sdfsfsdfsdfdsf \tag{68}$$

Since the right-hand-side of equation (68) is less than one, the special firm has a lower level of f' than the normal firm, and must be exerting effort less than e^*, since $f'' > 0$. Also since the participation constraint, (55), is satisfied as an equality, it must also be true that $p_S < s^*$. The special firm's subsidy is lower than under full information, although since its effort is also lower, its payoff stays the same. sdfadfasfdsdf

We must also see that the incentive compatibility constraint for the special firm is satisfied as a weak inequality; it is not near being tempted to pick the normal firm's contract. Setting the left-hand-side of the incentive compatibility constraint (56) equal to zero because the participation constraint is binding the special firm, substituting in for p_N from equation (59) and rearranging yields zcddgafd

$$asdfsafddsfdsf(-\underline{c}) - f(-x-\underline{c}) \geq f(-\overline{c}) - f(x-\overline{c}). \tag{69}$$

This is true, and true as a strict inequality, because $f'' > 0$ and the arguments of f on the left-hand-side of equation (69) take larger values than on the right-hand side.sdfasdfd

xx Compare with PG II.

xxxxx HERE, RESUME CHAPTER.

A little reflection will provide a host of additional ways to alter the Procurement Game. What if the firm discovers its costs only after accepting the contract? What if two firms bid against each other for the contract? What if the firm can bribe the government? What if the firm and the government bargain over the gains from the project instead of the government being able to make a take-it-or-leave-it contract offer? What if the game is repeated, so the government can use the information it acquires in the second period? If it is repeated, can the government commit to long-term contracts? Can it commit not to renegotiate? See Spulber (1989) and Laffont & Tirole (1993) if these questions interest you.

***10.6 The Groves Mechanism**

Hidden knowledge is particularly important in public economics, the study of government spending and taxation. We have just seen in Section 10.5 how important the information of private citizens is to the government in trying to decide what price to pay businesses for public services. Much the same issues arise when the government is trying to decide how much to tax or how much to spend. In the optimal taxation literature in which Mirrlees (1971) is the classic article, citizens differ in their income-producing ability, and the government wishes to demand higher taxes from the more able citizens, a clear problem of hidden knowledge. An even purer hidden knowledge problem is choosing the level of public goods based on private preferences. The government must decide whether it is worthwhile to buy a public good based on the combined preferences of all the citizens, but it needs to discover those preferences. Unlike in the previous games in this chapter, a group of agents will now be involved, not just one agent. Moreover, unlike in most games but similarly to the regulating principal of Section 10.4, the government is an altruistic principal who cares directly about the utility of the agents, rather than a car buyer or an insurance seller who cares about the agents' utility only in order to satisfy self-selection and participation constraints.

The example below is adapted from p. 426 of Varian (1992). The mayor of a town is considering installing a streetlight costing \$100. Each of the five houses near the light would be taxed exactly \$20, but the mayor will only install it if he decides that the sum of the residents' valuations for it is greater than the cost.

The problem is to discover the valuations. If the mayor simply asks them, householder Smith might say that his valuation is \$5,000, and householder Brown might say that he likes the darkness and would pay \$5,000 to *not* have a streetlight, but all the mayor could conclude would be that Smith's valuation exceeded \$20 and Brown's did not. Talk is cheap, and the dominant strategy is to overreport or underreport.

The flawed mechanism just described can be denoted by

$$M_1: \quad \left(20, \sum_{i=1}^{5} m_i \geq 100\right), \tag{70}$$

which means that each resident pays \$20, and the light is installed if the sum of the valuations exceeds 100.

An alternative is to make resident i pay the amount of his message, or pay zero if it is negative. This mechanism is

$$M_2: \quad \left(Max\{m_i, 0\}, \sum_{j=1}^{5} m_j \geq 100\right). \tag{71}$$

Mechanism M_2 has no dominant strategy. Player i would announce $m_i = 0$ if he thought the project would go through without his support, but he would announce up to his valuation if necessary. There is a continuum of Nash equilibria that attain the efficient result. Most of these are asymmetric, and there is a problem of how the equilibrium to be played out becomes common knowledge. This is a simple mechanism, however, and it already teaches a lesson: that people are more likely to report their true political preferences if they must bear part of the costs themselves.

Instead of just ensuring that the correct decision is made in a Nash equilibrium, it may be possible to design a mechanism which makes truthfulness a **dominant-strategy mechanism**. Consider the mechanism

$$M_3: \quad \left(100 - \sum_{j \neq i} m_j, \sum_{j=1}^{5} m_j \geq 100\right). \tag{72}$$

Under mechanism M_3, player i's message does not affect his tax bill except by its effect on whether or not the streetlight is installed. If player i's valuation is v_i, his full payoff is $v_i - 100 + \sum_{j \neq i} m_j$ if $m_i + \sum_{j \neq i} m_j \geq 100$, and zero otherwise. It is not hard to see that he will be truthful in a Nash equilibrium in which the other players are truthful, but we can go further: truthfulness is weakly dominant. Moreover, the players will tell the truth whenever lying would alter the mayor's decision.

Consider a numerical example. Suppose that Smith's valuation is 40 and the sum of the valuations is 110, so the project is indeed efficient. If the other players report their truthful sum of 70, Smith's payoff from truthful reporting is his valuation of 40 minus his tax of 30. Reporting more would not change his payoff, while reporting less than 30 would reduce it to 0.

If we are wondering whether Smith's strategy is dominant, we must also consider his best response when the other players lie. If they underreported, announcing 50 instead of the truthful 70, then Smith could make up the difference by overreporting 60, but his payoff would be -10 $(= 40 + 50 - 100)$ so he would do better to report the truthful 40, killing the project and leaving him with a payoff of 0. If the other players overreported, announcing 80 instead of the truthful 70, then Smith benefits if the project goes through, and he should report at least 20 to obtain his payoff of 40 minus 20. He is willing to report exactly 40, so there is an equilibrium with truth-telling.

The problem with a dominant-strategy mechanism like the one facing Smith is that it is not budget balancing. The government raises less in taxes than it spends on the project (in fact, the taxes would be negative). Lack of budget balancing is a crucial feature of dominant- strategy mechanisms. While the government deficit can be made either positive or negative, it cannot be made zero, unlike in the case of Nash mechanisms.

Notes

N10.1 The revelation principle and moral hazard with hidden knowledge

- The books by Fudenberg & Tirole (1991a), Laffont & Tirole (1993), Palfrey & Srivastiva (1993), Spulber (1989), and Baron's chapter in the *Handbook of Industrial Organization* edited by Schmalensee and Willig (1989) are good places to look for more on mechanism design.

- Levmore (1982) discusses hidden knowledge problems in tort damages, corporate freezeouts, and property taxes in a law review article.

- The revelation principle was named by Myerson (1979) and can be traced back to Gibbard (1973). A further reference is Dasgupta, Hammond & Maskin (1979). Myerson's game theory book is, as one might expect, a good place to look for further details (Myerson [1991, pp. 258-63, 294-99]).

- Moral hazard with hidden knowledge is common in public policy. Should the doctors who prescribe drugs also be allowed to sell them? The question trades off the likelihood of over-prescription against the potentially lower cost and greater convenience of doctor-dispensed drugs. See "Doctors as Druggists: Good Rx for Consumers?" *Wall Street Journal*, June 25, 1987, p. 24.

- For a careful discussion of the unravelling argument for information revelation, see Milgrom (1981b).

- A hidden knowledge game requires that the state of the world matter to one of the players' payoffs, but not necessarily in the same way as in Production Game VII. The Salesman Game of Section 10.2 effectively uses the utility function $U(e, w, \theta)$ for the agent and $V(q-w)$ for the principal. The state of the world matters because the agent's disutility of effort varies across states. In other problems, his utility of money might vary across states.

N10.2 An example of moral hazard with hidden knowledge: *The Salesman Game*

- Sometimes students know more about their class rankings than the professor does. One professor of labor economics used a mechanism of the following kind for grading class discussion. Each student i reports a number evaluating other students in the class. Student i's grade is an increasing function of the evaluations given i by other students and of the correlation between i's evaluations and the other students'. There are many Nash equilibria, but telling the truth is a focal point.

- In dynamic games of moral hazard with hidden knowledge the **ratchet effect** is important: the agent takes into account that his information-revealing choice of contract this period will affect the principal's offerings next period. A principal might allow high prices to a public utility in the first period to discover that its costs are lower than expected, but in the next period the prices would be reduced. The contract is ratcheted irreversibly to be more severe. Hence, the company might not choose a contract which reveals its costs in the first period. This is modelled in Freixas, Guesnerie & Tirole (1985).

  Baron (1989) notes that the principal might purposely design the equilibrium to be pooling in the first period so self selection does not occur. Having learned nothing, he can offer a more effective separating contract in the second period.

N10.4 Price Discrimination

- The names for price discrimination in Part 2, Chapter 17, Section 5 of Pigou (1920) are: (1) first-degree (perfect price discrimination), (2) second-degree (interquantity price discrimination), and (3) third- degree (interbuyer price discrimination). These arbitrary names have plagued generations of students of industrial organization, in parallel with the appalling Type I and Type II errors of statistics (better named as False Negatives or Rejections, and False Positives or Acceptances). I invented the terms **interbuyer price discrimination** and **interquantity price discrimination** for this edition, with the excuse that I think their meaning will be clear to anyone who already knows the concepts under their Pigouvian names.

- A narrower category of nonlinear pricing is the **quantity discount**, in which the price per unit declines with the quantity bought. Sellers are often constrained to this, since if the price per unit rises with the quantity bought, some means must be used to prevent a canny consumer from buying two batches of small quantities instead of one batch of a large quantity.

- Phlips's 1983 book, *The Economics of Price Discrimination*, is a good reference on the subject.

- In Varian's Nonlinear Pricing Game the probabilities of types for each player are not independent, unlike in most games. This does not make the game more complicated, though. If the assumption were "Nature assigns each buyer a utility function u or v with independent probabilities of 0.5 for each type," then there would be not just two possible states of the world in this game– uv and vu for Smith and Jones's types – but four – uv, vu, uu, and vv. How would the equilibrium change?

- The careful reader will think, "How can we say that Buyer V always gets higher utility than Buyer U for given x? Utility cannot be compared across individuals, and we could rescale Buyer V's utility function to make him always have lower utility without altering the essentials of the utility function." My reply is that more generally we could set up the utility functions as $v(x) + y$ and $u(x) + y$, with y denoting spending on all other goods (as Varian does in his book). Then to say that V always gets higher utility for a given x means that he always has a higher relative value than U does for good x relative to money. Rescaling to give V the utility function $.001v(x) + .001y$ would not alter that.

- The notation I used in Varian's Nonlinear Pricing Game is optimized for reading. If you wish to write this on the board or do the derivations for practice, use abbreviations like u_1 for $u_1(x_1)$, a for $v(x_1)$, and b for $v(x_2)$ to save writing. The tradeoff between brevity and transparency in notation is common, and must be made in light of whether you are writing on a blackboard or on a computer, for just yourself or for the generations.

N10.5 Rate-of-return regulation and government procurement

- I changed the notation from Laffont and Tirole and from my own previous edition. Rather than assign each type of firm a cost parameter β for a cost of $c = \beta - e$, I now assign each type of firm an ability parameter a, for a cost of $c = c_0 - a - e$. This will allow the desirable type of firm to be the one with the *High* value of the type parameter, as in most models.

N10.6 The Groves Mechanism

- Vickrey (1961) first suggested the nonbudget-balancing mechanism for revelation of preferences, but it was rediscovered later and became known as the Groves Mechanism (from Groves [1973]).

Problems

10.1. Unravelling
An elderly prospector owns a gold mine worth an amount θ drawn from the uniform distribution $U[0, 100]$ which nobody knows, including himself. He will certainly sell the mine, since he is too old to work it and it has no value to him if he does not sell it. The several prospective buyers are all risk neutral. The prospector can, if he desires, dig deeper into the hill and collect a sample of gold ore that will reveal the value of θ. If he shows the ore to the buyers, however, he must show genuine ore, since an unwritten Law of the West says that fraud is punished by hanging offenders from joshua trees as food for buzzards.

(a) For how much can he sell the mine if he is clearly too feeble to have dug into the hill and examined the ore? What is the price in this situation if, in fact, the true value is $\theta = 70$?

(b) For how much can he sell the mine if he can dig the test tunnel at zero cost? Will he show the ore? What is the price in this situation if, in fact, the true value is $\theta = 70$?

(c) For how much can he sell the mine if, after digging the tunnel at zero cost and discovering θ, it costs him an additional 10 to verify the results for the buyers? What is his expected payoff?

(d) Suppose that with probability 0.5 digging the test tunnel costs 5 for the prospector, but with probability 0.5 it costs him 120. Keep in mind that the 0-100 value of the mine is net of the buyer's digging cost. Denote the equilibrium price that buyers will pay for the mine after the prospector approaches them without showing ore by P. What is the buyer's posterior belief about the probability it costs 120 to dig the tunnel, as a function of P? Denote this belief by $B(P)$ (Assume, as usual, that all these parameters are common knowledge, although only the prospector learns whether the cost is actually 0 or 120.)

(e) What is the prospector's expected payoff in the conditions of part (d) if (i) the tunnel costs him 120, or (ii) the tunnel costs him 5?

10.2. Task Assignment
Table 1 shows the payoffs in the following game. Sally has been hired by Rayco to do either Job 1, to do Job 2, or to be a Manager. Rayco believes that Tasks 1 and 2 have equal probabilities of being the efficient ones for Sally to perform. Sally knows which task is efficient, but what she would like best is a job as Manager that gives her the freedom to choose rather than have the job designed for the task. The CEO of Rayco asks Sally which task is efficient. She can either reply "task 1," "task 2," or be silent. Her statement, if she makes one, is an example of "cheap talk," because it has no direct effect on anybody's payoff. See Farrell & Rabin (1996).

Table 1: The Right To Silence Game payoffs

		Sally's Job		
		Job 1	*Job 2*	*Manager*
Sally knows	Task 1 is efficient (0.5)	2, 5	1, −2	3, 3
	Task 2 is efficient (0.5)	1, −2	2, 5	3, 3

Payoffs to: (Sally, Rayco)

(a) If Sally did not have the option of speaking, what would happen?

(b) There exist perfect Bayesian equilibria in which it does not matter how Sally replies. Find one of these in which Sally speaks at least some of the time, and explain why it is an equilibrium. You may assume that Sally is not morally or otherwise bound to speak the truth.

(c) There exists a perverse variety of equilibrium in which Sally always tells the truth and never is silent. Find an example of this equilibrium, and explain why neither player would have incentive to deviate to out-of-equilibrium behavior.

10.3. Agency Law
Mr. Smith is thinking of buying a custom-designed machine from either Mr. Jones or Mr. Brown. This machine costs \$5,000 to build, and it is useless to anyone but Smith. It is common knowledge that with 90 percent probability the machine will be worth \$10,000 to Smith at the time of delivery, one year from today, and with 10 percent probability it will only be worth \$2,000. Smith owns assets of \$1,000. At the time of contracting, Jones and Brown believe there is a 20 percent chance that Smith is actually acting as an "undisclosed agent" for Anderson, who has assets of \$50,000.

Find the price be under the following two legal regimes: (a) An undisclosed principal is not responsible for the debts of his agent; and (b) even an undisclosed principal is responsible for the debts of his agent. Also, explain (as part [c]) which rule a moral hazard model like this would tend to support.

10.4. Incentive Compatibility and Price Discrimination
Two consumers have utility functions $u_1(x_1, y_1) = a_1 log(x_1) + y_1$ and $u_2(x_2, y_2) = a_2 log(x_2) + y_2$, where $1 > a_2 > a_1$. The price of the y-good is 1 and each consumer has an initial wealth of 15. A monopolist supplies the x-good. He has a constant marginal cost of 1.2 up to his capacity constraint of 10. He will offer at most two price-quantity packages, (r_1, x_1) and (r_2, x_2), where r_i is the total cost of purchasing x_i units. He cannot identify which consumer is which, but he can prevent resale.

(a) Write down the monopolist's profit maximization problem. You should have four constraints plus the capacity constraint.

(b) Which constraints will be binding at the optimal solution?

(c) Substitute the binding constraints into the objective function. What is the resulting expression? What are the first-order conditions for profit maximization? What are the profit-maximizing values of x_1 and x_2?

10.5. The Groves Mechanism

A new computer costing 10 million dollars would benefit existing Divisions 1, 2, and 3 of a company with 100 divisions. Each divisional manager knows the benefit to his division (variables $v_i, i = 1, ..., 3$), but nobody else does, including the company CEO. Managers maximize the welfare of their own divisions. What dominant strategy mechanism might the CEO use to induce the managers to tell the truth when they report their valuations? Explain why this mechanism will induce truthful reporting, and denote the reports by $x_i, i = 1, ..., 3$. (You may assume that any budget transfers to and from the divisions in this mechanism are permanent– that the divisions will not get anything back later if the CEO collects more payments than he gives, for example.)

10.6. The Two-Part Tariff (Varian 14.10, modified)

One way to price discriminate is to charge a lump sum fee L to have the right to purchase a good, and then charge a per-unit charge p for consumption of the good after that. The standard example is an amusement park where the firm charges an entry fee and a charge for the rides inside the park. Such a pricing policy is known as a **two-part tariff**. Suppose that all consumers have identical utility functions given by $u(x)$ and that the cost of production is cx. If the monopolist sets a two-part tariff, will it produce the socially efficient level of output, too little, or too much?

10.7. Selling Cars

A car dealer must pay \$10,000 to the manufacturer for each car he adds to his inventory. He faces three buyers. From the point of view of the dealer, Smith's valuation is uniformly distributed between \$11, 000 and \$21,000, Jones's is between \$9,000 and \$11,000, and Brown's is between \$4,000 and \$12,000. The dealer's policy is to make a separate take-it-or-leave-it offer to each customer, and he is smart enough to avoid making different offers to customers who could resell to each other. Use the notation that the maximum valuation is $\overline{V}$ and the range of valuations is R.

(a) What will the offers be?

(b) Who is most likely to buy a car? How does this compare with the outcome with perfect price discrimination under full information? How does it compare with the outcome when the dealer charges \$10,000 to each customer?

(c) What happens to the equilibrium prices if with probability 0.25 each buyer has a valuation of \$0, but the probability distribution remains otherwise the same? What happens to the equilibrium expected profit?

(d) What happens to the equilibrium price the seller offers to seller Jones if with probability 0.25 Jones has a valuation of \$30,000, but with probability 0.75 his valuation is uniformly distributed between \$9,000 and \$11,000 as before? Show the relation between price and profit on a rough graph.

Regulatory Ratcheting: A Classroom Game for Chapter 10

Electricity demand is perfectly inelastic, at 1 gigawatt per firm. The price is chosen by the regulator. The regulator cares about two things: (1) getting electrical service, and (2) getting it at the lowest price possible. The utilities like profit and dislike effort. Throughout the game, utility i has "cost reduction" parameter x_i, which it knows but the regulator does not. This parameter is big if the utility can reduce its costs with just a little effort.

Each year, the following events happen.

1. The regulator offers price P_i to firm i.

2. Firm i accepts or rejects.

3. If Firm i accepts, it secretly chooses its effort level e_i,

4. Nature secretly and randomly chooses the economywide shock u (uniform from 1 to 6) and Firm i's shock u_i (uniform from 1 to 6) and announces Firm i's cost, c_i. That cost equals

$$c_i = 20 + u + u_i - x_i e_i. \tag{73}$$

5. Firm i earns a period payoff of 0 if it rejects the contract. If it accepts, its payoff is

$$\pi_i = p_i(1) - c_i - e_i^2 \tag{74}$$

The regulator earns a period payoff of 0 from firm i if its contract is rejected. Otherwise, its payoff from that firm is

$$\pi_{regulator}(i) = 50 - p_i \tag{75}$$

All variables take integer values.

The game repeats for as many years as the class has time for, with each firm keeping the same value of x throughout.

For instructors' notes, go to http://www.rasmusen.org/GI/probs/10_regulation_game.pdf.

11 Signalling

October 3, 1999. January 18, 2000. November 30, 2003. 25 March 2005. Eric Rasmusen, Erasmuse@indiana.edu. Http://www.rasmusen.org. Footnotes starting with xxx are the author's notes to himself. Comments welcomed.

11.1: The Informed Player Moves First: Signalling

Signalling is a way for an agent to communicate his type under adverse selection. The signalling contract specifies a wage that depends on an observable characteristic — the signal — which the agent chooses for himself after Nature chooses his type. Figures 1d and 1e showed the extensive forms of two kinds of models with signals. If the agent chooses his signal before the contract is offered, he is signalling to the principal. If he chooses the signal afterwards, the principal is screening him. Not only will it become apparent that this difference in the order of moves is important, it will also be seen that signalling costs must differ between agent types for signalling to be useful, and the outcome is often inefficient.

We begin with signalling models in which workers choose education levels to signal their abilities. Section 11.1 lays out the fundamental properties of a signalling model, and Section 11.2 shows how the details of the model affect the equilibrium. Section 11.3 steps back from the technical detail to more practical considerations in applying the model to education. Section 11.4 turns the game into a screening model. Section 11.5 switches to diagrams and applies signalling to new stock issues to show how two signals need to be used when the agent has two unobservable characteristics. Section 11.6 addresses the rather different idea of signal jamming: strategic behavior a player uses to cover up information rather than to disclose it.

Spence (1973) introduced the idea of signalling in the context of education. We will construct a series of models which formalize the notion that education has no direct effect on a person's ability to be productive in the real world but useful for demonstrating his ability to employers. Let half of the workers have the type "high ability" and half "low ability," where ability is a number denoting the dollar value of his output. Output is assumed to be a noncontractible variable and there is no uncertainty. If output is contractible, it should be in the contract, as we have seen in Chapter 7. Lack of uncertainty is a simplifying assumption, imposed so that the contracts are functions only of the signals rather than a combination of the signal and the output.

Employers do not observe the worker's ability, but they do know the distribution of abilities, and they observe the worker's education. To simplify, we will specify that the players are one worker and two employers. The employers compete profits down to zero and the worker receives the gains from trade. The worker's strategy is his education level and his choice of employer. The employers' strategies are the contracts they offer giving wages as functions of education level. The key to the model is that the signal, education, is less costly for workers with higher ability.

In the first four variants of the game, workers choose their education levels before employers decide how pay should vary with education.

Education I

Players
A worker and two employers.

The Order of Play
0 Nature chooses the worker's ability $a \in \{2, 5.5\}$, the *Low* and *High* ability each having probability 0.5. The variable a is observed by the worker, but not by the employers.
1 The worker chooses education level $s \in \{0, 1\}$.
2 The employers each offer a wage contract $w(s)$.
3 The worker accepts a contract, or rejects both of them.
4 Output equals a.

Payoffs
The worker's payoff is his wage minus his cost of education, and the employer's is his profit.

$$\pi_{worker} = \begin{cases} w - 8s/a & \text{if the worker accepts contract } w. \\ 0 & \text{if the worker rejects both contracts.} \end{cases}$$

$$\pi_{employer} = \begin{cases} a - w & \text{for the employer whose contract is accepted.} \\ 0 & \text{for the other employer.} \end{cases}$$

The payoffs assume that education is more costly for a worker if his ability takes a lower value, which is what permits separation to occur.[1] As in any hidden knowledge game, we must think about both pooling and separating equilibria. Education I has both. In the pooling equilibrium, which we will call Pooling Equilibrium 1.1, both types of workers pick zero education and the employers pay the zero-profit wage of 3.75 regardless of the education level (3.75= [2+5.5]/2).

$$\textbf{Pooling Equilibrium 1.1} \quad \begin{cases} s(Low) = s(High) = 0 \\ w(0) = w(1) = 3.75 \\ Prob(a = Low|s = 1) = 0.5 \end{cases}$$

Pooling Equilibrium 1.1 needs to be specified as a perfect Bayesian equilibrium rather than simply a Nash equilibrium because of the importance of the interpretation that the uninformed player puts on out-of-equilibrium behavior. The equilibrium needs to specify the employer's beliefs when he observes $s = 1$, since that is never observed in equilibrium. In Pooling Equilibrium 1.1, the beliefs are passive conjectures (see Section 6.2): employers believe that a worker who chooses $s = 1$ is *Low* with the prior probability, which is 0.5. Given this belief, both types of workers realize that education is useless, and the model reaches the unsurprising outcome that workers do not bother to acquire unproductive education.

Under other beliefs, the pooling equilibrium breaks down. Under the belief $Prob(a = Low|s = 1) = 0$, for example, employers believe that any worker who acquired education

[1] xxx For each variant, formally ask whether the single-crossing property is satisfied.

is a *High*, so pooling is not Nash because the *High* workers are tempted to deviate and acquire education. This leads to the separating equilibrium for which signalling is best known, in which the high-ability worker acquires education to prove to employers that he really has high ability.

$$\textbf{Separating Equilibrium 1.2} \quad \begin{cases} s(Low) = 0, s(High) = 1 \\ w(0) = 2, w(1) = 5.5 \end{cases}$$

Following the method used in Chapters 7 to 10, we will show that Separating Equilibrium 1.2 is a perfect Bayesian equilibrium by using the standard constraints which an equilibrium must satisfy. A pair of separating contracts must maximize the utility of the *High*s and the *Low*s subject to two constraints: (a) the particpation constraints that the firms can offer the contracts without making losses; and (b) the self-selection constraints that the *Low*s are not attracted to the *High* contract, and the *High*s are not attracted by the *Low* contract. The participation constraints for the employers require that

$$w(0) \leq a_L = 2 \;\text{ and }\; w(1) \leq a_H = 5.5. \tag{1}$$

Competition between the employers makes the expressions in (1) hold as equalities. The self-selection constraint of the *Lows* is

$$U_L(s=0) \geq U_L(s=1), \tag{2}$$

which in Education I is

$$w(0) - 0 \geq w(1) - \frac{8(1)}{2}. \tag{3}$$

Since in Separating Equilibrium 1.2 the separating wage of the *Low*s is 2 and the separating wage of the *High*s is 5.5 from (1), the self- selection constraint (3) is satisfied.

The self-selection constraint of the *High*s is

$$U_H(s=1) \geq U_H(s=0), \tag{4}$$

which in Education I is

$$w(1) - \frac{8(1)}{5.5} \geq w(0) - 0. \tag{5}$$

Constraint (5) is satisfied by Separating Equilibrium 1.2.

There is another conceivable pooling equilibrium for Education I, in which $s(Low) = s(High) = 1$, but this turns out not to be an equilibrium, because the *Low*s would deviate to zero education. Even if such a deviation caused the employer to believe they were low - ability with probability 1 and reduce their wage to 2, the low - ability workers would still prefer to deviate, because

$$U_L(s=0) = 2 \geq U_L(s=1) = 3.75 - \frac{8(1)}{2}. \tag{6}$$

Thus, a pooling equilibrium with $s = 1$ would violate incentive compatibility for the *Low* workers.

Notice that we do not need to worry about a nonpooling constraint for this game, unlike in the case of the games of Chapter 9. One might think that because employers compete for workers, competition between them might result in their offering a pooling contract that the high-ability workers would prefer to the separating contract. The reason this does not matter is that the employers do not compete by offering contracts, but by reacting to workers who have acquired education. That is why this is signalling and not screening: the employers cannot offer contracts in advance that change the workers' incentives to acquire education.

We can test the equilibrium by looking at the best responses. Given the worker's strategy and the other employer's strategy, an employer must pay the worker his full output or lose him to the other employer. Given the employers' contracts, the *Low* has a choice between the payoff 2 for ignorance ($=2-0$) and 1.5 for education ($= 5.5-8/2$), so he picks ignorance. The *High* has a choice between the payoff 2 for ignorance ($= 2 - 0$) and 4.05 for education ($= 5.5 - 8/5.5$, rounded), so he picks education.

Unlike the pooling equilibrium, the separating equilibrium does not need to specify beliefs. Either of the two education levels might be observed in equilibrium, so Bayes's Rule always tells the employers how to interpret what they see. If they see that an agent has acquired education, they deduce that his ability is *High* and if they see that he has not, they deduce that it is *Low*. A worker is free to deviate from the education level appropriate to his type, but the employers' beliefs will continue to be based on equilibrium behavior. If a *High* worker deviates by choosing $s = 0$ and tells the employers he is a *High* who would rather pool than separate, the employers disbelieve him and offer him the *Low* wage of 2 that is appropriate to $s = 0$, not the pooling wage of 3.75 or the *High* wage of 5.5.

Separation is possible because education is more costly for workers if their ability is lower. If education were to cost the same for both types of worker, education would not work as a signal, because the low-ability workers would imitate the high-ability workers. This requirement of different signalling costs is known as the **single-crossing property**, since when the costs are depicted graphically, as they will be in Figure 1, the indifference curves of the two types intersect a single time (see also Section 10.3).

A strong case can be made that the beliefs required for the pooling equilibria are not sensible. Harking back to the equilibrium refinements of Section 6.2, recall that one suggestion (from Cho & Kreps [1987]) is to inquire into whether one type of player could not possibly benefit from deviating, no matter how the uninformed player changed his beliefs as a result. Here, the *Low* worker could never benefit from deviating from Pooling Equilibrium 1.1. Under the passive conjectures specified, the *Low* has a payoff of 3.75 in equilibrium versus -0.25 ($= 3.75 - 8/2$) if he deviates and becomes educated. Under the belief that most encourages deviation – that a worker who deviates is *High* with probability one – the *Low* would get a wage of 5.5 if he deviated, but his payoff from deviating would only be 1.5 ($= 5.5 - 8/2$), which is less than 2. The more reasonable belief seems to be that a worker who acquires education is a *High*, which does not support the pooling equilibrium.

The nature of the separating equilibrium lends support to the claim that education *per se* is useless or even pernicious, because it imposes social costs but does not increase total output. While we may be reassured by the fact that Professor Spence himself thought it worthwhile to become Dean of Harvard College, the implications are disturbing and suggest that we should think seriously about how well the model applies to the real world. We will do that in Section 11.3. For now, note that in the model, unlike most real-world situations, information about the agent's talent has no social value, because all agents would be hired and employed at the same task even under full information. Also, if side payments are not possible, Separating Equilibrium 1.2 is second-best efficient in the sense that a social planner could not make both types of workers better off. Separation helps the high-ability workers even though it hurts the low-ability workers.

11.2: Variants on the Signalling Model of Education

Although *Education I* is a curious and important model, it does not exhaust the implications of signalling. This section will start with*Education II*, which will show an alternative to the arbitrary assumption of beliefs in the perfect Bayesian equilibrium concept. *Education III* will be the same as *Education I* except for its different parameter values, and will have two pooling equilibrium rather than one separating and one pooling equilibrium.*Education IV* will allow a continuum of education levels, and will unify *Education I* and *Education III* by showing how all of their equilibria and more can be obtained in a model with a less restricted strategy space.

Education II: Modelling Trembles so Nothing is Out of Equilibrium

The pooling equilibrium of *Education I* required the modeller to specify the employers' out-of-equilibrium beliefs. An equivalent model constructs the game tree to support the beliefs instead of introducing them via the equilibrium concept. This approach was briefly mentioned in connection with the game of *PhD Admissions* in Section 6.2. The advantage is that the assumptions on beliefs are put in the rules of the game along with the other assumptions. So let us replace Nature's move in *Education I* and modify the payoffs as follows.

Education II

The Order of Play
0 Nature chooses worker ability $a \in \{2, 5.5\}$, each ability having probability 0.5. (a is observed by the worker, but not by the employer.) With probability 0.001, Nature endows a worker with free education.
...

Payoffs

$$\pi_{worker} = \begin{cases} w - 8s/a & \text{if the worker accepts contract } w \text{ (ordinarily)} \\ w & \text{if the worker accepts contract } w \text{ (with free education)} \\ 0 & \text{if the worker does not accept a contract} \end{cases}$$

With probability 0.001 the worker receives free education regardless of his ability. If the employer sees a worker with education, he knows that the worker might be one of this rare type, in which case the probability that the worker is *Low* is 0.5. Both $s = 0$

and $s = 1$ can be observed in any equilibrium and *Education II* has almost the same two equilibria as *Education I*, without the need to specify beliefs.[2] The separating equilibrium did not depend on beliefs, and remains an equilibrium. What was Pooling Equilibrium 1.1 becomes "almost" a pooling equilibrium — almost all workers behave the same, but the small number with free education behave differently. The two types of greatest interest, the $High$ and the Low, are not separated, but the ordinary workers are separated from the workers whose education is free. Even that small amount of separation allows the employers to use Bayes's Rule and eliminates the need for exogenous beliefs.

Education III: No Separating Equilibrium, Two Pooling Equilibria

Let us next modify *Education I* by changing the possible worker abilities from $\{2, 5.5\}$ to $\{2, 12\}$. The separating equilibrium vanishes, but a new pooling equilibrium emerges. In Pooling Equilibria 3.1 and 3.2, both pooling contracts pay the same zero-profit wage of 7 $(= [2 + 12]/2)$, and both types of agents acquire the same amount of education, but the amount depends on the equilibrium.

$$\textbf{Pooling Equilibrium 3.1} \quad \begin{cases} s(Low) = s(High) = 0 \\ w(0) = w(1) = 7 \\ Prob(a = Low|s = 1) = 0.5 \text{ (passive conjectures)} \end{cases}$$

$$\textbf{Pooling Equilibrium 3.2} \quad \begin{cases} s(Low) = s(High) = 1 \\ w(0) = 2, w(1) = 7 \\ Prob(a = Low|s = 0) = 1 \end{cases}$$

Pooling Equilibrium 3.1 is similar to the pooling equilibrium in *Education I* and *II*, but Pooling Equilibrium 3.2 is inefficient. Both types of workers receive the same wage, but they incur the education costs anyway. Each type is frightened to do without education because the employer would pay him not as if his ability were average, but as if he were known to be *Low*.

Examination of Pooling Equilibrium 3.2 shows why a separating equilibrium no longer exists. Any separating equilibrium would require $w(0) = 2$ and $w(1) = 7$, but this is the contract that leads to Pooling Equilibrium 3.2. The self-selection and zero-profit constraints cannot be satisfied simultaneously, because the *Low* type is willing to acquire $s = 1$ to obtain the high wage.

It is not surprising that information problems create inefficiencies in the sense that first-best efficiency is lost. Indeed, the surprise is that in some games with asymmetric information, such as Broadway Game I in Section 7.4, the first-best can still be achieved by tricks such as boiling-in-oil contracts. More often, we discover that the outcome is second-best efficient: given the informational constraints, a social planner could not alter the equilibrium without hurting some type of player. Pooling Equilibrium 3.2 is not even second-best efficient, because Pooling Equilibrium 3.1 and Pooling Equilibrium 3.2 result in the exact same wages and allocation of workers to tasks. The inefficiency is purely

[2] xxx Use Bayes Rule to show exactly what hte beliefs are here.

a problem of unfortunate expectations, like the inefficiency from choosing the dominated equilibrium in Ranked Coordination.

Pooling Equilibrium 3.2 also illustrates a fine point of the definition of pooling, because although the two types of workers adopt the same strategies, the equilibrium contract offers different wages for different education. The implied threat to pay a low wage to an uneducated worker never needs to be carried out, so the equilibrium is still called a pooling equilibrium. Notice that perfectness does not rule out threats based on beliefs. The model imposes these beliefs on the employer, and he would carry out his threats, because he believes they are best responses. The employer receives a higher payoff under some beliefs than under others, but he is not free to choose his beliefs.

Following the approach of *Education II*, we could eliminate Pooling Equilibrium 3.2 by adding an exogenous probability 0.001 that either type is completely unable to buy education. Then no behavior is never observed in equilibrium and we end up with Pooling Equilibrium 3.1 because the only rational belief is that if $s = 0$ is observed, the worker has equal probability of being $High$ or being Low. To eliminate Pooling Equilibrium 3.1 requires less reasonable beliefs; for example, a probability of 0.001 that a Low gets free education together with a probability of 0 that a $High$ does.

These first three games illustrate the basics of signalling: (a) separating and pooling equilibria both may exist, (b) out-of-equilibrium beliefs matter, and (c) sometimes one perfect Bayesian equilibrium can Pareto dominate others. These results are robust, but Education IV will illustrate some dangers of using simplified games with binary strategy spaces instead of continuous and unbounded strategies. So far education has been limited to $s = 0$ or $s = 1$; Education IV allows it to take greater or intermediate values.

Education IV: Continuous Signals and Continua of Equilibria

Let us now return to *Education I*, with one change: that education s can take any level on the continuum between 0 and infinity.[3]

The game now has continua of pooling and separating equilibria which differ according to the value of education chosen. In the pooling equilibria, the equilibrium education level is s^*, where each s^* in the interval $[0, \overline{s}]$ supports a different equilibrium. The out-of-equilibrium belief most likely to support a pooling equilibrium is $Prob(a = Low|s \neq s^*) = 1$, so let us use this to find the value of $\overline{s}$, the greatest amount of education that can be generated by a pooling equilibrium. The equilibrium is Pooling Equilibrium 4.1, where $s^* \in [0, \overline{s}]$.

Pooling Equilibrium 4.1 $\left\{ \begin{array}{l} s(Low) = s(High) = s^* \\ w(s^*) = 3.75 \\ w(s \neq s^*) = 2 \\ Prob(a = Low|s \neq s^*) = 1 \end{array} \right.$

The critical value $\overline{s}$ can be discovered from the incentive compatibility constraint of

[3]xxx Givre the entire game description again.

the *Low* type, which is binding if $s^* = \overline{s}$. The most tempting deviation is to zero education, so that is the deviation that appears in the constraint.

$$U_L(s=0) = 2 \leq U_L(s=\overline{s}) = 3.75 - \frac{8\overline{s}}{2}. \tag{7}$$

Equation (7) yields $\overline{s} = \frac{7}{16}$. Any value of s^* less than $\frac{7}{16}$ will also support a pooling equilibrium. Note that the incentive-compatibility constraint of the *High* type is not binding. If a *High* deviates to $s = 0$, he, too, will be thought to be a *Low*, so

$$U_H(s=0) = 2 \leq U_H(s=\frac{7}{16}) = 3.75 - \frac{8\overline{s}}{5.5} \approx 3.1. \tag{8}$$

In the separating equilibria, the education levels chosen in equilibrium are 0 for the *Low*'s and s^* for the *High*'s, where each s^* in the interval $[\overline{s}, \overline{\overline{s}}]$ supports a different equilibrium. A difference from the case of separating equilibria in games with binary strategy spaces is that now there are possible out-of-equilibrium actions even in a separating equilibrium. The two types of workers will separate to two education levels, but that leaves an infinite number of out-of-equilibrium education levels. As before, let us use the most extreme belief for the employers' beliefs after observing an out-of-equilibrium education level: that $Prob(a = Low|s \neq s^*) = 1$. The equilibrium is Separating Equilibrium 4.2, where $s^* \in [\overline{s}, s]$.

Separating Equilibrium 4.2 $\begin{cases} s(Low) = 0, \quad s(High) = s^* \\ w(s^*) = 5.5 \\ w(s \neq s^*) = 2 \\ Prob(a = Low|s \notin \{0, s^*\}) = 1 \end{cases}$

The critical value $\overline{s}$ can be discovered from the incentive-compatibility constraint of the *Low*, which is binding if $s^* = \overline{s}$.

$$U_L(s=0) = 2 \geq U_L(s=\overline{s}) = 5.5 - \frac{8\overline{s}}{2}. \tag{9}$$

Equation (9) yields $\overline{s} = \frac{7}{8}$. Any value of s^* greater than $\frac{7}{8}$ will also deter the *Low* workers from acquiring education. If the education needed for the wage of 5.5 is too great, the *High* workers will give up on education too. Their incentive compatibility constraint requires that

$$U_H(s=0) = 2 \leq U_H(s=\overline{\overline{s}}) = 5.5 - \frac{8\overline{\overline{s}}}{5.5}. \tag{10}$$

Equation (10) yields $\overline{\overline{s}} = \frac{77}{32}$. s^* can take any lower value than $\frac{77}{32}$ and the *High*'s will be willing to acquire education.

The big difference from *Education I* is that *Education IV* has Pareto-ranked equilibria. Pooling can occur not just at zero education but at positive levels, as in *Education III*, and the pooling equilibria with positive education levels are all Pareto inferior. Also, the separating equilibria can be Pareto ranked, since separation with $s^* = \overline{s}$ dominates separation with $s^* = \overline{\overline{s}}$. Using a binary strategy space instead of a continuum conceals this problem.

Education IV also shows how restricting the strategy space can alter the kinds of equilibria that are possible. *Education III* had no separating equilibrium because at the maximum possible signal, $s = 1$, the *Low*'s were still willing to imitate the *High*'s. *Education IV* would not have any separating equilibria either if the strategy space were restricted to allow only education levels less than $\frac{7}{8}$. Using a bounded strategy space eliminates possibly realistic equilibria.

This is not to say that models with binary strategy sets are always misleading. Education I is a fine model for showing how signalling can be used to separate agents of different types; it becomes misleading only when used to reach a conclusion such as "If a separating equilibrium exists, it is unique". As with any assumption, one must be careful not to narrow the model so much as to render vacuous the question it is designed to answer.

11.3 General Comments on Signalling in Education

Signalling and Similar Phenomena

The distinguishing feature of signalling is that the agent's action, although not directly related to output, is useful because it is related to ability. For the signal to work, it must be less costly for an agent with higher ability. Separation can occur in Education I because when the principal pays a greater wage to educated workers, only the *High*s, whose utility costs of education are lower, are willing to acquire it. That is why a signal works where a simple message would not: actions speak louder than words.

Signalling is outwardly similar to other solutions to adverse selection. The high-ability agent finds it cheaper than the low-ability one to build a reputation, but the reputation-building actions are based directly on his high ability. In a typical reputation model he shows ability by producing high output period after period. Also, the nature of reputation is to require several periods of play, which signalling does not.

Another form of communication is possible when some observable variable not under the control of the worker is correlated with ability. Age, for example, is correlated with reliability, so an employer pays older workers more, but the correlation does not arise because it is easier for reliable workers to acquire the attribute of age. Because age is not an action chosen by the worker, we would not need game theory to model it.

Problems in Applying Signalling to Education

On the empirical level, the first question to ask of a signalling model of education is, "What is education?". For operational purposes this means, "In what units is education measured?". Two possible answers are "years of education" and "grade point average." If the sacrifice of a year of earnings is greater for a low-ability worker, years of education can serve as a signal. If less intelligent students must work harder to get straight As, then grade-point-average can also be a signal.

Layard & Psacharopoulos (1974) give three rationales for rejecting signalling as an important motive for education. First, dropouts get as high a rate of return on education as those who complete degrees, so the signal is not the diploma, although it might be the years of education. Second, wage differentials between different education levels rise with age, although one would expect the signal to be less important after the employer has acquired more observations on the worker's output. Third, testing is not widely used for hiring, despite its low cost relative to education. Tests are available, but unused: students commonly take tests like the American SAT whose results they could credibly communicate to employers, and their scores correlate highly with subsequent grade point average. One would also expect an employer to prefer to pay an 18-year-old low wages for four years to determine his ability, rather than waiting to see what grades he gets as a history major.

Productive Signalling

Even if education is largely signalling, we might not want to close the schools. Signalling might be wasteful in a pooling equilibrium like Pooling Equilibrium 3.2, but in a separating equilibrium it can be second-best efficient for at least three reasons. First, it allows the employer to match workers with jobs suited to their talents. If the only jobs available were "professor" and "typist," then in a pooling equilibrium, both $High$ and *Low* workers would be employed, but they would be randomly allocated to the two jobs. Given the principle of comparative advantage, typing might improve, but I think, pridefully, that research would suffer.

Second, signalling keeps talented workers from moving to jobs where their productivity is lower but their talent is known. Without signalling, a talented worker might leave a corporation and start his own company, where he would be less productive but better paid. The naive observer would see that corporations hire only one type of worker (*Low*), and imagine there was no welfare loss.

Third, if ability is endogenous — moral hazard rather than adverse selection — signalling encourages workers to acquire ability. One of my teachers said that you always understand your next-to-last econometrics class. Suppose that solidly learning econometrics increases the student's ability, but a grade of A is not enough to show that he solidly learned the material. To signal his newly acquired ability, the student must also take "Time Series," which he cannot pass without a solid understanding of econometrics. "Time Series" might be useless in itself, but if it did not exist, the students would not be able to show he had learned basic econometrics.

11.4: The Informed Player Moves Second: Screening

In screening games, the informed player moves second, which means that he moves in response to contracts offered by the uninformed player. Having the uninformed player make the offers is important because his offer conveys no information about himself, unlike in a signalling model.

Education V: Screening with a Discrete Signal

Players

A worker and two employers.

The Order of Play
0 Nature chooses worker ability $a \in \{2, 5.5\}$, each ability having probability 0.5. Employers do not observe ability, but the worker does.
1 Each employer offers a wage contract $w(s)$.
2 The worker chooses education level $s \in \{0, 1\}$.
3 The worker accepts a contract, or rejects both of them.
4 Output equals a.

Payoffs

$$\pi_{worker} = \begin{cases} w - \frac{8s}{a} & \text{if the worker accepts contract } w. \\ 0 & \text{if the worker rejects both contracts.} \end{cases}$$

$$\pi_{employer} = \begin{cases} a - w & \text{for the employer whose contract is accepted.} \\ 0 & \text{for the other employer.} \end{cases}$$

Education V has no pooling equilibrium, because if one employer tried to offer the zero profit pooling contract, $w(0) = 3.75$, the other employer would offer $w(1) = 5.5$ and draw away all the *High*s. The unique equilibrium is

Separating Equilibrium 5.1 $\begin{cases} s(Low) = 0, s(High) = 1 \\ w(0) = 2, w(1) = 5.5 \end{cases}$

Beliefs do not need to be specified in a screening model. The uninformed player moves first, so his beliefs after seeing the move of the informed player are irrelevant. The informed player is fully informed, so his beliefs are not affected by what he observes. This is much like simple adverse selection, in which the uninformed player moves first, offering a set of contracts, after which the informed player chooses one of them. The modeller does not need to refine perfectness in a screening model. The similarity between adverse selection and screening is strong enough that Education V would not have been out of place in Chapter 9, but it is presented here because the context is so similar to the signalling models of education.

Education VI allows a continuum of education levels, in a game otherwise the same as Education V.

Education VI: Screening with a Continuous Signal

Players
A worker and two employers.

The Order of Play
0 Nature chooses worker ability $a \in \{2, 5.5\}$, each ability having probability 0.5. Employers do not observe ability, but the worker does.
1 Each employer offers a wage contract $w(s)$.
2 The worker choose education level $s \in [0, 1]$.

3 The worker chooses a contract, or rejects both of them.
4 Output equals a.

Payoffs.

$$\pi_{worker} = \begin{cases} w - 8s/a & \text{if the worker accepts contract } w. \\ 0 & \text{if the worker rejects both contracts.} \end{cases}$$

$$\pi_{employer} = \begin{cases} a - w & \text{for the employer whose contract is accepted.} \\ 0 & \text{for the other employer.} \end{cases}$$

Pooling equilibria generally do not exist in screening games with continuous signals, and sometimes separating equilibria in pure strategies do not exist either — recall Insurance Game III from Section 9.4. Education VI, however, does have a separating Nash equilibrium, with a unique equilibrium path.

Separating Equilibrium 6.1 $\begin{cases} s(Low) = 0, s(High) = 0.875 \\ \\ w = \begin{cases} 2 & \text{if } s < 0.875 \\ 5.5 & \text{if } s \geq 0.875 \end{cases} \end{cases}$

In any separating contract, the *Low*s must be paid a wage of 2 for an education of 0, because this is the most attractive contract that breaks even. The separating contract for the *High*s must maximize their utility subject to the constraints discussed in Education I. When the signal is continuous, the constraints are especially useful to the modeller for calculating the equilibrium. The participation constraints for the employers require that

$$w(0) \leq a_L = 2 \;\text{ and }\; w(s^*) \leq a_H = 5.5, \tag{11}$$

where s^* is the separating value of education that we are trying to find. Competition turns the inequalities in (11) into equalities. The self selection constraint for the low-ability workers is

$$U_L(s = 0) \geq U_L(s = s^*), \tag{12}$$

which in Education VI is

$$w(0) - 0 \geq w(s^*) - \frac{8s^*}{2}. \tag{13}$$

Since the separating wage is 2 for the *Low*s and 5.5 for the *High*s, constraint (13) is satisfied as an equality if $s^* = 0.875$, which is the crucial education level in Separating Equilibrium 6.1.

$$U_H(s = 0) = w(0) \leq U_H(s = s^*) = w(s^*) - \frac{8s^*}{5.5}. \tag{14}$$

If $s^* = 0.875$, inequality (14) is true, and it would also be true for higher values of s^*. Unlike the case of the continuous-strategy signalling game, Education IV, however, the equilibrium contract in Education VI is unique, because the employers compete to offer the most attractive contract that satisfies the participation and incentive compatibility constraints. The most attractive is the separating contract that Pareto dominates the other separating contracts by requiring the relatively low separating signal of $s^* = 0.875$.

Similarly, competition in offering attractive contracts rules out pooling contracts. The nonpooling constraint, required by competition between employers, is

$$U_H(s = s^*) \geq U_H(pooling), \tag{15}$$

which, for Education VI, is, using the most attractive possible pooling contract,

$$w(s^*) - \frac{8s^*}{5.5} \geq 3.75. \tag{16}$$

Since the payoff of *High*s in the separating contract is 4.23 (= $5.5 - 8 \cdot 0.875/5.5$, rounded), the nonpooling constraint is satisfied.

No Pooling Equilibrium in Education VI

Education VI lacks a pooling equilibrium, which would require the outcome $\{s = 0, w(0) = 3.75\}$, shown as C_1 in Figure 1. If one employer offered a pooling contract requiring more than zero education (such as the inefficient Pooling Equilibrium 3.2), the other employer could make the more attractive offer of the same wage for zero education. The wage is 3.75 to ensure zero profits. The rest of the wage function — the wages for positive education levels — can take a variety of shapes, so long as the wage does not rise so fast with education that the *High*s are tempted to become educated.

But no equilibrium has these characteristics. In a Nash equilibrium, no employer can offer a pooling contract, because the other employer could always profit by offering a separating contract paying more to the educated. One such separating contract is C_2 in Figure 1, which pays 5 to workers with an education of $s = 0.5$ and yields a payoff of 4.89 to the *High*s (= $5 - [8 \cdot 0.5]/5.5$, rounded) and 3 to the *Low*s (= $5 - 8 \cdot 0.5/2$). Only *High*s prefer C_2 to the pooling contract C_1, which yields payoffs of 3.75 to both *High* and *Low*, and if only *High*s accept C_2, it yields positive profits to the employer.

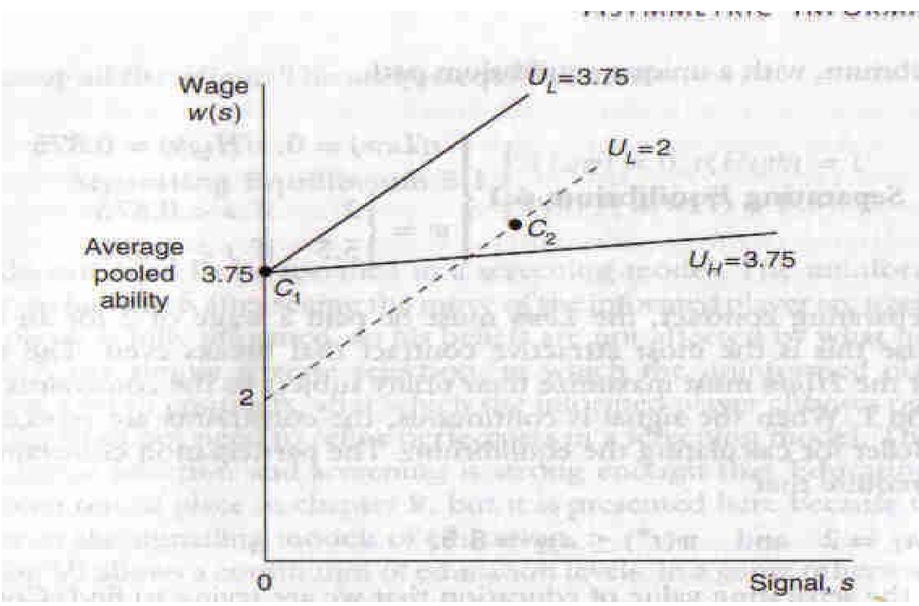


Figure 1: Education VI: no pooling Nash equilibrium

Nonexistence of a pooling equilibrium in screening models without continuous strategy spaces is a general result. The linearity of the curves in Education VI is special, but in any screening model the *Low*s would have greater costs of education, which is equivalent to steeper indifference curves. This is the **single-crossing property** alluded to in Education I. Any pooling equilibrium must, like C_1, lie on the vertical axis where education is zero and the wage equals the average ability. A separating contract like C_2 can always be found to the northeast of the pooling contract, between the indifference curves of the two types, and it will yield positive profits by attracting only the *High*s.

Education VII: No Nash Equilibrium in Pure Strategies

In Education VI we showed that screening models have no pooling equilibria. In Education VII the parameters are changed a little to eliminate even the separating equilibrium in pure strategies. Let the proportion of *High*s be 0.9 instead of 0.5, so the zero-profit pooling wage is 5.15 ($= 0.9[5.5] + 0.1[2]$) instead of 3.75. Consider the separating contracts C_3 and C_4, shown in Figure 2, calculated in the same way as Separating Equilibrium 5.1. The pair (C_3, C_4) is the most attractive pair of contracts that separates *High*s from *Low*s. *Low* workers accept contract C_3, obtain $s = 0$, and receive a wage of 2, their ability. *High*s accept contract C_4, obtain $s = 0.875$, and receive a wage of 5.5, their ability. Education is not attractive to *Low*s because the *Low* payoff from pretending to be *High* is 2 ($= 5.5 - 8 \cdot 0.875/2$), no better than the *Low* payoff of 2 from C_3 ($= 2 - 8 \cdot 0/2$).

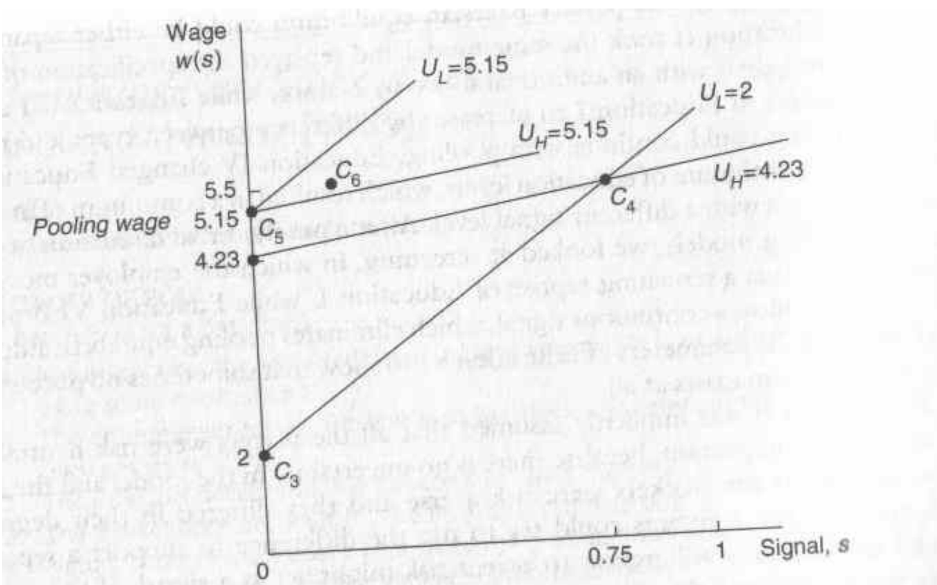


Figure 2: Education VII: No Nash Equilibrium

The wage of the pooling contract C_5 is 5.15, so that even the *High*s strictly prefer C_5 to (C_3, C_4). But our reasoning that no pooling equilibrium exists is still valid; some contract C_6 would attract all the *High*s from C_5. No Nash equilibrium in pure strategies exists, either separating or pooling.

A Summary of the Education Models

Because of signalling's complexity, most of this chapter has been devoted to elaboration of the education model. We began with Education I, which showed how with two types and two signal levels the perfect Bayesian equilibrium could be either separating or pooling. Education II took the same model and replaced the specification of out-of-equilibrium beliefs with an additional move by Nature, while Education III changed the parameters in Education I to increase the difference between types and to show how signalling could continue with pooling. Education IV changed Education I by allowing a continuum of education levels, which resulted in a continuum of inefficient equilibria, each with a different signal level. After a purely verbal discussion of how to apply signalling models, we looked at screening, in which the employer moves first. Education V was a screening reprise of Education I, while Education VI broadened the model to allow a continuous signal, which eliminates pooling equilibria. Education VII modified the parameters of Education VI to show that sometimes no pure-strategy Nash equilibrium exists at all.

Throughout it was implicitly assumed that all the players were risk neutral. Risk neutrality is unimportant, because there is no uncertainty in the model and the agents bear no risk. If the workers were risk averse and they differed in their degrees of risk aversion, the contracts could try to use the difference to support a separating equilibrium because willingness to accept risk might act as a signal. If the principal were risk averse he might offer a wage less than the average productivity in the pooling equilibrium, but he is under no risk at all in the separating equilibrium, because it is fully revealing. The models are also games of certainty, and this too is unimportant. If output were uncertain, agents would just make use of the expected payoffs rather than the raw payoffs and very little would change.

We could extend the education models further — allowing more than two levels of ability would be a high priority — but instead, let us turn to the financial markets and look graphically at a model with two continuous characteristics of type and two continuous signals.

*11.5. Two Signals: Game of Underpricing New Stock Issues

One signal might not be enough when there is not one but two characteristics of an agent that he wishes to communicate to the principal. This has been generally analyzed in Engers (1987), and multiple signal models have been especially popular in financial economics, for example, the multiple signal model used to explain the role of investment bankers in new stock issues by Hughes (1986). We will use a model of initial public offerings of stock as the example in this section.

Empirically, it has been found that companies consistently issue stock at a price so low that it rises sharply in the days after the issue, an abnormal return estimated to average 11.4 percent (Copeland & Weston [1988], p. 377). The game of Underpricing New Stock Issues tries to explain this using the percentage of the stock retained by the original owner and the amount of underpricing as two signals. The two characteristics being signalled are the mean of the value of the new stock, which is of obvious concern to the potential buyers, and the variance, the importance of which will be explained later.

Underpricing New Stock Issues
(Grinblatt & Hwang [1989])

Players
The entrepreneur and many investors.

The Order of Play
(See Figure 3a of Chapter 2 for a time line.)
0 Nature chooses the expected value (μ) and variance (σ^2) of a share of the firm using some distribution F.
1 The entrepreneur retains fraction α of the stock and offers to sell the rest at a price per share of P_0.
2 The investors decide whether to accept or reject the offer.
3 The market price becomes P_1, the investors' estimate of μ.
4 Nature chooses the value V of a share using some distribution G such that μ is the mean of V and σ^2 is the variance. With probability θ, V is revealed to the investors and becomes the market price.
5 The entrepreneur sells his remaining shares at the market price.

Payoffs

$$\pi_{entrepreneur} = U([1-\alpha]P_0 + \alpha[\theta V + (1-\theta)P_1]), \text{ where } U' > 0 \text{ and } U'' < 0.$$

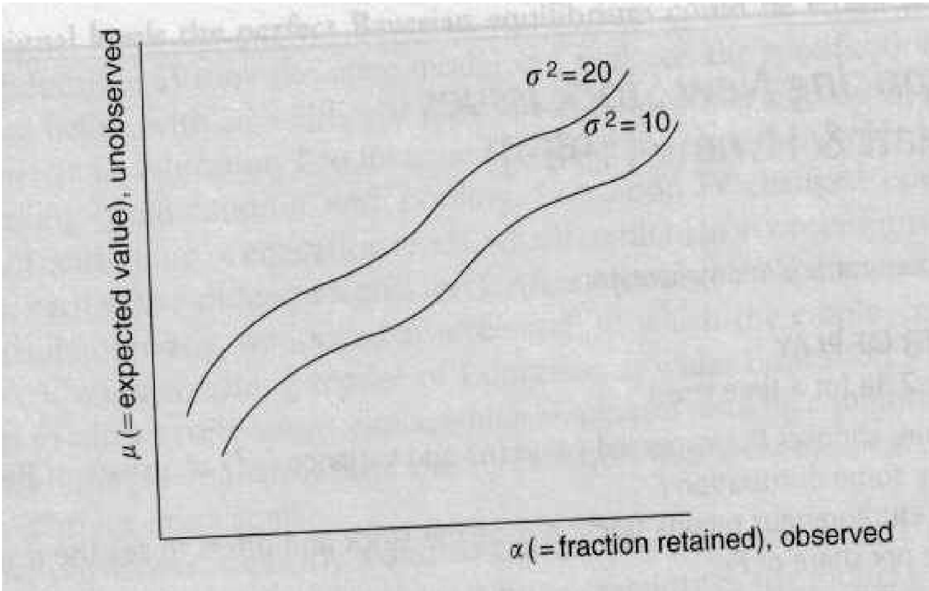
$$\pi_{investors} = (1-\alpha)(V - P_0) + \alpha(1-\theta)(V - P_1).$$

The entrepreneur's payoff is the utility of the value of the shares he issues at P_0 plus the value of those he sells later at the price P_1 or V. The investors' payoff is the true value of the shares they buy minus the price they pay.

Underpricing New Stock Issues subsumes the simpler model of Leland & Pyle (1977), in which σ^2 is common knowledge and if the entrepreneur chooses to retain a large fraction of the shares, the investors deduce that the stock value is high. The one signal in that model is fully revealing because holding a larger fraction exposes the undiversified entrepreneur to a larger amount of risk, which he is unwilling to accept unless the stock value is greater than investors would guess without the signal.

If the variance of the project is high, that also increases the risk to the undiversified entrepreneur, which is important even though the investors are risk neutral and do not care directly about the value of σ^2. Since the risk is greater when variance is high, the signal α is more effective and retaining a smaller amount allows the entrepreneur to sell the remainder at the same price as a larger amount for a lower-variance firm. Even though the investors are diversified and do not care directly about firm-specific risk, they are interested in the variance because it tells them something about the effectiveness of entrepreneur-retained shares as a signal of share value. Figure 3 shows the signalling schedules for two variance levels.

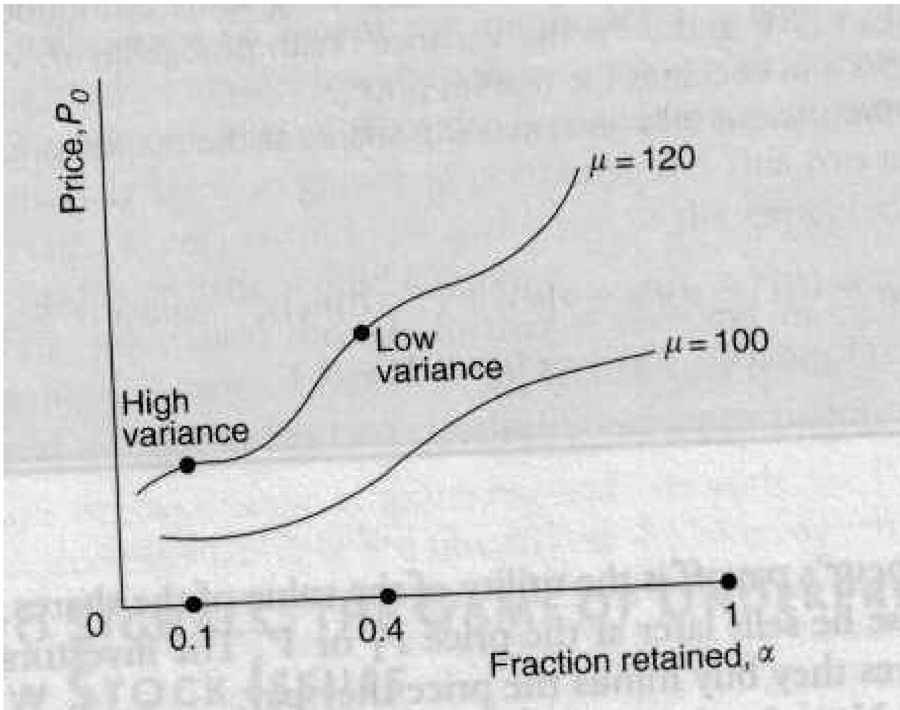
Figure 3: How the signal changes with the variance



In the game of Underpricing New Stock Issues, σ^2 is not known to the investors, so the signal is no longer fully revealing. An α equal to 0.1 could mean either that the firm has a low value with low variance, or a high value with high variance. But the entrepreneur can use a second signal, the price at which the stock is issued, and by observing α and P_0, the investors can deduce μ and σ^2.

I will use specific numbers for concreteness. The entrepreneur could signal that the stock has the high mean value, $\mu = 120$, in two ways: (a) retaining a high percentage, $\alpha = 0.4$, and making the initial offering at a high price of $P_0 = 90$, or (b) retaining a low percentage, $\alpha = 0.1$, and making the initial offering at a low price, $P_0 = 80$. Figure 4 shows the different combinations of initial price and fraction retained that might be used. If the stock has a high variance, he will want to choose behavior (b), which reduces his risk. Investors deduce that the stock of anyone who retains a low percentage and offers a low price actually has $\mu = 120$ and a high variance, so stock offered at the price of 80 rises in price. If, on the other hand, the entrepreneur retained $\alpha = .1$ and offered the high price $P_0 = 90$, investors would conclude that μ was lower than 120 but that variance was low also, so the stock would not rise in price. The low price conveys the information that this stock has a high mean and high variance rather than a low mean and low variance.

Figure 4: Different ways to signal a given μ.



This model explains why new stock is issued at a low price. The entrepreneur knows that the price will rise, but only if he issues it at a low initial price to show that the variance is high. The price discount shows that signalling by holding a large fraction of stock is unusually costly, but he is none the less willing to signal. The discount is costly because he is selling stock at less than its true value, and retaining stock is costly because he bears extra risk, but both are necessary to signal that the stock is valuable.

*11.6 Signal Jamming and Limit Pricing

This chapter has examined a number of models in which an informed player tries to convey information to an uninformed player by some means or other — by entering into an incentive contract, or by signalling. Sometimes, however, the informed party has the opposite problem: his natural behavior would convey his private information but he wants to keep it secret. This happens, for example, if one firm is informed about its poor ability to compete successfully, and it wants to conceal this information from a rival. The informed player may then engage in costly actions, just as in signalling, but now the costly action will be **signal jamming** (a term coined in Fudenberg & Tirole [1986c]): preventing information from appearing rather than generating information.

The model I will use to illustrate signal jamming is the limit pricing model of Rasmusen (1997). Limit pricing refers to the practice of keeping prices low to deter entry. Limit pricing can be explained in a variety of ways; notably, as a way for the incumbent to signal that he has low enough costs that rivals would regret entering, as in Problem 6.2 and Milgrom & Roberts [1982a]. Here, the explanation for limit pricing will be signal jamming: by keeping profits low, the incumbent keeps it unclear to the rival whether the market is big enough to accommodate two firms profitably. In the model, the incumbent can control S, a public signal of the size of the market. In the model below, this signal is the price that the incumbent charges, but it could equally well represent the incumbent's choice of advertising or capacity. The reason the signal is important is that the entrant must decide whether to enter based on his belief as to the probability that the market is large enough to support two firms profitably.

Limit Pricing as Signal Jamming

Players
The incumbent and the rival.

The Order of Play
0 Nature chooses the market size M to be M_{Small} with probability θ and M_{Large} with probability $(1-\theta)$, observed only by the incumbent.
1 The incumbent chooses the signal S to equal s_0 or s_1 for the first period if the market is small, s_1 or s_2 if it is large. This results in monopoly profit $\mu f(S) - C$, where $\mu > 2$. Both players observe the value of S.
2 The rival decides whether to be *In* or *Out* of the market.
3 If the rival chooses *In*, each player incurs cost C in the second period and they each earn the duopoly profit $M - C$. Otherwise, the incumbent earns $\mu f(S) - C$.

Payoffs
If the rival does not enter, the payoffs are $\pi_{incumbent} = (\mu f(S) - C) + (\mu f(S) - C)$ and $\pi_{rival} = 0$.
If the rival does enter, the payoffs are $\pi_{incumbent} = (\mu f(S) - C) + (M - C)$ and $\pi_{rival} = M - C$.
Assume that $f(s_0) < f(s_1) = M_{Small} < f(s_2) = M_{Large}$, $M_{Large} - C > 0$, and $M_{Small} - C < 0$.

Thus, if the incumbent chooses s_1, his profit will equal the maximum profit from a small market, even if the market is really large, but if he chooses s_2, his profit will be the maximum value for a large market – but that choice will have revealed that the market is large. The duopoly profit in a large market is large enough to sustain two firms, but the duopoly profit in a small market will result in losses for both firms.

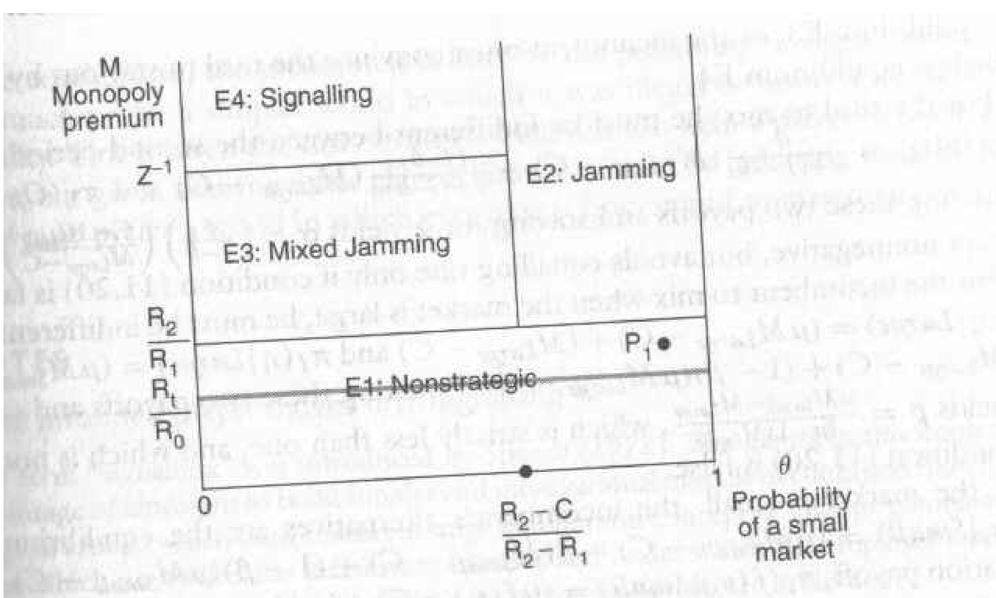


Figure 5: Signal jamming

There are four equilibria, each appropriate to a different parameter region in Figure 5. A small enough value of the parameter μ, which represents the value to being a monopoly, leads to a nonstrategic equilibrium exists, in which the incumbent simply maximizes profits in each period separately. This equilibrium is: (E1: Nonstrategic. $s_2|Large$, $s_1|Small$, $Out|s_0$, $Out|s_1$, $In|s_2$). The incumbent's equilibrium payoff in a large market is $\pi_I(s_2|Large) = (\mu M_{Large} - C) + (M_{Large} - C)$, compared with the deviation payoff of $\pi_I(s_1|Large) = (\mu M_{Small} - C) + (\mu M_{Large} - C)$. The incumbent has no incentive to deviate if $\pi_I(s_2|Large) - \pi_I(s_1|Large) = (1+\mu)M_{Large} - \mu(M_{Small} + M_{Large}) \geq 0$, which is equivalent to

$$\mu \leq \frac{M_{Large}}{M_{Small}}, \tag{17}$$

as shown in Figure 5. The rival will not deviate, because the incumbent's choice fully reveals the size of the market.

Signal jamming occurs if monopoly profits are somewhat higher, and if the rival would refrain from entering the market unless he decides it is more profitable than his prior beliefs would indicate. The equilibrium is (E2: Pure Signal-Jamming. $s_1|Large$, $s_1|Small$, $Out|s_0$, $Out|s_1$, $In|s_2$). The rival's strategy is the same as in E1, so the incumbent's optimal behavior remains the same, and he chooses s_1 if the opposite of condition (17) is true. As for the rival, if he stays out, his second-period payoff is 0, and if he enters, its expected value is $\theta(M_{Small} - C) + (1-\theta)(M_{Large} - C)$. Hence, as shown in Figure 5, he will follow the equilibrium behavior of staying out if

$$\theta \geq \frac{M_{Large} - C}{M_{Large} - M_{Small}}. \tag{18}$$

The intuition behind the signal jamming equilibrium is straightforward. The incumbent knows he will attract entry if he fully exploits the market when it is large, so he purposely dulls his efforts to conceal whether the market is large or small. If potential entrants place enough prior probability on the market being small, and are thus unwilling to enter without positive information that the market is large, the incumbent can thus deter entry.

Signal jamming shows up in other contexts. A wealthy man may refrain from buying a big house, landscaping his front yard, or wearing expensive clothing in order to avoid being a target for thieves or for political leaders in search of wealthy victims to tax or loot. A cabinet with shaky support may purposely take risky assertive positions because greater caution might hint to his rivals that his position was insecure and induce them to campaign actively against him. A general may advance his troops even when he is outnumbered, because to go on the defensive would provoke the enemy to attack. Note, however, that in each of these examples it is key that the uninformed player decide not to act aggressively if he fails to acquire any information.

A mixed form of signal jamming occurs if the probability of a small market is low, so if the signal of first-period revenues was jammed completely, the rival would enter anyway.

This equilibrium is (E3: Mixed Signal Jamming. ($s_1|Small$, $s_1|Large$ with probability α, $s_2|Large$ with probability $(1-\alpha)$, $Out|s_0$, $In|s_1$ with probability β, $Out|s_1$ with probability $(1-\beta)$, $In|s_2$). If the incumbent played $s_2|Large$ and $s_1|Small$, the rival would interpret s_1 as indicating a small market — an interpretation which would give the incumbent incentive to play $s_1|Large$. But if the incumbent always plays s_1, the rival would enter even after observing s_1, knowing there was a high probability that the market was really large. Hence, the equilibrium must be in mixed strategies, which is equilibrium E3, or the incumbent must convince the rival to stay out by playing s_0, which is equilibrium E4.

For the rival to mix, he must be indifferent between the second-period payoffs of $\pi_E(In|s_1) = \frac{\theta}{\theta+(1-\theta)\alpha}(M_{Small} - C) + \frac{(1-\theta)\alpha}{\theta+(1-\theta)\alpha}(M_{Large} - C)$ and $\pi_E(Out|s_1) = 0$. Equating these two payoffs and solving for α yields $\alpha = \left(\frac{\theta}{1-\theta}\right)\left(\frac{C-M_{Small}}{M_{Large}-C}\right)$, which is always nonnegative, but avoids equalling one only if condition (18) is false.

For the incumbent to mix when the market is large, he must be indifferent between $\pi_I(s_2|Large) = (\mu M_{Large}-C)+(M_{Large}-C)$ and $\pi_I(s_1|Large) = (\mu M_{Small}-C)+\beta(M_{Large}-C) + (1-\beta)(\mu M_{Large} - C)$. Equating these two payoffs and solving for β yields $\beta = \frac{\mu M_{Small}-M_{Large}}{(\mu-1)M_{Large}}$, which is strictly less than one, and which is nonnegative if condition (18) is false.

If the market is small, the incumbent's alternatives are the equilibrium payoff, $\pi_I(s_1|Small) = (\mu M_{Small}-C)+\beta(M_{Small}-C)+(1-\beta)(\mu M_{Small}-C)$, and the deviation payoff, $\pi_I(f(s_0)|Small) = (\mu f(s_0) - C) + (\mu M_{Small} - C)$. The difference is

$$\pi_I(s_1|Small)-\pi_I(f(s_0)|Small) = [\mu M_{Small}+\beta M_{Small}+(1-\beta)\mu M_{Small}]-[\mu f(s_0)+\mu M_{Small}] \tag{19}$$

Expression (19) is nonnegative under either of two conditions, both of which are found by substituting the equilibrium value of β into expression (19). The first is if $f(s_0)$ is small enough, a sufficient condition for which is

$$f(s_0) \leq M_{Small}\left(1 - \frac{M_{Small}}{M_{Large}}\right). \tag{20}$$

The second is if μ is no greater than some amount Z^{-1} defined so that

$$\mu \leq \left(\frac{M_{Small}}{M_{Large}} - 1 + \frac{f(s_0)}{M_{Small}}\right)^{-1} = Z^{-1}. \tag{21}$$

If condition (20) is false, then $Z^{-1} > \frac{M_{Large}}{M_{Small}}$, because $Z < \frac{M_{Small}}{M_{Large}}$ and $Z > 0$. Thus, we can draw region E3 as it is shown in Figure 5.

It follows that if condition (21) is replaced by its converse, the unique equilibrium is for the incumbent to choose $s_0|Small$, and the equilibrium is (E4: Signalling. $s_0|Small$, $s_2|Large$, $Out|s_0$, $In|s_1$, $In|s_2$). Passive conjectures will support this pooling signalling equilibrium, as will the out-of- equilibrium belief that if the rival observes s_1, he believes the market is large with probability $\frac{(1-\theta)\alpha}{\theta+(1-\theta)\alpha}$, as in equilibrium E3.

The signalling equilibrium is also an equilibrium for other parameter regions outside of E4, though less reasonable beliefs are required. Let the out-of-equilibrium belief be

$Prob(Large|s_1) = 1$. The equilibrium payoff is $\pi_I(s_0|Small) = (\mu f(s_0) - C) + (\mu M_{Small} - C)$ and the deviation payoff is $\pi_I(s_1|Small) = (\mu M_{Small} - C) + (M_{Small} - C)$. The signalling equilibrium remains an equilibrium so long as $\mu \geq \frac{M_{Small}}{f(s_0)}$.

The signalling equilibrium is an interesting one, because it turns the asymmetric information problem full circle. The informed player wants to conceal his private information by costly signal jamming if the information is *Large*, so when the information is *Small*, the player must signall at some cost that he is not signal jamming. If E4 is the equilibrium, the incumbent is hurt by the possibility of signal jamming; he would much prefer a simpler world in which it was illegal or nobody considered the possibility. This is often the case: strategic behavior can help a player in some circumstances, but given that the other players know he might be behaving strategically, everyone would prefer a world in which everyone is honest and nonstrategic.

Notes

N11.1 The informed player moves first: signalling

- The term "signalling" was introduced by Spence (1973). The games in this book take advantage of hindsight to build simpler and more rational models of education than in his original article, which used a rather strange equilibrium concept: a strategy combination from which no worker has incentive to deviate and under which the employer's profits are zero. Under that concept, the firm's incentives to deviate are irrelevant.

  The distinction between signalling and screening has been attributed to Stiglitz & Weiss (1989). The literature has shown wide variation in the use of both terms, and "signal" is such a useful word that it is often used in models that have no signalling of the kind discussed in this chapter.

- One convention sometimes used in signalling models is to call the signalling player (the agent), the **sender** and the player signalled to (the principal), the **receiver**.

- The applications of signalling are too many to properly list. A few examples are the use of prices in Wilson (1980) and Stiglitz (1987), the payment of dividends in Ross (1977), bargaining (Section 12.5), and greenmail (Section 15.2). Banks (1990) has written a short book surveying signalling models in political science. Empirical papers include Layard & Psacharopoulos (1974) on education and Staten & Umbeck (1986) on occupational diseases.

- Legal bargaining is one area of application for signalling. See Grossman & Katz (1983). Reinganum (1988) has a nice example of the value of pre-commitment in legal signalling. In her model, a prosecutor who wishes to punish the guilty and release the innocent wishes, if parameters are such that most defendants are guilty, to commit to a pooling strategy in which his plea bargaining offer is the same whatever the probability that a particular defendant would be found guilty.

- The peacock's tail may be a signal. Zahavi (1975) suggests that a large tail may benefit the peacock because, by hampering him, it demonstrates to potential mates that he is fit enough to survive even with a handicap.

- **Advertising.** Advertising is a natural application for signalling. The literature includes Nelson (1974), written before signalling was well known, Kihlstrom & Riordan (1984) and Milgrom & Roberts (1986). I will briefly describe a model based on Nelson's. Firms are one of two types, low quality or high quality. Consumers do not know that a firm exists until they receive an advertisement from it, and they do not know its quality until they buy its product. They are unwilling to pay more than zero for low quality, but any product is costly to produce. This is not a reputation model, because it is finite in length and quality is exogenous.

  If the cost of an advertisement is greater than the profit from one sale, but less than the profit from repeat sales, then high rates of advertising are associated with high product quality. A firm with low quality would not advertise, but a firm with high quality would.

  The model can work even if consumers do not understand the market and do not make rational deductions from the firm's incentives, so it does not have to be a signalling model. If consumers react passively and sample the product of any firm from whom they receive an advertisement, it is still true that the high quality firm advertises more, because the customers it attracts become repeat customers. If consumers do understand the firms' incentives, signalling reinforces the result. Consumers know that firms which advertise must have high quality, so they are willing to try them. This understanding is important, because if consumers knew that 90 percent of firms were low quality but did not understand that only high quality firms advertise, they would not respond to the advertisements which they received. This should bring to mind Section 6.2's game of PhD Admissions.

- If there are just two workers in the population, the model is different depending on whether:

  1 Each is $High$ ability with objective probability 0.5, so possibly both are $High$ ability; or
  2 One of them is $High$ and the other is Low, so only the subjective probability is 0.5.

  The outcomes are different because in case (2) if one worker credibly signals he is $High$ ability, the employer knows the other one must be Low ability.

Problems

11.1. Is Lower Ability Better?
Change Education I so that the two possible worker abilities are $a \in \{1, 4\}$.

(a) What are the equilibria of this game? What are the payoffs of the workers (and the payoffs averaged across workers) in each equilibrium?

(b) Apply the Intuitive Criterion (see N6.2). Are the equilibria the same?

(c) What happens to the equilibrium worker payoffs if the high ability is 5 instead of 4?

(d) Apply the Intuitive Criterion to the new game. Are the equilibria the same?

(e) Could it be that a rise in the maximum ability reduces the average worker's payoff? Can it hurt all the workers?

11.2. Productive Education and Nonexistence of Equilibrium
Change Education I so that the two equally likely abilities are $a_L = 2$ and $a_H = 5$ and education is productive: the payoff of the employer whose contract is accepted is $\pi_{employer} = a + 2s - w$. The worker's utility function remains $U = w - \frac{8s}{a}$.

(a) Under full information, what are the wages for educated and uneducated workers of each type, and who acquires education?

(b) Show that with incomplete information the equilibrium is unique (except for beliefs and wages out of equilibrium) but unreasonable.

11.3. Price and Quality
Consumers have prior beliefs that Apex produces low-quality goods with probability 0.4 and high-quality with probability 0.6. A unit of output costs 1 to produce in either case, and it is worth 11 to the consumer if it is high quality and 0 if low quality. The consumer, who is risk neutral, decides whether to buy in each of two periods, but he does not know the quality until he buys. There is no discounting.

(a) What is Apex's price and profit if it must choose one price, p^*, for both periods?

(b) What is Apex's price and profit if it can choose two prices, p_1 and p_2, for the two periods, but it cannot commit ahead to p_2?

(c) What is the answer to part (b) if the discount rate is $r = 0.1$?

(d) Returning to $r = 0$, what if Apex can commit to p_2?

(e) How do the answers to (a) and (b) change if the probability of low quality is 0.95 instead of 0.4? (There is a twist to this question.)

11.4. Signalling with a Continuous Signal
Suppose that with equal probability a worker's ability is $a_L = 1$ or $a_H = 5$, and the worker chooses any amount of education $y \in [0, \infty)$. Let $U_{worker} = w - \frac{8y}{a}$ and $\pi_{employer} = a - w$.

(a) There is a continuum of pooling equilibria, with different levels of y^*, the amount of education necessary to obtain the high wage. What education levels, y^*, and wages, $w(y)$, are paid in the pooling equilibria, and what is a set of out-of-equilibrium beliefs that supports them? What are the incentive compatibility constraints?

(b) There is a continuum of separating equilibria, with different levels of y^*. What are the education levels and wages in the separating equilibria? Why are out-of-equilibrium beliefs needed, and what beliefs support the suggested equilibria? What are the self selection constraints for these equilibria?

(c) If you were forced to predict one equilibrium which will be the one played out, which would it be?

11.5: Advertising.
Brydox introduces a new shampoo which is actually very good, but is believed by consumers to be good with only a probability of 0.5. A consumer would pay 11 for high quality and 0 for low quality, and the shampoo costs 6 per unit to produce. The firm may spend as much as it likes on stupid TV commercials showing happy people washing their hair, but the potential market consists of 110 cold-blooded economists who are not taken in by psychological tricks. The market can be divided into two periods.

(a) If advertising is banned, will Brydox go out of business?

(b) If there are two periods of consumer purchase, and consumers discover the quality of the shampoo if they purchase in the first period, show that Brydox might spend substantial amounts on stupid commercials.

(c) What is the minimum and maximum that Brydox might spend on advertising, if it spends a positive amount?

11.6. Game Theory Books
In the Preface I explain why I listed competing game theory books by saying, "only an author quite confident that his book compares well with possible substitutes would do such a thing, and you will be even more certain that your decision to buy this book was a good one."

(a) What is the effect of on the value of the signal if there is a possibility that I am an egotist who overvalues his own book?

(b) Is there a possible non strategic reason why I would list competing game theory books?

(c) If all readers were convinced by the signal of providing the list and so did not bother to even look at the substitute books, then the list would not be costly even to the author of a bad book, and the signal would fail. How is this paradox to be resolved? Give a verbal explanation.

(d) Provide a formal model for part (c).

11.7. The Single-Crossing Property
If education is to be a good signal of ability,
(a) Education must be inexpensive for all players.
(b) Education must be more expensive for the high ability player.
(c) Education must be more expensive for the low ability player.
(d) Education must be equally expensive for all types of players.
(e) Education must be costless for some small fraction of players.

11.8. A Continuum of Pooling Equilibria
Suppose that with equal probability a worker's ability is $a_L = 1$ or $a_H = 5$, and that the worker chooses any amount of education $y \in [0, \infty)$. Let $U_{worker} = w - \frac{8y}{a}$ and $\pi_{employer} = a - w$.

There is a continuum of pooling equilibria, with different levels of y^*, the amount of education necessary to obtain the high wage. What education levels, y^*, and wages, $w(y)$, are paid in the pooling equilibria, and what is a set of out-of-equilibrium beliefs that supports them? What are the self selection constraints?

11.9. Signal Jamming in Politics
A congressional committee has already made up its mind that tobacco should be outlawed, but it holds televised hearings anyway in which experts on both sides present testimony. Explain why these hearings might be a form of signalling, where the audience to be persuaded is congress as a whole, which has not yet made up its mind. You can disregard any effect the hearings might have on public opinion.

11.10. Salesman Clothing
Suppose a salesman's ability might be either $x = 1$ (with probability θ) or $x = 4$, and that if he dresses well, his output is greater, so that his total output is $x + 2s$ where s equals 1 if he dresses well and 0 if he dresses badly. The utility of the salesman is $U = w - \frac{8s}{x}$, where w is his wage. Employers compete for salesmen.

(a) Under full information, what will the wage be for a salesman with low ability?

(b) Show the self selection contraints that must be satisfied in a separating equilibrium under incomplete information.

(c) Find all the equilibria for this game if information is incomplete.

11.11. Crazy Predators (adapted from Gintis [2000], Problem 12.10.)
Apex has a monopoly in the market for widgets, earning profits of m per period, but Brydox has just entered the market. There are two periods and no discounting. Apex can either *Prey* on Brydox with a low price or accept *Duopoly* with a high price, resulting in profits to Apex of $-p_a$ or d_a and to Brydox of $-p_b$ or d_b. Brydox must then decide whether to stay in the market for the second period, when Brydox will make the same choices. If, however, Professor Apex, who owns 60 percent of the company's stock, is crazy, he thinks he will earn an amount $p^* > d_a$ from preying on Brydox (and he does not learn from experience). Brydox initially assesses the probability that Apex is crazy at θ.

(a) Show that under the following condition, the equilibrium will be separating, i.e., Apex will behave differently in the first period depending on whether the Professor is crazy or not:

$$-p_a + m < 2d_a \tag{22}$$

(b) Show that under the following condition, the equilibrium can be pooling, i.e., Apex will behave the same in the first period whether the Professor is crazy or not:

$$\theta \geq \frac{d_b}{p_b + d_b} \tag{23}$$

(c) If neither two condition (22) nor (23) apply, the equilibrium is hybrid, i.e., Apex will use a mixed strategy and Brydox may or may not be able to tell whether the Professor is crazy at the end of the first period. Let α be the probability that a sane Apex preys on Brydox in the first period, and let β be the probability that Brydox stays in the market in the second period after observing that Apex chose *Prey* in the first period. Show that the equilibrium values of α and β are:

$$\alpha = \frac{\theta p_b}{(1-\theta)d_b} \tag{24}$$

$$\beta = \frac{-p_a + m - 2d_a}{m - d_a} \tag{25}$$

(d) Is this behavior related to any of the following phenomenon?– Signalling, Signal Jamming, Reputation, Efficiency Wages.

11.12. Actions and Strategies

Explain the difference between an "action" and a "strategy," using a signal jamming game as an example.

11.13. Monopoly Quality

A consumer faces a monopoly. He initially believes that the probability that the monopoly has a high-quality product is H, and that a high-quality monopoly would be able to send him an advertisement at zero cost. With probability (1-H), though, the monopoly has low quality, and it would cost the firm A to send an ad. The firm does send an ad, offering the product at price P. The consumer's utility from a high-quality product is $X > P$, but from a low quality product it is 0. The production cost is C for the monopolist regardless of quality, where $C < P - A$. If the consumer does not buy the product, the seller does not incur the production cost.

You may assume that the high-quality firm always sends an ad, that the consumer will not buy unless he receives an ad, and that P is exogenous.

(a) Draw the extensive form for this game.

(b) What is the equilibrium if H is sufficiently high?

(c) If H is low enough, the equilibrium is in mixed strategies. The high-quality firm always advertises, the low quality firm advertises with probability M, and the consumer buys with probability N. Show using Bayes Rule how the consumer's posterior belief R that the firm is high-quality changes once he receives an ad.

(d) Explain why the equilibrium is not in pure strategies if H is too low (but H is still positive).

(e) Find the equilibrium probability of M. (You don't have to figure out N.)

PART III Applications

October 3, 1999. January 17, 2000. November 30, 2003. 24 March 2005. Eric Rasmusen, Erasmuse@indiana.edu. Http: www.rasmusen.org/GI. Footnotes starting with xxx are the author's notes to himself. Comments welcomed.

12 Bargaining

12.1 The Basic Bargaining Problem: Splitting a Pie

Part III of this book is designed to stretch your muscles by providing more applications of the techniques from Parts I and II. The next four chapters may be read in any order. They concern three ways that prices might be determined. Chapter 12 is about bargaining– where both sides exercise market power. Chapter 13 is about auctions– where the seller has market power, but sells a limited amount of a good and wants buyers to compete against each other. Chapter 14 is about fixed-price models with a variety of different features such as differentiated or durable goods. One thing all these chapters have in common is that they use new theory to answer old questions.

Bargaining theory attacks a kind of price determination ill described by standard economic theory. In markets with many participants on one side or the other, standard theory does a good job of explaining prices. In competitive markets we find the intersection of the supply and demand curves, while in markets monopolized on one side we find the monopoly or monopsony output. The problem is when there are few players on each side. Early in one's study of economics, one learns that under bilateral monopoly (one buyer and one seller), standard economic theory is inapplicable because the traders must bargain. In the chapters on asymmetric information we would have come across this repeatedly except for our assumption that either the principal or the agent faced competition.

Sections 12.1 and 12.2 introduce the archetypal bargaining problem, Splitting a Pie, ever more complicated versions of which make up the rest of the chapter. Section 12.2, where we take the original rules of the game and apply the Nash bargaining solution, is our one dip into cooperative game theory in this book. Section 12.3 looks at bargaining as a finitely repeated process of offers and counteroffers, and Section 12.4 views it as an infinitely repeated process. Section 12.5 returns to a finite number of repetitions (two, in fact), but with incomplete information. Finally, Section 12.6 approaches bargaining from a different level: how could people construct a mechanism for bargaining, a pre-arranged set of rules that would maximize their expected surplus.

Splitting a Pie

Players
Smith and Jones.

The Order of Play
The players choose shares θ_s and θ_j of the pie simultaneously.

Payoffs
If $\theta_s + \theta_j \leq 1$, each player gets the fraction he chose: $\left\{ \begin{array}{ll} \pi_s = & \theta_s. \\ \pi_j = & \theta_j. \end{array} \right.$
If $\theta_s + \theta_j > 1$, then $\pi_s = \pi_j = 0$.

Splitting a Pie resembles the game of Chicken except that it has a continuum of Nash equilibria: any strategy combination $(\theta_s,\, \theta_j)$ such that $\theta_s + \theta_j = 1$ is Nash. The Nash concept is at its worst here, because the assumption that the equilibrium being played is common knowledge is very strong when there is a continuum of equilibria. The idea of the focal point (section 1.5) might help to choose a single Nash equilibrium. The strategy space of Chicken is discrete and it has no symmetric pure-strategy equilibrium, but the strategy space of Splitting a Pie is continuous, which permits a symmetric pure-strategy equilibrium to exist. That equilibrium is the even split, (0.5, 0.5), which is a focal point.

If the players moved in sequence, the game would have a tremendous first-mover advantage. If Jones moved first, the unique Nash outcome would be (0,1), although only weakly, because Smith would be indifferent as to his action. (This is the same open-set problem that was discussed in Section 4.3.) Smith accepts the offer if he chooses θ_s to make $\theta_s + \theta_j = 1$, but if we added only an epsilon-worth of ill will to the model he would pick $\theta_s > 0$ and reject the offer.

In many applications this version of Splitting a Pie is unacceptably simple, because if the two players find their fractions add to more than 1 they have a chance to change their minds. In labor negotiations, for example, if manager Jones makes an offer which union Smith rejects, they do not immediately forfeit the gains from combining capital and labor. They lose a week's production and make new offers. The recent trend in research has been to model such a sequence of offers, but before we do that let us see how cooperative game theory deals with the original game.[1]

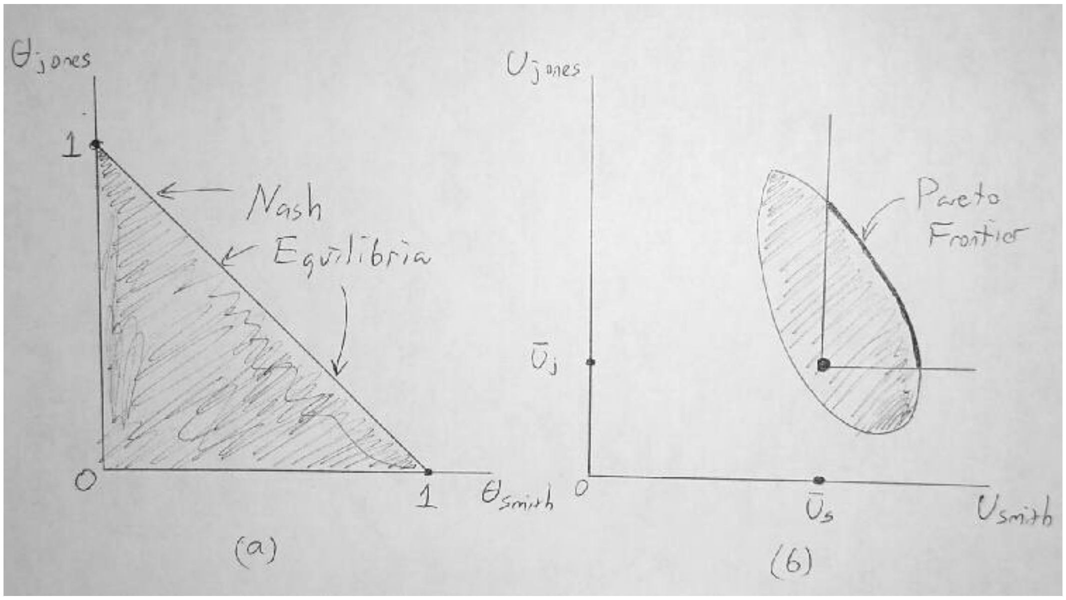
12.2 The Nash Bargaining Solution[2]

When game theory was young a favorite approach was to decide upon some characteristics an equilibrium should have based on notions of fairness or efficiency, mathematicize the characteristics, and maybe add a few other axioms to make the equilibrium turn out neatly. Nash (1950a) did this for the bargaining problem in what is perhaps the best-known application of cooperative game theory. Nash's objective was to pick axioms that would characterize the agreement the two players would anticipate making with each other. He used a game only a little more complicated than Splitting a Pie. In the Nash model, the two players can have different utilities if they do not come to an agreement, and the utility functions can be nonlinear in terms of shares of the pie. Figures 1a and 1b compare the two games.

Figure 1: (a) Nash Bargaining Game; (b) Splitting a Pie

[1] xxx Relate this to the take-it-or-leave it offer idea of earlier chapters, and note cahptr 7 on agency especailly.

[2] xxx This needs numerical examples.



In Figure 1, the shaded region denoted by X is the set of feasible payoffs, which we will assume to be convex. The disagreement point is $\bar{U} = (\bar{U}_s, \bar{U}_j)$. The Nash bargaining solution, $U^* = (U_s^*, U_j^*)$, is a function of $\bar{U}$ and X. The axioms that generate the concept are as follow

1 Invariance. For any strictly increasing linear function F,

$$U^*[F(\bar{U}), F(X)] = F[U^*(\bar{U}, X)]. \tag{1}$$

This says that the solution is independent of the units in which utility is measured.

2 Efficiency. The solution is Pareto optimal, so that not both players can be made better off. In mathematical terms,

$$(U_s, U_j) > U^* \Rightarrow (U_s, U_j) \notin X. \tag{2}$$

3 Independence of Irrelevant Alternatives. If we drop some possible utility combinations from X, leaving the smaller set Y, then if U^* was not one of the dropped points, U^* does not change.

$$U^*(\bar{U}, X) \in Y \subseteq X \Rightarrow U^*(\bar{U}, Y) = U^*(\bar{U}, X). \tag{3}$$

4 Anonymity (or Symmetry). Switching the labels on players Smith and Jones does not affect the solution.

The axiom of Independence of Irrelevant Alternatives is the most debated of the four, but if I were to complain, it would be about the axiomatic approach, which depends heavily on the intuition behind the axioms. Everyday intuition says that the outcome should be efficient and symmetric, so that other outcomes can be ruled out a priori. But most of the games in the earlier chapters of this book turn out to have reasonable but inefficient outcomes, and games like Chicken have reasonable asymmetric outcomes.

Whatever their drawbacks, these axioms fully characterize the Nash solution. It can be proven that if U^* satisfies the four axioms above, then it is the unique strategy combination such that

$$U^* = \underset{U \in X, U \geq \bar{U}}{argmax} \quad (U_s - \bar{U}_s)(U_j - \bar{U}_j). \tag{4}$$

Splitting a Pie is a simple enough game that not all the axioms are needed to generate a solution. If we put the game in this context, however, problem (12.4) becomes

$$\underset{\theta_s, \theta_j \mid \theta_s + \theta_j \leq 1}{Maximize} \quad (\theta_s - 0)(\theta_j - 0), \tag{5}$$

which generates the first order conditions

$$\theta_s - \lambda = 0, \text{ and } \theta_j - \lambda = 0, \tag{6}$$

where λ is the Lagrange multiplier on the constraint. From (12.6) and the constraint, we obtain $\theta_s = \theta_j = 1/2$, the even split that we found as a focal point of the noncooperative game.

Although Nash's objective was simply to characterize the anticipations of the players, I perceive a heavier note of morality in cooperative game theory than in noncooperative game theory. Cooperative outcomes are neat, fair, beautiful, and efficient. In the next few sections we will look at noncooperative bargaining models that, while plausible, lack every one of those features. Cooperative game theory may be useful for ethical decisions, but its attractive features are often inappropriate for economic situations, and the spirit of the axiomatic approach is very different from the utility maximization of economic theory.

It should be kept in mind, however, that the ethical component of cooperative game theory can also be realistic, because people are often ethical, or pretend to be. People very often follow the rules they believe represent virtuous behavior, even at some monetary cost. In bargaining experiments in which one player is given the ability to make a take-it-or-leave it offer, it is very commonly found that he offers a 50-50 split. Presumably this is because either he wishes to be fair or he fears a spiteful response from the other player to a smaller offer. If the subjects are made to feel that they had "earned" the right to be the offering party, they behave much more like the players in noncooperative game theory (Hoffman & Spitzer [1985]). Frank (1988) and Thaler (1992) describe numerous occasions where simple games fail to describe real-world or experimental results. People's payoffs include more than their monetary rewards, and sometimes knowing the cultural disutility of actions is more important than knowing the dollar rewards. This is one reason why it is helpful to a modeller to keep his games simple: when he actually applies them to the real world, the model must not be so unwieldy that he cannot combine it with his knowledge of the particular setting.

12.3 Alternating Offers over Finite Time

In the games of the next two sections, the actions are the same as in Splitting a Pie, but with many periods of offers and counteroffers. This means that strategies are no longer just actions, but rather rules for choosing actions based on the actions chosen in earlier periods.

Alternating Offers

Players
Smith and Jones.

The Order of Play
1 Smith makes an offer θ_1.
1* Jones accepts or rejects.
2 Jones makes an offer θ_2.
2* Smith accepts or rejects.
...
T Smith offers θ_T.
T* Jones accepts or rejects.

Payoffs
The discount factor is $\delta \leq 1$.
If Smith's offer is accepted by Jones in round m,

$$\pi_s = \delta^m \theta_m,$$
$$\pi_j = \delta^m (1 - \theta_m).$$

If Jones 's offer is accepted, reverse the subscripts.
If no offer is ever accepted, both payoffs equal zero.

When a game has many rounds we need to decide whether discounting is appropriate. If the discount rate is r then the discount factor is $\delta = 1/(1+r)$, so, without discounting, $r = 0$ and $\delta = 1$. Whether discounting is appropriate to the situation being modelled depends on whether delay should matter to the payoffs because the bargaining occurs over real time or the game might suddenly end (section 5.2). The game Alternating Offers can be interpreted in either of two ways, depending on whether it occurs over real time or not. If the players made all the offers and counteroffers between dawn and dusk of a single day, discounting would be inconsequential because, essentially, no time has passed. If each offer consumed a week of time, on the other hand, the delay before the pie was finally consumed would be important to the players and their payoffs should be discounted.

Consider first the game without discounting. There is a unique subgame perfect outcome — Smith gets the entire pie — which is supported by a number of different equilibria. In each equilibrium, Smith offers $\theta_s = 1$ in each period, but each equilibrium is different in terms of when Jones accepts the offer. All of them are weak equilibria because Jones is indifferent between accepting and rejecting, and they differ only in the timing of Jones's final acceptance.

Smith owes his success to his ability to make the last offer. When Smith claims the entire pie in the last period, Jones gains nothing by refusing to accept. What we have here is not really a first-mover advantage, but a last-mover advantage in offering, a difference not apparent in the one-period model.

In the game with discounting, the total value of the pie is 1 in the first period, δ in the second, and so forth. In period T, if it is reached, Smith would offer 0 to Jones, keeping 1 for himself, and Jones would accept under our assumption on indifferent players. In period

$T-1$, Jones could offer Smith δ, keeping $(1-\delta)$ for himself, and Smith would accept, although he could receive a greater share by refusing, because that greater share would arrive later and be discounted.

By the same token, in period $T-2$, Smith would offer Jones $\delta(1-\delta)$, keeping $1-\delta(1-\delta)$ for himself, and Jones would accept, since with a positive share Jones also prefers the game to end soon. In period $T-3$, Jones would offer Smith $\delta[1-\delta(1-\delta)]$, keeping $1-\delta[1-\delta(1-\delta)]$ for himself, and Smith would accept, again to prevent delay. Table 1 shows the progression of Smith's shares when $\delta = 0.9$.

Table 1: Alternating offers over finite time

Round	Smith's share	Jones's share	Total value	Who offers?
$T-3$	0.819	0.181	0.9^{T-4}	Jones
$T-2$	0.91	0.09	0.9^{T-3}	Smith
$T-1$	0.9	0.1	0.9^{T-2}	Jones
T	1	0	0.9^{T-1}	Smith

As we work back from the end, Smith always does a little better when he makes the offer than when Jones does, but if we consider just the class of periods in which Smith makes the offer, Smith's share falls. If we were to continue to work back for a large number of periods, Smith's offer in a period in which he makes the offer would approach $\frac{1}{1+\delta}$, which equals about 0.53 if $\delta = 0.9$. The reasoning behind that precise expression is given in the next section. In equilibrium, the very first offer would be accepted, since it is chosen precisely so that the other player can do no better by waiting.

12.4 Alternating Offers over Infinite Time

The Folk theorem of Section 5.2 says that when discounting is low and a game is repeated an infinite number of times, there are many equilibrium outcomes. That does not apply to the bargaining game, however, because it is not a repeated game. It ends when one player accepts an offer, and only the accepted offer is relevant to the payoffs, not the earlier proposals. In particular, there are no out-of-equilibrium punishments such as enforce the Folk Theorem's outcomes.

Let players Smith and Jones have discount factors of δ_s and δ_j which are not necessarily equal but are strictly positive and no greater than one. In the unique subgame perfect outcome for the infinite-period bargaining game, Smith's share is

$$\theta_s = \frac{1-\delta_j}{1-\delta_s\delta_j}, \tag{7}$$

which, if $\delta_s = \delta_j = \delta$, is equivalent to

$$\theta_s = \frac{1}{1+\delta}. \tag{8}$$

If the discount rate is high, Smith gets most of the pie: a 1,000 percent discount rate ($r = 10$) makes $\delta = 0.091$ and $\theta_s = 0.92$ (rounded), which makes sense, since under such extreme discounting the second period hardly matters and we are almost back to the simple game of Section 12.1. At the other extreme, if r is small, the pie is split almost evenly: if $r = 0.01$, then $\delta \approx 0.99$ and $\theta_s \approx 0.503$.

It is crucial that the discount rate be strictly greater than 0, even if only by a little. Otherwise, the game has the same continuum of perfect equilibria as in Section 12.1. Since nothing changes over time, there is no incentive to come to an early agreement. When discount rates are equal, the intuition behind the result is that since a player's cost of delay is proportional to his share of the pie, if Smith were to offer a grossly unequal split, such as (0.7, 0.3), Jones, with less to lose by delay, would reject the offer. Only if the split is close to even would Jones accept, as we will now prove.

Proposition 1 (Rubinstein [1982]) *In the discounted infinite game, the unique perfect equilibrium outcome is* $\theta_s = \frac{1-\delta_j}{(1-\delta_s\delta_j)}$, *where Smith is the first mover.*

Proof
We found that in the T-period game Smith gets a larger share in a period in which he makes the offer. Denote by M the maximum nondiscounted share, taken over all the perfect equilibria that might exist, that Smith can obtain in a period in which he makes the offer. Consider the game starting at t. Smith is sure to get no more than M, as noted in Table 2. (Jones would thus get 1 - M, but that is not relevant to the proof.)

Table 2: Alternating offers over infinite time

Round	Smith's share	Jones's share	Who offers?
$T-2$	$1-\delta_j(1-\delta_s M)$		Smith
$T-1$		$1-\delta_s M$	Jones
T	M		Smith

The trick is to find a way besides M to represent the maximum Smith can obtain. Consider the offer made by Jones at $t-1$. Smith will accept any offer which gives him more than the discounted value of M received one period later, so Jones can make an offer of $\delta_s M$ to Smith, retaining $1-\delta_s M$ for himself. At $t-2$, Smith knows that Jones will turn down any offer less than the discounted value of the minimum Jones can look forward to receiving at $t-1$. Smith, therefore, cannot offer any less than $\delta_j\,(1-\delta_s M)$ at $t-2$.

Now we have two expressions for "the maximum which Smith can receive," which we can set equal to each other:

$$M = 1 - \delta_j \left(1 - \delta_s M\right). \tag{9}$$

Solving equation (9) for M, we obtain

$$M = \frac{1-\delta_j}{1-\delta_s\delta_j}. \tag{10}$$

We can repeat the argument using m, the minimum of Smith's share. If Smith can expect at least m at t, Jones cannot receive more than $1-\delta_s m$ at $t-1$. At $t-2$ Smith knows that if he offers Jones the discounted value of that amount, Jones will accept, so Smith can guarantee himself $1-\delta_j\left(1-\delta_s m\right)$, which is the same as the expression we found for M. The smallest perfect equilibrium share that Smith can receive is the same as the largest, so the equilibrium outcome must be unique.

No Discounting, but a Fixed Bargaining Cost

There are two ways to model bargaining costs per period: as proportional to the remaining value of the pie (the way used above), or as fixed costs each period, which is analyzed next (again following Rubinstein [1982]). To understand the difference, think of labor negotiations during a construction project. If a strike slows down completion, there are two kinds of losses. One is the loss from delay in renting or selling the new building, a loss proportional to its value. The other is the loss from late-completion penalties in the contract, which often take the form of a fixed penalty each week. The two kinds of costs have very different effects on the bargaining process.

To represent this second kind of cost, assume that there is no discounting, but whenever Smith or Jones makes an offer, he incurs the cost c_s or c_j. In every subgame perfect equilibrium, Smith makes an offer and Jones accepts, but there are three possible cases.

Delay costs are equal

$$c_s = c_j = c.$$

The Nash indeterminacy of Section 12.1 remains almost as bad; any fraction such that each player gets at least c is supported by some perfect equilibrium.

Delay hurts Jones more

$$c_s < c_j.$$

Smith gets the entire pie. Jones has more to lose than Smith by delaying, and delay does not change the situation except by diminishing the wealth of the players. The game is stationary, because it looks the same to both players no matter how many periods have already elapsed. If in any period t Jones offered Smith x, in period $(t-1)$ Smith could offer Jones $(1-x-c_j)$, keeping $(x+c_j)$ for himself. In period $(t-2)$, Jones would offer Smith

$(x + c_j - c_s)$, keeping $(1 - x - c_j + c_s)$ for himself, and in periods $(t - 4)$ and $(t - 6)$ Jones would offer $(1 - x - 2c_j + 2c_s)$ and $(1 - x - 3c_j + 3c_s)$. As we work backwards, Smith's advantage rises to $\gamma(c_j - c_s)$ for an arbitrarily large integer γ. Looking ahead from the start of the game, Jones is willing to give up and accept zero.

Delay hurts Smith more

$$c_s > c_j.$$

Smith gets a share worth c_j and Jones gets $(1 - c_j)$. The cost c_j is a lower bound on the share of Smith, the first mover, because if Smith knows Jones will offer (0,1) in the second period, Smith can offer $(c_j, 1 - c_j)$ in the first period and Jones will accept.

12.5 Incomplete Information

Instant agreement has characterized even the multiperiod games of complete information discussed so far. Under incomplete information, knowledge can change over the course of the game and bargaining can last more than one period in equilibrium, a result that might be called inefficient but is certainly realistic. Models with complete information have difficulty explaining such things as strikes or wars, but if over time an uninformed player can learn the type of the informed player by observing what offers are made or rejected, such unfortunate outcomes can arise. The literature on bargaining under incomplete information is vast. For this section, I have chosen to use a model based on the first part of Fudenberg & Tirole (1983), but it is only a particular example of how one could construct such a model, and not a good indicator of what results are to be expected from bargaining.

Let us start with a one-period game. We will denote the price by p_1 because we will carry the notation over to a two-period version.

One-Period Bargaining with Incomplete Information

Players
A seller, and a buyer called Buyer_{100} or Buyer_{150} depending on his type.

The Order of Play
0 Nature picks the buyer's type, his valuation of the object being sold, which is $b = 100$ with probability γ and $b = 150$ with probability $(1 - \gamma)$.
1 The seller offers price p_1.
2 The buyer accepts or rejects p_1.

Payoffs
The seller's payoff is p_1 if the buyer accepts the offer, and otherwise 0.
The buyer's payoff is $b - p_1$ if he accepts the offer, and otherwise 0.

Equilibrium:
Buyer_{100}: accept if $p_1 \leq 100$.

Buyer$_{150}$: accept if $p_1 \leq 150$.
Seller: offer $p_1 = 100$ if $\gamma \geq 1/3$ and $p_1 = 150$ otherwise.

Both types of buyers have a dominant strategy for the last move: accept any offer $p_1 < b$. Accepting any offer $p_1 \leq b$ is a weakly best response to the seller's equilibrium strategy. No equilibrium exists in which a buyer rejects an offer of $p_1 = b$, because we would fall into the open-set problem: there would be no greatest offer in $[0, b)$ that the buyer would accept, and so we could not find a best response for the seller.

The only two strategies that make sense for the seller are $p_1 = 100$ and $p_1 = 150$, since prices lower than 100 would lead to a sale with the same probability as $p_1 = 100$, prices in $(100, 150]$ would have the same probability as $p_1 = 150$, and prices greater than 150 would yield zero profits. The seller will choose $p_1 = 150$ if it yields a higher payoff than $p_1 = 100$; that is, if

$$\pi(p_1 = 100) = \gamma(100) + (1-\gamma)(100) < \pi(p_1 = 150) = \gamma(0) + (1-\gamma)(150), \tag{11}$$

which requires that

$$\gamma < 1/3. \tag{12}$$

Thus, if less than a third of buyers have a valuation of 100, the seller will charge 150, gambling that he is not facing such a buyer.

This means, of course, that if $\gamma < 1/3$, sometimes no sale will be made. This is the most interesting feature of the model. By introducing incomplete information into a bargaining model, we have explained why bargaining sometimes breaks down and efficient trades fail to be carried out. This suggests that when wars occur because nations cannot agree, or strikes occur because unions and employers cannot agree, we should look to information asymmetry for an explanation.

Note also that this has some similarity to a mechanism design problem. It is crucial that only one offer can be made. Once the offer $p_1 = 150$ is made and rejected, the seller realizes that $b = 100$. At that point, he would like to make a second offer, of $p_1 = 100$. But of course if he could do that, then rejection of the first offer would not convey the information that $b = 100$.

Now let us move to a two-period version of the same game. This will get quite a bit more complex, so let us restrict ourselves to the case of $\gamma = 1/6$. Also, we will need to make an assumption on discounting— the loss that results from a delay in agreement. Let us assume that each player loses a fixed amount $D = 4$ if there is no agreement in the first period. (Notice that this means a player can end up with a negative payoff by playing this game, something experienced bargainers will find realistic.)

Two-Period Bargaining with Incomplete Information

Players

A seller, and a buyer called Buyer$_{100}$ or Buyer$_{150}$ depending on his type.

The Order of Play
0 Nature picks the buyer's type, his valuation of the object being sold, which is $b = 100$ with probability 1/6 and $b = 150$ with probability 5/6.
1 The seller offers price p_1.
2 The buyer accepts or rejects p_1.
3 The seller offers price p_2.
4 The buyer accepts or rejects p_2.

Payoffs
The seller's payoff is p_1 if the buyer accepts the first offer, $(p_2 - 4)$ if he accepts the second offer, and -4 if he accepts no offer.
The buyer's payoff is $(b - p_1)$ if he accepts the first offer, $(b - p_2 - 4)$ if he accepts the second offer, and -4 if he accepts no offer.

Equilibrium Behavior (Separating, in mixed strategies)
Buyer$_{100}$: Accept if $p_1 \leq 104$. Accept if $p_2 \leq 100$.
Buyer$_{150}$: Accept with probability 0.6 if $p_1 = 150$. Accept if $p_2 \leq 150$.
Seller: Offer $p_1 = 150$. If $p_1 = 150$ is rejected, offer $p_2 = 100$ with probability $\phi = 0.08$ and $p_2 = 150$ with probability 0.92.

If Buyer$_{100}$ deviates and rejects an offer of p_1 less than 104, his payoff will be -4, which is worse than $100 - p_1$. Rejecting p_2 does not result in any extra transactions cost, so he rejects any $p_2 < 100$.

Buyer$_{150}$'s equilibrium payoff is either

$$\pi_{Buyer\ 150}(Accept\ p_1 = 150) = b - 150 = 0, \tag{13}$$

or

$$\pi_{Buyer\ 150}(Reject\ p_1 = 150) = -4 + \phi(b - 100) + (1 - \phi)(b - 150) = -4 + \phi(b - 100), \tag{14}$$

which also equals zero if $\phi = 0.08$. Thus, Buyer$_{150}$ is indifferent and is willing to mix in the first period. In the second period, the game is just like the one-period game, so he will accept any offer of $p_2 \leq 150$.

To check on whether the seller has any incentive to deviate, let us work back from the end. If the game has reached the second period, he knows that the fraction of Buyer$_{100}$'s has increased, since there was some probability that a Buyer$_{150}$ would have accept $p_1 = 150$. The prior probability was $Prob(Buyer_{100}) = 1/6$, but the posterior is

$$\begin{aligned} Prob(Buyer_{100}|Rejected\ p_1 = 150) &= \frac{Prob(Rejected\ p_1{=}150|Buyer_{100})Prob(Buyer_{100})}{Prob(Rejected\ p_1{=}150)} \\ &= \frac{Prob(Rejected\ p_1{=}150|Buyer_{100})Prob(Buyer_{100})}{Prob(Rej|100)Prob(100)+Prob(Rej|150)Prob(150)} \\ &= \frac{(1)(1/6)}{(1)(1/6)+(0.4)(5/6)} \\ &= \frac{1}{3}. \end{aligned} \tag{15}$$

From the equilibrium of the one-period game, we know that if the probability of $b = 100$ is 1/3, the seller is indifferent between $p_2 = 100$ and $p_2 = 150$. Thus, he is willing to mix, and in particular to choose $p_2 = 100$ with probability .08.

How about in the first period? Well, I cheated a bit there in describing the equilibrium. I did not say what Buyer_{150} would do if the seller deviated to $p_1 \in (100, 150)$. What has to happen there is that if the seller deviates in that way, then some but not all of the Buyer_{150}'s accept the offer, and in the second period there is some probability of $p_2 = 100$ and some probability of $p_2 = 150$. The mixing probabilities, however, are not quite the same as the equilibrium mixing probabilities. Buyer_{150} is now comparing

$$\pi_{Buyer\ 150}(Accept\ p_1) = 150 - p_1 \tag{16}$$

with

$$\pi_{Buyer\ 150}(Reject\ p_1) = -4+\phi(p_1)(150-100)+(1-\phi(p_1))(150-150) = -4+50\phi(p_1), \tag{17}$$

which are equal if

$$\phi(p_1) = \frac{154 - p_1}{50}. \tag{18}$$

Thus, if p_1 is close to 100, $\phi(p_1)$ is close to 1, and the seller must almost certainly charge $p_2 = 100$.

The seller is willing to mix in the second period so long as 0.6 of the Buyer_{150}'s accept p_1, so that part of the strategy doesn't have to change. But if it doesn't, that means a deviation to $p_1 < 150$ won't change the seller's second-period payoff, or the probability that p_1 is accepted, so it will simply reduce his first-period revenues. Thus, neither player has incentive to deviate from the proposed equilibrium.[3] [4]

The most important lesson of this model is that bargaining can lead to inefficiency. Some of the Buyer_{150}s delay their transactions until the second period, which is inefficient since the payoffs are discounted. Moreover, there is at least some probability that the Buyer_{100}s never buy at all,as in the one-period game, and the potential gains from trade are lost.

Note, too, that this is a model in which prices fall over time as bargaining proceeds. The first period price is definitely $p_1 = 150$, but the second period price might fall to $p_1 = 100$. This can happen because high-valuation buyers know that though the price might fall if they wait, on the other hand it might not fall, and they will just incur an extra delay cost. This result has close parallels to the durable monopoly pricing problem that will be discussed in Chapter 14.

The price the buyer pays depends heavily on the seller's equilibrium beliefs. If the seller thinks that the buyer has a high valuation with probability 0.5, the price is 100, but

[3] xxx COMMITMENT EQUILIBRIUM . Suppose seller could commit to a second-period price. Would it affect anything? Yes. Mechanism design.

[4] xxx I thinkthere are other equilibira with first period prices of more than 150. They will not be very interesting equilibria though.

if he thinks the probability is 0.05, the price rises to 150. This implies that a buyer is unfortunate if he is part of a group which is believed to have high valuations more often; even if his own valuation is low, what we might call his bargaining power is low when he is part of a high-valuing group. Ayres (1991) found that when he hired testers to pose as customers at car dealerships, their success depended on their race and gender even though they were given identical predetermined bargaining strategies to follow. Since the testers did as badly even when faced with salesmen of their own race and gender, it seems likely that they were hurt by being members of groups that usually can be induced to pay higher prices.

*12.6 Setting up a Way to Bargain: The Myerson-Satterthwaite Mechanism

Let us now think about a different way to approach bargaining under incomplete information. This will not be a different methodology, for we will stay with noncooperative game theory, but now let us ask what would happen under different sets of formalized rules – different mechanisms.

Mechanisms were the topic of chapter 10, and bargaining is a good setting for considering them. We have seen in section 12.5 that under incomplete information it may easily happen that inefficiency arises in bargaining. This inefficiency surely varies depending on the rules of the game. Thus, if feasible, the players would like to bind themselves in advance to follow whichever rules are best at avoiding inefficiency.

Suppose a group of players in a game are interacting in some way. They would like to set up some rules for their interaction in advance, and this set of rules is what we call a mechanism. Usually, models analyze different mechanisms without asking how the players would agree upon them, taking that as exogenous to the model. This is reasonable – the mechanism may be assigned by history as an institution of the market. If it is not, then there is bargaining over which mechanism to use, a difficult extra layer of complexity.

Let us now consider the situation of two people trying to exchange a good under various mechanisms. The mechanism must do two things:

1 Tell under what circumstances the good should be transferred from seller to buyer; and

2 Tell the price at which the good should be transferred, if it is transferred at all.

Usually these two things are made to depend on **reports** of the two players – that is, on statements they make.

The first mechanisms we will look at are simple.

Bilateral Trading I: Complete Information

Players
A buyer and a seller.

The Order of Play
0 Nature independently chooses the seller to value the good at v_s and the buyer at v_b using the uniform distribution between 0 and 1. Both players observe these values.
1 The seller reports p_s.
2 The buyer chooses "Accept" or "Reject."
3 The good is allocated to the buyer if the buyer accepts and to the seller otherwise. The price at which the trade takes place, if it does, is $p = p_s$.

Payoffs
If there is no trade, both players have payoffs of 0. If there is trade, the seller's payoff is $p - v_s$ and the buyer's is $v_b - p$.

I have normalized the payoffs so that each player's payoff is zero if no trade occurs. I could instead have normalized to $\pi_s = v_s$ and to $\pi_b = 0$ if no trade occurred, a common alternative.

The unique subgame perfect Nash equilibrium of this game is for the seller to report $p_s = v_b$ and for the buyer to accept if $p_s \leq v_b$. Note that although it is Nash, it is not subgame perfect for the seller to charge $p_s = v_b$ and for the buyer to accept only if $p_s < v_b$, which would result in trade never occurring. The seller would deviate to offering a slightly higher price.

This is an efficient allocation mechanism, in the sense that the good ends up with the player who values it most highly. The problem is that the buyer would be unlikely to agree to this mechanism in the first place, before the game starts, because althought it is efficient it always gives all of the social surplus to the seller.

Note, too, that this is not a truth–telling mechanism, in the sense that the seller does not reveal his own value in his report of p_s. An example of a truth–telling mechanism for this game would replace move (3) with

(3′) The good is allocated to the seller if the buyer accepts and to the seller otherwise. The price at which the trade takes place, if it does, is $p = v_s + \frac{v_b - v_s}{2}$.

This new mechanism makes the game a bit silly. In it, the seller's action is irrelevant, since p_s does not affect the transaction price. Instead, the buyer decides whether or not trade is efficient and accepts if it is, at a price which splits the surplus evenly. In one equilibrium, the buyer accepts if $v_b \geq v_s$ and rejects otherwise, and the seller reports $p_s = v_s$. (In other equilibria, the buyer might choose to accept only if $v_b > v_s$, and the seller chooses other reports for p_s, but those other equilibria are the same as the first one in all important respects.) The mechanism would be acceptable to both players in advance, however, it gives no incentive to anyone to lie, and it makes sense if the game really has complete information.

Let us next look at a game of incomplete information and a mechanism which does depend on the players' actions.

Bilateral Trading II: Incomplete Information

Players
A buyer and a seller.

The Order of Play
0 Nature independently chooses the seller to value the good at v_s and the buyer at v_b using the uniform distribution between 0 and 1. Each player's value is his own private information.
1 The seller reports p_s and the buyer reports p_b.
2 The buyer accepts or rejects the seller's offer. The price at which the trade takes place, if it does, is p_s.

Payoffs
If there is no trade, the seller's payoff is 0 and the buyer's is 0.
If there is trade, the seller's payoff is $p_s - v_s$ and the buyer's is $v_b - p_s$.

This mechanism does not use the buyer's report at all, and so perhaps it is not surprising that the result is inefficient. It is easy to see, working back from the end of the game, that the buyer's equilibrium strategy is to accept the offer if $v_b \geq p_s$ and to reject it otherwise. If the buyer does that, the seller's expected payoff is

$$[p_s - v_s]\,[Prob\{v_b \geq p_s\}] + 0\,[Prob\{v_b \leq p_s\}] = [p_s - v_s]\,[1 - p_s]\,. \tag{19}$$

Differentiating this with respect to p_s and setting equal to zero yields the seller's equilibrium strategy of

$$p_s = \frac{1 + v_s}{2}. \tag{20}$$

This is not efficient because if v_b is just a little bigger than v_s, trade will not occur even though gains from trade do exist. In fact, trade will fail to occur whenever $v_b < \frac{1+v_s}{2}$.

Let us try another simple mechanism, which at least uses the reports of both players, replacing move (2) with (2′)

(2′) The good is allocated to the seller if $p_s > p_b$ and to the buyer otherwise. The price at which the trade takes place, if it does, is p_s.

Suppose the buyer truthfully reports $p_b = v_b$. What will the seller's best response be? The seller's expected payoff for the p_s he chooses is now

$$[p_s - v_s]\,[Prob\{p_b(v_b) \geq p_s\}] + 0\,[Prob\{p_b(v_b) \leq p_s\}] = [p_s - v_s][1 - p_s]. \tag{21}$$

where the expectation has to be taken over all the possible values of v_b, since p_b will vary with v_b.

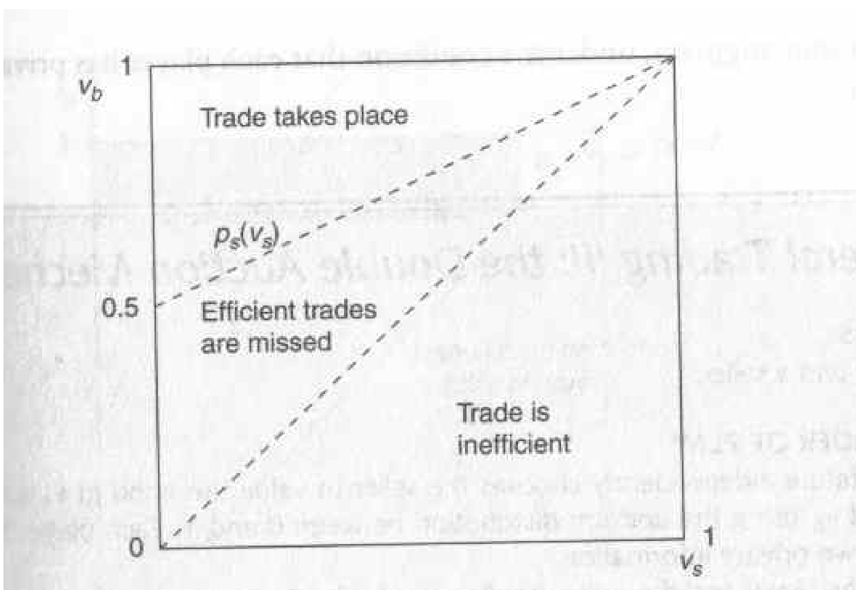
Maximizing this, the seller's strategy will solve the first-order condition $1-2p_s+v_s = 0$, and so will again be

$$p_s(v_s) = \frac{1+v_s}{2} = \frac{1}{2} + \frac{v_s}{2}. \tag{22}$$

Will the buyer's best response to this strategy be $p_b = v_b$? Yes, because whenever $v_b \geq \frac{1}{2} + \frac{v_s}{2}$ the buyer is willing for trade to occur, and the size of p_b does not affect the transactions price, only the occurrence or nonoccurrence of trade. The buyer needs to worry about causing trade to occur when $v_b < \frac{1}{2} + \frac{v_s}{2}$, but this can be avoided by using the truthtelling strategy. The buyer also needs to worry about preventing trade from occurring when $v_b > \frac{1}{2} + \frac{v_s}{2}$, but choosing $p_b = v_b$ prevents this from happening either.

Thus, it seems that either mechanism (2) or (2′) will fail to be efficient. Often, the seller will value the good less than the buyer, but trade will fail to occur and the seller will end up with the good anyway – whenever $v_b > \frac{1+v_s}{2}$. Figure 2 shows when trades will be completed based on the parameter values.

Figure 2: Trades in Bilateral Trading II



As you might imagine, one reason this is an inefficient mechanism is that it fails to make effective use of the buyer's information. The next mechanism will do better. Its trading rule is called the **double auction mechanism**. The problem is like that of the Groves Mechanism because we are trying to come up with an action rule (allocate the object to the buyer or to the seller) based on the agents' reports (the prices they suggest), under the condition that each player has private information (his value).

Bilateral Trading III: The Double Auction Mechanism

Players
A buyer and a seller.

The Order of Play
0 Nature independently chooses the seller to value the good at v_s and the buyer at v_b using the uniform distribution between 0 and 1. Each player's value is his own private information.
1 The buyer and the seller simultaneously decide whether to try to trade or not.
2 If both agree to try, the seller reports p_s and the buyer reports p_b simultaneously.
3 The good is allocated to the seller if $p_s \geq p_b$ and to the buyer otherwise. The price at which the trade takes place, if it does, is $p = \frac{(p_b+p_s)}{2}$.

Payoffs
If there is no trade, the seller's payoff is 0 and the buyer's is zero. If there is trade, then the seller's payoff is $p - v_s$ and the buyer's is $v_b - p$.

The buyer's expected payoff for the p_b he chooses is

$$\left[v_b - \frac{p_b + E[p_s|p_b \geq p_s]}{2}\right] [Prob\{p_b \geq p_s\}] , \tag{23}$$

where the expectation has to be taken over all the possible values of v_s, since p_s will vary with v_s.

The seller's expected payoff for the p_s he chooses is

$$\left[\frac{p_s + E(p_b|p_b \geq p_s)}{2} - v_s\right] [Prob\{p_b \geq p_s\}] , \tag{24}$$

where the expectation has to be taken over all the possible values of v_b, since p_b will vary with v_b.

The game has lots of Nash equilibria. Let's focus on two of them, a **one-price equilibrium** and the unique **linear equilibrium**.

In the **one-price equilibrium**, the buyer's strategy is to offer $p_b = x$ if $v_b \geq x$ and $p_b = 0$ otherwise, for some value $x \in [0, 1]$. The seller's strategy is to ask $p_s = x$ if $v_s \leq x$ and $p_s = 1$ otherwise. Figure 3 illustrates the one-price equilibrium for a particular value of x. Suppose $x = 0.7$. If the seller were to deviate and ask prices lower than 0.7, he would just reduce the price he receives. If the seller were to deviate and ask prices higher than 0.7, then $p_s > p_b$ and no trade occurs. So the seller will not deviate. Similar reasoning applies to the buyer, and to any value of x, including 0 and 1, where trade never occurs.

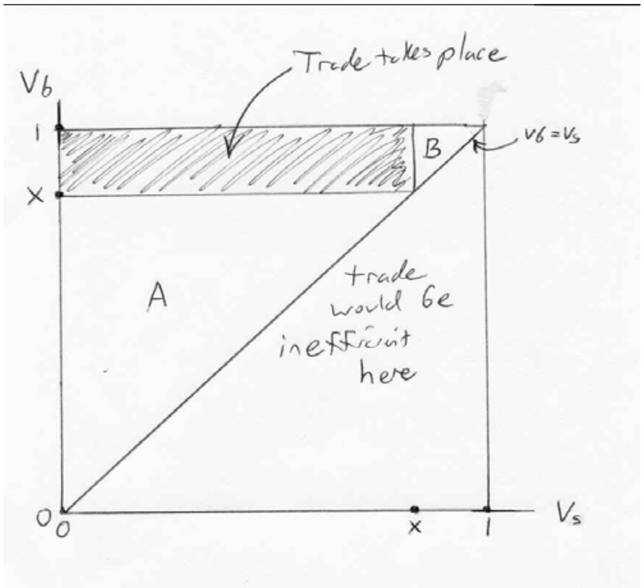


Figure 3: Trade in the one-price equilibrium

The **linear equilibrium** can be derived very neatly. Suppose the seller uses a linear strategy, so $p_s(v_s) = \alpha_s + c_s v_s$. Then from the buyer's point of view, p_s will be uniformly distributed from α_s to $\alpha_s + c_s$ with density $1/c_s$, as v_s ranges from 0 to 1. Since $E_b[p_s|p_b \geq p_s] = E_b(p_s|p_s \in [a_s, p_b]) = \frac{a_s+p_b}{2}$, the buyer's expected payoff (23) becomes

$$\left[v_b - \frac{p_b + \frac{\alpha_s + p_b}{2}}{2}\right]\left[\frac{p_b - \alpha_s}{c_s}\right]. \tag{25}$$

Maximizing with respect to p_b yields

$$p_b = \frac{2}{3}v_b + \frac{1}{3}\alpha_s. \tag{26}$$

Thus, if the seller uses a linear strategy, the buyer's best response is a linear strategy too! We are well on our way to a Nash equilibrium.

If the buyer uses a linear strategy $p_b(v_b) = \alpha_b + c_b v_b$, then from the seller's point of view p_b is uniformly distributed from α_b to $\alpha_b + c_b$ with density $1/c_b$ and the seller's payoff function, expression (24), becomes, since $E_s(p_b|p_b \geq p_s) = E_s(p_b|p_b \in [p_s, \alpha_b + c_b] = \frac{p_s+\alpha_b+c_b}{2}$,

$$\left[\frac{p_s + \frac{p_s+\alpha_b+c_b}{2}}{2} - v_s\right]\left[\frac{\alpha_b + c_b - p_s}{c_b}\right]. \tag{27}$$

Maximizing with respect to p_s yields

$$p_s = \frac{2}{3}v_s + \frac{1}{3}(\alpha_b + c_b). \tag{28}$$

Solving equations (26) and (28) together yields

$$p_b = \frac{2}{3}v_b + \frac{1}{12} \tag{29}$$

and

$$p_s = \frac{2}{3}v_s + \frac{1}{4}. \tag{30}$$

So we have derived a linear equilibrium. Manipulation of the equilibrium strategies shows that trade occurs if and only if $v_b \geq v_s + (1/4)$, which is to say, trade occurs if the valuations differ enough. The linear equilibrium does not make all efficient trades, because sometimes $v_b > v_s$ and no trade occurs, but it does make all trades with joint surpluses of 1/4 or more. Figure 4 illustrates this.

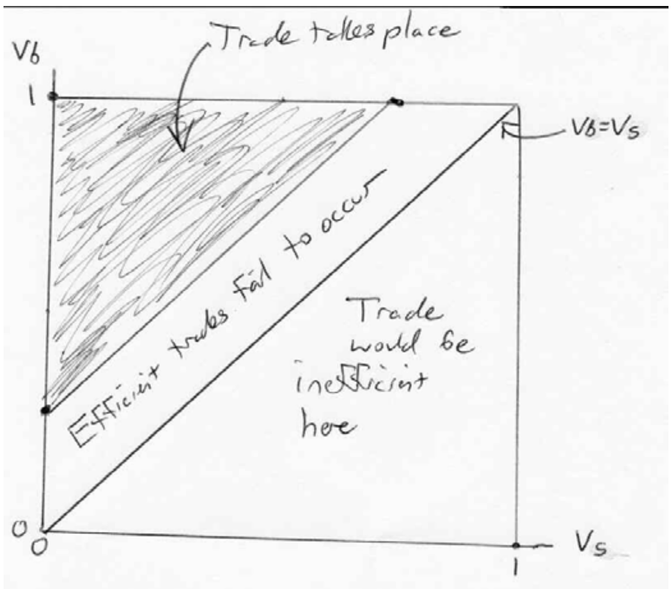


Figure 4: Trade in the linear equilibrium

One detail about equation (29) should bother you. The equation seems to say that if $v_b = 0$, the buyer chooses $p_b = 1/12$. If that happens, though, the buyer is bidding more than his value! The reason this can be part of the equilibrium is that it is only a weak Nash equilibrium. Since the seller never chooses lower than $p_s = 1/4$, the buyer is safe in choosing $p_b = 1/12$; trade never occurs anyway when he makes that choice. He could just as well bid 0 instead of 1/12, but then he wouldn't have a linear strategy.

The linear equilibrium is not a truth-telling equilibrium. The seller does not report his true value v_s, but rather reports $p_s = (2/3)v_s + 1/4$. But we could replicate the outcome in a truth-telling equilibrium. We could have the buyer and seller agree that they would make reports r_s and r_b to a neutral mediator, who would then choose the trading price p. He would agree in advance to choose the trading price p by (a) mapping r_s onto p_s just as in the equilibrium above, (b) mapping r_b onto p_b just as in the equilibrium above, and (c) using p_b and p_s to set the price just as in the double auction mechanism. Under this mechanism, both players would tell the truth to the mediator. Let us compare the original linear mechanism with a truth-telling mechanism.

The Chatterjee-Samuelson mechanism. *The good is allocated to the seller if $p_s \geq p_b$ and to the buyer otherwise. The price at which the trade takes place, if it does, is*

$p = \frac{(p_b+p_s)}{2}$

A direct incentive-compatible mechanism. *The good is allocated to the seller if* $\frac{2}{3}p_s + \frac{1}{4} \geq \frac{2}{3}p_b + \frac{1}{12}$, *which is to say, if* $p_s \geq p_b - 1/4$, *and to the buyer otherwise. The price at which the trade takes place, if it does, is*

$$p = \frac{(\frac{2}{3}p_b + \frac{1}{12}) + (\frac{2}{3}p_s + \frac{1}{4})}{2} = \frac{p_b + p_s}{3} + \frac{1}{6} \tag{31}$$

What I have done is substituted the equilibrium strategies of the two players into the mechanism itself, so now they will have no incentive to set their reports different from the truth. The mechanism itself looks odd, because it says that trade cannot occur unless v_b is more than 1/4 greater than v_s, but we cannot use the rule of trading if $v_b > v_s$ because then the players would start misreporting again. The truth-telling mechanism only works because it does not penalize players for telling the truth, and in order not to penalize them, it cannot make full use of the information to achieve efficiency.

In this game we have imposed a trading rule on the buyer and seller, rather than letting them decide for themselves what is the best trading rule. Myerson & Satterthwaite (1983) prove that of all the equilibria and all the mechanisms that are budget balancing, the linear equilibrium of the double auction mechanism yields the highest expected payoff to the players, the expectation being taken ex ante, before Nature has chosen the types. The mechanism is not optimal when viewed after the players have been assigned their types, and a player might not be happy with the mechanism once he knew his type. He will, however, at least be willing to participate.

What mechanism would players choose, ex ante, if they knew they would be in this game? If they had to choose after they were informed of their type, then their proposals for mechanisms could reveal information about their types, and we would have a model of bargaining under incomplete information that would resemble signalling models. But what if they chose a mechanism before they were informed of their type, and did not have the option to refuse to trade if after learning their type they did not want to use the mechanism?

In general, mechanisms have the following parts.
1 Each agent i simultaneously makes a report p_i.
2 A rule $x(p)$ determines the action (such as who gets the good, whether a bridge is built, etc.) based on the p.
3 Each agent i receives an incentive transfer a_i that in some way depends on his own report.
4 Each agent receives a budget-balancing transfer b_i that does not depend on his own report.

We will denote the agent's total transfer by t_i, so $t_i = a_i + b_i$.

In Bilateral Trading III, the mechanism had the following parts.
1 Each agent i simultaneously made a report p_i.
2 If $p_s \geq p_b$, the good was allocated to the seller, but otherwise to the buyer.
3 If there was no trade, then $a_s = a_b = 0$. If there was trade, then $a_s = \frac{(p_b+p_s)}{2}$ and $a_b = -(\frac{(p_b+p_s)}{2})$.
4 No further transfer b_i was needed, because the incentive transfers balanced the budget by themselves.

It turns out that if the players in Bilateral Trading can settle their mechanism and agree to try to trade in advance of learning their types, an efficient budget-balancing mechanism exists that can be implemented as a Nash equilibrium. The catch will be that after discovering his type, a player will sometimes regret having entered into this mechanism.

This would actually be part of a subgame perfect Nash equilibrium of the game as a whole. The mechanism design literature tends not to look at the entire game, and asks "Is there a mechanism which is efficient when played out as the rules of a game?" rather than "Would the players choose a mechanism that is efficient?"

Bilateral Trading IV: The Expected Externality Mechanism

Players
A buyer and a seller.

The Order of Play
-1 Buyer and seller agree on a mechanism $(x(p), t(p))$ that makes decisions x based on reports p and pays t to the agents, where p and t are 2-vectors and x allocated the good either to the buyer or the seller.
0 Nature independently chooses the seller to value the good at v_s and the buyer at v_b using the uniform distribution between 0 and 1. Each player's value is his own private information.
1 The seller reports p_s and the buyer reports p_b simultaneously.
2 The mechanism uses $x(p)$ to decide who gets the good, and $t(p)$ to make payments.

Payoffs
Player i's payoff is $v_i + t_i$ if he is allocated the good, t_i otherwise.

I was vague on how the two parties agree on a mechanism. The mechanism design literature is also very vague, and focuses on efficiency rather than payoff-maximization. To be more rigorous, we should have one player propose the mechanism and the other accept or reject. The proposing player would add an extra transfer to the mechanism to reduce the other player's expected payoff to his reservation utility.

Let me use the term **action surplus** to denote the utility an agent gets from the choice of action.

The **expected externality mechanism** has the following objectives for each of the parts of the mechanism.
1 Induce the agents to make truthful reports.
2 Choose the efficient action.
3 Choose the incentive transfers to make the agents choose truthful reports in equilibrium.
4 Choose the budget-balancing transfers so that the incentive transfers add up to zero.

First I will show you a mechanism that does this. Then I will show you how I came up with that mechanism. Consider the following three- part mechanism:

1 The seller announces p_s. The buyer announces p_b. The good is allocated to the seller if $p_s \geq p_b$, and to the buyer otherwise.

2 The seller gets transfer $t_s = \frac{(1-p_s^2)}{2} - \frac{(1-p_b^2)}{2}$.

3 The buyer gets transfer $t_b = \frac{(1-p_b^2)}{2} - \frac{(1-p_s^2)}{2}$.

Note that this is budget-balancing:

$$\frac{(1-p_s^2)}{2} - \frac{(1-p_b^2)}{2} + \frac{(1-p_b^2)}{2} - \frac{(1-p_s^2)}{2} = 0. \tag{32}$$

The seller's expected payoff as a function of his report p_s is the sum of his expected action surplus and his expected transfer. We have already computed his transfer, which is not conditional on the action taken.

The seller's action surplus is 0 if the good is allocated to the buyer, which happens if $v_b > p_s$, where we use v_b instead of p_b because in equilibrium $p_b = v_b$. This has probability $1 - p_s$. The seller's action surplus is v_s if the good is allocated to the seller, which has probability p_s. Thus, the expected action surplus is $p_s v_s$.

The seller's expected payoff is therefore

$$p_s v_s + \frac{(1-p_s^2)}{2} - \frac{(1-p_b^2)}{2}. \tag{33}$$

Maximizing with respect to his report, p_s, the first order condition is

$$v_s - p_s = 0, \tag{34}$$

so the mechanism is incentive compatible – the seller tells the truth.

The buyer's expected action surplus is v_b if his report is higher, e.g. if $p_b > v_s$, and zero otherwise, so his expected payoff is

$$p_b v_b + \frac{(1-p_b^2)}{2} - \frac{(1-p_s^2)}{2} \tag{35}$$

Maximizing with respect to his report, p_s, the first order condition is

$$v_b - p_b = 0, \tag{36}$$

so the mechanism is incentive compatible – the buyer tells the truth.

Now let's see how to come up with the transfers. The expected externality mechanism relies on two ideas.

The first idea is that to get the incentives right, each agent's incentive transfer is made equal to the sum of the expected action surpluses of the other agents, where the expectation is calculated conditionally on (a) the other agents reporting truthfully, and (b) our agent's report. This makes the agent internalize the effect of his externalities on the other agents. His expected payoff comes to equal the expected social surplus. Here, this means, for example, that the seller's incentive transfer will equal the buyer's expected action surplus. Thus, denoting the uniform distribution by F,

$$\begin{aligned} a_s &= \int_0^{p_s} 0 dF(v_b) + \int_{p_s}^1 v_b dF(v_b) \\ &= 0 + \frac{v_b^2}{2}\Big|_{p_s}^1 \\ &= \frac{1}{2} - \frac{p_s^2}{2}. \end{aligned} \tag{37}$$

The first integral is the expected buyer action surplus if no transfer is made because the buyer's value v_b is less than the seller's report p_s, so the seller keeps the good and the buyer's action surplus is zero. The second integral is the surplus if the buyer gets the good, which occurs whenever the buyer's value, v_b (and hence his report p_b), is greater than the seller's report, p_s.

We can do the same thing for the buyer's incentive, finding the seller's expected surplus.

$$\begin{aligned} a_b &= \int_0^{p_b} 0 dF(v_s) + \int_{p_b}^1 v_s dF(v_s) \\ &= 0 + \frac{v_s^2}{2}\Big|_{p_b}^1 \\ &= \frac{1}{2} - \frac{p_b^2}{2}. \end{aligned} \tag{38}$$

If the seller's value v_s is low, then it is likely that the buyer's report of p_b is higher than v_s, and the seller's action surplus is zero because the trade will take place. If the seller's value v_s is high, then the seller will probably have a positive action surplus.

The second idea is that to get budget balancing, each agent's budget- balancing transfer is chosen to help pay for the other agents' incentive transfers. Here, we just have two agents, so the seller's budget-balancing transfer has to pay for the buyer's incentive transfer. That is very simple: just set the seller's budget-balancing transfer b_s equal to the buyer's incentive transfer a_b (and likewise set b_b equal to a_s).

The intuition and mechanism can be extended to N agents. There are now N reports $p_1, ...p_N$. Let the action chosen be $x(p)$, where p is the N-vector of reports, and the action surplus of agent i is $W_i(x(p), v_i)$. To make each agent's incentive transfer equal to the sum of the expected action surpluses of the other agents, choose it so

$$a_i = E\left(\Sigma_{j \neq i} W_j(x(p), v_j)\right). \tag{39}$$

The budget balancing transfers can be chosen so that each agent's incentive transfer is paid for by dividing the cost equally among the other $N-1$ agents:

$$b_i = \frac{1}{N-1}\left(\Sigma_{j \neq i} E\left(\Sigma_{k \neq j} W_k(x(p), v_k)\right)\right). \tag{40}$$

There are other ways to divide the costs that will still allow the mechanism to be incentive compatible, but equal division is easiest to think about.

The expected externality mechanism does have one problem: the participation constraint. If the seller knows that $v_s = 1$, he will not want to enter into this mechanism. His expected transfer would be $t_s = 0 - (1 - 0.5)^2/2 = -0.125$. Thus, his payoff from the mechanism is $1 - 0.125 = 0.875$, whereas he could get a payoff of 1 if he refused to participate. We say that this mechanism fails to be **interim incentive compatible**, because at the point when the agents discover their own types, but not those of the other agents, the agents might not want to participate in the mechanism or choose the actions we desire.

Notes

N12.1 The Nash bargaining solution

- See Binmore, Rubinstein, & Wolinsky (1986) for a comparison of the cooperative and noncooperative approaches to bargaining. For overviews of cooperative game theory see Luce & Raiffa (1957) and Shubik (1982).

- While the Nash bargaining solution can be generalized to n players (see Harsanyi [1977], p. 196), the possibility of interaction between coalitions of players introduces new complexities. Solutions such as the Shapley value (Shapley [1953b] try to account for these complexities.

  The **Shapley value** satisfies the properties of invariance, anonymity, efficiency, and linearity in the variables from which it is calculated. Let S_i denote a **coalition** containing player i; that is, a group of players including i that makes a sharing agreement. Let $v(S_i)$ denote the sum of the utilities of the players in coalition S_i, and $v(S_i - \{i\})$ denote the sum of the utilities in the coalition created by removing i from S_i. Finally, let $c(s)$ be the number of coalitions of size s containing player i. The Shapley value for player i is then

$$\phi_i = \frac{1}{n} \sum_{s=1}^{n} \frac{1}{c(s)} \sum_{S_i} \left[v(S_i) - v(S_i - \{i\}) \right]. \tag{41}$$

  where the S_i are of size s. The motivation for the Shapley value is that player i receives the average of his marginal contributions to different coalitions that might form. Gul (1989) has provided a noncooperative interpretation.

N12.2 Alternating offers over infinite time

- The proof of Proposition 12.1 is not from the original Rubinstein (1982), but is adapted from Shaked & Sutton (1984). The maximum rather than the supremum can be used because of the assumption that indifferent players always accept offers.

- In extending alternating offers to three players, there is no obviously best way of specifiying how players make and accept offers. Haller (1986) shows that for at least one specification, the outcome is not similar to the Rubinstein (1982) outcome, but rather is a return to the indeterminacy of the game without discounting.

N12.3 Incomplete information.

- Bargaining under asymmetric information has inspired a large literature. In early articles, Fudenberg & Tirole (1983) uses a two-period model with two types of buyers and two types of sellers. Sobel & Takahashi (1983) builds a model with either T or infinite periods, a continuum of types of buyers, and one type of seller. Crampton (1984) uses an infinite number of periods, a continuum of types of buyers, and a continuum of types of sellers. Rubinstein (1985a) uses an infinite number of periods, two types of buyers, and one type of seller, but the types of buyers differ not in their valuations, but in their discount rates. Rubinstein (1985b) puts emphasis on the choice of out-of-equilibrium conjectures. Samuelson (1984) looks at the case where one bargainer knows the size of the pie better than the other bargainer. Perry (1986) uses a model with fixed bargaining costs and asymmetric information in which each bargainer makes an offer in turn, rather than one offering and the other accepting or rejecting. For overviews, see excellent surveys of Sutton (1986) and Kennan & Wilson (1993).

- The asymmetric information model in Section 12.5 has **one-sided** asymmetry in the information: only the buyer's type is private information. Fudenberg & Tirole (1983) and others have also built models with **two-sided** asymmetry, in which buyers' and sellers' types are both private information. In such models a multiplicity of perfect Bayesian equilibria can be supported for a given set of parameter values. Out-of-equilibrium beliefs become quite important, and provided much of the motivation for the exotic refinements mentioned in Section 6.2.

- There is no separating equilibrium if, instead of discounting, the asymmetric information model has fixed-size per-period bargaining costs, unless the bargaining cost is higher for the high-valuation buyer than for the low-valuation. If, for example, there is no discounting, but a cost of c is incurred each period that bargaining continues, no separating equilibrium is possible. That is the typical signalling result. In a separating equilibrium the buyer tries to signal a low valuation by holding out, which fails unless it really is less costly for a low-valuation buyer to hold out. See Perry (1986) for a model with fixed bargaining costs which ends after one round of bargaining.

N12.4 Setting up a way to bargain: the Myerson-Satterthwaite mechanism

- The Bilateral Trading model originated in Chatterjee & Samuelson (1983, p. 842), who also analyze the more general mechanism with $p = \theta p_s + (1-\theta)p_b$. I have adapted this description from Gibbons (1992, p.158).

- Discussions of the general case can be found in Fudenberg & Tirole (1991a, p. 273), and Mas-Colell, Whinston & Green (1994, p. 885). It is also possible to add extra costs that depend on the action chosen (for example, a transactions tax if the good is sold from buyer to seller). See Fudenberg and Tirole, p. 274. I have taken the term "expected externality mechanism" from MWG. Fudenberg and Tirole use "AGV mechanism" for the same thing, because the idea was first published in Arrow (1979) and D'Aspremont & Varet (1979). Myerson (1991) is also worth looking into.

Problems

12.1. A Fixed Cost of Bargaining and Incomplete Information

Smith and Jones are trying to split 100 dollars. In bargaining round 1, Smith makes an offer at cost c, proposing to keep S_1 for himself. Jones either accepts (ending the game) or rejects. In round 2, Jones makes an offer of S_2 for Smith, at cost 10, and Smith either accepts or rejects. In round 3, Smith makes an offer of S_3 at cost c, and Jones either accepts or rejects. If no offer is ever accepted, the 100 dollars goes to a third player, Parker.

(a) If $c = 0$, what is the equilibrium outcome?

(b) If $c = 80$, what is the equilibrium outcome?

(c) If Jones' priors are that $c = 0$ and $c = 80$ are equally likely, but only Smith knows the true value, what are the players' equilibrium strategies in rounds 2 and 3? (that is: what are S_2 and S_3, and what acceptance rules will each player use?)

(d) If Jones' priors are that $c = 0$ and $c = 80$ are equally likely, but only Smith knows the true value, what are the equilibrium strategies for round 1? (Hint: the equilibrium uses mixed strategies.)

12.2: Selling Cars

A car dealer must pay $10,000 to the manufacturer for each car he adds to his inventory. He faces three buyers. From the point of view of the dealer, Smith's valuation is uniformly distributed between $12,000 and $21,000, Jones's is between $9,000 and $12,000, and Brown's is between $4,000 and $12,000. The dealer's policy is to make a single take-it-or-leave-it offer to each customer, and he is smart enough to avoid making differerent offers to customers who could resell to each other. Use the notation that the maximum valuation is $\overline{V}$ and the range of valuations is R.

(a)] What will the offers be?

(b) Who is most likely to buy a car? How does this compare with the outcome with perfect price discrimination under full information? How does it compare with the outcome when the dealer charges $10,000 to each customer?

(c) What happens to the equilibrium prices if, with probability 0.25, each buyer has a valuation of $0, but the probability distribution remains otherwise the same?

12.3. The Nash Bargaining Solution

Smith and Jones, shipwrecked on a desert island, are trying to split 100 pounds of cornmeal and 100 pints of molasses, their only supplies. Smith's utility function is $U_s = C + 0.5M$ and Jones's is $U_j = 3.5C + 3.5M$. If they cannot agree, they fight to the death, with $U = 0$ for the loser. Jones wins with probability 0.8.

(a) What is the threat point?

(b) With a 50-50 split of the supplies, what are the utilities if the two players do not recontract? Is this efficient?

(c) Draw the threat point and the Pareto frontier in utility space (put U_s on the horizontal axis).

(d) According to the Nash bargaining solution, what are the utilities? How are the goods split?

(e) Suppose Smith discovers a cookbook full of recipes for a variety of molasses candies and corn muffins, and his utility function becomes $U_s = 10C + 5M$. Show that the split of goods in part (d) remains the same despite his improved utility function.

12.4:. Price Discrimination and Bargaining

A seller with marginal cost constant at c faces a continuum of consumers represented by the linear demand curve $Q^d = a - bP$, where $a > c$. Demand is at a rate of one or zero units per consumer, so if all consumers between points 1 and 2.5 on the consumer continuum make purchases at a price of 13, we say that a total of 1.5 units are sold at a price of 13 each.

(a) What is the seller's profit if he chooses one take-it-or-leave- it price?

Answer. This is the simple monopoly pricing problem. Profit is

$$\pi = Q(P - C) = Q(a/b - Q/b - c).$$

Differentiating with respect to Q yields

$$\frac{d\pi}{dQ} = a/b - 2Q/b - c = 0,$$

which can be solved to give us

$$Q_m = \frac{a - bc}{2}.$$

The price is then, using the demand curve,

$$P_m = \frac{a/b + c}{2},$$

which is to say that the price will be halfway between marginal cost and price which drives demand to zero. Profit is

$$\pi_m = \left(\frac{a/b - c}{2}\right)\left(\frac{a - cb}{2}\right).$$

(b) What is the seller's profit if he chooses a continuum of take-it-or-leave-it prices at which to sell, one price for each consumer? (You should think here of a pricing function, since each consumer is infinitesimal).

Answer. Under perfect price discrimination, the seller captures the entire area under the demand curve and over the marginal cost curve, because he charges each consumer exactly the reservation price. Since the price at which quantity demanded falls to zero is a/b and the quantity when price equals marginal cost is $a - bc$, the area of this profit triangle is

$$\pi_{ppd} = (1/2)(a/b - c)(a - bc)$$

Note that this is exactly twice the monopoly profit found earlier.

(c) What is the seller's profit if he bargains separately with each consumer, resulting in a continuum of prices? You may assume that bargaining costs are zero and that buyer and seller have equal bargaining power.

Answer. In this case, which I call "isoperfect price discrimination," profits are exactly half of what they are under perfect price discrimination, since the price charged to a consumer will exactly split the surplus he would have if the price equalled marginal cost. Thus, the profit is the same as using the simple monopoly price.

12.5. A Fixed Cost of Bargaining and Incomplete Information

Smith and Jones are trying to split 100 dollars. In bargaining round 1, Smith makes an offer at cost c, proposing to keep S_1 for himself. Jones either accepts (ending the game) or rejects. In round 2, Jones makes an offer of S_2 for Smith, at cost 10, and Smith either accepts or rejects. In round 3, Smith makes an offer of S_3 at cost c, and Jones either accepts or rejects. If no offer is ever accepted, the 100 dollars goes to a third player, Parker.

(a) If $c = 0$, what is the equilibrium outcome?

(b) If $c = 80$, what is the equilibrium outcome?

(c) If Jones' priors are that $c = 0$ and $c = 80$ are equally likely, but only Smith knows the true value, what is the equilibrium outcome? (Hint: the equilibrium uses mixed strategies.)

12.6. A Fixed Bargaining Cost, Again

Apex and Brydox are entering into a joint venture that will yield 500 million dollars, but they must negotiate the split first. In bargaining round 1, Apex makes an offer at cost 0, proposing to keep A_1 for itself. Brydox either accepts (ending the game) or rejects. In Round 2, Brydox makes an offer at cost 10 million of A_2 for Apex, and Apex either accepts or rejects. In Round 3, Apex makes an offer of A_3 at cost c, and Brydox either accepts or rejects. If no offer is ever accepted, the joint venture is cancelled.

(a) If $c = 0$, what is the equilibrium? What is the equilibrium outcome?

(b) If $c = 10$, what is the equilibrium? What is the equilibrium outcome?

(c) If $c = 300$, what is the equilibrium? What is the equilibrium outcome?

12.7. Myerson-Satterthwaite

The owner of a tract of land values his land at v_s and a potential buyer values it at v_b. The buyer and seller do not know each other's valuations, but guess that they are uniformly distributed between 0 and 1. The seller and buyer suggest p_s and p_b simultaneously, and they have agreed that the land will be sold to the buyer at price $p = \frac{(p_b+p_s)}{2}$ if $p_s \leq p_b$.

The actual valuations are $v_s = 0.2$ and $v_b = 0.8$. What is one equilibrium outcome given these valuations and this bargaining procedure? Explain why this can happen.

12.8. Negotiation (Rasmusen [2002])

Two parties, the Offeror and the Acceptor, are trying to agree to the clauses in a contract. They have already agreed to a basic contract, splitting a surplus 50- 50, for a surplus of Z for each player. The offeror can at cost C offer an additional clause which the acceptor can accept outright, inspect carefully (at cost M), or reject outright. The additional clause is either "genuine," yielding the Offeror X_g and the Acceptor Y_g if accepted, or "misleading," yielding the Offeror X_m (where $X_m > X_g > 0$) and the Acceptor $-Y_m < 0$.

What will happen in equilibrium?

January 17, 2000. February 26, 2003. November 10, 2004. 25 March 2005.
Eric Rasmusen, Erasmuse@indiana.edu, Http://www.rasmusen.org. [1]

13 Auctions

13.1 Auction Classification

Because auctions are stylized markets with well-defined rules, modelling them with game theory is particularly appropriate. Moreover, several of the motivations behind auctions are similar to the motivations behind the asymmetric information contracts of Part II of this book. Besides the mundane reasons such as speed of sale that make auctions important, auctions are useful for a variety of informational purposes. Often the buyers know more than the seller about the value of what is being sold, and the seller, not wanting to suggest a price first, uses an auction as a way to extract information. Art auctions are a good example, because the value of a painting depends on the buyer's tastes, which are known only to himself.

Auctions are also useful for agency reasons, because they hinder dishonest dealing between the seller's agent and the buyer. If the mayor were free to offer a price for building the new city hall and accept the first contractor who showed up, the lucky contractor would probably be the one who made the biggest political contribution. If the contract is put up for auction, cheating the public is more costly, and the difficulty of rigging the bids may outweigh the political gain.

We will spend most of this chapter on the effectiveness of different kinds of auction rules in extracting surplus from buyers, which requires considering the strategies with which they respond to the rules. Section 13.1 classifies auctions based on the relationships between different buyers' valuations of what is being auctioned, and explains the possible auction rules and the bidding strategies optimal for each rule. Section 13.2 compares the outcomes under the various rules. Section 13.3 looks at what happens when bidders do not know the value of the object to themselves, but that value is private, uncorrelated with the value to anyone else. Section 13.4 discusses optimal strategies under common-value information, which can lead bidders into the "winner's curse" if they are not careful. Section 13.5 is about information asymmetry in common-value auctions.

Private-Value and Common-Value Auctions

Auctions differ enough for an intricate classification to be useful. One way to classify auctions is based on differences in the values buyers put on what is being auctioned. We will call the dollar value of the utility that player i receives from an object its value to him, V_i, and we will call his *estimate* of its value his **valuation**, $\hat{V}_i$.

In a **private-value** auction, each player's valuation is independent of those of the other players. An example is the sale of antique chairs to people who will not resell them. Usually a player's value equals his valuation in a private-value auction.

[1] xxx Footnotes starting with xxx are the author's notes to himself. Comments are welcomed.

If an auction is to be private-value, it cannot be followed by costless resale of the object. If there were resale, a bidder's valuation would depend on the price at which he could resell, which would depend on the other players' valuations.

What is special about a private-value auction is that a player cannot extract any information about his own value from the valuations of the other players. Knowing all the other bids in advance would not change his valuation, although it might well change his bidding strategy. The outcomes would be similar even if he had to estimate his own value, so long as the behavior of other players did not help him to estimate it, so this kind of auction could just as well be called "private-valuation auction."

In a **common-value** auction, the players have identical values, but each player forms his own valuation by estimating on the basis of his private information. An example is bidding for US Treasury bills. A player's valuation would change if he could sneak a look at the other players' valuations, because they are all trying to estimate the same true value.

The values in most real-world auctions are a combination of private- value and common-value, because the valuations of the different players are correlated but not identical. This is sometimes called the **affiliated values** case. As always in modelling, we trade off descriptive accuracy against simplicity. It is common for economists to speak of mixed auctions as "common-value" auctions, since their properties are closer to those of common-value auctions.

Auction Rules and Private-Value Strategies

Auctions have as many different sets of rules as poker games do. We will begin with four different sets of auction rules, and in the private-value setting, since it is simplest. In teaching this material, I ask each student to pick a valuation between 80 and 100, after which we conduct the various kinds of auctions. I advise the reader to try this. Pick two valuations and try out sample strategy combinations for the different auctions as they are described. Even though the values are private, it will immediately become clear that the best-response bids still depend on the strategies the bidder thinks the other players have adopted.

The types of auctions to be described are:

1 English (first price open cry);
2 First price sealed bid;
3 Second price sealed bid (Vickrey);
4 Dutch (descending).

English (first price open cry)

Rules
Each bidder is free to revise his bid upwards. When no bidder wishes to revise his bid further, the highest bidder wins the object and pays his bid.

Strategies
A player's strategy is his series of bids as a function of (1) his value, (2) his prior estimate of other players' valuations, and (3) the past bids of all the players. His bid can therefore be updated as his information set changes.

Payoffs
The winner's payoff is his value minus his highest bid. The losers' payoffs are zero.

A player's dominant strategy in a private-value English auction is to keep bidding some small amount ϵ more than the previous high bid until he reaches his valuation, and then to stop. This is optimal because he always wants to buy the object if the price is less than its value to him, but he wants to pay the lowest price possible. All bidding ends when the price reaches the valuation of the player with the second-highest valuation. The optimal strategy is independent of risk neutrality if players know their own values with certainty rather than having to estimate them, although risk- averse players who must estimate their values should be more conservative in bidding.

In common-value open-cry auctions, the bidding procedure is important. Possible procedures include (1) the auctioneer to raise prices at a constant rate, (2) the bidders to raise prices as specified in the rules above, and (3) the **open-exit** auction, in which the price rises continuously and players must publicly announce that they are dropping out (and cannot re-enter) when the price becomes unacceptably high. In an open-exit auction the players have more evidence available about each others' valuations than when they can drop out secretly.

First-Price Sealed-Bid

Rules
Each bidder submits one bid, in ignorance of the other bids. The highest bidder pays his bid and wins the object.

Strategies
A player's strategy is his bid as a function of his value and his prior beliefs about other players' valuations.

Payoffs
The winner's payoff is his value minus his bid. The losers' payoffs are zero.

Suppose Smith's value is 100. If he bid 100 and won when the second bid was 80, he would wish that he had bid only less. If it is common knowledge that the second-highest value is 80, Smith's bid should be $80 + \epsilon$. If he is not sure about the second-highest value, the problem is difficult and no general solution has been discovered. The tradeoff is between bidding high– thus winning more often– and bidding low– thus benefiting more if the bid wins. The optimal strategy, whatever it may be, depends on risk neutrality and beliefs about the other bidders, so the equilibrium is less robust than the equilibria of English and second-price auctions.

Nash equilibria can be found for more specific first-price auctions. Suppose there are N risk-neutral bidders independently assigned values by Nature using a uniform density from 0 to some amount $\bar{v}$. Denote player i's value by v_i, and let us consider the strategy for player 1. If some other player has a higher value, then in a symmetric equilibrium, player 1 is going to lose the auction anyway, so he can ignore that possibility in finding his optimal bid. Player 1's equilibrium strategy is to bid ϵ above his expectation of the second-highest value, conditional on his bid being the highest (i.e., assuming that no other bidder has a value over v_1).

If we assume that v_1 is the highest value, the probability that Player 2's value, which is uniformly distributed between 0 and v_1, equals v is $1/v_1$, and the probability that v_2 is less than or equal to v is v/v_1. The probability that v_2 equals v and is the second-highest value is

$$Prob(v_2 = v) \cdot Prob(v_3 \leq v) \cdot Prob(v_4 \leq v) \cdots Prob(v_N \leq v), \tag{1}$$

which equals

$$\left(\frac{1}{v_1}\right)\left(\frac{v}{v_1}\right)^{N-2}. \tag{2}$$

Since there are $N-1$ players besides player 1, the probability that one of them has the value v, and v is the second-highest is $N-1$ times expression (2). Let us define the value of the second-highest valuer to be $v_{(2)}$) (as distinct from "v_2," the value of the second bidder). The expectation of $v_{(2)}$) is the integral of v over the range 0 to v_1,

$$\begin{aligned} Ev_{(2)} &= \int_0^{v_1} v(N-1)(1/v_1)[v/v_1]^{N-2}dv \\ &= (N-1)\frac{1}{v_1^{N-1}}\int_0^{v_1} v^{N-1}dv \\ &= \tfrac{(N-1)v_1}{N}. \end{aligned} \tag{3}$$

Thus we find that player 1 ought to bid a fraction $\frac{N-1}{N}$ of his own value, plus ϵ. If there are 2 bidders, he should bid $\frac{1}{2}v_1$, but if there are 10 he should bid $\frac{9}{10}v_1$.

A Mixed-Strategy Equilibrium in a First-Price Auction. The previous example is an elegant result but not a general rule. Often the equilibrium in a first-price auction is not even in pure strategies. Consider an auction in which each bidder's private value v is either 10 or 16 with equal probability and is known only to himself.

If the auction is first-price sealed bid, a bidder's optimal strategy is to set bid $b(v = 10) = 10$ and if $v = 16$ to use a mixed strategy, mixing over the support $[x, y]$, where it will turn out that $x = 10$ and $y = 13$, and his expected payoff will be

$$
\begin{aligned}
E\pi(v = 10) &= 0 \\
E\pi(v = 16) &= 3.
\end{aligned}
\tag{4}
$$

This will serve as an illustration of how to find an equilibrium mixed strategy when players mix over a continuum of pure strategies rather than just between two of them. The first step is to see why the equilibrium cannot be in pure strategies (though it can include pure strategies following some moves by Nature). Suppose both players are using the strategy $b(v = 10) = 10$, and $b_1(v_1 = 16) = z_1$ and $b_2(v_2 = 16) = z_2$, where z_1 and z_2 are in $(10, 16]$ because a bid of $10 + \epsilon$ will always defeat the $b(v = 10) = 10$ where a bid of 10 or less would not, and a bid of over 16 would yield a negative payoff. Either $z_1 = z_2$, or $z_1 \neq z_2$. If $z_1 = z_2$, then each player has incentive to deviate to $z_1 - \epsilon$ and win always instead of tying. If $z_1 < z_2$, then Player 2 will deviate to bid just $z_1 + \epsilon$. If he does that, hower, Player 1 would have incentive to deviate to bid $z_1 + 2\epsilon$, so he could win always at trivially higher cost. The same holds true if $z_2 < z_1$. Thus, there is no equilibrium in pure strategies.

The second step is to figure out what pure strategies will be mixed between by a player with $v = 16$. The bid, b, will not be less than 10 (which would always lose) or greater than 16 (which would always win and yield a negative payoff). In fact, since the pure strategy of $b = 10 + \epsilon | v = 16$ will win with probability of at least 0.50 (because the other player happens to have $v = 10$, thus yielding a payoff of at least $0.50(16 - 10) = 3$, the upper bound y must be no greater than 13.

Consider the following two possible features of an equilibrium mixing distribution. Suppose that Bidder 2's density has one or both of these features.

(a) The mixing density has an atom at point a in $[10, 13]$ – some particular point which has positive probability that we will denote by $T(a)$.

(b) The mixing density has a gap $[g, h]$ somewhere in $[10, 13]$– that is, there is zero probability of a bid in $[g, h]$.

(refutation of a) If Player 2 puts positive probability on point $a > 10$, then Player 1 should respond by putting positive probability on point $a - \epsilon$.xxx unfinished.

(refutation of b) xxx unfinishedSuppose $x > 10$ instead of $x = 10$. The bidder's expected payoff is, if the other player has no atom at x in his strategy,

$$0.5(16-x) + 0.5Prob.(win\ with\ x|v_2 = 16)(16-x) = 0.5(16-x). \tag{5}$$

If the bidder deviates to bidding 10 instead of x, his probability of winning is virtually unchanged, but his payoff increases to:

$$0.5(16-10) + 0.5Prob.(win\ with\ x|v_2 = 16)(16-x) = 0.5(16-10). \tag{6}$$

Thus, we can conclude that the mixing density $m(b)$ is positive over the entire interval $[10, 13]$, with no atoms. What will it look like? Let us confine ourselves to looking for a symmetric equilibrium, in which both players use the same function $m(b)$. We know the expected payoff from any bid b in the support must equal the payoff from $b = 10$ or $b = 13$, which is 3. Therefore,

$$0.5(16-b) + 0.5M(b)(16-b) = 3. \tag{7}$$

This implies that $(16-b) + M(b)(16-b) = 6$, so

$$M(b) = \frac{6}{16-b} - 1 = \frac{6-16+b}{16-b} = \frac{b-10}{16-b}, \tag{8}$$

so the mixing distribution is uniform, with density $m(b) = \frac{1}{16-b}$ on the support $[10, 13]$.

If the auction were second-price sealed-bid or English, a bidder's optimal strategy is to set bid or bid ceiling $b(v = 10) = 10$ and $b(v = 16) = 16$. His expected payoff is then

$$\begin{aligned} E\pi(v = 10) &= 0 \\ E\pi(v = 16) &= 0.5(16-10) + 0.5(16-10) = 3. \end{aligned} \tag{9}$$

The expected price is 13, the same as in the first-price auction, so the seller is indifferent about the rules. The buyer's expected payoff is also the same: 3 if $v = 16$ and 0 if $v = 10$. The only difference is that now the buyer's payoff ranges over the continuum $[0, 3]$ rather than being either 0 or 3.

Second-Price Sealed-Bid (Vickrey)

Rules
Each bidder submits one bid, in ignorance of the other bids. The bids are opened, and the highest bidder pays the amount of the second-highest bid and wins the object.

Strategies
A player's strategy is his bid as a function of his value and his prior belief about other players' valuations.

Payoffs
The winner's payoff is his value minus the second-highest bid that was made. The losers' payoffs are zero.

Second-price auctions are similar to English auctions. They are rarely used in reality, but are useful for modelling. Bidding one's valuation is the dominant strategy: a player who bids less is more likely to lose the auction, but pays the same price if he does win. The structure of the payoffs is reminiscent of the Groves mechanism of section 10.6, because in both games a player's strategy affects some major event (who wins the auction or whether the project is undertaken), but his strategy affects his own payoff only via that event. In the auction's equilibrium, each player bids his value and the winner ends up paying the second-highest value. If players know their own values, the outcome does not depend on risk neutrality.[2]

Dutch (descending)

Rules
The seller announces a bid, which he continuously lowers until some buyer stops him and takes the object at that price.

Strategies
A player's strategy is when to stop the bidding as a function of his valuation and his prior beliefs as to other players' valuations.

Payoffs
The winner's payoff is his value minus his bid. The losers' payoffs are zero.

The Dutch auction is **strategically equivalent** to the first-price sealed-bid auction, which means that there is a one-to-one mapping between the strategy sets and the equilibria of the two games. The reason for the strategic equivalence is that no relevant information is disclosed in the course of the auction, only at the end, when it is too late to change anybody's behavior. In the first-price auction a player's bid is irrelevant unless it is the highest, and in the Dutch auction a player's stopping price is also irrelevant unless it is the highest. The equilibrium price is calculated the same way for both auctions.

Dutch auctions are actually used. One example is the Ontario tobacco auction, which uses a clock four feet in diameter marked with quarter- cent gradations. Each of six or so buyers has a stop button. The clock hand drops a quarter cent at a time, and the stop buttons are registered so that ties cannot occur (tobacco buyers need reflexes like race-car drivers). The farmers who are selling their tobacco watch from an adjoining room and can later reject the bids if they feel they are too low (a form of reserve price); 2,500,000 lb. a day can be sold using the clock (Cassady [1967] p. 200).

[2] xxx I should make note of the odd 2nd price asymmetric equilibrium with zero price, in private value auctions.

Dutch auctions are common in less obvious forms. Filene's is one of the biggest stores in Boston, and Filene's Basement is its most famous department. In the basement are a variety of marked-down items formerly in the regular store, each with a price and date attached. The price customers pay at the register is the price on the tag minus a discount which depends on how long ago the item was dated. As time passes and the item remains unsold, the discount rises from 10 to 50 to 70 percent. The idea of predictable time discounting has also recently been used by bookstores ("Waldenbooks to Cut Some Book Prices in Stages in Test of New Selling Tactic," *Wall Street Journal*, March 29, 1988, p. 34).

13.2 Comparing Auction Rules

When one mentions auction theory to an economic theorist, the first thing that springs to his mind is the idea, in some sense, that different kinds of auctions are really the same. Milgrom & Weber (1982) give a good summary of how and why this is true. Regardless of the information structure, the Dutch and first-price sealed-bid auctions are the same in the sense that the strategies and the payoffs associated with the strategies are the same. That equivalence does not depend on risk neutrality, but let us assume that all players are risk neutral for the next few paragraphs.

In private, independent-value auctions, the second-price sealed- bid and English auctions are the same in the sense that the bidder who values the object most highly wins and pays the valuation of the bidder who values it the second highest, but the strategies are different in the two auctions. In all four kinds of private independent-value auctions discussed, the seller's expected price is the same. This fact is the biggest result in auction theory: the **revenue equivalence theorem** (Vickrey [1961]). We will show it using the following game, a mechanism design approach.

The Auctions Mechanism Game

Players: A seller and N buyers.

Order of Play:
0. Nature chooses buyer i's value for the object, v_i, using the strictly positive, atomless, density $f(v)$ on the interval $[\underline{v}, \overline{v}]$.
1. The seller chooses a mechanism $[x(a), t(a)]$ that takes payment t and gives the object with probability x to a player (himself, or one of the N buyers) who announces that his value is a.[3] He also chooses the procedure in which players select a (sequentially, simultaneously, etc.).
2. Each buyer simultaneously chooses to participate in the auction or to stay out.
3. The buyers and seller choose a according to the mechanism procedure.
4. The object is allocated and transfers are paid according to the mechanism, if it was accepted by all players.

Payoffs:
The seller's payoff is

$$\pi_s = \sum_{i=1}^{N} t(a_i, a_{-i}) \tag{10}$$

Buyer i's payoff is

$$\pi_i = x(a_i, a_{-i})v_i - t(a_i, a_{-i}) \tag{11}$$

Many auction procedures fit the mechanism paradigm. The x function could allocate the good with 70% probability to the high bidder and with 30% probability to the lowest bidder, for example; each bidder could be made to pay the amount he bids, even if he loses; t could include an entry fee; there could be a "reserve price" which is the minimum bid for which the seller will surrender the good. The seller must choose a mechanism that for each type v_i satisfies a participation constraint (bidder v_i will join the auction, so, for example,the entry fee is not too large) and an incentive compatibility constraint (the bidder will truthfully reveal his type). The game has multiple equilibria, because there is more than one mechanism that maximizes the seller's payoff.

[3] In equilibrium, the probability that more than one player will announce the same a is zero, so we will not bother to specify a tie-breaking rule. More properly, the seller should do so, but in this game, any rule would do equally well.

If the other $(N-1)$ buyers each choose $a_j = v_j$, as they will in a direct mechanism, which induces truthtelling, let us denote the expected maximized payoff of a buyer with value v_i as $\pi(v_i)$, so that

$$\pi(v_i) \equiv \overset{Max}{a_i} \; \{E_{v_{-i}}[x(a_i, v_{-i})v_i - t(a_i, v_{-i})]\}. \tag{12}$$

Another way to write $\pi(v_i)$ is as a base level, $\pi_i(\underline{v})$, plus the integral of its derivatives from $\underline{v}$ to v_i:

$$\pi(v_i) = \pi(\underline{v}) + \int_{v=\underline{v}}^{v_i} \frac{d\pi(v)}{dv} dv. \tag{13}$$

The seller will not give the lowest-valuing buyer type, $\underline{v}$, a positive expected payoff because then t_i could be increased by a constant amount for all types— an entry fee– without violating the participation or incentive compatibility constraints. Thus, $\pi(\underline{v}) = 0$ and we can rewrite the payoff as

$$\pi(v_i) = \int_{v=\underline{v}}^{v_i} \frac{d\pi(v)}{dv} dv. \tag{14}$$

To connect equations (12) and (14), our two expressions for $\pi(v_i)$, we will use a trick discovered by Mirrlees (1971) for mechanism design problems generally and use the Envelope Theorem to eliminate the t transfers. Differentiating $\pi(v_i)$ with respect to v_i, the Envelope Theorem says that we can ignore the indirect effect of v_i on π via its effect on a_i, since $\frac{da_i}{dv_i} = 0$ in the maximized payoff. herefore,

$$\begin{aligned} \tfrac{d\pi(v_i)}{dv_i} &= E_{v_{-i}}[\tfrac{\partial\pi(v_i)}{\partial v_i} + \tfrac{\partial\pi(v_i)}{\partial a_i}\tfrac{da_i}{dv_i}] \\ &= E_{v_{-i}}\tfrac{\partial\pi(v_i)}{\partial v_i} \\ &= E_{v_{-i}}x(a_i, v_{-i}), \end{aligned} \tag{15}$$

where the last line takes the partial derivative of equation (12).

Substituting back into equation (14) and using the fact that in a truthful direct mechanism $a_i = v_i$, we arrive at

$$\pi(v_i) = E_{v_{-i}} \int_{v=\underline{v}}^{v_i} x(v, v_{-i}) dv. \tag{16}$$

Now we can rearrange (12) and use our new $\pi(v_i)$ expression in (16) to solve for the expected transfer from a buyer of type v_i to the seller:

$$\begin{aligned} E_{v_{-i}} t(v_i, v_{-i}) &= E_{v_{-i}} x(v_i, v_{-i}) v_i - \pi(v_i) \\ &= E_{v_{-i}} \left[x(v_i, v_{-i}) v_i - \int_{v=\underline{v}}^{v_i} x(v, v_{-i}) dv \right]. \end{aligned} \tag{17}$$

Let $\pi_s(i)$ denote the seller's expected revenue from buyer i, not yet knowing any of the buyers' types, so

$$\begin{aligned} \pi_s(i) &= E_{v_i} E_{v_{-i}} t(v_i, v_{-i}) \\ &= E_{v_i} E_{v_{-i}} \left[x(v_i, v_{-i}) v_i - \int_{m=\underline{v}}^{v_i} x(m, v_{-i}) dm \right] \\ &= E_{v_{-i}} \int_{v=\underline{v}}^{\overline{v}} \left[x(v, v_{-i}) v - \int_{m=\underline{v}}^{v} x(m, v_{-i}) dm \right] f(v) dv \\ &= E_{v_{-i}} \int_{v=\underline{v}}^{\overline{v}} \left[x(v, v_{-i}) f(v) v - \int_{m=\underline{v}}^{v} x(m, v_{-i}) f(v) dm \right] dv \end{aligned} \tag{18}$$

At this point, we need to integrate by parts to deal with $\int x(m, v_{-i}) f(v) dm$. The formula for integration by parts is $\int_{z=a}^{b} g(z) h'(z) dz = g(z) h(z) \big|_{z=a}^{b} - \int_{z=a}^{b} h(z) g'(z) dz$. Let $z = v, g(z) = \int_{m=\underline{v}}^{v} x(m, v_{-i}) dm$ and $h'(z) = f(v)$. It follows that $g'(z) = x(v, v_{-i})$ and $h(z) = F(v)$ and we can write $\pi_s(i)$ as

$$\begin{aligned} & E_{v_{-i}} \int_{v=\underline{v}}^{\overline{v}} x(v, v_{-i}) f(v) v dv - E_{v_{-i}} \left(F(v) \int_{m=\underline{v}}^{v} x(m, v_{-i}) dm \big|_{v=\underline{v}}^{\overline{v}} - \int_{v=\underline{v}}^{\overline{v}} F(v) x(v, v_{-i}) dv \right) \\ & = E_{v_{-i}} \int_{v=\underline{v}}^{\overline{v}} x(v, v_{-i}) f(v) v dv - \\ & E_{v_{-i}} \left(F(\overline{v}) \int_{m=\underline{v}}^{\overline{v}} x(m, v_{-i}) dm - F(\underline{v}) \int_{m=\underline{v}}^{\underline{v}} x(m, v_{-i}) dm - \int_{v=\underline{v}}^{\overline{v}} F(v) x(v, v_{-i}) dv \right). \end{aligned} \tag{19}$$

Since $F(\overline{v}) = 1$ and $F(\underline{v}) = 0$ by definition, and since we can divide by f because of the assumption that the density is strictly positive, we can further rewrite $\pi_s(i)$ as

$$\begin{aligned} &= E_{v_{-i}} \int_{v=\underline{v}}^{\overline{v}} x(v, v_{-i}) f(v) v dv - E_{v_{-i}} \left(\int_{v=\underline{v}}^{\overline{v}} x(v, v_{-i}) dv - \int_{v=\underline{v}}^{\overline{v}} F(v) x(v, v_{-i}) dv \right) \\ &= E_{v_{-i}} \int_{v=\underline{v}}^{\overline{v}} x(v, v_{-i}) f(v) \left(v - \frac{1 - F(v)}{f(v)} \right) dv \end{aligned} \tag{20}$$

The seller's expected payoff from all N bidders sums $\pi_s(i)$ up over i:

$$E_v\pi_s = \sum_{i=1}^{N} \int_{v=\underline{v}}^{\overline{v}} x(v, v_{-i}) f(v) \left(v - \frac{1 - F(v)}{f(v)} \right) dv \tag{21}$$

The seller wants to choose $x()$ so as to maximize (21). The way to do this is to set $x = 1$ for the v which has the biggest $\left(v - \frac{1-F(v)}{f(v)}\right)$, which is what we will next examine.

MR approach

Think of a firm facing a demand curve made up of a continuum of bidders drawn from $f(v)$. The marginal revenue from this demand curve is[4]

$$MR(v) = v - \frac{1 - F(v)}{f(v)}. \tag{22}$$

If the value of the object to the seller is v_{seller}, then he should act like a monopolist with constant marginal cost of $v_{seller} = 0$ facing the demand curve based on $f(v)$, which means he should set a reserve price at the quantity where marginal revenue equals marginal cost. Figure 1 illustrates this. The analog to price is, of course, the value to the buyer, and the analog to quantity is the probability of sale at that price, which is $1 - F(v)$. This gives a good explanation for why the reserve price should be above the seller's use value for the object. In particular, if the seller gets no benefit from retaining the object ($v_{seller} = 0$), he should still set a reserve price where marginal revenue equals marginal cost. And the optimal reserve price is independent of the number of bidders— it just depends on the possible bidder values. He must publicize this, of course, because the point of the reserve price is then just to induce the buyers to bid higher.

[4]See Bulow and Roberts [1989] on risk-neutral private values; Bulow & Klemperer [1996] on common values and risk aversion.

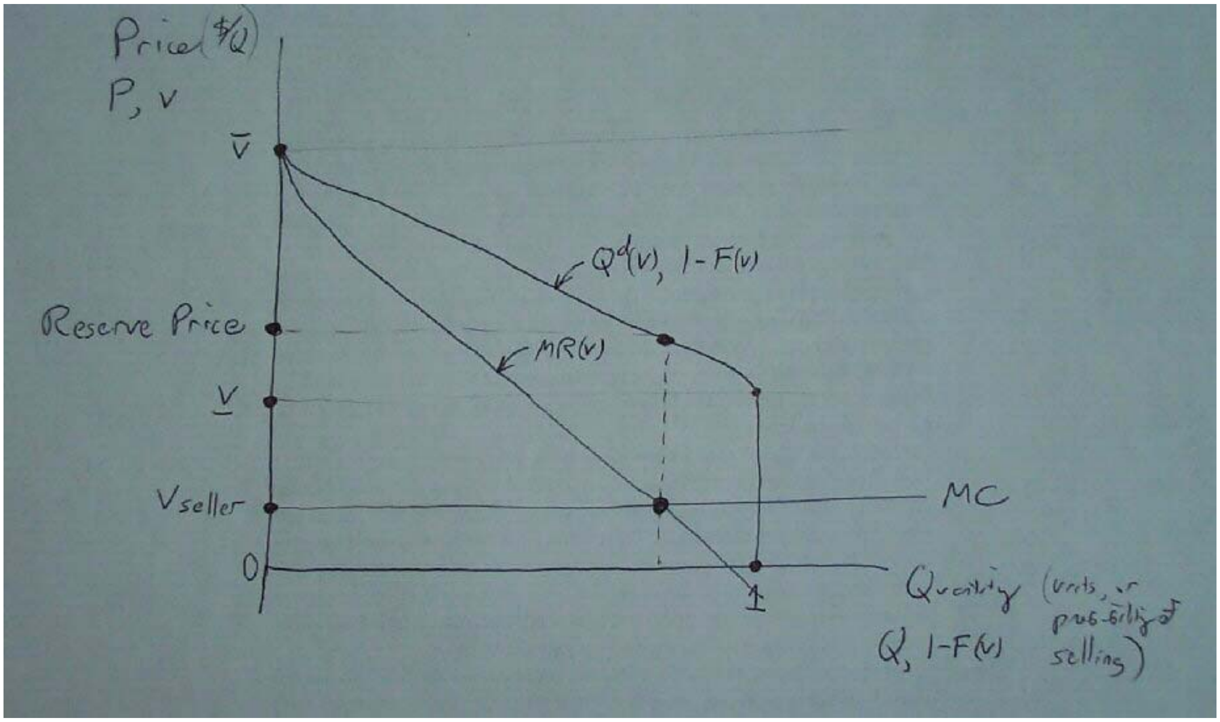


Figure 1: Auctions and Marginal Revenue

This expression has economic meaning, as Bulow & Klemperer (1996) show.[5] Suppose the seller had to make a single take-it-or-leave- it bargaining offer, instead of holding an auction. We can think of v as being like a price on a demand curve and $[1 - F(v)]$ as being like a quantity, since at price $p = v$ the single unit has probability $[1 - F(v)]$ of being sold. Revenue is then $R = pq = v[1 - F(v)]$ and marginal revenue is $\frac{dR}{dq} = p + q\frac{dp}{dq} = p + \frac{q}{dq/dp} = v + \frac{[1-F(v)]}{-f(v)}$. Thus, $\left(v_i - \frac{1-F(v_i)}{f(v_i)}\right)$ is like marginal revenue, and it makes sense to choose a price or set up an auction rule that elicits a price such that marginal revenue is maximized. Figure 1 illustrates this.[6]

Notice that t is not present in equation (21). It is there implicitly, however, because we have assumed we have found a truthful mechanism, and to satisfy the participation and incentive compatibility constraints, t has to be chosen appropriately. We can use equation (21) to deduce some properties of the optimal mechanism, and then we will find a number of mechanisms that satisfy those properties and all achieve the same expected payoff for the seller.

Suppose $\left(v_i - \frac{1-F(v_i)}{f(v_i)}\right)$ is increasing in v_i. This is a reasonable assumption, satisfied if the **monotone hazard rate** condition that $\frac{1-F(v_i)}{f(v_i)}$ decreases in v_i is true. In that case, the seller's best auction sells with probability one to the buyer with the biggest value of v_i. Thus, we have proved:

[5]xxx Also: conditional expecatoin of biggest minus second- biggest bid.

[6]In price theory, the monopolist chooses q or p so that marginal revenue equals not zero, but marginal cost. Here, we have assumed the seller gets zero value from not selling the good. If he did, there would be an opportunity cost to selling the good which would play the role of the marginal cost.

THE REVENUE EQUIVALENCE THEOREM. Suppose all bidders's valuations are drawn from the same strictly positive and atomless density $f(v)$ over $[\underline{v}, \overline{v}]$ and that f satisfies the monotone hazard rate condition. Any auction in which type $\underline{v}$ has zero expected surplus and the winner is the bidder with the highest value will have the same expected profit for the seller.

In particular, the following forms are optimal and yield identical expected profits:

Ascending auction. Everyone pays an entry fee (to soak up the surplus of the $\underline{v}$ type). The winner is the highest-value bidder, and he is refunded his entry fee but pays the value of the second- highest valuer.

Second-price sealed bid. The same.xxx

First-price sealed bid. The winner is the person who bids the highest. Is he is the highest valuer? A bidder's expected payoff is $\pi(v_i) = P(b_i)(v_i - b_i) - T$, where T is the entry fee and $P(b_i)$ is the probability of winning with bid b_i. The first order condition is $\frac{d\pi(v_i)}{db_i} = P'(v_i - b_i) - P = 0$, with second order condition $\frac{d^2\pi(v_i)}{db_i^2} \leq 0$. Using the implicit function theorem and the fact that $\frac{d^2\pi(v_i)}{db_i dv_i} = P' > 0$, we can conclude that $\frac{db_i)}{dv_i} \geq 0$. But it cannot be that $\frac{db_i)}{dv_i} = 0$, because then there would be values v_1 and v_2 such that $b_1 = b_2 = b$ and $\frac{d\pi(v_1)}{db_1} = P'(b)(v_1 - b) - P(b) = 0 = \frac{d\pi(v_2)}{db_2} = P'(b)(v_2 - b) - P(b)$, which cannot be true. So bidders with higher values bid higher, and the highest valuer will win the auction.

Dutch (descending) auction. Same as the first-price sealed-bid.

All-pay sealed-bid. Each player pays his bid. Whoever bids highest is awarded the object.xxx

Although the different auctions have the same expected payoff for the seller, they do not have the same realized payoff. In the first-price- sealed-bid auction, for example, the winning buyer's payment depends entirely on his own valuation. In the second-price-sealed-bid auction the winning buyer's payment depends entirely on the second-highest valuation, which is sometimes close to his own valuation and sometimes much less.

In the Dutch and first-price sealed-bid auctions, the winning bidder has estimated the value of the second-highest bidder, and that estimate, while correct on average, is above or below the true value in particular realizations. The variance of the price is higher in those auctions because of the additional estimation, which means that a risk-averse seller should use the English or second-price auction.

Hazard Rates

Suppose we have a density $f(v)$ for a bidder's value for an object being sold, with cumulative distribution $F(v)$ on support $[\underline{v}, \overline{v}]$. The hazard rate $h(v)$ is defined as

$$h(v) = \frac{f(v)}{1 - F(v)} \tag{23}$$

What this means is that $h(v)$ is the probability density of v for the distribution which is like $F(v)$ except cutting off the lower values, so its support is limited to $[v, \overline{v}]$. In economic terms, $h(v)$ is the probability density for the valuing equalling v given that we know that value equals at least v.

For most distributions we use, the hazard rate is increasing, including the uniform, normal, logistic, and exponential distributions, and any distribution with increasing density over its support (see Bagnoli & Bergstrom [1989]). Figure 2 shows three of them

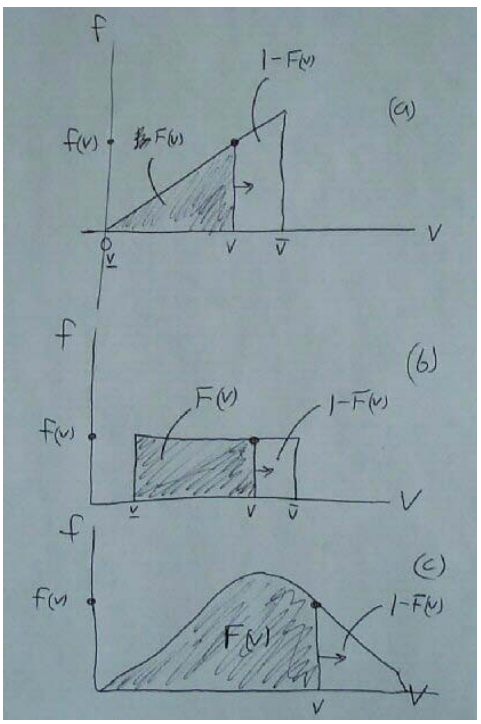


Figure 2: Three Densities to Illustrate Hazard Rates

Myerson (1981) and Bulow & Roberts (1989) look at optimal auctions when bidders are asymmetric or higher values do not imply that $(v_i - \frac{1-F(v_i)}{f(v_i)}$ is higher. Then the revenue-maximizing auction might not allocate the good to the bidder with the highest value. This is because that bidder might not be the bidder with the highest *expected* value. If there were two bidders, it could happen that $\left(\frac{1-F_1(v)}{f_1(v)}\right) > \left(\frac{1-F_2(v)}{f_2(v)}\right)$, in which case the auction should be biased in favor of Bidder 2 who should sometimes win even if $v_2 < v_1$.

Whether the auction is private-value or not, the Dutch and first- price sealed-bid auctions are strategically equivalent. If the auction is correlated-value and there are three or more bidders, the open-exit English auction leads to greater revenue than the second-price sealed-bid auction, and both yield greater revenue than the first-price sealed-bid auction (Milgrom & Weber [1982]). If there are just two bidders, however, the open-exit English auction is no better than the second-price sealed-bid auction, because the open-exit feature – knowing when nonbidding players drop out – is irrelevant.

A question of less practical interest is whether an auction form is Pareto optimal; that is, does the auctioned object end up in the hands of whoever values it most? In a common-value auction this is not an interesting question, because all bidders value the object equally. In a private-value auction, all the five auction forms – first-price, second-price, Dutch, English, and all-pay – are Pareto optimal. They are also optimal in a correlated-value auction if all players draw their information from the same distribution and if the equilibrium is in symmetric strategies.

Hindering Buyer Collusion

As I mentioned at the start of this chapter, one motivation for auctions is to discourage collusion between players. Some auctions are more vulnerable to this than others. Robinson (1985) has pointed out that whether the auction is private-value or common-value, the first-price sealed-bid auction is superior to the second-price sealed-bid or English auctions for deterring collusion among bidders.

Consider a buyer's cartel in which buyer Smith has a private value of 20, the other buyers' values are each 18, and they agree that everybody will bid 5 except Smith, who will bid 6. (We will not consider the rationality of this choice of bids, which might be based on avoiding legal penalties.) In an English auction this is self- enforcing, because if somebody cheats and bids 7, Smith is willing to go all the way up to 20 and the cheater will end up with no gain from his deviation. Enforcement is also easy in a second-price sealed-bid auction, because the cartel agreement can be that Smith bids 20 and everyone else bids 6.

In a first-price sealed-bid auction, however, it is hard to prevent buyers from cheating on their agreement in a one-shot game. Smith does not want to bid 20, because he would have to pay 20, but if he bids anything less than the other players' value of 18 he risks them overbidding him. The buyer will end up paying a price of 18, rather than the 6 he would receive in an English auction with collusion. The seller therefore will use the first-price sealed-bid auction if he fears collusion.[7]

13.3 Risk and Uncertainty over Values

[7] Even then, his fears may be realized. Bidding rings are not uncommon even though they are illegal and even though first-price auctions are used. See Sultan (1974) for the Electric Conspiracy, one famous example.

In a private-value auction, does it matter what the seller does, given the revenue equivalence theorem? Yes, because of risk aversion, which invalidates the theorem. Risk aversion makes it important which auction rule is chosen, because the seller should aim to reduce uncertainty, even in a private value auction. (In a common value auction, reducing uncertainty has the added benefit of ameliorating the winner's curse.)

Consider the following question:

If the seller can reduce bidder uncertainty over the value of the object being auctioned, should he do so?

We must assume that this is a precommitment on the part of the seller, since otherwise he would like to reveal favorable information and conceal unfavorable information. But it is often plausible that the seller can set up an auction system which reduces uncertainty - say, by a regular policy of allowing bidders to examine the goods before the auction begins. Let us build a model to show the effect of such a policy.

Suppose there are N bidders, each with a private value, in an ascending open-cry auction. Each measures his private value v with an independent error ϵ. This error is with equal probability $-x, +x$ or 0. The bidders have diffuse priors, so they take all values of v to be equally likely, ex ante. Let us denote the measured value by $\hat{v} = v + \epsilon$, which is an unbiased estimate of v. What should bidder i bid up to?

If bidder i is risk neutral, he should bid up to $p = \hat{v}$. If he pays $\hat{v}$, his expected utility is,

$$\pi(risk\ neutral, p = \hat{v}) = \frac{([\hat{v} - x] - \hat{v})}{3} + \frac{(\hat{v} - \hat{v})}{3} + \frac{(\hat{v} + x - \hat{v})}{3} = 0. \tag{24}$$

If bidder i is risk averse, however, and wins with bid p, his expected utility is, if his utility function is $\pi = U(v - p)$ for concave U,

$$\pi(risk\ averse, p) = \frac{U([\hat{v} - x] - p)}{3} + \frac{U(\hat{v} - p)}{3} + \frac{U([\hat{v} + x] - p)}{3} \tag{25}$$

Since the utility function U is concave,

$$\frac{U([\hat{v} - x] - p)}{3} + \frac{U([\hat{v} + x] - p)}{3} < \frac{2}{3}U(\hat{v} - p). \tag{26}$$

The implication is that a fair gamble of x has less utility than no gamble. This means that the middle term in equation (25) must be positive if it is to be true that $\pi = U(0)$, which means that $\hat{v} - p > 0$. In other words, bidder i will have a negative expected payoff unless his maximum bid is strictly less than his valuation.

Other auctions with risk-averse bidders are more difficult to analyze. The problem is that in a first-price sealed-bid auction or a Dutch auction, there is risk not only from uncertainty over the value but over how much the other players will bid. One finding is that in a private-value auction the first-price sealed-bid auction yields a greater expected revenue than the English or second-price auctions. That is because by increasing his bid from the level optimal for a risk-neutral bidder, the risk-averse bidder insures himself. If he wins, his surplus is slightly less because of the higher price, but he is more likely to win and avoid a surplus of zero. Thus, the buyers' risk aversion helps the seller.

Notice that the seller does not have control over all the elements of the model. The seller can often choose the auction rules unilaterally. This includes not just how bids are made, but such things as whether the bidders get to know how many potential bidders are in the auction, whether the seller himself is allowed to bid, and so forth. Also, the seller can decide how much information to release about the goods. The seller cannot, however, decide whether the bidders are risk averse or not, or whether they have common values or private values – no more than he can choose what their values are for the good he is selling. All of those things concern the utility functions of the bidders. At best, the seller can do things such as choose to produce goods to sell at the auction which have common values instead of private values.[8]

An error I have often seen is to think that the presence of uncertainty in valuations always causes the winner's curse. It does not, unless the auction is a common-value one. Uncertainty in one's valuation is a necessary but not sufficient condition for the winner's curse. It is true that risk-averse bidders should not bid as high as their valuations if they are uncertain about their valuations, even if the auction is a private-value one. That sounds a lot like a winner's curse, but the reason for the discounted bids is completely different, depending as it does on risk aversion. If bidders are uncertain about valuations but they are risk-neutral, their dominant strategy is still to bid up to their valuations. If the winner's curse is present, even if a bidder is risk-neutral he discounts his bid because if he wins on average his valuation will be greater than the value.

Risk Aversion in Private-Value Auctions

When buyers are risk averse, the ranking of seller revenues is:

1st price sealed-bid and Dutch: best and identical to each other
English: next
2nd-price sealed bid: worst

To understand this, consider the following example. Each bidder's private value is either 10 or 16, with equal probability, and is known only to himself.

[8]xxx Here add reference to my work on buying info on one's value.

If the auction is second-price sealed-bid or English, a bidder's optimal strategy is to set bid or bid ceiling $b(v=10)=10$ and $b(v=16)=16$. His expected payoff is then

$$\begin{aligned} E\pi(v=10) &= 0 \\ E\pi(v=16) &= 0.5(16-10)+0.5(16-10)=3. \end{aligned} \tag{27}$$

If the auction is first-price sealed bid or Dutch, a bidder's optimal strategy is to set bid $b(v=10)=10$ and if $v=16$ to use a mixed strategy, mixing over the support $[x,y]$, where it will turn out that $x=10$ and $y=13$. His expected payoff will be

$$\begin{aligned} E\pi(v=10) &= 0 \\ E\pi(v=16) &= 3. \end{aligned} \tag{28}$$

Here is how one finds the optimal strategy in a symmetric equilibrium. Suppose $x>10$ instead of $x=10$. The bidder's expected payoff is, if the other player has no atom at x in his strategy,

$$0.5(16-x)+0.5Prob.(win\ with\ x|v_2=16)(16-x)=0.5(16-x). \tag{29}$$

If the bidder deviates to bidding 10 instead of x, his probability of winning is virtually unchanged, but his payoff increases to:

$$0.5(16-10)+0.5Prob.(win\ with\ x|v_2=16)(16-x)=0.5(16-10). \tag{30}$$

What if Bidder 2 did have an atom at x, and bids x with probability P? Then Bidder 1 should have an atom at $x+\epsilon$ instead, for some small ϵ. Thus, it must be that $x=10$.

Suppose $y=16$. Then the bidder's expected payoff is zero. This, however, is a lower expected payoff than bidding 12, which has an expected payoff of at least $0.5(16-12)>0$ because its payoff is positive if the other player's value is just 10. To find the exact value of y, note that in a mixed strategy equilibrium, all pure strategies in the mixing support must have the same expected payoff. If 10 is in the mixing support, then its expected payoff (since it will almost surely win) is $\pi(b=10)=0.5(16-10)=0.5(16-10)$. The expected payoff from the pure strategy of bidding y, which always wins (being at the top of the support) is $\pi(b=16)=0.5(16-y)=0.5(16-y)$. Equating these yields $y=13$.

The expected price is 13, the same as in the second-price auction, so the seller is indifferent about the rules, as the Revenue Equivalence Theorem says. The buyer's expected payoff is also the same: 3 if $v=16$ and 0 if $v=10$. The only difference is that now the buyer's payoff ranges over the continuum $[0,3]$ rather than being either 0 or 3.

xxxI need to state the equilibruim and then derive it. What is the mixing distribution?

Now let us make the players risk averse. The optimal second-price strategy does not change.

But now the utility of 6 has shrunk relative to the utility of 3, so the optimal 1st-price strategy does change. That strategy made the expected payoff– not the expected dollar profit– equal for each pure strategy on the mixing support. As before, the optimal strategy if $v_1 = 16$ is to mix over $[x, y]$. with $x = 10$ and $y < 16$, but now y will increase. The payoff from bidding 10 is, if we normalize by setting $U(0) \equiv 0$,

$$\pi(b = 10) = 0.5U(16 - 10) + 0.5U(0) = 0.5U(6). \tag{31}$$

The payoff from bidding y is

$$\pi(b = 10) = 0.5U(16 - y) + 0.5U(16 - y). \tag{32}$$

This requires that y be set so that

$$0.5U(6) = U(16 - y) \tag{33}$$

The utility of winning with $b = 10$ has to equal twice the utility of winning with $b = y$. This means, given concave $U()$, that the dollar profit from winning with $b = 10$ has to equal *more* than twice the dollar profit of winning with $b = y$. Thus, we need $(16 - y) < 6$, and y must increase. Intuitively, if the buyer is risk averse, he becomes less willing to take a chance of losing the auction by bidding low, and more willing to bid high and get a lower payoff but with greater probability.

13.4: Common-Value Auctions and the Winner's Curse

In Section 13.2 we distinguished private-value auctions from common-value auctions, in which the values of the players are identical but their valuations may differ. All four sets of rules discussed there can be used for common-value auctions, but the optimal strategies are different. In common-value auctions, each player can extract useful information about the object's value to himself from the bids of the other players. Surprisingly enough, a buyer can use the information from other buyers' bids even in a sealed-bid auction, as will be explained below.

When I teach this material I bring a jar of pennies to class and ask the students to bid for it in an English auction. All but two of the students get to look at the jar before the bidding starts, and everybody is told that the jar contains more than 5 and less than 100 pennies. Before the bidding starts, I ask each student to write down his best guess of the number of pennies. The two students who do not get to see the jar are like "technical analysts," those peculiar people who try to forecast stock prices using charts showing the past movements of the stock while remaining ignorant of the stock's "fundamentals."

A common-value auction in which all the bidders knew the value would not be very interesting, but more commonly, as in the penny jar example, the bidders must estimate the common value. The obvious strategy, especially following our discussion of private-value auctions, is for a player to bid up to his unbiased estimate of the number of pennies in the jar. But this strategy makes the winner's payoff negative, because the winner is the bidder who has made the largest positive error in his valuation. The bidders who underestimated the number of pennies, on the other hand, lose the auction, but their payoff is limited to a downside value of zero, which they would receive even if the true value were common knowledge. Only the winner suffers from overbidding: he has stumbled into the **winner's curse.** When other players are better informed, it is even worse for an uninformed player to win. Anyone, for example, who wins an auction against 50 experts should worry about why they all bid less.

To avoid the winner's curse, players should scale down their estimates in forming their bids. The mental process is a little like deciding how much to bid in a private-value, first-price sealed-bid auction, in which bidder Smith estimates the second-highest value conditional upon himself having the highest value and winning. In the common-value auction, Smith estimates his own value, not the second-highest, conditional upon himself winning the auction. He knows that if he wins using his unbiased estimate, he probably bid too high, so after winning with such a bid he would like to retract it. Ideally, he would submit a bid of [X *if I lose, but* $(X-Y)$ *if I win*], where X is his valuation conditional upon losing and $(X-Y)$ is his lower valuation conditional upon winning. If he still won with a bid of $(X-Y)$ he would be happy; if he lost, he would be relieved. But Smith can achieve the same effect by simply submitting the bid $(X-Y)$ in the first place, since the size of losing bids is irrelevant.

Another explanation of the winner's curse can be devised from the Milgrom definition of "bad news" (Milgrom [1981b], appendix B). Suppose that the government is auctioning off the mineral rights to a plot of land with common value V and that bidder i has valuation $\hat{V}_i$. Suppose also that the bidders are identical in everything but their valuations, which are based on the various information sets Nature has assigned them, and that the equilibrium is symmetric, so the equilibrium bid function $b(\hat{V}_i)$ is the same for each player. If Bidder 1 wins with a bid $b(\hat{V}_1)$ that is based on his prior valuation $\hat{V}_1$, his posterior valuation $\tilde{V}_1$ is

$$\tilde{V}_1 = E(V|\hat{V}_1, b(\hat{V}_2) < b(\hat{V}_1), \ldots, b(\hat{V}_n) < b(\hat{V}_1)). \tag{34}$$

The news that $b(\hat{V}_2) < \infty$ would be neither good nor bad, since it conveys no information, but the information that $b(\hat{V}_2) < b(\hat{V}_1)$ is bad news, since it rules out values of b more likely to be produced by large values of $\hat{V}_2$. In fact, the lower the value of $b(\hat{V}_1)$, the worse is the news of having won. Hence,

$$\tilde{V}_1 < E(V|\hat{V}_1) = \hat{V}_1, \tag{35}$$

and if Bidder 1 had bid $b(\hat{V}_1) = \hat{V}_1$ he would immediately regret having won. If his winning bid were enough below $\hat{V}_1$, however, he would be pleased to win.

Deciding how much to scale down the bid is a hard problem because the amount depends on how much all the other players scale down. In a second-price auction a player calculates the value of $\tilde{V}_1$ using equation (34), but that equation hides considerable complexity under the disguise of the term $b(\hat{V}_2)$, which is itself calculated as a function of $b(\hat{V}_1)$ using an equation like (34).[9]

Oil Tracts and the Winner's Curse

The best known example of the winner's curse is from bidding for offshore oil tracts. Offshore drilling can be unprofitable even if oil is discovered, because something must be paid to the government for the mineral rights. Capen, Clapp & Campbell (1971) suggest that bidders' ignorance of the winner's curse caused overbidding in US government auctions of the 1960s. If the oil companies had bid close to what their engineers estimated the tracts were worth, rather than scaling down their bids, the winning companies would have lost on their investments. The hundredfold difference in the sizes of the bids in the sealed-bid auctions shown in Table 1 lends some plausibility to the view that this is what happened.

Table 1 Bids by Serious Competitors in Oil Auctions

[9] xxxx Here it would be nice to do a numerical examlpe of how much to scale down. Nobody seems to have one, thoguh— not Milgrom, Cramton, Dixit-Skeath, Myerson, Gintis. Some of them use the Wallet Game, which might be worth adding. Simply using the uniform distirubtion for f is tough because order statistics are tricky– this involves finding the expected value, an integral, given that I have valuation x1 and the second-highest valuation is x2. And the first-rpice bidding function is just characetrizied by a differnetial equation, not found by anyone. Deriving the advantage of Enlgish over second-price over first-rpice is easier, and might be worth doing.

Offshore Louisiana 1967 Tract SS 207	**Santa Barbara Channel 1968** Tract 375	**Offshore Texas 1968** Tract 506	**Alaska North Slope 1969** Tract 253
32.5	43.5	43.5	10.5
17.7	32.1	15.5	5.2
11.1	18.1	11.6	2.1
7.1	10.2	8.5	1.4
5.6	6.3	8.1	0.5
4.1		5.6	0.4
3.3		4.7	
		2.8	
		2.6	
		0.7	
		0.7	
		0.4	

Later studies such as Mead, Moseidjord & Sorason (1984) that actually looked at profitability conclude that the rates of return from offshore drilling were not abnormally low, so perhaps the oil companies did scale down their bids rationally. The spread in bids is surprisingly wide, but that does not mean that the bidders did not properly scale down their estimates. Although expected profits are zero under optimal bidding, realized profits could be either positive or negative. With some probability, one bidder makes a large overestimate which results in too high a bid even after rationally adjusting for the winner's curse. The knowledge of how to bid optimally does not eliminate bad luck; it only mitigates its effects.

Another consideration is the rationality of the other bidders. If bidder Apex has figured out the winner's curse, but bidders Brydox and Central have not, what should Apex do? Its rivals will overbid, which affects Apex's best response. Apex should scale down its bid even further than usual, because the winner's curse is intensified against overoptimistic rivals. If Apex wins against a rival who usually overbids, Apex has very likely overestimated the value.

Risk aversion affects bidding in a surprisingly similar way. If all the players are equally risk averse, the bids would be lower, because the asset is a gamble, whose value is lower for the risk averse. If Smith is more risk averse than Brown, then Smith should be more cautious for two reasons. The direct reason is that the gamble is worth less to Smith. The indirect reason is that when Smith wins against a rival like Brown who regularly bids more, Smith probably overestimated the value. Parallel reasoning holds if the players are risk neutral, but the private value of the object differs among them.

Asymmetric equilibria can even arise when the players are identical. Second-price, two-person, common-value auctions usually have many asymmetric equilibria besides the symmetric equilibrium we have been discussing (see Milgrom [1981c] and Bikhchandani [1988]). Suppose that Smith and Brown have identical payoff functions, but Smith thinks Brown is going to bid aggressively. The winner's curse is intensified for Smith, who would probably have overestimated if he won against an aggressive bidder like Brown, so Smith bids more cautiously. But if Smith bids cautiously, Brown is safe in bidding aggressively, and there is an asymmetric equilibrium. For this reason, acquiring a reputation for aggressiveness is valuable.

Oddly enough, if there are three or more players the sealed-bid, second-price, common-value auction has a unique equilibrium, which is also symmetric. The open-exit auction is different: it has asymmetric equilibria, because after one bidder drops out, the two remaining bidders know that they are alone together in a subgame which is a two- player auction. Regardless of the number of players, first-price sealed-bid auctions do not have this kind of asymmetric equilibrium. Threats in a first-price auction are costly because the high bidder pays his bid even if his rival decides to bid less in response. Thus, a bidder's aggressiveness is not made safer by intimidation of another bidder.

The winner's curse crops up in situations seemingly far removed from auctions. An employer must beware of hiring a worker passed over by other employers. Someone renting an apartment must hope that he is not the first visitor who arrived when the neighboring trumpeter was asleep. A firm considering a new project must worry that the project has been considered and rejected by competitors. The winner's curse can even be applied to political theory, where certain issues keep popping up. Opinions are like estimates, and one interpretation of different valuations is that everyone gets the same data, but they analyze it differently.

On a more mundane level, in 1987 there were four major candidates – Bush, Kemp, Dole, and Other – running for the Republican nomination for President of the United States. Consider an entrepreneur auctioning off four certificates, each paying one dollar if its particular candidate wins the nomination. If every bidder is rational, the entrepreneur should receive a maximum of one dollar in total revenue from these four auctions, and less if bidders are risk averse. But holding the auction in a bar full of partisans, how much do you think he would actually receive?

The Wallet Game

Players
Smith and Jones.

Order of Play
(0) Nature chooses the amounts s_1 and s_2 of the money in Smith's wallet and Jones's, using density functions $f_1(s_1)$ and $f_2(s_2)$. Each player observes only his own wallet contents.
(1) Each player chooses a bid ceiling p_1 or p_2. An auctioneer auctions off the two wallets by gradually raising the price until either p_1 or p_2 is reached and one player exits.

Payoffs:
The player that exits first gets zero. The winning player has a payoff of $(s_1 + s_2 - Min(p_1, p_2))$.

One equilibrium is for bidder i to choose bid ceiling $p_i = 2s_i$. This is an equilibrium because if he wins at that price, the value of the wallets is at least $2s_i$, since player j's signal must be $s_j = s_i$. If he were to choose a ceiling any lower, then he might pass up a chance to get the wallet at a price less than its value; if he chooses a ceiling any higher, the other player might drop out and i would overpay.

There are other equilibria, though–asymmetric ones.

In general, asymmetric equilibria are common in common-value auctions. That is because the severity of the winner's curse facing player i [10] depends on the bidding behavior of the other players. If other players bid aggressively, then if i wins anyway, he must have a big overestimate of the value of the object. So the more aggressive are the other players, the more conservative ought i to be— which in turn will make the other players more aggressive.

Here, another equilibrium is $(p_1 = 10s_1, p_2 = \frac{10}{9}s_2)$. If the two players tie, having choosen $p = p_1 = p_2$, then $10s_1 = \frac{10}{9}s_2$, which implies that $s_1 = \frac{1}{9}s_2$, which implies that $s_1 + s_2 = 10s_1 = p$, and $v = p$. This has a worse equilibrium payoff for bidder 2, because he hardly ever wins, and when he does win it is because s_1 was very low— so there is hardly any money in Bidder 1's wallet. Being the aggressive bidder in an equilibrium is valuable. If there is a sequence of auctions, this means establishing a reputation for aggressiveness can be worthwhile, as shown in Bikhchandani (1988).

Common Values

Milgrom and Weber found that when there is a common value element in an auction game ("affiliated values"), then the ranking of seller revenues is:

English: best
2nd-price sealed bid: next best
1st price sealed-bid and Dutch: worst and identical

It is actually hard to come up with classroom examples for common value auctions. Paul Klemperer has done so, however, at page 69 of his 1999 survey.

[10] xxx A term invented by Robert Wilson in the 60's , I think. Find source.

Suppose that n signals are independently drawn from the uniform distribution on $[\underline{s}, \overline{s}]$. Note that the expectation of the kth highest value is

$$Es_{(k)} = \underline{s} + \left(\frac{n+1-k}{n+1} \right) (\overline{s} - \underline{s}) \tag{36}$$

In particular, this means the expectation of the second-highest value is

$$Es_{(2)} = \underline{s} + \left(\frac{n+1-2}{n+1} \right) (\overline{s} - \underline{s}) = \underline{s} + \left(\frac{n-1}{n+1} \right) (\overline{s} - \underline{s}) \tag{37}$$

and the expectation of the lowest value is

$$Es_{(n)} = \underline{s} + \left(\frac{n+1-n}{n+1} \right) (\overline{s} - \underline{s}) = \underline{s} + \left(\frac{1}{n+1} \right) (\overline{s} - \underline{s}). \tag{38}$$

Suppose n risk-neutral bidders, $i = 1, 2, ...n$ each receive a signal s_i independently drawn from the uniform distribution on $[v - m, v + m]$, where v is the true value of the object to each of them. Assume that they have "diffuse priors" on v, which means they think any value is equally likely.

Denote the j^{th} highest signal by $s_{(j)}$. Note that

$$Ev|(s_1, s_2, ...s_n) = \frac{s_{(n)} + s_{(1)}}{2}. \tag{39}$$

This is a remarkable property of the uniform distribution: if you observe signals 6,7,11, and 24, the expected value of the object is 15 (=[6+24]/2), well above the mean of 12 and the median of 9, because only the extremes of 6 and 24 are useful information. A density that had a peak, like the normal density, would yield a different result, but here all we can tell from the data is that all values of v between $(6 + m)$ and $(24 - m)$ are equally probable.

Figure 3 illustrates this. Someone who saw just signals $s_{(n)}$ and $s_{(1)}$ could deduce that v could not be less than $(s_{(1)} - m)$ or greater than $(s_{(n)} + m)$. Learning $s_{(2)}$, for example, would be unhelpful, because the only information it conveys is that $v \leq (s_{(2)} + m)$ and $v \geq (s_{(2)} - m)$, and our observer has already figured that out.

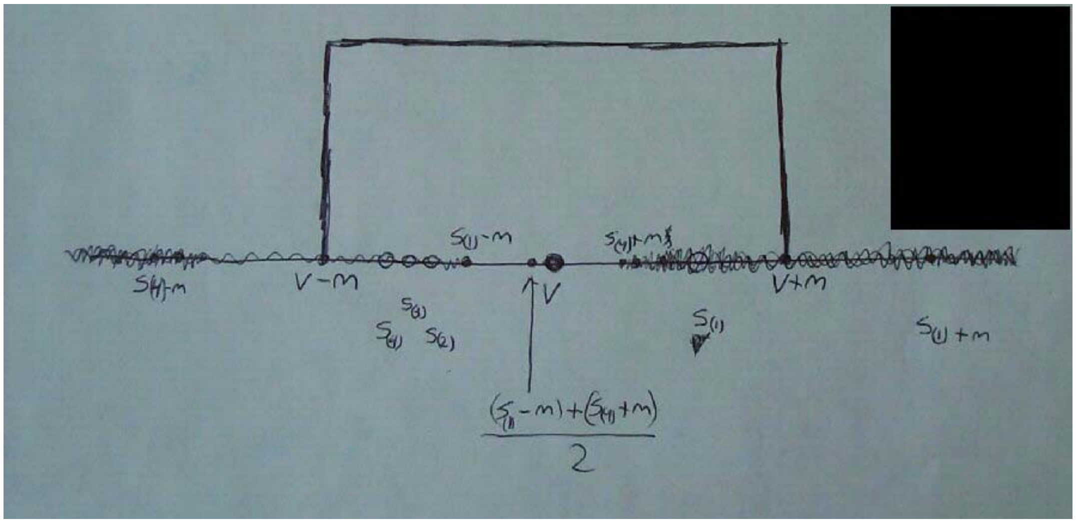


Figure 3: Extracting Information From Uniformly Distributed Signals

What are the strategies in symmetric equilibria for the different auction rules? (We will ignore possible asymmetric equilibria.

English, ascending, open-cry, open-exit auction

Equilibrium: If nobody else has quit yet, drop out when the price rises to s_i. Otherwise, drop out when the price rises to $p_i = \frac{p_{(n)}+s_i}{2}$, where $p_{(n)}$ is the price at which the first dropout occurred.

If nobody else has quit yet, then bidder i is safe in agreeing to pay the price. Either (a) he has the lowest signal, and will lose the auction, or (b) everybody else has signal s_i too, and they will all drop out at the same time, or (c) he will never drop out, and he will win. In case (b), his estimate of the value is s_i, and that is where he should drop out.

Once one person has dropped out at $p_{(n)}$, the other bidders can guess that he had the lowest signal, so they know that signal $s_{(n)}$ must equal $p_{(n)}$. Suppose bidder i has signal $s_i > s_{(n)}$. Either (a) someone else has a higher signal and bidder i will lose the auction, or (b) everybody else who has not yet dropped out has signal s_i too, and they will all drop out at the same time, or (c) he will never drop out and he will win. In case (b), his estimate of the value is $p_{(i)} = \frac{p_{(n)}+s_i}{2}$, since $p_{(n)}$ and s_i are the extreme signal values, and that is where he should drop out.

The price paid by the winner will be the price at which the second- highest bidder drops out, which is $\frac{s_{(n)}+s_{(2)}}{2}$. These expected values are

$$Es_{(n)} = (v-m) + \left(\frac{n+1-n}{n+1}\right)((v+m)-(v-m)) = v + \left(\frac{2-n}{n+1}\right)2m. \tag{40}$$

and

$$Es_{(2)} = (v-m) + \left(\frac{n+1-2}{n+1}\right)((v+m)-(v-m)) = v + \left(\frac{n-3}{n+1}\right)2m. \tag{41}$$

Averaging them yields the expected winning price,

$$Ep_{(2)} = \frac{\left[v+\left(\frac{2-n}{n+1}\right)2m\right]+\left[v+\left(\frac{n-3}{n+1}\right)2m\right]}{2} = v - \frac{1}{2}\left(\frac{1}{n+1}\right)2m. \tag{42}$$

Notice that the bigger m is, the lower the expected seller revenue. Also notice that the higher is n,the greater is the expected seller revenue. This will be true for all three auction rules we examine here.

2nd-Price Sealed-bid Auction

Equilibrium: Bid $p_i = s_i - \left(\frac{n-2}{n}\right)m$.

Bidder i thinks of himself as being tied for winner with one other bidder, and so having to pay his bid. So he thinks he is the highest of $(n-1)$ bidders drawn from $[v-m, v+m]$ and tied with one other, so on average if this happens, $s_i = (v-m) + \left(\frac{([n-1]+1-1)}{[n-1]+1}\right)([v+m]-[v-m]) = (v-m)+\left(\frac{n-1}{n}\right)(2m) = v+\frac{n-2}{n}(m)$. He will bid this value, which is, solving for v, $p_i = s_i - \left(\frac{n-2}{n}\right)(m)$.

On average, the second-highest bidder actually has the signal $Es_{(2)} = v + \left(\frac{n-3}{n+1}\right)m$, as found earlier. So the expected price, and hence the expected revenue from the auction is

$$Ep_{(2)} = [v+\left(\frac{n-3}{n+1}\right)m] - \left(\frac{n-2}{n}\right)(m) = v + \left(\frac{n(n-3)-(n+1)(n-2)}{(n+1)n}\right)m, \tag{43}$$

which equals

$$v - \left(\frac{n-1}{n}\right)\left(\frac{1}{n_+1}\right)2m. \tag{44}$$

Note that in this example, the expected revenue is lower. Why? It is because bidder 2 does not know the lower bound is so low when he makes his bid. He has to guess at the lower bound.

1st-price sealed-bid auction, Dutch descending auction

Equilibrium: Bid $(s_i - m)$.

Bidder i bids $(s_i - z)$ for some amount z that does not depend on his signal, because given the assumption of diffuse priors, he does not know whether his signal is a high one or a low one.[11] Define T so that $s_i \equiv v - m + T$. Bidder i has the highest signal and wins the auction if T is big enough, which has probability $\left(\frac{T}{2m}\right)^{n-1}$, because it is the probability that the $(n-1)$ other signals are all less than $(v - m + T)$. He earns v minus his bid of $(s_i - z)$ if he wins, which equals $(z + m - T)$. If, instead, he deviates and bids a small amount ϵ higher, he would win $(z+m-(T-\epsilon))$ with additional probability. Using a Taylor expansion $(g(T+\epsilon) \approx g(T) + g'(T)\epsilon)$ tells us that

$$\left(\frac{T+\epsilon}{2m}\right)^{n-1} - \left(\frac{T}{2m}\right)^{n-1} \approx (n-1)T^{N-2}\left(\frac{1}{2m}\right)^{n-1}\epsilon. \tag{45}$$

The disadvantage of bidding higher is that Bidder i would pay an additional ϵ in the $\left(\frac{T}{2m}\right)^{n-1}$ cases in which he would have won anyway. In equilibrium, he is indifferent about this small deviation, [12] so

$$\int_{T=0}^{2m}\left[\left((n-1)T^{N-2}\left(\frac{1}{2m}\right)^{n-1}\epsilon\right)(z+m-T) - \epsilon\left(\frac{T}{2m}\right)^{n-1}\right]dT = 0. \tag{46}$$

This implies that

$$\left(\frac{\epsilon}{2m}\right)^{n-1}\int_{T=0}^{2m}\left[\left((n-1)T^{N-2}\right)(z+m) - (n-1)T^{n-1} - T^{n-1}\right]dT = 0. \tag{47}$$

which in turn implies that

$$\left(\frac{\epsilon}{2m}\right)^{n-1}|_{T=0}^{2m} T^{N-1}(z+m) - T^n = 0, \tag{48}$$

so $(2m)^{n-1}(z+m) - (2m)^n - 0 + 0 = 0$ and $z = m$. Bidder i's optimal strategy in the symmetric equilibrium is to bid $p_i = s_i - m$. The winning bid is set by the bidder with the highest signal, and that highest signal's expected value is

$$\begin{aligned} Es_{(1)} &= \underline{s} + \left(\tfrac{n+1-1}{n+1}\right)(\overline{s} - \underline{s}) \\ &= v - m + \left(\tfrac{n}{n+1}\right)((v+m) - (v-m)) \\ &= v - m + \left(\tfrac{n}{n+1}\right)(2m) \end{aligned} \tag{49}$$

The expected revenue is therefore

$$Ep_{(1)} = v - (1)\left(\frac{1}{n+1}\right)2m. \tag{50}$$

Here, the revenue is even lower than under the first two auction rules.

[11] xxx This does not explain why he does not,f or example, shrink his signal, bidding zs_i. Think about that.

[12] xxx why? Property of continuous density?

A Mechanism to Extract All the Surplus (see Myerson [1981])

Ask bidder i to declare s_i, allocate the good to the high bidder at the price $\frac{s_{(1)}+s_{(n)}}{2}$, which is an unbiased estimate of v, and ensure truthtelling by the boiling-in- oil punishment of additional transfers of $t = -\infty$ if the reports are such that $s_{(n)} + m < s_{(1)}$, which cannot possibly occur if all bidders tell the truth.

Matthews (1987) takes the buyer's viewpoint and show that buyers with increasing absolute risk aversion and private values prefer first- price auctions to second-price, even though the prices are higher, because they are also less risky.

Myerson (1981) shows that if the bidders' private information is correlated, the seller can construct a mechanism that extracts all the information and all the surplus.

Bidders' signals are **affiliated** if a high value of one bidder's signal makes high values of the other bidders' signals more likely, roughly.

13.5 Information in Common-Value Auctions

The Seller's Information

Milgrom & Weber (1982) have found that honesty is the best policy as far as the seller is concerned. If it is common knowledge that he has private information, he should release it before the auction. The reason is not that the bidders are risk averse (though perhaps this strengthens the result), but the "No news is bad news" result of section 8.1. If the seller refuses to disclose something, buyers know that the information must be unfavorable, and an unravelling argument tells us that the quality must be the very worst possible.

Quite apart from unravelling, another reason to disclose information is to mitigate the winner's curse, even if the information just reduces uncertainty over the value without changing its expectation. In trying to avoid the winner's curse, bidders lower their bids, so anything which makes it less of a danger raises their bids.

Asymmetric Information among the Buyers

Suppose that Smith and Brown are two of many bidders in a common-value auction. If Smith knows he has uniformly worse information than Brown (that is, if his information partition is coarser than Brown's), he should stay out of the auction: his expected payoff is negative if Brown expects zero profits.

If Smith's information is not uniformly worse, he can still benefit by entering the auction. Having independent information, in fact, is more valuable than having good information. Consider a common-value, first-price, sealed-bid auction with four bidders. Bidders Smith and Black have the same good information, Brown has that same information plus an extra signal, and Jones usually has only a poor estimate, but one different from any other bidder's. Smith and Black should drop out of the auction – they can never beat Brown without overpaying. But Jones will sometimes win, and his expected surplus is positive. If, for example, real estate tracts are being sold, and Jones is quite ignorant of land values, he can still do well if, on rare occasions, he has inside information concerning the location of a new freeway, even though ordinarily he should refrain from bidding. If Smith and Black both use the same appraisal formula, they will compete each other's profits away, and if Brown uses the formula plus extra private information, he drives their profits negative by taking some of the best deals from them and leaving the worst ones.

In general, a bidder should bid less if there are more bidders or his information is absolutely worse (that is, if his information partition is coarser). He should also bid less if parts of his information partition are coarser than those of his rivals, even if his information is not uniformly worse. These considerations are most important in sealed-bid auctions, because in an open-cry auction information is revealed by the bids while other bidders still have time to act on it.

Notes

N13.1 Auction classification and private-value strategies

- McAfee & McMillan (1987) and Milgrom (1987) are excellent older surveys of the literature and theory of auctions. Both articles take some pains to relate the material to models of asymmetric information. More recent is Klemperer (1999). Milgrom & Weber (1982) is a classic article that covers many aspects of auctions. Paul Milgrom's consulting firm, Agora Market Design, has a website with many good working papers that can be found via http: www.market-design.com. Klemperer (2000) collects many of the most important articles.

- Cassady (1967) is an excellent source of institutional detail on auctions. The appendix to his book includes advertisements and sets of auction rules, and he cites numerous newspaper articles.

- Bargaining and auctions are two extremes in ways to sell goods. In between are various mixtures such as bargaining with an outside option of going to someone else, auctions with reserve prices, and so forth. For a readable comparison of the two sale methods, see Bulow & Klemperer (1996).

N13.2 Comparing auction rules

- Many (all?) leading auction theorists were involved in the seven- billion dollar spectrum auction by the United States government in 1994, either helping the government sell spectrum or helping bidders decide how to buy it. Paul Milgrom's 1999 book, *Auction Theory for Privatization*, tells the story. See also McAfee & McMillan (1996). Interesting institutional details have come in the spectrum auctions and stimulated new theoretical research. Ayres & Cramton (1996), for example, explore the possibility that affirmative action provisions designed to help certain groups of bidders may have actually increased the revenue raised by the seller by increasing the amount of competition in the auction.

- One might think that an ascending second-price, open-cry auction would come to the same results as an ascending first-price, open-cry auction, because if the price advances by ϵ at each bid, the first and second bids are practically the same. But the second-price auction can be manipulated. If somebody initially bids \$10 for something worth \$80, another bidder could safely bid \$1,000. No one else would bid more, and he would pay only the second price: \$10.

- In one variant of the English auction, the auctioneer announces each new price and a bidder can hold up a card to indicate he is willing to bid that price. This set of rules is more practical to administer in large crowds and it also allows the seller to act strategically during the course of the auction. If, for example, the two highest valuations are 100 and 130, this kind of auction could yield a price of 110, while the usual rules would only allow a price of $100 + \varepsilon$.

- Vickrey (1961) notes that a Dutch auction could be set up as a second-price auction. When the first bidder presses his button, he primes a buzzer that goes off when a second bidder presses a button.

- Auctions are especially suitable for empirical study because they are so stylized and generate masses of data. Hendricks & Porter (1988) is a classic comparison of auction theory with data. Tenorio (1993) is another nice example of empirical work using data from real auctions, in his case, the Zambian foreign exchange market. See Laffont (1997) for a survey of empirical work.

- Second-price auctions have actually been used in a computer operating system. An operating system must assign a computer's resources to different tasks, and researchers at Xerox Corporation designed the Spawn system, under which users allocate "money" in a second-price sealed bid auction for computer resources. See "Improving a Computer Network's Efficiency," *The New York Times*, (March 29, 1989) p. 35.

- After the last bid of an open-cry art auction in France, the representative of the Louvre has the right to raise his hand and shout "pre-emption de l'etat," after which he takes the painting at the highest price bid (*The Economist*, May 23, 1987, p. 98). How does that affect the equilibrium strategies? What would happen if the Louvre could resell?

- **Share Auctions.** In a share auction each buyer submits a bid for both a quantity and a price. The bidder with the highest price receives the quantity for which he bid at that price. If any of the product being auctioned remains, the bidder with the second-highest price takes the quantity he bid for, and so forth. The rules of a share auction can allow each buyer to submit several bids, often called a **schedule** of bids. The details of share auctions vary, and they can be either first-price or second-price. Models of share auctions are very complicated; see Wilson (1979).

- **Reserve prices.** A reserve price is one below which the seller refuses to sell. Reserve prices can increase the seller's revenue, and their effect is to make the auction more like a regular fixed-price market. For discussion, see Milgrom & Weber (1982). They are also useful when buyers collude, a situation of bilateral monopoly. See "At Many Auctions, Illegal Bidding Thrives as a Longtime Practice Among Dealers," *Wall Street Journal*, February 19, 1988 p. 21. In some real-world English auctions, the auctioneer does not announce the reserve price in advance, and he starts the bidding below it. This can be explained as a way of allowing bidders to show each other that their valuations are greater than the starting price, even though it may turn out that they are all lower than the reserve price.

- Concerning auctions with risk-averse players, see Maskin & Riley (1984).

- Che & Gale (1998) point out that if bidders differ in their willingness to pay in a private value auction because of budget constraints rather than tastes then the revenue equivalence theorem can fail. The following example from page 2 of their paper shows this. Suppose two budget-constrained bidders are bidding for one object. In auction 1, each buyer has a budget of 2 and knows only his own value, which is drawn uniformly from [0,1]. The budget constraints are never binding, and it turns out that the expected price is 1/3 under either a first-price or a second-price auction. In auction 2, however, each buyer knows only his own budget, which is drawn uniformly from [0,1], and both have values for the object of 2. The budget constraint is always binding, and the equilibrium strategy is to bid one's entire budget under either set of auction rules. The expected price is still 1/3 in the second- price auction, but now it is 2/3 in the first-price auction. The seller therefore prefers to use a first-price auction.

- **The Dollar Auction.** Auctions look like tournaments in that the winner is the player who chooses the largest amount for some costly variable, but in auctions the losers generally do not incur costs proportional to their bids. Shubik (1971), however, has suggested an auction for a dollar bill in which both the first- and second-highest bidders pay the second price. If both players begin with infinite wealth, the game illustrates why equilibrium might not exist if strategy sets are unbounded. Once one bidder has started bidding against another, both of them do best by continuing to bid so as to win the dollar as well as pay the bid. This auction may seem absurd, but it has considerable similarity to patent races (see section xxx) and arms races. See Baye & Hoppe (2003) for more on the equivalence between innovation games and auctions.

- The dollar auction is just one example of auctions in which more than one player ends up paying. It is an **all-pay auction**, in which every player ends up paying, not just the winner. Even odder is the **loser-pays auction**, a two-player auction in which only the loser pays. All-pay auctions are a standard way to model rentseeking: imagine that N players each exert e in effort simultaneously to get a prize worth V, the winner being whoever's effort is highest.

  Rentseeking is a bit different, though. One difference is that it is often realistic to model it as a contest in which the highest bidder has the best chance to win, but lower bidders might win instead. Tullock (1980) started a literature on this in an article that mistakenly argued that the expected amount paid might exceed the value of the prize. See Baye, Kovenock & de Vries (1999) for a more recent analysis of this **rent dissipation**). There is no obvious way to model contests, and the functional form does matter to behavior, as Jack Hirshleifer (1989) tells us. The most popular functional form is this one, in which P_1 and P_2 are the probability of winning of the two players, e_1 and e_2 are their efforts, and R and θ are parameters which can be used to increase the probability that the high bidder wins and to give one player an advantage over the other.

$$P_1 = \frac{\theta e_1^R}{\theta e_1^R + e_2^R} \quad P_2 = \frac{e_2^R}{\theta e_1^R + e_2^R} \tag{51}$$

  If $\theta = 0$ and R becomes large, this becomes close to the simple all-pay auction, because neither player has an advantage and the highest bidder wins with probability near one.

Once we depart from true auctions, it is also often plausible that the size of the prize increases with effort (when the contest is a mechanism used by a principal to motivate agents– see Chung [1996]) or that the prize shrinks with effort (see Alexeev & Leitzel [1996]).

N13.3 Common-value auctions and the winner's curse

- The winner's curse and the idea of common values versus private values have broad application. The winner's curse is related to the idea of regression to the mean discussed in section 2.4. Kaplow & Shavell (1996) use the idea to discuss property versus liability rules, one of the standard rule choices in law-and-economics. If someone violates a property rule, the aggrieved party can undo the violation, as when a thief is required to surrender stolen property. If someone violates a liability rule, the aggrieved party can only get monetary compensation, as when someone who breaches a contract is required to pay damages to the aggrieved party. Kaplow and Shavell argue that if a good has independent values, a liability rule is best because it gives efficient incentives for rule violation; but if it has common value and courts make errors in measuring the common value, a property rule may be better. See especially around page 761 of their article.

N13.4 Information in common-value auctions

- Even if valuations are correlated, the optimal bidding strategies can still be the same as in private-value auctions if the values are independent. If everyone overestimates their values by 10 percent, a player can still extract no information about his value by seeing other players' valuations.

- "Getting carried away" may be a rational feature of a common-value auction. If a bidder has a high private value and then learns from the course of the bidding that the common value is larger than he thought, he may well end up paying more than he had planned, although he would not regret it afterwards. Other explanations for why bidders seem to pay too much are the winner's curse and the fact that in every auction all but one or two of the bidders think that the winning bid is greater than the value of the object.

- Milgrom & Weber (1982) use the concept of **affiliated** variables in classifying auctions. Roughly speaking, random variables X and Y are affiliated if a larger value of X means that a larger value of Y is more likely, or at least, no less likely. Independent random variables are affiliated.

Problems

13.1. Rent Seeking
Two risk-neutral neighbors in sixteenth century England, Smith and Jones, have gone to court and are considering bribing a judge. Each of them makes a gift, and the one whose gift is the largest is awarded property worth £2,000. If both bribe the same amount, the chances are 50 percent for each of them to win the lawsuit. Gifts must be either £0, £900, or £2,000.

(a) What is the unique pure-strategy equilibrium for this game?

(b) Suppose that it is also possible to give a £1500 gift. Why does there no longer exist a pure-strategy equilibrium?

(c) What is the symmetric mixed-strategy equilibrium for the expanded game? What is the judge's expected payoff?

(d)) In the expanded game, if the losing litigant gets back his gift, what are the two equilibria? Would the judge prefer this rule?

13.2. The Founding of Hong Kong
The Tai-Pan and Mr. Brock are bidding in an English auction for a parcel of land on a knoll in Hong Kong. They must bid integer values, and the Tai-Pan bids first. Tying bids cannot be made, and bids cannot be withdrawn once they are made. The direct value of the land is 1 to Brock and 2 to the Tai-Pan, but the Tai-Pan has said publicly that he wants it, so if Brock gets it, he receives 5 in "face" and the Tai-Pan loses 10. Moreover, Brock hates the Tai-Pan and receives 1 in utility for each 1 that the Tai-Pan pays out to get the land.

(a) First suppose there were no "face" or "hate" considerations, just the direct values. What are the equilibria if the Tai-pan bids first?

(b) Continue supposing there were no "face" or "hate" considerations, just the direct values. What are the three possible equilibria if Mr. Brock bids first? (Hint: in one of them, Brock wins; in the other two, the Tai-pan wins.)

(c) Now fill in the entries in Table 13.2.

Table 13.2: The Tai-Pan Game

Winning bid:	1	2	3	4	5	6	7	8	9	10	11	12
If Brock wins:												
π_{Brock}												
$\pi_{Tai-Pan}$												
If Brock loses:												
π_{Brock}												
$\pi_{Tai-Pan}$												

(d) In equilibrium, who wins, and at what bid?

(e) What happens if the Tai-Pan can precommit to a strategy?

(f) What happens if the Tai-Pan cannot precommit, but he also hates Brock, and gets 1 in utility for each 1 that Brock pays out to get the land?

13.3. Government and Monopoly
Incumbent Apex and potential entrant Brydox are bidding for government favors in the widget market. Apex wants to defeat a bill that would require it to share its widget patent rights with Brydox. Brydox wants the bill to pass. Whoever offers the chairman of the House Telecommunications Committee more campaign contributions wins, and the loser pays nothing. The market demand curve is $P = 25 - Q$, and marginal cost is constant at 1.

(a) Who will bid higher if duopolists follow Bertrand behavior? How much will the winner bid?

(b) Who will bid higher if duopolists follow Cournot behavior? How much will the winner bid?

(c) What happens under Cournot behavior if Apex can commit to giving away its patent freely to everyone in the world if the entry bill passes? How much will Apex bid?

13.4. An Auction with Stupid Bidders
Smith's value for an object has a private component equal to 1 and component common with Jones and Brown. Jones's and Brown's private components both equal zero. Each player estimates the common component Z independently, and player i's estimate is either x_i above the true value or x_i below, with equal probability. Jones and Brown are naive and always bid their valuations. The auction is English. Smith knows X_i, but not whether his estimate is too high or too low.

(a) If $x_{Smith} = 0$, what is Smith's dominant strategy if his estimate of Z equals 20?

(b) If $x_i = 8$ for all players and Smith estimates $Z = 20$, what are the probabilities that he puts on different values of Z?

(c) If $x_i = 8$ but Smith knows that $Z = 13$ with certainty, what are the probabilities he puts on the different combinations of bids by Jones and Brown?

(d) Why is 9 a better upper limit on bids for Smith than 21, if his estimate of Z is 20, and $x_i = 8$ for all three players?

(e) Suppose Smith could pay amount 0.001 to explain optimal bidding strategy to his rival bidders, Jones and Brown. Would he do so?

13.5. A Teapot Auction with Incomplete Information
Smith believes that Brown's value v_b for a teapot being sold at auction is 0 or 100 with equal probability. Smith's value of $v_s = 400$ is known by both players.

(a) What are the players' equilibrium strategies in an open cry auction? You may assume that in case of ties, Smith wins the auction.

(b) What are the players' equilibrium strategies in a first-price sealed-bid auction? You may

14 Pricing

January 17, 2000. December 12, 2003. 24 May 2005. Eric Rasmusen, Erasmuse@indiana.edu. Http://www.rasmusen.org. Footnotes starting with xxx are the author's notes to himself. Comments welcomed.

14.1 Quantities as Strategies: Cournot Equilibrium Revisited

Chapter 14 is about how firms with market power set prices. Section 14.1 generalizes the Cournot Game of Section 3.5 in which two firms choose the quantities they sell, while Section 14.2 sets out the Bertrand model of firms choosing prices.[1] Section 14.3 goes back to the origins of product differentiation, and develops two Hotelling location models. Section 14.4 shows how to do comparative statics in games, using the differentiated Bertrand model as an example and supermodularity and the implicit function theorem as tools. Section 14.5 shows that even if a firm is a monopolist, if it sells a durable good it suffers competition from its future self.

Cournot Behavior with General Cost and Demand Functions

In the next few sections, sellers compete against each other while moving simultaneously. We will start by generalizing the Cournot Game of Section 3.5 from linear demand and zero costs to a wider class of functions. The two players are firms Apex and Brydox, and their strategies are their choices of the quantities q_a and q_b. The payoffs are based on the total cost functions, $c(q_a)$ and $c(q_b)$, and the demand function, $p(q)$, where $q = q_a + q_b$. This specification says that only the sum of the outputs affects the price. The implication is that the firms produce an identical product, because whether it is Apex or Brydox that produces an extra unit, the effect on the price is the same.

Let us take the point of view of Apex. In the Cournot-Nash analysis, Apex chooses its output of q_a for a given level of q_b as if its choice did not affect q_b. From its point of view, q_a is a function of q_b, but q_b is exogenous. Apex sees the effect of its output on price as

$$\frac{\partial p}{\partial q_a} = \frac{dp}{dq}\frac{\partial q}{\partial q_a} = \frac{dp}{dq}. \tag{1}$$

Apex's payoff function is

$$\pi_a = p(q)q_a - c(q_a). \tag{2}$$

To find Apex's reaction function, we differentiate with respect to its strategy to obtain

$$\frac{d\pi_a}{dq_a} = p + \frac{dp}{dq}q_a - \frac{dc}{dq_a} = 0, \tag{3}$$

which implies

$$q_a = \frac{\frac{dc}{dq_a} - p}{\frac{dp}{dq}}, \tag{4}$$

[1] xxxx This intro needs reworking because of moved sections.

or, simplifying the notation,

$$q_a = \frac{c' - p}{p'}. \tag{5}$$

If particular functional forms for $p(q)$ and $c(q_a)$ are available, equation (5) can be solved to find q_a as a function of q_b. More generally, to find the change in Apex's best response for an exogenous change in Brydox's output, differentiate (5) with respect to q_b, remembering that q_b exerts not only a direct effect on $p(q_a + q_b)$, but possibly an indirect effect via q_a.

$$\frac{dq_a}{dq_b} = \frac{(p - c')(p'' + p''\frac{dq_a}{dq_b})}{p'^2} + \frac{c''\frac{dq_a}{dq_b} - p' - p'\frac{dq_a}{dq_b}}{p'}. \tag{6}$$

Equation (6) can be solved for $\frac{dq_a}{dq_b}$ to obtain the slope of the reaction function,

$$\frac{dq_a}{dq_b} = \frac{(p - c')p'' - p'^2}{2p'^2 - c''p' - (p - c')p''} \tag{7}$$

If both costs and demand are linear, as in section 3.5, then $c'' = 0$ and $p'' = 0$, so equation (7) becomes

$$\frac{dq_a}{dq_b} = -\frac{p'^2}{2p'^2} = -\frac{1}{2}. \tag{8}$$

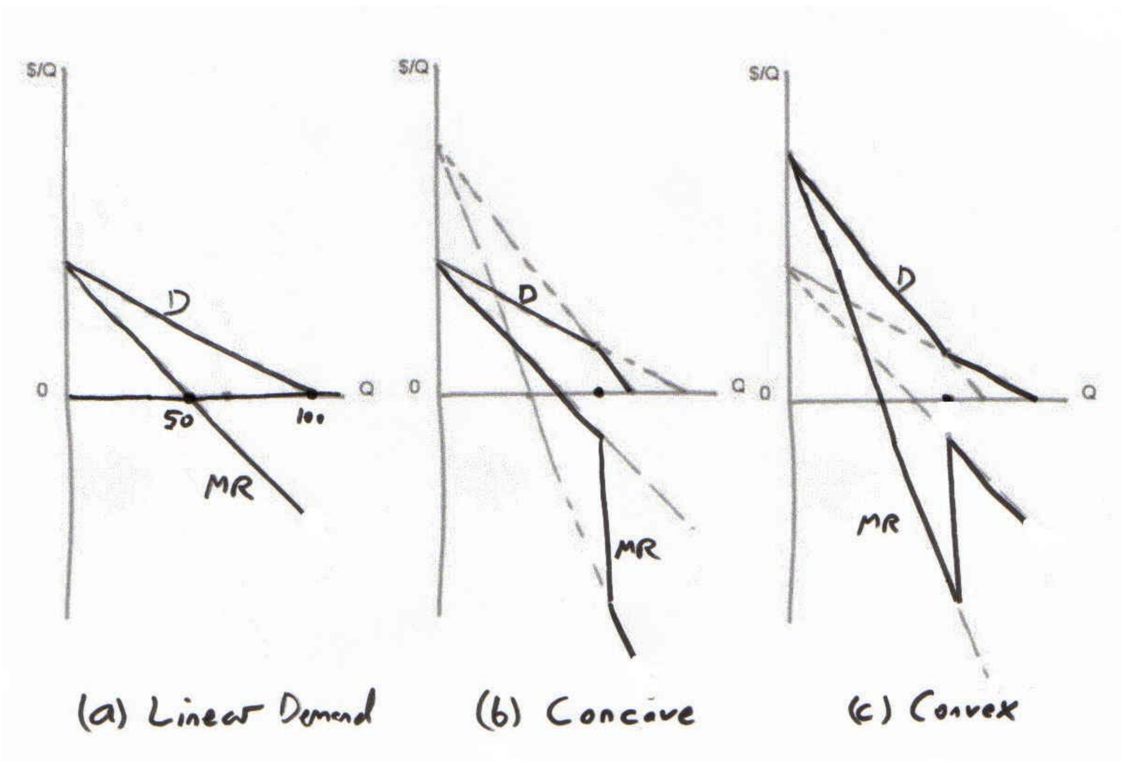


Figure X: Different Demand Curves

The general model faces two problems that did not arise in the linear model: nonuniqueness and nonexistence. If demand is concave and costs are convex, which implies that $p'' < 0$

and $c'' > 0$, then all is well as far as existence goes. Since price is greater than marginal cost ($p > c'$), equation (7) tells us that the reaction functions are downward sloping, because $2p'^2 - c''p' - (p - c')p''$ is positive and both $(p - c')p''$ and $-p'^2$ are negative. If the reaction curves are downward sloping, they cross and an equilibrium exists, as was shown in Chapter 3's Figure 1 for the linear case represented by equation (8). We usually do assume that costs are at least weakly convex, since that is the result of diminishing or constant returns, but there is no reason to believe that demand is either concave or convex. If the demand curves are not linear, the contorted reaction functions of equation (7) might give rise to multiple Cournot equilibria as in the present chapter's Figure 1.[2]

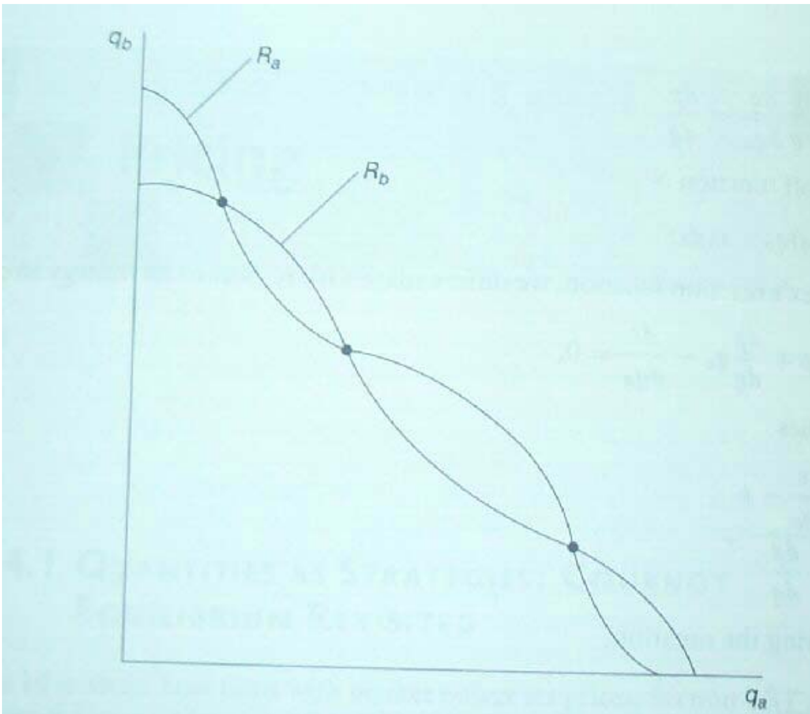


Figure 1: Multiple Cournot-Nash Equilibria

If demand is convex or costs are concave, so $p'' > 0$ or $c'' < 0$, the reaction functions can be upward sloping, in which case they might never cross and no equilibrium would exist. The problem can also be seen from Apex's payoff function, equation (2). If $p(q)$ is convex, the payoff function might not be concave, in which case standard maximization techniques break down. The problems of the general Cournot model teach a lesson to modellers: sometimes simple assumptions such as linearity generate atypical results.

Many Oligopolists[3]

Let us return to the simpler game in which production costs are zero and demand is linear. For concreteness, we will use the particular inverse demand function

$$p(q) = 120 - q. \tag{9}$$

[2]xxx Here goes fig14.demand.jpg.

[3]xxx Use positive marginal costs throughout.

Using (9), the payoff function, (2), becomes

$$\pi_a = 120q_a - q_a^2 - q_b q_a. \tag{10}$$

In section 3.5, firms picked outputs of 40 apiece given demand function (9). This generated a price of 40. With n firms instead of two, the demand function is

$$p\left(\sum_{i=1}^{n} q_i\right) = 120 - \sum_{i=1}^{n} q_i, \tag{11}$$

and firm j's payoff function is

$$\pi_j = 120q_j - q_j^2 - q_j \sum_{i \neq j} q_i. \tag{12}$$

Differentiating j's payoff function with respect to q_j yields

$$\frac{d\pi_j}{dq_j} = 120 - 2q_j - \sum_{i \neq j} q_i = 0. \tag{13}$$

The first step in finding the equilibrium is to guess that it is symmetric, so that $q_j = q_i$, $(i = 1, \ldots, n)$. This is an educated guess, since every player faces a first-order condition like (13). By symmetry, equation (13) becomes $120 - (n+1)q_j = 0$, so that

$$q_j = \frac{120}{n+1}. \tag{14}$$

Consider several different values for n. If $n = 1$, then $q_j = 60$, the monopoly optimum; and if $n = 2$ then $q_j = 40$, the Cournot output found in section 3.5. If $n = 5$, $q_j = 20$; and as n rises, individual output shrinks to zero. Moreover, the total output of $nq_j = \frac{120n}{n+1}$ gradually approaches 120, the competitive output, and the market price falls to zero, the marginal cost of production. As the number of firms increases, profits fall.

14.2 Prices as Strategies

Here we will explore the Bertrand model more.

Capacity Constraints: the Edgeworth Paradox

Let us start by altering the Bertrand model by constraining each firm to sell no more than $K = 70$ units. The industry capacity of 140 exceeds the competitive output, but do profits continue to be zero?

When capacities are limited we require additional assumptions because of the new possibility that a firm with a lower price might attract more consumers than it can supply. We need to specify a **rationing rule** telling which consumers are served at the low price and which must buy from the high-price firm. The rationing rule is unimportant to the payoff of the low-price firm, but crucial to the high-price firm. One possible rule is

Intensity rationing. *The consumers able to buy from the firm with the lower price are those who value the product most.*

The inverse demand function from equation (9) is $p = 120 - q$, and under intensity rationing the K consumers with the strongest demand buy from the low-price firm. Suppose that Brydox is the low-price firm, charging a price of 30, so that 90 consumers wish to buy from it, though only K can do so. The residual demand facing Apex is then

$$q_a = 120 - p_a - K. \tag{15}$$

This is the demand curve in Figure 2a.

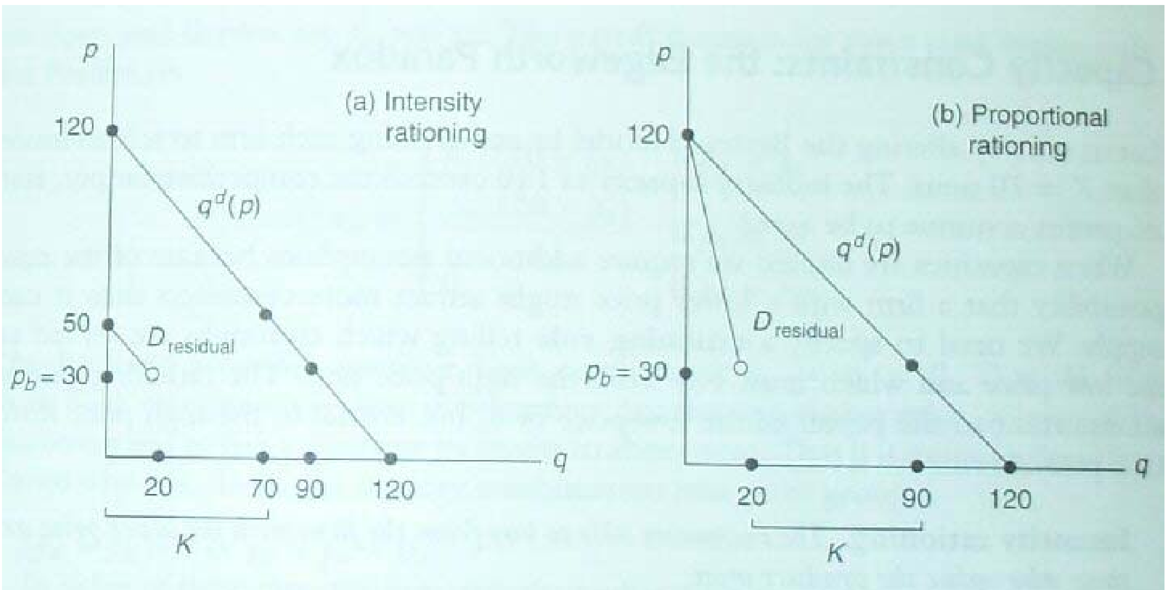


Figure 2: Rationing Rules when $p_b = 30$, $p_a > 30$, and $K = 70$

Under **intensity rationing**, if $K = 70$ the payoff functions are

$$\pi_a = \begin{cases} p_a \cdot Min\{120 - p_a, 70\} & \text{if } p_a < p_b \quad (a) \\ \frac{p_a(120-p_a)}{2} & \text{if } p_a = p_b \quad (b) \\ 0 & \text{if } p_a > p_b, p_b \geq 50 \quad (c) \\ p_a(120 - p_a - 70) & \text{if } p_a > p_b, p_b < 50 \quad (d) \end{cases} \tag{16}$$

Here is why equations (16c) and (16d) look the way they do. If Brydox has the lower price, all consumers will want to buy from Brydox if they buy at all, but only 70 will be able to. If Brydox's price is more than 50, then less than 70 will want to buy at all, and so 0 consumers will be left for Apex – which is equation (16c). If Brydox's price is less than 50, then Brydox will sell 70 units, and the residual demand curve facing Apex is as in equation (15), yielding equation (16d).

The appropriate rationing rule depends on what is being modelled. Intensity rationing is appropriate if buyers with more intense demand make greater efforts to obtain low prices. If the intense buyers are wealthy people who are unwilling to wait in line, the least intense buyers might end up at the low-price firm which is the case of **inverse-intensity rationing**. An intermediate rule is proportional rationing, under which every type of consumer is equally likely to be able to buy at the low price.

Proportional rationing. *Each consumer has the same probability of being able to buy from the low-price firm.*

Under proportional rationing, if $K = 70$ and 90 consumers wanted to buy from Brydox, 2/9 ($= \frac{q(p_b)-K}{q(p_b)}$) of each type of consumer will be forced to buy from Apex (for example, 2/9 of the type willing to pay 120). The residual demand curve facing Apex, shown in Figure 14.2b and equation (17), intercepts the price axis at 120, but slopes down at a rate three times as fast as market demand because there are only 2/9 as many remaining consumers of each type.

$$q_a = (120 - p_a)\left(\frac{120 - p_b - K}{120 - p_b}\right) \tag{17}$$

The capacity constraint has a very important effect: (0,0) is no longer a Nash equilibrium in prices. Consider Apex's best response when Brydox charges a price of zero. If Apex raises his price above zero, he retains most of his consumers (because Brydox is already producing at capacity), but his profits rise from zero to some positive number, regardless of the rationing rule. In any equilibrium, both players must charge prices within some small amount ϵ of each other, or the one with the lower price would deviate by raising his price. But if the prices are equal, then both players have unused capacity, and each has an incentive to undercut the other. No pure-strategy equilibrium exists under either rationing rule. This is known as the **Edgeworth paradox** after Edgeworth (1897, 1922).

Suppose that demand is linear, with the highest reservation price being $p = 100$ and the maximum market quantity $Q = 100$ at $p = 0$. Suppose also that there are two firms, Apex and Brydox, each having a constant marginal cost of 0 up to capacity of $Q = 80$ and infinity thereafter. We will assume intensity rationing of buyers.

Note that industry capacity of 160 exceeds market demand of 100 if price equals marginal cost. Note also that the monopoly price is 50, which with quantity of 50 yields industry profit of 2,500. But what will be the equilibrium?

Prices of $(p_a = 0, p_b = 0)$ are not an equilibrium. Apex's profit would be zero in that strategy combination. If Apex increased its price to 5, what would happen? Brydox would immediately sell $Q = 80$, and to the most intense 80 percent of buyers. Apex would be left with all the buyers between $p = 20$ and $p = 5$ on the demand curve, for $Q_a = 15$ and profit of $\pi_a = (5)(15) = 75$. So deviation by Apex is profitable. (Of course, $p = 5$ is not necessarily the most profitable deviation – but we do not need to check that; I looked for an *easy* deviation.)

Equal prices of (p_a, p_b) with $p_a = p_b > 0$ are not an equilibrium. Even if the price is close to 0, Apex would sell at most 50 units as its half of the market, which is less than its capacity of 80. Apex could deviate to just below p_b and have a discontinuous jump in sales for an increase in profit, just as in the basic Bertrand game.

Unequal prices of (p_a, p_b) are not an equilibrium. Without loss of generality, suppose $p_a > p_b$. So long as p_b is less than the monopoly price of 50, Brydox would deviate to a new price even close to but not exceeding p_a. And this is not *just* the open-set problem. Once Brydox is close enough to Apex, Apex would deviate by jumping to a price just below Brydox.

If capacities are large enough, the Edgeworth paradox disappears. Consider capacities of 150 per firm, for example. The argument made above for why equal prices of 0 is not an equilibrium fails, because if Apex were to deviate to a positive price, Brydox would be fully capable of serving the entire market, leaving Apex with no consumers.

If capacities are small enough, the Edgeworth paradox also disappears, but so does the Bertrand paradox. Suppose each firm has a capacity of 20. They each will choose to sell at a price of 60, in which case they will each sell 20 units, their entire capacities. Apex will have a payoff of 1,200. If Apex deviates to a lower price, it will not sell any more, so that would be unprofitable. If Apex deviates to a higher price, it will sell fewer, and since the monopoly price is 50, its profit will be lower; note that a price of 61 and a quantity of 19 yields profits of 1,159, for example.

We could have expanded the model to explain why the firms have small capacities by adding a prior move in which they choose capacity subject to a cost per unit of capacity, foreseeing what will happen later in the game.

A mixed-strategy equilibrium does exist, calculated using intensity rationing by Levitan & Shubik (1972) and analyzed in Dasgupta & Maskin (1986b). Expected profits are positive, because the firms charge positive prices. Under proportional rationing, as under intensity rationing, profits are positive in equilibrium, but the high-price firm does better with proportional rationing. The high-price firm would do best with **inverse-intensity rationing**, under which the consumers with the least intense demand are served at the low-price firm, leaving the ones willing to pay more at the mercy of the high-price firm.

Even if capacity were made endogenous, the outcome would be inefficient, either because firms would charge prices higher than marginal cost (if their capacity were low), or they would invest in excess capacity (even though they price at marginal cost).

14.3 Location Models

In Section 14.2 we analyzed the Bertrand model with differentiated products using demand functions whose arguments were the prices of both firms. Such a model is suspect because it is not based on primitive assumptions. In particular, the demand functions might not be generated by maximizing any possible utility function. A demand curve with a constant elasticity less than one, for example, is impossible because as the price goes to zero, the amount spent on the commodity goes to infinity. Also, demand curves (**??**) and (**??**) were restricted to prices below a certain level, and it would be good to be able to justify that restriction.

Location models construct demand functions like (??) and (??) from primitive assumptions. In location models, a differentiated product's characteristics are points in a space. If cars differ only in their mileage, the space is a one-dimensional line. If acceleration is also important, the space is a two-dimensional plane. An easy way to think about this approach is to consider the location where a product is sold. The product "gasoline sold at the corner of Wilshire and Westwood," is different from "gasoline sold at the corner of Wilshire and Fourth." Depending on where consumers live, they have different preferences over the two, but, if prices diverge enough, they will be willing to switch from one gas station to the other.

Location models form a literature in themselves. We will look at the first two models analyzed in the classic article of Hotelling (1929), a model of price choice and a model of location choice. Figure 4 shows what is common to both. Two firms are located at points x_a and x_b along a line running from zero to one, with a constant density of consumers throughout. In the Hotelling Pricing Game, firms choose prices for given locations. In the Hotelling Location Game, prices are fixed and the firms choose the locations.

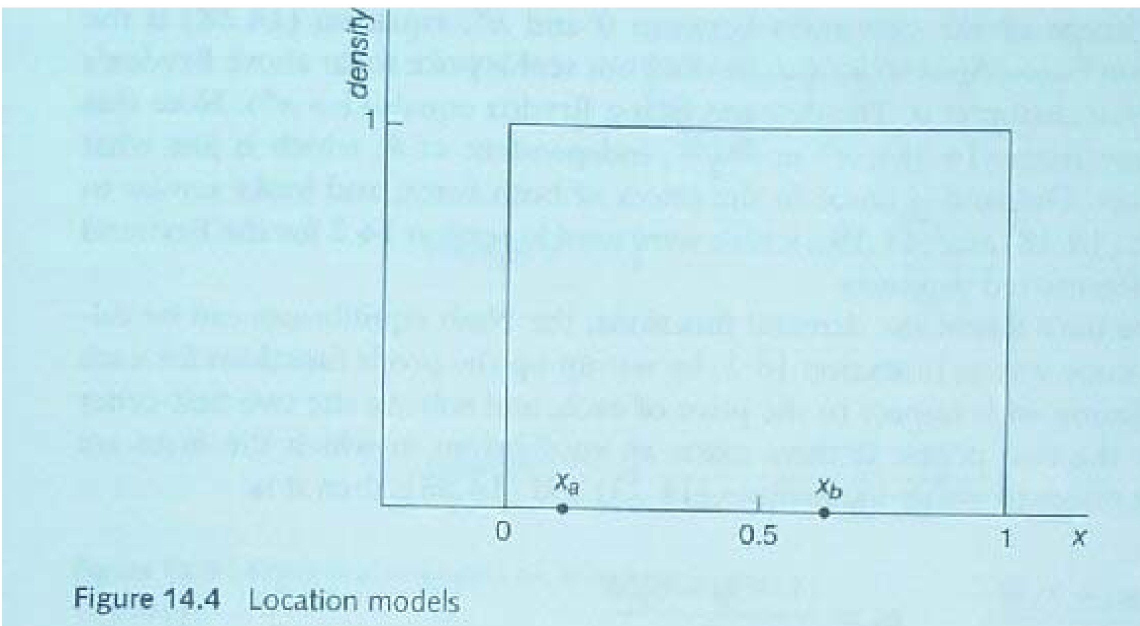


Figure 14.4 Location models

Figure 4: Location Models

The Hotelling Pricing Game

(Hotelling [1929])

Players

Sellers Apex and Brydox, located at x_a and x_b, where $x_a < x_b$, and a continuum of buyers indexed by location $x \in [0, 1]$.

The Order of Play

1 The sellers simultaneously choose prices p_a and p_b.
2 Each buyer chooses a seller.

Payoffs
Demand is uniformly distributed on the interval [0,1] with a density equal to one (think of each consumer as buying one unit). Production costs are zero. Each consumer always buys, so his problem is to minimize the sum of the price plus the linear transport cost, which is θ per unit distance travelled.

$$\pi_{buyer\ at\ x} = V - Min\{\theta|x_a - x| + p_a,\ \theta|x_b - x| + p_b\}. \tag{18}$$

$$\pi_a = \begin{cases} p_a(0) = 0 & \text{if } p_a - p_b > \theta(x_b - x_a) \quad \text{(a)} \\ & \text{(Brydox captures entire market)} \\ p_a(1) = p_a & \text{if } p_b - p_a > \theta(x_b - x_a) \quad \text{(b)} \\ & \text{(Apex captures entire market)} \\ p_a(\frac{1}{2\theta}\left[(p_b - p_a) + \theta(x_a + x_b)\right]) & \text{otherwise (the market is divided)} \quad \text{(c)} \end{cases} \tag{19}$$

Brydox has analogous payoffs.

The payoffs result from buyer behavior. A buyer's utility depends on the price he pays and the distance he travels. Price aside, Apex is most attractive of the two sellers to the consumer at $x = 0$ ("consumer 0") and least attractive to the consumer at $x = 1$ ("consumer 1"). Consumer 0 will buy from Apex so long as

$$V - (\theta x_a + p_)) > V - (\theta x_b + p_b), \tag{20}$$

which implies that

$$p_a - p_b < \theta(x_b - x_a), \tag{21}$$

which yields payoff (19a) for Apex. Consumer 1 will buy from Brydox if

$$V - [\theta(1 - x_a) + p_a] < V - [\theta(1 - x_b) + p_b], \tag{22}$$

which implies that

$$p_b - p_a < \theta(x_b - x_a), \tag{23}$$

which yields payoff (19b) for Apex.

Very likely, inequalities (21) and (23) are both satisfied, in which case Consumer 0 goes to Apex and Consumer 1 goes to Brydox. This is the case represented by payoff (19c), and the next task is to find the location of consumer x^*, defined as the consumer who is at the boundary between the two markets, indifferent between Apex and Brydox. First, notice that if Apex attracts Consumer x_b, he also attracts all $x > x_b$, because beyond x_b the consumers' distances from both sellers increase at the same rate. So we know that if there is an indifferent consumer he is between x_a and x_b. Knowing this, (18) tells us that

$$V - [\theta(x^* - x_a) + p_a = V - -[\theta(x_b - x^*) + p_b], \tag{24}$$

so that

$$p_b - p_a = \theta(2x^* - x_a - x_b), \tag{25}$$

and

$$x^* = \frac{1}{2\theta}\left[(p_b - p_a) + \theta(x_a + x_b)\right]. \tag{26}$$

Do remember that equation (26) is valid only if there really does exist a consumer who is indifferent – if such a consumer does not exist, equation (26) will generates a number for x^*, but that number is meaningless.

Since Apex keeps all the consumers between 0 and x^*, equation (26) is the demand function facing Apex so long as he does not set his price so far above Brydox's that he loses even consumer 0. The demand facing Brydox equals $(1 - x^*)$. Note that if $p_b = p_a$, then from (26), $x^* = \frac{x_a+x_b}{2}$, independent of θ, which is just what we would expect. Demand is linear in the prices of both firms, and looks similar to demand curves (**??**) and (**??**), which were used in Section 3.xxx for the Bertrand game with differentiated products.[4]

Now that we have found the demand functions, the Nash equilibrium can be calculated in the same way as in Section 14.2, by setting up the profit functions for each firm, differentiating with respect to the price of each, and solving the two first-order conditions for the two prices. If there exists an equilibrium in which the firms are willing to pick prices to satisfy inequalities (21) and (23), then it is

$$p_a = \frac{(2 + x_a + x_b)\theta}{3}, \quad p_b = \frac{(4 - x_a - x_b)\theta}{3}. \tag{27}$$

From (27) one can see that Apex charges a higher price if a large x_a gives it more safe consumers or a large x_b makes the number of contestable consumers greater. The simplest case is when $x_a = 0$ and $x_b = 1$, when (27) tells us that both firms charge a price equal to θ. Profits are positive and increasing in the transportation cost.

We cannot rest satisfied with the neat equilibrium of equation (27), because the assumption that there exists an equilibrium in which the firms choose prices so as to split the market on each side of some boundary consumer x^* is often violated. Hotelling did not notice this, and fell into a common mathematical trap. Economists are used to models in which the calculus approach gives an answer that is both the local optimum and the global optimum. In games like this one, however, the local optimum is not global, because of the discontinuity in the objective function. Vickrey (1964) and D'Aspremont, Gabszewicz & Thisse (1979) have shown that if x_a and x_b are close together, no pure-strategy equilibrium exists, for reasons similar to why none exists in the Bertrand model with capacity constraints. If both firms charge nonrandom prices, neither would deviate to a slightly different price, but one might deviate to a much lower price that would capture every single consumer. But if both firms charged that low price, each would deviate by raising his price slightly. It turns out that if, for example, Apex and Brydox are located symmetrically around the center of the interval, $x_a \geq 0.25$, and $x_b \leq 0.75$, no pure-strategy equilibrium exists (although a mixed-strategy equilibrium does, as Dasgupta & Maskin [1986b] show).

Hotelling should have done some numerical examples. And he should have thought about the comparative statics carefully. Equation (27) implies that Apex should choose a

[4]xxx Make the link to Section 14.2 clearer.

higher price if both x_a and x_b increase, but it is odd that if the firms are locating closer together, say at 0.90 and 0.91, that Apex should be able to charge a higher price, rather than suffering from more intense competition. This kind of odd result is a typical clue that the result has a logical flaw somewhere. Until the modeller can figure out an intuitive reason for his odd result, he should suspect an error. For practice, let us try a few numerical examples, illustrated in Figure 5.

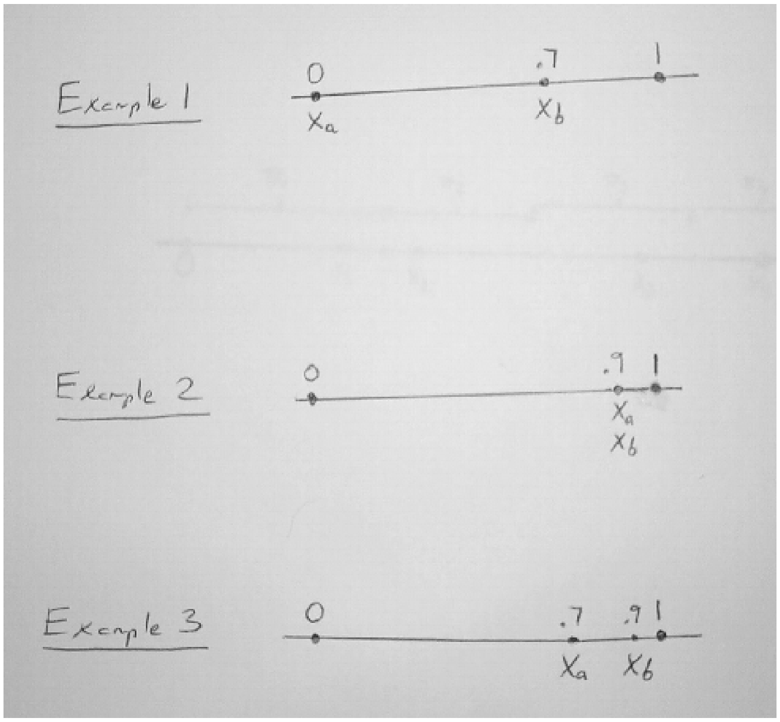


Figure 5: Numerical examples for Hotelling pricing

Example 1. Everything works out simply
Try $x_a = 0, x_b = 0.7$ and $\theta = 0.5$. Then equation (27) says $p_a = (2{+}0{+}0.7)0.5/3 = 0.45$ and $p_b = (4{-}0{-}0.7)0.5/3 = 0.55$. Equation (26) says that $x^* = \frac{1}{2*0.5}[(0.55 - 0.45) + 0.5(0.0 + 0.7)] = 0.45$.

In Example 1, there is a pure strategy equilibrium and the equations generated sensible numbers given the parameters we chose. But it is not enough to calculate just one numerical example.

Example 2. Same location – but different prices?
Try $x_a = 0.9, x_b = 0.9$ and $\theta = 0.5$. Then equation (27) says $p_a = (2.0 + 0.9 + 0.9)0.5/3 \approx 0.63$ and $p_b = (4.0 - 0.9 - 0.9)0.5/3 \approx 0.37$.

Example 2 shows something odd happening. The equations generate numbers that seem innocuous until one realizes that if both firms are located at 0.9, but $p_a = 0.63$ and $p_b = 0.37$, then Brydox will capture the entire market! The result is nonsense, because equation (27)'s derivation relied on the assumption that $x_a < x_b$, which is false in this example.

Example 3. Locations too close to each other.
$x^* < x_a < x_b$. Try $x_a = 0.7, x_b = 0.9$ and $\theta = 0.5$. Then equation (27) says that $p_a = (2.0 + 0.7 + 0.9)0.5/3 = 0.6$ and $p_b = (4 - 0.7 - 0.9)0.5/3 = 0.4$. As for the split of the market, equation (26) says that $x^* = \frac{1}{2*0.5}[(0.4 - 0.6) + 0.5(0.7 + 0.9)] = 0.6$.

Example 3 shows a serious problem. If the market splits at $x^* = 0.6$ but $x_a = 0.7$ and $x_b = 0.9$, the result violates our implicit assumption that the players split the market. Equation (26) is based on the premise that there does exist some indifferent consumer, and when that is a false premise, as under the parameters of Example 3, equation (26) will still spit out a value of x^*, but the value will not mean anything. And in fact the consumer at $x = 0.6$ is not really indifferent between Apex and Brydox. He could buy from Apex at a total cost of 0.6 + 0.1(0.5) = 0.65 or from Brydox, at a total cost of 0.4 + 0.3 (0.5) = 0.55. In fact, there exists no consumer who strictly prefers Apex. Even Apex's 'home' consumer at $x = 0.7$ would have a total cost of buying from Brydox of $0.4+0.5(0.9-0.7) = 0.5$ and would prefer Brydox. Similarly, the consumer at $x = 0$ would have a total cost of buying from Brydox of $0.4 + 0.5(0.9 - 0.0) = 0.85$, compared to a cost from Apex of $0.6 + 0.5(0.7 - 0.0) = 0.95$, and he, too, would prefer Brydox.

The problem in examples 2 and 3 is that the firm with the higher price would do better to deviate with a discontinuous price cut to just below the other firm's price. Equation (27) was derived by calculus, with the implicit assumption that a local profit maximum was also a global profit maximum, or, put differently, that if no small change could raise a firm's payoff, then it had found the optimal strategy. Sometimes a big change will increase a player's payoff even though a small change would not. Perhaps this is what they mean in business by the importance of "nonlinear thinking" or "thinking out of the envelope." The everyday manager or scientist as described by Schumpeter (1934) and Kuhn (1970) concentrates on analyzing incremental changes and only the entrepreneur or genius breaks through with a discontinuously new idea, the profit source or paradigm shift.

Let us now turn to the choice of location. We will simplify the model by pushing consumers into the background and imposing a single exogenous price on all firms.

The Hotelling Location Game

(Hotelling [1929])

Players
n Sellers.

The Order of Play
The sellers simultaneously choose locations $x_i \in [0, 1]$.

Payoffs
Consumers are distributed along the interval [0,1] with a uniform density equal to one. The price equals one, and production costs are zero. The sellers are ordered by their location so $x_1 \leq x_2 \leq \ldots \leq x_n$, $x_0 \equiv 0$ and $x_{n+1} \equiv 1$. Seller i attracts half the consumers from the gaps on each side of him, as shown in figure 14.6, so that his payoff is

$$\pi_1 = x_1 + \frac{x_2 - x_1}{2}, \tag{28}$$

$$\pi_n = \frac{x_n - x_{n-1}}{2} + 1 - x_n, \tag{29}$$

or, for $i = 2, \ldots n-1$,

$$\pi_i = \frac{x_i - x_{i-1}}{2} + \frac{x_{i+1} - x_i}{2}. \tag{30}$$

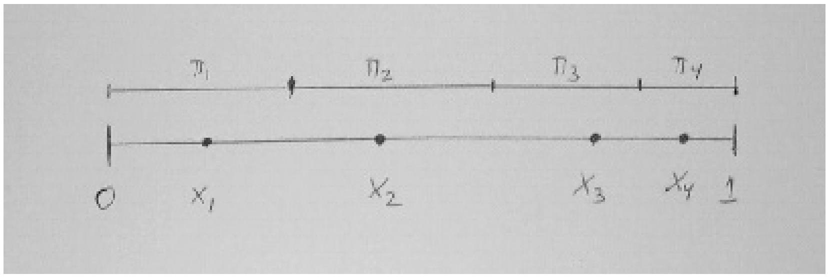


Figure 6: Payoffs in the Hotelling Location Game

With **one seller**, the location does not matter in this model, since the consumers are captive. If price were a choice variable and demand were elastic, we would expect the monopolist to locate at $x = 0.5$.

With **two sellers**, both firms locate at $x = 0.5$, regardless of whether or not demand is elastic. This is a stable Nash equilibrium, as can be seen by inspecting Figure 4 and imagining best responses to each other's location. The best response is always to locate ε closer to the center of the interval than one's rival. When both firms do this, they end up splitting the market since both of them end up exactly at the center.

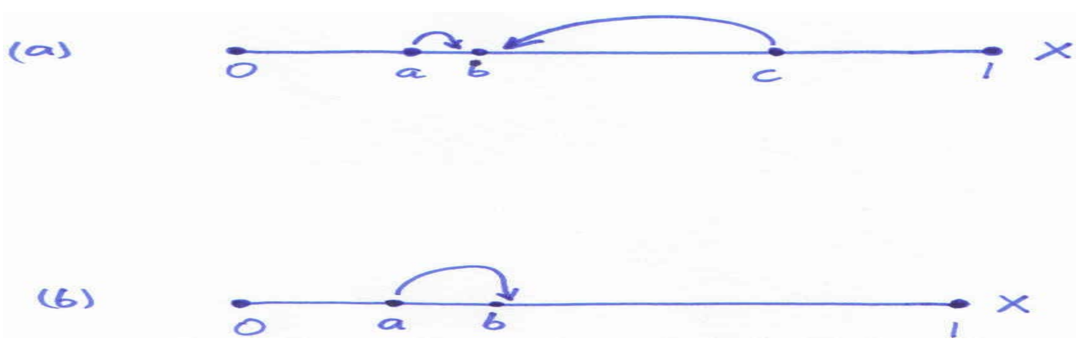


Figure 7: Nonexistence of pure strategies with three players

With **three sellers** the model does not have a Nash equilibrium in pure strategies. Consider any strategy combination in which each player locates at a separate point. Such a strategy combination is not an equilibrium, because the two players nearest the ends would edge in to squeeze the middle player's market share. But if a strategy combination has any two players at the same point a, as in Figure 7, the third player would be able to acquire a share of at least $(0.5 - \epsilon)$ by moving next to them at b; and if the third player's share is that large, one of the doubled-up players would deviate by jumping to his other side and capturing his entire market share. The only equilibrium is in mixed strategies.

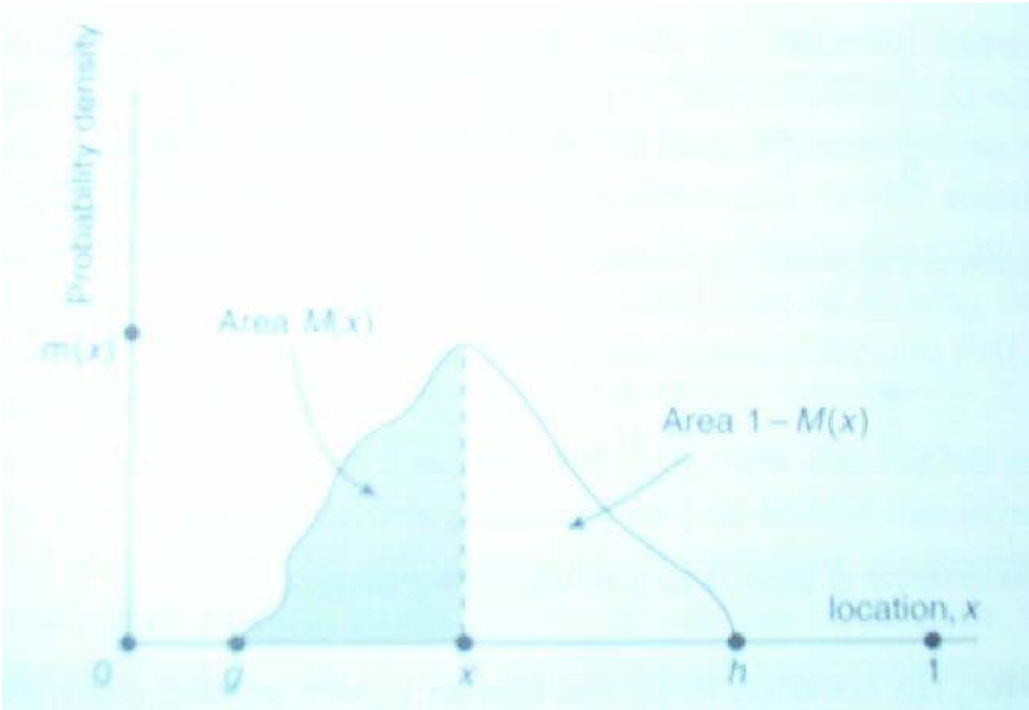


Figure 8: The Equilibrium Mixed-Strategy Density in the Three-Player Location Game

Suppose all three players use the same mixing density, with $m(x)$ the probability density for location x, and positive density on the support $[g, h]$, as depicted in Figure 8. We will need the density for the distribution of the minimum of the locations of Players 2 and 3. Player 2 has location x with density $m(x)$, and Player 3's location is greater than that with probability $1 - M(x)$, letting M denote the cumulative distribution, so the density for Player 2 having location x and it being smaller is $m(x)[1 - M(x)]$. The density for either Player 2 or Player 3 choosing x and it being smaller than the other firm's location is then $2m(x)[1 - M(x)]$.

If Player 1 chooses $x = g$ then his expected payoff is

$$\pi_1(x_1 = g) = g + \int_g^h 2m(x)[1 - M(x)] \left(\frac{x - g}{2}\right) dx, \tag{31}$$

where g is the safe set of consumers to his left, $2m(x)[1 - M(x)]$ is the density for x being the next biggest location of a firm, and $\frac{x-g}{2}$ is Player 1's share of the consumers between his own location of g and the next biggest location.

If Player 1 chooses $x = h$ then his expected payoff is, similarly,

$$\pi_1(x_1 = h) = (1 - h) + \int_g^h 2m(x)M(x) \left(\frac{h - x}{2}\right) dx, \tag{32}$$

where $(1 - h)$ is the set of safe consumers to his right

In a mixed strategy equilibrium, Player 1's payoffs from these two pure strategies must be equal, and they are also equal to his payoff from a location of 0.5, which we can plausibly guess is in the support of his mixing distribution. Going on from this point, the algebra and calculus start to become fierce. Shaked (1982) has computed the symmetric mixing probability density $m(x)$ to be as shown in Figure 9,

$$m(x) = \begin{cases} 2 & \text{if } \frac{1}{4} \leq x \leq \frac{3}{4} \\ 0 & \text{otherwise} \end{cases} \tag{33}$$

I do not know how Shaked came to his answer, but I would tackle the problem by guessing that $M(x)$ was a uniform distribution and seeing if it worked, which was perhaps his method too. (You can check that using this mixing density, the payoffs in equation (31) and (32) do equal each other.) Note also that this method has only shown what the symmetric equilibrium is like; it turns out that asymmetric equilibria also exist (Osborne & Pitchik [1986]).

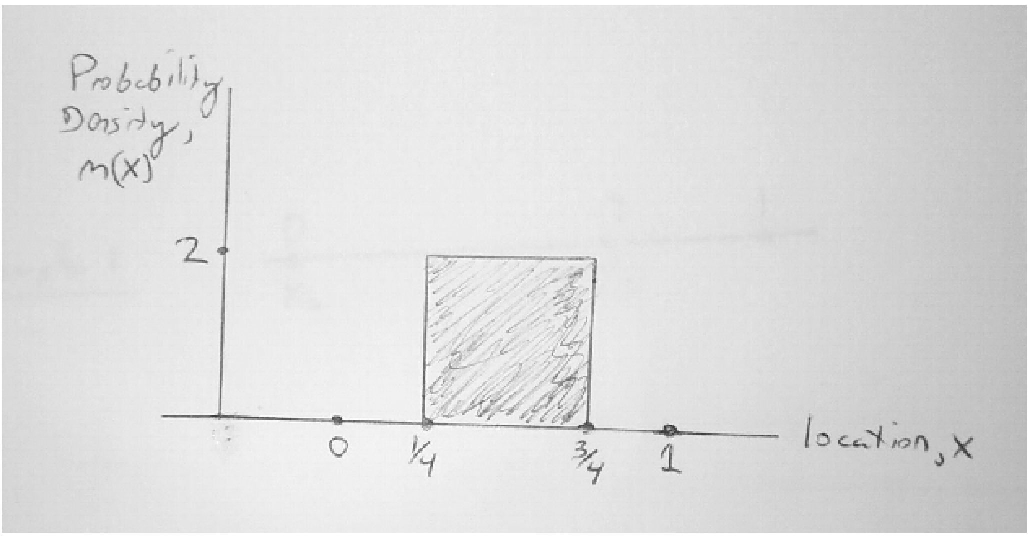


Figure 9: The Equilibrium Mixing Density for Location

Strangely enough, three is a special number. With **more than three sellers**, an equilibrium in pure strategies does exist if the consumers are uniformly distributed, but this is a delicate result (Eaton & Lipsey [1975]). Dasgupta & Maskin (1986b), as amended by Simon (1987), have also shown that an equilibrium, possibly in mixed strategies, exists for any number of players n in a space of any dimension m.

Since prices are inflexible, the competitive market does not achieve efficiency. A benevolent social planner or a monopolist who could charge higher prices if he located his outlets closer to more consumers would choose different locations than competing firms. In particular, when two competing firms both locate in the center of the line, consumers are no better off than if there were just one firm. The average distance of a consumer from a seller would be minimized by setting $x_1 = 0.25$ and $x_2 = 0.75$, the locations that would be chosen either by the social planner or the monopolist.[5]

[5]xxxInsert fig14.effieincy.jpg here.

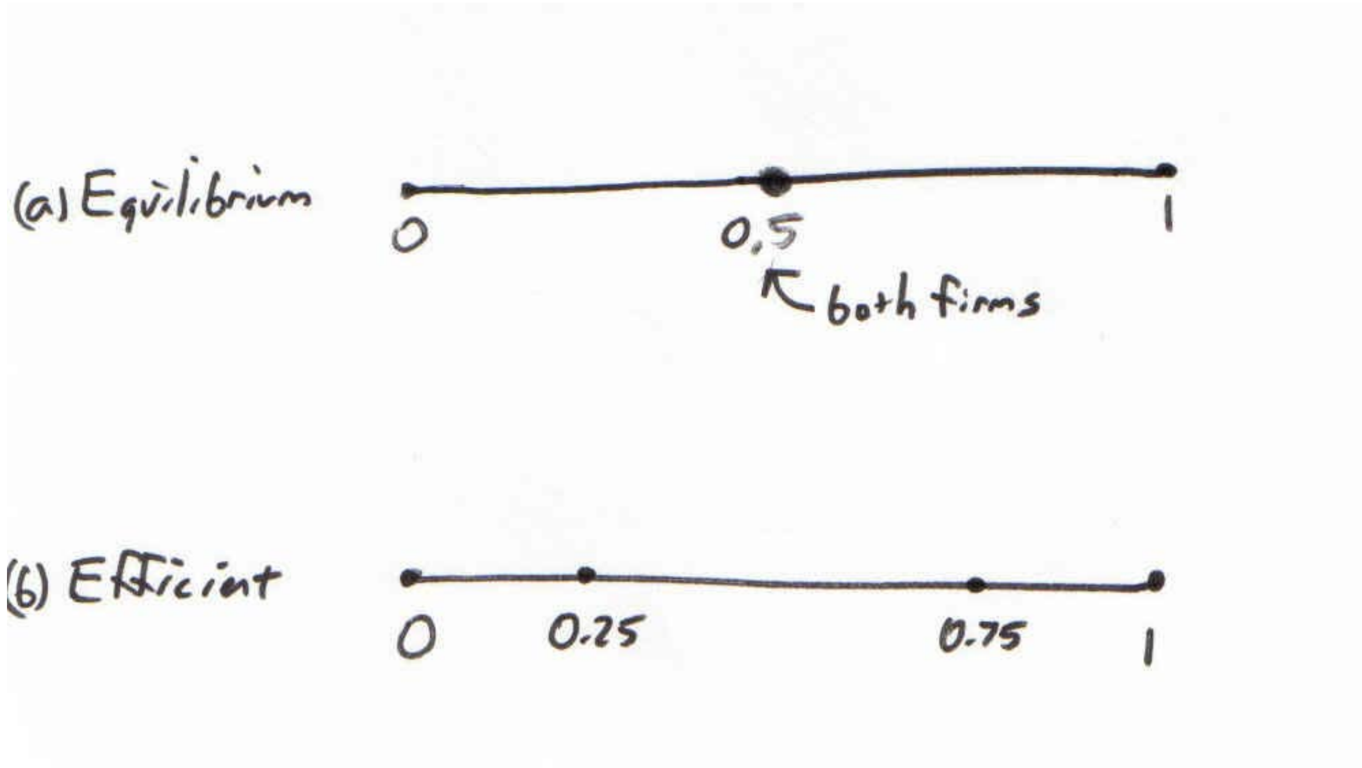


Figure 9X: Equilibrium versus Efficiency

The Hotelling Location Model, however, is very well suited to politics. Often there is just one dimension of importance in political races, and voters will vote for the candidate closest to their own position, so there is no analog to price. The Hotelling Location Model predicts that the two candidates will both choose the same position, right on top of the median voter. This seems descriptively realistic; it accords with the common complaint that all politicians are pretty much the same.

14.4 Comparative Statics and Supermodular Games

Comparative statics is the analysis of what happens to endogenous variables in a model when the exogenous variable change. This is a central part of economics. When wages rises, for example, we wish to know how the price of steel will change in response. Game theory presents special problems for comparative statics, because when a parameter changes, not only does Smith's equilibrium strategy change in response, but Jones's strategy changes as a result of Smith's change as well. A small change in the parameter might produce a large change in the equilibrium because of feedback between the different players' strategies.

Let us use a differentiated Bertrand game as an example. Suppose there are N firms, and for firm j the demand curve is

$$Q_j = Max\{\alpha - \beta_j p_j + \sum_{i \neq j} \gamma_i p_i, 0\}, \tag{34}$$

with $\alpha \in (0, \infty)$, $\beta_i \in (0, \infty)$, and $\gamma_i \in (0, \infty)$ for $i = 1, \ldots, N$. Assume that the effect of p_j on firm j's sales is larger than the effect of the other firms' prices, so that

$$\beta_j > \sum_{i \neq j} \gamma_i. \tag{35}$$

Let firm i have constant marginal cost κc_i, where $\kappa \in \{1, 2\}$ and $c_i \in (0, \infty)$, and let us assume that each firm's costs are low enough that it does operate in equilibrium. (The shift variable κ could represent the effect of the political regime on costs.)

The payoff function for firm j is

$$\pi_j = (p_j - \kappa c_j)(\alpha - \beta_j p_j + \sum_{i \neq j} \gamma_i p_i). \tag{36}$$

Firms choose prices simultaneously.

Does this game have an equilibrium? Does it have several equilibria? What happens to the equilibrium price if a parameter such as c_j or κ changes? These are difficult questions because if c_j increases, the immediate effect is to change firm j's price, but the other firms will react to the price change, which in turn will affect j's price. Moreover, this is not a symmetric game – the costs and demand curves differ from firm to firm, which could make algebraic solution of the Nash equilibrium quite messy. It is not even clear whether the equilibrium is unique.

Two approaches to comparative statics can be used here: the implicit function theorem, and supermodularity. We will look at each in turn.

The Implicit Function Theorem

The implicit-function theorem says that if $f(y, z) = 0$, where y is endogenous and z is exogenous, then

$$\frac{dy}{dz} = -\left(\frac{\frac{\partial f}{\partial z}}{\frac{\partial f}{\partial y}} \right). \tag{37}$$

It is worth knowing how to derive this. We start with $f(y, z) = 0$, which can be rewritten as $f(y(z), z)) = 0$, since y is endogenous. Using the calculus chain rule,

$$\frac{df}{dz} = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial y}\frac{dy}{dz} = 0. \tag{38}$$

where the expression equals zero because after a small change in z, f will still equal zero after y adjusts. Solving for $\frac{dy}{dz}$ yields equation (37).

The implicit function theorem is especially useful if y is a choice variable and z a parameter, because then we can use the first-order condition to set $f(y, z) \equiv \frac{\partial \pi}{\partial y} = 0$ and the second-order condition tells us that $\frac{\partial f}{\partial y} = \frac{\partial^2 \pi}{\partial y^2} \leq 0$. One only has to make certain that the solution is an interior solution, so the first- and second- order conditions are valid.

We do have a complication if the model is strategic: there will be more than one endogenous variable, because more than one player is choosing variable values. Suppose that instead of simply $f(y,z) = 0$, our implicit equation has two endogenous and two exogenous variables, so $f(y_1, y_2, z_1, z_2) = 0$. The extra z_2 is no problem; in comparative statics we are holding all but one exogenous variable constant. But the y_2 does add something to the mix. Now, using the calculus chain rule yields not equation (38) but

$$\frac{df}{dz_1} = \frac{\partial f}{\partial z_1} + \frac{\partial f}{\partial y_1}\frac{dy_1}{dz_1} + \frac{\partial f}{\partial y_2}\frac{dy_2}{dz_1} = 0. \tag{39}$$

Solving for $\frac{dy_1}{dz_1}$ yields

$$\frac{dy_1}{dz_1} = -\left(\frac{\frac{\partial f}{\partial z_1} + \frac{\partial f}{\partial y_2}\frac{dy_2}{dz_1}}{\frac{\partial f}{\partial y_1}}\right). \tag{40}$$

It is often unsatisfactory to solve out for $\frac{dy_1}{dz_1}$ as a function of both the exogenous variables z_1 and z_2 and the endogenous variable y_2 (though it is okay if all you want is to discover whether the change is positive or negative), but ordinarily the modeller will also have available an optimality condition for Player 2 also: $g(y_1, y_2, z_1, z_2) = 0$. This second condition yields an equation like (40), so that two equations can be solved for the two unknowns.

We can use the differentiated Bertrand game to see how this works out. Equilibrium prices will lie inside the interval $(c_j, \overline{p})$ for some large number $\overline{p}$, because a price of c_j would yield zero profits, rather than the positive profits of a slightly higher price, and $\overline{p}$ can be chosen to yield zero quantity demanded and hence zero profits. The equilibrium or equilibria are, therefore, interior solutions, in which case they satisfy the first-order condition

$$\frac{\partial \pi_j}{\partial p_j} = \alpha - 2\beta_j p_j + \sum_{i \neq j} \gamma_i p_i + \kappa c_j \beta_j = 0, \tag{41}$$

and the second-order condition,

$$\frac{\partial^2 \pi_j}{\partial p_j^2} = -2\beta_j < 0. \tag{42}$$

Next, apply the implicit function theorem by using p_i and c_i, $i = 1, \ldots, N$, instead of y_i and z_i, $i = 1, 2$, and by letting $\frac{\partial \pi_j}{\partial p_j} = 0$ from equation (41) be our $f(y_1, y_2, z_1, z_2) = 0$. The chain rule yields

$$\frac{df}{dc_j} = -2\beta_j \frac{dp_j}{dc_j} + \sum_{i \neq j} \gamma_i \frac{dp_i}{dc_j} + \kappa \beta_j = 0, \tag{43}$$

so

$$\frac{dp_j}{dc_j} = \frac{\sum_{i \neq j} \gamma_i \frac{dp_i}{dc_j} + \kappa \beta_j}{2\beta_j}. \tag{44}$$

Just what is $\frac{dp_i}{dc_j}$? For each i, we need to find the first-order condition for firm i and then use the chain rule again. The first-order condition for Player i is that the derivative of π_i with respect to p_i (*not* p_j) equals zero, so

$$g^i \equiv \frac{\partial \pi_i}{\partial p_i} = \alpha - 2\beta_i p_i + \sum_{k \neq i} \gamma_k p_k + \kappa c_i \beta_i = 0. \tag{45}$$

The chain rule yields (keeping in mind that it is a change in c_j that interests us, *not* a change in c_i),

$$\frac{dg^i}{dc_j} = -2\beta_i \frac{dp_i}{dc_j} + \sum_{k \neq i} \gamma_k \frac{dp_k}{dc_j} = 0. \tag{46}$$

With equation (44), the $(N-1)$ equations (46) give us N equations for the N unknowns $\frac{dp_i}{dc_j}$, $i = 1, \ldots, N$.

It is easier to see what is going on if there are just two firms, j and i. Equations (44) and (46) are then

$$\frac{dp_j}{dc_j} = \frac{\gamma_i \frac{dp_i}{dc_j} + \kappa \beta_j}{2\beta_j}. \tag{47}$$

and

$$-2\beta_i \frac{dp_i}{dc_j} + \gamma_j \frac{dp_j}{dc_j} = 0. \tag{48}$$

Solving these two equations for $\frac{dp_j}{dc_j}$ and $\frac{dp_i}{dc_j}$ yields

$$\frac{dp_j}{dc_j} = \frac{2\beta_i \beta_j \kappa}{4\beta_i \beta_j - \gamma_i \gamma_j} \tag{49}$$

and

$$\frac{dp_i}{dc_j} = \frac{\gamma_j \beta_j \kappa}{4\beta_i \beta_j - \gamma_i \gamma_j}. \tag{50}$$

Keep in mind that the implicit function theorem only tells about infinitesimal changes, not finite changes. If c_n increases enough, then the nature of the equilibrium changes drastically, because firm n goes out of business. Even if c_n increases a finite amount, the implicit function theorem is not applicable, because then the change in p_n will cause changes in the prices of other firms, which will in turn change p_n again.

We cannot go on to discover the effect of changing κ on p_n, because κ is a discrete variable, and the implicit function theorem only applies to continuous variables. The implicit function theorem is none the less very useful when it does apply. This is a simple example, but the approach can be used even when the functions involved are very complicated. In complicated cases, knowing that the second-order condition holds allows the modeller to avoid having to determine the sign of the denominator if all that interests him is the sign of the relationship between the two variables.

Supermodularity

The second approach uses the idea of the supermodular game, an idea related to that of strategic complements (Chapter 3.6). Suppose that there are N players in a game, subscripted by i and j, and that player i has a strategy consisting of $\overline{s}^i$ elements, subscripted by s and t, so his strategy is the vector $y^i = (y^i_1, \ldots, y^i_{\overline{s}^i})$. Let his strategy set be S^i and his payoff function be $\pi^i(y^i, y^{-i}; z)$, where z represents a fixed parameter. We say that the game is a **smooth supermodular game** if the following four conditions are satisfied for every player $i = 1, \ldots N$:

A1′ The strategy set is an interval in $R^{\overline{s}^i}$:

$$S^i = [\underline{y^i}, \overline{y^i}]. \tag{51}$$

A2′ π^i is twice continuously differentiable on S^i.

A3′ (Supermodularity) Increasing one component of player i's strategy does not decrease the net marginal benefit of any other component: for all i, and all s and t such that $1 \leq s < t \leq \overline{s}^i$,

$$\frac{\partial^2 \pi^i}{\partial y^i_s \partial y^i_t} \geq 0. \tag{52}$$

A4′ (Increasing differences in one's own and other strategies) Increasing one component of i's strategy does not decrease the net marginal benefit of increasing any component of player j's strategy: for all $i \neq j$, and all s and t such that $1 \leq s \leq \overline{s}^i$ and $1 \leq t \leq \overline{s}^j$,

$$\frac{\partial^2 \pi^i}{\partial y^i_s \partial y^j_t} \geq 0. \tag{53}$$

In addition, we will be able to talk about the comparative statics of smooth supermodular games if a fifth condition is satisfied, the increasing differences condition, (A5′).

A5′: (Increasing differences in one's own strategies and parameters) Increasing parameter z does not decrease the net marginal benefit to player i of any component of his own strategy: for all i, and all s such that $1 \leq s \leq \overline{s}^i$,

$$\frac{\partial^2 \pi^i}{\partial y^i_s \partial z} \geq 0. \tag{54}$$

The heart of supermodularity is in assumptions $A3'$ and $A4'$. Assumption $A3'$ says that the components of player i's strategies are all **complementary inputs**; when one component increases, it is worth increasing the other components too. This means that even if a strategy is a complicated one, one can still arrive at qualitative results about the strategy, because all the components of the optimal strategy will move in the same direction together. Assumption A4′ says that the strategies of players i and j are **strategic complements**; when player i increases a component of his strategy, player j will want to do so also. When the strategies of the players reinforce each other in this way, the feedback between them is less tangled than if they undermined each other.

I have put primes on the assumptions because they are the special cases, for smooth games, of the general definition of supermodular games in the Mathematical Appendix.

Smooth games use differentiable functions, but the supermodularity theorems apply more generally. One condition that is relevant here is condition A5:

A5: π^i has increasing differences in y^i and z for fixed y^{-i}; for all $y^i \geq y^{i\prime}$, the difference $\pi^i(y^i, y^{-i}, z) - \pi^i(y^{i\prime}, y^{-i}, z)$ is nondecreasing with respect to z.

Is the differentiated Bertrand game supermodular? The strategy set can be restricted to $[c_i, \overline{p}]$ for player i, so A1$'$ is satisfied. π_i is twice continuously differentiable on the interval$[c_i, \overline{p}]$, so A2$'$ is satisfied. A player's strategy has just one component, p_i, so A3$'$ is immediately satisfied. The following inequality is true,

$$\frac{\partial^2 \pi_i}{\partial p_i \partial p_j} = \gamma_j > 0, \tag{55}$$

so A4$'$ is satisfied. And it is also true that

$$\frac{\partial^2 \pi_i}{\partial p_i \partial c_i} = \kappa \beta_i > 0, \tag{56}$$

so A5$'$ is satisfied for c_i.

From equation (41), $\frac{\partial \pi_i}{\partial p_i}$ is increasing in κ, so $\pi_i(p_i, p_{-i}, \kappa) - \pi_i(p_i', p_{-i}, \kappa)$ is nondecreasing in κ for $p_i > p_i'$, and A5 is satisfied for κ.

Thus, all the assumptions are satisfied. This being the case, a number of theorems can be applied, including the following two.

Theorem 1. *If the game is supermodular, there exists a largest and a smallest Nash equilibrium in pure strategies.*

Theorem 2. *If the game is supermodular and assumption (A5) or (A5$'$) is satisfied, then the largest and smallest equilibrium are nondecreasing functions of the parameter z.*

Applying Theorems 1 and 2 yields the following results for the differentiated Bertrand game:

1. There exists a largest and a smallest Nash equilibrium in pure strategies (Theorem 1).

2. The largest and smallest equilibrium prices for firm i are nondecreasing functions of the cost parameters c_i and κ (Theorem 2).

Note that supermodularity, unlike the implicit function theorem, has yielded comparative statics on κ, the discrete exogenous variable. It yields weaker comparative statics on c_i, however, because it just finds the effect of c_i on p_i^* to be nondecreasing, rather than telling us its value or whether it is actually increasing.

14.5 Vertical Differentiation[6]

Vertical Differentiation I: Monopoly Quality Choice

Players
A seller and a continuum of buyers.

The Order of Play
0 There is a continuum of buyers of length 1 parametrized by quality desire θ_i distributed by Nature uniformly on $[0,1]$.
1 The seller picks quality s_1 from the interval $[0, \overline{s}]$.
2 The seller picks prices p_1 from the interval $[0, \infty)$.
3 Buyer i chooses one unit of a good, or refrains from buying. The seller produces it at constant marginal cost c, which does not vary with quality.

Payoffs
The seller maximizes

$$(p_1 - c)q_1 \tag{57}$$

Buyer i's payoff is zero if he does not buy, and if he does buy it is

$$(\underline{\theta} + \theta_i)s_1 - p_1, \tag{58}$$

where the parameter $\underline{\theta} \in (0,1)$ is the same for all buyers.

The participation constraint for consumer i is

$$(\underline{\theta} + \theta_i)s_1 - p_1 \geq 0, \tag{59}$$

which will be binding for some buyer type θ^* for which

$$(\underline{\theta} + \theta^*)s_1 = p_1, \tag{60}$$

so $\theta^* = \frac{p_1}{s_1} - \underline{\theta}$, and

$$q_i = (1 - \theta^*) = 1 + \underline{\theta} - \frac{p_1}{s_1}. \tag{61}$$

The seller maximizes

$$\pi = (p_1 - c)q_1 = (p_1 - c)[1 + \underline{\theta} - \frac{p_1}{s_1}]. \tag{62}$$

This is clearly maximized at the corner solution of $s_1 = \overline{s}$, since high quality has no extra cost. Then the first order condition for choosing p_1 is

$$\frac{d\pi}{dp_1} = 1 + \underline{\theta} - 2\frac{p_1}{\overline{s}} + \frac{c}{\overline{s}} = 0, \tag{63}$$

[6] xxx From Tirole, p. 296 and chatper 3 Price disc. Shaked-Sutton (1983) Econometrica. "Natural Oligopolies."

so

$$p_1 = \frac{(\underline{\theta}+1)\overline{s}+c}{2}; \tag{64}$$

that is, the price is halfway between c and $(\underline{\theta}+1)\overline{s}$, which is the valuation of the most quality-valuing buyer.

The seller's profit is then

$$\pi = (p_1 - c)q_1 = (\frac{(\underline{\theta}+1)\overline{s}+c}{2} - c)[1 + \underline{\theta} - \frac{\frac{(\underline{\theta}+1)\overline{s}+c}{2}}{s_1}]. \tag{65}$$

Next we will allow the seller to use two quality levels. A social planner would just use one– the maximal one of $s = \overline{s}$– since it is no cheaper to produce lower quality. The monopoly seller might use two, however, because it can help him to price discriminate between

Vertical Differentiation II: Price Discrimination Using Quality

Players
A seller and a continuum of buyers.

The Order of Play
0 There is a continuum of buyers of length 1 parametrized by quality desire θ_i distributed by Nature uniformly on $[0, 1]$.
1 The seller picks qualities s_1 and s_2 from the interval $[0, \overline{s}]$.
2 The seller picks prices p_1 and p_2 from the interval $[0, \infty)$.
3 Buyer i chooses one unit of a good, or refrains from buying. The seller produces it at constant marginal cost c, which does not vary with quality.

Payoffs
The seller maximizes

$$(p_1 - c)q_1 + (p_2 - c)q_2 \tag{66}$$

Buyer i's payoff is zero if he does not buy, and if he does buy it is

$$(\underline{\theta} + \theta_i)s - p, \tag{67}$$

where the parameter $\underline{\theta} \in (0, 1)$ is the same for all buyers.

This is a problem of mechanism design and price discrimination by quality. The seller needs to picks s_1, s_2, p_1, and p_2 to satisfy incentive compatibility and participation constraints if he wants to offer two qualities with positive sales of both, and he also needs to decide if that is more profitable than offering just one quality (e.g., choosing s_1 and p_1 to satisfy participation constraints and choosing s_2 and p_2 to violate them).

We already solved the one-quality problem, in Vertical Differentiation I.

Let us assume that the seller constructs his mechanism so that every type of buyer does make a purchase (xxx CHeck on this later.)

When two qualities are used, buyers with $\theta \in [0, q_1]$ buy the low quality and buyers with $\theta \in [q_1, 1]$ buy the high quality. Demand for the high-quality good will be $q_2 = q - q_1$. The high quality will be set to $s_2 = \overline{s}$, since if any lower s_2 were picked the seller could increase s_2 and p_2 could rise, increasing his payoff.

The buyer with $\theta \in [0, q_1]$ who has the greatest temptation to buy high quality instead of low is the one with $\theta = q_1$. He must satisfy the self-selection constraint

$$(\underline{\theta} + q_1)s_1 - p_1 \geq (\underline{\theta} + q_1)s_2 - p_2. \tag{68}$$

The buyer with $\theta \in [q_1, 1]$ who has the greatest temptation to buy low quality instead of high is the one with $\theta = q_1$. He must satisfy the self-selection constraint

$$(\underline{\theta} + q_1)s_1 - p_1 \leq (\underline{\theta} + q_1)s_2 - p_2. \tag{69}$$

Since this is the same buyer type, we see that the constraints have to be satisfied as equalities. Substituting $s_2 = \overline{s}$, we get

$$(\underline{\theta} + q_1)s_1 - p_1 = (\underline{\theta} + q_1)\overline{s} - p_2. \tag{70}$$

Which participation constraint will be satisfied as an equality? That for $\theta = 0$. If the lowest of types had a positive surplus from buying the low-quality good, p_1 being less than his valuation of $(\underline{\theta} + 0)s_1$, then so would all the other types. Thus, the seller could increase p_1 without losing any customers. Since the customer with $\theta = 0$ gets zero surplus, we know that

$$\underline{\theta}s_1 - p_1 = 0. \tag{71}$$

Putting together equations (70) and (71) gives us

$$(\underline{\theta} + q_1)\frac{p_1}{\underline{\theta}} - p_1 = (\underline{\theta} + q_1)\overline{s} - p_2, \tag{72}$$

which when solved for q_1 yields us a sort of demand curve. Here's the algebra.

$$p_1 - p_1 + p_2 = (\underline{\theta} + q_1)\overline{s} - \frac{q_1 p_1}{\underline{\theta}} \tag{73}$$

and

$$p_2 = \tag{74}$$

Then the seller maximizes his payoff function by choice of p1 and p2, giving us two first order conditions to solve out.

This problem is mathematically identical to price discriminating by quantity purchases.

ASSUMPTION 1:

$$\underline{\theta} + 1 \geq \underline{\theta} \tag{75}$$

or

$$\underline{\theta} \leq 1. \tag{76}$$

This says that there is enough consumer heterogeneity.

If Assumption 1 is violated, then one firm takes over the market.

ASSUMPTION 2:

$$c + \frac{\underline{\theta} + 1 - 2\underline{\theta}}{3}(s_2 - s_1) \leq \underline{\theta} s_1. \tag{77}$$

This will ensure that the market is covered–that every consumer does buy from one firm or the other.

Vertical Differentiation III: Duopoly Quality Choice

Players
Two sellers and a continuum of buyers.

The Order of Play
0 There is a continuum of buyers of length 1 parametrized by quality desire θ_i distributed by Nature uniformly on $[0, 1]$.
1 Sellers 1 and 2 simultaneously qualities s_1 and s_2 from the interval $[0, \overline{s}]$.
2 Sellers 1 and 2 simultaneously pick prices p_1 and p_2 from the interval $[0, \infty)$.
3 Buyer i chooses one unit of a good, or refrains from buying. The sellers produces at constant marginal cost c, which does not vary with quality.

Payoffs
Seller j's payoff is

$$(p_j - c)q_j. \tag{78}$$

Buyer i's payoff is zero if he does not buy, and if he does buy, from seller j, it is

$$(\underline{\theta} + \theta_i)s_j - p_j, \tag{79}$$

where the parameter $\underline{\theta} \in (0, 1)$ is the same for all buyers.

Work back from the end of the game. Let us assume that the firms have chosen qualities so that each has some sales. Then there is an indifferent consumer type θ such that the consumer's payoff from each firm's good is eual. The demands will then be

$$q_1 = \frac{p_2 - p_1}{s_2 - s_1} - \underline{\theta} \tag{80}$$

and

$$q_2 = \underline{\theta} + 1 - \frac{p_2 - p_1}{s_2 - s_1} \tag{81}$$

WHen firms maximize their payoffs by choice of price, the reaction curves turn out to be

$$p_1 = \frac{p_2 + c - \underline{\theta}(s_2 - s_1)}{2} \tag{82}$$

and

$$p_2 = \frac{p_1 + c + (\underline{\theta} + 1)(s_2 - s_1)}{2} \tag{83}$$

The prices are strategic complements.

When solved for equilibrium prices, it turns out that $p_2 > p_1$ and Firm 2 has higher profits also, since cost is independent of quality. The profits are

$$\pi_1 = \frac{1 - \underline{\theta})^2(s_2 - s_1)}{9} \tag{84}$$

and

$$\pi_2 = \frac{(\underline{\theta} + 2)^2(s_2 - s_1)}{9} \tag{85}$$

Note that for both firms, profits are increasing in $s_2 - s_1$.

How about quality choice,in the first stage? Well, both firms benefit from having more different qualities. So in equilibrium, they will be separated as far as possible–

$$s_1 = \underline{\theta} \quad s_2 = \overline{s}. \tag{86}$$

Qualities are strategic substitutes.

If we replace c by $c(s)$, increasing costs in quality, and add another technical assumption, then there is a finite number of firms in equilibrium ("natural oligopoly") even if we reduce the fixed cost to 0. (Is there positive profit?)

14.5 Durable Monopoly[7]

Introductory economics courses are vague on the issue of the time period over which transactions take place. When a diagram shows the supply and demand for widgets, the x-axis is labelled "widgets, " not "widgets per week" or "widgets per year." Also, the diagram splits off one time period from future time periods, using the implicit assumption that supply and demand in one period is unaffected by events of future periods. One problem with this on the demand side is that the purchase of a good which lasts for more than one use is an investment; although the price is paid now, the utility from the good continues into the future. If Smith buys a house, he is buying not just the right to live in the house tomorrow, but the right to live in it for many years to come, or even to live in it for a few years and then sell the remaining years to someone else. The continuing utility he receives from this durable good is called its **service flow**. Even though he may not intend to rent out the house, it is an investment decision for him because it trades off present expenditure for future utility. Since even a shirt produces a service flow over more than an instant of time,

[7] xxx I need to relate this closely to the auctions Auction Equivalence Theorem adn to the Bargaining chapter. THere is a deep linkage.

the durability of goods presents difficult definitional problems for national income accounts. Houses are counted as part of national investment (and an estimate of their service flow as part of services consumption), automobiles as durable goods consumption, and shirts as nondurable goods consumption, but all are to some extent durable investments.

In microeconomic theory, "durable monopoly" refers not to monopolies that last a long time, but to monopolies that sell durable goods. These present a curious problem. When a monopolist sells something like a refrigerator to a consumer, that consumer drops out of the market until the refrigerator wears out. The demand curve is, therefore, changing over time as a result of the monopolist's choice of price, which means that the modeller should not make his decisions in one period and ignore future periods. Demand is not **time separable**, because a rise in price at time t_1 affects the quantity demanded at time t_2.

The durable monopolist has a special problem because in a sense he does have a competitor – himself in the later periods. If he were to set a high price in the first period, thereby removing high-demand buyers from the market, he would be tempted to set a lower price in the next period to take advantage of the remaining consumers. But if it were known he would lower the price, the high-demand buyers would not buy at a high price in the first period. The threat of the future low price forces the monopolist to keep his current price low.

To formalize this situation, let the seller have a monopoly on a durable good which lasts two periods. He must set a price for each period, and the buyer must decide what quantity to buy in each period. Because this one buyer is meant to represent the entire market demand, the moves are ordered so that he has no market power, as in the principal-agent models in Section 7.3. Alternatively, the buyer can be viewed as representing a continuum of consumers (see Coase [1972] and Bulow [1982]). In this interpretation, instead of "the buyer" buying q_1 in the first period, q_1 of the buyers each buy one unit in the first period.

Durable Monopoly

Players
A buyer and a seller.

The Order of Play
1 The seller picks the first-period price, p_1.
2 The buyer buys quantity q_1 and consumes service flow q_1.
3 The seller picks the second-period price, p_2.
4 The buyer buys additional quantity q_2 and consumes service flow $(q_1 + q_2)$.

Payoffs
Production cost is zero and there is no discounting. The seller's payoff is his revenue, and the buyer's payoff is the sum across periods of his benefits from consumption minus his expenditure. His benefits arise from his being willing to pay as much as

$$B(q_t) = 60 - \frac{q_t}{2} \tag{87}$$

for the marginal unit service flow consumed in period t, as shown in Figure 10. The payoffs are therefore

$$\pi_{seller} = q_1 p_1 + q_2 p_2 \tag{88}$$

and

$$\begin{aligned}
\pi_{buyer} &= [consumer\ surplus_1] + [consumer\ surplus_2] \\
&= [total\ benefit_1 - expenditure_1] + [total\ benefit_2 - expenditure_2] \\
&= \left[\tfrac{(60-B(q_1))q_1}{2} + B(q_1)q_1 - p_1 q_1\right] \\
&\quad + \left[\tfrac{60-B(q_1+q_2)}{2}(q_1+q_2) + B(q_1+q_2)(q_1+q_2) - p_2 q_2\right]
\end{aligned} \tag{89}$$

Thinking about durable monopoly is hard because we are used to one-period models in which the demand curve, which relates the price to the quantity demanded, is identical to the marginal-benefit curve, which relates the marginal benefit to the quantity consumed. Here, the two curves are different. The marginal benefit curve is the same each period, since it is part of the rules of the game, relating consumption to utility. The demand curve will change over time and depends on the equilibrium strategies, depending as it does on the number of periods left in which to consume the good's services, expected future prices, and the quantity already owned. Marginal benefit is a given for the buyer; quantity demanded is his strategy.

The buyer's total benefit in period 1 is the dollar value of his utility from his purchase of q_1, which equals the amount he would have been willing to pay to rent q_1. This is composed of the two areas shown in figure 14.10a, the upper triangle of area $\left(\frac{1}{2}\right)(q_1+q_2)(60 - B(q_1+q_2))$ and the lower rectangle of area $(q_1+q_2)B(q_1+q_2)$. From this must be subtracted his expenditure in period 1, $p_1 q_1$, to obtain what we might call his consumer surplus in the first period. Note that $p_1 q_1$ will not be the lower rectange, unless by some strange accident, and the "consumer surplus" might easily be negative, since the expenditure in period 1 will also yield utility in period 2 because the good is durable.

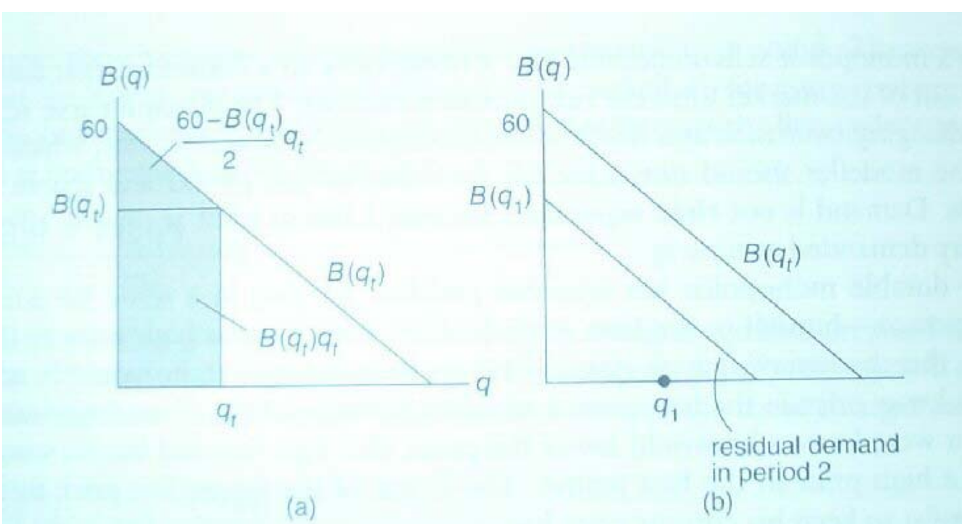


Figure 10: The Buyer's Marginal Benefit per Period in the Game of Durable Monopoly

To find the equilibrium price path one cannot simply differentiate the seller's utility with respect to p_1 and p_2, because that would violate the sequential rationality of the seller and the rational response of the buyer. Instead, one must look for a subgame perfect equilibrium, which means starting in the second period and discovering how much the buyer would purchase given his first-period purchase of q_1, and what second-period price the seller would charge given the buyer's second-period demand function.

In the first period, the marginal unit consumed was the $q_1 - th$. In the second period, it will be the $(q_1 + q_2) - th$. The residual demand curve after the first period's purchases is shown in Figure 10b. It is a demand curve very much like the demand curve resulting from intensity rationing in the capacity-constrained Bertrand game of Section 14.2, as shown in Figure 2a. The most intense portion of the buyer's demand, up to q_1 units, has already been satisfied, and what is left begins with a marginal benefit of $B(q_1)$, and falls at the same slope as the original marginal benefit curve. The equation for the residual demand is therefore, using equation (87),

$$p_2 = B(q_1) - \frac{1}{2}q_2 = 60 - \frac{1}{2}q_1 - \frac{1}{2}q_2. \tag{90}$$

Solving for the monopoly quantity, q_2^*, the seller maximizes $q_2 p_2$, solving the problem

$$\underset{q_2}{Maximize}\quad q_2\left(60 - \frac{q_1 + q_2}{2}\right), \tag{91}$$

which generates the first-order condition

$$60 - q_2 - \frac{1}{2}q_1 = 0, \tag{92}$$

so that

$$q_2^* = 60 - \frac{1}{2}q_1. \tag{93}$$

From equations (90) and (93), it can be seen that $p_2^* = 30 - q_1/4$.

We must now find q_1^*. In period one, the buyer looks ahead to the possibility of buying in period two at a lower price. Buying in the first period has two benefits: consumption of the service flow in the first period and consumption of the service flow in the second period. The price he would pay for a unit in period one cannot exceed the marginal benefit from the first-period service flow in period one plus the foreseen value of p_2, which from (93) is $30 - q_1/4$. If the seller chooses to sell q_1 in the first period, therefore, he can do so at the price

$$\begin{aligned} p_1(q_1) &= B(q_1) + p_2 \\ &= (60 - \tfrac{1}{2}q_1) + (30 - \tfrac{1}{4}q_1), \\ &= 90 - \tfrac{3}{4}q_1. \end{aligned} \tag{94}$$

Knowing that in the second period he will choose q_2 according to (93), the seller combines (93) with (94) to give the maximand in the problem of choosing q_1 to maximize profit over the two periods, which is

$$\begin{aligned} (p_1 q_1 + p_2 q_2) &= (90 - \tfrac{3}{4} q_1) q_1 + (30 - \tfrac{1}{4} q_1)(60 - \tfrac{1}{2} q_1) \\ &= 1800 + 60 q_1 - \tfrac{5}{8} q_1^2, \end{aligned} \tag{95}$$

which has the first-order condition

$$60 - \frac{5}{4} q_1 = 0, \tag{96}$$

so that

$$q_1^* = 48 \tag{97}$$

and, making use of (94), $p_1^* = 54$.

It follows from (93) that $q_2^* = 36$ and $p_2 = 18$. The seller's profits over the two periods are $\pi_s = 3,240$ $(= 54(48) + 18(36))$.

The purpose of these calculations is to compare the situation with three other market structures: a competitive market, a monopolist who rents instead of selling, and a monopolist who commits to selling only in the first period.

A *competitive market* bids down the price to the marginal cost of zero. Then, $p_1 = 0$ and $q_1 = 120$ from (87), and profits equal zero.

If the monopolist *rents* instead of selling, then equation (87) is like an ordinary demand equation, because the monopolist is effectively selling the good's services separately each period. He could rent a quantity of 60 each period at a rental fee of 30 and his profits would sum to $\pi_s = 3,600$. That is higher than 3,240, so profits are higher from renting than from selling outright. The problem with selling outright is that the first-period price cannot be very high or the buyer knows that the seller will be tempted to lower the price once the buyer has bought in the first period. Renting avoids this problem.

If the monopolist can *commit to not producing in the second period*, he will do just as well as the monopolist who rents, since he can sell a quantity of 60 at a price of 60, the sum of the rents for the two periods. An example is the artist who breaks the plates for his engravings after a production run of announced size. We must also assume that the artist can convince the market that he has broken the plates. People joke that the best way an artist can increase the value of his work is by dying, and that, too, fits the model.

If the modeller ignored sequential rationality and simply looked for the Nash equilibrium that maximized the payoff of the seller by his choice of p_1 and p_2, he would come to the commitment result. An example of such an equilibrium is ($p_1 = 60$, $p_2 = 200$, *Buyer purchases according to* $q_1 = 120 - p_1$, *and* $q_2 = 0$). This is Nash because neither player has incentive to deviate given the other's strategy, but it fails to be subgame perfect, because the seller should realize that if he deviates and chooses a lower price once the second period is reached, the buyer will respond by deviating from $q_2 = 0$ and will buy more units.

With more than two periods, the difficulties of the durable-goods monopolist become even more striking. In an infinite-period model without discounting, if the marginal cost of production is zero, the equilibrium price for outright sale instead of renting is constant – at zero! Think about this in the context of a model with many buyers. Early consumers foresee that the monopolist has an incentive to cut the price after they buy, in order to sell to the remaining consumers who value the product less. In fact, the monopolist would continue to cut the price and sell more and more units to consumers with weaker and weaker demand until the price fell to marginal cost. Without discounting, even the high-valuation consumers refuse to buy at a high price, because they know they could wait until the price falls to zero. And this is not a trick of infinity: a large number of periods generates a price close to zero.

We can also use the durable monopoly model to think about the durability of the product. If the seller can develop a product so flimsy that it only lasts one period, that is equivalent to renting. A consumer is willing to pay the same price to own a one-hoss shay that he knows will break down in one year as he would pay to rent it for a year. Low durability leads to the same output and profits as renting, which explains why a firm with market power might produce goods that wear out quickly. The explanation is not that the monopolist can use his market power to inflict lower quality on consumers– after all, the price he receives is lower too– but that the lower durability makes it credible to high-valuation buyers that the seller expects their business in the future and will not lower his price.

Notes

N14.1 Quantities as Strategies: the Cournot Equilibrium Revisited

- Articles on the existence of a pure-strategy equilibrium in the Cournot model include Novshek (1985) and Roberts & Sonnenschein (1976).

- **Merger in a Cournot model.** A problem with the Cournot model is that a firm's best policy is often to split up into separate firms. Apex gets half the industry profits in a duopoly game. If Apex split into firms $Apex_1$ and $Apex_2$, it would get two thirds of the profit in the Cournot triopoly game, even though industry profit falls.

  This point was made by Salant, Switzer & Reynolds (1983) and is the subject of problem 14.2. It is interesting that nobody noted this earlier, given the intense interest in Cournot models. The insight comes from approaching the problem from asking whether a player could improve his lot if his strategy space were expanded in reasonable ways.

- An ingenious look at how the number of firms in a market affects the price is Bresnahan & Reiss (1991), which looks empirically at a number of very small markets with one, two, three or more competing firms. They find a big decline in the price from one to two firms, a smaller decline from two to three, and not much change thereafter.

  Exemplifying theory, as discussed in the Introduction to this book, lends itself to explaining particular cases, but it is much less useful for making generalizations across industries. Empirical work associated with exemplifying theory tends to consist of historical anecdote rather than the linear regressions to which economics has become accustomed. Generalization and econometrics are still often useful in industrial organization, however, as Bresnahan & Reiss (1991) shows. The most ambitious attempt to connect general data with the modern theory of industrial organization is Sutton's 1991 book, *Sunk Costs and Market Structure*, which is an extraordinarily well-balanced mix of theory, history, and numerical data.

N14.2 Prices as strategies: the Bertrand equilibrium

- As Morrison (1998) points out, Cournot actually does (in Chapter 7) analyze the case of price competition with imperfect substitutes, as well as the quantity competition that bears his name. It is convenient to continue to contrast "Bertrand" and "Cournot" competition, however, though a case can be made for simplifying terminology to "price" and "quantity" competition instead. For the history of how the Bertrand name came to be attached to price competition, see Dimand & Dore (1999).

- Intensity rationing has also been called **efficient rationing**. Sometimes, however, this rationing rule is inefficient. Some low-intensity consumers left facing the high price decide not to buy the product even though their benefit is greater than its marginal cost. The reason intensity rationing has been thought to be efficient is that it is efficient if the rationed-out consumers are unable to buy at any price.

- OPEC has tried both price and quantity controls ("OPEC, Seeking Flexibility, May Choose Not to Set Oil Prices, but to Fix Output," *Wall Street Journal*, October 8, 1987, p. 2; "Saudi King Fahd is Urged by Aides To Link Oil Prices to Spot Markets," *Wall Street Journal*, October 7, 1987, p. 2). Weitzman (1974) is the classic reference on price versus quantity control by regulators, although he does not use the context of oligopoly. The decision rests partly on enforceability, and OPEC has also hired accounting firms to monitor prices ("Dutch Accountants Take On a Formidable Task: Ferreting Out 'Cheaters' in the Ranks of OPEC," *Wall Street Journal*, February 26, 1985, p. 39).

- Kreps & Scheinkman (1983) show how capacity choice and Bertrand pricing can lead to a Cournot outcome. Two firms face downward-sloping market demand. In the first stage of the game, they simultaneously choose capacities, and in the second stage they simultaneously choose prices (possibly by mixed strategies). If a firm cannot satisfy the demand facing it in the second stage (because of the capacity limit), it uses intensity rationing (the results depend on this). The unique subgame perfect equilibrium is for each firm to choose the Cournot capacity and price.

- Haltiwanger & Waldman (1991) have suggested a dichotomy applicable to many different games between players who are **responders**, choosing their actions flexibly, and those who are **nonresponders**, who are inflexible. A player might be a nonresponder because he is irrational, because he moves first, or simply because his strategy set is small. The categories are used in a second dichotomy, between games exhibiting **synergism**, in which responders choose to do whatever the majority do (upward sloping reaction curves), and games exhibiting **congestion**, in which responders want to join the minority (downward sloping reaction curves). Under synergism, the equilibrium is more like what it would be if all the players were nonresponders; under congestion, the responders have more influence. Haltiwanger and Waldman apply the dichotomies to network externalities, efficiency wages, and reputation.

- Section 14.3 shows how to generate demand curves (**??**) and (**??**) using a location model, but they can also be generated directly by a quadratic utility function. Dixit (1979) states with respect to three goods 0, 1, and 2, the utility function

$$U = q_0 + \alpha_1 q_1 + \alpha_2 q_2 - \frac{1}{2}\left(\beta_1 q_1^2 + 2\gamma q_1 q_2 + \beta_2 q_2^2\right) \tag{98}$$

(where the constants $\alpha_1, \alpha_2, \beta_1$, and β_2 are positive and $\gamma^2 \leq \beta_1\beta_2$) generates the inverse demand functions

$$p_1 = \alpha_1 - \beta_1 q_1 - \gamma q_2 \tag{99}$$

and

$$p_2 = \alpha_2 - \beta_2 q_2 - \gamma q_1. \tag{100}$$

- There are many ways to specify product differentation. This chapter looks at horizontal differentiation where all consumers agree that products A and B are more alike than A and C, but they disagree as to which is best. Another way horizontal differentiation might work is for each consumer to like a particular product best, but to consider all others as equivalent. See Dixit & Stiglitz (1977) for a model along those lines. Or, differentiation might be vertical: all consumers agree that A is better than B and B is better than C but they disagree as to how *much* better A is than B. Firms therefore offer different qualities at different prices. Shaked & Sutton (1983) have explored this kind of vertical differentation.

N14.3 Location models

- For a booklength treatment of location models, see Greenhut & Ohta (1975).

- Vickrey notes the possible absence of a pure-strategy equilibrium in Hotelling's model in pp.323-324 of his 1964 book *Microstatics*. D'Aspremont, Gabszewicz & Thirse (1979) work out the mixed-strategy equilibrium for the case of quadratic transportation costs, and Osborne & Pitchik (1987) do the same for Hotelling's original model.

- Location models and switching cost models are attempts to go beyond the notion of a market price. Antitrust cases are good sources for descriptions of the complexities of pricing in particular markets. See, for example, Sultan's 1974 book on electrical equipment in the 1950s, or antitrust opinions such as *US v. Addyston Pipe & Steel Co.*, 85 F. 271 (1898).

- It is important in location models whether the positions of the players on the line are moveable. See, for example, Lane (1980).

- The location games in this chapter model use a one-dimensional space with end points, i.e., a line segment. Another kind of one-dimensional space is a circle (not to be confused with a disk). The difference is that no point on a circle is distinctive, so no consumer preference can be called extreme. It is, if you like, Peoria versus Berkeley. The circle might be used for modelling convenience or because it fits a situation: e.g., airline flights spread over the 24 hours of the day. With two players, the Hotelling location game on a circle has a continuum of pure-strategy equilibria that are one of two types: both players locating at the same spot, versus players separated from each other by 180°. The three-player model also has a continuum of pure-strategy equilibria, each player separated from another by 120°, in contrast to the nonexistence of a pure-strategy equilibrium when the game is played on a line segment.

- Characteristics such as the color of cars could be modelled as location, but only on a player-by-player basis, because they have no natural ordering. While Smith's ranking of (red=1, yellow=2, blue=10) could be depicted on a line, if Brown's ranking is (red=1, blue=5, yellow=6) we cannot use the same line for him. In the text, the characteristic was something like physical location, about which people may have different preferences but agree on what positions are close to what other positions.

N14.6 Durable monopoly

- The proposition that price falls to marginal cost in a durable monopoly with no discounting and infinite time is called the "Coase Conjecture," after Coase (1972). It is really a proposition and not a conjecture, but alliteration was too strong to resist.

- Gaskins (1974) has written a well-known article on the problem of the durable monopolist who foresees that he will be creating his own future competition in the future because his product can be recycled, using the context of the aluminum market.

- Leasing by a durable monopoly was the main issue in the antitrust case *US v. United Shoe Machinery Corporation*, 110 F. Supp. 295 (1953), but not because it increased monopoly profits. The complaint was rather that long-term leasing impeded entry by new sellers of shoe machinery, a curious idea when the proposed alternative was outright sale. More likely, leasing was used as a form of financing for the machinery consumers; by leasing, they did not need to borrow as they would have to do if it was a matter of financing a purchase. See Wiley, Ramseyer, and Rasmusen (1990).

- Another way out of the durable monopolist's problem is to give best-price guarantees to consumers, promising to refund part of the purchase price if any future consumer gets a lower price. Perversely, this hurts consumers, because it stops the seller from being tempted to lower his price. The "most-favored-consumer" contract, which is the analogous contract in markets with several sellers, is analyzed by Holt & Scheffman (1987), for example, who demonstrate how it can maintain high prices, and Png & Hirshleifer (1987), who show how it can be used to price discriminate between different types of buyers.

- The durable monopoly model should remind you of bargaining under incomplete information. Both situations can be modelled using two periods, and in both situations the problem for the seller is that he is tempted to offer a low price in the second period after having offered a high price in the first period. In the durable monopoly model this would happen if the high-valuation buyers bought in the first period and thus were absent from consideration by the second period. In the bargaining model this would happen if the buyer rejected the first-period offer and the seller could conclude that he must have a low valuation and act accordingly in the second period. With a rational buyer, neither of these things can happen, and the models' complications arise from the attempt of the seller to get around the problem.

  In the durable-monopoly model this would happen if the high-valuation buyers bought in the first period and thus were absent from consideration by the second period. In the bargaining model this would happen if the buyer rejected the first-period offer and the seller could conclude that he must have a low valuation and act accordingly in the second period. For further discussion, see the survey by Kennan & Wilson (1993).

Problems

14.1. Differentiated Bertrand with Advertising
Two firms that produce substitutes are competing with demand curves

$$q_1 = 10 - \alpha p_1 + \beta p_2 \tag{101}$$

and

$$q_2 = 10 - \alpha p_2 + \beta p_1. \tag{102}$$

Marginal cost is constant at $c = 3$. A player's strategy is his price. Assume that $\alpha > \beta/2$.

(a) What is the reaction function for firm 1? Draw the reaction curves for both firms.

(b) What is the equilibrium? What is the equilibrium quantity for firm 1?

(c) Show how firm 2's reaction function changes when β increases. What happens to the reaction curves in the diagram?

(d) Suppose that an advertising campaign could increase the value of β by one, and that this would increase the profits of each firm by more than the cost of the campaign. What does this mean? If either firm could pay for this campaign, what game would result between them?

14.2. Cournot Mergers (See Salant, Switzer, & Reynolds [1983])
There are three identical firms in an industry with demand given by $P = 1 - Q$, where $Q = q_1 + q_2 + q_3$. The marginal cost is zero.

(a) Compute the Cournot equilibrium price and quantities.

(b) How do you know that there are no asymmetric Cournot equilibria, in which one firm produces a different amount than the others?

(c) Show that if two of the firms merge, their shareholders are worse off.

14.3. Differentiated Bertrand
Two firms that produce substitutes have the demand curves

$$q_1 = 1 - \alpha p_1 + \beta(p_2 - p_1) \tag{103}$$

and

$$q_2 = 1 - \alpha p_2 + \beta(p_1 - p_2), \tag{104}$$

where $\alpha > \beta$. Marginal cost is constant at c, where $c < 1/\alpha$. A player's strategy is his price.

(a) What are the equations for the reaction curves $p_1(p_2)$ and $p_2(p_1)$? Draw them.

(b) What is the pure-strategy equilibrium for this game?

(c) What happens to prices if α, β, or c increase?

(d) What happens to each firm's price if α increases, but only firm 2 realizes it (and firm 2 knows that firm 1 is uninformed)? Would firm 2 reveal the change to firm 1?

Problem 14.4. Asymmetric Cournot Duopoly
Apex has variable costs of q_a^2 and a fixed cost of 1000, while Brydox has variables costs of $2q_b^2$ and no fixed cost. Demand is $p = 115 - q_a - q_b$.

(a) What is the equation for Apex's Cournot reaction function?

(b) What is the equation for Brydox' Cournot reaction function?

(c) What are the outputs and profits in the Cournot equilibrium?

Problem 14.5. Omitted.

Problem 14.6. Price Discrimination
A seller faces a large number of buyers whose market demand is given by $P = \alpha - \beta Q$. Production marginal cost is constant at c.

(a) What is the monopoly price and profit?

(b) What are the prices under perfect price discrimination if the seller can make take-it-or-leave-it offers? What is the profit?

(c) What are the prices under perfect price discrimination if the buyer and sellers bargain over the price and split the surplus evenly? What is the profit?

The Kleit Oligopoly Game: A Classroom Game for Chapter 14

The widget industry in Smallsville has N firms. Each firm produces 150 widgets per month. All costs are fixed, because labor is contracted for on a yearly basis, so we can ignore production cost for the purposes of this case. Widgets are perishable; if they are not sold within the month, they explode in flames.

There are two markets for widgets, the national market, and the local market. The price in the national market is \$20 per widget, with the customers paying for delivery, but the price in the local market depends on how many are for sale there in a given month. The price is given by the following market demand curve:

$$P = 100 - \frac{Q}{N},$$

where Q is the total output of widgets sold in the local market. If, however, this equation would yield a negative price, the price is just zero, since the excess widgets can be easily destroyed.

\$20 is the **opportunity cost** of selling a widget locally– it is what the firm loses by making that decision. The benefit from the decision depends on what other firms do. All firms make their decisions at the sme time on whether to ship widgets out of town to the national market. The train only comes to Smallsville once a month, so firms cannot retract their decisions. If a firm delays making its decision till too late, then it misses the train, and all its output will have to be sold in Smallsville.

General Procedures

For the first seven months, each of you will be a separate firm. You will write down two things on an index card: (1) the number of the month, and (2) your LOCAL-market sales for that month. Also record your local and national market sales on your Scoresheet. The instructor will collect the index cards and then announce the price for that month. You should then calculate your profit for the month and add it to your cumulative total, recording both numbers on your Scoresheet.

For the last five months, you will be organized into five different firms. Each firm has a capacity of 150, and submits a single index card. The card should have the number of the firm on it, as well as the month and the local output. The instructor will then calculate the market price, rounding it to the nearest dollar to make computations easier. Your own computations will be easier if you pick round numbers for your output.

If you do not turn in an index card by the deadline, you have missed the train and all 150 of your units must be sold locally. You can change your decision up until the deadline by handing in a new card noting both your old and your new output, e.g., "I want to change from 40 to 90."

Procedures Each Month

1. Each student is one firm. No talking.

2. Each student is one firm. No talking.

3. Each student is one firm. No talking.

4. Each student is one firm. No talking.

5. Each student is one firm. No talking.

6. Each student is one firm. You can talk with each other, but then you write down your own output and hand all outputs in separately.

7. Each student is one firm. You can talk with each other, but then you write down your own output and hand all outputs in separately.

8. You are organized into Firms 1 through 5, so N=5. People can talk within the firms, but firms cannot talk to each other. The outputs of the firms are secret.

9. You are organized into Firms 1 through 5, so N=5. People can talk within the firms, but firms cannot talk to each other. The outputs of the firms are secret.

10. You are organized into Firms 1 through 5, so N=5. You can talk to anyone you like, but when the talking is done, each firm writes down its output secretly and hands it in.

11. You are organized into Firms 1 through 5, so N=5. You can talk to anyone you like, but when the talking is done, each firm writes down its output secretly and hands it in. Write the number of your firm with your output. This number will be made public once all the outputs have been received.

For instructors' notes, go to http://www.rasmusen.org/GI/probs/14_cournotgame.pdf.

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*15 Entry

*15.1 Innovation and Patent Races

How do firms come to enter particular industries? Of the many potential products that might be produced, firms choose a small number, and each product is only produced by a few firms. Most potential firms choose to remain potential, not actual. Information and strategic behavior are especially important in borderline industries in which only one or two firms are active in production.

This chapter begins with a discussion of innovation with the complications of imitation by other firms and patent protection by the government. Section 15.2 looks at a different way to enter a market: by purchasing an existing firm, something that also provides help against moral hazard on the part of company executives. Section 15.3 analyzes a more traditional form of entry deterrence, predatory pricing, using a Gang of Four model of a repeated game under incomplete information. Section 15.4 returns to a simpler model of predatory pricing, but shows how the ability of the incumbent to credibly engage in a price war can actually backfire by inducing entry for buyout.

Market Power as a Precursor of Innovation

Market power is not always inimical to social welfare. Although restrictive monopoly output is inefficient, the profits it generates encourage innovation, an important source of both additional market power and economic growth. The importance of innovation, however, is diminished because of imitation, which can so severely diminish its rewards as to entirely prevent it. An innovator generally incurs some research cost, but a discovery instantly imitated can yield zero net revenues. Table 15.1 shows how the payoffs look if the firm that innovates incurs a cost of 1 but imitation is costless and results in Bertrand competition. Innovation is a dominated strategy.

Table 15.1 Imitation with Bertrand pricing

		Brydox	
		Innovate	*Imitate*
Apex	*Innovate*	-1,-1	$-1, 0$
	Imitate	$0, -1$	**0,0**

Payoffs to: (Apex, Brydox)

Under different assumptions, innovation occurs even with costless imitation. The key is whether duopoly profits are high enough for one firm to recoup the entire costs of innovation. If they are, the payoffs are as shown in table 15.2, a version of Chicken. Although the firm that innovates pays the entire cost and keeps only half the benefit, imitation is not dominant. Apex imitates if Brydox innovates, but not if Brydox imitates. If Apex could move first, it would bind itself not to innovate, perhaps by disbanding its research laboratory.

Table 15.2 Imitation with profits in the product market

		Brydox	
		Innovate	*Imitate*
Apex	*Innovate*	1,1	**1,2**
	Imitate	**2,1**	0,0

Payoffs to: (Apex, Brydox)

Without a first-mover advantage, the game has two pure strategy Nash equilibria, (*Innovate*, *Imitate*) and (*Imitate*, *Innovate*), and a symmetric equilibrium in mixed strategies in which each firm innovates with probability 0.5. The mixed-strategy equilibrium is inefficient, since sometimes both firms innovate and sometimes neither.

History might provide a focal point or explain why one player moves first. Japan was for many years incapable of doing basic scientific research, and does relatively little even today. The United States therefore had to innovate rather than imitate in the past, and today continues to do much more basic research.

Much of the literature on innovation compares the relative merits of monopoly and competition. One reason a monopoly might innovate more is because it can capture more of the benefits, capturing the entire benefit if perfect price discrimination is possible (otherwise, some of the benefit goes to consumers). In addition, the monopoly avoids a second inefficiency: entrants innovating solely to steal the old innovator's rents without much increasing consumer surplus. The welfare aspects of innovation theory – indeed, all aspects – are intricate, and the interested reader is referred to the surveys by Kamien & Schwartz (1982) and Reinganum (1989).

Patent Races

One way that governments respond to imitation is by issuing patents: exclusive rights to make, use, or sell an innovation. If a firm patents its discovery, other firms cannot imitate, or even use the discovery if the make it independently. Research effort therefore has a discontinuous payoff: if the researcher is the first to make a discovery, he receives the patent; if he is second, nothing. Patent races are examples of the tournaments discussed in section 8.2 except that if no player exerts any effort, none of them will get the reward. Patents are also special because they lose their value if consumers find a substitute and stop buying the patented product. Moreover, the effort in tournaments is usually exerted over a fixed time period, whereas research usually has an endogenous time period, ending when the discovery is made. Because of this endogeneity, we call the competition a **patent race.**

We will consider two models of patents. On the technical side, the first model shows how to derive a continuous mixed strategies probability distribution, instead of just the single number derived in chapter 3. On the substantive side, it shows how patent races lead to inefficiency.

Patent Race for a New Market

Players
Three identical firms, Apex, Brydox, and Central.

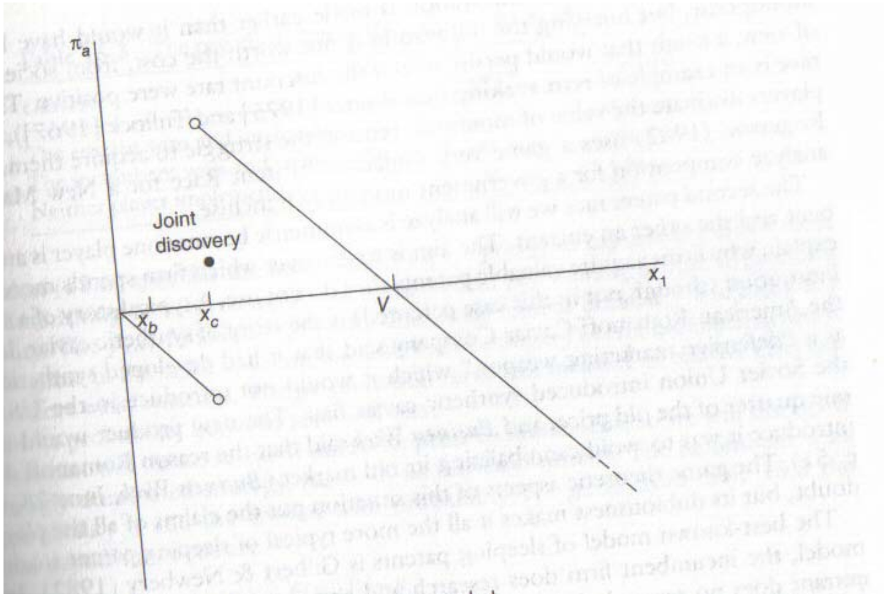
The Order of Play
Each firm simultaneously chooses research spending $x_i \geq 0$, $(i = a, b, c)$.

Payoffs
Firms are risk neutral and the discount rate is zero. Innovation occurs at time $T(x_i)$ where $T' < 0$. The value of the patent is V, and if several players innovate simultaneously they share its value.

$$\pi_i = \begin{cases} V - x_i & \text{if } T(x_i) < T(x_j), \ (\forall j \neq i) & \text{(Firm } i \text{ gets the patent)} \\ \frac{V}{1+m} - x_i & \text{if } T(x_i) = T(x_k), & \text{(Firm } i \text{ shares the patent with} \\ & & m = 1 \text{ or } 2 \text{ other firms)} \\ -x_i & \text{if } T(x_i) > T(x_j) \text{ for some } j & \text{(Firm } i \text{ does not get the patent)} \end{cases}$$

The game does not have any pure strategy Nash equilibria, because the payoff functions are discontinuous. A slight difference in research by one player can make a big difference in the payoffs, as shown in figure 15.1 on the next page for fixed values of x_b and x_c. The research levels shown in figure 15.1 are not equilibrium values. If Apex chose any research level x_a less than V, Brydox would respond with $x_a + \varepsilon$ and win the patent. If Apex chose $x_a = V$, then Brydox and Central would respond with $x_b = 0$ and $x_c = 0$, which would make Apex want to switch to $x_a = \varepsilon$.

Figure 15.1 The payoffs in Patent Race for a New Market



There does exist a symmetric mixed strategy equilibrium. We will derive $M_i(x)$, the cumulative density function for the equilibrium mixed strategy, rather than the density function itself. The probability with which firm i chooses a research level less than or equal to x will be $M_i(x)$. In a mixed-strategy equilibrium a player is indifferent between any of the pure strategies among which he is mixing. Since we know that the pure strategies $x_a = 0$ and $x_a = V$ yield zero payoffs, if Apex mixes over the support $[0, V]$ then the expected payoff for every strategy mixed between must also equal zero. The expected payoff from the pure strategy x_a is the expected value of winning minus the cost of research. Letting x stand for nonrandom and X for random variables, this is

$$V \cdot Pr(x_a \geq X_b, x_a \geq X_c) - x_a = 0, \tag{1}$$

which can be rewritten as

$$V \cdot Pr(X_b \leq x_a) Pr(X_c \leq x_a) - x_a = 0, \tag{2}$$

or

$$V \cdot M_b(x_a) M_c(x_a) - x_a = 0. \tag{3}$$

We can rearrange equation (15.3) to obtain

$$M_b(x_a) M_c(x_a) = \frac{x_a}{V}. \tag{4}$$

If all three firms choose the same mixing distribution M, then

$$M(x) = \left(\frac{x}{V}\right)^{1/2} \text{ for } 0 \leq x \leq V. \tag{5}$$

What is noteworthy about a patent race is not the nonexistence of a pure strategy equilibrium but the overexpenditure on research. All three players have expected payoffs of zero, because the patent value V is completely dissipated in the race. As in Brecht's

Threepenny Opera, "When all race after happiness/Happiness comes in last."[1] To be sure, the innovation is made earlier than it would have been by a monopolist, but hurrying the innovation is not worth the cost, from society's point of view, a result that would persist even if the discount rate were positive. The patent race is an example of **rent seeking** (see Posner [1975] and Tullock [1967]), in which players dissipate the value of monopoly rents in the struggle to acquire them. Indeed, Rogerson (1982) uses a game very similar to "Patent Race for a New Market" to analyze competition for a government monopoly franchise.

The second patent race we will analyze is asymmetric because one player is an incumbent and the other an entrant. The aim is to discover which firm spends more and to explain why firms acquire valuable patents they do not use. A typical story of a sleeping innovation (though not in this case patented) is the story of synthetic caviar. In 1976, the American Romanoff Caviar Company said that it had developed synthetic caviar as a "defensive marketing weapon" which it would not introduce in the US unless the Soviet Union introduced synthetic caviar first. The new product would sell for one quarter of the old price, and *Business Week* said that the reason Romanoff did not introduce it was to avoid cannibalizing its old market (*Business Week*, June 28, 1976, p. 51). The game theoretic aspects of this situation put the claims of all the players in doubt, but its dubiousness makes it all the more typical of sleeping patent stories.

The best-known model of sleeping patents is Gilbert & Newbery (1982). In that model, the incumbent firm does research and acquires a sleeping patent, while the entrant does no research. We will look at a slightly more complicated model which does not reach such an extreme result.

Patent Race for an Old Market

Players
An incumbent and an entrant.

The Order of Play
1 The firms simultaneously choose research spending x_i and x_e, which result in research achievements $f(x_i)$ and $f(x_e)$, where $f' > 0$ and $f'' < 0$.
2 Nature chooses which player wins the patent using a function g that maps the difference in research achievements to a probability between zero and one.

$$Prob(\text{incumbent wins patent}) = g[f(x_i) - f(x_e)], \tag{6}$$

where $g' > 0,\ g(0) = 0.5$, and $0 \leq g \leq 1$.
3 The winner of the patent decides whether to spend Z to implement it.

Payoffs
The old patent yields revenue y and the new patent yields v. The payoffs are shown in table 15.3.

[1] Act III, scene 7 of the *Threepenny Opera*, translated by John Willett (Berthold Brecht, *Collected Works*, London: Eyre Methuen (1987).

Table 15.3 The payoffs in Patent Race for an Old Market

Outcome	$\pi_{incumbent}$	$\pi_{entrant}$
The entrant wins and implements	$-x_i$	$v - x_e - Z$
The incumbent wins and implements	$v - x_i - Z$	$-x_e$
Neither player implements	$y - x_i$	$-x_e$

Equation (15.6) specifies the function $g[f(x_i) - f(x_e)]$ to capture the three ideas of (a) diminishing returns to inputs, (b) rivalry, and (c) winning a patent race as a probability. The $f(x)$ function represents dimishing returns because f increases at a decreasing rate in the input x. Using the difference between $f(x)$ for each firm makes it relative effort which matters. The $g(\cdot)$ function turns this measure of relative effective input into a probability between zero and one.

The entrant will do no research unless he plans to implement, so we will disregard the strongly dominated strategy, ($x_e > 0$, *no implementation*). The incumbent wins with probability g and the entrant with probability $1 - g$, so from table 15.3 the expected payoff functions are

$$\pi_{incumbent} = (1 - g[f(x_i) - f(x_e)])(-x_i) + g[f(x_i) - f(x_e)]Max\{v - x_i - Z, y - x_i\} \quad (7)$$

and

$$\pi_{entrant} = (1 - g[f(x_i) - f(x_e)])(v - x_e - Z) + g[f(x_i) - f(x_e)](-x_e). \quad (8)$$

On differentiating and letting f_i and f_e denote $f(x_i)$ and $f(x_e)$ we obtain the first order conditions

$$\frac{d\pi_i}{dx_i} = -(1 - g[f_i - f_e]) - g'f_i'(-x_i) + g'f_i'Max\{v - x_i - Z, y - x_i\} - g[f_i - f_e] = 0 \quad (9)$$

and

$$\frac{d\pi_e}{dx_e} = -(1 - g[f_i - f_e]) + g'f_e'(v - x_e - Z) - g[f_i - f_e] + g'f_e'x_e = 0. \quad (10)$$

Equating (15.9) and (15.10), which both equal zero, we obtain

$$-(1-g) - g'f_i'x_i + g'f_i'Max\{v - x_i - Z, y - x_i\} - g = -(1-g) + g'f_e'(v - x_e - Z) - g + g'f_e'x_e, \quad (11)$$

which simplifies to

$$f_i'[x_i + Max\{v - x_i - Z, y - x_i\}] = f_e'[v - x_e - Z + x_e], \quad (12)$$

or

$$\frac{f_i'}{f_e'} = \frac{v - Z}{Max\{v - Z, y\}}. \quad (13)$$

We can use equation (15.13) to show that different parameters generate two qualitatively different outcomes.

Outcome 1. *The entrant and incumbent spend equal amounts, and each implements if successful.*
This happens if there is a big gain from patent implementation, that is, if

$$v - Z \geq y, \tag{14}$$

so that equation (15.13) becomes

$$\frac{f_i'}{f_e'} = \frac{v - Z}{v - Z} = 1, \tag{15}$$

which implies that $x_i = x_e$.

Outcome 2. *The incumbent spends more and does not implement if he is successful (he acquires a sleeping patent).*
This happens if the gain from implementation is small, that is, if

$$v - Z < y, \tag{16}$$

so that equation (15.13) becomes

$$\frac{f_i'}{f_e'} = \frac{v - Z}{y} < 1, \tag{17}$$

which implies that $f_i' < f_e'$. Since we assumed that $f'' < 0$, f' is decreasing in x, and it follows that $x_i > x_e$.

This model shows that the presence of another player can stimulate the incumbent to do research he otherwise would not, and that he may or may not implement the discovery. The incumbent has at least as much incentive for research as the entrant because a large part of a successful entrant's payoff comes at the incumbent's expense. The benefit to the incumbent is the maximum of the benefit from implementing and the benefit from stopping the entrant, but the entrant's benefit can only come from implementing. Contrary to the popular belief that sleeping patents are bad, here they can help society by eliminating wasteful implementation.

*15.2 Takeovers and Greenmail

The Free Rider Problem

Game theory is well suited to modelling takeovers because the takeover process depends crucially on information and includes a number of sharply delineated actions and events. Suppose that under its current mismanagement, a firm has a value per share of v, but no shareholder has enough shares to justify the expense of a proxy fight to throw out the current managers, although doing so would raise the value to $(v + x)$. An outside bidder makes a tender offer conditional upon obtaining a majority. Any bid p between v and $(v+x)$ can make both the bidder and the shareholders better off. But do the shareholders accept such an offer?

We will see that they do not. Quite simply, the only reason the bidder makes a tender offer is that the value would rise higher than his bid, so no shareholder should accept his bid.

The Free Rider Problem in Takeovers
(Grossman & Hart [1980])

Players
A bidder and a continuum of shareholders, with amount m of shares.

The Order of Play
1 The bidder offers p per share for the m shares.
2 Each shareholder decides whether to accept the bid (denote by θ the fraction that accept).
3 If $\theta \geq 0.5$, the bid price is paid out, and the value of the firm rises from v to $(v+x)$ per share.

Payoffs
If $\theta < 0.5$, the takeover fails, the bidder's payoff is zero, and the shareholder's payoff is v per share. Otherwise,

$$\pi_{bidder} = \begin{cases} \theta m(v+x-p) & \text{if } \theta \geq 0.5. \\ 0 & \text{otherwise} \end{cases}$$

$$\pi_{shareholder} = \begin{cases} p & \text{if the shareholder accepts.} \\ v+x & \text{if the shareholder rejects.} \end{cases}$$

Bids above $(v+x)$ are dominated strategies, since the bidder could not possibly profit from them. But if the bid is any lower, an individual shareholder should hold out for the new value of $(v+x)$ rather than accepting p. To be sure, when they all do that, the offer fails and they end up with v, but no individual wants to accept if he thinks the offer will succeed. The only equilibria are the many strategy combinations that lead to a failed takeover, or a bid of $p = (v+x)$ accepted by a majority, which succeeds but yields a payoff of zero to the bidder. If organizing an offer has even the slightest cost, the bidder would not do it.

The free rider problem is clearest where there is a continuum of shareholders, so that the decision of any individual does not affect the success of the tender offer. If there were, instead, nine players with one share each, then in one asymmetric equilibrium five of them tender at a price just slightly above the old market price and four hold out. Each of the five tenderers knows that if he held out, the offer would fail and his payoff would be zero. This is an example of the discontinuity problem of section 8.6.

In practice, the free rider problem is not quite so severe even with a continuum of shareholders. If the bidder can quietly buy a sizeable number of shares without driving up the price (something severely restricted in the United States by the Williams Act), then his capital gains on those shares can make a takeover profitable even if he makes nothing from shares bought in the public offer. Dilution tactics such as freeze-out mergers also help the bidder (see Macey & McChesney [1985]). In a freeze-out, the bidder buys 51 percent of the shares and merges the new acquisition with another firm he owns, at a price below its full value. If dilution is strong enough, the shareholders are willing to sell at a price less than $v + x$.

Still another takeover tactic is the two-tier tender offer, a nice application of the Prisoner's Dilemma. Suppose the underlying value of the firm is 30, which is the initial stock price. A monopolistic bidder offers a price of 10 for 51 percent of the stock and 5 for the other 49 percent, conditional upon 51 percent tendering. It is then a dominant strategy to tender, even though all the shareholders would be better off refusing to sell.

Greenmail

Greenmail occurs when managers buy out some shareholders at an inflated stock price to stop them from taking over. Opponents of greenmail explain this using the Corrupt Managers model. Suppose that a little dilution is possible, or the bidder owns some shares to start with, so he can take over the firm but would lose most of the gains to the other shareholders. The managers are willing to pay the bidder a large amount of greenmail to keep their jobs, and both manager and bidder prefer greenmail to an actual takeover, despite the fact that the other shareholders are considerably worse off.

Managers often use what we might call the Noble Managers model to justify greenmail. In this model, current management knows the true value of the firm, which is greater than both the current stock price and the takeover bid. They pay greenmail to protect the shareholders from selling their mistakenly undervalued shares.

The Corrupt Managers model faces the objection that it fails to explain why the corporate charter does not prohibit greenmail. The Noble Managers model faces the objection that it implies either that shareholders are irrational or that stock prices rise after greenmail because shareholders know that the greenmail signal (giving up the benefits of a takeover) is more costly for a firm which really is not worth more than the takeover bid.

Shleifer & Vishny (1986) have constructed a more sophisticated model in which greenmail is in the interest of the shareholders. The idea is that greenmail encourages potential bidders to investigate the firm, eventually leading to a takeover at a higher price than the initial offer. Greenmail is costly, but for that very reason it is an effective signal that the manager thinks a better offer could come along later. (Like Noble Managers, this assumes that the manager acts in the interests of the shareholders.) I will present a numerical example in the spirit of Shleifer & Vishny rather than following them exactly, since their exposition is not directed towards the behavior of the stock price.

The story behind the model is that a manager has been approached by a bidder, and he must decide whether to pay him greenmail in the hopes that other bidders – "white knights" – will appear. The manager has better information than the market as a whole about the probability of other bidders appearing, and some other bidders can only appear after they undertake costly investigation, which they will not do if they think the takeover price will be bid up by competition with the first bidder. The manager pays greenmail to encourage new bidders by getting rid of their competition.

Greenmail to Attract White Knights

(Shleifer & Vishny [1986])

Players

The manager, the market, and bidder Brydox. (Bidders Raider and Apex do not make decisions.)

The Order of Play

Figure 15.2 shows the game tree. After each time t, the market picks a share price p_t.

0 Unobserved by any player, Nature picks the state to be (A), (B), (C), or (D), with probabilities 0.1, 0.3, 0.1, and 0.5, unobserved by any player.

1 Unless the state is (D), the Raider appears and offers a price of 15. The manager's information partition becomes $\{$(A), (B,C), (D)$\}$; everyone else's becomes $\{$(A,B,C), (D)$\}$.

2 The manager decides whether to pay greenmail and extinguish the Raider's offer at a cost of 5 per share.

3 If the state is (A), Apex appears and offers a price of 25 if greenmail was paid, and 30 otherwise.

4 If the state is (B), Brydox decides whether to buy information at a cost of 8 per share. If he does, then he can make an offer of 20 if the Raider has been paid greenmail, or 27 if he must compete with the Raider.

5 Shareholders accept the best offer outstanding, which is the final value of a share. If no offer is outstanding, the final value is 5 if greenmail was paid, 10 otherwise.

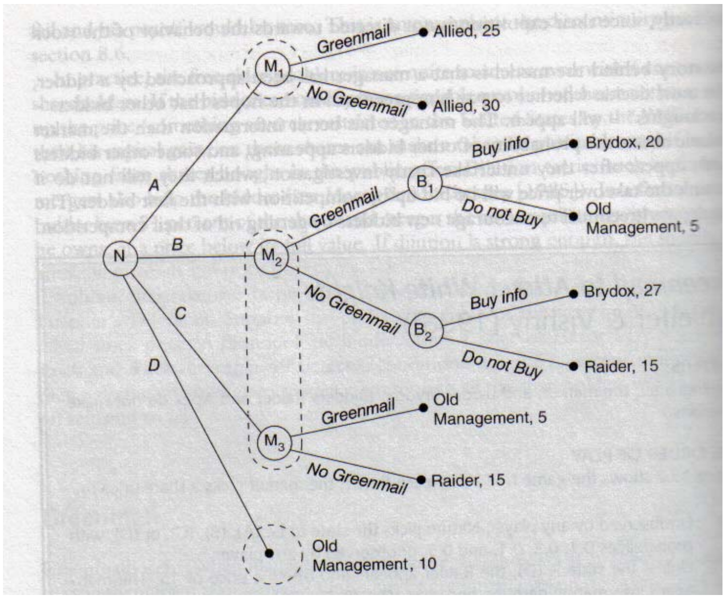
Payoffs

The manager maximizes the final value.

The market minimizes the absolute difference between p_t and the final value.

If he buys information, Brydox receives 23 $(= 31-8)$ minus the value of his offer; otherwise he receives zero.

Figure 15.2 The game tree for Greenmail to Attract White Knights



The payoffs specify that the manager should maximize the final value of the firm, rather than a weighted average of the prices p_0 through p_5. This assumption is reasonable because the only shareholders to benefit from a high value of p_t are those that sell their stock at t. The manager cannot say: "The stock is overvalued: Sell!", because the market would learn the overvaluation too, and refuse to buy.

The prices 15, 20, 27, and 30 are assumed to be the results of blackboxed bargaining games between the manager and the bidders. Assuming that the value of the firm to Brydox is 31 ensures that he will not buy information if he foresees that he would have to compete with the Raider. Since Brydox has a dominant strategy – buy information if the Raider has been paid greenmail and not otherwise – our focus will be on the market price and the decision of whether to pay greenmail. This model is also not designed to answer the question of why the Raider appears. His behavior is exogenous. As the model stands, his expected profit is positive since he is sometimes paid greenmail, but if he actually had to buy the firm he would regret it in states B and C, since the final value of the firm would be 10.

We will see that in equilibrium the manager pays greenmail in states (B) and (C), but not in (A) or (D). Table 15.4 shows the equilibrium price path of the market price.

Table 15.4 The equilibrium price in Greenmail to Attract White Knights

State	Probability	p_0	p_1	p_2	p_3	p_4	p_5	Final management
(A)	0.1	14.5	19	30	30	30	30	Allied
(B)	0.3	14.5	19	16.25	16.25	20	20	Brydox
(C)	0.1	14.5	19	16.25	16.25	5	5	Old management
(D)	0.5	14.5	10	10	10	10	10	Old management

The market's optimal strategy amounts to estimating the final value. Before the market receives any information, its prior beliefs estimate the final value to be 14.5 (= 0.1[30] + 0.3[20]+ 0.1[5] + 0.5[10]). If state (D) is ruled out by the arrival of the Raider, the price rises to 19 (= 0.2[30] + 0.6[20]+ 0.2[5]). If the Raider does not appear, it becomes common knowledge that the state is (D), and the price falls to 10.

If the state is (A), the manager knows it and refuses to pay greenmail in expectation of Apex's offer of 30. Observing the lack of greenmail, the market deduces that the state is (A), and the price immediately rises to 30.

If the state is (B) or (C) the manager does pay greenmail and the market, ruling out (A), uses Bayes's Rule to assign probabilities of 0.75 to (B) and 0.25 to (C). The price falls from 19 to 16.25 (= 0.75[20] + 0.25[5]).

It is clear that the manager should not pay greenmail in states (A) or (D), when the manager knows that Brydox is not around to investigate. What if the manager deviates in the information set (B,C) and refuses to pay greenmail? The market would initially believe that the state was (A), so the price would rise to $p_2 = 30$. But the price would fall again after Apex failed to make an offer and the market realized that the manager had deviated. Brydox would refuse to enter at time 3, and the Raider's offer of 15 would be accepted. The payoff of 15 would be less than the expected payoff of 16.25 from paying greenmail.

The model does not say that greenmail is always good for the shareholders, only that it can be good *ex ante*. If the true state turns out to be (C), then greenmail was a mistake, *ex post*, but since state (B) is more likely, the manager is correct to pay greenmail in information set (B,C). What is noteworthy is that greenmail is optimal even though it drives down the stock price from 19 to 16.25. Greenmail communicates the bad news that Apex is not around, but makes the best of that misfortune by attracting Brydox.

*15.3 Predatory Pricing: The Kreps-Wilson Model

One traditional form of monopolization and entry deterrence is predatory pricing, in which the firm seeking to acquire the market charges a low price to drive out its rival. We have looked at predation already in chapters 4, 5 and 6 in the "Entry Deterrence" games. The major problem with entry deterrence under complete information is the chainstore paradox. The heart of the paradox is the sequential rationality problem faced by an incumbent who wishes to threaten a prospective entrant with low post-entry prices. The incumbent can respond to entry in two ways. He can collude with the entrant and share the profits, or he can fight by lowering his price so that both firms make losses. We have seen that the incumbent would not fight in a perfect equilibrium if the game has complete information. Foreseeing the incumbent's accommodation, the potential entrant ignores the threats.

In Kreps & Wilson (1982a), an application of the gang of four model of chapter 6, incomplete information allows the threat of predatory pricing to successfully deter entry. A monopolist with outlets in N towns faces an entrant who can enter each town. In our adaption of the model, we will start by assuming that the order in which the towns can be entered is common knowledge, and that if the entrant passes up his chance to enter a town, he cannot enter it later. The incomplete information takes the form of a small probability that the monopolist is "strong" and has nothing but *Fight* in his action set: he is an uncontrolled manager who gratifies his passions in squelching entry instead of maximizing profits.

Predatory Pricing
(Kreps & Wilson [1982a])

Players
The entrant and the monopolist.

The Order of Play
0 Nature chooses the monopolist to be *Strong* with low probability θ and *Weak*, with high probability $(1-\theta)$. Only the monopolist observes Nature's move.
1 The entrant chooses *Enter* or *Stay Out* for the first town.
2 The monopolist chooses *Collude* or *Fight* if he is weak, *Fight* if he is strong.
3 Steps (1) and (2) are repeated for towns 2 through N.

Payoffs
The discount rate is zero. Table 15.5 gives the payoffs per period, which are the same as in table 4.1.

Table 15.5 Predatory Pricing

		Weak incumbent	
		Collude	*Fight*
Entrant	*Enter*	**40,50**	$-10, 0$
	Stay out	$0, 100$	**0,100**

Payoffs to: (Entrant, Incumbent)

In describing the equilibrium, we will denote towns by names such as i_{30} and i_5, where the numbers are to be taken purely ordinally. The entrant has an opportunity to enter town i_{30} before i_5, but there are not necessarily 25 towns between them. The actual gap depends on θ but not N.

Part of the Equilibrium for Predatory Pricing

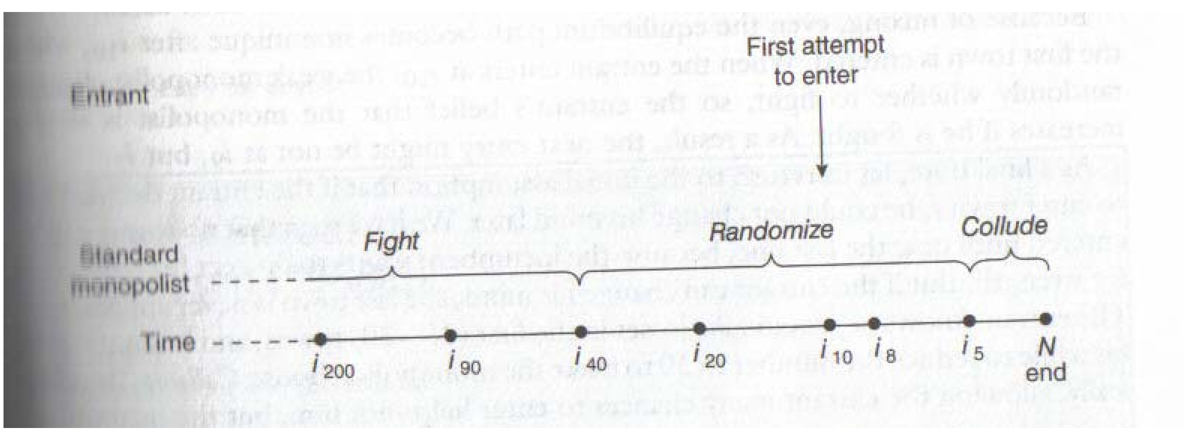
Entrant: Enter first at town i_{-10}. If entry has occurred before i_{10} and been answered with *Collude*, enter every town after the first one entered.

Strong monopolist: Always fight entry.

Weak monopolist: Fight any entry up to i_{30}. Fight the first entry after i_{-30} with a probability $m(i)$ that diminishes until it reaches zero at i_5. If *Collude* is ever chosen instead, always collude thereafter. If *Fight* was chosen in response to the first attempt at entry, increase the mixing probability $m(i)$ in subsequent towns.

This description, which is illustrated by figure 15.3, only covers the equilibrium path and small deviations. Note that out-of-equilibrium beliefs do not have to be specified (unlike in the original model of Kreps and Wilson), since whenever a monopolist colludes, in or out of equilibrium, Bayes's Rule says that the entrant must believe him to be *Weak*.

Figure 15.3 The equilibrium in Predatory Pricing



The entrant will certainly stay out until i_{30}. If no town is entered until i_5 and the monopolist is *Weak*, then entry at i_5 is undoubtedly profitable. But entry is attempted at i_{10}, because since $m(i)$ is diminishing in i, the weak monopolist probably would not fight even there.

Out of equilibrium, if an entrant were to enter at i_{90}, the weak monopolist would be willing to fight, to maintain i_{10} as the next town to be entered. If he did not, then the entrant, realizing that he could not possibly be facing a strong monopolist, would enter every subsequent town from i_{89} to i_1. If no town were entered until i_5, the weak monopolist would be unwilling to fight in that town, because too few towns are left to protect. If a town between i_{30} and i_5 has been entered and fought over, the monopolist raises the mixing probability that he fights in the next town entered, because he has a more valuable reputation to defend. By fighting in the first town he has increased the belief that he is strong and increased the gap until the next town is entered.

What if the entrant deviated and entered town i_{20}? The equilibrium calls for a mixed strategy response beginning with i_{30}, so the weak monopolist must be indifferent between fighting and not fighting. If he fights, he loses current revenue but the entrant's posterior belief that he is strong rises, rising more if the fight occurs late in the game. The entrant knows that in equilibrium the weak monopolist would fight with a probability of, say, 0.9 in town i_{20}, so fighting there would not much increase the belief that he was strong, but if he fought in town i_{13}, where the mixing probability has fallen to 0.2, the belief would rise much more. On the other hand, the gain from a given reputation diminishes as fewer towns remain to be protected, so the mixing probability falls over time.

The description of the equilibrium strategies is incomplete because describing what happens after unsuccessful entry becomes rather intricate. Even in the simultaneous-move games of chapter 3, we saw that games with mixed strategy equilibria have many different possible realizations. In repeated games like Predatory Pricing, the number of possible realizations makes an exact description very complicated indeed. If, for example, the entrant entered town i_{20} and the monopolist chose $Fight$, the entrant's belief that he was strong would rise, pushing the next town entered to i_{-8} instead of i_{10}. A complete description of the strategies would say what would happen for every possible history of the game, which is impractical at this book's level of detail.

Because of mixing, even the equilibrium path becomes nonunique after i_{10}, when the first town is entered. When the entrant enters at i_{10}, the weak monopolist chooses randomly whether to fight, so the entrant's belief that the monopolist is strong increases if he is fought. As a result, the next entry might be not at i_9, but i_7.

As a final note, let us return to the initial assumption that if the entrant decided not to enter town i, he could not change his mind later. We have seen that no towns will be entered until near the last one, because the incumbent wants to protect his reputation for strength. But if the entrant can change his mind, the last town is never approached. The entrant knows he would take losses in the first $(N - 30)$ towns, and it is not worth his while to reduce the number to 30 to make the monopolist choose *Collude*. Paradoxically, allowing the entrant many chances to enter helps not him, but the incumbent.

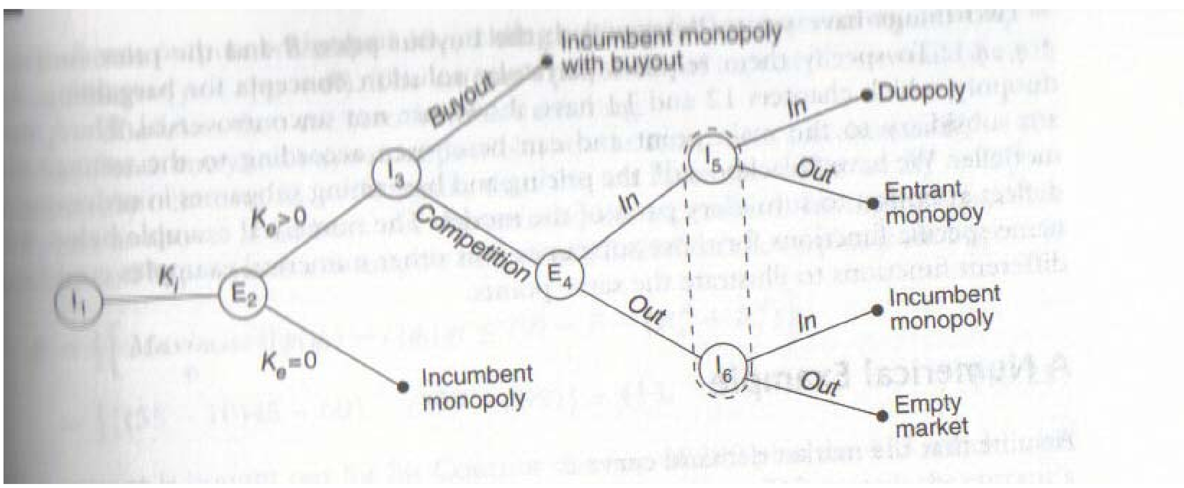
15.4 *Entry for Buyout

The previous section suggested that predatory pricing might actually be a credible threat if information were slightly incomplete, because the incumbent might be willing to makes losses fighting the first entrant to deter future entry. This is not the end of the story, however, because even if entry costs exceed operating revenues, entry might still be profitable if the entrant is bought out by the incumbent.

To see this most simply, let us start by thinking about how entry might be deterred under complete information. The incumbent needs some way to precommit himself to unprofitable post-entry pricing. Spence (1977) and Dixit (1980) suggest that the incumbent could enlarge his initial capacity to make the post-entry price naturally drop to below average cost. The post-entry price would still be above average variable cost, so having already sunk the capacity cost the incumbent fights entry without further expense. The entrant's capacity cost is not yet sunk, so he refrains from entry.

In the model with the extensive form of figure 15.4, the incumbent has the additional option of buying out the entrant. An incumbent who fights entry bears two costs: the loss from selling at a price below average total cost, and the opportunity cost of not earning monopoly profits. He can make the first a sunk cost, but not the second. The entrant, foreseeing that the incumbent will buy him out, enters despite knowing that the duopoly price will be less than average total cost. The incumbent faces a second perfectness problem, for while he may try to deter entry by threatening not to buy out the entrant, the threat is not credible.

Figure 15.4 Entry for Buyout



Entry for Buyout
(Rasmusen [1988a])

Players
The incumbent and the entrant.

The Order of Play
1 The incumbent selects capacity K_i.
2 The entrant decides whether to enter or stay out, choosing a capacity $K_e \geq 0$.
3 If the entrant picks a positive capacity, the incumbent decides whether to buy him out at price B.
4 If the entrant has been bought out, the incumbent selects output $q_i \leq K_i + K_e$.
5 If the entrant has not been bought out, each player decides whether to stay in the market or exit.
6 If a player has remained in the market, he selects the output $q_i \leq K_i$ or $q_e \leq K_e$.

Payoffs
Each unit of capacity costs a, the constant marginal cost is c, a firm that stays in the market incurs fixed cost F, and there is no discounting. There is only one period of production.
If no entry occurs, $\pi_{inc} = [p(q_i) - c]q_i - aK_i - F$ and $\pi_{ent} = 0$.
If entry occurs and is bought out, $\pi_{inc} = [p(q_i) - c]q_i - aK_i - B - F$ and $\pi_{ent} = B - aK_e$.
Otherwise,

$$\pi_{incumbent} = \begin{cases} [p(q_i, q_e) - c]q_i - aK_i - F & \text{if the incumbent stays.} \\ -aK_i & \text{if the incumbent exits.} \end{cases}$$

$$\pi_{entrant} = \begin{cases} [p(q_i, q_e) - c]q_e - aK_e - F & \text{if the entrant stays.} \\ -aK_e & \text{if the entrant exits.} \end{cases}$$

Two things have yet to be specified: the buyout price B and the price function $p(q_i, q_e)$. To specify them requires particular solution concepts for bargaining and duopoly, which chapters 12 and 14 have shown are not uncontroversial. Here, they are subsidiary to the main point and can be chosen according to the taste of the modeller. We have "blackboxed" the pricing and bargaining subgames in order not to deflect attention to subsidiary parts of the model. The numerical example below will name specific functions for those subgames, but other numerical examples could use different functions to illustrate the same points.

A Numerical Example

Assume that the market demand curve is

$$p = 100 - q_i - q_e. \tag{18}$$

Let the cost per unit of capacity be $a = 10$, the marginal cost of output be $c = 10$, and the fixed cost be $F = 601$. Assume that output follows Cournot behavior and the bargaining solution splits the surplus equally, in accordance with the Nash bargaining solution and Rubinstein (1982).

If the incumbent faced no threat of entry, he would behave as a simple monopolist, choosing a capacity equal to the output which solved

$$\underset{q_i}{Maximize} \; (100 - q_i)q_i - 10q_i - 10q_i. \tag{19}$$

Problem (15.19) has the first-order condition

$$80 - 2q_i = 0, \tag{20}$$

so the monopoly capacity and output would both equal 40, yielding a net operating revenue of 1,399 ($= [p - c]q_i - F$), well above the capacity cost of 400.

We will not go into details, but under these parameters the incumbent chooses the same output and capacity of 40 even if entry is possible but buyout is not. If the potential entrant were to enter, he could do no better than to choose $K_e = 30$, which costs 300. With capacities $K_i = 40$ and $K_e = 30$, Cournot behavior leads the two firms to solve

$$\underset{q_i}{Maximize} \; (100 - q_i - q_e)q_i - 10q_i \;\; s.t. \;\; q_i \leq 40 \tag{21}$$

and

$$\underset{q_e}{Maximize}\ (100 - q_i - q_e)q_e - 10q_e \ \ s.t.\ \ q_e \leq 30, \tag{22}$$

which have first order conditions

$$90 - 2q_i - q_e = 0 \tag{23}$$

and

$$90 - q_i - 2q_e = 0. \tag{24}$$

The Cournot outputs both equal 30, yielding a price of 40 and net revenues of $R_i^d = R_e^d = 299$ $(= [p - c]q_i - F)$. The entrant's profit net of capacity cost would be -1 $(= R_e^d - 30a)$, less than the zero from not entering.

What if both entry and buyout are possible, but the incumbent still chooses $K_i = 40$? If the entrant chooses $K_e = 30$ again, then the net revenues would be $R_e^d = R_i^d = 299$, just as above. If he buys out the entrant, the incumbent, having increased his capacity to 70, produces a monopoly output of 45. Half of the surplus from buyout is

$$\begin{aligned} B &= 1/2\left[\underset{q_i}{Maximize}\ \{[p(q_i) - c]q_i | q_i \leq 70\} - F - (R_e^d + R_i^d)\right] \\ &= 1/2[(55 - 10)45 - 601 - (299 + 299)] = 413. \end{aligned} \tag{25}$$

The entrant is bought out for his Cournot revenue of 299 plus the 413 which is his share of the buyout surplus, a total buyout price of 712. Since 712 exceeds the entrant's capacity cost of 300, buyout induces entry which would otherwise have been deterred. Nor can the incumbent deter entry by picking a different capacity. Choosing any K_i greater than 30 leads to the same Cournot output of 60 and the same buyout price of 712. Choosing K_i less than 30 allows the entrant to make a profit even without being bought out.

Realizing that entry cannot be deterred, the incumbent would choose a smaller initial capacity. A Cournot player whose capacity is less than 30 would produce right up to capacity. Since buyout will occur, if a firm starts with a capacity less than 30 and adds one unit, the marginal cost of capacity is 10 and the marginal benefit is the increase (for the entrant) or decrease (for the incumbent) in the buyout price. If it is the entrant who adds a unit of capacity, the net revenue R_e^d rises by at least $(40 - 10)$, the lowest possible Cournot price minus the marginal cost of output. Moreover, R_i^d falls because the entrant's extra output lowers the market price, so under our bargaining solution the buyout price rises by more than 15 $(= \frac{40-10}{2})$ and the entrant should add extra capacity up to $K_e = 30$. A parallel argument shows why the incumbent should build a capacity of at least 30. Increasing the capacities any further leaves the buyout price unchanged, because the duopoly net revenues are unaffected, so both firms choose exactly 30.

The industry capacity equals 60 when buyout is allowed, but after the buyout only 45 is used. Industry profits in the absence of possible entry would have been 999 $(= 1,399 - 400)$, but with buyout they are 824 $(= 1,424 - 600)$, so buyout has decreased industry profits by 175. Consumer surplus has risen from 800 ($= 0.5[100 - p(q|K = 40)][q|K = 40]$) to 1,012.5 $(= 0.5[100 - p(q|K = 60)][q|K = 60])$, a gain of 212.5, so buyout raises total welfare in this example. The increase in output outweighs the inefficiency of the entrant's investment in capacity, an outcome that depends on the particular parameters chosen.

The model is a tangle of paradoxes. The central paradox is that the ability of the incumbent to destroy industry profits after entry ends up hurting him rather than helping because it increases the buyout price. This has a similar flavor to the "judo economics" of Gelman & Salop (1983): the incumbent's very size and influence weighs against him. In the numerical example, allowing the incumbent to buy out the entrant raised total welfare, even though it solidified monopoly power and resulted in wasteful excess capacity. Under other parameters, the effect of excess capacity dominates, and allowing buyout would lower welfare – but only because it encourages entry, of which we usually approve. Adding more potential entrants would also have perverse effects. If the incumbent's excess capacity can deter one entrant, it can deter any number. We have seen that a single entrant might enter anyway, for the sake of the buyout price. But if there are many potential entrants, it is easier to deter entry. Buying out a single entrant would not do the incumbent much good, so he would only be willing to pay a small buyout price, and the small price would discourage any entrant from being the first. The game becomes complicated, but clearly the multiplicity of potential entrants makes entry more difficult for any of them.

Notes

N15.1 Innovation and patent races

- The idea of the patent race is described by Barzel (1968), although his model showed the same effect of overhasty innovation even without patents.

- Reinganum (1985) has shown that an important element of patent races is whether increased research hastens the arrival of the patent or just affects whether it is acquired. If more research hastens the innovation, then the incumbent might spend less than the entrant because the incumbent is enjoying a stream of profits from his present position that the new innovation destroys.

- **Uncertainty in innovation.** Patent Race for an Old Market, is only one way to model innovation under uncertainty. A more common way is to use continuous time with discrete discoveries and specifies that discoveries arrive as a Poisson process with parameter $\lambda(X)$, where X is research expenditure, $\lambda' > 0$, and $\lambda'' < 0$, as in Loury (1979) and Dasgupta & Stiglitz (1980). Then

$$\begin{array}{ll} Prob(\text{invention at } t) & = \lambda e^{-\lambda(X)t}; \\ Prob(\text{invention before } t) & = 1 - e^{-\lambda(X)t}. \end{array} \tag{26}$$

A little algebra gives us the current value of the firm, R_0, as a function of the innovation rate, the interest rate, the post-innovation value V_1, and the current revenue flow R_0. The return on the firm equals the current cash flow plus the probability of a capital gain.

$$rV_0 = R_0 - X + \lambda(V_1 - V_0), \tag{27}$$

which implies

$$V_0 = \frac{\lambda V_1 + R_0 - X}{\lambda + r}. \tag{28}$$

Expression (15.28) is frequently useful.

- A common theme in entry models is what has been called the **fat-cat effect** by Fudenberg & Tirole (1986a, p. 23). Consider a two-stage game, in the first stage of which an incumbent firm chooses its advertising level and in the second stage plays a Bertrand subgame with an entrant. If the advertising in the first stage gives the incumbent a base of captive customers who have inelastic demand, he will choose a higher price than the entrant. The incumbent has become a "fat cat." The effect is present in many models. In section 14.3's Hotelling Pricing Game a firm located so that it has a large "safe" market would choose a higher price. In section 5.5's Customer Switching Costs a firm that has old customers locked in would choose a higher price than a fresh entrant in the last period of a finitely repeated game.

N15.2 Predatory Pricing: the Kreps-Wilson Model

- For other expositions of this model see pages 77-82 of Martin (1993) 239-243 of Osborne & Rubinstein (1994).
- Kreps & Wilson (1982a) do not simply assume that one type of monopolist always chooses *Fight*. They make the more elaborate but primitive assumption that his payoff function makes fighting a dominant strategy. Table 15.6 shows a set of payoffs for the strong monopolist which generate this result.

Table 15.6 Predatory Pricing with a dominant strategy

		Strong Incumbent	
		Collude	*Fight*
Entrant	*Enter*	20,10	$-10, 40$
	Stay out	$0, 100$	**0,100**

Payoffs to: (Entrant, Incumbent)

Under the Kreps-Wilson assumption, the strong monopolist would actually choose to collude in the early periods of the game in some perfect Bayesian equilibria. Such an equilibrium could be supported by out-of-equilibrium beliefs that the authors point out are absurd: if the monopolist fights in the early periods, the entrant believes he must be a weak monopolist.

Problems

15.1: Crazy Predators (adapted from Gintis [forthcoming], Problem 12.10.)
Apex has a monopoly in the market for widgets, earning profits of m per period, but Brydox has just entered the market. There are two periods and no discounting. Apex can either *Prey* on Brydox with a low price or accept *Duopoly* with a high price, resulting in profits to Apex of $-p_a$ or d_a and to Brydox of $-p_b$ or d_b. Brydox must then decide whether to stay in the market for the second period, when Brydox will make the same choices. If, however, Professor Apex, who owns 60 percent of the company's stock, is crazy, he thinks he will earn an amount $p^* > d_a$ from preying on Brydox (and he doesn not learn from experience). Brydox initially assesses the probability that Apex is crazy at θ.

15.1a Show that under the following condition, the equilibrium will be separating, i.e., Apex will behave differently in the first period depending on whether the Professor is crazy or not:

$$-p_a + m < 2d \tag{29}$$

15.1b Show that under the following condition, the equilibrium can be pooling, i.e., Apex will behave the same in the first period whether the Professor is crazy or not:

$$\theta \geq \frac{d_b}{p_b + d_b} \tag{30}$$

15.1c If neither of the two conditions above applies, the equilibrium is hybrid, i.e., Apex will use a mixed strategy and Brydox may or may not be able to tell whether the Professor is crazy at the end of the first period. Let α be the probability that a sane Apex preys on Brydox in the first period, and let β be the probability that Brydox stays in the market in the second period after observing that Apex chose Prey in the first period. Show that equilibrium values of α and β are:

$$\alpha = \frac{\theta d_b}{(1-\theta)p_b} \tag{31}$$

$$\beta = \frac{-p_a + m - 2d_a}{m - d_a} \tag{32}$$

15.1d Is this behavior related to any of the following phenomenon: signalling, signal jamming, reputation, efficiency wages?

15.2: Rent Seeking
I mentioned that Rogerson (1982) uses a game very similar to "Patent Race for a New Market" to analyze competition for a government monopoly franchise. See if you can do this too. What can you predict about the welfare results of such competition?

15.3: A Patent Race
See what happens in Patent Race for an Old Market when specific functional forms and parameters are assumed. Set $f(x) = log(x)$, $g(y) = 0.5(1 + y/(1+y)$ if $y \geq 0$, $g(y) = 0.5(1 + y/(1-y)$ if $y \leq 0$, $y = 2$, and $z = 1$. Figure out the research spending by each firm for the three cases of (a) $v = 10$, (b) $v = 4$, (c) $v = 2$ and (d) $v = 1$.

15.4: Entry for Buyout
Find the equilibrium in Entry for Buyout if all the parameters of the numerical example are the same except that the marginal cost of output is $c = 20$ instead of $c = 10$.